Continuum Elastoplasticity - Finite Strain

We consider in the following equations continuum elastoplasticity with quasistatic finite strain deformation of an isotropic material, with isochoric plasticity and isotropic strain hardening.

1 Kinematics

We model continuum elastoplasticity with a multiplicative decomposition of the deformation gradient into elastic and plastic parts.

$$\boldsymbol{F} = \boldsymbol{F}^e \boldsymbol{F}^p \tag{1}$$

We also specify the following (expected) relations:

$$\boldsymbol{b}^e = \boldsymbol{F}^e \boldsymbol{F}^{e^T} \tag{2}$$

$$\boldsymbol{C}^p = \boldsymbol{F}^{p^T} \boldsymbol{F}^p \tag{3}$$

$$\implies \boldsymbol{b}^e = \boldsymbol{F} \boldsymbol{C}^{p^{-1}} \boldsymbol{F}^T \tag{4}$$

The eigenvalues of $\boldsymbol{b}^e$ are the squares of the principal elastic stretches (λ_e^A), and the associated spectral decomposition of $\boldsymbol{b}^e$ is

$$\boldsymbol{b}^e = \sum_{A=1}^{3} \lambda_e^{A^2} \boldsymbol{n}^A \otimes \boldsymbol{n}^A \tag{5}$$

2 Constitutive laws

For nonlinear hyperelastic models, we have

$$\boldsymbol{\tau} : \boldsymbol{d} = \dot{w} \tag{6}$$

where $\boldsymbol{\tau}$ is the Kirchhoff stress, $\boldsymbol{d}$ is the rate of deformation tensor, and w is the strain energy density function. For isotropic hyperelasticity, the strain energy density can be written in terms of $\boldsymbol{b}^e$ or in terms of the invariants or eigenvalues of $\boldsymbol{b}^e$. It is also common to represent energy stored in the plastic deformation using a scalar α to represent equivalent plastic strain, so that $w = \hat{w}(\boldsymbol{b}^e, \alpha) = \bar{w}(\lambda_e^{a^2}, \alpha)$. Recognizing that the dissipation of energy must be non-negative, this equation can then be written as

$$\boldsymbol{\tau} : \boldsymbol{d}^p - \frac{\partial w}{\partial \alpha} \dot{\alpha} \geq 0 \tag{7}$$

The eigenvalues of $\boldsymbol{\tau}$ are the principal stresses (β^A), and the eigenvectors of $\boldsymbol{\tau}$ are equal to those of $\boldsymbol{b}^e$, resulting in a spectral decomposition of

$$\boldsymbol{\tau} = \sum_{A=1}^{3} \beta^A \boldsymbol{n}^A \otimes \boldsymbol{n}^A \tag{8}$$

The following relation holds in this case:

$$\beta^A = \frac{\partial \bar{w}}{\partial \lambda_e^A} \lambda_e^A \tag{9}$$

2.1 Yield condition

We specify a yield function $f(\boldsymbol{\tau})$ with the yield condition $f \leq 0$. Isochoric plasticity implies that f is independent of $\mathrm{tr}(\boldsymbol{\tau})$, so we have

$$f(\boldsymbol{\tau}) = \bar{f}(\mathrm{dev}(\boldsymbol{\tau})) \tag{10}$$

$$\text{where } \mathrm{dev}(\boldsymbol{\tau}) = \boldsymbol{\tau} - \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\tau})\mathbb{1} \tag{11}$$

We define $q = -\frac{\partial w}{\partial \alpha}$ to be the conjugate equivalent stress associated with α and use a particular model $f(\boldsymbol{\tau}) = \tilde{f}(\boldsymbol{\tau}, q)$, where $q = \bar{q}(\alpha)$ is used to model the isotropic hardening. As an example, the isotropic von Mises yield function is

$$\tilde{f}(\boldsymbol{\tau}, q) = |\,\mathrm{dev}(\boldsymbol{\tau})| - \sqrt{\frac{2}{3}}(\tau_y - q) \tag{12}$$

$$= -\sqrt{\frac{2}{3}}(\tau_y - q) \tag{13}$$

where τ_y is the yield stress. If we define the following

$$\boldsymbol{\beta} = [\beta^1, \beta^2, \beta^3]^T \tag{14}$$

$$\mathbf{1} = [1, 1, 1]^T \tag{15}$$

$$\mathrm{Dev}(\boldsymbol{\beta}) = \boldsymbol{\beta} - \tfrac{1}{3}(\boldsymbol{\beta} \cdot \mathbf{1})\mathbf{1} \tag{16}$$

then we can write the yield function $f = \bar{f}(\boldsymbol{\beta}, q)$ as

$$\bar{f}(\boldsymbol{\beta}, q) = |\,\mathrm{Dev}(\boldsymbol{\beta})| - \sqrt{\frac{2}{3}}(\tau_y - q) \tag{17}$$

We could model linear isotropic hardening by defining $q = -K\alpha$.

2.2 Flow rules

Consider the associative flow rules

$$\boldsymbol{d}^p = \gamma \frac{\partial f}{\partial \boldsymbol{\tau}} \tag{18}$$

$$\dot{\alpha} = \gamma \frac{\partial f}{\partial q} \tag{19}$$

where $\gamma \geq 0$ is the plastic multiplier.

Then we have the following relation:

$$\implies \gamma \frac{\partial f}{\partial \boldsymbol{\tau}} : \boldsymbol{\tau} + \gamma \frac{\partial f}{\partial q} q \geq 0 \tag{20}$$

$$\iff \gamma \frac{\partial f}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{\beta} + \gamma \frac{\partial f}{\partial q} q \geq 0 \tag{21}$$

The loading/unloading (Kuhn-Tucker) and consistency conditions give the following relations:

$$\gamma f = 0 \tag{22}$$

$$\gamma \dot{f} = 0 \tag{23}$$

3 Algorithmic integration of the flow rules

We consider the case of displacment loading. The total applied displacement is discretized into pseudo-time steps. We solve for $\boldsymbol{F}$ iteratively within each pseudo-time step, so we have an assumed value for $\boldsymbol{F}_{n+1}$ when updating from t_n to t_{n+1}. We define a trial state by assuming that all deformation between t_n and t_{n+1} is elastic, namely

$$\alpha_{n+1}^{tr} = \alpha_n \tag{24}$$

$$\boldsymbol{b}_{n+1}^{e^{tr}} = \boldsymbol{F}_{n+1} \boldsymbol{C}_n^{p^{-1}} \boldsymbol{F}_{n+1}^T \tag{25}$$

The yield function is evaluated using the trial state. If $f_{n+1}^{tr} < 0$, then the trial state holds and no plastic flow has ocurred. Otherwise, the body has undergone plastic deformation. The flow rules are algorithmically integrated and stress and strain values are updated using the return mapping algorithm, as follows:

$$\alpha_{n+1} = \alpha_{n+1}^{tr} + \gamma_{n+1} \Delta t \Big(\frac{\partial f}{\partial q} \Big)_{n+1} \tag{26}$$

$$\lambda_{e_{n+1}}^{A^2} = \exp \Bigg(-2\gamma_{n+1} \Delta t \Big(\frac{\partial f}{\partial \beta^A} \Big)_{n+1} \Bigg) \lambda_{e_{n+1}}^{A^{tr^2}} \tag{27}$$

$$\boldsymbol{n}_{n+1}^A = \boldsymbol{n}_{n+1}^{A^{tr}} \tag{28}$$

Recall that the spectral decompositions can be used to relate $\boldsymbol{b}^e$ and $\boldsymbol{\tau}$ to the principal stretches and stresses, respectively.

$$\boldsymbol{b}^e = \sum_{A=1}^{3} \lambda_e^{A^2} \boldsymbol{n}^A \otimes \boldsymbol{n}^A \tag{29}$$

$$\boldsymbol{\tau} = \sum_{A=1}^{3} \beta^A \boldsymbol{n}^A \otimes \boldsymbol{n}^A \tag{30}$$

Depending on the yield function f, hardening function q, and strain energy density function w, it may be necessary to iteratively solve for $\gamma_{n+1}\Delta_t$ (which is treated as a single variable), q_{n+1}, and $\boldsymbol{\beta}_{n+1}$ using the following equations:

$$f_{n+1} = 0 \tag{31}$$

$$q = -\frac{\partial \bar{w}}{\partial \alpha} \tag{32}$$

$$\beta^A = \frac{\partial \bar{w}}{\partial \lambda_e^A} \lambda_e^A \tag{33}$$

4 Residual and jacobian

The element residual and jacobian have the same form as finite strain elasticity, namely

$$r_e = \int_{\Omega_e} \left(P^h_{iJ} \frac{\partial w^h_i}{\partial X_J} - w^h_i f_i \right) dV - \sum_i \int_{\partial \Omega_{e_{T_i}}} w^h_i T_i \, dS \tag{34}$$

$$j_e = \int_{\Omega_e} \left(\frac{\partial w^h_i}{\partial x_j} c^{ep}_{ijkl} \frac{\partial \Delta u^h_k}{\partial x_l} + \frac{\partial w^h_i}{\partial x_j} \tau^h_{jk} \frac{\partial \Delta u^h_i}{\partial x_k} \right) dV \tag{35}$$

However, note that instead of the spatial elastic tangent, we are using the spatial algorithmic elastoplastic tangent $\boldsymbol{c}^{ep}$. Assuming the yield function f can be written as $f(\boldsymbol{\beta}, q) = g(\boldsymbol{\beta}) + h(q)$ for some g,h, we can write $\boldsymbol{c}^{ep}$ as the following:

$$\boldsymbol{c}^{ep} = \sum_{A=1}^{3} \sum_{B=1}^{3} a^{ep}_{AB} \boldsymbol{m}^{A^{tr}} \otimes \boldsymbol{m}^{B^{tr}} + 2 \sum_{A=1}^{3} \beta^A \boldsymbol{c}^{A^{tr}} \tag{36}$$

where

$$\boldsymbol{a}^{ep} = \boldsymbol{h} - \frac{\left(1 - \gamma \Delta t \frac{\partial q}{\partial \alpha} \frac{\partial^2 f}{\partial q^2}\right) \left(\boldsymbol{h} \frac{\partial f}{\partial \boldsymbol{\beta}}\right) \otimes \left(\boldsymbol{h} \frac{\partial f}{\partial \boldsymbol{\beta}}\right)}{\left(1 - \gamma \Delta t \frac{\partial q}{\partial \alpha} \frac{\partial^2 f}{\partial q^2}\right) \frac{\partial f}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{h} \frac{\partial f}{\partial \boldsymbol{\beta}} + \frac{\partial q}{\partial \alpha} \left(\frac{\partial f}{\partial q}\right)^2} \tag{37}$$

$$\boldsymbol{h} = \left(\boldsymbol{a}^{e^{-1}} + \gamma \Delta t \frac{\partial^2 f}{\partial \boldsymbol{\beta}^2} \right)^{-1} \tag{38}$$

$$a^e_{AB} = \frac{\partial^2 w}{\partial \lambda^A_e \partial \lambda^B_e} \lambda^A_e \lambda^B_e + \frac{\partial w}{\partial \lambda^A_e} \lambda^A_e \delta_{AB} \tag{39}$$

$$\boldsymbol{m}^{A^{tr}} = \boldsymbol{n}^{A^{tr}} \otimes \boldsymbol{n}^{A^{tr}} \tag{40}$$

$$\begin{aligned} \boldsymbol{c}^{A^{tr}} = \frac{1}{d_A} \Bigg[& \mathbb{I}_{b^{tr}_e} - \boldsymbol{b}^{tr}_e \otimes \boldsymbol{b}^{tr}_e - \frac{\det(\boldsymbol{b}^{tr}_e)}{\lambda^{A^2}_e} \left(\mathbb{I} - (\mathbb{1} - \boldsymbol{m}^{A^{tr}}) \otimes (\mathbb{1} - \boldsymbol{m}^{A^{tr}}) \right) \\ & + \lambda^{A^2}_e \left(\boldsymbol{b}^{tr}_e \otimes \boldsymbol{m}^{A^{tr}} + \boldsymbol{m}^{A^{tr}} \otimes \boldsymbol{b}^{tr}_e + (\mathrm{tr}(\boldsymbol{b}^{tr}_e) - 4\lambda^{A^2}_e) \boldsymbol{m}^{A^{tr}} \otimes \boldsymbol{m}^{A^{tr}} \right) \Bigg] \end{aligned} \tag{41}$$

$$\mathbb{I}^{b^{tr}_e}_{ijkl} = \tfrac{1}{2} (b^{tr}_{e_{ik}} b^{tr}_{e_{jl}} + b^{tr}_{e_{il}} b^{tr}_{e_{jk}}) \tag{42}$$

$$d_A = \frac{\lambda^{A^{tr^2}}_e - \lambda^{B^{tr^2}}_e}{\lambda^{A^{tr^2}}_e - \lambda^{C^{tr^2}}_e}, \text{ with } A, B, C \text{ even permutations of } \{1,2,3\} \tag{43}$$

$$\tag{44}$$

References

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Simo, J.C., "Algorithms for statis and dynamic multiplicative plasticity that preserve the classical return mapping schemes of the infinitesimal theory," *Computer Methods in Applied Mechanics and Engineering*, 99 (1992), 61-112.