ENG1005 S2 2024 Workshop 2
Traffic Flow

20 marks total

This problem set is intended for you to apply the mathematical skills you are learning. It is also designed to practice communicating your work clearly.

It is expected that you will use the workshop to develop (rough) solutions. During the workshop, you should discuss the problems with your peers and the academic staff who are there to assist you. In particular, if you are uncertain about what the problems are asking or you are stuck on a particular point, this is the time to get assistance. The time between the end of the workshop and when the solutions are due is only meant to be for writing up your solutions and for this you should not need more than an hour or two at most.

General submission information:

1. Electronic submission of your solutions is due on Moodle by **11:55 pm on Sunday of the same week**.
2. **Your solutions should include a description/explanation of what you are doing at each step and relevant working.** Without these you will receive limited marks. The description should be in complete English sentences. All mathematics should be appropriately laid out and with appropriate notation. Your writing should be similar in style to the worked solutions from the Applied Class problem sheets, not the annotations from the videos. For more information and advice, please read the "Guidelines for writing in mathematics" document posted under the "Additional information and resources" section of the ENG1005 Moodle page.
3. Your solutions may be typed or handwritten and scanned (the latter is encouraged). The **final document should be submitted as a single pdf file that is clearly and easily legible**. If the marker is unable to read it (or any part of it) you may lose marks.

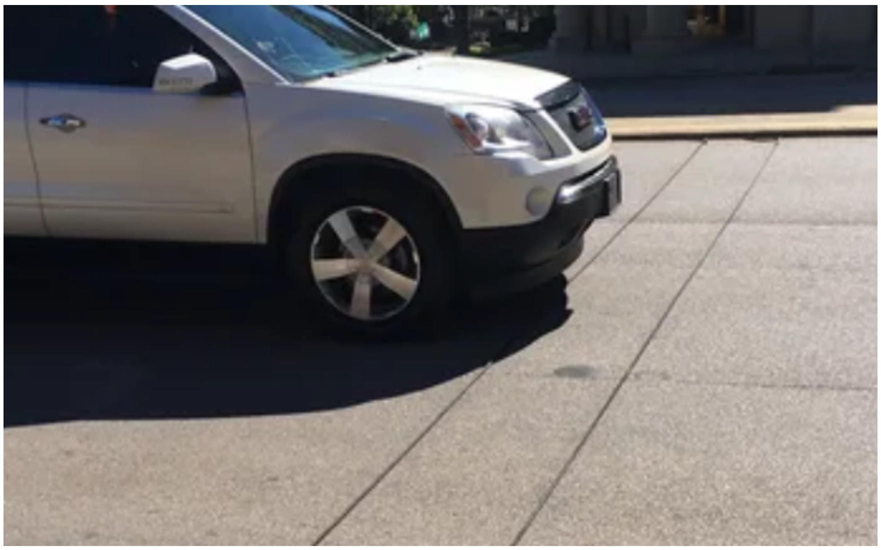
Academic integrity:

You can (and should!) discuss your solutions with the other students, but **you must write up your solutions by yourself**. Copying solutions is serious academic misconduct and will be penalised according to Monash University guidelines. Other examples of academic misconduct include asking a personal tutor to do any of your assessments and posting your assessments to a "homework" website. Please refer back to your Academic Integrity module if you are in any doubt about what constitutes academic misconduct. **Your integrity is an important part of who you are. It is much more important than any grade you could receive.**

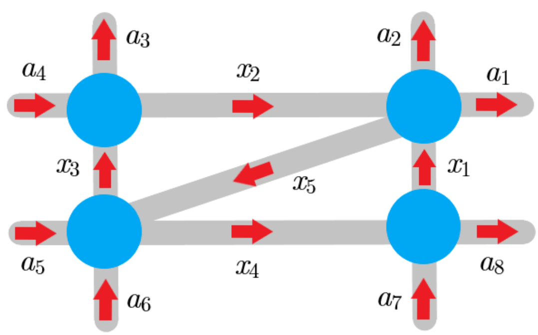
Networks and Traffic Flow

Networks are ubiquitous in the human-made environment, from physical ones such as road networks and electrical grids to virtual ones such as the world wide web. They also exist in the natural world, from connections between neurons in the brain to the structure of fungi. Many processes occur on such networks and they can often be represented using matrices and analysed using the tools of linear algebra. In this workshop, we'll explore a traffic network and see what Gaussian elimination can tell us.

If you have been driving, you might have noticed black cables on the road from time to time. These are portable traffic counters. They are used to measure the number of cars travelling on the road in a set period of time. By having two cables, the direction of each car can be determined by the order in which the cables are run over.



In this workshop, we will consider the following traffic network consisting of one-way streets:



The network consists of four junctions (coloured in blue), five "internal" streets between the junctions (labelled x_1 to x_5), and eight "external" streets that connects this network to the rest of the traffic grid (labelled a_1 to a_8). The direction of traffic on each street is indicated by the red arrow on the street.

Suppose traffic counters are placed on the eight external streets, and the average numbers of cars per hours (travelling in the indicated direction) are found to be

$$a_1 = 40, \quad a_2 = 5, \quad a_3 = 15, \quad a_4 = 25, \quad a_5 = 30, \quad a_6 = 15, \quad a_7 = 10, \quad a_8 = 20$$

Using these measurements, we would like to say something about the traffic on the internal streets. Assume that the cars are always in motion in the network, i.e., there are no cars entering/leaving garages on the street, or stopping to park on the street.

1. By considering the traffic at each junction, write down four equations involving x_1 to x_5. **[2 marks]**

   Solution: At each junction, we should equate the number of car coming in versus going out. Therefore we have the following four equations:

$$\begin{aligned} x_3 + 25 &= x_2 + 15 \\ x_1 + x_2 &= x_5 + 40 + 5 \\ x_4 + 10 &= x_1 + 20 \\ 30 + 15 + x_5 &= x_3 + x_4 \end{aligned}$$

2. Without further computation, explain why the eight traffic counters are *not* sufficent to determine the traffic flow on all streets. **[1 marks]**

   Solution: We have five unknowns but only four equations. Therefore the system is underdetermined. We cannot get a unique solution for all the traffic flows.

3. Without further computation, how many extra traffic counters do you expect you need to determine the traffic flow on all streets? Briefly justify your answer. **[1 marks]**

   Solution: Placing a traffic counter on a street has the effect of turning that variable into a constant (the measured number). Since we have four equations, if we place one extra counter and reduce the number of variables by one, we would have the same number of equations and variables. We expect to then be able to determine the four variables uniquely.

4. Write down the system of equations in matrix form, then compute the **reduced** row echelon form of the augmented matrix. You should do this by hand and submit your working, but you may (and are highly recommended to) verify your solution using CAS or Matlab. **[4 marks]**

   Solution: The augmented matrix for our system is

$$\left[\begin{array}{ccccc|c} 0 & -1 & 1 & 0 & 0 & -10 \\ 1 & 1 & 0 & 0 & -1 & 45 \\ -1 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & -1 & -1 & 1 & -45 \end{array}\right]$$

   We use row reduction to find the reduced row echelon form:

   $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 45 \\ 0 & -1 & 1 & 0 & 0 & -10 \\ -1 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & -1 & -1 & 1 & -45 \end{array}\right]$$

   $R_2 = -R_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 45 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ -1 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & -1 & -1 & 1 & -45 \end{array}\right]$$

   $R_3 = R_3 + R_1$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 45 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 & -1 & 55 \\ 0 & 0 & -1 & -1 & 1 & -45 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 45 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & -1 & -1 & 1 & -45 \end{array}\right]$$

$R_4 = R_4 + R_3$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 45 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 = R_1 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 35 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 = R_1 - R_3$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 = R_2 + R_3$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & 0 & 1 & -1 & 55 \\ 0 & 0 & 1 & 1 & -1 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. Hence write down the solution for the average number of cars per hour on each street (you may have parameters in your solution). **[2 marks]**

   Solution: The system has three pivots and two free variables. x_4 and x_5 columns do not contain pivots, therefore we set $x_4 = s$ and $x_5 = t$. From the RREF we can read off the solution $x_1 = s - 10$, $x_2 = 55 - s + t$, and $x_3 = 45 - s + t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 55 \\ 45 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

6. How many extra traffic counters do you need to completely determine all the traffic flow in the networks? Where should you place the extra counters? Make sure you explain and justify your answers. **[2 marks]**

   Solution: Since the system of equations has three pivots, the number of "real" independent equations is actually three. Therefore we need to reduce the number of variable to three. This means we need two extra traffic counters. Since all traffic flow can be determined from x_4 and x_5, we can put the two counters there.

7. Your solution should have some parameters. Give an explanation for what the vectors associated with these parameters represent. To answer this, you may find it useful to draw the road network and add the traffic on each road for different values of the parameters (varying them separately). **[2 marks]**

Solution: The vectors associated to the parameters represent loops in the network. $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ represents cars going in the loop consists of the streets x_3, x_2 and x_5. $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ represents cars going in the loop consists of the streets x_1, x_2 (opposite direction), x_3 (opposite direction), and x_4.

8. Not every solution in the full solution space corresponds to a realistic traffic flow. In particular we require that $x_1 \dots x_5$ to be all non-negative. What is the *minimum* number of cars per hour on the street x_4 in real life? **[2 marks]**

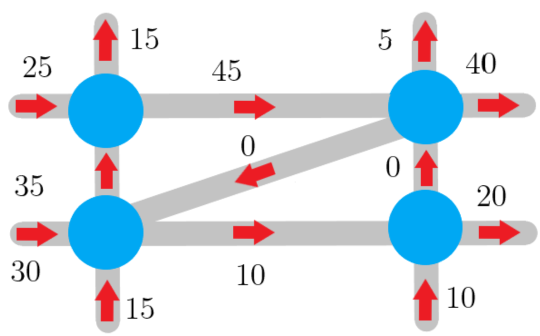
   Solution: We observe that $x_1 = x_4 - 10$. Since $x_1 \geq 0$, it follows that $x_4 \geq 10$. Furthermore, from the next question we actually see a traffic flow where $x_4 = 10$, so $x_4 = 10$ is the real life minimum.

9. In real life, what is the minimum total traffic possible in the network? In other words, what is the minimum of $x_1 + x_2 + x_3 + x_4 + x_5$? Find the traffic flow when the minimum is realized and sketch the network labelling the traffic on all the streets with direction. **[2 marks]**

   Solution: Since we need to find $x_1 + x_2 + x_3 + x_4 + x_5$, we use the solution in question 5 to express this quantity in terms of the free variables s and t.

$$x_1 + x_2 + x_3 + x_4 + x_5 = (s - 10) + (55 - s + t) + (45 - s + t) + s + t = 90 + 3t$$

   Since $t = x_5 \geq 0$, $90 + 3t \geq 90$. Therefore the $x_1 + x_2 + x_3 + x_4 + x_5 \geq 90$. Setting $x_4 = 10$ and $x_5 = 0$, we see that $x_1 = 0$, $x_2 = 45$, $x_3 = 35$. This can occur in real life. Therefore the minimum of $x_1 + x_2 + x_3 + x_4 + x_5$ is 90. A possible traffic flow with that minimum is



   Note: Other flows are possible! For example you can try $x_4 = 11$. A slightly harder problem would be to find all the possible minimum traffic flows. Can you do that?

There is also 1 additional mark given for the quality of the English and 1 additional mark for correct mathematical notation. These marks are easy to obtain but the markers will be instructed to be strict in awarding these marks.