ENG1005 S1 2024 Workshop 9
Bacteria Growth

26 marks total

This problem set is intended for you to apply the mathematical skills you are learning. It is also designed to practice communicating your work clearly.

It is expected that you will use the workshop to develop (rough) solutions. During the workshop, you should discuss the problems with your peers and the academic staff who are there to assist you. In particular, if you are uncertain about what the problems are asking or you are stuck on a particular point, this is the time to get assistance. The time between the end of the workshop and when the solutions are due is only meant to be for writing up your solutions and for this you should not need more than an hour or two at most.

General submission information:

1. Electronic submission of your solutions is due on Moodle by **11:55 pm (Melbourne time) on Friday of the same week**.
2. **Your solutions should include a description/explanation of what you are doing at each step and relevant working.** Without these you will receive limited marks. The description should be in complete English sentences. All mathematics should be appropriately laid out and with appropriate notation. Your writing should be similar in style to the worked solutions from the Applied Class problem sheets, not the annotations from the videos. For more information and advice, please read the "Guidelines for writing in mathematics" document posted under the "Additional information and resources" section of the ENG1005 Moodle page.
3. Your solutions may be typed or handwritten and scanned (the latter is encouraged). The **final document should be submitted as a single pdf file that is clearly and easily legible**. If the marker is unable to read it (or any part of it) you may lose marks.

Academic integrity:

You can (and should!) discuss your solutions with the other students, but **you must write up your solutions by yourself**. Copying solutions is serious academic misconduct and will be penalised according to Monash University guidelines. Other examples of academic misconduct include asking a personal tutor to do any of your assessments and posting your assessments to a "homework" website. Please refer back to your Academic Integrity module if you are in any doubt about what constitutes academic misconduct. **Your integrity is an important part of who you are. It is much more important than any grade you could receive.**

Bacteria Growth

Taylor and Maclaurin series are very power tools in the study of differential equations. In this workshop we will use ODE's to find a formula for the n-th term in a recurrence relation.

A bacteria culture test is commonly used to identify the bacteria responsible for an infection. A small sample containing bacteria cells is allowed to grow and multiply until there are enough cells to make an accurate identification. A particular bacteria divides once per day when a mature cell splits off a new cell in a process called mitosis. This particular bacteria has a maturation delay of one day, meaning that a newly divided out cell will not immediately start splitting off a cell the next day, but instead waits a day and only starts to split two days later.

Suppose there is 1 immature cell in a Petri dish in day 0. Then in day 1, the cell would mature but would not divide yet, so there is still only 1 cell in the dish. In day 2, the mature cell will divide and now there are 2 cells, 1 mature and 1 immature.

1. Find the numbers of cells in the dish in day 3, 4 and 5. Can you see a pattern in the sequence? **[2 marks]**

2. Let M_n denote the number of mature cells in day n, and I_n the number of immature cells. Express $M_{n+1}, M_{n+2}, I_{n+1}$ and I_{n+2} in terms of M_n and I_n. **[2 marks]**

3. Hence show that the total number of cells in day n satisfies the recurrence relation

$$T_{n+2} = T_{n+1} + T_n$$

[1 mark]

4. Show that $T_{n+1} \leq 2T_n$. **[1 mark]**

From the sequence of numbers $T_0, T_1, T_2, \ldots$ we construct a power series

$$f(x) = \sum_{i=0}^{\infty} \frac{T_i}{i!} x^i$$

$f(x)$ is known as the *exponential generating function* of the sequence $\{T_i, i = 0, 1, 2, \ldots\}$.

5. Find the radius of convergence of $f(x)$. Hint: Use the previous question. **[3 marks]**

6. Show that $f(x)$ satisfies the differential equation

$$f''(x) - f'(x) - f(x) = 0$$

[3 marks]

7. What are the values of the initial conditions $f(0)$ and $f'(0)$? **[1 mark]**

8. Solve the differential equation $f''(x) - f'(x) - f(x) = 0$ with the initial conditions to find $f(x)$. **[4 marks]**

9. Use the closed-form solution of $f(x)$ to write down the Maclaurin series of $f(x)$. You may use the series $e^t = \sum_0^\infty \frac{t^i}{i!}$. **[3 marks]**

10. Hence find a formula for T_n in terms of n. **[1 mark]**

11. What is the behaviour of T_n as n approaches infinity? **[2 marks]**

12. If you need 10000 cells to identify the bacteria, how many days do you have to wait? You should use the (very accurate) approximation from previous question. **[1 marks]**

There is also 1 additional mark given for the quality of the English and 1 additional mark for correct mathematical notation. These marks are easy to obtain but the markers will be instructed to be strict in awarding these marks.