Eng. Math

ENG 1005

Week 6: Multivariable Calculus II

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

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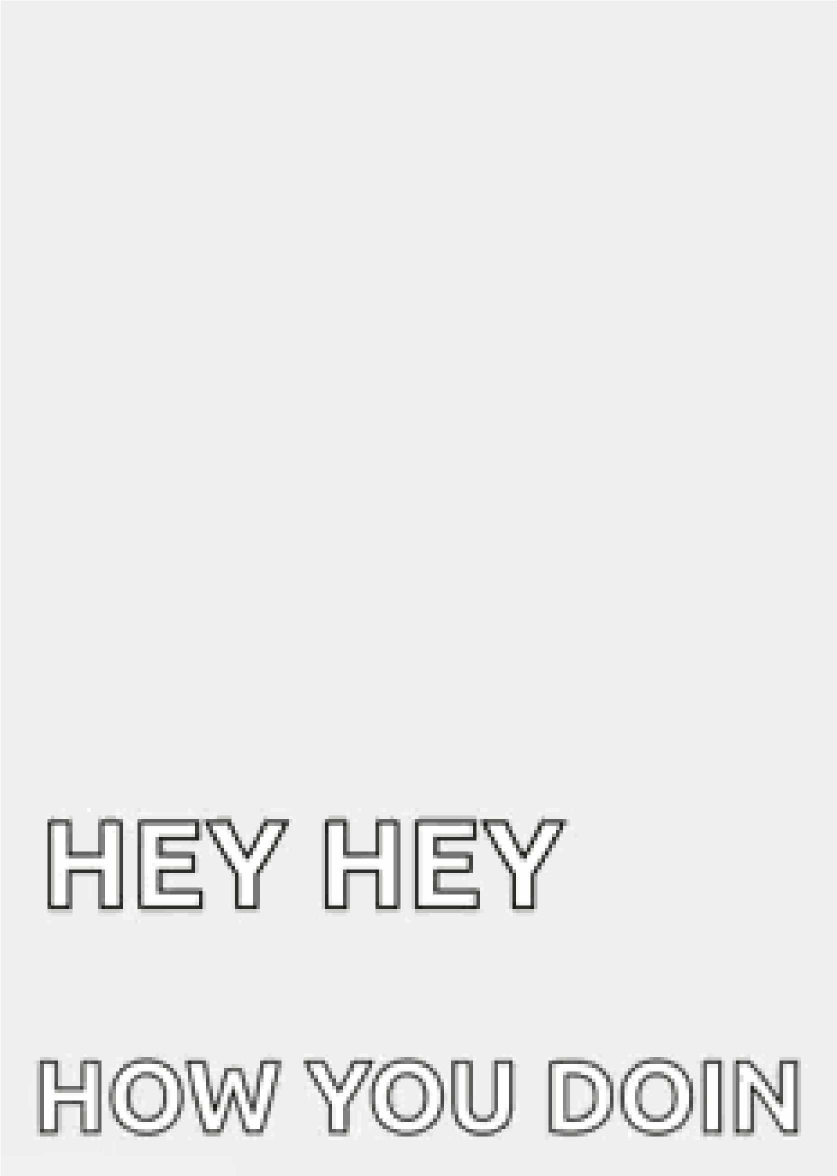
*Please bear with me, attending

to your messages soon*



Check-In

Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
3	Matrices
4	Eigenvalues & Eigenvectors
5	Multivariable Calculus 1
6	Multivariable Calculus 2
7	Integration techniques and hyperbolic functions
8	O.D.E 1
9	O.D.E 2
10	O.D.E 3
11	Series 1
12	Series 2

The Big Learning Outcomes for Week 6

After completing this week's task, you should be able to:

•Identify and characterize critical points (local maxima and minima, and saddle points).

•Calculate absolute maxima and minima of a function.

•Use Lagrange multipliers to find local maxima and minima of a function subject to an equality constraint.

•Parameterise basic curves and surfaces.

•Calculate tangents to curves.

Attendance Codes (Week 6)

International students

Tutorial	Wednesday, 28 Aug	02	8:00AM	3H6P6
Tutorial	Wednesday, 28 Aug	01	2:00PM	JH7NV

Workshop	Thursday, 29 Aug	01	1:00PM	CF6JJ
Workshop	Friday, 30 Aug	02	10:00AM	2Y4NW

Admin. Stuff (1)

1. Feedback on Workshop Submission: On Thur/Fri

2. How to Kill It In Mid-Term: Thur/Fri

3. Submissions:

Summary

ASSESSMENT	DUE
Kick Starting Week 3: Lecture Quiz 3 (Total mark for all 12 weeks of lecture quizzes is 5%)	Tuesday, 6 August 2024, 11:55 PM **Due in 4 days**
Applied class quiz week 3 (Total mark for all 12 weeks of applied quizzes is 5%)	Wednesday, 7 August 2024, 11:55 PM **Due in 5 days**
Workshop 3 problem set (Total mark for all 12 weeks of workshop sets is 20%)	Sunday, 11 August 2024, 11:55 PM

Admin. Stuff (2)

4. Consultation/Feedback hour

- Wed: 10 am till 11 am
- Fri: 8 am till 9 am
- Sat: 1030 am till 1130 am

Location: 5-4-68

NA

(every odd week, e.g. 3-Week, but msg me first at least a day before; appointment-based.)

Resources

1. PASS with Yvonne and Zi Wei

2. Mid-Term Mock Exams! Week 6

3. Additional: Videos (Mid-Term Prep.)

Resources (update comes every Fri.)

3. Additional: Videos (Mid-Term Prep)

Let us start!

Today’s Activity

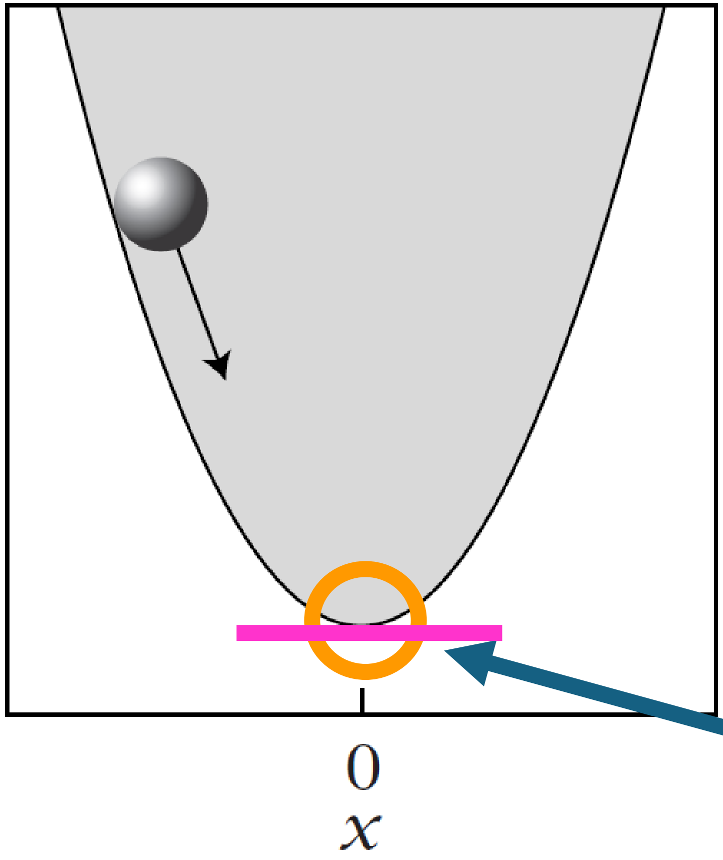
0. Refresher!

1. Applied Problem Set

2. Applied Quiz

Revisiting some Physics! (i)

$V(x)$ **Potential Energy**



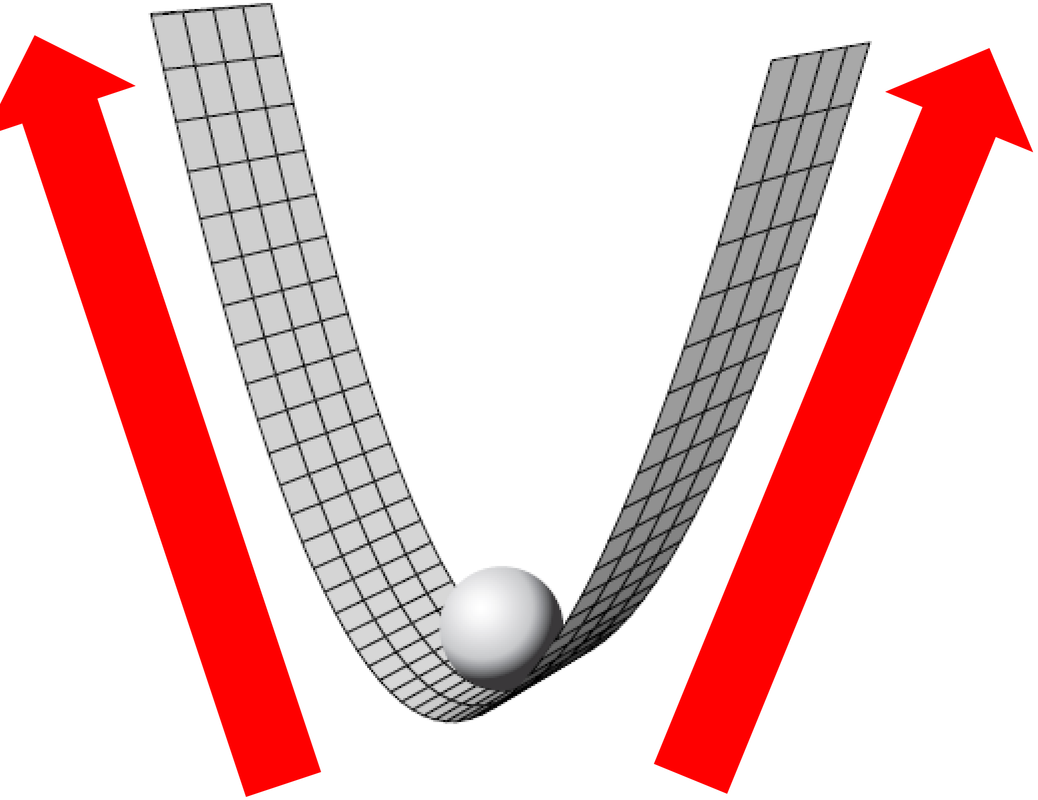
- Physics says that the ball will roll to wherever the potential energy **is a minimum**!

- We note that the minimum point is where the gradient is 0!

- $$\left.\frac{dV(x)}{dx}\right|_{x=0} = 0$$

Stable/Relaxed/Equilibrium/Critical Point

Stable State!

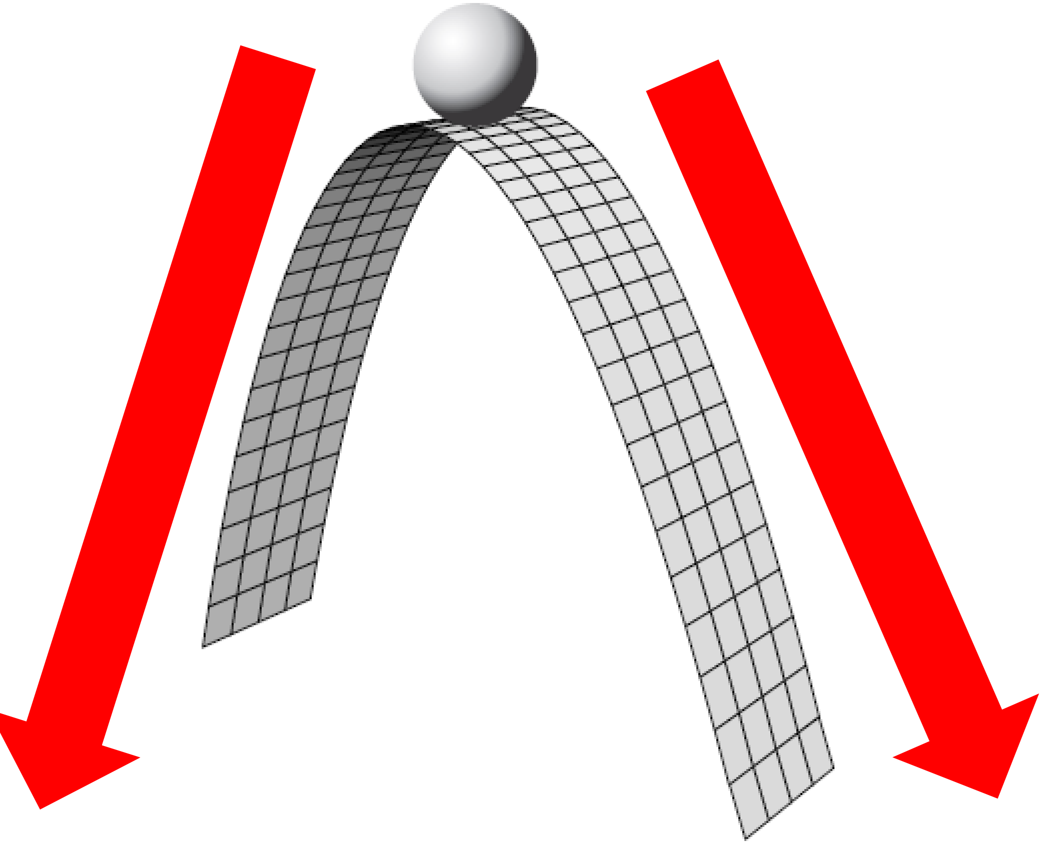


- $$\frac{dV(x)}{dx} = 0$$
- $$\frac{d^2V(x)}{dx^2} > 0$$
  - But why positive?
- $$\underbrace{\frac{d}{dx}\left(\frac{dV(x)}{dx}\right) > 0}$$

Gained in energy as it escapes from the stable/critical point.

Un-Stable State! **(My fav.!)**

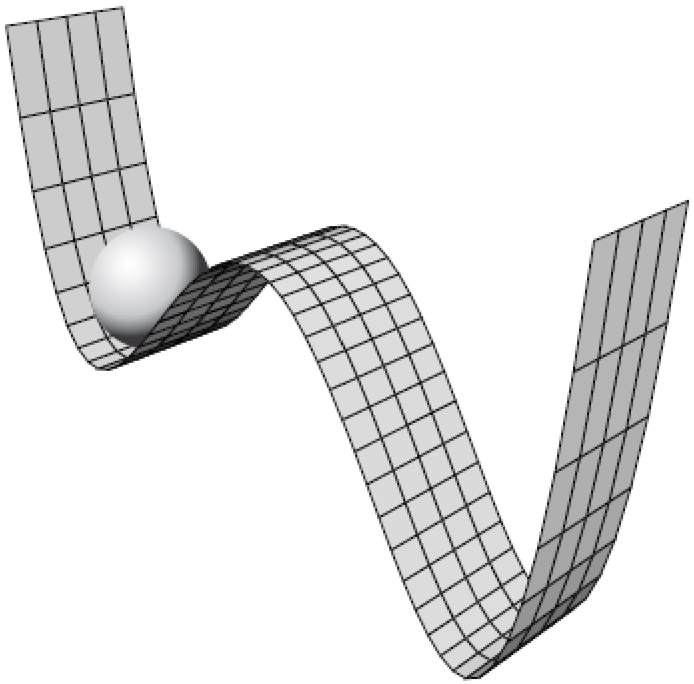
The physics are crazier!



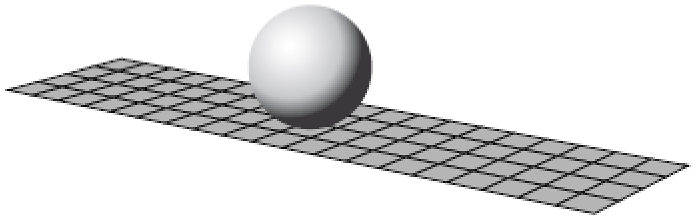
- $$\frac{dV(x)}{dx} = 0$$
- $$\underbrace{\frac{d}{dx}\left(\frac{dV(x)}{dx}\right)}_{\text{Lose energy as it escapes from the unstable/critical point.}} < 0$$

Lose energy as it escapes from the unstable/critical point.

Metastable State!



Neutral State!



Today's Activity

0. Refresher!

1. Applied Problem Set

2. Applied Quiz

Question 1

1. Find and classify all the critical points of $f(x, y) = x^3/3 - x + (y^3/3 - y)$.

Learning Outcomes?

Identify and characterize critical points

Step 1: Find the gradient!

$$\nabla f(x, y) = (x^2 - 1, y^2 - 1).$$

Step 2: Find the critical points!

$$x^2 - 1 = 0 \qquad x = \pm 1$$

$$y^2 - 1 = 0 \qquad y = \pm 1$$

- $(+1, -1)$
- $(-1, -1)$
- $(+1, +1)$
- $(-1, +1)$

Step 3: Second Derivative

$$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

Step 4: Calculate the Hessian! (5min)

$$\det\ H = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4xy$$

$$\boxed{\nabla f(x, y) = (x^2 - 1, y^2 - 1).}$$

Step 5: Classification!

The classification of each critical point depends on the sign of det H and $\frac{\partial^2 f}{\partial x^2}$:

- If det $H < 0$, the critical point is a saddle point.
- If det $H > 0$, the critical point is a local minimum if $\frac{\partial^2 f}{\partial x^2} > 0$, and a local maximum if $\frac{\partial^2 f}{\partial x^2} < 0$.
- If det $H = 0$, the test is inconclusive.

Step 6: Plug it in!

$$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\boxed{\det H = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4xy}$$

Critical Point	det H	$\frac{\partial^2 f}{\partial x^2}$	Classification
$(1, 1)$	4	2	Local Minimum
$(1, -1)$	-4	2	Saddle Point
$(-1, 1)$	-4	-2	Saddle Point
$(-1, -1)$	4	-2	Local Maximum

Question 2

2. Which of the following guarantees a saddle point of the function $f(x, y)$ at a critical point (x_0, y_0)?

   (a) $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ have the same sign at (x_0, y_0)

   (b) $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ have opposite signs at (x_0, y_0)

   (c) $\partial^2 f/\partial x \partial y$ is negative at (x_0, y_0)

   (d) none of the above

Learning Outcomes?

How to mathematically get saddle point

(a) For a function f (x, y)

$$\det H = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$\boxed{\text{(a) } \partial^2 f/\partial x^2 \text{ and } \partial^2 f/\partial y^2 \text{ have the same sign at } (x_0, y_0)}$

$$\det H = \underbrace{\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}}_{\text{Positive!}} - \underbrace{\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}_{\text{Positive/Negative/0}}$$

(b) For a function f (x, y)

$$\det H = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

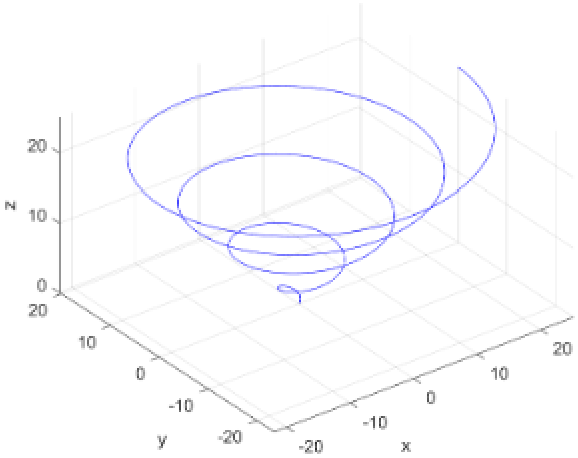
$\boxed{\text{(b) } \partial^2 f/\partial x^2 \text{ and } \partial^2 f/\partial y^2 \text{ have opposite signs at } (x_0, y_0)}$

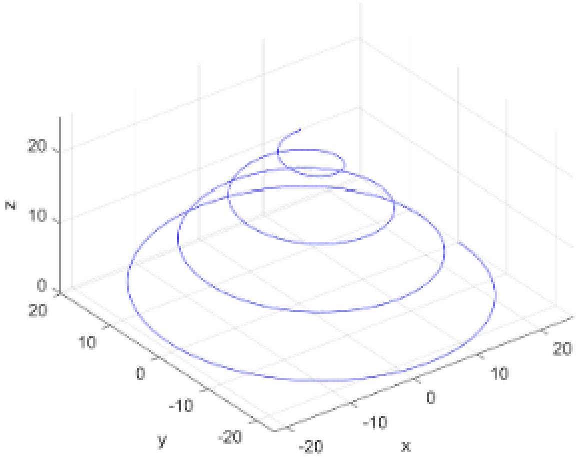
$$\det H = \underbrace{\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}}_{\text{Negative!}} - \underbrace{\left(\frac{\partial^2 f}{\partial x \partial y} \right)^2}_{\text{Positive/Negative/0}}$$

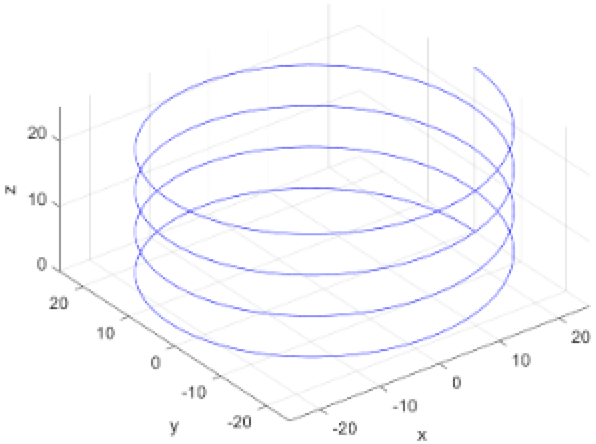
Question 3

3. Match the following parametric representations to the correct curve:

(a) $\mathbf{x}(t) = (t\cos(t), t\sin(t), t)$ for $0 \leq t < 8\pi$

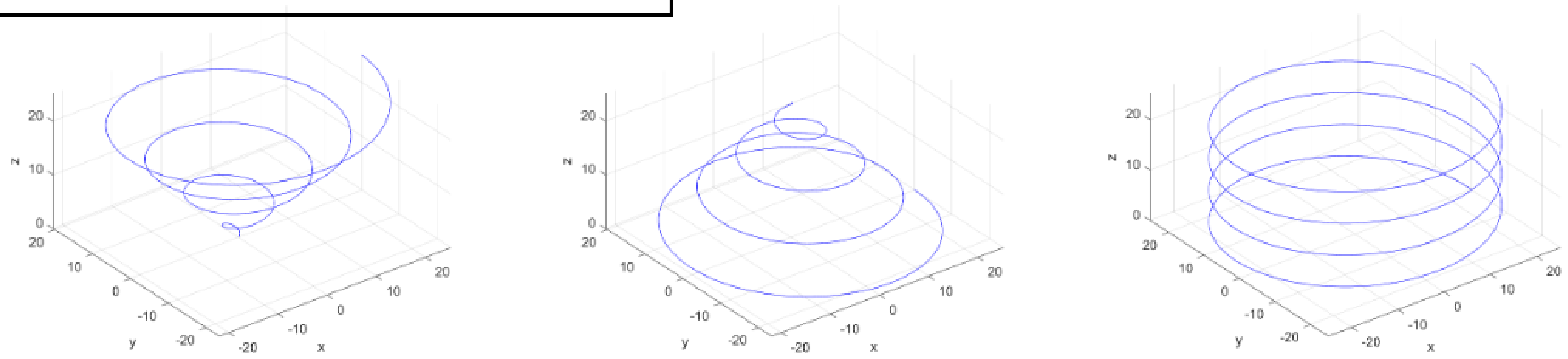






3. Match the following parametric representations to the correct curve:

(a) $\mathbf{x}(t) = (t\cos(t), t\sin(t), t)$ for $0 \leq t < 8\pi$



Leftmost graph shows a spiral that increases along the z-axis as t increases. The spiral expands outward in the x-y-plane and rises along the z-axis, consistent with the parametric equation.

Question 4

4. Use Lagrange multipliers to determine the point on the sphere $x^2 + y^2 + z^2 = 1$ that is furthest from the point $(1, 2, 2)$. *Hint: To solve the system of equations, you can first write x, y, z all in terms of λ, and then substitute into the constraint $x^2 + y^2 + z^2 = 1$ to find λ.*

Learning Outcomes?

Lagrange Multiplier!

Head-up: Sometimes it will be

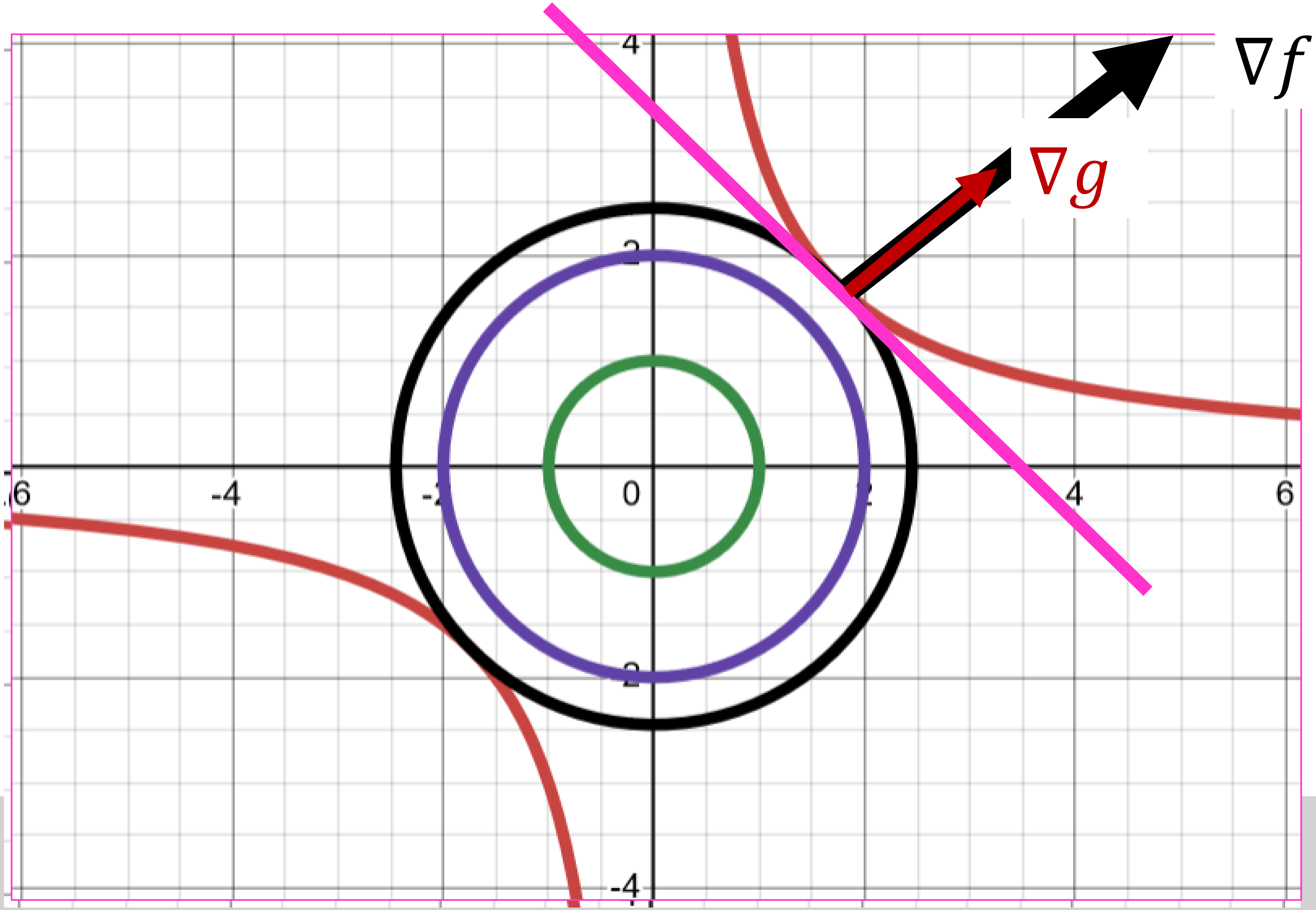
Trying and Erroring!

❑You do we keep finding this multiplier? For what?

- **Your objective function**
  - ❑ $f\ (x, y) = x^2 + y^2$
- **Your constraint (at a constant level)**
  - ❑ $g\ (x, y) = xy = 3$

A different example to explain Lagrange

❑You do we keep finding this multiplier? For what?



$$\overbrace{\nabla f \ /\!/ \ \nabla g}^{\text{Parallel}}$$

- $\nabla f \propto \nabla g$
- $\nabla f = \underbrace{\lambda}_{\text{L. Multi.}} \nabla g$

❑ $f\,(x, y, z) = x^2 + y^2$

❑ $g\,(x, y) = xy = 3$

Question 4

4. Use Lagrange multipliers to determine the point on the sphere $x^2 + y^2 + z^2 = 1$ that is furthest from the point $(1, 2, 2)$. *Hint: To solve the system of equations, you can first write x, y, z all in terms of λ, and then substitute into the constraint $x^2 + y^2 + z^2 = 1$ to find λ.*

- Step 1: Find your objective function
- Step 2: Find your constraint (at a level set)
- Step 3: Lagrange function
- Step 4: Systems of equations
- Step 5: Solving the multiplier (lambda)

- **Step 1: Find your objective function**

The distance between any point (x, y, z) on the sphere and the point $(1, 2, 2)$ is given by:

$$d(x, y, z) = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

To simplify calculations, we maximize the square of the distance:

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

$$\boxed{r = (x^2 + y^2 + z^2)^{1/2}}$$

- ## Step 2: Find your constraint

The constraint is given by the equation of the sphere:

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

This represents the surface of the sphere. The solution must satisfy this constraint.

- ## Step 3: Lagrange function

$$L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$

Substituting the expressions for $f(x, y, z)$ and $g(x, y, z)$, we get:

$$L(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z-2)^2 - \lambda(x^2 + y^2 + z^2 - 1)$$

- **Step 4: Systems of equations**

$$\frac{\partial L}{\partial x} = 2(x-1) - 2\lambda x = 0 \longrightarrow x = \frac{1}{1-\lambda},$$

$$\frac{\partial L}{\partial y} = 2(y-2) - 2\lambda y = 0 \longrightarrow y = \frac{2}{1-\lambda},$$

$$\frac{\partial L}{\partial z} = 2(z-2) - 2\lambda z = 0 \longrightarrow z = \frac{2}{1-\lambda}$$

$$\frac{\partial L}{\partial \lambda} = -(x^2 + y^2 + z^2 - 1) = 0$$

- **Step 5: Solving the multiplier (lambda)**

$$\frac{\partial L}{\partial \lambda} = -(x^2 + y^2 + z^2 - 1) = 0$$

$$\downarrow$$

$$\frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 1$$

$$\downarrow$$

$$\lambda = 4 \qquad \lambda = -2$$

$$x = \frac{1}{1-\lambda}, \quad y = \frac{2}{1-\lambda}, \quad z = \frac{2}{1-\lambda}$$

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- **Step 6: Finding the points**

For each value of λ, substitute back into the expressions for x, y, and z:

- For $\lambda = 4$:

$$x = -\frac{1}{3}, \quad y = -\frac{2}{3}, \quad z = -\frac{2}{3}$$

The point is $\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

- For $\lambda = -2$:

$$x = \frac{1}{3}, \quad y = \frac{2}{3}, \quad z = \frac{2}{3}$$

The point is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

$$x = \frac{1}{1-\lambda}, \quad y = \frac{2}{1-\lambda}, \quad z = \frac{2}{1-\lambda}$$

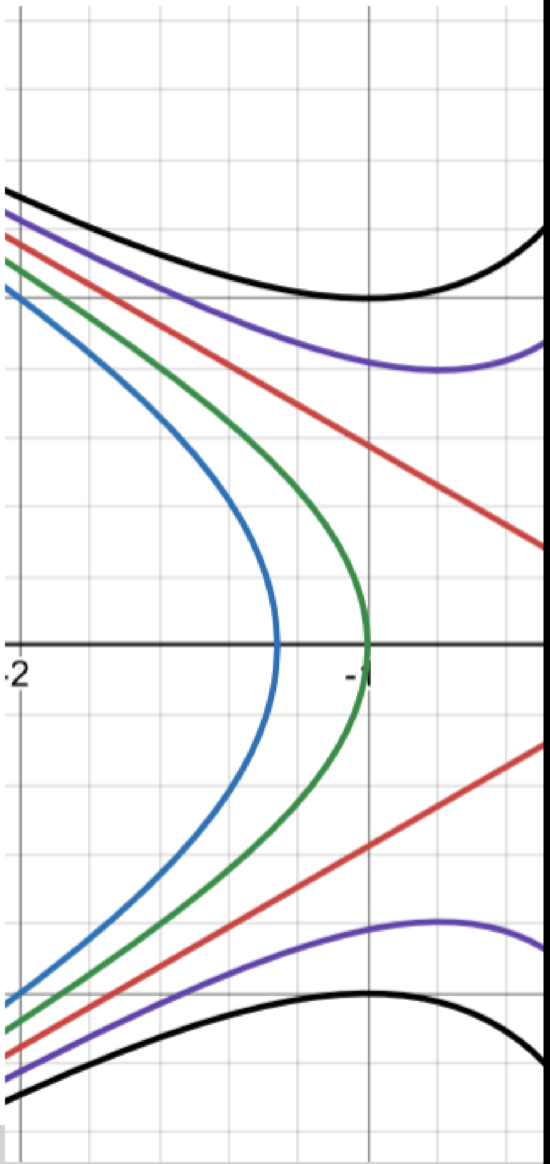
- **Step 7: Compare these coordinates into with (1,2,2) which is furthest?**

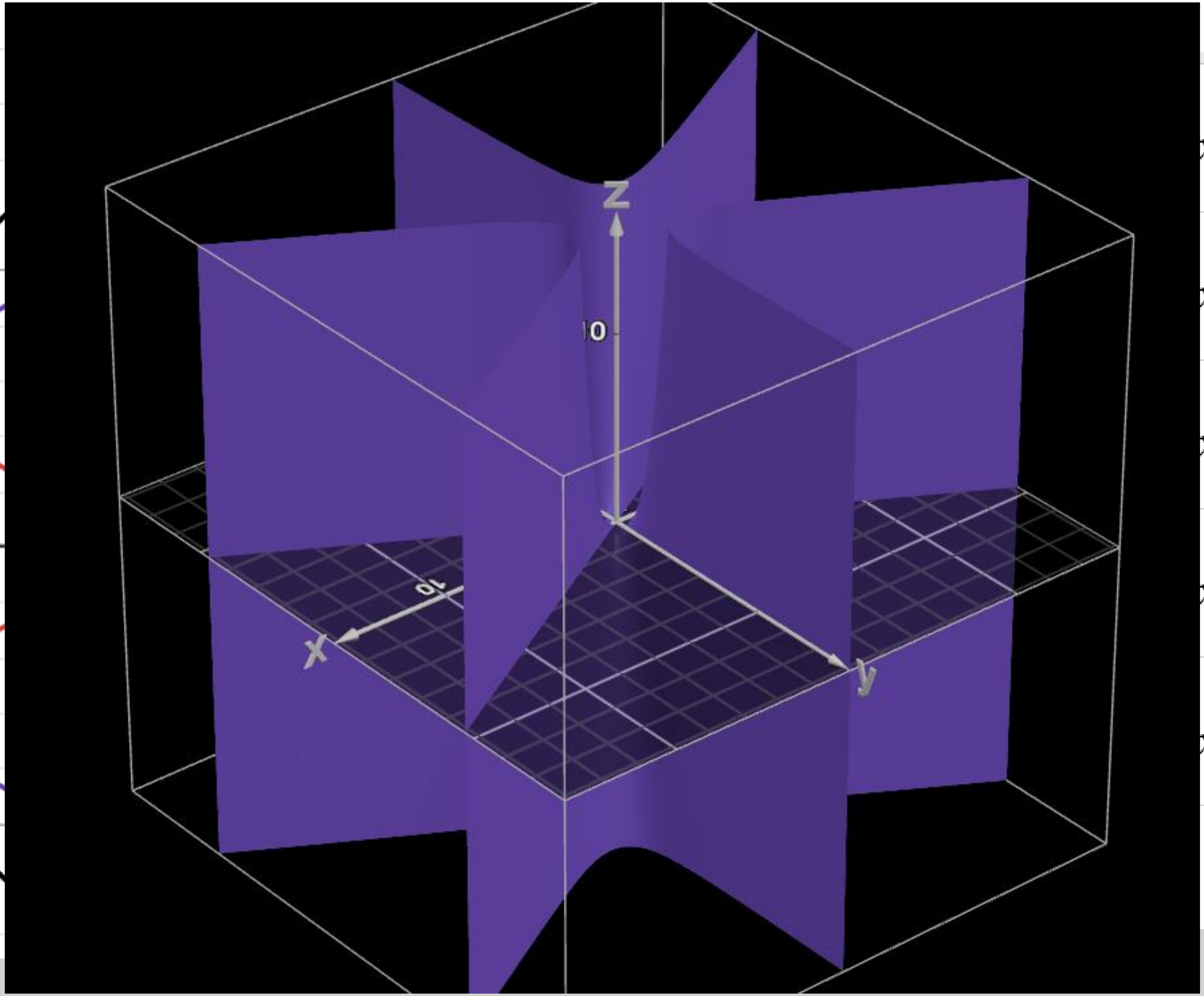
Question 5

5. Consider the function $f(x, y) = x^3 - 3xy^2$. Find the critical point(s) of the function. What happens if your try to classify the critical point(s) using the usual test we perform? In such cases, we must resort to other means to classify the critical point. Using your favourite program, plot a selection of both positive and negative contours. Why do you think this function might be called a "monkey saddle"?

Learning Outcomes?

Sometimes the second derivative tests are inconclusive, particularly for point 0,0.





$xy^2 = -2$

$xy^2 = -1$

$xy^2 = 0$

$xy^2 = 1$

$xy^2 = 2$

Question 6

6. Find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2 - 2x$ inside the triangular region with vertices at $(2, 0)$, $(0, 2)$ and $(0, -2)$.

Learning Outcomes?

Max and Min of a function!

Critical Points and Boundaries

- Step 1: Finding critical point in the region

$$\boxed{\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x-2, 2y)}$$

$$\nabla f(x,y) = 0$$

$\boxed{\text{The critical point inside the region is } (1,0)}$

Thus, $f(1,0) = -1$

- Min point!: second order derivative

6. Find the absolute maximum and absolute minimum of the function $f(x,y) = x^2 + y^2 - 2x$ inside the triangular region with vertices at $(2,0)$, $(0,2)$ and $(0,-2)$.

Attention: Let us do parametrization!

How to Evaluate on the Boundary? (1)

For each boundary line segment, we parameterize the coordinates x and y as functions of a parameter t. Then, we substitute these expressions into the function $f\,(x, y)$, turning it into a single-variable function of t. Finally, we evaluate the function at the endpoints of the line segment and check for any critical points along the segment.

Parameterizing the boundary lines allows us to reduce the two-variable function f (x, y) to a single-variable function f (t). This simplification makes it easier to identify critical points and evaluate the function's values along the boundary.

How to Evaluate on the Boundary? (2)

$$f\left(\underbrace{(g(y)}_{\text{Express x in terms of y (workshop)}},y\right)$$

$$f(x,y)$$

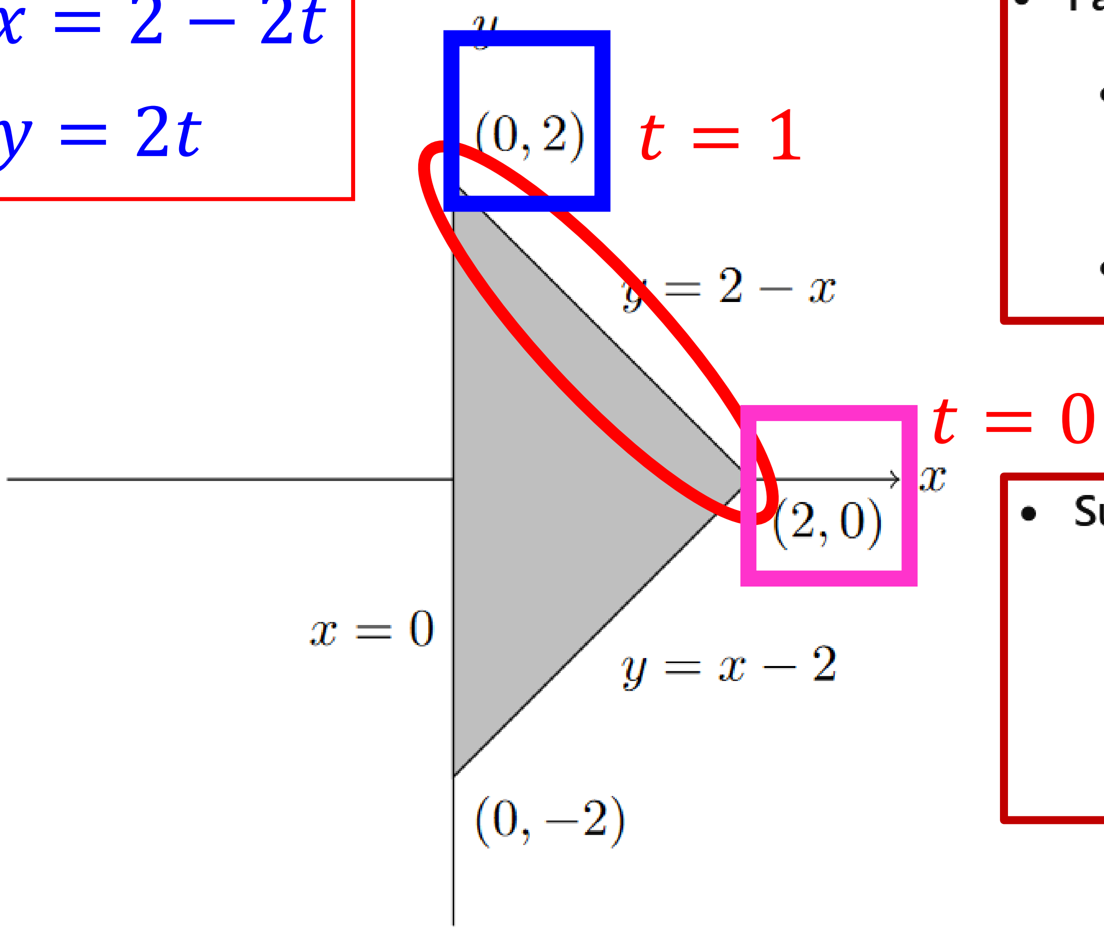
$$f(t)$$

Let's opt for this now!

Full Attention Is Appreciated Here!

- **Step 2.1: Find your constraint: function on the boundary 1: (2, 0) & (0, 2)** WHY???

$$x = 2 - 2t$$
$$y = 2t$$



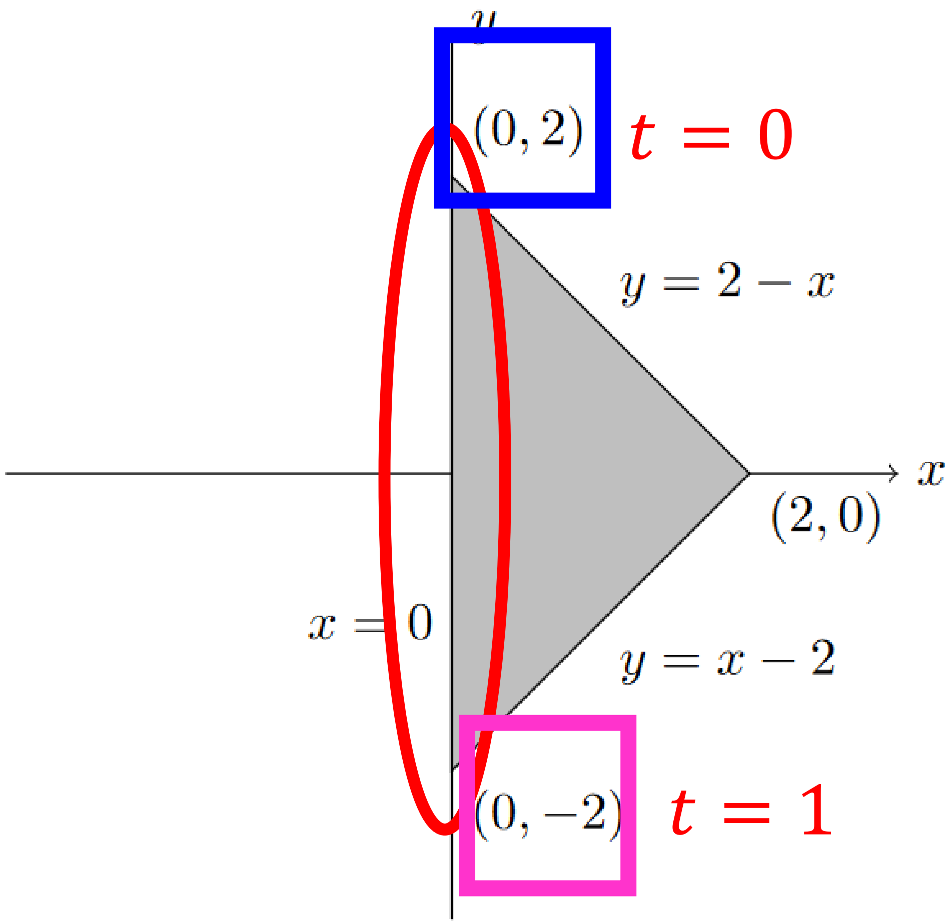
Logical Guessing

- **Parameterization**:
  - The equation of this line can be expressed as $x = 2 - 2t$ and $y = 2t$, where t ranges from 0 to 1.
  - When $t = 0$, the point is $(2, 0)$; when $t = 1$, the point is $(0, 2)$.

- **Substitute into the function**:
  - $f(x, y) = (2 - 2t)^2 + (2t)^2 - 2(2 - 2t)$
  - This gives you a single-variable function in terms of t, which you can then differentiate to find critical points on this boundary segment.

6. Find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2 - 2x$ inside the triangular region with vertices at $(2, 0)$, $(0, 2)$ and $(0, -2)$.

- Step 2.2: Find your constraint (function on the boundary 2: (0, -2) & (0, 2)

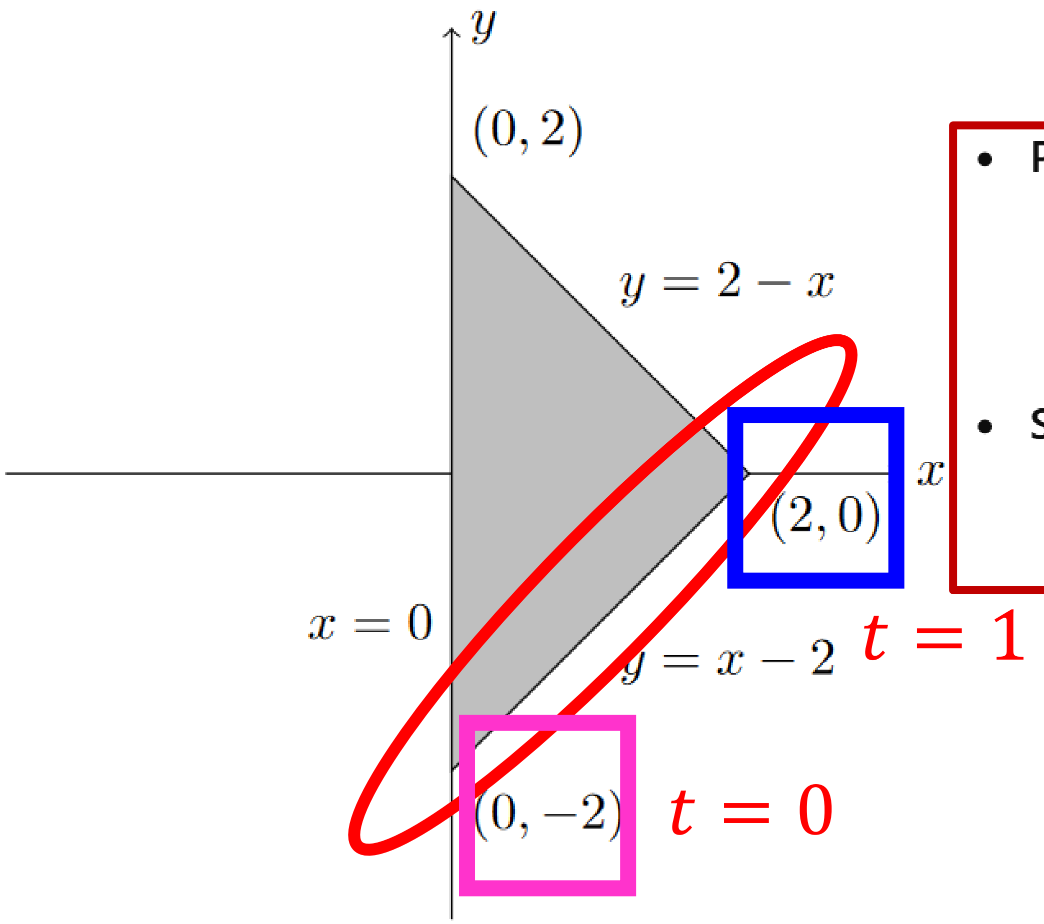


You try! (10min)

- **Parameterization**:
  - The equation of this line is $x = 0$ and $y = 2 - 4t$, where t ranges from 0 to 1.
- **Substitute into the function**:
  - $f(x, y) = (0)^2 + (2 - 4t)^2 - 2(0) = 4 - 16t + 16t^2$.

6. Find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2 - 2x$ inside the triangular region with vertices at $(2, 0)$, $(0, 2)$ and $(0, -2)$.

- **Step 2.3: Find your constraint (function on the boundary 3: (0, -2) & (2, 0)**



- **Parameterization**:
  - The equation of this line is $x = 2t$ and $y = -2 + 2t$, where t ranges from 0 to 1.
- **Substitute into the function**:
  - $f(x, y) = (2t)^2 + (-2 + 2t)^2 - 2(2t) = 4t^2 + (4t^2 - 8t + 4)$

6. Find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2 - 2x$ inside the triangular region with vertices at $(2, 0)$, $(0, 2)$ and $(0, -2)$.

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- # Step 3: Summarizing

- **Line segment from** $(2,0)$ **to** $(0,2)$:
  - **Parameterization**: $x = 2 - 2t$, $y = 2t$, where $t \in [0,1]$.
  - **Substitute into the function**:

$$f(x(t), y(t)) = (2-2t)^2 + (2t)^2 - 2(2-2t)$$
$$f(t) = 4(1-t)^2 + 4t^2 - 4(1-t)$$

  - **Evaluations**:
    - At $t = 0$: $f(2,0) = 0$
    - At $t = 1$: $f(0,2) = 4$

- **Line segment from** $(0,2)$ **to** $(0,-2)$:
  - **Parameterization**: $x = 0$, $y = 2 - 4t$, where $t \in [0,1]$.
  - **Substitute into the function**:

$$f(0, y(t)) = 0^2 + (2-4t)^2 - 2(0)$$
$$f(t) = (4t^2 - 16t + 16)$$

  - **Evaluations**:
    - At $t = 0$: $f(0,2) = 4$
    - At $t = 1$: $f(0,-2) = 4$

- **Line segment from** $(0,-2)$ **to** $(2,0)$:
  - **Parameterization**: $x = 2t$, $y = -2 + 2t$, where $t \in [0,1]$.
  - **Substitute into the function**:

$$f(x(t), y(t)) = (2t)^2 + (-2+2t)^2 - 2(2t)$$
$$f(t) = 4t^2 + 4t^2 - 8t$$

  - **Evaluations**:
    - At $t = 0$: $f(0,-2) = 4$
    - At $t = 1$: $f(2,0) = 0$

We now compare the function values from the critical point and the boundary evaluations:

- The smallest value is $f = -1$, occurring at the critical point $(1,0)$.
- The largest value is $f = 4$, occurring on the boundary at the points $(0,2)$ and $(0,-2)$.

Question 7/8

Try at Home, following the slide examples!

Pass sessions will be answering these in detailed!

Thank You

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Week 6: Multivariable Calculus II

(MEC) Senior Lecturer: K.B. Goh, Ph.D.

Tutor: (a) Ian Keen & (b) Jack

Pass Leader: (i) Zi Wei and (ii) Yvonne

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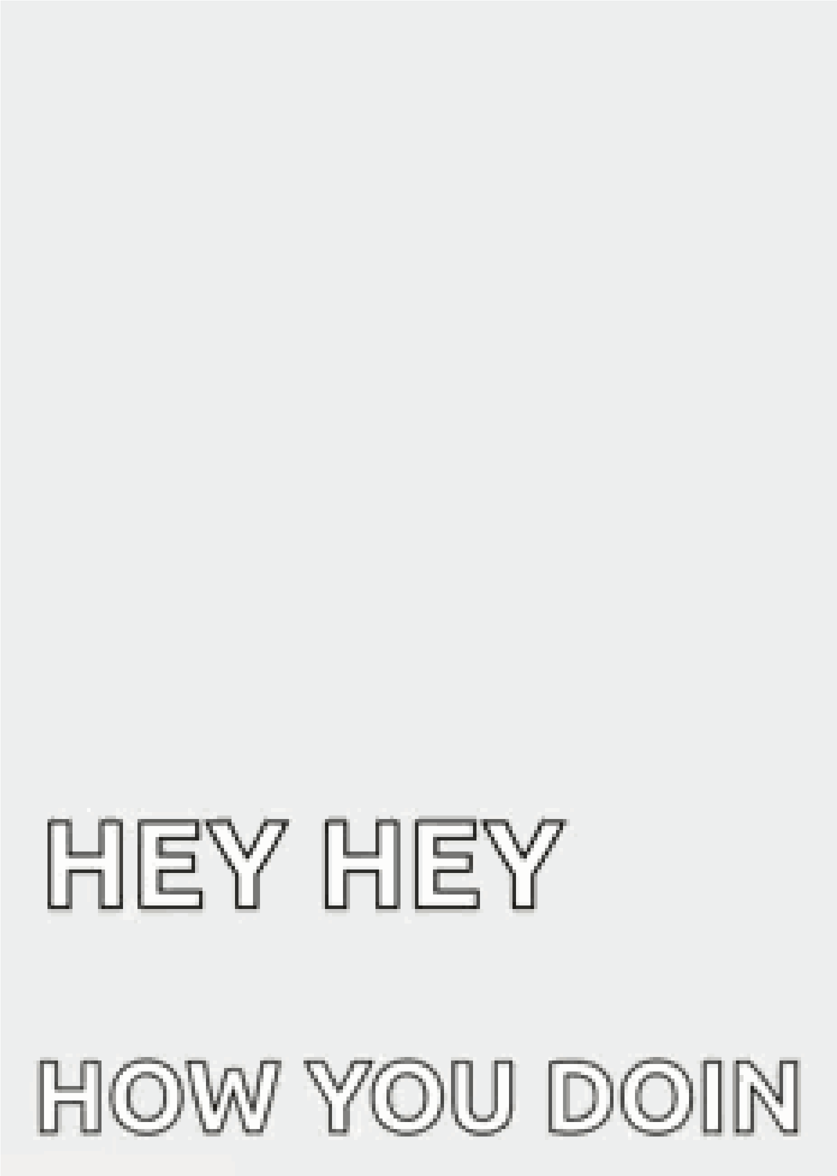
*Please bear with me, attending

to your messages soon*



Check-In

Resources

1. PASS with Yvonne and Zi Wei

2. Math Centre

3. Additional: Videos (Mid-Term Prep.)

Resources

3. Additional: Videos

4. Pass materials

Topics

Week	Topic
1	Vectors, Lines, and Planes
2	Systems of Linear Equations
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5	Multivariable Calculus 1
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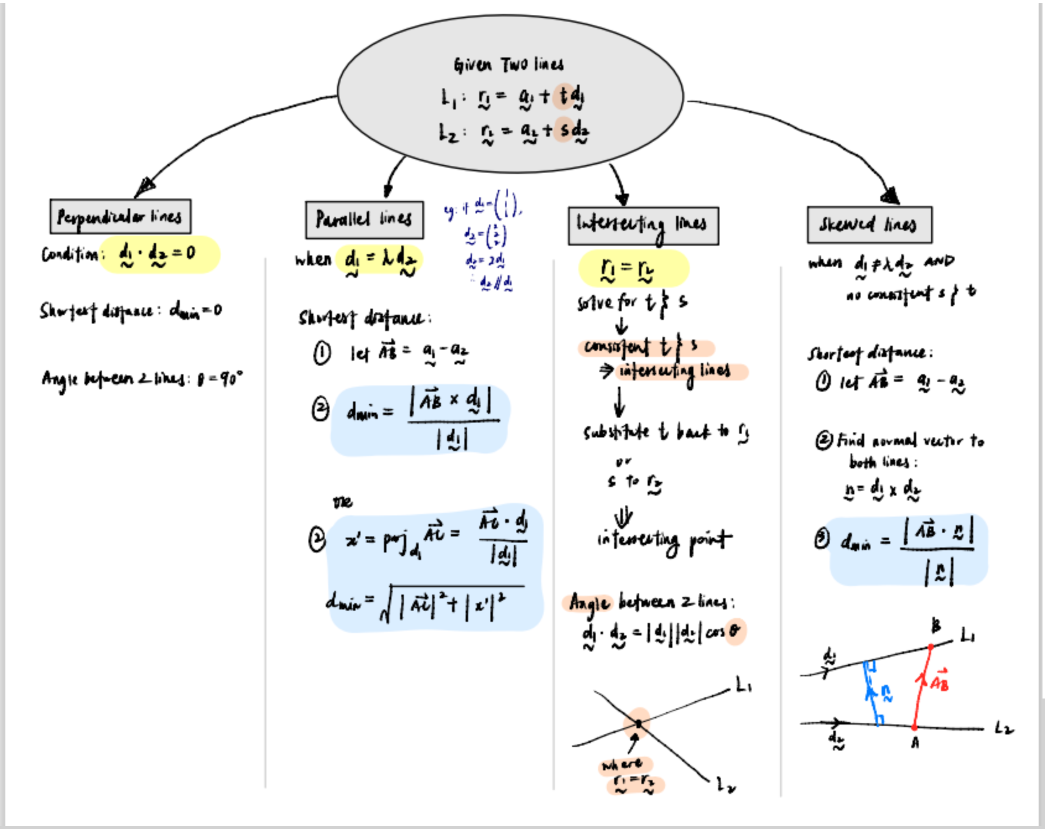
Mid-Term: How to Kill It in Mid Term?

- Step 1: RoadMap given in the Pass Sessions (MUM campus only).



- Step 2: Videos in the Google Drive.
- Step 3: Pass Solved Questions.
- Step 4: Mock Exams/Revision Quiz

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Assessments breakdown

Task description	*Value*	*Due date*
Lecture quizzes	5%	Weekly 11:55pm the night before your applied class (except Monday classes) except in Week 1 when the due date is Friday at 11:55 pm (Malaysia time)
Applied class quizzes	5%	Weekly during your applied class
Workshop problems	20%	Weekly at 11:55pm Sunday, except for Weeks 7 (midsem)
Mid-semester exam	20%	During your workshop in Week 7

The Big Learning Outcomes for Week 6

After completing this week's task, you should be able to:

- Identify and characterize critical points (local maxima and minima, and saddle points).
- Calculate absolute maxima and minima of a function.
- Use Lagrange multipliers to find local maxima and minima of a function subject to an equality constraint.
- Parameterise basic curves and surfaces.
- Calculate tangents to curves.

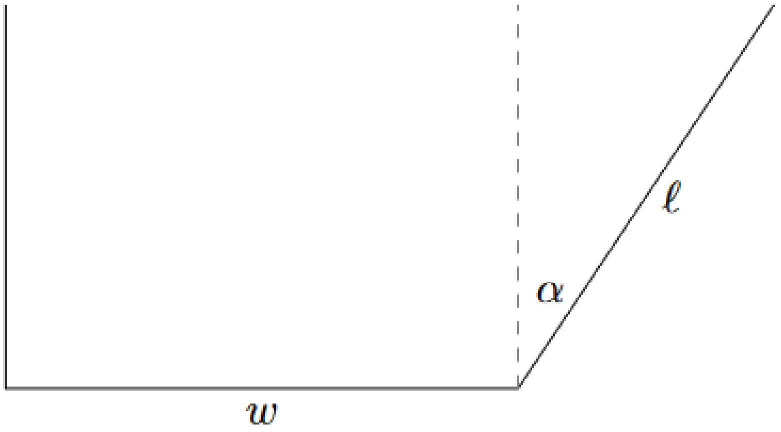
Today’s Activity

0. Applied Problem 9

1. Workshop Problem Set

Question 9

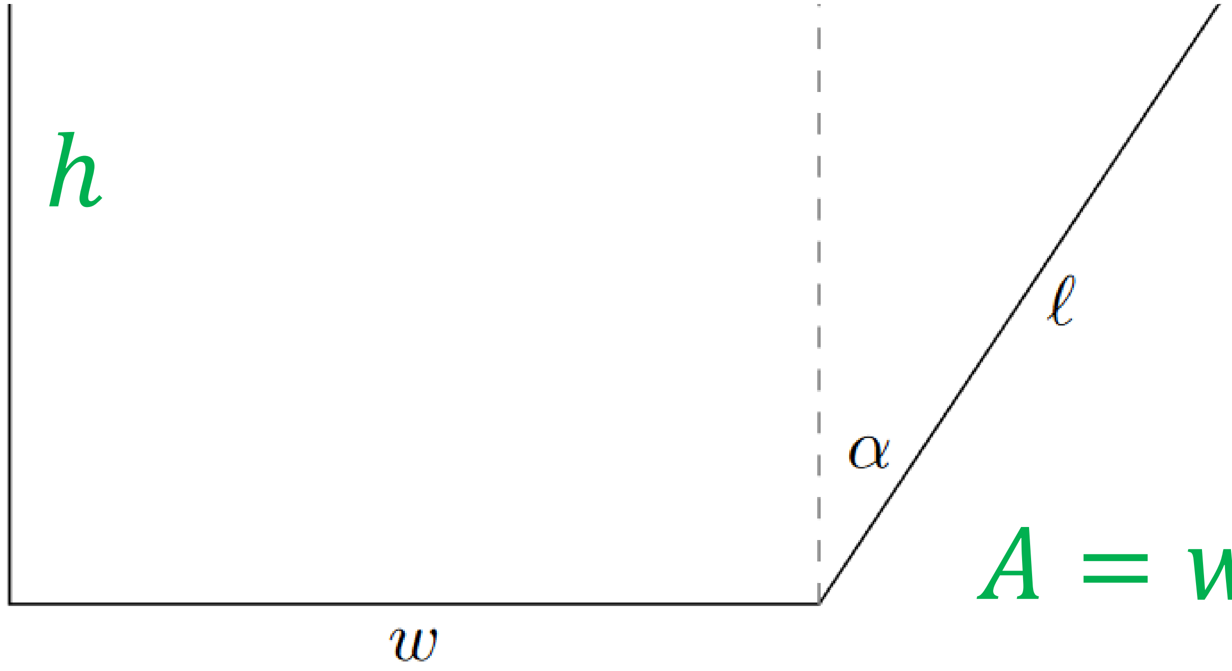
9. (Challenging!) You are asked to make a rain gutter for a house. You have a sheet of metal that is $L = 15 + 10\sqrt{3}\,\text{cm}$ wide and you can bend it to have a profile as shown below where w, l and α are values you can choose.



Write down an expression for the cross-sectional area of the gutter in terms of w, α and ℓ. Now write down a constraint connecting L, w, α and ℓ. Using Lagrange multipliers, if your aim is to maximise the capacity of the gutter, what should ℓ and α be? *Hint: You can assume that there is a single interesting solution and this doesn't have $\alpha = 0$ or $\alpha = \pi/2$.*

Learning Outcomes? *Lagrange*

- Step 1: Find your objective function (Area)



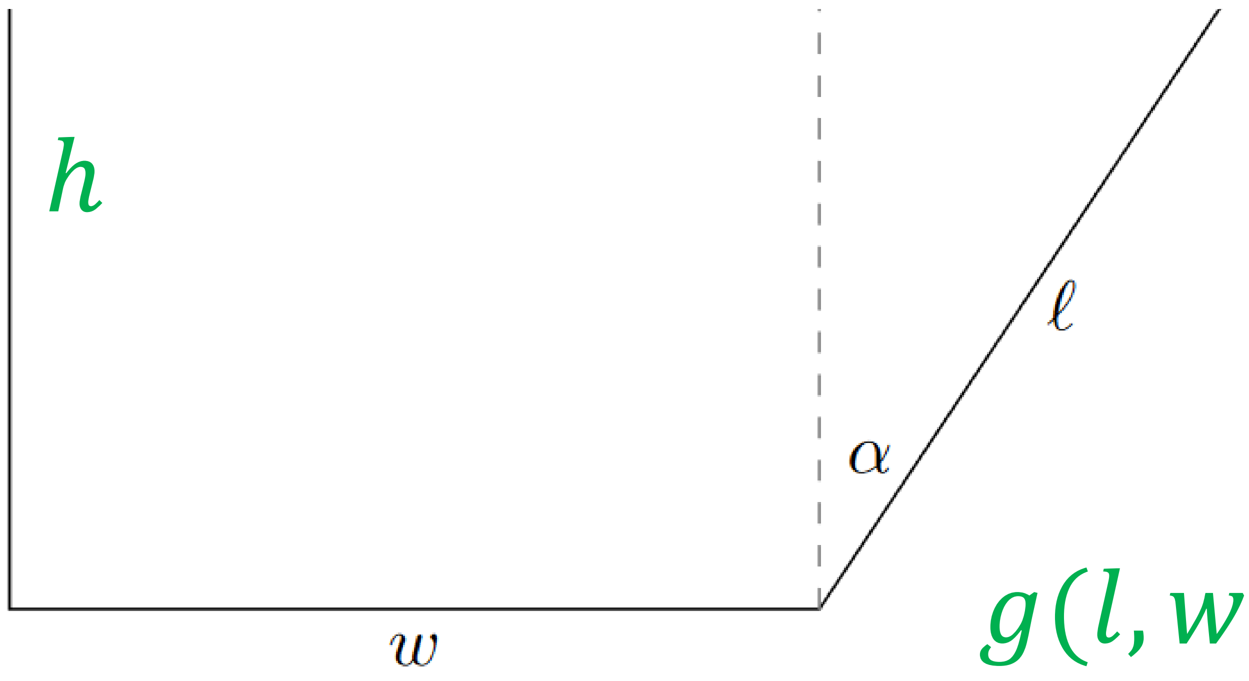
$$= wh + 0.5hl\sin\alpha$$

- Let's reduce the parameter space!

$$h = l\cos\alpha$$

$$A = wl\cos\alpha + 0.5l\cos\alpha\, l\sin\alpha$$

- Step 2: Find your constraint (length)



$$= h + w + l = L$$

$$\downarrow$$

- **Let's reduce the parameter space!**

$$\downarrow$$

$$h = l \cos \alpha$$

$$\downarrow$$

$$g(l, w, \alpha) = l \cos \alpha + w + l = L$$

- **Step 3: Lagrange function**

$$\frac{\partial A}{\partial \ell} = \lambda \frac{\partial g}{\partial \ell} \qquad \Longrightarrow \qquad w \cos\alpha + \ell \cos\alpha \, \sin\alpha = \lambda(1 + \cos\alpha)$$

$$\frac{\partial A}{\partial w} = \lambda \frac{\partial g}{\partial w} \qquad \Longrightarrow \qquad \ell \cos\alpha = \lambda$$

$$\frac{\partial A}{\partial \alpha} = \lambda \frac{\partial g}{\partial \alpha} \qquad \Longrightarrow \qquad -w\ell \sin\alpha + \frac{1}{2}\ell^2(\cos^2\alpha - \sin^2\alpha) = -\lambda\ell \sin\alpha$$

$$A = wl\cos\alpha + 0.5l\cos\alpha\, l\sin\alpha$$

$$g(l, w, \alpha) = l\cos\alpha + w + l$$

Are You Camera ready?

Try on your own, by referring to these!

- **Step 4(a): Mix and Match!**

Step a: Solve for w

We start by substituting from the second equation into the first one. The equation becomes:

$$w + \ell \sin\alpha = \ell(1 + \cos\alpha)$$

To isolate w, we rearrange the terms:

$$\boxed{w = \ell(1 + \cos\alpha - \sin\alpha)}$$

This expression represents w in terms of ℓ and α.

- **Workings:** $w\cos\alpha + l\cos\alpha\sin\alpha = (l\cos\alpha)(1 + \cos\alpha)$

$$w + l\sin\alpha = l(1 + \cos\alpha)$$

- **Divide the cos!**

$$\frac{\partial A}{\partial \ell} = \lambda\frac{\partial g}{\partial \ell} \implies w\cos\alpha + \ell\cos\alpha\,\sin\alpha = \lambda(1 + \cos\alpha)$$

$$\frac{\partial A}{\partial w} = \lambda\frac{\partial g}{\partial w} \implies \ell\cos\alpha = \lambda$$

$$\frac{\partial A}{\partial \alpha} = \lambda\frac{\partial g}{\partial \alpha} \implies -w\ell\sin\alpha + \frac{1}{2}\ell^2(\cos^2\alpha - \sin^2\alpha) = -\lambda\ell\sin\alpha$$

- **Step 4(b): Mix and Match!**

$$\boxed{w = l(1 + \cos\alpha - \sin\alpha)}$$

Step b: Substitute w and Solve the Third Equation

Next, we substitute the expression for w into the third equation: **& lambda too**

$$-\ell^2(1 + \cos\alpha - \sin\alpha)\sin\alpha + \frac{1}{2}\ell^2(\cos^2\alpha - \sin^2\alpha) = -\ell^2\cos\alpha\sin\alpha$$

This equation contains trigonometric identities. By simplifying and equating both sides, we find that:

$$\boxed{\sin\alpha = \frac{1}{2}}$$

The solution $\sin\alpha = \frac{1}{2}$ corresponds to the angle $\alpha = 30°$.

- **Workings:**

$$\frac{\partial A}{\partial \ell} = \lambda\frac{\partial g}{\partial \ell} \quad \Longrightarrow \quad w\cos\alpha + \ell\cos\alpha\,\sin\alpha = \lambda(1 + \cos\alpha)$$

$$\frac{\partial A}{\partial w} = \lambda\frac{\partial g}{\partial w} \quad \Longrightarrow \quad \boxed{\ell\cos\alpha = \lambda}$$

$$\frac{\partial A}{\partial \alpha} = \lambda\frac{\partial g}{\partial \alpha} \quad \Longrightarrow \quad \boxed{-w\ell\sin\alpha} + \frac{1}{2}\ell^2(\cos^2\alpha - \sin^2\alpha) = \boxed{-\lambda}\ell\sin\alpha$$

ASH
rsity

- **Step 4(c): Mix and Match!**

Step c: Calculate L

Now that we have $\alpha = 30°$, we can substitute this value into the equation for L:

$$\boxed{\ell(2 + 2\cos\alpha - \sin\alpha)} = \ell\left(2 + \sqrt{3} - \frac{1}{2}\right)$$

Simplifying further, we get:

$$\boxed{L = \ell\left(\frac{3}{2} + \sqrt{3}\right)}$$

This represents the relationship between L and ℓ.

- Step 4(a) $w = l(1 + \cos\alpha - \sin\alpha)$
- Constraint: $L = l\cos\alpha + w + l$

Step 4(d): Mix and Match!

Step d Final Substitution and Solution for ℓ

Finally, we are given that:

$$\boxed{L = 15 + 10\sqrt{3}}$$

$$\underbrace{L = \ell\left(\frac{3}{2} + \sqrt{3}\right)}_{\boxed{4(c)}}$$

Using this, we solve for ℓ. After calculating, we find:

$$\ell = 10 \text{ cm}$$

(Challenging!) You are asked to make a rain gutter for a house. You have a sheet of metal that is $\boxed{L = 15 + 10\sqrt{3}\,\text{cm}}$ wide and you can bend it to have a profile as shown below where w, l and α are values you can choose.

Today's Activity

0. Applied Problem 9

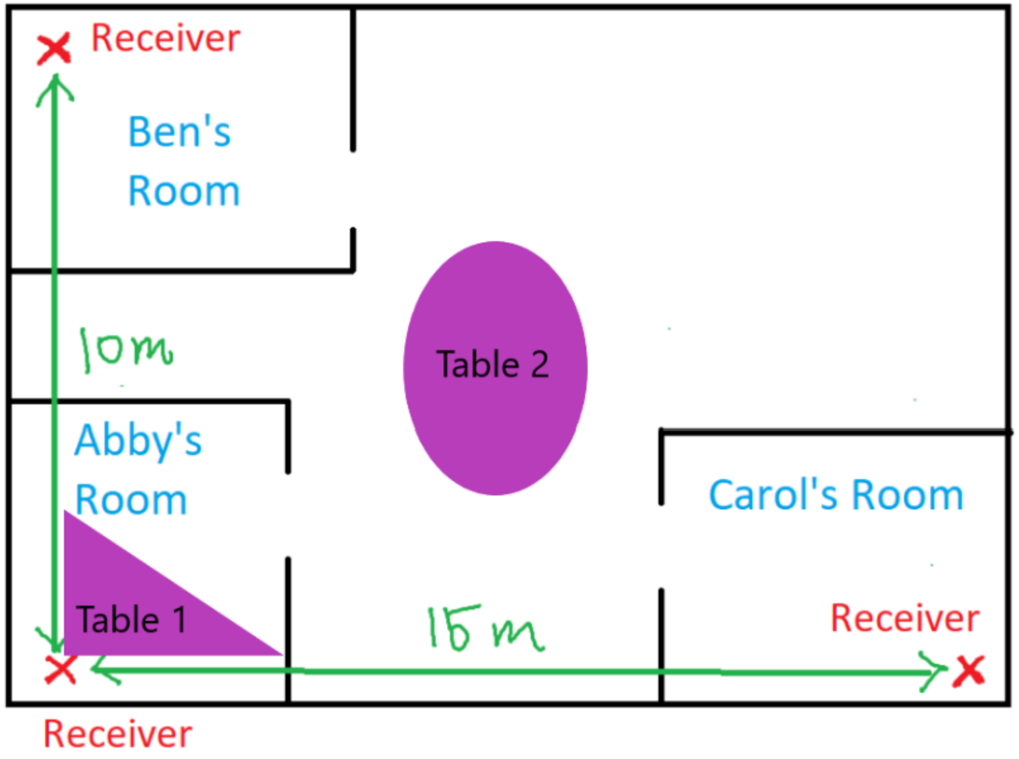
1. Workshop Problem Set

One worked-out example!

Wifi Access Point

In this week's workshop, we will continue with last week's problem of finding the optimum location for a WiFi access point in a house. Recall that if we set up a coordinate system where Abby's receiver is at the origin $(0,0)$, Ben's receiver is on the y-axis at $(0,10)$, and Carol's receiver is on the x-axis at$(15,0)$, then the power consumption of the access point in watts is given by

$$\boxed{T(x,y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)}$$



1. Find all the critical points of $T(x, y)$ (you should have already done this in the previous workshop!). **[1 mark]**

$$\nabla T = \begin{bmatrix} \frac{1}{50}(6x - 30) \\ \frac{1}{50}(6y - 20) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{25}x - \frac{3}{5} \\ \frac{3}{25}y - \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(x, y) = (5, 10/3)

Wifi Access Point

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$$T(x, y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)$$

2. Use the second derivative test to classify each critical point of $T(x, y)$.

Step 1: We need to calculate this!

$$= \frac{\partial^2 T}{\partial x^2} \cdot \frac{\partial^2 T}{\partial y^2} - \left(\frac{\partial^2 T}{\partial x \partial y}\right)^2$$

Step 2: Calculate the second derivate w.r.t (5, 10/3)

$$\frac{\partial^2 T}{\partial x^2} = \frac{6}{50}, \quad \frac{\partial^2 T}{\partial y^2} = \frac{6}{50}, \quad \frac{\partial^2 T}{\partial x \partial y} = 0$$

Step 3: Solve step 1!

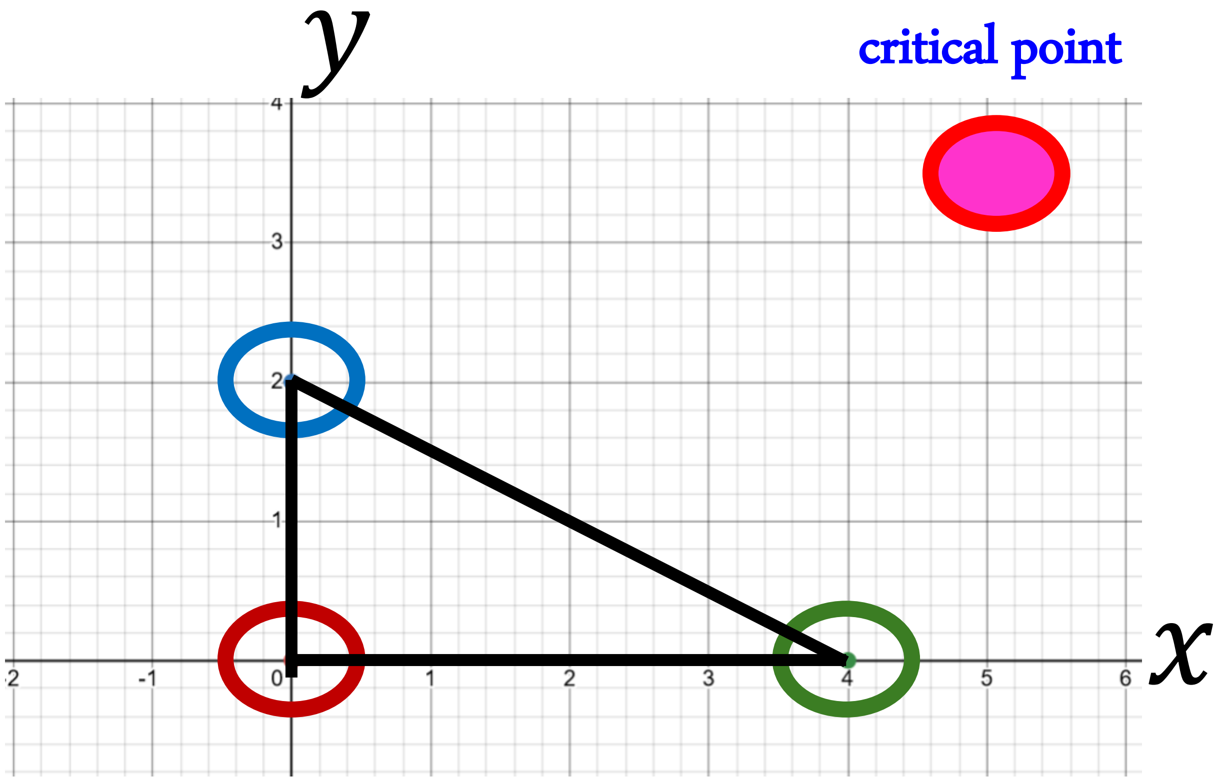
$$= \frac{\partial^2 T}{\partial x^2} \cdot \frac{\partial^2 T}{\partial y^2} - \left(\frac{\partial^2 T}{\partial x \partial y}\right)^2 > 0$$

Local Min.!

$$T(x, y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)$$

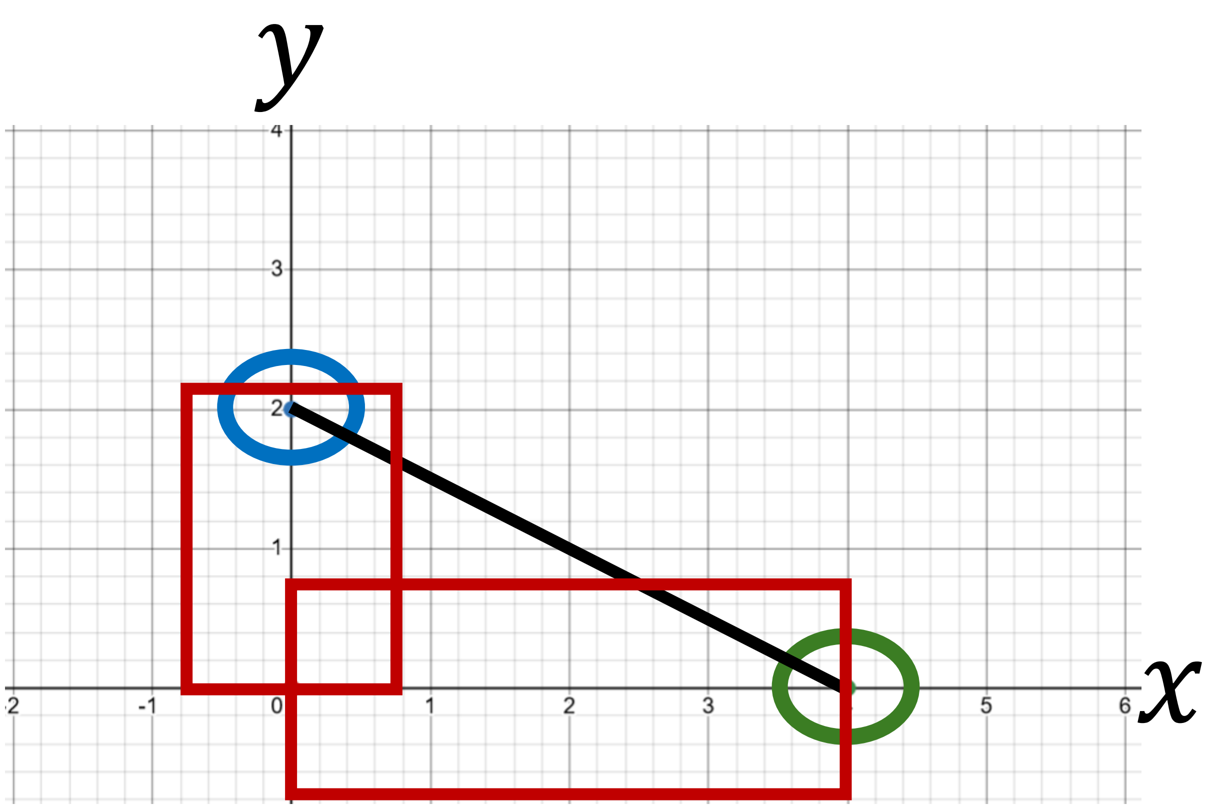
Abby wants to put the WiFi access point on a corner table in her room. The corner table is triangular in shape with vertices at $(0,0)$, $(4,0)$ and $(0,2)$.

3. Explain why the access point should be placed along the edge of the table and not in the interior of the table. **[2 marks]**



- The only critical point (5, 10/3) is outside the triangular table.
- ✓ Any extremum of a function can occur either along the boundary of a region or at interior critical points.
- ○ In this case, the minimum of T(x, y) on the table must occur at the boundary of the table.

4. Write down an equation for the diagonal edge between $(4, 0)$ and $(0, 2)$, make sure you include the range of the variables. **[2 marks]**



Red Boxes=Ranges

Step 1: Find $y = mx + c$

5 min!

- $m = -2/4$
- $c = 2$

Step 2: Plug it in

$$y = -0.5x + 2$$

$$\boxed{2y + x = 4}$$

How to Evaluate on the Boundary?

$$f\,(x, y)$$

$$f\,(\underbrace{(g(y)}_{\text{Express x in terms of y (workshop, Now)}}\,, y)$$

$$f\,(\,t\,)$$

Applied tutorial!

5. Using the equation from the previous part, express the power consumption T along the diagonal edge as a function of a single variable y, and hence find the location which minimises T along this edge.

[2 marks]

Step 1: Converting to a single variable y

$$\boxed{T(x,y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)}$$

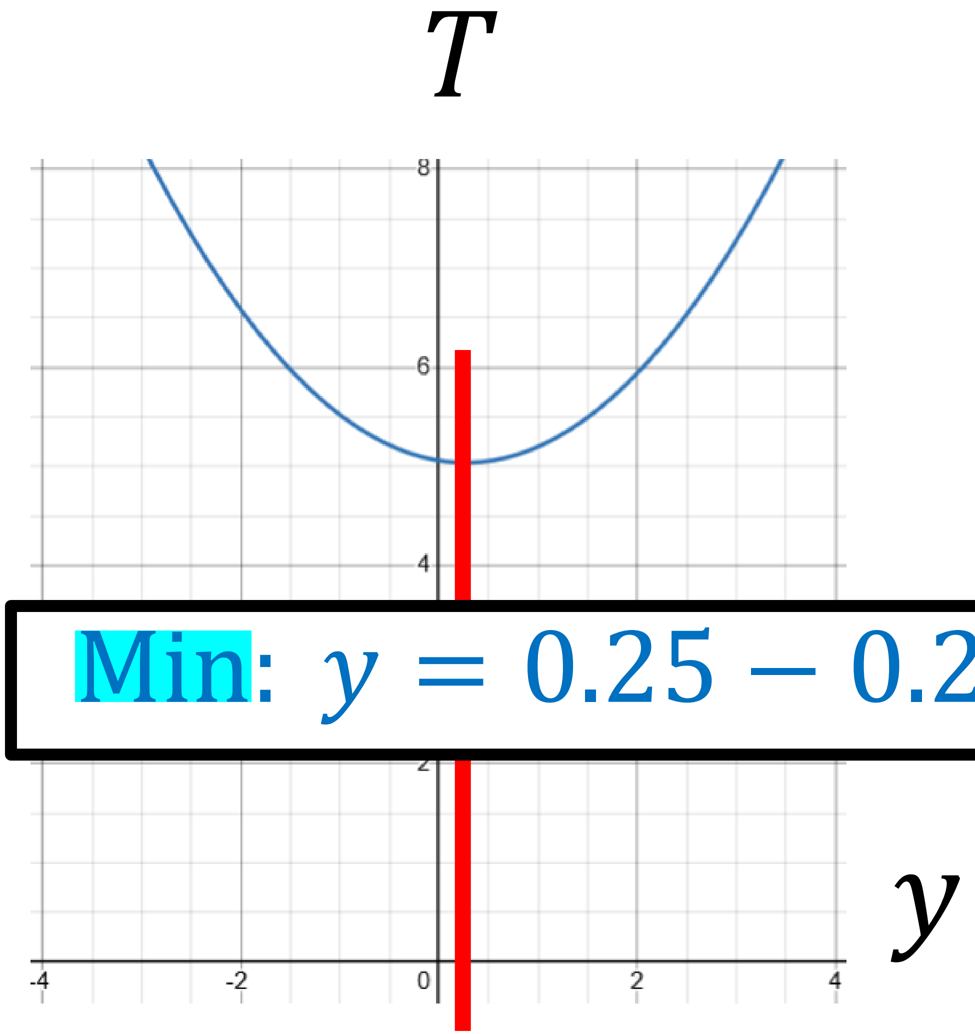
$$x = -2y + 4$$

Step 2: A single variable function

5 mins to try!

$$T(4-2y, y) = 1/50\ (15y^2 - 8y + 253)$$

Step 3.1: Visualizing



Min: $y = 0.25 - 0.27$

Step 3.2: Numeric

$$\boxed{\frac{d}{dy}(15y^2 - 8y + 253) = 30y - 8 = 0}$$

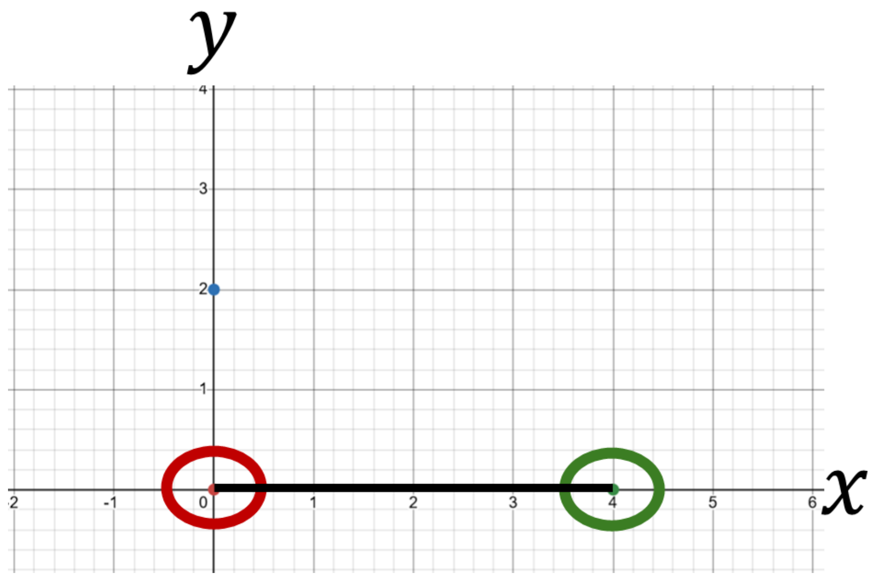
$$y = \frac{8}{30} = \frac{4}{15}$$

Min: $y = 0.27$

$T = 1/50\,(15y^2 - 8y + 253)$

6. Repeat the calculation for the remaining two edges, and hence find the location on the table which minimises T. **[3 marks]**

6.1) 1-Dimensional Horizontal Line



$$T(x,y) = 0.02(3x^2 + 3y^2 - 30x - 20y + 325)$$

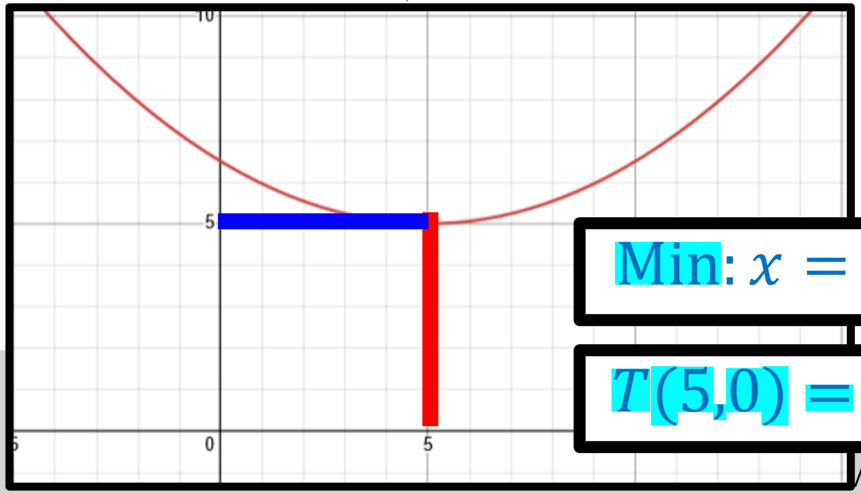
$$\downarrow$$

$$T(x,0) = 0.02(3x^2 + 3(0) - 30x - 20(0) + 325)$$

$$\downarrow$$

$$T(x,0) = 0.02(3x^2 - 30x + 325)$$

$$\downarrow$$



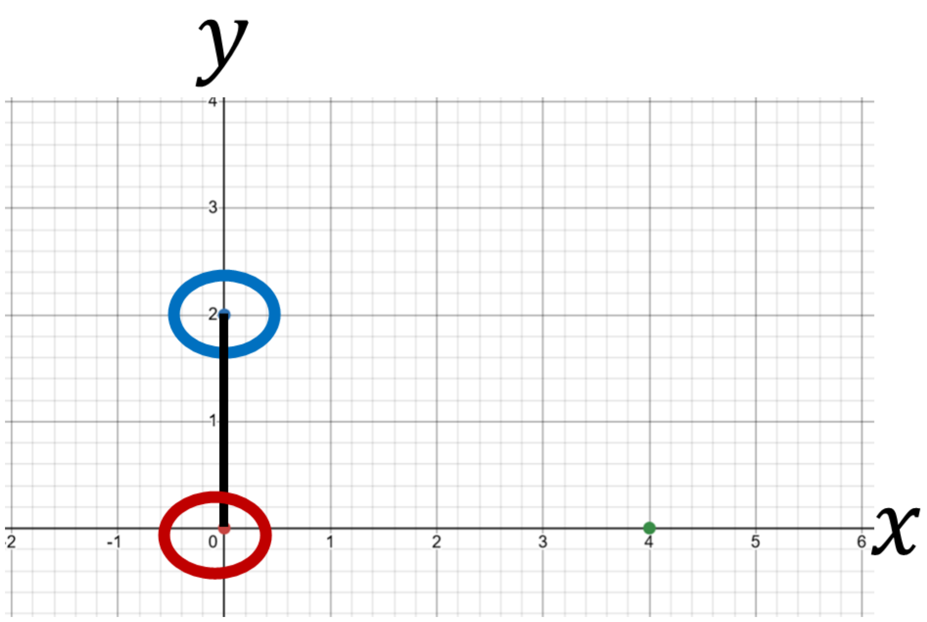
$$\boxed{\text{Min}: x = 5}$$

$$\boxed{T(5,0) = 5}$$

$$\boxed{T(x,y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)}$$

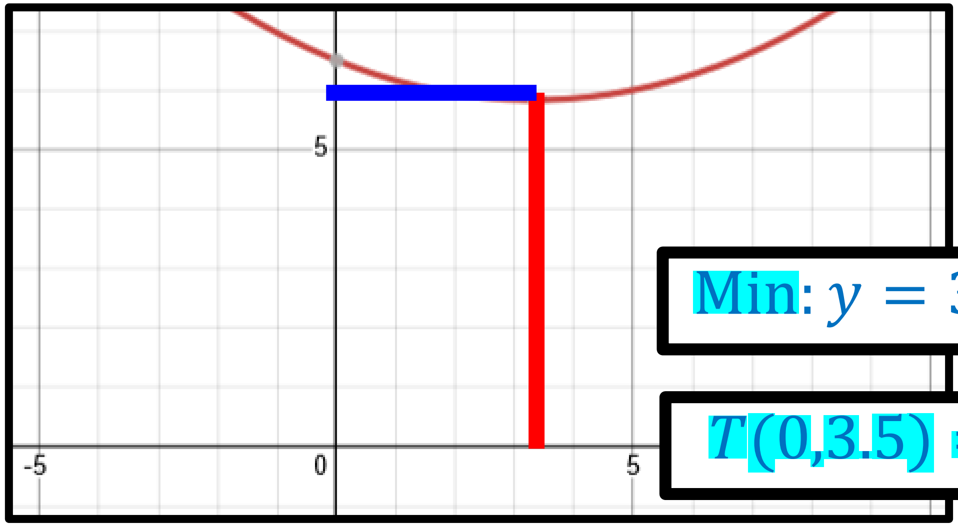
6. Repeat the calculation for the remaining two edges, and hence find the location on the table which minimises T. **[3 marks]**

6.2) 1-Dimensional Vertical Line



$$T(x,y) = 0.02(3x^2 + 3y^2 - 30x - 20y + 325)$$

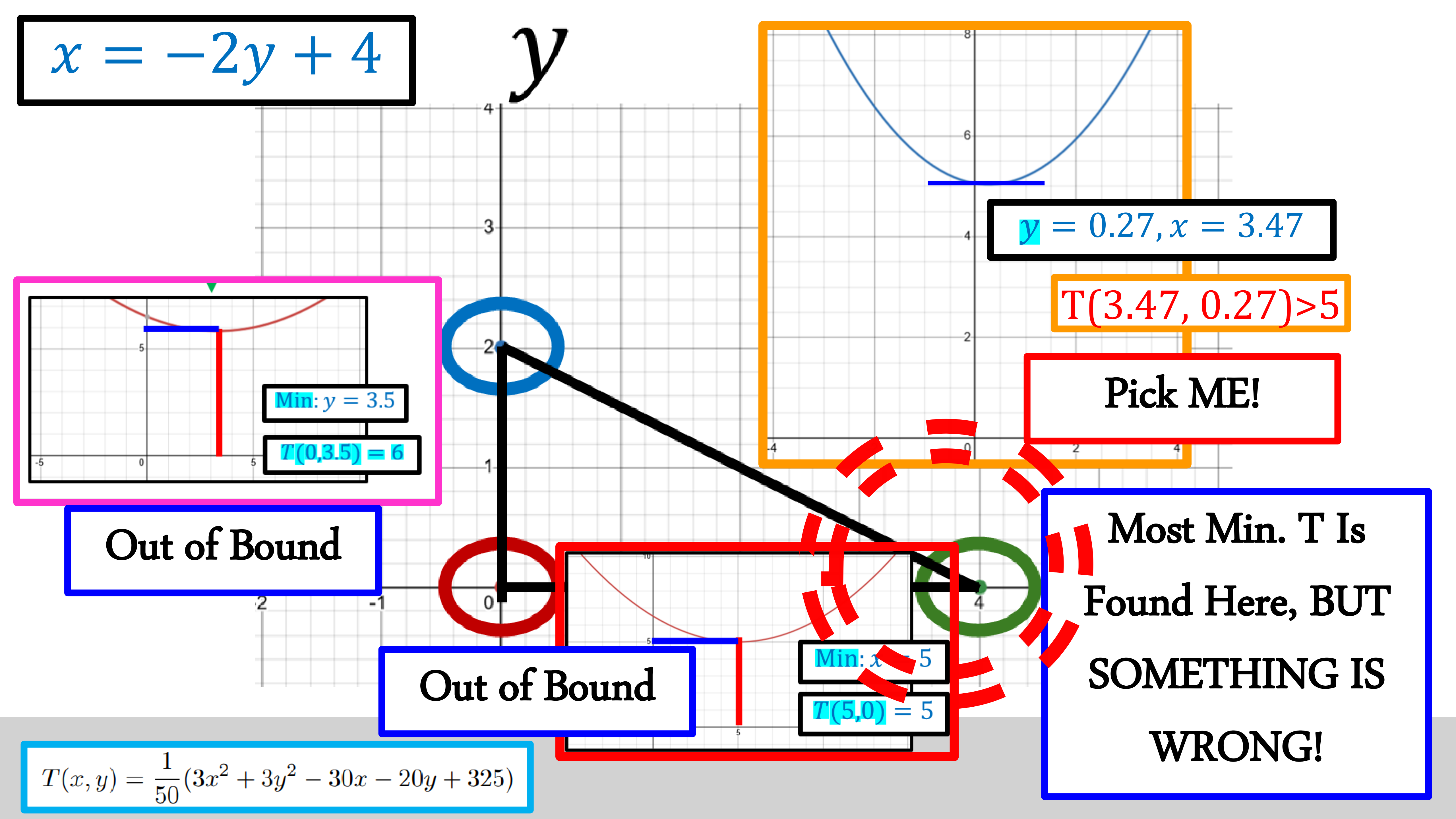
$$T(0,y) = 0.02(3y^2 - 20y + 325)$$



Min: $y = 3.5$

$T(0,3.5) = 6$

$$T(x,y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)$$



$x = -2y + 4$
y
$y = 0.27, x = 3.47$
T(3.47, 0.27)>5
Pick ME!
Min: $y = 3.5$
$T(0,3.5) = 6$
Out of Bound
Out of Bound
Min: x 5
$T(5,0) = 5$
Most Min. T Is Found Here, BUT SOMETHING IS WRONG!
$T(x,y) = \frac{1}{50}(3x^2 + 3y^2 - 30x - 20y + 325)$

7. To minimise power consumption, should the WiFi access point be place somewhere in the interior of the table or on the boundary of the table? Justify your answer. **[1 marks]**

A repeat of Q3!

- The only critical point is outside the table.

✓ Any extremum of a function can occur either along the boundary of a region or at interior critical points.

- In this case, the minimum of $T(x, y)$ on the table must occur at the boundary of the table.

It is decided that the WiFi access point should be placed on a more central table. The second table is oval shaped and can be described by the set of points

$$\{(x, y) : (x-7)^2 + \frac{1}{2}(y-5)^2 \leq 1\}$$

We will use the method of Lagrange multipliers to minimise the power consumption along the boundary of the table.

8. Write down the *constraint* function $g(x, y)$. **[1 marks]**

$$(x-7)^2 + \frac{1}{2}(y-5)^2 \underbrace{= 1}_{\text{Level Set}}$$

$$\boxed{g(x,y) = (x-7)^2 + \frac{1}{2}(y-5)^2.}$$

9. Write down the three simultaneous equations that x, y, and the Lagrange multiplier λ must satisfy at an extreme value of power consumption along the boundary of the table. **[2 marks]**

$$\nabla T = \lambda \nabla g$$

$$g(x, y) = 1$$

step1

$$\nabla T = \begin{bmatrix} \frac{1}{50}(6x - 30) \\ \frac{1}{50}(6y - 20) \end{bmatrix}$$

$$\nabla g(x, y) = \begin{bmatrix} 2(x - 7) \\ y - 5 \end{bmatrix}$$

step 2

$$\frac{1}{50}(6x - 30) = 2\lambda(x - 7)$$

$$\frac{1}{50}(6y - 20) = \lambda(y - 5)$$

$$(x - 7)^2 + \frac{1}{2}(y - 5)^2 = 1$$

step 3

10. Use two of the equations to express x and y in terms of λ.

$$x = \frac{15 - 350\lambda}{3 - 50\lambda}, \; y = \frac{20 - 250\lambda}{6 - 50\lambda}$$

$$\frac{1}{50}(6x - 30) = 2\lambda(x - 7)$$

$$\frac{1}{50}(6y - 20) = \lambda(y - 5)$$

11. Using Matlab or CAS, solve the last equation (after substitution) for all possible real solutions of λ, rounded to 2 decimal places. **[2 marks]**

- Write the constraint expression:

$$\boxed{(x-7)^2 + \frac{1}{2}(y-5)^2 = 1}$$

- Rewrite the above, expressing in terms of Lambda!

$$\boxed{x = \frac{15 - 350\lambda}{3 - 50\lambda},\ y = \frac{20 - 250\lambda}{6 - 50\lambda}}$$

- Lambda are?

-0.1 & 0.29

12. Determine the most power efficient position to place the WiFi access point on the oval table, round your answer to 2 decimal places. **[2 marks]**

Pls, try it together with your team!

The most efficient point is: $(6.24, 4.09)$

The Big Learning Outcomes for Week 6

After completing this week's task, you should be able to:

- Identify and characterize critical points (local maxima and minima, and saddle points).

- Calculate absolute maxima and minima of a function.

- Use Lagrange multipliers to find local maxima and minima of a function subject to an equality constraint.

- Parameterise basic curves and surfaces.

- Calculate tangents to curves.

Thank You