A3 Eryk Halicki

1.a) $v_1 = a_1$ $\quad W_1 = \text{span}(a_1)$

$v_2 = a_2 - \text{proj}_{v_1}(a_2)$

$= a_2 - \left[\frac{(a_2 \cdot v_1)}{\|v_1\|^2} v_1\right]$

$= \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} - \frac{(4 \cdot 3 + (-1 \cdot -2) + 2 \cdot 0)}{3^2 + -1^2 + 2^2} v_1$

$= \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} - \frac{14}{14} v_1$

$= \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$W = \text{span}\left\{\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right\}$

b) $y = c_1 v_1 + c_2 v_2$

$c_1 = \frac{y \cdot v_1}{\|v_1\|^2} = \frac{(7 \cdot 3 + -1 \cdot 1 + 2 \cdot 18)}{14}$

$= \frac{56}{14} = 4$

$$c_2 = \frac{y \cdot v_2}{\|v_2\|^2} = \frac{7 \cdot 1 + 1 \cdot -1 + 18 \cdot -2}{6}$$

$$= -5$$

$$y = 4v_1 - 5v_2$$

c) i) $$P = \frac{1}{\|v_1\|^2} v_1 v_1^T + \frac{1}{\|v_2\|^2} v_2 v_2^T$$

$$= \frac{1}{14}\begin{bmatrix} 9 & -3 & 6 \\ -3 & 1 & -2 \\ 6 & -2 & 4 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.81 & -0.38 & 0.10 \\ -0.38 & 0.23 & 0.19 \\ 0.10 & 0.19 & 0.95 \end{bmatrix}$$

ii) $P\cancel{x} = x$

$$= \begin{bmatrix} -8.24 \\ 3.52 \\ -2.38 \end{bmatrix}$$

iii) $u = u_1 + u_2$

$u_1 = x$

$u_2 = u - u_1$

$$= \begin{bmatrix} 1.24 \\ 2.48 \\ -0.62 \end{bmatrix}$$

2. a) $Pu = u_1$, $u = u_1 + u_2$

u_1 is projection of u onto k

u_2 is projection of u onto $k^\perp$

$(I-P)u = x$

$u - Pu = x$

$u - x = Pu$

$Pu - x = u_1$

$u - x = u - u_2$

$x = u_2$

so, $(I-P)u = u_2$ and u_2 is orthogonal to u_1, so $(I-P)$ is the projection matrix onto $k^\perp$

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b) i) $A = uu^T + vv^T$

$Ax = (uu^T + vv^T)x$

$= uu^T x + vv^T x$

$= u(u \cdot x) + v(v \cdot x)$

so, Ax produces vectors that are linear combinations of u and v, meaning Ax is in the subspace spanned by $\{u, v\}$. $\therefore$ $\dim(\text{Col}(A)) = 2$

$\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A))$

$\dim(\mathrm{Row}(A)) + \dim(\mathrm{Null}(A)) = n$

$\dim(\mathrm{Null}(A)) = n-2$

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ii) Q contains a set of orthonormal basis vectors spanning $\mathrm{Col}(A)$

using these, we can create a projection matrix P, which projects a vector x onto $\mathrm{Col}(A)$.

$P = Q_1 Q_1^T + Q_2 Q_2^T + Q_3 Q_3^T$

$= Q Q^T$ (no need to normalize since Q is already normalized)

$$= \begin{bmatrix} 0.67 & -0.33 & 0 & 0.33 \\ -0.33 & 0.67 & 0 & 0.33 \\ 0 & 0 & 1 & 0 \\ 0.33 & 0.33 & & 0.67 \end{bmatrix} = P$$

now using Pb we can find $proj_{Col(A)}(b)$.

However, we need $proj_{Null(A^T)}(b)$. Q2.a)

$Null(A^T) = Col(A)^{\perp}$, meaning $(I-P)$ will be the projection matrix onto $Null(A^T)$.

$$(I-P)b = \begin{bmatrix} 0.33 \\ 0.33 \\ 0.00 \\ -0.33 \end{bmatrix}$$

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3. step 1: find orthonormal basis for Col(A).

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$v_2 = a_2 - proj_{W_1}(a_2)$$

$$= a_2 - (a_2 \cdot q_1) q_1$$

$$= a_2 - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\rightarrow q_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} -1/\sqrt{10} \\ 2/\sqrt{10} \\ -1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$$

$$v_3 = a_3 - \text{proj}_{W_2}(a_3)$$

$$= a_3 - (a_3 \cdot q_1) q_1 - (a_3 \cdot q_2) q_2$$

$$= a_3 - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \\ 2 \end{bmatrix}$$

$$q_3 = \frac{v_3}{\|v_3\|}$$

$$= \begin{bmatrix} -3/\sqrt{23} \\ -3/\sqrt{23} \\ 1/\sqrt{23} \\ 2/\sqrt{23} \end{bmatrix} \qquad Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{10} & -3/\sqrt{23} \\ 0 & 2/\sqrt{10} & -3/\sqrt{23} \\ 1/\sqrt{3} & -1/\sqrt{10} & 1/\sqrt{23} \\ 1/\sqrt{3} & 2/\sqrt{10} & 2/\sqrt{23} \end{bmatrix}$$

$Q^T Q = I$, since $Q^T_i \, Q_j = 1$ if $i = j$

$Q^T_i \, Q_j = 0$ if $i \neq j$

$Q_i \cdot Q_i = 1$

since they are the same unit vector

$Q_i \cdot Q_j = 0$

since all basis vectors are orthogonal in Q

so, $A = QR$

$$Q^T A = \cancel{Q^T Q} R$$

$$Q^T A = R$$

$$Q^T A = \begin{bmatrix} 0.58 & 0 & 0.58 & 0.58 \\ -0.32 & 0.63 & -0.32 & 0.63 \\ -0.63 & -0.63 & 0.21 & 0.42 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.73 & 1.73 & 1.73 \\ 0 & 3.16 & 3.16 \\ 0 & 0 & 4.8 \end{bmatrix}$$

4. a) $A^{-1} = A^T$

$\therefore A^T A = I$ 1.

$A A^T = I$ 2.

1. $A^T A = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & \cdots \\ a_2 \cdot a_1 & a_2 \cdot a_2 & \\ \vdots & & \ddots \end{bmatrix}$

if $A^T A \neq I$, then $a_i \cdot a_i \neq 1$ or $a_i \cdot a_j \neq 0$ $(j \neq i)$

Neither of which are possible if a matrix is orthogonal, since $a_i \cdot a_i = 1$ for any unit vector, and $a_i \cdot a_j = 0$ for all orthognal vectors. So A^TA must $= I$ if A is orthogonal.

2. $AA^T = I$

Since A's columns are an orthonormal set, we know the columns of A are linearly independent. Additionally, A is square. LI columns + Square = invertible. So, we know A is invertible, and that $A^TA = I$.

$A^{-1}A = I$

$A^{-1}A = A^TA$

$\cancel{A^{-1}AA^{-1} = A^TAA^{-1}}$

$A^{-1} = A^T$

b) False.

$A = QR$ implies Q's columns are an orthonormal set, but A may not be sqare, resulting in Q not being square, which disqualifies Q from being orthogonal.

c) True

if A^2 is orthogonal $A^2(A^T)^2 = I$.

$A^2(A^T)^2 = I$

$A(AA^T)A^T = I$

AIA^T

$AA^T = I$ ✓ $\therefore A^2$ is also orthogonal

d) True.

if A is $m \times n$, but an orthonormal set then $A^TA = I$, since $A^T = n \times m$ $A = m \times n$

$A^TA = n \times \cancel{m \times m} \times n$

$= n \times n.$ and the diagonal entries will be the dot products of each column with themselves, which $= 1$. All remaining entries will be 0, since $a_i \cdot a_j = 0$ for a_i and a_j that are orthogonal.

So, A can be a non-square matrix with orthonormal columns, ~~and~~ with $A^T A = I$, but not be orthogonal since it isn't square.

5. a) True.

$$\text{proj}_v(\text{proj}_u(y)) = \text{proj}_v\left(\left(\frac{y \cdot u}{\|u\|^2}\right) u\right)$$

So $\text{proj}_u(y)$ is a scalar multiple of vector u, lets call this cu.

$$\text{proj}_v(cu) = c\,\text{proj}_v(u)$$
$$= c \cdot 0$$
$$= 0$$

since v and u are orthogonal, $\text{proj}_v(u) = 0$

b) False

~~We~~ We can still find a Q with orthonormal vectors by doing QR factorization, and choosing an arbitrary orthogonal vector for the a_i's in A that are dependant (since $a_i - \text{proj}_{W_{i-1}}(a_i)$ will $= 0$, since a_i will already be in W_{i-1})

However, unlike with a set of linearly independant vectors, there will be diagonal entries in R that are 0, since

linearly dependant columns will just produce

$$r_i = \begin{bmatrix} cr_{i1} \\ cr_{i2} \\ \vdots \\ cr_{ij} \\ 0 \\ \\ 0 \end{bmatrix}$$ for some dependant a_j

regardless, A will be upper triangular, just not satisfying $r_{ii} \neq 0$

c) True.

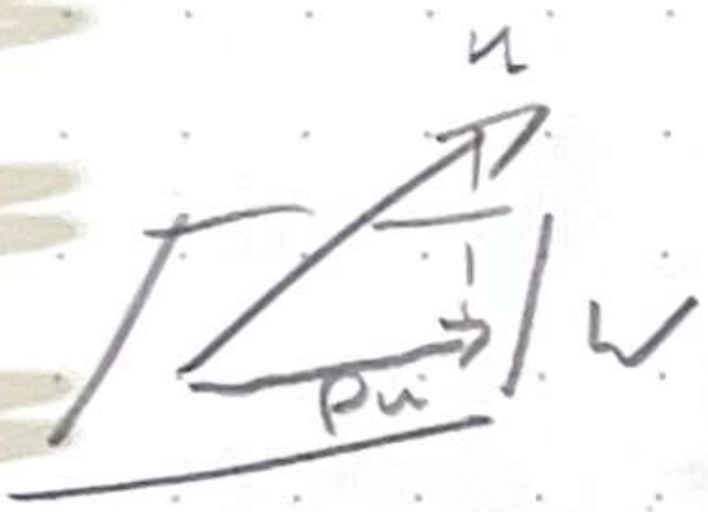
if A has linearly independant columns, $r_{ii} > 0$, in Matrix R.

since R is upper triangular, the determinant of R is the product of diagonal entries.

Furthermore, $r_{ii} > 0$, so $\det(R) > 0$,

so ~~A~~ R is invertible

d) True.



~~Pu~~ let u be a vector in $\mathbb{R}^n$.

Pu will be in subspace W. if we apply P again, PPu, the resulting vector doesnt change, since its already in W.

$\therefore$ $P^2 = P$, and even $P^n = P$ for $n \geq 1$

6. $A = PBP^{-1}$ (similar)

a) $A_1 = RQ \quad A = QR$

$A = QA_1Q^{-1}$

$= QRQ\cancel{Q^{-1}}$ Q is invertible since A is square

$= QR$ ✓

$\therefore$ A and A_1 are similar, since A can be represented as QA_1Q^{-1}

$Ax = \lambda x$

$(QA_1Q^{-1})x = \lambda x$

$\cancel{Q^{-1}}QA_1Q^{-1}x = \lambda Q^{-1}x$

$A_1(Q^{-1}x) = \lambda(Q^{-1}x)$

$\therefore A_1$ has same eigenvalues as A, with corresponding eigenvectors of form $x \to Q^{-1}x$

b) $A_0 = Q_0 R_0 = Q_0 \overbrace{R_0 Q_0}^{A_1} Q_0^{-1}$

$A_1 = R_0 Q_0 = Q_1 \overbrace{R_1 Q_1}^{A_2} Q_1^{-1}$

$A_2 = R_1 Q_1 = Q_2 \overbrace{R_2 Q_2}^{A_3} Q_2^{-1}$

$\vdots$

for any k, we can represent A as $A = Q_0 Q_1 \cdots \underbrace{(R_{k-1} Q_{k-1})}_{A_k} Q_{k-2}^{-1} \cdots Q_0^{-1}$

so A is similar to A_k via the similarity matrix

$Q_0 Q_1 Q_2 \cdots Q_{k-2}$

which is invertible

by $(Q_0 Q_1 Q_2 \cdots Q_{k-2})^T$

$= Q_{k-2}^T Q_{k-3}^T \cdots Q_0^T$

$= Q_{k-2}^{-1} Q_{k-3}^{-1} \cdots Q_0^{-1}$ since Q is orthogonal $Q^T = Q^{-1}$

```python
import numpy as np

arr_a = np.array([[2,3],[2,1]], dtype=float)
arr_b = np.array([[1,0,-1],[1,2,1],[-4,0,1]], dtype=float)

def qr_algorithm(A, iterations):
    Ak = A
    for i in range(iterations):
        Q,R = np.linalg.qr(Ak)
        Ak = R@Q
    return Ak

print(f"Eigenvalues of \n{arr_a}\nare\n {np.diagonal(qr_algorithm(arr_a, 100).round(2))}")
print(f"Eigenvalues of \n{arr_b}\nare\n {np.diagonal(qr_algorithm(arr_b, 100).round(2))}")
```

```
erykhalicki@mac A3 % python3 Q6.py
Eigenvalues of
[[2. 3.]
 [2. 1.]]
are
 [ 4. -1.]
Eigenvalues of
[[ 1.  0. -1.]
 [ 1.  2.  1.]
 [-4.  0.  1.]]
are
 [ 3.  2. -1.]
```