5.3 Integration by Substitution
Solutions to the Selected Problems

Formula

If

$$\int f(x)\,dx = F(x)$$

$$\boxed{\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big)}$$

Method of ***u*-substitution**

If $u = g(x)$ then $du = g'(x)dx$

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du = F(u) = F\big(g(x)\big) + C$$

15–56. Evaluate the integrals using appropriate substitutions.

15.

$$\int (4x-3)^9\,dx$$

Solution

Let us assume

$$u = 4x - 3$$

$$du = 4dx$$

$$\int (4x-3)^9\,dx = \int u^9 \frac{1}{4} du$$

$$= \frac{1}{4}\int u^9\,du$$

$$= \frac{1}{4}\frac{u^{10}}{10} + C$$

$$= \frac{1}{40}(4x-3)^{10} + C$$

$$\boxed{\int (4x-3)^9\,dx = \frac{1}{40}(4x-3)^{10} + C}$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University

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20.

$$\int \sec^2 5x \, dx$$

Solution

Let us change the variable with the following transformation:

$$u = 5x$$

$$du = 5dx$$

$$\int \sec^2 5x \, dx = \int \sec^2 u \frac{1}{5} du = \frac{1}{5}\int \sec^2 u \, du = \frac{1}{5}\tan 5x + C$$

$$\boxed{\int \sec^2 5x \, dx = \frac{1}{5}\tan 5x + C}$$

23.

$$\int \frac{1}{\sqrt{1-4x^2}} dx$$

Solution

$$\int \frac{1}{\sqrt{1-4x^2}} dx = \int \frac{1}{\sqrt{1-(2x)^2}} dx$$

Let us assume,

$$u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2} du$$

$$\begin{aligned} \int \frac{1}{\sqrt{1-4x^2}} dx &= \int \frac{1}{\sqrt{1-u^2}} \frac{1}{2} du \\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{1}{2}\sin^{-1} u + C \end{aligned}$$

$$\left[\text{Formula: } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\boxed{\int \frac{1}{\sqrt{1-4x^2}} dx = \frac{1}{2}\sin^{-1}(2x) + C}$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University

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29.

$$\int \frac{x^3}{(5x^4+2)^3}\,dx$$

Solution

Let us assume,

$$u = 5x^4 + 2 \Rightarrow du = 20x^3 dx \Rightarrow x^3 dx = \frac{1}{20} du$$

$$\begin{aligned}
\int \frac{x^3}{(5x^4+2)^3}\,dx &= \int \frac{1}{(5x^4+2)^3} x^3 dx \\
&= \int \frac{1}{u^3}\left(\frac{1}{20} du\right) \\
&= \frac{1}{20}\int \frac{1}{u^3} du \\
&= \frac{1}{20}\int u^{-3}\, du \\
&= \frac{1}{20}\cdot\frac{u^{-3+1}}{-3+1} + C \qquad \left[\text{Formula: } \int x^n\,dx = \frac{x^{n+1}}{n+1}; n \neq -1\right] \\
&= -\frac{1}{40}\cdot\frac{1}{u^2} + C \\
&= -\frac{1}{40(5x^4+2)^2} + C
\end{aligned}$$

$$\boxed{\int \frac{x^3}{(5x^4+2)^3}\,dx = -\frac{1}{40(5x^4+2)^2} + C}$$

31.

$$\int e^{\sin x}\cos x\,dx$$

Solution

Let us assume,

$$u = \sin x \Rightarrow du = \cos x\,dx$$

$$\int e^{\sin x}\cos x\,dx = \int e^u\,du = e^u + C$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University

$$\boxed{\int e^{\sin x}\cos x\,dx = e^{\sin x} + C}$$

33.

$$\int x^2 e^{-2x^3}\,dx$$

Solution

Let us assume,

$$u = -2x^3 \Rightarrow du = -6x^2dx \Rightarrow x^2dx = -\frac{1}{6}du$$

$$\int x^2 e^{-2x^3}\,dx = \int e^{-2x^3}\,x^2dx$$

$$= \int e^u\left(-\frac{1}{6}du\right)$$

$$= -\frac{1}{6}\int e^u\,du$$

$$= -\frac{1}{6}e^u + C$$

$$= -\frac{1}{6}e^{-2x^3} + C$$

$$\boxed{\int x^2 e^{-2x^3}\,dx = -\frac{1}{6}e^{-2x^3} + C}$$

34.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$$

Solution

Let us assume,

$$u = e^x - e^{-x} \Rightarrow du = (e^x + e^{-x})dx$$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \int \frac{1}{u}du$$

$$= \ln|u| + C \qquad \left[\text{Formula: } \int \frac{1}{x}dx = \ln|x|\right]$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University

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$$\boxed{\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln|e^x - e^{-x}| + C}$$

35.

$$\int \frac{e^x}{1+e^{2x}}\,dx$$

Solution

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx$$

Let us assume,

$$u = e^x \Rightarrow du = e^x dx$$

$$\int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{1}{1+u^2}\,du$$

$$= \tan^{-1} u + C \qquad \left[\text{Formula: } \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\boxed{\int \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1}(e^x) + C}$$

36.

$$\int \frac{t}{1+t^4}\,dt$$

Solution

$$\int \frac{t}{1+t^4}\,dt = \int \frac{t}{1+(t^2)^2}\,dt$$

Let us assume,

$$u = t^2 \Rightarrow du = 2t\,dt \Rightarrow t\,dt = \frac{1}{2}du$$

$$\int \frac{t}{1+(t^2)^2}\,dt = \frac{1}{2}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\tan^{-1} u + C \qquad \left[\text{Formula: } \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\boxed{\int \frac{t}{1+t^4}\,dt = \frac{1}{2}\tan^{-1}(t^2) + C}$$

37.

$$\int \frac{\sin(5/x)}{x^2}\,dx$$

Solution

Let us assume,

$$u = \frac{5}{x} \Rightarrow du = -\frac{5}{x^2}\,dx \Rightarrow \frac{1}{x^2}\,dx = -\frac{1}{5}\,du$$

$$\begin{aligned}
\int \frac{\sin(5/x)}{x^2}\,dx &= \int \sin\left(\frac{5}{x}\right) \cdot \frac{1}{x^2}\,dx \\
&= \int \sin u \left(-\frac{1}{5}\right) du \\
&= -\frac{1}{5}\int \sin u\,du \\
&= -\frac{1}{5}(-\cos u) + C \\
&= \frac{1}{5}\cos\left(\frac{5}{x}\right) + C
\end{aligned}$$

$$\boxed{\int \frac{\sin(5/x)}{x^2}\,dx = \frac{1}{5}\cos(5/x) + C}$$

39.

$$\int \cos^4 3t \sin 3t\,dt$$

Solution

Let us assume,

$$u = \cos 3t \Rightarrow du = -3\sin 3t\,dt \Rightarrow \sin 3t\,dt = -\frac{1}{3}\,du$$

$$\int \cos^4 3t \sin 3t\,dt = -\frac{1}{3}\int u^4\,du = -\frac{1}{3}\frac{u^5}{5} + C$$

$$= -\frac{1}{15}u^5 + C$$

$$= -\frac{1}{15}\cos^5 3t + C$$

$$\boxed{\int \cos^4 3t \sin 3t\, dt = -\frac{1}{15}\cos^5 3t + C}$$

41.

$$\int x \sec^2(x^2)\, dx$$

Solution

Let us assume,

$$u = x^2 \Rightarrow du = 2x\, dx \Rightarrow x\, dx = \frac{1}{2}du$$

$$\int x \sec^2(x^2)\, dx = \frac{1}{2}\int \sec^2(u)\, du$$

$$= \frac{1}{2}\tan u + C \qquad \left[\text{Formula: } \int \sec^2\theta\, d\theta = \tan\theta\right]$$

$$= \frac{1}{2}\tan(x^2) + C$$

$$\boxed{\int x \sec^2(x^2)\, dx = \frac{1}{2}\tan(x^2) + C}$$

45.

$$\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}}\, dx$$

Solution

Let us assume,

$$u = \tan x \Rightarrow du = \sec^2 x\ dx \Rightarrow x\, dx = \frac{1}{2}du$$

$$\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}}\, dx = \int \frac{1}{\sqrt{1 - \tan^2 x}} \sec^2 x\, dx$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(\tan x) + C$$

$$\boxed{\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = \sin^{-1}(\tan x) + C}$$

53.

$$\int \frac{y}{\sqrt{2y+1}}\,dy$$

Solution

Let us assume,

$$u = 2y + 1 \Rightarrow du = 2dy \Rightarrow dy = \frac{1}{2}du$$

$$\int \frac{y}{\sqrt{2y+1}}\,dy = \int \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\left(\frac{1}{2}du\right)$$

$$= \frac{1}{4}\int \frac{(u-1)}{\sqrt{u}}\,du$$

$$= \frac{1}{4}\int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du$$

$$= \frac{1}{4}\int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du$$

$$= \frac{1}{4}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$$

$$= \frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) + C$$

$$= \frac{1}{6}u^{\frac{3}{2}} - \frac{1}{2}u^{\frac{1}{2}} + C$$

$$\int \frac{y}{\sqrt{2y+1}}\,dy = \frac{1}{6}(2y+1)^{\frac{3}{2}} - \frac{1}{2}(2y+1)^{\frac{1}{2}} + C$$

$$= \frac{1}{6}[2y+1-3]\sqrt{2y+1} + C$$

$$= \frac{1}{3}(y-1)\sqrt{2y+1} + C$$

$$\boxed{\int \frac{y}{\sqrt{2y+1}}\,dy = \frac{1}{3}(y-1)\sqrt{2y+1} + C}$$

Alternative

Let us assume,

$$u^2 = 2y+1 \Rightarrow 2u\,du = 2dy \Rightarrow dy = u\,du$$

$$\int \frac{y}{\sqrt{2y+1}}\,dy = \int \frac{\frac{1}{2}(u^2-1)}{\sqrt{u^2}}\,u\,du$$

$$= \frac{1}{2}\int \frac{u^2-1}{u}u\,du$$

$$= \frac{1}{2}\int (u^2-1)\,du$$

$$= \frac{1}{2}\left(\frac{u^3}{3} - u\right) + C$$

$$= \frac{u^3}{6} - \frac{u}{2} + C$$

$$= \frac{1}{6}\left(\sqrt{2y+1}\right)^3 - \frac{1}{2}\sqrt{2y+1} + C$$

$$= \frac{1}{3}(y-1)\sqrt{2y+1} + C$$

$$\boxed{\int \frac{y}{\sqrt{2y+1}}\,dy = \frac{1}{3}(y-1)\sqrt{2y+1} + C}$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University

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54.

$$\int x\sqrt{4-x}\,dx$$

Solution

Let us assume,

$$u^2 = 4 - x \Rightarrow 2u\,du = -dx \Rightarrow dx = -2u\,du$$

$$\int x\sqrt{4-x}\,dx = \int (4-u^2)\sqrt{u^2}\,(-2u\,du) = -2\int (4-u^2)|u|u\,du$$

Since,

$$4 - x \geq 0 \Rightarrow u \geq 0, \qquad \sqrt{u^2} = |u| = u$$

$$\int x\sqrt{4-x}\,dx = -2\int (4-u^2)u^2\,du$$

$$= \int (2u^4 - 8u^2)\,du$$

$$= 2\frac{u^5}{5} - 8\frac{u^3}{3} + C$$

$$\int x\sqrt{4-x}\,dx = \frac{2}{5}\left(\sqrt{4-x}\right)^5 - \frac{8}{3}\left(\sqrt{4-x}\right)^3 + C$$

$$= \frac{2}{5}\left(\sqrt{4-x}\right)^4\sqrt{4-x} - \frac{8}{3}\left(\sqrt{4-x}\right)^2\sqrt{4-x} + C$$

$$= \frac{2}{5}(4-x)^2\sqrt{4-x} - \frac{8}{3}(4-x)\sqrt{4-x} + C$$

$$\boxed{\int x\sqrt{4-x}\,dx = \frac{1}{15}(6x^2 - 8x - 64)\sqrt{4-x} + C}$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University