TIME EVOLUTION of FLUID FLOWS using DYNAMIC MODE DECOMPOSITION

Exploratory Project Report submitted in the partial fulfillment for the award of

Bachelor's Degree

By

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THESIS CERTIFICATE

This is to certify that the thesis entitled **TIME EVOLUTION of FLUID FLOWS using DYNAMIC MODE DECOMPOSITION**, submitted by **Aman Srivastava** (Roll No. 21065012), to the Indian Institute of Technology (Banaras Hindu University), Varanasi, in the partial fulfillment for the award of Bachelor's degree, is a bona fide record of work done by him/her under my/our supervision. It is certified that the statement made by the student in his/her declaration is correct to the best of my/our knowledge.

Date of Submission: 5^{th} May, 2023

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DECLARATION BY THE CANDIDATE

I, **Aman Srivastava**, certify that the work embodied in this thesis is my own bona fide work carried out by me under the supervisions of **Dr. Basuraj Bhowmik**, from January'23 to May'23 at the Department of Civil Engineering, Indian Institute of Technology (BHU), Varanasi. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and/or cited to the researchers wherever their works have been utilized in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, etc., reported in journals, books, magazines, reports dissertation, thesis, etc., or available at websites.

Date: 5^{th} May , 2023

Place: IITBHU Aman Srivastava

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ABSTRACT

DMD originated in the fluid dynamics community , and has since been applied to a wide range of flow geometries (jets, cavity flow, wakes, channel flow, boundary layers, etc.), to study mixing, acoustics, and combustion, among other phenomena. In the original paper of Schmid both a cavity flow and a jet were considered. In the original paper of Rowley, a jet in cross-flow was investigated. It is no surprise that DMD has subsequently been used widely in both cavity flows and jets . **DMD has also been applied to wake flows**, including to investigate frequency lockon , the wake past a gurney flap, the cylinder wake , and dynamic stall . Dynamic Mode Decomposition (DMD) has emerged as a powerful tool for analyzing and modeling complex fluid flows. **DMD can identify the dominant modes of fluid behavior from experimental or simulation data and provide insights into the underlying dynamics.** In this report, we review the application of DMD to fluid mechanics and highlight its advantages and limitations. We discuss the mathematical foundations of DMD, its computational implementation, and its use in modeling fluid flows. We conclude by discussing the current challenges and future directions of DMD research in engineering. Overall, DMD is a promising approach for understanding the complex dynamics of fluid flows and has the potential to lead to significant advances in fluid mechanics and related fields.

KEY WORDS : DMD, modelling complex fluids flow, cylinder wake, fluid mechanics

CONTENTS

INTRODUCTION

Dynamic mode decomposition (DMD) is a purely data-driven technique that estimates a locally linear representation of complex nonlinear dynamics. The critical point about the method is that it does not require any prior information about the system or its internal physics to capture its dynamics. Hence, it is comparable to gray box models of system identification. DMD and related algorithms provide approximate system identification, unlike purely data-driven statistical models, and machine learning algorithms. The identification of complex dynamical systems as approximate linear dynamics has several benefits. Among them is the simplicity in understanding the system, short-term prediction, pattern identification, and the applicability of linear control algorithms, [1]

Fluid mechanical devices abound in engineering. From pumps to compressors to airplanes to rockets, it is easy to think of devices that operate in fluid environments. In a sense, we are lucky, in that most fluid mechanical systems of interest are described by the same governing equations: the Navier–Stokes equations. Unfortunately, the Navier–Stokes equations are a set of nonlinear partial differential equations (PDEs) that give rise to all manner of dynamics, including those characterized by bifurcations, limit cycles, resonances, and full-blown turbulence. As a result, the complex geometries and challenging flow regimes (e.g., high speeds, pressures, or temperatures) typical in engineered systems can easily confound our ability to generate analytic solutions. This forces us to rely on experiments and high-performance computations when studying such systems, [2]

In engineering flows, nonlinear dynamics may be unavoidable. Furthermore, as we continue to push for better performance and higher efficiency in engineered devices, an understanding of transient dynamics, not just steady-state behaviors, becomes critical. As such, it seems there is a clear role for DMD in engineering analysis going forward. In this study, I have tried to provide an example of DMD analysis.

LITERATURE REVIEW

Dynamic Mode Decomposition (DMD) has gained significant attention in recent years as a powerful tool for analyzing and modeling complex systems. Here is a brief literature review of some of the key papers on DMD I used for my model.

Schmid, Peter J. et.al[4] paper introduced the concept of DMD and demonstrated its application to fluid mechanics. The author showed that DMD can be used to identify the dominant modes of fluid flow and provide insights into the underlying dynamics.

Tu, Jonathan H., Clarence W. Rowley, and Dirk M. Luchtenburg. et al.[5] This paper introduced the concept of Optimal Dynamic Mode Decomposition (ODMD), which is a variant of DMD that identifies the most informative modes of a system. The authors showed that ODMD can be used to identify the most important features of a system and provide insights into its behavior.

Kutz, J. Nathan, et al. [6] paper provided a comprehensive overview of DMD and its applications. The authors discussed the mathematical foundations of DMD, its computational implementation, and its applications in fluid mechanics, structural health monitoring, and other fields.

METHODOLOGY

Singular Value Decomposition (SVD)

The singular value decomposition (SVD) is among the most important matrix factorizations of the computational era, providing a foundation for nearly all of the data methods in this book. The SVD provides a numerically stable matrix decomposition that can be used for a variety of purposes and is guaranteed to exist. We will use the SVD to obtain low-rank approximations to matrices and to perform pseudo-inverses of non-square matrices to find the solution of a system of equations.[1]

$$\mathbf{Ax=b}$$

The SVD provides a systematic way to determine a low-dimensional approximation to high-dimensional data in terms of dominant patterns. This technique is data-driven in that patterns are discovered purely from data, without the addition of expert knowledge or intuition. The SVD is numerically stable and provides a hierarchical representation of the data in terms of a new coordinate system defined by dominant correlations within the data.

Generally, we are interested in analyzing a large data set $\mathbf{X} \in \mathbf{C}^{\mathbf{n} \times \mathbf{m}}$:

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_m \\ | & | & & | \end{bmatrix}.$$

The columns $\mathbf{x_k} \in \mathbf{C^n}$ may be measurements from simulations or experiments .The index k is a label indicating the kth distinct set of measurements

The SVD is a unique matrix decomposition that exists for every complex-valued matrix $\mathbf{X} \in \mathbf{C}^{\mathbf{n} \times \mathbf{m}}$

$$\mathbf{X} = \mathbf{U}\, \boldsymbol{\Sigma}\, \mathbf{V}^*$$

where $\mathbf{U} \in \mathbf{C}^{\mathbf{n} \times \mathbf{n}}$ and $\mathbf{V} \in \mathbf{C}^{\mathbf{m} \times \mathbf{m}}$ are unitary matrices with orthonormal columns, and $\sum \in \mathbf{R}^{\mathbf{n}} \mathbf{x}\ \mathbf{m}$ is a matrix with real, nonnegative entries on the diagonal and zeros off the diagonal. Here $*$ denotes the complex conjugate transpose

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^* = \begin{bmatrix} \hat{\mathbf{U}} & \hat{\mathbf{U}}^{\perp} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Sigma}} \\ \mathbf{0} \end{bmatrix} \mathbf{V}^* = \hat{\mathbf{U}}\hat{\boldsymbol{\Sigma}}\mathbf{V}^*.$$

The columns of U are called left singular vectors of X and the columns of V are right singular vectors. The diagonal elements of $\boldsymbol{\Sigma} \in \mathbf{C}^{\mathbf{n} \times \mathbf{m}}$ are called singular values and they are ordered from largest to smallest. The rank of X is equal to the number of nonzero singular values.[2]

Truncation

Truncated **Singular Value Decomposition** (**SVD**) is a matrix factorization technique that factors a matrix M into the three matrices U, Σ, and V. This is very similar to PCA, excepting that the factorization for SVD is done on the data matrix, whereas for PCA, the factorization is done on the covariance matrix. Typically, SVD is used under the hood to find the principle components of a matrix.[3]

Dynamical Systems

Dynamical systems provide a mathematical framework to describe the world around us, modeling the rich interactions between quantities that co-evolve in time. Formally, dynamical systems concerns the analysis, prediction, and understanding of the behavior of systems of differential equations or iterative mappings that describe the evolution of the state of a system. This formulation is general enough to encompass a staggering range of phenomena, including those observed in classical mechanical systems, electrical circuits, turbulent fluids, climate science, finance, ecology, social systems, neuroscience, epidemiology, and nearly every other system that evolves in time.[1]

Goals and Challenges in Modern Dynamical Systems[3]

As we generally use dynamical systems to model real-world phenomena, there are a number of high-priority goals associated with the analysis of dynamical systems:

1) **Future state prediction**. In many cases, such as meteorology and climatology, we seek predictions of the future state of a system. Long-time predictions may still be challenging.

2) **Design and optimization**. We may seek to tune the parameters of a system for improved performance or stability, for example through the placement of fins on a rocket.

3) **Estimation and control**. It is often possible to actively control a dynamical system through feedback, using measurements of the system to inform actuation to modify the behavior. In this case, it is often necessary to estimate the full state of the system from limited measurements.

4) **Interpretability and physical understanding**. Perhaps a more fundamental goal of dynamical systems is to provide physical insight and interpretability into a system's behavior through analyzing trajectories and solutions to the governing equations of motion

Dynamic Mode Decomposition

Several algorithms have been proposed for DMD, although here I present the exact DMD framework developed . Whereas earlier formulations required uniform sampling of the dynamics in time, the approach presented here works with irregularly sampled data and with concatenated data from several different experiments or numerical simulations. Moreover, the exact formulation of Tu et al. provides a precise mathematical definition of DMD that allows for rigorous theoretical results. Finally, exact DMD is based on the efficient and numerically well-conditioned singular value decomposition, as is the original formulation by Schmid.[4]

DMD is inherently data-driven, and the first step is to collect a number of pairs of snapshots of the state of a system as it evolves in time.

As before, a snapshot may be the state of a system, such as a three-dimensional fluid velocity field sampled at a number of discretized locations, that is reshaped into a high-dimensional column vector. These snapshots are then arranged into two data matrices, X and X:

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}(t_1) & \mathbf{x}(t_2) & \cdots & \mathbf{x}(t_m) \\ | & | & & | \end{bmatrix}$$

$$\mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}(t'_1) & \mathbf{x}(t'_2) & \cdots & \mathbf{x}(t'_m) \\ | & | & & | \end{bmatrix}.$$

The DMD algorithm seeks the leading spectral decomposition (i.e., eigenvalues and eigenvectors) of the best-fit linear operator A that relates the two snapshot matrices in time:

$$\mathbf{X}' \approx \mathbf{A}\mathbf{X}.$$

The best fit operator A then establishes a linear dynamical system that best advances snapshot measurements forward in time. If we assume uniform sampling in time, this becomes:

$$\mathbf{x}_{k+1} \approx \mathbf{A}\mathbf{x}_k.$$

The exact DMD algorithm used in this report is given by the following steps:[1]

Step 1 : Compute the singular value decomposition of X

$$\mathbf{X} \approx \tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}^*,$$

Step 2: The full matrix A may be obtained by computing the pseudo-inverse of X:

$$\mathbf{A} = \mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{U}}^*.$$

However, we are only interested in the leading r eigenvalues and eigenvectors of A, and we may thus project A onto the POD modes in U

$$\tilde{\mathbf{A}} = \tilde{\mathbf{U}}^*\mathbf{A}\tilde{\mathbf{U}} = \tilde{\mathbf{U}}^*\mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}.$$

The key observation here is that the reduced matrix Atilde has the same nonzero eigenvalues as the full matrix A. Thus, we need only compute the reduced Atilde directly, without ever working with the high-dimensional A matrix. The reduced-order matrix Atilde defines a linear model for the dynamics of the vector of POD coefficients x:

$$\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{A}}\tilde{\mathbf{x}}_k.$$

Step 3: The spectral decomposition of Atilde is computed:

$$\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\mathbf{\Lambda}.$$

The entries of the diagonal matrix are the DMD eigenvalues, which also correspond to eigenvalues of the full A matrix. The columns of W are eigenvectors of Atilde and provide a coordinate transformation that diagonalizes the matrix. These columns may be thought of as linear combinations of POD mode amplitudes that behave linearly with a single temporal pattern given by λ.

Step 4. The high-dimensional DMD modes Φ are reconstructed using the eigenvectors W of the reduced system and the time-shifted snapshot matrix X′ according to:

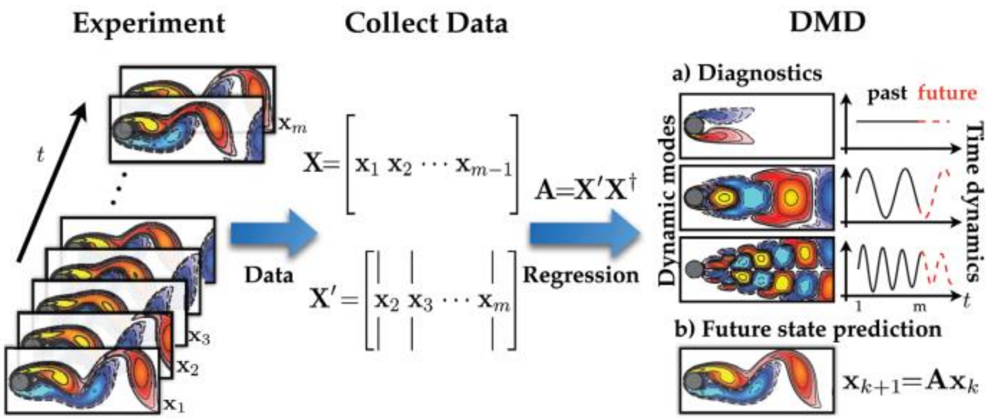
$$\mathbf{\Phi} = \mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\mathbf{W}.$$

Remarkably, these DMD modes are eigenvectors of the high-dimensional A matrix corresponding to the eigenvalues.

$$\begin{aligned}\mathbf{A}\mathbf{\Phi} &= (\mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\underbrace{\tilde{\mathbf{U}}^*)(\mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}}_{\tilde{\mathbf{A}}}\mathbf{W}) \\ &= \mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{A}}\mathbf{W} \\ &= \mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}\mathbf{W}\mathbf{\Lambda} \\ &= \mathbf{\Phi}\mathbf{\Lambda}.\end{aligned}$$

RESULTS AND DISCUSSION

A short visual for the processes involved in DMD for fluid flow past cylinder at R=100



- The Dynamic Mode Decomposition is performed on the fluids data provided by a book on Data Driven Science and Engineering (Machine Learning, Dynamical Systems, and Control) by Nathan Kutz and Stevel L. Brunton
- The time step chosen for the approximation is 0.01.
- The rank r chosen for the analysis is 21.

In Dynamic Mode Decomposition (DMD), spatial modes refer to the spatial distribution of the system's dynamics. DMD spatial modes represent the spatial patterns of the system's behavior over time, and they can provide insights into the system's underlying dynamics.

The spatial modes are obtained by decomposing the system's snapshot matrix using Singular Value Decomposition (SVD) and then projecting the resulting modes onto the system's physical space. Each spatial mode is associated with a particular frequency and growth rate, and it represents the spatial distribution of the corresponding dynamic mode.

Final DMD Spatial modes for different values of r

The X-axis represents the spatial location along the length of the domain of interest, such as the length of a pipe or the width of a channel. The Y-axis represents the spatial location along the height or depth of the domain.

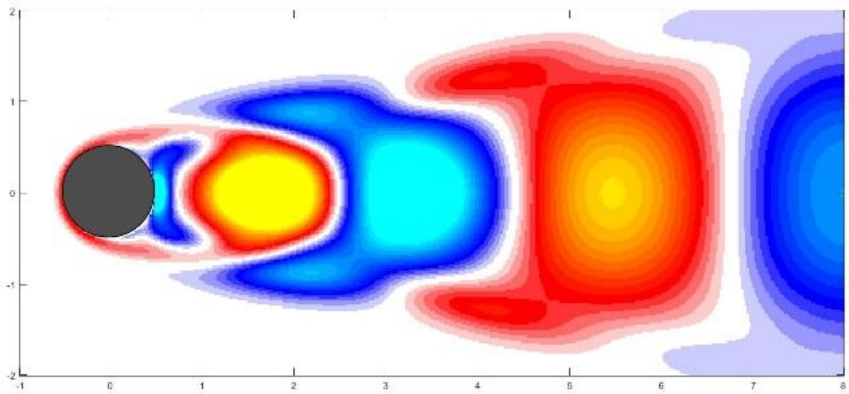


Fig.1 Spatial Model for r=1

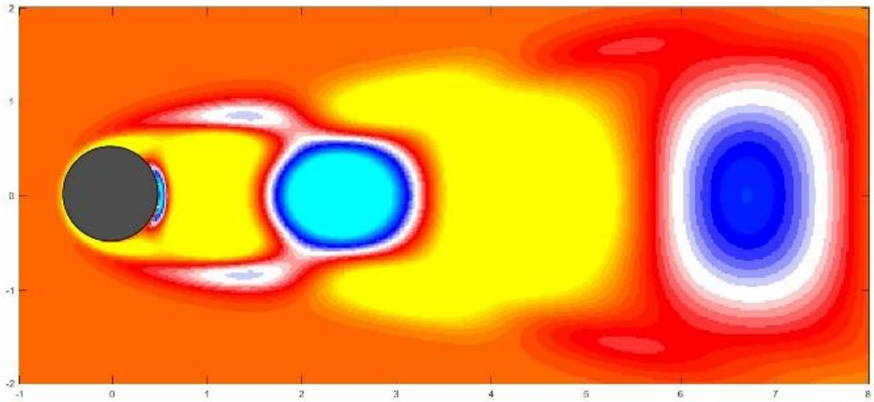


Fig.2 Spatial Model for r=2

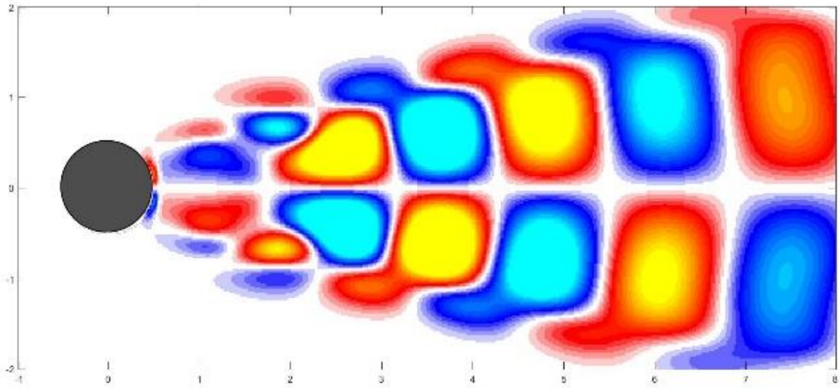


Fig.3 Spatial Model for r=3

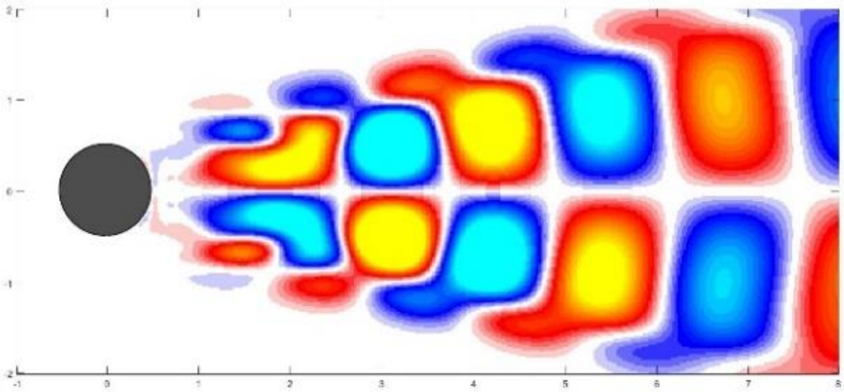


Fig.4 Spatial Model for r=4

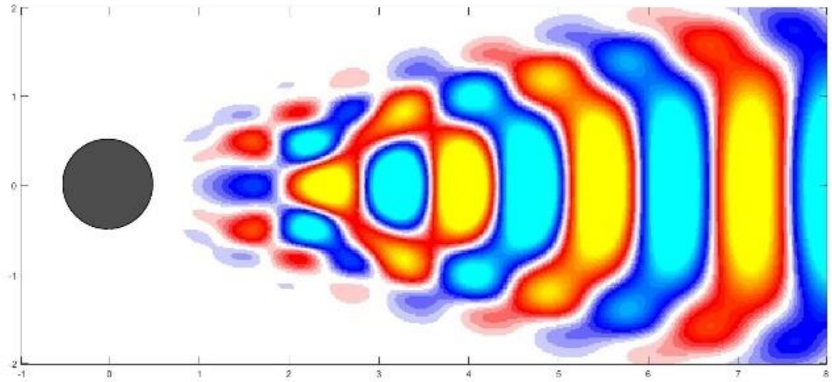


Fig.5 Spatial Model for r=5

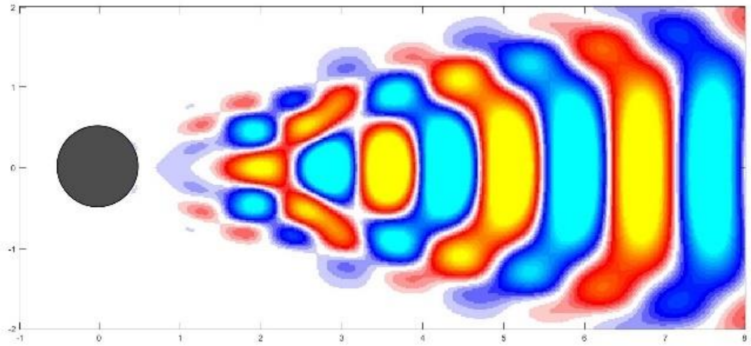


Fig.6 Spatial Model for r=6

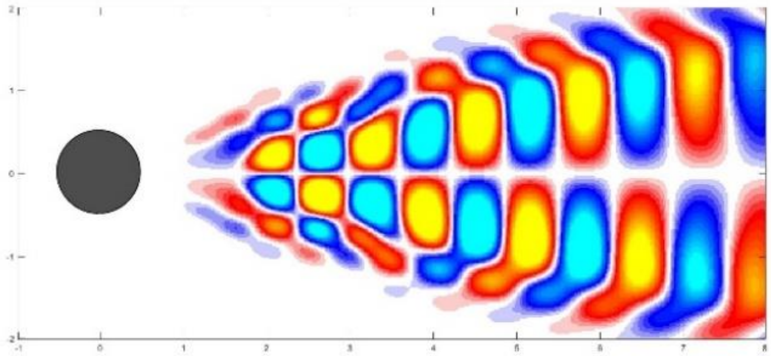


Fig.7 Spatial Model for r=7

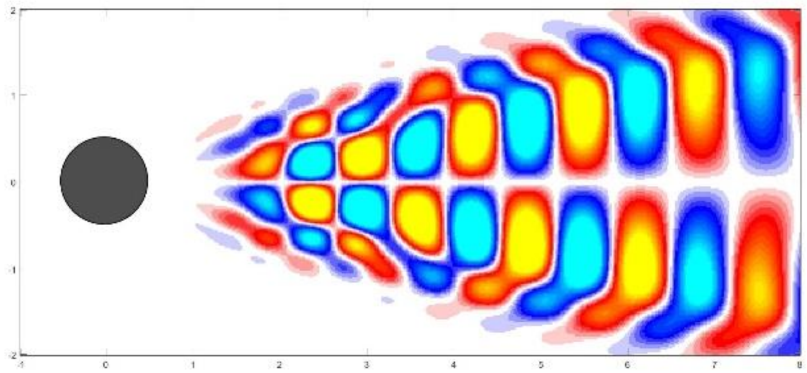


Fig.8 Spatial Model for r=8

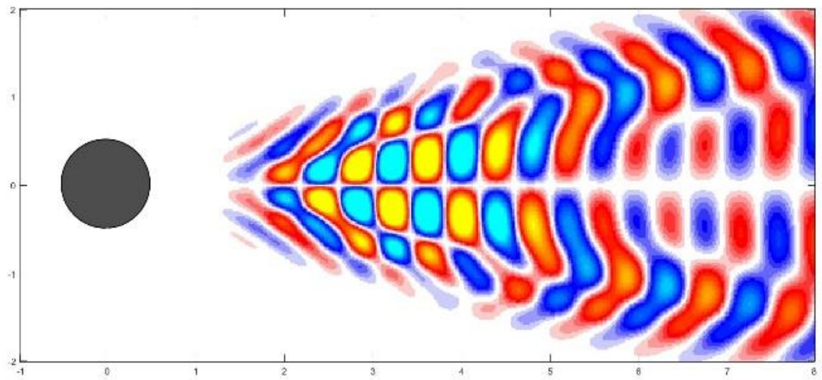


Fig.9 Spatial Model for r=9

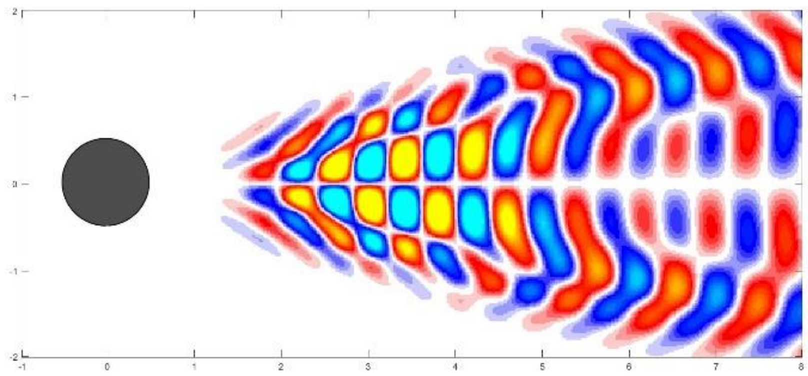


Fig.10 Spatial Model for r=10

CONCLUSIONS AND FUTURE WORK

In this study, I have tried to achieve visuals of DMD spatial modes evolved for different values of r (truncation). The vortex flow pattern can be clearly seen behind the cylinder.

Dynamic Mode Decomposition (DMD) is a mathematical technique used to analyze and model complex systems. In civil engineering, DMD can be used in several application.

Structural Health Monitoring: DMD can be used to monitor the health of structures such as bridges, buildings, and dams. By analyzing the vibration data from these structures, DMD can identify changes in the structure's dynamic behavior, which can indicate the presence of damage or defects.

Wind and Seismic Analysis: DMD can be used to analyze the behavior of structures under wind and seismic loads. By analyzing the vibration data from these loads, DMD can identify the dominant modes of vibration, which can help engineers design more resilient structures.

Traffic Flow Analysis: DMD can be used to analyze traffic flow data to identify patterns and predict congestion. By decomposing the traffic data into modes, DMD can identify the most significant factors affecting traffic flow and provide insights into how to optimize traffic management.

Fluid Flow Analysis: DMD can be used to analyze the behavior of fluid flows in civil engineering applications, such as water treatment plants and drainage systems. By analyzing the flow data, DMD can identify the dominant modes of fluid behavior and provide insights into how to optimize the flow efficiency

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