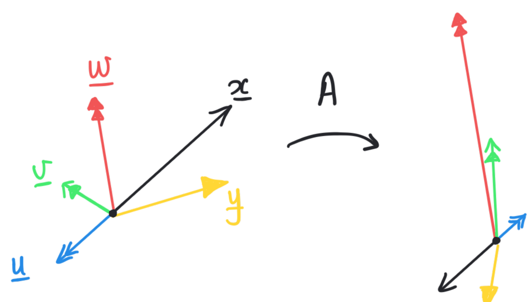
ENG1005 Week 4: Applied class problem sheet

This problem sheet is intended for you to work through in your Applied Class in a small-group setting with the help of your instructor and your peers. **At the end of the applied class you will be asked to complete a quiz for credit. The quiz questions are based on the questions on this problem sheet. You may (and should!) ask your group members and your instructor for guidance if needed.**

1. Which of the vectors **u**, **v**, **w**, **x**, **y** shown below are eigenvectors of the matrix A (whose transformation of the vectors is also shown below)? If the vector is an eigenvector, then determine the corresponding eigenvalue.



2. Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where a is a real number with $a \geq 0$.

   (a) Find the eigenvalues and eigenvectors of this matrix.

   (b) For one value of a there is only one distinct eigenvalue. What is this value? How many eigenvector directions are there for this value of a?

   (c) For one value of a the eigenvectors are orthogonal (perpendicular). What is this value?

   (d) For what value(s) of a is the matrix diagonalisable? Write down a diagonalisation where possible.

   (e) For what value(s) of a is it possible to write down a diagonalisation of the form $\mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^T$? Write down such a diagonalisation where possible, where D has increasing values on the diagonal and v_{11} and v_{12} are both positive.

3. A 2×2 matrix has eigenvalues 9 and -18. It has corresponding eigenvectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. What is the matrix? *Hint: Once you have an answer, you should check it, either using your favourite calculator or double checking the eigenvectors given are indeed eigenvectors of the matrix you have found.*

4. Consider the matrix
$$\mathsf{A} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$
Calculate the eigenvalues and eigenvectors of A.

5. Suppose the matrix A has eigenvector $\mathbf{v}$ with corresponding eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of A^n. What is its corresponding eigenvalue? If A is an invertible matrix, can you deduce the eigenvalues and eigenvectors of A^{-1}?

6. For the matrix
$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$
what are the eigenvalues? Can you find a corresponding eigenvector for one of these eigenvalues? What might you guess is a method for finding the eigenvalues of a general upper triangular matrix?

7. Construct a 3×3 non-zero matrix with real entries that has all three eigenvalues equal to zero.

8. (a) Calculate the characteristic polynomial, eigenvalues and corresponding eigenvectors of the matrix
$$\mathsf{A} = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$
   (b) Let D be the diagonal matrix whose diagonal entries are the eigenvalues of A and let V be the matrix whose columns are the corresponding eigenvectors of A, written in the same order. Verify that $\mathsf{AV} = \mathsf{VD}$ and calculate V^{-1}.
   (c) Use the expression $\mathsf{A} = \mathsf{VDV}^{-1}$ to calculate the matrix A^{50}.

9. *Bonus.* The Cayley–Hamilton theorem states that the matrix satisfies its own characteristic equation. In other words, if $p(\lambda)$ is the characteristic polynomial of the matrix A, then $p(\mathsf{A}) = \mathsf{0}$ holds. Verify that this is true for the matrix A given in question 8. *Note:* You should treat the constant term in the characteristic polynomial as the same constant times the identity.