Summer 2020

MAT 216

Solutions

Problem Sheet - 2

1. (i) The domain and codomain of the given expression: $\Re^3$ and $\Re^2$ respectively, that is $\Re^3 \to \Re^2$. The transformation is linear.

   (ii) The domain and codomain of the given expression: $\Re^2$ and $\Re^3$ respectively, that is $\Re^2 \to \Re^3$. The transformation is nonlinear.

2. The standard matrix for *T* is

$$\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}.$$

We know that $T:\Re^n \to \Re^m$ is multiplying by *A* and it is important to state that *A* is the standard matrix for *T*, then the linear transformation $T:\Re^n \to \Re^m$ by $T_A:\Re^n \to \Re^m$, thus $T_A(x) = Ax$, where the vector *x* in $\Re^n$ is expressed as a column matrix.

$$T_A(x) = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$$T(-1,2,4) = (3,-2,-3).$$

3. The standard basis vectors e_1, e_2 and e_3 are

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(i) $T_A(e_1) = Ae_1 = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, T_A(e_2) = Ae_2 = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

$$T_A(e_3) = Ae_3 = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}.$$

(ii) $T_A(e_1 + e_2 + e_3) = T_A(e_1) + T_A(e_2) + T_A(e_3) = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}.$

(iii) $T_A(7e_3) = 7T_A(e_3) = \begin{bmatrix} 0 \\ 14 \\ -21 \end{bmatrix}.$

4. Let $u_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ & $u_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

Then $u_1 + u_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$

$$T(u_1 + u_2) = \begin{pmatrix} x_1 + x_2 - y_1 - y_2 \\ x_1 + x_2 - z_1 - z_2 \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_1 - z_1 \end{pmatrix} + \begin{pmatrix} x_2 - y_2 \\ x_2 - z_2 \end{pmatrix} = T(u_1) + T(u_2).$$

If c is a scalar then $cu_1 = c\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} c\,x_1 \\ c\,y_1 \\ c\,z_1 \end{pmatrix}$

$$T\left(cu_1\right)=T\begin{pmatrix} c\,x_1 \\ c\,y_1 \\ c\,z_1 \end{pmatrix}=\begin{pmatrix} c\,x_1-c\,y_1 \\ c\,x_1-c\,z_1 \end{pmatrix}=c\begin{pmatrix} x_1-y_1 \\ x_1-z_1 \end{pmatrix}=cT\left(u_1\right).$$

Therefore, T is a linear transformation.

5. Let $u_1=\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ & $u_2=\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

Then $u_1+u_2=\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$

$$T\left(u_1\right)=\begin{pmatrix} 3x_1-2y_1+z_1 \\ x_1-3y_1-2z_1 \end{pmatrix}, \qquad T\left(u_2\right)=\begin{pmatrix} 3x_2-2y_2+z_2 \\ x_2-3y_2-2z_2 \end{pmatrix}$$

$$T\left(u_1+u_2\right)=\begin{pmatrix} 3\left(x_1+x_2\right)-2\left(y_1+y_2\right)+\left(z_1+z_2\right) \\ \left(x_1+x_2\right)-3\left(y_1+y_2\right)-2\left(z_1+z_2\right)_2 \end{pmatrix}=\begin{pmatrix} 3x_1-2y_1+z_1 \\ x_1-3y_1-2z_1 \end{pmatrix}+\begin{pmatrix} 3x_2-2y_2+z_2 \\ x_2-3y_2-2z_2 \end{pmatrix}=T\left(u_1\right)+T\left(u_2\right)$$

.

If c is a scalar then $cu_1=c\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} c\,x_1 \\ c\,y_1 \\ c\,z_1 \end{pmatrix}$

$$T\left(cu_1\right)=T\begin{pmatrix} c\,x_1 \\ c\,y_1 \\ c\,z_1 \end{pmatrix}=\begin{pmatrix} 3c\,x_1-2c\,y_1+c\,z_1 \\ c\,x_1-3c\,y_1-2c\,z_1 \end{pmatrix}=c\begin{pmatrix} 3x_1-2y_1+z_1 \\ x_1-3\,y_1-2\,z_1 \end{pmatrix}=cT\left(u_1\right)$$

Therefore, T is a linear transformation.

6. (i) Since $T=\left(0,0,0\right)=\left(1,1\right)\neq\left(0,0\right)$

Hence T is not a linear transformation.

(ii) T is a linear transformation.

7. (i) The system of linear equations is:

$$\begin{aligned} 3x+0y-2z&=5\\ 7x+y+4z&=-3\\ 0x-2y+z&=7 \end{aligned}$$

(ii) The system of linear equations is:

$$\begin{aligned} x \qquad\qquad &=7\\ y \qquad &=-2\\ z \quad &=3\\ w&=4 \end{aligned} \; .$$

8. (a) Since the general solution: $x=5+2t,\ y=t.$

So, the linear equation is: $x-2y=5$.

(b) Since $x-2y=5$ Let $x=t,\ 2y=x-5,$ so, $y=\frac{1}{2}t-\frac{5}{2}$.

9. Given that

$$\begin{aligned} x - y&=3\\ 2x-2y&=k \end{aligned}$$

So, $0=6-k,\ \therefore 6-k=0$

If $k\neq 6$, the system has no solutions.

If $k=6$, the system has many solutions.

No value of k yields one solution.