Eng. Math

ENG 1005

Week 8: O.D.E 1

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

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Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

The Big Learning Outcomes for Week 8

After completing this week's task, you should be able to:

- Categorize ODEs as linear/nonlinear and identify their order.
- Solve first-order separable ODEs.
- Solve first-order linear ODEs.

Attendance Codes (Week 8)

International students

Workshop	Thursday, 12 Sep	01	1:00PM	XLFFY
Workshop	Friday, 13 Sep	02	10:00AM	XLZ5M

Tutorial	Wednesday, 11 Sep	02	8:00AM	G4N92
Tutorial	Wednesday, 11 Sep	01	2:00PM	5JEAW

Admin. Stuff (1)

1. Def. Mid Term: 26 Sept; 8am-10am (location TBC)

2. Submissions:

Summary

ASSESSMENT	DUE
Kick Starting Week 3: Lecture Quiz 3 (Total mark for all 12 weeks of lecture quizzes is 5%)	Tuesday, 6 August 2024, 11:55 PM Due in 4 days
Applied class quiz week 3 (Total mark for all 12 weeks of applied quizzes is 5%)	Wednesday, 7 August 2024, 11:55 PM Due in 5 days
Workshop 3 problem set (Total mark for all 12 weeks of workshop sets is 20%)	Sunday, 11 August 2024, 11:55 PM

Admin. Stuff (2)

3. Consultation/Feedback hour

- Wed: 10 am till 11 am — Location: 9-4-01
- Fri: 8 am till 9 am — Location: 5-4-68
- Sat: 1030 am till 1130 am — NO

(every odd week, e.g. 3-Week, but msg me first at least a day before; appointment-based.)

Resources

1. Additional: Videos

2. Pass materials

Let us start!

Refresher!

LIATE RULE!

L	**Logarithmic** functions
I	Inverse trig. functions
A	Algebraic functions
T	Trig. functions
E	Exponential functions

$$\boxed{\int f\,dg = fg - \int g\,df}$$

ODE vs PDE

- Variables

  ODE: **1** independent

  PDE: **2** or more independent

- Complexity

  ODE: **Long hand solution is often possible**

  PDE: **Numerical methods (approximation)**

Today’s Activity

1. Applied Problem Set

2. Applied Quiz

- ✓ Q1
- ✓ Q2
- ✓ Q3
- ✓ Q4
- ✓ Q6

Question 1

For each of the following differential equations, identify whether the ODE is linear or nonlinear and its order. If it is first order, identify all techniques that would be appropriate to use to solve the ODE: separation of variables (the technique for separable equations), integrating factors, the method of undetermined coefficients (breaking the solution into a homogeneous part plus particular solution) or none of these.

(a) $\dfrac{dy}{dx} = e^{2x+y}$

(b) $\dfrac{dx}{dt} = xt^2 - 4x$

(c) $e^{2u'x^2} = e^{u^3}$ where $'$ indicates differentiation with respect to x

(d) $\dfrac{df}{dx} = 2f + e^{3x}$

Learning Outcomes?

Categorize the ODEs

Q1(a)

$$\frac{dy}{dx} = \exp(2x + y)$$

✓ Non-linear, because?:

y appears in exponential

✓ First–order because?: $\frac{dy}{dx}$

✓ Can we write in this form?: $\frac{dy}{dx} = \frac{f(x)}{g(y)}$?

YES

$$\boxed{\frac{dy}{dx} = \frac{\exp(2x)}{\exp(-y)}}$$

✓ Go for S.V!

Q1(b)

$$\frac{dx}{dt} = xt^2 - 4x = x(t^2 - 4)$$

✓ Linear, because?:

x and its derivative power 1

✓ First–order because?: $\frac{dx}{dt}$

✓ Can we write in this form?: $\frac{dx}{dt} = \frac{f(t)}{g(x)}$?

YES

$$\boxed{\frac{dx}{dt} = \frac{(t^2-4)}{1/x}}$$

✓ Go for S.V!

Q1(c)

✓ Non-linear, because?:

$$\exp(2u'x^2) = \exp(u^3) \longrightarrow 2\frac{du}{dx} = \frac{u^3}{x^2}$$

u power 3

✓ First–order because?: $\frac{du}{dx}$

✓ Can we write in this form?: $\frac{du}{dx} = \frac{f(x)}{g(u)}$?

YES

$$\boxed{\frac{du}{dx} = \frac{x^{-2}}{u^{-3}}}$$

✓ Go for S.V!

Q1(d)

$$\frac{df}{dx} = 2f + \exp(3x)$$

✓ Linear, because?:

f power 1

✓ First–order because?: $\dfrac{df}{dx}$

we write in this form?: $\dfrac{df}{dx} = \dfrac{f(x)}{g(f)}$?

NO

✓ Go for I.F!

WHO WANT TO ATTEMPT THIS IN REAL TIME?

Q1(d)

$$\frac{df}{dx} = 2f + \exp(3x)$$

Question 2(c)

2. Solve the following three Initial Value Problems:

(a) $y'(t) = y^2, \qquad y(0) = 1.$

(b) $y'(t) = y^3, \qquad y(0) = 1.$

(c) $y'(t) = e^y, \qquad y(0) = 1.$

Which solution "blows up", i.e., $y \to \infty$, first? At what time?

Learning Outcomes?

Solve ODE

$$\frac{dy}{dt} = \exp(y)$$

- ✓ First–order: $\dfrac{dy}{dt}$
- ✓ Linear: $No!$
- ✓ Can we write in this form for S.V: $\dfrac{dy}{dt} = \dfrac{f(t)}{g(x)}$?

yes $$\boxed{\frac{dy}{dt} = \frac{t^0}{\exp(-y)}}$$

You Try!

$$\frac{dy}{dt} = \frac{t^0}{\exp(-y)}$$

$$-\exp(-y) = t + C$$

Apply I.C. $y(0) = 1$

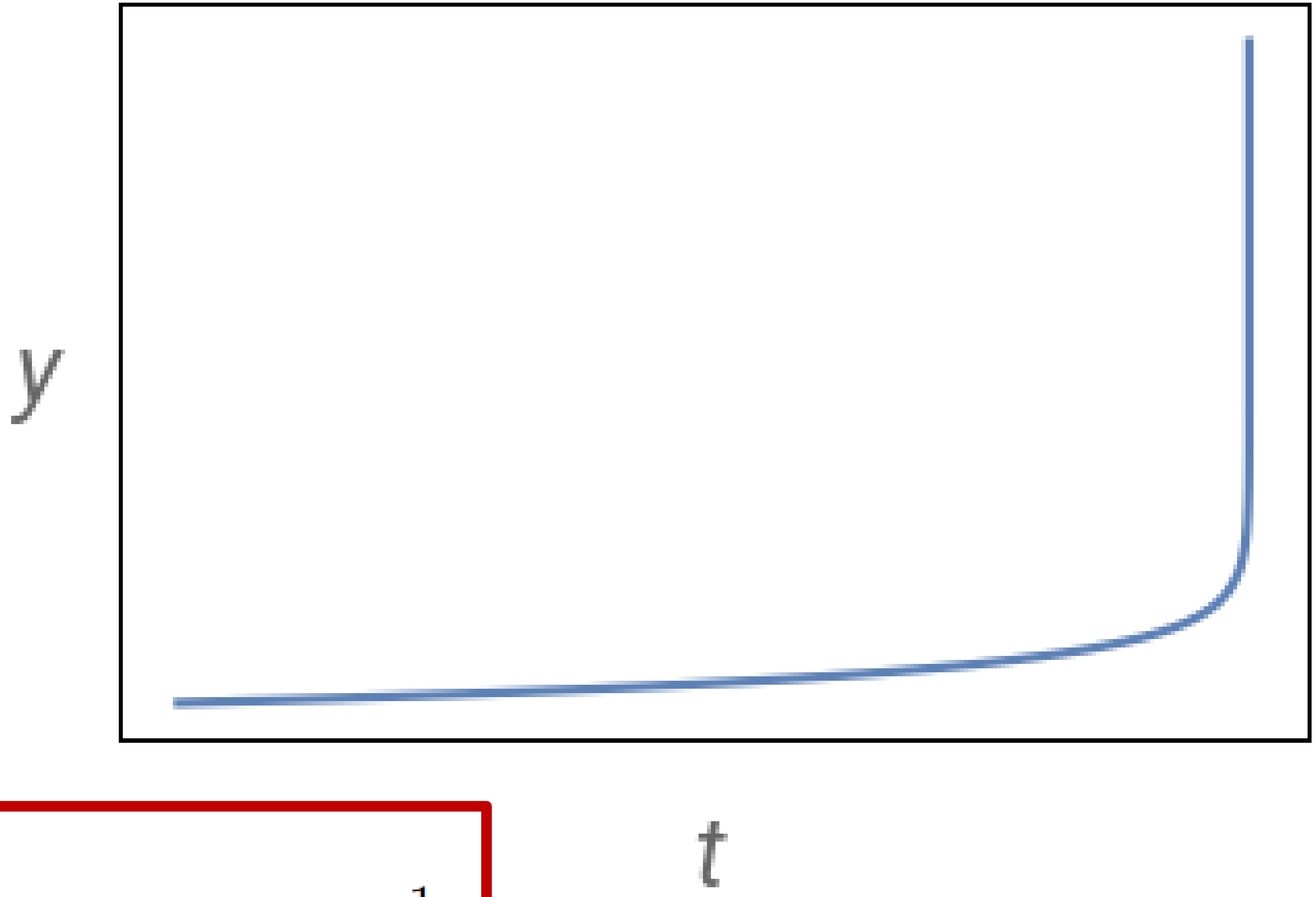
$$-\exp(-\mathbf{1}) = \mathbf{0} + C$$

$$-\exp(-y) = t + C$$

$$\exp(-y) = \exp(-1) - t$$

$$\boxed{y = \ln\left(\frac{1}{e^{-1} - t}\right)}$$

$$y = \ln\left(\frac{1}{e^{-1} - t}\right)$$



The solution diverges as $t \to e^{-1}$.

Question 3

3. Consider the ODE given by

$$ty' + 3y = 5t^2 - 3$$

(a) Find the general solution for this ODE.

(b) Find the solution for this ODE given the initial condition $y(1) = 1$. For this solution $y(t)$, what is the value $y(2)$ of the solution at time $t = 2$?

✓ Can we write in this form?: $\dfrac{dy}{dt} = \dfrac{f(t)}{g(y)}$?

NO

Learning Outcomes? I.F

✓ Case 1:

$$\frac{dy}{dx} + ay = bx$$

✓ Case 2:

$$\frac{dy}{dx} + a\frac{y}{x} = bx$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The integrating factor (I.F.) is given as: $I.F. = e^{\int P(x)dx}$

I.F $\exp(ax)$

I.F x^a

$$\boxed{y(x) = \frac{bx}{a} - \frac{b}{a^2} + C_1 e^{-ax}}$$

$$\boxed{y(x) = \frac{bx^2}{a+2} + C_1 x^{-a}}$$

✓ Case 1:

$$\frac{dy}{dx} + ay = bx$$

✓ Case 2:

$$\frac{dy}{dx} + a\frac{y}{x} = bx$$

$$y' + \frac{3}{t}y = 5t - \frac{3}{t}$$

- Identify the pattern! CASE 2

✓ I.F.: t^3

Step 1: X with the I.F

$$(t^3)y' + (t^3)\frac{3}{t}y = (t^3)5t - (t^3)\frac{3}{t}$$

Step 2: Simplify

$$t^3y' + 3t^2\,y = 5t^4 - 3t^2$$

Step 3: Further Simplify

$$\frac{d}{dt}(t^3y) = 5t^4 - 3t^2$$

Step 4: Solving

$$y = t^2 - 1 + Ct^{-3}$$

$$y = t^2 - 1 + Ct^{-3}$$

- ✓ Invoke the I.C. : y(1)=1 $\boxed{C = 1}$
- ✓ y(2) ? $\boxed{25/8}$

Question 4(b)

4. Consider the following two differential equations:

(a) $y'x\sin(x) + [\sin(x) + x\cos(x)]y = xe^x$

(b) $y'\tan(x) + y = 1$

Find the general solution for each of these ODEs.

Learning Outcomes?

Sometimes S.V and I.F work!

ALWAYS USE S.V!

$y'\tan(x) + y = 1$

$$\frac{dy}{dx} = \frac{\cot(x)}{1/(1-y)}$$

✓ First–order

✓ Linear

✓ How we write in this form: $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$?

You Try!

$$\frac{dy}{dx} = \frac{\cot(x)}{1/(1-y)}$$

$$\frac{dy}{-y+1} = \cot(x)\,dx$$

10min!

$$-\ln(-y+1) = \ln\big(\sin(x)\big) + c$$

$$\boxed{y(x) = c_1 \csc(x) + 1}$$

5min!

Full Working for 4(b)

Additional Question: From the slide (IF)

$$y' + \tan(x)y = \cos(x)$$

- You attempt with this additional guide/resources
- Come talk to me after class to check your workings!

Question 6

6. Pursuing a criminal, Sherlock Holmes discovers a cup of tea at a temperature of $50\,°C$. Holmes methodically examines the surrounding scene and is led to the conclusion that the tea belonged to the criminal. He returns to the cup of tea and discovers it has cooled to $40\,°C$ in the intervening 10 minutes. How long ago did the criminal make the tea?

   You may assume that the cup of tea cools according to Newton's law of cooling:

$$\frac{dT}{dt} = -k\,(T - T_{\text{ambient}})$$

   where T is the temperature of the cup of tea at time t, k is a constant and T_{ambient} is the ambient air temperature. You may also assume that $T_{\text{ambient}} = 20\,°C$ and that the tea began at $100\,°C$.

Rewrite

$$\frac{dT}{dt} = -k(T\text{-}T_{amb})$$

$$\frac{dT}{(T - T_{amb})} = -k\ dt$$

Try to Solve!

$$\ln(T - T_{amb}) = -kt + C$$

Simplified!

$$T - T_{amb} = \exp(-kt + C)$$

$$T = \mathrm{A}\exp(-kt) + T_{amb}$$

exp(C)=A

$$T = A\exp(-kt) + T_{amb}$$

✓ Invoke the I.C. : T(0) =100

$$\boxed{A = 80}$$

- Step 1

$$50 = 80\exp(-kt) + 20$$

- Step 2

$$40 = 80\exp(-k(t + 10)) + 20$$

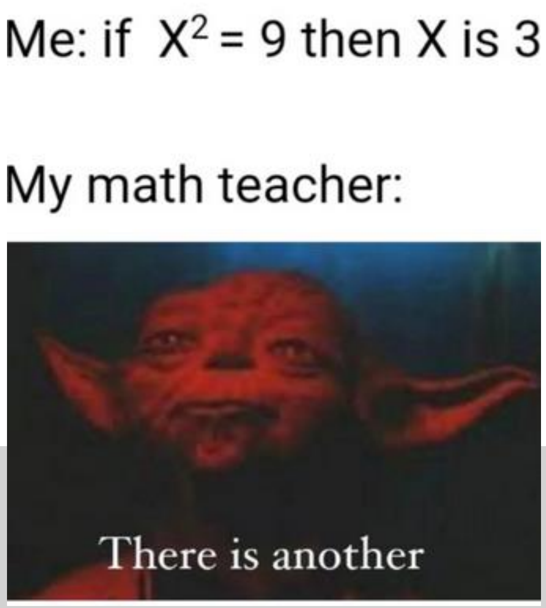
Two unknows with 2 Eqs.
Solving k & t are possible!
You try!

$$\boxed{t = 24.2\ mins}$$

Today's Activity

1. Applied Problem Set

2. Applied Quiz



Thank You

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Assessments breakdown

Task description	*Value*	*Due date*
Lecture quizzes	5%	Weekly 11:55pm the night before your applied class (except Monday classes) except in Week 1 when the due date is Friday at 11:55 pm (Malaysia time)
Applied class quizzes	5%	Weekly during your applied class
Workshop problems	20%	Weekly at 11:55pm Sunday, except for Weeks 7 (midsem)
Mid-semester exam	20%	During your workshop in Week 7

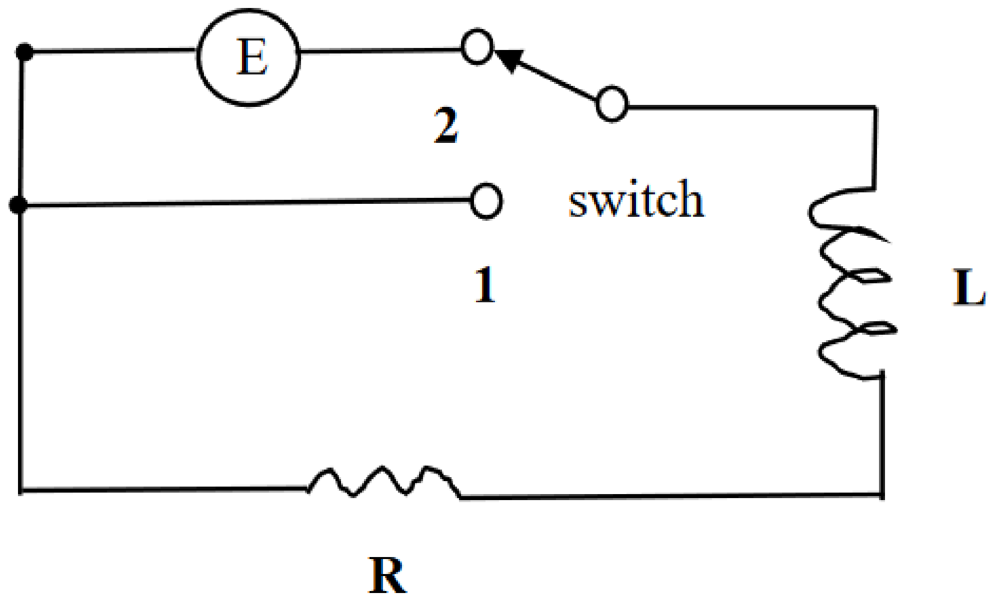
Today’s Activity

1. Workshop Problem Set

Electrical Circuit

A series circuit shown below containing a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a source E of electromotive force (emf). This system is governed by the first-order equation

$$\boxed{LI'(t) + RI(t) = E(t).}$$



where I is the current which depends on the time t, and the prime symbol indicates differentiation with respect to t.

1. Identify whether the ODE is linear or nonlinear. Justify your answer.

$$L\frac{dI(t)}{dt} + R\,I(t) = E(t)$$

- ✓ Linear or Non?:

Linear: I power 1 and L & R are constants

Electrical Circuit

A series circuit shown below containing a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a source E of electromotive force (emf). This system is governed by the first-order equation

$$LI'(t) + RI(t) = E(t).$$

Assume that the switch has been in position 2 for a while, so that a steady current of 5 A is flowing in the circuit. At time $t = 0$, the switch is changed to position 1, so that $I(0) = 5$ and $E = 0$ for $t \geq 0$

2. Show that the ODE is a separable ODE for $t \geq 0$. **[1 mark]**

$\boxed{t \geq 0}$

$$L\frac{dI}{dt} + R\,I = 0$$

✓ Can we write in this form?: $\frac{dI}{dt} = \frac{f(t)}{g(I)}$? Yaa!

$$\frac{dI}{dt} = -\frac{R\,t^0}{L\,I^{-1}}$$

3. Suppose that $L = 7$ H and $R = 33\ \Omega$. Find $I(t)$ using separation of variables and evaluate its value at $t = 0.1$. (Give the final answers in 3 significant figures.) **[4 marks]**

You Solve, 5min

Rewrite

$$\frac{dI}{dt} = -\frac{R}{L}\frac{t^0}{I^{-1}} \rightarrow \frac{1}{I}dI = -\frac{R}{L}dt \rightarrow$$

$$\ln(I) = -\frac{R}{L}t + C$$

Add I.C.

$$I(0) = 5$$

$$C = \ln(5)$$

With L and R, solve it

$$\boxed{I = 5e^{-33(0.1)/7} = 3.12\text{ A}}$$

4. Suppose that L and R are unknown. After some time, the current decays to 80% of its initial value. It takes another 0.15 seconds for the current to decay to 38% of its initial value. How long does it take for the current to reach 80% of its initial value and what is the ratio of L/R? (Give the final answers in 3 significant figures.) **[4 marks]**

$$I = 5\exp(-kt)$$

- After some t (lets call this a time), it reduces from 5A to 4A (80% state)
- And, a+0.15s, I becomes 1.9 A (0.38*5)

✓ **5mins to write the 2 system of Eqs.**

$$\boxed{\frac{R}{L} = k}$$

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4. Suppose that L and R are unknown. After some time, the current decays to 80% of its initial value. It takes another 0.15 seconds for the current to decay to 38% of its initial value. How long does it take for the current to reach 80% of its initial value and what is the ratio of L/R? (Give the final answers in 3 significant figures.) **[4 marks]**

✓ **EQ 1:** $4 = 5\exp(-ak)$

✓ **EQ 2:** $1.9 = 5\exp(-ak)\exp(-0.15k)$

$\boxed{k=4.96}$

$\boxed{a=0.045}$

$\boxed{\frac{R}{L} = k}$

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Assume that the switch remains in position 2. Suppose that $E(t) = 150e^{-10t}\cos(40t)$, $R = 25$, $L = 5$, and $I(0) = 5$.

5. Identify all techniques that would be appropriate to use to solve the ODE. **[1 mark]**

A: Separation Variables;

B: Integration Factors;

C: Method of Undetermined Coefficients.

Assume that the switch remains in position 2. Suppose that $E(t) = 150e^{-10t}\cos(40t)$, $R = 25$, $L = 5$, and $I(0) = 5$.

5. Identify all techniques that would be appropriate to use to solve the ODE. **[1 mark]**

6. Find $I(t)$ using all the techniques mentioned in the previous question. **[10 marks]**

B: Integration Factors; $\rightarrow$ NOW

C: Method of Undetermined Coefficients. $\rightarrow$ LATER

B: Integration Factors;

I.F

$$5\frac{dI(t)}{dt} + 25\,I(t) = 150\exp(-10t)\cos(40t)$$

Simplify

$$\frac{dI(t)}{dt} + 5\,I(t) = 30\exp(-10t)\cos(40t)$$

$\boxed{\mathrm{I.F} : \exp(5t)}$

$$\frac{d}{dt}\{\exp(5t)I\} = 30\exp(-5t)\cos(40t)$$

I.C,

B=331/65

Integration by Parts

$$I = \frac{6}{65}e^{-10t}[8\sin(40t) - \cos(40t)] + Be^{-5t}$$

Assume that the switch remains in position 2. Suppose that $E(t) = 150e^{-10t}\cos(40t)$, $R = 25$, $L = 5$, and $I(0) = 5$.

C: Method of Undetermined Coefficients.

Method of Undetermined Coefficients

$$5\frac{dI(t)}{dt} + 25\,I(t) = 150\exp(-10t)\cos(40t)$$

Simplify

$$\boxed{\frac{dI(t)}{dt} + 5\,I(t)} = \boxed{30\exp(-10t)\cos(40t)}$$

Gives rise to

Homogeneous solution, I_H

Gives rise to

Particular solution, I_p

Total Solution encompasses **homogeneous** + **particular** components

$$I(t) = I_h(t) + I_p(t)$$

1 2

① Calculating Homogeneous Solution

Let $\frac{dI(t)}{dt} + 5\,I(t) = 0$ → Assume $I_h(t) = Ae^{\lambda t}$

Substitute $I_h(t)$ expression

$$\frac{d(Ae^{\lambda t})}{dt} + 5\,(Ae^{\lambda t}) = 0$$

Expansion

$$\lambda \cancel{Ae^{\lambda t}} + 5\,(\cancel{Ae^{\lambda t}}) = 0$$

$$\lambda = -5$$

$$\frac{dI}{I} = -5\,dt$$

$$\ln I = -5t + C$$

$$I = \exp(-5t + C)$$

$$\boxed{I_h(t) = Ae^{-5t}}$$

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② Calculating Particular Solution

Assume $I_p(t) = \underbrace{\exp(-10t)}_{\text{Exponential function}} \underbrace{[B\sin(40t) + C\cos(40t)]}_{\text{Oscillatory function}}$

Note:

$I_p(t)$ has similar form compared to $E(t) = 30\exp(-10t)\cos(40t)$

Substitute assumed $I_p(t)$ into ODE

$$\frac{dI_p(t)}{dt} + 5\,I_p(t) = 30\exp(-10t)\cos(40t)$$

$$\frac{d(\exp(-10t)\,[B\sin(40t) + C\cos(40t)])}{dt} + 5(\exp(-10t)\,[B\sin(40t) + C\cos(40t)]) = 30\exp(-10t)\cos(40t)$$

⋮ Expansion & Simplification

② Calculating Particular Solution (Cont'd)

Expansion and term grouping,

$$(40B - 5C)\cos(40t) + (-5B - 40C)\sin(40t) = 30\cos(40t) + 0\sin(40t)$$

Comparing coefficients for cos and sin,

$$40B - 5C = 30; \qquad -5B - 40C = 0$$

Solving for B and C,

$$B = 48/65; \ C = -6/65 \longrightarrow I_p(t) = \exp(-10t)\,[B\sin(40t) + C\cos(40t)]$$

3 Satisfying Initial Condition (To obtain A in I_H)

Expression for total solution,

$$I(t) = I_H(t) + I_p(t)$$

$$I(t) = Ae^{-5t} + \exp(-10t)\left[\frac{48}{65}\sin(40t) - \frac{6}{65}\cos(40t)\right]$$

Substituting initial condition $I(0) = 5$,

$5 = A - \frac{6}{65}$; $A = \frac{331}{65}$

4 Final Answer

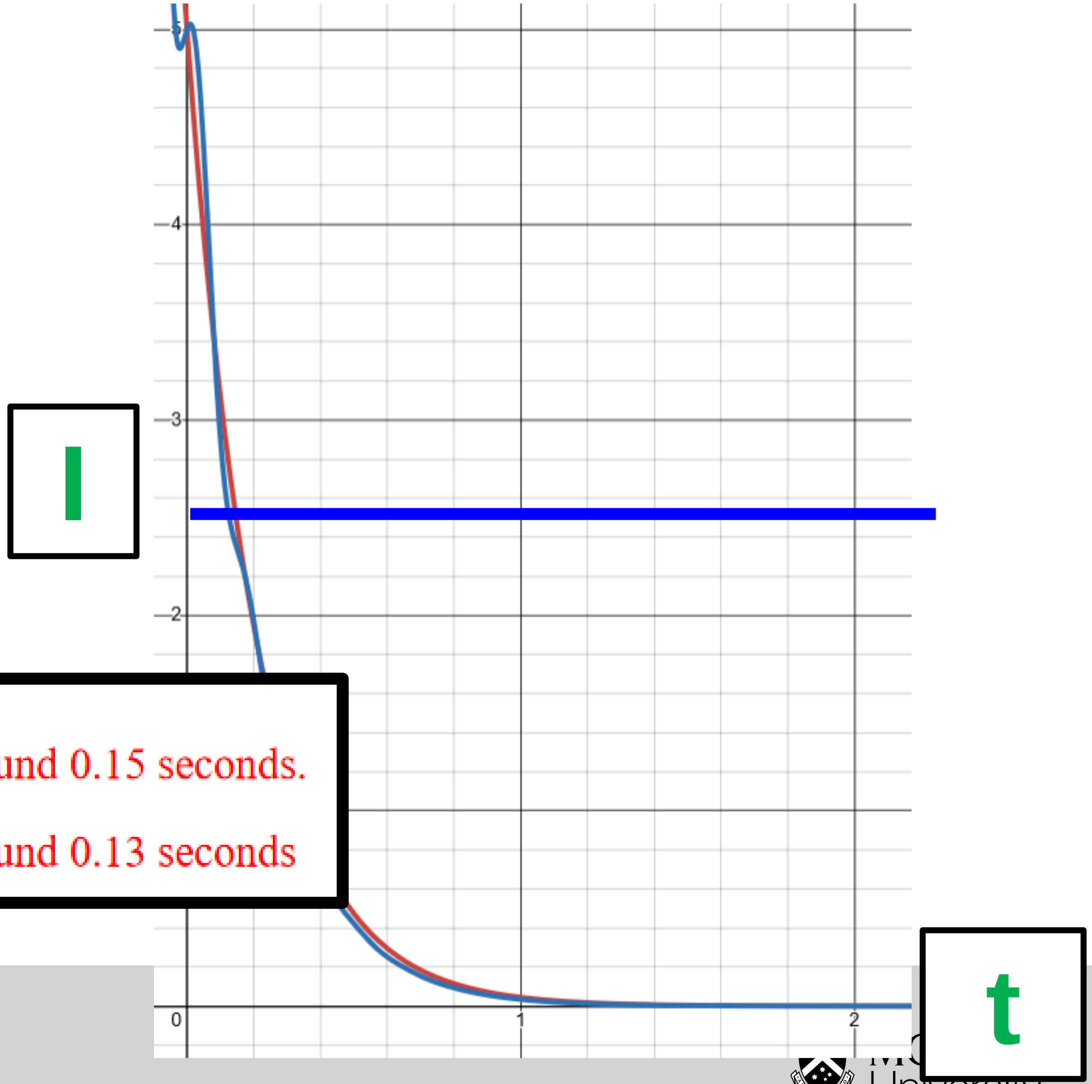
$$I(t) = \frac{331}{65}e^{-5t} + \exp(-10t)\left[\frac{48}{65}\sin(40t) - \frac{6}{65}\cos(40t)\right]$$

I.F vs. M.U.C?

7. Plot the currents $I(t)$ of Questions 3 and 6 in the same graph. From the graph, find the times for the two currents decay to 50% of their initial values. **[4 marks]**

$$I = 5\exp\left(-\frac{33}{7}t\right)$$

$$I = \frac{6}{65}\exp(-10t)\left(8\sin(40t) - \cos(40t)\right) + \frac{331}{65}\exp(-5t)$$



The current I(t) in Question 3 decays to 50% of its initial value at around 0.15 seconds.

The current I(t) in Question 6 decays to 50% of its initial value at around 0.13 seconds

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Thank You