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Part I
Basic Magic Square

We define a magic square to be a square matrix of distinct positive integers from to where the sum of any row, column or diagonal of same length is always equal to the same number. The magic constant. Mathematically, by a magic square of degree n we mean $\mathcal{A}_{n\times n} = [a_{ij}]$ such that

1. All a_{ij} are distinct
2. $\max_{i,j}\{a_{ij}\} - \min_{i,j}\{a_{ij}\} = n^2 - 1$
3. For $1 \leq k$, $l \leq n$.

$$\sum_{i=1}^{n} a_{ki} = \sum_{j=1}^{n} a_{jl} = \sum_{i=1}^{n} = a_{ii} = \sum_{j=1}^{n} a_{j(n-j+1)} \tag{1}$$

Algorithm 1: Siamese Method for Odd Magic Squares

Input : n: Size of the magic square, which is an odd integer
Output : A magic square of size $n \times n$

Initialize $n \times n$ matrix M with zeros;
Set the starting position at the middle of the first row: $(i, j) = (0, \frac{n-1}{2})$;
for $k = 1$ ***to*** n^2 **do**
    Set $M[i][j] = k$;
    Compute the next position (i', j') as follows;
    if $(i - 1 < 0)$ **then** $i' \leftarrow n - 1$;
        else $i' \leftarrow i - 1$;
    if $(j + 1 \geq n)$ **then** $j' \leftarrow 0$;
        else $j' \leftarrow j + 1$;
    if $M[i'][j'] \neq 0$ **then**
        $i \leftarrow i + 1$;
        if $(i \geq n)$ **then**
            $i \leftarrow 0$;
        $j \leftarrow j$;
    else
        $i \leftarrow i'$;
        $j \leftarrow j'$;

Algorithm 2: Scaling Odd-Degree Magic Square to Target Sum

Input : M: The original $n \times n$ magic square
Input : S: The target sum for each row, column and diagonal
Output : A scaled magic square with sum S

Calculate the current Magic Constant M_c: $M_c = \frac{n(n^2+1)}{2}$;
Compute the scaling factor f as follows $f = \frac{S}{M_c}$;
for $i = 1$ ***to*** n **do**
    for $j = 1$ ***to*** n **do**
        Scale each element $M[i][j] \leftarrow M[i][j] \times f$;

The algorithm can be thought of as

- Start by placing the number 1 in middle of the top row.
- For each subsequent number, move *diagonally up and right* from the current position. If this move takes you **outside**, wrap around to the opposite side of matrix (Thus, you will see 2 at the bottom row and offset by 1)
- If target cell is already occupied, move straight down from the current position (6 is below 5).
- Continue this process until all numbers from 1 to n^2 are placed

$$\mathcal{M}_{n=5} = \begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix} \tag{2}$$

Total sum of above is 325, for $\mathcal{M}_{n=3}$ it is 45 and for $n = 7$ it is 1225. By the way, it is possible to create magic squares with 1 elsewhere. Say, we go two towards left and one up with 1 at $(0, 0)$ so we get the following magic square for $n = 5$ (Backtrack by 2 down and 2 left)

$$\mathcal{M}_{n=5} = \begin{bmatrix} 1 & 24 & 17 & 15 & 8 \\ 14 & 7 & 5 & 23 & 16 \\ 22 & 20 & 13 & 6 & 4 \\ 10 & 3 & 21 & 19 & 12 \\ 18 & 11 & 9 & 2 & 25 \end{bmatrix} \tag{3}$$

Also possible to rotate or reflect it. Regardless, all these which start with 1 and add on subsequent numbers will always sum to $\sum_{i=1}^{n^2} i$, notice that the central number is equal to

$$\left[a_{(n-1)/2}\right]_{\mathcal{M}} = \frac{\sum_{i=1}^{n^2}}{n^2} \tag{4}$$

Further, central number c equals $n^2 + 1/2$. Moreover, the magic number (Sum) is equal to $c \cdot n$. Also, the total sum of numbers present in such magic squares is equal to $c \cdot n^2$. The correctness may be proved in various ways like induction.

If in $\mathcal{M}_{n=5}$ the sum S you want is not 65 then you can *scale* the matrix. By scaling I mean multiplying every element by a certain number. This gives us the scaling factor as

$$f = \frac{S}{m_c} \tag{5}$$

Now, it is impossible to make a magic square with $m = 19$ where $n = 3$ with distinct naturals. However, the sum need not be a multiple of n, as we can have $\mathcal{M}_{n=5}$ with $m = 131$ via scaling by 2 and adding 1 to selected cells (Specifically, $\{(0, 1), (1, 0), (2, 2), (3, 4), (4, 3)\}$). In fact,

> For any integer S, an $n \times n$ magic square with sum (Magic Sum) S exists for all $n \geq 4$.

So, 3×3 only has scaling. Regardless, the scaling argument OR adding certain number k to all entries works in all cases. Thus, both of algorithms provided do in fact, work.

```python
def constructMagicSquare(n):
  if n%2 == 0:
    raise ValueError("Only odd degree allowed")
  M = [[0] * n for _ in range(n)]
  i, j = 0,  n // 2 # Position of 1
  for num in range(1, n*n+1):
    M[i][j] = num
    new_i, new_j = (i-1)%n, (j+1)%naturals
    if M[new_i][new_j]: # If occupied
      new_i, new_j = (i+1)%n, j # Move down
    i, j = new_i, new_J
  return m
```

```python
def scaleMagicSquare(M, sum):
  n = len(M)
  curr = n * (n*n + 1) // 2
  factor = sum / curr
  scaledM = [[int(round(factor * M[i][j])) for j in range(n)] for i in range(n)]
  return scaled M
```

Part II
Appendix

1: Symbols

- $\ni$: Such that
- $\mathrm{tr}\mathcal{A}$: Trace of matrix $\mathcal{A}$
- m: Contextually, magic sum
- $\mathcal{M}_n$: Here, magic square of degree n