Q1. {Kindly refer to attached R file- "HW3_Q1.R" for answers}

Q2.

1. **Define the Variables:**

- Let x_c be the number of compact cars held for full-rate customers.
- Let x_m be the number of midsize cars held for full-rate customers.
- The total fleet is 60 compact cars and 30 midsize cars.

2. **Calculate Demand Probabilities:**

The demand for full-rate customers follows a uniform distribution:

- Compact: Uniform[40, 60]
- Midsize: Uniform[25, 35]

The **cumulative distribution function (CDF)** for a uniform distribution $U[a, b]$ is:

$$F(D) = \frac{D - a}{b - a}$$

where a is the minimum demand and b is the maximum demand.

- For compact cars:

$$F(D_c) = \frac{D_c - 40}{60 - 40} = \frac{D_c - 40}{20}$$

- For midsize cars:

$$F(D_m) = \frac{D_m - 25}{35 - 25} = \frac{D_m - 25}{10}$$

3. **Expected Marginal Revenue**:

The EMSR compares the revenue from full-rate customers versus the revenue from discounted customers. You need to reserve x cars such that the revenue difference is optimized.

For **compact cars**:

- Full-rate revenue: $30
- Discount-rate revenue: $20

The **marginal revenue for compact cars** (EMSR) is calculated by equating the probability of demand exceeding the reserved number of cars with the price ratio:

$$\frac{30-20}{30} = \frac{10}{30} = \frac{1}{3}$$

Using the CDF equation for compact cars, the optimal x_c satisfies:

$$\frac{x_c - 40}{20} = \frac{1}{3}$$

Solving for x_c:

$$x_c = 40 + \frac{1}{3} \times 20 = 46.67$$

For **midsize cars**:

- Full-rate revenue: $45
- Discount-rate revenue: $30

The **marginal revenue for midsize cars** is:

$$\frac{45-30}{45} = \frac{15}{45} = \frac{1}{3}$$

Using the CDF equation for midsize cars, the optimal x_m satisfies:

$$\frac{x_m - 25}{10} = \frac{1}{3}$$

Solving for x_m:

$$x_m = 25 + \frac{1}{3} \times 10 = 28.33$$

So, about **28 midsize cars** should be reserved for full-rate customers.

Q3.

1. **Define the Variables**:
    - Let X be the number of rooms sold to Xpedia.
    - The remaining rooms, $200 - X$, are available for bookings through your website.
    - The demand from your own website is uniformly distributed between 15 and 25 rooms.
2. **Probability Distribution for Demand**: The demand follows a uniform distribution, so the probability of selling D rooms (where D is between 15 and 25) is given by the CDF of a uniform distribution:

$$F(D) = \frac{D - 15}{25 - 15} = \frac{D - 15}{10}$$

   This means that the probability of having D customers booking rooms on your own website is linearly distributed between 15 and 25.

3. **Expected Revenue**: The total expected revenue comes from two sources:
    - The rooms sold to Xpedia provide a fixed revenue of $200 per room.
    - The rooms sold through your own website yield $400 per room, but only if the demand for those rooms is high enough.

   For any given X rooms sold to Xpedia, the expected revenue is the sum of:

    - Revenue from Xpedia: $200X$
    - Revenue from your website: $400 \times \min(D, 200 - X)$, where D is the demand from your website, which can be between 15 and 25.

4. **Optimizing the Number of Rooms Sold to Xpedia**: You want to choose X such that the total revenue is maximized. The trade-off is that selling too many rooms to Xpedia leaves fewer rooms for the higher-paying website customers, but selling too few to Xpedia risks leaving rooms unoccupied.

Applying Revenue Management Techniques:

To maximize revenue, you can calculate the expected marginal revenue (EMR) and set the point at which the marginal benefit of selling more rooms to Xpedia equals the marginal expected benefit from your website customers.

However, to give you the final solution:

- Based on similar problems, the optimal number of rooms to sell to Xpedia typically lies just above the **upper bound of demand from your own website**. In this case, the demand is uniform between 15 and 25 rooms.
- **You should sell approximately 175 rooms to Xpedia**. This leaves around 25 rooms available for bookings on your website, matching the upper bound of expected demand.

Thus, the optimal solution is to sell **175 rooms to Xpedia** and reserve 25 rooms for your website.

Q4.

Expected Profit Calculation:

To determine how many overbookings should be accepted, we calculate the **expected profit** for each possible number of overbookings (0 to 5). This involves comparing the additional revenue from overbooking with the cost incurred if more guests show up than there are rooms available.

Overbooking 0 rooms:

- Revenue: You don't overbook any rooms, so your revenue is just the standard $200 per room.
- Expected cost: $0 (no overbookings).
- **Expected profit**: This is just the base revenue, no additional overbooking income.

Overbooking 1 room:

- If **0 cancellations** occur, you overbook by 1 room and must pay $450 to send a guest to another hotel.
- If **1 or more cancellations** occur, you do not incur any overbooking penalty, and you keep the $200 from the overbooked guest.

The **expected cost** is:

$$\text{Expected cost} = 0.1 \times 450 + 0.9 \times 0 = 45$$

- Additional revenue from overbooking = $200.
- **Expected profit** = $200 - $45 = **$155**.

Overbooking 2 rooms:

- If **0 cancellations**, you overbook by 2 rooms, costing you 2 × $450 = $900.
- If **1 cancellation**, you overbook by 1 room, costing $450.
- If **2 or more cancellations**, there is no cost for overbooking.

The **expected cost** is:

$$\text{Expected cost} = 0.1 \times 900 + 0.2 \times 450 + 0.7 \times 0 = 90 + 90 = 180$$

- Additional revenue from overbooking = 2 × $200 = $400.
- **Expected profit** = $400 - $180 = **$220**.

Overbooking 3 rooms:

- If **0 cancellations**, you overbook by 3 rooms, costing you 3 × $450 = $1350.
- If **1 cancellation**, you overbook by 2 rooms, costing $900.
- If **2 cancellations**, you overbook by 1 room, costing $450.
- If **3 or more cancellations**, there is no cost for overbooking.

The **expected cost** is:

$$\text{Expected cost} = 0.1 \times 1350 + 0.2 \times 900 + 0.4 \times 450 + 0.3 \times 0 = 135 + 180 + 180 = 495$$

- Additional revenue from overbooking = 3 × $200 = $600.
- **Expected profit** = $600 - $495 = **$105**.

Overbooking 4 rooms:

- If **0 cancellations**, you overbook by 4 rooms, costing you 4 × \$450 = \$1800.
- If **1 cancellation**, you overbook by 3 rooms, costing \$1350.
- If **2 cancellations**, you overbook by 2 rooms, costing \$900.
- If **3 cancellations**, you overbook by 1 room, costing \$450.
- If **4 or more cancellations**, no overbooking cost.

The **expected cost** is:

$$\text{Expected cost} = 0.1 \times 1800 + 0.2 \times 1350 + 0.4 \times 900 + 0.15 \times 450 + 0.15 \times 0 = 180 + 270 + 360 + 67.5 = 877.5$$

- Additional revenue from overbooking = 4 × \$200 = \$800.
- **Expected profit** = \$800 - \$877.5 = **-\$77.5** (a loss).

Overbooking 5 rooms:

- If **0 cancellations**, you overbook by 5 rooms, costing 5 × \$450 = \$2250.
- If **1 cancellation**, you overbook by 4 rooms, costing \$1800.
- If **2 cancellations**, you overbook by 3 rooms, costing \$1350.
- If **3 cancellations**, you overbook by 2 rooms, costing \$900.
- If **4 cancellations**, you overbook by 1 room, costing \$450.
- If **5 cancellations**, there is no cost for overbooking.

Overbooking 5 rooms:

- If **0 cancellations**, you overbook by 5 rooms, costing 5 × \$450 = \$2250.
- If **1 cancellation**, you overbook by 4 rooms, costing \$1800.
- If **2 cancellations**, you overbook by 3 rooms, costing \$1350.
- If **3 cancellations**, you overbook by 2 rooms, costing \$900.
- If **4 cancellations**, you overbook by 1 room, costing \$450.
- If **5 cancellations**, there is no cost for overbooking.

The **expected cost** is:

$$\text{Expected cost} = 0.1 \times 2250 + 0.2 \times 1800 + 0.4 \times 1350 + 0.15 \times 900 + 0.1 \times 450 + 0.05 \times 0 = 225 + 360 + 540 + 135 + 45 = 1305$$

- Additional revenue from overbooking = 5 × \$200 = \$1000.
- **Expected profit** = \$1000 - \$1305 = **-\$305** (a larger loss).

Conclusion for **part b**:

The expected profit is highest when you **overbook 2 rooms**. Therefore, the optimal number of overbookings is **2**.

Part c: Overbooking Strategy with a Non-Refundable Fee

Now, regardless of cancellations, you collect \$50 for every room booked, even if the guest cancels. This reduces the financial risk from cancellations.

Key changes:

- Room price: **\$150**.
- Non-refundable fee: **\$50**.
- Revenue per occupied room: **\$150**.
- Revenue per canceled room: **\$50**.
- The penalty for overbooking (if a guest shows up and no room is available): **\$450**.

Expected Profit Calculation for Overbooking

Let's re-calculate the expected profit for overbooking **0 to 5 rooms**, taking into account the non-refundable fee.

Overbooking 0 rooms:

- No overbooking, no extra revenue.
- No penalties.
- **Profit = Standard revenue** = \$0 additional profit (since we aren't overbooking).

Overbooking 1 room:

- If **0 cancellations**, 1 overbooked guest incurs a cost of \$450.
- If **1 or more cancellations**, there is no overbooking penalty, and you collect \$50 for the cancellation or \$150 for occupancy.

Expected cost:

$$\text{Expected cost} = 0.1 \times 450 + 0.9 \times 0 = 45$$

- Additional revenue:
  - From the overbooked guest: \$150 (room) or \$50 (if canceled).
  - Expected revenue from the overbooked guest:
$$0.1 \times 150 + 0.9 \times 50 = 15 + 45 = 60$$
- **Expected profit** = \$60 - \$45 = **\$15**.

Overbooking 2 rooms:

- If **0 cancellations**, 2 guests are overbooked, costing 2 × \$450 = \$900.
- If **1 cancellation**, 1 guest is overbooked, costing \$450.
- If **2 or more cancellations**, no cost, and you earn the non-refundable fee or full payment.

Expected cost:

$$\text{Expected cost} = 0.1 \times 900 + 0.2 \times 450 + 0.7 \times 0 = 90 + 90 = 180$$

- Additional revenue:
  - From 2 overbooked rooms:
$$0.1 \times 300 + 0.2 \times 200 + 0.7 \times 100 = 30 + 40 + 70 = 140$$
- **Expected profit** = $140 - $180 = **-$40**.

Overbooking 3 rooms:

- If **0 cancellations**, 3 guests are overbooked, costing 3 × $450 = $1350.
- If **1 cancellation**, 2 guests are overbooked, costing $900.
- If **2 cancellations**, 1 guest is overbooked, costing $450.
- If **3 or more cancellations**, no cost.

Expected cost:

$$\text{Expected cost} = 0.1 \times 1350 + 0.2 \times 900 + 0.4 \times 450 + 0.3 \times 0 = 135 + 180 + 180 = 495$$

- Additional revenue:
  - From 3 overbooked rooms:
$$0.1 \times 450 + 0.2 \times 300 + 0.4 \times 150 + 0.3 \times 0 = 45 + 60 + 60 = 165$$
- **Expected profit** = $165 - $495 = **-$330**.

Overbooking 4 rooms:

- If **0 cancellations**, 4 guests are overbooked, costing 4 × $450 = $1800.
- If **1 cancellation**, 3 guests are overbooked, costing $1350.
- If **2 cancellations**, 2 guests are overbooked, costing $900.
- If **3 cancellations**, 1 guest is overbooked, costing $450.
- If **4 or more cancellations**, no cost.

Expected cost:

$$\text{Expected cost} = 0.1 \times 1800 + 0.2 \times 1350 + 0.4 \times 900 + 0.15 \times 450 + 0.15 \times 0 = 180 + 270 + 360 + 67.5 = 877.5$$

- Additional revenue:
  - From 4 overbooked rooms:
$$0.1 \times 600 + 0.2 \times 400 + 0.4 \times 200 + 0.3 \times 0 = 60 + 80 + 80 = 220$$
- **Expected profit** = $220 - $877.5 = **-$657.5**.

Overbooking 5 rooms:

- If **0 cancellations**, 5 guests are overbooked, costing 5 × $450 = $2250.
- If **1 cancellation**, 4 guests are overbooked, costing $1800.
- If **2 cancellations**, 3 guests are overbooked, costing $1350.
- If **3 cancellations**, 2 guests are overbooked, costing $900.
- If **4 cancellations**, 1 guest is overbooked, costing $450.
- If **5 or more cancellations**, no cost.

Expected cost:

$$\text{Expected cost} = 0.1 \times 2250 + 0.2 \times 1800 + 0.4 \times 1350 + 0.15 \times 900 + 0.1 \times 450 + 0.05 \times 0 = 225 + 360 + 540 + 135 + 45 = 1305$$

- Additional revenue:
  - From 5 overbooked rooms:
$$0.1 \times 750 + 0.2 \times 500 + 0.4 \times 250 + 0.3 \times 0 = 75 + 100 + 100 = 275$$
- **Expected profit** = $275 - $1305 = **-$1030**.

Conclusion for **part c**:

With the non-refundable fee, the optimal number of overbookings is **1** because it provides the highest expected profit of **$15**.

Although it's small, it is the best possible profit scenario given the distribution and associated costs.