ENG1005 Week 3: Applied class problem sheet solutions

Question 1

Matrix	A	B	C	D	E
Rows	2	3	3	3	2
Columns	2	1	3	3	1

Matrices can only be added together if they have the same dimensions. The only pair of matrices with the same dimensions are C and D, so only $\mathsf{C}+\mathsf{D}$ and $\mathsf{D}+\mathsf{C}$ are defined.

The product of two matrices FG only exists if the number of columns in F is the same as the number of rows in G. Thus the matrix products that are defined are AE, CB, CD, DB and DC.

Question 2

For the following transformation matrices

$$\mathsf{A} = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathsf{C} = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}, \quad \mathsf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

A corresponds to a 30° rotation anticlockwise around the origin, B corresponds to a shear, C corresponds to a projection onto the vector $(3/5, 4/5)$ and D corresponds to a reflection over the x-axis. The image is transformed by rotating 30° anticlockwise around the origin, followed by a reflection over the x-axis. Therefore the transformation matrix required is DA.

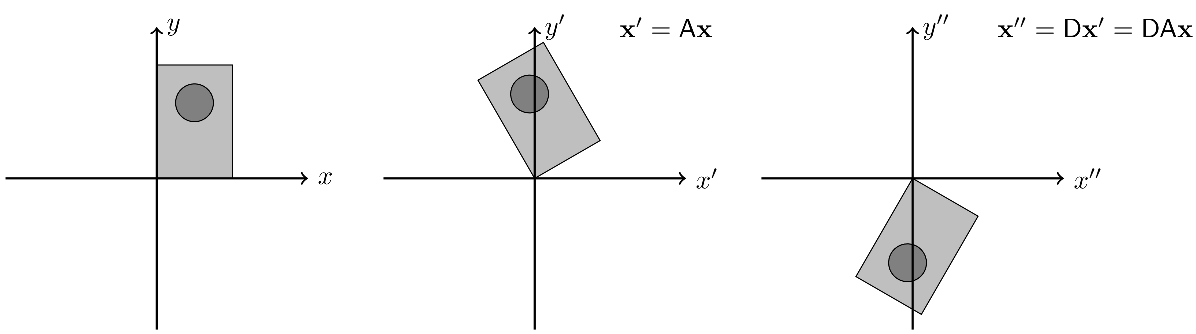

Figure 1: Sequence of transformations for Q1.

Question 3

We are given that a transformation matrix T satisfies

$$\mathsf{T}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathsf{T}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

We set

$$\mathsf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Substituting into both equations we have that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

This gives a system of four simultaneous equations,

$$a = 2, \quad c = 4, \quad -a + b = 0, \quad -c + d = -2.$$

Solving these gives $a = 2, b = 2, c = 4, d = 2$ so the transformation matrix is

$$\mathsf{T} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}.$$

Question 4

For the matrix

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

with $\det(\mathsf{A}) = 4$, using properties of the determinant we can find determinants of the following matrices,

$$\mathsf{B} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \quad \mathsf{C} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathsf{D} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}, \quad \mathsf{E} = \begin{bmatrix} 4 & -1 & 1 \\ -2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}.$$

To get matrix B, the second row of A is replaced with the sum of the second and third rows of A. Adding a multiple of one row to another does not change the determinant so $\det(\mathsf{B}) = 4$.

To get matrix C, the first column of A is replaced with the sum of the first and second columns of A. Adding a multiple of one column to another does not change the determinant so $\det(\mathsf{C}) = 4$.

To get matrix D, the first and third rows of A are switched. Each row or column swap changes the sign of the determinant so $\det(\mathsf{D}) = -4$.

To get matrix E, the first column of A is multiplied by 2. Multiplying a row or column by a scalar multiplies the determinant by the same scalar, so $\det(\mathsf{E}) = 8$.

Question 5

We have that for $n \geq 1$,

$$\mathsf{A} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad \mathsf{A}^2 = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathsf{A}^3 = \begin{bmatrix} 27 & 0 \\ 0 & -8 \end{bmatrix}, \quad \mathsf{A}^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}.$$

Question 6

We have that

$$\mathsf{A} = \begin{bmatrix} 0 & -2 & 7 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{A}^2 = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{A}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For an $n \times n$ upper triangular matrix with zeros on the leading diagonal $\mathsf{A}^m = \mathsf{0}$ if $m \geq n$ as each multiplication results in one more diagonal of zeros. Since there are $n-1$ non-zero diagonals in A, multiplying A by itself n times gives the zero matrix with the same dimensions as A.

Question 7

We have that for $n \geq 1$,

$$\mathsf{A} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad \mathsf{A}^2 = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}, \quad \mathsf{A}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}, \quad \mathsf{A}^n = \begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}.$$

Question 8

The determinant of a 2D/3D matrix represents the scaling factor of the area/volume when the matrix is applied as a linear transformation. If we apply two transformations in order, the scaling factor of the combined transformation is the product of the scaling factors of each transformation. Thus we would expect that $\det(\mathsf{AB}) = \det(\mathsf{A})\det(\mathsf{B})$. Using this property, for an invertible matrix, $\det(\mathsf{AA}^{-1}) = \det(\mathsf{A})\det(\mathsf{A}^{-1})$. As $\mathsf{AA}^{-1} = \mathsf{I}$ and $\det(\mathsf{I}) = 1$ since applying the identity matrix as a linear transformation returns the original matrix, it follows that $\det(\mathsf{A}^{-1}) = 1/\det(\mathsf{A})$.

Question 9

For the matrix

$$\mathsf{A} = \begin{bmatrix} -8 & 0 \\ -1 & 2 \end{bmatrix},$$

the determinant is $\det(\mathsf{A}) = -8 \times 2 - 0 \times (-1) = -16$. The inverse using the cofactor method is

$$\mathsf{A}^{-1} = \frac{1}{\det(\mathsf{A})} \begin{bmatrix} 2 & 0 \\ 1 & -8 \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} 2 & 0 \\ 1 & -8 \end{bmatrix}.$$

For the matrix

$$\mathsf{B} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 4 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

the determinant can be obtained by expanding along the first column,

$$\det(\mathsf{B}) = 4\det\left(\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}\right) = -15.$$

To find the inverse, we can use the cofactor method. The matrix of minors M has elements $m_{ij} = \det(\mathsf{S}_{ij})$ where S_{ij} is the matrix with row i and column j removed from the original matrix. For this matrix

$$\mathsf{M} = \begin{bmatrix} -5 & -1 & 1 \\ -5 & -4 & 4 \\ -15 & 0 & 15 \end{bmatrix}.$$

The cofactor matrix B has elements $c_{ij} = (-1)^{i+j} m_{ij}$. For this matrix

$$\mathsf{B} = \begin{bmatrix} -5 & 1 & 1 \\ 5 & -4 & -4 \\ -15 & 0 & 15 \end{bmatrix}.$$

The inverse of B is

$$\mathsf{B}^{-1} = \frac{1}{\det(\mathsf{B})} \mathsf{B}^{\mathsf{T}} = -\frac{1}{15} \begin{bmatrix} 5 & -5 & 15 \\ -1 & 4 & 0 \\ -1 & 4 & -15 \end{bmatrix}.$$

Alternatively, Gauss-Jordan elimination could be used on the augmented matrix $[\,\mathsf{B} \mid \mathsf{I}\,]$, which after reduction to reduced row echelon form gives the augmented matrix $[\,\mathsf{I} \mid \mathsf{B}^{-1}\,]$.