ENG1005: Week 12 Applied class problem sheet

This problem sheet is intended for you to work through in your Applied Class in a small-group setting with the help of your instructor and your peers. **At the end of the applied class you will be asked to complete a quiz for credit. The quiz questions are based on the questions on this problem sheet. You may (and should!) ask your group members and your instructor for guidance if needed.**

1. (a) Calculate the first four terms of the Taylor series for $f(x) = e^x$ about $x = 2$.

   (b) Calculate the first four terms of the Taylor series for $f(x) = \sqrt{x}$ at $x = 1$.

   (c) Not for credit: Write down a formula for the general term for part (b).

2. If $f(x)$ has a Maclaurin series / Taylor series about $x = 0$ given by $f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots$, then provided $g(0) = 0$ the Maclaurin series / Taylor series about $x = 0$ for $f(g(x))$ is $f_0 + f_1 g(x) + f_2[g(x)]^2 + \ldots$.

   (a) Write down the Taylor series about $x = 0$ for the functions e^x, $\sin(x)$ and $1/(1-x)$ at $x = 0$. (You are strongly encouraged to remember these; you can look them up for the purposes of this question.)

   (b) Use suitable substitutions to deduce the Taylor series about $x = 0$ for the functions e^{-x}, $\sin(2x)$ and $1/(1+x^3)$.

3. Use L'Hôpital's rule to compute the following limits:

$$\text{(a)} \lim_{x\to 0} \frac{e^{3x}-1}{x} \qquad \text{and} \qquad \text{(b)} \lim_{x\to 0} \frac{e^x - 1 - x}{\sin^2(x)}\,.$$

4. We can also compute complicated Maclaurin series using multiple simple ones.

   (a) Compute the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$.

   (b) Hence obtain a Maclaurin series for

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)\,.$$

   (c) Compute the radius of convergence for the Maclaurin series in part (b).

5. This question continues Q2.

   (a) Write down the Maclaurin series for e^x.

   (b) Hence, write down the Maclaurin series for e^{-x^2}.

   (c) Use these to obtain an infinite series for the function

$$s(x) = \int_0^x e^{-u^2}\,du\,.$$

6. Prove that for any $n > 0$ $\lim_{x\to\infty} x^n e^{-x} = 0$.

7. Consider a function of two variables $f(x, y)$. The Taylor expansion about a point $\mathbf{x}_0 = (x_0, y_0)$ is given by

$$f(\mathbf{x}_0 + \mathbf{u}) = f(\mathbf{x}_0) + \mathbf{u} \cdot \boldsymbol{\nabla} f\big|_{\mathbf{x}=\mathbf{x}_0} + \frac{1}{2}(\mathbf{u} \cdot \boldsymbol{\nabla})^2 f\big|_{\mathbf{x}=\mathbf{x}_0} + \dots .$$

By carefully writing out the second-order term, show that it is equivalent to

$$\frac{1}{2}(\mathbf{u} \cdot \boldsymbol{\nabla})^2 f\big|_{\mathbf{x}=\mathbf{x}_0} = \frac{1}{2}\mathbf{u}^T \mathsf{H} \mathbf{u}\,,$$

where H is the Hessian matrix of f at $\mathbf{x}_0$.