Calculus Assignment 02

Date:

Q Evaluate limit using L Hopital Rule.

a) $\lim\limits_{x\to\infty} \left(1-\frac{3}{x}\right)^{2x}$ $\quad (1^\infty)$

$$y = (1 - 3/x)^{2x}$$

$$\ln y = 2x \ln(1 - 3/x)$$

$$\lim_{x\to\infty} 2x\ln(1-3/x) \quad (\infty \cdot 0)$$

$$\lim_{x\to\infty} \frac{\ln(1-3/x)}{1/2x} \quad \left(\frac{0}{0}\right)$$

L-Hop:- $\frac{1}{1-3/x}\cdot(-3/x^2) \div \frac{-1}{2x^2} = \frac{-6x}{x-3}$; $\left(\frac{\infty}{\infty}\right)$

L-Hop:- -6

$\ln y = -6 \longrightarrow \boxed{y = e^{-6}}$

b) $\lim\limits_{x\to 0} \frac{\tan(5x)-\sin(5x)}{x^3}$; $\left(\frac{0}{0}\right)$

L-Hop:- $\lim\limits_{x\to 0} \frac{5\sec^2(5x) - 5\cos(5x)}{3x^2} = \frac{5-5}{0}$; $\left(\frac{0}{0}\right)$

L-Hop:- $\frac{25(2)\sec(5x)\cdot\sec(5x)\cdot\tan(5x) + 25\sin(5x)}{6x}$

$$\lim_{x\to 0} \frac{50\sec^2(5x)\tan(5x) + 25\sin(5x)}{6x} = \left(\frac{0}{0}\right)$$

L-Hop:- $\frac{1}{6}\left[50\sec^2(5x)5\cdot\sec^2(5x) + \tan(5x)\cdot 500\sec^2(5x)\tan(5x) + 125\cos 5x\right]$

$$\lim_{x\to 0} \frac{125}{3}\sec^4(5x) + \frac{250}{3}\tan^2(5x)\sec^2(5x) + \frac{125}{6}\cos(5x)$$

$$= \boxed{\frac{125}{2}}$$

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c) $\lim_{x\to+\infty} \left(\frac{x}{x+1}\right)^x \quad (1^\infty)$

let $y = \left(\frac{x}{x+1}\right)^x \longrightarrow \ln y = \ln\left(\frac{x}{x+1}\right)^x$

$\ln y = x\ln\left(\frac{x}{x+1}\right) \quad (\infty \cdot 0)$

L Hop:-

$\lim_{x\to 0} \frac{1/x}{x+1} \cdot \frac{1(x+1)-(x)(1)}{(x+1)^2} \div -\frac{1}{x^2}$

$\frac{(\cancel{x}+1-\cancel{x})\cancel{(x+1)}}{\cancel{x}\cancel{(x+1)^2}\ 1} \times -\cancel{x^2}\ 1$

$-\frac{x}{x+1} \ ; \ \left(\frac{\infty}{\infty}\right)$

L Hop:- $\frac{-1}{1} = -1$

$\ln y = -1$

$y = e^{-1} \Longrightarrow \boxed{y = \frac{1}{e}}$

d) $\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2} \qquad \frac{1-1-0}{0} \ ; \ \left(\frac{0}{0}\right)$

L-Hop:- $\frac{e^{2x}\cdot 2-2}{2x} = \frac{2e^{2x}-2}{2x} = \frac{e^{2x}-1}{x} \ ; \ \frac{0}{0}$

L-Hop :- $\frac{2e^{2x}}{1} = \boxed{2}$

e) $\lim_{x\to 0^+} x^{\sin x} \quad (0^0)$

$\ln y = \ln x^{\sin x}$

$\ln y = \sin x \ln x \ ; \ (0 \cdot \infty)$

L-Hop:- $\frac{\ln x}{1/\sin x}$

$= \frac{1}{x} \div -1(\sin x)^{-2}\cos x$

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$$= \frac{-\sin^2 x}{x\cos x} \quad ; \left(\frac{0}{0}\right)$$

$$\text{L-Hop} = \frac{-2\sin x\cdot\cos x}{(1)(\cos x)+(x)(-\sin x)} = \frac{-2\sin x\cos x}{\cos x - x\sin x} = \frac{0}{1} = 0$$

$\ln y = 0$

$y = e^0 \rightarrow \boxed{y = 1}$

f) $$\lim_{x\to 0} \frac{\ln(x^4+1) - \sin^2 x}{x^4} = \left(\frac{0}{0}\right)$$

$$\text{L-Hop:-} \frac{[(-4x^3)/x^4+1] - 2\sin x\cos x}{4x^3}$$

$$\frac{-4x^3 - 2\sin x\cos x\,(x^4+1)}{x^4+1} \div 4x^3$$

$$\lim_{x\to 0} \frac{-4x^3 - 2\sin x\cos x\,(x^4+1)}{4x^3(x^4+1)}$$

$$\lim_{x\to 0} \frac{-1}{(x^4+1)} - \frac{2\sin x\cos x}{4x^3} = \infty - \frac{0}{0}$$

$$\cancel{\lim_{x\to 0}} \quad \frac{-1}{0} - \frac{2\sin x\cos x}{4x^3}$$

$$\lim_{x\to 0} \frac{-1}{0} - \frac{\sin 2x}{4x^3}$$

$$\lim_{x\to 0} -\infty - \left(\frac{4\sin 2x}{24x}\right) \Rightarrow -\infty - \left(\frac{-8\cos 2x}{24}\right)$$

$$\lim_{x\to 0} = \boxed{-\infty}$$

Q2. It's because L-Hopital is only applied indeterminate forms of limits in quotient form $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

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Q3. L-Hopital rule is a direct application of derivatives to evaluate indeterminate limits. L Hopital rule works using derivative of numerator and denominator to compare how quickly each function is approaching its limit.

Q4. differentiable at $(0, \pi)$

continous at $[0, \pi]$

$f(0) = 0 \qquad f(\pi) = 0$

$f'(x) = 2\cos 2x$

$\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2} \rightarrow \boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}}$

Q5. Continuity:

$f(-1) = |-1| = 1 \quad , \quad f(1) = |1| = 1$

$\because$ RHS = LHS = function is continuos

Differentiability:-

$$f'(a) = \lim_{x \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{x \to 0^+} \frac{|0+h| - 0}{h} = \frac{h}{h} = 1$$

$$\lim_{x \to 0^-} \frac{|0+h| - 0}{h} \rightarrow \text{as } h \to 0^- \text{ so } h < 0$$

$$\frac{|-h|}{-h} = \frac{h}{-h} = -1$$

$f'(0^-) \neq 1$, hence function is not differenciable at $(-1, 1)$

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Q6. Countinous at $[1, e]$, differentiable at $[1, e]$

$f(1) = \ln 1 = 0$, $f(e) = \ln e = 1$

$f'(x) = 1/x$

slope of tangent = slope of secant

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{c} = \frac{1 - 0}{e - 1} \rightarrow \frac{1}{c} = \frac{1}{e - 1}$$

$\boxed{c = e - 1}$ $\because$ the value satisfies mean value theorem

Q7. $f(a) = f(1) = 2$

$f(b) = f(2) = ?$

$f'(x) = 3x^2 + 2x$

$f(x) = \int 3x^2 + 2x\,dx = \frac{3x^3}{3} + \frac{2x^2}{2} + c$

$f(x) = x^3 + x^2 + c$

for c: $f(1) = 1^3 + 1^2 + c$

$2 = 1 + 1 + c \rightarrow c = 0$

$f(2) = (2)^3 + (2)^2 = 8 + 4$

$\boxed{f(2) = 12}$

Q8. $\omega = \frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$

$x^2 = R^2 + d^2 - 2Rd\cos\theta$

$x^2 = R^2 + (25)^2 - 2(25)R\cos\theta$

$x^2 = R^2 + 625 - 50R\cos\theta$

$\because \frac{d}{dt}(x^2) = \frac{d}{dt}(R^2) + \frac{d}{dt}(625) - \frac{d}{dt}(50R\cos\theta)$

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$$2x\frac{dx}{dt} = 2R\frac{dR}{dt} + 0 - 50\left[\frac{dR}{dt}\cos\theta + R(-\sin\theta)\frac{d\theta}{dt}\right]$$

$$2x\frac{dx}{dt} = 2R\frac{dR}{dt} - 50\frac{dR}{dt}\cos\theta + R\sin\theta\frac{d\theta}{dt}$$

$\because \theta = \pi$ assuming as boat is farthest from shore to find x and $\frac{dx}{dt}$

$$x^2 = R^2 + 625 + 50R\cos(\pi)$$

$$x^2 = 20^2 + 652 + 50(20)(-1)$$

$\frac{dx}{dt}$: $2x\frac{dx}{dt} = 2R\frac{dR}{dt} - 50\frac{dR}{dt}\cos\theta + R\sin\theta\frac{d\theta}{dt}$

$$2(45)\frac{dx}{dt} = 2(20)(4) - 50(4)\cos\pi + 4\sin(\pi)(\pi/5)$$

$$90\frac{dx}{dt} = 160 - 200(-1) + 0$$

$$\frac{dx}{dt} = \frac{36\cancel{0}}{9\cancel{0}} = \boxed{4cm/s}$$

Q9. $\frac{dv}{dt} = 3\,l/s = 3\times10^{-3}\ m^3/s$

$v = \frac{1}{3}\pi r^2 h \quad \because r = \sqrt{h}$

$$v = \frac{1}{3}\pi(\sqrt{h})^2 h = \frac{1}{3}\pi h^2$$

$$\frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{1}{3}\pi h^2\right)$$

$$\frac{d(v)}{dt} = \frac{1}{3}\pi\, 2h\frac{dh}{dt}$$

$$3\times10^{-3} = \frac{1}{3}\pi\, 2h(4)\frac{dh}{dt}$$

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$$\frac{10^{-3}}{8\pi} = \frac{dh}{dt} \Rightarrow \boxed{3.58\times10^{-4}\ m/s}$$

Q10 $\theta = 90^\circ + 15^\circ = 105^\circ$ (as wall is vertical)

$$l^2 = x^2 + y^2 - 2xy\cos\theta$$
$$12^2 = 5^2 + y^2 - 2(5)y\cos\theta$$
$$144 = 25 + y^2 - 10y(-0.24)$$
$$119 = y^2 + 2.4y \rightarrow y^2 + 2.4y - 119 = 0$$
$$y = \frac{-(2.4) \pm \sqrt{(2.4)^2 - 4(1)(-119)}}{2(1)}$$
$$y = \frac{-2.4 + 21.94}{2}, \quad y = \frac{-2.4 - 21.94}{2} \text{ (Reject)}$$
$$y = \frac{19.64}{2} = \boxed{9.82m}$$

$$144 = x^2 + y^2 - 2xy\cos 105$$
$$\frac{d}{dt}(144) = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) - 2(-0.24)\frac{d}{dt}(xy)$$
$$0 = \frac{dx}{dt}(2x + 0.48y) + \frac{dy}{dt}(2y + 0.48x)$$
$$\frac{dy}{dt} = \boxed{-1.33\ m/s} \quad \because \text{Negative bcz of direction}$$

Q11 $\frac{dx}{dt} = 80$ km/hr.

$\frac{dz}{dt} = ?$ $\quad \because z^2 = x^2 + y^2$

$z = 1.12$ km

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$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2)$$

$$\frac{dz}{dt} = \frac{160+60}{2.24} = \frac{220}{2.24} = \boxed{98.2 \text{ m/s}}$$

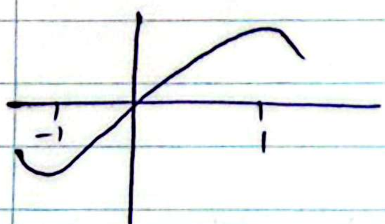
Q12 i $f(x) = \frac{x}{x^2+1}$; $(-\infty, \infty)$

a) $$f'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$f'(x) = 0 \longrightarrow -x^2+1 = 0$

$x = \pm 1$

$$\begin{array}{ccccc} - & & + & & - \\ \hline & -1 & & +1 & \end{array}$$



$\boxed{\text{Increasing: } -1 < x < 1 \quad \text{decreasing: } x < -1,\ x > 1}$

b) critical points ; $\boxed{x = \pm 1}$

stationary points ; $\boxed{x = \pm 1}$

Non stationary points ; $\boxed{\text{None}}$

c) $f''(x) = 0$

$$\because f''(x) = \frac{-2x(x^2+1)^2 + (x^2-1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

$2x(x^2-3) = 0$

$x = 0 \ , \ \pm\sqrt{3}$

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$$\boxed{\begin{array}{l}\text{concave down:- } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \\ \text{concave up: } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)\end{array}}$$

d) $x = \boxed{-\sqrt{3}\ , 0, \sqrt{3}}$

e) i) $f(-1) = \dfrac{-1}{(-1)^2+1} = \boxed{-\dfrac{1}{2}\ \text{(minima)}}$, $f(1) = \dfrac{1}{1^2+1} = \boxed{\dfrac{1}{2}\ \text{(maxima)}}$

ii) $f(x) = e^x \sin x$; $[-\pi, \pi]$

a) $f'(x) = e^x(\cos x + \sin x)$

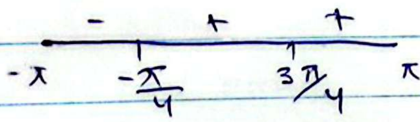
$\because f'(x) = 0$

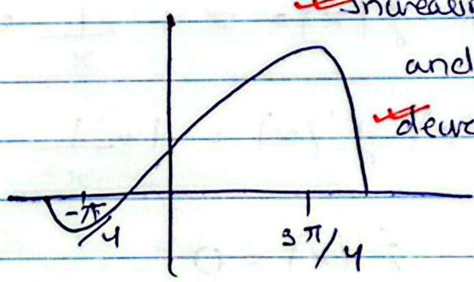
$e^x(\cos x + \sin x) = 0 \rightarrow \cos x + \sin x = 0$

$\sin x = -\cos x \rightarrow \sin x = \cos x \cdot -1$

$\tan x = -1$

$x = \dfrac{-\pi}{4}$, $\dfrac{3\pi}{4}$





Increasing; $-\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$ and $\dfrac{3\pi}{4} < x < \pi$

decreasing; $-\pi < x < -\pi/4$

b) critical points; $x = \boxed{-\pi/4\ , 3\pi/4}$

stationary ; $x = \boxed{-\pi/4\ , 3\pi/4}$

Non-stationary points; $\boxed{\text{None}}$

c) $f''(x) = 0$

$\because f''(x) = \dfrac{d}{dx}\left[e^x(\cos x + \sin x)\right]$

$= e^x \cos x \dfrac{d}{dx} + e^x \sin x \dfrac{d}{dx}$

$= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$

$f''(x) = e^x \cos x + e^x \cos x = 2e^x \cos x$

$2e^x \cos x = 0$

$\cos x = 0 \rightarrow x \pm \pi/2$

concave up ; $\boxed{(-\pi/2, \pi/2)}$

concave down; $\boxed{[-\pi, -\pi/2) \cup (\pi/2, \pi]}$

d) inflection points : $x = \boxed{-\pi/2, \pi/2}$

e) $f(-\pi/4) = e^{-\pi/4} \sin(-\pi/4) = \boxed{0.32}$ (minima)

$f(3\pi/4) = e^{3\pi/4} \sin(3\pi/4) = \boxed{7.46}$ (maxima)

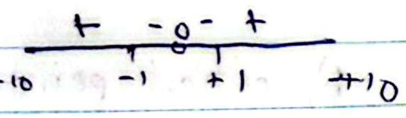
iii $f(x) = x + \frac{1}{x}$; $[-10, 10]$, $x \neq 0$

a) $f'(x) = 1 - \frac{1}{x^2}$, $x \neq 0$

$f'(x) = 0$

$1 - \frac{1}{x^2} = 0 \rightarrow x^2 - 1 = 0$

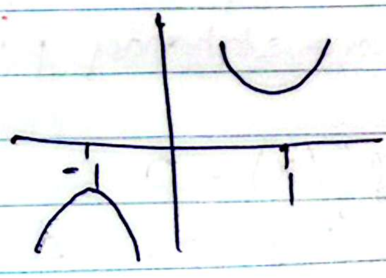
$x = \pm 1$



b) critical points ; $\boxed{x = \pm 1}$

stationary points ; $\boxed{x = \pm 1}$

Non-stationary points ; $\boxed{x = 0}$



c) $f''(x) = 0$

$\because f''(x) = \frac{d}{dx}\left(1 - 1/x^2\right)$

$= 0 + \frac{2}{x^2} \rightarrow f''(x) = \frac{2}{x^2}$

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concave up : $\boxed{(0, 10]}$
concave down : $\boxed{[-10, 0)}$

d) As $f(x)$ is undefined thus no inflection points

e) $f(-1) = 1 - 1/1 = \boxed{-2 \text{ (minima)}}$
$f(1) = 1 + 1/1 = \boxed{2 \rightarrow \text{(maxima)}}$

End points: $f(-10) = 10 - \frac{1}{10} = \boxed{-10.1 \text{ (minima)}}$
$f(10) = 10 + \frac{1}{10} = \boxed{10.1 \text{ (maxima)}}$

Q13. $f(x) = x^4$
$f'(x) = 4x^3$
$f''(x) = 12x^2$
$12x^2 = 0 \rightarrow x^2 = 0 \rightarrow x = 0$
$12(0)^2$ is neither >0 nor <0
Inflection point occurs when both up and down concavities are present, but here only concave up is present, thus no inflection point is present.

Q14. $f(x) = 3x^3$
$f'(x) = 9x^2 \rightarrow 9x^2 = 0$
x has a stationary point at 0
Type of function:
$(-\infty, 0) \rightarrow x = -2 = 9(-2)^2 = 36$
$(0, \infty) \rightarrow x = 2 = 9(2)^2 = 36$
} both increasing
Due to the function giving same sign results (before and after) inflection point, there exists no maxima or minima.