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PRICING OF HIGH-DIMENSIONAL AMERICAN OPTIONS BY NEURAL NETWORKS

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Pricing of American options in discrete time is considered, where the option is allowed to be based on several underlyings. It is assumed that the price processes of the underlyings are given Markov processes. We use the Monte Carlo approach to generate artificial sample paths of these price processes, and then we use the least squares neural networks regression estimates to estimate from this data the so-called continuation values, which are defined as mean values of the American options for given values of the underlyings at time t subject to the constraint that the options are not exercised at time t. Results concerning consistency and rate of convergence of the estimates are presented, and the pricing of American options is illustrated by simulated data.

KEY WORDS: American options, consistency, neural networks, nonparametric regression, optimal stopping, rate of convergence, regression-based Monte Carlo methods.

1. INTRODUCTION

In this paper, we consider American options in discrete time, which are also called Bermuda options. The price V_0 of such options can be defined as a solution of an optimal stopping problem

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\dots,T)} \mathbf{E}\{f_\tau(X_\tau)\}. \tag{1.1}$$

Here f_t is the (discounted) payoff function, $X_0, X_1, \dots, X_T$ is the underlying stochastic process describing, e.g., the prices of the underlyings and the financial environment (like interest rates, etc.), and $\mathcal{T}(0, \dots, T)$ is the class of all $\{0, \dots, T\}$-valued stopping times, i.e., $\tau \in \mathcal{T}(0, \dots, T)$ is a measurable function of $X_0, \dots, X_T$ satisfying

$$\{\tau = \alpha\} \in \mathcal{F}(X_0, \dots, X_\alpha) \quad \text{for all } \alpha \in \{0, \dots, T\}.$$

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As a simple example consider pricing of an American put option with strike K on the arithmetic mean of several correlated underlyings, where the stock values are modeled via Black–Scholes theory by

$$X_{i,t} = x_{i,0} \cdot e^{r \cdot t} \cdot e^{\sum_{j=1}^{m} (\sigma_{i,j} \cdot W_j(t) - \frac{1}{2}\sigma_{i,j}^2 t)} \quad (i = 1, \ldots, m). \tag{1.2}$$

Here $r > 0$ is the (given) riskless interest rate, $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,m})^T$ is the (given) volatility of the ith stock, $x_{i,0}$ is the initial stock price of the ith stock, and $\{W_j(t) : t \in \mathbb{R}_+\}$ $(j = 1, \ldots, m)$ are independent Wiener processes.

If we sell the option at time $t > 0$ and the stock prices are at this point $x = (x_1, \ldots, x_m)$ (i.e., the arithmetic mean of the stock prices is $\frac{1}{m} \sum_{j=1}^{m} x_j$), we get the payoff

$$\max \left\{ K - \frac{1}{m} \sum_{j=1}^{m} x_j, 0 \right\},$$

and if we discount this payoff toward time zero, we get the discounted payoff function

$$f_t(x_1, \ldots, x_m) = e^{-r \cdot t} \cdot \max \left\{ K - \frac{1}{m} \sum_{j=1}^{m} x_j, 0 \right\}. \tag{1.3}$$

But even if all the parameters are known (i.e., if $x_{i,0}$ $(i = 1, \ldots, m)$ and K are given and if we estimate the volatilities σ_i $(i = 1, \ldots, m)$ and the riskless interest rate from observed data from the past), it is not obvious how we can compute the price

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E} \left\{ e^{-r \cdot \tau} \cdot \max \left\{ K - \frac{1}{m} \sum_{i=1}^{m} X_{i,\tau}, 0 \right\} \right\}$$

of the corresponding American option.

In the above Black–Scholes model, we can reformulate the whole problem as a free boundary problem for partial differential equations (cf., e.g., Chapter 8 in Elliott and Kopp 1999), but the numerical solution of this free boundary problem gets very complicated if the number m of underlyings gets large. In addition, for $m \leq 2$ binomial trees (cf., e.g., Chapter 1 in Elliott and Kopp 1999) are able to produce very good estimates of V_0, but for $m > 3$ it is basically impossible to model with this method the correlation structure of the stocks correctly.

The purpose of this paper is to develop a Monte Carlo algorithm which is able to compute an approximation of the price (1.1) even in cases when the option is based on a large number of correlated stocks, when the stock prices are not modeled by a simple Black–Scholes model as in (1.2), and when the payoff function is not as simple as in (1.3). In particular, the method developed in this paper is also applicable in cases when the process $X_{i,t}$ is adjusted to observed data by time series estimation as described, e.g., in Franke and Diagne (2002).

In the sequel, we assume that $X_0, X_1, \ldots, X_T$ is a $\mathbb{R}^d$-valued Markov process recording all necessary information about financial variables including prices of the underlying assets as well as additional risk factors driving stochastic volatility or stochastic interest rates. Neither the Markov property nor the form of the payoff as a function of the state X_t is restrictive and can always be achieved by including supplementary variables.

The computation of (1.1) can be done by determination of an optimal stopping rule $\tau^* \in \mathcal{T}(0, \ldots, T)$ satisfying

$$V_0 = \mathbf{E}\{f_{\tau^*}(X_{\tau^*})\}. \tag{1.4}$$

Let

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1,\ldots,T)} \mathbf{E}\{f_\tau(X_\tau) \mid X_t = x\} \tag{1.5}$$

be the so-called continuation value describing the value of the option at time t given $X_t = x$ and subject to the constraint of holding the option at time t rather than exercising it. Here $\mathcal{T}(t+1, \ldots, T)$ is the class of all $\{t+1, \ldots, T\}$-valued stopping times. It can be shown that

$$\tau^* = \inf\{s \geq 0 : q_s(X_s) \leq f_s(X_s)\} \tag{1.6}$$

satisfies (1.4), i.e., τ^* is an optimal stopping time (cf., e.g., Chow, Robbins, and Siegmund 1971 or Shiryayev 1978). Therefore, it suffices to compute the continuation values (1.5) to solve the optimal stopping problem (1.1).

The continuation values satisfy the dynamic programming equations

$$\begin{aligned} &q_T(x) = 0,\\ &q_t(x) = \mathbf{E}\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} \mid X_t = x\} \quad (t = 0, 1, \ldots, T-1). \end{aligned} \tag{1.7}$$

Indeed, by analogy to (1.6) we have

$$q_t(x) = \mathbf{E}\{f_{\tau_t^*}(X_{\tau_t^*}) \mid X_t = x\} \quad \text{where} \quad \tau_t^* = \inf\{s \geq t+1 \mid q_s(X_s) \leq f_s(X_s)\},$$

hence by using the Markov property of $\{X_s\}_{s=0,\ldots,T}$ we get

$$\begin{aligned} q_t(X_t) &= \mathbf{E}\left\{f_{t+1}(X_{t+1}) \cdot I_{\{q_{t+1}(X_{t+1}) \leq f_{t+1}(X_{t+1})\}} + f_{\tau_{t+1}^*}(X_{\tau_{t+1}^*}) \cdot I_{\{q_{t+1}(X_{t+1}) > f_{t+1}(X_{t+1})\}} \middle| X_t\right\}\\ &= \mathbf{E}\{\mathbf{E}\{\ldots \mid X_0, \ldots, X_{t+1}\} \mid X_0, \ldots, X_t\}\\ &= \mathbf{E}\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} \mid X_t\}. \end{aligned}$$

Unfortunately, the conditional expectation in (1.7) in general cannot be computed in applications. The basic idea of regression-based Monte Carlo methods for pricing American options is to apply recursively regression estimates to artificially created samples of

$$(X_t, \max\{f_{t+1}(X_{t+1}), \hat{q}_{t+1}(X_{t+1})\})$$

(so-called Monte Carlo samples) to construct estimates $\hat{q}_t$ of q_t. The algorithm based on linear regression was proposed by Tsitsiklis and Van Roy (1999). They studied consistency and the rates in Tsitsiklis and Van Roy (1999, 2001). Another algorithm based on a different regression estimation than (1.7) was originally proposed by Carrière (1996) and further investigated by Longstaff and Schwartz (2001). Its consistency and the rates of convergence were investigated by Clément, Lamberton, and Protter (2002). Nonparametric least squares regression estimates have been applied and investigated in this context in Egloff (2005) and Egloff, Kohler, and Todorovic (2007), smoothing spline regression estimates have been analyzed by Kohler (2008), and recursive kernel regression estimates have been considered by Barty et al. (2008). The approaches mentioned earlier provide a lower bound for the true optimal stopping value. Upper bounds based on the dual problem are derived, e.g., in Rogers (2002) or in Haugh and Kogan (2004). The paper Andersen and Broadie (2004) derives upper and lower bounds for American options based on duality. More details and further references can be found in Glasserman (2004). Laprise et al. (2001) compare several Monte Carlo approaches empirically. Other Monte Carlo methods, e.g., ones based on the Malliavin calculus can be found in Fournié et al. (1999, 2001), and ones based on the state-space partitioning in Glasserman (2004). A quantization method can be found in Bally, Pagès, and Printems (2005).

In this paper, we propose to use the least squares neural networks regression estimates to compute the conditional expectations in (1.7), which is particularly promising for options based on several underlyings, where high-dimensional regression problems have to be solved in order to compute approximations of the continuation values. Because of the well-known curse of dimensionality it is difficult to choose here a reasonable nonparametric regression estimate, and neural networks belong (together with regression trees [cf., e.g., Breiman et al. 1984] or interaction models [cf., e.g., Stone 1994]) to standard estimates in this field.

There are key differences between our approach presented in this paper and the TR and LS algorithms proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001). First, we generate new data to construct samples of $\hat{Y}_{i,t}^{(t)}$. Second, we use a nonparametric approach rather then a linear regression model, and finally, we show convergence for arbitrary continuation values q_t and the rates for the continuation values with bounded Fourier transforms. This is generally not true for the TR and LS algorithms as they use linear regression models and their results apply only to linear continuation values. As demonstrated in Section 4 our estimate (implemented on a standard personal computer) produces better results than the TR and LS algorithms can produce on any other computer regardless of the CPU speed and amount of memory used on that computer.

In what follows we define the least squares neural networks regression estimates of the continuation values where all parameters of the estimates are selected using the given data only. We show that these estimates are universally consistent, i.e., that their L_2 errors converge to zero in probability and almost surely for all distributions. Furthermore, under regularity conditions on the smoothness of the continuation values we investigate the rates of convergence of the estimates. Finally, we validate the estimates in practice by applying them to simulated data.

The precise definition of the estimates and the main theoretical results concerning consistency and rate of convergence of the estimate are given in Sections 2 and 3, respectively. The application of the estimates to simulated data is described in Section 4, and the proofs are given in Section 5.

2. DEFINITION OF THE ESTIMATE

Let $\sigma : \mathbb{R} \to [0, 1]$ be a sigmoid function, i.e., monotonically increasing function satisfying

$$\sigma(x) \to 0 \quad (x \to -\infty) \quad \text{and} \quad \sigma(x) \to 1 \quad (x \to \infty).$$

An example of such a sigmoid function is the logistic squasher defined by $\sigma(x) = \frac{1}{1+e^{-x}} (x \in \mathbb{R})$. In the sequel we estimate the continuation values by neural networks with $k \in \mathbb{N}$ hidden neurons and a sigmoid function σ. We will use the principle of least squares to fit such a function to the data, and for technical reasons we restrict the sum of the absolute values of the output weights. The selection of the number of hidden neurons k will be data-driven by making use of sample splitting.

Let $\beta_n > 0$ (which will be chosen later subject to the constraint $\beta_n \to \infty$ $(n \to \infty)$) and let $\mathcal{F}_k(\beta_n)$ be a class of neural networks defined by

$$\mathcal{F}_k(\beta_n) = \left\{ \sum_{i=1}^{k} c_i \cdot \sigma\left(a_i^T x + b_i\right) + c_0 : a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R},\ \sum_{i=0}^{k} |c_i| \le \beta_n \right\}, \tag{2.1}$$

where σ is the sigmoid function from earlier.

In the sequel we describe an algorithm to estimate the continuation values q_t recursively. To do this we generate artificial independent Markov processes $\{X_{i,t}^{(l)}\}_{t=0,\ldots,T}$ $(l = 0, 1, \ldots, T-1,\ i = 1, 2, \ldots, n)$ which have the same distribution as $\{X_t\}_{t=0,\ldots,T}$. Then we use these so-called Monte Carlo samples to generate the data recursively and estimate q_t using the regression representation given in (1.7).

We start with

$$\hat{q}_{n,T}(x) = 0 \quad (x \in \mathbb{R}^d).$$

Fix $t \in \{0, 1, \ldots, T-1\}$. Given an estimate $\hat{q}_{n,t+1}$ of q_{t+1}, we estimate

$$q_t(x) = \mathbf{E}\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} \mid X_t = x\}$$

by a neural networks regression estimate using an "approximative" sample of

$$(X_t, \max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\}).$$

With the notation

$$\hat{Y}_{i,t}^{(t)} = \max\left\{ f_{t+1}\left(X_{i,t+1}^{(t)}\right), \hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)\right\}$$

(where we have suppressed the dependency of $\hat{Y}_{i,t}^{(t)}$ on n) this "approximative" sample is given by

$$\left\{\left(X_{i,t}^{(t)}, \hat{Y}_{i,t}^{(t)}\right) : i = 1, \ldots, n\right\}. \tag{2.2}$$

Observe that this sample depends on the tth sample of $\{X_s\}_{s=0,\ldots,T}$ and $\hat{q}_{n,t+1}$, i.e., for each time step t we use a new sample of the stochastic process $\{X_s\}_{s=0,\ldots,T}$ in order to define our data (2.2).

We choose the parameter k of the neural networks regression estimate fully automatically by sample splitting. Thus, we subdivide (2.2) in a learning sample of size $n_l = \lceil n/2 \rceil$ and a testing sample of size $n_t = n - n_l$ and define for a given $k \in \mathcal{P}_n = \{1, \ldots, n\}$ a regression estimate of q_t by

$$\hat{q}_{n_l,t}^k(\cdot) = \arg\min_{f \in \mathcal{F}_k(\beta_n)} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} \left| f\left(X_{i,t}^{(t)}\right) - \hat{Y}_{i,t}^{(t)} \right|^2 \right), \tag{2.3}$$

where $z = \arg\min_{x \in \mathcal{D}} f(x)$ is an abbreviation for $z \in \mathcal{D}$ and $f(z) = \min_{x \in \mathcal{D}} f(x)$. Here we assume for simplicity that the above minima exist, however we do not require them to be unique.

We choose the value of the parameter k by minimizing the empirical L_2 risk on the testing sample. So we choose

$$\hat{k} = \arg\min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} \left| \hat{q}^k_{n_l,t}\left(X^{(t)}_{i,t}\right) - \hat{Y}^{(t)}_{i,t} \right|^2 \tag{2.4}$$

and define our final neural networks regression estimate of q_t by

$$\hat{q}_{n,t}(x) = \hat{q}^{\hat{k}}_{n_l,t}(x) \quad (x \in \mathbb{R}^d). \tag{2.5}$$

3. THEORETICAL RESULTS

We say that $a_n = O_{\mathbf{P}}(b_n)$ if $\limsup_{n\to\infty} \mathbf{P}(a_n > c \cdot b_n) = 0$ for some finite constant c.

Our main theoretical result is the following theorem.

THEOREM 3.1. *Let $L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is an $\mathbb{R}^d$-valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L. Define the estimate $\hat{q}_{n,t}$ by (2.3), (2.4), and (2.5) for some $\beta_n > 0$. Let $k_n \in \mathcal{P}_n$ and assume that k_n, β_n satisfy*

$$\beta_n \to \infty \quad (n \to \infty), \quad k_n \to \infty \quad (n \to \infty), \quad \frac{\beta_n^4 \cdot k_n \cdot \log n}{n} \to 0 \quad (n \to \infty).$$

Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx)$$
$$= O_{\mathbf{P}}\left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t,t+1,\ldots,T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) \right)$$

for all $t \in \{0, 1, \ldots, T\}$.

As a first consequence we get consistency of the estimate.

COROLLARY 3.2. *Let $L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is an $\mathbb{R}^d$-valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L, i.e.,*

$$|f_t(x)| \le L \text{ for } x \in \mathbb{R}^d \text{ and } t \in \{0, 1, \ldots, T\}. \tag{3.1}$$

Define the estimate $\hat{q}_{n,t}$ by (2.3), (2.4), and (2.5). Let $\beta_n > 0$ and assume that β_n satisfies

$$\beta_n \to \infty \quad (n \to \infty), \quad \frac{\beta_n^4 \cdot \log n}{n} \to 0 \quad (n \to \infty).$$

Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \to 0 \quad \textit{in probability}$$

for all $t \in \{0, 1, \ldots, T\}$.

Proof of Corollary 3.2. Conditions of Corollary 3.2 allow us to choose $k_n \in \mathcal{P}_n$ such that $k_n \to \infty$ ($n \to \infty$) and $\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} \to 0$ ($n \to \infty$). By Theorem 3.1 we get

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx)$$
$$= O_{\mathbf{P}}\left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t, t+1, \ldots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \right)$$

for all $t \in \{0, 1, \ldots, T\}$. Condition (3.1) implies that q_t is bounded, hence we get by Lemma A.9 in the Appendix

$$\max_{s \in \{t, t+1, \ldots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \to 0 \quad (n \to \infty).$$ □

REMARK 3.3. Convergence in probability showed in Corollary 3.2 can be strengthened to a.s. convergence under slightly stronger constraints on k_n. This follows from the proofs of Lemmas 5.2 and 5.3 by noticing that the bounds on probabilities there converge to zero exponentially fast and by applying the Borel–Cantelli lemma.

The above corollary shows that the L_2 error of our estimate converges to zero in probability when the number of Monte Carlo samples tends to infinity. In view of an application with necessarily finite sample size it would be nice to know how quickly the error converges to zero for sample size tending to infinity. It is well known in nonparametric regression that assumptions on the underlying distribution, in particular on the smoothness of the regression function, are necessary in order to be able to derive nontrivial rates of convergence results (see, e.g., Cover 1968, Devroye 1982, or chapter 3 in Györfi et al. 2002). For our neural networks estimate, we restrict the smoothness of the continuation values by imposing constraints on their Fourier transformation.

In addition, we assume that the stochastic process is bounded. Usually, in practice financial processes are modeled by unbounded processes. In this case we choose a large value $A > 0$ and replace X_t by their bounded approximations

$$X_t^A = X_{\min\{t, \tau_A\}} \quad \text{where} \quad \tau_A = \inf\{s \geq 0 \,:\, X_s \notin [-A, A]^d\}.$$

(Here we assume for simplicity that the stochastic process has continuous paths in order to be able to neglect an additional truncation of X_t^A.) This boundedness assumption enables us to estimate the price of the American option from samples of polynomial size in the number of free parameters, in contrast to Monte Carlo estimation from standard (unbounded) Black–Scholes models, where Glasserman and Yu (2004) showed that samples of exponential size in the number of free parameters are needed. For many industrial models, the localization error can be estimated explicitly. For instance, Section 4 in Egloff et al. (2007) contains a priori bounds for the localization and payoff truncation error in case of discretely sampled jump diffusion processes. So in particular we can assume in the sequel that $X_t \in [-A, A]^d$ a.s. Note that this assumption implies that the payoffs $f_t(x)$, $x \in [-A, A]^d$ are bounded as well for the usual payoff functions.

Next we analyze the rate of convergence of the estimate. To this end we need to introduce the class of functions having Fourier transform with the first absolute moment finite. The Fourier transform $\tilde{F}$ of a function $f \in L_1(\mathbb{R}^d)$ is defined by

$$\tilde{F}(\nu) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\nu^T x} f(x)\, dx \quad (\nu \in \mathbb{R}^d).$$

If $\tilde{F} \in L_1(\mathbb{R}^d)$ then the inverse formula

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\nu^T x} \tilde{F}(\nu)\, d\nu \tag{3.2}$$

holds almost everywhere with respect to the Lebesgue measure. Let $0 < C < \infty$ and consider the class of functions $\mathcal{F}_C$ for which (3.2) holds on $\mathbb{R}^d$ and, in addition,

$$\int_{\mathbb{R}^d} \|\nu\| |\tilde{F}(\nu)|\, d\nu \le C. \tag{3.3}$$

A class of functions satisfying (3.3) is a subclass of functions with Fourier transform having first absolute moment finite, i.e., $\int_{\mathbb{R}^d} \|\nu\| |\tilde{F}(\nu)|\, d\nu < \infty$ (these functions are continuously differentiable on $\mathbb{R}^d$). The next corollary provides the rate of convergence of the estimate.

COROLLARY 3.4. *Let $L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is an $\mathbb{R}^d$-valued Markov process, $X_t \in [-A, A]^d$ almost surely for some $A > 0$ and all $t \in \{0, 1, \ldots, T\}$, that the discounted payoff function f_t is bounded in absolute value by L, i.e.,*

$$|f_t(x)| \le L \text{ for } x \in \mathbb{R}^d \text{and } t \in \{0, 1, \ldots, T\},$$

and that the Fourier transform $\tilde{Q}_t$ of q_t satisfies (3.2) and (3.3) for all $x \in \mathbb{R}^d$ and all $t \in \{0, \ldots, T\}$. Let $\beta_n = const \cdot \log n$ and define the estimate $\hat{q}_{n,t}$ by (2.3), (2.4), and (2.5). Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}}\left(\left(\frac{\log^5 n}{n}\right)^{1/2}\right)$$

for all $t \in \{0, 1, \ldots, T\}$.

Proof of Corollary 3.4. Set $k_n = (\frac{n}{\log^5 n})^{1/2}$. Using Lemma A.10 in the Appendix we have for n sufficiently large

$$\max_{s \in \{t, t+1, \ldots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \le \frac{(2\sqrt{d}AC)^2}{k_n}.$$

Then Theorem 3.1 implies

$$\begin{aligned}
&\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\
&= O_{\mathbf{P}}\left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t, t+1, \ldots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx)\right) \\
&= O_{\mathbf{P}}\left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \frac{(2\sqrt{d}AC)^2}{k_n}\right) \\
&= O_{\mathbf{P}}\left(\sqrt{\frac{\log^5 n}{n}}\right)
\end{aligned}$$

for all $t \in \{0, 1, \ldots, T\}$. □

REMARK 3.5. Assume $X_0 = x_0$ a.s. for some $x_0 \in \mathbb{R}$. We can estimate the price

$$V_0 = \max\{f_0(x_0), q_0(x_0)\}$$

(cf. (1.1) and (1.5)) of the American option by

$$\hat{V}_0 = \max\{f_0(x_0), \hat{q}_{n,0}(x_0)\}.$$

Since the distribution of X_0 is concentrated on x_0, under the assumptions of Corollary 3.4 we have the following error bound:

$$\begin{aligned} |\hat{V}_0 - V_0|^2 &= |\max\{f_0(x_0), \hat{q}_{n,0}(x_0)\} - \max\{f_0(x_0), q_0(x_0)\}|^2 \\ &\le |\hat{q}_{n,0}(x_0) - q_0(x_0)|^2 \\ &= O_{\mathbf{P}}\left(\left(\frac{\log^5 n}{n}\right)^{1/2}\right). \end{aligned}$$

Note that the rate in Corollary 3.4 depends on dimension in a sense that the class of functions satisfying condition (3.3) shrinks with dimension.

4. APPLICATION TO SIMULATED DATA

In this section, we illustrate the finite sample behavior of our algorithm by comparing it with the Tsitsiklis–Van Roy (TR) algorithm and Longstaff–Schwartz (LS) algorithm proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively.

We simulate the paths of the underlying stocks with a simple Black–Scholes model. The time to maturity is assumed to be 1 year. We discretize the time interval [0, 1] by dividing it into m equidistant time steps with $t_0 = 0 < t_1 < \cdots < t_m = 1$. In the first two examples we consider an option on a single stock. The prices of the underlying stock at time points t_j $(j = 0, \ldots, m)$ are then given by

$$X_{i,t_j} = x_0 \cdot \exp\left((r - 1/2 \cdot \sigma^2) \cdot t_j + \sigma \cdot W_{t_j}\right) \quad (i = 1, \ldots, n\ , j = 0, \ldots, m).$$

We choose $x_0 = 100$, $r = 0.05$, $m = 12$ and discount factors e^{-rt_j} for $j = 0, \ldots, m$. For our algorithm we use sample size of 2,000 while for the other algorithms sample size of 10,000.

For our algorithm we set the number of learning and training samples to $n_l = n_t =$ 1,000. To simplify the implementation we select the number of hidden neurons by sample splitting (as described in Section 2) from the set $\{2^0, 2^1, \ldots, 2^5\}$. The neural networks least squares estimate is computed approximately by backpropagation (i.e., by gradient descent). For the LS and TR algorithms, we use in the first example polynomials of degree 3, in the second example polynomials of degrees 3 and 1, and in the third example (high-dimensional case) polynomial of degree 1, since these choices yield the best results.

We apply all three algorithms to 100 independently generated sets of paths. We would like to stress that all three algorithms provide lower bounds to the optimal stopping value. Since we evaluate the approximative optimal stopping rule on newly generated data, a higher Monte Carlo estimate (MCE) indicates a better performance of the algorithm. We compare the algorithms using boxplots. Observe that the higher the boxplot of the MCE the better the performance of the corresponding algorithm.

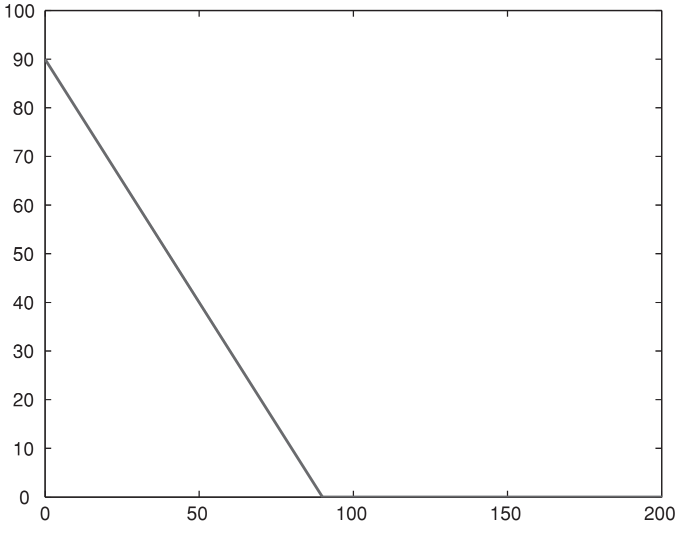

FIGURE 4.1. Put-payoff with exercise price 90.

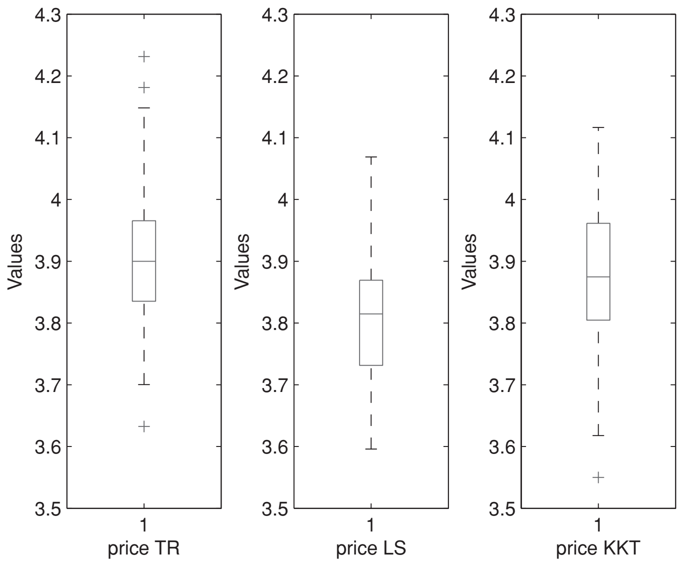

FIGURE 4.2. Boxplots for the realized option prices for the put-payoff of the Tsitsiklis–Van Roy (price TR), Longstaff–Schwartz (price LS) algorithms, and our algorithm (price KKT). In the boxplot the box stretches from the 25th percentile to the 75th percentile and the median is shown as a line across the box.

In our first example we analyze a standard put-payoff with exercise price 90 as illustrated in Figure 4.1, and simulate the paths of the underlying stock with a volatility of $\sigma = 0.25$. As we can see from Figure 4.2, our algorithm is slightly better than the LS algorithm and comparable to the TR algorithm. This is not surprising, since it is well known that for simple payoff functions the LS as well as the TR algorithms perform very well.

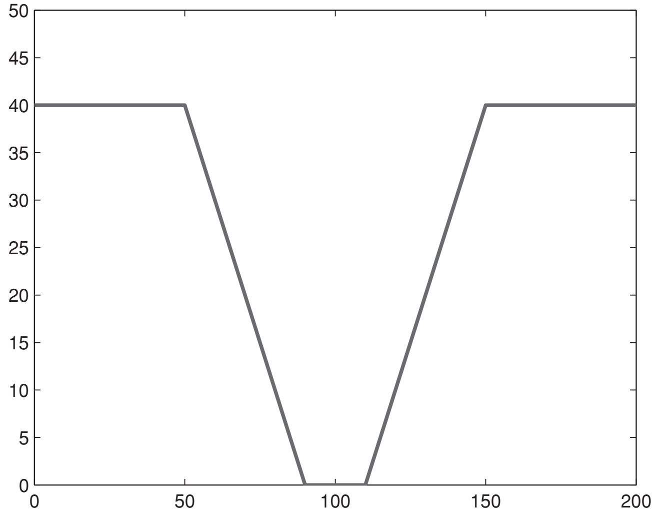

FIGURE 4.3. Strangle spread payoff with strike prices 50, 90, 110, and 150.

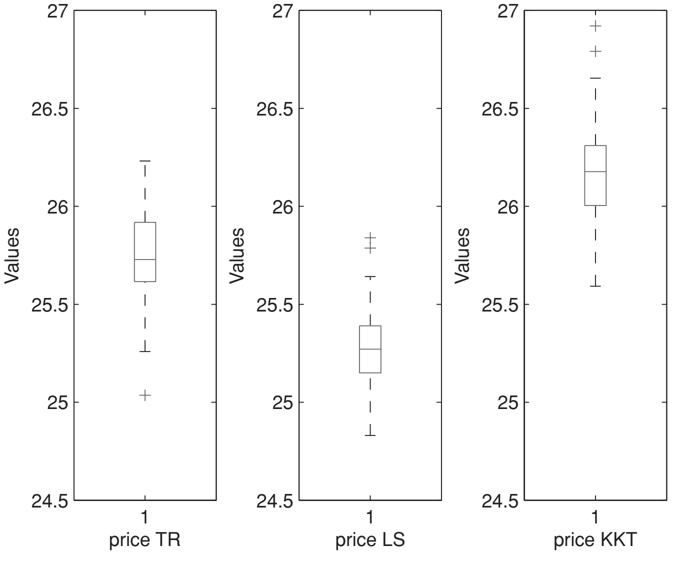

FIGURE 4.4. Realized option prices for the strangle spread-payoff of the Tsitsiklis–Van Roy (price TR), Longstaff–Schwartz (price LS), and our algorithm (price KKT) in a 1-dimensional case.

In our second example we make the pricing problem more difficult. We consider $m = 48$ time steps, a strangle spread payoff with strikes 50, 90, 110, and 150 as illustrated in Figure 4.3, and a large volatility of $\sigma = 0.5$. Figure 4.4 shows that our algorithm provides a higher MCE of the option price than the LS and the TR algorithms.

Finally, in our third example we consider the high-dimensional case and use for the pricing problem a strangle spread function with strikes 75, 90, 110, and 125 for the average of five correlated stock prices. The stocks are ADECCO R, BALOISE R, CIBA, CLARIANT, and CREDIT SUISSE R. The stock prices were observed from November

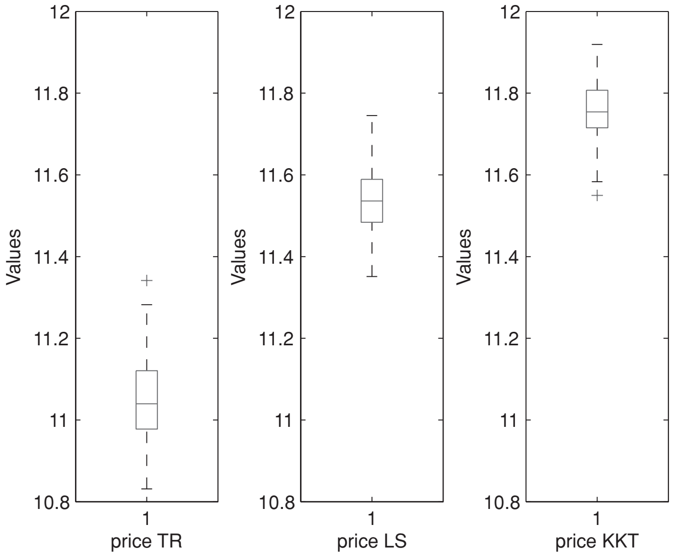

FIGURE 4.5. Realized option prices for the strangle spread-payoff of the Tsitsiklis–Van Roy (price TR), Longstaff–Schwartz (price LS), and our algorithm (price KKT) in a five-dimensional case.

10, 2000 until October 3, 2003 on weekdays when the stock market was open for the total of 756 days. We estimate the volatility from data observed in the past by the historical volatility

$$\sigma = \begin{pmatrix} 0.3024 & 0.1354 & 0.0722 & 0.1367 & 0.1641 \\ 0.1354 & 0.2270 & 0.0613 & 0.1264 & 0.1610 \\ 0.0722 & 0.0613 & 0.0717 & 0.0884 & 0.0699 \\ 0.1367 & 0.1264 & 0.0884 & 0.2937 & 0.1394 \\ 0.1641 & 0.1610 & 0.0699 & 0.1394 & 0.2535 \end{pmatrix}.$$

Again we used $x_0 = 100$, $r = 0.05$, and $m = 48$. As we can see in Figure 4.5, our algorithm is superior to the LS and the TR algorithms, since the higher boxplot of the MCE again indicates better performance.

REMARK 4.1. The algorithms have been implemented in Matlab. Computation of one of the values in Figures 4.2, 4.4, and 4.5 requires less than 1 minute for the LS and the TR algorithms and for our algorithm between 1.5 and 2.5 hours. Clearly this favors the use of the LS and TR methods for much higher sample size. However, due to the problems with the bias of the estimate the results produced by the LS and TR algorithms do not substantially improve regardless of the CPU speed and amount of memory used. So these algorithms will never produce results as good as the results of our algorithm in Figures 4.4 and 4.5, regardless of how much time or memory we allow them to use. Furthermore, our method has been implemented in Matlab in a straightforward manner and could be computed much faster by an efficient implementation in Matlab or C++.

5. PROOFS

5.1. Auxiliary Results

In the sequel we formulate auxiliary results which will be needed in the derivation of the rate of convergence. We start by defining so-called covering numbers.

Let $x_1, \ldots, x_n \in \mathbb{R}^d$ and set $x_1^n = (x_1, \ldots, x_n)$. Define the distance $d_2(f, g)$ between $f, g : \mathbb{R}^d \to \mathbb{R}$ by

$$d_2(f, g) = \left(\frac{1}{n} \sum_{i=1}^{n} |f(x_i) - g(x_i)|^2 \right)^{1/2}.$$

An ϵ-cover of $\mathcal{F}$ (w.r.t. the distance d_2) is a set of functions $f_1, \ldots, f_\kappa : \mathbb{R}^d \to \mathbb{R}$ with the property

$$\min_{1 \le j \le \kappa} d_2(f, f_j) < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

Let $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n)$ denote the size κ of the smallest ϵ-cover of $\mathcal{F}$ w.r.t. the distance d_2, and set $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n) = \infty$ if there does not exist any ϵ-cover of $\mathcal{F}$ of a finite size. $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n)$ is called L_2-ϵ-covering number of $\mathcal{F}$ on x_1^n.

In the Appendix, we prove the following bound on the covering number of $\mathcal{F}_k(\beta_n)$, where $\mathcal{F}_k(\beta_n)$ is defined by (2.1).

LEMMA 5.1. *Let $\mathcal{F}_k(\beta_n)$ be defined by (2.1), let $\epsilon > 0$ and let $x_1^n \in (\mathbb{R}^d)^n$. Then*

$$\mathcal{N}_2\big(\epsilon, \mathcal{F}_k(\beta_n), x_1^n\big) \le \left(\frac{12 e \beta_n (k+1)}{\epsilon} \right)^{(4d+9)k+1}.$$

In the proof we will use results concerning regression estimation in case of additional measurement errors in the dependent variable, which we describe in the sequel.

Let $(X, Y), (X_1, Y_1), \ldots$ be independent and identically distributed $\mathbb{R}^d \times \mathbb{R}$ valued random variables with $\mathbf{E} Y^2 < \infty$. Let $m(x) = \mathbf{E}\{Y | X = x\}$ be the corresponding regression function. Assume that we want to estimate m from observed data, but instead of a sample

$$\mathcal{D}_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$$

of (X, Y) we have only available a set of data

$$\bar{\mathcal{D}}_n = \{(X_1, \hat{Y}_{1,n}), \ldots, (X_n, \hat{Y}_{n,n})\}$$

where the only assumption on $\hat{Y}_{1,n}, \ldots, \hat{Y}_{n,n}$ is that the measurement error

$$\frac{1}{n} \sum_{i=1}^{n} |Y_i - \hat{Y}_{i,n}|^2 \tag{5.1}$$

is small. In particular, we do not assume that the random variables in $\bar{\mathcal{D}}_n$ are independent or identically distributed. In the sequel we are interested in the influence of the measurement error (5.1) on the L_2 error of a regression estimate applied to the data $\bar{\mathcal{D}}_n$.

As we do not assume anything on the difference between the true y-values Y_i and the observed values $\hat{Y}_{i,n}$ besides the assumption that (5.2) is small, it is clear that there is no chance to get rid of this measurement error completely. But a natural conjecture is that a small measurement error (5.2) does only slightly influence the L_2 error of suitably defined regression estimates. That this conjecture is indeed true was proven for the least squares estimates in Kohler (2006). Next we describe the part of this result, which will be needed in the proof of our main result.

Assume

$$Y_i,\, \hat{Y}_{i,n} \in [-L, L] \text{ a.s.}$$

($i = 1, \dots, n$) and define the estimate m_n by

$$m_n(\cdot) = \arg\min_{f \in \mathcal{F}_n} \left(\frac{1}{n} \sum_{i=1}^{n} |f(X_i) - \hat{Y}_{i,n}|^2 \right),$$

where $\mathcal{F}_n$ is a set of functions $f : \mathbb{R}^d \to \mathbb{R}$. Then the following result holds.

LEMMA 5.2. *Assume that $Y - m(X)$ is sub-Gaussian in the sense that*

$$C^2 \mathbf{E}\left\{ e^{(Y-m(X))^2/C^2} - 1 \mid X \right\} \le \sigma_0^2 \quad \textit{almost surely} \tag{5.2}$$

for some $C, \sigma_0 > 0$. Let $\beta_n, L \ge 1$ and assume that the regression function is bounded in absolute value by L and that β_n satisfies $\beta_n \to \infty$ $(n \to \infty)$. Let $\mathcal{F}_n$ be a set of functions $f : \mathbb{R}^d \to [-\beta_n, \beta_n]$ and define the estimate m_n as above. Then there exist constants $c_1, c_2, c_3 > 0$ depending only on σ_0 and C such that for any δ_n which satisfies

$$\delta_n \to 0 \quad (n \to \infty) \quad \textit{and} \quad \frac{n \cdot \delta_n}{\beta_n^2} \to \infty \quad (n \to \infty)$$

and

$$c_1 \frac{\sqrt{n}\delta}{\beta_n^2} \ge \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left(\log \mathcal{N}_2 \left(\frac{u}{4\beta_n}, \left\{ f - g : f \in \mathcal{F}_n, \frac{1}{n} \sum_{i=1}^{n} |f(x_i) - g(x_i)|^2 \le \frac{\delta}{\beta_n^2} \right\}, x_1^n \right) \right)^{1/2} du$$

for all $\delta \ge \delta_n$, all $x_1, \dots, x_n \in \mathbb{R}^d$ and all $g \in \mathcal{F}_n \cup \{m\}$ we have

$$\mathbf{P}\left\{ \int |m_n(x) - m(x)|^2 \mu(dx) > c_3 \left(\frac{1}{n} \sum_{i=1}^{n} |Y_i - \hat{Y}_{i,n}|^2 + \delta_n + \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \mu(dx) \right) \right\} \to 0$$

for $n \to \infty$.

Proof. See proof of theorem 1 in Kohler (2006) and observe that we can assume $\beta_n \ge L$ (since $\beta_n \to \infty$ for $n \to \infty$). □

The above lemma enables us to analyze the rate of convergence of the estimate for fixed function space. Next we explain how we can use the data to choose an appropriate function space from a finite collection

$$\{\mathcal{F}_{n,k} : k \in \mathcal{P}_n\}$$

of function spaces. To do this we split the sample into a learning sample

$$\hat{\mathcal{D}}_{n_l} = \left\{(X_1, \hat{Y}_{1,n}), \dots, (X_{n_l}, \hat{Y}_{n_l,n})\right\}$$

of size $n_l = \lceil n/2 \rceil$ and a testing sample

$$\{(X_{n_l+1}, \hat{Y}_{n_l+1,n}), \dots, (X_n, \hat{Y}_{n,n})\}$$

of size $n_t = n - n_l$. For fixed $k \in \mathcal{P}_n$ we use the learning sample to define an estimate $m_{n_l}^k$ by

$$m_{n_l}^k(\cdot) = \arg\min_{f \in \mathcal{F}_{n,k}} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |f(X_i) - \hat{Y}_{i,n}|^2 \right).$$

Next we choose $\hat{k} \in \mathcal{P}_n$ by minimizing the empirical L_2 risk on the testing sample, i.e., we set

$$m_n(x) = m_{n_l}^{\hat{k}}(x) \quad (x \in \mathbb{R}^d),$$

where

$$\hat{k} = \arg\min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} \left| m_{n_l}^k(X_i) - \hat{Y}_{i,n} \right|^2.$$

Then the following result holds.

LEMMA 5.3. *Assume that $Y - m(X)$ is sub-Gaussian in the sense that (5.3) holds for some $C, \sigma > 0$ and assume $|\mathcal{P}_n| \to \infty \, (n \to \infty)$. Assume furthermore that conditioned on $X_1, \dots, X_n$ the data sets*

$$\hat{\mathcal{D}}_{n_l} \quad \textit{and} \quad \{Y_{n_l+1}, \dots, Y_n\}$$

are independent. Let for each $k \in \mathcal{P}_n$ a set $\mathcal{F}_{n,k}$ of functions $f : \mathbb{R}^d \to \mathbb{R}$ be given and let the estimate m_n be defined as above. Then

$$\begin{aligned}
&\frac{1}{n_t} \sum_{i=n_l+1}^{n} |m_n(X_i) - m(X_i)|^2 \\
&\quad = O_{\mathbf{P}} \left(\frac{\log |\mathcal{P}_n|}{n_t} + \frac{1}{n_t} \sum_{i=n_l+1}^{n} |Y_i - \hat{Y}_{i,n}|^2 + \min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} \left| m_{n_l}^k(X_i) - m(X_i) \right|^2 \right).
\end{aligned}$$

Proof. The result follows by applying lemma 2 in Kohler (2006) (see Lemma A.12 in the Appendix) conditioned on $\hat{\mathcal{D}}_{n_l}$ and $X_1, \dots, X_n$ and with

$$\mathcal{F}_n = \left\{ m_{n_l}^k \; : \; k \in \mathcal{P}_n \right\}.$$

Here we bound the covering number by the finite cardinality $|\mathcal{P}_n|$ of the set of estimates. □

5.2. Proof of Theorem 3.1

Before we start with the proof, observe that the boundedness of the discounted payoff function f_t by L implies $|q_t(x)| \le L$ for $x \in \mathbb{R}^d$. W.l.o.g. assume $\beta_n \ge L$ (since $\beta_n \to \infty$ for n tending to infinity).

In the sequel we will show

$$\text{(5.3)} \quad \int |\hat{q}_{n,s}(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) = O_{\mathbf{P}}\left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{t \in \{s,s+1,\dots,T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \right)$$

for all $s \in \{0, 1, \dots, T\}$.

For $s = T$ we have $\hat{q}_{n,T}(x) = 0 = q_T(x)$, so the assertion is trivial. So let $t < T$ and assume that the assertion holds for $s \in \{t+1, \dots, T\}$. By induction it suffices to show (5.3) for $s = t$, which we will show in the sequel in seven steps.

In the first step of the proof we show

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}}\left(\frac{1}{n_t} \sum_{i=n_l+1}^{n} \left|\hat{q}_{n,t}\left(X_{i,t}^{(t)}\right) - q_t\left(X_{i,t}^{(t)}\right)\right|^2 + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_t} \right).$$

Let $\mathcal{D}_{n,t}$ be the set of all $X_{j,s}^{(r)}$ with either $r \ge t+1$, $s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n\}$ or $r = t$, $s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n_l\}$. Conditioned on $\mathcal{D}_{n,t}$,

$$\left\{\hat{q}_{n_l,t}^k : k \in \mathcal{P}_n\right\}$$

consists of $|\mathcal{P}_n|$ different functions. Furthermore, because of boundedness of $\hat{q}_{n_l,t}^k$ and q_t by β_n we have

$$\begin{aligned}
\sigma_k^2 &:= \mathbf{Var}\left\{\left|\hat{q}_{n_l,t}^k\left(X_{n_l+1,t}^{(t)}\right) - q_t\left(X_{n_l+1,t}^{(t)}\right)\right|^2 \middle| \mathcal{D}_{n,t}\right\} \\
&\le \mathbf{E}\left\{\left|\hat{q}_{n_l,t}^k\left(X_{n_l+1,t}^{(t)}\right) - q_t\left(X_{n_l+1,t}^{(t)}\right)\right|^4 \middle| \mathcal{D}_{n,t}\right\} \\
&\le 4\beta_n^2 \int \left|\hat{q}_{n_l,t}^k(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx).
\end{aligned}$$

Using this and the Bernstein inequality (see Lemma A.11 in the Appendix) we get using the notation $\epsilon_n = c_4 \cdot (\beta_n^4 \log |\mathcal{P}_n|)/n_t$

$$\begin{aligned}
&\mathbf{P}\left\{ \int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) > \frac{2}{n_t} \sum_{i=n_l+1}^{n} \left|\hat{q}_{n,t}\left(X_{i,t}^{(t)}\right) - q_t\left(X_{i,t}^{(t)}\right)\right|^2 + \epsilon_n \mid \mathcal{D}_{n,t} \right\} \\
&\le |\mathcal{P}_n| \cdot \max_{k \in \mathcal{P}_n} \mathbf{P}\left\{ \int \left|\hat{q}_{n_l,t}^k(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx) \right. \\
&\qquad \left. > \frac{2}{n_t} \sum_{i=n_l+1}^{n} \left|\hat{q}_{n_l,t}^k\left(X_{i,t}^{(t)}\right) - q_t\left(X_{i,t}^{(t)}\right)\right|^2 + \epsilon_n \mid \mathcal{D}_{n,t} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \mathbf{P}\left\{ 2\left(\int \left|\hat{q}^k_{n_l,t}(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx) - \frac{1}{n_t}\sum_{i=n_l+1}^{n} \left|\hat{q}^k_{n_l,t}\big(X^{(t)}_{i,t}\big) - q_t\big(X^{(t)}_{i,t}\big)\right|^2 \right) \right. \\
&\qquad\qquad \left. > \int \left|\hat{q}^k_{n_l,t}(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx) + \epsilon_n \,\middle|\, \mathcal{D}_{n,t} \right\} \\
&= |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \mathbf{P}\left\{ \int \left|\hat{q}^k_{n_l,t}(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx) - \frac{1}{n_t}\sum_{i=n_l+1}^{n} \left|\hat{q}^k_{n_l,t}\big(X^{(t)}_{i,t}\big) - q_t\big(X^{(t)}_{i,t}\big)\right|^2 \right. \\
&\qquad\qquad \left. > \frac{1}{2}\int \left|\hat{q}^k_{n_l,t}(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx) + \frac{\epsilon_n}{2} \,\middle|\, \mathcal{D}_{n,t} \right\} \\
&\le |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \mathbf{P}\left\{ \int \left|\hat{q}^k_{n_l,t}(x) - q_t(x)\right|^2 \mathbf{P}_{X_t}(dx) - \frac{1}{n_t}\sum_{i=n_l+1}^{n} \left|\hat{q}^k_{n_l,t}\big(X^{(t)}_{i,t}\big) - q_t\big(X^{(t)}_{i,t}\big)\right|^2 \right. \\
&\qquad\qquad \left. > \frac{\sigma_k^2}{8\beta_n^2} + \frac{\epsilon_n}{2} \,\middle|\, \mathcal{D}_{n,t} \right\} \\
&\le |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \exp\left(\frac{-n_t \cdot \left(\frac{\sigma_k^2}{8\beta_n^2} + \frac{\epsilon_n}{2} \right)^2}{2\sigma_k^2 + 2\left(\frac{\sigma_k^2}{8\beta_n^2} + \frac{\epsilon_n}{2} \right) \cdot \frac{4\beta_n^2}{3}} \right) \\
&= |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \exp\left(\frac{-n_t \cdot \left(\left(\sigma_k^2 + 4\epsilon_n\beta_n^2\right)/8\beta_n^2 \right)^2}{2\sigma_k^2 + 2\left(\frac{\sigma_k^2 + 4\epsilon_n\beta_n^2}{8\beta_n^2} \right) \cdot \frac{4\beta_n^2}{3}} \right) \\
&= |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \exp\left(\frac{-n_t \cdot \left(\sigma_k^2 + 4\epsilon_n\beta_n^2\right)}{2\left(8\beta_n^2\right)^2 \frac{\sigma_k^2}{\sigma_k^2 + 4\epsilon_n\beta_n^2} + 2\left(8\beta_n^2\right) \cdot \frac{4\beta_n^2}{3}} \right) \\
&\le |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \exp\left(\frac{-4n_t \cdot \epsilon_n\beta_n^2}{128\beta_n^4 + 64\beta_n^4/3} \right) \\
&= |\mathcal{P}_n| \cdot \max_{k\in\mathcal{P}_n} \exp\left(-\frac{3}{112} c_4 \beta_n^2 \log|\mathcal{P}_n| \right) \\
&\to 0 \qquad (n \to \infty).
\end{aligned}
$$

In the second step of the proof, we show

$$\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n,t}\left(X_{i,t}^{(t)}\right)-q_t\left(X_{i,t}^{(t)}\right)\right|^2$$
$$=O_{\mathbf{P}}\left(\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)-q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2+\frac{\log|\mathcal{P}_n|}{n_t}\right.$$
$$\left.+\min_{k\in\mathcal{P}_n}\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|q_{n_l,t}^{k}\left(X_{i,t}^{(t)}\right)-q_t\left(X_{i,t}^{(t)}\right)\right|^2\right).$$

To do this we apply Lemma 5.3. In the context of Lemma 5.3 we have $X_i = X_{i,t}^{(t)}$,

(5.4)
$$Y_i=\max\left\{f_{t+1}\left(X_{i,t+1}^{(t)}\right),q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right\}\text{ and }\hat{Y}_{i,n}=\max\left\{f_{t+1}\left(X_{i,t+1}^{(t)}\right),\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)\right\}.$$

Observing

$$\frac{1}{n_t}\sum_{i=n_l+1}^{n}|Y_i-\hat{Y}_{i,n}|^2\leq\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|q_{t+1}\left(X_{i,t+1}^{(t)}\right)-\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2$$

the assertion follows from Lemma 5.3 if we apply it conditioned on $\mathcal{D}_{n,t}$.

In the third step of the proof, we show

$$\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)-q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2$$
$$=O_{\mathbf{P}}\left(\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)+\frac{\beta_n^4\cdot\log|\mathcal{P}_n|}{n_t}\right).$$

Using

$$\mathbf{P}\left\{\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)-q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2\right.$$
$$\left.>2\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)+\epsilon_n\mid\mathcal{D}_{n,t}\right\}$$
$$=\mathbf{P}\left\{\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right)-q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2-\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)\right.$$
$$\left.>\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)+\epsilon_n\mid\mathcal{D}_{n,t}\right\}$$

this follows as in the first step by an application of the Bernstein inequality.

In the fourth step of the proof, we show

$$\min_{k\in\mathcal{P}_n}\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n_l,t}^{k}\left(X_{i,t}^{(t)}\right)-q_t\left(X_{i,t}^{(t)}\right)\right|^2=O_{\mathbf{P}}\left(\int\left|\hat{q}_{n_l,t}^{k_n}(x)-q_t(x)\right|^2\mathbf{P}_{X_t}(dx)+\frac{\beta_n^4\cdot\log|\mathcal{P}_n|}{n_t}\right).$$

To see this, we observe that we have as in the third step of the proof

$$\begin{aligned}
&\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n_l,t}^{k_n}\left(X_{i,t}^{(t)}\right)-q_t\left(X_{i,t}^{(t)}\right)\right|^2\\
&\quad= O_{\mathbf{P}}\left(\int\left|\hat{q}_{n_l,t}^{k_n}(x)-q_t(x)\right|^2\mathbf{P}_{X_t}(dx)+\frac{\beta_n^4\cdot\log|\mathcal{P}_n|}{n_t}\right),
\end{aligned}$$

hence the assertion follows from

$$\begin{aligned}
&\min_{k\in\mathcal{P}_n}\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n_l,t}^{k}\left(X_{i,t}^{(t)}\right)-q_t\left(X_{i,t}^{(t)}\right)\right|^2\\
&\quad\le\frac{1}{n_t}\sum_{i=n_l+1}^{n}\left|\hat{q}_{n_l,t}^{k_n}\left(X_{i,t}^{(t)}\right)-q_t\left(X_{i,t}^{(t)}\right)\right|^2.
\end{aligned}$$

In the fifth step of the proof, we show

$$\begin{aligned}
&\int\left|\hat{q}_{n_l,t}^{k_n}(x)-q_t(x)\right|^2\mathbf{P}_{X_t}(dx)\\
&\quad= O_{\mathbf{P}}\left(\frac{1}{n_l}\sum_{i=1}^{n_l}|Y_i-\hat{Y}_{i,n}|^2+\delta_n+\inf_{f\in\mathcal{F}_{k_n}(\beta_n)}\int|f(x)-q_t(x)|^2\mathbf{P}_{X_t}(dx)\right),
\end{aligned}$$

where $\delta_n = c_6\cdot\beta_n^4\cdot k_n\cdot\log(n)/n$ and Y_i and $\hat{Y}_{i,n}$ are defined in (5.5). To do this we show that with this choice of δ_n the conditions of Lemma 5.2 are satisfied.

Observe that Y_1 is bounded in absolute value by L, hence (5.2) holds. By Lemma 5.1 we get for $g\in\mathcal{F}_{k_n}(\beta_n)\cup\{q_t\}$

$$\begin{aligned}
&\mathcal{N}_2\left(\frac{u}{4\beta_n},\left\{f-g: f\in\mathcal{F}_{k_n}(\beta_n),\ \frac{1}{n}\sum_{i=1}^{n}|f(X_i)-g(X_i)|^2\le\frac{\delta}{\beta_n^2}\right\},\ X_1^n\right)\\
&\quad\le\mathcal{N}_2\left(\frac{u}{4\beta_n},\mathcal{F}_{k_n}(\beta_n),\ X_1^n\right)\\
&\quad\le\left(\frac{48e\beta_n^2(k_n+1)}{u}\right)^{(4d+9)k_n+1},
\end{aligned}$$

thus

$$\begin{aligned}
&\int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}}\left\{\log\mathcal{N}_2\left(\frac{u}{4\beta_n},\left\{f-g: f\in\mathcal{F}_{k_n}(\beta_n),\ \frac{1}{n}\sum_{i=1}^{n}|f(X_i)-g(X_i)|^2\le\frac{\delta}{\beta_n^2}\right\},\ X_1^n\right)\right\}^{1/2}du\\
&\le\int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}}\left\{\log\left(\frac{48e\beta_n^2(k_n+1)}{u}\right)^{(4d+9)k_n+1}\right\}^{1/2}du.
\end{aligned}$$

Let $\delta > 1/n$. Then by bounding u from below by $(c_2\delta)/\beta_n^2$ and using constant $c_5>0$ we get

$$\begin{aligned}
&\int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log\left(\frac{48e\beta_n^2(k_n+1)}{u} \right)^{(4d+9)k_n+1} \right\}^{1/2} du \\
&\leq \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log\left(\frac{48e\beta_n^4(k_n+1)}{c_2\delta} \right)^{(4d+9)k_n+1} \right\}^{1/2} du \\
&\leq \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log\left(\frac{48e\beta_n^4(k_n+1)n}{c_2} \right)^{(4d+9)k_n+1} \right\}^{1/2} du \\
&\leq c_5\sqrt{\delta}\sqrt{k_n}(\log(n))^{1/2}.
\end{aligned}$$

This together with

$$\begin{aligned}
\frac{c_1\sqrt{n}\delta}{\beta_n^2} &\geq c_5\sqrt{\delta}\sqrt{k_n}\sqrt{\log(n)} \\
\Leftrightarrow \delta &\geq c_6\beta_n^4 k_n \frac{\log(n)}{n}
\end{aligned}$$

shows that

$$\delta_n := c_6\beta_n^4 k_n \frac{\log(n)}{n}$$

satisfies the condition of Lemma 5.2.

In the sixth step of the proof, we show

$$\frac{1}{n_l}\sum_{i=1}^{n_l} |Y_i - \hat{Y}_{i,n}|^2 = O_{\mathbf{P}}\left(\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \frac{\beta_n^4 \cdot \log|\mathcal{P}_n|}{n_l} \right).$$

First we observe

$$\frac{1}{n_l}\sum_{i=1}^{n_l} |Y_i - \hat{Y}_{i,n}|^2 \leq \frac{1}{n_l}\sum_{i=1}^{n_l} \left|\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right) - q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2.$$

To show

$$\begin{aligned}
&\frac{1}{n_l}\sum_{i=1}^{n_l} \left|\hat{q}_{n,t+1}\left(X_{i,t+1}^{(t)}\right) - q_{t+1}\left(X_{i,t+1}^{(t)}\right)\right|^2 \\
&\quad = O_{\mathbf{P}}\left(\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \frac{\beta_n^4 \cdot \log|\mathcal{P}_n|}{n_l} \right),
\end{aligned}$$

we condition on all data points $X_{j,s}^{(r)}$ with $r \geq t+1, s \in \{0, \ldots, T\}$ and $j \in \{1, \ldots, n\}$. Then the assertion follows by an application of Bernstein inequality as in steps 1 and 3.

In the seventh (and last) step of the proof, we observe that we get by induction

$$\begin{aligned}
&\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) \\
&\quad = O_{\mathbf{P}}\left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s\in\{t+1,\ldots,T-1\}} \inf_{f\in\mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \right).
\end{aligned}$$

We complete the proof by gathering the above results. □

APPENDIX

LEMMA A.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of real functions on $\mathbb{R}^m$. If $\mathcal{F} \oplus \mathcal{G}$ denotes the set of functions $\{f + g : f \in \mathcal{F}, g \in \mathcal{G}\}$, then for any $z_1^n \in \mathbb{R}^{n \cdot m}$ and $\epsilon, \delta > 0$, we have*

$$\mathcal{N}_2(\epsilon + \delta, \mathcal{F} \oplus \mathcal{G}, z_1^n) \leq \mathcal{N}_2(\epsilon, \mathcal{F}, z_1^n)\mathcal{N}_2(\delta, \mathcal{G}, z_1^n) .$$

Proof of Lemma A.1. Let $\{f_1, \ldots, f_{\mathcal{K}}\}$ and $\{g_1, \ldots, g_\Lambda\}$ be an ϵ-cover and a δ-cover of $\mathcal{F}$ and $\mathcal{G}$, respectively, on z_1^n of minimal size. Then, for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, there exist $\kappa \in \{1, \ldots, \mathcal{K}\}$ and $\lambda \in \{1, \ldots, \Lambda\}$ such that

$$\left(\frac{1}{n}\sum_{i=1}^{n} |f(z_i) - f_\kappa(z_i)|^2\right)^{1/2} < \epsilon$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n} |g(z_i) - g_\lambda(z_i)|^2\right)^{1/2} < \delta .$$

By the triangle inequality for norms we have

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{i=1}^{n} |f(z_i) + g(z_i) - (f_\kappa(z_i) - g_\lambda(z_i))|^2\right)^{1/2} \\
&\quad \leq \left(\frac{1}{n}\sum_{i=1}^{n} |f(z_i) - f_\kappa(z_i)|^2\right)^{1/2} + \left(\frac{1}{n}\sum_{i=1}^{n} |g(z_i) - g_\lambda(z_i)|^2\right)^{1/2} \\
&\quad \leq \epsilon + \delta
\end{aligned}$$

which proves that $\{f_\kappa + g_\lambda : 1 \leq \kappa \leq \mathcal{K}, 1 \leq \lambda \leq \Lambda\}$ is an $(\epsilon + \delta)$-cover of $\mathcal{F} \oplus \mathcal{G}$ on z_1^n. $\square$

LEMMA A.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of real functions on $\mathbb{R}^m$ such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in \mathbb{R}^m$, $f \in \mathcal{F}$, $g \in \mathcal{G}$. If $\mathcal{F} \odot \mathcal{G}$ denotes the set of functions $\{f \cdot g : f \in \mathcal{F}, g \in \mathcal{G}\}$ then, for any $z_1^n \in \mathbb{R}^{n \cdot m}$ and $\epsilon, \delta > 0$ we have*

$$\mathcal{N}_2(\epsilon + \delta, \mathcal{F} \odot \mathcal{G}, z_1^n) \leq \mathcal{N}_2(\epsilon / M_2, \mathcal{F}, z_1^n)\mathcal{N}_2(\delta / M_1, \mathcal{G}, z_1^n).$$

Proof of Lemma A.2. Let $\{f_1, \ldots, f_{\mathcal{K}}\}$ and $\{g_1, \ldots, g_\Lambda\}$ be an ϵ / M_2-cover and a δ / M_1-cover of $\mathcal{F}$ and $\mathcal{G}$, respectively, on z_1^n of minimal size. By the boundedness of f and g we can assume w.l.o.g. $|f_\kappa(z)| \leq M_1$, $|g_\lambda(z)| \leq M_2$. For every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, there exist $\kappa \in \{1, \ldots, \mathcal{K}\}$ and $\lambda \in \{1, \ldots, \Lambda\}$ such that

$$\left(\frac{1}{n}\sum_{i=1}^{n} |f(z_i) - f_\kappa(z_i)|^2\right)^{1/2} < \frac{\epsilon}{M_2}$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n} |g(z_i) - g_\lambda(z_i)|^2\right)^{1/2} < \frac{\delta}{M_1} .$$

We have, by the triangle inequality of norms

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{i=1}^{n}|f(z_i)\cdot g(z_i)-f_\kappa(z_i)\cdot g_\lambda(z_i)|^2\right)^{1/2}\\
&\quad=\left(\frac{1}{n}\sum_{i=1}^{n}|f(z_i)\cdot(g_\lambda(z_i)+g(z_i)-g_\lambda(z_i))-f_\kappa(z_i)\cdot g_\lambda(z_i)|^2\right)^{1/2}\\
&\quad\leq\left(\frac{1}{n}\sum_{i=1}^{n}|g_\lambda(z_i)\cdot(f(z_i)-f_\kappa(z_i))|^2\right)^{1/2}+\left(\frac{1}{n}\sum_{i=1}^{n}|f(z_i)\cdot(g(z_i)-g_\lambda(z_i))|^2\right)^{1/2}\\
&\quad\leq M_2\left(\frac{1}{n}\sum_{i=1}^{n}|f(z_i)-f_\kappa(z_i)|^2\right)^{1/2}+M_1\left(\frac{1}{n}\sum_{i=1}^{n}|g(z_i)-g_\lambda(z_i)|^2\right)^{1/2}\\
&\quad\leq\epsilon+\delta
\end{aligned}$$

which implies that $\{f_\kappa\cdot g_\lambda : 1\leq\kappa\leq\mathcal{K}, 1\leq\lambda\leq\Lambda\}$ is an $(\epsilon+\delta)$-cover of $\mathcal{F}\odot\mathcal{G}$ on z_1^n. □

Before presenting proof of Lemma 5.1 we define a few required quantities and cite a few results from the literature.

DEFINITION A.3. Let $\epsilon>0$, let $\mathcal{G}$ be a set of functions $\mathbb{R}^d\to\mathbb{R}$, $1\leq p<\infty$, and let ν be a probability measure on $\mathbb{R}^d$. Define for a function $f:\mathbb{R}^d\to\mathbb{R}$,

$$\|f\|_{L_p(\nu)}:=\left\{\int|f(z)|^p d\nu\right\}^{\frac{1}{p}}.$$

(a) Every finite collection of functions $g_1,\ldots,g_N\in\mathcal{G}$ with

$$\|g_j-g_k\|_{L_p(\nu)}\geq\epsilon$$

for all $1\leq j<k\leq N$ is called an **ϵ-packing of $\mathcal{G}$** w.r.t. $\|\cdot\|_{L_p(\nu)}$.

(b) Let $\mathcal{M}(\epsilon,\mathcal{G},\|\cdot\|_{L_p(\nu)})$ be the size of the largest ϵ-packing of $\mathcal{G}$ w.r.t. $\|\cdot\|_{L_p(\nu)}$. Take $\mathcal{M}(\epsilon,\mathcal{G},\|\cdot\|_{L_p(\nu)})=\infty$ if there exists an ϵ-packing of $\mathcal{G}$ w.r.t. $\|\cdot\|_{L_p(\nu)}$ of size N for every $N\in\mathbb{N}$. Then $\mathcal{M}(\epsilon,\mathcal{G},\|\cdot\|_{L_p(\nu)})$ is called an **ϵ-packing number of $\mathcal{G}$** w.r.t. $\|\cdot\|_{L_p(\nu)}$.

(c) Let $z_1^n=(z_1,\ldots,z_n)$ be n fixed points in $\mathbb{R}^d$. Let ν_n be the corresponding empirical measure, i.e.,

$$\nu_n(A)=\frac{1}{n}\sum_{i=1}^{n}I_A(z_i)\quad(A\subseteq\mathbb{R}^d).$$

Then

$$\|f\|_{L_p(\nu_n)}=\left\{\frac{1}{n}\sum_{i=1}^{n}|f(z_i)|^p\right\}^{\frac{1}{p}}$$

and any ϵ-packing of $\mathcal{G}$ w.r.t. $\|\cdot\|_{L_p(\nu_n)}$ will be called an L_p **ϵ-packing of $\mathcal{G}$ on** z_1^n and the ϵ-packing number of $\mathcal{G}$ w.r.t. $\|\cdot\|_{L_p(\nu_n)}$ will be denoted by

$$\mathcal{M}_p\left(\epsilon, \mathcal{G}, z_1^n\right).$$

In other words, $\mathcal{M}_p(\epsilon, \mathcal{G}, z_1^n)$ is the maximal $N \in \mathbb{N}$ such that there exist functions $g_1, \ldots, g_N \in \mathcal{G}$ with

$$\left\{\frac{1}{n}\sum_{i=1}^{n} |g_j(z_i) - g_k(z_i)|^p\right\}^{\frac{1}{p}} \geq \epsilon$$

for all $1 \leq j < k \leq N$.

DEFINITION A.4. Let $\mathcal{A}$ be a class of subsets of $\mathbb{R}^d$ and let $n \in \mathbb{N}$.

(a) For $z_1, \ldots, z_n \in \mathbb{R}^d$ define

$$s\left(\mathcal{A}, \{z_1, \ldots, z_n\}\right) = \left|\{A \cap \{z_1, \ldots, z_n\} : A \in \mathcal{A}\}\right|,$$

that is, $s(\mathcal{A}, \{z_1, \ldots, z_n\})$ is the number of different subsets of $\{z_1, \ldots, z_n\}$ of the form $A \cap \{z_1, \ldots, z_n\}$, $A \in \mathcal{A}$.

(b) Let G be a subset of $\mathbb{R}^d$ of size n. One says that $\mathcal{A}$ **shatters** G if $s(\mathcal{A}, G) = 2^n$, i.e., if each subset of G can be represented in the form $A \cap G$ for some $A \in \mathcal{A}$.

(c) The nth **shatter coefficient** of $\mathcal{A}$ is

$$S(\mathcal{A}, n) = \max_{\{z_1,\ldots,z_n\}\subseteq\mathbb{R}^d} s\left(\mathcal{A}, \{z_1, \ldots, z_n\}\right).$$

That is, the shatter coefficient is the maximal number of different subsets of n points that can be picked out by sets from $\mathcal{A}$.

DEFINITION A.5. Let $\mathcal{A}$ be a class of subsets of $\mathbb{R}^d$ with $\mathcal{A} \neq \emptyset$. The **VC dimension** (or Vapnik–Chervonenkis dimension) $V_{\mathcal{A}}$ of $\mathcal{A}$ is defined by

$$V_{\mathcal{A}} = \sup\left\{n \in \mathbb{N} : S(\mathcal{A}, n) = 2^n\right\},$$

i.e., the VC dimension $V_{\mathcal{A}}$ is the largest integer n such that there exists a set of n points in $\mathbb{R}^d$ which can be shattered by $\mathcal{A}$.

THEOREM A.6. *Let $\mathcal{G}$ be a class of functions $g : \mathbb{R}^d \to [0, B]$ with $V_{\mathcal{G}^+} \geq 2$, let $p \geq 1$, let ν be a probability measure on $\mathbb{R}^d$, and let $0 < \epsilon < \frac{B}{4}$. Then*

$$\mathcal{M}\left(\epsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}\right) \leq 3\left(\frac{2eB^p}{\epsilon^p}\log\frac{3eB^p}{\epsilon^p}\right)^{V_{\mathcal{G}^+}}.$$

THEOREM A.7. *Let $\mathcal{G}$ be an r-dimensional vector space of real functions on $\mathbb{R}^d$, and set*

$$\mathcal{A} = \left\{\{z : g(z) \geq 0\} : g \in \mathcal{G}\right\}.$$

Then

$$V_{\mathcal{A}} \leq r.$$

LEMMA A.8. *Let $\mathcal{F}$ be a family of real functions on $\mathbb{R}^m$, and let $g : \mathbb{R} \to \mathbb{R}$ be a fixed nondecreasing function. Define the class $\mathcal{G} = \{g \circ f : f \in \mathcal{F}\}$. Then*

$$V_{\mathcal{G}^+} \leq V_{\mathcal{F}^+}.$$

The proofs of Theorems A.6, A.7, and Lemma A.8 can be found in Györfi et al. (2002).

Proof of Lemma 5.1. Define the following classes of functions:

$$\begin{aligned}
\mathcal{G}_1 &= \{a^T x + b : a \in \mathbb{R}^d, b \in \mathbb{R}\},\\
\mathcal{G}_2 &= \{\sigma(a^T x + b) : a \in \mathbb{R}^d, b \in \mathbb{R}\},\\
\mathcal{G}_3 &= \{c \cdot \sigma(a^T x + b) : a \in \mathbb{R}^d, b \in \mathbb{R}, c \in [-\beta_n, \beta_n]\},
\end{aligned}$$

where $\sigma : \mathbb{R} \to [0, 1]$ is a sigmoid function (i.e., σ is a nondecreasing function with the property $\lim_{x\to-\infty} \sigma(x) = 0$ and $\lim_{x\to\infty} \sigma(x) = 1$) and $\beta_n > 0$.

$\mathcal{G}_1$ is a linear vector space of dimension $d + 1$, thus Theorem A.7 implies

$$V_{\mathcal{G}_1^+} \leq d + 2,$$

where $\mathcal{G}^+$ denotes the set

$$\mathcal{G}^+ = \{\{(z, t) \in \mathbb{R}^d \times \mathbb{R}, t \leq g(z)\}; g \in \mathcal{G}\}$$

for all subgraphs of functions of $\mathcal{G}$ and $V_{\mathcal{G}^+}$ is VC-dimension of $\mathcal{G}^+$ (see Definition A.5).

Since σ is a nondecreasing function, Lemma A.8 yields

$$V_{\mathcal{G}_2^+} \leq d + 2\ .$$

Thus, by Theorem A.6, we have for $0 < \epsilon < 1/4$

$$\begin{aligned}
\mathcal{N}_2\big(\epsilon, \mathcal{G}_2, x_1^n\big) &\leq \mathcal{M}_2\big(\epsilon, \mathcal{G}_2, x_1^n\big)\\
&\leq 3\left(\frac{2e}{\epsilon^2}\log\frac{3e}{\epsilon^2}\right)^{d+2}\\
&\leq 3\left(\frac{3e}{\epsilon^2}\right)^{2d+4}.
\end{aligned}$$

By Lemma A.2 we have for $0 < \epsilon/2\beta_n < 1/4$ or equivalently $0 < \epsilon < \beta_n/2$

$$\begin{aligned}
\mathcal{N}_2\big(\epsilon, \mathcal{G}_3, x_1^n\big) &\leq \mathcal{N}_2\left(\frac{\epsilon}{2}, \{c : |c| \leq \beta_n\}, x_1^n\right)\mathcal{N}_2\left(\frac{\epsilon}{2\beta_n}, \mathcal{G}_2, x_1^n\right)\\
&\leq \frac{2\beta_n}{(\epsilon/2)} \cdot 3\left(\frac{3e}{(\epsilon/(2\beta_n))^2}\right)^{2d+4}\\
&\leq \left(\frac{12e\beta_n}{\epsilon}\right)^{4d+9}.
\end{aligned}$$

By applying Lemma A.1 we obtain for $0 < \epsilon < (k+1)\cdot\beta_n/2$

$$\begin{aligned}\mathcal{N}_2\big(\epsilon, \mathcal{F}_k(\beta_n), x_1^n\big) &\le \mathcal{N}_2\left(\frac{\epsilon}{k+1}, \{c_0 : |c_0| \le \beta_n\}, x_1^n\right)\left(\mathcal{N}_2\left(\frac{\epsilon}{k+1}, \mathcal{G}_3, x_1^n\right)\right)^k \\ &\le \frac{2\beta_n(k+1)}{\epsilon}\left(\frac{12e\beta_n(k+1)}{\epsilon}\right)^{(4d+9)k} \\ &\le \left(\frac{12e\beta_n(k+1)}{\epsilon}\right)^{(4d+9)k+1}.\end{aligned}$$

By boundedness of $\mathcal{F}_n(\beta_n)$, the proof is trivial for $\epsilon \ge (k+1)\cdot\beta_n/2 \ge \beta_n$, which completes the proof. □

LEMMA A.9. *Let σ be a squashing function. Then for every probability measure μ on $\mathcal{R}^d$, every measurable function $f : \mathcal{R}^d \to \mathbb{R}$ with $\int |f(x)|^2\mu(dx) < \infty$, and every $\epsilon > 0$, there exists a neural network*

$$h(x) = \sum_{i=1}^{k} c_i\sigma\big(a_i^T x + b_i\big) + c_0 \quad (k \in \mathbb{N}, a_i \in \mathcal{R}^d, b_i, c_i \in \mathbb{R})$$

such that

$$\int |f(x) - h(x)|^2\mu(dx) < \epsilon.$$

For the proof refer to lemma 16.2 in Györfi et al. (2002).

LEMMA A.10. *Let σ be a squashing function. Then, for every probability measure μ on $\mathcal{R}^d$, every measurable $f \in \mathcal{F}_C$, and every $k \ge 1$, there exists a neural network f_k in*

$$\mathcal{F}_k = \left\{\sum_{i=1}^{k} c_i\sigma\big(a_i^T x + b_i\big) + c_0; \quad k \in \mathbb{N},\ a_i \in \mathcal{R}^d,\ b_i, c_i \in \mathbb{R}\right\} \tag{A.1}$$

such that

$$\int_{S_r} (f(x) - f_k(x))^2\mu(dx) \le \frac{(2rC)^2}{k} \tag{A.2}$$

where S_r is the sphere with radius r centered at 0. *The coefficients of the linear combination in (A.1) may be chosen so that $\sum_{i=0}^{k}|c_i| \le 3rC + f(0)$.*

For the proof refer to lemma 16.8 in Györfi et al. (2002).

LEMMA A.11 (Bernstein inequality). *Let $X_1, \ldots, X_n$ be independent real-valued random variables, let $a, b \in \mathbb{R}$ with $a < b$, and assume that $X_i \in [a, b]$ with probability one* ($i = 1, \ldots, n$). *Let*

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Var}\{X_i\} > 0.$$

Then, for all $\epsilon > 0$,

$$\mathbf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbf{E}\{X_i\})\right| > \epsilon\right\} \le 2\exp\left(-\frac{n\epsilon^2}{2\sigma^2 + 2\epsilon(b-a)/3}\right).$$

For the proof refer to lemma A.2 in Györfi et al. (2002).

LEMMA A.12. *Let*

$$Y_i = m(x_i) + W_i \quad (i = 1, \ldots, n)$$

for some $x_1, \ldots, x_n \in \mathbb{R}^d$, $m : \mathbb{R}^d \to \mathbb{R}$ *and some random variables* $W_1, \ldots, W_n$ *which are independent, have expectation zero and satisfy*

$$\max_{i=1,\ldots,n} K^2 \cdot \mathbf{E}\left\{e^{W_i^2/K^2} - 1\right\} \le \sigma_0^2$$

for some $K, \sigma_0 > 0$. *Set*

$$\bar{m}_n(\cdot) = \arg\min_{f \in \mathcal{F}_n} \frac{1}{n}\sum_{i=1}^{n} |f(x_i) - \bar{Y}_i|^2$$

for some set $\mathcal{F}_n$ *of functions* $f : \mathbb{R}^d \to \mathbb{R}$ *and some random variables* $\bar{Y}_1, \ldots, \bar{Y}_n$. *Then there exists constants* $c_7, c_8 > 0$ *which depend only on* σ_0 *and* K *such that for any* $\delta_n > 0$ *with*

$$\delta_n \to 0 \quad (n \to \infty) \quad \textit{and} \quad n \cdot \delta_n \to \infty \quad (n \to \infty)$$

and

(A.3)
$$\sqrt{n} \cdot \delta \ge c_7 \int_{\delta/(2^9\sigma_0)}^{\sqrt{\delta}} \left(\log \mathcal{N}_2\left(u, \{f - g : f \in \mathcal{F}_n, \frac{1}{n}\sum_{i=1}^{n}|f(x_i) - g(x_i)|^2 \le \delta\}, x_1^n\right)\right)^{1/2} du$$

for all $\delta \ge \delta_n$ *and all* $g \in \mathcal{F}_n$ *we have*

$$\mathbf{P}\left\{\frac{1}{n}\sum_{i=1}^{n}|\bar{m}_n(x_i) - m(x_i)|^2 \right.$$
$$\left. > c_8\left(\frac{1}{n}\sum_{i=1}^{n}|Y_i - \bar{Y}_i|^2 + \delta_n + \min_{f \in \mathcal{F}_n}\frac{1}{n}\sum_{i=1}^{n}|f(x_i) - m(x_i)|^2\right)\right\}$$
$$\to 0 \quad (n \to \infty).$$

For the proof refer to Kohler (2006).

REFERENCES

ANDERSEN, L., and M. BROADIE (2004): Primal-Dual Simulation Algorithm for Pricing Multidimensional American Options, *Manage. Sci.* 50, 1222–1234.

BALLY, V., G. PAGÈS, and J. PRINTEMS (2005): A Quantization Method for Pricing and Hedging Multi-Dimensional American Style Options, *Math. Finance* 15, 119–168.

BARTY, K., P. GIRARDEAU, J.-S. ROY, and C. STRUGAREK (2008): Application of Kernel-based Stochastic Gradient Algorithms to Option Pricing, *Monte Carlo Methods and Applications* 14, 99–127.

BREIMAN, L., J. H. FRIEDMAN, R. OLSHEN, and C. J. STONE (1984): *Classification and Regression Trees*. Belmont, CA: Wadsworth.

CARRIÈRE, J. F. (1996): Valuation of the Early-Exercise Price for Options Using Simulations and Nonparametric Regression, *Insur. Math. Econ.* 19, 19–30.

CHOW, Y. S., H. ROBBINS, and D. SIEGMUND (1971): *Great Expectations: The Theory of Optimal Stopping*. Boston: Houghton Mifflin.

CLÉMENT, E., D. LAMBERTON, and P. PROTTER (2002): An Analysis of a Least Squares Regression Method for American Option Pricing, *Finance Stoch.* 6, 449–471.

COVER, T. (1968): Rates of Convergence for Nearest Neighbor Procedures. *Proceedings of the Hawaii International Conference on System Sciences*, Honolulu, HI, pp. 413–415.

DEVROYE, L. (1982): Necessary and Sufficient Conditions for the Almost Everywhere Convergence of Nearest Neighbor Regression Function Estimates, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 61, 467–481.

EGLOFF, D. (2005): Monte Carlo Algorithms for Optimal Stopping and Statistical Learning, *Ann. Appl. Probab.* 15, 1–37.

EGLOFF, D., M. KOHLER, and N. TODOROVIC (2007): A Dynamic Look-Ahead Monte Carlo Algorithm for Pricing American Options, *Ann. Appl. Probab.* 17, 1138–1171.

ELLIOTT, R. J., and P. E. KOPP (1999): *Mathematics of Financial Markets*, New York: Springer.

FOURNIÉ, E., J. M. LASRY, J. LEBUCHOUX, and P. L. LIONS (1999): Applications of Malliavin Calculus to Monte-Carlo Methods in Finance I, *Finance Stoch.* 3, 391–412.

FOURNIÉ, E., J. M. LASRY, J. LEBUCHOUX, and P. L. LIONS (2001): Applications of Malliavin Calculus to Monte-Carlo Methods in Finance II, *Finance Stoch.* 5, 201–236.

FRANKE, J., and M. DIAGNE (2002): Estimating Market Risk with Neural Networks, *Stat. Decis.* 24, 233–253.

GLASSERMAN, P. (2004): *Monte Carlo Methods in Financial Engineering*. New York: Springer.

GLASSERMAN, P., and B. YU (2004): Number of Paths Versus Number of Basis Functions in American Option Pricing, *Ann. Appl. Probab.* 14, 1–30.

GYÖRFI, L., M. KOHLER, A. KRZYŻAK, and H. WALK (2002): *A Distribution-Free Theory of Nonparametric Regression. Springer Series in Statistics*. New York: Springer.

HAUGH, M., and L. KOGAN (2004): Pricing American Options: A Duality Approach, *Operat. Res.* 52, 258–270.

KOHLER, M. (2006): Nonparametric Regression with Additional Measurement Errors in the Dependent Variable, *J. Stat. Plan. Inference* 136, 3339–3361.

KOHLER, M. (2008): A Regression Based Smoothing Spline Monte Carlo Algorithm for Pricing American Options, *Adv. Stat. Anal.* 92, 153–178.

LAPRISE, S. B., Y. SU, R. WU, M. C. FU, and D. B. MADAN (2001): Pricing American Options: A Comparision of Monte Carlo Simulation Approaches, *J. Computat. Finance* 4, 39–88.

LONGSTAFF, F. A., and E. S. SCHWARTZ (2001): Valuing American Options by Simulation: A Simple Least-Squares Approach, *Rev. Financ. Stud.* 14, 113–147.

ROGERS, L. C. G. (2002): Monte Carlo Valuing of American Options, *Math. Finance* 12, 271–286.

SHIRYAYEV, A. N. (1978): *Optimal Stopping Rules. Applications of Mathematics*, New York: Springer.

STONE, C. J. (1994): The Use of Polynomial Splines and their Tensor Products in Multivariate Function Estimation, *Ann. Stat.* 22, 118–184.

TSITSIKLIS, J. N., and B. VAN ROY (1999): Optimal Stopping of Markov Processes: Hilbert Space Theory, Approximation Algorithms, and an Application to Pricing High-Dimensional Financial Derivatives, *IEEE Trans. Autom. Control* 44, 1840–1851.

TSITSIKLIS, J. N., and B. VAN ROY (2001): Regression Methods for Pricing Complex American-Style Options, *IEEE Trans. Neural Networks* 12, 694–703.