MAT 216

Solutions (Problem Sheet - 3):

1. (i) $p(x)=x-2$ and $A=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$

$$p(A)=A-2I=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix},$$

$$(ii)\ p(A)=\begin{bmatrix}20 & 7\\ 14 & 6\end{bmatrix},\quad (iii)\ p(A)=\begin{bmatrix}39 & 13\\ 26 & 13\end{bmatrix}.$$

2.

$$(i)\quad \begin{aligned} 2x_1+3x_2-6x_3-5x_4+2x_5 &= 7\\ x_3+3x_4-7x_5 &= 6\\ x_4-2x_5 &= 1, \end{aligned}$$

(a) In echelon form, the leading unknowns are the pivot variables and the others are the free variables.

Here the pivot variables are $x_1,\ x_3,\ x_4$, and the free variables are $x_2,\ x_5$.

$$(ii)\quad \begin{aligned} 2x-6y+7z &= 1\\ 4y+3z &= 8\\ 2z &= 4, \end{aligned}$$

(a) Here the leading unknowns are $x,\ y,\ z$, so they are the pivot variables. There is no free variable.

$$(ii)\quad \begin{aligned} x+2y-3z &= 2\\ 2x+3y+z &= 4\\ 3x+4y+5z &= 8 \end{aligned}$$

(a) The notion of pivot and free variables applies only to a system in echelon form.

$$\begin{aligned}(i)\quad 2x_1+3x_2-6x_3-5x_4+2x_5&=7\\ x_3+3x_4-7x_5&=6\\ x_4-2x_5&=1,\end{aligned}$$

(b) Assign parameters to the free variables. Let $x_2=s$, and $x_5=t$ and solve for the pivot variables by back substitution.

Substitute $x_5=t$ in the last Eq., we obtain, $x_4-2t=1$, or $x_4=1+2t$

Substitute $x_5=t$ & $x_4=1+2t$ in the second Eq., to get

$x_3+3(1+2t)-7t=6$ or $x_3+3+6t-7t=6$ or $x_3=3+t$

Substitute $x_5=t$, $x_4=1+2t$, $x_3=3+t$ & $x_2=s$ in the first Eq., to get

$2x_1+3s-6(3+t)-5(1+2t)+2t=7$ or $2x_1+3s-18-6t-5-10t+2t=7$

or $x_1=15+\dfrac{3s}{2}+7t$.

$$\begin{aligned}(ii)\quad 2x-6y+7z&=1\\ 4y+3z&=8\\ 2z&=4,\end{aligned}$$

(b) The last Eq. gives $z=2$.

Substitute … $y=\dfrac{1}{2}$

…. $x=-5$.

3. Find the coefficient matrix A and the augmented matrix M of the following systems:

$$\begin{aligned}(i)\quad x+2y-3z&=4\\ 3y-4z+7x&=5\\ 6z+8x-9y&=1,\end{aligned}\qquad \begin{aligned}(ii)\quad x_1+2x_2-3x_3-2x_4+4x_5&=1\\ 2x_1+5x_2-8x_3-x_4+6x_5&=4\\ x_1+4x_2-7x_3+5x_4+2x_5&=8\end{aligned}$$

4. Gaussian elimination:

(i) $x+y+2z=9$

$2x+4y-3z=1$

$3x+6y-5z=0,$

The augmented matrix is

$$\begin{bmatrix}1 & 1 & 2 & 9\\ 2 & 4 & -3 & 1\\ 3 & 6 & -5 & 0\end{bmatrix} \xrightarrow[R_3'=-3R_1+R_3]{R_2'=-2R_1+R_2} \begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 0 & 3 & -11 & -27\end{bmatrix} \xrightarrow[R_2'=(1/2)R_2]{} \begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 1 & -7/2 & -17/2\\ 0 & 3 & -11 & -27\end{bmatrix}$$

$$\xrightarrow[R_3'=-3R_2+R_3]{} \begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 1 & -7/2 & -17/2\\ 0 & 0 & -1/2 & -3/2\end{bmatrix} \xrightarrow[R_3'=-2R_3]{} \begin{bmatrix}1 & 1 & 2 & 9\\ 0 & 1 & -7/2 & -17/2\\ 0 & 0 & 1 & 3\end{bmatrix}$$

The corresponding system of linear equations is

$$x+y+2z=9$$
$$y-\frac{7z}{2}=\frac{-17}{2}$$
$$z=3$$

Substitute $z=3$ in the second Eq., to get $y=2$.

Substitute $z=3$ & $y=2$ in the first Eq., we obtain $x=1$.

Similarly you can solve *(ii)* & *(iii)*.

Ans.: (i) $x=1,\ y=2,\ z=3$, (ii) $x=3,\ y=1,\ z=2$, (iii) $x=t,\ y=2s,\ z=s, w=t$.

5. Gauss-Jordan elimination:

$$(i)\quad \begin{aligned} x_1+3x_2-2x_3\qquad\quad +2x_5\qquad\quad &=0\\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6 &=-1\\ 5x_3+10x_4\qquad +15x_6 &=5\\ 2x_1+6x_2\qquad\quad +8x_4+4x_5+18x_6 &=6, \end{aligned}$$

The augmented matrix is:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \xrightarrow[R_4'=-2R_1+R_4]{R_2'=-2R_1+R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & -1 & -2 & 0 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2'=(-1)R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \xrightarrow[R_4'=-4R_2+R_4]{R_3'=-5R_2+R_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 6 & 2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3'=(1/6)R_3]{R_1'=2R_2+R_1} \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2'=-3R_3+R_2} \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1+3x_2\qquad +2x_5 &=0\\ x_3+2x_4 &=0\\ x_6 &=1/3 \end{aligned}$$

Here the free variables are $x_2,\ x_4$ & x_5 . Let the arbitrary values be $x_2=r,\ x_4=s$ & $x_5=t$.

So the solution is:

$x_1=-3r-4s-2t,\ x_2=r,\ \ x_3=-2s,\ x_4=s,\ x_5=t$ & $x_6=1/3$.

Similarly, you can solve problem No. *(ii)*.

Ans.: $(i)\ \ x_1 = -3r - 4s - 2t,\ \ x_2 = r,\ \ x_3 = -2s,\ \ x_4 = s,\ \ x_5 = t,\ \ x_6 = 1/3$

$(ii)\ \ x = (5/8) - (3/5)t - (3/5)s,\ y = (1/10) + (2/5)t - (1/10)s,\ \ z = t, w = s$.

6. Similarly you can solve prob. *(i)* & *(ii)*.

   Ans.: $(i)\ \ x_1 = -s,\ x_2 = -t - s\,,\ x_3 = 4s,\ \ x_4 = t\,,\ (ii)\ \ x = 7s - 5t, y = -6s + 4t,\ z = 2s, w = 2t$.

7. $(i)\ \ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ We are going to find A^{-1}.

   Let us consider $(A|I)$ where I is the identity matrix.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3' = -R_1 + R_3]{R_2' = -2R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3' = 2R_2 + R_3]{R_1' = -2R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \xrightarrow{R_3' = (-1)R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow[R_2' = 3R_3 + R_2]{R_1' = -9R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

   The presentation of last matrix is $(I|A)$

   So the inverse matrix is:

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}.$$

   Now you are able to find A^{-1} of Problem No. *(ii)*

Ans.: $(i)\ \ A^{-1}=\begin{pmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{pmatrix}$, $\quad (ii)\ \ B^{-1}=\dfrac{1}{2}\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & 1\\ 1 & 1 & -1\end{pmatrix}$.

8. The augmented matrix is:

$$\left(\begin{array}{ccc|c}1 & 1 & 2 & b_1\\ 1 & 0 & 1 & b_2\\ 2 & 1 & 3 & b_3\end{array}\right)\xrightarrow[R_3'=-2R_1+R_3]{R_2'=-R_1+R_2}\left(\begin{array}{ccc|c}1 & 1 & 2 & b_1\\ 0 & -1 & -1 & b_2-b_1\\ 0 & -1 & -1 & b_3-2b_1\end{array}\right)\xrightarrow{R_2'=(-1)R_2}\left(\begin{array}{ccc|c}1 & 1 & 2 & b_1\\ 0 & 1 & 1 & b_1-b_2\\ 0 & -1 & -1 & b_3-2b_1\end{array}\right)$$

$$\xrightarrow[R_3'=R_2+R_3]{R_1'=-R_2+R_1}\left(\begin{array}{ccc|c}1 & 0 & 1 & b_2\\ 0 & 1 & 1 & b_1-b_2\\ 0 & 0 & 0 & b_3-b_2-b_1\end{array}\right)$$

If $b_3-b_2-b_1=0$ then the system is consistent.

$\therefore\ b_3=b_1+b_2$

So that $b=\begin{pmatrix}b_1\\ b_2\\ b_1+b_2\end{pmatrix}$.

Ans.: $(i)\ \ b=\begin{pmatrix}b_1\\ b_2\\ b_1+b_2\end{pmatrix}$ or $b_3=b_1+b_2$, $\quad (ii)\ \ b=\begin{pmatrix}b_2+b_3\\ b_2\\ b_3\end{pmatrix}$ or $b_1=b_2+b_3$.

9. Firstly we need to identify matrices:

$A=\begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{pmatrix}$, $x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ & $b=\begin{pmatrix}5\\ 3\\ 17\end{pmatrix}$

Already you know how to find the inverse, so find A^{-1}.

Here $A^{-1}=\begin{pmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{pmatrix}$.

Now solving the system we can $x=A^{-1}b$

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Ans.: $x_1 = 1,\ \ x_2 = -1,\ \ x_3 = 2.$