MONASH University

Eng. Math

ENG 1005

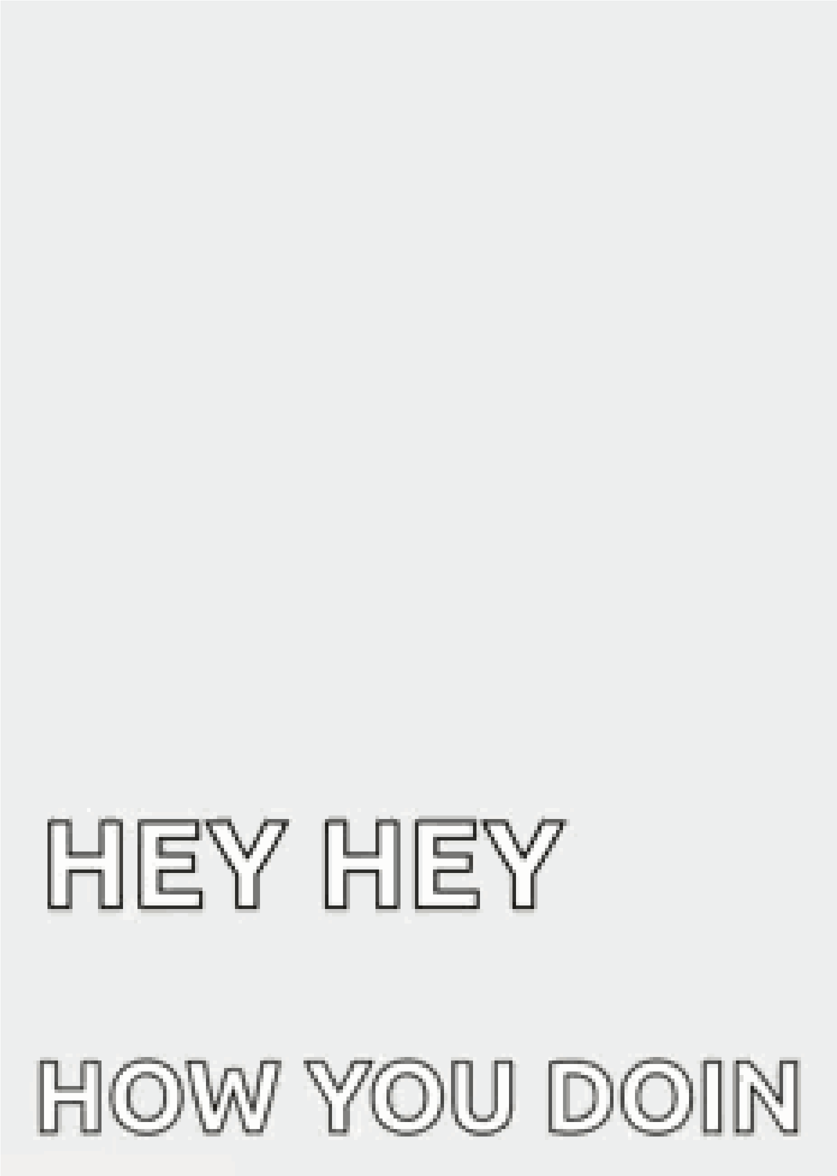
Week 4: Eigenvalues/Eigenvectors

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

kekboon.goh@monash.edu



Check-In

Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

The Big Learning Outcomes for Week 4

After completing this week's task, you should be able to:

•Understand what eigenvalues and eigenvectors are in the framework of matrices as linear transformations of space.

•Calculate eigenvalues and eigenvectors for 2x2 and 3x3 systems.

•Understand geometric and algebraic multiplicity.

•Identify diagonalisable matrices.

•Diagonalise 2x2 and 3x3 diagonalisable matrices.

Attendance Codes (Week 4)

International students

Tutorial	Wednesday, 14 Aug	02	8:00AM	5DBJ8
Tutorial	Wednesday, 14 Aug	01	2:00PM	KNZDK

Workshop	Thursday, 15 Aug	01	1:00PM	H27CE
Workshop	Friday, 16 Aug	02	10:00AM	DEW3D

Admin. Stuff (1)

1. Feedback on Workshop Submission: On Thur/Fri

2. Pls join our MS TEAM group: (90%)_Email me

3. Submissions:

Summary

ASSESSMENT	DUE
Kick Starting Week 3: Lecture Quiz 3 (Total mark for all 12 weeks of lecture quizzes is 5%)	Tuesday, 6 August 2024, 11:55 PM **Due in 4 days**
Applied class quiz week 3 (Total mark for all 12 weeks of applied quizzes is 5%)	Wednesday, 7 August 2024, 11:55 PM **Due in 5 days**
Workshop 3 problem set (Total mark for all 12 weeks of workshop sets is 20%)	Sunday, 11 August 2024, 11:55 PM

Admin. Stuff (2)

4. Consultation/Feedback hour

- Wed: 10 am till 11 am
- Fri: 8 am till 9 am

Location: 5-4-68

- Sat: 1030 am till 1130 am

None

(every odd week, e.g. 3-Week, but msg me first at least a day before; appointment-based.)

Resources

1. PASS with Yvonne and Zi Wei

2. Math Centre

Let us start!

Today’s Activity

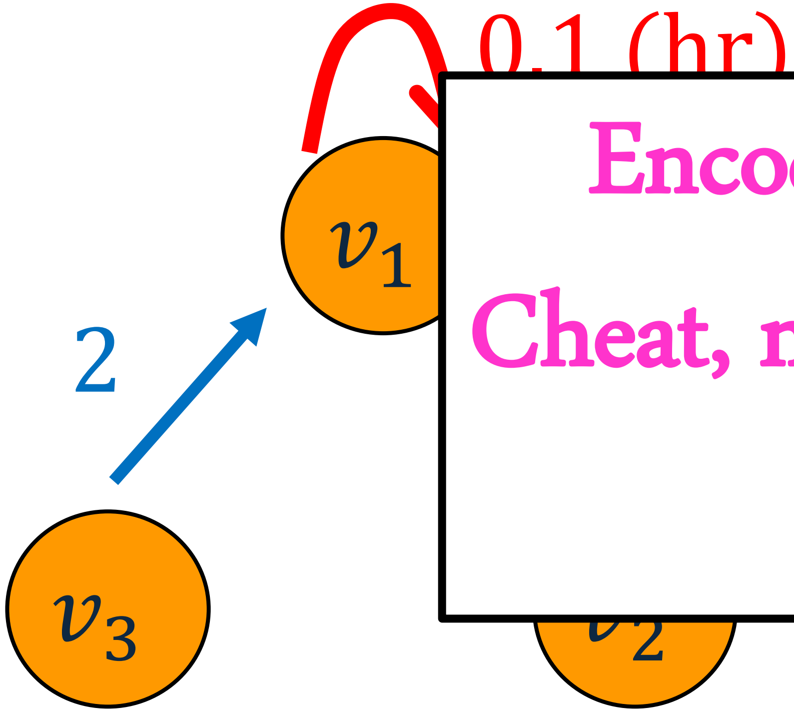
0. Refresher!

1. Applied Problem Set

2. Applied Quiz

Some facts!

As an underrated fact of linear algebra: Matrices=Graphs



(final) To

Encoding Matrices as Graphs is a Cheat, making Complex Stuff Easier to Study.

(init

v_3 2

Graph

Matrix

What is Eigen?

German word: “Own” or “itself”

- Eigenvector captures the “direction” of the vector after space transformation;

- Being it old self (eigen), or a new vector (direction).

The general formula (ii)

$$\underbrace{A}_{\text{Transformation matrix}}\mathbf{v} = \underbrace{\lambda\mathbf{v}}_{\text{Scalar multiplication}}$$

Eigenvector

$$A\mathbf{v} = \underbrace{(\lambda I)}_{\text{Multiple with I does nothing}}\mathbf{v}$$

$$(A - \lambda I)\mathbf{v} = 0$$

MONASH University

The general formula (i)

$$(A - \lambda I)\mathbf{v} = 0$$

- Given that **v** is non-zero, making life more interesting/difficult, what we can do next?

❖ If there are eigen, same direction, the determinant is?

❖ Area/volume is 0, same line.

❖ Is this invertible?

No; no unique solutions

$$\det(A - \lambda I) = 0$$

Some terminology (1)

- Linearly Independent:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In $\mathbb{R}^2$, $\mathbf{v}_1$ and $\mathbf{v}_2$ correspond to the standard basis vectors. $\mathbf{v}_1$ points along the x-axis, and $\mathbf{v}_2$ points along the y-axis. Since they are not collinear (they don't lie on the same line), they are independent.

Some terminology (2)

- **Algebraic multiplicity** of an eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial.

➢ For instance, in the eigenvalues λ_1 and λ_2, distinct values, each eigenvalue has an (one) associated algebraic multiplicity.

➢ In another instance, suppose A has a (twice) repeated eigenvalue λ, it is associated with algebraic multiplicity 2.

Some terminology (3)

- **Geometric multiplicity** of an eigenvalue λ is the dimension of the eigenspace corresponding to λ, which is the null space of $A-\lambda I$.

In simpler words, how many linearly independent eigenvectors correspond to λ

For repeated eigenvalues, they can have **single or infinity many directions**, but more productive to report linearly independent eigenvectors

Some terminology (4)

- A matrix can only be diagonalized if the algebraic multiplicity is equal to the geometric multiplicity for every eigenvalue

Example 2

Diagonalise the matrix $A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

Repeated eigenvalues

This matrix has a single eigenvalue $\lambda = 1$ and a single eigenvector direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and hence a single linearly independent eigenvector

This matrix cannot be diagonalised

- This example 2: the algebraic multiplicity, 2, is unequal to the geometric multiplicity that is 1.

Why Study Eigens? Comp. Science

$$A^n = VD^nV^{-1}$$

$$D^n = \begin{Bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{Bmatrix}$$

- ***V: columns of eigenvectors***
- ***D: diagonal matrix with eigenvalues***

Question 1

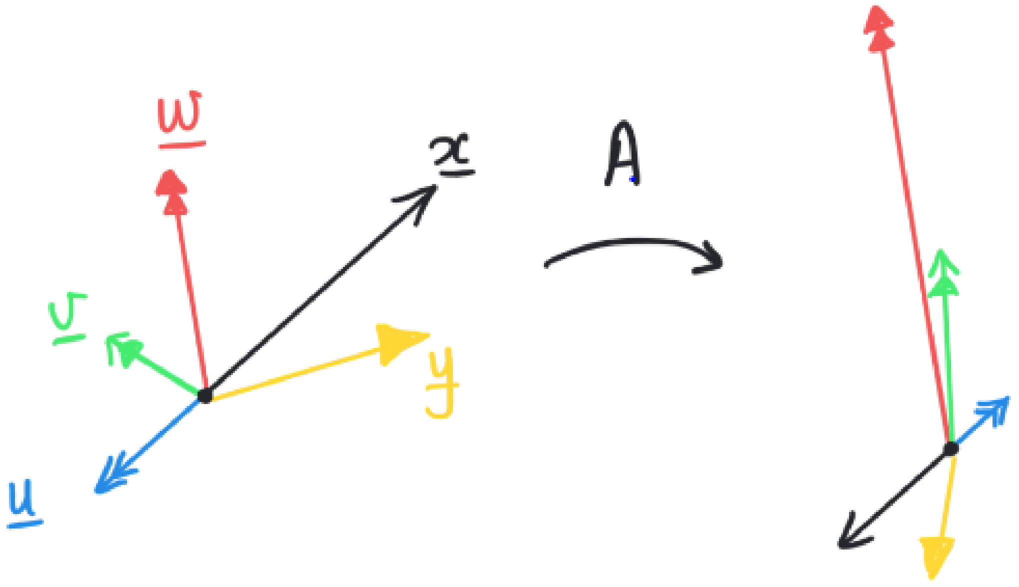
The vector **u** points along the same line after transformation by A, although in the opposite sense. The fact it points along the same line means it is an eigenvector. The fact it has the opposite sense means the eigenvalue is negative. The vector is also half the length so the eigenvalue is -0.5. The same is true for the vector **x**.

The vector **v** does not point along the same line after transformation. Therefore it is not an eigenvector. The same is true for the vector **y**.

The vector **w** points along the same line after transformation. Therefore it is an eigenvector. The vector has not changed sense, so the eigenvalue is positive. It is twice the length so the eigenvalue is 2.

Learning Outcomes?

Understanding eigenvalues and eigenvectors are in the framework of matrices as linear transformations of space.



- The vector u/x points along the same line after transformation by A, although in the opposite sense: the eigenvalue is negative. The vector is also half the length so the eigenvalue is −0.5.

- The vector w points along the same line after transformation: an eigenvector with a positive 2 eigenvalue.

- The vector v/y does not point along the same line after transformation. Therefore it is not an eigenvector.

Question 2(a)

2. Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where a is a real number with $a \geq 0$.

(a) Find the eigenvalues and eigenvectors of this matrix.

(b) For one value of a there is only one distinct eigenvalue. What is this value? How many eigenvector directions are there for this value of a?

(c) For one value of a the eigenvectors are orthogonal (perpendicular). What is this value?

(d) For what value(s) of a is the matrix diagonalisable? Write down a diagonalisation where possible.

(e) For what value(s) of a is it possible to write down a diagonalisation of the form $\mathsf{A} = \mathsf{VDV}^T$? Write down such a diagonalisation where possible, where D has increasing values on the diagonal and v_{11} and v_{12} are both positive.

Learning Outcomes?

Calculating the Eigens.

The Eigens

1. Values $\det(A - \boldsymbol{\lambda I}) = \mathbf{0}$

2. Vectors $(A - \boldsymbol{\lambda I})\mathbf{v} = \mathbf{0}$

- The eigenvalues **λ** are found by solving the characteristic equation:

$$\boxed{\det(A - \lambda I) = 0}$$

- Solve this matrix:

$$\boxed{A - \lambda I} = \underbrace{\begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}}_{\textbf{A}} - \lambda \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\textbf{Identity Matrix}} = \begin{pmatrix} 1-\lambda & 2 \\ 2a^2 & 1-\lambda \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}}$$

- Combining the two steps, we get

$$\det(A - \lambda I) = 0$$

We know that:

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 2a^2 & 1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda) - (2)(2a^2) = 0$$

- We arrive to:

$$(1-\lambda)^2 - 4a^2 = 0$$

$$\lambda^2 - 2\lambda + (1 - 4a^2) = 0$$

Use your quadratic formula:

$$\lambda_1 = 1 + 2a \quad \text{and} \quad \lambda_2 = 1 - 2a$$

$$\boxed{A = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}}$$

The Eigens

1. Values $\det(A - \boldsymbol{\lambda I}) = 0$

2. Vectors $(A - \boldsymbol{\lambda I})\mathbf{v} = 0$

$$\boxed{A = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}}$$

- The eigenvectors we now solve:

$$(A - \lambda I)\mathbf{v} = 0$$

- For : $\lambda_1 = 1 + 2a \quad \text{and} \quad \lambda_2 = 1 - 2a$

$$A = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}$$

$$(A - \lambda I)\mathbf{v} = 0$$

- For : $\lambda_1 = 1 + 2a$

➢ Expanding:

$$A - \lambda_1 I = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix} - \begin{pmatrix} 1+2a & 0 \\ 0 & 1+2a \end{pmatrix} = \begin{pmatrix} -2a & 2 \\ 2a^2 & -2a \end{pmatrix}$$

➢ Multiply with **v**:

$$\begin{pmatrix} -2a & 2 \\ 2a^2 & -2a \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

➢ Solving:

$$-2av_1 + 2v_2 = 0 \quad \text{(Equation 1)}$$

$$2a^2 v_1 - 2av_2 = 0 \quad \text{(Equation 2)}$$

➢ Either eq 1 or 2

$$\longrightarrow \quad v_2 = av_1 \quad \longrightarrow$$

➢ Eigenvector 1:

$$\mathbf{v}_1 = \begin{pmatrix} v_1 \\ av_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

➢ Now you try for the second one!

$$\lambda_1 = 1 + 2a \quad \text{and} \quad \lambda_2 = 1 - 2a$$

$$A = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}$$

$$(A - \lambda I)\mathbf{v} = 0$$

- For : $\lambda_2 = 1 - 2a$

➢ Expanding:

$$A - \lambda_2 I = \begin{pmatrix} 2a & 2 \\ 2a^2 & 2a \end{pmatrix}$$

➢ Multiply with v:

$$\begin{pmatrix} 2a & 2 \\ 2a^2 & 2a \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

➢ Solving:

$$2av_1 + 2v_2 = 0 \quad \text{(Equation 3)}$$

$$2a^2v_1 + 2av_2 = 0 \quad \text{(Equation 4)}$$

➢ Either eq 3/4

$$v_2 = -av_1$$

➢ Eigenvector 2:

$$\mathbf{v}_2 = \begin{pmatrix} v_1 \\ -av_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -a \end{pmatrix}$$

In Short

The eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix}$ are:

$$\text{For } \lambda_1 = 1 + 2a, \text{ the eigenvector is } \mathbf{v}_1 = v_1 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

$$\text{For } \lambda_2 = 1 - 2a, \text{ the eigenvector is } \mathbf{v}_2 = v_1 \begin{pmatrix} 1 \\ -a \end{pmatrix}$$

where v_1 is an arbitrary scalar.

Question 2(b)

2. Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where a is a real number with $a \geq 0$.

(a) Find the eigenvalues and eigenvectors of this matrix.

(b) For one value of a there is only one distinct eigenvalue. What is this value? How many eigenvector directions are there for this value of a?

For $\lambda_1 = 1 + 2a$, the eigenvector is $\mathbf{v}_1 = v_1 \begin{pmatrix} 1 \\ a \end{pmatrix}$

For $\lambda_2 = 1 - 2a$, the eigenvector is $\mathbf{v}_2 = v_1 \begin{pmatrix} 1 \\ -a \end{pmatrix}$

- ❖ A value of **a**, giving us only one distinct eigenvalue? Given we have two calculates values.
  - ➢ With $a = 0$, we have λ_1 and λ_2 reduce to 1.

$$A = \begin{pmatrix} 1 & 2 \\ 2\cancel{a^2}\,\mathbf{0} & 1 \end{pmatrix}$$

$$(A - \lambda I)\mathbf{v} = 0$$

- For : $\lambda = 1$ with $a = 0$

➢ Expanding: ↓

$$A - \lambda I = \begin{pmatrix} 1-1 & 2 \\ 0 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

➢ Multiply with v: ↓

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

↓

➢ We know: $v_2 = 0$ ⟶ $\mathbf{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

➢ Sub. value v_2=0, allowing **v** to be non-zero:

Question 2(c)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where a is a real number $\boxed{\text{with } a \geq 0.}$

(a) Find the eigenvalues and eigenvectors of this matrix.

(b) For one value of a there is only one distinct eigenvalue. What is this value? How many eigenvector directions are there for this value of a?

(c) For one value of a the eigenvectors are orthogonal (perpendicular). What is this value?

For $\lambda_1 = 1 + 2a$, the eigenvector is $\mathbf{v}_1 = v_1 \begin{pmatrix} 1 \\ a \end{pmatrix}$

For $\lambda_2 = 1 - 2a$, the eigenvector is $\mathbf{v}_2 = v_1 \begin{pmatrix} 1 \\ -a \end{pmatrix}$

Recall Dot Product Rule!

$$\overbrace{\begin{pmatrix} 1 \\ -a \end{pmatrix}}^{\text{Direction Vector 1}} \cdot \underbrace{\begin{pmatrix} 1 \\ a \end{pmatrix}}_{\text{Direction Vector 2}} = 0$$

$$a = ?$$

$$a = 1$$

Question 2(d)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where a is a real number with $a \geq 0$.

(d) For what value(s) of a is the matrix diagonalisable? Write down a diagonalisation where possible.

Learning Outcomes?

What is diagonalizable?

PITSTOP: A Refresher

Examples using a 2 by 2 matrix:

$$\boxed{A = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}}$$

- Eigenvalues: $\lambda_1=4$ and $\lambda_2=3$.

- With distinct eigenvalues, the matrix is diagonalizable.

$$\boxed{B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}$$

- Eigenvalues: $\lambda=2$ (repeated)

There is a repeated eigenvalue, whether or not the matrix can be diagonalized depends on the eigenvectors now.

- This matrix does not have two linearly independent eigenvectors, so it is not diagonalizable.

❑Practical Implications?

Diagonalization is useful because diagonal matrices are easier to work with, especially for computing powers of large matrices or solving huge systems of linear equations.

For instance, when a matrix is diagonalizable, it simplifies the process of exponentiation and can be applied in various fields, such as differential equations and quantum mechanics.

Return: Question 2(d)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where a is a real number with $a \geq 0$.

(d) For what value(s) of a is the matrix diagonalisable? Write down a diagonalisation where possible.

Learning Outcomes?

What is diagonalizable?

Diagonalization is a process of converting a matrix into a diagonal matrix that is similar to the original matrix. A matrix ***A*** is diagonalizable if it can be written in the form:

$$\boxed{A = VDV^{-1}}$$

- ❑ ***V: columns of eigenvectors***
- ❑ ***D: diagonal matrix with eigenvalues***

where:

- D is a diagonal matrix (a matrix with non-zero elements only on its main diagonal).
- V is an invertible matrix (a matrix that has an inverse).
- V^{-1} is the inverse of V.

We require that the matrix has two distinct eigenvalues.

$$\det\left(\begin{pmatrix} 1 & 2 \\ 2a^2 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 2 \\ 2a^2 & 1-\lambda \end{pmatrix}\right)$$

- Equate to zero : ↓

$$\lambda^2 - 2\lambda + 1 - 4a^2 = 0$$

The discriminant of this quadratic equation is?

is diagonalizable for any a values except a=0.

$$\det(A - \lambda I) = 0$$

- Two distinct roots: ↓

$$16a^2 > 0$$

$$A = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

For the quadratic equation $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the discriminant. The value of the discriminant shows how many roots f(x) has:

- If $b^2 - 4ac > 0$ then the quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then the quadratic function has one repeated real root.

Question 3

A 2×2 matrix has eigenvalues 9 and -18. It has corresponding eigenvectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. What is the matrix? *Hint: Once you have an answer, you should check it, either using your favourite calculator or double checking the eigenvectors given are indeed eigenvectors of the matrix you have found.*

Learning Outcomes?

Inverse engineering,
using the diagonalizable formula!

- We can use: $\boxed{A = VDV^{-1}}$
  - A reminder:
    - ***V: columns of eigenvectors***
    - ***D: diagonal matrix with eigenvalues***
- We then have
  - V inverse?:

$$\underbrace{\boxed{D = \begin{bmatrix} 9 & 0 \\ 0 & -18 \end{bmatrix}}}_{\textbf{Eigenvalues}} \qquad \underbrace{\boxed{V = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}}}_{\textbf{Eigenvectors}} \Rightarrow \boxed{V^{-1} = \frac{1}{9}\begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}}$$

A 2×2 matrix has eigenvalues 9 and -18. It has corresponding eigenvectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

❑ Mix them well

$$A = VDV^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -18 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -3 & -6 \\ -30 & -6 \end{bmatrix}}$$

$$\boxed{D = \begin{bmatrix} 9 & 0 \\ 0 & -18 \end{bmatrix}} \quad \boxed{V = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}} \quad \boxed{V^{-1} = \frac{1}{9} \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}}$$

Question 4

Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

Calculate the eigenvalues and eigenvectors of A.

Learning Outcomes?

3 by 3 Systems

$$\boxed{\det(A-\lambda I)=0}$$

❑ Eigenvalues

$$\det(A-\lambda I)=\begin{vmatrix}4-\lambda & 1 & -1\\ 1 & 2-\lambda & 1\\ -1 & 1 & 4-\lambda\end{vmatrix}$$

➢ Expanding: ↓

$$\det(A-\lambda I)=(4-\lambda)\begin{vmatrix}2-\lambda & 1\\ 1 & 4-\lambda\end{vmatrix}-1\begin{vmatrix}1 & 1\\ -1 & 4-\lambda\end{vmatrix}+(-1)\begin{vmatrix}1 & 2-\lambda\\ -1 & 1\end{vmatrix}$$

➢ Solv. Det.: ↓

$$\det(A-\lambda I)=(4-\lambda)(\lambda^2-6\lambda+7)-(5-\lambda)-(3-\lambda)$$

➢ Equate to 0: ↓

$$-\lambda^3+10\lambda^2-29\lambda+20=0$$

$$\boxed{\mathrm{A}=\begin{bmatrix}4 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 4\end{bmatrix}}$$

The eigenvalues are 1, 4 and 5.

MONASH University

Try At Home for the Next One!

$$\boxed{\det(A - \lambda I) = 0}$$

❑ Eigenvectors

- ➢ See the worked example in Q2
- ➢ You can find the step in Vid number 7
- ➢ Pls have a look at it to guide you in reaching the solutions

$$\boxed{A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}}$$

Question 5

Suppose the matrix A has eigenvector $\mathbf{v}$ with corresponding eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of A^n. What is its corresponding eigenvalue? If A is an invertible matrix, can you deduce the eigenvalues and eigenvectors of A^{-1}?

Learning Outcomes?

To prove:

$\mathbf{v}$ is also an eigenvector of A^n.

The corresponding eigenvalue is λ^n.

We know that $A\mathbf{v} = \lambda\mathbf{v}$

For: A^1

$$A^1\mathbf{v} = A\mathbf{v} = \lambda\mathbf{v}$$

For: A^2

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v})$$

$$= \lambda(A\mathbf{v})$$

$$= \lambda(\lambda\mathbf{v})$$

$$A^2\mathbf{v} = \lambda^2\mathbf{v}$$

$$\boxed{A^n\mathbf{v} = \lambda^n\mathbf{v}}$$

This shows that $\mathbf{v}$ is indeed an eigenvector of A^n with eigenvalue λ^n.

Question 6

For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

what are the eigenvalues? Can you find a corresponding eigenvector for one of these eigenvalues? What might you guess is a method for finding the eigenvalues of a general upper triangular matrix?

Learning Outcomes?

Upper triangular, eigenvalues

Pens down!
Full attention is appreciated.

- Define the 2x2 Upper Triangular Matrix: $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

- Compute the Characteristic Polynomial, namely

$$A - \lambda I = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ 0 & d-\lambda \end{pmatrix}$$

- Compute the Determinant:

$$\det(A - \lambda I) = (a-\lambda)(d-\lambda) - (0)(b) = (a-\lambda)(d-\lambda)$$

- Find the Eigenvalues $\lambda_1 = a$ and $\lambda_2 = d$

The eigenvalues of the 2x2 *upper triangular matrix A* are precisely the entries on its diagonal: *a* and *d*.

Up for a mini challenge?

Prove it using 3 by 3 matrices.

- Upper triangular
- Lower triangular

(Optional, submit with P.S 4)

$\det(A - \lambda I) = 0$

$$\det(\mathsf{A} - \lambda \mathsf{I}) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda)(6-\lambda).$$

$$\lambda = 1, 4, 6.$$

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

Question 7

Construct a 3×3 non-zero matrix with real entries that has all three eigenvalues equal to zero.

Learning Outcomes?

Can we use the trick from Q6?

For instance:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of an upper triangular *n* x *n* *matrix* are precisely the entries on its diagonal.

Question 8

(a) Calculate the characteristic polynomial, eigenvalues and corresponding eigenvectors of the matrix

$$\mathsf{A} = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

(b) Let D be the diagonal matrix whose diagonal entries are the eigenvalues of A and let V be the matrix whose columns are the corresponding eigenvectors of A, written in the same order. Verify that $\mathsf{AV} = \mathsf{VD}$ and calculate V^{-1}.

(c) Use the expression $\mathsf{A} = \mathsf{VDV}^{-1}$ to calculate the matrix A^{50}.

Learning Outcomes?

A repeat of Q2

Try at Home for 8a and 8b!
Refer to Q2

(c) Use the expression $A = VDV^{-1}$ to calculate the matrix A^{50}.

Lets us do a simple prove!

$$A^3 = (VDV^{-1})(VDV^{-1})(VDV^{-1})$$

$$A^3 = VD\overbrace{(V^{-1}V)}^{\text{Identity Matrix}}D\overbrace{(V^{-1}V)}DV^{-1}$$

$$A^3 = VDDDV^{-1} \longrightarrow A^n = VD^nV^{-1}$$

$$A^{50} = VD^{50}V^{-1}$$

Question 9

Bonus. The Cayley–Hamilton theorem states that the matrix satisfies its own characteristic equation. In other words, if $p(\lambda)$ is the characteristic polynomial of the matrix A, then $p(\mathsf{A}) = 0$ holds. Verify that this is true for the matrix A given in question 8. *Note:* You should treat the constant term in the characteristic polynomial as the same constant times the identity.

Learning Outcomes?

Cayley-Hamilton's rule

$$A = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

- Define the 2x2 Matrix: $A = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$

  - Compute the Characteristic equation, namely

$$\begin{bmatrix} -2-\lambda & 4 \\ 4 & -2-\lambda \end{bmatrix}$$

- Compute the Determinant:

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -2-\lambda & 4 \\ 4 & -2-\lambda \end{bmatrix}\right) = (\lambda+2)^2 - 16$$

- Find the Eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -6$.

From the determinant

$$A = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -2-\lambda & 4 \\ 4 & -2-\lambda \end{bmatrix}\right) = (\lambda+2)^2 - 16$$

C.H's instruction

$$p(\lambda) = \lambda^2 + 4\lambda - 12.$$

$$p(\mathrm{A}) = \mathrm{A}^2 + 4\mathrm{A} - 12\mathrm{I}$$

Based on C.H, can we recover zero matrix

$$= \underbrace{\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}}_{A^2} + 4\underbrace{\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}}_{A} - 12\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

$$\boxed{= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

Proven!

Thank You

MONASH University

Eng. Math

(Workshop) ENG 1005

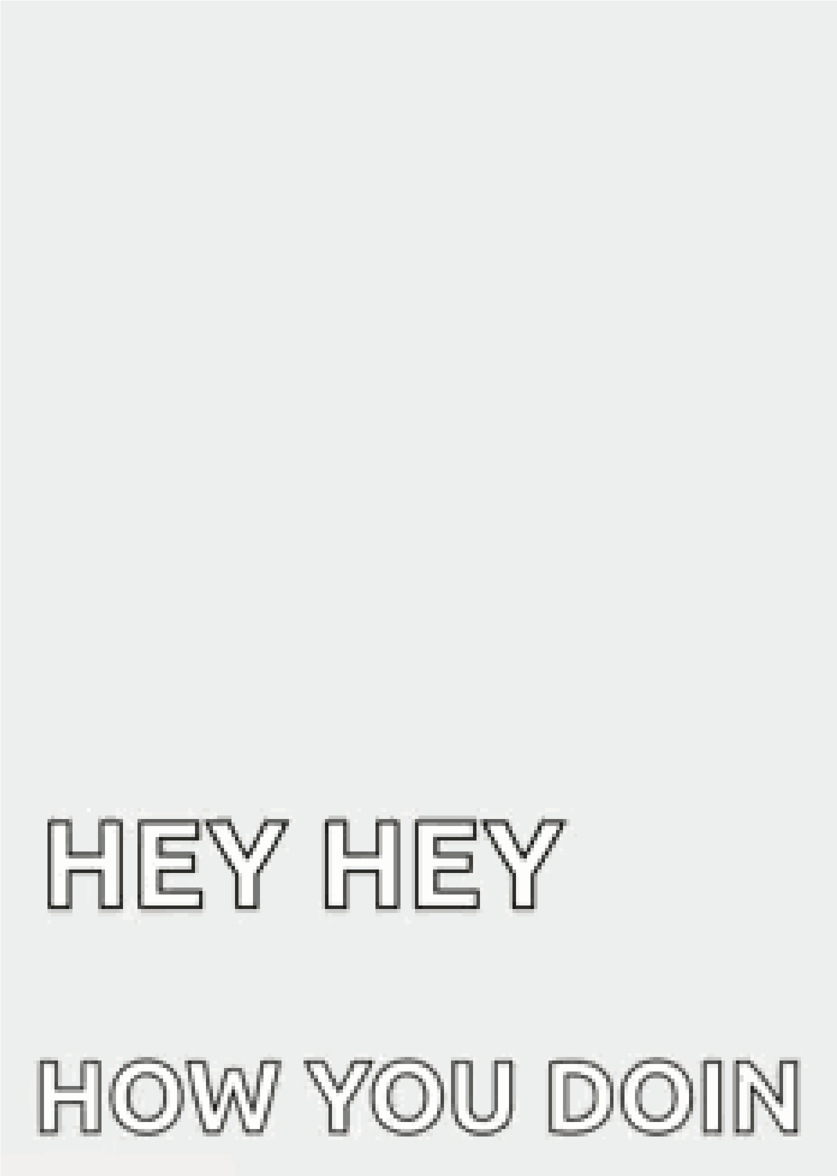
Week 4: Eigenvalues/Eigenvectors

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

kekboon.goh@monash.edu



Check-In

Resources

1. PASS with Yvonne and Zi Wei

2. Math Centre

3. Additional: Videos (Mid-Term Prep.)

Resources

3. Additional: Videos (Mid-Term Prep.)

Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

Feedback: Workshop 3

- comment 1: writing/drawing & resolution.

Assessments breakdown

Task description	*Value*	*Due date*
Lecture quizzes	5%	Weekly 11:55pm the night before your applied class (except Monday classes) except in Week 1 when the due date is Friday at 11:55 pm (Malaysia time)
Applied class quizzes	5%	Weekly during your applied class
Workshop problems	20%	Weekly at 11:55pm Sunday, except for Weeks 7 (midsem)
Mid-semester exam	20%	During your workshop in Week 7

The Big Learning Outcomes for Week 4

After completing this week's task, you should be able to:

- Understand what eigenvalues and eigenvectors are in the framework of matrices as linear transformations of space.
- Calculate eigenvalues and eigenvectors for 2x2 and 3x3 systems.
- Understand geometric and algebraic multiplicity.
- Identify diagonalisable matrices.
- Diagonalise 2x2 and 3x3 diagonalisable matrices.

Today's Activity

0. Algebraic and Geometric Multiplicities

1. Workshop Problem Set

Algebraic and Geometric Multiplicities

$$\boxed{B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}$$

- Eigenvalues: λ=2 & 2 (repeated)
  - Algebraic=2

- $\overbrace{\text{One Eigenvectors}}^{\text{Geometric=1}}$

- $\overbrace{\text{Two Eigenvectors}}^{\text{Geometric=2}}$

When Algebraic=Geometric, diagonalizing can happen

Webpage Ranking

Eigenvalues and eigenvectors of matrices are extremely important tools with applications in every field of engineering. In this workshop we will study the Google page rank algorithm which revolutionised the search engine, leading to Google becoming one of the largest companies in the world and almost monopolising the internet search market.

You are given the task of analysing internet traffic between Google, Intel, and Microsoft's webpages. After some data analysis, you find that for people visiting Google's page, after each hour, 20% of them would stay on Google's page, 40% would follow a link and go to Intel's page, and 40% to Microsoft's page. For those visiting Intel's page, 50% would stay, 10% would go to Google's page, and 40% to Microsoft's page. For those visiting Microsoft's page, 40% would stay, 30% would go to Google's page, and 30% to Intel's page.

1. Suppose initially there are 1000 visitors to each company's webpage. Assuming there are no additional visitors, how many people are on each company's webpage after 1 hour? **[1 mark]**

After one hour, the number of visitors to Google, Intel and Microsoft will be?

$$G=\underbrace{0.2\times 1000}_{\boxed{G}}+\underbrace{0.1\times 1000}_{\boxed{I}}+\underbrace{0.3\times 1000}_{\boxed{M}}=600$$

$$I=0.4\times 1000+0.5\times 1000+0.3\times 1000=1200$$

$$M=0.4\times 1000+0.4\times 1000+0.4\times 1000=1200$$

After some data analysis, you find that for people visiting Google's page, after each hour, 20% of them would stay on Google's page, 40% would follow a link and go to Intel's page, and 40% to Microsoft's page. For those visiting Intel's page, 50% would stay, 10% would go to Google's page, and 40% to Microsoft's page. For those visiting Microsoft's page, 40% would stay, 30% would go to Google's page, and 30% to Intel's page.

2. Let G_n, I_n, M_n denote the number of people on Google, Intel and Microsoft's webpage after n hours respectively, and $\mathbf{x}_n = \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$. Write down a matrix $\mathbf{A}$ such that $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$. This is called the transition matrix. **[2 marks]**

$G_{n+1}=0.2G_n+0.1I_n+0.3M_n$

$I_{n+1}=0.4G_n+0.5I_n+0.3M_n$

$M_{n+1}=0.4G_n+0.4I_n+0.4M_n$

The matrix equation is:

$$\underbrace{\begin{bmatrix} G_{n+1} \\ I_{n+1} \\ M_{n+1} \end{bmatrix}}_{\text{Current}} = \underbrace{\begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}}_{\text{Transformation/transition matrix } A} \underbrace{\begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}}_{\text{Previous}}$$

G=0.2x1000+0.1x1000+0.3x1000

I=0.4x1000+0.5x1000+0.3x1000

M=0.4x1000+0.4x1000+0.4x1000

3. Using the transition matrix **A** and the help of Matlab or CAS, find out how many people are on each company's webpage after 3 hours. What about after 24 hours? Round your answer to the nearest integer. **[1 marks]**

Let's think simply:

For n=2 (after hour 2)

For n=1 (after hour 1)

$$\begin{bmatrix} G_{1+1} \\ I_{1+1} \\ M_{1+1} \end{bmatrix} = A \begin{bmatrix} G_1 \\ I_1 \\ M_1 \end{bmatrix}$$

$$\begin{bmatrix} G_{2+1} \\ I_{2+1} \\ M_{2+1} \end{bmatrix} = A \underbrace{\begin{bmatrix} G_{1+1} \\ I_{1+1} \\ M_{1+1} \end{bmatrix}}_{A \begin{bmatrix} G_1 \\ I_1 \\ M_1 \end{bmatrix}}$$

$$= AA \begin{bmatrix} G_1 \\ I_1 \\ M_1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} G_{n+1} \\ I_{n+1} \\ M_{n+1} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}}$$

You try next

For n=3 (after hour 3)

$$\begin{bmatrix} G_{3+1} \\ I_{3+1} \\ M_{3+1} \end{bmatrix} = A \begin{bmatrix} G_3 \\ I_3 \\ M_3 \end{bmatrix}$$

$$=?$$

$$\begin{bmatrix} G_3 \\ I_3 \\ M_3 \end{bmatrix} = \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdot \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix}$$

For n=24 (after hour 24)

$$\begin{bmatrix} G_{24} \\ I_{24} \\ M_{24} \end{bmatrix} = \mathbf{A}^{24} \cdot \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix}$$

The visitor numbers seem unchanging!

$$\begin{bmatrix} G_{n+1} \\ I_{n+1} \\ M_{n+1} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$$

1. Suppose initially there are 1000 visitors to each company's webpage. Assuming there are no additional visitors, how many people are on each company's webpage after 1 hour? **[1 mark]**

4. Now suppose initially that Google's webpage receives 2000 visitors, and that each of Intel and Microsoft's webpages receives 500 visitors. How many people are on each company's webpage after 3 and 24 hours respectively? Round your answer to the nearest integer. **[1 marks]**

For n=3 (after hour 3)

Again, the visitor numbers seem unchanging!

$$\begin{bmatrix} G_3 \\ I_3 \\ M_3 \end{bmatrix} = \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdot \boxed{\begin{bmatrix} 2000 \\ 500 \\ 500 \end{bmatrix}} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix}$$

For n=24 (after hour 24)

$$\begin{bmatrix} G_{24} \\ I_{24} \\ M_{24} \end{bmatrix} = \mathbf{A}^{24} \cdot \begin{bmatrix} 2000 \\ 500 \\ 500 \end{bmatrix} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix}$$

You notice that the numbers of visitors to the company's webpages seem to stabilise as time passes, regardless of where the 3000 visitors start initially! Let's investigate this further.

5. Show that if the number of visitors stabilises to some time independent constant $\mathbf{x} = \begin{bmatrix} G \\ I \\ M \end{bmatrix}$, then $\mathbf{x}$ is in fact an eigenvector of $\mathbf{A}$ with eigenvalue 1. **[2 marks]**

$$\begin{bmatrix} G_3 \\ I_3 \\ M_3 \end{bmatrix} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix} = \begin{bmatrix} G_{24} \\ I_{24} \\ M_{24} \end{bmatrix}$$

Let's go back to some basic concepts!

Use this trick to visualize the mapping from the old to the new location!

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{(new time)}} = \underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{\text{Transition Matrix, } A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{(old time)}}$$

- If x'/y' and x/y are equal, what can we deduce about the eigenvalue of A ?

You notice that the numbers of visitors to the company's webpages seem to stabilise as time passes, regardless of where the 3000 visitors start initially! Let's investigate this further.

5. Show that if the number of visitors stabilises to some time independent constant $\mathbf{x} = \begin{bmatrix} G \\ I \\ M \end{bmatrix}$, then $\mathbf{x}$ is in fact an eigenvector of $\mathbf{A}$ with eigenvalue 1. **[2 marks]**

$$\begin{bmatrix} G_3 \\ I_3 \\ M_3 \end{bmatrix} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix} = \begin{bmatrix} G_{24} \\ I_{24} \\ M_{24} \end{bmatrix}$$

$$A\mathbf{x} = \lambda \mathbf{x}$$

- **x** being an eigenvector of A with eigenvalue 1.

6. Write down the characteristic equation and find the eigenvalues of the transition matrix $\mathbf{A}$.

[2 marks]

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{bmatrix} 0.2-\lambda & 0.1 & 0.3 \\ 0.4 & 0.5-\lambda & 0.3 \\ 0.4 & 0.4 & 0.4-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 1.1\lambda^2 - 0.1\lambda$$

Eigenvalues:

$\boxed{1,}$ $0.1,$ and $0.$

$$\begin{bmatrix} G_{n+1} \\ I_{n+1} \\ M_{n+1} \end{bmatrix} = \overbrace{\begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}}^{A} \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$$

7. For each eigenvalue of $\mathbf{A}$, find the corresponding eigenvector(s).

- **When λ is 1**

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -0.8 & 0.1 & 0.3 \\ 0.4 & -0.5 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{bmatrix}$$

Row operations: $\left(\begin{array}{ccc|c} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} v_1 - 0.5\, v_3 = 0 \\ v_2 - v_3 = 0 \end{cases}$$

$$\begin{cases} v_1 = 0.5\, v_3 \\ v_2 = v_3 \end{cases}$$

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.5\, v_3 \\ v_3 \\ v_3 \end{pmatrix}$$

Eigenvector: $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} G_{n+1} \\ I_{n+1} \\ M_{n+1} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & \\ 0.4 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$$

7. For each eigenvalue of $\mathbf{A}$, find the corresponding eigenvector(s).

When λ is 1:

Eigenvector:

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

When λ is 0, 0.1, respectively?

Try at Home!

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G_{n+1} \\ I_{n+1} \\ M_{n+1} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$$

8. Diagonalise the matrix $\mathbf{A}$, i.e., find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. **[2 marks]**

Get it from Q7

$$\mathbf{D} = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{0.1} & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

- ❑ ***V: columns of eigenvectors***
- ❑ ***D: diagonal matrix with eigenvalues***

9. Show that $\mathbf{A}^n = \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$ in terms of $\mathbf{D}$ and $\mathbf{V}$.

Now you do it with a simple prove!

$$A^3 = (VDV^{-1})(VDV^{-1})(VDV^{-1})$$

$$A^3 = VD\overbrace{(V^{-1}V)}^{\text{Identity Matrix}}D\overbrace{(V^{-1}V)}DV^{-1}$$

$$A^3 = VDDDV^{-1} \longrightarrow A^n = VD^nV^{-1}$$

10. For the particular $\mathbf{D}$ you have, what happens to $\mathbf{D}^n$ as n becomes large? Use your observation and the previous part to explain why the numbers of visitors to the company's webpages stabilise as time passes, regardless of where the 3000 visitors start initially. **[3 marks]**

$$A^n = VD^nV^{-1}$$

For large n values!

$$D^n = \begin{matrix} 1^n \to 1 & 0 & 0 \\ 0 & 0.1^n \to 0 & 0 \\ 0 & 0 & 0^n \end{matrix} \Rightarrow D^n = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From Q2

2. Let G_n, I_n, M_n denote the number of people on Google, Intel and Microsoft's webpage after n hours respectively, and $\mathbf{x}_n = \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$. Write down a matrix $\mathbf{A}$ such that $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$. This is called the transition matrix. **[2 marks]**

$$\mathbf{x}_n = \mathbf{A}^n \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix} = \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1} \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$$

Rule of thumb: When the matrix is raised to a higher *n*, use dragonized form!

$$\mathbf{V} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$D^n = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$A^n = VD^nV^{-1}$$

Continuing

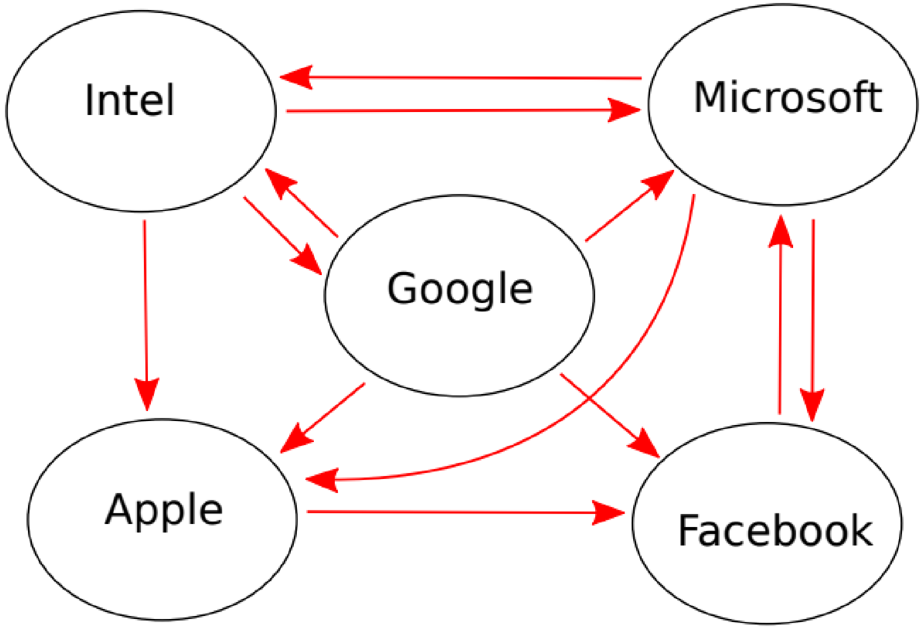
10. For the particular $\mathbf{D}$ you have, what happens to $\mathbf{D}^n$ as n becomes large? Use your observation and the previous part to explain why the numbers of visitors to the company's webpages stabilise as time passes, regardless of where the 3000 visitors start initially. **[3 marks]**

$$= \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}\begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \frac{1}{5}(G_n + I_n + M_n) \\ \frac{2}{5}(G_n + I_n + M_n) \\ \frac{2}{5}(G_n + I_n + M_n) \end{bmatrix} = \begin{bmatrix} 600 \\ 1200 \\ 1200 \end{bmatrix}}$$

$$\underbrace{G_n + I_n + M_n}_{3000}$$

You can now analyse a more complicated network of webpages. The diagram below shows the webpages of Apple, Facebook, Google, Intel and Microsoft. A directed arrow indicates that there is a link on the starting webpage to the ending webpage. Suppose that after each hour, a person visiting a webpage has an equal chance of staying on the webpage or following any one of its links. For example, in this diagram, anyone visiting Google has a 20% chance of being on Apple, Facebook, Google, Intel and Microsoft's webpage after an hour.



11. Write down the transition matrix **A** for this network of webpages, ordering the companies alphabetically for your variables. **[1 mark]**

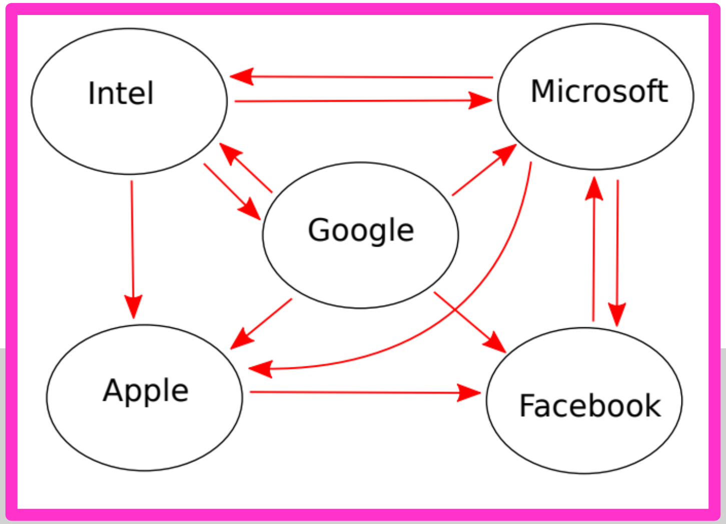
Apple → Apple (stay) or Facebook → 50%

Facebook → Facebook (stay) or Micro → 50%

Google → Google (stay) or rest → 20%

Intel → Intel (stay) or Micro/Apple/Google → 25%

Micro → Micro (stay) or Intel/Apple/Facebook → 25%



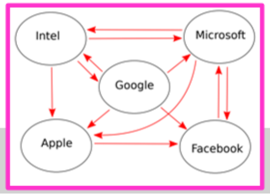
Apple → Apple (stay) or Facebook → 50%

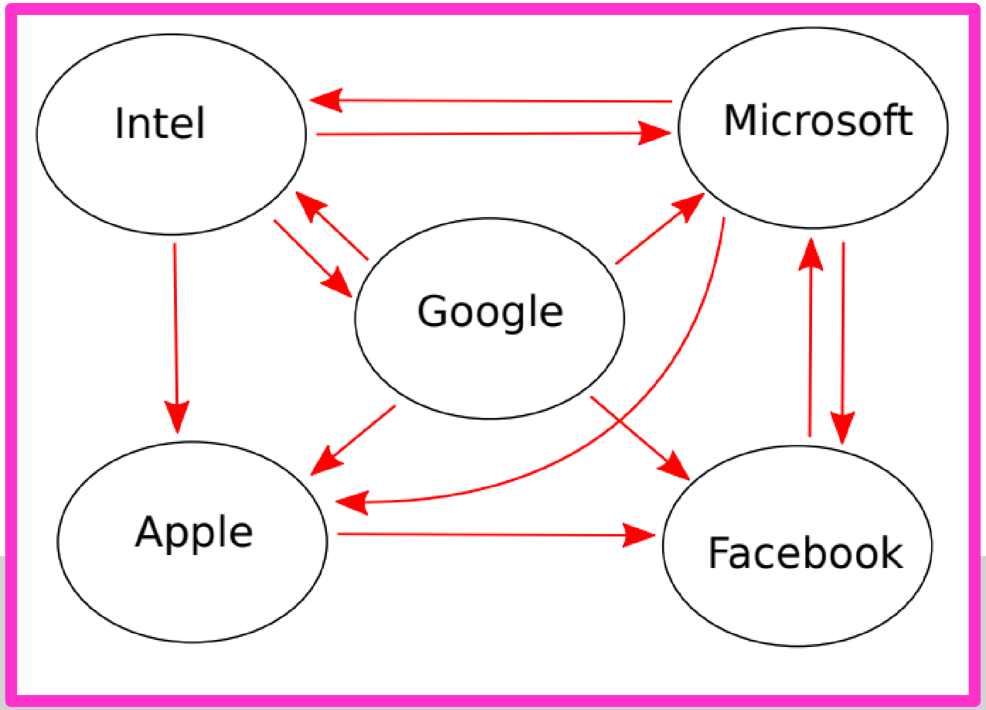
Facebook → Facebook (stay) or Micro → 50%

Google → Google (stay) or rest → 20%

Intel → Intel (stay) or Micro/Apple/Google → 25%

Micro → Micro (stay) or Intel/Apple/Facebook → 20%





From

To

	A	F	G	I	M
A	0.5	0	0.2	0.25	0.25
F	0.5	0.5	0.2	0	0.25
G	0	0	0.2	0.25	0
I	0	0	0.2	0.25	0.25
M	0	0.5	0.2	0.25	0.25

12. For the same reason as in Question (10), the number of visitors to each page will stabilise regardless of initial distribution. With the help of Matlab or CAS, rank the webpages by the stabilised number of visitors in descending order. **[2 marks]**

2. Let G_n, I_n, M_n denote the number of people on Google, Intel and Microsoft's webpage after n hours respectively, and $\mathbf{x}_n = \begin{bmatrix} G_n \\ I_n \\ M_n \end{bmatrix}$. Write down a matrix $\mathbf{A}$ such that $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$. This is called the transition matrix. **[2 marks]**

When n=1

$$x_{1+1} = Ax_1$$

When n=2

$$x_{2+1} = Ax_2 = A^2x_1$$

When n=3

$$x_{3+1} = Ax_3 = A^3x_1$$

$$x_{n+1} = A^n x_n$$

$$x_{n+1} = A^n x_n$$

$$A^n = VD^nV^{-1}$$

Assume n is very big

Assume *total visitors 3k*

Increasing number

Google < Intel < Apple < Micro < Facebook

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0 & 0.2 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.2 & 0 & 0.25 \\ 0 & 0 & 0.2 & 0.25 & 0 \\ 0 & 0 & 0.2 & 0.25 & 0.25 \\ 0 & 0.5 & 0.2 & 0.25 & 0.25 \end{bmatrix}$$

Conc … on!

The page ranking is Facebook, Microsoft, Apple, Intel, and Google.

H

University

The Big Learning Outcomes for Week 4

After completing this week's task, you should be able to:

- Understand what eigenvalues and eigenvectors are in the framework of matrices as linear transformations of space.
- Calculate eigenvalues and eigenvectors for 2x2 and 3x3 systems.
- Understand geometric and algebraic multiplicity.
- Identify diagonalisable matrices.
- Diagonalise 2x2 and 3x3 diagonalisable matrices.

Thank You