Lecture 23 — Rearrangement Inequality

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1 The Inequalities

We start with an example. Suppose there are four boxes containing \$10, \$20, \$50 and \$100 bills, respectively. You may take 2 bills from one box, 3 bills from another, 4 bills from another, and 5 bills from the remaining box. What is the maximum amount of money you can get?

Clearly, you'd want to take as many bills as possible from the box with largest-value bills! So you would take 5 \$100 bills, 4 \$50 bills, 3 \$20 bills, and 2 \$10 bills, for a grand total of

$$5 \cdot \$100 + 4 \cdot \$50 + 3 \cdot \$20 + 2 \cdot \$10 = \$780. \tag{1}$$

Suppose instead that your arch-nemesis (who isn't very good at math) is picking the bills instead, and he asks you how many bills he should take from each box. In this case, to minimize the amount of money he gets, you'd want him to take as many bills as possible from the box with lowest-value bills. So you tell him to take 5 \$10 bills, 4 \$20 bills, 3 \$50 bills, and 2 \$100 bills, for a grand total of

$$5 \cdot \$10 + 4 \cdot \$20 + 3 \cdot \$50 + 2 \cdot \$100 = \$480. \tag{2}$$

The maximum is attained when the number of bills taken and the denominations are *similarly sorted* as in (1) and the minimum is attained when they are *oppositely sorted* as in (2). The Rearrangement Inequality formalizes this observation.

Theorem 1.1 (Rearrangement)**:** Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers (not necessarily positive) with

$$x_1 \le x_2 \le \cdots \le x_n, \text{ and } y_1 \le y_1 \le \cdots \le y_n,$$

and let σ be a permutation of $\{1, 2, \ldots, n\}$. (That is, σ sends each of $1, 2, \ldots, n$ to a different value in $\{1, 2, \ldots, n\}$.) Then the following inequality holds:

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \le x_1 y_{\sigma 1} + x_2 y_{\sigma 2} + \cdots + x_n y_{\sigma n} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Proof. We prove the inequality on the right by induction on n. The statement is obvious for $n = 1$. Suppose it true for $n - 1$. Let m be an integer such that $\sigma m = n$. Since $x_n \ge x_m$ and $y_n \ge y_{\sigma n}$,

$$\begin{aligned} 0 &\le (x_n - x_m)(y_n - y_{\sigma n}) \qquad (3) \\ \implies x_m y_n + x_n y_{\sigma n} &\le x_m y_{\sigma n} + x_n y_n. \end{aligned}$$

Hence

$$x_1 y_{\sigma 1} + \cdots + x_m \underbrace{y_{\sigma m}}_{y_n} + \cdots + x_n y_{\sigma n} \le x_1 y_{\sigma 1} + \cdots + x_m y_{\sigma n} + \cdots + x_n y_n. \tag{4}$$

By the induction hypothesis,

$$x_1 y_{\sigma 1} + \cdots + x_m y_{\sigma n} + \cdots + x_{n-1} y_{\sigma(n-1)} \le x_1 y_1 + \cdots + x_m y_m + \cdots + x_{n-1} y_{n-1}.$$

Thus the RHS of (4) is at most $x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n y_n$, as needed.

To prove the LHS, apply the above with $-y_i$ instead of y_i (noting that negating an inequality reverses the sign). $\square$

Remark 1.2: Equality is attained on the RHS if and only if, for every r, the following are equal as multisets:

$$\{y_m \mid x_m = r\} = \{y_{\sigma m} \mid x_m = r\}.$$

To see this, note that otherwise, using the procedure above, at some time we will have to switch two unequal numbers y'_k, y'_m with unequal corresponding x's, $x_k \neq x_m$, and we get inequality (see (3)). Similarly, equality is attained on the LHS if and only if for every r,

$$\{y_{n+1-m} \mid x_m = r\} = \{y_{\sigma m} \mid x_m = r\}.$$

In particular, if the $a_1, \ldots, a_n$ are all distinct and $b_1, \ldots, b_n$ are all distinct, then equality on the right-hand side occurs only when $\sigma(m) = m$ for all m, and equality on the left-hand side occurs only when $\sigma(m) = n + 1 - m$ for all m.

The rearrangement inequality can be used to prove the following.

Theorem 1.3 (Chebyshev)**:** Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two similarly sorted sequences. Then

$$\frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \le \frac{a_1 b_1 + \cdots + a_n b_n}{n}$$

Proof. Add up the following inequalities (which hold by the Rearrangement Inequality):

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ a_1 b_2 + a_2 b_3 + \cdots + a_n b_1 &\le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &\vdots \\ a_1 b_n + a_2 b_1 + \cdots + a_n b_{n-1} &\le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$

After factoring the left-hand side and dividing by n^2, we get the right-hand inequality.

By replacing b_i with $-b_i$ and using the above result we get the left-hand inequality. $\square$

2 Problems

1. Given that $a, b, c \ge 0$, prove $a^3 + b^3 + c^3 \ge a^2 b + b^2 c + c^2 a$.

2. Powers: For $a, b, c > 0$ prove that

(a) $a^a b^b c^c \geq a^b b^c c^a$.

(b) $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

3. Suppose $a_1, a_2, \ldots, a_n > 0$ and let $s = a_1 + \cdots + a_n$. Prove that

$$\frac{a_1}{s-a_1} + \cdots + \frac{a_n}{s-a_n} \geq \frac{n}{n-1}.$$

In particular, conclude Nesbitt's Inequality

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

for $a, b, c > 0$.

4. Prove the following for $x, y, z > 0$:

(a) $\frac{x^2}{y} + \frac{y^2}{x} \geq x + y$.

(b) $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$.

(c) $\frac{xy}{z^2} + \frac{yz}{x^2} + \frac{zx}{y^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

5. (IMO 1978/2) Let $a_1, \ldots, a_n$ be pairwise distinct positive integers. Show that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

6. (modified ISL 2006/A4) Prove that for all positive a, b, c,

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ac}{a+c} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}.$$

7. Prove that for any positive real numbers a, b, c the following inequality holds:

$$\frac{a^2+bc}{b+c} + \frac{b^2+ac}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c.$$

8. (MOSP 2007) Let k be a positive integer, and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1+x_i}\right)\left(\sum_{i=1}^{n} x_i\right) \leq \left(\sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i}\right)\left(\sum_{i=1}^{n} \frac{1}{x_i^k}\right).$$

9. The numbers 1 to 100 are written on a 10×10 board (1–10 in the first row, etc.). We are allowed to pick any number and two of its neighbors (horizontally, vertically, or diagonally—but our choice must be consistent), increase the number with 2 and decrease the neighbors by 1, or decrease the number by 2 and increase the neighbors by 1. At some later time the numbers in the table are again $1, 2, \ldots, 100$. Prove that they are in the original order.

3 Solutions

1. The sequences a, b, c and a^2, b^2, c^2 are similarly sorted. Therefore, by the rearrangement inequality,

$$a^2 \cdot a + b^2 \cdot b + c^2 \cdot c \geq a^2 \cdot b + b^2 \cdot c + c^2 \cdot a.$$

2. Since ln is an increasing function, we take the ln of both sides to find that the inequalities are equivalent to

$$\begin{aligned} a \ln a + b \ln b + c \ln c &\geq b \ln a + c \ln b + a \ln c \\ a \ln a + b \ln b + c \ln c &\geq \frac{a+b+c}{3}(\ln a + \ln b + \ln c). \end{aligned}$$

Note the sequences (a, b, c) and $(\ln a, \ln b, \ln c)$ are similarly sorted, since ln is an increasing function. Then the first inequality follows from Rearrangement and the second from Chebyshev.

3. Since both sides are symmetric, we may assume without loss of generality that $a_1 \leq \cdots \leq a_n$. Then $s - a_1 \geq \cdots \geq s - a_n$ and $\frac{1}{s-a_1} \leq \cdots \leq \frac{1}{s-a_n}$. By Chebyshev's inequality with $(a_1, \ldots, a_n)$ and $\left(\frac{1}{s-a_1}, \ldots, \frac{1}{s-a_n}\right)$, we get

$$\begin{aligned} \frac{a_1}{s-a_1} + \cdots + \frac{a_n}{s-a_n} &\geq \frac{1}{n}(a_1 + \cdots + a_n)\left(\frac{1}{s-a_1} + \cdots + \frac{1}{s-a_n}\right) \\ &= \frac{1}{n}\left(\frac{s}{s-a_1} + \cdots + \frac{s}{s-a_n}\right) \\ &= \frac{1}{n}\left(\frac{a_1}{s-a_1} + \cdots + \frac{s}{s-a_n} + n\right). \end{aligned}$$

This gives

$$\begin{aligned} \frac{n-1}{n}\left(\frac{a_1}{s-a_1} + \cdots + \frac{a_n}{s-a_n}\right) \geq 1 \implies \\ \frac{a_1}{s-a_1} + \cdots + \frac{a_n}{s-a_n} \geq \frac{n}{n-1}. \end{aligned}$$

4. (a) Without loss of generality, $x \geq y$. Then $x^2 \geq y^2$ and $\frac{1}{y} \geq \frac{1}{x}$, i.e. (x^2, y^2) and $(\frac{1}{y}, \frac{1}{x})$ are similarly sorted. Thus

$$x^2 \cdot \frac{1}{y} + y^2 \cdot \frac{1}{x} \geq x^2 \cdot \frac{1}{x} + y^2 \cdot \frac{1}{y}.$$

   (b) Letting $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, the inequality is equivalent to

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

   This is true by the rearrangement inequality applied to the similarly sorted sequences (a, b, c) and (a, b, c).

(c) Let $a = \frac{x^{\frac{1}{3}}y^{\frac{1}{3}}}{z^{\frac{2}{3}}}$, $b = \frac{x^{\frac{1}{3}}z^{\frac{1}{3}}}{y^{\frac{2}{3}}}$, and $c = \frac{y^{\frac{1}{3}}z^{\frac{1}{3}}}{x^{\frac{2}{3}}}$. Then the inequality to prove becomes

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

which was proved in problem 1.

5. Let $b_1, \ldots, b_n$ be the numbers $a_1, \ldots, a_n$ in increasing order. Since $b_1 \leq \cdots \leq b_n$ and $\frac{1}{1^2} \geq \cdots \geq \frac{1}{n^2}$, by the rearrangement inequality,

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} \geq \frac{b_1}{1^2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{n^2}.$$

However, since the positive integers b_m are distinct and in increasing order, we must have $b_m \geq m$. This gives the RHS is at least $\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n}$.

6. Since the inequality is symmetric we may assume $a \leq b \leq c$. Then

$$a + b \leq a + c \leq b + c. \tag{5}$$

We claim that

$$\frac{ab}{a+b} \leq \frac{ac}{a+c} \leq \frac{bc}{b+c}. \tag{6}$$

Indeed, the two inequalities are equivalent to

$$a^2b + abc \leq a^2c + abc$$
$$abc + ac^2 \leq abc + bc^2$$

both of which hold.

Thus by Chebyshev applied to (5) and (6), we get

$$\left(\frac{ab}{a+b} + \frac{ac}{a+c} + \frac{bc}{b+c}\right)((a+b) + (a+c) + (b+c)) \leq 3(ab + bc + ca). \tag{7}$$

Dividing by $2(a + b + c)$ gives the desired inequality.

(Note: The original problem asked to prove

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + \cdots + a_n)} \sum_{i<j} a_i a_j$$

when $a_1, \ldots, a_n$. This can be proved by summing (7) over all 3-element subsets $\{a, b, c\}$ of (the multiset) $\{a_1, \ldots, a_n\}$, then dividing. This is a rare instance of a general inequality following directly from the 3-variable case!)

7. Since the inequality is symmetric, we may assume without loss of generality that $a \leq b \leq c$. Then

$$a^2 \leq b^2 \leq c^2$$
$$\frac{1}{b+c} \leq \frac{1}{a+c} \leq \frac{1}{a+b}.$$

Hence by the rearrangement inequality,

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{b^2}{b+c}+\frac{c^2}{c+a}+\frac{a^2}{a+b}.$$

Adding $\frac{bc}{b+c}+\frac{ac}{c+a}+\frac{ab}{a+b}$ to both sides gives

$$\begin{aligned}\frac{a^2+bc}{b+c}+\frac{b^2+ac}{c+a}+\frac{c^2+ab}{a+b}&\geq\frac{b^2+bc}{b+c}+\frac{c^2+ac}{c+a}+\frac{a^2+ab}{a+b}\\&=\frac{b(b+c)}{b+c}+\frac{c(c+a)}{c+a}+\frac{a(a+b)}{a+b}\\&=a+b+c.\end{aligned}$$

8. We apply Chebyshev's inequality twice:

$$\begin{aligned}\left(\sum_{i=1}^{n}\frac{1}{1+x_i}\right)\left(\sum_{i=1}^{n}x_i\right)&\leq\left[\frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{x_i^k}\right)\left(\sum_{i=1}^{n}\frac{x_i^k}{1+x_i}\right)\right]\left(\sum_{i=1}^{n}x_i\right)\\&=\left(\sum_{i=1}^{n}\frac{1}{x_i^k}\right)\left[\frac{1}{n}\left(\sum_{i=1}^{n}\frac{x_i^k}{1+x_i}\right)\left(\sum_{i=1}^{n}x_i\right)\right]\\&\leq\left(\sum_{i=1}^{n}\frac{1}{x_i^k}\right)\left(\sum_{i=1}^{n}\frac{x_i^{k+1}}{1+x_i}\right).\end{aligned}$$

Indeed, without loss of generality $x_1\leq\cdots\leq x_n$. In the first application of Chebyshev we use that the following are oppositely sorted:

$$\frac{1}{x_1^k}\geq\cdots\geq\frac{1}{x_n^k}$$
$$\frac{x_1^k}{1+x_1}\leq\cdots\leq\frac{x_n^k}{1+x_n}.$$

The second inequality comes from the fact that $f(x)=\frac{x^k}{1+x}$ is an increasing function for $k\geq 1$, $x\geq 0$: if $x\leq y$ then $x^k\leq y^k$ and $x^{k-1}\leq y^{k-1}$ together give

$$x^k+x^ky\leq y^k+xy^k$$
$$\frac{x^k}{1+x}\leq\frac{y^k}{1+y}.$$

(A simple derivative calculation also does the trick.)

In the second application of Chebyshev we use that the following are similarly sorted:

$$\frac{x_1^k}{1+x_1}\leq\cdots\leq\frac{x_n^k}{1+x_n}$$
$$x_1\leq\cdots\leq x_n.$$

9. Look for an invariant! Let $a_{ij}=10(i-1)+j$, the original number in the (i,j) position in the array. Let b_{ij} be the numbers after some transformations. Then

$$P=\sum_{1\leq i,j\leq 10}a_{ij}b_{ij}$$

is invariant. (Indeed, two opposite neighbors of a_{ij} are $a_{ij} \pm d$ for some d; the sum changes by $\pm(2a_{ij} - (a_{ij} - d) - (a_{ij} + d)) = 0$ at each step.)

Initially, $P = \sum_{1\le i,j\le 10} a_{ij}^2$. Suppose that b_{ij} are a permutation of the a_{ij}'s. Then

$$\sum_{1\le i,j\le 10} a_{ij}^2 = \sum_{1\le i,j\le 10} a_{ij}b_{ij}.$$

By the equality case of the rearrangement inequality, since the a_{ij} are all distinct and the b_{ij} are all distinct, the a_{ij} and b_{ij} must be sorted similarly, i.e. $a_{ij} = b_{ij}$ for all i, j.

References

[1] A. Engel. *Problem-Solving Strategies*. Springer, 1998, New York.