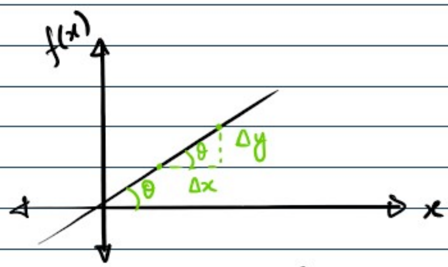
Numerical Differentiation

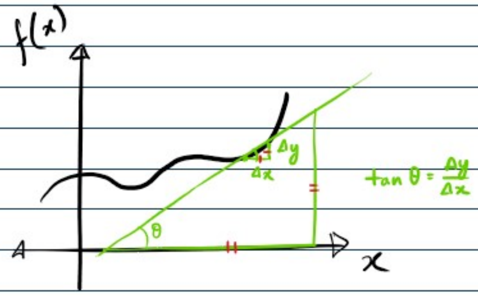
Wednesday, April 7, 2021 4:26 PM

Chapter 03

General idea of differentiation:



$$f'(x) = \frac{\Delta y}{\Delta x} = \tan\theta$$



Formal definition of derivative:

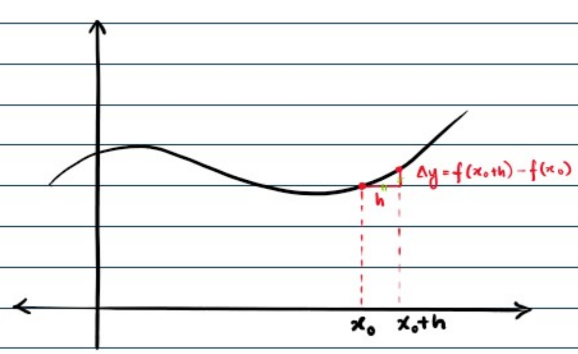
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

→ Trivial Approximation

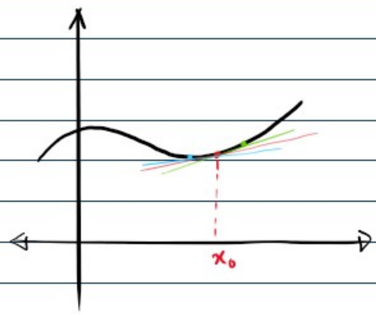
→ eqt^n ①

The smaller the 'h', the answer will be more accurate



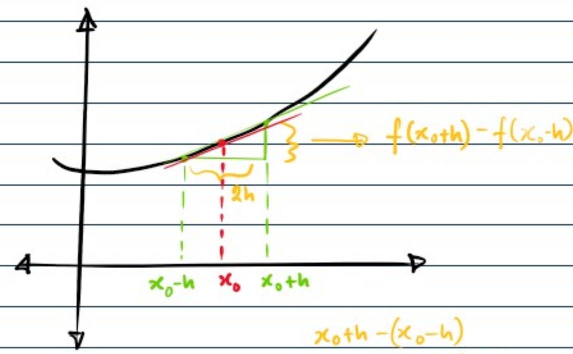
In equation ①

- if $h > 0$, forward difference
- if $h < 0$, backward difference



Central Difference (formula)

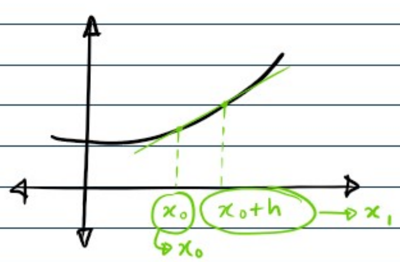
$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$



$x_0+h-(x_0-h)$
$= \cancel{x_0}+h-\cancel{x_0}+h$
$= 2h$

Important Note: Central difference is more accurate than forward and backward difference because it provides a parallel line to the original line.

Forward Difference



$$P_1(x) = \frac{x-x_1}{x_0-x_1}\cdot f(x_0) + \frac{x-x_0}{x_1-x_0}\cdot f(x_1)$$

Error from Cauchy's Theorem

$$f(x) = \frac{x-x_0}{x_0-x_1}\cdot f(x_0) + \frac{x-x_0}{x_1-x_0}\cdot f(x_1) + \underbrace{\frac{f''(\xi)}{2!}}_{u}\cdot\underbrace{(x-x_0)\cdot(x-x_1)}_{v}$$

$\xi \in [x_0, x_0+h]$

$$\frac{d}{dx}(uv) = \frac{du}{dx}\cdot v + \frac{dv}{dx}\cdot u$$

$$f'(x) = \frac{1}{x_0-x_1}\cdot f(x_0) + \frac{1}{x_1-x_0}\cdot f(x_1) + \frac{f'''(\xi)}{2}\cdot\frac{d\xi}{dx}\cdot(x-x_0)(x-x_1) +$$

$$\frac{f''(\xi)}{2}(2x-x_0-x_1)$$

$$\therefore f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{f''(\xi)}{2}(x_0 - x_1)$$

as $x_1 = x_0 + h$

$$= \frac{f(x_0+h) - f(x_0)}{h} + \frac{f''(\xi)}{2}(-h)$$

Truncation Error

$\boxed{\text{Error} \propto h}$ → Linearly proportional to h

$f''(\xi)$

Example :

$\ln(x)$ at $x_0 = 2$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

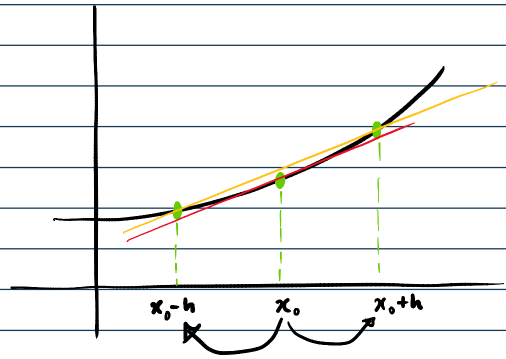
$$f(x_0) = \frac{1}{x_0}$$

$$f(2) = \frac{1}{2} = 0.5$$

h	Forward difference * $\frac{\ln(2+h) - \ln(2)}{h}$	Truncation Error
1	0.40547	0.9453
0.1	0.487902	0.0120989
0.01	0.498754	0.00124585
0.001	0.499875	0.000124958

Error $\sim (-h)$

Central Difference



x_0 , $x_1 = x_0+h$, $x_2 = x_0+2h$

$$f(x) = \frac{\boxed{(x-x_1)(x-x_2)}}{(x_0-x_1)(x_0-x_2)}\cdot f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\cdot f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\cdot f(x_2)$$

$$+ \underbrace{\frac{f^{(3)}(\xi)}{3!}}_{u}\underbrace{(x-x_0)(x-x_1)(x-x_2)}_{v}$$

Rough

$x^2 - xx_2 - x_0x - x_0x_2$

$2x - x_2 - x_0$

$$f'(x) = \frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)}\cdot f(x_0) + \frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)}\cdot f(x_1) + \frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)}\cdot f(x_2)$$

$$+ \frac{f^{(3)}(\xi)}{3!}\left[(x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_1)(x-x_2)\right]$$

$$+ \frac{f^{(4)}(\xi)}{3!}\cdot\frac{d(\xi)}{dx}\cdot(x-x_0)(x-x_1)(x-x_2)$$

$$f'(x_1) = \frac{x_1-x_2}{(x_0-x_1)(x_0-x_2)}\cdot f(x_0) + \frac{2x_1-x_0-x_2}{(x_1-x_0)(x_1-x_2)}\cdot f(x_1) + \frac{x_1-x_0}{(x_2-x_0)(x_2-x_1)}\cdot f(x_2)$$

$$+ \frac{f^{(3)}(\xi)}{3!}\cdot(x-x_0)(x-x_2)$$

By putting values of x_0 x_1 and x_2 $x_1 = x_0+h$; $x_2 = x_0+2h$,

$$f'(x_1) = -\frac{1}{2h}f(x_0) + \frac{1}{2h}f(x_0+2h) - \frac{f^{(3)}(\xi)}{3!}h^2$$

$$= \frac{f(x_0+2h)-f(x_0)}{2h} - \boxed{\frac{f^{(3)}(\xi)}{3!}h^2}$$ — Truncation Error

Error $\propto h^2$

$\ln(x)$ $\quad \frac{1}{x}$ $\quad 0.5$

Example → Derivative of $f(x) = \log(x)$ at $\boxed{x = 2.}$

h	Forward difference	Truncation error	Central difference	Truncation erro
1	0.405465	0.0945349	0.549306	-0.0493061
0.1	0.487902	0.0120984	0.500417	-0.000417293
0.01	0.498754	0.00124585	0.500004	-4.16673e-06
0.001	0.499875	0.000124958	0.500000	-4.16666e-08

$\propto h$ $\qquad \propto h^2$