Q1 Compute the generalized inverse of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}_{3\times 4}$$

$\therefore r(A) \le \min(3,4)$

$\therefore r(A) \le 3$

$$M = \begin{bmatrix} 2 & 5 & 2 \\ 7 & 12 & 4 \\ 1 & -3 & -2 \end{bmatrix}$$

$|M| = 2(-24+12) - 5(-14-4) + 2(-21-12)$

$|M| = -24 + 90 - 66$

$|M| = -90 + 90$

$|M| = 0$

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

$|M| = 7-6$

$= 1$

$M^{-1} = \frac{1}{|M|} \text{adj}.M$

$= \frac{1}{1}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

$(M^{-1})^T = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$$\therefore H = \begin{bmatrix} 7 & -3 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G = H^T$$

$$\therefore G = \begin{bmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 12 & 4 \\ -3 & -2 \end{bmatrix}$$

$$|M| = -24 + 12$$
$$= -12$$

$$M^{-1} = \frac{1}{|M|} \text{adj.} M$$
$$= \frac{-1}{12} \begin{bmatrix} -2 & -4 \\ 3 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 1/6 & 1/3 \\ -1/4 & -1 \end{bmatrix}$$

$$(M^{-1})^T = \begin{bmatrix} 1/6 & -1/4 \\ 1/3 & -1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & -1/4 \\ 0 & 0 & 1/3 & -1 \end{bmatrix}$$

$G = H^T$

$$\therefore G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/6 & 1/3 \\ 0 & -1/4 & -1 \end{bmatrix}$$

verify that $AGA = A$

$$\begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-6+0+0 & -2+2+0+0 & 0+0+0+0 \\ 21-21+0+0 & -6+7+0+0 & 0+0+0+0 \\ 0-3+0+0 & 0+1+0+0 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 2+0+0 & 5+0+0 & 2+0+0 \\ 0+3+0 & 0+7+0 & 0+12+0 & 0+4+0 \\ -3+3+0 & -6+7+0 & -15+12+0 & -6+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

Q2 Find generalized inverse of the following matrix,

$$A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

& verify that, $AGA = A$

$\rightarrow$ $$A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore r(A) \le \min(3,3)$

$\therefore r(A) \le \cancel{\min}\ 3$

$$M = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|M| = 3(0-1) - 8(0-1) + 1(-4-1)$$
$$= -3 + 8 - 5$$
$$= -8 + 8$$
$$= 0$$

$$M = \begin{bmatrix} 3 & 8 \\ -4 & 1 \end{bmatrix}$$

$|M| = 3 + 32$

$|M| = 35$

$\therefore r(A) \le 2$

$$\therefore M^{-1} = \frac{1}{|M|} \text{adj}. M$$

$$= \frac{1}{35} \text{adj} \begin{bmatrix} 3 & 8 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{35}\begin{bmatrix} 1 & -8 \\ 4 & 3 \end{bmatrix}$$

$$\therefore M^{-1} = \begin{bmatrix} 1/35 & -8/35 \\ 4/35 & 3/35 \end{bmatrix}$$

$$(M^{-1})^T = \begin{bmatrix} 1/35 & 4/35 \\ -8/35 & 3/35 \end{bmatrix}$$

$$\therefore H = \begin{bmatrix} 1/35 & 4/35 & 0 \\ -8/35 & 3/35 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G = (H)^T$$

$$\therefore G = \begin{bmatrix} 1/35 & -8/35 & 0 \\ 4/35 & 3/35 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

verify that $AGA = A$

$$\therefore A \cdot G = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/35 & -8/35 & 0 \\ 4/35 & 3/35 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3/35 + \frac{32}{35} + 0 & -\frac{24}{35} + \frac{24}{35} + 0 & 0+0+0 \\ -4/35 + 4/35 + 0 & \frac{32}{35} + \frac{3}{35} + 0 & 0+0+0 \\ 1/35 + 4/35 + 0 & -\frac{8}{35} + \frac{3}{35} + 0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{35}{35} & 0 & 0 \\ 0 & \frac{35}{35} & 0 \\ \frac{5}{35} & -\frac{5}{35} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/7 & -1/7 & 0 \end{bmatrix}$$

$$\therefore A \cdot G \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/7 & -1/7 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+0 & 8+0+0 & 1+0+0 \\ 0-4+0 & 0+1+0 & 0+1+0 \\ \frac{3}{7}+\frac{4}{7}+0 & \frac{8}{7}-\frac{1}{7}+0 & \frac{1}{7}-\frac{1}{7}+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ \frac{7}{7} & \frac{7}{7} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore AGA = A$

Q3 find g-inverse for the following matrix,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Also verify that $AGA = A$

→ $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$r(A) \leq \min(3,3)$

$r(A) \leq 3$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$|M| = 1(45-48) - 2(36-42) + 3(32-35)$

$= 1(-3) - 2(-6) + 3(-3)$

$= -3 + 12 - 9$

$= -12 + 12$

$= 0$

$$M = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$|M| = 5 - 8$

$|M| = -3$

$\therefore r(A) \leq 2$

$M^{-1} = \dfrac{1}{|M|} \text{adj. } M$

$= \dfrac{-1}{3} \begin{bmatrix} 5 & -2 \\ -4 & 1 \end{bmatrix}$

$$\therefore M^{-1} = \begin{bmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{bmatrix}$$

$$(M^{-1})^T = \begin{bmatrix} -5/3 & 4/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$$\therefore H = \begin{bmatrix} -5/3 & 4/3 & 0 \\ 2/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G = (H)^T$$

$$\therefore G = \begin{bmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

verify that $AGA = A$

$$\therefore A \cdot G = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{3} + \frac{8}{3} + 0 & \frac{2}{3} - \frac{2}{3} + 0 & 0+0+0 \\ -\frac{20}{3} + \frac{20}{3} + 0 & \frac{8}{3} - \frac{5}{3} + 0 & 0+0+0 \\ -\frac{35}{3} + \frac{32}{3} + 0 & \frac{14}{3} - \frac{8}{3} + 0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3/3 & 0 & 0 \\ 0 & 3/3 & 0 \\ -3/3 & 6/3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$

$$AGA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 2+0+0 & 3+0+0 \\ 0+4+0 & 0+5+0 & 0+6+0 \\ -1+8+0 & -2+10+0 & -3+12+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Q4 Find different solutions to $Ax = y$, where,

$$A = \begin{bmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 2 \\ 3 & 2 & 7 & 7 \end{bmatrix}$$

-> $$y = \begin{bmatrix} 6 \\ 8 \\ 22 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 2 \\ 3 & 2 & 7 & 7 \end{bmatrix}$$

$$M = \begin{bmatrix} 5 & 3 & 1 \\ 8 & 5 & 2 \\ 21 & 13 & 5 \end{bmatrix}$$

$$|M| = 5(25-26) - 3(40-42) + 1(104-105)$$
$$= -5 \quad + 6 \quad - 1$$
$$= -6 + 6$$
$$= 0$$

$$M = \begin{bmatrix} 3 & 1 & -4 \\ 5 & 2 & 3 \\ 13 & 5 & 2 \end{bmatrix}$$

$$|M| = 3(4-15) - 1(10-39) - 4(25-26)$$
$$= -36 \quad + 29 \quad + 4$$
$$= -29 + 29$$
$$= 0$$

$$M = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$$

$|M| = 25 - 24$

$|M| = 1$

$\therefore r(A) \leq 2$

$$\therefore M^{-1} = \frac{1}{|M|} \text{adj}.M$$

$$= \frac{1}{1} \begin{bmatrix} 5 & -3 \\ -8 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ -8 & 5 \end{bmatrix}$$

$$(M^{-1})^T = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$$

$$H = \begin{bmatrix} 5 & -8 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$G = (H)^T$

$$\therefore G = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -8 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$GY = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -8 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 8 \\ 22 \\ 2 \end{bmatrix}$$

$$Gy = \begin{bmatrix} 30-24+0+0 \\ -48+40+0+0 \\ 0+0+0+0 \\ 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ 0 \\ 0 \end{bmatrix}$$

$$GA = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -8 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 3 & 1 & -4 \\ 8 & 5 & 2 & 3 \\ 21 & 13 & 5 & 2 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24+0+0 & 15-15+0+0 & 5-6+0+0 & -20-9+0+0 \\ -40+40+0+0 & -24+25+0+0 & -8+10+0+0 & 32+15+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 & -29 \\ 0 & 1 & 2 & 47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore GA - I = \begin{bmatrix} 1 & 0 & -1 & -29 \\ 0 & 1 & 2 & 47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore GA-I = \begin{bmatrix} 0 & 0 & -1 & -29 \\ 0 & 0 & 2 & 47 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$(GA-I)Z = \begin{bmatrix} 0 & 0 & -1 & -29 \\ 0 & 0 & 2 & 47 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$= \begin{bmatrix} -z_3 - 29z_4 \\ 2z_3 + 47z_4 \\ -z_3 \\ -z_4 \end{bmatrix}$$

$$\tilde{x} = Gy + (GA-I)Z$$

$$= \begin{bmatrix} 6 \\ -8 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -z_3 - 29z_4 \\ 2z_3 + 47z_4 \\ -z_3 \\ -z_4 \end{bmatrix}$$

$$\tilde{x} = \begin{bmatrix} 6 - z_3 - 29z_4 \\ -8 + 2z_3 + 47z_4 \\ -z_3 \\ -z_4 \end{bmatrix}$$

where z_3 & z_4 are arbitrary

$\therefore$ putting $z_3 = z_4 = 0$

$$\therefore \tilde{X} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \tilde{X}' = [6 \ -8 \ 0 \ 0]$$

$\therefore$ putting $z_3 = -1$ & $z_4 = 2$

$$\therefore \tilde{X} = \begin{bmatrix} 6-(-1)-29(2) \\ -8+2(-1)+47(2) \\ -(-1) \\ -2 \end{bmatrix}$$

$$\therefore \tilde{X} = \begin{bmatrix} 6+1-58 \\ -8-2+94 \\ 1 \\ -2 \end{bmatrix}$$

$$\therefore \tilde{X} = \begin{bmatrix} -51 \\ 84 \\ 1 \\ -2 \end{bmatrix}$$

$$\tilde{X}' = [-51 \ 84 \ 1 \ -2]$$