Constant maturity swaps, forward measure and LIBOR market model

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Constant maturity swaps can be regarded as generalizations of vanilla interest rate swaps. In a vanilla swap one exchanges the fixed swap rate against a floating LIBOR, which involves an interest rate relevant for that particular settlement period only. In a CMS swap this will be generalized. One will exchange the fixed legs against floating legs – usually the swap rate.

It this note we give a new (for our knowledge) approximate formula for convexity adjustment based on forward measure approach and LIBOR market model. This link is interesting itself – showing that convexity adjustment is model and calibration dependent. Forward measure approach can be considered as a version of change of numeraire concept [GER, MR]. We follow [LN].

1. Forward measure paradigm

This section is devoted to the forward measure approach – a technique absolutely fundamental in pricing of interest rate derivatives. Although almost all results are known, I decided to collect them here for the reason of completeness and clarity. Some proofs are simplified and new.

Let $B(t,T)$ be discount factors on the period $[t,T]$ and $D(t)$ be the instantaneous discounting i.e. a positive process with finite variation such that $D(0) = 1$. Sometimes $D(t)^{-1}$ is called savings account. There exists a market practice of pricing of linear products by the formula

$$Price = ED(T)\xi(T) = B(0,T)\xi(0),$$

where $\xi(t)$ is a rate (swap rate, LIBOR rate) and $\xi(0)$ is its prediction at 0 (forward rate). Sometimes the formula is not true and the difference between $ED(T)\xi(t)$ and $B(0,T)\xi(0)$ is called convexity adjustment. There are practical tricks [Hu] how to calculate convexity adjustment. However, many of these rules are theoretically inconsistent and cannot be used to derive convexity corrections for general products. Jamshidian [J] initiated systematic treatment of formulas of the type $ED(T)\xi(t)$. Let us start with collection of facts on general approach to interest rate derivatives pricing.

The market is arbitrage-free if

$$M(t,T) = \frac{B(t,T)D(t)}{B(0,T)}$$

is a positive continuous martingale. Hence there exists a stochastic process $\sigma(t,T)$ such that

$$dB(t,T) = -B(t,T)\big(D(dt) + \sigma(t,T)dW(t)\big)$$

and

$$dM(t,T) = -M(t,T)\sigma(t,T)dW(t)$$

Therefore

$$M(t,T) = \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(s,T)ds - \int_0^t \sigma(s,T)dW(s)\right\}$$

Since $B(T,T) = 1$, $D(T) = B(0,T)M(T,T)$.

Pricing of European interest rate derivatives consists of finding

$$E(D(T)\xi),$$

where ξ is an F_T-measurable random variable – the intrinsic value of the claim.

Define E_T by

$$E_T\varsigma = E\varsigma M(T,T)$$

for any random variable ξ. By the Girsanov theorem E_T is a probability measure under which the process

$$W_T(t) = W(t) + \int_0^t \sigma(s,T)dW(s)$$

is a Wiener process. Now

$$ED(T)\xi = B(0,T)EM(T,T)\xi = B(0,T)E_T\xi$$

Let $\xi(t)$ be stochastic process. If $\xi(t)$ is a martingale under the measure E_T, then $ED(T)\xi(t) = B(0,T)\xi(0)$.

Lemma. Let $\xi(t)$ be a stochastic processes such that $\xi(t)M(t,T)$ is a martingale under the measure E, then $\xi(t)$ is a martingale under the measure E_T.

Proof. Let ζ be an F_t-measurable random variable.

$$\begin{aligned}
&E_T\xi(t)\zeta = EM(T,T)\xi(t)\zeta = EE\big(M(T,T)\xi(t)\varsigma\big|F_t\big) \\
&= E\xi(t)\varsigma E\big(M(T,T)\big|F_t\big) = E\varsigma\xi(t)M(t,T) = E\varsigma E\big(\xi(T)M(T,T)\big|F_t\big) \\
&= EE\big(\varsigma\xi(T)M(T,T)\big|F_t\big) = EM(T,T)\xi(T)\zeta = E_T\xi(T)\varsigma.
\end{aligned}$$

Example. For the forward LIBOR rate

$$L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T-\delta)}{B(t,T)} - 1\right).$$

$L(t,T)M(t,T)$ is a martingale.

If $\xi(t)$ is not a martingale, it may happen that $ED(T)\xi(t) \neq B(0,T)\xi(0)$ and we have to introduce a convexity adjustment what is the topic of the next section.
We may take discounting with respect to multiple cash flows as in the case of swaptions. Let δ be accrual period for both interest rates and swaps. We assume it constant for simplicity. Define consecutive swap points as $T_{i+1} = T_i + \delta$ for certain initial $T = T_0 < \delta$. Accept notation $E_n = E_{T_n}$ and $W_n = W_{T_n}$. Forward LIBOR rates are defined as $L_n(t) = L(t,T_n)$ and forward swap rates as

$$S_{nN}(t) = \frac{\sum_{i=n+1}^{N} B(t,T_i)L_i(t)}{\sum_{i=n+1}^{N} B(t,T_i)} = \frac{B(t,T_n) - B(t,T_N)}{\delta \sum_{i=n+1}^{N} B(t,T_i)}. \tag{1.2}$$

Let

$$D(S_{nN}) = \sum_{i=n+1}^{N} D(T_i),$$

Pricing of European swap derivative consists of finding

$$E\big(D(S_{nN})\xi\big),$$

where ξ is an $F_{T_{n+1}}$-measurable random variable – the intrinsic value of the claim. Since $M(t,T)$ is a positive continuous martingale,

$$M(t,S_{nN}) = \frac{\sum_{i=n+1}^{N} B(0,T_i)M(t,T_i)}{\sum_{i=n+1}^{N} B(0,T_i)} = D(t)\frac{\sum_{i=n+1}^{N} B(t,T_i)}{\sum_{i=n+1}^{N} B(0,T_i)}$$

also is. Moreover

$$dM(t,S_{nN}) = -\frac{\sum_{i=n+1}^{N} B(0,T_i)M(t,T_i)}{\sum_{i=n+1}^{N} k_i B(0,T_i)} \cdot \frac{\sum_{i=n+1}^{N} B(0,T_i)M(t,T_i)\sigma(t,T_i)}{\sum_{i=n+1}^{N} B(0,T_i)M(t,T_i)} dW(t)$$

$$= -M(t,S_{nN}) \frac{\sum_{i=n+1}^{N} B(t,T_i)\sigma(t,T_i)}{\sum_{i=n+1}^{N} B(t,T_i)} dW(t)$$

Therefore E_{nN} defined by

$$E_{nN}\varsigma = E\varsigma M(T_{n+1}, S_{nN})$$

is a probability measure under which the process

$$W_{nN}(t) = W(t) + \int_0^t \frac{\sum_{i=n+1}^{N} B(s,T_i)\sigma(s,T_i)}{\sum_{i=n+1}^{N} B(s,T_i)} ds$$

is a Wiener process. Hence

$$ED(S_{nN})\xi = \sum_{i=n+1}^{N} B(0,T_i) EM(T_i,T_i)\xi = \sum_{i=n+1}^{N} B(0,T_i) EM(T_{n+1},T_i)\xi = E_{nN}\xi \sum_{i=n+1}^{N} B(0,T_i).$$

Moreover

$$M(t,S_{nN})S_{nN}(t) = D(t) \frac{\sum_{i=n+1}^{N} B(t,T_i)}{\sum_{i=n+1}^{N} B(0,T_i)} \frac{B(t,T_n) - B(t,T_N)}{\sum_{i=n+1}^{N} B(t,T_i)} = D(t) \frac{B(t,T_n) - B(t,T_N)}{\sum_{i=n+1}^{N} B(0,T_i)}.$$

Hence $S_{nN}(t)M(t,S_{nN})$ is a martingale under the measure *E*, and then the forward swap $S_{nN}(t)$ rate is a martingale under E_{nN}.

2. Constant Maturity Swaps in the LIBOR market model

In the LIBOR market model of interest rate dynamics we assume that the LIBOR rates satisfy the following stochastic equations:

(1.4) $$dL_n(t) = \sigma(t,T_n)\gamma_n(t)L_n(t)dt + \gamma_n(t)L_n(t)dW(t),$$

where $\gamma_n(t)$ is instantaneous volatility and

$$\sigma(t,T_n) = \sum_{T_j > t+\delta}^{n} \frac{\delta L_j(t)\gamma_j(t)}{1+\delta L_j(t)}.$$

Let us examine relation between convexity adjustment on the scholar example of settled in arrears swaps. The floating leg payment at time T_n is equal $\delta L_{n+1}(T_n)$. By the forward measure paradigm

$$Claim = \delta ED(T_n)L_{n+1}(T_n) = \delta B(0,T_n)E_n L_{n+1}(T_n).$$

$L_{n+1}(t)$ satisfies the equation

$$dL_{n+1}(t) = \gamma_{n+1}(t)L_{n+1}(t)dW_n(t) + \frac{\delta\gamma_{n+1}^2(t)L_{n+1}^2(t)}{1+\delta L_{n+1}(t)}dt.$$

This equation may be solved numerically. If we linearise the drift we may get an approximate solution

$$L_{n=1}(T_n) \cong L_{n+1}(0)\exp\left(\int_0^{T_n}\gamma_{n+1}(t)dW_n(t) + \left(\frac{\delta L_{n+1}(0)}{1+\delta L_{n+1}(0)} - \frac{1}{2}\right)\int_0^{T_n}\gamma_{n+1}^2(t)dt\right).$$

Taking into consideration that

$$T_n\sigma_{n+1}^2 = \int_0^{T_n}\gamma_{n+1}^2(t)dt,$$

where σ_{n+1} is the market volatility of the caplet starting at T_n, we get the well known convexity adjustment formula

$$E_n L_{n+1}(T_n) \cong L_{n+1}(0)\exp\left(\frac{\delta L_{n+1}(0)T_n\sigma_{n+1}^2}{1+\delta L_{n+1}(0)}\right).$$

Return to the CMS example. The floating leg payment at time T_{n+1} is equal $\delta S_{nN}(T_n)$. By the forward measure paradigm

$$CMS = \delta E\big(D(T_{n+1})S_{nN}(T_n)\big) = \delta B(0,T_{n+1})E_{n+1}S_{nN}(T_n).$$

By [G,R] forward swap rate $S_{nN}(t)$ satisfies

$$dS_{nN}(t) = S_{nN}(t)\sum_{k=n+1}^{N} R_{nN}^{k}(t)\gamma_k(t)dW_{nN}(t)$$

$$= S_{nN}(t)\sum_{k=n+1}^{N} R_{nN}^{k}(t)\gamma_k(t)\left(dW(t) + \frac{\sum_{i=n+1}^{N} B(t,T_i)\sigma(t,T_i)}{\sum_{i=n+1}^{N} B(t,T_i)} dt \right)$$

$$= S_{nN}(t)\sum_{k=n+1}^{N} R_{nN}^{k}(t)\gamma_k(t)\left(dW_{n+1}(t) + \sum_{i=n+2}^{N} Q_{kN}^{i}(t)\gamma_i(t)dt \right),$$

where

$$R_{nN}^{i}(t) = \frac{\delta L_i(t)}{1+\delta L_i(t)} \frac{B(t,T_n)\sum_{k=i}^{N} B(t,T_k) + B(t,T_N)\sum_{k=n+1}^{i-1} B(t,T_k)}{\left(B(t,T_n) - B(t,T_N)\right)\sum_{k=n+1}^{N} B(t,T_k)}$$

and

$$Q_{kN}^{l}(t) = \frac{\delta L_l(t)\sum_{j=l}^{N} B(t,T_j)}{\left(1+\delta L_l(t)\right)\sum_{i=n+1}^{N} B(t,T_i)}.$$

By the approximation

$$dS_{nN}(t) \approx S_{nN}(t)\sum_{k=n+1}^{N} R_{nN}^{k}(0)\gamma_k(t)\left(dW_{n+1}(t) + \sum_{i=n+2}^{N} Q_{kN}^{i}(0)\gamma_i(t)dt \right).$$

Hence

$$E_{n+1}S_{nN}(T_n) \approx S_{nN}(0)\exp\left(\sum_{l=n+2}^{N}\sum_{i=n+1}^{N} R_{nN}^{i}(0)\varphi_{il}^{n} Q_{nN}^{l}(0) \right),$$

where

$$\varphi_{kl}^{n} = \int_0^{T_n} \gamma_l(t)\gamma_k(t)dt .$$

Hence the convexity adjustment is model and even calibration dependent! Take the simplest case of single dimensional LIBOR market model. By [G,R]

$$T_n\sigma_{nN}^2 \approx \sum_{l=n+1}^{N}\sum_{i=n+1}^{N} R_{nN}^{i}(0)\varphi_{il}^{n} R_{nN}^{l}(0),$$

where σ_{nN} is the market volatility of the swap on the time period $\left[T_n, T_N\right]$. By [G,R]

$$R_{nN}^{i}(t) \approx \frac{B(t,T_i)L_i(t)}{\sum_{k=n+1}^{N} B(t,T_k)L_k(t)}$$

and hence

$$\sum_{i=n+1}^{N} R_{nN}^{i}(0) \approx 1 .$$

If we assume that $\varphi_{il}^{n} = T_n \sigma_{nN}^2 =$ const, what is equivalent to taking single factor model we have

$$E_n S_{nN}(T_n) \approx S_{nN}(0) \exp\left(T_n \sigma_{nN}^2 \sum_{i=n+2}^{N} Q_{nN}^{i}(0) \right) \approx S_{nN}(0)\left(1 + T_n \sigma_{nN}^2 \sum_{i=n+2}^{N} Q_{nN}^{i}(0) \right).$$

Analogously, the price of call option on $S_{nN}(T_n)$ settled at T_{n+1} is equal

$$Option \cong \delta B(0, T_{n+1})\left(S_{nN}(0) N(h) - K N\left(h - \sigma_{nN} \sqrt{T_n} \right) \right),$$

where

$$h = \frac{\ln S_{nN}(0) - \ln K + \left(\sum_{i=n+2}^{N} Q_{nN}^{i}(0) + \frac{1}{2} \right) T_n \sigma_{nN}^2}{\sigma_{nN} \sqrt{T_n}} .$$

3. Example

Convexity adjustments were calculated for $N = 10$Y and forward interest rates and volatilites given in the table. Correction equal $\sum_{i=n+2}^{N} Q_{nN}^{i}(0)$ is listed in the last row.

Volatility	**40,0%**	**39,0%**	**38,0%**	**37,0%**	**36,0%**	**35,0%**	**34,0%**	**33,0%**	**32,0%**	**31,0%**
Rate	2,1%	2,2%	2,3%	2,4%	2,5%	2,6%	2,7%	2,8%	2,9%	3,0%
Time	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**
Bond	97,9%	95,8%	93,7%	91,5%	89,3%	87,0%	84,7%	82,4%	80,1%	77,7%
Correction	**1,7%**	**2,6%**	**2,9%**	**2,8%**	**2,4%**	**1,8%**	**1,2%**	**0,7%**	**0,2%**	

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