7.3 Integrating Trigonometric Functions
Solutions to the Selected Problems

1–52. Evaluate the integral.

1. $\int \cos^3 x \sin x \, dx$

Solution

Let

$$u = \cos x \Rightarrow du = -\sin x \, dx$$

$$\int \cos^3 x \sin x \, dx = \int u^3 \, (-du) = -\frac{u^4}{4} + C$$

$$\boxed{\int \cos^3 x \sin x \, dx = -\frac{1}{4}\cos^4 x + C}$$

2. $\int \sin^5 3x \cos 3x \, dx$

Solution

Let

$$u = \sin 3x \Rightarrow du = \frac{1}{3}\cos 3x \, dx$$

$$\int \sin^5 3x \cos 3x \, dx = \int u^5 \left(\frac{1}{3} du\right) = \frac{1}{3} \times \frac{u^6}{6} + C$$

$$\boxed{\int \sin^5 3x \cos 3x \, dx = \frac{1}{18}\sin^6 3x + C}$$

3. $\int \sin^2 5\theta \, d\theta$

Solution

Using the following trigonometric identity

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\int \sin^2 5\theta \, d\theta = \frac{1}{2}\int (1 - \cos(10\theta)) \, d\theta = \frac{1}{2}\theta - \frac{1}{20}\sin(10\theta) + C$$

$$\boxed{\int \sin^2 5\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{20}\sin(10\theta) + C}$$

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9. $\int \sin^2 t \cos^3 t \, dt$

Solution

$$\int \sin^2 t \cos^3 t \, dt = \int \sin^2 t \cos^2 t \cos t \, dt = \int \sin^2 t \,(1 - \sin^2 t) \cos t \, dt$$

Let

$$u = \sin t \Rightarrow du = \cos t \, dt$$

$$\int \sin^2 t \,(1 - \sin^2 t) \cos t \, dt = \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$\boxed{\int \sin^2 t \cos^3 t \, dt = \frac{1}{3} \sin^3 t - \frac{1}{5} \sin^5 t + C}$$

10. $\int \sin^3 x \cos^2 x \, dx$

Solution

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Let

$$u = \cos x \Rightarrow du = -\sin x \, dx$$

$$\int (1 - \cos^2 x) \cos^2 x \sin x \, dx = \int (1 - u^2) u^2 \,(-du) = \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$\boxed{\int \sin^3 x \cos^2 x \, dx = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C}$$

11. $\int \sin^2 x \cos^2 x \, dx$

Solution

Using the following trigonometric identities,

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x) \, dx$$

$$= \frac{1}{4}\int (1 - \cos^2 2x) \, dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{8}\int \quad dx - \frac{1}{8}\int \cos 4x \, dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x$$

$$\boxed{\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

Alternative

Using the following trigonometric identity,

$$2\sin x \cos x = \sin 2x$$

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{8}\int (1 - \cos 4x) \, dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x$$

$$\boxed{\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

13. $\int \sin 2x \cos 3x \, dx$

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Solution

Using the following trigonometric identity,

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\int \sin 2x \cos 3x \, dx = \frac{1}{2}\int [\sin(-x) + \sin(5x)] \, dx$$

$$= -\frac{1}{2}\int \sin x \, dx + \frac{1}{2}\int \sin 5x \, dx$$

$$= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x$$

$$\boxed{\int \sin 2x \cos 3x \, dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C}$$

15. $\int \sin x \cos\left(\frac{x}{2}\right) dx$

Solution

Using the following trigonometric identity,

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\int \sin x \cos\left(\frac{x}{2}\right) dx = \frac{1}{2}\int \left[\sin\left(\frac{x}{2}\right) + \sin\left(\frac{3x}{2}\right)\right] dx$$

$$= \frac{1}{2}\int \sin\left(\frac{x}{2}\right) dx + \frac{1}{2}\int \sin\left(\frac{3x}{2}\right) dx$$

$$= \frac{1}{2}\left[-2\cos\left(\frac{x}{2}\right) - \frac{2}{3}\cos\left(\frac{3x}{2}\right)\right] + C$$

$$\boxed{\int \sin x \cos\left(\frac{x}{2}\right) dx = -\cos\left(\frac{x}{2}\right) - \frac{1}{3}\cos\left(\frac{3x}{2}\right) + C}$$

16. $\int (\cos x)^{\frac{1}{3}} \sin x \, dx$ or, $\int \sqrt[3]{\cos x} \sin x \, dx$

Solution

Let

$$u = \cos x \Rightarrow du = -\sin x\, dx$$

$$\int (\cos x)^{\frac{1}{3}} \sin x\, dx = \int u^{\frac{1}{3}} (-du) = -\int u^{\frac{1}{3}}\, du = -\frac{3}{4} u^{\frac{4}{3}}$$

$$\boxed{\int (\cos x)^{\frac{1}{3}} \sin x\, dx = -\frac{3}{4} (\cos x)^{\frac{4}{3}} + C}$$

Or,

$$\boxed{\int \sqrt[3]{\cos x} \sin x\, dx = -\frac{3}{4} \cos x \sqrt[3]{\cos x} + C}$$

17. $\int_0^{\frac{\pi}{2}} \cos^3 x\, dx$

Solution

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx = \sin x - \frac{1}{3} \sin^3 x$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \sin\left(\frac{\pi}{2}\right) - \frac{1}{3} \sin^3\left(\frac{\pi}{2}\right) = \frac{2}{3}$$

$$\boxed{\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \frac{2}{3}}$$

28. $\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx$

Solution

Let

$$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow \frac{1}{\sqrt{x}} dx = 2du$$

$$\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx = \int \sec u\, (2du) = 2 \ln|\sec u + \tan u|$$

$$\boxed{\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx = 2 \ln\left|\sec \sqrt{x} + \tan \sqrt{x}\right| + C}$$

29. $\int \tan^2 x \sec^2 x\, dx$

Solution

$$\int \tan^2 x \sec^2 x \, dx = \int u^2 \, du = \frac{1}{3}u^3 + C$$

$$\boxed{\int \tan^2 x \sec^2 x \, dx = \frac{1}{3}\tan^3 x + C}$$

30. $\int \tan^5 x \sec^4 x \, dx$

Solution

$$\int \tan^5 x \sec^4 x \, dx = \int \tan^5 x \sec^2 x \sec^2 x \, dx = \int \tan^5 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^5 (1 + u^2) \, du = \frac{1}{6}u^6 + \frac{1}{7}u^7 + C$$

$$\boxed{\int \tan^5 x \sec^4 x \, dx = \frac{1}{6}\tan^6 x + \frac{1}{7}\tan^7 x + C}$$

33. $\int \sec^5 x \tan^3 x \, dx$

Solution

$$\int \sec^5 x \tan^3 x \, dx = \int \sec^4 x \tan^2 x \sec x \tan x \, dx$$

$$= \int \sec^4 x \, (1 + \sec^2 x) \sec x \tan x \, dx$$

$$= \int u^4 (1 + u^2) \, du$$

$$= \frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$

$$\boxed{\int \sec^5 x \tan^3 x \, dx = \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C}$$

34. $\int \tan^5 \theta \sec \theta \, d\theta$

Solution

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$$\int \tan^5\theta \sec\theta\, d\theta = \int \tan^4\theta \sec\theta \tan\theta\, d\theta$$

$$= \int (1 + \sec^2\theta)^2 \sec\theta \tan\theta\, d\theta$$

$$= \int (1 + 2\sec^2\theta + \sec^4\theta) \sec\theta \tan\theta\, d\theta$$

$$= \int (1 + 2u^2 + u^4)\, du$$

$$= u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$\boxed{\int \tan^5\theta \sec\theta\, d\theta = \sec\theta + \frac{2}{3}\sec^3\theta + \frac{1}{5}\sec^5\theta + C}$$

35. $\int \tan^4 x \sec x\, dx$

Solution

$$\int \tan^4 x \sec x\, dx = \int (\sec^2 x - 1)^2 \sec x\, dx$$

$$= \int (\sec^5 x - 2\sec^3 x + \sec x)\, dx$$

$$= \int \sec^5 x\, dx - 2\int \sec^3 x\, dx + \int \sec x\, dx$$

Using the following reduction formula

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

we obtain,

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$$

$$\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left[\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|\right]$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x|$$

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$$\boxed{\int \tan^4 x \sec x \, dx = \frac{1}{4}\sec^3 x \tan x - \frac{5}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C}$$

43. $\int \sqrt{\tan x} \sec^4 x \, dx$

Solution

$$\int \sqrt{\tan x} \sec^4 x \, dx = \int \sqrt{\tan x}\,(1 + \tan^2 x) \sec^2 x \, dx$$

Let

$$u = \tan x \Rightarrow du = \sec^2 x \, dx$$

$$\int \sqrt{\tan x}\,(1 + \tan^2 x) \sec^2 x \, dx = \int \sqrt{u}(1 + u^2) \, du = \frac{2}{3}u^{\frac{3}{2}} + \frac{2}{7}u^{\frac{7}{2}} = \frac{2}{3}u\sqrt{u} + \frac{2}{7}u^3\sqrt{u}$$

$$\boxed{\int \sqrt{\tan x} \sec^4 x \, dx = \frac{2}{3}\tan x \sqrt{\tan x} + \frac{2}{7}\tan^3 x \sqrt{\tan x} + C}$$

57. Let m, n be nonnegative distinct integers. Prove that,

(a) $\int_0^{2\pi} \sin mx \cos nx \, dx = 0$

(b) $\int_0^{2\pi} \cos mx \cos nx \, dx = 0$

(c) $\int_0^{2\pi} \sin mx \sin nx \, dx = 0$

Solution

(a) $\int_0^{2\pi} \sin mx \cos nx \, dx$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$$

$$\int \sin mx \cos nx \, dx = \frac{1}{2}\int [\sin(m+n)x + \sin(m-n)x] \, dx$$

$$= -\frac{1}{2(m+n)}\cos(m+n)x - \frac{1}{2(m-n)}\cos(m-n)x$$

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$$\int_0^{2\pi} \sin mx \cos nx \, dx$$

$$= \left[-\frac{1}{2(m+n)}\cos 2(m+n)\pi - \frac{1}{2(m-n)}\cos 2(m-n)\pi\right] - \left[-\frac{1}{2(m+n)}\cos 0 - \frac{1}{2(m-n)}\cos 0\right]$$

$$\boxed{\int_0^{2\pi} \sin mx \cos nx \, dx = 0}$$

Optional but Important Problems

58. Let m be an integer. Prove that,

(a) $\int_0^{2\pi} \sin mx \cos mx \, dx = 0$

(b) $\int_0^{2\pi} \cos^2 mx \, dx = \begin{cases} \pi & m \neq 0 \\ 2\pi & m = 0 \end{cases}$

(b) $\int_0^{2\pi} \sin^2 mx \, dx = \begin{cases} \pi & m \neq 0 \\ 0 & m = 0 \end{cases}$

Solution

(a) Case I: m is an positive integer.

$$\sin mx \cos mx = \frac{1}{2}\sin(2mx)$$

$$\int_0^{2\pi} \sin mx \cos mx \, dx = -\frac{1}{4m}[\cos(4m\pi) - \cos 0] = 0$$

Case II: $m = 0$.

$$\int_0^{2\pi} \sin mx \cos mx \, dx = \int_0^{2\pi} 0 \, dx = 0$$

(b) Case I: m is an positive integer.

$$\cos^2 mx = \frac{1}{2}(1 + \cos 2mx)$$

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$$\int_0^{2\pi} \cos^2 mx\,dx = \frac{1}{2}\int_0^{2\pi} (1+\cos 2mx)\,dx = \frac{1}{2}\int_0^{2\pi} 1\,dx + \frac{1}{2}\int_0^{2\pi} \cos 2mx\,dx$$

$$= \frac{1}{2}\times 2\pi + \frac{1}{4m}\times \sin 2mx\Big|_0^{2\pi} = \pi + \frac{1}{4m}\times(\sin 4m\pi - 0) = \pi$$

Case II: $m = 0$.

$$\int_0^{2\pi} \cos^2 mx\,dx = \int_0^{2\pi} 1\,dx = 2\pi$$

(c) Case I: m is an positive integer.

$$\sin^2 mx = \frac{1}{2}(1-\cos 2mx)$$

$$\int_0^{2\pi} \sin^2 mx\,dx = \frac{1}{2}\int_0^{2\pi} (1-\cos 2mx)\,dx = \frac{1}{2}\int_0^{2\pi} 1\,dx - \frac{1}{2}\int_0^{2\pi} \cos 2mx\,dx$$

$$= \frac{1}{2}\times 2\pi - \frac{1}{4m}\times \sin 2mx\Big|_0^{2\pi} = \pi - \frac{1}{4m}\times(\sin 4m\pi - 0) = \pi$$

Case II: $m = 0$.

$$\int_0^{2\pi} \sin^2 mx\,dx = \int_0^{2\pi} 0\,dx = 0$$

Problem. Let m be a nonnegative integer and c is any real number. Prove that,

(a) $\displaystyle\int_c^{c+2\pi} \sin mx \cos mx\,dx = 0$

(b) $\displaystyle\int_c^{c+2\pi} \cos^2 mx\,dx = \begin{cases} \pi & m \neq 0 \\ 2\pi & m = 0 \end{cases}$

(c) $\displaystyle\int_c^{c+2\pi} \sin^2 mx\,dx = \begin{cases} \pi & m \neq 0 \\ 0 & m = 0 \end{cases}$

Solution

(a) $\displaystyle\int_c^{c+2\pi} \sin mx \cos mx\,dx$

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$$\sin mx \cos mx = \frac{1}{2}\sin(2mx)$$

$$\int_c^{c+2\pi} \sin mx \cos mx \, dx = -\frac{1}{4m}\left[\cos\big(2m(c+2\pi)\big) - \cos(2mc)\right]$$

$$= -\frac{1}{4m}[\cos(2mc)\cos(4m\pi) + \sin(2mc)\sin(4m\pi) - \cos(2mc)]$$

$$= -\frac{1}{4m}[\cos(2mc) + \sin(2mc) \times 0 - \cos(2mc)]$$

$$= 0$$

(b) Case I: $m \neq 0$

$$\int_c^{c+2\pi} \cos^2 mx \, dx = \int_c^{c+2\pi} \frac{1}{2}(1 + \cos 2mx)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2m}\sin 2mx\right)\Big|_c^{c+2\pi}$$

$$= \frac{1}{2}\times\left[(c+2\pi) + \frac{1}{2m}\sin 2m(c+2\pi)\right] - \frac{1}{2}\times\left[c + \frac{1}{2m}\sin 2mc\right]$$

$$= \frac{1}{2}\times 2\pi + \frac{1}{4m}\times[\sin 2m(c+2\pi) - \sin 2mc]$$

$$= \pi + \frac{1}{4m}\times[\sin 2mc \cos 4m\pi + \cos 2mc \sin 4m\pi - \sin 2mc]$$

$$= \pi + \frac{1}{4m}\times\left[\sin 2mc\left(\underbrace{\cos 4m\pi}_{1} - 1\right) + \cos 2mc\, \underbrace{\sin 4m\pi}\right]$$

$$= \pi$$

Case II: $m = 0$

$$\int_c^{c+2\pi} \cos^2 mx \, dx = \int_c^{c+2\pi} dx = 2\pi$$

(c) Case I: $m \neq 0$

$$\int_c^{c+2\pi} \sin^2 mx \, dx = \int_c^{c+2\pi} \frac{1}{2}(1 - \cos 2mx)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2m}\sin 2mx\right)\Big|_c^{c+2\pi}$$

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$$= \frac{1}{2} \times \left[(c+2\pi) - \frac{1}{2m}\sin 2m(c+2\pi)\right] - \frac{1}{2} \times \left[c - \frac{1}{2m}\sin 2mc\right]$$

$$= \frac{1}{2} \times 2\pi - \frac{1}{4m} \times [\sin 2m(c+2\pi) - \sin 2mc]$$

$$= \pi - \frac{1}{4m} \times [\sin 2mc \cos 4m\pi + \cos 2mc \sin 4m\pi - \sin 2mc]$$

$$= \pi - \frac{1}{4m} \times \left[\sin 2mc \left(\underbrace{\cos 4m\pi}_{1} - 1\right) + \cos 2mc \underbrace{\sin 4m\pi}_{0}\right]$$

$$= \pi$$

Case II: $m = 0$

$$\int_c^{c+2\pi} \sin^2 mx\, dx = \int_c^{c+2\pi} 0\, dx = 0$$

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