Separable Variables — 2.2

A differential equation is an equation which involves two or more variable together with their derivatives or it may involve differential coeffecient only.

Ex: ① $\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - y = \cos 2x$. order — 4.

(ii) $\frac{dy}{dx} = \frac{x^2+1}{y+1}$. order — 1.

Order: The highest power of derivative/differential coeffecient is called the order of the D·E

Degree: The highest power of the highest order derivative is the degree of the DE.

Ex: i. $\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + 5y = 0$. order → 2, degree → 2.

ii. $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = 1$. order → 3, degree → 1.

1st order D·E → $p(y)\frac{dy}{dx} = g(x)$

$$\Rightarrow \int p(y)\,dy = \int g(x)\,dx$$

$$\therefore H(y) = G(x) + c$$

Example 1: $(1+x)dy = y\,dx$

or, $\int \frac{dy}{y} = \int \frac{dx}{1+x}$

or, $\ln|y| = \ln|1+x| + c$.

Example 2: $\frac{dy}{dx} = \frac{-x}{y}$. $y(4) = 3$.

or. $dy \cdot y = dx(-x)$

or. $\int y\,dy = -\int x\,dx$

or. $\frac{y^2}{2} = -\frac{x^2}{2} + c$.

using the initial condition:

$\frac{(-3)^2}{2} = -\frac{4^2}{2} + c$.

or. $\frac{9}{2} + \frac{16}{2} = c$.

$\therefore\ c = \frac{25}{2}$.

Hence, $\frac{y^2}{2} = \frac{-x^2}{2} + \frac{25}{2}$

$\therefore\ x^2 + y^2 = 25$.

Example 3: $\frac{dy}{dx} = y^2 - 4$. or. $\frac{dy}{y^2-4} = dx$

or. $\int \frac{dy}{y^2-4} = \int dx$

$\frac{1}{y^2-4} = \frac{1}{(y+2)(y-2)} = \frac{A}{y+2} + \frac{B}{y-2}$.

or, $1 = A(y-2) + B(y+2)$.

$A = -\frac{1}{4}$, $B = \frac{1}{4}$.

or $\int \left(\frac{-1/4}{y+2} + \frac{1/4}{y-2} \right) dy = \int dx$

or. $\frac{-1}{4} \ln|y+2| + \frac{1}{4} \ln|y-2| = x + c$

$\therefore\ \frac{1}{4} \ln\left| \frac{y-2}{y+2} \right| = x + c$.

Example 4: $\left(e^{2y}-y\right)\cos x \frac{dy}{dx} = e^{y}\sin 2x$. $\quad y(0)=0$.

or, $\left(\frac{e^{2y}-y}{e^{y}}\right)dy = \frac{\sin 2x}{\cos x}dx$. $\quad$ [$\sin 2x = 2\sin x \cdot \cos x$.]

or, $\left(e^{y}-\frac{y}{e^{y}}\right)dy = 2\sin x \, dx$.

or, $\int\left(e^{y}-ye^{-y}\right)dy = \int 2\sin x \, dx$

or, $e^{y}+ye^{-y}+e^{-y} = -2\cos x + c$.

Using the initial condition $c=4$.

$\therefore e^{y}+ye^{-y}+e^{-y} = -2\cos x + 4$.

Exercise Set 2.2:

1. $\frac{dy}{dx} = \sin 5x$.

or, $\int dy = \int \sin 5x \, dx$

$\therefore y = -\frac{1}{5}\cos 5x + c$.

2. $\frac{dy}{dx} = (x+1)^2$

or, $\int dy = \int (x+1)^2 dx$

$\therefore y = \frac{1}{3}(x+1)^3$.

3. $dx + e^{3x}dy = 0$.

or, $dx = -e^{3x}dy$

or, $\frac{-dx}{e^{3x}} = dy$

or, $(-1)\int e^{-3x}dx = \int dy$

or, $(-1)\left(-\frac{1}{3}e^{-3x}\right) = y$

$\therefore y = \frac{1}{3}e^{-3x} + c$.

4. $dy - (y-1)^2 dx = 0$.

or, $dy = (y-1)^2 dx$.

or, $\int \frac{dy}{(y-1)^2} = \int dx$

or, $\frac{-1}{y-1} = x + c$.

or, $-y+1 = \frac{1}{x} + c$.

or, $y = 1 - \frac{1}{x} + c$.

5. $x \cdot \frac{dy}{dx} = 4y$

or, $\int \frac{dy}{4y} = \int \frac{dx}{x}$

or, $\frac{1}{4}\ln y = \ln x + c$

or, $\ln y = \ln x^4 + c$

$\therefore\ y = x^4 + c$.

6. $\frac{dy}{dx} + 2xy^2 = 0$.

or, $\frac{dy}{dx} = -2xy^2$

or, $\frac{dy}{y^2} = -\int 2x\,dx$

or, $-\frac{1}{y} = -x^2 + c$

$\therefore\ y = \frac{1}{x^2} + c$.

7. $\frac{dy}{dx} = e^{3x+2y} = e^{3x}\cdot e^{2y}$.

or, $e^{-2y}\,dy = e^{3x}\,dx$

or, $-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + c$.

8. $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y} = e^{-y} + e^{-2x}\cdot e^{-y}$

or, $e^x y \frac{dy}{dx} = e^{-y}(1 + e^{-2x})$.

or, $\int y e^y\,dy = \int (e^{-x} + e^{-3x})\,dx$.

or, $ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + c$

or, $ye^y - e^y + e^{-x} + \frac{1}{3}e^{-3x} = c$.

9. $y\ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$.

or, $\int x^2 \ln x\,dx = \int \frac{(y+1)^2}{y}\,dy$

or, $\int x^2 \ln x\,dx = \int \frac{y^2+2y+1}{y}\,dy$

or, $\frac{x^3}{3}\ln x - \frac{1}{9}x^3 = \frac{y^2}{2} + 2y + \ln y + c$.

$\int \ln x \cdot x^2$

$= \ln x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}$

$= \frac{1}{3}x^3 \ln x - \int \frac{x^2}{3}$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{x^3}{3}$.

10. $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$.

Here
$2y+3=z$
$2\,dy = dz$
$dy = \frac{1}{2}dz$
$4x+5=u$
$4\,dx = du$
$dx = \frac{du}{4}$

or, $\displaystyle\int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$

or, $\frac{1}{2}(-1)\frac{1}{(2y+3)} = \frac{1}{4}(-1)\frac{1}{(4x+5)} + c$

or, $\dfrac{1}{2(2y+3)} = \dfrac{1}{4(4x+5)} + c$.

11. $\operatorname{cosec} y\, dx + \sec^2 x\, dy = 0$.

or, $\operatorname{cosec} y\, dx = -\sec^2 x\, dy$.

$2\cos^2 x = 1+\cos 2x$.

or, $\displaystyle\int \frac{dy}{\operatorname{cosec} y} = \int -\frac{dx}{\sec^2 x}$

or, $\displaystyle\int \sin y\, dy = (-1)\frac{1}{2}\int 2\cos^2 x\, dx$.

or, $\displaystyle\int \sin y\, dy = -\frac{1}{2}\int (1+\cos 2x)\,dx$.

or, $-\cos y = -\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + c$

or, $\cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$.

12. $\sin 3x\, dx = -2y\cos^3 3x\, dy$.

$\tan 3x = z$.
$3\sec^2 3x\, dx = dz$.

or, $\displaystyle\int \frac{\sin 3x}{\cos 3x}\sec^2 3x\, dx = -2\int y\, dy$.

or, $\displaystyle\int \tan 3x \cdot \sec^2 3x\, dx = -y^2 + c$

or, $\displaystyle 3\int z\, dz = -y^2 + c$.

or, $\dfrac{3z^2}{2} = -y^2 + c$.

or, $\frac{3}{2}(\tan 3x)^2 = -y^2 + c$.

13. $(e^y+1)^2 e^{-y}\,dx + (e^x+1)^3 e^{-x}\,dy = 0.$

$e^y+1 = z$
$e^y\,dy = dz.$

or, $\dfrac{(e^y+1)^2}{e^y}\,dx = -\dfrac{(e^x+1)^3}{e^x}\,dy.$

or, $\displaystyle\int \frac{e^y}{(e^y+1)^2}\,dy = -\int \frac{e^x\,dx}{(e^x+1)^3}.$

or, $(-1)\cdot\left(\dfrac{1}{e^y+1}\right) = +2\cdot\dfrac{1}{(e^x+1)^2} + c.$

14. $x(1+y^2)^{1/2}\,dx = y\,(1+x^2)^{1/2}\,dy.$

$1+x^2 = z.$
$2x\,dx = dz.$

or, $\displaystyle\int \frac{x\,dx}{(1+x^2)^{1/2}} = \int \frac{y\,dy}{(1+y^2)^{1/2}}$

or, $(1+y^2)^{1/2} = (1+x^2)^{1/2} + c.$

19. $\dfrac{dy}{dx} = \dfrac{xy+3x-y-3}{xy-2x+4y-8} = \dfrac{x(y+3)-1(y+3)}{x(y-2)+4(y-2)}.$

or, $\dfrac{dy}{dx} = \dfrac{(y+3)(x-1)}{(y-2)(x+4)}.$

or, $\displaystyle\int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx$

or, $\displaystyle\int \frac{y-2+5-5}{y+3}\,dy = \int \frac{x-1+5-5}{x+4}\,dx$

or, $\displaystyle\int \left(1-\frac{5}{y+3}\right)dy = \int \left(1-\frac{5}{x+4}\right)dx$

or, $y - 5\ln|y+3| = x - 5\ln|x+4| + c.$

20. $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$

$= \frac{y(x+2)-1(x+2)}{y(x-3)+1(x-3)}$

or, $\frac{dy}{dx} = \frac{(x+2)(y-1)}{(x-3)(y+1)}$

or, $\int \frac{y+1}{y-1} dy = \int \frac{x+2}{x-3} dx$

or, $\int \frac{y-1+2}{y-1} dy = \int \frac{x-3+5}{x-3} dx$

or, $y + 2\ln(y-1) = x + 5\ln(x-3) + c$

21. $\frac{dy}{dx} = x\sqrt{1-y^2}$.

$\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int x\,dx$.

$\Rightarrow \sin^{-1} y = \frac{x^2}{2} + c$

$\therefore y = \sin\left(\frac{1}{2}x^2\right) + c$

22. $(e^x + e^{-x}) \frac{dy}{dx} = y^2$

or, $\frac{dy}{y^2} = \frac{dx}{e^x + e^{-x}}$.

or, $-\frac{1}{y} = \int \frac{e^x dx}{(e^x)^2+1}$

$\therefore -\frac{1}{y} = \tan^{-1} e^x + c$.

Now,

$e^x = u$

$\therefore e^x dx = du$.

$dx = \frac{du}{e^x}$.

$= \int \frac{du}{(e^x)^2+1}$

$= \int \frac{du}{u^2+1}$

$= \tan^{-1} u + c$

$= \tan^{-1}(e^x) + c$.

23. $\frac{dx}{dt} = 4(x^2+1)$. $x(\pi/4) = 1$.

$\Rightarrow \int \frac{dx}{x^2+1} = 4\int dt$

$\Rightarrow \tan^{-1} x = 4t + c$.

$\therefore \tan^{-1} \pi/4 = 4 + c$

$\therefore c = -3\pi/4$.

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24. $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$. $\quad y(2)=2$.

or, $\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1}$.

or, $\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$.

or, $\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| + c_1$

$y=2, x=2$.

$c=0$.

25. $x^2 \frac{dy}{dx} = y - xy$ $\quad y(-1) = -1$.

or, $x^2 \frac{dy}{dx} = y(1-x)$

or, $\frac{dy}{y} = \left(\frac{1-x}{x^2}\right)dx$

or, $\frac{dy}{y} = \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$ $\qquad 0 = 1+c$

or, $\int \frac{dy}{y} = \int\left(\frac{1}{x^2} - \frac{1}{x}\right)dx$. $\qquad \therefore c = -1$.

or, $\ln y = -\frac{1}{x} - \ln x + c$.

$\therefore \ln y = -\frac{1}{x} - \ln x - 1$.

26. $\frac{dy}{dt} + 2y = 1$, $\quad y(0) = \frac{5}{2}$

or, $\frac{dy}{dt} = 1 - 2y$

or, $\frac{dy}{1-2y} = dt$

or, $\int \frac{dy}{1-2y} = \int dt$

or, $\ln \frac{|1-2y|}{-2} = t + c$.

$\frac{\ln|-4|}{-2} = c$

$\Rightarrow -\frac{1}{2}\ln 4 = c$.

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* $(e^y+1)^2 e^x dx + (e^x+1)^3 e^y dy = 0.$

similar to no "13"

or, $\int \frac{e^x}{(e^x+1)^3} dx = -\int \frac{e^y}{(e^y+1)^2} dy$

$u = e^x + 1$
$du = e^x dx.$

or, $\int \frac{du}{u^3} = -\int \frac{dv}{v^2}.$

$v = e^y + 1$
$dv = e^y dy$

or, $-\frac{1}{2u^2} = \frac{1}{v} + c.$

$\therefore \frac{1}{-2(e^x+1)^2} = \frac{1}{(e^y+1)} + c.$

Linear Equation → 2.3

The general form of a linear 1st order D.E is,

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$, then it is called homogenous linear 1st order DE;

If $(g(x)) \neq 0$, then it is called no-homogenous liner 1st order DE;

Solving procedure :-

① Make the derivative term free of coefficients.

i.e, dividing the equation ① by $a_1 x$

$$\frac{dy}{dx} + P(x)y = f(x) \rightarrow \text{Standard form.}$$

(ii) Find the Integrating factor (I.F)

$$I.F = e^{\int P(x)dx}$$

(iii) Solution: $y \times IF = \int (IF \times f(x))dx + c.$

Example-1: $\frac{dy}{dx} - 3y = 0.$

Here, $P(x) = -3$

$\therefore I.F = e^{\int -3dx}$

$= e^{-3x}.$

Solution: $y \cdot e^{-3x} = \int e^{-3x} \times 0 \; dx + c$

or, $y \cdot e^{-3x} = c$

$\therefore y = (e^{3x}) \cdot c$

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Example 2: $\frac{dy}{dx} - 3y = 6$

$$I.F = e^{-3x}$$

$$\text{Solution} \rightarrow y \cdot e^{-3x} = \int e^{-3x} \cdot 6 \, dx + c$$

$$= 6 \frac{e^{-3x}}{-3} = -2e^{-3x}$$

$$\text{or, } y \cdot e^{-3x} = -2e^{-3x} + c$$

$$\text{or, } \frac{y}{e^{3x}} = -2e^{-3x} + c$$

$$\therefore y = -2 + e^{3x} c.$$

* Example 3: $x \frac{dy}{dx} - 4y = x^6 e^x$.

Dividing by x, we get the standard form.

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x. \quad \text{—} (1)$$

$$P(x) = -\frac{4}{x}; \quad f(x) = x^5 e^x.$$

$$I.F = e^{\int -\frac{4}{x} dx}$$

$$= e^{-4 \ln x} = e^{\ln x^{-4}} = x^{-4}.$$

Multiply (1) with $x^{-4} \Rightarrow$

$$x^{-4} \frac{dy}{dx} - 4x^{-5} y = x e^x$$

$$\text{or } \frac{d}{dx}\left[x^{-4} y\right] = x e^x.$$

$$\text{Now, } y \cdot x^{-4} = \int e^x x \, dx + c$$

$$= x e^x - e^x + c$$

$$\therefore y = x^5 e^x - x^4 e^x + c x^4$$

Example-4: $(x^2-9)\frac{dy}{dx}+xy=0$.

or, $\frac{dy}{dx}+\frac{x}{x^2-9}y=0$

$I.F = e^{\int \frac{x}{x^2-9}dx}$

$= e^{\frac{1}{2}\int 2x/x^2-9 \, dx}$

$= e^{\frac{1}{2}\ln|x^2-9|}$

$= \sqrt{x^2-9}$.

Solution → $y\cdot\sqrt{x^2-9} = \int \sqrt{x^2-9}\; 0 + c$

or, $y\sqrt{x^2-9} = c$

$\therefore\; y = \frac{c}{\sqrt{x^2-9}}$.

Example-5: $\frac{dy}{dx}+y=x$ $\qquad y(0)=4$.

$IF = e^{\int 1dx} = e^{x}$

$y\cdot e^{x} = \int e^{x}\cdot x + c$

or, $y\cdot e^{x} = e^{x}(x-1)+c$

or, $y\cdot e^{x} = e^{x}x - e^{x}+c$

or, $y = x-1+ce^{-x}$. From initial condition $y=4$ when $x=0$.

$c=5$

$\therefore\; y = x-1+5e^{-x}$.

Note: $\frac{dy}{dx}=\frac{1}{x+y^2}$ is not linear in the variable y.

But its reciprocal $\frac{dx}{dy}=x+y^2$

or, $\frac{dx}{dy}-x=y^2$. is recognized as a liner in

the variable x.

$e^{\int(-1)dy} = e^{-y}$.

$e^{-y}\cdot x = \int e^{-y}\cdot y^2 + c$.

$\therefore\; x = -y^2-2y-2+ce^{y}$.

Exercise — 2.3 ⇒.

3. $\frac{dy}{dx}+y=e^{3x}$.

$I.F = e^{\int 1 dx} = e^{x}$

Solution → $y \cdot e^{x} = \int e^{x} \cdot e^{3x} dx + c$

or, $ye^{x} = \frac{1}{4}e^{4x} + c$

or, $y = \frac{e^{4x}}{4e^{x}} + ce^{-x}$

or, $y = \frac{e^{3x} \cdot e^{x}}{4e^{x}} + ce^{-x}$

$\therefore y = \frac{1}{4}e^{3x} + ce^{-x}$.

4. $3\frac{dy}{dx} + 12y = 4$. or, $\frac{dy}{dx} + 4y = \frac{4}{3}$

$IF = e^{\int 4 dx} = e^{4x}$

Solution → $y \cdot e^{4x} = \int e^{4x} \cdot \frac{4}{3} dx + c$

$= \frac{e^{4x}}{4} \cdot \frac{4}{3} + c$

or, $y \cdot e^{4x} = \frac{e^{4x}}{3} + c$

$\therefore y = \frac{1}{3} + ce^{-4x}$.

5. $y' + 3x^2y = x^2 \Rightarrow \frac{dy}{dx} + 3x^2y = x^2$

$IF = e^{\int 3x^2 dx}$

$= e^{x^3}$.

Sol^n: $y \cdot e^{x^3} = \int e^{x^3} x^2 dx + c$

or, $y \cdot e^{x^3} = e^{x^3}/3 + c$

$\therefore y = \frac{1}{3} + ce^{-x^3}$.

$\int x^2 e^{x^3}$

$= \frac{1}{3}\int e^{u} du$

$= \frac{e^{x^3}}{3}$

$u = x^3$

$\frac{du}{3} = x^2 dx$.

$\frac{x^3}{3}$

6. $y' + 2xy = x^3$ or, $\frac{dy}{dx} + 2xy = x^3$.

$IF = e^{\int 2x\,dx} = e^{x^2}$.

Soln: $ye^{x^2} = \int (e^{x^2}\cdot x^3)\,dx + c$.

or, $ye^{x^2} = \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + c$

or, $y = \frac{x^2}{2} - \frac{1}{2} + ce^{-x^2}$.

$\int e^{x^2}\cdot x^3\,dx$ $\quad u = x^2$, $du = 2x\,dx$, $\therefore \frac{du}{2} = x\,dx$.

$= \int e^{x^2}\cdot x^2\cdot x\,dx$

$= \frac{1}{2}\int e^u u\,du$

Now, $\int e^u u\,du$

$= ue^u - e^u$.

$\therefore \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + c$.

7. $x^2y' + xy = 1$. or, $\frac{dy}{dx}x^2 + xy = 1$.

or, $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}$.

$I.F = e^{\int \frac{1}{x}dx}$

$= e^{\ln x} = x$.

Soln: $yx = \int (x\cdot\frac{1}{x^2})\,dx + c$

$= \int \frac{1}{x}\,dx + c$

or, $yx = \ln x + c$

$\therefore y = \frac{\ln x}{x} + cx^{-1}$.

8. $y' = 2y + x^2 + 5$ or, $\frac{dy}{dx} - 2y = +x^2 + 5$.

$I.f = e^{\int -2\,dx} = e^{-2x}$.

Soln: $y\cdot e^{-2x} = \int [e^{-2x}\cdot(x^2+5)]\,dx + c$.

or, $ye^{-2x} = (-\frac{1}{2}e^{-2x})(x^2+5) - \frac{1}{2}e^{-2x}x + \frac{e^{-2x}}{2} + c$.

$x^2+5\,(-\frac{1}{2}e^{-2x}) - \int 2x(-\frac{1}{2}e^{-2x})\,dx$

$= (x^2+5)(-\frac{1}{2}e^{-2x}) + \int e^{-2x}x\,dx$

$= (-\frac{1}{2}e^{-2x})(x^2+5) - \frac{1}{2}e^{-2x}x + \frac{e^{-2x}}{2}$.

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9. $x\frac{dy}{dx} - y = x^2 \sin x$

or, $\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = x\sin x$

$I.F = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$.

Sol^n: $y\cdot\frac{1}{x} = \int x\sin x\cdot\frac{1}{x}\,dx + c$

or, $\frac{y}{x} = -\cos x + c$

$\therefore y = -x\cos x + cx$.

10. $x\cdot\frac{dy}{dx} + 2y = 3$. or, $\frac{dy}{dx} + \frac{2}{x}y = \frac{3}{x}$ —(1)

$I.F = e^{\int\frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.

Sol^n: $y\cdot x^2 = \int x^2\cdot\frac{3}{x}\,dx + c$

$= \int 3x\,dx + c$

or, $yx^2 = \frac{3}{2}x^2 + c$

or, $y = \frac{3}{2} + \frac{c}{x^2}$.

11. $x\cdot\frac{dy}{dx} + 4y = x^3 - x$. or, $\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1$.

$I.F = e^{\int\frac{4}{x}dx} = e^{4\ln x} = x^4$.

Sol^n: $y\cdot x^4 = \int x^4(x^2-1)\,dx + c$

$= \int(x^6 - x^4)\,dx + c$

or, $y\cdot x^4 = \frac{x^7}{7} - \frac{x^5}{5} + c$

$\therefore y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$.

12. $(1+x)\dfrac{dy}{dx} - xy = x(1+x)$.

or, $\dfrac{dy}{dx} - \dfrac{x}{1+x}y = x$.

$$\text{I.F} = e^{\int -\frac{x}{1+x}dx} = e^{-\int \frac{x+1-1}{x+1}dx}$$
$$= e^{-\int\left(1-\frac{1}{x+1}\right)dx}$$
$$= e^{-[x-\ln(x+1)]}$$
$$= e^{-x+\ln(x+1)}$$
$$= e^{-x}\cdot e^{\ln(x+1)}$$
$$= e^{-x}\cdot(x+1).$$

Soln: $y\cdot e^{-x}(x+1) = \int x\cdot e^{-x}(x+1)\,dx + c = \int (x^2+x)e^{-x}\,dx + c$

$$= \int x^2 e^{-x}\,dx + \int x e^{-x}\,dx + c$$
$$= -e^{-x}(x^2+2x+2) - xe^{-x} + e^{-x} + c$$

$\therefore\ ye^{-x}(x+1) = -e^{-x}(x^2+2x+2) - e^{-x}(x-1) + c$.

13. $x^2y' + x(x+2)y = e^x$

or, $x^2\dfrac{dy}{dx} + x(x+2)y = e^x$

or, $\dfrac{dy}{dx} + \dfrac{x+2}{x}y = \dfrac{e^x}{x^2}$.

or, $\dfrac{dy}{dx} + \left(1+\dfrac{2}{x}\right)y = e^x x^{-2}$

$$\text{I.F} = e^{\int\left(1+\frac{2}{x}\right)dx} = e^{x+2\ln x}$$
$$= e^x\cdot e^{(\ln x)^2}$$
$$= e^x\cdot x^2$$

Soln: $y\cdot e^x x^2 = \int (e^x\cdot x^2\cdot e^x\cdot x^{-2})\,dx + c$

$$= \int (e^x)^2\,dx + c$$

$\therefore\ y\cdot x^2 e^x = \dfrac{e^{2x}}{2} + c$

14. $xy' + (1+x)y = e^{-x}\sin 2x$.

or, $x \cdot \frac{dy}{dx} + (1+x)y = e^{-x}\sin 2x$

or, $\frac{dy}{dx} + \left(\frac{1}{x}+1\right)y = \frac{e^{-x}\sin 2x}{x}$.

$I \cdot F = e^{\int\left(\frac{1}{x}+1\right)dx}$

$= e^{\ln x + x} = xe^{x}$.

Soln: $y \cdot xe^{x} = \int \left(xe^{x} \cdot \frac{e^{-x} \cdot \sin 2x}{x}\right) dx + c$

$= \int \sin 2x \, dx + c$

$\therefore y \cdot xe^{x} = -\frac{\cos 2x}{2} + c$.

15. $y\,dx - 4(x+y^6)\,dy = 0$.

or, $y\,dx - (4x + 4y^6)\,dy = 0$

or, $\frac{dx}{dy} - \frac{4}{y}x - 4y^5 = 0$

or, $\frac{dx}{dy} - \frac{4}{y}x = 4y^5$

$I \cdot F = e^{-\int \frac{4}{y} dy} = e^{-4\ln y} = e^{\ln y^{-4}} = y^{-4}$.

Soln: $xy^{-4} = \int (4y^5 \cdot y^{-4})\,dy + c$

$= \int 4y \cdot dy + c$

$\therefore xy^{-4} = 2y^2 + c$

16. $y\,dx = (ye^{y} - 2x)\,dy$.

$\Rightarrow \frac{dx}{dy} = e^{y} - \frac{2x}{y}$.

$\Rightarrow \frac{dx}{dy} + \frac{2}{y}x = e^{y}$.

$I \cdot F = e^{\int \frac{2}{y} dy} = e^{2\ln y} = y^2$.

P.T.O

Soln: $xy^2 = \int (y^2 e^y)\, dy + c$

$= y^2 e^y - 2\int y\cdot e^y\, dy + c$

$\therefore xy^2 = y^2 e^y - 2(y\cdot e^y + e^y) + c.$

17. $\cos x \frac{dy}{dx} + (\sin x)y = 1.$

or, $\frac{dy}{dx} + \tan x\, y = \sec x.$

I.F $= e^{\int \tan x\, dx}$

$= e^{\ln \sec x} = \sec x.$

Soln: $y\cdot \sec x = \int (\sec x\cdot \sec x)\, dx + c$

$= \int (\sec^2 x)\, dx + c$

$\therefore y\cdot \sec x = \tan x + c$

19. $(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}.$

$\Rightarrow \frac{dy}{dx} + \left(1+\frac{1}{x+1}\right)y = \frac{2xe^{-x}}{x+1}$

I.F $= e^{\int \left(1+\frac{1}{x+1}\right) dx} = e^{x+\ln x} = e^x . x$

$u = x+1.$

Soln: $y\cdot e^x x = \int \left(xe^x \cdot \frac{2xe^{-x}}{x+1}\right) dx + c$ $\qquad \frac{du}{dx} = 1.$

$= \int \frac{2x^2}{x+1} + c$

$= 2\int \frac{(u-1)^2}{u} \cdot du + c$

$= 2\int \frac{u^2 - 2u + 1}{u} \cdot du + c$

$= 2\int \left(u - 2 + \frac{1}{u}\right) du + c.$

$\therefore y\cdot e^x x = 2\ln(x+1) + (x+1)^2 - 4(x+1) + c.$

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24. $(x^2-1)\frac{dy}{dx}+2y=(x+1)^2$

or, $\frac{dy}{dx}+\frac{2}{x^2-1}y=\frac{x+1}{x-1}$.

$I.F = e^{\int \frac{2}{x^2-1}dx}$.

$= e^{2\int \frac{1}{(x+1)(x-1)}dx}$

$= e^{2\cdot\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)}$

$= \frac{x-1}{x+1}$.

$\therefore y\cdot\frac{x-1}{x+1}=\int\left(\frac{x-1}{x+1}\cdot\frac{x+1}{x-1}\right)dx+c$

$= x+c$.

$*\quad \frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{(x-1)}$

or, $1=A(x-1)+B(x+1)$.

$\therefore A=-\frac{1}{2},\ B=\frac{1}{2}$.

$\therefore \int\frac{-1/2}{x+1}+\int\frac{1/2}{x-1}$

$=-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|$

$=\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)$.

25. $xy' + y = e^x$. $y(1) = 2$.

or, $x \cdot \frac{dy}{dx} + y = e^x$

or, $\frac{dy}{dx} + \frac{y}{x} = e^x x^{-1}$.

$I.F = e^{\int 1/x \, dx} = e^{\ln x} = x$.

Soln: $yx = \int (x \cdot x^{-1} e^x) \, dx + c$.

$= e^x + c$.

now, $2 \cdot 1 = e^1 + c$

$\therefore c = -0.72$.

26. $y \frac{dx}{dy} - x = 2y^2$ $y(1) = 5$.

$\Rightarrow \frac{dx}{dy} - \frac{1}{y} x = 2y$.

$I.F = e^{\int -1/y \, dy} = e^{-\ln y} = y^{-1}$.

Soln: $-xy = \int (-y \cdot 2y) \, dy + c$

$= -y^2 + c$.

$\therefore -5 = -25 + c$

$\therefore c = 20$.

$\therefore y^2 - xy - 20 = 0$. Ans.

29. $(x+1) \frac{dy}{dx} + y = \ln x$. $y(1) = 10$.

or, $\frac{dy}{dx} + \frac{1}{x+1} \cdot y = \frac{\ln x}{x+1}$.

$I.F = e^{\int \frac{1}{x+1} dx}$

$= e^{\ln(x+1)}$

$= x+1$.

Sol^n: $y(x+1) = \int \left[(x+1) \cdot \frac{\ln x}{(x+1)} \right] dx + c$

$= \int \ln x + c$

$= x\ln x - x + c.$

now, $10 \times 2 = \ln 1 - 1 + c$

$\therefore c = 21.$

$\therefore y(x+1) - x\ln x + x - 21 = 0.$

30. $y' + (\tan x)y = \cos^2 x . \; y(0) = -1.$

I.F $= e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

Soln: $y \cdot \sec x = \int (\sec x \cdot \cos^2 x) dx + c.$

$= \int \left(\cancel{\sec x} \cdot \frac{1}{\cancel{\sec x}} \cdot \frac{1}{\sec x} \right) dx + c$

$= \int \cos x \, dx + c$

$= \sin x + c.$

$c = -1.$

$\therefore y\sec x - \sin x + 1 = 0.$

Differentional equations.

Exact equation — 2.4

* A differential equation,

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{—— (I)}$$

is called exact D.E if, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solving procedure:-

If (I) is an exact D.E, then there exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y) \text{ —— (II) and } \frac{\partial f}{\partial y} = N(x,y) \text{ —— (III)}$$

Integrating (III) W.R.T only 'y'. Keeping x constant.

$$f(x,y) = \int N(x,y)\,dy + g(x) \text{ —— (IV)}$$

Now, differentiating (IV) w.r.to x only.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\int N(x,y)\,dy\right] + g'(x) \text{ —— (V)}$$

comparing (II) and (V),

$$g'(x) = \square$$
$$g(x) = \square \text{ —— (VI)}$$

Now, putting the value of $g(x)$ in (IV)

$$f(x,y) = \square.$$

Solution:- $f(x,y) = c$.

Example - 1: $2xy\,dx + (x^2-1)dy = 0$ —— (1)

where, $M(x,y) = 2xy$; $N(x,y) = (x^2-1)$

$\frac{\partial M}{\partial y} = 2x$. $\frac{\partial N}{\partial x} = 2x$

$\therefore$ the D.E is exact

There exists a function $f(x,y)$ such that,

$$\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y)$$

or, $\frac{\partial f}{\partial x} = 2xy$ —— (2) $\frac{\partial f}{\partial y} = x^2-1$ —— (3)

Integrating (3) w.r.to y only keeping x constant

$$f(x,y) = \int (x^2-1)\,dy + g(x)$$

$$\therefore f(x,y) = x^2y - y + g(x) \text{ —— (4)}$$

Differentiating (4) with respect to x only,

$$\frac{\partial f}{\partial x} = 2xy + g'(x) \text{ —— (5)}$$

Comparing (2) and (5) $\Rightarrow$

$$g'(x) = 0$$

$$g(x) = c$$

Now putting the value of $g(x)$ in (4).

$$f(x,y) \Rightarrow x^2y - y = c.$$

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Example-2: $\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos xy + 2y\right)dy = 0$ —— (i)

where, $M(x,y) = e^{2y} - y\cos xy$; $N(x,y) = 2xe^{2y} - x\cos xy + 2y$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \left[\frac{d}{dy}y \cdot \cos xy + y \cdot \frac{d}{dy}\cos xy\right]$$

$$= 2e^{2y} - \left[1 \cdot \cos xy + (-\sin xy) \cdot \frac{d}{dy}(xy) \cdot y\right]$$

$$= 2e^{2y} - \cos xy + \sin xy \cdot xy$$

$$= 2e^{2y} + xy\sin xy - \cos xy$$

$$\frac{\partial N}{\partial x} = 2e^{2y} - \left[\frac{d}{dx}x \cdot \cos xy + x \cdot \frac{d}{dx}(\cos xy)\right]$$

$$= 2e^{2y} - \left[\cos xy + (-\sin xy)\,\frac{d}{dx}(xy)\,x\right]$$

$$= 2e^{2y} - \cos xy + xy\sin xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \left[\text{So the D.E is exact}\right]$$

There exist a function $f(x,y)$, such that

$$\frac{\partial f}{\partial x} = M(x,y), \qquad \frac{\partial f}{\partial y} = N(x,y)$$

or, $\frac{\partial f}{\partial x} = \left(e^{2y} - y\cos xy\right)$ —— (ii) or, $\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$ —— (iii)

Integrating (iii) w.r.to y only, keeping x constant

$$f(x,y) = \int\left(2xe^{2y} - x\cos xy + 2y\right)dy + g(x)$$

$$= 2x\int e^{2y}dy - x\int \cos xy\, dy + 2\int y\, dy + g(x)$$

$$= 2x \cdot \frac{e^{2y}}{2} - x \cdot \frac{\sin xy}{x} + 2 \cdot \frac{y^2}{2} + g(x)$$

$$= xe^{2y} - \sin xy + y^2 + g(x) \quad \text{—— (iv)}$$

Now differentiating (iv) w.r.to x only.

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + g'(x) \quad \text{—— (v)}$$

Comparing (ii) and (v) $\Rightarrow$ $g'(x) = 0$.

$\therefore \; g(x) = c$

Now from equation (iv), $xe^{2y} - \sin xy + y^2 + c = 0$.

Example 3 : $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$.

Now, $(xy^2 - \cos x \sin x)dx = y(1-x^2)dy$.

or, $(\cos x \sin x - xy^2)dx + y(1-x^2)dy = 0$.

$\dfrac{\partial M}{\partial y} = -2xy$, $\dfrac{\partial N}{\partial x} = -2xy$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ [so the D.E is exact]

There exist a function $f(x,y)$ such that

$\dfrac{\partial f}{\partial x} = \cos x \sin x - xy^2$ —(ii), $\dfrac{\partial f}{\partial y} = y(1-x^2)$ —(iii)

Integrating (iii) w.r.to y only, keeping x constant

$$f(x,y) = \int y(1-x^2)\, dy + g(x)$$
$$= \frac{y^2}{2}(1-x^2) + g(x) \quad \text{—(iv)}$$

Differentiating (iv) w.r.to x

$$\frac{\partial f}{\partial x} = -xy^2 + g'(x) \quad \text{—(v)}$$

comparing (ii) and (v), $g'(x) = \cos x \sin x$

$g(x) = -\int (\cos x)(-\sin x\, dx)$

$= -\frac{1}{2}\cos^2 x$

Thus, $\dfrac{y^2}{2}(1-x^2) - \dfrac{1}{2}\cos^2 x = c$

or, $y^2(1-x^2) - \cos^2 x = c$

The initial condition $y=2$ when $x=0$.

$$4(1-0) - \cos(0) = c$$
$$\therefore c = 3$$

Solution $\rightarrow$ $y^2(1-x^2) - \cos x = 3$.

Example-4: $xy\,dx + (2x^2+3y^2-20)\,dy = 0$.

$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 4x$$

Hence not exact.

* $\frac{M_y - N_x}{N}$, if the term is only a function of x. Then. $I.F = e^{\int f(x)dx}$.

* $g(y) = \frac{N_x - M_y}{M}$, if the term is only a function of y. Then. $I.F = e^{\int g(y)dy}$.

$M_y = x$ $\quad$ $N_x = 4x$

$M_x = y$ $\quad$ $N_y = 6y$.

$$\therefore f(x) = \frac{x-4x}{2x^2+3y^2-20} \quad g(y) = \frac{4x-x}{xy}$$
$$\Rightarrow \frac{3x}{xy} = \frac{3}{y}$$

So the, $I.F = e^{\int \frac{3}{y}dy}$
$$\Rightarrow e^{3\ln y} = y^3$$

Multiply I.F with the given equation,

$$xy^4dx + (2x^2y^3 + 3y^5 - 20y^3) = 0.$$

$$\frac{\partial M}{\partial y} = 4xy^3. \quad \frac{\partial N}{\partial x} = 4xy^3 \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ Hence the D.E is exact.

There exist a function $f(x,y)$ such that.

$$\frac{\partial f}{\partial x} = xy^4 \;\text{—(ii)}. \quad \frac{\partial f}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3 \;\text{—(iii)}$$

$$f(x,y) = \int (2x^2y^3 + 3y^5 - 20y^3)\,dy + g(x)$$
$$= \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 + g(x)$$

$$\therefore \frac{\partial f}{\partial x} = \frac{2xy^4}{2} = xy^4. \quad \therefore \text{Now} \Rightarrow g'(x) = 0, \; g(x) = c.$$

$$\therefore \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c.$$

Exercise set 2.4:

1. $(2x-1)dx + (3y+7)dy = 0$ — (i)

$\frac{\partial M}{\partial y} = 0 \cdot \quad \frac{\partial N}{\partial x} = 0.$

$\frac{\partial f}{\partial x} = 2x-1$ — (ii) $\quad \frac{\partial f}{\partial y} = 3y+7$ — (iii)

$f(x,y) = \int (3y+7)\,dy + g(x)$

$= \frac{3}{2}y^2 + 7y + g(x)$ — (iv)

differentiating (iv): $\frac{\partial f}{\partial x} = 0 \cdot$ — (v)

$g'(x) = 2x-1$

$g(x) = x^2 - x$

$\therefore \frac{3}{2}y^2 + 7y + x^2 - x = c.$

2. $(2x+y)dx - (x+6y)dy = 0$

$\Rightarrow (-2x-y)dx + (-x-6y)dy = 0.$

$\frac{\partial M}{\partial y} = -1. \quad \frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore$ D.E is exact.

$\frac{\partial f}{\partial x} = -2x-y$ — (ii) $\cdot \quad \frac{\partial f}{\partial y} = -x-6y$ — (iii)

$f(x,y) = \int (-x-6y)\,dy + g(x)$

$= -xy - 3y^2 + g(x)$ — (iv)

$\frac{\partial f}{\partial x} = -y + g'(x)$ — (v)

$g'(x) = -2x$

$g(x) = -x^2$

$\therefore\ -xy - 3y^2 - x^2 = c$

Solution.

$(5x+4y)dx + (4x-8y^3)dy = 0$ — ①

$\frac{\partial M}{\partial y} = 4, \quad \frac{\partial N}{\partial x} = 4.$

$\frac{\partial f}{\partial x} = 5x+4y$ — ⑪ $\quad \frac{\partial f}{\partial y} = 4x - 8y^3$ — ⑪⑪

$$f(x,y) = \int (4x-8y^3)\,dy + g(x)$$

$$= 4xy - 2y^4 + g(x) \text{ — (iv)}$$

differentiating (iv) with R.to x.

$$\frac{\partial f}{\partial x} = 4y + g'(x)$$

$\therefore\ g'(x) = 5x.$

$g(x) = \frac{5}{2}x^2$

$\therefore\ 4xy - 2y^4 + \frac{5}{2}x^2 = c.$

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4. $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)\,dy = 0$. $y(\frac{\pi}{2}) = 0$

$\frac{\partial M}{\partial y} = \cos y - \sin x$ $\quad \frac{\partial N}{\partial x} = -\sin x + \cos y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. [so, the D.E is exact].

$\frac{\partial f}{\partial x} = \sin y - y\sin x$ — (ii). $\frac{\partial f}{\partial y} = \cos x + x\cos y - y$ — (iii)

$f(x,y) = \int (\cos x + x\cos y - y)\,dy + g(x)$

$= y\cos x + x\sin y - \frac{y^2}{2} + g(x)$ — (iv)

Differentiating (iv) w.r.to x

$\frac{\partial f}{\partial x} = -y\sin x + \sin y + g'(x)$ — (v)

Now, $g'(x) = 0$, $g(x) = c$

from (iv) $\Rightarrow$ $y\cos x + x\sin y - \frac{y^2}{2} - 0 = c$.

when $x = \frac{\pi}{2}$, $y = 0$.

$0 \cdot \cos\frac{\pi}{2} + \frac{\pi}{2}\sin 0 - 0 = c$

$\therefore c = 0$.

Now, $y\cos x + x\sin y - \frac{y^2}{2} = 0$.

5. $(2xy^2 - 3)dx + (2x^2y + 4)dy = 0$.

$\frac{\partial M}{\partial y} = 4xy$ $\quad \frac{\partial N}{\partial x} = 4xy$

$\frac{\partial f}{\partial x} = 2xy^2 - 3$, — (ii). $\frac{\partial f}{\partial y} = 2x^2y + 4$ — (iii)

$f(x,y) = \int (2x^2y + 4)dy + g(x)$

$= x^2y^2 + 4y + g(x)$ — (iv)

Differentiating (4) with respect to x.

$$\frac{\partial f}{\partial x} = 2xy^2 \text{ ——(v)}$$

Now. $g'(x) = -3$

$g(x) = -3x$

$\therefore \quad x^2y^2 + 4y - 3x = c.$

7. $(x^2 - y^2)dx + (x^2 - 2xy)dy = 0$

$\frac{\partial M}{\partial y} = -2y. \quad \frac{\partial N}{\partial x} = 2x - 2y.$ No exact.

$M_y = -2y \qquad N_x = 2x - 2y$

$M_x = 2x \qquad N_y = -2x$

$$f(x) = \frac{-2y - 2x + 2y}{x^2 - y^2} = \frac{-2x}{(x+y)(x-y)}. \qquad g(x) = \frac{2x - 2y + 2y}{x^2 - 2xy} = \frac{2x}{x(x-2y)} = \frac{2}{x-2y}.$$

Impossible to solve.

If anyone can, I will give him "treat"!

9. $(x - y^3 + y^2 \sin x)\,dx = (3xy^2 + 2y\cos x)\,dy$.

$\Rightarrow (x - y^3 + y^2\sin x)\,dx + (-3xy^2 - 2y\cos x)\,dy = 0$ — ①

$\frac{\partial M}{\partial y} = -3y^2 + 2y\sin x$ $\frac{\partial N}{\partial x} = -3y^2 + 2y\sin x$.

So the D.E is exact.

$\frac{\partial f}{\partial x} = (x - y^3 + y^2\sin x)$ — (ii) $\frac{\partial f}{\partial y} = (-3xy^2 - 2y\cos x)$ — (iii).

$f(x,y) = \int(-3xy^2 - 2y\cos x)\,dy + g(x)$

$= -xy^3 - y^2\cos x + g(x)$ — (iv)

Differentiating (iv) w.r.to x,

$\frac{\partial f}{\partial x} = -y^3 + y^2\sin x + g'(x)$

$g'(x) = x$. $g(x) = \frac{x^2}{2}$

$\therefore -xy^3 - y^2\cos x + \frac{x^2}{2} = c$

10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$ · — ①

$\frac{\partial M}{\partial y} = 3y^2$. $\frac{\partial N}{\partial x} = 3y^2$. $\frac{\partial f}{\partial x} = x^3 + y^3$ — (ii). $\frac{\partial f}{\partial y} = 3xy^2$ — (iii)

$f(x,y) = \int 3xy^2\,dy + g'(x)$.

$= xy^3 + g(x)$ — (iv)

$\frac{\partial f}{\partial x} = y^3 + g'(x)$ — (v)

$g(x)' = x^3$ $g(x) = \frac{1}{4}x^4$.

$\therefore xy^3 + \frac{1}{4}x^4 = c$.

12. $(3x^2y+e^y)\,dx+(x^3+xe^y-2y)\,dy=0$ —(i)

$\frac{\partial M}{\partial y} = 3x^2+e^y$. $\quad \frac{\partial N}{\partial x} = 3x^2+e^y$

∴ D.E is exact.

$\frac{\partial f}{\partial x} = 3x^2y+e^y$ —(ii). $\quad \frac{\partial f}{\partial y} = x^3+xe^y-2y$ —(iii).

$f(x,y) = \int (x^3+xe^y-2y)\,dy + g(x)$

$= x^3y + xe^y - y^2 + g(x)$. —(iv)

Now, differentiating (iv) with respect to x.

$\frac{\partial f}{\partial x} = 3x^2y + e^y + g'(x)$ ∴ $g'(x)=0$. $g(x)=c$.

∴ $x^3y + xe^y - y^2 = c$.

13. $x\frac{dy}{dx} = 2xe^x - y + 6x^2$

$\Rightarrow x\,dy = (2xe^x - y + 6x^2)\,dx$

∴ $x\,dy + (-2xe^x + y - 6x^2)\,dx = 0$.

$\frac{\partial M}{\partial y} = 1$. $\quad \frac{\partial N}{\partial x} = 1$. ∴ D.E is exact.

$\frac{\partial f}{\partial x} = -2xe^x + y - 6x^2$ —(ii) $\quad \frac{\partial f}{\partial y} = x$ —(iii)

$f(x,y) = \int x\,dy + g(x)$

$= xy + g(x)$ —(iv)

∴ $\frac{\partial f}{\partial x} = y + g'(x)$.

∴ $g'(x) = -2xe^x - 6x^2$

$g(x) = -2e^x x - 2e^x - 2x^3$

∴ $xy - 2xe^x - 2e^x - 2x^3 = c$

$\int xe^x\,dx$

$= x\int e^x\,dx - \int \frac{d}{dx}(x)\cdot e^x\,dx$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x$

$= e^x(x-1)$.

17. $(\tan x - \sin x \sin y)\,dx + \cos x \cdot \cos y\,dy = 0 \quad \text{—(i)}$

$$\frac{\partial M}{\partial y} = -\sin x \cos y \qquad \frac{\partial N}{\partial x} = -\sin x \cos y$$

$$\frac{\partial f}{\partial x} = \tan x - \sin x \sin y \text{ —(ii)} \qquad \frac{\partial f}{\partial y} = \cos x \cdot \cos y \text{ —(iii)}$$

$$f(x,y) = \int (\cos x \cdot \cos y)\,dy + g(x)$$

$$= \sin y \cos x + g(x) \text{ —(iv)}$$

$$\frac{\partial f}{\partial x} = -\sin x \cdot \sin y + g'(x) \text{ —(v)}$$

$$g'(x) = \tan x$$

$$g(x) = -\ln|\cos x|$$

$$\therefore \sin y \cos x - \ln|\cos x| = c$$

21. $(x+y)^2\,dx + (2xy + x^2 - 1)\,dy = 0. \qquad y(1) = 1$

or, $(x^2 + 2xy + y^2)\,dx + (2xy + x^2 - 1)\,dy = 0$

$$\frac{\partial M}{\partial y} = 2x + 2y \qquad \frac{\partial N}{\partial x} = 2y + 2x \qquad \therefore \text{D.E is exact}$$

$$\frac{\partial f}{\partial x} = (x^2 + 2xy + y^2) \text{ —(ii)} \qquad \frac{\partial f}{\partial y} = (2xy + x^2 - 1) \text{ —(iii)}$$

$$f(x,y) = \int (2xy + x^2 - 1)\,dy + g(x)$$

$$= xy^2 + x^2 y - y + g(x) \text{ —(iv)}$$

$$\frac{\partial f}{\partial x} = y^2 + 2xy + g'(x) \text{ —(v)}$$

$$g'(x) = x^2 \qquad g(x) = \frac{1}{3}x^3$$

$$\therefore xy^2 + x^2 y - y + \frac{1}{3}x^3 = c$$

now, $1 + 1 - 1 + \frac{1}{3} - c = 0$

$$1 + \frac{1}{3} - c = 0$$

$$\frac{4}{3} - c = 0 \qquad c = +\frac{4}{3}$$

$$\therefore xy^2 + x^2 y - y + \frac{1}{3}x^3 = +\frac{4}{3}$$

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22. $(e^{x}+y)\,dx+(2+x+ye^{y})\,dy=0$. $y(0)=1$.

$\frac{\partial M}{\partial y}=1$. $\frac{\partial N}{\partial x}=1$. $\therefore$ D.E is exact.

$\frac{\partial f}{\partial x}=e^{x}+y$ —(ii) $\frac{\partial f}{\partial y}=2+x+ye^{y}$ —(iii)

$f(x,y)=\int(2+x+ye^{y})\,dy+g(x)$

$=2y+xy+e^{y}(y-1)+g(x)$

$=2y+xy+e^{y}y-e^{y}+g(x)$. —(iv)

$\frac{\partial f}{\partial x}=y+g'(x)$ —(v)

$g'(x)=e^{x}$

$g(x)=e^{x}$. $\therefore 2y+xy+e^{y}y-e^{y}+e^{x}=c$.

$\therefore$ when, $x=0$, $y=1$.

$2+e^{1}-e^{1}+1=c$

$\therefore c=+3$.

Solution. $2y+xy+e^{y}y-e^{y}+e^{x}=+3$.

23. $(4y+2t-5)\,dt+(6y+4t-1)\,dy=0$. $y(-1)=2$

$\frac{\partial M}{\partial y}=4$. $\frac{\partial N}{\partial t}=4$. $\therefore$ D.E is exact.

$\frac{\partial f}{\partial t}=4y+2t-5$ —(ii) $\frac{\partial f}{\partial y}=6y+4t-1$ —(iii)

$f(t,y)=\int(6y+4t-1)\,dy+g(n)$

$=3y^{2}+4ty-y+g(n)$. —(iv)

$\frac{\partial f}{\partial t}=4y+g'(n)$ —(v) $\therefore g'(n)=2t-5$

$g(n)=t^{2}-5t$.

$\therefore 3y^{2}+4ty-y+t^{2}-5t=c$

$\Rightarrow 12-8-2+1+5=c$ $\therefore c=8$.

$\therefore 3y^{2}+4ty-y+t^{2}-5t=8$.

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25. $(y^2\cos x - 3x^2y - 2x)\,dx + (2y\sin x - x^3 + \ln y)\,dy = 0$. $\quad y(0) = e$.

$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2$. $\quad \frac{\partial N}{\partial x} = 2y\cos x - 3x^2$

$\frac{\partial f}{\partial x} = y^2\cos x - 3x^2y - 2x$ — (ii). $\quad \frac{\partial f}{\partial y} = 2y\sin x - x^3 + \ln y$ — (iii)

$f(x,y) = \int (2y\sin x - x^3 + \ln y)\,dy + g(x)$

$= y^2\sin x - x^3y + y(\ln y - 1) + g(x)$

$\frac{\partial f}{\partial x} = y^2\cos x - 3x^2y + g'(x)$.

$g'(x) = -2x \cdot \; g(x) = -x^2$.

$\therefore y^2\sin x - x^3y + y(\ln y - 1) - x^2 = c$.

when, $x = 0$, $y = e$.

$0 - 0 + e\ln e - e - 0 = c$.

or, $e - e = c$

$\therefore c = 0$.

$\therefore y^2\sin x - x^3y + y(\ln y - 1) - x^2 = 0$.

26. $\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y + \sin x)$. $\quad y(0) = 1$.

or, $y(y + \sin x)\,dx + \left(-\frac{1}{1+y^2} - \cos x + 2xy\right)dy = 0$

or, $(y^2 + y\sin x)\,dx + \left(\frac{-1}{1+y^2} - \cos x + 2xy\right)dy = 0$.

$\frac{\partial M}{\partial y} = 2y + \sin x$. $\quad \frac{\partial N}{\partial x} = \sin x + 2y$.

$\frac{\partial f}{\partial x} = y^2 + y\sin x$ — (ii). $\quad \frac{\partial f}{\partial y} = \left(\frac{-1}{1+y^2} - \cos x + 2xy\right)$ — (iii).

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$$f(x,y) = \int (y^2 + y\sin x)\,dx + g(y)$$

$$= xy^2 - y\cos x + g(y) \quad \text{---(iv)}$$

$$\frac{\partial f}{\partial y} = 2xy - \cos x + g'(y) \quad \text{---(v)}$$

$$g'(y) = -\frac{1}{1+y^2}$$

$$g(y) = -\tan^{-1}y$$

$$\therefore\ xy^2 - y\cos x - \tan^{-1}y = c$$

when $x=0,\ y=1$

$$0 - 1 - \tan^{-1}(1) = c$$

$$\therefore\ xy^2 - y\cos x - \tan^{-1}y = -1 - \tan^{-1}(1) \qquad \tan^{-1} = 45^\circ$$

$$= -1 - \frac{\pi}{4}$$

A

24. $\left(\frac{3y^2 - t^2}{y^5}\right)\frac{dy}{dt} + \frac{t}{2y^4} = 0 \qquad y(1) = 1$

or. $\left(\frac{3y^2 - t^2}{y^5}\right)\frac{dy}{dt} = \frac{-t}{2y^4}$

or. $\left(\frac{3y^2 - t^2}{y^5}\right)dy = \left(\frac{-t}{2y^4}\right)dt$

or. $\left(\frac{t}{2y^4}\right)dt + \left(\frac{3y^2 - t^2}{y^5}\right)dy = 0 \quad \text{---(i)}$

$M = \frac{t}{2y^4} \qquad N = \frac{3y^2 - t^2}{y^5}$

$\frac{\partial M}{\partial y} = \frac{t}{2}(-4y^{-5}) \qquad , \qquad \frac{\partial N}{\partial t} = \frac{1}{y^5}\left(\frac{d}{dx}(3y^2) - \frac{d}{dx}(x^2)\right)$

$= -2ty^{-5}$

$$= \frac{0 - 2x}{y^5}$$

$$= -2xy^{-5}$$

$\therefore$ Exact

There exists a function $f(t,y)$, such that,

$$\frac{\partial f}{\partial t} = M(t,y) \qquad \frac{\partial f}{\partial y} = N(t,y)$$

$$= \frac{t}{2y^4} \text{ --- (ii)} \qquad = \frac{3y^2 - t^2}{y^5} \text{ --- (iii)}$$

$$f(t,y) = \int \frac{3y^2 - t^2}{y^5}\, dy + g(x)$$

$$= \int \left(\frac{3}{y^3} - \frac{t^2}{y^5}\right) dy + g(x)$$

$$= -3/2y^2 + \frac{t^2}{4y^4} + g(x)$$

$$= \frac{t^2}{4y^4} - \frac{3}{2y^2} + g(x) \text{ --- (iv)}$$

Now, differentiating 4 with respect to 't'

$$\frac{\partial f}{\partial t} = \frac{2t}{4y^4} + g'(x)$$

$$= \frac{t}{2y^4} + g'(x) \text{ --- (v)}$$

Comparing equation (ii) and (v).

$$g'(x) = 0 \quad \therefore g(x) = c$$

Now, puting the value of $g(x)$ in equation (4).

$$\frac{t^2}{4y^4} - \frac{3}{2y^2} = c$$

when, $y = 1, x = 1$,

$$\frac{1}{4} - \frac{3}{2} = c$$

$$\frac{1-6}{4} = c$$

$$\text{or, } -5 = 4c$$

$$\therefore c = \frac{-5}{4}$$

Now, $\frac{t^2}{4y^4} - \frac{3}{2y^2} = \frac{-5}{4}$.

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Example 1: $(x^2+y^2)dx + (x^2-xy)dy = 0$.   Let, $y = ux$

$dy = udx + xdu$.

or, $(x^2+u^2x^2)dx + (x^2-ux^2)(udx+xdu) = 0$

or, $(x^2+u^2x^2)dx + (ux^2-u^2x^2)dx + (x^3-ux^3)du = 0$.

or, $(x^2+ux^2)dx + (x^3-ux^3)du = 0$.

or, $x^2(1+u)dx + x^3(1-u)du = 0$.

or, $x^2(1+u)dx = x^3(u-1)du$.

or, $\frac{x^2}{x^3}dx = \frac{u-1}{u+1}du$

or, $\int \frac{x^2}{x^3}dx = \int \frac{u-1}{u+1}du$.

or, $\int \frac{1}{x}dx = \int\left(\frac{u}{u+1} - \frac{1}{u+1}\right)dx$

or, $\ln|x| = \int \frac{u+1-1}{u+1}dx - \ln|u+1|$

$= \int\left(1-\frac{1}{u+1}\right)du - \ln|u+1|$

$= u - \ln|u+1| - \ln|u+1|$.

$= u - 2\ln|u+1| + c$ $y = ux$

$\therefore \ln x = \frac{y}{2} - 2\ln\left|\frac{y}{x}+1\right| + c$. $\therefore u = \frac{y}{x}$.

Exercise Set 2.5 ⇒

$y = ux$, $dy = udx + xdu$.

1. $(x-y)dx + xdy = 0$.

or, $(x-ux)dx + x(udx+xdu) = 0$.

or, $(x-ux)dx + (xudx + x^2du) = 0$.

or, $xdx + x^2du = 0$.

or, $\frac{dx}{x} + du = 0$

or, $\ln x + u = c$

$\therefore \ln x + \frac{y}{x} = c$.

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2. $(x+y)dx + xdy = 0$

or. $(x+ux)dx + x(udx + xdu) = 0$

or, $(x+2ux)dx + x^2 du = 0$

or, $x(1+2u)\,dx + x^2\,du = 0$

or, $\dfrac{dx}{x} + \dfrac{du}{1+2u} = 0$

or, $\ln|x| + \frac{1}{2}\ln|1+2u| = 0$

$\therefore \ln|x| + \frac{1}{2}\ln\left|1+\frac{2y}{x}\right| = 0$

3. $xdx + (y-2x)dy = 0$

or, $xdx + (ux-2x)(udx + xdu) = 0$

or, $xdx + (u^2x - 2ux)dx + (ux^2 - 2x^2)du = 0$

or, $(u^2x - 2ux + x)\,dx + x^2(u-2)\,du = 0$

or, $x(u^2-2u+1)\,dx = -x^2(u-2)\,du$

or, $(u^2-2u+1)\,dx = x(2-u)\,du$

or, $(u-1)^2\,dx = x(2-u)\,du$

or, $\int \frac{1}{x}\,dx = \int \dfrac{2-u}{(u-1)^2}\,du$

or, $\ln x = 2\int \dfrac{1}{(u-1)^2}\,du - \int \dfrac{u}{(u-1)^2}\,du$

$u-1=z$
$du = dz$
$u = z-1$

$= 2(-1)\dfrac{1}{u-1} - \int \dfrac{u+1-1}{(u-1)^2}\,du$

$= 2(-1)\dfrac{1}{u-1} - \left[\int \dfrac{1}{u-1}\,du + \dfrac{1}{(u-1)^2}\,du\right]$

$= -2\dfrac{1}{u-1} - \left[\ln(u-1) - \dfrac{1}{u-1}\right]$

$\therefore \ln x = \dfrac{-2}{u-1} - \ln(u-1) + \dfrac{1}{u-1}$ A.

5. $(y^2+yx)dx - x^2dy = 0$. Let, $y = ux$

$dy = udx + xdu$

or, $(u^2x^2 + ux^2)dx - x^2(udx + xdu) = 0$.

or, $(u^2x^2 + ux^2 - ux^2)dx - x^3du = 0$.

or, $u^2x^2dx = x^3du$

or, $\int \frac{x^2}{x^3}dx = \int \frac{1}{u^2}du$

or, $\ln x = -\frac{1}{u} + c$

or, $\ln x = -1/y/x + c$

$\therefore \ln x = -\frac{x}{y} + c$.

6. $(y^2+yx)dx + x^2dy = 0$. Let, $y = ux$

$dy = udx + xdu$.

or, $(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0$.

or, $(u^2x^2 + 2ux^2)dx = -x^3du$

or, $x^2(u^2 + 2u)dx = -x^3du$.

or, $-\int \frac{1}{x}dx = \int -\frac{1}{u^2+2u}du = \int \frac{1}{u(u+2)}du$.

or, $-\ln x = \frac{1}{2}\ln u - \frac{1}{2}\ln(u+2) + c$

or, $-\ln x = \frac{1}{2}\ln(y/x) - \frac{1}{2}\ln(\frac{y}{x}+2) + c$.

$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$

or, $1 = A(u+2) + Bu$.

$u = -2$, $1 = -2B$ $\therefore B = -\frac{1}{2}$.

$u = 0$, $1 = 2A$ $\therefore A = \frac{1}{2}$.

Now, $\int \frac{1/2}{u} + \frac{-1/2}{u+2}$.

$= \frac{1}{2}\ln u - 1/2\ln(u+2)$.

7. $\frac{dy}{dx} = \frac{y-x}{y+x}$

or, $(y+x)dy = (y-x)dx$.

or, $(y+x)dy = -(x-y)dx$

or, $(x-y)dx + (x+y)dy = 0$.

or, $(x-ux)dx + (x+ux)(udx+xdu) = 0$.

or, $(x-ux)dx + uxdx + x^2du + u^2xdx + u^2xdu = 0$.

or, $xdx + u^2xdx = -x^2du - ux^2du$

or, $xdx + u^2xdx = -x^2(1+u)du$.

or, $x(1+u^2)dx = -x^2(1+u)du$.

or, $\int \frac{-x}{x^2}dx = \int \frac{1+u}{1+u^2}du$

or, $-\ln x = \int \frac{1}{1+u^2}du + \int \frac{u}{1+u^2}du$

$1+u^2 = dy$
$2u\,du = dy$
$\therefore u\,du = \frac{dy}{2}$

$= \tan^{-1}(u) + \frac{1}{2}\int \frac{1}{y}dy$

$= \tan^{-1}(u) + \frac{1}{2}\ln(1+u^2)du$.

$= \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left\{\left(\frac{y}{x}\right)^2+1\right\} + c$.

8. $\frac{dy}{dx} = \frac{x+3y}{3x+y}$.

or. $(3x+y)dy = (x+3y)dx$

or, $(3x+ux)(udx+xdu) = (x+3ux)dx$

or, $(-x-3ux)dx + 3uxdx + 3x^2du + u^2xdx + ux^2du = 0$.

or, $-xdx + u^2xdx + x^2(3+u)du = 0$.

or. $x(u^2-1)dx = -x^2(u+3)du$.

or. $\int \frac{-x}{x^2}dx = \int \frac{u+3}{u^2-1}du$

or. $-\ln x = \int \frac{u}{u^2-1}du + 3\int \frac{1}{u^2-1}du$

or. $-\ln x = \frac{1}{2}\ln(u^2-1) + 3\ln\frac{u-1}{u+1} + c$

$\therefore -\ln x = \frac{1}{2}\ln\left(\frac{y^2}{x^2}-1\right) + 3\ln\frac{y/x-1}{y/x+1} + c$

9. $-y\,dx + (x + \sqrt{xy})\,dy = 0$

or, $-ux\,dx + (x + \sqrt{x \cdot ux})(u\,dx + x\,du) = 0$

or, $-ux\,dx + ux\,dx + x^2\,du + xu\sqrt{u}\,dx + x^2\sqrt{u}\,du = 0$

or, $xu\sqrt{u}\,dx = -x^2(1+\sqrt{u})\,du$

or, $\int \frac{x}{-x^2}\,dx = \int \frac{1+\sqrt{u}}{u\sqrt{u}}\,du$

or, $-\ln x = \int \frac{1}{u^{3/2}}\,du + \frac{1}{u}\,du$.

or, $-\ln x = \frac{u^{-1/2}}{-1/2} + \ln u + c$

or, $-\ln x = -2\,(y/x)^{-1/2} + \ln(y/x) + c$.

10. $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$. $x > 0$.

or, $x\,dy = (y + \sqrt{x^2 - y^2})\,dx$

or, $x(u\,dx + x\,du) = (ux + \sqrt{x^2 - u^2x^2})\,dx$

or, $\cancel{ux\,dx} + x^2\,du = \cancel{ux\,dx} + x\sqrt{1-u^2}\,dx$

or, $x^2\,du = x\sqrt{1-u^2}\,dx$.

or, $\int \frac{1}{\sqrt{1-u^2}}\,du = \int \frac{x}{x^2}\,dx$

or, $\sin^{-1}u + c = \ln x$

$\therefore \ln x = \sin^{-1}(y/x) + c$.

11. $xy^2\frac{dy}{dx} = y^3 - x^3$, $(y(1) = 2)$.

or, $xu^2x^2\,dy = (y^3 - x^3)\,dx$

or, $u^2x^3(u\,dx + x\,du) = (y^3 - x^3)\,dx$

or, $u^3x^3\,dx + u^2x^4\,du = u^3x^3\,dx - x^3\,dx$

or, $u^2x^4\,du = -x^3\,dx$

or, $\int u^2\,du = -\int \frac{x^3}{x^4}\,dx$

or, $\frac{1}{3}u^3 + c = -\ln x$

or, $\frac{1}{3}\left(\frac{y}{x}\right)^3 + c = -\ln x$

$\therefore c = -8/3$ $\quad\therefore \frac{1}{3}\left(\frac{y}{x}\right)^3 - 8/3 = -\ln x$.

12. $(x^2+2y^2)\frac{dx}{dy} = xy$ $\qquad y(-1)=1$

or, $(x^2+2u^2x^2)dx - xux(udx+xdu)=0$

or, $(x^2+2u^2x^2)dx - u^2x^2dx - x^3udu=0$

or, $x^2dx + u^2x^2dx - x^3udu=0$.

or, $x^2(1+u^2)dx = ux^3du$

or, $\frac{x^2}{x^3}dx = \frac{u}{1+u^2}du$

or, $\int \frac{1}{x}dx = \int \frac{u}{1+u^2}du$.

or, $\ln|x| = \frac{1}{2}\ln|1+u^2)+c$

or, $\ln|x| = \frac{1}{2}\ln \ln\left|1+\frac{y^2}{x^2}\right|+c$

Now, $\ln|-1| = 1/2 \ln\left|1+\frac{1}{(-1)^2}\right|+c$

or, $0 = \frac{1}{2}\ln 2 + c$

$\therefore c = -\frac{1}{2}\ln 2$

$\therefore \ln|x| = \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right| - \frac{1}{2}\ln 2$.

13. $(x+ye^{y/x})dx - xe^{y/x}dy = 0$. $\qquad y(1)=0$.

or, $(x+uxe^u)dx - xe^u(udx+xdu)=0$

or, $xdx - x^2e^u du=0$.

or, $\int\frac{1}{x}dx - \int e^u du=0$.

or, $\ln x - e^u = c$

or, $\ln x - e^{y/x} = c$

$\ln|x| - e^{y/x} = -1$.

$\therefore \ln x - e^{y/x} = -1$.

14. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0$. $(y(1) = e)$

$x = vy$

$dx = v\,dy + y\,dv$.

or, $y(v\,dy + y\,dv) + vy(\ln vy - \ln y - 1)\,dy = 0$

or, $\cancel{vy\,dy} + y^2\,dv + vy\ln vy\,dy - vy\ln y\,dy - \cancel{vy\,dy} = 0$

or, $vy(\ln vy - \ln y)\,dy + y^2\,dv = 0$.

or, $v\ln\frac{vy}{y}\,dy + y\,dv = 0$

or. $v\ln v\,dy + y\,dv = 0$.

or, $\frac{dy}{y} + \frac{dv}{v\ln v} = 0$

or. $\ln y + \ln|\ln|v|| = c$

$\therefore$ $\ln|\ln|\frac{x}{y}|| + \ln y = c$.

$c = -1$.

$y = ux$.

4. $y\,dx = 2(x+y)\,dy$.

$dy = u\,dx + x\,du$.

or, $ux\,dx = 2(x+ux)(u\,dx + x\,du)$

or, $ux\,dx = (2x + 2ux)(u\,dx + x\,du)$

$= (2ux + 2u^2x)\,dx + (2x^2 + 2ux^2)\,du$

or. $(ux + 2u^2x)\,dx + (2x^2 + 2ux^2)\,du = 0$.

or, $x(u + u^2)\,dx + x^2(2 + 2u)\,du = 0$.

or, $\frac{x}{x^2}\,dx = -\frac{2(1\cancel{+u})}{u(1\cancel{+u})}\,du$

or, $\int \frac{1}{x}\,dx = \int -\frac{2}{u}\,du$

or, $\ln x = -2\ln u + c$

$\therefore$ $\ln x = -2\ln y/x + c$.

Linear Models – 3.1

Growth and decay Model: $y(t) = e^{k \cdot t} \cdot A$

If $k > 0$, it is called exponential Growth law.

$k < 0$, it is called Decay law.

$$\boxed{y = Ae^{kt}} \text{ — } (*)$$

A culture of Bacteria contains 100 cells, after 60 min later. There are 450 cells. Find the number of cells at any time t, and doubling time.

Soln: From the exponential Growth law,

$$y(t) = Ae^{kt},$$

when, $t = 0$, $y(t) = 100$.

$$\therefore 100 = Ae^{k \cdot 0} \Rightarrow A = 100.$$

Hence, $y(t) = 100e^{kt}$.

when, $t = 60$, $y(t) = 450$.

$$450 = 100e^{k \cdot 60}$$

$$\text{or, } 4.5 = e^{k60}$$

$$\text{or, } \ln 4.5 = 60k$$

$$\therefore k = \frac{\ln(4.5)}{60} = 0.02507.$$

when, $y(t) = 200$, $200 = 100e^{(0.02507)t}$

$$\text{or, } 2 = e^{(0.02507)t}.$$

$$\text{or, } \ln 2 = 0.02507t$$

$$\text{or, } t = \ln 2 / 0.02507$$

$$\therefore t = 27.65 \text{ min}.$$

Example 1: From exponential Growth law,

$$P(t) = Ae^{kt}.$$

when, $t = 0$, $P(t) = P_0$.

$$\therefore P_0 = Ae^{k \cdot 0}$$

or, $P_0 = Ae^0$

$\therefore A = P_0$.

Hence, $P(t) = P_0 e^{kt}$.

when, $t = 1$, $P(t) = \frac{3}{2} P_0$

$$\therefore \frac{3}{2} P_0 = P_0 e^{k \cdot 1}$$

or, $\frac{3}{2} = e^k$

$\therefore k = \ln \frac{3}{2} = 0.4055$

$$P(t) = P_0 e^{(0.4055)t}$$

tripled time, $3P_0 = P_0 e^{(0.4055)t}$

or, $3 = e^{(0.4055)t}$

or, $\ln 3 = 0.4055t$

or, $t = \frac{\ln 3}{0.4055}$

$\therefore t = 2.71$ h.

Ans: 2.71 h.

Example 2: $A(t) = ce^{kt}$.

when, $t = 0$, $A(t) = A_0$

$\therefore A_0 = ce^{k \cdot 0}$

$\therefore c = A_0$

Hence, $A(t) = A_0 e^{kt}$.

If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance reemains.

when, $t = 15$, $A(t) = 0.99957 A_0$.

$$\therefore 0.99957 A_0 = A_0 e^{k \cdot 15}$$

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or, $k = \frac{1}{15} \ln 0.99957 = -0.00002867.$

$A(t) = A_0 e^{-0.0000287t}.$

Now for half life,

$\frac{1}{2} A_0 = A_0 e^{-0.00002867 \times t}.$

or, $\frac{1}{2} = e^{-0.00002867 \times t}$

or, $t = \ln 1/2 / -0.00002867$

$\therefore t = 24180$ yr.

Example 3: $A(t) = A_0 e^{kt}$ at initial stage.

at, $t = 5600$,

$\frac{1}{2} A_0 = A_0 e^{5600k}$

or, $\frac{1}{2} = e^{5600k}$

or, $\ln \frac{1}{2} = 5600k$

$\therefore k = -0.000012378.$

$\therefore A(t) = A_0 e^{-0.000012378t}.$

now, $A(t) = \frac{1}{1000} A_0.$

$\therefore \frac{1}{1000} A_0 = A_0 e^{-0.000012378t}.$

or, $\frac{1}{1000} = e^{-0.000012378t}$

or, $\ln\left(\frac{1}{1000}\right) = -0.000012378t$

or, $t = \frac{\ln 1000}{0.000012378}$

$\therefore t = 55,800$ yr.

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* If you have 50 grms of 'C-14' today. How much will be left in 100 years? (Half life of ^{14}C is 5730 yr).

Solution → $A(t) = ce^{kt}$

when $t=0$, $y(t)=50$.

$\therefore 50 = ce^{k \cdot 0}$

or. $50 = ce^{0}$

$\therefore c = 50$.

using half life, $25 = 50e^{k(5730)}$

or, $\frac{1}{2} = e^{k5730}$

or, $k = \frac{1}{5730} \ln\left(\frac{1}{2}\right)$

$\therefore k \approx -1.20968 \times 10^{-4}$.

when, $t=100$,

$\therefore y(100) = 50e^{(-1.20968 \times 10^{-4}) \times 100}$

$\therefore y(100) = 49.3988$ grms.

Ans: 49.3988 grms.

Exercise Set 3.1:

1. $P(t) = Ae^{kt}$

when $t=0$, $P(0) = P_0$

$\therefore P_0 = Ae^{k\cdot 0}$

or, $A = P_0$. $\therefore P(t) = P_0 e^{kt}$

when $t=5$, $P(5) = 2P_0$

$\therefore 2P_0 = P_0 e^{k5}$

or, $2 = e^{k5}$

or, $\ln 2 = 5k$ $\therefore k = 0.13863$

tripled time,

$3P_0 = P_0 e^{0.13863 \times t}$

or, $\ln 3 = 0.13863t$

$\therefore t = 7.9248$ yr. Ans: 7.9248 yr

2. $P(t) = ce^{\frac{1}{c}kt}$

when $t=0$, $P(0) = P_0$ $\therefore P_0 = ce^{k\cdot 0}$ $\therefore c = P_0$

$2P_0 = ce^{kt}$

or, $2P_0 = P_0 e^{k5}$

or, $k = \frac{\ln 2}{5}$ 1.386294

$P(3) = 10000$,

or, $P_0 e^{3k} = 10{,}000$

or, $P_0 = \frac{10000}{e^{\frac{1}{5}\ln 2 \times 3}} = \frac{10000}{1.51}$

$= 6622.5$ people.

$\therefore P(10) = P_0 e^{\left(\frac{\ln 2}{5} \times 10\right)}$

$= 6622.5 \times e^{\left(\frac{\ln 2}{5} \times 10\right)}$

$= 26489.99043$.

$\therefore 26489.99043 - 6622.5 = 19867.49043$. $19867.49043 / 6622.5 = 2.99 \times 100$

$= 300\%$.

3. $Y(t) = Ae^{kt}$ —— ①

Initial condition $Y(0) = 500$

$\therefore \; 500 = Ae^{k\cdot 0}$

or, $A = 500$.

when $t = 10$, $Y(10) = 500 \times 15\%$

$= 575$

$Y(10) = 500e^{kt}$

or, $\frac{575}{500} = e^{10k}$

or, $\ln\left(\frac{575}{500}\right) = 10k$

$\therefore \; k = \frac{1}{10}\ln\left(\frac{575}{500}\right) \approx 0.013$

$Y(30) = 500e^{(0.013)\times 30}$

≈ 760.43.

$\therefore \; 760 - 500 = 260$. $\frac{260}{500} = 0.52 \times 100$

$= 52\%$

$\therefore$ The population has increased 52% during 30 years.

5. $P(t) = ce^{+kt}$ —— *

Inition condition,

$P_0 = c\,e^{-k \cdot 0}$

$\therefore c = P_0.$

$P(3.3) = \frac{P_0}{2}$

$\therefore \frac{P_0}{2} = P_0 e^{+k \times 3.3}$

or, $\frac{1}{2} = e^{+k \times (3.3)}$

or, $k = -0.2100446.$

$P(t) = P_0 - \frac{90}{100} P_0$

$= \left(\frac{100-90}{100}\right) P_0$

$= \frac{10}{100} P_0 = \frac{1}{10} P_0$

$\therefore \frac{1}{10} P_0 = P_0 e^{-0.2100446 \times t}$

or, $\ln(1/10) = -.2100446 \times t$

$\therefore t = 10.96$ hours.

REDUCTION OF ORDER — 4.2

Example-1: $y'' - y = 0$, $y_1 = e^x$. $\quad y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}$

Standard format $\rightarrow y'' + p(x)y' + q(x)y = 0$ — (*)

Now, Linear combination, $y = y_1c_1 + c_2y_2$.

Here, $p(x) = 0$.
$$y_2 = e^x \int \frac{e^{-\int 0\,dx}}{(e^x)^2}dx$$
$$= e^x \cdot \int e^{-2x}dx$$
$$= e^x \cdot \frac{e^{-2x}}{-2}$$
$$= -\frac{1}{2}e^{-x}.$$

Solution: $y = c_1y_1 + c_2y_2$

or, $y_1 = c_1e^x + c_2\frac{e^{-x}}{-2}$.

Example-2: $x^2y'' - 3xy' + 4y = 0$. $\quad y_1 = x^2$.

or, $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$.

$$p(x) = -\frac{3}{x}.$$
$$y_2 = x^2 \int \frac{e^{-\int -\frac{3}{x}dx}}{(x^2)^2}dx$$
$$= x^2 \int \frac{e^{\ln x^3}}{x^4}$$
$$= x^2 \int \frac{1}{x} = x^2 \ln x.$$

$\therefore y_1 = c_1x^2 + c_2x^2\ln x$.

Exercise Set 4.2 ⇒

1. $y'' - 4y' + 4y = 0$. $\quad y_1 = e^{2x}$.

$p(x) = -4$. $\quad y_2 = e^{2x}\int \frac{e^{-\int -4dx}}{e^{4x}}dx$
$$= e^{2x}\int \frac{e^{4x}}{e^{4x}}dx$$
$$= e^{2x}\int 1\,dx$$
$$= e^{2x}\cdot x$$

$\therefore y = c_1e^{2x} + c_2x\cdot e^{2x}$.

2. $y'' + 2y' + y = 0$; $y_1 = xe^{-x}$

$P(x) = 2$.

$$y_2 = xe^{-x} \int \frac{e^{-\int 2dx}}{(xe^{-x})^2} dx$$

$$= xe^{-x} \int \frac{e^{-2x}}{x^2 \cdot e^{-2x}} dx$$

$$= xe^{-x} \int 1/x^2 \, dx$$

$$= xe^{-x} \frac{x^{-1}}{-1}$$

$$= -e^{-x}$$

$\therefore y = c_1 xe^{-x} - e^{-x} c_2$. Ans

3. $y'' + 16y = 0$ $y_1 = \cos 4x$.

$p(x) = 0$.

$$y_2 = \cos 4x \int \frac{e^{-\int 0 dx}}{\cos^2 4x} dx$$

$$= \cos 4x \int \sec^2 4x \, dx$$

$$= \frac{1}{4} \cos 4x \cdot \tan 4x$$

$$= \frac{1}{4} \sin 4x.$$

$\therefore y = c_1 \cos 4x + c_2 \frac{1}{4} \sin 4x$.

4. $y'' + 9y = 0$. $y_1 = \sin 3x$

$p(x) = 0$.

$$y_2 = \sin 3x \int \frac{e^{-\int 0 dx}}{\sin^2 3x} dx$$

$$= \sin 3x \int \operatorname{cosec}^2 3x \, dx$$

$$= \sin 3x \cdot \frac{1}{3}(-1) \cot 3x$$

$$= -\frac{1}{3} \cos 3x.$$

$$y = y_1 c_1 + y_2 c_2$$

$$= c_1 \sin 3x - \frac{1}{3} \cos 3x \, c_2$$

5. $y'' - y = 0$. $\quad y_1 = \cosh x$.

$p(x) = 0$. $\quad y_2 = \cosh x \int \frac{e^{-\int 0\,dx}}{\cosh^2 x}\,dx$

$= \cosh x \int \operatorname{sech}^2 x\,dx$

$= \frac{1}{n} \cosh x \cdot \tanh x$

$= \frac{1}{n} \sinh x$.

$\therefore y = c_1 \cosh x + \frac{1}{n} \sinh x\, c_2$.

6. $y'' - 25y = 0$. $\quad y_1 = e^{5x}$.

$p(x) = 0$, $\quad y_2 = e^{5x} \int \frac{e^{-\int 0\cdot dx}}{e^{10x}}\,dx$

$= e^{5x} \int \frac{1}{e^{10x}}\,dx$

$= e^{5x} \int e^{-10x}\,dx$

$= -\frac{1}{10} e^{5x} \cdot e^{-10x}$

$= -\frac{1}{10} e^{-5x}$.

$\therefore y = c_1 e^{5x} - \frac{1}{10} e^{-5x} c_2$

7. $9y'' - 12y' + 4y = 0$ $\quad y_1 = e^{2x/3}$.

or, $y'' - 4/3\, y' + \frac{4}{9} y = 0$.

$P(x) = -4/3$. $\quad y_2 = e^{2x/3} \int \frac{e^{-\int 4/3\,dx}}{e^{4x/3}}\,dx$

$= e^{2x/3} \int \frac{e^{4x/3}}{e^{4x/3}}\,dx$

$= e^{2x/3} \int 1\,dx$

$= e^{2x/3} \cdot x$

$\therefore y = c_1 e^{2x/3} + c_2 e^{2x/3} \cdot x$.

10. $x^2y'' + 2xy' + 16y = 0$ —— (1) $\qquad y_1 = x^2.$

or, $y'' + \frac{2}{x}y' + \frac{16}{x^2}y = 0.$

$P(x) = \frac{2}{x}.$ $\qquad y_2 = x^2 \int \frac{e^{-2\int \frac{1}{x}dx}}{x^4}dx$

$= x^2 \int \frac{e^{-\ln x^2}}{x^4}dx$

$= x^2 \int \frac{e^{\ln(\frac{1}{x^2})}}{x^4}dx$

$= x^2 \int \frac{1}{x^6}dx$

$= \frac{1}{-5}x^{-3}.$

$\therefore y = c_1x^2 - \frac{1}{5}x^{-3}c_2.$

11. $xy'' + y' = 0.$ $\qquad y_1 = \ln x$

$y'' + \frac{1}{x}y' = 0.$

$P(x) = \frac{1}{x}.$ $\qquad y_2 = \ln x \int \frac{e^{-\int 1/x\,dx}}{(\ln x)^2}dx$

$= \ln x \int \frac{e^{-\ln x}}{(\ln x)^2}dx$

$= \ln x \int \frac{e^{\ln(1/x)}}{(\ln x)^2}dx$

$= \ln x \int \frac{1/x}{(\ln x)^2}dx.$ $\qquad$ let $\ln x = z$ $\quad \therefore \frac{1}{x}dx = dz$

$= \ln x \int \frac{dz}{z^2}$

$= \ln x\left(-\frac{1}{z}\right)$

$= \ln x\left(\frac{-1}{\ln x}\right).$

$= -1.$

$\therefore y = c_1\ln x - c_2.$

16. $(1-x^2)y'' + 2xy' = 0$. $\qquad y_1 = 1$

or, $y'' + \frac{2x}{1-x^2}y' = 0$ $\qquad p(x) = \frac{2x}{1-x^2}$.

$\therefore y_2 = 1\int \frac{e^{-\int \frac{2x}{1-x^2}dx}}{1^2}\,dx$

$= \int e^{\ln(1-x^2)}\,dx$ $\qquad$ Let, $1-x^2 = z$

or, $-2x\,dx = dz$.

$= \int (1-x^2)\,dx$

$= \int 1\,dx - \int x^2\,dx$

$= x - \frac{x^3}{3}$

$\therefore y = c_1 1 + \left(x - \frac{x^3}{3}\right)c_2$

15. $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$. $\qquad y_1 = x+1$.

or, $y'' + \frac{2(1+x)}{(1-2x-x^2)}y' - \frac{2}{(1-2x-x^2)}y = 0$. $\qquad p(x) = \frac{2(1+x)}{1-2x-x^2}$.

$y_2 = (x+1)\int \frac{e^{-\int \frac{2(1+x)}{1-2x-x^2}dx}}{(x+1)^2}\,dx$

*

$\boxed{z = 1-2x-x^2 \quad dz = (-2-2x)\,dx}$

$= (x+1)\int \frac{e^{\int (-2-2x)/(1-2x-x^2)\,dx}}{(x+1)^2}\,dx$

$= (x+1)\int \frac{e^{\ln(1-2x-x^2)}}{(x+1)^2}\,dx$

**

$\boxed{\begin{aligned}&-(x^2+2x-1)\\&= -(x^2+2x+1-2)\\&= -(x^2+2x+1)+2\end{aligned}}$

$= (x+1)\int \frac{1-2x-x^2}{1+2x+x^2}\,dx$

$= (x+1)\int \frac{-(x^2+2x+1)}{1+2x+x^2}\,dx + 2\int \frac{1}{(x+1)^2}\,dx$

$= (x+1)\int (-1)\,dx + 2\int \frac{1}{(x+1)^2}\,dx$.

***

$\boxed{\begin{aligned}&x+1 = z\\&dx = dz.\end{aligned}}$

$= (x+1)\left[-x + 2\left(\frac{z^{-2+1}}{-2+1}\right)\right]$

$= (x+1)\left[-x - 2\frac{1}{(x+1)}\right]$

$= \cancel{(x+1)}\,\frac{-x^2-x-2}{\cancel{(x+1)}} = (-x^2-x-2)$

$y = c_1(x+1) + c_2(-x^2-x-2)$

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3

Homogeneous Linear Equations With Constant coefficient — 4.3

$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x)$ — ①.

$a_0, a_1, a_2 \dots a_n \longrightarrow$ constant.

① will be homogenous if $F(x) = 0$

① will be non homogenous if $F(x) \neq 0$

Now, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \longrightarrow$ Homogenous.

$\frac{d^2y}{dx^2} - 2\cdot\frac{dy}{dx} + 3y = 2e^{4x} \longrightarrow$ non Homogenous.

Homogenous differential equation:—

① Find auxiliary equation by taking $y = e^{mx}$ as a solution.

$\frac{d^2}{dx^2}(e^{mx}) - 3\frac{d}{dx}(e^{mx}) + 2e^{mx} = 0$.

or, $m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$

or, $e^{mx}(m^2 - 3m + 2) = 0$.

So the auxiliary equation, $m^2 - 3m + 2 = 0$.

* Case-1: If m are distinct real roots $(m_1, m_2, m_3 \dots)$

General Soln: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.

or, $y = c_1 e^{2x} + c_2 e^{x}$.

* Case-2: If m are repeated real roots (k times).

$y = (c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{mx}$.

* Case-3: If m has conjugated complex roots, $(a+ib, a-ib)$.

$y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$.

$m = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Excercise Set-4.3 $\Rightarrow$

1: $4y''+y'=0$

$y=e^{mx}$
$y'=me^{mx}$
$y''=m^2e^{mx}$.

or, $4m^2e^{mx}+me^{mx}=0$

or, $e^{mx}(4m^2+m)=0$.

or, $e^{mx}=0$. $4m^2+m=0$.

or, $m(4m+1)=0$

$\therefore m=0,-\frac{1}{4}$.

$\therefore$ General solution $=c_1e^{0x}+c_2e^{-\frac{1}{4}x}$

$=c_1+c_2e^{-\frac{x}{4}}$.

3. $y''-y'-6y=0$.

or, $m^2e^{mx}-me^{mx}-6e^{mx}=0$.

or, $e^{mx}(m^2-m-6)=0$.

$\therefore m^2-m-6=0$.

or, $m^2-3x+2x-6=0$

or, $m(m-3)+2(x-3)=0$

or, $(m+2)(m-3)$

$\therefore m=-2,3$.

$\therefore$ General solution $=c_1e^{-2x}+c_2e^{3x}$.

5. $y''+8y'+16y=0$

or, $m^2e^{mx}+8me^{mx}+16e^{mx}=0$.

or, $e^{mx}(m^2+8m+16)=0$.

$\therefore m^2+8m+16=0$.

or, $m^2+4m+4m+16=0$.

or, $(m+4)(m+4)=0$

$\therefore m=-4,-4$.

$\therefore$ General Solution; $y=c_1e^{-4x}+c_2e^{-4x}.x$.

$* \; e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$

9. $y'' + 9y = 0$

or, $m^2 e^{mx} + 9e^{mx} = 0$.

or, $e^{mx}(m^2+9) = 0$.

$\therefore m^2 + 9 = 0$.

or, $m^2 = -9$

$\therefore m = \pm\sqrt{-9} = \pm 3i$.

$\therefore \alpha + i\beta = 0 \pm 3i$ Now, $\alpha = 0$, $\beta = 3$.

$\therefore$ General Solution $= e^{0\cdot x}\left[A\cos 3x + B\sin 3x\right]$.

11. $y'' - 4y' + 5y = 0$

or, $m^2 e^{mx} - 4me^{mx} + 5e^{mx} = 0$.

or, $e^{mx}(m^2 - 4m + 5) = 0$.

$\therefore m^2 - 4m + 5 = 0$.

or, $m = \dfrac{4 \pm \sqrt{(4)^2 - 4\cdot 1\cdot 5}}{2\cdot 1}$

$= \dfrac{4 \pm \sqrt{-4}}{2}$

$= \dfrac{4 \pm 2i}{2} = 2 \pm i$.

$\therefore \alpha = 2$, $\beta = 1$.

$\therefore$ General Solution $= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$

$= e^{2x}\left[A\cos x + B\sin x\right]$.

13. $3y'' + 2y' + y = 0$

or, $3m^2 e^{mx} + 2me^{mx} + e^{mx} = 0$.

or, $e^{mx}(3m^2 + 2m + 1) = 0$.

$\therefore 3m^2 + 2m + 1 = 0$.

or, $m = \dfrac{-2 \pm \sqrt{4 - 4\cdot 3\cdot 1}}{2\cdot 3}$

or, $m = \dfrac{-2 \pm \sqrt{8}}{6}$

or, $m = -\dfrac{1}{3} \pm \dfrac{\sqrt{4}}{3} i$. p.t.o

$\therefore \alpha = -\frac{1}{3} \quad \beta = \frac{\sqrt{4}}{3}$

General Solution $= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$

$= e^{-\frac{1}{3}x}\left[A\cos\frac{\sqrt{4}}{3}x + B\sin\frac{\sqrt{4}}{3}\right]$

14. $2y'' - 3y' + 4y = 0$.

or, $2m^2e^{mx} - 3me^{mx} + 4e^{mx} = 0$.

or, $e^{mx}(2m^2 - 3m + 4) = 0$.

Now, $2m^2 - 3m + 4 = 0$.

or, $m = \frac{3 \pm \sqrt{9 - 4\cdot2\cdot4}}{2\cdot2}$

$= \frac{3 \pm \sqrt{23}\,i}{4}$

$= \frac{3}{4} \pm \frac{\sqrt{23}}{4}i$.

$\therefore$ General Solution $= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$

$= e^{\frac{3}{4}x}\left[A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right]$.

15. $y''' - 4y'' - 5y' = 0$

or, $m^3e^{mx} - 4m^2e^{mx} - 5me^{mx} = 0$.

or, $e^{mx}(m^3 - 4m^2 - 5m) = 0$.

Now, $m^3 - 4m^2 - 5m = 0$

or, $m^2 - 4m - 5 = 0$.

or, $m^2 - 5m + m - 5 = 0$

or, $m(m-5) + 1(m-5) = 0$

or, $(m+1)(m-5) = 0$

$\therefore m = -1, 5$

Hence, $m_1 = -1$, $m_2 = 5$

General Solution: $c_1e^{m_1x} + c_2e^{m_1x}$

$= c_1e^{-x} + c_2e^{5x}$.

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17. $y''' - 5y'' + 3y' + 9y = 0$. or, $m^3e^{mx} - 5m^2e^{mx} + 3me^{mx} + 9e^{mx} = 0$

or, $e^{mx}(m^3 - 5m^2 + 3m + 9) = 0$

Now, $m^3 - 5m^2 + 3m + 9 = 0$.

or, $m^3 + m^2 - 6m^2 - 6m + 9m + 9 = 0$

or, $m^2(m+1) - 6m(m+1) + 9(m+1) = 0$

or, $(m^2 - 6m + 9)(m+1) = 0$

or, $(m-3)^2(m+1) = 0$.

now, $(m-3)^2 = 0$. $m+1 = 0$

$\therefore m_2 = 3$. $\therefore m_1 = -1$

General Solution: $c_1e^{x} + c_2e^{3x}$.

23. $y^4 + y''' + y'' = 0$.

or, $m^4e^{mx} + m^3e^{mx} + m^2e^{mx} = 0$.

or, $e^{mx}(m^4 + m^3 + m^2) = 0$.

Now, $m^4 + m^3 + m^2 = 0$.

or, $m^2 + m + 1 = 0$.

or, $m = \frac{-1 \pm \sqrt{1-4}}{2 \cdot 1}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$.

General Solution: $e^{-\frac{1}{2}x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$

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25. $16\frac{d^4y}{dx^4}+24\frac{d^2y}{dx^2}+9y=0$

or, $16m^4e^{mx}+24m^2e^{mx}+9e^{mx}=0$

or, $e^{mx}(16m^4+24m^2+9)=0$

Now, $16m^4+24m^2+9=0$

or, $(4m^2)^2+2\cdot4m^2\cdot3+3^2=0$

or, $(4m^2+3)^2=0$

or, $4m^2+3=0$

or, $m=\sqrt{\frac{-3}{4}}=\pm\frac{\sqrt{3}i}{2}$

$\alpha=0$, $\beta=\frac{\sqrt{3}}{2}$

General Solution $=e^{0x}\left[A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right]$

$=A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x$

29. $y''+16y=0$. $y(0)=2$, $y'(0)=-2$.

Now, $m^2+16=0$

or, $m^2=-4^2$

or, $m=\pm4i$. $\alpha=0$, $\beta=4$.

General Solution; $A\cos4x+B\sin4x$

Now, $y=A\cos4x+B\sin4x$

$\therefore y(0)=2$

or, $A\cos4x+B\sin4x=2$

$\therefore A=2$

$y'(0)=-2$

or, $-A\sin4x+B\cos4x=-2$

$\therefore B=-\frac{1}{2}$

Now, The final equation $=2\cos4x-\frac{1}{2}\sin4x$.

$$\left[\begin{array}{l} y_p = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x \\ y_p' = -Ae^{\alpha x}\sin\beta x + Be^{\alpha x}\cos\beta x \end{array}\right]$$

30. $\frac{d^2y}{d\theta^2} + y = 0$. or, $y'' + y = 0$. $\quad y\left(\frac{\pi}{3}\right) = 0$ $y'\left(\frac{\pi}{3}\right) = 2$.

or, $e^{mx}(m^2+1) = 0$

Now, $m^2+1=0$. or, $m = \sqrt{-1}$.

or, $m = \pm i$

$\alpha = 0$, $\beta = 1$.

$\therefore$ General Solution $= A\cos x + B\sin x$

$y\left(\frac{\pi}{3}\right) = 0$. $\qquad y'\left(\frac{\pi}{3}\right) = 0$.

or, $A\cos x + B\sin x = 0$. $\qquad$ or, $-A\sin x + B\cos x = 2$

or, $\frac{A}{2} + B\frac{\sqrt{3}}{2} = 0$. $\qquad$ or, $-\frac{\sqrt{3}}{2}A + \frac{B}{2} = 2$

or, $A + \sqrt{3}B = 0$. $\qquad$ or, $-\sqrt{3}A + B = 4$

— (i) $\qquad$ — (ii)

(i)+(ii) $\Rightarrow$ $\sqrt{3}A + 3B = 0$

$-\sqrt{3}A + B = 4$

or, $4B = 4$ $\therefore B = 1$. $\quad A = -\sqrt{3}$

$\therefore y = -\sqrt{3}\cos x + \sin x$.

31. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0$. or, $y'' - 4y' - 5y = 0$

Now, $m^2 - 4m - 5 = 0$.

or, $m = \frac{4 \pm \sqrt{36}}{2}$

$= \frac{4 \pm 6}{2}$

$= -1, 5$.

General Solution $= c_1e^{5x} + c_2e^{-x}$.

Now,

$y(1) = 0$

or, $c_1 e^{5x} + c_2 e^{-1} = 0$

or, $c_1 e^5 + c_2 \frac{1}{e^1} = 0$

or, $c_1 \cdot e^5 \cdot e^1 + c_2 = 0$

or, $c_1 e^6 + c_2 = 0$

Again, $y'(1) = 2$

or, $\frac{d}{dx}(c_1 e^{5x} + c_2 e^{-x}) = 2$

or, $c_1 5 e^{5x} - c_2 e^{-x} = 2$

or, $5c_1 e^{5x} - (-c_1 e^6) \cdot e^{-x} = 2$

or, $5c_1 e^{5x} + c_1 e^6 \cdot e^{-x} = 2$

or, $5c_1 e^5 + c_1 e^{6-1} = 2$

or, $6c_1 e^5 = 2$

or, $c_1 e^5 = \frac{1}{3}$

or, $e^5 = \frac{1}{3}\left(\frac{1}{c_1}\right)$

or, $\ln e^5 = \ln\left(\frac{1}{3} \times \frac{1}{c_1}\right)$

or, $5 = -1.0986 + (\ln 1 - \ln c_1)$

or, $6.0986 = -\ln c_1$

or, $e^{-6.0986} = c_1$

$\therefore c_1 = 2.246 \times 10^{-3}$

$c_2 = -2.246 \times 10^{-3} \times e^6$

$= -906.10 \times 10^{-3}$

Solution: $y = 2.246 \times 10^{-3} \cdot e^{5x} - 906.10 \times 10^{-3} e^{-x}$.

Example 3: $y''' + 3y'' - 4y = 0$.

Now, $m^3 + 3m^2 - 4 = 0$.

or, $m^3 + 2m^2 + m^2 - 4 = 0$.

or, $m^2(m+2) + (m+2)(m-2) = 0$.

or, $(m+2)(m^2 + m - 2) = 0$

or, $(m+2)(m+2)(m-1) = 0$.

$\therefore m = -2, -2, 1$

$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-2x}.x$.

33. $y'' + y' + 2y = 0$. $y(0) = y'(0) = 0$.

$m^2 + m + 2 = 0$. $m = \frac{-1 \pm \sqrt{7} i}{2}$

$\therefore m = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

General Solution $= e^{-\frac{1}{2}x}\left[A \cos \frac{\sqrt{7}}{2} x + B \sin \frac{\sqrt{7}}{2} x\right]$

$y(0) = 0$

or, $e^{-\frac{1}{2} \times 0}(A \cos 0 + B \sin 0) = 0$.

or, $A = 0$.

$y'(0) = 0$. $\frac{d}{dx}\left\{e^{-\frac{1}{2}x}\left(A \cos \frac{\sqrt{7}}{2} x + B \sin \frac{\sqrt{7}}{2} x\right)\right\} = 0$.

$\Rightarrow e^{-\frac{1}{2}x} . \frac{d}{dx}\left(A \cos \frac{\sqrt{7}}{2} x + B \sin \frac{\sqrt{7}}{2}\right) + \left(A \cos \frac{\sqrt{7}}{2} x + B \sin \frac{\sqrt{7}}{2} x\right) \frac{d}{dx} e^{-\frac{1}{2}x} = 0$.

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Undetermined Coefficients — Suspension Approach.

$Y = Y_c + Y_p$ → Particular Integral.

Y_c → Complementary.

Y_c = assuming the eqn as a Homogenous eqn

$g(x)$	form of Y_p
1. 10. 100	A.
$5x+7$	$Ax+B$.
$3x^2-2$	Ax^2+Bx+C.
x^3-x+1	Ax^3+Bx^2+Cx+D.
$\sin ax$	$A\cos ax+B\sin ax$
$\cos ax$	$A\cos ax+B\sin ax$.
e^{4x}	Ae^{4x}.
$(9x-4)e^{4x}$	$(Ax+B)e^{4x}$.
x^2e^{4x}	$(Ax^2+Bx+C)e^{4x}$.
$e^x \sin 4x$	$(A\cos 4x+B\sin 4x)e^x$.

Exercise 4.4:

1: $y'' + 3y' + 2y = 6$ —— (i)

homogeneous lineare equation,

$$y'' + 3y' + 2y = 0 \text{ —— (ii)}$$

Now, $m^2 + 3m + 2 = 0$.

$\therefore m = -1, -2$.

General Solution: $y_c = c_1 e^{-x} + c_2 e^{-2x}$ —— (iii)

$y_p = A \quad y'_p = 0, \quad y''_p = 0$.

Now putting those values into equation (i)

$$y'' + 3y' + 2y = 6$$

or, $0 + 0 + 2A = 6$.

$\therefore A = 3$

$\therefore y_p = 3$.

Solution: $y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3$. Ans

3. $y'' - 10y' + 25y = 30x + 3$ —— (i)

$y'' - 10y' + 25y = 0$ —— (ii) [Assoiating Homogeneous linear equn]

Now, $m^2 - 10m + 25 = 0$.

or, $(m-5)^2 = 0$.

$\therefore m = 5, 5$.

$y_c = c_1 e^{5x} + c_2 e^{5x}$ —— (iii)

$y_p = Ax + B$

$y'_p = A$

$y''_p = 0$.

(i) $\rightarrow$ $0 - 10A + 25(Ax + B) = 30x + 3$

or, $-10A + 25B + 25Ax = 30x + 3$.

p.t.o

Equity co-efficients of x and constant.

$25A = 30$. $\quad 25B - 10A = 3$

or, $A = \frac{6}{5}$. $\quad$ or, $25B - 10\cdot\frac{6}{5} = 3$

or $\cdot$ $25B = 15$

$\therefore B = \frac{3}{5}$.

$\therefore y_p = Ax + B = \frac{6}{5}x + \frac{3}{5}$.

$\therefore$ General equation, $y = y_c + y_p$

$= c_1e^{5x} + c_2e^{5x} + 6/5\,x + \frac{3}{5}$.

5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$ —— (i)

$\frac{1}{4}y'' + y' + y = 0$ —— (ii)

Now, $\frac{1}{4}m^2 + m + 1 = 0$.

or $m^2 + 4m + 4 = 0$. or. $(m+2)^2 = 0$. $\therefore m = -2, -2$.

$y_c = c_1e^{-2x} + c_2e^{-2x}$. —— (iii)

$y_p = Ax^2 + Bx + c$

$y'_p = 2Ax + B$

$y''_p = 2A$.

(i) $\rightarrow$ $\frac{1}{4}\times 2A + (2Ax + B) + Ax^2 + Bx + c = x^2 - 2x$.

or, $Ax^2 + (2A + B)x + (B + c + \frac{1}{2}A) = x^2 - 2x$.

Equity co-efficient of x and constant.

$Ax^2 = x^2$ $\therefore A = 1$

$B + 2A = -2$ or, $B + 2\cdot1 = -2$ $\therefore B = -4$.

$B + c + \frac{1}{2}A = 0$ or, $-4 + c + \frac{1}{2} = 0$ $\therefore c = \frac{7}{2}$.

$\therefore y_p = x^2 - 4x + \frac{7}{2}$

$\therefore y = y_c + y_p = c_1e^{-2x} + c_2e^{-2x} + x^2 - 4x + \frac{7}{2}$.