第四章 不定积分

4.1 不定积分的概念与性质

一、填空题：

1. 已知 $\int x^3 f(x)dx = x + \frac{1}{x} + C$ 则 $\int f(x)dx = \underline{-\frac{1}{2x^2} - \frac{1}{4x^4} + C}$

2. $f(x)$ 满足 $f'(x)(1+x^2) = x^2$ 则 $f(x) = \underline{x - \arctan x + C}$

二、计算下列不定积分：

1. $\int (1-\frac{1}{x^2})\sqrt{x\sqrt{x}}dx = \frac{4}{7}x^{\frac{7}{4}} + \frac{4}{\sqrt[4]{x}} + C$

2. $\int \frac{1+2x^2}{1+x^2}dx = 2x - \arctan x + C$

3. $\int x^4(1+x^2)^3 dx = \frac{1}{5}x^5 + \frac{3}{7}x^7 + \frac{1}{3}x^9 + \frac{1}{11}x^{11} + C$

4. $\int (e^x - 1)^2 \, 2^x dx = \frac{(2e^2)^x}{\ln(2e^2)} - \frac{2(2e)^x}{\ln(2e)} + \frac{2^x}{\ln 2} + C$

5. $\int \frac{\cos 2x}{\cos x - \sin x}dx = \sin x - \cos x + C$

三、验证：$\int \frac{1}{\sqrt{1+x^2}}dx = \ln(x + \sqrt{1+x^2}) + C$

4.2 换元积分法

一、填空题：

1. 已知 $\int f(x)dx = F(x) + c$，则 $\int x^2 f(x^3)dx = \underline{\frac{1}{3}F(x^3) + C}$

2. $F(x)$为$f(x)$的一个原函数，则 $\int \frac{f(x)}{1+4F^2(x)}dx = \underline{\frac{1}{2}\arctan[2F(x)] + C}$

3. $F(x)$为$f(x)$的一个原函数，$f(x) = \frac{F(x)}{1+x^2}$，则 $f(x) = \underline{\frac{C\arctan x}{1+x^2}}$

二、计算下列不定积分:

1. $\int \frac{6x-5}{\sqrt{3x^2-5x+7}}dx = 2\sqrt{3x^2-5x+7}+C$

2. $\int \frac{\cot x}{\ln \sin x}dx = \ln\left|\ln \sin x\right|+C$

3. $\int \frac{x^3}{9+x^2}dx = \frac{1}{2}x^2-\frac{9}{2}\ln(x^2+9)+C$

4. $\int \cos^2(3x+4)dx = \frac{1}{2}[x+\frac{1}{6}\sin(6x+8)]+C$

5. $\int \frac{1}{\sqrt{1-x^2}(\arccos x)^2}dx = \frac{1}{\arccos x}+C$

6. $\int \frac{e^x+7}{4e^x-1}dx = \frac{1}{4}\ln(4e^x-1)+7\ln(4-e^{-x})+C$

7. $\int \frac{1+\sin\sqrt{x}}{\sqrt{x}}dx = 2\sqrt{x}-2\cos\sqrt{x}+C$

8. $\int (x^3+x)\sqrt{1+x^2}dx = \frac{1}{5}(1+x^2)^{\frac{5}{2}}+C$

9. $\int \frac{1}{\sin 2x}dx = \frac{1}{2}\ln\left|\tan x\right|+C = \frac{1}{2}\ln\left|\csc 2x-\cot 2x\right|+C$

三、计算下列不定积分:

1. $\int x^2(1-x)^{20}dx = \frac{1}{21}(x-1)^{21}+\frac{1}{11}(x-1)^{22}+\frac{1}{23}(x-1)^{23}+C$

2. $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx$ （令 $x=t^6$ ）

3. $\int \sqrt{1+e^x}dx = 2\sqrt{1+e^x}+2\ln(\sqrt{1+e^x}-1)-x+C$

4. $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}dx = \frac{x}{\sqrt{1-x^2}}+C$

5. $\int \frac{1}{(4+x^2)^2}dx = \frac{1}{16}(\arctan\frac{x}{2}+\frac{2x}{4+x^2})+C$

4.3 分部积分法

一、计算下列不定积分:

1. $\int x^2 \arcsin x dx = \frac{1}{3}x^3 \arcsin x + \frac{1}{9}(x^2+2)\sqrt{1-x^2} + C$

2. $\int \frac{x}{e^{3x}} dx = -\frac{1}{9}(3x+1)e^{-3x} + C$

3. $\int \frac{\ln x}{(1-x)^2} dx = \frac{\ln x}{1-x} + \ln(1-x) - \ln x + C$

4. $\int \frac{x \arcsin^2 x}{\sqrt{1-x^2}} dx = 2x \arcsin x + 2\sqrt{1-x^2} - (\arcsin x)^2 \sqrt{1-x^2} + C$

5. $\int \frac{x \arcsin^2 x}{\sqrt{1-x^2}} dx = x \tan \frac{x}{2} + C$

6. $\int e^{3x} \sin^2 x dx = \frac{1}{6}e^{3x} - \frac{1}{26}e^{3x}(2\sin 2x + 3\cos 2x) + C$

二、求 $I_n = \int \frac{1}{x^n \sqrt{x+1}} dx$ 的递推公式. （$I_n = -\frac{\sqrt{x+1}}{(n-1)x^{n-1}} - \frac{2n-3}{2(n-1)} I_{n-1}$）

4.4 有理函数的积分

一、计算下列不定积分

1. $\int \frac{1}{x^5(x^6+1)} dx = -\frac{1}{4x^4} - \frac{1}{6}\ln(x^2+1) + \frac{1}{12}\ln(x^4-x^2+1) - \frac{\sqrt{3}}{6}\arctan\frac{2x^2-1}{\sqrt{3}} + C$

2. $\int \frac{3x+5}{x^2+4x+7} dx = \frac{3}{2}\ln(x^2+4x+7) - \frac{1}{\sqrt{3}}\arctan\frac{x+2}{\sqrt{3}} + C$

3. $\int \frac{x^2}{(x+1)^{10}} dx$ （令 $x+1=u$ ）

4. $\int \frac{\cos^3 x}{1+\sin^2 x} dx = 2\arctan(\sin x) - \sin x + C$

5. $\int \frac{1}{3+\cos x} dx = \frac{1}{\sqrt{2}}\arctan\frac{\tan\frac{x}{2}}{\sqrt{2}} + C$

6. $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}} dx = \ln\frac{1-\sqrt{1-x^2}}{|x|} - \arcsin x + C$