ENG1005 Week 7: Applied class problem sheet solutions

Question 1

We start by setting

$$I = \int_0^{2\pi} \sin(4x)\cos(5x)\, dx.$$

Using integration by parts with

$$f(x) = \sin(4x), \quad g(x) = \frac{1}{5}\sin(5x),$$
$$f'(x) = 4\cos(4x), \quad g'(x) = \cos(5x),$$

the integral becomes

$$I = \left[\frac{1}{5}\sin(4x)\sin(5x)\right]_0^{2\pi} - \frac{4}{5}\int_0^{2\pi} \cos(4x)\sin(5x)\, dx.$$

As $\sin(mx)$ has a period of 2π for any integer m, the terms in the brackets cancel out, thus

$$I = -\frac{4}{5}\int_0^{2\pi} \cos(4x)\sin(5x)\, dx,$$

Applying integration by parts again with

$$f(x) = \cos(4x), \quad g(x) = -\frac{1}{5}\cos(5x),$$
$$f'(x) = -4\sin(4x), \quad g'(x) = \sin(5x),$$

the integral becomes

$$I = \left[\frac{4}{25}\cos(4x)\cos(5x)\right]_0^{2\pi} + \frac{16}{25}\int_0^{2\pi} \sin(4x)\cos(5x)\, dx.$$

As $\cos(nx)$ has a period of 2π for any integer n, the terms in the brackets cancel out. Using the definition of the original integral I we have that

$$I = \frac{16}{25}I,$$

hence $I = 0$. For the general case with

$$I = \int_0^{2\pi} \sin(mx)\cos(nx)\, dx.$$

where m and n are integers, if $m = 0$ then $I = 0$ since $\sin(0) = 0$. If $n = 0, m \neq 0$ then we have

$$I = \int_0^{2\pi} \sin(mx)\, dx = \left[-\frac{1}{m}\cos(mx)\right]_0^{2\pi} = 0,$$

as $\cos(mx)$ has a period of 2π. If $m \neq 0, n \neq 0$ using integration by parts once we have that

$$I = \left[\frac{1}{n}\sin(mx)\sin(nx)\right]_0^{2\pi} - \frac{m}{n}\int_0^{2\pi}\cos(mx)\sin(nx)\,dx.$$

The terms in the brackets cancel out as $\sin(mx)$ and $\sin(nx)$ have a period of 2π. For the case where $m = n$ this simplifies to $I = -I$, so $I = 0$. If $m \neq n$, a second integration by parts gives

$$I = \left[\frac{m}{n^2}\cos(mx)\cos(nx)\right]_0^{2\pi} + \frac{m^2}{n^2}\int_0^{2\pi}\sin(mx)\cos(nx)\,dx.$$

The terms in the brackets cancel out as $\cos(mx)$ and $\cos(nx)$ have a period of 2π. Using the definition of the original integral I we have that

$$I = \frac{m^2}{n^2}I,$$

hence $I = 0$. Therefore the integral I is 0 for any integers m and n.

Question 2

Part A

For the integral

$$\int \sin(x)\ln(\cos(x))\,dx,$$

using integration by parts with

$$f(x) = \ln(\cos(x)), \quad g(x) = -\cos(x),$$
$$f'(x) = -\frac{\sin(x)}{\cos(x)}, \quad g'(x) = \sin(x),$$

the integral becomes

$$\int \sin(x)\ln(\cos(x))\,dx = -\cos(x)\ln(\cos(x)) - \int \sin(x)\,dx.$$

which simplifies to

$$\int \sin(x)\ln(\cos(x))\,dx = -\cos(x)\ln(\cos(x)) + \cos(x) + C,$$

where C is a constant of integration.

Part B

For the integral

$$\int \sin^{-1}(x)\,dx,$$

using integration by parts with

$$f(x) = \sin^{-1}(x), \quad g(x) = x,$$
$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad g'(x) = 1,$$

the integral becomes

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

Making the substitution $u = 1 - x^2$ with $du = -2x\, dx$,

$$\begin{aligned}\int \sin(x)\ln(\cos(x))\, dx &= x\sin^{-1}(x) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du.\\ &= x\sin^{-1}(x) + \sqrt{u} + C.\end{aligned}$$

Hence,

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C,$$

where C is a constant of integration.

Question 3

Given that $\tanh(x) = -15/17$, using the hyperbolic function identity $\operatorname{sech}^2(x) = 1 - \tanh^2(x)$ and noting that $\operatorname{sech}(x) > 0$ for all real x we have that

$$\operatorname{sech}(x) = \sqrt{1 - \left(-\frac{15}{17}\right)^2} = \sqrt{\frac{64}{289}} = \frac{8}{17}$$

Using the definitions of the other hyperbolic functions,

$$\begin{aligned}\cosh(x) &= \frac{1}{\operatorname{sech}(x)} = \frac{17}{8},\\ \coth(x) &= \frac{1}{\tanh(x)} = -\frac{17}{15},\\ \sinh(x) &= \tanh(x)\cosh(x) = -\frac{15}{8},\\ \operatorname{cosech}(x) &= \frac{1}{\sinh(x)} = -\frac{8}{15}.\end{aligned}$$

Question 4

We want to find

$$\frac{d}{dx}(\cosh^{-1}(x)).$$

If we set $y = \cosh^{-1}(x)$ then $x = \cosh(y)$. Using implicit differentiation,

$$1 = \sinh(y)\frac{dy}{dx}.$$

For the inverse function to be defined, we set $y \geq 0$. Using the identity $\cosh^2(y) - \sinh^2(y) = 1$ and noting that $\sinh(y) \geq 0$ if $y \geq 0$ then

$$\sinh(y) = \sqrt{\cosh^2(y) - 1} = \sqrt{x^2 - 1}.$$

Hence, the derivative is

$$\frac{d}{dx}(\cosh^{-1}(x)) = \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}.$$

Question 5

Part A

For the integral

$$\int x^3 e^{-x^2}\,dx,$$

we can make the substitution $u = -x^2$ with $du = -2x\,dx$ to get

$$\int x^3 e^{-x^2}\,dx = -\frac{1}{2}\int x^2 e^u\,du = \frac{1}{2}\int ue^u\,du.$$

Using integration by parts with

$$\begin{aligned} f(u) &= u, \quad & g(u) &= e^u, \\ f'(u) &= 1, \quad & g'(u) &= e^u, \end{aligned}$$

the integral becomes

$$\begin{aligned} \frac{1}{2}\int ue^u\,du &= \frac{1}{2}\left(ue^u - \int e^u\,du\right), \\ &= \frac{1}{2}\left(ue^u - e^u\,du + D\right). \end{aligned}$$

where D is an arbitrary constant. Simplifying and substituting back $u = -x^2$, the integral is

$$\int x^3 e^{-x^2}\,dx = -\frac{1}{2}(x^2+1)e^{-x^2} + C,$$

where C is a constant of integration.

Part B

For the integral

$$\int \sin(\ln(x))\,dx,$$

we can make the substitution $u = \ln(x)$. Here $du = (1/x)\,dx$ so $x\,du = dx$ and since $x = e^u$ then $e^u\,du = dx$. Making this substitution,

$$\int \sin(\ln(x))\,dx = \int \sin(u)e^u\,du.$$

Using integration by parts with

$$\begin{aligned} f(u) &= \sin(u), \quad & g(u) &= e^u, \\ f'(u) &= \cos(u), \quad & g'(u) &= e^u, \end{aligned}$$

the integral becomes

$$\int \sin(u)e^u\,du = \sin(u)e^u - \int \cos(u)e^u\,du.$$

Applying integration by parts again with

$$\begin{aligned} f(u) &= \cos(u), \quad & g(u) &= e^u, \\ f'(u) &= -\sin(u), \quad & g'(u) &= e^u, \end{aligned}$$

the integral becomes

$$\int \sin(u) e^u \, du = \sin(u) e^u - \cos(u) e^u - \int \sin(u) e^u \, du.$$

Rearranging and accounting for the integration constant, we have that

$$\int \sin(u) e^u \, du = \frac{1}{2} \left(\sin(u) - \cos(u)\right) e^u + C.$$

Substituting back $u = \ln(x), x = e^u$, the integral is

$$\int \sin(\ln(x)) \, dx = \frac{1}{2} x \left(\sin(\ln(x)) - \cos(\ln(x))\right) + C,$$

where C is a constant of integration.

Question 6

Using the definition of the hyperbolic functions to show the first identity,

$$\begin{aligned} \sinh(-x) &= \frac{e^{(-x)} - e^{-(-x)}}{2} \\ &= -\frac{e^x - e^{-x}}{2} = -\sinh(x). \end{aligned}$$

Similarly for the second identity,

$$\begin{aligned} \cosh(-x) &= \frac{e^{(-x)} + e^{-(-x)}}{2} \\ &= \frac{e^x + e^{-x}}{2} = \cosh(x). \end{aligned}$$

For the third identity, we have that

$$\cosh(x+y) = \frac{e^{x+y} + e^{-(x+y)}}{2}.$$

Expanding the other side,

$$\begin{aligned} \cosh(x)\cosh(y) + \sinh(x)\sinh(y) &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4} \\ &= \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y). \end{aligned}$$

Question 7

If $y = \sinh^{-1}(x)$, we can set $x = \sinh(y)$ and solve for y to find an explicit expression for $\sinh^{-1}(x)$. Using the definition of the hyperbolic functions,

$$x = \frac{e^y - e^{-y}}{2}.$$

Multiplying both sides by e^y and rearranging gives

$$(e^y)^2 - 2xe^y - 1 = 0.$$

This is a quadratic equation in e^y. Solving for e^y using the quadratic formula and noting that $e^y > 0$ for all real y,

$$e^y = \frac{2x + \sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}.$$

Solving for y,

$$y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

Similarly, if $y = \tanh^{-1}(x)$, which has domain $-1 < x < 1$, we can set $x = \tanh(y)$ and solve for y to find an explicit expression for $\tanh^{-1}(x)$. Using the definition of the hyperbolic functions,

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Rearranging gives

$$\begin{aligned}(e^y + e^{-y})x &= (e^y - e^{-y}),\\ e^{-y}(1+x) &= e^y(1-x),\\ \frac{1+x}{1-x} &= e^{2y}.\end{aligned}$$

Solving for y,

$$y = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1.$$

Question 8

For the parametric equation

$$x(t) = \cosh(t), \quad y(t) = \sinh(t),$$

at $t = 0$, $x = 1$ and $y = 0$. As $t \to \infty$, $x \to e^t/2$ and $y \to e^t/2$, so $y \to x$. As $t \to -\infty$, $x \to e^{-t}/2$ and $y \to -e^{-t}/2$, so $y \to -x$. Hence the curve corresponding to this parametric equation is a hyperbola passing through the x-axis at $x = 1$ with asymptotes $y = x$ and $y = -x$. Using the identity $\cosh^2(t) - \sinh^2(t) = 1$, an equation of the curve is

$$x^2 - y^2 = 1, \quad x \geq 1,$$

which corresponds to the right branch of a hyperbola. This is shown in Figure 1.

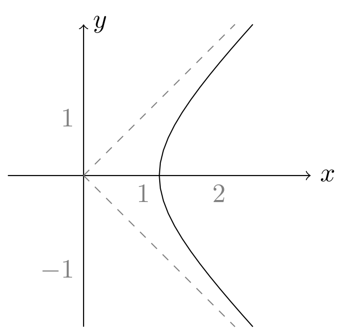

Figure 1: Hyperbola corresponding to the parametric equation $x(t) = \cosh(t), y(t) = \sinh(t)$. Asymptotes correspond to the dashed lines.

Question 9

For the integral

$$I_n(x) = \int \cosh^n(x)\,dx = \int \cosh^{n-1}(x)\cosh(x)\,dx,$$

using integration by parts with

$$f(x) = \cosh^{n-1}(x), \quad g(x) = \sinh(x),$$
$$f'(x) = (n-1)\sinh(x)\cosh^{n-2}(x), \quad g'(x) = \cosh(x),$$

the integral becomes

$$I_n(x) = \sinh(x)\cosh^{n-1}(x) - (n-1)\int \sinh^2(x)\cosh^{n-2}(x)\,dx.$$

Using the identity $\sinh^2(x) = \cosh^2(x) - 1$,

$$\begin{aligned} I_n(x) &= \sinh(x)\cosh^{n-1}(x) - (n-1)\int (\cosh^2(x) - 1)\cosh^{n-2}(x)\,dx, \\ &= \sinh(x)\cosh^{n-1}(x) - (n-1)\int \cosh^n(x)\,dx + (n-1)\int \cosh^{n-2}(x)\,dx \\ &= \sinh(x)\cosh^{n-1}(x) - (n-1)I_n(x) + (n-1)I_{n-2}(x). \end{aligned}$$

Rearranging for $I_n(x)$ we have that

$$nI_n(x) = \sinh(x)\cosh^{n-1}(x) + (n-1)I_{n-2}(x).$$

To find the integral

$$I_{-4}(x) = \int \frac{1}{\cosh^4(x)}\,dx,$$

using the recurrence relation above with $n = -2$ gives

$$-2I_{-2}(x) = \sinh(x)\cosh^{-3}(x) - 3I_{-4}(x).$$

We know that

$$I_{-2}(x) = \int \frac{1}{\cosh^2(x)}\,dx = \int \operatorname{sech}^2(x)\,dx = \tanh(x) + D,$$

where D is an arbitrary constant. Solving for $I_{-4}(x)$ gives

$$I_{-4}(x) = \frac{\sinh(x)\cosh^{-3}(x) + 2\tanh(x)}{3} + C,$$

where C is a constant of integration, which we can set to zero when evaluating the definite integral. Now evaluating the definite integral

$$I_{-4}(\ln(2)) - I_{-4}(0) = \int_0^{\ln(2)} \frac{1}{\cosh^4(x)}\,dx,$$

at $x = \ln(2)$, $\sinh(\ln(2)) = 3/4$, $\cosh(\ln(2)) = 5/4$ and $\tanh(\ln(2)) = 3/5$, so

$$I_{-4}(\ln(2)) = \frac{1}{3}\left(\frac{3}{4}\left(\frac{5}{4}\right)^{-3} + 2\left(\frac{3}{5}\right)\right) = \frac{66}{125}.$$

As $\sinh(0) = 0$ and $\tanh(0) = 0$, $I_{-4}(0) = 0$. Hence,

$$\int_0^{\ln(2)} \frac{1}{\cosh^4(x)}\,dx = \frac{66}{125}.$$

Question 10

Assuming the Earth is spherical, the north-south extent is given by the arc length

$$L = \frac{\pi}{180} R \Delta\theta^\circ,$$

where R is the radius of the Earth and $\Delta\theta^\circ$ is the angle in degrees between the north and south extents. For Australia, this distance is

$$L = \frac{\pi}{180}(6400)(43 - 12) = 3.5 \times 10^3 \text{ km}.$$

Using the Mercator projection $y = R \tanh^{-1}(\sin\phi)$ where ϕ is the latitudinal angle, the distance is

$$D = 6400 \left(\tanh^{-1}(\sin(43^\circ)) - \tanh^{-1}(\sin(12^\circ))\right) = 4.0 \times 10^3 \text{ km}.$$

Repeating the calculations for the UK ($49^\circ - 59^\circ$) gives an extent of 1.1×10^3 km using the arc length and 1.9×10^3 km using the Mercator projection. For Greenland ($61^\circ - 69^\circ$) the extent is 9.0×10^2 km using the arc length and 2.1×10^3 km using the Mercator projection.