2.4 Exact Differential Equations
Solutions to the Selected Problems

Standard Form

$$M(x,y)dx + N(x,y)dy = 0$$

Exactness Condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Example Test whether the following differential equation is *exact* or not. If exact then solve.

$$2xydx + x^2dy = 0$$

Solution

Let us denote

$$M(x,y) = 2xy, \qquad N(x,y) = x^2$$

Now differentiating $M(x,y)$ and $N(x,y)$ partially with respect to y and x respectively, we obtain,

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2x$$

Since,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

the given differential equation is an *exact* one.

To find the solution let us have a function f of x and y such that,

$$\frac{\partial f}{\partial x} = 2xy \tag{1}$$

$$\frac{\partial f}{\partial y} = x^2 \tag{2}$$

Now integrating the eq. (1) partially with respect to x we get,

$$f(x,y) = x^2y + g(y) \tag{3}$$

where $g(y)$ is an arbitrary function of y only.

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Now differentiating above equation partially with respect to y we get,

$$\frac{\partial f}{\partial y} = x^2 + \frac{dg}{dy} \tag{4}$$

Now comparing the eqs. (2) and (4), we get,

$$\frac{dg}{dy} = 0$$

$$g(y) = A$$

where A is an arbitrary constant of integration.

Now the eq. (3) becomes,

$$f(x, y) = x^2 y + A$$

Hence, the solution is

$$x^2 y + A = K$$
$$\boxed{x^2 y = C}$$

Example Test whether the following differential equation is *exact* or not. If exact then solve.

$$(x + 2xy^3)dx + (1 + 3x^2y^2)dy = 0$$

Solution (Detailed)

Let us denote

$$M(x, y) = x + 2xy^3, \qquad N(x, y) = 1 + 3x^2y^2$$

Now differentiating $M(x, y)$ and $N(x, y)$ partially with respect to y and x respectively, we obtain,

$$\frac{\partial M}{\partial y} = 0 + (2x)(3y^2) = 6xy^2$$

$$\frac{\partial N}{\partial x} = 0 + (6x)(y^2) = 6xy^2$$

Since,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

the given differential equation is an *exact* one.

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Solutions to the Selected Problems

To find the solution let us have a function f of x and y such that,

$$\frac{\partial f}{\partial x} = x + 2xy^3 \tag{1}$$

$$\frac{\partial f}{\partial y} = 1 + 3x^2y^2 \tag{2}$$

Now integrating the eq. (1) partially with respect to x we get,

$$f(x,y) = \frac{1}{2}x^2 + x^2y^3 + g(y) \tag{3}$$

where $g(y)$ is an arbitrary function of y only.

Now differentiating above equation partially with respect to y we get,

$$\frac{\partial f}{\partial y} = 0 + 3x^2y^2 + \frac{dg}{dy}$$

$$\therefore \frac{\partial f}{\partial y} = 3x^2y^2 + \frac{dg}{dy} \tag{4}$$

Now comparing the eqs. (2) and (4), we get,

$$\frac{dg}{dy} = 1$$

$$g(y) = y$$

Now the eq. (3) becomes,

$$f(x,y) = \frac{1}{2}x^2 + x^2y^3 + y$$

Hence, the solution is

$$\boxed{\frac{1}{2}x^2 + x^2y^3 + y = C}$$

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2.4 Exact Differential Equations
Solutions to the Selected Problems

An alternative and faster way to solve an exact differential equation follows.

Example Test whether the following differential equation is *exact* or not. If exact then solve.

$$(x + 2xy^3)dx + (1 + 3x^2y^2)dy = 0$$

Solution (Shortcut)

Let us denote

$$M(x,y) = x + 2xy^3, \qquad N(x,y) = 1 + 3x^2y^2$$

Now differentiating $M(x,y)$ and $N(x,y)$ partially with respect to y and x respectively, we obtain,

$$\frac{\partial M}{\partial y} = 0 + (2x)(3y^2) = 6xy^2$$

$$\frac{\partial N}{\partial x} = 0 + (6x)(y^2) = 6xy^2$$

Since,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

the given differential equation is an *exact* one.

Now the solution of the given exact differential equation may be found from the following equation:

$$\int M(x,y)\partial x + \int N_1(y)dy = C \tag{1}$$

where $\int M(x,y)\partial x$ denotes the partial integration of $M(x,y)$ with respect to x only and $N_1(y)$ contains the x-free terms of $N(x,y)$. In our example,

$$M(x,y) = x + 2xy^3, \qquad N(x,y) = 1 + 3x^2y^2$$

$$N_1(y) = 1$$

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Therefore, the eq. (1) becomes

$$\int (x + 2xy^3)\partial x + \int 1\, dy = C$$

$$\boxed{\frac{1}{2}x^2 + x^2y^3 + y = C}$$

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