ENG1005 S2 2024 Workshop 3
Missile Trajectory

27 marks total

This problem set is intended for you to apply the mathematical skills you are learning. It is also designed to practice communicating your work clearly.

It is expected that you will use the workshop to develop (rough) solutions. During the workshop, you should discuss the problems with your peers and the academic staff who are there to assist you. In particular, if you are uncertain about what the problems are asking or you are stuck on a particular point, this is the time to get assistance. The time between the end of the workshop and when the solutions are due is only meant to be for writing up your solutions and for this you should not need more than an hour or two at most.

General submission information:

1. Electronic submission of your solutions is due on Moodle by **11:55 pm on Sunday of the same week**.
2. **Your solutions should include a description/explanation of what you are doing at each step and relevant working.** Without these you will receive limited marks. The description should be in complete English sentences. All mathematics should be appropriately laid out and with appropriate notation. Your writing should be similar in style to the worked solutions from the Applied Class problem sheets, not the annotations from the videos. For more information and advice, please read the "Guidelines for writing in mathematics" document posted under the "Additional information and resources" section of the ENG1005 Moodle page.
3. Your solutions may be typed or handwritten and scanned (the latter is encouraged). The **final document should be submitted as a single pdf file that is clearly and easily legible**. If the marker is unable to read it (or any part of it) you may lose marks.

Academic integrity:

You can (and should!) discuss your solutions with the other students, but **you must write up your solutions by yourself**. Copying solutions is serious academic misconduct and will be penalised according to Monash University guidelines. Other examples of academic misconduct include asking a personal tutor to do any of your assessments and posting your assessments to a "homework" website. Please refer back to your Academic Integrity module if you are in any doubt about what constitutes academic misconduct. **Your integrity is an important part of who you are. It is much more important than any grade you could receive.**

Missile Trajectory

Radars have long been used to track flying objects. However it is typically not possible to continuously track a object in real time. Instead, locations of the object are taken at different times. An interpolation process is then used to estimate the trajectory and make predictions about the future location of the object. In this workshop, we will study the methods of polynomial interpolation and least square approximation.

Suppose a missile is fired from sea level and travels directly east. Three radar stations are situated 1km, 2km and 3km due east of the launch site. As the missile passes directly overhead, each radar station measures the altitude of the missile. A short time after launch, the missile engine is switched off. For simplicity, ignore any aerodynamic forces, and assume that without any thrust from its engine, the only force acting on the missile is gravity. From Newtonian mechanics, we know that the missile will follow a parabolic trajectory, i.e., if we let x be the horizontal distance of the missile from the launch site, and y be its altitude, then we have

$$y(x) = Ax^2 + Bx + C$$

where A, B, C are constants.

From the three radar station measurements, we have the follow data about the location of the missile, with both x and y in kilometres.

x	1	2	3
y	6	7	5

We will try to find the constants A, B and C such that the three points $(1, 6)$, $(2, 7)$, and $(3, 5)$ all lie on the parabola $y = Ax^2 + Bx + C$.

1. Using the given measurements from the table, write down a **matrix equation** for the unknown variables A, B, and C. **[1 marks]**

   Solution: We have an equation at each point, therefore the system of equations is

$$\begin{aligned} A + B + C &= 6 \\ 4A + 2B + C &= 7 \\ 9A + 3B + C &= 5 \end{aligned}$$

   In matrix form, this beomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix}$$

2. Calculate the determinant of the matrix in your equation. It should be non-zero. What does this tell you about whether you can fit a unique quadratic curve through the three data points? **[2 marks]**

   Solution: Since the matrix is only 3 by 3, we can expand along the top row to find the determinant.

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} = 1 \times (2 \times 1 - 3 \times 1) - 1 \times (4 \times 1 - 9) + 1 \times (4 \times 3 - 9 \times 1) = -2$$

   Since the determinant is non-zero, the matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}$ is invertible. This mean the system of equations has a unique solution, and there is a unique quadratic through the three points.

3. Calculate the **inverse of the matrix**, and hence find the values of A, B and C. **[3 marks]**

Solution: We use row reduction to find the inverse (you may also use alternative methods for inversion)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 9 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 4R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -4 & 1 & 0 \\ 9 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 - 9R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -4 & 1 & 0 \\ 0 & -6 & -8 & -9 & 0 & 1 \end{array}\right]$$

$R_3 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

$R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 3 & -1 \\ 0 & -2 & -3 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

$R_2 + 3R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 3 & -1 \\ 0 & -2 & 0 & 5 & -8 & 3 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

$R_2/(-2)$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

$R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

Therefore the inverse is

$$\begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ 3 & -3 & 1 \end{bmatrix}$$

To solve for the matrix equation, we can multiply both sides by the inverse matrix. We have

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{11}{2} \\ 2 \end{bmatrix}$$

4. Where do you predict the missile will land (i.e. hit sea level)? **[1 mark]**

   Solution: The equation of the trajectory is $y = -\frac{3}{2}x^2 + \frac{11}{2}x + 2$. We need to solve for $y = 0$.

$$y = -\frac{3}{2}x^2 + \frac{11}{2}x + 2 = -\frac{1}{2}(3x^2 - 11x - 4) = -\frac{1}{2}(x-4)(3x+1)$$

   Since the missile is travelling in the positive direction, the missile will land at $x = 4$.

Usually having more data will give a more accurate prediction. Suppose a fourth radar station located 4km east also made a measurement, giving the following table:

x	1	2	3	4
y	6	7	5	4

5. Without doing any calculation, do you expect to be able to find a parabola through those four data points? Explain your answer. **[1 marks]**

   Solution: If we were to set up the equations again, we will get four equations, but still only the three unknows A, B, and C. The system is overdetermined and we expect no solution. Therefore we do not expect to find a parabola through four points.

We will now try to find a cubic $y = Ax^3 + Bx^2 + Cx + D$ that goes through the four data points.

6. Using the given measurements from the new table, write down a matrix equation for the unknown variables A, B, C and D. **[1 marks]**

   Solution: Similar to question 1, the matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \\ 4 \end{bmatrix}$$

7. Using appropriate row and column operations and properties of determinants with those operations, find the determinant of the matrix in your equation. What does this tell you about whether there is a unique cubic through the four data points? **[4 marks]**

   Solution: Let D be the determinant of the matrix. We use row and column operations to bring the matrix into upper triangular form, keeping track of the determinant along the way.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix}, \qquad \det = D$$

   $C_1 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 2 & 8 \\ 1 & 9 & 3 & 27 \\ 1 & 16 & 4 & 64 \end{bmatrix}, \qquad \det = -D$$

   $R_2 = R_2 - R_1, R_3 = R_3 - R_1, R_4 = R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 7 \\ 0 & 8 & 2 & 26 \\ 0 & 15 & 3 & 63 \end{bmatrix}, \qquad \det = -D$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}, \qquad \det = D$$

$R_3 = R_3 - 2R_2, R_4 = R_4 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 6 & 42 \end{bmatrix}, \qquad \det = D$$

$R_4 = R_4 - 3R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \qquad \det = D$$

The final upper triangular matrix has determinant $1 \times 1 \times 2 \times 6 = 12$. Therefore the determinant of the original matrix is also $D = 12$. The determinant is non-zero, hence the matrix is invertible, and the system of equations has a unique solution. This means there is a unique cubic passing through the four data points.

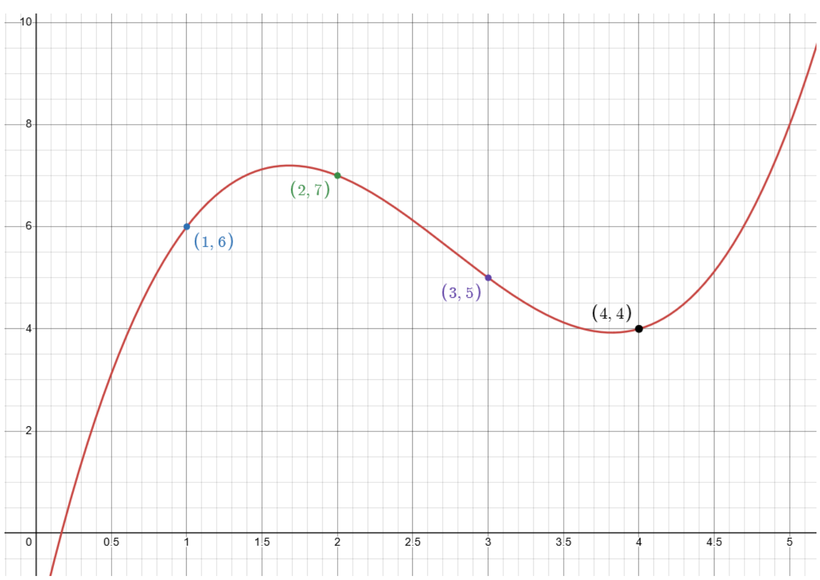
8. Using Matlab or CAS, find A, B, C and D. You might find the commands on the last page useful. **[2 marks]**

Solution: Using the Matlab command from the last page, we find

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{11}{2} \\ \frac{77}{6} \\ -2 \end{bmatrix}$$

9. Plot your data points and the interpolating polynomial through them. Be sure to include axis labels. **[1 mark]**

Solution:



10. Where does the missile land? Does this plot seem a realistic trajectory for a missile? **[1 mark]**

    Solution: The missile does not land. The plot suggests that the missile will fly off into space. This is not a realistic trajectory.

In practice, measurements are never exact and almost always contain some error. For example, you may find the data points don't fit a parabola precisely. In theses situations, the best we can do is to find the curve that best fits the data points. For simplicity, we will investigate the (straight) line of best fit, but similar ideas can be applied to find the parabola or any curve of best fit.

Suppose the missile did not turn of its engine, so that it flies in a straight line instead of having a parabolic trajectory. Let's return to the first set of measurements.

x	1	2	3
y	6	7	5

11. Show that the three observed points do not lie on a straight line. **[1 mark]**

    Solution: The straight line passing through the first two points $(1, 6)$ and $(2, 7)$ has a positive gradient. However the line passing through $(2, 7)$ and $(3, 5)$ has a negative gradient, so the two lines are distinct and the three points are not on the same line.

Let $y(x) = Sx + T$ denote a straight line. We allow S and T to vary over all real numbers to cover all possible straight lines. For each pair (S, T), we let the triple $(y(1), y(2), y(3))$ be the *predicted value* of the radar station measurements. The actual *observed value* from the radar stations is $(6, 7, 5)$. In the method of least squares, the idea is to minimize the distance (as points in $\mathbf{R}^3$) between the predicted value and the observed value. The values if S and T that achieve this minimum will give us the line of best fit.

12. Express $(y(1), y(2), y(3))$ in terms of S and T. **[1 mark]**

    Solution: Substitute $x = 1$, $x = 2$ and $x = 3$ into the straight line equation, we get

$$((y(1), y(2), y(3)) = (S + T, 2S + T, 3S + T)$$

13. Explain that the set of all predicted values, as S and T vary, form a plane in $\mathbf{R}^3$, the space of all triples. **[1 marks]**

    Solution: Writing $(y(1), y(2), y(3))$ in vector form we have

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \end{bmatrix} = S \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

    with S and T varying over all real numbers.

    We observe that this is the parametric equation of a plane passing through $(0, 0, 0)$, with vectors $(1, 2, 3)$ and $(1, 1, 1)$ on the plane.

14. Using ideas from previous weeks, find the minimum distance between the plane of all predicted values and $(6, 7, 5)$. **[3 marks]**

    Solution: A normal vector to the plane is $(1, 2, 3) \times (1, 1, 1) = (-1, 2, -1)$. Since $(0, 0, 0)$ is a point on the plane, the distance from $(6, 7, 5)$ to the plane is the absolute value of the scalar projection of $(6, 7, 5) - (0, 0, 0) = (6, 7, 5)$ onto the normal vector.

$$\text{Distance} = \left| \frac{(6, 7, 5) \cdot (-1, 2, -1)}{|(-1, 2, -1)|} \right| = \frac{3}{\sqrt{6}}$$

15. Find the line of best fit for the given data set. **[2 marks]**

Solution: The vector projection of $(6, 7, 5)$ onto the normal vector is

$$\frac{3}{\sqrt{6}}\left(\frac{(-1, 2, -1)}{\sqrt{6}}\right) = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right)$$

The closest point on the plane to $(6, 7, 5)$ is $(6, 7, 5) - (-\frac{1}{2}, 1, -\frac{1}{2}) = (\frac{13}{2}, 6, \frac{11}{2})$. This is our "best" prediction for $(y(1), y(2), y(3))$. From this point we can solve for S and T. Solving the equations

$$S + T = \frac{13}{2}$$
$$2S + T = 6$$
$$3S + T = \frac{11}{2}$$

we get $S = -\frac{1}{2}, T = 7$. The line of best fit is $y = -\frac{1}{2}x + 7$.

There is also 1 additional mark given for the quality of the English and 1 additional mark for correct mathematical notation. These marks are easy to obtain but the markers will be instructed to be strict in awarding these marks.

Some Matlab commands

You may find these helpful in your calculations.

Try the following matrix-related commands:

1. Enter a matrix A in Matlab/Octave using

   ```
   A = [1 2 3; 6 7 8; 1 1 1]
   ```

2. Enter a column vector **b** in Matlab/Octave using

   ```
   b = [2; 3; 5]
   ```

3. Solve the system of equations $\mathsf{A}\mathbf{x} = \mathbf{b}$ using

   ```
   x = A\b
   ```

4. Create a matrix whose columns are powers of the vector **b**

   ```
   A = vander(b)
   ```

5. Calculate the determinant of a matrix A

   ```
   det(A)
   ```

For plotting, if you have your values of a_i in a vector **a** you can use

```
ti = [1 2 3]
di = [3 1 1]
t = [1:0.01:3];
d = polyval(a,t);
plot(ti,di,'bo')
hold on
plot(t,d,'r-')
hold off
```