ENG1005 Week 4: Applied class problem sheet solutions

Question 1

The vector $\mathbf{u}$ points along the same line after transformation by A, although in the opposite sense. The fact it points along the same line means it is an eigenvector. The fact it has the opposite sense means the eigenvalue is negative. The vector is also half the length so the eigenvalue is -0.5. The same is true for the vector $\mathbf{x}$.

The vector $\mathbf{v}$ does not point along the same line after transformation. Therefore it is not an eigenvector. The same is true for the vector $\mathbf{y}$.

The vector $\mathbf{w}$ points along the same line after transformation. Therefore it is an eigenvector. The vector has not changed sense, so the eigenvalue is positive. It is twice the length so the eigenvalue is 2.

Question 2

Part A

For the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 2a^2 & 1 \end{bmatrix},$$

where $a \geq 0$ is a real number, to find the eigenvalues we solve the characteristic equation $\det(\mathsf{A} - \lambda \mathsf{I}) = 0$ for λ. We have that

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2a^2 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - 4a^2.$$

Solving $(1-\lambda)^2 - 4a^2 = 0$ gives $\lambda = 1 - 2a$ or $\lambda = 1 + 2a$. To find the corresponding eigenvectors, we solve the linear system $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, which will have at least one free parameter. Provided that $a \neq 0$, for the eigenvalue $\lambda = 1 - 2a$ we have that

$$\begin{bmatrix} 2a & 2 \\ 2a^2 & 2a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system has one free parameter, since the second row of $\mathsf{A} - \lambda\mathsf{I}$ is a multiple of the first row. Setting $v_2 = s$ as a free parameter, we require that $2av_1 + 2s = 0$, so $v_1 = -s/a$. Thus the eigenvectors corresponding to $\lambda = 1 - 2a$ are

$$\mathbf{v} = s\begin{bmatrix} -1/a \\ 1 \end{bmatrix},$$

where s is any non-zero number. For notational convenience we choose an eigenvector in this direction to be

$$\mathbf{v} = \begin{bmatrix} 1 \\ -a \end{bmatrix}.$$

Repeating this procedure for the eigenvalue $\lambda = 1 + 2a$ we have that

$$\begin{bmatrix} -2a & 2 \\ 2a^2 & -2a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Setting $v_2 = s$ as a free parameter, we require that $-2av_1 + 2s = 0$, so $v_1 = s/a$. Thus the eigenvectors corresponding to $\lambda = 1 + 2a$ are

$$\mathbf{v} = s \begin{bmatrix} 1/a \\ 1 \end{bmatrix},$$

where s is any non-zero number. For notational convenience we choose an eigenvector in this direction to be

$$\mathbf{v} = \begin{bmatrix} 1 \\ a \end{bmatrix}.$$

Part B

If $a = 0$, there is only one distinct eigenvalue $\lambda = 1$. The corresponding eigenvectors satisfy

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We require that $v_2 = 0$. Here v_1 is a free parameter, so the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{v} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where s is any non-zero number. Hence, there is only one eigenvector direction in this case.

Part C

The eigenvectors are orthogonal if the dot product between an eigenvector in each direction is zero. We require that

$$\begin{bmatrix} 1 \\ -a \end{bmatrix} \cdot \begin{bmatrix} 1 \\ a \end{bmatrix} = 0$$

Solving for a we require that $1 - a^2 = 0$, so $a = 1$ for the eigenvectors to be orthogonal (remembering the restriction $a \geq 0$).

Part D

The matrix is diagonalisable if each eigenvalue has the same algebraic multiplicity as geometric multiplicity. The eigenvalues are not repeated unless $a = 0$. Hence if $a \neq 0$, the algebraic multiplicity is one and the geometric must also be one for both eigenvalues and hence the matrix is diagonalisable. If $a = 0$, the algebraic multiplicity is two and the geometric is one and the matrix is not diagonalisable.

The matrix V has columns that are the eigenvectors of A. The matrix D has entries on the leading diagonal that are the corresponding eigenvalues. Hence for $a \neq 0$

$$\mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^{-1}$$

with

$$\mathsf{V} = \begin{bmatrix} 1 & 1 \\ -a & a \end{bmatrix} \qquad \text{and} \qquad \mathsf{D} = \begin{bmatrix} 1 - 2a & 0 \\ 0 & 1 + 2a \end{bmatrix}$$

Part E

For $\mathsf{V}^{-1} = \mathsf{V}^T$, the matrix A must be symmetric. This requires $a = 1$. Note that the two eigenvectors were orthogonal for this value of a as shown in part (c).

The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_1 = -1$ and $\lambda_2 = 3$ respectively. We still need to scale them to have unit length. The length of both vectors is $\sqrt{2}$, hence

$$\mathsf{V} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \mathsf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Question 3

We can use diagonalisation to create the matrix using

$$\mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^{-1}$$

where the columns of V are eigenvectors and the matrix D is a diagonal matrix with the corresponding eigenvalues on the diagonal. Hence

$$\mathsf{V} = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathsf{D} = \begin{bmatrix} 9 & 0 \\ 0 & -18 \end{bmatrix}.$$

We now still need to invert V. The determinant is $\det \mathsf{V} = 1 \times 5 - (-2) \times 2 = 9$. Hence

$$\mathsf{V}^{-1} = \frac{1}{9} \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$$

Finally we get

$$\begin{aligned} \mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^{-1} &= \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -18 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -6 \\ -30 & -6 \end{bmatrix} \end{aligned}$$

Question 4

The eigenvalues are 1, 4 and 5. Corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

respectively.

Question 5

If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ then $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$. For $n > 1$ we have that

$$\mathsf{A}^n\mathbf{v} = \mathsf{A}^{n-1}(\mathsf{A}\mathbf{v}) = \mathsf{A}^{n-1}\lambda\mathbf{v}.$$

Repeating this n times gives $\mathsf{A}^n\mathbf{v} = \lambda^n\mathbf{v}$. Hence $\mathbf{v}$ is an eigenvector of A^n with eigenvalue λ^n.

Question 6

For the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

the eigenvalues satisfy $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$. Expanding along the first column,

$$\det(\mathsf{A} - \lambda\mathsf{I}) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda)(6-\lambda).$$

The eigenvalues satisfy $(1-\lambda)(4-\lambda)(6-\lambda) = 0$, so $\lambda = 1, 4, 6$. In general, for an upper triangular or lower triangular matrix, the eigenvalues are the entries on the main diagonal.

Question 7

From the previous question, we can see that the eigenvalues of an upper triangular matrix are just the values on the diagonal. Hence a non-zero matrix with three zero eigenvalues could take the form

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 8

Part A

The characteristic polynomial is

$$p(\lambda) = \det(\mathsf{A} - \lambda\mathsf{I}) = \det\begin{bmatrix} -2-\lambda & 4 \\ 4 & -2-\lambda \end{bmatrix} = (-2-\lambda)^2 - 16.$$

The eigenvalues satisfy $p(\lambda) = 0$. Hence $-2-\lambda = \pm 4$ and $\lambda = -2 \mp 4 = -6$ or 2. The eigenvectors for $\lambda = -6$ must satisfy

$$(\mathsf{A} + 6\mathsf{I})\mathbf{v} = \mathbf{0}$$

Hence

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can using Gaussian elimination to solve, or we can read off (from lots of experience we've built up on solving such systems) that an eigenvector is

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, the eigenvectors for $\lambda = 2$ must satisfy

$$(\mathsf{A} - 2\mathsf{I})\mathbf{v} = \mathbf{0}$$

Hence

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can using Gaussian elimination to solve, or we can read off (from lots of experience we've built up on solving such systems) that an eigenvector is

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Part B

Using the eigenvalues and eigenvectors calculated above, we set

$$\mathsf{V} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathsf{D} = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$$

The matrices do obey the relationship given.

The determinant of V is 2. Hence using the standard expression for the inverse of a 2×2 matrix, we have

$$\mathsf{V}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Part C

$$\mathsf{A}^{50} = \mathsf{V}\mathsf{D}^{50}\mathsf{V}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6^{50} & 0 \\ 0 & 2^{50} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Question 9

The characteristic polynomial is

$$p(\lambda) = (-2 - \lambda)^2 - 16 = \lambda^2 + 4\lambda - 12.$$

Then

$$\begin{aligned} p(\mathsf{A}) = \mathsf{A}^2 + 4\mathsf{A} - 12\mathsf{I} &= \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix} + 4 \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix} - 12 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -16 \\ -16 & 20 \end{bmatrix} + \begin{bmatrix} -8 & 16 \\ 16 & -8 \end{bmatrix} + \begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Hence the identity holds for this matrix.