Chapter 3

Continuous Charge Distribution

The general philosophy of computing the electric field due to an arbitrary charge distribution is explained. Several examples with high degree of symmetry are explicitly computed as demonstration of the general philosophy.

3.1 Continuous Charge Distribution

So far we have only consider the electric field due to point charges. To find out the electric field due to multiple point charges we have used the superposition principle. In this way we found out what the electric field is on the axis of a dipole. We saw that at large distances (i.e, distances large in comparison to the size of the dipole) the electric field of the dipole falls off as $1/r^3$. This Result holds for all directions around the dipole.

Although we can continue this exercise with more than two charges what we really want to know is how to calculate the electric field due to a continuous distribution of charge. The reason for this is that primarily we are interested in systems and length scales which are much larger than the length scales of atoms and molecules.

So, although the fundamental charges are approximated by point charges we shall be dealing with cases where the charge density of a region will be averaged over many fundamental charges (see figure 3.1). With this in mind we shall assume that the charge distribution can the treated as a continuous quantity.

3.2 The General Setup

Suppose we have an arbitrary charge distribution. We can describe such a charge distribution by a function $\rho(x, y, z)$. We are assuming

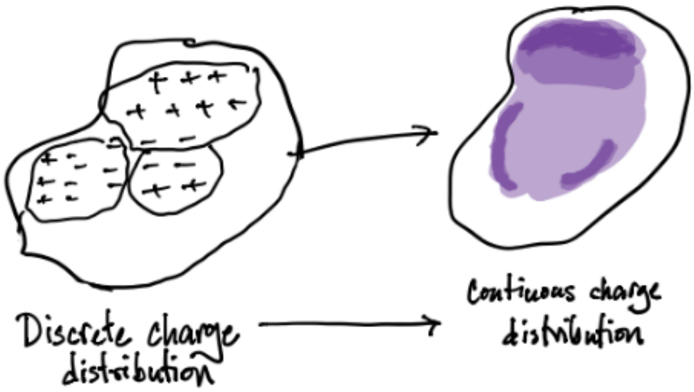

Figure 3.1: By taking the average of discrete charge distributions we are lead to a continuous charge distribution.

our charge distribution depends on x, y, z which implies that we have introduced a coordinate system. Then the amount of charge contained in a small region of space around the point whose coordinate is x, y, z would be given by:

$$dq = \rho(x, y, z)\, dx\, dy\, dz. \tag{3.1}$$

The problem is to find the electric field at a point P, at a distance from the charge distribution.

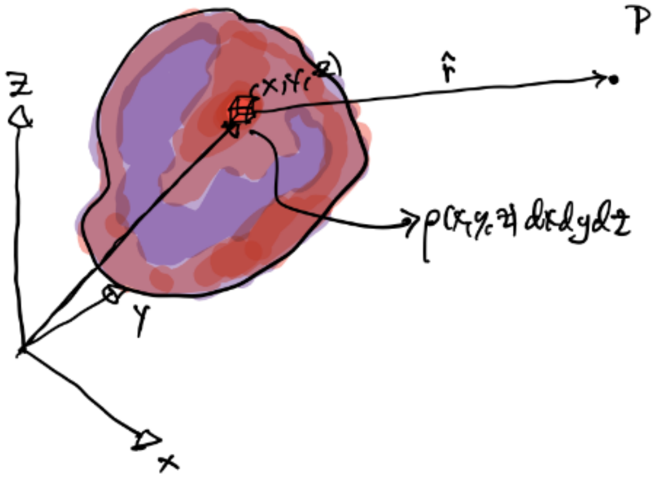

Figure 3.2: Dividing the charge distribution into small cubic cells.

The strategy we adopt is to divide the charge distribution into small cubic cells with volume $dx\,dy\,dz$. The distance r from such cells to the point P is large compared to the size of the cells so that from the point P the cells all look like point charges. So we can apply Coulomb's law to each cell and then add up all the contributions. But since we are considering cells whose sizes are infinitesimal, instead of adding up the cells we have to integrate with the limits that are appropriate to the shape of the charge distribution.

This is the general philosophy, but after we describe the general case, we shall do a few concrete examples to see how this philosophy is implemented in concrete situations.

The contribution to the electric field at point P due to the charge element $dq = \rho(x,y,z)\,dx\,dy\,dz$ is given by

$$\begin{aligned} \mathrm{d}\mathbf{E} &= \frac{k\,dq\hat{\mathbf{r}}}{r^2} \\ &= \frac{\rho(x,y,z)\hat{\mathbf{r}}\,dx\,dy\,dz}{r^2}. \end{aligned} \tag{3.2}$$

To find the net electric field at the point P due to all the change in our charge distribution we have to integrate over x, y, z with the appropriate limits. These limits will depend on the specifics of the geometry of our charge distribution. If we denote the region of space that the charge occupies by Σ then we can write the electric field at P due to all the charge in Σ by:

$$\mathbf{E}(p) = \iiint_{\text{source}} \mathrm{d}\mathbf{E} \tag{3.3}$$

$$= \iiint_{\Sigma} \frac{\rho(x,y,z)\hat{\mathbf{r}}(x,y,z,P)\,dx\,dy\,dz}{r^2(x,y,z,P)} \tag{3.4}$$

where we have explicitly denoted the (x,y,z) and P-dependency of the unit vector $\hat{\mathbf{r}}$ and the distance r between the charge element position and the point P.

If we explicitly introduce the coordinates of P as x', y', z' then we would have to write $\hat{\mathbf{r}}(x,y,z;x',y',z')$ and $P(x,y,z;x',y',z')$ but we shall refrain from doing so in the interest of brevity.

In the integral above, as we move about the charge distribution ρ, $\hat{\mathbf{r}}$ and r changes. And so for a general charge distribution we may not be able to do the integral explicitly. Below we shall see some simple examples where the integral can be performed exactly. These days, with the advent, of powerful computers one may evaluate the integral numerically for complicated charge distributions.

3.3 Examples

Let us do a few examples.

3.3.1 The electric field due to an infinite line of charge.

Although there are no infinite line of charge in the real world, we can approximate a long line of charge by an infinite line of charge. The geometry of the problem Is shown in the figure below.

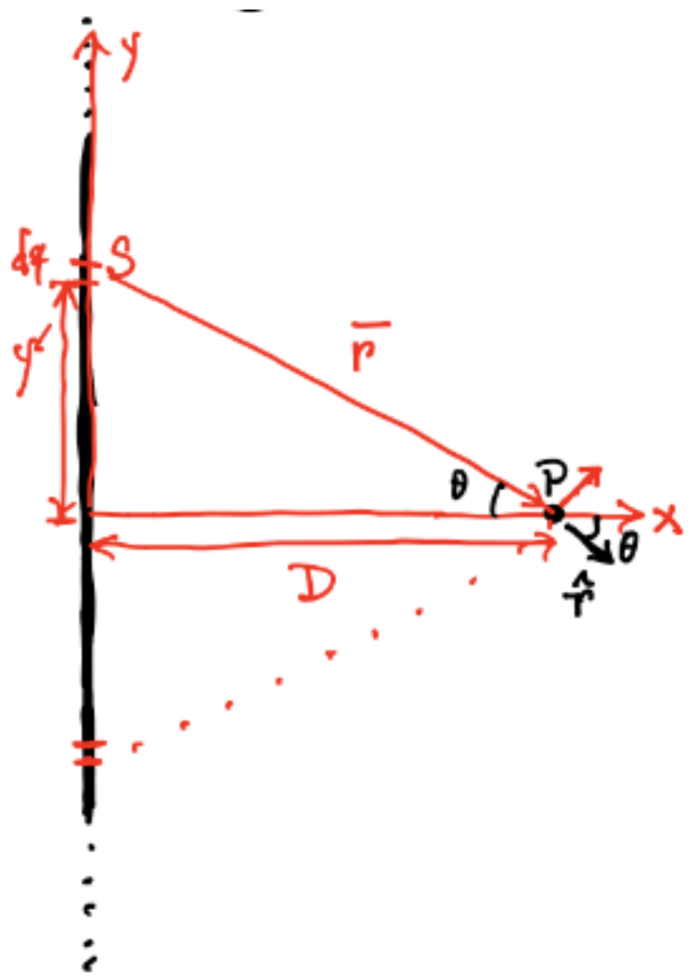

Figure 3.3: An infinite line of charge.

We assume that the charge per unit length is a constant and we denote it by ρ. The contribution to the electric field at point P due to the element of charge dq placed at point S at y' on the positive y-axis will be denoted by $d\mathbf{E}(P)$.

At this point, we want to make clear that y' will be our integration variable and the small charge element dq at the source S at distance y' is given by

$$dq = \rho\, dy'. \tag{3.5}$$

Then the contribution to the electric field at P due to dq at the point S will be given by

$$\mathrm{d}\mathbf{E}(p) = \frac{k\, dq\hat{\mathbf{r}}}{r^2} \tag{3.6}$$

where r is given by $r = \sqrt{y'^2 + D^2}$ and $\hat{\mathbf{r}} = \hat{\imath}\cos\theta - \hat{\jmath}\sin\theta$ with $\cos\theta = \frac{D}{\sqrt{y'^2+D^2}}$

Note that we have chosen our coordinate system in a way such that on the other side of the origin there a charge element dq placed at $-y'$ on the axis which contributes the following to the electric field at P:

$$\mathrm{d}\mathbf{E}(p) = \frac{k\, dq}{r'^2}\hat{\mathbf{r}}' \tag{3.7}$$

where dq is given by the same expression as for the d$\vec{E}$ element above except $\hat{\mathbf{r}}'$ is now given by $\hat{\mathbf{r}}' = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$ but $r' = r = \sqrt{y'^2 + D^2}$ and $\cos\theta = \frac{D}{\sqrt{y'^2+D^2}}$

Thus we see that if we consider the two elements together the $\hat{\mathbf{j}}$ components of the electric field will cancel each other out. So the resultant electric field will be in the $\hat{\mathbf{i}}$ direction. So the total electric field at the point P will be

$$\mathbf{E} = \int_{sources} \mathrm{d}\mathbf{E} \tag{3.8}$$

$$= 2\int \frac{k\rho(\hat{\mathbf{i}}\cos\theta)}{\sqrt{y'^2 + D^2}}\mathrm{d}q' \tag{3.9}$$

$$= 2k\rho\hat{\mathbf{i}}D\int \frac{\mathrm{d}y'}{(y'^2 + D^2)^{3/2}} \tag{3.10}$$

$$\vec{E} = \frac{2k\rho}{D}\hat{\mathbf{i}} \tag{3.11}$$

3.3.2 The electric field due to a charged ring

Let us consider a ring with uniform charge density and we have to find the electric field on a point on the axis of the ring.

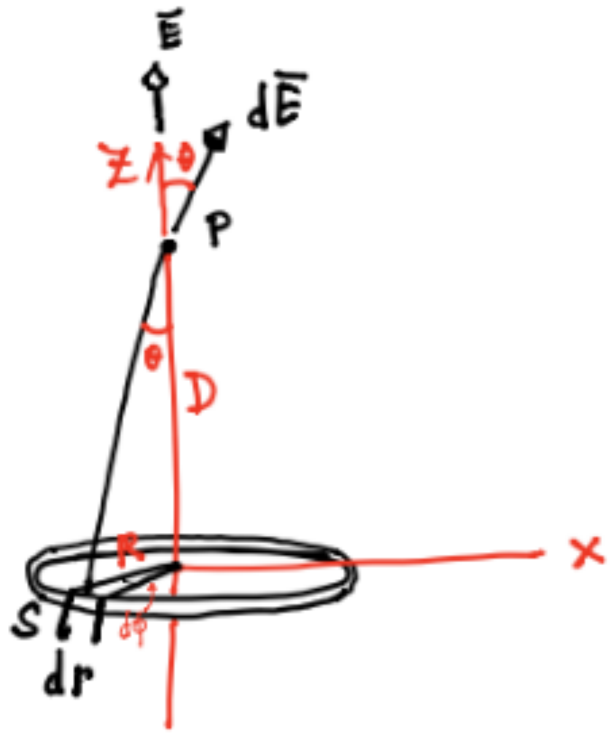

Figure 3.4: The electric field due to a charged ring

By the symmetry argument similar to the previous problem we can see that the resultant electric field will along the z axis. The component of the electric field d**E** due to the element of charge dr at S in the z direction will be given by:

$$\mathrm{d}E_z = \frac{k\lambda \mathrm{d}r}{(R^2+D^2)} \frac{D}{(R^2+D^2)^{1/2}} \tag{3.12}$$

$$= \frac{k\lambda D}{(R^2+D^2)^{3/2}} \mathrm{d}r \tag{3.13}$$

The element dr is given by d$r = R\mathrm{d}\phi$ and so we have

$$\mathbf{E} = \hat{\mathbf{j}} E_z = \hat{\mathbf{j}} \frac{k\lambda DR}{(R^2+D^2)^{3/2}} \int_0^{2\pi} \mathrm{d}\phi \tag{3.14}$$

$$= \hat{\mathbf{j}} 2\pi \frac{k\lambda DR}{(R^2+D^2)^{3/2}} \tag{3.15}$$

note $2\pi R\lambda = Q$ is the total charge and so:

$$\mathbf{E} = \frac{\hat{\mathbf{j}}kQD}{(R^2 + D^2)^{3/2}}. \tag{3.16}$$

It is interesting to note that when we are at large distances from the charged ring, i.e. $D \gg R$, then we obtain the result of a point charge:

$$\mathbf{E} = \frac{kQ\hat{\mathbf{j}}}{D^2} \tag{3.17}$$

3.3.3 Electric field due to a uniformly charged disk

Suppose we have to find the electric field due to a charged disk on a point on its axis. This problems can be easily solved by observing that we can think of a disk as being made out of a large number of thin concentric rings of charges.

suppose we have a disk of radius R where the charge per unit area is σ. Then consider a thin disk of thickness $\mathrm{d}r$ at radius r . Then using the previous section's result we can see that the contribution to the electric field at point P is

$$\mathrm{dE} = \frac{\hat{\mathbf{j}}kD(2\pi r\sigma \mathrm{d}r)}{(r^2 + D^2)^{3/2}} \tag{3.18}$$

where $(2\pi r\sigma \mathrm{d}r)$ is what we called the total charge for the ring. To find the field due to the whole disk we just integrate the above expression from zero to R:

$$\begin{aligned} \vec{E} &= \hat{\mathbf{j}}\, kD2\pi\sigma \int_0^R \frac{r\mathrm{d}r}{(r^2 + D^2)^{3/2}} \\ &= \hat{\jmath}\, 2\pi\sigma k \left(1 - \frac{D}{(R^2 + D^2)^{1/2}}\right) \end{aligned} \tag{3.19}$$

There are two interesting limits of this formula. One is when

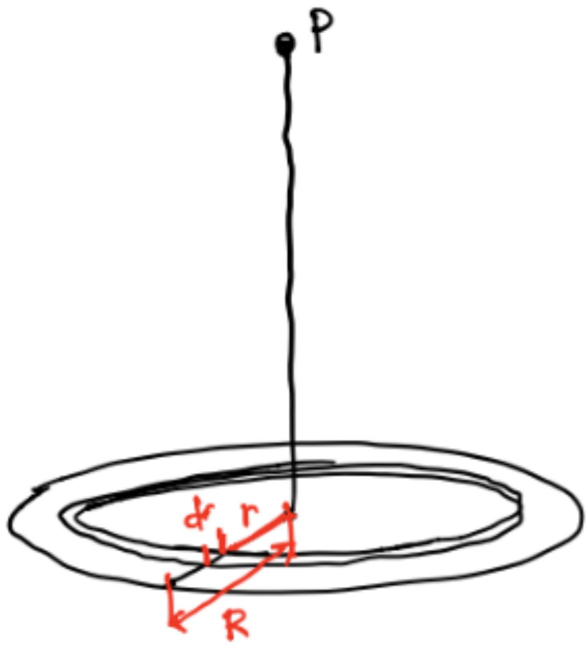

Figure 3.5: figure 5: The electric field due to a charged disk

$D \gg R$ then the factor inside the brackets become:

$$\begin{aligned} & 1-\frac{D}{(R^2+D^2)^{1/2}} \\ =& 1-\frac{1}{(1+R^2/D^2)^{1/2}} \\ =& 1-\left(1+\frac{R^2}{D^2}\right)^{-1/2} \\ =& 1-1+\frac{1}{2}\frac{R^2}{D^2}+\dots \\ \approx& \frac{1}{2}\frac{R^2}{D^2}. \end{aligned} \tag{3.20}$$

Thus far away from the disk we recover the result for the point charge.

But more interesting when $D \ll R$, i.e, very close to the disk we get (by taking $R \to \infty$)

$$\vec{E} = \hat{\mathbf{j}} 2\pi\sigma k. \tag{3.21}$$

This is the electric field due to an infinite sheet of uniform charge. It is independent of any distance.