BRAC UNIVERSITY

MAT215

MATHEMATICS III: COMPLEX VARIABLES & LAPLACE TRANSFORMATIONS

Monthly Assignment

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Assignment Set: A

Submission Date: 27 December

Ans To The Question No. [1(a)]

Let, $f(x) = sin(x)$
Now let us calculate the first few derivatives of f(x)

$$f(0) = sin(0) = 0$$
$$f^{'}(0) = cos(0) = 1$$
$$f^{''}(0) = -sin(0) = 0$$
$$f^{'''}(0) = -cos(0) = -1$$
$$f^{''''}(0) = sin(0) = 0$$
$$f^{'''''}(0) = cos(0) = 1$$

We can see that the coefficient alters between 0,1 and -1. So, for n^{th} derivative, we have to determine whether the n^{th} coefficient is 0,1 or -1.

Thus the Maclaurin Series expansion for sin(x) is -

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(x-0)^k$$

$$= f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + ...$$
$$= 0 + \frac{1}{1!}x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + ...$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$$

Now, we can see a pattern that allows us to derive an expansion for the n^{th} term in the series, which is $\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ where, n = 0,1,2,3,4,.......
Substituting this into the formula for Taylor Series expansion, we obtain,

$$sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

[Showed]

Ans To The Question No. [1(b)]

Let, $f(x) = cos(x)$

Now let us calculate the first few derivatives of f(x)

$$f(0) = cos(0) = 1$$

$$f^{'}(0) = -sin(0) = 0$$

$$f^{''}(0) = -cos(0) = -1$$

$$f^{'''}(0) = sin(0) = 0$$

$$f^{''''}(0) = cos(0) = 1$$

$$f^{'''''}(0) = -sin(0) = 0$$

We can see that the coefficient alters between 0,1 and -1. So, for n^{th} derivative, we have to determine whether the n^{th} coefficient is 0,1 or -1.

Thus the Maclaurin Series expansion for cos(x) is -

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(x-0)^k$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\ldots$$

$$= 1 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 + \ldots\ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots$$

Now, we can see a pattern that allows us to derive an expansion for the n^{th} term in the series, which is $\frac{(-1)^n}{(2n)!}x^{2n}$ where, n = 0,1,2,3,4,....... Substituting this into the formula for Taylor Series expansion, we obtain,

$$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

[Showed]

Ans To The Question No. [1(c)]

From equation (1),

$$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ....$$

From equation (2),

$$cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ....$$

Now, if we integrate equation (2), we get,

$$\int cos\theta \cdot d\theta$$

$$= \int 1 \cdot d\theta - \int \frac{\theta^2}{2!} \cdot d\theta + \int \frac{\theta^4}{4!} - ....$$

$$= \left(\theta - \frac{\theta^3}{2! \cdot 3} + \frac{\theta^5}{4! \cdot 5} - ....\right) + c$$

$$= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ....\right) + c$$

$$= sin\theta + c \ \ [from\ equation\ (1)]$$

Therefore, we can see that,

$$\int cos\theta \cdot d\theta = sin\theta + c \qquad [Showed]$$

Ans To The Question No. [1(d)]

We know, $sin\theta = tan\theta \cdot cos\theta$

So, now from equation (i) and (ii), we can say that,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\theta^{2k+1} = tan\theta \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\theta^{2k}$$

$$\Rightarrow (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...) = tan\theta(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...)....(iv)$$

Now, we know that, $\quad tan\theta = a_0 + a_1\theta + a_2\theta^2 + ...(iii)$

Putting the value of $tan\theta$ in equation (iv), we get,

$$\Rightarrow (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...) = (a_0 + a_1\theta + a_2\theta^2 + ...)(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...)$$

$$\Rightarrow (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...) = a_1\theta - (\frac{a_1}{2} - a_3)\theta^3 + (a_5 - \frac{a_3}{2} + \frac{a_1}{24})\theta^5 - .....$$

Now, comparing L.H.S and R.H.S, we get,
$a_1 = 1$
$a_3 = \frac{a_1}{2} - \frac{1}{6} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$
$a_5 = \frac{1}{120} - \frac{a_1}{24} + \frac{a_3}{2} = \frac{1}{120} - \frac{1}{24} + \frac{1}{6} = \frac{2}{15}$

Putting these values in equation (iii), we get,

$$tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + ...$$

[Showed]

Ans To The Question No. [2(a)(i)]

For n^{th} root of unity, here is given, $w^n = 1$

$$\hookrightarrow w = \sqrt[n]{1}$$
$$= (1)^{1/n}$$
$$= (cos(0^\circ) + isin(0)^\circ)^{\frac{1}{n}}$$
$$= [cos(2\pi k + 0^\circ) + isin(2\pi k + 0^\circ)]^{\frac{1}{n}}$$
$$= [cos(2\pi k) + isin(2\pi k)]^{\frac{1}{n}}$$
$$= cos\left(\frac{2\pi k}{n}\right) + isin\left(\frac{2\pi k}{n}\right) \quad [by\ using\ De\ Moivre's\ theorem]$$

[Showed]

Ans To The Question No. [2(a)(ii)]

When $n = 3$, the n^{th} root of unity is,

$$w = (1)^{1/3}$$
$$= (cos2\pi k + isin2\pi k)^{1/3}$$
$$= cos\left(\frac{2\pi k}{3}\right) + isin\left(\frac{2\pi k}{3}\right)$$
$$[where\ k\ =\ 0, 1, ..., n-1]$$

if $k = 0, \quad w = cos(0) + isin(0) = 1$
if $k = 1, \quad w = cos(\frac{2\pi}{3}) + isin(\frac{2\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
if $k = 2, \quad w = cos(\frac{4\pi}{3}) + isin(\frac{4\pi}{3}) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

When $n = 4$, the n^{th} root of unity is,

$$w = (1)^{1/4}$$

$$= (cos2\pi k + isin2\pi k)^{1/4}$$

$$= cos\left(\frac{\pi k}{2}\right) + isin\left(\frac{\pi k}{2}\right)$$

$$[where\ k\ =\ 0, 1, ..., n-1]$$

if $k = 0,\ \ w = cos(0) + isin(0) = 1$
if $k = 1,\ \ w = cos(\frac{\pi}{2}) + isin(\frac{\pi}{2}) = i$
if $k = 2,\ \ w = cos(\pi) + isin(\pi) = -1$
if $k = 3,\ \ w = cos(\frac{3\pi}{2}) + isin(\frac{3\pi}{2}) = -i$

When $n = 5$, the n^{th} root of unity is,

$$w = (1)^{1/5}$$

$$= (cos2\pi k + isin2\pi k)^{1/5}$$

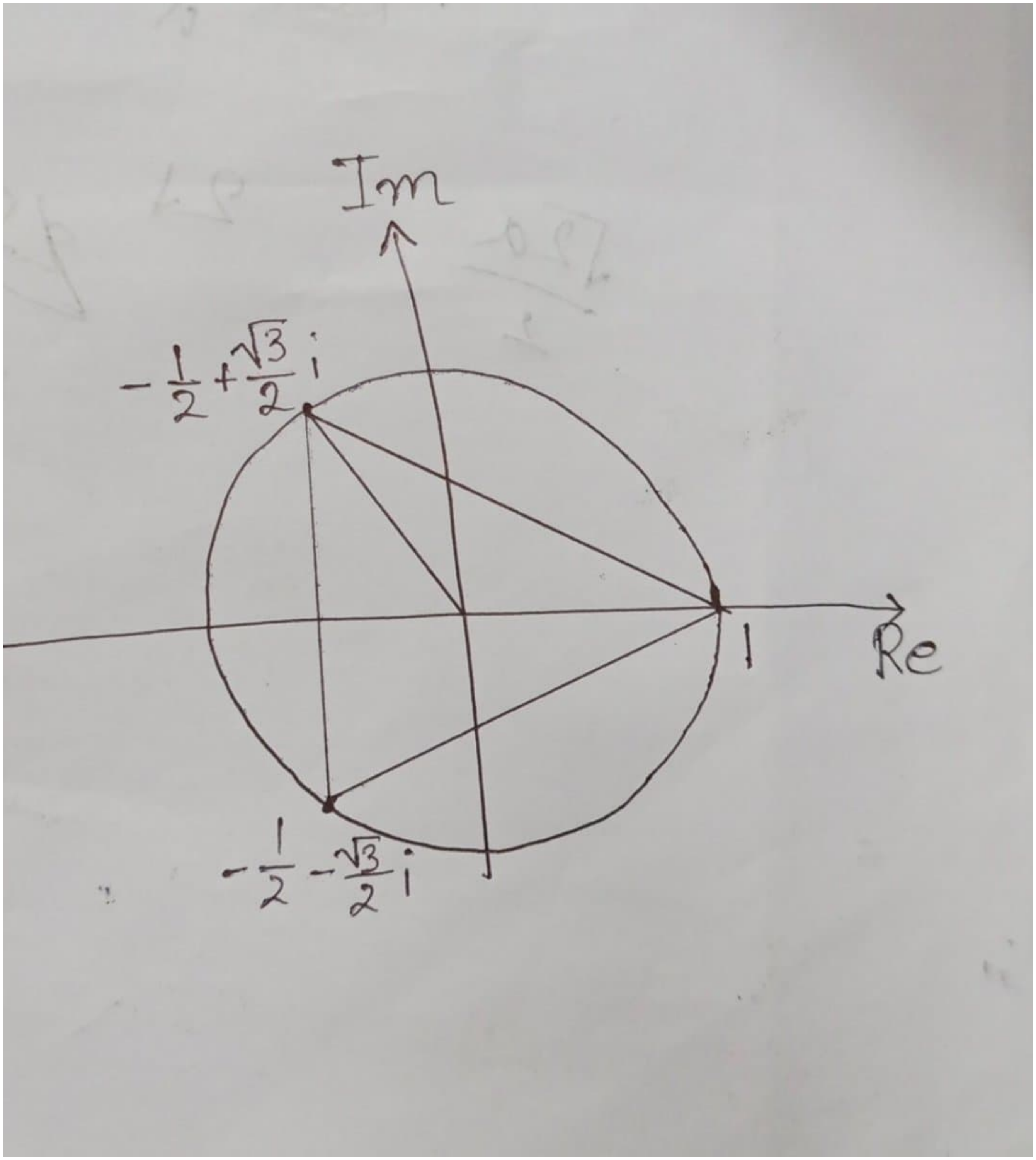
$$= cos\left(\frac{2\pi k}{5}\right) + isin\left(\frac{2\pi k}{5}\right)$$

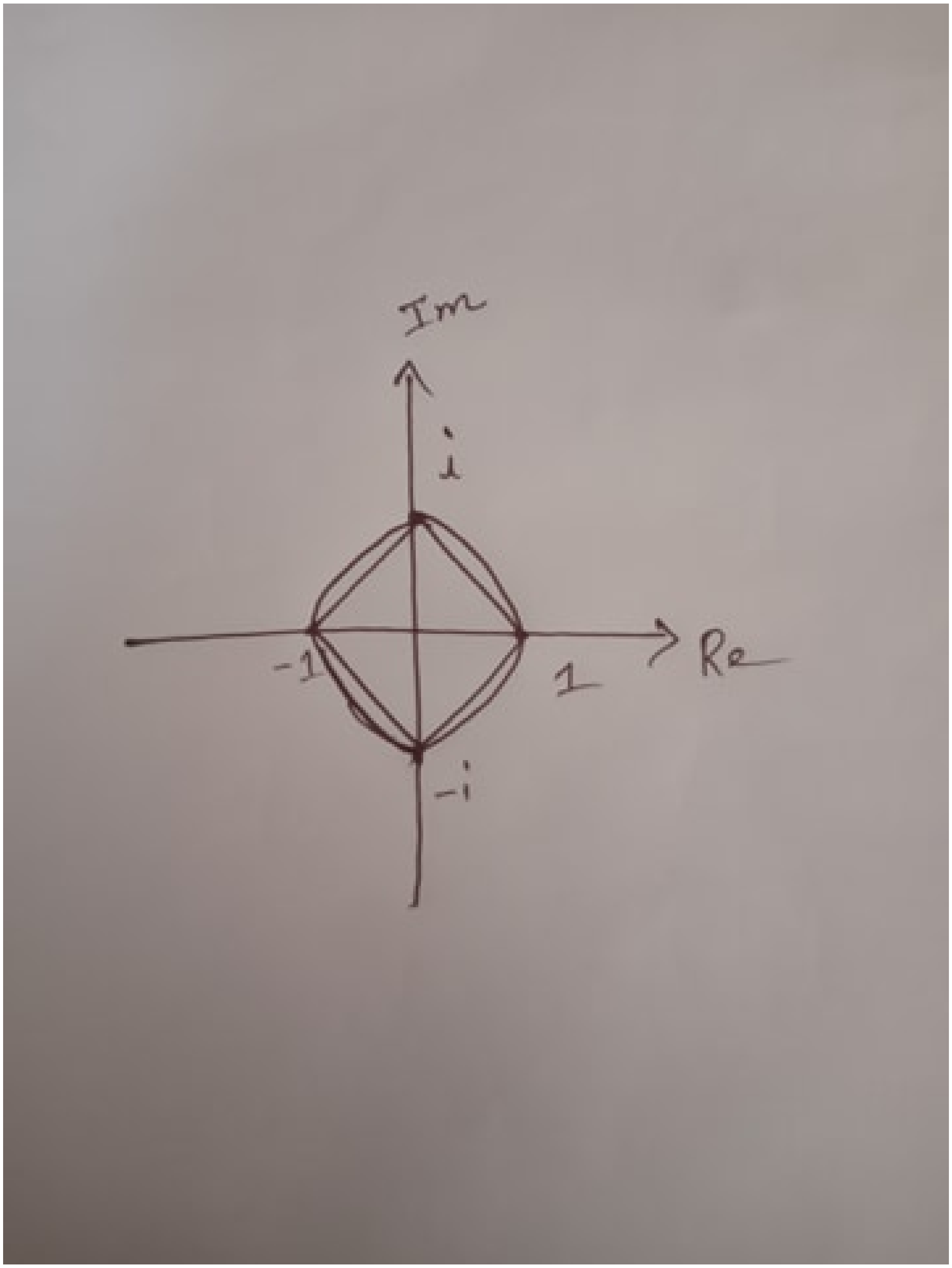
$$[where\ k\ =\ 0, 1, ..., n-1]$$

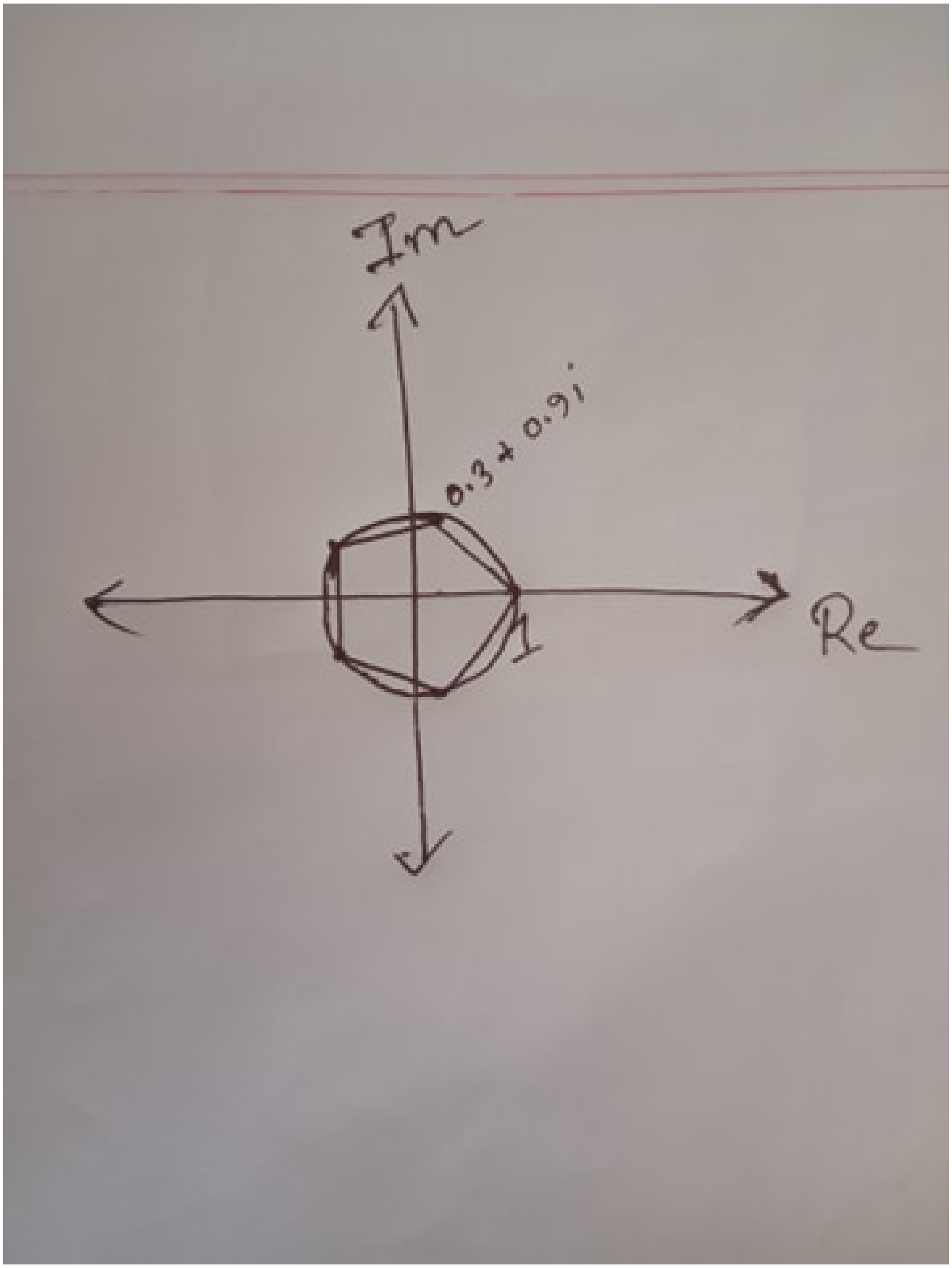
if $k = 0,\ \ w = cos(0) + isin(0) = 1$
if $k = 1,\ \ w = cos(\frac{2\pi}{5}) + isin(\frac{2\pi}{5}) = 0.3 + 0.9i$
if $k = 2,\ \ w = cos(\frac{4\pi}{5}) + isin(\frac{4\pi}{5}) = -0.8 + 0.5i$
if $k = 3,\ \ w = cos(\frac{6\pi}{5}) + isin(\frac{6\pi}{5}) = -0.8 - 0.5i$
if $k = 4,\ \ w = cos(\frac{8\pi}{5}) + isin(\frac{8\pi}{5}) = 0.3 - 0.9i$

That's how we find n n^{th} roots of unity for n = 3, n = 4 and n = 5.

Ans To The Question No. [2(a)(iii)]




Im
i
-1
1
Re
-i



Im
0.3+0.9i
Re
1

Ans To The Question No. [2(b)]

We can say that, $|z_3 - z_2| = |z_3 - z_1| \qquad [Since, |z_1| = |z_2|]$
and, $z_1 + z_2 + z_3 = 0$
$\longrightarrow z_3 = -z_1 - z_2$

Now, $z_3 - z_2 = -z_1 - z_2 - z_2 = -2z_2 - z_1$
and, $z_3 - z_1 = -z_1 - z_2 - z_1 = -2z_1 - z_2$

So,

$$|z_3 - z_1|^2$$
$$= |2z_1 + z_2|^2$$
$$= (2z_1 + z_2)(2z_1 + z_2)$$
$$= 4z_1z_1 + 2z_1z_2 + 2z_1z_2 + z_2 \cdot z_2$$
$$= 4|z_1|^2 + 4z_1z_2 + |z_2|^2$$
$$= 4 \cdot 1 + 4 \cdot Re(z_1z_2) + 1$$
$$= 5 + 4 \cdot Re(z_1z_2)$$

Similarly,

$$|z_3 - z_2|^2$$
$$= |2z_2 + z_1|^2$$
$$= (2z_2 + z_1)(2z_2 + z_1)$$
$$= 4z_2z_2 + 2z_1z_2 + 2z_1z_2 + z_1 \cdot z_1$$
$$= 4 + 4 \cdot Re(z_1z_2) + 1$$
$$= 5 + 4 \cdot Re(z_1z_2)$$

Thus, $|z_3 - z_1|^2 = |z_3 - z_2|^2$
So, $|z_3 - z_1| = |z_3 - z_2|$ ......(1)

Again, $|z_2 - z_1| = |z_3 - z_2|$
and, $z_1 + z_2 + z_3 = 0$
$\longrightarrow z_2 = -z_1 - z_3$

Now, $z_2 - z_1 = -z_1 - z_3 - z_1 = -2z_1 - z_3$

So,

$$|z_2 - z_1|^2$$
$$= |2z_1 + z_3|^2$$
$$= (2z_1 + z_3)(2z_1 + z_3)$$
$$= 4z_1z_1 + 2z_1z_3 + 2z_1z_3 + z_3 \cdot z_3$$
$$= 4|z_1|^2 + 4 \cdot Re(z_1z_3) + 1$$
$$= 4 \cdot 1 + 4 \cdot Re(z_1z_3) + 1$$
$$= 5 + 4 \cdot Re(z_1z_3)$$

Thus,

$$z_3 - z_2 = z_3 - (-z_1 - z_3) = z_3 + z_1 + z_3 = 2z_3 + z_1$$

Now,

$$|z_3 - z_2|^2 = |2z_3 + z_1|^2$$
$$= (2z_3 + z_1)(2z_3 + z_1)$$
$$= 4z_3z_3 + 2z_1z_3 + 2z_1z_3 + z_1 \cdot z_1$$
$$= 4|z_3|^2 + 4 \cdot Re(z_1z_3) + |z_1|^2$$
$$= 4 + 4 \cdot Re(z_1z_3) + 1$$
$$= 5 + 4 \cdot Re(z_1z_3)$$

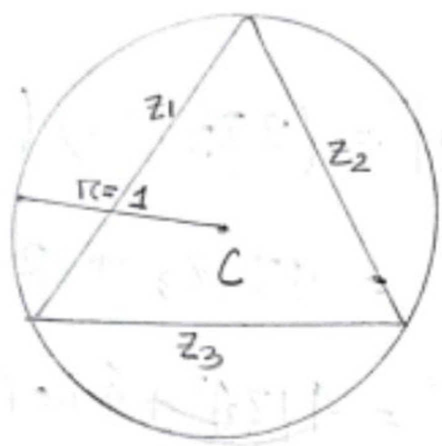
Thus,

$$|z_2 - z_1|^2 = |z_3 - z_2|^2$$

Therefore, $|z_2 - z_1| = |z_3 - z_2|$ ....(2)

Finally, from equation (1) and (2) we can say,

$$|z_3 - z_1| = |z_3 - z_2| = |z_2 - z_1|$$



So, these numbers z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the unit circle.

BONUS QUESTIONS

Ans To The Question No. [1(a)]

Let,

$$s = (1 + z + z^2 + ... + z^n) ....(i)$$

$$\Rightarrow zs = z + z^2 + z^3 + ... + z^{n+1}$$

From equation (i), we can derive, $(s - 1) = z + z^2 + z^3 + ... + z^n$

Therefore,

$$zs = (s - 1) + z^{n+1}$$

$$\Rightarrow zs - s = -1 + z^{n+1}$$

$$\Rightarrow s(z - 1) = -1 + z^{n+1}$$

$$\Rightarrow s = \frac{-1 + z^{n+1}}{(z - 1)}$$

$$\Rightarrow s = \frac{1 - z^{n+1}}{(1 - z)} \quad [z \; not \; equals \; to \; 1]$$

Putting back the value of s, we get,

$$1 + z + z^2 + ... + z^n = \frac{1 - z^{n+1}}{(1 - z)}$$

[Verified]

Ans To The Question No. [1(b)]

In 1(a), we verified,

$$1 + z + z^2 + ... + z^n = \frac{1 - z^{n+1}}{(1 - z)}$$

Substituting $z = e^{i\theta}$, we get,

$$1 + e^{i\theta} + e^{2i\theta} + ... + e^{ni\theta} = \frac{1 - e^{(n+1)i\theta}}{(1 - e^{i\theta})}$$

Multiplying the numerator and denominator by $ie^{-i\theta/2}$, we get,

$$1 + e^{i\theta} + e^{2i\theta} + ... + e^{ni\theta} = \frac{ie^{-i\theta/2} - ie^{(2n+1)i\theta/2}}{sin(\frac{\theta}{2})}$$

Then,

$$\begin{aligned}
&1 + cos\theta + cos2\theta + ... + cosn\theta \\
&= Re(1 + e^{i\theta} + e^{2i\theta} + ... + e^{ni\theta}) \\
&= Re\left(\frac{ie^{-i\theta/2} - ie^{(2n+1)i\theta/2}}{sin(\frac{\theta}{2})}\right) \\
&= \frac{sin(\theta/2) + sin[(2n+1)\theta/2]}{2sin(\theta/2)} \\
&= \frac{1}{2} + \frac{sin[(2n+1)\theta/2]}{2sin(\theta/2)}
\end{aligned}$$

[Established]