7.5 Integrating Rational Functions by Partial Fractions
Solutions to the Selected Problems

Formula

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C; \qquad a > 0$$

Problem Prove that,

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C; \;\; a > 0$$

Proof

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\left(\int \frac{1}{x-a}\,dx - \int \frac{1}{x+a}\,dx\right)$$

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|)$$

$$= \frac{1}{2a}\ln\frac{|x-a|}{|x+a|}$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

9–34. Evaluate the integrals.

9. $\int \frac{1}{x^2 - 3x - 4}\,dx$

Solution

$x^2 - 3x - 4 \equiv (x+1)(x-4)$

$\frac{1}{(x+1)(x-4)} \equiv \frac{A}{x+1} + \frac{B}{x-4}$

$1 \equiv A(x-4) + B(x+1) = (A+B)x + (B-4A)$

$$A + B = 0$$

$$B - 4A = 1$$

Solving these two equations simultaneously, we get,

$$A = -\frac{1}{5}, \qquad B = \frac{1}{5}$$

$$\int \frac{1}{x^2-3x-4}\,dx = -\frac{1}{5}\int \frac{1}{x+1}\,dx + \frac{1}{5}\int \frac{1}{x-4}\,dx$$

$$\boxed{\int \frac{1}{x^2-3x-4}\,dx = -\frac{1}{5}\ln|x+1| + \frac{1}{5}\ln|x-4| + C}$$

10. $\displaystyle\int \frac{1}{x^2-6x-7}\,dx$

Solution

$x^2-6x-7 \equiv (x+1)(x-7)$

$\dfrac{1}{(x+1)(x-7)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x-7}$

$1 \equiv A(x-7) + B(x+1) = (A+B)x + (B-7A)$

$$A+B=0$$

$$B-7A=1$$

Solving these two equations simultaneously, we get,

$$A=-\frac{1}{8}, \qquad B=\frac{1}{8}$$

$$\int \frac{1}{x^2-6x-7}\,dx = -\frac{1}{8}\int \frac{1}{x+1}\,dx + \frac{1}{8}\int \frac{1}{x-7}\,dx$$

$$\boxed{\int \frac{1}{x^2-6x-7}\,dx = -\frac{1}{8}\ln|x+1| + \frac{1}{8}\ln|x-7| + C}$$

11. $\displaystyle\int \frac{11x+17}{2x^2+7x-4}\,dx$

Solution

$2x^2+7x-4 \equiv (x+4)(2x-1)$

$\dfrac{11x+17}{(x+4)(2x-1)} \equiv \dfrac{A}{x+4} + \dfrac{B}{2x-1}$

$11x+17 \equiv A(2x-1) + B(x+4) = (2A+B)x + (4B-2A)$

$$2A+B=11$$

$$4B-A=17$$

Solving these two equations simultaneously, we get,

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$$A = 3, \qquad B = 5$$

$$\frac{11x+17}{(x+4)(2x-1)} \equiv \frac{3}{x+4} + \frac{5}{2x-1}$$

$$\int \frac{11x+17}{(x+4)(2x-1)}\,dx = 3\int \frac{1}{x+4}\,dx + 5\int \frac{1}{2x-1}\,dx$$

$$\boxed{\int \frac{11x+17}{(x+4)(2x-1)}\,dx = 3\ln|x+4| + \frac{5}{2}\ln|2x-1| + C}$$

21. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$

Solution

$$x^3 - x = x(x-1)(x+1)$$

By long division or other way,

$$\frac{x^5+x^2+2}{x^3-x} \equiv (x^2+1) + \frac{x^2+x+2}{x^3-x}$$

Again,

$$\frac{x^2+x+2}{x^3-x} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Using cover-up method,

$$A = -2, \qquad B = 2, \qquad C = 1$$

$$\frac{x^5+x^2+2}{x^3-x} \equiv (x^2+1) - \frac{2}{x} + \frac{2}{x-1} + \frac{1}{x+1}$$

$$\boxed{\int \frac{x^5+x^2+2}{x^3-x}\,dx = \frac{x^3}{3} + x - 2\ln|x| + 2\ln|x-1| + \ln|x+1| + C}$$

23. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$

Solution

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Solutions to the Selected Problems

$$\frac{2x^2+3}{x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$2x^2+3 \equiv A(x-1)^2 + Bx(x-1) + Cx$$

$$2x^2+3 \equiv A(x^2-2x+1) + B(x^2-x) + Cx$$

$$2x^2+3 \equiv (A+B)x^2 + (-2A-B+C)x + A$$

Collecting like powers in x, we get,

$$A+B=2$$

$$-2A-B+C=0$$

$$A=3$$

Solving these equations simultaneously, we obtain,

$$A=3, \qquad B=-1, \qquad C=5$$

[Or using cover-up method we could determine A, C]

$$\frac{2x^2+3}{x(x-1)^2} \equiv \frac{3}{x} - \frac{1}{(x-1)} + \frac{5}{(x-1)^2}$$

$$\boxed{\int \frac{2x^2+3}{x(x-1)^2}dx = 3\ln|x| - 2\ln|x-1| - \frac{5}{x-1} + C}$$

29. $\displaystyle\int \frac{2x^2-1}{(4x-1)(x^2+1)}dx$

Solution

Decomposing the integrand into partial fractions, we get,

$$\frac{2x^2-1}{(4x-1)(x^2+1)} \equiv \frac{A}{4x-1} + \frac{Bx+C}{x^2+1}$$

$$2x^2-1 \equiv A(x^2+1) + (Bx+C)(4x-1)$$

$$2x^2-1 \equiv (A+4B)x^2 + (-B+4C)x + (A-C)$$

Collecting like powers in x, we get,

$$A+4B=2$$

$$-B+4C=0$$

$$A-C=-1$$

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Solving these equations simultaneously, we obtain,

$$A = -\frac{14}{17}, \qquad B = \frac{12}{17}, \qquad C = \frac{3}{17}$$

$$\frac{2x^2 - 1}{(4x-1)(x^2+1)} \equiv -\frac{14}{17}\frac{1}{4x-1} + \frac{12}{17}\frac{x}{x^2+1} + \frac{3}{17}\frac{1}{x^2+1}$$

$$\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}dx \equiv -\frac{14}{17}\int \frac{1}{4x-1}dx + \frac{12}{17}\int \frac{x}{x^2+1}dx + \frac{3}{17}\int \frac{1}{x^2+1}dx$$

$$\boxed{\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}dx \equiv -\frac{7}{34}\ln|4x-1| + \frac{6}{17}\ln(x^2+1) + \frac{3}{17}\tan^{-1}x + C}$$

33. $\int \frac{x^3 - 2x^2 + 2x - 2}{x^2+1}dx$

Solution

By polynomial division,

$$\frac{x^3 - 2x^2 + 2x - 2}{x^2+1} = x - 2 + \frac{x}{x^2+1}$$

Now integrating,

$$\int \frac{x^3 - 2x^2 + 2x - 2}{x^2+1}dx = \frac{x^2}{2} - 2x + \frac{1}{2}\ln|x^2+1|$$

$$\boxed{\int \frac{x^3 - 2x^2 + 2x - 2}{x^2+1}dx = \frac{x^2}{2} - 2x + \frac{1}{2}\ln|x^2+1| + C}$$

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