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BENGALURU, INDIA

WHERE THE WORLD COMES TO LEARN

Module-1

Probability Distributions: Review of basic probability theory. Random variables (discrete and continuous), probability mass and density functions. Mathematical expectation, mean and variance. Binomial, Poisson and normal distributions- problems (derivations for mean and standard deviation for Binomial and Poisson distributions only)-Illustrative examples. Exponential distribution.

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Objectives:

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- To introduce the concept of random variables, probability distributions.
- To explore discrete and continuous distributions with practical application in Computer Science Engineering and social life situations.

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Applications:

- Probability distributions, such as the Poisson distribution is used to model network traffic, predict latency, and optimize network performance.
- Probability distributions used in the design and analysis of randomized algorithms. Common randomized algorithms are things like Quicksort and Quick select.
- Exponential distributions, are used to predict the failure rates of hardware and software components, helping engineers design more reliable and fault-tolerant systems

Outcomes:

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At the end of the module, we will be able to

- Explain the basic concepts of probability, random variables, probability distribution.
- Apply suitable probability distribution models for the given scenario.

Discrete and Continuous probability distributions

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There are two types of discrete probability distributions.

- Binomial Distribution.
- Poisson Distribution.

There are two types of continuous probability distributions.

- Exponential Distributions.
- Normal Distributions.

Binomial Distribution

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This distribution can be used under the following conditions:

- Binomial distribution is also known as the 'Bernoulli' distribution.
- The random experiment is performed repeatedly a finite and fixed numbers of times (say n).
- The outcome of each trial may be classified into two mutually disjoint categories, called success (the occurrence of the event) and failure (the non-occurrence of the event).
- All the trials are independent i.e the result of any trial, is not affected in any way by the preceding trials and doesn't affect the result of succeeding trials.
- The probability of success (happening of an event) in any trial is p and is constant for each trial. And $q = 1 - p$, is then termed as the probability of failure and is constant for each trial.

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- If p is the success and q is the failure, the probability of x success out of n trials is given by b(n ,p ,x) = $nC_x p^x q^{n-x}$, $x = 0,1,2,3 \ldots \ldots n$
- Consider the probability distribution table

x	0	1	2	….	n
P(x)	q^n	$nC_1 p^1 q^{n-1}$	$nC_2 p^2 q^{n-2}$	….	p^n

- Where n is a given positive integer, p is a real number such that $0 \leq p < 1$ $and\ q = 1 - p.$
- This probability function is called Binomial probability function and corresponding distribution is called Binomial distribution.
- Mean $\mu = np$, Variance (V) = npq and
- Standard deviation (σ) $\sqrt{npq}$

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Mean of binomial distribution

Mean $(\mu) = \sum_{x=0}^{n} xp(x)$

$$(\mu) = \sum_{x=0}^{n} x\, nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n.(n-1)!}{(x-1)!(n-x)!} p.\, p^{x-1} q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} \cdot p^{x-1} q^{(n-1)-(x-1)}$$

$$= \sum_{x=0}^{n} (n-1)\, C_{(x-1)} p^{x-1} q^{(n-1)-(x-1)}$$

$$= \mathrm{np}\, (q+p)^{n-1}$$

$$= \mathrm{np}$$

Mean $(\mu) = \mathrm{np}$

Variance of binomial distribution

Variance (V) $= \sum_{x=0}^{n} x^2 p(x) - \mu^2$

$$\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} x(x-1)p(x) + \sum_{x=0}^{n} xp(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, nC_x p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1)\, \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1)\, \frac{n(n-1)(n-2)!}{(x-2)!(n-x)!} p^2 \,.p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=0}^{n} \frac{(n-2)!}{(x-)![(n-2)-(x-2)]!} \,.p^{x-2}\, q^{(n-2)-(x-2)+np}$$

$$= n(n-1)p^2 \sum_{x=0}^{n} (n-2)\, C_{(x-2)} p^{x-2} q^{(n-2)-(x-2)}$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

Variance (V) $= n(n-1)p^2 + np - (np)^2$

$$= npq$$

S.D (σ) $= \sqrt{npq}$

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1) The probability that a pen manufactured by a factory be defective is 1/10.If 12 such pens are manufactured, what is the probability that (i) exactly 2 are defective (ii)at least 2 are defective(iii)none of them are defective.

- Solution: Probability of a defective pen is p = 1/10 = 0.1

Probability of a non-defective pen is q = 1-p = 1-0.1 = 0.9

We have P(x) = $nC_x p^x q^{n-x}$, where we have n = 12

(i) Prob. (exactly two defective) is P(x=2)

P(x)= $12C_2\, 0.1^2\, 0.9^{10}$

(ii) Prob.(at least 2 defectives) is P(x>2)

$P(x>2)=1-P(x<2)$

$= 1-[P(x=0)+P(x=1)]$

$=1-[12C_2\, 0.1^2\, 0.9^{10} + 12C_1\, 0.1^1\, 0.9^{11}]=0.341$

(iii) Prob. (no defective) is P(x=0)

P(x)= $12C_0\, 0.1^0\ 0.9^{12} = (0.9)^{12} = 0.2824$

2) The Probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a laboratory, find the probability that (i) 8 seeds germinate (ii) at least 8 seeds germinate (iii) at most 8 seeds germinate.

- Solution: Probability of a germination of a seed is p = 0.7

Probability of a non- germination of a seed is q = 1-p = 1-0.7 = 0.3

We have P(x) = $nC_x p^x q^{n-x}$, where we have n = 10

(i) Prob. of (exactly 8 seed germinate) P(x=8)

P(x=8)= $10C_8\, 0.7^8\, 0.3^2$ =0.2335

(ii) Prob. Of (at least 8 seed germinate) P(x≥8)

P(x≥8)= [P(x=8)+P(x = 9)+P(= 10)]

= $10C_8\, 0.7^8\ \ 0.3^2 + 10C_9\, 0.7^9 0.3 + 10C_{10}\, 0.7^{10}$

= 0.2335+0.1211+0.0282=0.3828

(iii) Prob. (of at most 8 seeds germinate) P(x<8)

P(x<8) = 1-[P(x≥8)] = 1-0.3828 = 0.6172

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4) If the mean and standard deviation of the number of correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$. Find an estimate of the number of candidates answering correctly (i)8 or more questions (ii)2 or less questions.

Solution: We have mean (μ) = np and S.D $(\sigma) = \sqrt{npq}$ for a binomial distribution.

Given that mean np = 2.5 and $\sqrt{1.875} = \sqrt{npq}$ or 1.875 = npq

(2.5)q = 1.875 on simplification we get q = 0.75 and p = 1- q = 1- 0.75 = 0.25

Since np = 2.5 we have n (0.25) = 2.5 gives n = 10

Let x denotes the number of correctly answered questions.

$P(x) = nc_x p^x q^{n-x} = 10c_x(1/4)^x(3/4)^{10-x} = \frac{1}{4^{10}}[10c_x(3)^{10-x}]$

Since the estimate is need for 4096 students we have

$f(x) = 4096\, P(x) = \frac{4096}{4^{10}}[10c_x(3)^{10-x}] = \frac{2^{12}}{2^{20}}[10c_x(3)^{10-x}]$

i..e $4096P(x) = \frac{1}{256}[10c_x(3)^{10-x}] = f(x)$

(i) Prob. of 8 or more question $f(x \geq 8) = f(8) + f(9) + f(10)$

$$= \frac{1}{256}[10c_8(3)^{10-8} + 10c_9(3)^{10-9} + 10c_{10}(3)^{10-10}]$$

$$= \frac{1}{256}(436) = 1.703 \cong 2$$

Number of students correctly answered 8 or more questions is 2

(ii) Prob. of 2 or less question $f(x \leq 2) = f(2) + f(1) + f(0)$

$$= \frac{1}{256}[10c_2(3)^{10-2} + 10c_1(3)^{10-1} + 10c_0(3)^{10-0}]$$

$$= 2152.8 \cong 2153$$

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5) The Probability of a shooter hitting a target is 1/3. How many times he should shoot so that the probability of hitting the target at least once is more than 3/4.

Solution:

Probability of hitting the target is p = 1/3

Probability of missing the target is q = 2/3

$P(x) = nc_x p^x q^{n-x} = nc_x (1/3)^x (2/3)^{n-x}$

We have to find n such that

$P(x \geq 1) > 3/4$

$1 - P(x<1) > 3/4$

$1 - P(x=0) > 3/4$

$1 - [nc_0 (1/3)^0 (2/3)^{n-0}] > 3/4$

$1 - (2/3)^n > 3/4$

$(2/3)^n < \frac{1}{4} = 0.25$

We can find n by inspection as we have

$(2/3) = 0.67$, $(2/3)^2 = 0.44$, $(2/3)^3 = 0.3$, $(2/3)^4 = 0.2$

Hence the required n is 4

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Poisson Distribution

- Poisson distribution is regarded as the limiting case of Binomial distribution.
- when number of trials is very large ($n \to \infty$).
- probability of success for each trial is indefinitely small ($p \to 0$).
- Mean (μ) = np = m (say) is finite.
- The Poisson probability function is given by $p(x) = \frac{m^x e^{-m}}{x!}$

Here x is called as Poisson variate. (x = 0,1,2,3......)

- The distribution of probabilities for x = 0,1,2,3....

x	0	1	2	3	....
P(x)	e^{-m}	$\frac{m^1 e^{-m}}{1!}$	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^3 e^{-m}}{3!}$	.....

- Mean (μ) = m and Variance (V) =$\sqrt{m}$,
- Standard Deviation (σ)= $\sqrt{m}$

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Mean of Poisson distribution

Mean $(\mu) = \sum_{x=0}^{n} x.\, p(x)$

$$\mu = \sum_{x=0}^{n} x \frac{m^x e^{-m}}{x!}$$

$$= \sum_{x=0}^{n} \frac{m^x e^{-m}}{(x-1)!}$$

$$= me^{-m} \sum_{x=1}^{n} \frac{m^{x-1}}{(x-1)!}$$

$$= me^{-m} \left[1+\frac{m}{1!} + \frac{m^2}{2!}+ \frac{m^2}{3!}+ \ldots\right]$$

$$= me^{-m}\, e^{m} = m$$

Mean $(\mu) = m$

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Variance of Poisson distribution

Variance (V) $= \sum x^2 p(x) - \mu^2$

$$\sum x^2 p(x) = \sum x(x-1)p(x) + \sum xp(x)$$

$$= \sum x(x-1)\frac{m^x e^{-m}}{(x-1)!} + m$$

$$= \sum \frac{m^x e^{-m}}{(x-2)!} + \sum \frac{m^x e^{-m}}{(x-1)!}$$

$$= m^2 e^{-m} \sum \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m}\, [1+\frac{m}{1!} + \frac{m^2}{2!}+ \frac{m^2}{3!}+ \, . \, . \, . \,] + m$$

$$= m^2 e^{-m}\, e^m + m = m^2 + m$$

$$\sum x^2 p(x) \; = m^2 + m$$

Variance (V) $= m^2 + m$- $m^2 = m$

S.D $(\sigma) = \sqrt{m}$

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Problems on Poisson Distribution

1) Between the hours 2pm and 4pm the average number of phone calls per minute coming into the switch board of a company is 2,35. Find the probability that during one particular minute, there will be at most 2 phone calls. [Given e^-2.35=0.095374]

- Solution: If the random variable x denotes the number of telephone calls per minute, then x will follow Poisson distribution with parameter μ=2.35 and probability function.

$$p(x) = \frac{m^x e^{-m}}{x!} = \frac{2.35^x e^{-2.35}}{x!}$$

The probability that during one particular minute there will be at most 2 phone calls is given by. P(x≤ 2) = $P(x = 0) + P(x = 1) + P(x = 2)$

$$= e^{-2.35} \frac{2.35^1 e^{-2.35}}{1!} + \frac{2.35^2 e^{-2.35}}{2!} = 0.5828543.$$

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2) In a certain factory turning out razor blades there is a small probability of 1/1500 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective (iii) two defective blades in a consignment of 10,000 packets.

Solution: Probability of a defective blade =1/1500=0.002

In a packet of 10, the mean number of defective blades is

m =np=10*0.002=0.02

Poisson distribution is $\frac{m^x e^{-m}}{x!} = \frac{0.02^x e^{-0.02}}{x!}$

Let f(x) = 10000 P(x) ; Also $e^{-0.02}$=0.9802

$$f(x) = \frac{0.9802(0.02)^x}{x!}$$

(i) Probability of no defective = f (0) = 9802

(ii) Probability of one defective = f (1) = 9802*(0.02) = 196

(iii) Probability of two defective = f (2) = $\frac{0.9802(0.02)^2}{2!} = 2$

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3) If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals, more than two will get a bad reaction.

Solution:

As the probability of occurrence of bad reaction is very small, this follows Poisson distribution and we have $P(x) = \frac{m^x e^{-m}}{x!}$

Mean (μ) = m = np = 2000*0.001 = 2

We have to find P (x > 2)

$$P(x > 2) = 1 - P(x \leq 2)$$

$$= 1 - [P(x = 0) + P(x = 1) + P(x = 2)]$$

$$= 1 - e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!}\right]$$

$$= 1 -- e^{-2}[1 + 2 + 2]$$

$$P(x > 2) = 0.3222$$

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4) The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of the drivers with
   (i) No accident in a year
   (ii) More than 3 accidents in a year.

Solution: Given that, mean (μ) = m = 3. We have Poisson distribution (μ) = m = 3

The Poisson distribution is given by P(x) = $\frac{m^x e^{-m}}{x!} = \frac{m^x e^{-3}}{3!}$

Let f (x) = 1000P(x)

i..e f(x) = $1000\frac{3^x e^{-3}}{x!} = 50.\frac{3^x}{x!}$ since $e^{-3} = 0.05$

(i) Number of driver with no accidents in a year = f(x = 0) = $50.\frac{3^0}{0!} = 50$

(ii) Prob. of more than 3 accidents in a year = P(x > 3) = 1- P(x≤ 3)

= 1- [P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)]

=1- $e^{-3}\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right] = 0.35$

Number of driver out of 1000 with more than 3 accidents in a year = 1000 * 0.35 = 350

5) 2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses contains

(i) No defective fuses

(ii) 3 or more defective fuses.

Solution: Probability of defective fuse is 2% = 2/100 = 0.02

Mean number of defectives (μ) = m = np = 200*0.02 = 4

The Poisson distribution is given by $P(x) = \frac{m^x e^{-m}}{x!}$

$$P(x) = \frac{4^x e^{-4}}{x!} \qquad \text{But } e^{-4} = 0.0183$$

(i) Probability of no defective fuse = $P(0) = \frac{4^0 e^{-4}}{0!} = 0.0183$

(ii) Probability of 3 or more defective fuses $P(x \leq 3) = 1 - [P(x < 3)]$

$$= 1 - [P(x = 0) + P(x = 1) + P(x = 2)]$$

$$= 1 - (0.0183)\left[1 + \frac{4^1}{1!} + \frac{4^2}{2!}\right]$$

$$= 0.7621$$

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More Examples to solve

- X is binomial distributed random variable. If the mean and variance of X are 2 and 3/2 respectively, find the distribution function.
- A switch board can handle only 4 telephone calls per minute. If the incoming calls per minute follow a Poisson distribution with parameter 3,find the probability that the switch board is over taxed in any one minute.
- A travel agency has 2 cars which it hires daily. The number of demands for a car on each day is distributed as a Poisson variant with mean1.5.find the probability that on a particular day (i)there was no demand (ii)a demand is refused.
- When a coin is tossed 4 times, find the probability of getting (i)exactly one head (ii) at most 3 heads (iii)at least 2 heads

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Thank you