Assignment 02

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Section: 11

Ans. to Q. No - 1

a

Here $A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$

Using Gaussian Elimination Method,

$$= \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \quad [R_3 \leftarrow R_3 + (0.5)R_1]$$

$$= \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad [R_3 \leftarrow R_3 + (+2)R_2]$$

$= U$

Now, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad [R_3 - 2R_2 \rightarrow R_3]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -2 & 1 \end{bmatrix} \quad [R_3 \leftarrow R_3 - (0.5)R_1]$$

$$= L$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= LU$$

$\therefore$ This is the LU Decomposition

ii

We put the system into the form $Ax = b$ where,

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

Now, we use the LU factorization of A we found in (i),

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

Here, $Ux = y$ with $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Then the system reduces to $Ly = b$

This is just a linear system in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

or, in the equation form,

$$y_1 = -4$$

$$y_2 = -2$$

$$-\frac{1}{2} y_1 - 2y_2 + y_3 = 6$$

forward substitution yields

$y_1 = -4$, $y_2 = -2$ and

$y_3 = 0$

Therefore, $y = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$. Now we solve the system $Ux = y$ which is in matrix form

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$$

or, $5x_3 = 0$

$-2x_2 + 2x_3 = -2$

$2x_1 - 2x_2 - 2x_3 = -4$

Backward substitution yields,

$x_3 = 0, \quad x_2 = 1, \quad x_1 = -1$

Therefore, $x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Ans to Q No - 2

i

The augmented matrix for the given system is:

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right]$$

ii

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \quad [R_1 \leftarrow (-1)R_1]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -3 & -2 & -4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \quad \begin{array}{l} [R_2 \leftarrow R_2 + (-2)R_1] \\ [R_3 \leftarrow R_3 + (-1)R_1] \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \quad [R_2 \leftarrow (-\tfrac{1}{3})R_2]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} [R_1 \leftarrow R_1 + (-2)R_2] \\ [R_3 \leftarrow R_3 + (-2)R_2] \end{array}$$

iii

Leading variables $= x_1, x_3$

Free variables $= x_2, x_4$

iv

The reduced row echelon form :-

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

$\therefore \quad x_1 + (-3)x_2 = -2$

$\Rightarrow x_1 = 3x_2 - 2$

and, $x_3 + (-2)x_4 = 1$

$\Rightarrow x_3 = 2x_4 + 1$

Let, $x_2 = s$ and $x_4 = t$

Therefore, the solution to this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s-2 \\ s \\ 2t+1 \\ t \end{bmatrix} ; \quad s, t \in \mathbb{R}$$

(Ans.)

Ans. to Q No - 3

(i)

$$[A \mid I] = \left[\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+(-2)R_1 \to R_2]{R_1 \rightleftarrows R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1+(-1)R_2 \to R_2]{R_2 \rightleftarrows R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + (-3)R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 3 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow R_4 + (-1)R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 3 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_4 \leftarrow R_4 + \cancel{2R_2}\,(-2)R_3]{R_3 \leftarrow (-\frac{1}{3})R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 3 & \frac{2}{3} & -\frac{4}{3} & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2 + R_3]{R_1 \leftarrow R_1 + (-2)R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{4}{3} & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & -2 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 3 & \frac{2}{3} & -\frac{4}{3} & 1 \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow 3R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{4}{3} & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & -2 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \leftarrow R_2 + \left(\frac{2}{3}\right)R_4 \\ R_3 \leftarrow R_3 + \left(\frac{2}{3}\right)R_4}]{R_1 \leftarrow R_1 + \left(-\frac{4}{3}\right)R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -11 & -2 & 5 & -4 \\ 0 & 1 & 0 & 0 & 4 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right]$$

$$\underbrace{\hphantom{-11 \quad -2 \quad 5 \quad -4}}_{\text{Inverse } (A^{-1}) \text{ Matrix}}$$

ii

Here, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -3 \end{bmatrix}$

Since, A ~~B~~ is invertible, the given System has a unique solution,

$$x = A^{-1} b$$

So, $x = \begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 2 \\ -3 \end{bmatrix}$

$$= \begin{bmatrix} -11\times 2 + (-2)\times 2 + 5\times 2 + (-4)(-3) \\ (4)(2) + (1)(2) + (-2)(2) + (2)(-3) \\ (5)(2) + (1)(2) + (-2)(2) + (2)(-3) \\ (9)(2) + (2)(2) + (-4)(2) + (3)(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 0 \\ 2 \\ 5 \end{bmatrix} \quad \text{(Ans.)}$$

(iii)

For verifying we need to prove that $Ax = b$

So,
$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} -4 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} (0)(-4) + (-1)(0) + (1)(2) + (0)(5) \\ (2)(-4) + (1)(0) + (0)(2) + (2)(5) \\ (1)(-4) + (-1)(0) + (3)(2) + (0)(5) \\ (0)(-4) + (1)(0) + (1)(2) + (-1)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ -3 \end{bmatrix} = b$$

[Verified]

Ans. to Q No - 4

i

Let us take an arbitrary vector $V = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$ where $a, b, c \in R$. We want to solve the following equation for c_1, c_2, c_3.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Now, we will take the augmented matrix and perform reduced row reduction

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 2 & 1 & -3 & b \\ -1 & -1 & 1 & c \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 + (+1)R_1]{R_2 \leftarrow R_2 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & -1 & -1 & c+a \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & b+c-\text{[illegible]}a \end{array}\right]$$

This shows that the system has no solution (incosistent) if

$b + c - a \neq 0$.

$\therefore$ Span$(S) \neq R^3$.

ii

If, $a = 2$, $b = 3$, $c = -1$,

then $\cancel{c_1 +}\ c_3 = \cancel{0 = 3 + (-1) - 2 = 0}$ free variable

$c_2 = 3 - (2 \times 2) - 0 = -1 - c_3$

$c_1 = 2 + 2c_3$

Therefore, the vector $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ can be written as

$$(2c_3 + 2)\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \underset{-c_3}{(-1)}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

(Ans)

(iii)

We have to find a vector V such that $b+c-a \neq 0$.

Let, $b=2$, $c=3$, $a=1$

$\therefore 2+3-1 \neq 0$.

$\therefore V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can be a vector $\in R^3$ and it is not a linear combination of the vectors in S.

Ans to Q No - 5

i

Let us take an arbitrary ~~vector~~ 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}$ where $a, b, c, d \in R$.

We want to solve the following equation for, c_1, c_2, c_3, c_4.

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Now,

$$c_1 + c_3 + c_4 = a$$

$$c_2 + c_3 + c_4 = b$$

$$c_2 + c_4 = c$$

$$c_1 + c_4 = d$$

Now, we will take the augment matrix of the above system and perform reduced row reduction.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 1 & 0 & 1 & c \\ 1 & 0 & 0 & 1 & d \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_2 - R_3]{R_4 \leftarrow R_1 - R_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & 0 & a-d \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 - R_3}]{R_4 \leftarrow R_3 - R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a-b+c \\ 0 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & 0 & b-c \\ 0 & 0 & 0 & 0 & b-c-a+d \end{array}\right]$$

This shows that the system is inconsistent or has no solution if

$$b - c - a + d \neq 0$$

$$\therefore \quad Span(S) \neq M_{2,2}$$

ii

if, $a = 2$, $b = 3$, $c = 5$, $d = 4$

then, ~~$0c_4 = 3-5-2+4 = 0$~~ c_4 = free variable

$c_3 = 3 - 5 = -2$

$c_2 = 5 - c_4$

$c_1 = 2 - 3 + 5 = 4 - c_4$

Therefore the Matrix $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ can be written

as

$$-c_4 + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 5 - c_4\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$$

(Ans)

(iii)

We have to find atleast one vectors

$A \in M_{2,2}$ such that

$b - c - a + d \neq 0$

Let $a = 1,\ b = 2,\ c = 3,\ d = 4$

then $2 - 3 - 1 + 4 = 2 \neq 0$

$\therefore\ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ can be a vectors $\in M_{2,2}$

and it is not a linear combination

of the vectors in Set S.