2.5 Solutions by Substitution
Solutions to the Selected Problems

Standard Form

$$\frac{dy}{dx} = f(x, y)$$

where f is a homogeneous function of degree 0, i.e.,

$$f(tx, ty) = f(x, y).$$

Example. Solve the given differential equation by using an appropriate substitution.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Solution

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} \tag{1}$$

Let

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{(tx)(ty)} = \frac{t^2x^2 + t^2y^2}{t^2xy} = \frac{x^2 + y^2}{xy} = f(x, y)$$

Let us make the following substitution:

$$y(x) = xv(x) \tag{2}$$

where $v(x)$ is an arbitrary function of x only.

Now, using eq. (2) into the eq. (1), one gets,

$$\frac{d}{dx}(xv) = \frac{x^2 + x^2v^2}{x^2v}$$

$$x\frac{dv}{dx} + v = \frac{1 + v^2}{v}$$

$$x\frac{dv}{dx} + v = \frac{1}{v} + v$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

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$$vdv = \frac{1}{x}dx$$

$$\int v\,dv = \int \frac{1}{x}dx$$

$$\frac{v^2}{2} = \ln|x| + C$$

$$v^2 = 2\ln|x| + 2C$$

$$v^2 = \ln|x|^2 + 2C$$

$$\frac{y^2}{x^2} = \ln(x^2) + 2C$$

$$\boxed{y^2 = x^2(\ln(x^2) + K)}$$

7. Solve the given differential equation by using an appropriate substitution.

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

Solution

$$\frac{dy}{dx} = \frac{y - x}{y + x} \tag{1}$$

Let

$$f(x, y) = \frac{y - x}{y + x}$$

$$f(tx, ty) = \frac{ty - tx}{ty + tx} = \frac{y - x}{y + x} = f(x, y)$$

Let us make the following substitution:

$$y(x) = xv(x) \tag{2}$$

where $v(x)$ is an arbitrary function of x only.

Now, using eq. (2) into the eq. (1), one gets,

$$\frac{d}{dx}(xv) = \frac{xv - x}{xv + x}$$

$$x\frac{dv}{dx} + v = \frac{v - 1}{v + 1}$$

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$$x\frac{dv}{dx}=\frac{v-1}{v+1}-v$$

$$x\frac{dv}{dx}=-\frac{v^2+1}{v+1}$$

$$\frac{v+1}{v^2+1}dv=-\frac{1}{x}dx$$

$$\int\frac{v+1}{v^2+1}dv=-\int\frac{1}{x}dx$$

$$\int\frac{v}{v^2+1}dv+\int\frac{1}{v^2+1}dv=-\int\frac{1}{x}dx$$

$$\frac{1}{2}\ln|v^2+1|+\tan^{-1}v=-\ln|x|+C$$

$$\ln|v^2+1|+2\tan^{-1}v=-2\ln|x|+2C$$

$$\ln|v^2+1|+2\ln|x|+2\tan^{-1}v=2C$$

$$\ln\left(\frac{y^2}{x^2}+1\right)+\ln|x|^2+2\tan^{-1}\left(\frac{y}{x}\right)=2C$$

$$\ln\left(\frac{y^2+x^2}{x^2}\right)+\ln(x^2)+2\tan^{-1}\left(\frac{y}{x}\right)=2C$$

$$\ln\left[\left(\frac{y^2+x^2}{x^2}\right)x^2\right]+2\tan^{-1}\left(\frac{y}{x}\right)=2C$$

$$\ln(y^2+x^2)+2\tan^{-1}\left(\frac{y}{x}\right)=2C$$

$$\boxed{\ln(y^2+x^2)+2\tan^{-1}\left(\frac{y}{x}\right)=K}$$

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