4.3 Homogeneous Linear Equations with Constant Coefficients
Solutions to the Selected Problems

Formula

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c=0$$

Auxiliary equation

$$am^2+bm+c=0$$

1–14. Find the general solution of the given second-order differential equation.

1. $4y''+y'=0$

Solution

The auxiliary equation of the given second-order homogeneous linear differential equation is,

$$4m^2+m=0$$

which has the following solutions

$$m=0, \text{or } m=-\frac{1}{4}$$

Therefore, the general solution is

$$y(x)=C_1e^{0x}+C_2e^{-\frac{1}{4}x}$$

$$\boxed{y(x)=C_1+C_2e^{-\frac{1}{4}x}}$$

5. $y''+8y'+16y=0$

Solution

The auxiliary equation of the given second-order homogeneous linear differential equation is,

$$m^2+8m+16=0$$

which has the solutions $m=-4$ with multiplicity 2.

Therefore, the general solution is

$$\boxed{y(x)=C_1e^{-4x}+C_2xe^{-4x}}$$

Mohammad Hassan Murad
Senior Lecturer
Department of Mathematics and Natural Sciences
BRAC University

4.3 Homogeneous Linear Equations with Constant Coefficients
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11. $y'' - 4y' + 5y = 0$

Solution

The auxiliary equation of the given second-order homogeneous linear differential equation is,

$$m^2 - 4m + 5 = 0$$

which has the complex solutions

$$m = 1 \pm 2i$$

Therefore, the general solution is

$$y(x) = e^{1x}(C_1 \cos 2x + C_2 \sin 2x)$$

$$\boxed{y(x) = e^{x}(C_1 \cos 2x + C_2 \sin 2x)}$$

15–28. Find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y = 0$

Solution

The auxiliary equation of the given higher-order homogeneous linear differential equation is,

$$m^3 - 4m^2 - 5m = 0$$

$$m(m^2 - 4m - 5) = 0$$

which has the following solutions

$$m = 0, \text{or } m = 1 \pm 2i$$

Therefore, the general solution is

$$y(x) = C_1 e^{0x} + e^{x}(C_2 \cos 2x + C_3 \sin 2x)$$

$$\boxed{y(x) = C_1 + e^{x}(C_2 \cos 2x + C_3 \sin 2x)}$$

17. $y''' - 5y'' + 3y' + 9y = 0$

Solution

4.3 Homogeneous Linear Equations with Constant Coefficients
Solutions to the Selected Problems

The auxiliary equation of the given higher-order homogeneous linear differential equation is,

$$m^3 - 5m^2 + 3m + 9 = 0$$

$$(m + 1)(m^2 - 6m + 9) = 0$$

$$(m + 1)(m - 3)^2 = 0$$

which has the solutions $m = 0$ (multiplicity 1), or $m = 3$ (multiplicity 2)

Therefore, the general solution is

$$y(x) = C_1 e^{0x} + C_2 e^{3x} + C_3 x e^{3x}$$

$$\boxed{y(x) = C_1 + C_2 e^{3x} + C_3 x e^{3x}}$$

Example $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

Solution

The auxiliary equation of the given higher-order homogeneous linear differential equation is,

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$

which has the solutions $m = 2i$ (multiplicity 2), or $m = -2i$ (multiplicity 2)

Therefore, the general solution is

$$y(x) = e^{0x}[C_1 \cos(2x) + C_2 \sin(2x) + C_3 x \cos(2x) + C_4 x \sin(2x)]$$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 x \cos 2x + C_4 x \sin 2x}$$

25. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$

Solution

The auxiliary equation of the given higher-order homogeneous linear differential equation is,

$$16m^4 + 24m^2 + 9 = 0$$

$$(4m^2+3)^2=0$$

Ans. $\boxed{y(x)=C_1\cos\left(\frac{\sqrt{3}}{2}x\right)+C_2\sin\left(\frac{\sqrt{3}}{2}x\right)+C_3x\cos\left(\frac{\sqrt{3}}{2}x\right)+C_4x\sin\left(\frac{\sqrt{3}}{2}x\right)}$