we return to the topic of differential equations and present additional examples. However, a more thorough examination of this vast field will need to wait for a course focused on this topic.

8.1 MODELING WITH DIFFERENTIAL EQUATIONS

Growth and Decay Problems

Time (hours)	*Number of Bacteria (millions per ml)*
0	1.2
0.5	2.5
1	5.1
1.5	11.0
2	23.0
2.5	45.0
3	91.0
3.5	180.0
4	350.0

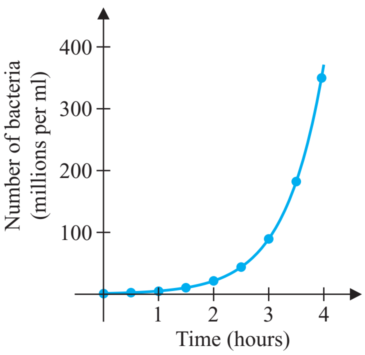

FIGURE 8.1
Growth of bacteria

In this age, we are all keenly aware of how infection by microorganisms such as *Escherichia coli* (*E. coli*) causes disease. Many organisms (such as *E. coli*) produce a toxin that can cause sickness or even death. Some bacteria can reproduce in our bodies at a surprisingly fast rate, overwhelming our bodies' natural defenses with the sheer volume of toxin they are producing. The table shown in the margin indicates the number of *E. coli* bacteria (in millions of bacteria per ml) in a laboratory culture measured at half-hour intervals during the course of an experiment. We have plotted the number of bacteria per milliliter versus time in Figure 8.1. What would you say the graph most resembles? If you said, "an exponential," you guessed right. Careful analysis of experimental data has shown that many populations grow at a rate proportional to their current level. This is quite easily observed in bacterial cultures, where the bacteria reproduce by binary fission (i.e., each cell reproduces by dividing into two cells). In this case, the rate at which the bacterial culture grows is directly proportional to the current population (until such time as resources become scarce or overcrowding becomes a limiting factor). If we let $y(t)$ represent the number of bacteria in a culture at time t, then the rate of change of the population with respect to time is $y'(t)$. Thus, since $y'(t)$ is proportional to $y(t)$, we have

$$y'(t) = ky(t), \tag{1.1}$$

for some constant of proportionality k (the **growth constant**). Since equation (1.1) involves the derivative of an unknown function, we call it a **differential equation.** Our aim is to *solve* the differential equation, that is, find the *function* $y(t)$. Assuming that $y(t) > 0$ (this is a reasonable assumption, since $y(t)$ represents a population), we have

$$\frac{y'(t)}{y(t)} = k. \tag{1.2}$$

Integrating both sides of equation (1.2) with respect to t, we obtain

$$\int \frac{y'(t)}{y(t)}\,dt = \int k\,dt. \tag{1.3}$$

Substituting $y = y(t)$ in the integral on the left-hand side, we have $dy = y'(t)\,dt$ and so, (1.3) becomes

$$\int \frac{1}{y}\,dy = \int k\,dt.$$

Evaluating these integrals, we obtain

$$\ln|y| + c_1 = kt + c_2,$$

where c_1 and c_2 are constants of integration. Subtracting c_1 from both sides yields

$$\ln|y| = kt + (c_2 - c_1) = kt + c,$$

for some constant c. Since $y(t) > 0$, we have

$$\ln y(t) = kt + c$$

and taking exponentials of both sides, we get

$$y(t) = e^{\ln y(t)} = e^{kt+c} = e^{kt}e^{c}.$$

Since c is an arbitrary constant, we write $A = e^c$ and get

$$y(t) = Ae^{kt}. \tag{1.4}$$

We refer to (1.4) as the **general solution** of the differential equation (1.1). For $k > 0$, equation (1.4) is called an **exponential growth law** and for $k < 0$, it is an **exponential decay law.** (Think about the distinction.)

In example 1.1, we examine how an exponential growth law predicts the number of cells in a bacterial culture.

EXAMPLE 1.1 Exponential Growth of a Bacterial Colony

A freshly inoculated bacterial culture of *Streptococcus A* (a common group of microorganisms that cause strep throat) contains 100 cells. When the culture is checked 60 minutes later, it is determined that there are 450 cells present. Assuming exponential growth, determine the number of cells present at any time t (measured in minutes) and find the doubling time.

Solution Exponential growth means that

$$y'(t) = ky(t)$$

and hence, from (1.4),

$$y(t) = Ae^{kt}, \tag{1.5}$$

where A and k are constants to be determined. If we set the starting time as $t = 0$, we have

$$y(0) = 100. \tag{1.6}$$

Equation (1.6) is called an **initial condition.** Setting $t = 0$ in (1.5), we now have

$$100 = y(0) = Ae^0 = A$$

and hence,

$$y(t) = 100\,e^{kt}.$$

We can use the second observation to determine the value of the growth constant k. We have

$$450 = y(60) = 100\,e^{60k}.$$

Dividing both sides by 100 and taking the natural logarithm of both sides, we have

$$\ln 4.5 = \ln e^{60k} = 60k,$$

so that

$$k = \frac{\ln 4.5}{60} \approx 0.02507.$$

We now have a formula representing the number of cells present at any time t:

$$y(t) = 100\,e^{kt} = 100\exp\left(\frac{\ln 4.5}{60}t\right).$$

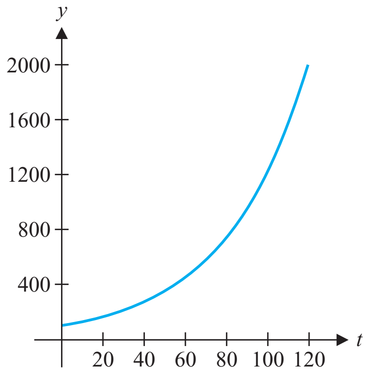

FIGURE 8.2
$y = 100\,e^{\left(\frac{\ln 4.5}{60}t\right)}$

See Figure 8.2 for a graph of the projected bacterial growth over the first 120 minutes. One further question of interest to microbiologists is the **doubling time,** that is, the time it takes for the number of cells to double. We can find this by solving for the time t for which $y(t) = 2y(0) = 200$. We have

$$200 = y(t) = 100 \exp\left(\frac{\ln 4.5}{60}t\right).$$

Dividing both sides by 100 and taking logarithms, we obtain

$$\ln 2 = \frac{\ln 4.5}{60}t,$$

so that
$$t = \frac{60 \ln 2}{\ln 4.5} \approx 27.65.$$

So, the doubling time for this culture of *Streptococcus A* is about 28 minutes. The doubling time for a bacterium depends on the specific strain of bacteria, as well as the quality and quantity of the food supply, the temperature and other environmental factors. However, it is not dependent on the initial population. Here, you can easily check that the population reaches 400 at time

$$t = \frac{120 \ln 2}{\ln 4.5} \approx 55.3$$

(exactly double the time it took to reach 200).

That is, the initial population of 100 doubles to 200 in approximately 28 minutes and it doubles again (to 400) in another 28 minutes and so on. ■

Numerous physical phenomena satisfy exponential growth or decay laws. For instance, experiments have shown that the rate at which a radioactive element decays is directly proportional to the amount present. (Recall that radioactive elements are chemically unstable elements that gradually decay into other, more stable elements.) Let $y(t)$ be the amount (mass) of a radioactive element present at time t. Then, we have that the rate of change (rate of decay) of $y(t)$ satisfies

$$y'(t) = ky(t). \tag{1.7}$$

Note that (1.7) is precisely the same differential equation as (1.1), encountered in example 1.1 for the growth of bacteria and hence, from (1.4), we have that

$$y(t) = Ae^{kt},$$

for some constants A and k (here, the **decay constant**) to be determined.

It is common to discuss the decay rate of a radioactive element in terms of its **half-life,** the time required for half of the initial quantity to decay into other elements. For instance, scientists have calculated that the half-life of carbon-14 (^{14}C) is approximately 5730 years. That is, if you have 2 grams of ^{14}C today and you come back in 5730 years, you will have approximately 1 gram of ^{14}C remaining. It is this long half-life and the fact that living creatures continually take in ^{14}C that make ^{14}C measurements useful for radiocarbon dating. (See the exercise set for more on this important application.)

EXAMPLE 1.2 Radioactive Decay

If you have 50 grams of ^{14}C today, how much will be left in 100 years?

Solution Let $y(t)$ be the mass (in grams) of ^{14}C present at time t. Then, we have

$$y'(t) = ky(t)$$

and as we have already seen, $y(t) = Ae^{kt}$.

The initial condition is $y(0) = 50$, so that

$$50 = y(0) = Ae^0 = A$$

and $y(t) = 50\,e^{kt}$.

To find the decay constant k, we use the half-life:

$$25 = y(5730) = 50\,e^{5730k}.$$

Dividing both sides by 50 and taking logarithms gives us

$$\ln \frac{1}{2} = \ln\ e^{5730k} = 5730k,$$

so that
$$k = \frac{\ln \frac{1}{2}}{5730} \approx -1.20968 \times 10^{-4}.$$

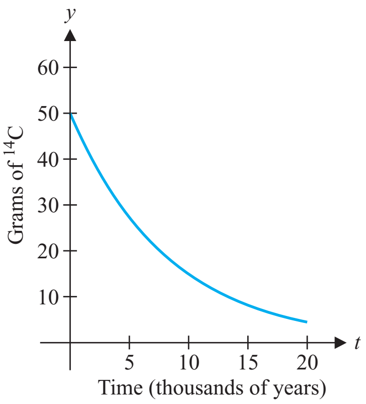

FIGURE 8.3
Decay of ^{14}C

A graph of the mass of ^{14}C as a function of time is seen in Figure 8.3. Notice the extremely large time scale shown. This should give you an idea of the incredibly slow rate of decay of ^{14}C. Finally, notice that if we start with 50 grams, then the amount left after 100 years is

$$y(100) = 50e^{100k} \approx 49.3988 \text{ grams.} \blacksquare$$

A mathematically similar physical principle is **Newton's Law of Cooling.** If you introduce a hot object into cool surroundings (or equivalently, a cold object into warm surroundings), the rate at which the object cools (or warms) is not proportional to its temperature, but rather, to the difference in temperature between the object and its surroundings. Symbolically, if we let $y(t)$ be the temperature of the object at time t and let T_a be the temperature of the surroundings (the **ambient temperature,** which we assume to be constant), we have the differential equation

$$y'(t) = k[y(t) - T_a]. \tag{1.8}$$

Notice that (1.8) is not the same as the differential equation describing exponential growth or decay. (Compare these; what's the difference?) Even so, we can approach finding a solution in the same way. In the case of cooling, we assume that

$$T_a < y(t).$$

(Why is it fair to assume this?) If we divide both sides of equation (1.8) by $y(t) - T_a$ and then integrate both sides, we obtain

$$\int \frac{y'(t)}{y(t) - T_a}\,dt = \int k\,dt = kt + c_1. \tag{1.9}$$

Notice that we can evaluate the integral on the left-hand side by making the substitution $u = y(t) - T_a$, so that $du = y'(t)\,dt$. Thus, we have

$$\begin{aligned}\int \frac{y'(t)}{y(t) - T_a}\,dt = \int \frac{1}{u}\,du &= \ln|u| + c_2 = \ln|y(t) - T_a| + c_2\\ &= \ln[y(t) - T_a] + c_2,\end{aligned}$$

since $y(t) - T_a > 0$. From (1.9), we now have

$$\ln[y(t) - T_a] + c_2 = kt + c_1 \quad \text{or} \quad \ln[y(t) - T_a] = kt + c,$$

where we have combined the two constants of integration. Taking exponentials of both sides, we obtain

$$y(t) - T_a = e^{kt+c} = e^{kt}e^c.$$

Finally, for convenience, we write $A = e^c$, to obtain

$$y(t) = Ae^{kt} + T_a,$$

where A and k are constants to be determined.

We illustrate Newton's Law of Cooling in example 1.3.

EXAMPLE 1.3 Newton's Law of Cooling for a Cup of Coffee

A cup of fast-food coffee is 180°F when freshly poured. After 2 minutes in a room at 70°F, the coffee has cooled to 165°F. Find the temperature at any time t and find the time at which the coffee has cooled to 120°F.

Solution Letting $y(t)$ be the temperature of the coffee at time t, we have

$$y'(t) = k[y(t) - 70].$$

Proceeding as above, we obtain

$$y(t) = Ae^{kt} + 70.$$

Observe that the initial condition here is the initial temperature, $y(0) = 180$. This gives us

$$180 = y(0) = Ae^0 + 70 = A + 70,$$

so that $A = 110$ and

$$y(t) = 110\,e^{kt} + 70.$$

We can now use the second measured temperature to solve for the constant k. We have

$$165 = y(2) = 110\,e^{2k} + 70.$$

Subtracting 70 from both sides and dividing by 110, we have

$$e^{2k} = \frac{165 - 70}{110} = \frac{95}{110}.$$

Taking logarithms of both sides yields

$$2k = \ln\left(\frac{95}{110}\right)$$

and hence,

$$k = \frac{1}{2}\ln\left(\frac{95}{110}\right) \approx -0.0733017.$$

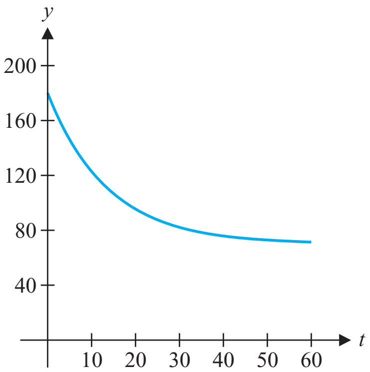

FIGURE 8.4
Temperature of coffee

A graph of the projected temperature against time is shown in Figure 8.4. From Figure 8.4, you might observe that the temperature appears to have fallen to 120°F after about 10 minutes. We can solve this symbolically by finding the time t for which

$$120 = y(t) = 110\,e^{kt} + 70.$$

It is not hard to solve this to obtain

$$t = \frac{1}{k}\ln\frac{5}{11} \approx 10.76 \text{ minutes.}$$

The details are left as an exercise. ■