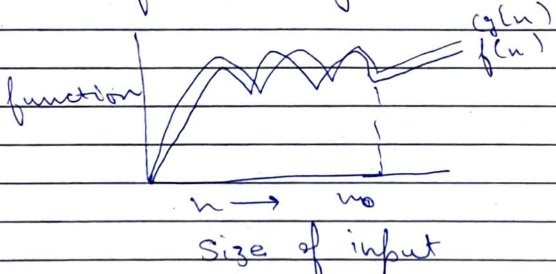
Tutorial - 1

1. **Asymptotic Notations** - They are the mathematical notations used to describe the running time of an alogorithm when the input tends towards a particular value or a limiting value.

Different asymptoitic notations

1) Big O(n)

$$f(n) = O(g(n))$$



$$f(n) = O(g(n))$$
iff $f(n) \leq cg(n)$
$\forall \; n \geq n_0$

for some constant, $c > 0$

g(n) is "tight" upper bound of f(n)

eg. $f(n) = n^2 + n$
$g(n) = n^3$
$n^2 + n \leq cn^3$
$n^2 + n = O(n^3)$

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ii) Big omega (Ω)

$$f(n) = \Omega(g(n))$$

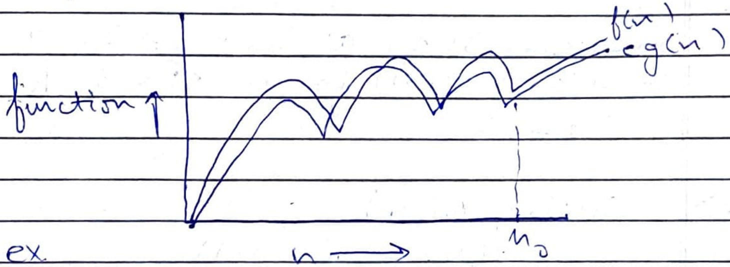
g(n) is "tight" lower bound of function f(n)

$$f(n) = \Omega(g(n))$$

iff $f(n) \geq c\,g(n)$

$\forall \; n \geq n_0$

for some constant $c > 0$



ex.

$$f(n) = n^3 + 4n^2$$

$$g(n) = n^2$$

$$n^3 + 4n^2 = \Omega(n^2)$$

iii) Big Theta (Θ)

$$f(n) = \Theta(g(n))$$

g(n) is both "tight" upper and "tight" lower bound of function f(n).

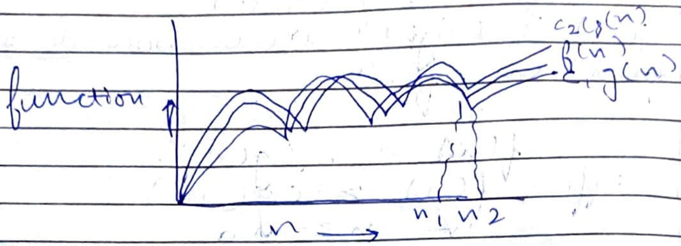
$$f(n) = \Theta(g(n))$$

iff

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

$$\forall \; n \geq \max(n_1, n_2)$$

for some constant $c_1 > 0$ and $c_2 > 0$



<u>Ex</u>

$3n+2 = \theta(n)$ as $3n+2 \geq 3n$ and
$3n+2 \leq 4(n)$ for n, $K_1 = 3$, $K_2 = 4$
$n_0 = 2$

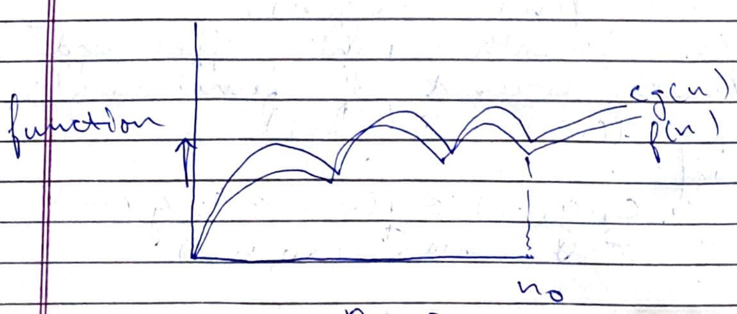
iv> <u>Small</u> $o(\theta)$

$f(n) = o(g(n))$

$g(n)$ is upper bound of function $f(n)$

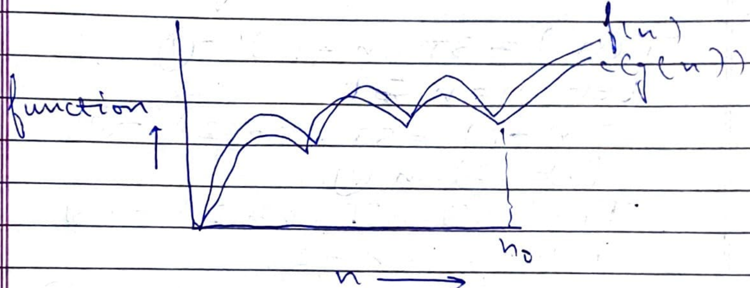
$f(n) = o(g(n))$

when $f(n) < c\, g(n)$
$\forall\ n > n_0$
and $\forall$ constants, $c > 0$



Ex — $f(n) = n^2$
$g(n) = n^3$
$n^2 = O(n^3)$

v) Small omega (ω)
$f(n) \geq \omega(g(n))$
$g(n)$ is lower bound of $f(n)$
$f(n) = \omega(g(n))$
when $f(n) > c\, g(n)$
$\forall\ n > n_0$
and $\forall$ constants, $c > 0$



$f(n) = 4n + 6 \quad g(n) = (1)$

2) for ($i = 1$ to n)
{ $i = i * 2$ }

$i = \underbrace{1, 2, 4, 8, 16, \ldots, n}_{k}$ (G.P)

$- O(k)$

$a = 1,\ r = 2 = 2$

GP K^{th} value $= t_K = ar^{K-1}$

$$n = 1 \times 2^{K-1}$$

$$n = \frac{2^K}{2}$$

$$2n = 2^K$$

$$\log(2n) = K \log 2$$

$$K = \log_2 2n$$

$$K = \log_2 2 + \log_2 n$$

$$K = 1 + \log n$$

Time Comp $= O(1 + \log_2 n)$

$= O(\log_2 n)$

3> $T(n) = 3T(n-1)$ — ①

Let $n = n - 1$

$T(n-1) = 3T(n-2)$ — ②

Put ② in ①

$T(n) = 3 \times 3T(n-2)$ — ③

Put $n = n-2$

$T(n-2) = 3T(n-3)$ — ④

Put ④ in ③

$T(n) = 3 \times 3 \times 3T(n-3)$ — ⑤

$T(n) = 3^n + (n-n)$

$= 3^n + (0)$

$= 3^n$

$= O(3^n)$

4> $T(n) = 2T(n-1) - 1$

$= 2(2T(n-2) - 1) - 1$

$= 2^2(T(n-2)) - 2 - 1$

$= 2^3 + (n-3) - 2^2 - 2^1 - 2^0$

$- - - -$

$$= 2^n + (n-n) - 2^{n-1} - 2^{n-2} - 2^{n-3} - \dots - 2^2 - 2^1 - 2^0$$

$$= 2^n - 2^{n-1} - 2^{n-2} - 2^{n-3} - \dots - 2^2 - 2^1 - 2^0$$

$$= 2^n - (2^n - 1)$$

$$T(n) = 1$$

5)
```
int i=1, s=1;
while (s<=n) {
    i++ ; s=s+i;
    printf("#");
}
```

$$S_i = S_{i-1} + i$$

i is incrementing by one step
s in incrementing by value of i
following will be values after few iterations –

$i=2,\ S=3$ — 1st iteration
$i=3,\ S=6$ — 2nd iteration
$i=4,\ S=10$ — 3rd iteration

Let the value of n be K values of s => 1, 3, 6, 10, --- s represents a series of sum of first n natural numbers for $i=k$, $S = \frac{k(k+1)}{2}$

for stopping loop.

$$\frac{k(k+1)}{2} > n = \frac{k^2+k}{2} > n$$

$$T(n) = O(\sqrt{n})$$

6. ```
void function (int n) {
    int i, count = 0;
    for (i=1, i*i<n; i++)
        count ++;
}
```

$i = 1, 2, 3, \ldots n$

$i^2 = 1, 4, 9, \ldots n$

So

$i^2 <= n$ or $i <= \sqrt{n}$

$a_k = a + (k-1)d$

$a = 1 \quad d = 1$

$a_k \leq \sqrt{n}$

$\sqrt{n} = 1 + (k-1) * 1$

$\sqrt{n} = k$

$T(n) = O(\sqrt{n})$

7. ```
void function (int n) {
    int i, j, k, count = 0;
    for (i=n/2; i<=n; i++)
        for (j=1; j<=n; j=j*2)
            for (k=1; k<=n; k=k*2)
                count ++;
        }
    }
}
}
```

$i = n/2 \qquad j = \log_2 n \qquad k = \log_2 n$

$\left(\frac{n}{2}+1\right)$ times $\quad \log_2 n \quad \log_2 n$

$O(i * j * k) = O\left(\left(\frac{n}{2}+1\right) * \log_2 n * \log_2 n\right)$

$= O\left(\left(\frac{n}{2}+1\right) \times (\log n)^2\right)$

$T(n) = O(n(\log n)^2)$

8.
```
function (int n) {
    if (n == 1) return;
    for (i = 1 to n) {
        for (j = 1 to n) {
            print("*");
        }
        function (n-3);
    }
```

$T(n) = T(n-3) + n^2 \quad — (1)$

$T(1) = 1 \quad — (2)$

put $n = n-3$ in (1)

$T(n-3) = T(n-6) + (n-3)^2 \quad — (3)$

Put (3) in (1)

$T(n) = T(n-6) + (n-3)^2 + n^2 \quad — (4)$

put $n = n-6$ in (1)

$T(n-6) = T(n-9) + (n-6)^2 \quad — (5)$

Put (5) in (4)

$T(n) = T(n-9) + (n-6)^2 + (n-3)^2 + n$

Generalising

$T(n) = T(n-3k) + (n-3(k-1))^2 + (n-3(k-2))^2 + \cdots + n^2$

Let $n - 3k = 1$

$\frac{n-1}{3} = k$

$$T(n) = T(1) + \left(n - 3\left(\frac{n-1}{3} - 1\right)\right)^2 + \left(n - 3\left(\frac{n-1}{3}\right)\right)^2 + \cdots n^2$$

$$T(n) = T(1) + (n - ((n-1) - 3]^2 + [n - (n-1-6)]^2 + [n - [n-1-9]]^2 + \cdots n^2$$

$$T(n) = 1 + (3+1)^2 + (6+1)^2 + \cdots n^2$$

$$T(n) = 1^2 + 4^2 + 7^2 \cdots n^2$$

$$T(n) = n^2 + \cdots 1$$

$$\boxed{T(n) = O(n^2)}$$

Q.

```
void functions (int n) {
    for (i=1 to n) {
        for (j=1; j<=n; j=j+i) {
            printf ("*");
        }
    }
}
```

for $i = 1$, $j \to n$ times

for $i = 2$, $j = 1 + 3 + 5 - - - + n$

$$a_n = a + (k-1)d$$

$$a = 1 \quad d = 2$$

$$n = 1 + (k-1) * 2$$

$$\frac{n-1}{2} = k - 1$$

$$k = \frac{n-1}{2} + 1$$

$$\boxed{k = \frac{n+1}{2}}$$ No. of terms

for $i = 2$, $j \rightarrow \frac{n+1}{2}$ times

for $i=3$, $j = 1+4+7+ \dots n$

$n = 1 + (K-1) \times 3$

$$\boxed{\frac{n-1}{3} + 1 = K}$$

for $i=3$, $j = \frac{n+2}{3}$ times

Generalising

for $i=n$, $j = \frac{n+K-1}{K}$ times

Time complexity is

$$\underbrace{n + \frac{n+1}{2} + \frac{n+2}{3} + \dots + \frac{n+K-1}{K}}_{n \text{ terms}}$$

General term $= \frac{n+K-1}{K}$

$$\sum_{K=1}^{n} \frac{n+K-1}{K} = \frac{\sum_1^n n + \sum_1^n K - \sum 1}{K}$$

$$= \frac{n\left(\frac{n+1}{2}\right) + nK - n}{K}$$

$$= \frac{n^2 + \frac{n}{2} + nK - n}{K}$$

$$T(n) = \frac{n^2 + \frac{n}{2} + nK - n}{K}$$

Neglecting constant terms

$$\boxed{T(n) = O(n^2)}$$

10. as given n^k & c^n
relation b/w n^k & c^n is

$$n^k = O(c^n)$$

as $n^k \le a c^n$

$\forall$ $n \ge n_0$ & some constant $a > 0$

for $n_0 = 1$

$c = 2$

$$1^k \le a2$$

$n_0 = 1$ & $c = 2$