7.8 Improper Integrals
Solutions to the Selected Problems

Formula

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{c\to a^+}\int_c^b f(x)\,dx$$

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{c\to b^-}\int_a^c f(x)\,dx$$

$$\int_a^{+\infty} f(x)\,dx \stackrel{\text{def}}{=} \lim_{c\to +\infty}\int_a^c f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{c\to -\infty}\int_c^b f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx \stackrel{\text{def}}{=} \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$$

3–32. Evaluate the integrals that converge.

3. $\displaystyle\int_0^{+\infty} e^{-2x}\,dx$

Solution

$$\int_0^{+\infty} e^{-2x}\,dx \stackrel{\text{def}}{=} \lim_{b\to +\infty}\int_0^b e^{-2x}\,dx$$

$$\int_0^b e^{-2x}\,dx = \frac{1}{2}(1-e^{-2b})$$

$$\lim_{b\to +\infty}\int_0^b e^{-2x}\,dx = \frac{1}{2}\times\lim_{b\to +\infty}(1-e^{-2b}) = \frac{1}{2}(1-0) = \frac{1}{2}$$

$$\boxed{\int_0^{+\infty} e^{-2x}\,dx = \frac{1}{2}}$$

4. $\displaystyle\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx$

Solution

$$\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx \stackrel{\text{def}}{=} \lim_{b\to +\infty}\int_{-1}^b \frac{x}{1+x^2}\,dx$$

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$$\int_{-1}^{b} \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)$$

$$\int_{-1}^{b} \frac{x}{1+x^2}\,dx = \frac{1}{2}\left[\ln(1+b^2) - \ln 2\right]$$

$$\lim_{b\to+\infty} \int_{-1}^{b} \frac{x}{1+x^2}\,dx = \frac{1}{2} \times \lim_{b\to+\infty} \left[\ln(1+b^2) - \ln 2\right] = \frac{1}{2}(+\infty - \ln 2) = +\infty$$

$$\boxed{\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx \text{ diverges.}}$$

5. $\displaystyle\int_{3}^{+\infty} \frac{2}{x^2-1}\,dx$

Solution

$$\int_{3}^{+\infty} \frac{2}{x^2-1}\,dx \stackrel{\text{def}}{=} \lim_{b\to+\infty} \int_{3}^{b} \frac{2}{x^2-1}\,dx$$

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$$

$$\int_{3}^{b} \frac{2}{x^2-1}\,dx = \ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2}{4}\right| = \ln\left|\frac{1-\frac{1}{b}}{1+\frac{1}{b}}\right| + \ln 2$$

$$\lim_{b\to+\infty} \int_{3}^{b} \frac{2}{x^2-1}\,dx = \lim_{b\to+\infty} \left[\ln\left|\frac{1-\frac{1}{b}}{1+\frac{1}{b}}\right| + \ln 2\right] = \ln 1 + \ln 2 = \ln 2$$

$$\boxed{\int_{3}^{+\infty} \frac{2}{x^2-1}\,dx = \ln 2}$$

6. $\displaystyle\int_{0}^{+\infty} xe^{-x^2}\,dx$

Solution

$$\int_{0}^{+\infty} xe^{-x^2}\,dx \stackrel{\text{def}}{=} \lim_{b\to+\infty} \int_{0}^{b} xe^{-x^2}\,dx$$

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$$

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$$\int_0^b xe^{-x^2}\,dx = -\frac{1}{2}e^{-b^2} + \frac{1}{2} = \frac{1}{2}\left(1 - e^{-b^2}\right)$$

$$\lim_{b\to+\infty}\int_0^b xe^{-x^2}\,dx = \frac{1}{2}\lim_{b\to+\infty}\left(1 - e^{-b^2}\right) = \frac{1}{2}(1-0) = \frac{1}{2}$$

$$\boxed{\int_0^{+\infty} xe^{-x^2}\,dx = \frac{1}{2}}$$

9. $\displaystyle\int_{-\infty}^{0}\frac{1}{(2x-1)^3}\,dx$

Solution

$$\int_{-\infty}^{0}\frac{1}{(2x-1)^3}\,dx \overset{\text{def}}{=} \lim_{a\to-\infty}\int_a^0 \frac{1}{(2x-1)^3}\,dx$$

$$u = 2x - 1 \Rightarrow du = 2dx$$

$$\int \frac{1}{(2x-1)^3}\,dx = \frac{1}{2}\int\frac{1}{u^3}\,du = -\frac{1}{4}\frac{1}{u^2} = -\frac{1}{4}\frac{1}{(2x-1)^2}$$

$$\int_a^0 \frac{1}{(2x-1)^3}\,dx = -\frac{1}{4}\left[1 - \frac{1}{(2a-1)^2}\right]$$

$$\lim_{a\to-\infty}\int_a^0 \frac{1}{(2x-1)^3}\,dx = -\frac{1}{4}\lim_{a\to-\infty}\left[1 - \frac{1}{(2a-1)^2}\right] = -\frac{1}{4}(1-0) = -\frac{1}{4}$$

$$\boxed{\int_{-\infty}^{0}\frac{1}{(2x-1)^3}\,dx = -\frac{1}{4}}$$

10. $\displaystyle\int_{-\infty}^{3}\frac{1}{x^2+9}\,dx$

Solution

$$\int_{-\infty}^{3}\frac{1}{x^2+9}\,dx \overset{\text{def}}{=} \lim_{a\to-\infty}\int_a^3 \frac{1}{x^2+9}\,dx$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3}$$

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$$\int_a^3 \frac{1}{x^2+9}\,dx = \frac{1}{3}\left(\tan^{-1} 1 - \tan^{-1}\frac{a}{3}\right) = \frac{\pi}{12} - \frac{1}{3}\tan^{-1}\frac{a}{3}$$

$$\lim_{a\to-\infty}\int_a^3 \frac{1}{x^2+9}\,dx = \lim_{a\to-\infty}\left(\frac{\pi}{12} - \frac{1}{3}\tan^{-1}\frac{a}{3}\right) = \frac{\pi}{12} - \frac{1}{3}\times\lim_{a\to-\infty}\left(\tan^{-1}\frac{a}{3}\right) = \frac{\pi}{12} - \frac{1}{3}\times\left(-\frac{\pi}{2}\right)$$

$$\boxed{\int_{-\infty}^3 \frac{1}{x^2+9}\,dx = \frac{\pi}{4}}$$

14. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

Solution

$$\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx \overset{\text{def}}{=} \int_{-\infty}^{0} \frac{x}{\sqrt{x^2+2}}\,dx + \int_{0}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx$$

$$\overset{\text{def}}{=} \lim_{a\to-\infty}\int_a^0 \frac{x}{\sqrt{x^2+2}}\,dx + \lim_{b\to+\infty}\int_0^b \frac{x}{\sqrt{x^2+2}}\,dx$$

$$\int \frac{x}{\sqrt{x^2+2}}\,dx = \sqrt{x^2+2}$$

$$\int_0^b \frac{x}{\sqrt{x^2+2}}\,dx = \sqrt{b^2+2} - \sqrt{2}$$

$$\lim_{b\to+\infty}\int_0^b \frac{x}{\sqrt{x^2+2}}\,dx = \lim_{b\to+\infty}\left(\sqrt{b^2+2} - \sqrt{2}\right)$$

Similarly,

$$\int_a^0 \frac{x}{\sqrt{x^2+2}}\,dx = \sqrt{2} - \sqrt{a^2+2}$$

$$\lim_{a\to-\infty}\int_a^0 \frac{x}{\sqrt{x^2+2}}\,dx = \lim_{a\to-\infty}\left(\sqrt{2} - \sqrt{a^2+2}\right)$$

Hence,

$$\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{x}{\sqrt{x^2+2}}\,dx + \lim_{b\to+\infty}\int_0^b \frac{x}{\sqrt{x^2+2}}\,dx$$

$$= \lim_{a\to-\infty}\left(\sqrt{2} - \sqrt{a^2+2}\right) + \lim_{b\to+\infty}\left(\sqrt{b^2+2} - \sqrt{2}\right)$$

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$$= \lim_{-a\to+\infty}\left(\sqrt{2}-\sqrt{a^2+2}\right)+\lim_{b\to+\infty}\left(\sqrt{b^2+2}-\sqrt{2}\right)$$

$$= \lim_{b\to+\infty}\left(\sqrt{2}-\sqrt{b^2+2}\right)+\lim_{b\to+\infty}\left(\sqrt{b^2+2}-\sqrt{2}\right)$$

$$= \lim_{b\to+\infty}\left(\sqrt{2}-\sqrt{b^2+2}+\sqrt{b^2+2}-\sqrt{2}\right)$$

$$= \lim_{b\to+\infty} 0$$

$$\boxed{\int_{-\infty}^{+\infty}\frac{x}{\sqrt{x^2+2}}\,dx = 0.}$$

15. $\displaystyle\int_{-\infty}^{+\infty}\frac{x}{(x^2+3)^2}\,dx$

Solution

$$\int_{-\infty}^{+\infty}\frac{x}{(x^2+3)^2}\,dx \overset{\text{def}}{=} \int_{-\infty}^{0}\frac{x}{(x^2+3)^2}\,dx+\int_{0}^{+\infty}\frac{x}{(x^2+3)^2}\,dx$$

$$\overset{\text{def}}{=} \lim_{a\to-\infty}\int_{a}^{0}\frac{x}{(x^2+3)^2}\,dx+\lim_{b\to+\infty}\int_{0}^{b}\frac{x}{(x^2+3)^2}\,dx$$

$$\int\frac{x}{(x^2+3)^2}\,dx = -\frac{1}{2}\frac{1}{x^2+3}$$

$$\int_{0}^{b}\frac{x}{(x^2+3)^2}\,dx = \frac{1}{6}-\frac{1}{2(b^2+3)}$$

$$\lim_{b\to+\infty}\int_{0}^{b}\frac{x}{(x^2+3)^2}\,dx = \lim_{b\to+\infty}\left(\frac{1}{6}-\frac{1}{2(b^2+3)}\right) = \frac{1}{6}-0 = \frac{1}{6}$$

Since

$$f(x) = \frac{x}{(x^2+3)^2}$$

is an odd function, we can get,

$$\int_{-\infty}^{0}\frac{x}{(x^2+3)^2}\,dx = -\frac{1}{6}$$

Hence,

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$$\boxed{\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\,dx = 0}$$

17. $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$

Solution

$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx \overset{\text{def}}{=} \lim_{a\to 0^+} \int_a^8 \frac{1}{\sqrt[3]{x}}\,dx$$

$$\int \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}x^{\frac{2}{3}}$$

$$\int_a^8 \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\left(4 - a^{\frac{2}{3}}\right)$$

$$\lim_{a\to 0^+} \int_a^8 \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\lim_{a\to 0^+}\left(4 - a^{\frac{2}{3}}\right) = \frac{3}{2}\times 4 - \frac{3}{2}\lim_{a\to 0^+} a^{\frac{2}{3}} = 6$$

$$\boxed{\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = 6}$$

26. $\displaystyle\int_{-2}^2 \frac{1}{x^2}\,dx$

Solution

$$\int_{-2}^2 \frac{1}{x^2}\,dx \overset{\text{def}}{=} \int_{-2}^0 \frac{1}{x^2}\,dx + \int_0^2 \frac{1}{x^2}\,dx \overset{\text{def}}{=} \lim_{b\to 0^-}\int_{-2}^b \frac{1}{x^2}\,dx + \lim_{a\to 0^+}\int_a^2 \frac{1}{x^2}\,dx$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$\int_a^2 \frac{1}{x^2}\,dx = \left(-\frac{1}{2}\right) - \left(-\frac{1}{a}\right) = \frac{1}{a} - \frac{1}{2}$$

$$\int_{-2}^b \frac{1}{x^2}\,dx = \left(-\frac{1}{b}\right) - \left(-\frac{1}{-2}\right) = -\frac{1}{b} - \frac{1}{2}$$

$$\lim_{a\to 0^+}\int_a^2 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left(\frac{1}{a} - \frac{1}{2}\right) = +\infty - \frac{1}{2} = +\infty$$

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Since,

$$\int_0^2 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\int_a^2 \frac{1}{x^2}\,dx = +\infty$$

diverges, hence,

$$\boxed{\int_{-2}^2 \frac{1}{x^2}\,dx \text{ diverges.}}$$

27. $\displaystyle\int_{-1}^8 \frac{1}{\sqrt[3]{x}}\,dx$

Solution

$$\int_{-1}^8 \frac{1}{\sqrt[3]{x}}\,dx \stackrel{\text{def}}{=} \int_{-1}^0 \frac{1}{\sqrt[3]{x}}\,dx + \int_0^8 \frac{1}{\sqrt[3]{x}}\,dx \stackrel{\text{def}}{=} \lim_{b\to 0^-}\int_{-1}^b \frac{1}{\sqrt[3]{x}}\,dx + \lim_{a\to 0^+}\int_a^8 \frac{1}{\sqrt[3]{x}}\,dx$$

$$\int \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}x^{\frac{2}{3}}$$

$$\int_{-1}^b \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\left(b^{\frac{2}{3}} - (-1)^{\frac{2}{3}}\right) = \frac{3}{2}\left(b^{\frac{2}{3}} - 1\right), \qquad \int_a^8 \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\left(4 - a^{\frac{2}{3}}\right)$$

$$\lim_{b\to 0^-}\int_{-1}^b \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\lim_{b\to 0^-}\left(b^{\frac{2}{3}} - 1\right) = \frac{3}{2}\lim_{b\to 0^-}\left(b^{\frac{2}{3}}\right) - \frac{3}{2} = -\frac{3}{2}$$

$$\lim_{a\to 0^+}\int_a^8 \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\lim_{a\to 0^+}\left(4 - a^{\frac{2}{3}}\right) = \frac{3}{2}\times 4 - \frac{3}{2}\lim_{a\to 0^+} a^{\frac{2}{3}} = 6$$

$$\boxed{\int_{-1}^8 \frac{1}{\sqrt[3]{x}}\,dx = \frac{9}{2}}$$

30. $\displaystyle\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

Solution

$$\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx \stackrel{\text{def}}{=} \lim_{b\to +\infty}\left[\lim_{a\to 1^+}\int_a^b \frac{1}{x\sqrt{x^2-1}}\,dx\right]$$

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$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x|$$

$$\int_a^b \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|b| - \sec^{-1}|a| = \sec^{-1} b - \sec^{-1} a$$

$$\lim_{a\to 1^+}\int_a^b \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{a\to 1^+}(\sec^{-1} b - \sec^{-1} a) = \lim_{a\to 1^+}\left(\cos^{-1}\frac{1}{b} - \cos^{-1}\frac{1}{a}\right) = \cos^{-1} b - 0$$

$$= \cos^{-1}\left(\frac{1}{b}\right)$$

$$\lim_{b\to +\infty}\left[\lim_{a\to 1^+}\int_a^b \frac{1}{x\sqrt{x^2-1}}\,dx\right] = \lim_{b\to +\infty}\left[\cos^{-1}\frac{1}{b}\right] = \lim_{\frac{1}{b}\to 0^+}\left[\cos^{-1}\frac{1}{b}\right] = \frac{\pi}{2}$$

$$\boxed{\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \frac{\pi}{2}}$$

31. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx$

Solution

$$\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx \overset{\text{def}}{=} \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(x+1)}\,dx$$

$$\int \frac{1}{\sqrt{x}(x+1)}\,dx = \int \frac{1}{u(u^2+1)}(2u\,du) = \int \frac{2}{u^2+1}\,du = 2\tan^{-1} u = 2\tan^{-1}\sqrt{x}$$

$$\int_a^1 \frac{1}{\sqrt{x}(x+1)}\,dx = 2\left(\frac{\pi}{4} - \tan^{-1}\sqrt{a}\right)$$

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(x+1)}\,dx = 2\times\lim_{a\to 0^+}\left(\frac{\pi}{4} - \tan^{-1}\sqrt{a}\right) = \frac{\pi}{2}$$

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx = \frac{\pi}{2}}$$

32. $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x}(x+1)}\,dx$

Solution

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$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x+1)}\,dx \overset{\text{def}}{=} \lim_{b\to+\infty}\left[\lim_{a\to0^+}\int_a^b \frac{1}{\sqrt{x}(x+1)}\,dx\right]$$

$$\int \frac{1}{\sqrt{x}(x+1)}\,dx = \int \frac{1}{u(u^2+1)}(2u\,du) = \int \frac{2}{u^2+1}\,du = 2\tan^{-1}u = 2\tan^{-1}\sqrt{x}$$

$$\int_a^b \frac{1}{\sqrt{x}(x+1)}\,dx = 2\left(\tan^{-1}\sqrt{b} - \tan^{-1}\sqrt{a}\right)$$

$$\lim_{a\to0^+}\int_a^b \frac{1}{\sqrt{x}(x+1)}\,dx = 2\times\lim_{a\to0^+}\left(\tan^{-1}\sqrt{b} - \tan^{-1}\sqrt{a}\right) = 2\tan^{-1}\sqrt{b}$$

$$\lim_{b\to+\infty}\left[\lim_{a\to0^+}\int_a^b \frac{1}{\sqrt{x}(x+1)}\,dx\right] = 2\times\lim_{b\to+\infty}\left[\tan^{-1}\sqrt{b}\right] = 2\times\frac{\pi}{2}$$

$$\boxed{\int_0^{+\infty} \frac{1}{\sqrt{x}(x+1)}\,dx = \pi}$$

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