MAT216 Fall 2020

Assignment 3 on Week 6–7

Problem 1. (Coordinates and Change of Basis)

Consider

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}, \qquad B' = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

i) Show that B and B' are bases for $\mathbb{R}^3$.

ii) Find the coordinate matrices of $\mathbf{v} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$ relative to the bases B and B'.

iii) Find the transition matrix $P_{B \to B'}$.

iv) Verify that $[\mathbf{x}(\mathbf{v})]_{B'} = P_{B \to B'}[\mathbf{x}(\mathbf{v})]_B$.

Problem 2. (Coordinates and Change of Basis)

Consider

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}, \qquad B' = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \right\}.$$

i) Show that B and B' are bases for $M_{2,2}$.

ii) Find the coordinate matrices of $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ relative to the bases B and B'.

iii) Find the transition matrix $P_{B \to B'}$.

iv) Verify that $[\mathbf{x}(A)]_{B'} = P_{B \to B'}[\mathbf{x}(A)]_B$.

Problem 3. (Coordinates and Change of Basis)

Consider

$$B = \{1 + x, x + x^2, 1 + x^2\}, \qquad B' = \{1 + x^2, 1 - x^2, 1 - 2x\}.$$

i) Show that B and B' are bases for $\mathbf{P}_2$.

ii) Find the coordinate matrices of $p(x) = -x^2 + 4x + 7$ relative to B and B'

iii) Find the transition matrix $P_{B \to B'}$.

iv) Verify that $[\mathbf{x}(p)]_{B'} = P_{B \to B'}[\mathbf{x}(p)]_B$.

Problem 4. (Row Space, Column Space of a Matrix)

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix}$$

i) Find row space $R(A)$ and column space $C(A)$ of A.

ii) Find the bases for row space and column space of A obtained in i).

iii) Find $\dim\big(R(A)\big)$ and $\dim\big(C(A)\big)$.

iv) Find the $\text{rank}(A)$.

Problem 5. (Homogeneous and Nonhomogeneous Systems of Equations)

Consider the matrix A in Problem 4.

i) Find the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$, that is $N(A)$, the nullspace of A.

ii) Find the basis and dimension of $N(A)$.

iii) What is $\dim\big(C(A)\big) + \dim\big(N(A)\big)$? Explain by referring it to an appropriate theorem.

iv) If $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}$, determine whether the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent.

v) If the system $A\mathbf{x} = \mathbf{b}$ is consistent, find the complete solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ denotes the particular solution and $\mathbf{x}_h$ denotes a solution associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Note: It is recommended to use information and results obtained in Problem 4 to solve Problem 5.