Learning Objectives

In this tutorial you will explore some modelling with both ODEs and PDEs and you will study the stability of equilibrium solutions.

These problems relate to the following course learning objectives:

- *Model with ODEs.*
- *Analyze equilibria and their stability.*
- *Model with PDEs.*

Problems

1. [1] The von Bertalanffy-Pütter type tumour growth model is

$$\frac{dv}{dt} = pv^a - qv^b \quad , \quad v(0) = v_0,$$

in which $v(t)$ is the volume of the tumour in mm^3 at time t in days. The parameters are $p > 0$, $q > 0$, and $b > a > 0$. The first term pv^a corresponds to resources (nutrients, oxygen, energy, etc) entering the tumour, while the second term $-qv^b$ corresponds to resource consumption by the tumour.

(a) What are the two equilibrium tumour volumes? What are their stabilities?

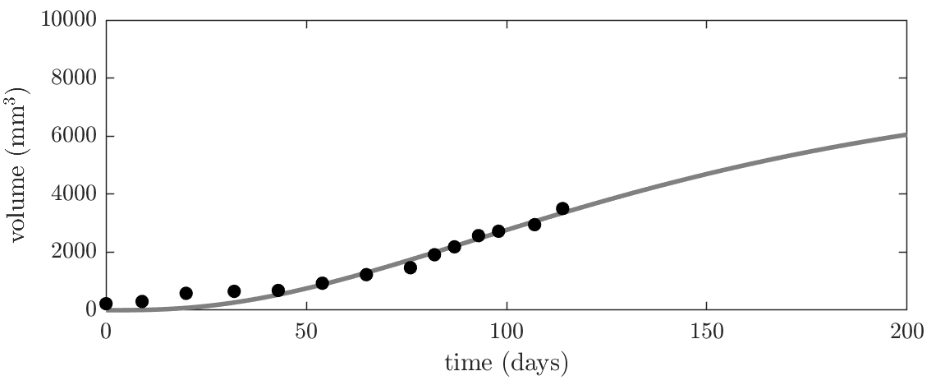
(b) Explain why this model predicts that the tumour volume grows monotonically.
Hint: Use the results from part (1a) to help you.

(c) What is the sensitivity of the (non-trivial) equilibrium tumour volume with respect to p? How does this sensitivity depend on the nominal value of p and q?

(d) If we assume that resources are consumed proportionally to the number of cells, what is the value of b? Call this value b_0.

(e) If we assume that the resources enter the tumour through its surface area and that the tumour is approximately spherical, what is the value of a? Call this value a_0.

(f) The volume of a breast tumour was measured from medical images. The data is shown below in circles. The curve is the least squares best-fit of the model using $a = a_0$ and $b = b_0$, which gives $p = 0.72$ and $q = 0.036$. What ultimate tumour size does this predict? How trustworthy is this prediction?



[1]Based on the 2020 test 2.

2. [2] A particular highway has two lanes in the same direction. The density of vehicles in lane $i = 1, 2$ is $\rho_i(x,t)$ at position x down the highway and time t. The continuum traffic model with the Greenshields velocity model is

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}\left[\rho_1 v_{\max}(1-\rho_1/\rho_{\max})\right] = \alpha(\rho_2 - \rho_1)$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}\left[\rho_2 v_{\max}(1-\rho_2/\rho_{\max})\right] = \alpha(\rho_1 - \rho_2)$$

(a) What does

$$Q = \int_{-\infty}^{\infty} \rho_1 + \rho_2 \, dx$$

represent? Assume that the flux goes to 0 as x approaches $\pm\infty$ and show that Q is constant in time.

(b) What do the terms $\alpha(\rho_2 - \rho_1)$ and $\alpha(\rho_1 - \rho_2)$ represent? Discuss the reasonableness of these terms as a model for what they represent.

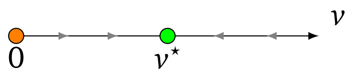
(c) How would you change this model to account for the fact that $n(t) > 0$ cars get into the highway at the position $x = 2$. What do you expect to happen to the quantity Q from part (2a)?

[2] Based on the 2024 test 2.

1. (a) The two equilibria are $v = 0$ and

$$\begin{aligned} pv^a - qv^b = 0 &\Leftrightarrow pv^a = qv^b \\ &\Leftrightarrow \frac{p}{q} = v^{b-a} \\ &\Leftrightarrow v = \left(\frac{p}{q}\right)^{\frac{1}{b-a}} = v^\star \end{aligned}$$

We can study the stability of these equilibria by sketching the 1D phase portrait:



We can deduce that:

- $v = 0$ is unstable
- $v = v^\star$ is stable

(b) From the phase portrait of the previous part, we can tell that the solutions are going to be monotonic:

- If $v_0 < v^\star$, then the tumour will increase monotonically
- If $v_0 > v^\star$, then the tumour will decrease monotonically

(c)

$$\begin{aligned} S(v^\star, p) &= \frac{\partial v^\star}{\partial p} \frac{p}{v^\star} \\ &= \frac{1}{q} \frac{1}{b-a} \left(\frac{p}{q}\right)^{\frac{1}{b-a}-1} \frac{p}{\left(\frac{p}{q}\right)^{\frac{1}{b-a}}} \\ &= \frac{1}{b-a} \end{aligned}$$

So the sensitivity depends only on the difference between a and b and not on p or q.

(d) This means that $qv^b = KN$, where N is the number of cells. If the cells are all uniform, then the number of cells should be proportional to the volume, so we get $qv^b = kv$ and $b_0 = 1$.

(e) This implies $pv^a = KA$, where A is the surface area.

Then we have $v = k_1 r^3$ and $A = k_2 r^2$, where r is the radius of the area occupied by the tumour. This implies that $A = k_3 v^{2/3}$, so $pv^a = Kv^{2/3}$, thus $a_0 = 2/3$.

(f) Since the tumour is increasing, we can deduce that its size will approach $v^\star$:

$$v^\star = \left(\frac{p}{q}\right)^{\frac{1}{b-a}} = \left(\frac{0.72}{0.036}\right)^{\frac{1}{1-2/3}} = 8000\text{mm}^3 = 8\text{cm}^3$$

We have data until the tumour reached the size of 4000 mm^3, or about half of our long term prediction, so our extrapolation is well outside the data we have. It is not very trustworthy.

2. (a) The quantity Q represents the total number of cars in the highway.
We have

$$
\begin{aligned}
\frac{dQ}{dt} &= \int_{-\infty}^{\infty} \frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_2}{\partial t}\, dx \\
&= \int_{-\infty}^{\infty} -\frac{\partial}{\partial x}\left[\rho_1 v_{\max}(1-\rho_1/\rho_{\max})\right] - \frac{\partial}{\partial x}\left[\rho_2 v_{\max}(1-\rho_2/\rho_{\max})\right]\, dx \\
&= \lim_{x\to-\infty}\left\{\frac{\partial}{\partial x}\left[\rho_1 v_{\max}(1-\rho_1/\rho_{\max})\right] + \frac{\partial}{\partial x}\left[\rho_2 v_{\max}(1-\rho_2/\rho_{\max})\right]\right\} \\
&\quad - \lim_{x\to+\infty}\left\{\frac{\partial}{\partial x}\left[\rho_1 v_{\max}(1-\rho_1/\rho_{\max})\right] + \frac{\partial}{\partial x}\left[\rho_2 v_{\max}(1-\rho_2/\rho_{\max})\right]\right\} \\
&= 0
\end{aligned}
$$

(b) These terms represent cars changing lanes. When lane 1 has more cars than lane 2, then $\rho_1 > \rho_2$ and $\alpha(\rho_2 - \rho_1) < 0$ meaning cars are leaving lane 1 to go to lane 2. This seems to be a reasonable way to model cars changing lanes.

(c) This means that at $x = 2$, there should be an influx of cars added to the "lane-changing-term".
Also, usually the cars join the right-most lane, so assuming that lane 1 is the right-lane and lane 2 is the left-lane, we get a new equation 1:

$$
\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}\left[\rho_1 v_{\max}(1-\rho_1/\rho_{\max})\right] = \alpha(\rho_2-\rho_1) + n(t)\underbrace{\left(H_{2.5}(x) - H_{3.5}(x)\right)}_{\text{uniformly in } [2.5,3.5]} \tag{S_1}
$$

where $H_a(x)$ is the Heaviside function.
Alternatively we could have

$$
\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}\left[\rho_1 v_{\max}(1-\rho_1/\rho_{\max})\right] = \alpha(\rho_2-\rho_1) + n(t)\underbrace{\frac{1}{\sqrt{\frac{\pi}{8}}} e^{-8(x-3)^2}}_{\substack{\text{normally distributed} \\ \text{around } x=3 \text{ with } \sigma=\frac{1}{4}}} \tag{S_2}
$$

In this case, since cars are coming in, I expect that Q will be increasing. In fact:

$$
\frac{dQ}{dt} = n(t) > 0,
$$

which means that the total number of cars in the highway will keep increasing by $n(t)$.

Learning Objectives

Students need to be able to...

- *Model with ODEs.*
- *Analyze equilibria and their stability.*
- *Model with PDEs.*

Context

In lecture we studied some examples of finding the stability of equilibrium solutions as well as some modelling with PDEs, both the transport equation and the traffic model.

In this tutorial, we will continue practicing studying the stability of the equilibrium solution as well as studying a model given by an ODE and then we will delve deeper into the traffic model.

Before Tutorial

The midterm is next week on Friday (during lecture), so:

- Make a "when to meet" to decide on the best time for 2h office hours before the test on Wednesday (after 2pm) or Thursday anytime.
- Let Bernardo know the date+time chosen so it can be included in the midterm info announcement on Friday.

Send an announcement to students letting them know that they will **NOT** need to bring a laptop.

Students should review how to find the stability of an equilibrium solution of an ODE as well as how to sketch a phase portrait of an ODE (both a single ODE and a system of ODEs).

What to Do

Introduce the learning objectives for the day's tutorial.