ENG1005 Week 5: Applied class problem sheet solutions

Question 1

Part A

For the function $f(x,y) = \sin\left(\sqrt{x^2+y^2}\right)$, we want to find the contours $f(x,y) = 0$, $f(x,y) = 1$ and $f(x,y) = -1$. Solving $f(x,y) = 0$ for x and y,

$$\sin\left(\sqrt{x^2+y^2}\right) = 0 \implies \sqrt{x^2+y^2} = n\pi, \quad n = 0, 1, 2, \ldots$$

This gives the equation $x^2 + y^2 = (n\pi)^2$, so the contours $f(x,y) = 0$ are circles with radius $n\pi$ centred around the origin. Solving $f(x,y) = 1$ for x and y,

$$\sin\left(\sqrt{x^2+y^2}\right) = 1 \implies \sqrt{x^2+y^2} = \frac{(4n+1)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

This gives the equation $x^2+y^2 = ((4n+1)\pi/2)^2$, so the contours $f(x,y) = 1$ are circles with radius $(4n+1)\pi/2$ centred around the origin. Solving $f(x,y) = -1$ for x and y,

$$\sin\left(\sqrt{x^2+y^2}\right) = -1 \implies \sqrt{x^2+y^2} = \frac{(4n+3)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

This gives the equation $x^2 + y^2 = ((4n+3)\pi/2)^2$, so the contours $f(x,y) = -1$ are circles with radius $(4n+3)\pi/2$ centred around the origin. The contours are shown in Figure 1.

Parts B-D

Using the chain rule to calculate the partial derivatives,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos\left(\sqrt{x^2+y^2}\right)\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) \\
\frac{\partial f}{\partial x} &= \frac{x\cos\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}. \\
\frac{\partial f}{\partial y} &= \frac{y\cos\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}.
\end{aligned}$$

The gradient vector is $\nabla f(x,y) = (\partial f/\partial x, \partial f/\partial y)$, which is

$$\nabla f(x,y) = \left(\frac{x\cos\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}, \frac{y\cos\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}\right).$$

Evaluating at the point $(\pi/\sqrt{2}, \pi/\sqrt{2})$ gives

$$\nabla f\left(\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right) = \left(\frac{\pi \cos\left(\sqrt{\pi^2}\right)}{\sqrt{2}\sqrt{\pi^2}}, \frac{\pi \cos\left(\sqrt{\pi^2}\right)}{\sqrt{2}\sqrt{\pi^2}}\right).$$

$$\nabla f\left(\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

The direction of this vector is shown in Figure 1.

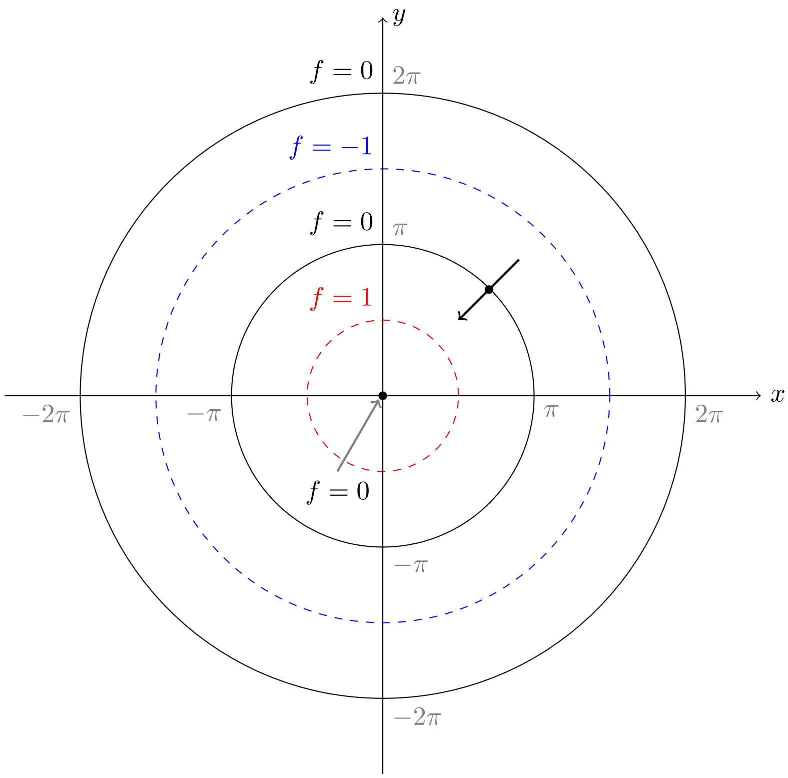

Figure 1: Contours of $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$ with direction of gradient shown at $(\pi/\sqrt{2}, \pi/\sqrt{2})$.

Question 2

To determine the sign of $\partial f/\partial x$ at the given point, we hold y constant while varying x. Graphically, we observe how f changes with x on a horizontal line around the point. Similarly, to determine the sign of $\partial f/\partial y$, we hold x constant while varying y and observe how f changes with y on a vertical line around the point.

At the given point on the contour plot, f decreases when x increases and f increases when y increases. Hence, $\partial f/\partial x < 0$ and $\partial f/\partial y > 0$ at the point.

Question 3

For a 3D surface in Cartesian coordinates $z = z(x, y)$ where gravity acts in the negative z-direction, a ball placed on the surface which is initially stationary will initially roll in the direction of steepest descent, which is in the opposite direction to the gradient vector. For the surface $z(x, y) = -xy^2 \cos(\pi x)$ and a starting position of $(1, 1)$, the gradient is

$$\nabla z(x, y) = \left(-y^2 \cos(\pi x) + \pi x y^2 \sin(\pi x), -2xy \cos(\pi x)\right).$$

A vector in the direction of steepest descent at $(1, 1)$ is

$$-\nabla z(1, 1) = -(1, 2).$$

Hence the ball will initially roll in the direction of $(-1, -2)$ or any positive scalar multiple of this.

Question 4

The mixed partial derivatives are $\dfrac{\partial^2 f}{\partial x\, \partial y}$ and $\dfrac{\partial^2 f}{\partial y\, \partial x}$. These are equal (subject to some very mild conditions that are usually true). Therefore we could just calculate one of the two derivatives. However, we will calculate both here. First of all

$$\frac{\partial f}{\partial x} = yx^{y-1}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}\left(\mathrm{e}^{y \log(x)}\right) = \log(x)\mathrm{e}^{y \log(x)} = \log(x)x^y.$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial f}{\partial y}\left(yx^{y-1}\right) &= x^{y-1} + y\frac{\partial f}{\partial y}\left(x^{y-1}\right) \\
&= x^{y-1} + y\,\frac{\partial f}{\partial y}\left(\mathrm{e}^{(y-1)\log(x)}\right) \\
&= x^{y-1} + y\log(x)\left(\mathrm{e}^{(y-1)\log(x)}\right) \\
&= x^{y-1} + y\log(x)x^{y-1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial f}{\partial x}\left(\log(x)x^y\right) &= \frac{1}{x}x^y + \log(x)yx^{y-1} \\
&= x^{y-1} + y\log(x)x^{y-1}
\end{aligned}$$

Question 5

First statement: If all the contours of a function $f(x, y)$ are parallel lines, then the surface given by $z = f(x, y)$ is a plane. This is **false**. For example, the function $f(x, y) = (x + y)^2$ has contours $x + y = \sqrt{C}$ where C is any non-negative constant, which correspond to parallel lines in the $x - y$ plane, but the surface is not a plane. If the contours given by $f(x, y) = C$ with equal increments in C are equally spaced parallel lines, then $f(x, y)$ is a plane.

Second statement: If $\partial f/\partial x = \partial f/\partial y$ everywhere, then $f(x, y)$ must be a constant. This is **false**. For example, the function $f(x, y) = x + y$ has partial derivatives $\partial f/\partial x = 1, \partial f/\partial y = 1$, but is not a constant function.

Third statement: There exists a function $f(x, y)$ with $\partial f/\partial x = 2y$ and $\partial f/\partial y = 2x$. This is **true** for the function $f(x, y) = 2xy + C$, where C is any constant.

Question 6

For the function $V(x, y, z) = 2y^2 - 3xy + yz^2$, the rate of change of V at $(1, 1, 1)$ in the direction of $\mathbf{v} = (1, 1, -1)$ is the directional derivative

$$\nabla_{\mathbf{v}} V(1, 1, 1) = \hat{\mathbf{v}} \cdot \nabla V(1, 1, 1),$$

where $\hat{\mathbf{v}}$ is a unit vector in the direction of $\mathbf{v}$. The length of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$, so

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}(1, 1, -1).$$

The gradient of V is

$$\nabla V(x, y, z) = (-3y, 4y - 3x + z^2, 2yz).$$

The directional derivative is

$$\hat{\mathbf{v}} \cdot \nabla V(1, 1, 1) = \frac{1}{\sqrt{3}}(1, 1, -1) \cdot (-3, 2, 2) = -\sqrt{3}.$$

Hence $\nabla_{\mathbf{v}} V(1, 1, 1) = -\sqrt{3}$. The function V increases most rapidly in the direction of the gradient, so at $(1, 1, 1)$ it increases most rapidly in the direction of $(-3, 2, 2)$, or any positive scalar multiple of this. The maximum rate of change is equal to the magnitude of the gradient, which is

$$|\nabla V(1, 1, 1)| = \sqrt{(-3)^2 + 2^2 + 2^2} = \sqrt{17}.$$

Therefore the maximum rate of change of V at $(1, 1, 1)$ is $\sqrt{17}$.

Question 7

We consider a point in 3D space $\mathbf{x} = (x, y, z)$, a constant vector $\mathbf{a} = (a_1, a_2, a_3)$ and define $r = |\mathbf{x}|$. To show the first statement,

$$\begin{aligned}
\nabla(r^3) &= \left(\frac{\partial}{\partial x} \left(\left(x^2 + y^2 + z^2\right)^{3/2} \right), \frac{\partial}{\partial y} \left(\left(x^2 + y^2 + z^2\right)^{3/2} \right), \frac{\partial}{\partial z} \left(\left(x^2 + y^2 + z^2\right)^{3/2} \right) \right), \\
\nabla(r^3) &= \left(3x \left(x^2 + y^2 + z^2\right)^{1/2}, 3y \left(x^2 + y^2 + z^2\right)^{1/2}, 3z \left(x^2 + y^2 + z^2\right)^{1/2} \right), \\
\nabla(r^3) &= 3r\mathbf{x}.
\end{aligned}$$

To show the second statement,

$$\begin{aligned}
\nabla(\mathbf{a} \cdot \mathbf{x}) &= \left(\frac{\partial}{\partial x}(a_1 x), \frac{\partial}{\partial y}(a_2 y), \frac{\partial}{\partial z}(a_3 z) \right), \\
\nabla(\mathbf{a} \cdot \mathbf{x}) &= (a_1, a_2, a_3) = \mathbf{a}.
\end{aligned}$$

Question 8

We need to calculate $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ for

$$u(x, t) = 1 - \operatorname{erf}(x/2\sqrt{t}) = 1 - \frac{2}{\pi} \int_0^{x/2\sqrt{t}} \mathrm{e}^{-y^2}\, \mathrm{d}y.$$

We have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(1) - \frac{2}{\pi}\frac{\partial}{\partial t}\left(\int_0^{x/2\sqrt{t}} \mathrm{e}^{-y^2}\,\mathrm{d}y\right)$$

Now, treating x as a constant in the derivative, this becomes

$$\frac{\partial u}{\partial t} = -\frac{2}{\pi}\frac{\partial}{\partial t}\left(\frac{x}{2\sqrt{t}}\right)\mathrm{e}^{-x^2/4t} = -\frac{2}{\pi}\left(-\frac{1}{2}\frac{x}{2t^{3/2}}\right)\mathrm{e}^{-x^2/4t} = \frac{2}{\pi}\left(\frac{x}{4t^{3/2}}\right)\mathrm{e}^{-x^2/4t}$$

Further, we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(1) - \frac{2}{\pi}\frac{\partial}{\partial x}\left(\int_0^{x/2\sqrt{t}} \mathrm{e}^{-y^2}\,\mathrm{d}y\right)$$

This time treating t as a constant in the derivative, we have

$$\frac{\partial u}{\partial x} = -\frac{2}{\pi}\frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{t}}\right)\mathrm{e}^{-x^2/4t} = -\frac{2}{\pi}\left(\frac{1}{2\sqrt{t}}\right)\mathrm{e}^{-x^2/4t}$$

Finally

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[-\frac{2}{\pi}\left(\frac{1}{2\sqrt{t}}\right)\mathrm{e}^{-x^2/4t}\right] = -\frac{2}{\pi}\left(\frac{1}{2\sqrt{t}}\right)(-2x)/(4t)\,\mathrm{e}^{-x^2/4t} = \frac{2}{\pi}\left(\frac{x}{4t^{3/2}}\right)\mathrm{e}^{-x^2/4t}$$

Comparing $\partial u/\partial t$ and $\partial^2 u/\partial x^2$, we see that they are equal and hence the function $u(x,t)$ given satisfies the equation.

Question 9

The mixed partial derivative can be written as

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

To determine the sign of the mixed partial derivative, we can observe how $\partial f/\partial x$ changes as y increases, or how $\partial f/\partial y$ changes as x increases. We choose a small and large value of y and observe how $\partial f/\partial x$ changes. From the contour plot in Figure 2, for $y = 0.5$, f decreases from 0 to -1 over the interval $0 < x < 2$, so $\partial f/\partial x$ is negative. When y increases to $y = 1.5$, f decreases from 0 to -3 over the same interval so $\partial f/\partial x$ is negative, with a larger rate of decrease compared to $y = 0.5$. Thus as y increases, $\partial f/\partial x$ decreases so the mixed partial derivative $\partial^2 f/\partial x\,\partial y$ is negative.

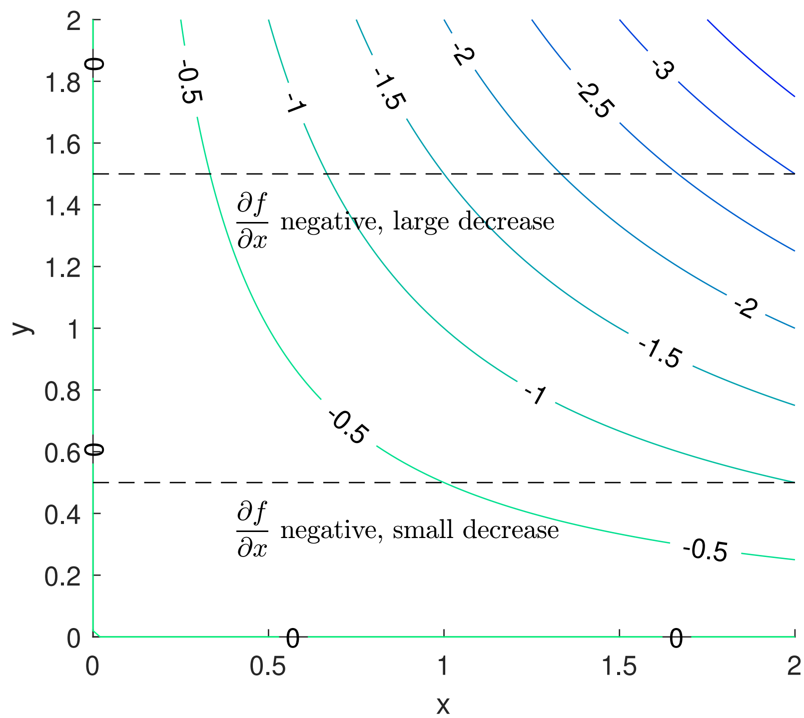

Figure 2: Contour plot for Q1 with sign and magnitude of $\partial f/\partial x$ indicated for small y and large y.