MAT 216

Problem Sheet - 3:

Example: Let $p(x)=2x^2-3x+4,$ and $A=\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}$. Find $p(A)$.

Solution: $p(A)=2A^2-3A+4I=2\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}^2-3\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}+4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}2 & 8\\ 0 & 18\end{bmatrix}-\begin{bmatrix}-3 & 6\\ 0 & 9\end{bmatrix}+\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}=\begin{bmatrix}9 & 2\\ 0 & 13\end{bmatrix}$$

Solve the following Problems:

1. Let $(i)\ p(x)=x-2,\ \ (ii)\ p(x)=2x^2-x+1,\ \ (iii)\ p(x)=x^3-2x+4,$ and $A=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$.

   Find $p(A)$ in each part.

2. Consider the following systems:

   $$(i)\ \begin{aligned}2x_1+3x_2-6x_3-5x_4+2x_5&=7\\ x_3+3x_4-7x_5&=6\\ x_4-2x_5&=1,\end{aligned}\qquad (ii)\ \begin{aligned}2x-6y+7z&=1\\ 4y+3z&=8\\ 2z&=4,\end{aligned}\qquad (ii)\ \begin{aligned}x+2y-3z&=2\\ 2x+3y+z&=4\\ 3x+4y+5z&=8\end{aligned}.$$

   (a) Determine the pivot and free variables in each of the above systems.

   (b) Solve the above systems *(i)* & *(ii)*.

3. Find the coefficient matrix A and the augmented matrix M of the following systems:

   $$(i)\ \begin{aligned}x+2y-3z&=4\\ 3y-4z+7x&=5\\ 6z+8x-9y&=1,\end{aligned}\qquad (ii)\ \begin{aligned}x_1+2x_2-3x_3-2x_4+4x_5&=1\\ 2x_1+5x_2-8x_3-x_4+6x_5&=4\\ x_1+4x_2-7x_3+5x_4+2x_5&=8\end{aligned}$$

4. Solve the following systems of linear equations by Gaussian elimination method:

(i) $\begin{aligned} x+y+2z&=9\\ 2x+4y-3z&=1\\ 3x+6y-5z&=0, \end{aligned}$ (ii) $\begin{aligned} x+y+2z&=8\\ -x-2y+3z&=1\\ 3x-7y+4z&=10, \end{aligned}$ (iii) $\begin{aligned} x-y+2z-w&=-1\\ 2x+y-2z-2w&=-2\\ -x+2y-4z+w&=1\\ 3x\qquad\quad -3w&=-3. \end{aligned}$

Ans.: (i) $x=1,\ y=2,\ z=3$, (ii) $x=3,\ y=1,\ z=2$, (iii) $x=t,\ y=2s,\ z=s, w=t$.

5. Solve the following systems of linear equations by Gauss-Jordan elimination:

(i) $\begin{aligned} x_1+3x_2-2x_3\qquad\quad +2x_5\qquad\quad &=0\\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6 &=-1\\ 5x_3+10x_4\qquad +15x_6 &=5\\ 2x_1+6x_2\qquad\quad +8x_4+4x_5+18x_6 &=6, \end{aligned}$

(ii) $\begin{aligned} 10y-4z+w&=\ 1\\ x+4y-z+w&=\ 2\\ 3x+2y+z+2w&=\ 5\\ -2x-8y+2z-2w&=-4\\ x-6y+3z\qquad &=1. \end{aligned}$

Ans.: (i) $x_1=-3r-4s-2t,\ x_2=r,\ x_3=-2s,\ x_4=s,\ x_5=t,\ x_6=1/3$

(ii) $x=(5/8)-(3/5)t-(3/5)s,\ y=(1/10)+(2/5)t-(1/10)s,\ z=t, w=s$.

6. Solve the following homogeneous systems of linear equations by any method:

(i) $\begin{aligned} 3x_1+x_2+x_3+x_4&=0\\ 5x_1-x_2+x_3-x_4&=0, \end{aligned}$

(ii) $\begin{aligned} y+3z-2w&=0\\ 2x+y-4z+3w&=0\\ 2x+3y+2z-w&=0\\ -4x-3y+5z-4w&=0. \end{aligned}$

Note: Homogeneous linear systems: A system of linear equations is said to be homogeneous if the constant terms are all zero; that is the system has the form

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=0\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=0\\ \vdots\qquad\quad &\ \ \vdots\\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n&=0 \end{aligned}$$

Ans.: (i) $x_1=-s,\ x_2=-t-s,\ x_3=4s,\ x_4=t$, (ii) $x=7s-5t, y=-6s+4t,\ z=2s, w=2t$.

7. Find the inverse of the following matrices using row operations:

$$(i)\quad A=\begin{pmatrix}1&2&3\\2&5&3\\1&0&8\end{pmatrix},\qquad (ii)\quad B=\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}.$$

Ans.: $(i)\quad A^{-1}=\begin{pmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{pmatrix},\qquad (ii)\quad B^{-1}=\frac{1}{2}\begin{pmatrix}1&-1&1\\-1&1&1\\1&1&-1\end{pmatrix}.$

8. Find conditions that b 's must satisfy for the system to be consistent:

$$(i)\quad \begin{aligned}x_1+x_2+2x_3&=b_1\\x_1\qquad+x_3&=b_2\\2x_1+x_2+3x_3&=b_3,\end{aligned}\qquad (ii)\quad \begin{aligned}x_1-2x_2+5x_3&=b_1\\4x_1-5x_2+8x_3&=b_2\\-3x_1+3x_2-3x_3&=b_3.\end{aligned}$$

Ans.: $(i)\quad b=\begin{pmatrix}b_1\\b_2\\b_1+b_2\end{pmatrix}$ or $b_3=b_1+b_2$, $(ii)\quad b=\begin{pmatrix}b_2+b_3\\b_2\\b_3\end{pmatrix}$ or $b_1=b_2+b_3$.

9. Solve the following system of linear equations using $x=A^{-1}b$:

$$\begin{aligned}x_1+2x_2+3x_3&=5\\2x_1+5x_2+3x_3&=3\\x_1\qquad\quad+8x_3&=17.\end{aligned}$$

Ans.: $x_1=1,\ x_2=-1,\ x_3=2.$