Eng. Math

ENG 1005

Week 10: O.D.E 3

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

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Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

The Big Learning Outcomes for Week 10

<u>After completing this week's task, you should be able to:</u>

- Solve second-order constant coefficient boundary value problems and eigenvalue problems.
- Solve linear systems of ODEs.

Some parts are not examinable!

Attendance Codes (Week 10)

International students

Workshop	Thursday, 3 Oct	01	1:00PM	ZTM47
Workshop	Friday, 4 Oct	02	10:00AM	PBNTP

Tutorial	Wednesday, 2 Oct	02	8:00AM	NGY9A
Tutorial	Wednesday, 2 Oct	01	2:00PM	JGVYT

Admin. Stuff (1)

1. Mid Term Results Discussion Thur/Fri

2. Submissions:

Summary

ASSESSMENT	DUE
Kick Starting Week 3: Lecture Quiz 3 (Total mark for all 12 weeks of lecture quizzes is 5%)	Tuesday, 6 August 2024, 11:55 PM Due in 4 days
Applied class quiz week 3 (Total mark for all 12 weeks of applied quizzes is 5%)	Wednesday, 7 August 2024, 11:55 PM Due in 5 days
Workshop 3 problem set (Total mark for all 12 weeks of workshop sets is 20%)	Sunday, 11 August 2024, 11:55 PM

Admin. Stuff (2)

3. Consultation/Feedback hour

- Wed: 10 am till 11 am
- Fri: 8 am till 9 am
- Sat: 1030 am till 1130 am

Location: 9-4-01

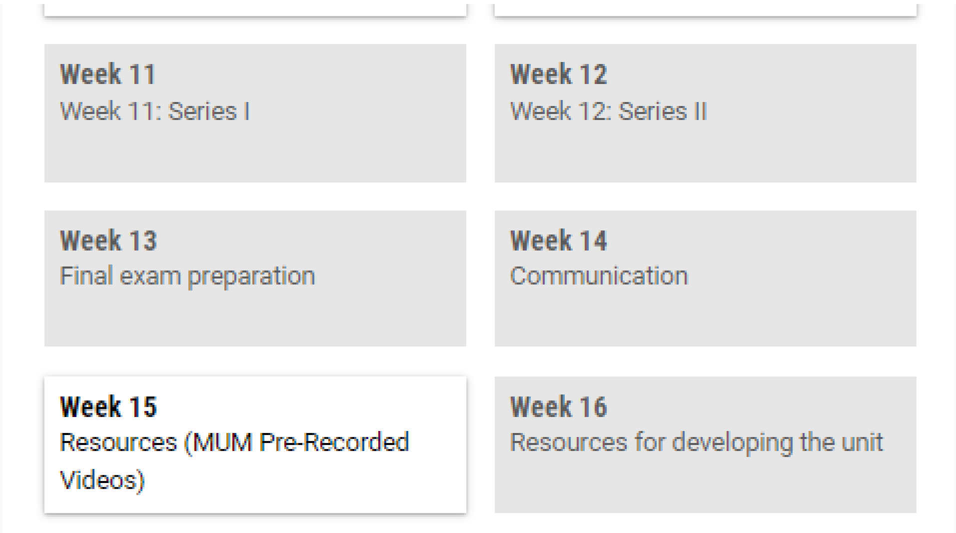
Location: 5-4-68

NO

(every odd week, e.g. 3-Week, but msg me first at least a day before; appointment-based.)

Resources

1. Additional: Videos



2. Pass materials

Let us start!

Two Tricks

- ✓ Euler Rule for complex numbers
- ✓ Hyperbolic cosh/sinh in terms of exponentials

Trick: (1)

$$\sin(-t) = -\sin(t)$$
$$\cos(-t) = \cos(t)$$

- ✓ Euler Identity

$$e^{it} = \cos(t) + i\sin(t)$$

- ✓ Example:

$$e^{-2it} = \cos(-2t) + i\sin(-2t)$$
$$= \cos(2t) - i\sin(2t)$$

Trick (2.1)

$$\boxed{y(x) = Ae^{\omega x} + Be^{-\omega x}}$$

✓ Useful Identity

$$\cosh(\omega x) = \frac{e^{\omega x} + e^{-\omega x}}{2}, \quad \sinh(\omega x) = \frac{e^{\omega x} - e^{-\omega x}}{2}$$

✓ Example

$$\cosh(wx) + \sinh(wx) = e^{wx}$$

$$\cosh(wx) - \sinh(wx) = e^{-wx}$$

$$\boxed{y = A\{\cosh(wx) + \sinh(wx)\} + B\{\cosh(wx) - \sinh(wx)\}}$$

Trick (2.2)

$$\boxed{y = A\{\cosh(wx) + \sinh(wx)\} + B\,\{\cosh(wx) - \sinh(wx)\}}$$

$$\Downarrow$$

$$\mathrm{y = a * cosh(wx) + b * sinh(wx)}$$

- ✓ A+B=a
- ✓ A-B=b

Why rewrite in trigonometric forms?

- ✓ The use of trigonometry functions can make boundary conditions simpler to apply, especially for problems involving symmetry.

Today's Activity

1. Applied Problem Set
2. Applied Quiz

- ✓ Q1
- ✓ Q3
- ✓ Q5
- ✓ Q6

Differential Equation: $y'' + \lambda y = 0$

If $\lambda > 0$:

- The general solution is:
$$y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x).$$
- **Non-Trivial Solutions:** If either $A \neq 0$ or $B \neq 0$, the solution represents oscillatory behavior and is non-trivial.
- **Trivial Solution:** If both $A = 0$ and $B = 0$, then $y(x) = 0$, which is the trivial solution.

If $\lambda = 0$:

- The equation simplifies to:
$$y'' = 0.$$
- The general solution is:
$$y(x) = C_1 + C_2 x.$$
- **Non-Trivial Solutions:** If $C_1 \neq 0$ or $C_2 \neq 0$, the solution represents a constant or a linear function, which is non-trivial.
- **Trivial Solution:** If both $C_1 = 0$ and $C_2 = 0$, then $y(x) = 0$, which is the trivial solution.

If $\lambda < 0$:

- Let's set $\lambda = -\mu$ (where $\mu > 0$). The equation becomes:
$$y'' - \mu y = 0.$$
- The general solution is:
$$y(x) = C_1 e^{\sqrt{\mu}\,x} + C_2 e^{-\sqrt{\mu}\,x}.$$
- **Non-Trivial Solutions:** If either $C_1 \neq 0$ or $C_2 \neq 0$, this solution can represent exponential growth or decay, making it non-trivial.
- **Trivial Solution:** If both $C_1 = 0$ and $C_2 = 0$, then $y(x) = 0$, which is the trivial solution.

Question 1

1. Consider the boundary value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \lambda y = 0 \quad \text{subject to} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \text{ at } x = 0 \text{ and } x = 2$$

For which values λ does this have a non-trivial solution?

Learning Outcomes?

BVP!

Generic Cases!

$$\frac{d^2y}{dx^2} \pm \overbrace{\lambda}^{\text{Constant}} y = 0 \longrightarrow \frac{d^2y}{dx^2} \pm w^2 y = 0$$

Trick from some textbook:
Change the constant
into a square form

A nice trigonometric solutions!

In short (4-step solution)

$$\frac{d^2y}{dx^2} \pm w^2 y = 0$$

- Step 1: Sub. $y = e^{kt}$ into the ODE.
- Step 2: The characteristic equation $k^2 e^{kt} \pm w^2 e^{kt} = 0$
- Step 3: General solutions;
- Step 4: BC+ Back substitution.

Full Solution with explanations! (10 mins)

- Take notes of the complex number trick and the hyperbolic cosh and sinh forms

Question 3

3. Find the general solution of the equations

$$\dot{x} = 2y - x, \qquad \dot{y} = x$$

using the eigenvalue-eigenvector method.

Learning Outcomes?

Eigenvalue-Eigenvector Method

- Step 1: Write in a matrix form!

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

- Step 2: Find the Eigens for *A* (5mins)

$$\boxed{\mathsf{A} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}} \quad \left\{ \begin{array}{l} \det(\mathsf{A} - \lambda \mathsf{I}) = 0 \\[2ex] (\mathsf{A} - \lambda \mathsf{I})\mathbf{v} = \mathbf{0} \end{array} \right.$$

$$\boxed{\dot{x} = 2y - x, \qquad \dot{y} = x}$$

- Step 3: Write the general solution using the Eigens

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = Ae^{-2t} \boxed{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} + Be^{t} \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Eigenvalue!

Eigenvector!

Question 5

5. The third order homogeneous ODE

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0$$

can be written as a system of first order equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathsf{A}\mathbf{x}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

and $x_1(t) = y(t)$, $x_2(t) = \mathrm{d}y/\mathrm{d}t$, $x_3(t) = \mathrm{d}^2 y/\mathrm{d}t^2$. Find A, its eigenvalues, eigenvectors and hence find the general solution for $\mathbf{x}$ and therefore y.

Learning Outcomes?

Higher order ODE!

- Step 1: Rewrite a System of First-Order ODEs

Given:

$$x_1(t) = y(t), \quad x_2(t) = \frac{dy}{dt}, \quad x_3(t) = \frac{d^2y}{dt^2}$$

Manipulate:

$$\boxed{\frac{dx_1}{dt} = \frac{dy}{dt} = x_2}$$

$$\boxed{\frac{dx_2}{dt} = \frac{d^2y}{dt^2} = x_3}$$

$$\boxed{\frac{dx_3}{dt} = \frac{d^3y}{dt^3} = -2x_3 - x_2 - 2x_1}$$

$$\boxed{\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0}$$

- Step 2: Write in Matrix Form!

$$\frac{dx_1}{dt} = x_2(t), \quad \frac{dx_2}{dt} = x_3(t), \quad \frac{dx_3}{dt} = -2x_3(t) - x_2(t) - 2x_1(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Step 3: Find the Eigens!

Eigenvalue	Eigenvector
$\lambda_1 = -2$	$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$
$\lambda_2 = i$	$\begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}$
$\lambda_3 = -i$	$\begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}$

- Step 4: General Solution

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = Ae^{-2t}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + Be^{it}\begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} + Ce^{-it}\begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix}.}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}$$

- # Step 5: Our Solution

As we only require the solution for y(t)

$$\boxed{y(t) = x_1(t) = Ae^{-2t} + Be^{it} + Ce^{-it}}$$

$$\downarrow$$

$$\boxed{y(t) = Ae^{-2t} + b\cos(t) + c\sin(t)}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = Ae^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + Be^{it} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} + Ce^{-it} \begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix}.$$

Combine Complex Exponentials Using Euler's Formula: Recall Euler's formula:

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-it} = \cos(t) - i\sin(t)$$

Using these, we can rewrite the terms involving Be^{it} and Ce^{-it}:

$$Be^{it} + Ce^{-it} = B(\cos(t) + i\sin(t)) + C(\cos(t) - i\sin(t))$$

Expand this expression:

$$Be^{it} + Ce^{-it} = (B + C)\cos(t) + i(B - C)\sin(t)$$

Define New Real Constants: Let's define two new real constants, b and c:

- $b = B + C$
- $c = i(B - C)$

Note that b and c are both real since B and C are arbitrary complex constants.

Question 6(a)

Let $R(t)$ be Romeo's love for Juliet at time t, with positive values meaning love and negative meaning hate. Similarly, let $J(t)$ be Juliet's love for Romeo. Then Strogatz' model for how Romeo and Juliet's love changes over time is:

$$\dot{R} = aR + bJ \qquad \dot{J} = cR + dJ$$

where a, b, c and d are constants that describe their 'romantic style'. If we focus on Romeo's love, we see that if $a > 0$, then Romeo is encouraged by his own feelings, while a negative value would mean the opposite. Similarly, a positive value of b would mean that Romeo is encourage by Juliet's feelings, while a negative value would mean he is discouraged by them.

(a) Assume that Romeo and Juliet are exactly alike and that $a = d = -1$ and $b = c = 1$. Find the general solution to this system of equations. What happens to their love at long times?

Learning Outcomes? Systems of ODE

- Step 1: Rewrite a System of First-Order ODEs

$$\begin{bmatrix} \dot{R} \\ \dot{j} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

- Step 2: Find the Eigens!

$$\lambda_1 = 0, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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- Step 3: At long scale

$$\begin{bmatrix} R \\ J \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cancel{Be^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

For long times, R and J approach the same constant A!

$\boxed{\text{Boring Love Story?}}$

Question 6(b)

Let $R(t)$ be Romeo's love for Juliet at time t, with positive values meaning love and negative meaning hate. Similarly, let $J(t)$ be Juliet's love for Romeo. Then Strogatz' model for how Romeo and Juliet's love changes over time is:

$$\dot{R} = aR + bJ \qquad \dot{J} = cR + dJ$$

where a, b, c and d are constants that describe their 'romantic style'. If we focus on Romeo's love, we see that if $a > 0$, then Romeo is encouraged by his own feelings, while a negative value would mean the opposite. Similarly, a positive value of b would mean that Romeo is encourage by Juliet's feelings, while a negative value would mean he is discouraged by them.

(a) Assume that Romeo and Juliet are exactly alike and that $a = d = -1$ and $b = c = 1$. Find the general solution to this system of equations. What happens to their love at long times?

(b) Now assume that Romeo and Juliet are opposites and that $a = -d = 1$ and $b = -c = 2$. Again find the general solution this this system of equations. What happens at long times now?

Learning Outcomes? Systems of ODE

- Step 1: Rewrite a System of First-Order ODEs

$$\begin{bmatrix} \dot{R} \\ \dot{j} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

- Step 2: Find the Eigens!

$$\lambda_1 = \sqrt{3}i, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 + \sqrt{3}i \end{bmatrix}, \quad \lambda_2 = -\sqrt{3}i, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 - \sqrt{3}i \end{bmatrix}$$

- Step 3: At long scale

$$\begin{bmatrix} R \\ J \end{bmatrix} = Ae^{\sqrt{3}i} \begin{bmatrix} 2 \\ -1+\sqrt{3}i \end{bmatrix} + Be^{-\sqrt{3}i} \begin{bmatrix} 2 \\ -1-\sqrt{3}i \end{bmatrix}$$

For long times, R and J will oscillate

Exciting Love Story?

Thank You

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Eng. Math

ENG 1005

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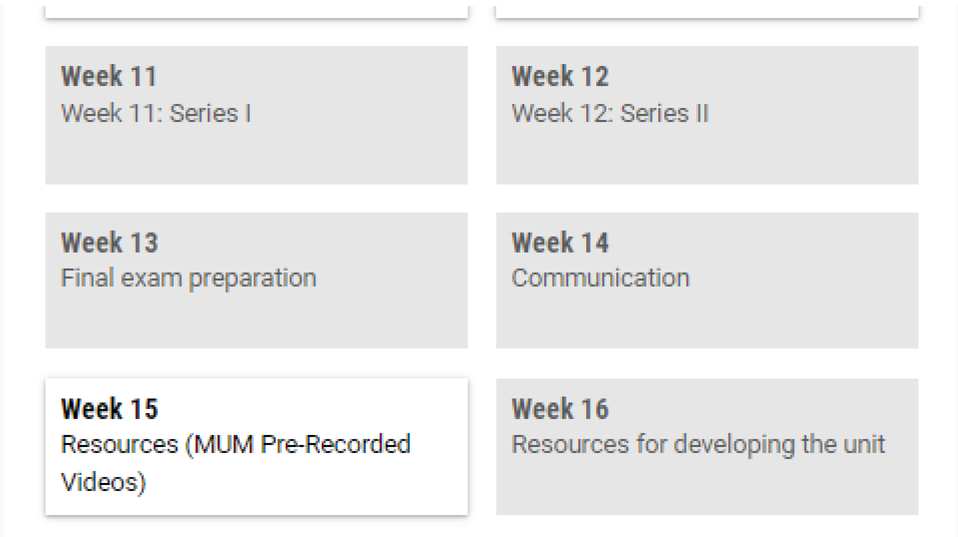
Resources

1. Additional: Videos



2. Pass materials

Let us start!

Assessments breakdown

Task description	*Value*	*Due date*
Lecture quizzes	5%	Weekly 11:55pm the night before your applied class (except Monday classes) except in Week 1 when the due date is Friday at 11:55 pm (Malaysia time)
Applied class quizzes	5%	Weekly during your applied class
Workshop problems	20%	Weekly at 11:55pm Sunday, except for Weeks 7 (midsem)
Mid-semester exam	20%	During your workshop in Week 7

The Big Learning Outcomes for Week 10

After completing this week's task, you should be able to:

- Solve second-order constant coefficient boundary value problems and eigenvalue problems.
- Solve linear systems of ODEs.

Some parts are not examinable!

Today's Activity

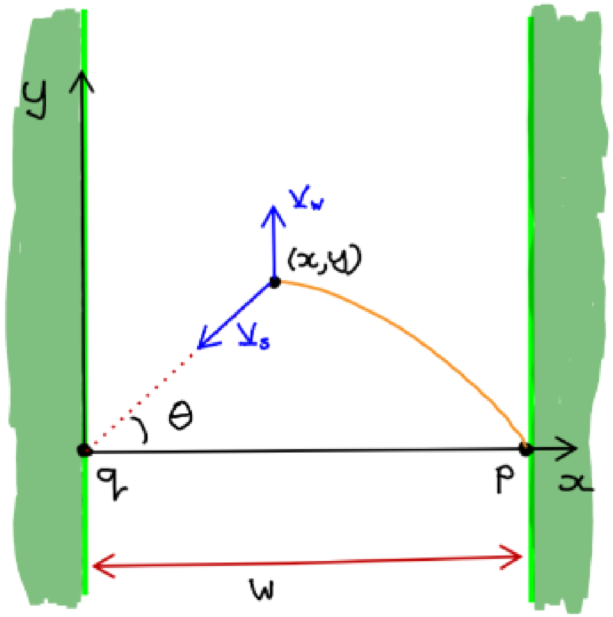
1. Workshop Problem Set

Some quick reminders:

$$\mathbf{x} = (x, y, z)$$

$$\mathbf{v} = (v_x, v_y, v_z)$$

Pursuit problems involve determining the trajectory needed by one object to intercept another. This could be a rocket carrying astronauts to the international space space station, a missile launched at an aircraft, or a police car in pursuit of a fleeing criminal. In this workshop, you will explore the following pursuit problem: consider a canal of width $w > 0$, see diagram below.



Relative to the xy-coordinate system indicated on the diagram, assume that the water in the canal is flowing in the positive y-direction with a speed $s \geq 0$ and that a swimmer enters the canal at the point $p = (w, 0)$. The swimmer then swims towards the point $q = (0, 0)$ always facing in the direction of q. Letting $(x, y) = (x(t), y(t))$ denote the position of the swimmer at time t and $\mathbf{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ their velocity, your objective is to determine the trajectory of swimmer as they move through the canal and attempt to get to the point q on the other side. You may assume that swimmer can swim at a constant speed $c > 0$ in still water and swims at this speed in the canal. In the above diagram, $\mathbf{v}_s$ denotes the velocity of the swimmer in still water and $\mathbf{v}_w$ is the velocity of the water in the canal, so that the total velocity of the swimmer is $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_w$.

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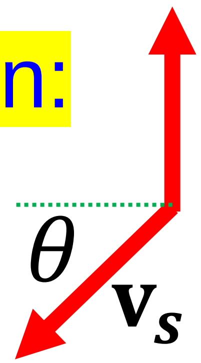
1. Express the velocity vector $\mathbf{v}$ in terms of x, y and the constants s, c.

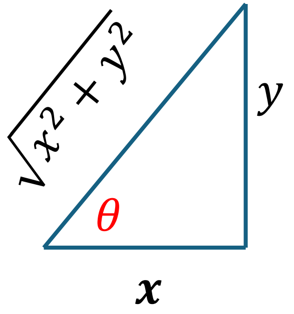
Given:



$\mathbf{v_w} = (v_{w,x}, v_{w,y})$
$= (0, s)$

$\mathbf{v_s} = (v_{s,x}, v_{s,y})$
$= (c, c)$

$$v_y = s - c\sin(\theta) = s - \frac{cy}{\sqrt{x^2+y^2}}$$



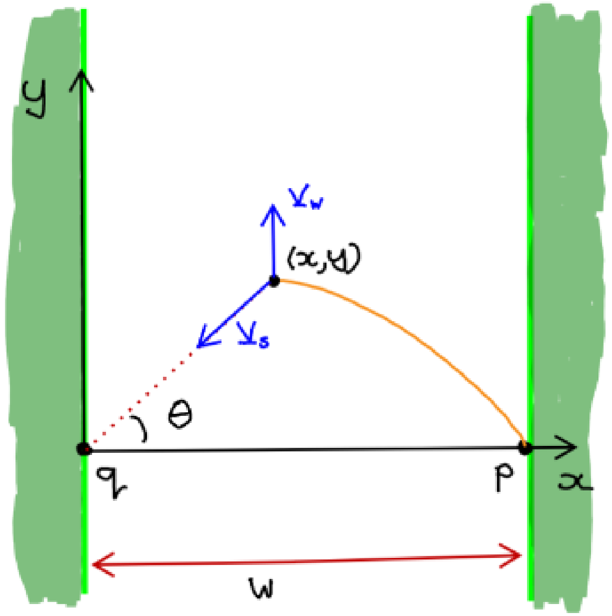
Change theta to x, y

$$v_x = -c\cos(\theta) = \frac{-cx}{\sqrt{x^2+y^2}}$$

$$\boxed{\mathbf{v} = \mathbf{v}_s + \mathbf{v}_w = \left(\frac{-cx}{\sqrt{x^2+y^2}}, s - \frac{cy}{\sqrt{x^2+y^2}}\right)}$$

$$\mathbf{v} = (v_x, v_y)$$

Relative to the xy-coordinate ... at the water in the canal is flowing in the positive y-direction ... nters the canal at the point $p = (w, 0)$. The swimmer then swims towards the point $q = (0, 0)$ always facing in the direction of q. Letting $(x, y) = (x(t), y(t))$ denote the position of the swimmer at time t and $\mathbf{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ their velocity, your objective is to determine the trajectory of swimmer as they move through the canal and attempt to get to the point q on the other side. You may assume that swimmer can swim at a constant speed $c > 0$ in still water and swims at this speed in the canal. In the above diagram, $\mathbf{v}_s$ denotes the velocity of the swimmer in still water and $\mathbf{v}_w$ is the velocity of the water in the canal, so that the total velocity of the swimmer is $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_w$.



2. Use the formula for $\mathbf{v}$ and the chain rule to calculate $\frac{dy}{dx}$. This will yield a first order differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

for an appropriate function $f(x, y)$ that also contains the constants s and c. **[3 marks]**

You know that: $v_y = \frac{dy}{dt}$ $v_x = \frac{dx}{dt}$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{cy - s\sqrt{x^2 + y^2}}{cx}$

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_w = \left(\frac{-cx}{\sqrt{x^2 + y^2}}, s - \frac{cy}{\sqrt{x^2 + y^2}} \right)$$

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3. Is this differential equation linear? Is it separable? Make sure you explain your answers.

$$\frac{dy}{dx} = \frac{-s\sqrt{x^2 + y^2} + cy}{cx}$$

- Non-linear
- Not Separable!

Why?

Show that the differential equation can be written as

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

where g is the function $g(u) = u - \frac{s}{c}\sqrt{1+u^2}$.

$$u = \frac{y}{x}, \qquad y = u \cdot x$$

$$\frac{dy}{dx} = \frac{-s\sqrt{x^2 + (ux)^2} + c(u \cdot x)}{cx}$$

$$\frac{dy}{dx} = u - \frac{s}{c}\sqrt{1+u^2}$$

$$\frac{dy}{dx} = \frac{-s\sqrt{x^2+y^2} + cy}{cx}$$

5. To solve the differential equation we make the substitution $u = \frac{y}{x}$. Show that with this substitution, your original first order differential equation of y and x turns into a *separable* differential equation of u and x. **[3 marks]**

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{dy}{dx}\frac{1}{x} - y\frac{1}{x^2}$$

$$\frac{du}{dx} = \frac{u - \frac{s}{c}\sqrt{1+u^2}}{x} - \frac{ux}{x^2} = \frac{-\frac{s}{c}\sqrt{1+u^2}}{x}$$

$$\frac{dy}{dx} = u - \frac{s}{c}\sqrt{1+u^2}$$

6. Solve the separable equation and use the initial condition $y(w) = 0$ to show that the trajectory of the swimmer is given by

$$y = y(x) = \frac{w}{2}\left(\left(\frac{x}{w}\right)^{1-\frac{s}{c}} - \left(\frac{x}{w}\right)^{1+\frac{s}{c}}\right)$$

Make sure you explain the main steps in the derivation of the solution. **[7 marks]**

$$\frac{du}{dx} = \frac{-\frac{s}{c}\sqrt{1+u^2}}{x}$$

$$\frac{dx}{x} = -\frac{c}{s}\frac{du}{\sqrt{1+u^2}}$$

$$\int \frac{dx}{x} = -\frac{c}{s}\int \frac{du}{\sqrt{1+u^2}}$$

- Useful

$$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\cosh(a)\,da}{\sqrt{1+\sinh^2(a)}} = \int \frac{\cosh(a)\,da}{\cosh(a)} = a = \sinh^{-1}(u) + C$$

- Step 6 (10mins)

7. Is it always possible for the swimmer to reach the point q for any choice of $s \geq 0$ and $c > 0$? If not, then determine the set of speeds (s, c) for which the swimmer is able to reach q. Can you provide an intuitive explanation of your findings? **[4 marks]**

In order for the swimmer to reach q, the trajectory must cross the point (0, 0). In other words y(0) = 0.

Let w=1

$$\boxed{\frac{s}{c} = 10, \frac{1}{10}, 1}$$

"So, there will be three cases to test:

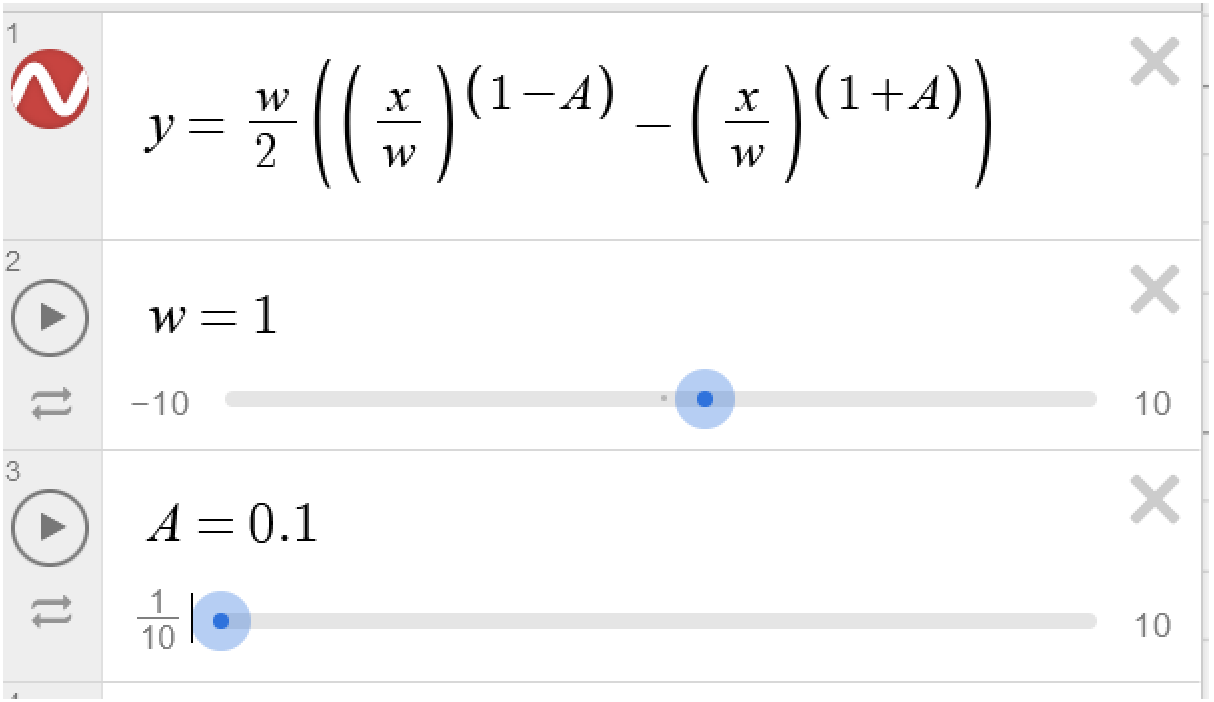
1. $s > c$
2. $s < c$
3. $s = c$"

$$\boxed{y = y(x) = \frac{w}{2}\left(\left(\frac{x}{w}\right)^{1-\frac{s}{c}} - \left(\frac{x}{w}\right)^{1+\frac{s}{c}}\right)}$$

Let w=1

$$\boxed{\frac{s}{c} = 10, \frac{1}{10}, 1}$$

Arbitrarily



5mins

What happen if w is not 1

$$\boxed{y = y(x) = \frac{w}{2}\left(\left(\frac{x}{w}\right)^{1-\frac{s}{c}} - \left(\frac{x}{w}\right)^{1+\frac{s}{c}}\right)}$$

$$w \neq 1$$

The Big Learning Outcomes for Week 10

After completing this week's task, you should be able to:

- Solve second-order constant coefficient boundary value problems and eigenvalue problems.
- Solve linear systems of ODEs.

Some parts are not examinable!

Thank You