7.4 Trigonometric Substitutions
Solutions to the Selected Problems

1–26. Evaluate the integral.

1. $\int \sqrt{4-x^2}\,dx$

Solution

Let

$$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta$$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,(2\cos\theta\,d\theta) = 4\int \cos^2\theta\,d\theta = 2\int (1+\cos 2\theta)\,d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) = 2\theta + 2\sin\theta\cos\theta = 2\sin^{-1}\left(\frac{x}{2}\right) + 2\frac{x}{2}\,\frac{\sqrt{4-x^2}}{2}$$

$$\boxed{\int \sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C}$$

2. $\int \sqrt{1-4x^2}\,dx$

Solution

Let

$$x = \frac{1}{2}\sin\theta \Rightarrow dx = \frac{1}{2}\cos\theta\,d\theta$$

$$\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-\sin^2\theta}\left(\frac{1}{2}\cos\theta\,d\theta\right) = \frac{1}{2}\int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta = \frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\times 2x \times \frac{\sqrt{1-4x^2}}{2}$$

$$\boxed{\int \sqrt{1-4x^2}\,dx = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$$

3. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Solution

Let

$$x = 4\sin\theta \Rightarrow dx = 4\cos\theta\,d\theta$$

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$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}}(4\cos\theta\,d\theta) = 16\int \sin^2\theta\,d\theta = 8\int(1-\cos 2\theta)\,d\theta$$

$$= 8\left(\theta - \frac{1}{2}\sin 2\theta\right) = 8\theta - 8\sin\theta\cos\theta = 8\sin^{-1}\left(\frac{x}{4}\right) - 8\times\frac{x}{4}\times\frac{\sqrt{16-x^2}}{4}$$

$$\boxed{\int \frac{x^2}{\sqrt{16-x^2}}\,dx = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C}$$

5. $\int \frac{1}{(4+x^2)^2}\,dx$

Solution

Let

$$x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{1}{(4+x^2)^2}\,dx = \int \frac{1}{(4+4\tan^2\theta)^2}(2\sec^2\theta\,d\theta) = \frac{1}{8}\int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta$$

$$= \frac{1}{8}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta$$

$$= \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{16}\frac{x}{\sqrt{4+x^2}}\frac{2}{\sqrt{4+x^2}}$$

$$\boxed{\int \frac{1}{(4+x^2)^2}\,dx = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\,\frac{x}{4+x^2} + C}$$

7. $\int \frac{\sqrt{x^2-9}}{x}\,dx;\, x \geq 3$

Solution

Let

$$x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\,d\theta$$

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta) = \int 3\sqrt{\tan^2\theta}\tan\theta\,d\theta$$

Since, $x \geq 3$ therefore, $\theta \in [0, \pi/2)$ and $\tan\theta \geq 0$, hence,

$$\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$$

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = 3\int \tan^2\theta\, d\theta = 3\int (\sec^2\theta - 1)\, d\theta = 3(\tan\theta - \theta)$$

$$= 3\,\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right)$$

$$\boxed{\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C, \qquad x \geq 3}$$

Or,

$$\boxed{\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3\tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right) + C, \quad x \geq 3}$$

9. $\displaystyle\int \frac{3x^3}{\sqrt{1-x^2}}\,dx$

Solution

Method 1 (trigonometric substitution)

Let

$$x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$$

$$\int \frac{3x^3}{\sqrt{1-x^2}}\,dx = \int \frac{3\sin^3\theta}{\sqrt{1-\sin^2\theta}}(\cos\theta\, d\theta) = 3\int \sin^3\theta\, d\theta = 3\int \sin^2\theta \sin\theta\, d\theta$$

$$= 3\int (1-\cos^2\theta)\sin\theta\, d\theta = 3\int (1-u^2)\,(-du) = u^3 - 3u = \cos^3\theta - 3\cos\theta$$

$$= \left(\sqrt{1-x^2}\right)^3 - 3\sqrt{1-x^2}$$

$$\boxed{\int \frac{3x^3}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}(2+x^2) + C}$$

Method 2 (u-substitution)

Let

$$u = x^2 \Rightarrow du = 2x\, dx$$

$$\int \frac{3x^3}{\sqrt{1-x^2}}\,dx = \frac{3}{2}\int \frac{x^2}{\sqrt{1-x^2}}2xdx = \frac{3}{2}\int \frac{u}{\sqrt{1-u}}du = -\frac{3}{2}\int \frac{1-u-1}{\sqrt{1-u}}du$$

$$= \frac{3}{2}\int \frac{1}{\sqrt{1-u}}\,du - \frac{3}{2}\int \frac{1-u}{\sqrt{1-u}}\,du = -3\sqrt{1-u} + \frac{3}{2}\times\frac{2}{3}(1-u)^{\frac{3}{2}} = -3\sqrt{1-x^2} + (1-x^2)^{\frac{3}{2}}$$

$$\boxed{\int \frac{3x^3}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}(2+x^2) + C}$$

19. $\int e^x\sqrt{1-e^{2x}}\,dx$

Solution

$$u = e^x \Rightarrow du = e^x dx$$

$$\int e^x\sqrt{1-e^{2x}}\,dx = \int \sqrt{1-u^2}\,du$$

Let

$$u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$$

$$\int \sqrt{1-u^2}\,du = \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\sin^{-1}u + \frac{1}{2}u\sqrt{1-u^2}$$

$$\boxed{\int e^x\sqrt{1-e^{2x}}\,dx = \frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C}$$

37–48. Evaluate the integral.

37. $\int \frac{1}{x^2-4x+5}\,dx$

Solution

$x^2 - 4x + 5 \equiv (x-2)^2 + 1$

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x-2)^2+1}\,dx$$

Let

$$x = 2 + \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$$

$$\int \frac{1}{(x-2)^2+1}\,dx = \int \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta = \int \quad d\theta = \theta = \tan^{-1}(x-2)$$

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$$\boxed{\int \frac{1}{x^2-4x+5}\,dx = \tan^{-1}(x-2)}$$

38. $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$

Solution

$2x - x^2 \equiv -(x^2-2x+1)+1 \equiv 1-(x-1)^2$

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

Let

$$x = 1+\sin\theta \Rightarrow dx = \cos\theta\,d\theta$$

$$\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int d\theta = \theta = \sin^{-1}(x-1)$$

$$\boxed{\int \frac{1}{\sqrt{2x-x^2}}\,dx = \sin^{-1}(x-1)+C}$$

40. $\displaystyle\int \frac{1}{16x^2+16x+5}\,dx$

Solution

$16x^2+16x+5 \equiv 16(x^2+x)+5 \equiv 16\left(x^2+2x\frac{1}{2}+\frac{1}{4}-\frac{1}{4}\right)+5$

$\equiv 16\left(x^2+2x\frac{1}{2}+\frac{1}{4}\right)+1 \equiv 16\left[\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^2\right]$

$$\int \frac{1}{16x^2+16x+5}\,dx = \frac{1}{16}\int \frac{1}{\left(\frac{1}{4}\right)^2+\left(x+\frac{1}{2}\right)^2}\,dx$$

Let

$$x = \frac{1}{2}+u \Rightarrow dx = du$$

$\displaystyle\int \frac{1}{\left(\frac{1}{4}\right)^2+\left(x+\frac{1}{2}\right)^2}\,dx = \frac{1}{1/4}\tan^{-1}\left(\frac{u}{1/4}\right) = 4\tan^{-1}(4x+2)$

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$$\boxed{\int \frac{1}{16x^2+16x+5}\,dx = \frac{1}{4}\tan^{-1}(4x+2)+C}$$

41. $\displaystyle\int \frac{1}{\sqrt{x^2-6x+10}}\,dx$

Solution

$x^2-6x+10 \equiv (x-3)^2+1$

$$\int \frac{1}{\sqrt{x^2-6x+10}}\,dx = \int \frac{1}{\sqrt{1+(x-3)^2}}\,dx$$

Let

$$x = 3+u \Rightarrow dx = du$$

$$\int \frac{1}{\sqrt{1+(x-3)^2}}\,dx = \int \frac{1}{\sqrt{1+u^2}}\,du$$

Let

$$u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$$

$$\int \frac{1}{\sqrt{1+u^2}}\,du = \int \frac{1}{\sec\theta}\sec^2\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| = \ln\left|u+\sqrt{1+u^2}\right|$$

$$\int \frac{1}{\sqrt{x^2-6x+10}}\,dx = \ln\left|(x-3)+\sqrt{1+(x-3)^2}\right| = \ln\left|x-3+\sqrt{x^2-6x+10}\right|$$

$$\boxed{\int \frac{1}{\sqrt{x^2-6x+10}}\,dx = \ln\left|x-3+\sqrt{x^2-6x+10}\right|+C}$$

47. $\displaystyle\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx$

Solution

$4x-x^2 \equiv -(x-2)^2+4$

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{2^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{2}\right) = \sin^{-1}\left(\frac{x-2}{2}\right)$$

$$\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx = \sin^{-1}0-\sin^{-1}\left(-\frac{1}{2}\right) = 0-\left(-\frac{\pi}{6}\right)$$

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$$\boxed{\int_1^2 \frac{1}{\sqrt{4x - x^2}}\, dx = \frac{\pi}{6}}$$

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