4.2 Reduction of Order
Solutions to the Selected Problems

Formula

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = 0 \tag{1}$$

If $y_1(x)$ is a known solution of the eq. (1) then the other *linearly independent solution* $y_2(x)$ is

$$y_2(x) = u(x)y_1(x)$$

In Problems **1–16** the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0;\ y_1(x) = e^{2x}$

Solution

$$y'' - 4y' + 4y = 0$$

where

$$P(x) = -4, \qquad Q(x) = 4$$

The other linearly independent solution $y_2(x)$ is defined by

$$y_2(x) = u(x)y_1(x)$$

$$y_2(x) = u(x)e^{2x}$$

where $u(x)$ may be found from the following formula:

$$u(x) = \int \frac{e^{-\int P dx}}{y_1^2(x)} dx$$

$$= \int \frac{e^{\int 4dx}}{(e^{2x})^2} dx$$

$$= \int \frac{e^{4x}}{e^{4x}} dx$$

$$= x$$

$$y_2(x) = xe^{2x}$$

Therefore, the general solution is

$$\boxed{y(x) = C_1e^{2x} + C_2xe^{2x}}$$

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3. $y'' + 16y = 0;\ y_1(x) = \cos 4x$

Solution

$$y'' + 16y = 0$$

where

$$P(x) = 0, \qquad Q(x) = 16$$

The other linearly independent solution $y_2(x)$ is defined by

$$y_2(x) = u(x)y_1(x)$$

$$y_2(x) = u(x)e^{2x}$$

where $u(x)$ may be found from the following formula:

$$\begin{aligned} u(x) &= \int \frac{e^{-\int P\,dx}}{y_1^2(x)}\,dx \\ &= \int \frac{e^C}{(\cos 4x)^2}\,dx \\ &= A\int \sec^2 4x\,dx \\ &= A\tan 4x \end{aligned}$$

$$\begin{aligned} y_2(x) &= A\tan 4x \cos 4x \\ &= A\sin 4x \end{aligned}$$

Therefore, the general solution is

$$\boxed{y(x) = C_1\cos 4x + C_2\sin 4x}$$

9. $x^2y'' - 7xy' + 16y = 0;\ y_1(x) = x^4$

Solution

In standard form

$$y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0; \quad x > 0$$

where

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$$P(x) = -\frac{7}{x}, \qquad Q(x) = \frac{16}{x^2}$$

The other linearly independent solution $y_2(x)$ is defined by

$$y_2(x) = u(x)y_1(x)$$

$$y_2(x) = u(x)e^{2x}$$

where $u(x)$ may be found from the following formula:

$$u(x) = \int \frac{e^{-\int -\frac{7}{x}dx}}{y_1^2(x)}dx$$

$$= \int \frac{e^{7\ln x}}{(x^4)^2}dx$$

$$= \int \frac{x^7}{x^8}dx$$

$$= \int \frac{1}{x}dx$$

$$= \ln x$$

$$\therefore y_2(x) = x^4 \ln x$$

Therefore, the general solution is

$$\boxed{y(x) = C_1x^4 + C_2x^4 \ln x}$$

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