MONASH University

Eng. Math

ENG 1005

Week 7: Integration techniques & hyperbolic functions

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Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

The Big Learning Outcomes for Week 7

After completing this week's task, you should be able to:

- Identify when integration by parts is a relevant integration technique and apply it.
- Know the definitions of hyperbolic functions and their properties.
- Calculate derivatives of hyperbolic functions.
- Identify when a hyperbolic function substitution is a relevant integration technique.

Resources

1. Additional: Videos

2. Pass materials

Let us start!

Today’s Activity

1. Applied Problem Set

2. Applied Quiz

- ✓ Q1
- ✓ Q2
- ✓ Q4
- ✓ Q6
- ✓ Q7
- ✓ Q9

For Your Information! (1)

Understanding $\log(x)$ in Different Communities:

- Mathematics: $\log(x) = \ln(x)$ (base e).
- Sciences (e.g., Chemistry, Biology): $\log(x) = \log_{10}(x)$ (base 10).
- Computer Science: $\log(x) = \log_2(x)$ (base 2).

In Your Case: $\log(x) = \ln(x)$ (base e).

For Your Information! (2)

$$\boxed{\ln x = \log_e x = \log x}$$

$$\frac{d(\log x)}{dx} = \frac{1}{x}$$

$$\boxed{\log_{10} x = \lg x}$$

$$\frac{d(\lg x)}{dx} = \frac{1}{x \log(10)}$$

For Your Information! (3)

$$fg = \int f\,dg + \int g\,df$$

Re-Arranging

$$\boxed{\int f\,dg = fg - \int g\,df}$$

$$\int_0^{2\pi} \underbrace{\sin(4x)}_{f}\, \underbrace{\cos(5x)\,dx}_{dg}. = I$$

$$\boxed{\int f\,dg = I}$$

Question 1

1. Use integration by parts twice to find

$$\int_0^{2\pi} \sin(4x)\cos(5x)\,dx.$$

As an extension, can you use the same approach to find the value of

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx$$

for all integers m and n?

Learning Outcomes?

Step 1: First Integration of Part

$$I = \int_0^{2\pi} \sin(4x)\cos(5x)\,dx \rightarrow I = uv - \int v\,du$$

Using integration by parts, we start by choosing:

$$u = \sin(4x) \quad \text{and} \quad dv = \cos(5x)\,dx$$

Differentiating and integrating, we get:

$$du = 4\cos(4x)\,dx \quad \text{and} \quad v = \frac{1}{5}\sin(5x)$$

Substituting into the integration by parts formula:

$$I = uv - \int v\,du = \left[\frac{1}{5}\sin(4x)\sin(5x)\right]_0^{2\pi} - \frac{4}{5}\int_0^{2\pi} \cos(4x)\sin(5x)\,dx$$

You Try!

The boundary term evaluates to zero since:

$$\left[\frac{1}{5}\sin(4x)\sin(5x)\right]_0^{2\pi} = \frac{1}{5}(\sin(8\pi) - \sin(0)) = 0$$

Thus, the integral simplifies to:

$$I = -\frac{4}{5}\int_0^{2\pi} \cos(4x)\sin(5x)\,dx$$

Step 2: Second Integration of Part

$$I = -\frac{4}{5}\int_0^{2\pi} \cos(4x)\sin(5x)\,dx$$

$$I = uv - \int v\,du$$

We now evaluate the new integral:

$$J = \int_0^{2\pi} \cos(4x)\sin(5x)\,dx$$

Choosing:

$$u = \cos(4x) \quad \text{and} \quad dv = \sin(5x)\,dx$$

Differentiating and integrating:

$$du = -4\sin(4x)\,dx \quad \text{and} \quad v = -\frac{1}{5}\cos(5x)$$

Substituting into the integration by parts formula:

$$J = uv - \int v\,du = -\frac{1}{5}\left[\cos(4x)\cos(5x)\right]_0^{2\pi} - \frac{4}{5}I$$

The boundary term again evaluates to zero:

$$\left[\cos(4x)\cos(5x)\right]_0^{2\pi} = 0$$

So, we have:

$$J = -\frac{4}{5}I$$

Original I

$$I = \int_0^{2\pi} \sin(4x)\cos(5x)\,dx$$

Step 3: Solve for I

From Step 1: $I = -\frac{4}{5}\int_0^{2\pi} \cos(4x)\sin(5x)\,dx$ → $\boxed{I = -\frac{4}{5}J}$

From Step 2: $\boxed{J = -\frac{4}{5}I}$

Substituting for J, we get:

$$I = -\frac{4}{5}\left(-\frac{4}{5}I\right) = \frac{16}{25}I$$

Solving for I:

$$I = \frac{16}{25}I \quad \Rightarrow \quad I - \frac{16}{25}I = 0 \quad \Rightarrow \quad \frac{9}{25}I = 0 \quad \Rightarrow \quad I = 0$$

Step 4: Conclusion!

Substituting for J, we get:

$$I = -\frac{4}{5}\left(-\frac{4}{5}I\right) = \frac{16}{25}I$$

Solving for I:

$$I = \frac{16}{25}I \quad \Rightarrow \quad I - \frac{16}{25}I = 0 \quad \Rightarrow \quad \frac{9}{25}I = 0 \quad \Rightarrow \quad I = 0$$

This result follows from the periodic nature of the sine and cosine functions, where their products over a complete period cancel out

Question 2

2. Use integration by parts (potentially combined with a substitution), to find

(a) $\int \sin(x)\ \ln(\cos(x))\, dx$

Learning Outcomes?

LIATE RULE!

f: Logarithmic

g: Trigonometric

L	Logarithmic functions
I	Inverse trig. functions
A	Algebraic functions
T	Trig. functions
E	Exponential functions

$$\int f\, dg = fg - \int g\, df$$

Step 1: Integration by Part

$$\int \sin(x)\ \ln(\cos(x))\, dx$$

$$\int f\, dg = fg - \int g\, df$$

$$f(x) = \ln\left(\cos\left(x\right)\right), \quad g(x) = -\cos\left(x\right),$$
$$f'(x) = -\frac{\sin\left(x\right)}{\cos\left(x\right)}, \quad g'(x) = \sin\left(x\right),$$

Step 2: Integration by Part

$$\int \sin\left(x\right) \ln\left(\cos\left(x\right)\right) dx = -\cos\left(x\right) \ln\left(\cos\left(x\right)\right) - \int \sin\left(x\right) dx.$$

$$\int \sin\left(x\right) \ln\left(\cos\left(x\right)\right) dx = -\cos\left(x\right) \ln\left(\cos\left(x\right)\right) + \cos\left(x\right) + C,$$

Question 4

4. Evaluate

$$\frac{d}{dx}(\cosh^{-1} x)$$

[Hint: you might find implicit differentiation useful.]

Step 1

$$\frac{d}{dx}(\cosh^{-1} x)$$

To do this, we start by letting $y = \cosh^{-1}(x)$. By the definition of the inverse hyperbolic cosine function, this implies:

$$x = \cosh(y).$$

Next, we differentiate both sides of the equation $x = \cosh(y)$ with respect to x using implicit differentiation:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cosh(y)).$$

Step 2

The derivative of x with respect to x is simply 1, and using the chain rule, the derivative of $\cosh(y)$ with respect to x is $\sinh(y) \cdot \frac{dy}{dx}$. Therefore, we have:

$$1 = \sinh(y)\frac{dy}{dx}.$$

To find $\frac{dy}{dx}$, we solve for it by dividing both sides of the equation by $\sinh(y)$:

$$\frac{dy}{dx} = \frac{1}{\sinh(y)}.$$

Step 3

Next, we need to express $\sinh(y)$ in terms of x. Using the identity $\cosh^2(y) - \sinh^2(y) = 1$, we can solve for $\sinh^2(y)$:

$$\sinh^2(y) = \cosh^2(y) - 1.$$

Since $x = \cosh(y)$, we substitute to get:

$$\sinh^2(y) = x^2 - 1.$$

Taking the square root of both sides gives:

$$\sinh(y) = \sqrt{x^2 - 1}.$$

Step 4

$$\boxed{x = \cosh(y).}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sinh(y)}.}$$

Finally, substitute this expression for $\sinh(y)$ back into the equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}.$$

Thus, the derivative of the inverse hyperbolic cosine function is:

$$\frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{x^2 - 1}}.$$

We show that:

$$\boxed{\frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{x^2-1}}.}$$

You show this at home:

$$\boxed{\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}}$$

Question 6(c)

6. Show that

(a) $\sinh(-x) = -\sinh(x)$

(b) $\cosh(-x) = \cosh(x)$

(c) $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Step 1: Expanding the Terms!

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \cosh(y) = \frac{e^y + e^{-y}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \sinh(y) = \frac{e^y - e^{-y}}{2}$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Step 2: Sub. Into the pink-boxed Eq.

$$\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)$$

$$\downarrow$$

$$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-(x+y)}}{4}+\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-(x+y)}}{4}$$

$$\downarrow$$

$$\boxed{=\frac{e^{x+y}+e^{-(x+y)}}{2}=\cosh(x+y).}$$

$$\cosh(x)=\frac{e^x+e^{-x}}{2},\quad \cosh(y)=\frac{e^y+e^{-y}}{2}$$

$$\sinh(x)=\frac{e^x-e^{-x}}{2},\quad \sinh(y)=\frac{e^y-e^{-y}}{2}$$

$$\cosh(x+y)=\cosh(x)\cosh(y)+\sinh(x)\sinh(y)$$

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Question 7

7. Find an explicit expression for inverse sinh and inverse tanh.

Learning Outcomes?

Hyperbolic functions

Step 1: By definition, we write (anyone did not get this?)

$$x = \frac{e^y - e^{-y}}{2} = \sinh\ y$$

Step 2: By rearranging above, we can then write: (anyone did not get this?)

$$(e^y)^2 - 2xe^y - 1 = 0.$$

Step 3: Solve the quadratic Eq.

$$e^y = \frac{2x + \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2 + 1}$$

$$\boxed{y = \sinh^{-1}(x)}$$

$$\boxed{y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)}$$

Question 9

9. Use the identity $\cosh^2(x) - \sinh^2(x) = 1$ and integration by parts to express

$$I_n(x) = \int \cosh^n(x)\,dx$$

in terms of $I_{n-2}(x)$. Using this relationship with $n = -2$, find

$$\int_0^{\ln(2)} \frac{1}{\cosh^4(x)}\,dx.$$

Learning Outcomes?

Complex Integration by Parts

Try At Home!

- **Step 1: Given Integral (do some manipulation)**

$$I_n(x) = \int \cosh^n(x)\,dx = \boxed{\int \cosh^{n-1}(x)\cosh(x)\,dx}$$

- **Step 2: Integration by Parts**

$$\text{Let } f(x) = \cosh^{n-1}(x), \quad g(x) = \sinh(x)$$

$$\downarrow$$

$$f'(x) = (n-1)\sinh(x)\cosh^{n-2}(x), \quad g'(x) = \cosh(x)$$

$$\downarrow$$

$$I_n(x) = \sinh(x)\cosh^{n-1}(x) - (n-1)\int \sinh^2(x)\cosh^{n-2}(x)\,dx$$

- **Step 3: Invoke Identity**

$$\boxed{I_n(x) = \sinh(x)\cosh^{n-1}(x) - (n-1)\int \sinh^2(x)\cosh^{n-2}(x)\,dx}$$

Use the identity $\sinh^2(x) = \cosh^2(x) - 1$

$$I_n(x) = \sinh(x)\cosh^{n-1}(x) - (n-1)\int (\cosh^2(x) - 1)\cosh^{n-2}(x)\,dx$$

- **Step 4: Cleaning up your equation (expand and reduce)**

$$I_n(x) = \sinh(x)\cosh^{n-1}(x) - (n-1)\boxed{\int \cosh^n(x)\,dx} + (n-1)\boxed{\int \cosh^{n-2}(x)\,dx}$$

$$\boxed{I_n(x) = \int \cosh^n(x)\,dx}$$

$$I_n(x) = \sinh(x)\cosh^{n-1}(x) - (n-1)I_n(x) + (n-1)I_{n-2}(x)$$

$$\boxed{I_n(x) = \int \cosh^n(x)\,dx}$$

- **Step 5: Rearranging** $\cancel{I_n}(x) = \sinh(x)\cosh^{n-1}(x) - (n - \cancel{1})I_n(x) + (n-1)I_{n-2}(x)$

$$nI_n(x) = \sinh(x)\cosh^{n-1}(x) + (n-1)I_{n-2}(x)$$

- **Step 6: Find a specific Integral (n=4)**

$$\text{To find } I_{-4}(x) = \int \frac{1}{\cosh^4(x)}\,dx$$

Use the recurrence relation with $n = -2$

$$-2I_{-2}(x) = \sinh(x)\cosh^{-3}(x)\boxed{-3I_{-4}(x)}$$

- ## Step 7: Know your integral

$$I_{-2}(x) = \int \frac{1}{\cosh^2(x)}\,dx = \int \operatorname{sech}^2(x)\,dx = \tanh(x) + D$$

- ## Step 8: Can you find this?

$$I_{-4}(x) = \frac{\sinh(x)\cosh^{-3}(x) + 2\tanh(x)}{3} + C$$

where C is a constant of integration.

$$\boxed{\int_0^{\ln(2)} \frac{1}{\cosh^4(x)}\,dx = \frac{66}{125}}$$

- ## Step 9: Final!

$$I_{-4}(\ln(2)) - I_{-4}(0) = \int_0^{\ln(2)} \frac{1}{\cosh^4(x)}\,dx$$

$$I_{-4}(\ln(2)) = \frac{1}{3}\left(\frac{3}{4}\left(\frac{5}{4}\right)^{-3} + 2\left(\frac{3}{5}\right)\right) = \frac{66}{125}$$

$$I_{-4}(0) = 0$$

$$\boxed{-2I_{-2}(x) = \sinh(x)\cosh^{-3}(x) - 3I_{-4}(x)}$$

$$\boxed{I_n(x) = \int \cosh^n(x)\,dx}$$

Thank You