Proofs Using Logical Equivalences

Rosen (6th Ed.) 1.2

Note: These are all **Direct Proofs**

Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive
$\Leftrightarrow (q \vee p) \wedge T$	Negation
$\Leftrightarrow q \vee p$	Identity
$\Leftrightarrow p \vee q$	Commutative

Begin with exactly the left-hand side statement
End with exactly what is on the right
Justify EVERY step with a logical equivalence

Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive

Why did we need this step?

Our logical equivalence specified that ∨ is distributive on the right. This does not guarantee distribution on the left!

Ex.: Matrix multiplication

(Note that whether or not ∨ is distributive on the left is not the point here.)

Prove: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Contrapositive

$p \rightarrow q$	
$\Leftrightarrow \neg p \vee q$	Implication Equivalence
$\Leftrightarrow q \vee \neg p$	Commutative
$\Leftrightarrow \neg(\neg q) \vee \neg p$	Double Negation
$\Leftrightarrow \neg q \rightarrow \neg p$	Implication Equivalence

Prove: $p \rightarrow p \vee q$ is a tautology

Must show that the statement is true for any value of p and q.

$p \rightarrow p \vee q$	
$\Leftrightarrow \neg p \vee (p \vee q)$	Implication Equivalence
$\Leftrightarrow (\neg p \vee p) \vee q$	Associative
$\Leftrightarrow (p \vee \neg p) \vee q$	Commutative
$\Leftrightarrow T \vee q$	Negation
$\Leftrightarrow q \vee T$	Commutative
$\Leftrightarrow T$	Domination

This tautology is called the addition rule of inference.

Why do I have to justify everything?

- Note that your operation must have the same order of operands as the rule you quote unless you have already proven (and cite the proof) that order is not important.
  - $3+4 = 4+3$
  - $3/4 \neq 4/3$

A*B ≠ B*A for everything!

Prove: $(p \wedge q) \rightarrow p$ is a tautology

$(p \wedge q) \rightarrow p$	
$\Leftrightarrow \neg(p \wedge q) \vee p$	Implication Equivalence
$\Leftrightarrow (\neg p \vee \neg q) \vee p$	DeMorgan's
$\Leftrightarrow (\neg q \vee \neg p) \vee p$	Commutative
$\Leftrightarrow \neg q \vee (\neg p \vee p)$	Associative
$\Leftrightarrow \neg q \vee (p \vee \neg p)$	Commutative
$\Leftrightarrow \neg q \vee T$	Negation
$\Leftrightarrow T$	Domination

Prove or Disprove

$p \rightarrow q \Leftrightarrow p \wedge \neg q$???

To prove that something is not true it is enough to provide one counter-example. (Something that is true must be true in every case.)

p	q	$p \rightarrow q$	$p \wedge \neg q$
F	T	T	F

The statements are not logically equivalent

Prove: $\neg p \leftrightarrow q \Leftrightarrow p \leftrightarrow \neg q$

$\neg p \leftrightarrow q$	
$\Leftrightarrow (\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Biconditional Equivalence
$\Leftrightarrow (\neg\neg p \vee q) \wedge (\neg q \vee \neg p)$	Implication Equivalence (x2)
$\Leftrightarrow (p \vee q) \wedge (\neg q \vee \neg p)$	Double Negation
$\Leftrightarrow (q \vee p) \wedge (\neg p \vee \neg q)$	Commutative
$\Leftrightarrow (\neg\neg q \vee p) \wedge (\neg p \vee \neg q)$	Double Negation
$\Leftrightarrow (\neg q \rightarrow p) \wedge (p \rightarrow \neg q)$	Implication Equivalence (x2)
$\Leftrightarrow p \leftrightarrow \neg q$	Biconditional Equivalence

Class Exercise: Without using truth tables, prove that $((p \vee q) \wedge \neg p) \rightarrow q$ is a <u>tautology</u>.

$p \wedge T \Leftrightarrow p$; $p \vee F \Leftrightarrow p$	**Identity Laws**
$p \vee T \Leftrightarrow T$; $p \wedge F \Leftrightarrow F$	**Domination Laws**
$p \vee p \Leftrightarrow p$; $p \wedge p \Leftrightarrow p$	**Idempotent Laws**
$\neg(\neg p) \Leftrightarrow p$	**Double Negation Law**
$p \vee q \Leftrightarrow q \vee p$; $p \wedge q \Leftrightarrow q \wedge p$	**Commutative Laws**
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$; $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	**Associative Laws**
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	**Distributive Laws**
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	
$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$	**De Morgan's Laws**
$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$	
$p \vee (p \wedge q) \Leftrightarrow p$	**Absorption Laws**
$p \wedge (p \vee q) \Leftrightarrow p$	
$p \vee \neg p \Leftrightarrow T$	**Negation Laws**
$p \wedge \neg p \Leftrightarrow F$	
$(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$	**Implication Equivalence**

Class Exercise: Without using truth tables, prove that $((p \vee q) \wedge \neg p) \rightarrow q$ is a <u>tautology</u>.

$((p \vee q) \wedge \neg p) \rightarrow q$	
$\Leftrightarrow \neg((p \vee q) \wedge \neg p) \vee q$	Implication Equivalence
$\Leftrightarrow (\neg(p \vee q) \vee \neg\neg p) \vee q$	DeMorgan
$\Leftrightarrow (\neg(p \vee q) \vee p) \vee q$	Double Negation
$\Leftrightarrow \neg(p \vee q) \vee (p \vee q)$	Associative
$\Leftrightarrow (p \vee q) \vee \neg(p \vee q)$	Commutative
$\Leftrightarrow T$	Negation

Normal or Canonical Forms

Rosen (6th Ed.) 1.2 (exercises)

Logical Operators

$\vee$ - Disjunction
$\wedge$ - Conjunction
$\neg$ - Negation
$\rightarrow$ - Implication
$\oplus$ - Exclusive or
$\leftrightarrow$ - Biconditional

Do we need all these?

$p \rightarrow q = \neg p \vee q$
$(p \wedge \neg q) \vee (\neg p \wedge q)$
$p \leftrightarrow q \Leftrightarrow$
$(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow$
$(\neg p \vee q) \wedge (\neg q \vee p)$

Functionally Complete

- A set of logical operators is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.
- $\wedge$, $\vee$, and $\neg$ form a functionally complete set of operators.

Are $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ equivalent?

$\neg(p \vee (\neg p \wedge q))$	
$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$	DeMorgan
$\Leftrightarrow \neg p \wedge (\neg\neg p \vee \neg q)$	DeMorgan
$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	Double Negation
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distribution
$\Leftrightarrow (p \wedge \neg p) \vee (\neg p \wedge \neg q)$	Commutative
$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	Negation
$\Leftrightarrow (\neg p \wedge \neg q) \vee F$	Commutative
$\Leftrightarrow (\neg p \wedge \neg q)$	Identity

Are $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ equivalent?

- Even though both are expressed with only $\wedge$, $\vee$, and $\neg$, it is still hard to tell without doing a proof.
- What we need is a unique representation of a compound proposition that uses $\wedge$, $\vee$, and $\neg$.
- This unique representation is called the **Disjunctive Normal Form**.

Disjunctive Normal Form

- A disjunction of conjunctions where every variable or its negation is represented once in each conjunction (a *minterm*)
  - each minterm appears only once

Example: DNF of $p \oplus q$ is

$(p \wedge \neg q) \vee (\neg p \wedge q)$

Truth Table

p	q	$p \oplus q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

Method to construct DNF

- Construct a truth table for the proposition.
- Use the rows of the truth table where the proposition is True to construct minterms
  - If a variable is false, use the negation of the variable in the minterm
  - If the variable is true, use the propositional variable in the minterm
- Connect the minterms with ∨'s.

How to find the DNF of $(p \vee q) \rightarrow \neg r$

p	q	r	(p ∨ q)	¬r	(p ∨ q)→¬r
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

There are five sets of input that make the statement true. Therefore there are five minterms.

p	q	r	(p ∨ q)	¬r	(p ∨ q)→¬r
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

From the truth table we can set up the DNF

(p ∨ q)→¬r ⇔ (p∧q∧¬r) ∨ (p∧¬q∧¬r) ∨ (¬p∧q∧¬r) ∨ (¬p∧¬q∧r) ∨ (¬p∧¬q∧¬r)

Can we show that just ¬ and ∧ form a set of functionally complete operands?

Use DeMorgan's Laws on the DNF.

Example:

(p ∨ q)→¬r

⇔ (p∧q∧¬r) ∨ (p∧¬q∧¬r) ∨ (¬p∧q∧¬r) ∨ (¬p∧¬q∧r) ∨ (¬p∧¬q∧¬r) DNF

⇔ ¬ ¬[(p∧q∧¬r) ∨ (p∧¬q∧¬r) ∨ (¬p∧q∧¬r) ∨ (¬p∧¬q∧r) ∨ (¬p∧¬q∧¬r)] Double Neg

⇔ ¬[¬(p∧q∧¬r) ∧ ¬(p∧¬q∧¬r) ∧ ¬(¬p∧q∧¬r) ∧ ¬(¬p∧¬q∧r) ∧ ¬(¬p∧¬q∧¬r)] DeMorgan

Find an expression equivalent to p → q that uses only conjunctions and negations.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

How many minterms in the DNF?

The DNF of p → q is (p∧q) ∨ (¬p∧ q) ∨ (¬p∧ ¬q).

Then, applying DeMorgan's Law, we get that this is equivalent to

¬[¬(p∧q) ∧ ¬(¬p∧ q) ∧ ¬(¬p∧ ¬q)].

Now can we write an equivalent statement to p → q that uses only disjunctions and negations?

p→q	
⇔¬[¬(p∧q) ∧ ¬(¬p∧ q) ∧ ¬(¬p∧ ¬q)]	From Before
⇔¬[(¬p∨¬q) ∧ (¬¬p∨¬q) ∧ (¬¬p ∨ ¬¬q)]	DeMorgan
⇔¬[(¬p∨¬q) ∧ (p∨¬q) ∧ (p∨q)]	Doub. Neg.
⇔¬(¬p∨¬q) ∨ ¬(p∨¬q) ∨ ¬(p∨q)	DeMorgan