9.2: Let $A \in C^m$ be the Toeplitz matrix defined as follows:

$A_{jj} = 1$

$A_{j,j+1} = 2$

i.e,

$$A^{[3]} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad A^{[4]} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a. What are the eigenvalues, determinant, and rank of A?

We prove all three

The matrix is upper triangular.

THM : An upper triangular matrix A of size n has determinant $\prod_{i=1}^{n} a_{i,i}$ and therefore has full rank.

We prove by induction.

Suppose A of size n. If $n = 1$, . If $n = 2$,

$$det(A) = a_{11} * a_{22} - 0 = \prod_{i=1}^{2} a_{i,i}$$

Now suppose true for $n = 1 : .. N - 1$. We prove true for size N.

For a given element $a_{i,j} \in A$, let $\hat{A}_{i,j}$ designate the sub-block which does not include its row or column.

$$det(A) = \sum_{i=1}^{N} a_{i,1} * det(\hat{A}_{1,1}), = a_{1,1} * det(\hat{A}_{1,1}) + \sum_{i=2}^{N} 0 * det(\hat{A}_{i,2}) = a_{1,1} * det(\hat{A}_{1,1}).$$

$$\rightarrow a_{11} \prod_{i=2}^{N} a_{ii} = \prod_{i=1}^{N} a_{ii}$$

I. **What are the eigenvalues?**

The characteristic polynomial is $det(A - \lambda I)$. Since $A - \lambda I$ is itself upper triangular,

$$det(A - \lambda I) = \prod_{i=1}^{m} (1 - \lambda) \leftrightarrow \lambda_i = 1 \ \forall \ i$$

II. **What is the determinant?**

$$det(A) = \prod_{i=1}^{m} 1 = 1.$$

III. **What is the rank?**

m.

b. What is A^{-1}?

We can convert this matrix to the identity matrix by the following algorithm: let $\overline{a_j}$ be the jth row of A.

$for\, j = 1 : m-1$

$$\overline{a}_{m-j} = \overline{a}_{m-j} - 2 * \overline{a}_{m-j+1}.$$

end

By applying this algorithm to the identity matrix itself, we can produce the inverse.

Let's visually apply this to the simple example of $A \in C^3$.

```
A =
     1     2     0
     0     1     2
     0     0     1


A^-1 =
     1     0     0
     0     1     0
     0     0     1

 step 1
A =
     1     2     0
     0     1     0
     0     0     1

A^-1 =
     1     0     0
     0     1    -2
     0     0     1


 step 2
A =
     1     0     0
     0     1     0
     0     0     1

A^-1 =
     1    -2     4
     0     1    -2
     0     0     1
```

We claim the inverse is then:

$$A^{-1} = [\overline{b}_1, \overline{b}_2, \ldots \overline{b_m}] \ : \ b_{j,k} = -2^{k-j} \ : \ k > j, \ ELSE\ 0$$