ENG1005: Week 5 Applied class problem sheet

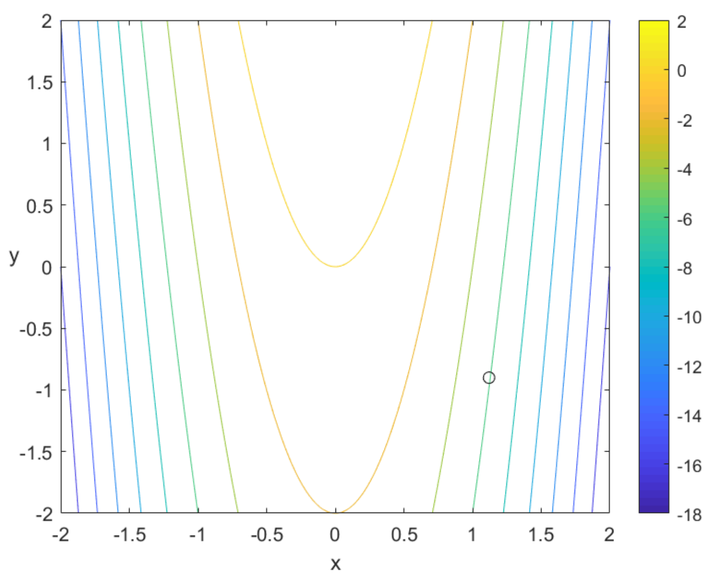
This problem sheet is intended for you to work through in your Applied Class in a small-group setting with the help of your instructor and your peers. **At the end of the applied class you will be asked to complete a quiz for credit. The quiz questions are based on the questions on this problem sheet. You may (and should!) ask your group members and your instructor for guidance if needed.**

1. Consider the function

$$f(x, y) = \sin \sqrt{x^2 + y^2}$$

   (a) Sketch the contours with levels 0, ± 1 for this function
   (b) Calculate the partial derivatives of the function, $\partial f/\partial x$ and $\partial f/\partial y$
   (c) Calculate the gradient vector at the point $(\pi/\sqrt{2}, \pi/\sqrt{2})$
   (d) Indicate the direction of the gradient vector at this point on your contour sketch

2. Which of the following are true at the point indicated by an open circle on the contour plot of the function $f(x, y)$ below



   (a) $\partial f/\partial x > 0$ and $\partial f/\partial y > 0$
   (b) $\partial f/\partial x > 0$ and $\partial f/\partial y < 0$
   (c) $\partial f/\partial x < 0$ and $\partial f/\partial y > 0$
   (d) $\partial f/\partial x < 0$ and $\partial f/\partial y < 0$

3. Consider the surface $z = -xy^2 \cos(\pi x)$. In which direction will a ball begin to roll if it is placed on the surface at the point $x = y = 1$?

4. Consider the function $f(x,y) = x^y$ for $x > 0$. Calculate the mixed partial derivatives of this function.

5. Which of the following are true and which are false?
   (a) If all the contours of a function $f(x,y)$ are parallel lines, then the surface given by $z = f(x,y)$ is a plane.
   (b) If $\partial f/\partial x = \partial f/\partial y$ everywhere, then $f(x,y)$ must be a constant.
   (c) There exists a function $f(x,y)$ with $\partial f/\partial x = 2y$ and $\partial f/\partial y = 2x$.

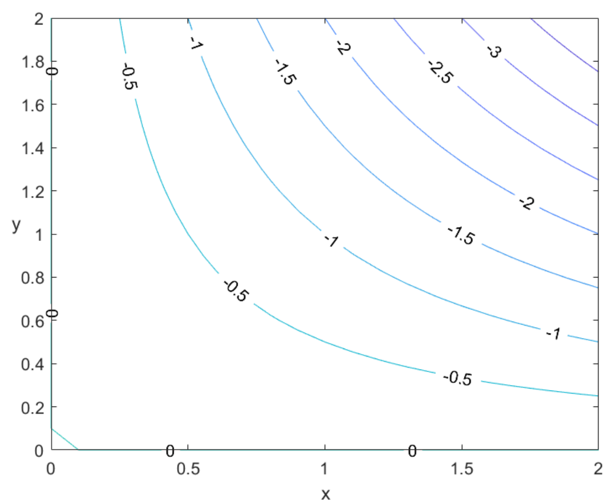
6. Consider the function
$$V(x,y,z) = 2y^2 - 3xy + yz^2$$
   (a) Find the rate of change of V at the point $(1,1,1)$ in the direction of the vector $\mathbf{v} = (1,1,-1)$.
   (b) In which direction does V increase most rapidly at $(1,1,1)$?
   (c) What is the maximum rate of change of V as $(1,1,1)$?

7. Consider the point in three-dimensional space $\mathbf{x} = (x,y,z)$ and a constant vector $\mathbf{a} = (a_1,a_2,a_3)$. Let $r = |\mathbf{x}|$. Show that
$$\boldsymbol{\nabla}(r^3) = 3r\mathbf{x} \quad \text{and} \quad \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}$$

8. The error function $\operatorname{erf}(z)$ is given by
$$\operatorname{erf}(z) = \frac{2}{\pi}\int_0^z \mathrm{e}^{-y^2}\,\mathrm{d}y.$$
   Show that
$$u(x,t) = 1 - \operatorname{erf}(x/2\sqrt{t})$$
   is a solution of the equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
   (*i.e.*, substitute this function into the equation given and show that the equation holds).

   This equation is known as the one-dimensional *heat equation* and it is a mathematical model for how heat diffuses. The solution given has $u(0,t) = 1$ and $u \to 0$ as $x \to \infty$. It could describe how heat diffuses in a semi-infinite rod that is initially at temperature zero and the end at $x = 0$ is kept at temperature 1 for times $t \geq 0$.

9. (Bonus challenge question) In the contour plot below, what is the sign of the mixed partial derivative?



(a) positive

(b) negative

(c) approximately zero

(d) cannot be determined from the contour plot