Assignment — 03

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Section : 11

$\boxed{\text{Ans to the Q. No} - 1}$

i

The augmented matrix of the system of equations from set "B" is such –

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 2 & 2 & 4 & b \\ 4 & 0 & 0 & c \end{array}\right] \xrightarrow[R_3 = R_2 - 4R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 4 & 0 & b-2a \\ 0 & 4 & -8 & c-4a \end{array}\right]$$

$$\xrightarrow[R_2 = \frac{1}{4}R_2]{R_3 = R_2 - R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 1 & 0 & \frac{b-2a}{4} \\ 0 & 0 & 8 & b+2a-c \end{array}\right]$$

$$\xrightarrow[\substack{R_1 = R_1 + R_2 \\ R_3 = \frac{1}{8}R_3}]{R_1 = \cancel{4}R_1 - \frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\cancel{4a-2b}+c}{4} \\ 0 & 1 & 0 & \frac{b-2a}{4} \\ 0 & 0 & 1 & \frac{b+2a-c}{8} \end{array}\right]$$

Therefore, it can be said that B spans R_3. Now, we have to prove that B is linearly independent.

The augmented matrix takes the form

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 4 & 0 & 0 & 0 \end{array}\right]$$

Doing exactly the same Gauss-Jordan elimination ~~as previ~~ we get

$$\xrightarrow{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ It is seen that B ~~spans R3~~. is a base for R3.

Again augmented matrix of B' is

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & a \\ 2 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right] \xrightarrow[R_2 = R_2 - 2R_1]{R_1 \rightleftarrows R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & c \\ 0 & 1 & -1 & b-2c \\ 0 & -2 & 1 & a \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & c \\ 0 & 1 & -1 & b-2c \\ 0 & 0 & -1 & a+2b-4c \end{array}\right]$$

$$\xrightarrow[\substack{R_3 = -R_3 \\ R_1 = R_1 - R_3 \\ R_2 = R_2 + R_3}]{\cancel{R_2 = R_2 + R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & c+a+2b-4c \\ 0 & 1 & 0 & -a-b+2c \;\; \cancel{3b-7c+a} \\ 0 & 0 & 1 & 4c-a-2b \end{array}\right]$$

$$\therefore\ c_1 = \cancel{3c} \; a + 2b - 3c$$

$$c_2 = \cancel{a + 3b - 6c} \; -a - b + 2c$$

$$c_3 = \cancel{4c - a + 2b} \; \cancel{b - 2c} \; 4c - a - 2b$$

Therefore, it can be said that B' spans $R3$ and is also linearly independent because if ~~a~~ $(a, b, c) = 0$ then $(c_1, c_2, c_3) = 0$.

$\therefore$ B' is a base for R^3.

(Ans)

$\ddot{u}$

Given, $V = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$

Now,

the augmented matrix relative to base B is :-

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 2 & 4 & 2 \\ 4 & 0 & 0 & 4 \end{array}\right]$$

From (i), we found that, for base B,

$$c_1 = \frac{c}{4}, \quad c_2 = \frac{b-2a}{4}, \quad c_3 = \frac{b+2a-c}{8}$$

Here, $a = 9$, $b = 2$, $c = 4$

$\therefore c_1 = 1$, $c_2 = -4$, $c_3 = 2$

$\therefore$ The coordinate matrix of v relative to base B is $[x(v)]_B = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$

Again,

From 1(i) we found that, for base B',

$c_1 = a + 2b - 3c$; $c_2 = -a - b + 2c$

$c_3 = -a - 2b + 4c$

Now, $a = 9$, $b = 2$, $c = 4$.

$\therefore c_1 = 1$; $c_2 = -3$; $c_3 = 3$

$\therefore$ Coordinate matrix of $v = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$ ~~wrt~~

relative to base B' is

$$[x(v)]_{B'} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}.$$

(Ans)

$$\underline{\underline{iii}}$$

$$[B' \mid B] = \cancel{\begin{matrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 4 & 0 & 0 \end{matrix}} \left[\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 & 2 & 4 \\ 1 & 0 & 1 & 4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - 2R_1]{R_1 \rightleftharpoons R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & 0 & 0 \\ 0 & 1 & -1 & -6 & 2 & 4 \\ 0 & -2 & 1 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = -R_3]{R_3 = R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & 0 & 0 \\ 0 & 1 & -1 & -6 & 2 & 4 \\ 0 & 0 & 1 & 11 & -3 & -10 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - R_3]{R_2 = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 3 & 10 \\ 0 & 1 & 0 & 5 & -1 & -6 \\ 0 & 0 & 1 & 11 & -3 & -10 \end{array}\right]$$

$$\therefore\ P_{B\to B'} = \begin{bmatrix} -7 & 3 & 10 \\ 5 & -1 & -6 \\ 11 & -3 & -10 \end{bmatrix}$$

(Ans)

iv

$$P_{B\to B'}\,[x(v)]_B$$

$$= \begin{bmatrix} -7 & 3 & 10 \\ 5 & -1 & -6 \\ 11 & -3 & -10 \end{bmatrix} \times \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7\times1 + 3\times(-4) + 10\times(2) \\ 5\times1 + (-1)\times(-4) + (-6)\times2 \\ 11\times1 + (-3)\times(-4) + (-10)\times2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

$$= [x(v)]_{B'}$$

[verified]

Ans to Q.No- 2

i

The augmented matrix from set B is such

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & -1 & b \\ 0 & 1 & 0 & 1 & c \\ 1 & 0 & 0 & 1 & d \end{array}\right] \xrightarrow[R_3 = R_3 - R_2]{R_4 = R_4 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & -1 & b \\ 0 & 0 & -1 & 2 & c-b \\ 0 & 0 & -1 & 0 & d-a \end{array}\right]$$

$$\xrightarrow[\substack{R_1 = R_1 - R_3 \\ R_2 = R_2 - R_3 \\ R_4 = R_4 - R_3}]{R_3 = -R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & a-b+c \\ 0 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & -2 & b-c \\ 0 & 0 & 0 & -2 & d-a-c+b \end{array}\right]$$

$$\xrightarrow[\substack{R_2 = R_2 - R_4 \\ R_3 = R_3 + 2R_4}]{\substack{R_4 = -\frac{1}{2}R_4 \\ R_1 = R_1 - 3R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{-a+b-c+3d}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b+c+d}{2} \\ 0 & 0 & 1 & 0 & a-d \\ 0 & 0 & 0 & 1 & \frac{a-b+c-d}{2} \end{array}\right]$$

Therefore, it can be said that B spans $M_{2,2}$ and is also ~~line~~ linearly independent because if $(a,b,c,d)=0$, then, $(c_1, c_2, c_3, c_4)=0$.

$\therefore$ B is a base for $M_{2,2}$.

Again, augmented matrix from set B' is such

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & a \\ 1 & 1 & 2 & 2 & b \\ 0 & 1 & 1 & 1 & c \\ 1 & 0 & 0 & 1 & d \end{array}\right] \xrightarrow[R_4 = R_4 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & a \\ 0 & 1 & 1 & 0 & b-a \\ 0 & 1 & 1 & 1 & c \\ 0 & 0 & -1 & -1 & d-a \end{array}\right]$$

$$\xrightarrow[\substack{\cancel{R_3 \rightleftharpoons R_4} \\ R_4 = -R_4}]{R_3 = R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & a \\ 0 & 1 & 1 & 0 & b-a \\ 0 & 0 & 0 & 1 & c-b+a \\ 0 & 0 & 1 & 1 & a-d \end{array}\right] \xrightarrow[R_1 = R_1 - R_3]{R_3 \rightleftharpoons R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & d \\ 0 & 1 & 1 & 0 & b-a \\ 0 & 0 & 1 & 1 & a-d \\ 0 & 0 & 0 & 1 & c-b+a \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - R_4]{R_2 = R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -a+b-c+d \\ 0 & 1 & 0 & -1 & -2a+b+d \\ 0 & 0 & 1 & 1 & a-d \\ 0 & 0 & 0 & 1 & c-b+a \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - R_4]{R_2 = R_2 + R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -a+b-c+d \\ 0 & 1 & 0 & 0 & -a+c+d \\ 0 & 0 & 1 & 0 & b-c-d \\ 0 & 0 & 0 & 1 & c-b+a \end{array}\right]$$

Therefore, it can be said that B' spans $M_{2,2}$ and is also linearly independent because if $(a,b,c,d) = 0$ then $(c_1, c_2, c_3, c_4) = 0$

$\therefore$ B' is a basis for $M_{2,2}$.

(Ans.)

ii

Here, $a = 1$, $b = -1$, $c = -2$, $d = 3$

$\therefore$ Coordinate matrix relative to basis

B is $[x(v)]_B$.

From (i) we got,

$$c_1 = \frac{-a+b-c+3d}{2} = \frac{9}{2}$$

$$c_2 = \frac{-a+b+c+d}{2} = -\frac{1}{2}$$

$$c_3 = a-d = -2$$

$$c_4 = \frac{a-b+c-d}{2} = -\frac{3}{2}$$

$$\therefore [x(A)]_B = \begin{bmatrix} 9/2 \\ -1/2 \\ -2 \\ -3/2 \end{bmatrix}$$

Again, for B' we get from (i)

$c_1 = -a+b-c+d = 3$

$c_2 = -a+c+d = 0$

$c_3 = b-c-d = -2$

$c_4 = a-b+c = 0$

$\therefore$ Coordinate matrix relative to basis

B' is $[x(A)]_{B'} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ (Ans)

iii

$$[B' \mid B] = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_4 = R_4 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 = -R_4]{R_3 = R_3 - R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - R_3]{R_3 \rightleftarrows R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - R_4]{R_2 = R_2 - R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 & -1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - R_4]{R_2 = R_2 + R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 3 \end{array}\right]$$

$\therefore$ $\cancel{P_{B \to B'} [x(A)]_B}$

$$P_{B \to B'} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -3 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

iv

$$P_{B \to B'} [x(A)]_B = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -3 \\ 1 & 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 9/2 \\ -1/2 \\ -2 \\ -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 9/2 + 0 \times (-1/2) + 0(-2) + (-2)(-3/2) \\ 0 \times 9/2 + 1(-1/2) + (-1)(-2) + (1)(-3/2) \\ -1 \times 9/2 + 0 \times (-1/2) + (1)(-2) + (-3)(-3/2) \\ 1 \times 9/2 + 0 + 0 + 3(-3/2) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} = [x(A)]_{B'}$$

[verified]

4.5

Ans. to Q No - 3

i

The augmented matrix of set B is such

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow[R_3 = R_3 - R_2]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 2 & c-b+a \end{array}\right]$$

$$\xrightarrow{R_3 = \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 1 & \frac{c-b+a}{2} \end{array}\right] \xrightarrow[R_2 = R_2 + R_3]{R_1 = R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a-c+b}{2} \\ 0 & 1 & 0 & \frac{-a+b+c}{2} \\ 0 & 0 & 1 & \frac{c-b+a}{2} \end{array}\right]$$

Therefore, it can be said that B spans P_2 and is also linearly independent.

$\therefore$ B is a basis for P_2.

The augmented matrix of set B' is.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 0 & -2 & b \\ 1 & -1 & 0 & c \end{array}\right] \xrightarrow{R_2 \rightleftarrows R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & c \\ 0 & 0 & -2 & b \end{array}\right]$$

$$\xrightarrow[R_3 = -\frac{R_3}{2}]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & c-a \\ 0 & 0 & 1 & \frac{-b}{2} \end{array}\right] \xrightarrow{R_2 = -\frac{R_2}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & \frac{1}{2} & \frac{a-c}{2} \\ 0 & 0 & 1 & \frac{-b}{2} \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{a+c}{2} \\ 0 & 1 & \frac{1}{2} & \frac{a-c}{2} \\ 0 & 0 & 1 & -\frac{b}{2} \end{array}\right] \xrightarrow[R_2 = R_2 - \frac{R_3}{2}]{R_1 = R_1 - \frac{R_3}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2a+2c+b}{4} \\ 0 & 1 & 0 & \frac{2a-2c+b}{4} \\ 0 & 0 & 1 & \frac{-b}{2} \end{array}\right]$$

Therefore, it is seen that B' spans P_2 and is also linearly independent.

$\therefore$ B' is a basis for P_2.

ii

Given, $P(x) = 7 + 4x - x^2$

$\therefore a = 7,\ b = 4,\ c = -1$

From (i) we found that, for B,

$c_1 = \dfrac{a-c+b}{2}$; $c_2 = \dfrac{-a+b+c}{2}$; $c_3 = \dfrac{a-b+c}{2}$

$= 6$ $\qquad = -2$ $\qquad = 1$

$$\therefore [x(p)]_B = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

Again, from (i) we found that, for B'

$c_1 = \dfrac{2a+2c+b}{4}$; $c_2 = \dfrac{2a-2c+b}{4}$; $c_3 = \dfrac{-b}{2}$

$= 4$ $\qquad = 5$ $\qquad = -2$

$$\therefore [x(p)]_{B'} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}.$$

$\underline{iii}$

$$[B' \mid B] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrows R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = -\frac{R_3}{2}]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{R_2}{2}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 1/2 & 1 \\ 0 & 1 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - \frac{R_3}{2}]{R_1 = R_1 - \frac{R_3}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 3/4 & 1 \\ 0 & 1 & 0 & 3/4 & -1/4 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 \end{array}\right]$$

$$\therefore P_{B \to B'} = \begin{bmatrix} 3/4 & 3/4 & 1 \\ 3/4 & -1/4 & 0 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

(iv)

$$P_{B \to B'} \times [x(p)]_B = \begin{bmatrix} 0.75 & 0.75 & 1 \\ 0.75 & -0.25 & 0 \\ -0.50 & -0.50 & 0 \end{bmatrix} \times \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \times 6 - 0.75 \times 2 + 1 \times 1 \\ 0.75 \times 6 + 0.25 \times 2 + 0 \times 1 \\ -0.50 \times 6 + 0.50 \times 2 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

$$= [x(p)]_{B'}$$

[verified]

Ans. to Q No - 4

(i)

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix} \xrightarrow[\substack{R_3 = R_3 - 3R_1 \\ R_4 = R_4 - 4R_1}]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow[R_4 = R_4 - R_2]{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{R_4 = R_4 + 2R_3} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

~~Rest~~ The non-zero row vectors of B in row echelon form are: $\omega_1 = (1, 2, 1, 0, 0)$, $\omega_2 = (0, 1, -1, 1, 0)$, $\omega_3 = (0, 0, 0, 1, -2)$

$\therefore R(A) = \text{rowspace}(A) = R(B) = R^3.$

Again, we know, $C(A) = R(A^T)$

Now, $A^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 9 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

$\xrightarrow[R_4 = R_4 - R_2]{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow[R_4 \rightleftharpoons R_5]{R_3 \rightleftharpoons R_4} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_4 = R_4 + 2R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= B$

The non zero row vectors of B in row echelon form are: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

$\therefore C(A) = C(B) = R(A^T)$
$= R^3$

(Ans)

ii

The basis for "row space" of A is

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\} \quad [\text{from (i)}]$$

The basis for "column space" of A

is $$S' = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\} \quad [\text{from (i)}]$$

iii

From (ii) we can see that,

$$\dim(R(A)) = 3 \quad [\because S \text{ contains 3 vectors}]$$

and,

$$\dim(C(A)) = 3 \quad [\because S' \text{ contains 3 vectors}]$$

iv

$$\dim(R(A)) = \dim(C(A)) = 3$$

Hence, ~~rak~~ rank $(A) = 3$.

(Ans)

Ans. to Q No - 5

From 4(i), we get,

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the required augmented matrix will be

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 = R_1 + 2R_3 \\ R_2 = R_2 - R_3}]{R_1 = R_1 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now, the matrix gives us the following homogenous system.

$$x_1 + 3x_3 - 4x_5 = 0$$

$$x_2 - x_3 + 2x_5 = 0$$

$$x_4 - 2x_5 = 0$$

Since, x_3 and x_5 are free variables, we set, $x_3 = s$ and $x_5 = t$ where, $s, t \in R$.

Then, $x_1 = -3s + 4t$

$x_2 = s - 2t$

$x_4 = 2t$

Therefore, the solution vector x can be written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3s+4t \\ s-2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

which gives us the nullspace, $N(A) = \left\{ s\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

(Ans)

ii

A basis of $N(A)$ is the set of linearly independent vectors in $N(A)$

$$\therefore \text{ Basis, } B = \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$\therefore$ Dimension of $N(A) = 2$ (Ans) iii

From Definition 11.2.4,

$$\dim(R(A)) = \dim(C(A)) = \text{rank}(A)$$

From theoream 2,

$$\text{row Rank}(A) = \text{colum Rank}(A) = \text{rank}(A)$$

We know, $\dim(N(A)) = \text{nullity}(A)$

$$\therefore \dim(C(A)) + \dim(N(A))$$

$$= \boxed{\text{rank}(A) + \text{nullity}(A)}$$

$$= \boxed{n}$$; where n is the number of colums of given matrix

$\boxed{\text{rank-nullity theoream}}$

iv

The required augmented matrix is –

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 5 & 1 & 1 & 0 & -1 \\ 3 & 7 & 2 & 2 & -2 & 2 \\ 4 & 9 & 3 & -1 & 4 & -3 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 = R_3 - 3R_1 \\ R_4 = R_4 - 4R_1}]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & -3 \\ 0 & 1 & -1 & 2 & -2 & -1 \\ 0 & 1 & -1 & -1 & 4 & -7 \end{array}\right] \xrightarrow[\substack{R_3 = R_3 - R_2 \\ R_4 = R_4 - R_2}]{R_1 = R_1 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & -2 & 0 & 7 \\ 0 & 1 & -1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -2 & 4 & -4 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 = R_2 - R_3 \\ R_4 = R_4 + 2R_3}]{R_1 = R_1 + 2R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -4 & 11 \\ 0 & 1 & -1 & 0 & 2 & -5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here, $x_3, x_5 \rightarrow$ free

$x_1 = 11 - 3x_3 + 4x_5$

$x_2 = -5 + x_3 - 2x_5$

$x_4 = 2 + 2x_5$

$\therefore$ The system is consistent.

V

From (iv) we got,

$$x_1 = 11 - 3x_3 + 4x_5$$

$$x_2 = -5 + x_3 - 2x_5$$

$$x_4 = 2 + 2x_5$$

If we consider $x_3 = s$ and $x_5 = t$, then the complete solution is given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 11 - 3s + 4t \\ -5 + s - 2t \\ s \\ 2 + 2t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -5 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right) = x_p + x_h$$

(Ans)