Probability and Statistics

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Textbook

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

References

Readings for these lecture notes:

- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer

These notes contain material from the above book.

Negative Binomial Distribution [1]

- Let us consider an experiment where the properties are the same as those listed for a binomial experiment, with the exception that the trials will be repeated **until a fixed number of successes occur**.

- Therefore, instead of finding the probability of **x successes** in **n trials**, where n is fixed, we are now interested in the probability that, the **k^{th} success** occurs on the **r^{th} trial**.

- Experiments of this kind are called **negative binomial experiments**.

Negative Binomial Distribution [2]

- Consider the use of a drug that is known to be effective in **60%** of the cases where it is used. The drug will be considered a **success** if it is **effective** in bringing some degree of relief to the patient.

- We are interested in finding the **probability** that the **fifth patient** to experience relief is the **seventh patient** to receive the drug during a given week.

Negative Binomial Distribution [3]

Designating a **success by *S*** and a **failure by *F***, a **possible order** of achieving the desired result is ***SFSSSFS***, which occurs with probability

$$(0.6)(0.4)(0.6)(0.6)(0.6)(0.4)(0.6) = (0.6)^5(0.4)^2$$

Negative Binomial Distribution [4]

We could list all possible orders by rearranging the ***F*'s** and ***S*'s** except for the **last outcome**, which must be the **fifth success**. The total number of possible orders is equal to the number of partitions of the **first six trials** into two groups with **2 failures assigned** to the one group and **4 successes assigned** to the other group.

This can be done in $_6C_4 = 15$ mutually exclusive ways.

Hence, if ***X* represents the outcome** on which the **fifth success** occurs, then

$$P(X = 7) = {_6C_4}\,(0.6)^5(0.4)^2 = 0.1866.$$

Negative Binomial Distribution [5]

If **repeated independent trials** can result in a success with probability **p** and **a failure** with probability **q = 1— p**, then the probability distribution of the random variable **x**, the number of the trial on which the **k^th** success occurs, is

$$b^*(x; k, p) = {}_{x-1}C_{k-1}\ p^k q^{x-k}, \quad x = k, k+1, k+2, \ldots$$

Negative Binomial Distribution [6]

Example: In an NBA (National Basketball Association) championship series, the team who wins **four games** out of **seven** will be the winner. Suppose that **team A** has probability **0.55** of winning over **team *B*** and both teams A and *B* face each other in the championship games.

(a) What is the probability that **team A** will win the series in **six games**?

(b) What is the probability that team **A will win the series**?

Negative Binomial Distribution [7]

(c) If both teams face each other in a regional playoff series and the winner is decided by winning **three** out of **five games**, what is the probability that team A will win a playoff?

Negative Binomial Distribution [8]

Solution:

$b^*(x; k, p) = {}_{x-1}C_{k-1}\ p^k q^{x-k}$, $x = k, k+1, k+2, \dots$

Let x denotes **number of matches** until the **k^{th} success** occur.

Here $x = 6$, $k = 4$

(a) $b^*(6; 4, 0.55)$ $= {}_{6-1}C_{4-1}\ (0.55)^4\ (0.45)^{6-4}$

$= {}_5C_3\ (0.55)^4\ (0.45)^2$

$= 0.1853$

Negative Binomial Distribution [9]

$b^{*}(x; k, p) = {}_{x-1}C_{k-1}\ p^{k}\, q^{x-k}$, $x = k, k+1, k+2, \ldots$

Let x denotes **number of matches** until the **k^{th} success** occur.

Here k = 4

(b) P(team A wins the championship series) = P(X≥4)

$P(X \geq 4) = b^{*}(4; 4, 0.55) + b^{*}(5; 4, 0.55)$

$+ b^{*}(6; 4, 0.55) + b^{*}(7; 4, 0.55)$

$= 0.0915 + 0.1647 + 0.1853 + 0.1668$

$= 0.6083.$

Negative Binomial Distribution [10]

$b^*(x; k, p) = {}_{x\text{-}1}C_{k\text{-}1}\ p^k q^{x-k}$, x = k, k+1, k+2, …

Let x denotes **number of matches** until the **k^{th} success** occur.

Here k = 3

(c) P(team A wins the playoff) =

$P(X \geq 3) = b^*(3; 3, 0.55) + b^*(4; 3, 0.55) + b^*(5; 3, 0.55)$

$= 0.1664 + 0.2246 + 0.2021$

$= 0.5931.$

Negative Binomial Distribution [11]

- If we consider the special case of the **negative binomial distribution where $k = 1$**, we have a probability distribution for the number of trials required for a **single success**. An example would be the tossing of a **coin until a head occurs**. We might be interested in the probability that the first head occurs on the fourth toss. The **negative binomial distribution** reduces to the form
- Here k = 1

$$b^*(x; 1, p) = p\, q^{x-1}, \quad x = 1,2,3,\ldots$$

Negative Binomial Distribution [12]

- Since the successive terms constitute a **geometric progression**, it is customary to refer to this special case as the **geometric distribution** and denote its values by **g(x; p)**.

Negative Binomial Distribution [13]

Example An oil drilling company ventures into various locations, and its success or failure is **independent** from one location to another. Suppose the probability of a success at any specific location is **0.25**.

(a) What is the probability that a driller drills **10 locations** and finds **1 success**?

(b) The driller feels that he will go **bankrupt** if he drills **10 times** before the **first success occurs**. What are the driller's prospects for bankruptcy?

(c) What is the probability that the driller gets the first success on the **11^{th} trial**?

Negative Binomial Distribution [14]

Solution: Let x denotes **number of successful drill**

(a) $b(x; n, p) = {}_nC_x \; p^x q^{n-x}$, x= 1, 1, 2,..,n

$b(1; 10, 0.25) = {}_{10}C_1 \; 0.25^1 \; 0.75^{10-1}$

$= 0.1877$

(b) Since trials are independent

Probability of bankruptcy $= q \times q \times q \times q \times q \times q \times q \times q \times q$

$= q^9$

$= (0.75)^9$

$= 0.056$

(c) Let x denotes **number of locations explore** until the **1^{th} success**.

Here x = 11

g(x; p) = $p\, q^{x-1}$, x= 1,2,3,…

g(**11**; 0.25) = $(0.25)\ (0.75)^{11-1}$

= $(0.25)\ (0.75)^{10}$

= **0.0141**

Negative Binomial Distribution [15]

Example Consider the information of Review **Exercise in previous question**. The driller feels that he will **"hit it big"** if the **second success** occurs on or before **the sixth attempt**. What is the probability that the driller will **"hit it big?"**

Negative Binomial Distribution [16]

Solution: $b^*(x; k, p) = {}_{x-1}C_{k-1}\ p^k q^{x-k}$, $x = k, k+1, k+2, \ldots$

Let x denotes **number of locations explore** to get **k^th^ successes**. **Here x = 6, k = 2, p = 0.25**

$$b^*(6; 2, 0.25) = {}_{6-1}C_{2-1}\ (0.25)^2\ (0.75)^{6-2}$$

$$= {}_5C_1 (0.25)^2\ (0.75)^4$$

$$= 0.0989$$

Negative Binomial Distribution [17]

Example: A couple decides they will continue to have children **until** they have **two males**. Assuming that **P(male) = 0.5**, what is the probability that their **second male** is their **fourth child**?

Negative Binomial Distribution [18]

Solution: $b^*(x; k, p) = {}_{x-1}C_{k-1}\ p^k q^{x-k}$, $x = k, k+1, k+2, \ldots$

Let x denotes **number of children** on which **k^{th} male** born.

Here **x = 4**, p = 0.5, **k = 2 (number of male)**

$$b^*(4; 2, 0.5) = {}_{4-1}C_{2-1}\ (0.5)^2\ (0.5)^{4-2}$$

$$= {}_{3}C_{1}\ (0.5)^2\ (0.5)^2$$

$$= 0.1875$$

Negative Binomial Distribution [19]

Example An oil company conducts a geological study that indicates that an exploratory oil well should have a 20% chance of striking oil. What is the probability that the **first strike** comes on the **third well drilled?**

To find the requested probability, we need to find **P(X = 3).** Note that X is technically a geometric random variable, since we are only looking for one success. Since a geometric random variable is just a special case of a negative binomial random variable, we'll try finding the probability using the negative binomial p.m.f.

Negative Binomial Distribution [20]

Let x denotes **number of drills** on which **kth strike** occur.

Here p = 0.20, 1−p = 0.80

k = 1, and **x = 3**

$b^*(x; k, p) = {}_{x-1}C_{k-1}\ p^k q^{x-k}$, x = k, k+1, k+2, ...

$b^*(3; 1, 0.2) = {}_{3-1}C_{1-1}\ (0.2)^1\ (0.8)^{3-1}$

$= {}_2C_0\ (0.2)^1\ (0.8)^2$

$= 0.128$

Negative Binomial Distribution [21]

Example: What is the probability that the **third strike** comes on the **seventh well** drilled?

Solution:

Let x denotes **number of drills** on which **k^{th} strike** occur.

$p = 0.20$, $1-p = 0.80$, $x = 7$ and $k = 3$

$$b^*(x;\, k,\, p) = {}_{x-1}C_{k-1}\; p^k\, q^{x-k},\ x = k, k+1, k+2, \ldots$$

$$b^*(7;\, 3, 0.2) = {}_{7-1}C_{3-1}\; (0.2)^3\, (0.8)^{7-3}$$

$$= {}_{6}C_{2}\; (0.2)^3\, (0.8)^4$$

$$= 0.049$$