Mathematical Modelling

APM348 Slides

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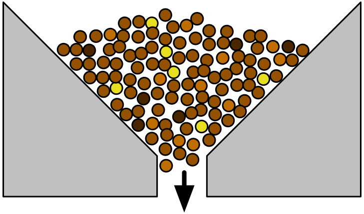
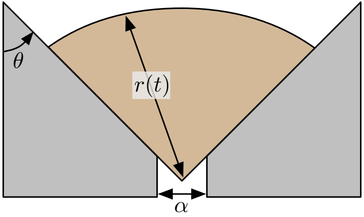
Exercise 1

What is modelling?

Exercise 2





The following ordinary differential equation models a crowd leaving a stadium through an exit

$$2\theta r \frac{dr}{dt} = -k\alpha\sqrt{r}$$

based on the premise

(TL) Torricelli's Law: The area of the region occupied by the crowd decreases proportionally to the width of the exit times the square root of its radius.

2.1 How is the premise expressed in the differential equation?

2.2 Sketch a slope field for this model

`https://www.desmos.com/calculator/lxb4g6cuiz`

and use it to study how the time it would take to evacuate that section depends on the parameters.

2.3 Using Euler's method, estimate how long it would take to evacuate a stadium with $\alpha = k = 1$, $\theta = \frac{\pi}{5}$ and $r(0) = 2$.

Ladd Peebles Stadium

According to the paper ‘‘A study of stadium exit design on evacuation performance’’ studying the Ladd Peebles stadium:

- The average person occupies 0.15m^2.
- The stadium fits 1200 people in one section.

3.1 According to an experiment in the paper, it took 8 minutes to evacuate the stadium. Use this to estimate k for Ladd Peebles.

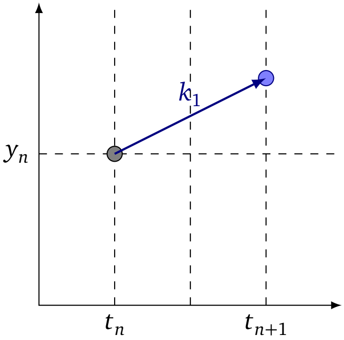
3.2 In the same paper, “for safety, the maximum flow through an exit is 109 people per meter-width per minute.” Does Ladd Peebles satisfy this safety concern?

Numerical Methods for:

$$y' = f(t, y)$$

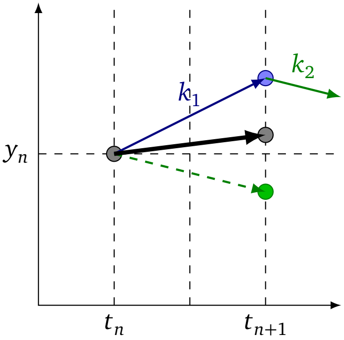
4.1 Euler Method:

$$y_{n+1} = y_n + hk_1$$
$$k_1 = f(t_n, y_n)$$



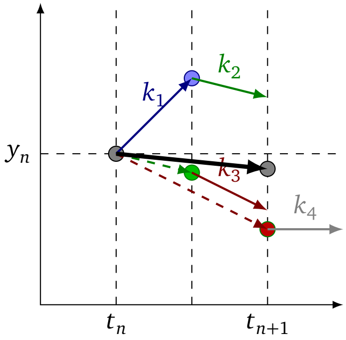
4.2 Heun Method (Improved Euler):

$$y_{n+1} = y_n + h\frac{k_1 + k_2}{2}$$
$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + h, y_n + hk_1)$$



4.3 Runge-Kutta Method:

$$y_{n+1} = y_n + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$
$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$
$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$
$$k_4 = f(t_n + h, y_n + hk_3)$$



Desmos with all these three methods:
`https://www.desmos.com/calculator/haolaltd9s`

Dimensional Analysis

There are seven fundamental dimensions:

Dimension	Symbol	SI Unit	SI Unit Symbol
length	L	metre	m
mass	M	kilogram	kg
time	T	second	s
electric current	I	ampere	A
temperature	Θ	kelvin	K
amount	N	mole	mol
light intensity	J	candela	cd

Note: Sometimes, we use charge Q (SI Unit coulomb C) as a fundamental dimension instead of current.

5.1 When can we add/subtract quantities? With different dimensions? With the same dimensions?

5.2 When can we equate quantities? With different dimensions? With the same dimensions?

5.3 When can we multiply/divide quantities? With different dimensions? With the same dimensions?

5.4 It is convenient to define some functions as a power series (e.g. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$). What conditions on the dimension of x is required to be able to do this?

5.5 What are the dimensions of a derivative $\frac{dy}{dx}$? What are the dimensions of an integral $\int y\,dx$?

Modelling: Relationship between the variables in a model must be dimensionally consistent.

Exercise 6

Consider the model for a mass undergoing radioactive decay:

$$\frac{dm}{dt} = -km$$

with $m(0) = m_0$.

6.1 What are the units of k? What are the units of $t_c = \frac{1}{k}$?

6.2 Introduce new variables: $\tau = \frac{t}{t_c}$ and $\overline{m}(\tau) = \frac{m(t)}{m_0}$. What is the ODE satisfied by $\overline{m}(\tau)$? What are its units? What are the parameters for this equation?

Consider the model for spruce budworm outbreak in Eastern Canada.[a]

$$\frac{dN}{dt} = RN\left(1-\frac{N}{K}\right) - \frac{BN^2}{A^2+N^2}.$$

The first term accounts for resource-limited population growth within a tree and the second term accounts for the predation of the budworms by birds.

7.1 What are the units of N, A, B, K?

7.2 Consider the new variables[b]:

- $x = N/A$ the non-dimensional budworm population
- $\tau = \frac{Bt}{A}$ the non-dimensional time
- $r = \frac{RA}{B}$ the non-dimensional growth rate
- $k = \frac{K}{A}$ the non-dimensional carrying capacity

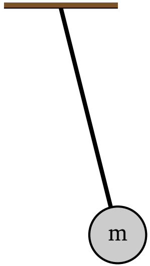
What is the ODE satisfied by $x(\tau)$?

[a] See "Nonlinear Dynamics and Chaos" by Strogatz.
[b] This is not the only way to do this.

Exercise 8

Buckingham Pi Theorem. Any physical relation involving N dimensional variables can be written in terms of a complete set of $N-r$ independent dimensionless variables, where r is the rank of the dimensional matrix $\mathcal{D}$.
The notational convention for the Buckingham Pi Theorem is that the "pi's", $\Pi_1, \ldots, \Pi_{N-r}$ represent dimensionless variables and a relation between them is given by $F(\Pi_1, \ldots, \Pi_{N-r}) = 0$.

Consider a pendulum. We make assumptions:



- The pivot is frictionless
- The rod is massless
- Air resistance is neglected
- The gravitational field is uniform
- The ceiling is infinitely rigid
- ...

8.1 What are the units of the following variables of interest?

(a) Period of the swing $[P] =$

(b) Pendulum mass $[m] =$

(c) Pendulum rod length $[l] =$

(d) Gravitational acceleration $[g] =$

(e) Amplitude of the swing $[\Theta] =$

Exercise 8

8.2 Let us create the dimensional matrix:

- One column for each variable of interest
- One row for each dimension
- Each term contains the power of the corresponding dimension for the corresponding variable

$$
\begin{array}{cc}
 & \begin{array}{ccccc} [P] & [m] & [l] & [g] & [\Theta] \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\mathcal{D} = & \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \end{array}\right] \begin{array}{l} \leftarrow M \\ \leftarrow L \\ \leftarrow T \end{array}
\end{array}
$$

8.3 What is the rank of this matrix?

8.4 What is the dimension of the null space?

8.5 Find a basis for the null space.

Exercise 8

For each vector of the null space basis,

$$\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Buckingham Pi Theorem states that these correspond to non-dimensional variables Π_1 and Π_2:

$$\Pi_1 = \frac{P^2 g}{l} \quad \text{and} \quad \Pi_2 = \Theta$$

and that there is a relation between them:

$$F(\Pi_1, \Pi_2) = 0 \quad \text{or} \quad \Pi_1 = f(\Pi_2) \quad \Longleftrightarrow \quad \frac{P^2 g}{l} = f(\Theta)$$

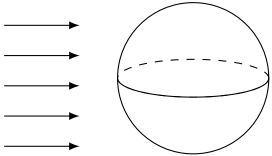
which implies that

$$P = \sqrt{\frac{l}{g}\, g(\Theta)},$$

or in other words, the fact that the *period of the pendulum is proportional to the square root of its length* is a consequence of a pure dimensional analysis of the variables in the problem.

Exercise 9

Consider the flow past a sphere.



You don't need to know much about fluid dynamics to be able to deduce some properties of the flow.

The sphere is in a fluid (water) and we measure the force necessary to keep the sphere from moving downstream.

We want to understand how the drag force depends on the stream velocity.

9.1 We choose the variables of interest. What are their units?

(a) drag force $[F] =$

(b) upstream velocity $[v] =$

(c) fluid density $[\rho] =$

(d) sphere diameter $[D] =$

(e) fluid viscosity (its resistance to deformation by shear stress) $[\mu] =$

9.2 Create a dimension matrix $\mathcal{D}$.

9.3 What is its rank? What is the dimension of its null space? Find a basis for its null space.

9.4 What are the non-dimensional variables Π's from Buckingham Pi Theorem?

9.5 What relations do you obtain?