USEFUL DEFINITIONS AND FORMULAS

1. $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x > 0$

2. $\Gamma(n+1) = n\Gamma(n) = n!$

3. $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$

4. $\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

5. $\Gamma(1) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

6. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m,n > 0$

7. $B(m,n) = B(n,m)$

8. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

9. $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

10. $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad p,q > -1$

11. $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad 0 < n < 1$

12. $B(m,n) = \int_0^{+\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^{+\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx \quad m,n > 0$

13. $\int_0^{+\infty} \frac{x^{-n}}{(1+x)}dx = B(n, 1-n) = \frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(1)} = \frac{\pi}{\sin n\pi} \quad 0 < n < 1$

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PROBLEMS

Evaluate in terms of gamma function.

1. $\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}dx$

2. $\int_0^b y^5\sqrt{b^2-y^2}\,dy$

3. $\int_0^\infty e^{-ax^2}dx\,;a>0$

4. $\int_0^\infty x^5e^{-4x}dx$

5. $\int_0^\infty x^6e^{-3x}dx$

6. $\int_0^\infty x^5e^{-x^2}dx$

7. $\int_0^\infty x^9e^{-x^2}dx$

8. $\int_0^\infty \sqrt{x}e^{-x^2}dx$

9. $\int_0^1 \frac{x^3}{\sqrt{1-x^3}}dx$

10. $\int_0^1 \frac{1}{\sqrt{\ln\left(\frac{1}{x}\right)}}dx$

11. $\int_0^1 \frac{1}{\sqrt{x\ln\left(\frac{1}{x}\right)}}dx$

12. $\int_0^1 \left(1-\frac{1}{x}\right)^{\frac{1}{3}}dx$

Evaluate in terms of beta function.

13. $\int_0^1 \frac{x^2}{\sqrt{1-x}}dx$

14. $\int_0^1 x^7(1-x)^3dx$

15. $\int_0^1 \frac{1}{\sqrt{1-x^3}}dx$

16. $\int_0^1 x^3\sqrt{1-x}\,dx$

17. $\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}}dx$

18. $\int_0^a y^7\sqrt{a^4-y^4}\,dy$

19. $\int_0^4 y^3\sqrt{64-y^3}\,dy$

20. $\int_0^1 x^2(1-x^3)^{\frac{3}{2}}dx$

21. $\int_0^\infty \frac{1}{1+x^4}dx$

Evaluate the following integrals:

22. $\int_0^\pi \sin^5\theta\cos^4\theta\,d\theta$

23. $\int_0^\pi \sin^6\theta\cos^7\theta\,d\theta$

24. $\int_0^{\frac{\pi}{6}} \sin^2 6\theta\cos^4 3\theta\,d\theta$

25. $\int_0^{\frac{\pi}{4}} \sin^2 4\theta\cos^3 2\theta\,d\theta$

26. $\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta\,d\theta$

27. $\int_0^{\frac{\pi}{8}} \sin^2 8\theta\cos^4 4\theta\,d\theta$

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1.

$$\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}dx$$

Solution

By the use of the following transformation,

$$x = 4y \Rightarrow dx = 4dy$$

and when

x	0	4
y	0	1

the given integral can be transformed as,

$$\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}dx = \int_0^1 4^{\frac{3}{2}}4^{\frac{5}{2}}y^{\frac{3}{2}}(1-y)^{\frac{5}{2}}\,4dy = 4^5\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{5}{2}}\,dy = 4^5 B\left(\frac{5}{2},\frac{7}{2}\right)$$

$$= 4^5\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)} = 4^5\frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)}{5!} = 12\pi$$

$$\boxed{\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}dx = 12\pi}$$

3.

$$\int_0^\infty e^{-ax^2}dx, \qquad a > 0$$

Solution

$$\int_0^{+\infty} e^{-ax^2}dx = \int_0^{+\infty} e^{-\left(\sqrt{a}\,x\right)^2}dx$$

Using the transformation

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$$y = \sqrt{a}\,x \Rightarrow dy = \sqrt{a}\,dx$$

x	0	$\to +\infty$
y	0	$\to +\infty$

$$\int_0^{+\infty} e^{-(\sqrt{a}x)^2} dx = \int_0^{+\infty} e^{-(\sqrt{a}x)^2} dx = \frac{1}{\sqrt{a}} \int_0^{+\infty} e^{-y^2} dy = \frac{1}{\sqrt{a}} \times \frac{\sqrt{\pi}}{2}$$

Ans.

$$\boxed{\int_0^{+\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}};\ a > 0}$$

Similarly,

$$\boxed{\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}};\ a > 0}$$

4.

$$\int_0^{+\infty} x^5 e^{-4x} dx$$

Solution

Using the transformation

$$y = 4x \Rightarrow dy = 4\,dx$$

x	0	$\to +\infty$
y	0	$\to +\infty$

$$\int_0^{+\infty} x^5 e^{-4x} dx = \frac{1}{4^6} \int_0^{+\infty} y^5 e^{-y} dx = \frac{\Gamma(6)}{4^6} = \frac{5!}{4^6} = \frac{5 \times 4 \times 3 \times 2 \times 1}{4^6}$$

Ans.

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$$\boxed{\int_{0}^{+\infty} x^5 e^{-4x} dx = \frac{15}{512}}$$

11. Evaluate the following integral in terms of gamma function:

$$\int_{0}^{1} \frac{1}{\sqrt{x \ln\left(\frac{1}{x}\right)}} dx \quad \text{or,} \int_{0}^{1} \frac{1}{\sqrt{-x \ln x}} dx$$

Solution

By the use of the following transformation,

$$y = \ln\left(\frac{1}{x}\right) \Rightarrow y = -\ln x \Rightarrow \ln x = -y \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y} dy$$

and when

x	0^+	$\to 1^-$
y	$+\infty$	$\to 0$

the given integral can be transformed as,

$$\int_{0}^{1} \frac{1}{\sqrt{x \ln\left(\frac{1}{x}\right)}} dx = -\int_{+\infty}^{0} y^{-\frac{1}{2}} e^{-\frac{y}{2}} dy = \int_{0}^{+\infty} y^{-\frac{1}{2}} e^{-\frac{y}{2}} dy$$

Again let,

$$u = \frac{1}{2} y \Rightarrow y = 2u \Rightarrow dy = 2du$$

and when

y	0	$\to +\infty$

$$u \left| \quad 0 \quad \right| \to +\infty$$

$$\int_0^{+\infty} y^{-\frac{1}{2}} e^{-\frac{y}{2}} dy = 2^{-\frac{1}{2}} \int_0^{+\infty} u^{-\frac{1}{2}} e^{-u}\, 2du = \sqrt{2}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}$$

$$\boxed{\int_0^1 \frac{1}{\sqrt{x \ln\left(\frac{1}{x}\right)}} dx = \sqrt{2\pi}}$$

17.

Solution

$$\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}} dx = B\left(\frac{7}{2}, \frac{5}{2}\right) = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)} = \frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2}}{5!}\pi = \frac{3}{256}\pi$$

$$\boxed{\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}} dx = \frac{3}{256}\pi}$$

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