From decimal expansions to reduced fractions

Rachid Hamadi, CSE, UNSW

COMP9021 Principles of Programming, Term 3, 2019

[1]:
```
from math import gcd
```

A real number is rational if and only if a pattern eventually appears in its decimal expansion that repeats forever. So π, being irrational, is such that no finite sequence of consecutive digits in 3.14159265358979... eventually repeats forever. On the other hand,

- $\frac{25}{12} = 2.08333...3...$
- $\frac{97}{21} = 4.619047619047619047...619047...$
- $\frac{11941}{49950} = 0.23905905905...905...$

The decimal expansion is unique except for fractions that in reduced form, have a power of 10 as denominator: those fractions have two decimal expansions, one that ends in 0 repeating forever, another one that ends in 9 repeating forever. For instance, $\frac{1234567}{1000} = 1234.567000...0... = 1234.566999...9....$

We want to, given two nonempty strings of digits σ and τ (that we treat as strings or numbers depending on the context), find out the unique natural numbers p and q such that the decimal expansion of $\frac{p}{q}$ reads as $0.\sigma\tau\tau\tau...\tau...$ and

- either $p = 0$ and $q = 1$ (case where σ and τ consist of nothing but 0's), or
- p and q are coprime, so $\frac{p}{q}$ is in reduced form (including the case where $p = 1$ and $q = 1$ because σ and τ consist of nothing but 9's).

For instance, if $\sigma = 23$ and $\tau = 905$, then $p = 11941$ and $q = 49950$.
Writing $|\sigma|$ for the length (number of digits) in a string of digits σ, we compute:

$$\begin{aligned}
0.\sigma\tau\tau\tau\ldots\tau\ldots &= \sigma 10^{-|\sigma|} + \tau(10^{-|\sigma|-|\tau|} + 10^{-|\sigma|-2|\tau|} + 10^{-|\sigma|-3|\tau|} + \ldots) \\
&= \sigma 10^{-|\sigma|} + \frac{\tau 10^{-|\sigma|-|\tau|}}{(1-10^{-|\tau|})} \\
&= \sigma 10^{-|\sigma|} + \frac{\tau 10^{-|\sigma|}}{(10^{|\tau|}-1)} \\
&= \frac{\sigma 10^{-|\sigma|}(10^{|\tau|}-1) + \tau 10^{-|\sigma|}}{(10^{|\tau|}-1)} \\
&= \frac{\sigma(10^{|\tau|}-1) + \tau}{(10^{|\tau|}-1)10^{|\sigma|}}
\end{aligned}$$

Reducing the last fraction if needed provides the desired answer.
The result of the previous computation immediately translates to the function that follows:

[2]:
```python
def compute_fraction(sigma, tau):
    numerator = int(sigma) * (10 ** len(tau) - 1) + int(tau)
    denominator = (10 ** len(tau) - 1) * 10 ** len(sigma)
    return numerator, denominator

compute_fraction('23', '905')
compute_fraction('000', '97')
compute_fraction('97', '000')
compute_fraction('01234', '543210')
```

[2]: (23882, 99900)

[2]: (97, 99000)

[2]: (96903, 99900)

[2]: (1234541976, 99999900000)

To reduce a fraction, it suffices to divide its numerator and its denominator by their gcd (greatest common divisor). The `math` module has a `gcd` function:

[3]:
```python
gcd(1234541976, 99999900000)
```

[3]: 24

Let us implement the gcd function ourselves, following Euclid's algorithm, which is based on the following reasoning. Let a and b be two natural numbers with $b > 0$. Since $a = \lfloor \frac{a}{b} \rfloor b + a \bmod b$:

- if n divides both a and b then it divides both a and $\lfloor \frac{a}{b} \rfloor b$, hence it divides $a - \lfloor \frac{a}{b} \rfloor b$, hence it divides $a \bmod b$;
- conversely, if n divides both b and $a \bmod b$ then it divides $\lfloor \frac{a}{b} \rfloor b + a \bmod b$, hence it divides a.

Hence n divides both a and b iff n divides both b and $a \bmod b$. So $\gcd(a, b) = \gcd(b, a \bmod b)$.

Since $a \bmod b < b$, we get a sequence of equalities of the form: $\gcd(a, b) = \gcd(a_1, b_1) = \gcd(a_2, b_2) = \cdots = \gcd(a_{k-1}, b_{k-1}) = \gcd(a_k, 0)$ with $k \geq 1$ and $b > b_1 > b_2 > \cdots > b_{k-1} > 0$; as $\gcd(a_k, 0) = a_k$, a_k is the gcd of a and b.

To compute $\lfloor \frac{a}{b} \rfloor$, Python offers the `//` operator; to compute $a \bmod b$, the `%` operator:

[4]:
```python
# True division.
# The result is always a floating point number.
8 / 2, 8.0 / 2, 8 / 2.0, 8.0 / 2.0

# Integer division.
# The result is an integer iff both arguments are integers.
9 // 2, 9.0 // 2, 9 // 2.0, 9.0 // 2.0

# Remainder.
# The result is an integer iff both arguments are integers.
9 % 2, 9.0 % 2, 9 % 2.0, 9.0 % 2.0
```

[4]: (4.0, 4.0, 4.0, 4.0)

[4]: (4, 4.0, 4.0, 4.0)

[4]: (1, 1.0, 1.0, 1.0)

If a and b are arbitrary numbers (not necessarily integers) with $b \neq 0$, then the equality $a = qb + r$ together with the conditions

- q is an integer
- $|r| < |b|$
- $r \neq 0 \to (r > 0 \leftrightarrow b > 0)$

determine q and r uniquely; `//` and `%` operate accordingly:

[5]:
```
5 // 2, 5 % 2
-5 // 2, -5 % 2
5 // -2, 5 % -2
-5 // -2, -5 % -2
print()

7.5 // 2, 7.5 % 2
-7.5 // 2, -7.5 % 2
7.5 // -2, 7.5 % -2
-7.5 // -2, -7.5 % -2
```

[5]: (2, 1)

[5]: (-3, 1)

[5]: (-3, -1)

[5]: (2, -1)

[5]: (3.0, 1.5)

[5]: (-4.0, 0.5)

[5]: (-4.0, -0.5)

[5]: (3.0, -1.5)

The `divmod()` function offers an alternative to the previous combined use of `//` and `%`:

[6]:
```
divmod(5, 2)
divmod(-5, 2)
divmod(5, -2)
divmod(-5, -2)
print()

divmod(7.5, 2)
divmod(-7.5, 2)
divmod(7.5, -2)
divmod(-7.5, -2)
```

[6]: (2, 1)

[6]: (-3, 1)

[6]: (-3, -1)

[6]: (2, -1)

[6]: (3.0, 1.5)

[6]: (-4.0, 0.5)

[6]: (-4.0, -0.5)

[6]: (3.0, -1.5)

Let us get back to Euclid's algorithm, so assume again that a and b are two natural numbers with $b > 0$. To implement the algorithm and compute $\gcd(a, b)$, it suffices to have two variables, say `a` and `b`, initialised to a and b, and then change the value of `a` to b and change the value of `b` to $a \bmod b$, and do that again and again until `b` gets the value 0. To change the value of `a` to $a \bmod b$ and change the value of `b` to b, it seems necessary to introduce a third variable:

[7]:
```
a = 30
b = 18
c = a % b
a = b
b = c
a, b
```

[7]: (18, 12)

But Python makes it easier:

[8]:
```
a = 30
b = 18
# Evaluate the expression on the right hand side;
# the result is the tuple (18, 12).
# Then assign that result to the tuple on the left,
# component by component.
a, b = b, a % b
a, b
```

[8]: (18, 12)

Note that when the value of `a` is strictly smaller than the value of `b`, then `a, b = b, a % b` exchanges the values of `a` and `b`:

[9]:
```
a = 12
b = 18
a, b = b, a % b
a, b
```

[9]: (18, 12)

On the other hand, if the value of `a` is at least equal to the value of `b`, then this holds too after `a, b = b, a % b` has been executed. Let us trace all stages in the execution of Euclid's algorithm.

The code makes use of a `while` statement whose condition is not a boolean expression. Applying `bool()` to an expression reveals which one of `True` or `False` the expression evaluates to in contexts where one or the other is expected:

[10]:
```
bool(None)
bool(0), bool(5), bool(-3)
bool(0.0), bool(0.1), bool(-3.14)
bool([]), bool([0]), bool([[]])
bool({}), bool({0: 0}), bool({0: None, 1: None})
bool(''), bool(' '), bool('0000')
```

[10]: `False`

[10]: `(False, True, True)`

[10]: `(False, True, True)`

[10]: `(False, True, True)`

[10]: `(False, True, True)`

[10]: `(False, True, True)`

[11]:
```
def trace_our_gcd(a, b):
    while b:
        a, b = b, a % b
        print(a, b)

for a, b in (1233, 1233), (1233, 990), (990, 1233):
    print(f'\nTracing the computation of gcd of {a} and {b}:')
    trace_our_gcd(a, b)
```

```

Tracing the computation of gcd of 1233 and 1233:
1233 0

Tracing the computation of gcd of 1233 and 990:
990 243
243 18
18 9
9 0

Tracing the computation of gcd of 990 and 1233:
1233 990
990 243
243 18
18 9
9 0
```

The gcd is the value of a when exiting the `while` loop:

[12]:
```
def our_gcd(a, b):
    while b:
```

```
        a, b = b, a % b
    return a
```

compute_fraction() returns the numerator and denominator of a fraction that another function, say reduce(), can easily reduce thanks to our_gcd(). It is natural to let reduce() take two arguments, the numerator and the denominator of the fraction to simplify, respectively. But compute_fraction() returns those as the first and second elements of a tuple; a function always returns a single value. Between the parentheses that surround the arguments of a function f(), one can insert an expression that evaluates to a tuple and precede it with the * symbol, which "unpacks" the members of the tuple and make them the arguments of f():

[13]:
```
def f(a, b):
    return 2 * a, 2 * b

# Makes a equal to (1, 3), and provides no value to b.
f((1, 3))
```

```
↪---------------------------------------------------------------------------

    TypeError                                 Traceback (most recent call␣
↪last)

    <ipython-input-13-81290db4c2eb> in <module>
      3
      4 # Makes a equal to (1, 3), and provides no value to b.
----> 5 f((1, 3))


    TypeError: f() missing 1 required positional argument: 'b'
```

[14]:
```
f(1, 3)
# f(f(1, 3)) would be f((2, 6)); f(*f(1, 3)) is f(2, 6)
f(*f(1, 3))
f(*f(*f(1, 3)))
```

[14]: (2, 6)

[14]: (4, 12)

[14]: (8, 24)

The * symbol can also be used in the definition of a function and precede the name of a parameter. It then has the opposite effect, namely, it makes a tuple out of all arguments that are provided to the function:

[15]:
```
# x is the tuple of all arguments passed to f().
def f(*x):
    return x * 2
```

```
f()
f(0)
f(f(0))
f(*f(0))
f(f(f(0)))
f(f(*f(0)))
f(*f(*f(0)))
```

[15]: ()

[15]: (0, 0)

[15]: ((0, 0), (0, 0))

[15]: (0, 0, 0, 0)

[15]: (((0, 0), (0, 0)), ((0, 0), (0, 0)))

[15]: ((0, 0, 0, 0), (0, 0, 0, 0))

[15]: (0, 0, 0, 0, 0, 0, 0, 0)

Thanks to this syntax, it is possible to let `reduce()` as well as another function `output()` take two arguments `numerator` and `denominator`, and "pipe" `compute_fraction()`, `reduce()` and `output()` together so that the unpacked returned value of one function becomes the arguments of the function that follows:

[16]:
```
def reduce(numerator, denominator):
    if numerator == 0:
        return 0, 1
    the_gcd = our_gcd(numerator, denominator)
    return numerator // the_gcd, denominator // the_gcd
```

[17]:
```
def output(numerator, denominator):
    print(f'{numerator}/{denominator}')
```

[18]:
```
output(*reduce(*compute_fraction('23', '905')))
output(*reduce(*compute_fraction('000', '97')))
output(*reduce(*compute_fraction('97', '000')))
output(*reduce(*compute_fraction('01234', '543210')))
```

```
11941/49950
97/99000
97/100
51439249/4166662500
```