

Multi-Player Multi-Armed Bandits

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December 10, 2012

Abstract

We study the exploratory Multi-Armed Bandit (MAB) problem in a setting where k players collaborate in order to identify an ε -optimal arm. Our motivation comes from recent employment of MAB models in large-scale, distributed web applications. Our results demonstrate a trade-off between the number of arm pulls required for the task, and the amount of communication between the players. In particular, our main result shows that by communicating only once, the k players are able to learn using only \sqrt{k} times more arm pulls than required in the single player setting. We complement this result with a lower bound showing this is in general the best possible. On the other extreme, we present an algorithm that uses the same amount of arm pulls as needed for a single player with communication logarithmic in $1/\varepsilon$, as well as an algorithm that explicitly tradeoffs arm pulls with communication.

1 Introduction

Over the past few years, reinforcement learning in general and multi-armed bandit (MAB) algorithms in particular have been employed in an increasing amount of web applications. MAB algorithms choose between stories to showcase on media sites (Agarwal et al., 2009, 2008), among ranking algorithms of search results (Moon et al., 2010, Radlinski et al., 2008, Yue and Joachims, 2009), trigger ads (Chakrabarti et al., 2008), and more. In many of these applications, websites cannot be served from a single front-end machine. The sheer volume of user requests they face, along with the geographical diversity those requests come from, require sites to use multiple front-end machines that are often dispersed geographically among multiple data centers.

In this work we assume that a single machine cannot accommodate the throughput of requests; rather, k different servers are needed. We will refer to these as *players*. Each player is responsible for deciding which arm to pull for its share of requests, and for updating its model once rewards can be inferred from user interactions. This paper studies

the tradeoffs between the communication among the k players, and their overall learning performance. At one extreme, if all players broadcast to each other each and every (pull, reward) pair as they happen, they can each simulate the decisions of a centralized algorithm. However, the load on each player would then be similar to the load of a centralized model, which we assumed to be prohibitive. At the other extreme, if the players never communicate, each will suffer the learning curve (i.e. number of pulls) required in order to attain good performance, thereby multiplying the cost of learning by k . We aim to quantify this tradeoff between inter-player communication and the cost of learning.

Following Audibert et al. (2010), Even-Dar et al. (2006), we focus on the problem of identifying a “good” arm with few arm pulls as possible (that is, minimizing the *simple regret*). Our objective is thus purely explorative, and specifically we do not care about the incurred costs or the involved regret. As discussed by Bubeck et al. (2009), Audibert et al. (2010), this is indeed the natural goal in many situations. In this work, we build up upon the results known in the single-player exploratory setup, and in particular the Successive Elimination algorithm (Even-Dar et al., 2006), showing how they can be used in a multi-player setting in a communication-efficient way.

Our setting is accordingly as follows. The players aim to identify an ε -best (i.e., ε -optimal) bandit arm with high confidence. They are evaluated by the number of total arm pulls, aggregated across all k players, required for them to do so. The arm pulls are evenly divided among the k players, and are assumed to take place synchronously. In order to reach their common goal, players may communicate with each other. Following Balcan et al. (2012), we measure communication by counting the number of *communication rounds* required by them in a protocol, where in a single round of communication each player broadcasts a single message to all the other players.

Our main results deal with the case where players are allowed to communicate only *once*. For this case, we present k -player algorithms that need only \sqrt{k} times more arm pulls for identifying the best and ε -best bandit arm, as compared to the standard, single-player setting. A single round of communication thus leads to an improvement of \sqrt{k} factor in the number of required pulls over a naïve, zero communication approach. We complement our algorithmic results for the single round case with a lower bound on the required number of pulls, implying that our results are essentially optimal.

In addition, we investigate how many communication rounds are required for competing with the learning performance of a single-player algorithm. We show a k -player algorithm achieving the *same* performance as a centralized algorithm with communication depending only logarithmically on $1/\varepsilon$. Interestingly, Balcan et al. (2012) established a similar result in a distributed PAC setting using boosting algorithms. Our algorithm achieves the same logarithmic dependence of communication on $1/\varepsilon$ for the multi-player MAB problem, while enjoying the optimal, distribution-dependent sample complexity upper bound.

1.1 Related Work

The classic (single-player) Multi-Armed Bandits problem was first considered under the PAC model by [Even-Dar et al. \(2002\)](#), who proposed the *Successive Elimination* (SE) strategy for ε -optimal arm identification which inspires large parts of our work. [Mannor and Tsitsiklis \(2004\)](#) provided tight, distribution-dependent sample complexity lower bounds for several variants of the PAC model, showing that the SE algorithm is essentially optimal in terms of its sample complexity. The setting was revisited by [Bubeck et al. \(2009\)](#), that investigated the links between the simple and cumulative regret measures and indicated that regret-minimizing algorithms (like UCB of [Auer et al. 2002](#)) are not well-suited for pure exploration tasks. More recently, [Audibert et al. \(2010\)](#) presented algorithms for best-arm identification, proved their optimality and demonstrated their effectiveness in simulations.

In a recent work, [Gabillon et al. \(2011\)](#) consider a multi-player (termed “multi-bandit”) MAB problem similar to ours, in which each player has to identify the best arm. However, in their setting each player gets a different set of arms (with its own rewards) and the challenge is to distribute the limited resources at hand (namely T arm pulls) so as to maximize the confidence of all players simultaneously. In our setting, each player gets access to the same set of arms, with the joint goal of identifying the (ε -)best arm while communicating with each other as few times as possible. Another variant of a multi-player MAB problem is studied by [Liu and Zhao \(2010\)](#). In their setup, where the goal is to minimize the cumulative regret, players do not communicate but rather compete with each other: once two players pull the same arm at the same time, a “collision” occurs and they share the reward in some way. In contrast, we study how sharing observations helps players achieve their goal jointly.

A related setting was considered by [Balcan et al. \(2012\)](#), who considered a distributed PAC model and investigated the involved communication trade-offs. In their model, data is distributed between the various players and the goal is to learn well with respect to the overall (i.e. mixture) distribution of the data. While their results do not have direct implication in our setting, the main theme is similar.

1.2 Organization

The rest of this paper is organized as follows. The problem setup is formally defined in Section 2. Section 3 presents an upper bound and a matching lower bound for the required number of arm pulls, as a function of the number of players, when the players are allowed a single round of communication. Section 4 extends the discussion to multiple rounds of communication. We conclude in Section 5.

2 Problem Setup and Background

2.1 Model and Notation

Our basic model is a generalization of the classic Multi-Armed Bandit (MAB) model. In the Multi-Player Multi-Armed Bandit setting, there are $k \geq 1$ individual players. The players are given n arms, enumerated by $[n] := \{1, 2, \dots, n\}$. Each arm $i \in [n]$ is associated with a reward, which is a binary random variable with expectation $p_i \in [0, 1]$. For convenience, we assume that the arms are ordered by their expected rewards, that is $p_1 \geq p_2 \geq \dots \geq p_n$. At every *time step* $t = 1, 2, \dots, T$, each player may pull one arm of his choice and observe an independent sample of its reward. Each player may choose any of the arms, regardless of the other players and their actions. At the end of the game, each player must commit to a single arm.

In the *best-arm identification* version of the problem, the goal of a multi-player algorithm given some target confidence level $\delta > 0$, is that with probability at least $1 - \delta$ all players correctly identify the best arm (i.e. the arm having the maximal expected reward). For simplicity, we assume in this setting that the best arm is unique. Similarly, in the (ε, δ) -PAC variant the goal is that each player finds an ε -optimal (or “ ε -best”) arm, that is an arm i with $p_i \geq p_1 - \varepsilon$, with high probability. In this paper we focus on the more general (ε, δ) -PAC setup, which also includes best-arm identification for $\varepsilon = 0$. We use the term *sample complexity* to refer to the total number of arm pulls needed for achieving the players’ goal (at the desired confidence).

During a *communication round*, each player may broadcast a message to all other players. A round may take place at any predefined time step, and assumed not to cause any delay to the players, nor consume any time steps. While we do not restrict the size (in bits) of each message, in a reasonable implementation a message should not be larger than $O(n)$ bits.

We introduce the notation $\Delta_i := p_1 - p_i$ to denote the suboptimality gap of arm i , and occasionally use $\Delta^* := \Delta_2$ for denoting the minimal gap. In the best-arm version of the problem, where we assume that the best arm is unique, we have $\Delta_i > 0$ for all i . In this work, we are interested in deriving distribution-dependent bounds over the sample complexity of algorithms. Namely, our bounds are stated as a function of ε, δ and also the distribution-specific values $\Delta := \Delta_2, \dots, \Delta_n$ ¹. Note, however, that our algorithms do *not* require any distribution-specific parameters.

The \tilde{O} notation in our sample complexity bounds hides poly-logarithmic factors in $n, k, \varepsilon, \delta$ and $\Delta_2, \dots, \Delta_n$.

Clearly, a multi-player problem with $k = 1$ players is just the standard exploratory MAB problem. When $k > 1$, the problem becomes more challenging as we constrain the

¹If one is interested in distribution-free bounds, then the problem at hand is trivial as the (one-round) uniform sampling strategy is optimal in this setting (up to poly-logarithmic factors); see also Mannor and Tsitsiklis (2004) for a discussion.

communication between the players. In the rest of the section we review relevant results known in the single-player case, and give an overview of our results for the k -player MAB setting.

2.2 Pure Exploration in Multi-Armed Bandits

The classic MAB problem was first considered under the PAC model by [Even-Dar et al. \(2002\)](#), who proposed the *Successive Elimination* (SE) strategy for ε -best arm identification. This state-of-the-art algorithm, as the more recent *Successive Rejects* algorithm ([Audibert et al., 2010](#)), is based on sequential elimination of suboptimal arms. The algorithm works in phases, each of which consists of uniform sampling of the remaining arms, followed by some elimination of the “worst” arms; see Algorithm 3 of [Even-Dar et al. \(2006\)](#). The SE algorithm enjoys the following sample complexity upper bound.

Theorem 2.1 ([Even-Dar et al. 2006](#), Theorem 8). *With probability at least $1 - \delta$, the Successive Elimination algorithm needs at most*

$$O\left(\sum_{i=2}^n \frac{1}{(\Delta_i^\varepsilon)^2} \log \frac{n}{\delta \Delta_i^\varepsilon}\right) \quad (1)$$

arm pulls for terminating and identifying an ε -best arm, where $\Delta_i^\varepsilon := \max\{\Delta_i, \varepsilon\}$.

The above result is essentially tight (up to poly-logarithmic factors), as implied by Theorem 5 of [Mannor and Tsitsiklis \(2004\)](#). Intuitively, the hardness of the task is therefore captured by the quantity

$$H(\Delta, \varepsilon) := \sum_{i=2}^n \frac{1}{(\Delta_i^\varepsilon)^2} \quad (\text{with } \Delta_i^\varepsilon = \max\{\Delta_i, \varepsilon\}), \quad (2)$$

which is roughly the number of arm pulls needed to find an ε -best arm with a reasonable probability; see also [Audibert et al. \(2010\)](#) for a discussion.

2.3 Summary of Our Results

Our results are summarized in Table 1 below, which compares the different algorithms on both the sample complexity and communication fronts. As a baseline for evaluating our results, we consider two trivial methods. In the first (“1P baseline”), the k -players simulate the centralized SE strategy with confidence δ , thus communicating upon each time step. In the second approach (“ k P naïve”) each player independently executes SE with target confidence δ/k , which ensures that with probability $1 - \delta$ all players find an ε -best arm but pays an $\tilde{O}(k)$ factor in the sample complexity.

Our first algorithmic result (Algorithm 1) deals with the case where only one round of communication is allowed, and shows an $\tilde{O}(\sqrt{k} H(\Delta, 0))$ upper bound on the number of

Algorithm	Sample Complexity	Communication
1P baseline	$\tilde{O}(H(\Delta, \varepsilon))$	every time step
k P naïve	$\tilde{O}(k H(\Delta, \varepsilon))$	none
Algorithms 1,2	$\tilde{O}(\sqrt{k} H(\Delta, \varepsilon))$	1 round
Algorithm 3	$\tilde{O}(H(\Delta, \varepsilon))$	$O(\log \frac{1}{\varepsilon})$ rounds
Algorithm 3'	$\tilde{O}(\varepsilon^{-2/r} H(\Delta, \varepsilon))$	r rounds

Table 1: Summary of our results.

pulls required for best-arm identification, thus improving upon the naïve approach by a \sqrt{k} factor. We then extend this result to the (ε, δ) -PAC setup (in Algorithm 2), establishing an $\tilde{O}(\sqrt{k} H(\Delta, \varepsilon))$ upper bound for the one-round case. We complement these results with a lower bound, showing that in general it is impossible to lower the \sqrt{k} gap between the sample complexity of single-player and one-round k -player algorithms.

We then relax the communication constraint and allow multiple communication rounds to take place. We show that k players can reproduce the optimal $\tilde{O}(H(\Delta, \varepsilon))$ bound achievable in the single-player setup, using only $\log_2(1/\varepsilon) + O(1)$ rounds. A rather simple modification of this result yields an algorithm that uses at most r rounds and $\tilde{O}(\varepsilon^{-2/r} H(\Delta, \varepsilon))$ arm pulls overall, for any integer r .

3 One Communication Round

This section considers the most basic variant of the multi-player MAB problem, where only one round of communication is allowed at the end of the game. Here, given a budget of allowed arm pulls, each player decides which arms to pull according to a policy based solely on his results. After exhausting the given budget, the players share their results and compute the output. Section 3.1 presents an algorithm for best-arm identification under this setting. Section 3.2 deals with the (ε, δ) -PAC setup. We demonstrate the tightness of our result in Section 3.3 with a lower bound for the required budget of arm pulls in the one round setting.

3.1 Best-Arm Identification Algorithm

We now describe a one-round, multi-player MAB algorithm that needs only a $\tilde{O}(\sqrt{k})$ times more arm pulls than a single-player algorithm, thus improving upon the trivial $\tilde{O}(k)$ factor of the naïve approach. Our algorithm is akin to a majority vote among the multiple players, where each player pulls arms in two stages. In the first stage, each player independently solves a “smaller” MAB instance on a random subset of the arms using Successive Elim-

ination². In the second stage, each player exploits the arm identified as best in the first stage, and communicates that arm and its observed reward. See Algorithm 1 below for a precise description. An appealing feature of our algorithm is that it requires each player to communicate merely 2 numerical values to the other players.

Although our goal is pure exploration, each player in the algorithm follows an explore-exploit strategy. The idea here is that a player should sample his recommended arm as much as his budget permits, even if it was easy to identify in his small-sized problem (i.e., did not require many pulls). This way we can guarantee that the top arms are sampled sufficiently-many times by the time the players have to agree on a single best arm.

Algorithm 1 ONE-ROUND BEST ARM

Input: time horizon T

Output: an arm

- 1: **for** player $j = 1$ to k **do**
 - 2: choose a subset A_j of $6n/\sqrt{k}$ arms uniformly at random
 - 3: **explore:** execute $i_j \leftarrow \text{SE}(A_j, 0, \frac{1}{3})$ using at most $\frac{1}{2}T$ pulls (and halting the algorithm early if necessary); if the algorithm fails to identify any arm or does not terminate gracefully, let i_j be an arbitrary arm
 - 4: **exploit:** pull arm i_j for $\frac{1}{2}T$ times and let \hat{q}_j be its average reward
 - 5: communicate the numbers i_j, \hat{q}_j
 - 6: **end for**
 - 7: let k_i be the number of players j with $i_j = i$, and define $A = \{i : k_i > \sqrt{k}\}$
 - 8: let $\hat{p}_i = (1/k_i) \sum_{\{j: i_j=i\}} \hat{q}_j$ for all i
 - 9: output $\arg \max_{i \in A} \hat{p}_i$; if the set A is empty, return an arbitrary arm.
-

In Theorem 3.1 we prove that Algorithm 1 indeed achieves the promised upper bound.

Theorem 3.1. *Algorithm 1 identifies the best arm correctly with probability at least $2/3$ using no more than*

$$O \left(\sqrt{k} \cdot \sum_{i \neq i^*} \frac{1}{\Delta_i^2} \log \frac{n}{\Delta_i} \right) \quad (3)$$

arm pulls, provided that $6 \leq \sqrt{k} \leq n$. The algorithm uses a single communication round, in which each player communicates 2 numerical values.

By repeating the algorithm $O(\log(1/\delta))$ times and taking the majority vote of the independent instances, we can amplify the success probability to $1 - \delta$ for any given $\delta > 0$.

²We note that any best-arm identification strategy can be used as a building-block in our algorithm, and the choice of Successive Elimination here is rather arbitrary.

Note that we can still do that with one communication round (at the end of all executions), but each player now has to communicate $O(\log(1/\delta))$ values³.

Theorem 3.2. *There exists a k -player algorithm that given*

$$O\left(\sqrt{k} \cdot \sum_{i \neq i^*} \frac{1}{\Delta_i^2} \log \frac{n}{\Delta_i} \log \frac{1}{\delta}\right)$$

arm pulls, identifies the best arm correctly with probability at least $1 - \delta$. The algorithm uses a single communication round, in which each player communicates $O(\log(1/\delta))$ numbers.

3.1.1 Analysis

We will show that a budget T of samples (arm pulls) per player, where

$$T \geq \frac{24c}{\sqrt{k}} \cdot \sum_{i \neq i^*} \frac{1}{\Delta_i^2} \log \frac{3n}{\Delta_i} \quad (4)$$

suffices for the players to jointly identify the best arm i^* with the desired probability. Here, c denotes the constant under the big O notation in eq. (1), and without loss of generality we assume that $c \geq 1$. Clearly, this would imply the sample complexity stated in Theorem 3.1.

We begin by analyzing the **explore** phase of the algorithm. Our first lemma shows that each player chooses the global best arm and identifies it as the local best arm with sufficiently large probability.

Lemma 3.3. *When (4) holds, each player identifies the (global) best arm correctly after the **explore** phase with probability at least $2/\sqrt{k}$.*

Proof. Let $H = \sum_{i \neq i^*} (1/\Delta_i^2) \log(3n/\Delta_i)$ and $H_j = \sum_{i^* \neq i \in A_j} (1/\Delta_i^2) \log(3n/\Delta_i)$ for all j . Then $\mathbf{E}[H_j \mid i^* \in A_j] \leq 6H/\sqrt{k}$ by the linearity of expectation, and Markov's inequality thus gives that $\Pr[H_j \leq 12H/\sqrt{k} \mid i^* \in A_j] \geq 1/2$. Clearly, we also have $\Pr[i^* \in A_j] = 6/\sqrt{k}$ which implies that

$$\Pr[i^* \in A_j \text{ and } H_j \leq 12H/\sqrt{k}] \geq 3/\sqrt{k}. \quad (5)$$

Now consider the “local” MAB problem facing player j , over the subset of arms A_j . If the (global) best arm i^* is amongst the arms in A_j , then by Theorem 2.1 the **SE** algorithm player j executes needs no more than

$$T_j := c \sum_{i^* \neq i \in A_j} \frac{1}{\Delta_i^2} \log \frac{3n}{\Delta_i} = cH_j$$

³In fact, by letting each player pick a bit larger subset of $O(\sqrt{\log(1/\delta)} \cdot n/\sqrt{k})$ arms, we can amplify the success probability to $1 - \delta$ without needing to communicate more than 2 values per player. However, this approach only works when $k = \Omega(\log(1/\delta))$.

pulls in order to identify i^* successfully with probability $2/3$. In case that $H_j \leq 12H/\sqrt{k}$, we have $T_j \leq 12cH/\sqrt{k} \leq T/2$, which means that the pulls budget of player j suffices for identifying the best arm. Together with (5), we conclude that with probability at least $2/\sqrt{k}$ player j identifies the best arm correctly. \square

We next address the **exploit** phase. The next simple lemma shows that the popular arms (i.e. those selected by many players) are estimated to a sufficient precision.

Lemma 3.4. *Provided that (4) holds, with probability at least $5/6$ we have $|\hat{p}_i - p_i| \leq \frac{1}{2}\Delta_*$ for all arms $i \in A$.*

Proof. Consider some arm $i \in A$. The estimate \hat{p}_i is the average reward of $k_i T/2 \geq \sqrt{k}T/2 \geq (2/\Delta_*^2) \log(12n)$ arm pulls (of the **exploit** phase). Hoeffding's inequality now gives that

$$\Pr[|\hat{p}_i - p_i| > \frac{1}{2}\Delta_*] \leq 2 \exp(-\frac{1}{2}\Delta_*^2 \cdot k_i T/2) \leq 1/6n,$$

and the lemma follows via a union bound. \square

We can now prove Theorem 3.1.

Proof. Let us first show that with probability at least $5/6$, the best arm i is contained in the set A . To this end, notice that k_{i^*} is the sum of k i.i.d. Bernoulli random variables $\{X_j\}_{j \in [k]}$ where X_j is the indicator of whether player j chooses arm i^* after the **explore** phase. By Lemma 3.3 we have that $\mathbf{E}[X_j] \geq 2/\sqrt{k}$ for all j , hence by Hoeffding's inequality,

$$\Pr[k_{i^*} \leq \sqrt{k}] \leq \Pr\left[\frac{1}{k} \sum_{j=1}^k (X_j - \mathbf{E}[X_j]) \leq \frac{-1}{\sqrt{k}}\right] \leq \exp(-2k/k) \leq 1/6$$

which implies that $i^* \in A$ with probability at least $5/6$.

Next, note that with probability at least $5/6$ the arm $i \in A$ having the highest empirical reward \hat{p}_i is the one with the highest expected reward p_i . Indeed, this follows directly from Lemma 3.4 that shows that with probability at least $5/6$, for all arms $i \in A$ the estimate \hat{p}_i is within $\frac{1}{2}\Delta$ of the true bias p_i .

Hence, via a union bound we conclude that with probability at least $2/3$, the best arm is in A and has the highest empirical reward. In other words, with probability at least $2/3$ the algorithm outputs the best arm i^* . \square

3.2 (ε, δ) -PAC Algorithm

We now present Algorithm 2 whose purpose is to recover an ε -optimal arm. The algorithm is similar to Algorithm 1 of the previous section, but now each player attempts to find an $O(\varepsilon)$ -best arm in a subsampled problem. Note, however, that there may be more than

one ε -best arm, so each “successful” player might come up with a different ε -best arm. Nevertheless, our analysis shows that – with high probability – a subset of the players can still arrive to a consensus on a single ε -best arm.

In Theorem 3.5 we prove that our algorithm needs only $\tilde{O}(\sqrt{k})$ times more arm pulls than a single-player algorithm for identifying an $O(\varepsilon)$ -best arm.

Algorithm 2 ONE-ROUND ε -ARM

Input: time horizon T , accuracy ε

Output: an arm

- 1: **for** player $j = 1$ to k **do**
 - 2: choose a subset A_j of $12n/\sqrt{k}$ arms uniformly at random
 - 3: **explore:** execute $i_j \leftarrow \text{SE}(A_j, \varepsilon, \frac{1}{3})$ using at most $\frac{1}{2}T$ pulls (and halting the algorithm early if necessary); if the algorithm fails to identify any arm or does not terminate gracefully, let i_j be an arbitrary arm
 - 4: **exploit:** pull arm i_j for $\frac{1}{2}T$ times, and let \hat{q}_j be the average reward
 - 5: communicate the numbers i_j, \hat{q}_j
 - 6: **end for**
 - 7: let k_i be the number of players j with $i_j = i$
 - 8: let $t_i = \frac{1}{2}k_iT$ and $\hat{p}_i = (1/k_i) \sum_{\{j: i_j=i\}} \hat{q}_j$ for all i
 - 9: define $A = \{i \in [n] : t_i \geq (4/\varepsilon^2) \log(12n)\}$
 - 10: output $\arg \max_{i \in A} \hat{p}_i$; if the set A is empty, return an arbitrary arm.
-

Theorem 3.5. *Algorithm 2 identifies a 2ε -best arm with probability at least $2/3$ using no more than*

$$O\left(\sqrt{k} \cdot \sum_{i \neq i^*} \frac{1}{(\Delta_i^\varepsilon)^2} \log \frac{n}{\Delta_i^\varepsilon}\right) \quad (6)$$

arm pulls, provided that $24 \leq \sqrt{k} \leq n$. The algorithm uses a single communication round, in which each player communicates 2 numerical values.

3.2.1 Analysis

Before proving the theorem, we first state several key lemmas. In the following, let n_ε and $n_{2\varepsilon}$ denote the number of ε -best and 2ε -best arms respectively. Our analysis considers two different regimes: $n_{2\varepsilon} \leq \frac{1}{50}\sqrt{k}$ and $n_{2\varepsilon} > \frac{1}{50}\sqrt{k}$, and shows that in any case,

$$T \geq \frac{400c}{\sqrt{k}} \sum_{i \neq i^*} \frac{1}{(\Delta_i^\varepsilon)^2} \log \frac{24n}{\Delta_i^\varepsilon} \quad (7)$$

suffices for identifying a 2ε -best arm with the desired probability. Again, we use c to denote the constant under the big O notation in eq. (1) and assume that $c \geq 1$. Clearly, this implies the sample complexity stated in eq. (6).

The first lemma shows that at least one of the players is able to find an ε -best arm. As we later show, this is sufficient for the success of the algorithm in case there are many 2ε -best arms.

Lemma 3.6. *When (7) holds, at least one player successfully identifies an ε -best arm in the **explore** phase, with probability at least $5/6$.*

The next lemma is more refined and states that in case there are few 2ε -best arms, the probability of each player to successfully identify an ε -best arm grows linearly with n_ε .

Lemma 3.7. *Assume that $n_{2\varepsilon} \leq \frac{1}{50}\sqrt{k}$. When (7) holds, each player identifies an ε -best arm in the **explore** phase, with probability at least $2n_\varepsilon/\sqrt{k}$.*

The last lemma we need analyzes the accuracy of the estimated rewards of arms in the set A .

Lemma 3.8. *With probability at least $5/6$, we have $|\hat{p}_i - p_i| \leq \varepsilon/2$ for all arms $i \in A$.*

We now turn to prove Theorem 3.5.

Proof. We shall prove that with probability $5/6$ the set A contains at least one ε -best arm. This would complete the proof, since Lemma 3.8 assures that with probability $5/6$, the estimates \hat{p}_i of all arms $i \in A$ are at most $\varepsilon/2$ -away from the true reward p_i , and in turn implies (via a union bound) that with probability $2/3$ the arm $i \in A$ having the maximal empirical reward \hat{p}_i must be a 2ε -best arm.

First, consider the case $n_{2\varepsilon} > \frac{1}{50}\sqrt{k}$. Lemma 3.6 shows that with probability $5/6$ there exists a player j that identifies an ε -best arm i_j . Since for at least $n_{2\varepsilon}$ arms $\Delta_i \leq 2\varepsilon$, we have

$$t_{i_j} \geq \frac{1}{2}T \geq \frac{400}{2\sqrt{k}} \cdot \frac{n_{2\varepsilon} - 1}{(2\varepsilon)^2} \log \frac{24n}{2\varepsilon} \geq \frac{1}{\varepsilon^2} \log(12n),$$

that is, $i_j \in A$.

Next, consider the case $n_{2\varepsilon} \leq \frac{1}{50}\sqrt{k}$. Let N denote the number of players that identified some ε -best arm. The random variable N is a sum of Bernoulli random variables $\{N_j\}_j$ where N_j is an indicator to whether player j identified some ε -best arm. By Lemma 3.7, $\mathbf{E}[N_j] \geq 2n_\varepsilon/\sqrt{k}$ and thus by Hoeffding's inequality,

$$\Pr \left[N < n_\varepsilon \sqrt{k} \right] = \Pr \left[\frac{1}{k} \sum_{j=1}^k (N_j - \mathbf{E}[N_j]) \leq -\frac{n_\varepsilon}{\sqrt{k}} \right] \leq \exp(-2n_\varepsilon^2) \leq \frac{1}{6}.$$

That is, with probability $5/6$, at least $n_\varepsilon \sqrt{k}$ players found an ε -best arm. A pigeon-hole argument shows that in this case there exists an ε -best arm i^* selected by at least \sqrt{k} players. Hence, with probability $5/6$ the number of samples of this arm collected in the **exploit** phase is at least $t_{i^*} \geq \sqrt{k}T/2 > (1/\varepsilon^2) \log(12n)$, which means that $i^* \in A$. \square

3.2.2 Proofs of Lemmas

For the proofs in this section, we need some additional notation. For any player j , let i_j^* denote the best arm in A_j , with ties broken arbitrarily. Let $\Delta_j^* := \Delta_{i_j^*} = \min_{i \in A_j} \Delta_i$ and $\Delta_{i,j} := \max\{\Delta_i - \Delta_j^*, \varepsilon\}$ for all $i \in A_j$. Finally, define

$$H := \sum_{i^* \neq i \in [n]} \frac{1}{(\Delta_i^{2\varepsilon})^2} \log \frac{6n}{\Delta_i^{2\varepsilon}}$$

and

$$H_j^{\text{local}} := \sum_{i_j^* \neq i \in A_j} \frac{1}{\Delta_{i,j}^2} \log \frac{3n}{\Delta_{i,j}^2}, \quad H_j^{\text{global}} := \sum_{i_j^* \neq i \in A_j} \frac{1}{(\Delta_i^{2\varepsilon})^2} \log \frac{6n}{\Delta_i^{2\varepsilon}}$$

for all players j .

Proof (of Lemma 3.6). The proof is analogical to that of Lemma 3.3. Let i^* be the best arm and let j be a player that chose it. The funds required by player j in order to succeed choosing an ε -best arm with probability at least $2/3$ is at most $H_j := \sum_{i^* \neq i \in A_j} (1/\Delta_i^\varepsilon)^2 \log(6n/\Delta_i^\varepsilon)$. Given Eq. (7), $\mathbf{E}[H_j] \leq T/4$, meaning that by Markov $\Pr[H_j \leq T/2] \geq 1/2$. Since the probability of choosing the best arm is $12/\sqrt{k}$ it follows that for any fixed player j , with probability at least $\frac{2}{3} \cdot \frac{1}{2} \cdot 12/\sqrt{k} = 4/\sqrt{k}$ the player identified an ε -best arm. Thus, the probability that all of the players fail to identify an ε -best arm is bounded by

$$\left(1 - \frac{4}{\sqrt{k}}\right)^k \leq e^{-4\sqrt{k}} < 1/6,$$

and the lemma follows. \square

Proof (of Lemma 3.7). For convenience, let $\alpha = 12/\sqrt{k}$ and note that by our assumptions $\alpha \leq \frac{1}{2}$. Also, since we assume $\sqrt{k} \leq n$ we have $n_{2\varepsilon} \leq \frac{1}{2}n$.

Fix some player j and let X_ε and $X_{2\varepsilon}$ denote the number of ε -best and 2ε -best arms chosen by this player, respectively. Consider the event $B = \{X_\varepsilon = X_{2\varepsilon} = 1\}$ in which the player chooses *exactly one* ε -optimal arm but no other 2ε -optimal arm. The probability

that this event occurs is

$$\begin{aligned}
\Pr[B] &= \frac{n_\varepsilon \cdot \binom{n-n_{2\varepsilon}}{\alpha n-1}}{\binom{n}{\alpha n}} = \alpha n_\varepsilon \frac{\binom{n-n_{2\varepsilon}}{\alpha n-1}}{\binom{n-1}{\alpha n-1}} \\
&\geq \alpha n_\varepsilon \left(\frac{n-n_{2\varepsilon}-\alpha n}{n-\alpha n} \right)^{\alpha n} = \alpha n_\varepsilon \left(1 - \frac{n_{2\varepsilon}}{n-\alpha n} \right)^{\alpha n} \\
&\geq \alpha n_\varepsilon \left(1 - \frac{2n_{2\varepsilon}}{n} \right)^{\alpha n} && (\text{since } \alpha \leq \frac{1}{2}) \\
&\geq \alpha n_\varepsilon (1 - 2\alpha n_{2\varepsilon}) && ((1-x)^a \geq 1-ax \text{ for } x \leq 1) \\
&\geq \frac{1}{2} \alpha n_\varepsilon = 6n_\varepsilon / \sqrt{k}. && (\text{since } \alpha n_{2\varepsilon} \leq \frac{1}{4})
\end{aligned}$$

On the other hand, given the event B , for player j we have $\Delta_j^* \leq \varepsilon$ and $\Delta_i \geq 2\varepsilon$ for all $i \neq i_j^*$. Consequently, $\Delta_{i,j} \geq \frac{1}{2} \Delta_i = \frac{1}{2} \Delta_i^{2\varepsilon}$ for all $i \in A_j$. Hence,

$$H_j^{\text{local}} = \sum_{i_j^* \neq i \in A_j} \frac{1}{\Delta_{i,j}^2} \log \frac{3n}{\Delta_{i,j}} \leq 4 \sum_{i_j^* \neq i \in A_j} \frac{1}{(\Delta_i^{2\varepsilon})^2} \log \frac{6n}{\Delta_i^{2\varepsilon}} = 4 H_j^{\text{global}}$$

Denoting $\Delta_{>2\varepsilon} := \{i : \Delta_i > 2\varepsilon\}$, we now have

$$\begin{aligned}
\mathbf{E}[H_j^{\text{local}} \mid B] &\leq 4 \mathbf{E}[H_j^{\text{global}} \mid B] \leq 4 \frac{12n}{\sqrt{k}} \cdot \frac{1}{|\Delta_{>2\varepsilon}|} \sum_{i \in \Delta_{>2\varepsilon}} \frac{1}{\Delta_i^2} \log \frac{6n}{\Delta_i} \\
&< \frac{50n}{\sqrt{k}} \cdot \frac{1}{n} \sum_{i \neq i^*} \frac{1}{(\Delta_i^{2\varepsilon})^2} \log \frac{6n}{\Delta_i^{2\varepsilon}} = \frac{50}{\sqrt{k}} H
\end{aligned}$$

Markov's inequality now gives $\Pr[H_j^{\text{local}} \leq 100H/\sqrt{k} \mid B] \geq \frac{1}{2}$, and together with $\Pr[B] \geq 6n_\varepsilon/\sqrt{k}$ we obtain that

$$\Pr[B \text{ and } H_j^{\text{local}} \leq 100H/\sqrt{k}] \geq 3n_\varepsilon/\sqrt{k}.$$

Continuing as in the proof of Lemma 3.3, we get that when (7) holds, with probability at least $2n_\varepsilon/\sqrt{k}$ (i) A_j contains an ε -best arm which is the only 2ε -best arm in A_j , and (ii) player j successfully identifies an arm which is ε -best with respect to the best arm in A_j . This implies that with probability at least $2n_\varepsilon/\sqrt{k}$, the arm i_j selected by player j is ε -best. \square

Proof (of Lemma 3.8). Since each estimate \hat{p}_i is the empirical average of at least t_i samples of arm i , Hoeffding's inequality gives

$$\Pr[|\hat{p}_i - p_i| > \varepsilon/2] \leq 2 \exp(-\varepsilon^2 t_i/2) \leq 1/6n,$$

and a union bound concludes the proof. \square

3.3 Lower Bound

The following theorem suggests that in general, for identifying the best arm k players need at least $\tilde{\Omega}(\sqrt{k})$ times the budget required for a single player to do so, when only allowed one round of communication (at the end of the game). Clearly, this also implies that a similar lower bound holds in the PAC setup, and proves that our algorithmic results for the one-round case are essentially tight.

Theorem 3.9. *there exist rewards p_1, \dots, p_n and integer T such that*

- *each player in the algorithm must use at least T/\sqrt{k} arm pulls for identifying the best arm with probability at least $2/3$;*
- *there exist a single-player algorithm that needs at most $\tilde{O}(T)$ pulls for identifying the best arm with probability at least $2/3$.*

Our proof for the theorem is based on a simple lower bound for the MAB problem, that follows directly from Lemma 5.1 of [Anthony and Bartlett \(1999\)](#).

Lemma 3.10. *Consider a MAB problem with two arms and rewards $\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon$. There exists a constant c_0 such that any algorithm that with probability at least $2/3$ identifies the best arm, needs at least c_0/ε^2 pulls in expectation.*

Proof (of Theorem 3.9). Let $c_1 = \sqrt{c_0}/2$, where c_0 is the constant of Lemma 3.10. Consider a MAB instance over n arms, with the rewards being a random permutation σ of $\frac{1}{2} + \Delta, \frac{1}{2} - \Delta, 0, \dots, 0$, where $\Delta := 1/\sqrt{n}$. By Theorem 2.1, the Successive Elimination algorithm is able to identify the best arm in this setting with probability $2/3$ using at most $\tilde{O}(n + \Delta^2) = \tilde{O}(n)$ arm pulls.

Assume that the players follow some algorithm, each using no more than $c_1 n / \sqrt{k}$ pulls. Without loss of generality, we may assume that the sequence of rewards, as well as the internal random bits of the algorithm (if it is randomized), were drawn before the execution has started. We shall denote this sequence of random variables by h .

First, fix some arbitrary sequence h . Let $T_{i,j}$ denote the number of times arm i (in decreasing order of rewards) was pulled by player j , and T_i denote the total number of pulls of arm i . Since the budget of each player is $c_1 n / \sqrt{k}$, we have $\Pr_\sigma[T_{1,j} > 0] \leq c_1 / \sqrt{k}$ and consequently $\mathbf{E}_\sigma[T_{1,j}] \leq \mathbf{E}_\sigma[T_{1,j} \mid T_{1,j} > 0] \cdot \Pr[T_{1,j} > 0] \leq c_1^2 n / k$ for any player j . Similarly, $\mathbf{E}_\sigma[T_{2,j}] \leq c_1^2 n / k$ and we get that $\mathbf{E}_\sigma[T_1 + T_2] \leq 2c_1^2 n = 2c_1^2 / \Delta^2 < c_0 / \Delta^2$.

Since the above holds for any given sequence h , this implies that $\mathbf{E}_\sigma[\mathbf{E}_h[T_1 + T_2]] < c_0 / \Delta^2$, which means that there exists a permutation σ_0 under which $\mathbf{E}_h[T_1 + T_2] < c_0 / \Delta^2$. For the permutation σ_0 and its corresponding reward setting, the algorithm does not sample the top two arms enough (in expectation), and by Lemma 3.10 it cannot succeed with probability $2/3$. \square

4 Multiple Communication Rounds

In this section we establish the general dependence of communication on $1/\varepsilon$, and show a tradeoff between the sample complexity of a multi-player algorithm and the number of communication rounds it uses, in terms of $1/\varepsilon$.

4.1 $O(\log(1/\varepsilon))$ Rounds Algorithm

We first show that by allowing $O(\log(1/\varepsilon))$ rounds of communication, we are able to attain the optimal sample complexity as in the single-player scenario. That is, we do not gain any improvement in learning performance from allowing more than $O(\log(1/\varepsilon))$ rounds.

Our algorithm is based on a variant of Successive Elimination. The idea is to eliminate in each round r (i.e., right after the r th communication round) all 2^{-r} -suboptimal arms. We accomplish that by letting each player sample uniformly all remaining arms and communicate the results to other players. Then, players are able to eliminate suboptimal arms with high confidence. If each such round is successful, after $\log_2(1/\varepsilon)$ rounds only the best arm survives. The algorithm is given in detail as Algorithm 3 below, and in Theorem 4.1 we state its sample complexity guarantee.

Algorithm 3 MULTI-PLAYER SUCCESSIVE ELIMINATION

Input: (ε, δ)

Output: an arm

- 1: define $\varepsilon_r = 2^{-r}$
 - 2: let $t_0 = 0$ and $t_r = (2/k\varepsilon_r^2) \log(4nr^2/\delta)$ for $r \geq 1$
 - 3: initialize $S_0 \leftarrow [n]$, $r \leftarrow 0$
 - 4: **repeat**
 - 5: set $r \leftarrow r + 1$
 - 6: **for** player $j = 1$ to k **do**
 - 7: sample each arm $i \in S_{r-1}$ for $t_r - t_{r-1}$ times
 - 8: let $\hat{p}_{j,i}^r$ be the average reward of arm i (in all rounds so far of player j)
 - 9: communicate the numbers $\hat{p}_{j,1}^r, \dots, \hat{p}_{j,n}^r$
 - 10: **end for**
 - 11: let $\hat{p}_i^r = (1/k) \sum_{j=1}^k \hat{p}_{j,i}^r$ for all $i \in S_{r-1}$, and let $\hat{p}_\star^r = \max_{i \in S_{r-1}} \hat{p}_i^r$
 - 12: set $S_r \leftarrow S_{r-1} \setminus \{i \in S_{r-1} : \hat{p}_i^r < \hat{p}_\star^r - \varepsilon_r\}$
 - 13: **until** $\varepsilon_r \leq \varepsilon/2$ or $|S_r| = 1$
 - 14: output S_r
-

Theorem 4.1. *With probability at least $1 - \delta$, Algorithm 3*

- identifies the optimal arm using

$$O\left(\sum_{i=2}^n \frac{1}{(\Delta_i^\varepsilon)^2} \log\left(\frac{n}{\delta} \log \frac{1}{\Delta_i^\varepsilon}\right)\right)$$

arm pulls, where $\Delta_i^\varepsilon := \max\{\Delta_i, \varepsilon\}$;

- terminates after at most $1 + \lceil \log_2(1/\varepsilon) \rceil$ rounds of communication (or after $1 + \lceil \log_2(1/\Delta) \rceil$ rounds for $\varepsilon = 0$).

Proof. Without loss of generality, we may assume that the rewards of all arms are drawn before the algorithm is executed, so that the empirical averages \hat{p}_i^r are defined for all arms at all rounds (even for arms that were eliminated prior to some round). Since \hat{p}_i^r is the empirical average of kt_r samples of arm i (aggregated from all players), for any round r and arm i we have by Hoeffding's inequality,

$$\Pr[|\hat{p}_i^r - p_i| \geq \varepsilon_r/2] \leq 2 \exp(-\varepsilon_r^2 kt_r/2) = \delta/2nr^2.$$

Hence, a union bound gives that $|\hat{p}_i^r - p_i| < \varepsilon_r/2$ for all i and r with probability at least

$$1 - \sum_{r=1}^{\infty} \sum_{i=1}^n \frac{\delta}{2nr^2} = 1 - \sum_{r=1}^{\infty} \frac{\delta}{2r^2} \geq 1 - \delta.$$

That is, with probability at least $1 - \delta$, an ε -optimal arm i is never eliminated by the algorithm, as the event $\hat{p}_i^r < \hat{p}_*^r - \varepsilon_r$ implies that either $\hat{p}_i^r < p_i - \varepsilon_r/2$ or $\hat{p}_j^r > p_j + \varepsilon_r/2$ for some arm j . In addition, any suboptimal arm i does not survive round $r_i = \lceil \log_2(1/\Delta_i^\varepsilon) \rceil + 1$, since $\Delta_i^\varepsilon \geq 2\varepsilon_{r_i}$ and so for $r = r_i$,

$$\begin{aligned} \hat{p}_i^r &< p_i + \varepsilon_r/2 = p_1 + \varepsilon_r/2 - \Delta_i \leq \hat{p}_1^r + \varepsilon_r - \Delta_i \leq \hat{p}_1^r - \varepsilon_r \\ &\leq \hat{p}_*^r - \varepsilon_r. \end{aligned}$$

That is, with probability at least $1 - \delta$, after $\lceil \log_2(1/\varepsilon) \rceil + 1$ rounds (when the algorithm terminates) all remaining arms are ε -optimal. When $\varepsilon = 0$, the algorithm terminates once only a single arm survives, and with high probability this occurs after at most $\lceil \log_2(1/\Delta) \rceil + 1$ rounds.

We conclude by computing the sample complexity. Let T_i be the total number of times arm $i \neq 1$ is pulled. Since $r_i \leq \log_2(4/\Delta_i^\varepsilon)$, we have

$$\begin{aligned} T_i &\leq 2(2^{r_i})^2 \log \frac{4nr_i^2}{\delta} \leq 2 \left(\frac{4}{\Delta_i^\varepsilon}\right)^2 \log \left(\frac{4n}{\delta} \log_2 \frac{4}{\Delta_i^\varepsilon}\right) \\ &= O\left(\frac{1}{(\Delta_i^\varepsilon)^2} \log \left(\frac{n}{\delta} \log \frac{1}{\Delta_i^\varepsilon}\right)\right). \end{aligned}$$

Consequently, the total number of arm pulls is $T_2 + \sum_{i=2}^n T_i$, which gives the theorem. \square

4.2 General Round Algorithm

By properly tuning the elimination thresholds ε_r of Algorithm 3 in accordance with the target accuracy ε , we can trade off between the number of communication rounds and the sample complexity of the algorithm. In particular, we can design a multi-player algorithm that terminates after at most R communication rounds, for any given parameter $R > 0$. This, however, comes at the cost of a compromise in sample complexity as quantified in the following theorem.

Theorem 4.2. *Given a parameter $R > 0$, set $\varepsilon_r = \varepsilon^{r/R}$ for all $r \geq 0$ in Algorithm 3. With probability at least $1 - \delta$, the modified algorithm⁴*

- *identifies an ε -optimal arm using*

$$O\left(\frac{1}{\varepsilon^{2/R}} \cdot \sum_{i=2}^n \frac{1}{(\Delta_i^\varepsilon)^2} \log \frac{nR}{\delta}\right)$$

arm pulls;

- *terminates after at most R rounds of communication.*

Proof. For all arms $i \in [n]$, let

$$r_i = \left\lceil R \frac{1 + \log_2(1/\Delta_i^\varepsilon)}{\log_2(1/\varepsilon)} \right\rceil.$$

Since $\Delta_i \geq 2\varepsilon_{r_i}$, if the algorithm is successful any arm i which is not ε_{r_i} -optimal is eliminated after at most r_i rounds. Clearly, after R rounds only ε -optimal arms survive. This happens with probability at least $1 - \delta$.

It remains to calculate the sample complexity. It is easy to verify that $2\varepsilon_{r_i} \geq \varepsilon^{1/R} \cdot \Delta_i^\varepsilon$, thus the number of times arm i was pulled is

$$T_i = \frac{4}{\varepsilon_{r_i}^2} \log \frac{2nr_i^2}{\delta} = O\left(\frac{1}{\varepsilon^{2/R}} \cdot \frac{1}{(\Delta_i^\varepsilon)^2} \log \frac{nR}{\delta}\right),$$

and the theorem follows. \square

5 Conclusions and Future Work

We have considered a multi-player multi-armed bandits problem, where arms are pulled – and rewards are observed – by several independent players with a common goal. In order

⁴In contrast to other algorithms discussed in this paper, this algorithm does not apply to $\varepsilon = 0$. In order to apply it in a best-arm identification setting, one should set $0 < \varepsilon \leq \Delta$, but this would require knowing the value of Δ (or some lower bound thereof) beforehand.

to learn efficiently, the players must communicate. We show an inherent tradeoff between the communication among the players, and the overall number of arm pulls required to solve the problem (less communication implies more arm pulls).

We leave the following for future work. Most importantly, we wish to translate our findings to the regret minimization setting. Furthermore, it would be interesting to extend our single-round results to the case where r communication rounds are allowed, for any constant $r > 1$. Finally, we wish to investigate lower bounds on arm pulls for the case of multiple rounds of communication.

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