Al Eryk Halicki: 80922826 E1

1. $\begin{bmatrix} 3 & -4 & 2 & -2 \\ -2 & 1 & 0 & -1 \\ 1 & -3 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 + \frac{2}{3} R_1 \to R_2$

$$\begin{bmatrix} 3 & -4 & 2 & -2 \\ 0 & \frac{-5}{3} & \frac{4}{3} & \frac{-7}{3} \\ 1 & -3 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 - \frac{1}{3} R_1 \to R_3$

$$\begin{bmatrix} 3 & -4 & 2 & -2 \\ 0 & \frac{-5}{3} & \frac{4}{3} & \frac{-7}{3} \\ 0 & \frac{-5}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

$R_3 - R_2 \to R_3$

$$\begin{bmatrix} 3 & -4 & 2 & -2 \\ 0 & \frac{-5}{3} & \frac{4}{3} & \frac{-7}{3} \\ 0 & 0 & -1 & 3 \end{bmatrix} = U \quad \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix} = L$$

2. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & -2 & -3 \end{bmatrix} \neq A$

<u>Correct approach</u>

$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 - R_1 \to R_2$

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 - 2R_1 \to R_3$

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$R_3 - 2R_2 \to R_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U \quad \Bigg| \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} = L$$

3. a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad a \neq 0$

if $a \neq 0$ then we can obtain an upper triangular matrix of form $\begin{bmatrix} a & b \\ c-xa & d-xb \end{bmatrix}$ by solving for $c - xa = 0$.

in this case, we have $L = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$.

for any real c and a, there is only one solution to $c - xa = 0$, since it is a linear system of one variable and one equation.

$\therefore L$ is unique.

b) $U = \begin{bmatrix} a & b \\ 0 & d-xb \end{bmatrix}$ $\quad L = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$ for $c - xa = 0$

4. $$A = \begin{bmatrix} 0 & -2 & 3 & 0 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 5 \\ -4 & 4 & -7 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 6 \\ 9 \\ -20 \end{bmatrix}$$

$R_3 - 2R_2 \to R_3$
$R_4 + 4R_2 \to R_4$

$$\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow[R_3 \Leftrightarrow R_4]{R_1 \Leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \Leftrightarrow R_4]{R_1 \Leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = Q$$

$$Q^{-1} = P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$QA = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -2 & 3 & 0 \\ -4 & 4 & -7 & 1 \\ 2 & -2 & 4 & 5 \end{bmatrix}$$

$$QA = \begin{bmatrix} \dots & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \left| \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right.$$

$R_3 + 4R_1 \to R_3$

$R_4 - 2R_1 \to R_4$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix} = U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} = L$$

$$Qb = \begin{bmatrix} 6 \\ 21 \\ -20 \\ 9 \end{bmatrix}$$

$QA = LU$

$A = PLU \qquad P = Q^{-1}$

$Ux = y$

$Ax = b$

$Ly = Qb$

$LUx = Qb$

$y_1 = 6 \qquad y_3 = 4$

$y_2 = 21 \qquad y_4 = -3$

$$y = \begin{bmatrix} 6 \\ 24 \\ 4 \\ -3 \end{bmatrix}$$

$y_3 - 4y_1 = -20$

$y_4 + 2y_1 = 9$

$Ux = y$

$$x_1 - x_2 + 2x_3 + x_4 = 6$$
$$-2x_2 + 3x_3 = 21$$
$$x_3 \cancel{+} 5x_4 = 4$$
$$3x_4 = -3$$

$x_4 = -1$

$x_3 = 9$

$x_2 = 3$

$x_1 = 6 + x_2 - 2x_3 - x_4$

$= 6 + 3 - 18 + 1$

$= -8$

$$x = \begin{bmatrix} -8 \\ 3 \\ 9 \\ -1 \end{bmatrix}$$

---

5. $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ $\quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = Ax_0 / m$

$= \begin{bmatrix} 6 \\ 5 \end{bmatrix} 1/6$

$= \begin{bmatrix} 1 \\ \frac{5}{6} \end{bmatrix}$

$x_2 = Ax_1 / m$

$= \begin{bmatrix} 6 + 10/6 \\ 5 + 15/6 \end{bmatrix} \cdot \frac{1}{m}$

$= \begin{bmatrix} 7.6667 \\ 7.5 \end{bmatrix} \cdot \frac{1}{7.6667}$

$= \begin{bmatrix} 1 \\ 0.9783 \end{bmatrix}$

$$x_3 = Ax_2 \cdot \frac{1}{m}$$

$$= \begin{bmatrix} 6 + 1.9566 \\ 5 + 2.9349 \end{bmatrix} \cdot \frac{1}{m}$$

$$= \begin{bmatrix} 7.9566 \\ 7.9349 \end{bmatrix} \cdot \frac{1}{7.9566}$$

$$= \begin{bmatrix} 1 \\ 0.9972 \end{bmatrix}$$

$$x_4 = Ax_3 \cdot \frac{1}{m}$$

$$= \begin{bmatrix} 6 + 1.9944 \\ 5 + 2.9916 \end{bmatrix} \cdot \frac{1}{m}$$

$$= \begin{bmatrix} 7.9944 \\ 7.9916 \end{bmatrix} \cdot \frac{1}{7.9944}$$

$$= \begin{bmatrix} 1 \\ 0.9996 \end{bmatrix} = e_1$$

$$7.9944 = \lambda_1$$

6. $f(x) = \dfrac{x^T A x}{x^T x}$

$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

a)

$x_1 = A x_0 \frac{1}{m} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \frac{1}{m} = \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix}$

$x_2 = A x_1 \frac{1}{m} = \begin{bmatrix} 5.8 \\ 2.8 \end{bmatrix} \frac{1}{5.8} = \begin{bmatrix} 1 \\ 0.483 \end{bmatrix}$

$x_3 = A x_2 \frac{1}{m} = \begin{bmatrix} 5.966 \\ 2.966 \end{bmatrix} \frac{1}{5.966} = \begin{bmatrix} 1 \\ 0.497 \end{bmatrix}$

$x_4 = A x_3 \frac{1}{3} = \begin{bmatrix} 5.994 \\ 2.994 \end{bmatrix} \frac{1}{5.994} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

$m_4 = 5.994$

b)

$$\ell(x_1) = \frac{\begin{bmatrix} 1 & 0.4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}}{\begin{bmatrix} 1 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 5.8 & 2.8 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}}{1.16}$$

$$= \frac{6.92}{1.16} = 5.966$$

$$\ell(x_2) = \frac{\begin{bmatrix} 1 & 0.483 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.483 \end{bmatrix}}{\begin{bmatrix} 1 & 0.483 \end{bmatrix} \begin{bmatrix} 1 \\ 0.48 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 5.966 & 2.966 \end{bmatrix} \begin{bmatrix} 1 \\ 0.483 \end{bmatrix}}{1.233} \approx 6.0$$

$\det(A-\lambda I)=0$

$$A-\lambda I=\begin{bmatrix}5-\lambda & 2\\ 2 & 2-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=(5-\lambda)(2-\lambda)-4$$
$$=10-7\lambda+\lambda^2-4=0$$
$$\lambda^2-7\lambda+6=0$$
$$\lambda=6,1$$

$R(x_2)$ is closer to λ_1

7. a) False, there can be LU factorizations of non square matrices, which are by definition not invertible

b) False, rows 2 and 3 need to be swapped, meaning we need a PLU factorization

c) False, you can obtain LU factorizations in multiple ways, for example, using a unit factorization v.s. a non unit factorization

d) False, if you select an x_0 that is orthogonal to e_1 corresponding to λ_1, then it will not converge to λ_1, since x_0 has no component in the direction of e_1.

$$A^k x_0 = 0 \lambda_1^k e_1 \dots + c_n \lambda_n^k e_n$$

$c_1 = 0$, since they are orthogonal

e) True, the eigen vectors of A are the standard basis vectors, and their corresponding eigen values are 2 and 1. Since $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ is not orthogonal to e_1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the sequence will converge

8. a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.\overline{3} & 1 & 0 & 0 & 0 \\ 0 & -0.375 & 1 & 0 & 0 \\ 0 & 0 & -.381 & 1 & 0 \\ 0 & 0 & 0 & -.382 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ 0 & 2.\overline{6} & -1 & 0 & 0 \\ 0 & 0 & 2.625 & -1 & 0 \\ 0 & 0 & 0 & 2.619 & -1 \\ 0 & 0 & 0 & 0 & 2.618 \end{bmatrix}$$

b)

$$t_1 = \begin{bmatrix} 6.\overline{5} \\ 9.\overline{6} \\ 10.\overline{4} \\ 9.\overline{6} \\ 6.\overline{5} \end{bmatrix} \quad t_2 = \begin{bmatrix} 4.74 \\ 7.\overline{6} \\ 8.59 \\ 7.\overline{6} \\ 4.74 \end{bmatrix} \quad t_3 = \begin{bmatrix} 3.6 \\ 6.\overline{05} \\ 6.9 \\ 6.\overline{05} \\ 3.6 \end{bmatrix}$$

$$t_4 = \begin{bmatrix} 2.79 \\ 4.\overline{7} \\ 5.49 \\ 4.\overline{7} \\ 2.79 \end{bmatrix}$$