A4 Eryk Halicki

1\. a) i) False.

$x = \hat{x}$ in the case where

b is in $Col(A)$, since $proj_{Col(A)} b = b$

ii) if b is orthogonal to $Col(A)$

then we are solving for

$A\hat{x} = 0$, since $proj_{Col(A)}(b) = 0$.

So, $\hat{x}$ is in $Null(A)$.

$\because Null(A) = Row(A)^{\perp}$, $\hat{x}$ is orthogonal

to $Row(A)$.

True

iii) take the case where A has L.I

columns but is not square. Then

A^{-1} does not exist, so the

equation doesn't make sense.

False

iv) if a system has infinite solutions, we know $Ax=b$ has a solution, by definition. so $b \in Col(A)$, and the least squares solution is the same as the normal solution. I.E. $\hat{x}=x$. And since there are infinite choices of x, there are infinite LS solutions.

True

v) If $\hat{x}_1$ and $\hat{x}_2$ are LS solutions to $Ax=b$, then $A\hat{x}_1=\hat{b}$ and $A\hat{x}_2=\hat{b}$

so if $\hat{x}_1+\hat{x}_2$ is also an LS solution, then $A(\hat{x}_1+\hat{x}_2)=\hat{b}=A\hat{x}_1+A\hat{x}_2=2\hat{b}$

but, $\hat{b}=2\hat{b}$ is only true if $\hat{b}=0$, so this False b)

i) $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

$A^T A x = A^T b$

$\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} x = A^T b$

$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} x = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

$14x_1 - 3x_2 = 1$

$-3x_1 + 21x_2 = 10$

$-98x_1 + 21x_2 = -7$

$-\ -3x_1 + 21x_2 = 10$

$-95x_1 = -17$

$x_1 = \frac{17}{95} \approx 0.1789$

$x_2 \approx 0.50$

=

$x = \begin{bmatrix} 0.1789 \\ 0.5017 \end{bmatrix}$

$Ax = b$ is overconstrained, with 3 equations and 2 variables, and has linearly independant columns, so we know LS will only have one solution.

ii) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = a_1$ $\quad a_2 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

$$v_2 = a_2 - \text{proj}_{W_1}(a_2)$$

$$= a_2 - \frac{v_1 \cdot a_2}{\|v_1\|^2} v_1$$

$$= a_2 - \frac{[-1+6-8]}{14} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -3/14 \\ -9/14 \\ 6/14 \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.786 \\ 2.643 \\ 3.571 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -0.786 \\ 2.643 \\ 3.571 \end{bmatrix} \right\}$ basis for Col(A)

$$\operatorname{proj}_{\operatorname{Col}(A)}(b) = \frac{v_1 \cdot b}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot b}{v_2 \cdot v_2} v_2$$

$$\approx \begin{bmatrix} -0.323 \\ 1.540 \\ 1.649 \end{bmatrix} = \hat{b}$$

$A\hat{x} = \hat{b}$

$x_1 - x_2 = -0.323$

$3x_1 + 2x_2 = 1.540$

$-2x_1 + 4x_2 = 1.649$

$x_1 = -0.323 + x_2$

$3(-0.323 + x_2) + 2x_2 = 1.54$

$-0.969 + 3x_2 + 2x_2 = 1.54$

$5x_2 = 2.509$

$x_2 = 0.5018$

$x_1 = 0.1788$

$$x \approx \begin{bmatrix} 0.1788 \\ 0.5018 \end{bmatrix}$$

iii) $\|A\hat{x} - b\| = \left\| \begin{bmatrix} -0.323 \\ 1.540 \\ 1.649 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\| = 4.56$

A4 cont.

1. c) $-x_1 + 2x_2 = 7$

$3x_1 - 6x_2 = -1$

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 10 & -20 \\ -20 & 40 \end{bmatrix} \hat{x} = \begin{bmatrix} -10 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -20 \\ 0 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

$10x_1 - 20x_2 = -10$

$x_1 - 2x_2 = -1$

$x_1 = -1 + 2x_2 \quad t = x_2$

$$\boxed{\hat{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t}$$

2. a)

$Q = [q_1, q_2, q_3, \dots, q_n]$

$q_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} / \|a_1\|$

$= \begin{bmatrix} 1/\sqrt{4} \\ -1/\sqrt{4} \\ 1/\sqrt{4} \\ -1/\sqrt{4} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$q_2 = \dfrac{a_2 - proj_{W_1}(a_2)}{\|a_2 - \dots\|}$

$= \dfrac{a_2 - \frac{a_2 \cdot q_1}{\|q_1\|^2} q_1}{\dots}$

$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

$q_3 = \dfrac{a_3 - proj_{W_2}(a_3)}{\|a_3 - \dots\|}$

$= \dfrac{a_3 - \left(\frac{a_3 \cdot q_1}{q_1 \cdot q_1} q_1 + \frac{a_3 \cdot q_2}{q_2 \cdot q_2} q_2\right)}{norm(\dots)}$

$= \begin{bmatrix} 0 \\ -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$

$Q = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & -0.7071 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0.7071 \end{bmatrix}$

$A = QR$

$Q^T A = R$

$$Q^T A = \begin{bmatrix} 2 & -1 & -0.5 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0.7071 \end{bmatrix} = R$$

b) True. If M has LI columns, so does R.

$M = QR$

$M\hat{x} = b$

$M^T M\hat{x} = M^T b$

$(QR)^T (QR)\hat{x} = (QR)^T b$

$R^T \cancel{Q^T Q} R\hat{x} = R^T Q^T b$

$R^T R\hat{x} = R^T Q^T b$

$\cancel{(R^T)^{-1}} R^T R\hat{x} = \cancel{(R^T)^{-1} R^T} Q^T b$

$R\hat{x} = Q^T b$ ✓

since R is square and has LI columns, it is invertible.

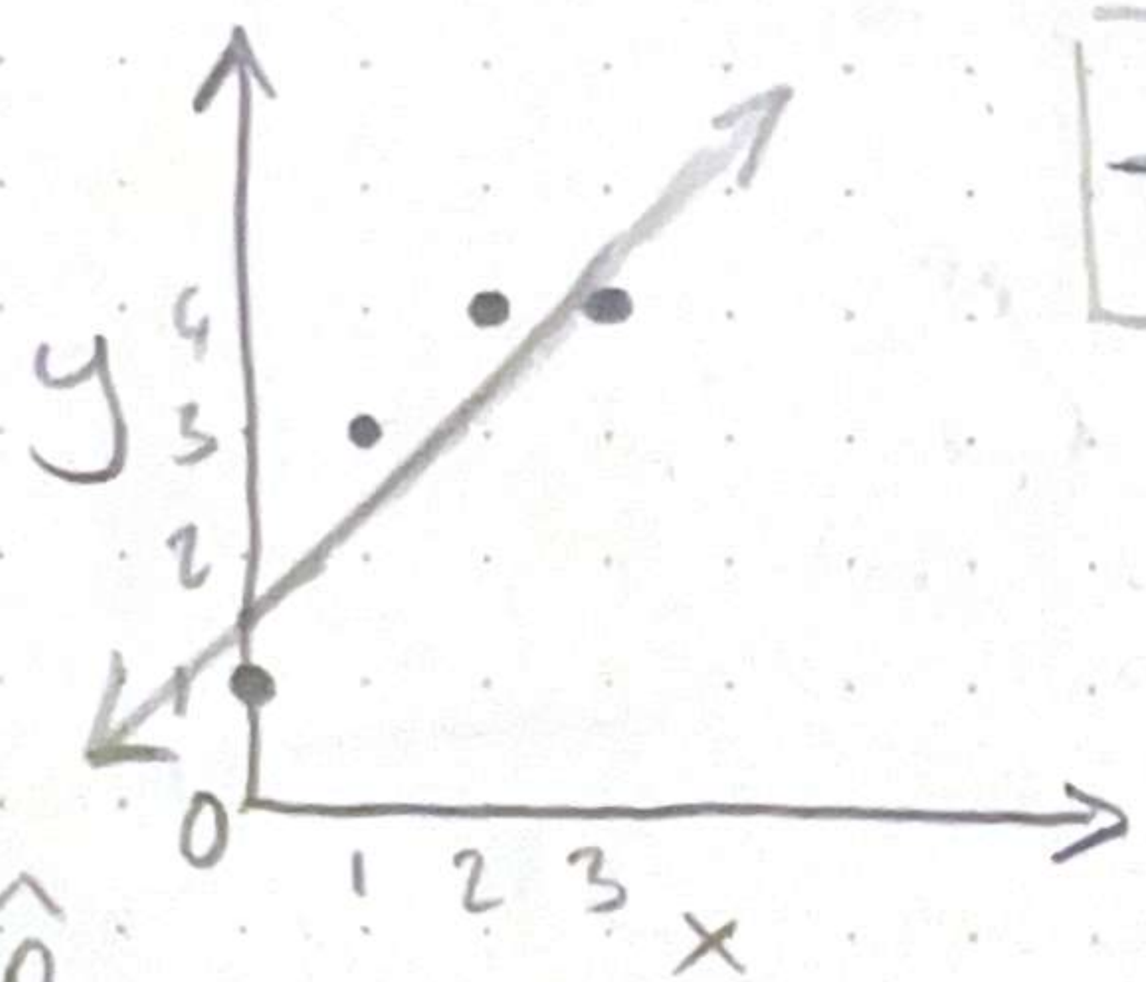
c) $Ax=b$

$R\hat{x}=Q^T b$

$$\begin{bmatrix} 2 & -1 & -0.5 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0.7071 \end{bmatrix} \hat{x} = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & -0.7071 & 0 & 0.7071 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

3. a)



$\boxed{—— = 1.5 + x}$

$$\begin{array}{ccc} X & \hat{B} & & Y \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} \hat{B}_0 \\ \hat{B}_1 \end{bmatrix} & = & \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} \end{array}$$

$$\hat{B} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$X^T X \hat{B} = X^T Y$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \hat{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

$\underbrace{\text{b)} \qquad \text{c)}}_{\text{digital}}$

4. a) In order to be symetric,

$A^T = A$.

in the case $B = A^T A$

$C = A A^T$

we can show both B and C are symetric matrices by the following.

$B^T = B$

$(A^T A)^T = A^T A$

$A^T \cancel{A^{T^T}} = A^T A$ ✓

$C^T = C$

$(A A^T)^T = A A^T$

$\cancel{A^{T^T}} A^T = A A^T$ ✓

b) i) True.

~~Ek~~ as per part a), A^TA and AA^T are both symmetric. And according to the spectral theorum, if a matrix is symmetric, it is orthogonally diagonalizable.

ii) False

A matrix is symmetric IFF it is orthogonally diagonilizable.

~~Bor~~ Not all orthogonal matrices are symmetric, eg. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

so not all orthogonal matrices are orthogonally diagonilizable.

iii) True

If A is orthogonally diagonalizable, we automatically know it is symetric and Real.

Additionally, if a symetric matrix has all real entries, all its eigenvalues are real, so we know A has all real eigenvalues

iv) True

If we use gram-schmidt, we can find an, real, orthogonal basis for W, such that

$Q = [q_1, q_2 \cdots q_k]$, where q_i are the basis vectors for W.

Once we have Q, we can normalize each column, and have a matrix B, containing orthonormal basis vectors for W.

It can be shown that projection matrix $P = BB^T$.

BB^T is automatically symetric since $(BB^T)^T = BB^T$.

Since B is real, BB^T is real and symetric, and has all real eigenvalues.

c) $A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

i) A is symetric, so it is orthogonally diagonalizable. As such, in PDP^{-1}, P ~~ea~~ is orthogonal, meaning all eigenvectors of A corresponding to different eigenvalues are orthogonal!

ii) $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$ $\det(A - \lambda I) = (6-\lambda)(9-\lambda) - 4$

$= 54 - 6\lambda - 9\lambda + \lambda^2 - 4$

$= \lambda^2 - 15\lambda + 50$

$= (\lambda - 10)(\lambda - 5) = 0$

$\lambda = 10, 5$

$(A - 10I)x = 0$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1 = 2x_2$

$x_1 = \frac{-x_2}{2} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = x / \|x\| = x = \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix}$

$(A - 5I)x = 0$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} x = 0$$

$x_1 - 2x_2 = 0 \quad x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} / \|x\| = x = \begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix}$

$x_1 = 2x_2$

$$P = \begin{bmatrix} 0.894 & -0.447 \\ 0.447 & 0.894 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

ii) by spectral theorum,

$uu^T = QDQ^T$ since uu^T is symmetric.

Through spectral decomposition,

$$uu^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T \dots + \lambda_n q_n q_n^T$$

Since $(uu^T)u = u(u^T u)$, u is an eigenvector of uu^T with eigenvalue $u^T u$. $(\|u\|^2)$

let $q_1 = \frac{u}{\|u\|}$ and $\lambda_1 = \|u\|^2$

then, $uu^T = \|u\|^2 q_1 q_1^T + \lambda_2 q_2 q_2^T \dots$

but $q_1 q_1^T = u \frac{1}{\|u\|} u^T \frac{1}{\|u\|}$

$$= \frac{uu^T}{\|u\|^2}$$

so the equation becomes

$$uu^T = \cancel{\|u\|^2} \frac{uu^T}{\cancel{\|u\|^2}}$$

implying uu^T only has one non zero eigenvalue. This means $\text{rank}(D) = 1$, and by extension, $\text{rank}(QDQ^T) = 1 = \text{rank}(uu^T)$

ii) $A = 5 \cdot \begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix} \begin{bmatrix} 0.894 & 0.447 \end{bmatrix}$

$+ 10 \cdot \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix} \begin{bmatrix} -0.447 & 0.894 \end{bmatrix}$

```python
import numpy as np
import math

w = np.array([[44,61,81,113,131]]).T
p = np.array([[91,98,103,110,112]]).T

ln_w = np.log(w)

W = np.hstack((np.ones((w.shape[0],1)), ln_w))
print(W)

B = np.linalg.solve(W.T@W, W.T@p)
print(B)

child_estimate = B[0,0] + B[1,0] * math.log(100)
print(child_estimate)
```

3b.py 12,

```
[[1.         3.78418963]
 [1.         4.11087386]
 [1.         4.39444915]
 [1.         4.72738782]
 [1.         4.87519732]]
[[17.92434012]
 [19.38499925]]
107.19556073585245
```

```python
import numpy as np

t = np.atleast_2d(np.arange(13)).T
p = np.array([[0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, 809.2]]).T

T = np.hstack((np.ones((t.shape[0],1)), t, t*t, t*t*t))
print(T)

B = np.linalg.solve(T.T@T, T.T@p)
print(B)

velocity_estimate = B.T@np.array([[1, 4.5, 4.5**2, 4.5**3]]).T

print(velocity_estimate)

# It was predictable that there would be a unique least squares solution
# because the system is over determined (more equations than variables)
# since there are 13 equations and 4 variables (B0, B1, B2, B3)

```

3c.py 14,24 All

```
 [1.000e+00 6.000e+00 3.600e+01 2.160e+02]
 [1.000e+00 7.000e+00 4.900e+01 3.430e+02]
 [1.000e+00 8.000e+00 6.400e+01 5.120e+02]
 [1.000e+00 9.000e+00 8.100e+01 7.290e+02]
 [1.000e+00 1.000e+01 1.000e+02 1.000e+03]
 [1.000e+00 1.100e+01 1.210e+02 1.331e+03]
 [1.000e+00 1.200e+01 1.440e+02 1.728e+03]]
[[-0.85576923]
 [ 4.70248501]
 [ 5.55536963]
 [-0.02736014]]
[[130.30845561]]
```