12. SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The “linear” attribute means, just as it did in the first-order situation, that the unknown function and its derivatives are not multiplied together, are not raised to powers, and are not the arguments of other function. So, for example,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

is second order linear.

The “constant coefficient” attribute means that the coefficients in the equation are not functions—they are constants. Thus a second-order linear equation with constant coefficient will have the form

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)}$$

where a, b, c are constants.

For the homogeneous case, this is the situation in which $f(x) = 0$. Certainly exponentials fit this description. Thus we guess a solution of the form

$$y = e^{mx}$$

Plugging this guess we find that

$$am^2 + bm + c = 0$$

Solve:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \qquad (1)$$

Let us consider the solution is

$$y = e^{mx}$$

By plugging this into (1) we find the auxiliary equation

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

Solving the above equation, we get,

$$\therefore m = 2, 3$$

The solutions are,

$$y_1 = e^{2x} \text{ and}$$

$$y_2 = e^{3x}$$

Therefore the general solution is

$$\boxed{y = C_1 e^{2x} + C_2 e^{3x}}$$

Solve:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \tag{1}$$

Let us consider the solution is

$$y = e^{mx}$$

By plugging this into (1) we find the auxiliary equation

$$m^2 - 6m + 9 = 0$$

Solving the above equation, we get,

$$\therefore m = 3, 3$$

Since the auxiliary equation has repeated roots, therefore, the general solution is

$$\boxed{y = C_1 e^{3x} + C_2 x e^{3x}}$$

Solve:

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Solution: Given differential equation is,

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \tag{1}$$

Let us consider the solution is

$$y = e^{mx}$$

By plugging this into (1) we find the auxiliary equation

$$2m^2 - 5m + 6 = 0$$

Solving the above equation, we get,

$$m = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times 6}}{2 \times 2}$$

$$= \frac{5 \pm \sqrt{25 - 48}}{4} = \frac{5 \pm \sqrt{-23}}{4} = \frac{5 \pm i\sqrt{23}}{4} = \frac{5}{4} \pm i\frac{\sqrt{23}}{4}$$

$$\therefore\ m_1 = \frac{5}{4} + i\frac{\sqrt{23}}{4}, \qquad m_2 = \frac{5}{4} - i\frac{\sqrt{23}}{4}$$

Therefore the general solution is

$$y = C_1 e^{\left(\frac{5}{4}+i\frac{\sqrt{23}}{4}\right)x} + C_2 e^{\left(\frac{5}{4}-i\frac{\sqrt{23}}{4}\right)x} = C_1 e^{\frac{5}{4}x} e^{i\frac{\sqrt{23}}{4}x} + C_2 e^{\frac{5}{4}x} e^{-i\frac{\sqrt{23}}{4}x} = e^{\frac{5}{4}x}\left(C_1 e^{i\frac{\sqrt{23}}{4}x} + C_2 e^{-i\frac{\sqrt{23}}{4}x}\right)$$

$$= e^{\frac{5}{4}x}\left(C_1 e^{i\frac{\sqrt{23}}{4}x} + C_2 e^{-i\frac{\sqrt{23}}{4}x}\right) \qquad e^{i\theta} = \cos\theta + i\sin\theta\,;\ e^{-i\theta} = \cos\theta - i\sin\theta$$

$$= e^{\frac{5}{4}x}\left[C_1\left(\cos\frac{\sqrt{23}}{4}x + i\sin\frac{\sqrt{23}}{4}x\right) + C_2\left(\cos\frac{\sqrt{23}}{4}x - i\sin\frac{\sqrt{23}}{4}x\right)\right]$$

$$= e^{\frac{5}{4}x}\left[(C_1 + C_2)\cos\frac{\sqrt{23}}{4}x + i(C_1 - C_2)\sin\frac{\sqrt{23}}{4}x\right]$$

$$= e^{\frac{5}{4}x}\left[A_1\cos\frac{\sqrt{23}}{4}x + A_2\sin\frac{\sqrt{23}}{4}x\right] \qquad A_1 = (C_1 + C_2), \quad A_2 = i(C_1 - C_2)$$

$$\boxed{y = e^{\frac{5}{4}x}\left[A_1\cos\left(\frac{\sqrt{23}}{4}x\right) + A_2\sin\left(\frac{\sqrt{23}}{4}x\right)\right]}$$

Problems

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0, \quad \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0, \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0, \quad 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 6y = 0$$

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0, \quad 3\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0, \quad 4\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0, \quad 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$