4.4 Undetermined Coefficients—Superposition Approach
Solutions to the Selected Problems

Standard Form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Associated homogeneous equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Auxiliary equation

$$y = e^{mx}$$

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)e^{mx} = 0$$

Since $e^{mx} \neq 0$

$$\boxed{am^2 + bm + c = 0}$$

Example 1. Solve the given differential equation by undetermined coefficients.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^x$$

Solution

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^x \tag{1}$$

Auxiliary equation to the associated homogeneous equation is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

which has the following solutions

$$m = 2 \text{ or } m = 3$$

Complementary function

$$\boxed{y_c(x) = C_1e^{2x} + C_2e^{3x}}$$

Particular solution

Let's assume the particular solution to be

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$$y_p(x) = Ae^x \tag{2}$$

Substituting (2) into the eq. (1) we get,

$$Ae^x - 5Ae^x + 6Ae^x = 2e^x$$

$$2Ae^x = 2e^x$$

which gives,

$$2A = 2 \Rightarrow \boxed{A = 1}$$

Now, the eq. (1) becomes

$$\boxed{y_p(x) = e^x}$$

Therefore, the general solution of the given differential equation (1) is

$$y(x) = y_c + y_p$$

$$\boxed{y(x) = C_1e^{2x} + C_2e^{3x} + e^x}$$

Example 2. Solve the given differential equation by undetermined coefficients.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6x^2 + 2x + 10$$

Solution

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6x^2 + 2x + 10 \tag{1}$$

From the **Example 1** we obtain the following complementary function

$$\boxed{y_c(x) = C_1e^{2x} + C_2e^{3x}}$$

Particular solution

Let's assume the particular solution to be

$$y_p(x) = Ax^2 + Bx + C \tag{2}$$

Differentiating eq. (2), we get,

$$y_p'(x) = 2Ax + B \tag{3}$$

$$y_p''(x) = 2A \tag{4}$$

Substituting eqs. (2)-(4) into the eq. (1) we get,

$$6Ax^2 + (-10A + 6B)x + (2A - 5B + 6C) = 6x^2 + 2x + 10$$

Equating the coefficients of various powers in x, we get,

$$6A = 6$$

$$-10A + 6B = 2$$

$$2A - 5B + 6C = 10$$

which gives,

$$\boxed{A = 1, \qquad B = 2, \qquad C = 3}$$

Now, the eq. (1) becomes

$$\boxed{y_p(x) = x^2 + 2x + 3}$$

Therefore, the general solution of the given differential equation (1) is

$$y(x) = y_c + y_p$$

$$\boxed{y(x) = C_1 e^{2x} + C_2 e^{3x} + x^2 + 2x + 3}$$

Example 3. Solve the given differential equation by undetermined coefficients.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 5\sin x$$

Solution

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 5\sin x \tag{1}$$

From the **Example 1** we obtain the following complementary function

$$\boxed{y_c(x) = C_1 e^{2x} + C_2 e^{3x}}$$

Particular solution

Let's assume the particular solution to be

$$y_p(x) = A\sin x + B\cos x \tag{2}$$

Differentiating eq. (2), we get,

$$y_p'(x) = A\cos x - B\sin x \tag{3}$$

$$y_p''(x) = -A\sin x - B\cos x \tag{4}$$

Substituting eqs. (2)-(4) into the eq. (1) we get,

$$(5A + 5B)\sin x + (5A - 5B)\cos x = 5\sin x$$

Equating the coefficients of $\sin x$ and $\cos x$, we get,

$$5A + 5B = 5$$

$$5A - 5B = 0$$

which gives,

$$\boxed{A = \frac{1}{2}, \qquad B = \frac{1}{2}, \qquad C = 3}$$

Now, the eq. (1) becomes

$$\boxed{y_p(x) = \frac{1}{2}\sin x + \frac{1}{2}\cos x}$$

Therefore, the general solution of the given differential equation (1) is

$$y(x) = y_c + y_p$$

$$\boxed{y(x) = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2}\sin x + \frac{1}{2}\cos x}$$

Example 4. Solve the given differential equation by undetermined coefficients.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3e^{2x}$$

Solution

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3e^{2x} \tag{1}$$

From the **Example 1** we obtain the following complementary function

$$\boxed{y_c(x) = C_1 e^{2x} + C_2 e^{3x}}$$

Particular solution

Let's assume the particular solution to be

$$y_p(x) = Ae^{2x} \tag{2}$$

Since, the particular solution Ae^{2x} is already present in the complementary function, we need to revise the assumption made in eq. (2) as follows:

$$y_p(x) = Axe^{2x} \tag{3}$$

Differentiating eq. (3), we get,

$$y_p'(x) = Ae^{2x} + 2Axe^{2x} \tag{4}$$

$$y_p''(x) = 4Ae^{2x} + 4Axe^{2x} \tag{5}$$

Substituting eqs. (3)-(5) into the eq. (1) we get,

$$-Ae^{2x} = 3e^{2x}$$

which gives,

$$\boxed{A = -3}$$

Now, the eq. (1) becomes

$$\boxed{y_p(x) = -3xe^{2x}}$$

Therefore, the general solution of the given differential equation (1) is

$$y(x) = y_c + y_p$$

$$\boxed{y(x) = C_1e^{2x} + C_2e^{3x} - 3xe^{2x}}$$

Example 5. Solve the given differential equation by undetermined coefficients.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = -4e^{2x}$$

Solution

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = -4e^{2x} \tag{1}$$

We obtain the following complementary function

$$\boxed{y_c(x) = C_1e^{2x} + C_2xe^{2x}}$$

Particular solution

Let's assume the particular solution to be

$$y_p(x) = Ae^{2x} \tag{2}$$

Since, the particular solution Ae^{2x} is already present in the complementary function, we need to revise the assumption made in eq. (2) as follows:

$$y_p(x) = Axe^{2x} + Be^{2x} \tag{3}$$

The particular solution Be^{2x} is already present in the complementary function, we need to revise the assumption made in eq. (2) once again as follows:

$$y_p(x) = Ax^3e^{2x} + Bx^2e^{2x} \tag{3}$$

Differentiating eq. (3), we get,

$$y_p'(x) = 2Bxe^{2x} + (3A + 2B)x^2e^{2x} + 2Ax^3e^{2x} \tag{4}$$

$$y_p''(x) = 2Be^{2x} + (6A + 8B)xe^{2x} + (12A + 4B)x^2e^{2x} + 4Ax^3e^{2x} \tag{5}$$

Substituting eqs. (3)-(5) into the eq. (1) we get,

$$2Be^{2x} + 6Axe^{2x} = -4e^{2x}$$

Equating the coefficients, we get,

$$2Be^{2x} = -4e^{2x}$$

$$6Axe^{2x} = 0$$

which gives,

$$\boxed{A = 0, \qquad B = -2}$$

Now, the eq. (1) becomes

$$\boxed{y_p(x) = -2x^2e^{2x}}$$

Therefore, the general solution of the given differential equation (1) is

$$y(x) = y_c + y_p$$

$$\boxed{y(x) = C_1e^{2x} + C_2xe^{2x} - 2x^2e^{2x}}$$

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