Assignment # 2

Q1 a) $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{2x}$; $\left(1-3/\infty\right)^{2\cdot\infty} = (1^{\infty})$

let. $y = \left(1-3/x\right)^{2x}$

$\ln y = \ln\left(1-3/x\right)^{2x} \rightarrow \ln y = 2x\ln\left(1-3/x\right)$

$\lim_{x\to\infty} 2x\ln\left(1-3/x\right)$; $2(\infty)\ln(1-3/\infty) = (\infty\cdot 0)$

$\lim_{x\to\infty}\frac{\ln\left(1-3/x\right)}{1/2x}$; $\frac{\ln\left(1-3/\infty\right)}{1/2(\infty)} = 0/0$

$\lim_{x\to\infty}\frac{\frac{1}{1-3/x}\cdot\left(-3/x^2\right)}{-1/2x^2}$

$\lim_{x\to\infty}\cancel{\frac{-3x}{x-3(x^2)}}\ \frac{\frac{1}{\frac{x-3}{x}}\cdot\left(-\frac{3}{x^2}\right)}{-1/2x^2}$

$\lim_{x\to\infty}\frac{\frac{-3x}{x-3(x^2)}}{-1/2x^2} \rightarrow \frac{-3x\cdot 2x^2}{x^3-3x^2}$

$\lim_{x\to\infty}\frac{-6x^3}{x^3-3x^2} \rightarrow \lim_{x\to\infty}\frac{-6\cancel{x^3}}{\cancel{x^3}(1-3/x)} = \frac{-6}{1-3/\infty}$

$\frac{-6}{1} = -6$

$\rightarrow \ln y = -6$

$\boxed{y = e^{-6}}$

b) $\lim_{x\to 0} \frac{\tan(5x)-\sin(5x)}{x^3}$; $\frac{\tan 0 - \sin 0}{0^3} = (0/0)$

$\lim_{x\to 0} \frac{5\sec^2(5x)-5\cos(5x)}{3x^2}$; $\frac{5\sec^2(0)-5\cos(0)}{3(0)^2} = (0/0)$

$\lim_{x\to 0} \frac{25(2)\sec(5x).\sec(5x).\tan(5x)+25\sin(5x)}{6x}$

$\lim_{x\to 0} \frac{50\sec^2(5x)\tan(5x)+25\sin(5x)}{6x}$; $\frac{50\sec^2(0)\tan 0+25\sin(0)}{6(0)}$
$= 0/0$

$\lim_{x\to 0} \frac{1}{6}\left[50\sec^2(5x)5.\sec^2(5x)+\tan(5x).500\sec^2(5x)\tan(5x)+ 125\cos(5x)\right]$

$\frac{1}{\cancel{6}3}\left[\overset{125}{\cancel{250}}\sec^4(5x)+\overset{250}{\cancel{500}}\tan^2(5x)\sec^2(5x)+125\cos(5x)\right]$

$\lim_{x\to 0} \frac{125}{3}\sec^4(5x)+\frac{250}{3}\tan^2(5x)\sec^2(5x)+\frac{125}{6}\cos(5x)$

$\frac{125}{3}\sec^4 0+\frac{250}{3}\tan^2(0)\sec^2(0)+\frac{125}{6}\cos(0)$

$\frac{125(1)}{3}+\frac{250(0)(1)}{3}+\frac{125(1)}{6} = \frac{125}{3}+\frac{125}{6} = \frac{250+125}{6} = \frac{375}{6}$

$= \boxed{125/2}$

c) $\lim_{x\to +\infty} \left(\frac{x}{x+1}\right)^x$; $\left(\frac{\infty}{\infty+1}\right)^\infty = (1)^\infty$

let $y = \left(\frac{x}{x+1}\right)^x \rightarrow \ln y = \ln\left(\frac{x}{x+1}\right)^x$

$\ln y = x\ln\left(\frac{x}{x+1}\right)$

$\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)$; $\infty\ln\left(\frac{\infty}{\infty+1}\right) = (\infty\cdot 0)$

$$\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}x \cot x}$$

$$\lim_{x\to 0} \frac{-1}{x \cdot \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} \to \frac{-1}{\frac{x \cdot \cos x}{\sin^2 x}}$$

$$\lim_{x\to 0} \frac{-\sin^2 x}{x \cdot \cos x} ; \frac{\sin^2 0}{0 \cos 0} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{-2\sin x \cos x}{\cos x(1) + x(-\sin x)} = \frac{-2\sin x \cos x}{\cos x - x\sin x}$$

$$\frac{-2\sin 0 \cos 0}{\cos 0 - 0 \sin 0} = \frac{0}{1} = 0$$

$$\ln y = 0$$

$$y = e^0 \to \boxed{y = 1}$$

f) $$\lim_{x\to 0} \frac{\ln(x^4+1) - \sin^2 x}{x^4} ; \frac{\ln(0^4+1) - \sin^2 0}{0^4} = \frac{0-0}{0} = \left(0/0\right)$$

$$\lim_{x\to 0} \frac{[(-4x^3)/x^4+1] - 2\sin x \cos x}{4x^3}$$

$$\frac{-4x^3 - 2\sin x \cos x(x^4+1)}{x^4+1} \div 4x^3$$

$$\lim_{x\to 0} \frac{-4x^3 - 2\sin x \cos x(x^4+1)}{4x^3(x^4+1)} \quad \because 2\sin x \cos x = \sin 2x$$

$$\frac{-\cancel{4x^3}}{\cancel{4x^3}(x^4+1)} - \frac{\cancel{(x^4+1)}2\sin x \cos x}{4x^3\cancel{(x^4+1)}}$$

$$\lim_{x\to 0} \frac{-1}{(x^4+1)} - \frac{2\sin x \cos x}{4x^3}$$

$$\lim_{x\to 0} \frac{-1}{0^4+1} - \frac{2\sin 0 \cos 0}{4(0)} = \infty - \frac{0}{0}$$

$$-\frac{1}{0} - \frac{\cancel{\sin 2x}^{\,2\sin x\cos x}}{4x^3} \quad \because 2\sin x\cos x = \sin 2x$$

$$\lim_{x\to 0} -\frac{1}{0} - \frac{\sin 2x}{4x^3}$$

$$-\infty - \frac{2\cos 2x}{12x^2} \rightarrow -\infty - \frac{2(-\sin(2x))2}{24x}$$

$$\lim_{x\to 0} -\infty - \left(\frac{-4\sin 2x}{24x}\right) \rightarrow -\infty - \left(-\frac{8\cos 2x}{24}\right)$$

$$\lim_{x\to 0} -\infty - \left(\frac{-8\cos 0}{24}\right) \rightarrow -\infty + \frac{1}{3} = \boxed{-\infty}$$

Q2. Because L-Hopital is only applied on indeterminate forms of limits in quotient form ($0/0$ or ∞/∞).

Q3. L Hopital rule is a direct application of derivatives to evaluate indeterminate limits. L Hopital rule ~~is used~~ works using derivative of numerator and denominator to compare ~~to~~ how quickly each function is approaching its limit.

Q4. differentiable at $(0, \pi)$

continuous at $[0, \pi]$

$f(0) = 0$ $\quad f(\pi) = 0$

$f'(x) = 2\cos 2x$

$\cos 2x = 0$

$2x = \pi/2, 3\pi/2 \rightarrow \boxed{x = \pi/4, 3\pi/4}$

Q5. Continuity:

$f(-1) = |-1| = 1$, $f(1) = |1| = 1$

$\because$ RHS = LHS & thus function continuous.

Differentiability:

$$f'(a) = \lim_{x \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\rightarrow \lim_{x \to 0^+} \frac{|0+h| - 0}{h} = \frac{\cancel{K}}{\cancel{K}} = 1$$

$$\lim_{x \to 0^-} \frac{|0+h| - |0|}{h} \rightarrow \text{as } h \to 0^- \text{ so } h < 0.$$

$$\frac{|-h|}{-h} = \frac{\cancel{K}}{-\cancel{K}} = -1$$

$f'(0^-) \neq 1$; hence function is not differentiable at $(-1, 1)$.

Q6. continuous at $[1, e]$, differentiable at $[1, e]$.

$f(1) = \ln 1 = 0$, $f(e) = \ln(e) = 1$.

$f'(x) = 1/x$

$\rightarrow$ slope of tangent = slope of secant

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{c} = \frac{1-0}{e-1} \rightarrow \frac{1}{c} = \frac{1}{e-1}$$

$\boxed{c = e-1}$ $\therefore$ the value satisfies mean value theorem.

Q7. $f(a) = f(1) = 2$

$f(b) = f(2) = ?$

$f'(x) = 3x^2 + 2x$

$$f(x) = \int 3x^2 + 2x \, dx = \frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{2}x^2}{\cancel{2}} + c$$

$f(x) = x^3 + x^2 + C$

for c: $f(1) = 1^3 + 1^2 + c$

$2 = 1 + 1 + c$

$c = 0$

$f(2) = (2)^3 + (2)^2 = 8 + 4$

$\boxed{f(2) = 12}$

Q8. $\omega = \frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$

$d = 25\text{ cm}, R = 20\text{ cm}, \frac{dx}{dt} = ?$

$x^2 = R^2 + d^2 - 2Rd\cos\theta$

$x^2 = R^2 + (25)^2 - 2(25)R\cos\theta$

$x^2 = R^2 + 625 - 50R\cos\theta$

$\therefore \frac{d}{dt}(x^2) = \frac{d}{dt}(R^2) + \frac{d}{dt}(625) - \frac{d}{dt}(50R\cos\theta)$

$2x\frac{dx}{dt} = 2R\frac{dR}{dt} + 0 - 50\left[\frac{dR}{dt}\cos\theta + R(-\sin\theta)\,d\theta/dt\right]$

$2x\, dx/dt = 2R\, dR/dt - 50\, dR/dt \cdot \cos\theta + R\sin\theta\, d\theta/dt$

$\therefore \theta = \pi$ assuming as boat is farthest from shore to find x and dx/dt.

$x^2 = R^2 + 625 + 50R\cos(\pi)$

$x^2 = 20^2 + 625 + 50(20)(-1)$

$x^2 = 2025 \rightarrow x = 45\text{cm}$.

$\frac{dx}{dt}$: $2x\frac{dx}{dt} = 2R\frac{dR}{dt} - 50\frac{dR}{dt}\cos\theta + R\sin\theta\frac{d\theta}{dt}$

$2(45)\frac{dx}{dt} = 2(20)(4) - 50(4)\cos\pi + 4\sin(\pi)(\pi/5)$

$90\, dx/dt = 160 - 200(-1) + 0$

$\frac{dx}{dt} = \frac{360}{90} = \boxed{4\text{ cm/s}}$

Q9. $\frac{dv}{dt} = 3L/s = 3 \times 10^{-3} \ m^3/s$

$V = \frac{1}{3} \pi r^2 h \qquad \because r = \sqrt{h}$

$V = \frac{1}{3} \pi (\sqrt{h})^2 h = \frac{1}{3} \pi h^2$

$\frac{d(V)}{dt} = \frac{d}{dt}\left(\frac{1}{3} \pi h^2\right)$

$\frac{d(V)}{dt} = \frac{1}{3} \pi 2h \frac{dh}{dt}$

$\cancel{3} \times 10^{-3} = \frac{1}{\cancel{3}} \pi 2h(4) \frac{dh}{dt}$

$\frac{10^{-3}}{8\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = 3.58 \times 10^{-4} \ m/s}$

Q10. $\theta = 90^\circ + 15^\circ = 105^\circ$ (as wall is vertical)

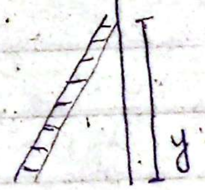
$\rightarrow L^2 = x^2 + y^2 - 2xy\cos\theta$

$12^2 = 5^2 + y^2 - 2(5) y \cos\theta$

$144 = 25 + y^2 - 10y\cos(105)$

$144 = 25 + y^2 - 10y(-0.24)$

$119 = y^2 + 2.4y \rightarrow y^2 + 2.4y - 119 = 0$



$y = \frac{-(2.4) \pm \sqrt{(2.4)^2 - 4(1)(-119)}}{2(1)}$

$y = \frac{-2.4 \pm \sqrt{481.76}}{2} = \frac{-2.4 \pm 21.94}{2}$

$y = \frac{-2.4 + 21.94}{2}, \quad \frac{-2.4 - 21.94}{2}$ (reject)

$y = \frac{19.64}{2} = 9.82 m$

$144 = x^2 + y^2 - 2xy \cos 105$

$\frac{d}{dt}(144) = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) - 2(-0.24)\frac{d}{dt}(xy)$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 0.48\left(y\frac{dx}{dt} + x\, dy/dt\right)$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 0.48y\frac{dx}{dt} + 0.48x\frac{dy}{dt}$

$0 = \frac{dx}{dt}(2x + 0.48y) + \frac{dy}{dt}(2y + 0.48x)$

$0 = 2[2(5) + 0.48(9.82)] + \frac{dy}{dt}[2(9.82) + 0.48(5)]$

$0 = 2[14.71] + \frac{dy}{dt}[22.04]$

$\frac{-29.42}{22.04} = \frac{dy}{dt}$

$\frac{dy}{dt} = -1.33 \, m/s$ ∴ Negative bcz of direction

Q11. $\frac{dx}{dt} = 80 \, km/hr$, $\frac{dy}{dt} = 60 \, km/hr$, $x = 1 \, km$, $y = 0.5 \, km$

$\frac{dz}{dt} = ?$

∴ $z^2 = x^2 + y^2$

$z^2 = 1^2 + 0.5^2 \rightarrow z = \sqrt{1 + 0.25} = 1.12 \, km$

$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2)$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow 2(1.12)\frac{dz}{dt} = 2(1)(80) + 2(0.5)(60)$

$\frac{dz}{dt} = \frac{160 + 60}{2.24} = \frac{220}{2.24} = \boxed{98.2 \, m/s}$

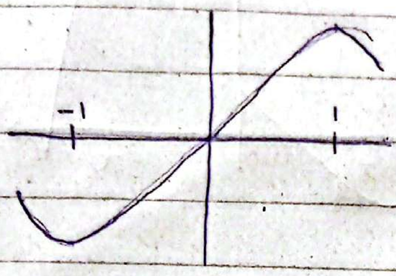
Q12: $f(x) = \dfrac{x}{x^2+1}$; $(-\infty, \infty)$

a) $f'(x) = \dfrac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

$f'(x) = 0 \rightarrow -x^2+1 = 0$

$x = \pm 1$

$$\begin{array}{ccccccc} & - & & + & & - & \\ -\infty & & -1 & & +1 & & +\infty \end{array}$$



b) critical points ; $x = \pm 1$

stationary points ; $x = \pm 1$

Non-stationary points; None.

c) $f''(x) = 0$

$\therefore f''(x) = \dfrac{-2x(x^2+1)^2 + (x^2-1)(2)(x^2+1)(2x)}{(x^2+1)^4}$

$f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

$\rightarrow 2x(x^2-3) = 0$

$x = 0, \pm\sqrt{3}$

Concave down: $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

Concave up: $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

d) $x = -\sqrt{3}, 0, \sqrt{3}$

e). $f(-1) = \dfrac{-1}{(-1)^2+1} = \dfrac{-1}{2}$, $f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$

$\dfrac{-1}{2} \downarrow$ minima

$\dfrac{1}{2} \downarrow$ maxima

ii) $f(x) = e^x \sin x$; $[-\pi, \pi]$

a) $f'(x) = e^x \cos x + e^x \sin x$

$= e^x(\cos x + \sin x)$

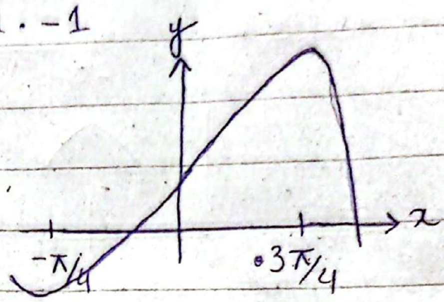
$\because f'(x) = 0$

$e^x(\cos x + \sin x) = 0 \rightarrow \cos x + \sin x = 0$

$\sin x = -\cos x \rightarrow \sin x = \cos x \cdot -1$

$\tan x = -1$

$x = \frac{-\pi}{4}, \frac{3\pi}{4}$



$-\pi$		$-\pi/4$		$3\pi/4$		π
	$-$		$+$		$+$	

b) critical points; $x = -\pi/4, 3\pi/4$

Stationary points; $x = -\pi/4, 3\pi/4$

Non-stationary points; None

c) $f''(x) = 0$

$\because f''(x) = \frac{d}{dx}[e^x(\cos x + \sin x)]$

$= e^x \cos x \frac{d}{dx} + e^x \sin x \frac{d}{dx}$

$= \cancel{e^x \sin x} + e^x \cos x + e^x \cos x - \cancel{e^x \sin x}$

$f''(x) = e^x \cos x + e^x \cos x = 2e^x \cos x$

~~2x~~ $2e^x \cos x = 0$

$\cos x = 0 \rightarrow x = \pm \pi/2$

Concave up; $(-\pi/2, \pi/2)$

Concave down; $[-\pi, -\pi/2) \cup (\pi/2, \pi]$

d) inflection points : $x = -\pi/2, \pi/2$

e) $f(-\pi/4) = e^{-\pi/4}\sin(-\pi/4) = -0.32$ → minima

$f(3\pi/4) = e^{3\pi/4}\sin(3\pi/4) = 7.46$ → maxima

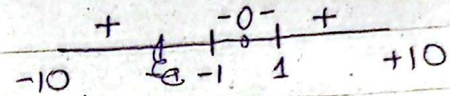
iii $f(x) = x + \dfrac{1}{x}$; $[-10, 10]$; $x \neq 0$.

a) $f'(x) = 1 - \dfrac{1}{x^2}$, $x \neq 0$

$f'(x) = 0$

$1 - 1/x^2 = 0 \Rightarrow x^2 - 1 = 0$

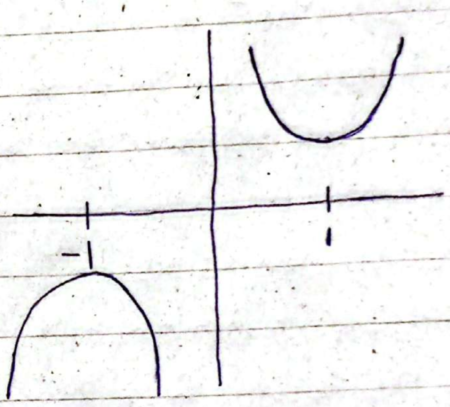
$x = \pm 1$



b) Critical points; $x = \pm 1$

Stationary points; $x = \pm 1$

Non-stationary points; $x = 0$



c) $f''(x) = 0$

$\therefore f''(x) = \dfrac{d}{dx}\left(1 - 1/x^2\right)$

$= 0 + 2/x^2$

$f''(x) = \dfrac{2}{x^2}$

Concave up; $(0, 10]$

Concave down; $[-10, 0)$

d) As $f(x)$ is undefined thus no inflection points.

e) $f(-1) = 1 - 1/1 = -2$ → minima

$f(1) = 1 + 1/1 = 2$ → maxima

Endpoints; $f(-10) = 10 - 1/10 = -10.1$ → minima

$f(10) = 10 + 1/10 = 10.1$ → maxima

Q13. $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$\rightarrow 12x^2 = 0$

$x^2 = 0 \Rightarrow x = 0$

$\rightarrow 12(0)^2$ is neither >0 nor <0.

~~Infle~~ Inflection point occurs when both up and down concavities are present, but here only concave up is present, thus no inflection point is present.

Q14. $f(x) = 3x^3$

$f'(x) = 9x^2$.

$\cancel{f''(x) = 18x}$ $9x^2 = 0$

~~9(0)² is neither >0 nor <0.~~

x has a stationary point at 0.

Type of function:

$(-\infty, 0) \Rightarrow x = -2 = 9(-2)^2 = 36$ } both

$(0, \infty) \Rightarrow x = 2 = 9(2)^2 = 36$ } increasing.

Due to the function giving same sign results (before & after) inflection point, there exists no maxima or minima.