4.6 Variation of Parameters
Solutions to the Selected Problems

Standard Form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)$$

$$\boxed{y'' + P(x)y' + Q(x)y = f(x)}$$

If $P(x)$ and $Q(x)$ are constant functions, then

Complementary Function

$$y_c(x) = C_1 y_1(x) + C_2 y_1(x)$$

Particular solution

$$\boxed{y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)}$$

$$u_1'(x) = \frac{W_1}{W}, \qquad u_2'(x) = \frac{W_2}{W}$$

$$\boxed{u_1(x) = \int \frac{W_1}{W}dx, \qquad u_2'(x) = \int \frac{W_2}{W}dx}$$

where, W is the *Wronskian*, defined by,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

and

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

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Example. Solve the given differential equation by variation of parameters.

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Solution

Complementary function

Associated homogeneous equation is

$$y'' - 4y' + 4y = 0$$

From the auxiliary equation

$$m^2 - 4m + 4 = 0$$

one gets the following complementary function

$$y_c = C_1 e^{2x} + C_2 x e^{2x}$$

where C_1 and C_2 are arbitrary constants.

Particular solution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Identifying $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ we compute the Wronskian

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{2x}(e^{2x} + 2xe^{2x}) - (2e^{2x})(xe^{2x}) = e^{4x} \neq 0$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -(x+1)e^{2x}(xe^{2x}) = -(x^2+x)e^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x} - 0 = (x+1)e^{4x}$$

Now,

$$u_1'(x) = \frac{W_1}{W} = -\frac{(x^2+x)e^{4x}}{e^{4x}} = -(x^2+x)$$

$$u_1(x) = -\int (x^2+x)\,dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2'(x) = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = (x+1)$$

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$$u_2(x) = \int (x+1)\,dx = \frac{x^2}{2} + x$$

$$y_p(x) = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

$$= \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

$$y(x) = y_c + y_p$$

$$\boxed{y(x) = \left(\frac{x^3}{6} + \frac{x^2}{2} + C_1 + C_2 x\right)e^{2x}}$$

1–18. Solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

Solution

Complementary function

Associated homogeneous equation is

$$y'' + y = 0$$

From the auxiliary equation

$$m^2 + 1 = 0$$

one gets the following complementary function

$$y_c = e^{0x}(C_1 \cos x + C_2 \sin x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

where C_1 and C_2 are arbitrary constants.

Particular solution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Identifying $y_1(x) = \cos x$ and $y_2(x) = \sin x$ we compute the Wronskian

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - \sin^2 x = 1 \neq 0$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sec x \sin x = -\tan x$$

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$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

Now,

$$u_1'(x) = \frac{W_1}{W} = -\frac{\tan x}{1} = -\tan x$$

$$u_1(x) = -\int \tan x \, dx = \ln|\cos x|$$

$$u_2'(x) = \frac{W_2}{W} = \frac{1}{1} = 1$$

$$u_2(x) = \int 1 \, dx = x$$

$$y_p(x) = \ln|\cos x| \cos x + x \sin x$$

$$y(x) = y_c + y_p = (C_1 \cos x + C_2 \sin x) + \ln|\cos x| \cos x + x \sin x$$

$$\boxed{y(x) = (\ln|\cos x| + C_1) \cos x + (x + C_2) \sin x}$$

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