A2 Ersyfa Halida,

1. $A = \begin{bmatrix} 2 & 7 \\ 1 & 0.5 \end{bmatrix}$ $\lambda_1 = 4$

$A - \lambda_1 I = \begin{bmatrix} 2-4 & 7 \\ 1 & 0.5-4 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 1 & -3.5 \end{bmatrix}$

$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x_1 = A x_0 \frac{1}{m}$

$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{-1}{2}$

$= \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

$x_2 = A x_1 \frac{1}{m}$

$= \begin{bmatrix} -2+(-3.5) \\ 1+1.75 \end{bmatrix} \frac{1}{m}$

$= \begin{bmatrix} -5.5 \\ 2.75 \end{bmatrix} \frac{-1}{5.5}$

$= \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

$x_3 = A x_2 \frac{1}{m}$

$= A x_1 \frac{1}{m}$

$= \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$ $m = -5.5$

$x_4 = A x_3 \frac{1}{m}$

$= \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$ $m = -5.5$

$\lambda_n = \lambda_1 + m$

$= 4 - 5.5$

$= -1.5$

$e_n = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

2. a) From theorem 0.1, we know that for an eigenvalue λ of A, $\lambda - \alpha$ is an eigenvalue of $A - \alpha I$.

Furthermore, from i) of theorem 0.2, we know $A - \alpha I$ is invertible.

As such: $(A - \alpha I)x = (\lambda - \alpha)x$

$$\cancel{(A-\alpha I)^{-1}(A - \alpha I)}\, x = (A - \alpha I)^{-1}(\lambda - \alpha)x$$

$$\frac{1}{\lambda - \alpha}\, x = (A - \alpha I)^{-1} x$$

From this, we see that $\frac{1}{\lambda - \alpha}$ is an eigenvalue of $(A - \alpha I)^{-1}$, with eigenvector x.

b) $A = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 2 & 0 \\ 3 & -3 & -2 \end{bmatrix}$ $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

b1) $A - I = \left[\begin{array}{ccc|ccc} -2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & -3 & -3 & 0 & 0 & 1 \end{array}\right]$

$R_3 + \frac{3}{2} R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 3 & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

$R_3 + \frac{3}{2} R_2 \to R_3$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = U \quad \Bigg| \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix} = L$$

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b2, $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad (A - \alpha I)\, x_1 = x_0$

$LU \, x_1 = x_0$

$Ly = b$ $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-3}{2} & \frac{-3}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= x_0$

$y_1 = 1 \quad y_2 = 1 \quad \frac{-3}{2} y_1 - \frac{3}{2} y_2 + y_3 = 1 \rightarrow \frac{-3}{2} - \frac{3}{2} + y_3 = 1$

$-3 + y_3 = 1$

$y_3 = 4$

$Ux_1 = y$ $\quad \begin{bmatrix} -2 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

$-2x_{11} + x_{12} + 4x_{13} = 1$

$x_{12} = 1$

$3x_{13} = 4$

$x_{13} = \frac{4}{3}$

$-2x_{11} + 1 + \frac{16}{3} = 1$

$-2x_{11} = \frac{-16}{3}$

$x_{11} = \frac{8}{3}$

$\vec{x}_1 = \begin{bmatrix} \frac{8}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix} \frac{1}{B_1}$

$$B_k = \frac{8}{3} \rightarrow M_1 = 1 + \frac{1}{B_1} = 1 + \frac{3}{8} = \frac{11}{8}$$

$$x_1 = \begin{bmatrix} \frac{8}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix} \frac{3}{8} = \begin{bmatrix} 1 \\ \frac{3}{8} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.38 \\ 0.5 \end{bmatrix}$$

iteration 2:

$(A - \alpha I) x_2 = x_1$

$LU \, x_2 = x_1$

$Ly = x_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-3}{2} & \frac{-3}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0.38 \\ 0.5 \end{bmatrix}$$

$y_1 = 1$

$y_2 = 0.38$

$\frac{-3}{2} y_1 - \frac{3}{2} y_2 + y_3 = 0.5$

$\frac{-3}{2} - \frac{3(0.38)}{2} - 0.5 = -y_3$

$2.57 = y_3$

$$y = \begin{bmatrix} 1 \\ 0.38 \\ 2.57 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 0.38 \\ 2.57 \end{bmatrix} \qquad U x_2 = y$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x_2 = y$$

$$-2x_{21} + x_{22} + 4x_{23} = 1$$

$$x_{22} = 0.38$$

$$3x_{23} = 2.57$$

$$x_{23} = 0.857$$

$$\frac{0.38 + 4(0.857) - 1}{2} = x_{21}$$

$$= 1.404$$

$$\vec{x}_2 = \begin{bmatrix} 1.404 \\ 0.38 \\ 0.857 \end{bmatrix} \frac{1}{\beta_2}$$

$$= \begin{bmatrix} 1 \\ 0.271 \\ 0.61 \end{bmatrix}$$

iteration 3: $LUx_3 = x_2$

$Ly = x_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0.271 \\ 0.61 \end{bmatrix}$

$y_1 = 1 \quad y_2 = 0.271$

$-\frac{3}{2} - \frac{3(0.271)}{2} - 0.61 = -y_3$

$= 2.52$

$y = \begin{bmatrix} 1 \\ 0.271 \\ 2.52 \end{bmatrix}$ $\quad Ux_3 = y$

$\begin{bmatrix} -2 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x_3 = y$

$x_{33} = \frac{2.52}{3} = 0.84$

$x_{32} = 0.271$

$x_{31} = \frac{0.271 + 4(0.84) - 1}{2}$

$= 1.32$

$B_3 = 1.32$

$M_3 = \alpha + \frac{1}{B_3}$

$= 1 + \frac{1}{1.32}$

1.76

$M_3 = 1.76$

3. a)

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -3 & 4 & 1 & -7 \\ 2 & 2 & 4 & 0 \end{bmatrix}$$

$$\overset{\substack{R_2+3R_1 \to R_2 \\ R_3-2R_1 \to R_3}}{=} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

$$\overset{\substack{R_3-4R_2 \to R_3 \\ R_1+R_2 \to R_1}}{=} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Row(A) = span\left(\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right)$$

$$= span(A_1^T, A_2^T)$$

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & 2 \\ 0 & 1 & 4 \\ 2 & -7 & 0 \end{bmatrix} \xrightarrow[R_4 - 2R_1 \to R_4]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \end{bmatrix}$$

$R_1 + 3R_2 \to R_1$

$R_4 + R_2 \to R_4$

$R_3 - R_2 \to R_3$

$$\begin{bmatrix} 1 & 0 & 14 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$Col(A) = span(A_1, A_2)$

Null(A^T)

RREF of A^T

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 + x_4 = 0$

$x_1 = -x_3 - x_4$

$x_2 + x_3 - x_4 = 0$

$x_2 = -x_3 + x_4$

$$Null(A^T) = span\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Null(A) =

RREF of A

$$= \left[\begin{array}{ccc|c} 1 & 0 & 14 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 14x_3 = 0$

$x_2 + 4x_3 = 0$

$x_2 = -4x_3$

$x_1 = -14x_3$

$$Null(A) = span\left\{\begin{bmatrix} -14 \\ -4 \\ 1 \end{bmatrix}\right\}$$

b) $\dim Null(A) + \dim Row(A) = 3$ ✓

$\dim Null(A^T) + \dim Col(A) = 4$ ✓

$$Row(A) = span\left(\underset{a}{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}}, \underset{b}{\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}}\right) \quad Null(A) = span\left(\underset{c}{\begin{bmatrix} -14 \\ -4 \\ 1 \end{bmatrix}}\right)$$

$a \cdot c = -14 + 12 + 2 = 0$ ✓

$b \cdot c = 14 - 16 + 2 = 0$ ✓

$$\text{Col}(A) = \text{span}\left(\underset{a}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}}, \underset{b}{\begin{bmatrix} -3 \\ 4 \\ 1 \\ -7 \end{bmatrix}} \right) \quad \text{Null}(A^T) = \left(\underset{c}{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{d}{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}} \right)$$

$a \cdot c = -1 + 1 + 0 + 0 = 0$ ✓

$a \cdot d = -1 - 1 + 0 + 2 = 0$ ✓

$b \cdot c = 3 - 4 + 1 + 0 = 0$ ✓

$b \cdot d = 3 + 4 + 0 - 7 = 0$ ✓

C) $B = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$

essentially looking for Null(A)
since, Row(A) = W

$$\text{Null}(A) = W^{\perp} = (\text{Row}(A))^{\perp}$$

$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] R_1 + R_2 \to R_1$

$x_1 + x_3 + x_4 = 0 \quad \to x_1 = -x_3 - x_4$

$x_2 - 2x_3 + x_4 = 0 \quad x_2 = 2x_3 - x_4$

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. a) true.

P is $n \times n$, and a stochastic column matrix, $\therefore$, its columns sum to 1.
if we take P^T, the rows sum to 1.

This means if we take $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ (x is $n \times 1$)

$P^T x = x$, since $P^T x$ is essentially just summing the rows of P^T.

$\therefore$, 1 is an eigenvalue of P^T, and since the eigenvalues of A and A^T are the same for any square matrix A, 1 is also an eigenvalue of P.

b) False, $\dim W + \dim W^{\perp} = n$ when W is in $\mathbb{R}^n$.
However $3 + 3 \neq 7$, so $W^{\perp} \neq V$.

c) $\dim Null(A) + \dim Row(A) = 10$

$\dim Row(A) = 3 = \dim Col(A)$

$\dim Null(A^T) + \dim Col(A) = 12$

$\dim Null(A^T) = 9$ ✓

True

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```
erykhalicki@mac A2 % python3 A2_Q5.py
Q5. A)
------------------------
m = 13.000000000000522
x = [[ 1.        ]
 [ 0.66666667]
 [-0.16666667]
 [ 0.11111111]
 [ 0.44444444]]
------------------------
Q5. B)
------------------------
m = -2.9999999999825917
x = [[ 1.        ]
 [ 0.71428571]
 [-0.14285714]
 [ 0.14285714]
 [ 0.42857143]]
------------------------
Q5. D)
------------------------
m = 1.000000000000421
x = [[ 1.        ]
 [ 0.69230769]
 [-0.15384615]
 [ 0.07692308]
 [ 0.46153846]]
------------------------
Q5. E)
------------------------
m = 6.000000000000037
x = [[-1.00000000e+00]
 [-5.00000000e-01]
 [-2.69808199e-14]
 [ 1.74375580e-14]
 [-5.00000000e-01]]
------------------------
```

```
#part C
def shifted_inverse_power_iteration(A, x0, alpha):
    y = x0
    B = A-np.eye(5,dtype=float)*alpha
    last_m=0
    m=max(x0, key=abs)[0]
    iteration = 0
    while not math.isclose(m, last_m, rel_tol=1e-14): # matching to 14 digits
        iteration+=1
        last_m = m
        x = np.linalg.solve(B,y)
        beta=max(x, key=abs)[0]
        m = alpha + 1./beta
        y = 1./beta*x

    return x,m
```

```
erykhalicki@mac A2 % python3 A2_Q6.py
Q6. A)
P =
[[0.         0.5        0.         0.         0.33333333 0.25      ]
 [0.         0.         1.         0.         0.33333333 0.25      ]
 [1.         0.5        0.         0.5        0.         0.25      ]
 [0.         0.         0.         0.         0.         0.25      ]
 [0.         0.         0.         0.         0.         0.        ]
 [0.         0.         0.         0.5        0.33333333 0.        ]]

x(50) =
[[0.2]
 [0.4]
 [0.4]
 [0. ]
 [0. ]
 [0. ]]
------------------------
Q6. B)
M =
[[0.025      0.45       0.025      0.025      0.30833333 0.2375    ]
 [0.025      0.025      0.875      0.025      0.30833333 0.2375    ]
 [0.875      0.45       0.025      0.45       0.025      0.2375    ]
 [0.025      0.025      0.025      0.025      0.025      0.2375    ]
 [0.025      0.025      0.025      0.025      0.025      0.025     ]
 [0.025      0.025      0.025      0.45       0.30833333 0.025     ]]

x(50) =
[[0.18918]
 [0.34617]
 [0.35777]
 [0.03498]
 [0.025  ]
 [0.04695]]

[('importance', '<f8'), ('index', '<i8')]
[[(0.35777, 2)]
 [(0.34617, 1)]
 [(0.18918, 0)]
 [(0.04695, 5)]
 [(0.03498, 3)]
 [(0.025  , 4)]]
------------------------
Q6. C)
----------------------
m = 0.9999501422160576
x =
[[0.52873995]
 [0.96760931]
 [1.        ]
 [0.0977674 ]
 [0.06988072]
 [0.13123141]]
----------------------
```

```python
import numpy as np
import math



adjacency_matrix = np.array([[0,0,1,0,0,0],
                             [1,0,1,0,0,0],
                             [0,1,0,0,0,0],
                             [0,0,1,0,0,1],
                             [1,1,0,0,0,1],
                             [1,1,1,1,0,0]]).T

inv_col_sum = 1/np.sum(adjacency_matrix, axis=0)

P = adjacency_matrix*inv_col_sum

x0 = np.ones((P.shape[0],1), dtype=float)/P.shape[0]
x = x0

# PART A
for i in range(50):
    x = np.round(P@x, 5)

print(f"Q6. A)")
print(f"P = \n{P}\n")
print(f"x(50) = \n{x}")
print("--------------------------")


#PART B
dtype = [('importance', float), ('index', int)]
alpha = 0.85

M = np.ones(P.shape, dtype=float)*(1-alpha)/P.shape[0] + P*alpha
x = x0

for i in range(50):
    x = np.round(M@x, 5)

print(f"Q6. B)")
print(f"M = \n{M}\n")
print(f"x(50) = \n{x}\n")

#adding index and sorting matrix
indexed_results = np.empty(M.shape[0], dtype=dtype)
indexed_results['importance'] = x.flatten()
indexed_results['index'] = np.arange(M.shape[0])

sorted_results = np.sort(indexed_results, order='importance')[::-1].reshape(M.shape[0],1)
print(sorted_results.dtype)
print(sorted_results)
print("--------------------------")

#PART C
def power_iteration(A, x0,max_iterations):
    last_m=0
    m=max(x0, key=abs)[0]
    x = x0
    iteration = 0

    while iteration<max_iterations:
        last_m = m
        iteration += 1
        x = A@x
        m=max(x, key=abs)[0]
        x=x/m
    return x,m

x,m = power_iteration(M,x0,15)
print(f"Q6. C)")
print(f"--------------------------")
print(f"m = {m}")
print(f"x = \n{x}")
print(f"--------------------------")
```