BRAC UNIVERSITY

MAT215

MATHEMATICS III: COMPLEX VARIABLES & LAPLACE TRANSFORMATIONS

Assignment 03

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SECTION: 09

ASSIGNMENT SET: F

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Ans To The Question No. (1)

Given,

$$f(z) = y - 2xy + i(-x + x^2 - y^2) + z^2$$
$$= y - 2xy - ix + ix^2 - iy^2 + x^2 + 2ixy - y^2$$
$$= (y - 2xy + x^2 - y^2) + i(x^2 - x + 2xy - y^2)$$

Therefore,

$$u(x, y) = x^2 - 2xy + y - y^2$$
$$v(x, y) = x^2 + 2xy - x - y^2$$

Now,

$$u_x = 2(x - y)$$
$$u_y = -2y - 2x + 1$$
$$v_x = 2x + 2y - 1$$
$$v_y = 2(x - y)$$

So, it can be said that all the Cauchy-Riemann equations are satisfied here which means that f'(z) exists for ALL VALUES of z, i.e., the function f is an entire function.

Answer: All values of z

Ans To The Question No. (2)

Given,

$$u(x, y) = 2 + 3x - y + x^2 - y^2 - 4xy$$

The function u is called harmonic if it satisfies the Laplace's differential equation in two-dimension, $u_{xx} + u_{yy} = 0$ ...(1)
Differentiating u partially with respect to x and y we obtain,

$$u_x = 2x - 4y + 3 \quad ....(2a)$$

$$u_y = -2y - 4x - 1 \quad ....(2b)$$

Similarly,

$$u_{xx} = 2$$

$$u_{yy} = -2$$

Hence, eq. (1) is satisfied. To obtain the conjugate harmonic function v we use the Cauchy–Riemann equations,

$$u_x = v_y \quad ....(3a)$$

$$u_y = -v_x \quad ....(3b)$$

From the eqs. (2a) and (3a) we get,

$$v_y = 2x - 4y + 3 \quad ....(4a)$$

and from the eqs. (2b) and (3b) we get,

$$v_x = 2y + 4x + 1 \quad ....(4b)$$

Now integrating eq. (4a) partially with respect to y we get,

$$v(x, y) = -2y^2 + 2xy + 3y + g(x) \quad ....(5)$$

where, g is the constant function (but may depend on x) of integration.
Again, differentiating eq. (5) with respect to x we get,

$$v_x = 2y + g'(x)) \quad ....(6)$$

From eq. (4b) and (6) we get,

$$g'(x) = 4x + 1$$

Integrating we get,

$$g(x) = 2x^2 + x + C$$

Feeding this into the eq. (5) we get,

$$v(x, y) = -2y^2 + 2xy + 3y + 2x^2 + x + C$$

Therefore, the harmonic conjugate of u(x,y) is
$v(x, y) = -2y^2 + 2xy + 3y + 2x^2 + x + C$ (Ans)

Ans To The Question No. (3)

Given,

$$f(z) = (\overline{z} + 1)^3 - 3\overline{z}$$
$$= (x + 1 - iy)^3 - 3(x - iy)$$
$$= (x + 1)^3 - 3(x + 1)y^2 - 3(x + 1)^2 iy + iy^3 - 3x + 3iy$$
$$= (x + 1)^3 - 3(x + 1)y^2 - 3x + i(y^3 + 3y - 3y(x + 1)^2)$$

Here,

$$u_x = 3(x + 1)^2 - 3y^2 - 3$$
$$u_y = -6y(x + 1)$$
$$v_x = -6y(x + 1)$$
$$v_y = -3(x + 1)^2 + 3y^2 + 3$$

Here, $u_x = -v_y$ and $u_y = v_x$
But, $u_x = -v_y$ holds true only when $u_x = v_y = 0$
So it means that $6y(x + 1) = 0$
i.e $x = -1$ or $y = 0$
So, Cauchy–Riemann equations are satisfied only at (x,y)=(-1,0)
And also, the partial derivatives are continuous at (x,y)=(-1,0), so f is differentiable only at the lines x = -1 and y = 0
As, f is not differentiable at any points in the neighborhood of each point on those lines, so f is not analytic at any point.

Ans To The Question No. (4)

Given, $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$

Here,

$$u_x = 3x^2 + 3y^2$$

$$u_y = 6xy$$

$$v_x = 6xy$$

$$v_y = 3x^2 + 3y^2$$

Now, $u_x = v_y$ is satisfied, but to satisfy $u_y = -v_x$, we must have $12xy = 0$ i.e $x = 0$ or $y = 0$

Thus the points where f could be differentiable are the points on the lines or coordinate axes (i.e $x = 0$ or $y = 0$)

As $u_x = v_y$ is satisfied everywhere and the partial derivatives of u and v are continuous at every point, therefore it can be said that f is differentiable at each point of the coordinate axes.

Ans To The Question No. (5)

Given, $v(x, y) = tan^{-1}\left(\frac{y}{x}\right)$
Now,

$$v_x = -\frac{y}{x^2 + y^2}$$

$$v_y = \frac{x}{y^2 + x^2}$$

Using the Cauchy-Riemann equations, we get,

$$u_x = \frac{x}{y^2 + x^2}$$

$$u_y = \frac{y}{x^2 + y^2}$$

These are the partial derivatives of u(x, y).

Now, to verify that $v(x, y)$ satisfies the Laplace's equation, we need to check if

$$v_{xx} + v_{yy} = 0$$

So,

$$v_{xx} = \frac{2yx}{(x^2 + y^2)^2}$$

$$v_{yy} = -\frac{2xy}{(y^2 + x^2)^2}$$

From the above derivatives, it can be discerned that $v_{xx} + v_{yy} = 0$.

Therefore, we can conclude stating that $v(x, y)$ satisfies the Laplace's equation.