Eng. Math

ENG 1005

Week 2: Systems of Linear Eqs.

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

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Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

The Big Learning Outcomes for Week 2

After completing this week's task, you should be able to:

- Solve systems of linear equations using Gaussian elimination with back substitution.
- Understand augmented matrix notation for systems of linear equations.
- Use elementary row operations to put a matrix into (i) *row echelon form (REF)* and (ii) *reduced row echelon form (RREF).*
- Find the solution space for a linear system of equations.
- Compute the rank of a matrix and understand its meaning.

Attendance Codes (Week 2)

International students

Tutorial	Wednesday, 31 Jul	02	8:00AM	M9X5B
Tutorial	Wednesday, 31 Jul	01	2:00PM	5GN86

Workshop	Thursday, 1 Aug	01	1:00PM	X5LZD
Workshop	Friday, 2 Aug	02	10:00AM	SFZ4P

Admin. Stuff (1)

1. Feedback on Workshop Submission: On Thur/Fri

2. PASS Sessions: More on Thur/Fri

PASS	01	Mon	12:00	5,
PASS	02	Wed	10:00	7,

3. Pls join our MS TEAM group: (90%)_Email me

4. Submissions:

Summary

ASSESSMENT	DUE
Lecture Quiz 2 (Total mark for all 12 weeks of lecture quizzes is 5%)	Tuesday, 30 July 2024, 11:55 PM **Due in 2 days**
Applied class quiz week 2 (Total mark for all 12 weeks of applied quizzes is 5%)	Saturday, 3 August 2024, 5:55 AM
Workshop 2 problem set (Total mark for all 12 weeks of workshop sets is 20%)	Sunday, 4 August 2024, 11:55 PM

Admin. Stuff (2)

4. Consultation/Feedback hour

- Wed: 10 am till 11 am
- Fri: 8 am till 9 am
- Sat: 1030 am till 1130 am

(every odd week, e.g. 3-Week, but msg me first at least a day before; appointment-based.)

Location: 5-4-68

Admin. Stuff (3)

5. Typo in your submitted quizzes, let me know

Pit-Stop: Gaussian Elimination

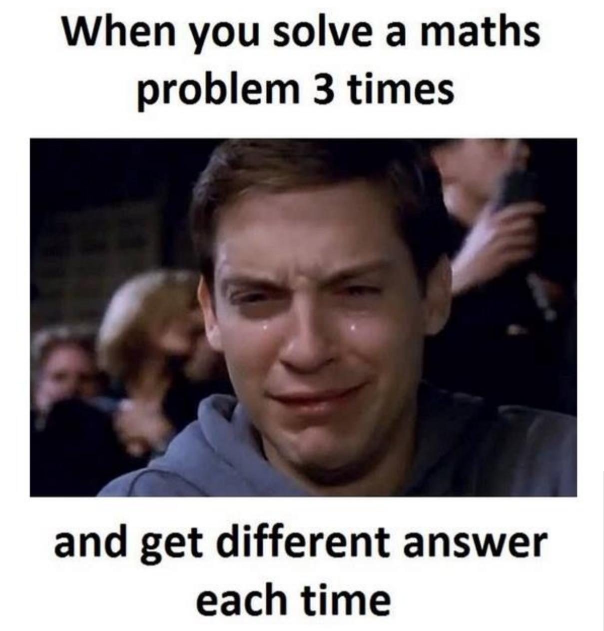
Basic Algorithm in Computational Software

- COMSOL
- Ansys
- SolidWorks

Today's Activity

1. Applied Problem Set

2. Applied Quiz



Question 1

Three shoppers bought fruit at a grocery store. Hewey bought 3 figs, 2 guavas, and 1 honeydew and paid 8 dollars. Dewey bought 1 fig, 1 guava, and 1 honeydew and paid 4 dollars. Louis bought 10 figs, 2 guavas, and 2 honeydews and paid 16 dollars.

How much does each individual item cost?

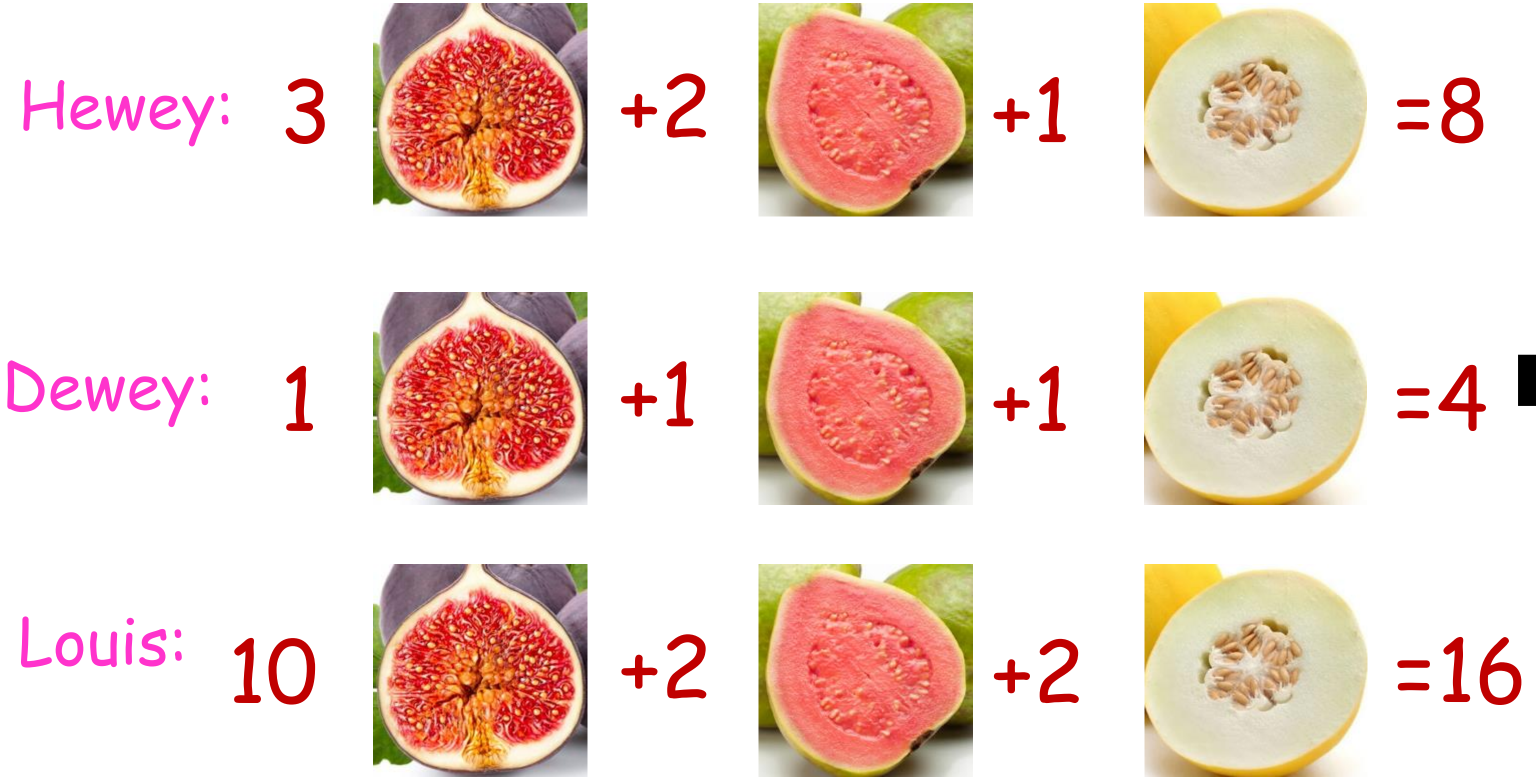
Learning Outcomes?

Construct & Solve: a system of linear equations via RREF.

In short (4-step solution)

- Step 1: Setting up (your stall);
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination;
- Step 4: Back substitution.

- Step 1: Setting up (your stall)



Systems of equations

$$\boxed{\begin{aligned} 3f + 2g + h &= 8, \\ f + g + h &= 4, \\ 10f + 2g + 2h &= 16. \end{aligned}}$$

Three shoppers bought fruit at a grocery store. Hewey bought 3 figs, 2 guavas, and 1 honeydew and paid 8 dollars. Dewey bought 1 fig, 1 guava, and 1 honeydew and paid 4 dollars. Louis bought 10 figs, 2 guavas, and 2 honeydews and paid 16 dollars.

- Step 2: Augmented Form

$$\begin{aligned} 3f + 2g + h &= 8, \\ f + g + h &= 4, \\ 10f + 2g + 2h &= 16. \end{aligned} \quad \Longrightarrow \quad \begin{array}{c} \begin{array}{ccc|c} f & g & h & \end{array} \\ \left[\begin{array}{ccc|c} 3 & 2 & 1 & 8 \\ 1 & 1 & 1 & 4 \\ 10 & 2 & 2 & 16 \end{array}\right] . \end{array}$$

- Put it inside a [] bracket;
- A vertical line between the equal sign =.
- Working with number only, no variables in it!

- Step 3: Gaussian Elimination (REF)

$$\begin{bmatrix} 3 & 2 & 1 & | & 8 \\ 1 & 1 & 1 & | & 4 \\ 10 & 2 & 2 & | & 16 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

(REF)

As a step i, do $\boxed{R_1 \leftrightarrow R_2}$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 3 & 2 & 1 & | & 8 \\ 10 & 2 & 2 & | & 16 \end{bmatrix}$$

$R_1 \rightarrow R_2$

(NEW) (OLD)

$R_2 \rightarrow R_1$

(NEW) (OLD)

As the step ii, maybe

From step i

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 8 \\ 10 & 2 & 2 & 16 \end{array}\right]$$

(NEW)

$$\boxed{\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 10R_1 \end{array}}$$

(OLD)

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -4 \\ 0 & -8 & -8 & -24 \end{array}\right]$$

Previously:

From step i

$$R_2 \to R_2 - 3R_1$$
$$R_3 \to R_3 - 10R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 8 \\ 10 & 2 & 2 & 16 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -4 \\ 0 & -8 & -8 & -24 \end{array}\right]$$

As the step iii, try

From step ii

$$R_3 \to R_3 - 8R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -4 \\ 0 & -8 & -8 & -24 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & 8 & 8 \end{array}\right]$$

As the step iv, maybe

From step iii

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & 8 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to -R_2 \\ R_3 \to R_3/8 \end{array}$$

(REF)

- Step 4: Back substitution

Inverse the top and bottom sets, before solving from top to bottom!



$$
\begin{array}{ccc}
f & g & h \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 4 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 1
\end{array}\right]
\rightarrow
\boxed{\begin{aligned}
h &= 1, \\
g + 2h &= 4, \\
f + g + h &= 4.
\end{aligned}}
$$

Advantage of REF?

$\boxed{\text{honeydews cost \$1}}$

$\boxed{\text{guavas cost \$2}}$

fig?

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In short (4-step solution)

- Step 1: Setting up (your stall);
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination;
- Step 4: Back substitution.

Question 2

Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= 12 \\ 2x_1 + 4x_2 + 12x_3 &= -18 \\ x_1 - 4x_2 - 12x_3 &= b \end{aligned}$$

For what value(s) of b (if any) is this system (i) consistent and (ii) inconsistent? For what value(s) of b (if any) does the system have a unique solution?

Learning Outcomes?

Consistent or Inconsistent?

Let's us judge using its REF

In short (3-step solution)

$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= 12 \\ 2x_1 + 4x_2 + 12x_3 &= -18 \\ x_1 - 4x_2 - 12x_3 &= b \end{aligned}$$

- Step 1: Setting up;
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination;

(REF)

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & 8 & 24 & -42 \\ 0 & 0 & 0 & 4b-90 \end{array}\right]$$

As a first step, do

$$\boxed{\begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 - R_1 \end{aligned}}$$

As the 2nd step, maybe

$$\boxed{R_3 \to 4R_3}$$

As the last step, try

$$\boxed{R_3 \to R_3 + R_2}$$

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- What b value to adopt?

$$\boxed{\left[\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & 8 & 24 & -42 \\ 0 & 0 & 0 & 4b-90 \end{array}\right]}$$

Consistent: $4b - 90 = 0$

Inconsistent: $4b - 90 \neq 0$

Question 3

The reduced row echelon form of an augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

for a system with unknowns x, y and z. Give the solution space.

Learning Outcomes?

All possible solutions

- Step 1: Understand the augmented echelon form;

❑ We have (x, y, z) 3 variables but only 2 equations

➢ Not Good! Not enough info: it must be *at least* 3 set of eqs.

❑ But we know that: x=4 & y=-5, let's write a possible ans.

➢ x=(4, -5, 0)

❑ Is that all the possible ans.?

➢ x(*s*)=(4, -5, 0)+*s*(0, 0, 1)

This matrix: infinitely many solutions, but why?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Question 4

Bruce is twice the weight of Alex. Doris balances Alex and Chris. Doris and Bruce combined weigh twice as much as Chris. Altogether, the group weighs 500 kg. Do you have enough information to determine everyone's weight? If so, how much does each person weigh?

Learning Outcomes?

English to Math

In short (4-step solution)

$$
\begin{aligned}
B &= 2A \\
D &= A + C \\
D + B &= 2C \\
A + B + C + D &= 500
\end{aligned}
$$

- Step 1: Setting up; *A=Alex; B=Bruce; C=Chris; D=Doris*
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination (try at home),

$$
\left[\begin{array}{cccc|c}
1 & 0 & 1 & -1 & 0 \\
0 & -1 & -2 & 2 & 0 \\
0 & 0 & -4 & 3 & 0 \\
0 & 0 & 0 & 5 & 1000
\end{array}\right]
$$

- Step 4: Back substitution, ***Alex weighs 50kg***

Bruce is twice the weight of Alex. Doris balances Alex and Chris. Doris and Bruce combined weigh twice as much as Chris. Altogether, the group weighs 500 kg. Do you have enough information to determine everyone's weight? If so, how much does each person weigh?

Question 5

Determine whether or not there exists a 3-row, 5-column augmented matrix

(a) whose corresponding system of linear equations is consistent and
(b) whose corresponding system of linear equations is inconsistent
(c) in row echelon form with one pivot
(d) in row echelon form with four pivots
(e) in row echelon form with 12 non-zero entries
(f) in reduced row echelon form with one non-zero entry
(g) in reduced row echelon form with 10 non-zero entries
(h) that generates a unique solution

In each case, if such an augmented matrix exists, write down an example, and if such an augmented matrix does not exist, give a clear explanation why.

Learning Outcomes?

Type of systems of linear eqs.

- Inconsistent

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} \end{array}\right]$$

- Consistent

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

- But?

Question 5

Determine whether or not there exists a 3-row, 5-column augmented matrix

(a) whose corresponding system of linear equations is consistent and
(b) whose corresponding system of linear equations is inconsistent
(c) in row echelon form with one pivot
(d) in row echelon form with four pivots
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(g) in reduced row echelon form with 10 non-zero entries
(h) that generates a unique solution

In each case, if such an augmented matrix exists, write down an example, and if such an augmented matrix does not exist, give a clear explanation why.

See lecture video: Topic 6 Pivot & 9 *Rank*

Question 6(a)

(a) Compute the rank of the matrix

$$\begin{pmatrix} 1 & -1 & -4 & 5 \\ 3 & -1 & -11 & 12 \\ 1 & -5 & 2k-5 & 11 \end{pmatrix}$$

How does it depend on k?

Learning Outcomes?

Determining non-zero rows; solvable?

In short (3-step solution)

- Step 1: Setting up;
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination;

$$\begin{bmatrix} 1 & -1 & -4 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 2k+1 & 0 \end{bmatrix}$$

rank is equal to the number of non-zero rows in the matrix after Gaussian elimination

$$\boxed{\begin{bmatrix} 1 & -1 & -4 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 2k+1 & 0 \end{bmatrix}}$$

Rank 3: $k \neq -1/2$

$$\begin{bmatrix} 1 & -1 & -4 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & \neq 0 & 0 \end{bmatrix}$$

Rank 2: $k = -1/2$

$$\begin{bmatrix} 1 & -1 & -4 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank is equal to the number of non-zero rows in the matrix after Gaussian elimination

Question 7

You are given the system of equations

$$\begin{aligned} v + x - 2y + z &= 0 \\ 2v - 3z &= 0 \\ 3v + 2x + y &= 0 \end{aligned}$$

for the four unknowns v, x, y and z. *Without doing any calculations*, how many solutions would you expect this system to have? *[Hint: what do you expect happens with two versus three lines in 2D; two versus three versus four planes in 3D?]* Now solve the system.

Learning Outcomes?

No variables vs. No eqs.

In short (3-step solution)

$$\begin{aligned} v + x - 2y + z &= 0 \\ 2v - 3z &= 0 \\ 3v + 2x + y &= 0 \end{aligned}$$

- Step 1: Setting up;
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination;

$$\boxed{\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 7 & -3 & 0 \\ 0 & 0 & -10 & 1 & 0 \end{array}\right]}$$

$$\left\{\begin{array}{rl} \begin{array}{r} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} & \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 0 & -2 & 4 & -5 & 0 \\ 0 & -1 & 7 & -3 & 0 \end{array}\right] \\ R_2 \leftrightarrow R_3 & \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 7 & -3 & 0 \\ 0 & -2 & 4 & -5 & 0 \end{array}\right] \\ R_3 \to R_3 - 2R_2 & \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 7 & -3 & 0 \\ 0 & 0 & -10 & 1 & 0 \end{array}\right] \end{array}\right.$$

- Step ***:Understand the augmented echelon form;

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 7 & -3 & 0 \\ 0 & 0 & -10 & 1 & 0 \end{array}\right] \longrightarrow \begin{aligned} v + x - 2y + z &= 0, \\ -x + 7y - 3z &= 0, \\ -10y + z &= 0. \end{aligned}$$

Since there are three equations with four unknowns
after Gaussian elimination, one free parameter is required.

$$\boxed{\begin{aligned} v + x - 2y + z &= 0, \\ -x + 7y - 3z &= 0, \\ -10y + z &= 0. \end{aligned}}$$

- We have (v, x, y, z) 4 variables but only 3 eqs.
  - Not enough info; at least 4 vs. 4
- Lets have intro. one free parameter t, assuming
  - t=z
- 10y=z
  - y = z/10 = t/10
- x=7y-3z
  - x = 7(t/10)-3t= -23t/10
- v=-x+2y-z
  - v = -(23t/10)+2(t/10)-t=3t/2

$$\boxed{(v, x, y, z) = 3t/2,\ -23t/10,\ t/10,\ t}$$

Question 8

Solve the system

$$(1+i)x - iy = -3 \quad \text{and} \quad 2x - 2y = 3i$$

where $i = \sqrt{-1}$.

Learning Outcomes?

The 4 steps also applied for complex numbers

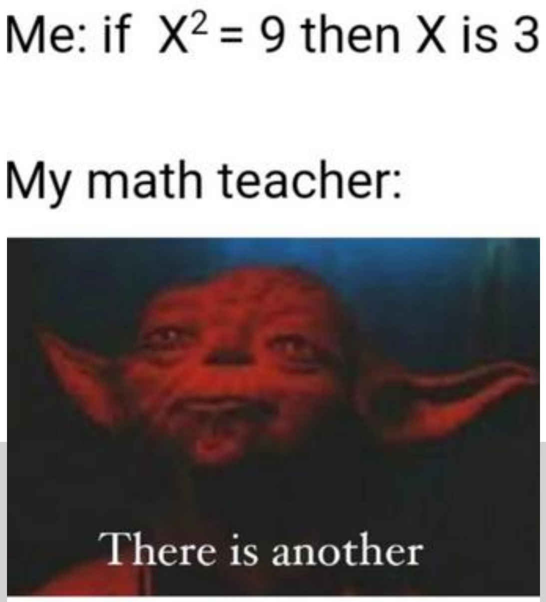
In short (4-step solution)

- Step 1: Setting up;
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination;
- Step 4: Back substitution,

Today's Activity

1. Applied Problem Set

2. Applied Quiz



Thank You

Eng. Math

(Workshop) ENG 1005

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11	Series 1
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Assessments breakdown

Task description	*Value*	*Due date*
Lecture quizzes	5%	Weekly 11:55pm the night before your applied class (except Monday classes) except in Week 1 when the due date is Friday at 11:55 pm (Malaysia time)
Applied class quizzes	5%	Weekly during your applied class
Workshop problems	20%	Weekly at 11:55pm Sunday, except for Weeks 7 (midsem)
Mid-semester exam	20%	During your workshop in Week 7

Admin. Stuff (1)

1. ENG 1090

2. Pls join our MS TEAM group: (90%)_Email me

3. Submissions:

Summary

ASSESSMENT	DUE
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- Wed: 10 am till 11 am
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(every odd week, e.g. 3-Week, but msg me first at least a day before; appointment-based.)

Location: 5-4-68

Admin. Stuff (3)

5. Typo in your submitted quizzes, let me know

Pass Session: Monday or Wednesday

Monday: 12 pm – 2 pm (6302)

Wednesday: 10 am – 12 pm (LT6008)

Feedback: Workshop 1

- comment 1: writing/drawing & resolution.

- comment 2: Arrows

- Comment 3: $\underbrace{2(\hat{\mathbf{a}}_i \cdot}_{}\, \hat{\mathbf{n}})\hat{\mathbf{n}} = \hat{\mathbf{a}}_i - \hat{\mathbf{a}}_r$

*(i) *scalar*; and (ii) operation between two vectors; either dot or cross, i.e., we can't solve*

$$\overbrace{2(\hat{\mathbf{a}}_i \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}^{}$$

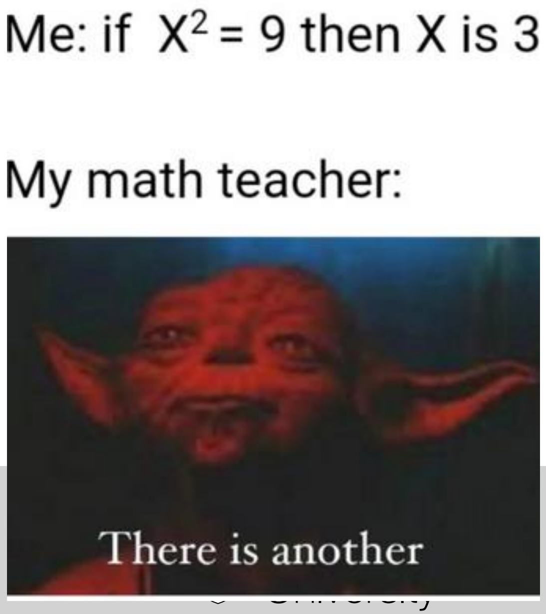
The Big Learning Outcomes for Week 2

After completing this week's task, you should be able to:

- Solve systems of linear equations using Gaussian elimination with back substitution.
- Understand augmented matrix notation for systems of linear equations.
- Use elementary row operations to put a matrix into (i) *row echelon form (REF)* and (ii) *reduced row echelon form (RREF).*
- Find the solution space for a linear system of equations.
- Compute the rank of a matrix and understand its meaning.

Today’s Activity

1. Applied Problem Set



Reminder: REF

- A REF is not unique, depending on how you do it!

- But the pivot, the first number of each row, must be the same.

Question 8

Solve the system

$$(1+i)x - iy = -3 \quad \text{and} \quad 2x - 2y = 3i$$

where $i = \sqrt{-1}$.

Learning Outcomes?

The 4 steps also applied for complex numbers

In short (3-step solution)

$$(1+i)x - iy = -3$$
$$2x - 2y = 3i$$

- Step 1: Setting up;
- Step 2: Augmented Form;

$$\left[\begin{array}{cc|c} 1+i & -i & -3 \\ 2 & -2 & 3i \end{array}\right]$$

- Step 3: Gaussian Elimination;

One step, do

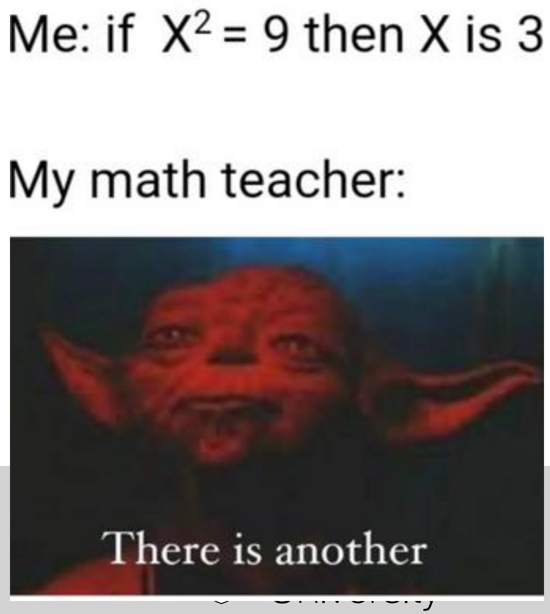
$$R_2 \to (1+i)R_2 - 2R_1$$

(REF) $\left[\begin{array}{cc|c} 1+i & -i & -3 \\ 0 & -2 & 3+3i \end{array}\right]$

$$y = -\frac{3}{2}(1+i) \quad \Rightarrow \quad x = -3/2.$$

Today’s Activity

2. Workshop Problem Set



Networks and Traffic Flow

Networks are ubiquitous in the human-made environment, from physical ones such as road networks and electrical grids to virtual ones such as the world wide web. They also exist in the natural world, from connections between neurons in the brain to the structure of fungi. Many processes occur on such networks and they can often be represented using matrices and analysed using the tools of linear algebra. In this workshop, we'll explore a traffic network and see what Gaussian elimination can tell us.

If you have been driving, you might have noticed black cables on the road from time to time. These are portable traffic counters. They are used to measure the number of cars travelling on the road in a set period of time. By having two cables, the direction of each car can be determined by the order in which the cables are run over.

In this workshop, we will consider the following traffic network consisting of one-way streets:

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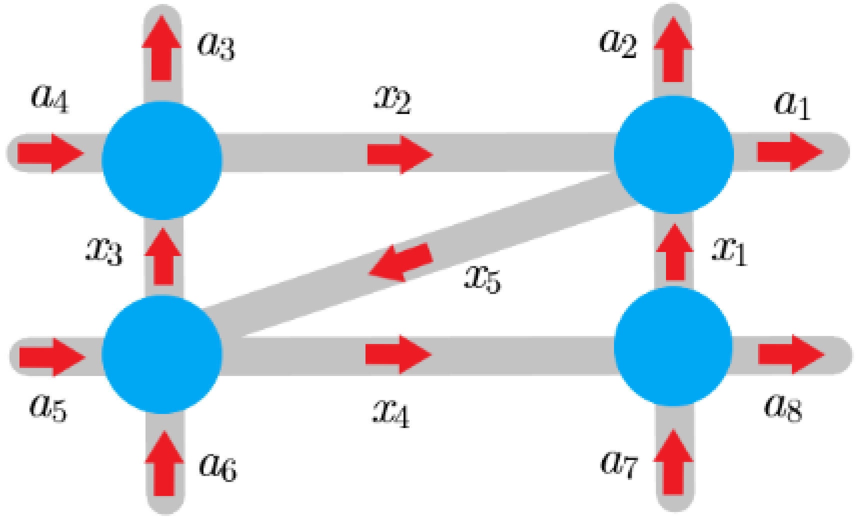


The network consists of four junctions (coloured in blue), five "internal" streets between the junctions (labelled x_1 to x_5), and eight "external" streets that connects this network to the rest of the traffic grid (labelled a_1 to a_8). The direction of traffic on each street is indicated by the red arrow on the street.

Suppose traffic counters are placed on the eight external streets, and the average numbers of cars per hours (travelling in the indicated direction) are found to be

$$a_1 = 40, \quad a_2 = 5, \quad a_3 = 15, \quad a_4 = 25, \quad a_5 = 30, \quad a_6 = 15, \quad a_7 = 10, \quad a_8 = 20$$

Using these measurements, we would like to say something about the traffic on the internal streets. Assume that the cars are always in motion in the network, i.e., there are no cars entering/leaving garages on the street, or stopping to park on the street.

1. By considering the traffic at each junction, write down four equations involving x_1 to x_5.

- Junction 1

$$x_3 + a_4 = x_2 + a_3$$

$$x_3 + 25 = x_2 + 15$$

- Junction 2

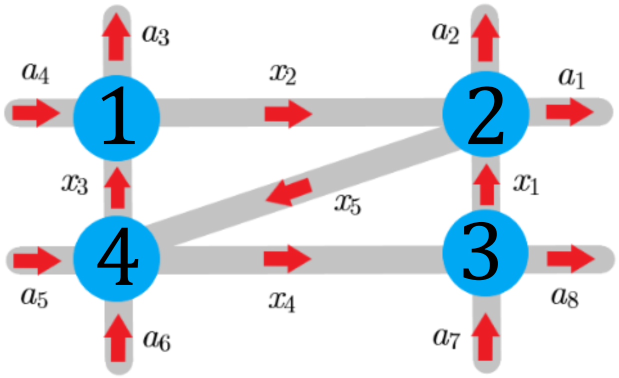
$$x_1 + x_2 = x_5 + 40 + 5$$

- Junction 3

$$x_4 + 10 = x_1 + 20$$

- Junction 4

$$x_5 + 30 + 15 = x_3 + x_4$$



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2. Without further computation, explain why the eight traffic counters are *not* sufficent to determine the traffic flow on all streets. **[1 marks]**

A traffic counter is a device or system used to measure and record the number of vehicles or pedestrians passing a certain point on a road or pathway

$$
\begin{aligned}
x_3 + 25 &= x_2 + 15 \\
x_1 + x_2 &= x_5 + 40 + 5 \\
x_4 + 10 &= x_1 + 20 \\
30 + 15 + x_5 &= x_3 + x_4
\end{aligned}
$$

- We have 5 variables but only 4 eqs.
  - Not Good! Not enough info: it must be *at least* 5 set of eqs.
- Underdetermined

3. Without further computation, how many extra traffic counters do you expect you need to determine the traffic flow on all streets? Briefly justify your answer. **[1 marks]**

❑ What do you think?

$$x_3 + 25 = x_2 + 15$$
$$x_1 + x_2 = x_5 + 40 + 5$$
$$x_4 + 10 = x_1 + 20$$
$$30 + 15 + x_5 = x_3 + x_4$$

4. Write down the system of equations in matrix form, then compute the **reduced** row echelon form of the augmented matrix. You should do this by hand and submit your working, but you may (and are highly recommended to) verify your solution using CAS or Matlab. **[4 marks]**

In short (3-step solution)

- Step 1: Setting up;
- Step 2: Augmented Form;
- Step 3: Gaussian Elimination.

$$
\begin{aligned}
x_3 + 25 &= x_2 + 15 \\
x_1 + x_2 &= x_5 + 40 + 5 \\
x_4 + 10 &= x_1 + 20 \\
30 + 15 + x_5 &= x_3 + x_4
\end{aligned}
$$

- Step 2+ 3: Augmented Form + (RREF) Gaussian Elimination.

$$\begin{aligned} x_3 + 25 &= x_2 + 15 \\ x_1 + x_2 &= x_5 + 40 + 5 \\ x_4 + 10 &= x_1 + 20 \\ 30 + 15 + x_5 &= x_3 + x_4 \end{aligned}$$

Augmented Form

Step 2

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\left[\begin{array}{ccccc|c} 0 & -1 & 1 & 0 & 0 & -10 \\ 1 & 1 & 0 & 0 & -1 & 45 \\ -1 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & -1 & -1 & 1 & -45 \end{array}\right]$$

RREF

Step 3

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & 0 & 1 & -1 & 55 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. Hence write down the solution for the average number of cars per hour on each street (you may have parameters in your solution). **[2 marks]**

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & 0 & 1 & -1 & 55 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- We have 5 variables but only 3 equations

- Not Good! Not enough info: it must be *at least* 5 set of eqs.

- 3 pivots and 2 free variables

5. Hence write down the solution for the average number of cars per hour on each street (you may have parameters in your solution). **[2 marks]**

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & 0 & 1 & -1 & 55 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 55 \\ 45 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

6. How many extra traffic counters do you need to completely determine all the traffic flow in the networks? Where should you place the extra counters? Make sure you explain and justify your answers.

- Refer to Q5.

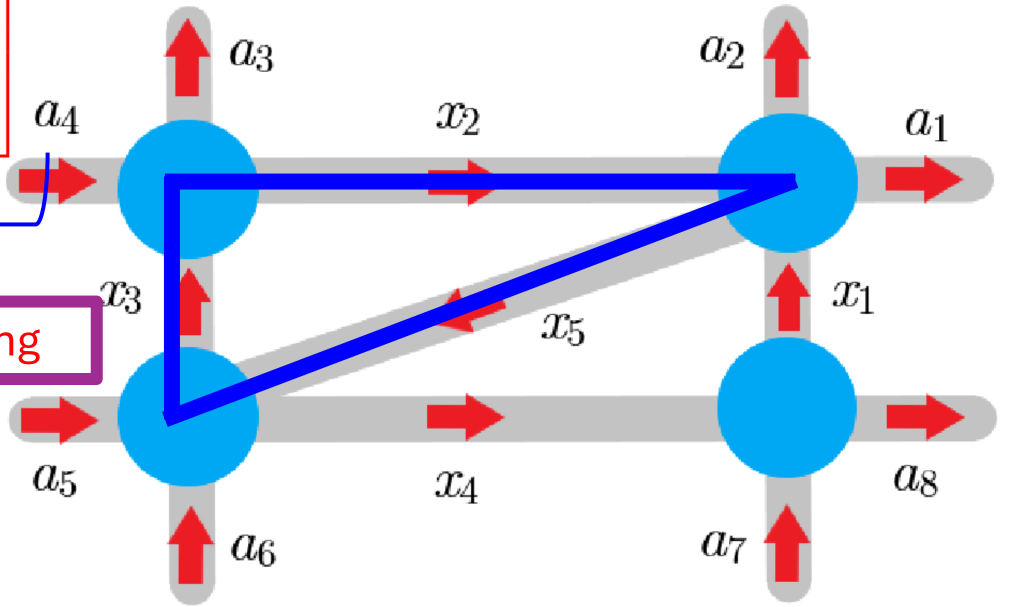
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 55 \\ 45 \\ 0 \\ 0 \end{bmatrix} + \boxed{s} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \boxed{t} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- All traffic flow can be determined if *s* and *t* are known!

7. Your solution should have some $\boxed{\text{parameters.}}$ Give an explanation for what the vectors associated with these parameters represent. To answer this, you may find it useful to draw the road network and add the traffic on each road for different values of the parameters (varying them separately). **[2 marks]**

- 2 Loops? How can it help us?
- Connection between the internal roads

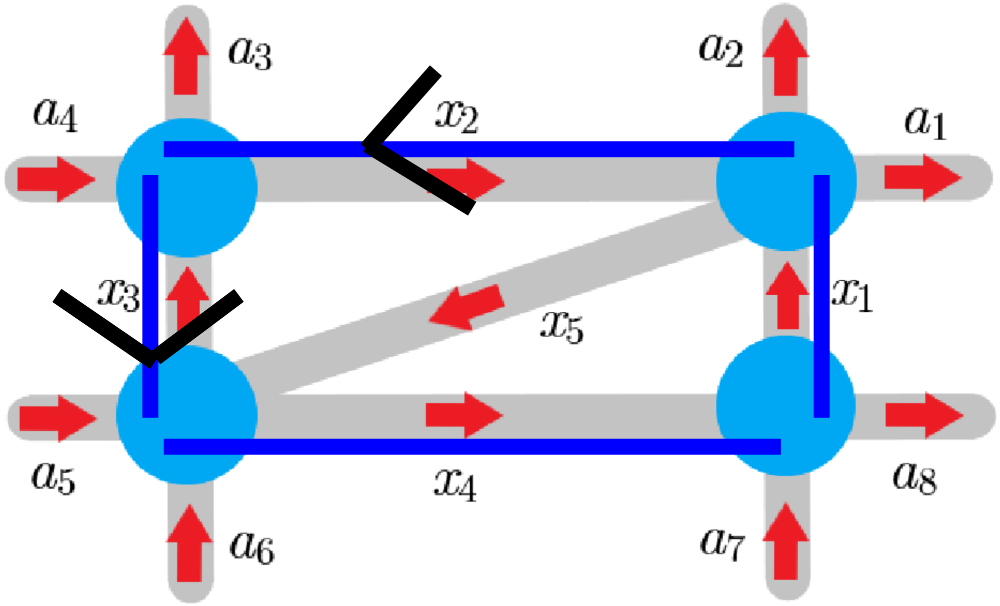
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -10 \\ 55 \\ 45 \\ 0 \\ 0 \end{bmatrix}}_{\text{Constant}} + s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \underbrace{t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Varying}}$$



$$\boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

Parameter: a set that defines a system

❑ Next loop?



$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 55 \\ 45 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

8. Not every solution in the full solution space corresponds to a realistic traffic flow. In particular we require that $x_1 \ldots x_5$ to be all non-negative. What is the *minimum* number of cars per hour on the street x_4 in real life? **[2 marks]**

- All the x_i to be ≥ 0; $i \subset \{1, \ldots, 5\}$;

- Minimum car for x_4 to be realized.

$$\begin{bmatrix} x_1 \\ \\ \\ \\ x_5 \end{bmatrix} \quad \begin{bmatrix} -10 \\ \\ \\ \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ \\ \\ \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = x_4 - 10.$$

9. In real life, what is the minimum total traffic possible in the network? In other words, what is the minimum of $x_1 + x_2 + x_3 + x_4 + x_5$? Find the traffic flow when the minimum is realized and sketch the network labelling the traffic on all the streets with direction. **[2 marks]**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 55 \\ 45 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- Find: *s* & *t* values
- $x_1 + x_2 + x_3 + x_4 + x_5 = ?$
- i.e., *s* & *t* values that make eq. above minimum.
- You can use the *s* from Q8!

- Check and illustrate it, as the final step!

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & 0 & 1 & -1 & 55 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The Big Learning Outcomes for Week 2

After completing this week's task, you should be able to:

- Solve systems of linear equations using Gaussian elimination with back substitution.
- Understand augmented matrix notation for systems of linear equations.
- Use elementary row operations to put a matrix into row echelon form and reduced row echelon form.
- Find the solution space for a linear system of equations.
- Compute the rank of a matrix and understand its meaning.

Thank You