Gauge/Gravity Duality

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Abstract:

The aim of the graduation thesis is to review the basics of Gauge/Gravity duality and the developments that led to it. We start by reviewing the large N expansion of gauge theories (particularly U(N) gauge theories). We show that the generating functional is given by a sum over topologies of 2d surfaces which is the first hint to the duality. We also review black hole thermodynamics which provides the second hint to the duality. Motivated by these two hints, we give a heuristic argument to show that gauge theories in d dimensions are dual to gravitational theories in d+1 dimensions.

After establishing the Gauge/Gravity duality, we provide a dictionary to go from gauge theory to gravity and vise versa. Using the dictionary, we show how to calculate correlation functions and expectation values of operators within the framework of the Gauge/Gravity duality.

Contents

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1 Introduction

Gauge/Gravity duality has proven to be one of the greatest developments in physics in the last 20 years. The duality indicates that certain strongly interacting quantum field theories(QFTs) can be described by weakly coupled classical gravity in some geometry and vice versa. Roughly speaking, a theory of gravity coupled to some fields inside the bulk of a spacetime geometry is dual to a quantum field theory that lives on the boundary of this spacetime. In terms of partition functions, we can summarize the duality into the following equation:

$$Z_{Gravity}[\phi] \equiv Z_{Gauge}[\varphi] \tag{1.1}$$

Where ϕ and φ indicate the classical field which is coupled to gravity in the bulk and its dual field operator of the QFT on the boundary respectively. The meaning of this equation is vague and one goal of this thesis is to define this equation in a more precise manner.

Gauge/Gravity duality was first developed by Juan Maldecena in 1997 [1]. In his paper, he considered the large N limit of supersymmetric Yang-Mills field theory, particularly, $\mathcal{N} = 4$ SU(N) Yang-Mills theory. By studying the Large N limit of super Yang-Mills, Maldacena managed to show that the structure of the Feynman diagram expansion is similar to that of type IIB string theories on D3-branes. He, then, showed that the corresponding classical theory in this Large N limit is -in fact- classical supergravity in 10 dimensions. By considering appropriate limits and compactifying the extra 5 dimensional space, he managed to establish the first and best known version of Gauge/Gravity duality, AdS/CFT duality. In the subsequent years, a huge influx of literature developed Maldacena's ideas further and extended his results to other areas of physics, spanning particle physics, condensed matter physics, hydrodynamics, black hole physics, etc.

Gauge/Gravity duality allows the study of strongly interacting QFTs by studying their gravity duals which are weakly coupled theories, and vice versa. This weak-strong relation is precisely the reason why this duality is so powerful. It provides a more tractable way to study theories such as quantum chromodynamics which is famously known for its difficulty. It can also be used to study unconventional superconductors which are characterized by the absence of quasi-particle descriptions and it can also open a new way to formulate quantum gravity and to study black hole microstates.

In this thesis, we attempt to give an introduction to the basics of this duality. We start by reviewing the basics of general relativity and Yang-Mills theory in section 2 on preliminary materials.

Before Maldacena developed his conjecture, there were many evidences and hints that such a relation may be hiding in physics. Of particular interest to us are the hints from black hole thermodynamics and the large N limit. We will give a quick review of these topics in section 3 of this thesis.

In section 4, we will cover some basic notions from string theory and attempt to justify the Gauge/Gravity duality. We will also provide a dictionary to precisely move from the gravity side of the relation to the field theory side.

In section 5, we show how to compute quantities such as general correlation functions in Gauge/Gravity duality. We then preform simple calculations of expectation values and linear response functions for scalar operators.

In this thesis, we set $c = \hbar = K_B = 1$ and we only restore the constants whenever it is illuminating the discussion. We also adopt the signature $(-, +, \ldots, +)$ for the metric. For indices, we follow the common convention of general relativity by adopting Greek letters throughout the first two chapters but starting from chapter 4, we follow two conventions depending on which spacetime we are considering. The conventions are:

- Greek indices $\mu, \nu, \ldots$ run from 0 to d and represents the d-dimensional spacetime where the field theory live. They are also used to write the coordinates of the AdS boundary.
- Latin letters M,N,... run from 0 to d+1 and represent the coordinates of the spacetime where the gravitational theory lives.

2 Preliminary Materials

2.1 Review of General Relativity

General relativity is inherently a geometric theory. The gravitational force is understood as a force that curves the spacetime geometry. The main object of interest in general relativity is the spacetime metric g which encodes the spacetime curvature, and thus, encodes gravity. In order to understand how the metric evolves locally in spacetime, one needs to understand some basic notions in differential geometry. This subsection attempts to provide these notions. For a more complete reference on differential geometry we recommend [3], and for general relativity we recommend [4–6].

2.1.1 Mathematical preliminary

In differential geometry, we have two main entities, vectors and 1-forms. Vectors behave like derivatives while 1-forms are like differentials. Particularly, if we have an n-dimensional manifold M with local coordinate system $\{x^1, x^2, ..., x^n\}$ then, we can write a vector v as

$$v = v^\mu \partial_\mu \tag{2.1}$$

While a 1-form w can be written as

$$w = w_\mu dx^\mu \tag{2.2}$$

One can also construct more complicated objects by taking tensor products between differentials and derivatives. For example, a tensor of type (r,s)

$$T = T^{\nu_1,.....,\nu_s}_{\mu_1,......,\mu_r} dx^{\mu_1} \otimes ..... \otimes dx^{\mu_r} \otimes \partial_{\nu_1} \otimes ..... \otimes \partial_{\nu_s} \tag{2.3}$$

An example of a tensor of type (2,0) that is of interest to us is the metric tensor g. The metric tensor defines the inner product between two vectors v and u as follows:

$$\langle v, u \rangle = g_{\mu\nu} v^\mu u^\nu \tag{2.4}$$

Differential geometry is made of coordinate independent objects. So, we require that under a change of basis $x^\mu \to x'^\mu$, the quantities described above must stay invariant. This imposes the following transformation laws:

$$v^\mu \to v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} v^\nu, \tag{2.5a}$$

$$w_\mu \to w'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} v_\nu \tag{2.5b}$$

The transformation law for a type(r,s) tensor is defined similarly

$$T^{\mu_1,.....,\mu_s}_{\nu_1,......,\nu_r} \to T'^{\mu_1,.....,\mu_s}_{\nu_1,......,\nu_r} = \frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}} .... \frac{\partial x'^{\mu_s}}{\partial x^{\sigma_s}} \frac{\partial x^{\rho_1}}{\partial x'^{\nu_1}} .... \frac{\partial x^{\rho_r}}{\partial x'^{\nu_r}} T^{\sigma_1,.....,\sigma_s}_{\rho_1,......,\rho_r} \tag{2.5c}$$

From now on, we will call the components $v^{\mu}, u_{\mu}, T^{\mu_1,\ldots,\mu_s}_{\nu_1,\ldots\ldots,\nu_r}$ as vector, 1-form, and tensor respectively. We will work with these quantities in the following discussion.

In general, derivatives of a tensor (such as $\partial_\mu v^\nu$) don't transform like a tensor. To circumvent this, we define the covariant derivative (also known as a connection) in the following manner:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + v^\alpha \Gamma^\nu_{\alpha\mu} \tag{2.6a}$$
$$\nabla_\mu v_\nu = \partial_\mu v_\nu + v_\alpha \Gamma^\alpha_{\nu\mu} \tag{2.6b}$$

Where $\Gamma^\alpha_{\beta\gamma}$ are known as Christoffel's symbols. These symbols encodes how the basis vectors curve. For example, in Euclidean space with curvilinear coordinates $\{\boldsymbol{e}_i\}_i$ the Christoffel's symbols are defined as

$$\Gamma^k_{ij} = \frac{\partial \boldsymbol{e}_i}{\partial x^j} \cdot \boldsymbol{e}^k = \frac{\partial \boldsymbol{e}_i}{\partial x^j} \cdot g^{km} \boldsymbol{e}_m \tag{2.7}$$

where g^{km} is the inverse of the metric tensor.

For a general manifold, the Christoffel's symbols are not unique and hence the connection is not unique and we have a freedom to define the connection. A commonly used connection in physics is the Levi-Civita connection. This connection satisfies the following two conditions:

For any two vectors X, Y defined as in eq (2.1), we have

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{2.8a}$$
$$\nabla_\sigma g_{\mu\nu} = 0 \tag{2.8b}$$

where ∇_X is the directional covariant derivative which is defined as

$$(\nabla_X Y)^\nu = X^\mu (\partial_\mu Y^\nu + Y^\alpha \Gamma^\nu_{\alpha\mu}) \tag{2.9}$$

The first condition is called the torsion-free condition while the second condition is about the compatibility of the connection with the metric. These two conditions define the connection uniquely and thus, they give us a unique set of Christoffell symbols. The Christoffel symbols satisfying the two conditions are given below

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}) \tag{2.10}$$

Before we discuss general relativity, we need to define a very important quantity known as the Riemann tensor R. The Riemann tensor is an object in differential geometry that encodes all information about the curvature of the manifold. It measures the uncommutativity between covariant derivatives. The Riemann tensor is a (3,1) tensor; it takes three different vectors and linearly map them to another vector. If we consider the vectors X, Y, Z then the Riemann tensor is defined in terms of Levi-Civita connection as

$$R(X, Y, Z) = \big([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\big) Z \tag{2.11}$$

The components of the Riemann tensor can be written as

$$R^{\alpha}_{\mu\nu\rho} = \partial_{\nu}\Gamma^{\alpha}_{\mu\rho} - \partial_{\rho}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho} - \Gamma^{\alpha}_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu} \tag{2.12}$$

We can construct different tensors by contracting two or more indices of the Riemann tensor. One such contraction that is often used in formulating general relativity is the Ricci tensor $R_{\mu\nu}$. the Ricci tensor is defined by contracting the first and third indices of the Riemann tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{2.13}$$

Taking the trace of the Ricci tensor gives the Ricci scalar R

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.14}$$

2.1.2 Building The General Relativity Lagrangian

To build a Lagrangian description of general relativity, we need to find a suitable scalar quantity to serve as the Lagrangian. This scalar quantity must have all information about the curvature of spacetime and hence, it encodes all information about gravity while being invariant to coordinate transformation. The simplest choice is a linear function of the Ricci scalar. So, the Lagrangian takes the following form:

$$\mathcal{L} = aR + b \tag{2.15}$$

In general relativity, $a = \frac{1}{16\pi G}$ where G is the gravitational constant, while $b = \mathcal{L}_{matter}$ where $\mathcal{L}_{matter}$ is the Lagrangian of any matter fields present in spacetime.

Now, after finding the Lagrangian, we need to establish an action principle. The action is the volume integral of the Lagrangian. The coordinate transformation invariant n-dimensional volume element is given by $dV = \sqrt{-g}d^nx$ where g denotes the determinant of the metric tensor. So we see that the action of general relativity in the absence of matter fields is

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \tag{2.16}$$

The main dynamical object in general relativity is the metric tensor so it follows that the equations of motion are nothing but the following variational principle:

$$\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}} = 0 \tag{2.17}$$

Variating the action with respect to $g^{\mu\nu}$, we get

$$\delta\mathcal{S} = \frac{1}{16\pi G}\int d^4x\{\sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu}) + (\delta\sqrt{-g})R_{\mu\nu}g^{\mu\nu} + \sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu}\} \tag{2.18}$$

By using Palatini identity [7], it is easy to see that the last term in 2.18 is a total derivative and hence, it doesn't affect the action.[1]

[1]In general, surface terms like the last term in 2.18 don't necessarily vanish and one needs to be careful when dealing with such terms in oder to have a well-defined variational problem.

It is easy to show that for an $n \times n$ matrix M, the following relation hold:

$$\delta(\det M) = \det M \, \mathrm{Tr}(M^{-1}\delta M) \tag{2.19}$$

We will only proof 2.19 for the case when M is both invertable and diagonalizable however 2.19 can be shown to be true for more general matrices.(check [8] for a more complete reference on matrix analysis).

Proof Let the eignvalues of M be $\{M_1, \dots, M_n\}$, then,we have:

$$\delta(\det M) = \delta\big(\prod_{i=1}^{n} M_i\big) = \det M \sum_{i=1}^{n} M_i^{-1}\delta M_i = \det M \, \mathrm{Tr}(M^{-1}\delta M)$$

□

Using 2.19, we can evaluate the second term in 2.18 to finally get

$$\delta\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} \tag{2.20}$$

Now, imposing that $\delta\mathcal{S} = 0$ regardless of how one varies the metric, we arrive at

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0 \tag{2.21}$$

Equation 2.21 Are known as Einstein Field Equations. In the presence of matter fields, 2.21 take the following form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.22}$$

where $T_{\mu\nu} \equiv \dfrac{2}{\sqrt{-g}}\dfrac{\delta\mathcal{S}_{matter}}{\delta g^{\mu\nu}}$ is the stress-energy-momentum tensor of the matter fields.

General relativity deals with many different matter fields but one of particular importance to Gauge/Gravity duality is described by the following energy tensor

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}$$

where Λ is the cosmological constant. Λ act as a constant energy density in spacetime. $\Lambda > 0$ is dark energy candidate. AdS spacetime is a solution to Einstein field equations with $\Lambda < 0$

If $\Lambda = 0$, the simplest black hole solution we can get is the Schwarzschild black hole solution. This solution has the following metric

$$ds^2 = -\big(1 - \frac{2GM}{r}\big)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2 \tag{2.23}$$

where $d\Omega$ is the line element of the unit sphere S^2.[2]

[2]Note that this metric approach the flat Minkowski metric in the limit $r \to \infty$

2.1.3 Gravitational Redshift

The idea of the gravitational redshift is similar to Doppler effect. As the gravitational field curve the spacetime geometry, the geodesic light takes is stretched and so, light waves are gravitationally redshifted.

Assume that we have two observers, observer A and observer B. Observer A sends a light signal to observer B. Light follows null geodesics so $ds^2 = 0$. Assuming spherical symmetry and isotropy of space and time direction, we have

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 = 0 \tag{2.24a}$$

or

$$\int_A^B dt = \int_A^B \sqrt{-\frac{g_{rr}}{g_{00}}}dr \tag{2.24b}$$

For static spacetimes(such as the Schwarzschild black hole spacetime), the metric $g_{\mu\nu}$ doesn't depend on time. Thus, it follows from 2.24b that the time taken from observer A to observer B is always the same. This means that both observers agree on time. However, the proper time for each observer is different since $d\tau^2 = |g_{00}|dt^2$

$$d\tau_A^2 = |g_{00}(r_A)|dt_A^2 \qquad\qquad d\tau_B^2 = |g_{00}(r_B)|dt_B^2 \tag{2.25}$$

But they should agree on the number of oscillations, so

$$\omega_A d\tau_A = \omega_B d\tau_B \Longleftrightarrow E_A d\tau_A = E_B d\tau_B \tag{2.26}$$

So, we finally get

$$E_B = \sqrt{\frac{|g_{00}(r_A)|}{|g_{00}(r_B)|}}E_A \tag{2.27}$$

For Schwarzschild black hole geometry, $r_B = \infty$, and $r_A \gg 2GM$, we have

$$E_\infty = \sqrt{|g_{00}(r_A)|}E_A \approx \big(1 - \frac{GM}{r_A}\big)E_A < E_A \tag{2.28}$$

Thus, the energy of the photon decreases at infinity. The energy of the photon decreases because the photon has to climb up the gravitational potential. Indeed, the second term of the above equation takes the form of the Newtonian potential for the photon. Also, suppose that the observer A is located at the horizon $r_A = 2GM$. Since $g_{00}(r_A) = 0$ at the horizon, $E_\infty \to 0$, namely light gets an infinite redshift.

2.2 Yang-Mills Field Theory

Yang-Mills field theory started as a mathematical extension of Abelian gauge field theories to the non-Abelian case. The most common Yang-Mills theory in the area of particle physics is the SU(N) gauge field theory which is used to describe both the strong and the weak nuclear forces. In this subsection, we will consider a general gauge field theory. Yang-Mills theory is a rich subject with many intricate non-perturbative objects such as solitons[9]. The mathematics of Yang-Mills also has many open problems such as the famous mass gap problem[10] and the theoretical proof of confinement[11]. For a more complete reference on Yang-Mills theory, we recommend [12–14]

A compact Lie group G has a Lie algebra $\mathfrak{g}$ whose generators T^a satisfy the Lie algebra commutation relations:

$$[T^a, T^b] = if^{abc}T^c \qquad\qquad a, b, c \in \{1, 2, \ldots, \dim G\} \tag{2.29}$$

f^{abc} is the fully anti-symmetric structure tensor. The factor i is required to make the generators hermitian. In particle physics, we are interested in $G = SU(N)$ or $G = U(N)$. For these groups, we often normalize the generators to $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$

In order to construct a Yang-Mills theory, let's start with a simple field ϕ that is described by a field theory which has a global symmetry group G. To make this symmetry local, we need to deal with terms involving derivatives in the Lagrangian. The simplest such term has the form $\mathcal{L} = |\partial_\mu\phi|^2$. We need to modify this term so that the symmetry doesn't break locally but before that, let us establish the transformation law of the field ϕ.

In the standard model, we always deal with fields in the fundamental representation of the group G (see [15]) however in AdS/CFT we often deal with fields in the adjoint representation so we will only consider this case. An adjoint matter field ϕ is defined as $\phi = \phi^aT^a$. The field ϕ follows this transformation law

$$\phi \to U\phi U^{-1} \tag{2.30a}$$

$$\phi^\dagger \to U\phi^\dagger U^{-1} \tag{2.30b}$$

From these transformation laws, we see that a derivative will transform as

$$\begin{aligned}\partial_\mu\phi \to \partial_\mu(U\phi U^{-1}) &= (\partial_\mu U)\phi U^{-1} + U(\partial_\mu\phi)U^{-1} + U\phi(\partial_\mu U^{-1}) \\ &= U(\partial_\mu\phi)U^{-1} + \left[(\partial_\mu U)U^{-1}, U\phi U^{-1}\right]\end{aligned} \tag{2.31}$$

We see from 2.31 that a derivative of a field in the adjoint representation doesn't transform in the adjoint representation. We need to modify the differential operator ∂_μ to an operator D_μ that transforms in the adjoint representation so that $D_\mu\phi \to U(D_\mu\phi)U^{-1}$. To do that, we introduce a Lie-algebra valued gauge field $A_\mu = A^a_\mu T^a$. The gauge field satisfies the transformation law:

$$A_\mu \to UA_\mu U^{-1} - i(\partial_\mu U)U^{-1} \tag{2.32}$$

Notice that A_μ doesn't transform in the adjoint representation because of the second term in 2.32. This curious term represents the gauge freedom.

Now, using the gauge field A_μ, we can construct gauge covariant derivatives. We define the covariant derivative[3] as

$$D_\mu \phi = \partial_\mu \phi - i[A_\mu, \phi] \tag{2.33}$$

It is easy to check that D_μ transforms in the adjoint representation so that $|D_\mu \phi|^2$ is gauge invariant.

Analogous to electromagnetism, one can construct a field strength tensor $F_{\mu\nu}$[4] as

$$[D_\mu, D_\nu]\phi = -i[F_{\mu\nu}, \phi] \tag{2.34}$$

More generally, $[D_\mu, D_\nu] = -iF_{\mu\nu}$ where it's understood that $F_{\mu\nu}$ acts on fields to the right according to their representations.

The classical action of pure Yang-Mill theory without matter is taken to be

$$\mathcal{S}_{YM} = \frac{-1}{2g^2} \int d^4x \, \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) \tag{2.35}$$

where g is the coupling constant. It is easy to check that the classical equation of motion is

$$D_\mu F^{\mu\nu} = 0 \tag{2.36}$$

Note that due to the non-commutativity of the gauge fields, they self-interact. This self-interaction is clear in the nonlinear nature of the equation of motion. This makes the analysis of Yang-Mills theory difficult to do analytically and one often relies on other methods such as lattice simulations.

[3] Note that a Lie group G can be described as a differential manifold. The covariant derivative D_μ plays the role of the Levi-Civita connection on this manifold

[4] The field strength tensor plays the role of the curvature tensor $F_{\mu\nu}$ in the geometric description of the group G

3 Hints of AdS/CFT

3.1 Large N limit of field theory

When dealing with field theories, one can hardly find analytic solutions. Field theory lack analytical techniques and so, the most common way to study field theory is via perturbative expansion. While the perturbative techniques are great in probing theories such as quantum electrodynamics and other weakly coupled field theories, it suffers from many shortcomings. Perturbative techniques fail to work if the field theory is strongly interacting due to the lack of a small parameter to use as the expansion parameter. It also fails to capture non-perturbative effects such as 't Hooft–Polyakov monopoles, domain walls, flux tubes, and instantons.[16]

The large N limit of field theories provide a way to probe field theories while overcoming the problems of perturbation theory. The "N" in the large N limit represent the dimension of the symmetry group of field theory. In a field theory with N fields, a composite scalar[5] quantity tend to have very small fluctuations as N grows. The reason for that comes from the central limit theorem. Composite scalars are made up of a sum of many terms and these terms scale with N. As N gets larger, the fluctuation of each term tend to cancel each other and we end up with vanishing fluctuations for very large N. Because of this, we can construct an effective field theory for the composite scalar by integrating out the original degrees of freedom. Since the new degrees of freedom have zero fluctuations, the resulting effective theory is classical.

There are two types of theories we will consider in this section, vector theories and matrix theories. In the case of vector theories, the effective field theory at large N tend to be easier to deal with as the new degrees of freedom are finite and independent of N. In the case of matrix theories, the effective theory is hard to solve, albeit easier than the original theory, this is because the number of the new degrees of freedom scale with N. In this section, we will discuss the $O(N)$ vector model and the $U(N)$ matrix model to illustrate the large N limit. This section is mostly based on the discussion given in [2, 17].

3.1.1 O(N) vector model

Let's consider an N component vector field ϕ with Euclidean action $\mathcal{S}(\phi)$ given by

$$\mathcal{S}(\phi) = \int \big[\frac{1}{2}(\partial_\mu \phi)^2 + NU(\phi(x)^2/N)\big]\, d^d x \tag{3.1}$$

where $U(\rho)$ is a general polynomial and the explicit N dependence has been chosen to lead to a large N limit. .
The partition function of this system is then:

$$\mathcal{Z} = \int \mathcal{D}\phi \exp[-\mathcal{S}(\phi)] \tag{3.2}$$

[5]Scalar in the sense of group transformation

It is expected that for very large N, $O(N)$ invariant quantities (such as $(\phi)^2$ self-average and so they have vanishing fluctuations. For example, $\langle\phi^2(x)\phi^2(y)\rangle \underset{N\to\infty}{\sim} \langle\phi^2(x)\rangle\langle\phi^2(y)\rangle$. This suggests that we can treat ϕ^2 as a dynamical variable in an effective field theory. To do this, we introduce two auxiliary fields λ, ρ via

$$1 = N\int \mathcal{D}\rho\delta(\phi^2 - N\rho) = \int \mathcal{D}\rho\mathcal{D}\lambda e^{\int d^dx \frac{\lambda(\phi^2 - N\rho)}{2}} \tag{3.3}$$

where the integral over λ is taken over a contour parallel to the imaginary axis.[6]

The delta function in eq 3.3 insures that $phi^2 = N\rho$. So, by inserting 3.3 into 3.2, we get

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}\rho\mathcal{D}\lambda \exp[-\mathcal{S}(\phi,\rho,\lambda)] \tag{3.4}$$

with

$$\mathcal{S}(\phi,\rho,\lambda) = \int d^dx\left[\frac{1}{2}(\partial_\mu\phi)^2 + NU(\rho) + \frac{\lambda(\phi^2 - N\rho)}{2}\right] \tag{3.5}$$

Notice that the path integral turned into a Gaussian in ϕ and so, we can preform the ϕ integral easily. Instead of integrating over all components of ϕ, we are going to split ϕ into one component field σ and N-1 component field π. We integrate over π only while leaving σ intact. In order to calculate correlation functions of σ, we also add to the action a source field $H(x)$[7]. We finally arrive at the following partition function:

$$\mathcal{Z} = \int \mathcal{D}\sigma\mathcal{D}\rho\mathcal{D}\lambda \exp\left[-\mathcal{S}_N(\sigma,\rho,\lambda) + \int d^dx H(x)\sigma(x)\right] \tag{3.6}$$

with

$$\mathcal{S}_N(\sigma,\rho,\lambda) = \int d^dx\left[\frac{1}{2}(\partial_\mu\sigma)^2 + NU(\rho) + \frac{\lambda(\phi^2 - N\rho)}{2}\right] + \frac{1}{2}(N-1)\,\mathrm{Tr}\left[\ln(-\partial_\mu\partial^\mu + \lambda)\right] \tag{3.7}$$

To move further, we have to assume an ansatz. Assume that the expectation values of the fields scale with N like

$$\langle\sigma\rangle \sim O(N^{\frac{1}{2}}), \quad \langle\rho\rangle \sim O(1), \quad \langle\lambda\rangle \sim O(1) \tag{3.8}$$

Using this ansatz, it is easy to see that $\mathcal{S}_N \sim N$ and so, taking the limit $N \to \infty$ in the path integral 3.6 is akin to taking the classical limit of the theory since $1/N$ and $\hbar$ have the same placement in the path integral. In the $N \to \infty$ limit, we can calculate the path integral 3.6 by the saddle point method which render the theory classical.

[6]Note that there should be a normalization constant in front of the integral however this constant is irrelevant in our discussion

[7]If ϕ represent the spin orientation then σ is the magnetization, H is the magnetic field, and ρ is the energy

3.1.2 SU(N) matrix model

In the $O(N)$ vector model, we arrived at a classical theory of three fields σ, ρ, λ which is a finite number of fields. In general, the large N limit of vector models is a classical theory of finite number of fields. This isn't necessarily the case in matrix models since we are dealing with $N \times N$ matrix fields.

The field theories that appear in the Gauge/Gravity duality are matrix field theories in the large N limit. In this section, we will consider $SU(N)$ Yang-Mills theory. The first one to discuss the large N limit of $SU(N)$ was 't Hooft in 1974 (see [18]). In his paper, he managed to show that Feynman diagrams arrange themselves according to their topologies and in the large N limit only those diagrams with the simplest topologies dominate. This statement will be made more concrete in this section.

First, let us consider pure Yang-Mills action

$$\mathcal{S}_{YM} = -\frac{1}{2g^2} \int d^4x \, \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \tag{3.9}$$

Taking the large N limit here is a bit subtle. If one employed a UV-cutoff Λ_{UV}, we find that the confinement and mass gap occurs at the strong coupling scale Λ_{QCD} [9] which, at one-loop, is given by

$$\Lambda_{QCD} = \Lambda_{UV} \exp\left(-\frac{3}{22} \frac{(4\pi)^2}{g^2 N} \right) \tag{3.10}$$

If we naively take $N \to \infty$ while keeping everything else in the theory constant, we see that $\Lambda_{QCD} = \Lambda_{UV}$ which means that this naive Large N limit doesn't have confinement nor a mass gap.

To go around this problem, we define the 't Hooft coupling $\lambda = g^2 N$. The large N limit is defined by taking $N \to \infty$ while keeping λ fixed. In terms of the 't Hooft coupling, we can write the Yang-Mills action as

$$\mathcal{S}_{YM} = -\frac{N}{2\lambda} \int d^4x \, \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \tag{3.11}$$

Already from here, we can see that $\mathcal{S}_{YM} \sim N$ and so, the Large N limit in this case also gives a classical theory. Later we will see that this classical theory is classical gravity in higher dimensions however in this section we will not attempt to show that. Instead, we will consider a perturbative expansion of Yang-Mills theory.

To establish a perturbative description, we need to derive Feynman rules. First, we note that the gauge field A_μ is a Lie algebra valued field and so, it transforms in the adjoint representation or like a quark-antiquark pair since it has two indices. So, we can adopt the double line notation to draw the propagator where each line represent one index.

Schematically, the Yang-Mills Lagrangian looks like

$$\mathcal{L} \sim \frac{N}{\lambda} \left[\partial A \partial A + A^2 \partial A + A^4 \right] \tag{3.12}$$

So, we can guess the Feynman rules from the form of the Lagrangian. The Feynman rules are summarized below

- Associate the factor λ/N to each propagator

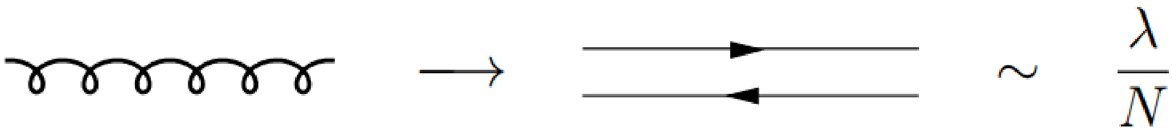


Figure 1. The propagator of SU(N) gauge field in the double line notation

- Associate the factor N/λ for each vertex

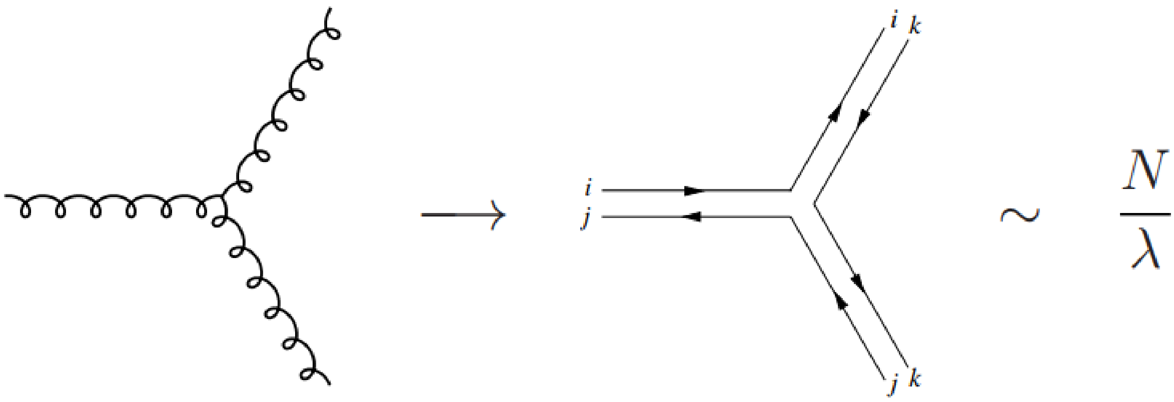


Figure 2. 3-vertex in double line Notation

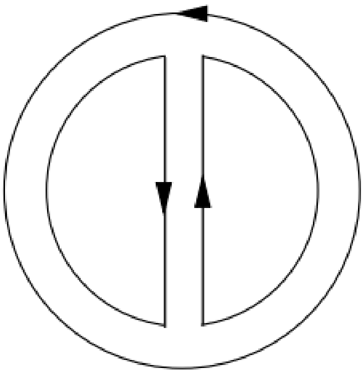
- Associate the factor N for each loop[8]

[8]This factor appears because each loop sums over N colors

Now, if we denote the number of vertices, propagators, and loops by V, E, and F respectively, we find that

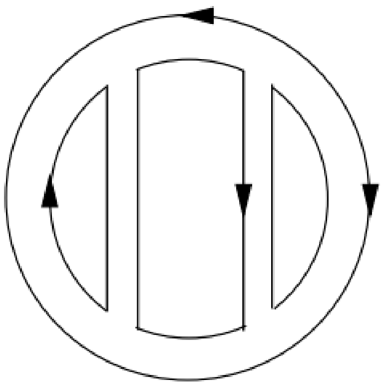
$$\text{Feynman Diagram} \sim (\frac{N}{\lambda})^V (\frac{\lambda}{N})^E N^F = \lambda^{E-V} N^{V-E+F} \tag{3.13}$$

To understand the behavior of the perturbative expansion, let us consider vacuum bubble diagrams. The leading order contribution is a diagram that looks like this Here, the



$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N^3 \sim \lambda N^2$$

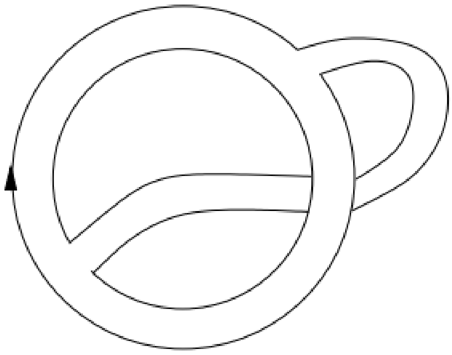
diagram has 3 propagators, 2 vertices, and 3 loops so we find that the propagator is of order λN^2. Similarly, at the next order of λ, we have the diagram We notice that this diagram



$$\sim \left(\frac{\lambda}{N}\right)^6 \left(\frac{N}{\lambda}\right)^4 N^4 \sim \lambda^2 N^2$$

is second order in λ but has the same N^2 behavior as the one-loop diagram.

Now let us consider a different type of diagrams. Consider the following diagram If you follow the loop in the above diagram, you will find that this diagram has only one loop. In other words, this diagram's contribution to the vacuum bubbles is suppressed by a factor of $1/N^2$ relative to the first two diagrams. So as $N \to \infty$ the contribution of this diagram becomes subleading.



$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N \sim \lambda$$

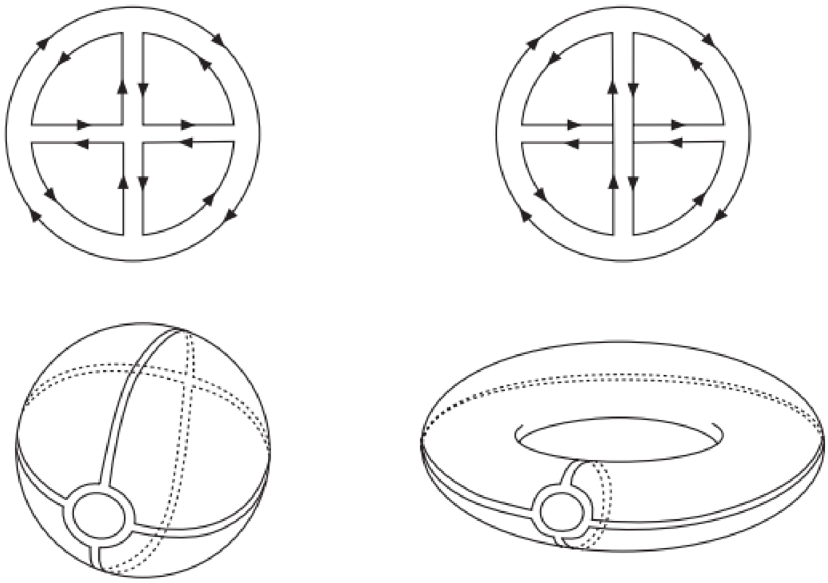


Figure 3. To the left is a planar

The diagrams that are proportional to N^2 are known as planar diagrams. These diagrams can be drawn on a plane[9] without crossing. On the other hands, there are diagrams that can only be drawn on more complicated topologies such as a torus. Such diagrams are called non-planar diagrams. In fact, the exponent of N in each diagram gives the Euler characteristic $\chi = V - E + F$. The figure below is an example of planner and non-planar diagrams drawn on a sphere and a torus respectively.

[9]It can also be any geometry that is topologically equivalent to the plane i.e. having the same Euler characteristic as the plane.

We conclude that the vacuum bubbles can be written as

$$\ln(\mathcal{Z}) = f_0(\lambda)N^2 + f_1(\lambda)N^0 + f_2(\lambda)N^{-2} + \cdots \tag{3.14}$$

More concretely, if we write Euler characteristic in terms of the genus h[10] of the Feynman diagram topologies, we get

$$\ln(\mathcal{Z}) = \sum_{h=0}^{\infty} f_h(\lambda)N^{\chi} = \sum_{h=0}^{\infty} f_h(\lambda)N^{2-2h} \tag{3.15}$$

As we can see, the partition function is given by a sum over the topologies of two dimensional surfaces. This has the same form as the string theory partition function! This suggest that there might be a relation between string theory and Large N theories. Perhaps we can find a relation that looks like the one below:

$$\mathcal{Z}_{gauge} \stackrel{?}{=} \mathcal{Z}_{string}$$

Later in section 4, we will see that this resemblance is actually a duality and the the planar diagrams will correspond to the classical gravity limit of string theory.[11]

3.2 Black Hole Thermodynamics

It was shown in the 70s by Bekenstein, Hawking, and others that black holes are in fact thermodynamic objects(see [26]). In there seminal work, they managed to prove a set of laws that govern black hole physics that have striking resemblence to thermodynamics laws. Perhaps the most remarkable out of them is that black holes have entropy that is proportional to the area at the horizon. Unlike regular thermodynamic systems, the entropy of black holes doesn't scale with volume but it rather scale with the area. This curious phenomena hints that black holes should have a mictroscopic description that lives in one dimension less. This is exactly what we see in the Gauge/Gravity duality where one can compute Black hole thermodynamic quantities from the dual field theory description and vice versa. In this section, we review the basics of black hole thermodynamics based on the reviews given in [25, 26].

Zeroth law: Black holes can be perturbed as objects fall into them however after a while the black hole will stabilize and reach an equilibrium that is characterized by a constant surface gravity.

Surface Gravity: surface gravity is defined as the acceleration an observer at infinity needs to provide in order to keep a particle located at the horizon at rest.

[10] number of holes

[11] Note that if $\lambda \ll 1$, then only low order diagrams (diagrams with a few loops) are important. On the other hand, if $\lambda \gg 1$, then all diagrams diagrams are important. In this case, the diagrams will completely cover the 2-dimensional surface they are drawn on. For example, planar diagrams will cover the entire sphere. This will be relevant when we discuss the relation between the large N limit and string theory

To compute the surface gravity, consider a particle motion in a black hole geometry with the following metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.16}$$

where $f(r)$ satisfies $f(r) = 0$ at the horizon $r = r_0$, and $f(r) = 1$ far from the horizon $r \gg r_0$

If the particle has zero angular momentum, then, we can write the particle 4-momentum as:

$$P = (m\varepsilon, m\frac{dr}{d\tau}, 0, 0) \tag{3.17}$$

where ε is the energy per unit mass and τ is the proper time.

The momentum vector satisfies $P^2 = -m^2$. Evaluating P^2 using the metric 3.16, we get

$$P^2 = -m^2 = -f(r)m^2\varepsilon^2 + m^2\left(\frac{dr}{d\tau}\right)^2 \frac{1}{f(r)} \tag{3.18}$$

For a particle moveing far from the horizon, this implies that:

$$\left(\frac{dr}{d\tau}\right)^2 = \varepsilon^2 - f(r) \tag{3.19}$$

Differentiating this last expression, we get

$$\frac{d^2 r}{d\tau^2} = -\frac{1}{2}\partial_r f(r) \tag{3.20}$$

For a particle at rest, the acceleration along the time direction is zero, so the proper acceleration is given by

$$a^\mu = (0, \frac{d^2 r}{d\tau^2}, 0, 0) \tag{3.21}$$

Then, we see that

$$a = \sqrt{g_{\mu\nu}a^\mu a^\nu} = \frac{\partial_r f}{2\sqrt{f}} \tag{3.22}$$

We see that the proper acceleration diverges at the horizon $r = r_0$ which is expected since a particle can't escape a black hole. What we want to compute is the acceleration at the horizon measured by an observer at infinity.

Consider a particle at a distance r and an observer at infinity. If the observer magically pulled the particle by a proper distance δs then the work done by the observer is

$$W_\infty = a_\infty \delta s \qquad \text{at} \quad \infty \tag{3.23}$$

$$W_r = a\delta s \qquad \text{at} \quad r \tag{3.24}$$

Now if we somehow converted the work to radiation and measured the radiation at infinity, the energy would be redshifted. We set $E_r = W_r, E_\infty = W_\infty$. Now, using equation 2.27, we get

$$E_\infty = \sqrt{\frac{f(r)}{f(\infty)}}E_r \rightarrow a_\infty = \sqrt{f}a = \frac{\partial_r f(r)}{2} \tag{3.25}$$

The surface gravity is defined as a_∞ evaluated at the horizon so we finally get

$$\kappa = a_\infty(r_0) = \frac{\partial_r f(r_0)}{2} \tag{3.26}$$

Surface gravity plays the role of temperature for black hole systems. In fact, when we compute the Hawking temperature, we will see that it is proportional to the surface gravity.

First Law

The horizon radius is proportional to the mass. This implies that the area of a black hole is proportional to the mass

$$dM \propto dA \tag{3.27}$$

Now, since the surface gravity plays the role of temperature and the mass plays the role of energy, then by analogy to the first law of thermodynamics ($dE = TdS$), we arrive at the first law of black hole thermodynamics:

$$dM \propto \kappa dA \tag{3.28}$$

The coefficient of proportionality will be clear once we define the Hawking temperature and black hole entropy in terms of the surface gravity and area.

Hawking Radiation

In 1974, Hawking showed that a black hole has black body radiation and he managed to calculate the temperature of this radiation [27]. He showed that by considering the quantum effects near the black hole horizon. He studied quantum field theory in a curved geometry near the horizon and he managed to show that the quantum vacuum is not unique and there are different vacuums for different observers. So, while an observer far in the past before a black hole is formed would see the vacuum state with no radiation, an observer far in the future and far away from the horizon will see a black hole radiating particle-antiparticle pairs.[12]

We will not attempt to compute Hawking temperature using Hawking's method however we will use a simple method that is quite relevant to the kind of calculations done in AdS/CFT. This method was presented in [25]. The method can be summarized by preforming a Wick rotation $t \to -it_E$ and requiring that the resultant Euclidean geometry should become smooth.

Let us first consider the Euclidean metric

$$ds_E^2 = +f(r)dt_E^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.29}$$

[12]Since different observers(frames) are generally not equivalent in curved spacetime, different observes can diagonalize the hamiltonian of field theory differently. This results in different creation and annihilation operators for different observers and hence each observer has its own vacuum. The creation and annihilation operators for different observers are related to each other by a Bogoliubov transformation (see [28])

Here, $f(r)$ is a function satisfying $f(r_0) = 0, f(\infty) = 1$. Since the metric is spherically symmetric, we can ignore the spherical coordinates and focus on the radial and time coordinates. Furthermore, if we assume that $\partial_r f(r_0) \neq 0$, we can expand the function $f(r)$ near the horizon. We get:

$$ds_E^2 \approx \frac{dr^2}{\partial_r f(r_0)(r - r_0)} + \partial_r f(r_0)(r - r_0) dt_E^2 \tag{3.30}$$

Now by defining $\rho \equiv 2\sqrt{(r - r_0)/\partial_r f(r_0)}$, the metric becomes"

$$ds_E^2 = d\rho^2 + \rho^2 d\left(\frac{\partial_r f(r_0)}{2} t_E\right)^2 \tag{3.31}$$

If the time coordinate t_E is periodic, then this form of the metric is exactly the same as the flat metric. In fact, we need to make the time coordinate periodic or else we will have a conical singularity as $\rho \to 0$. Hence if we define the periodicity of the time coordinate as β, we get

$$\beta = \frac{4\pi}{\partial_r f(r_0)} \tag{3.32}$$

Now, we can identify the periodicity β with the inverse temperature. So, we finally get the black hole temperature of the Hawking temperature to be

$$T_H = \frac{\partial_r (r_0)}{4\pi} \tag{3.33a}$$

or in terms the surface gravity, we get

$$T_H = \frac{\kappa}{2\pi} \tag{3.33b}$$

Black Hole Partition Function

As we saw, black holes are thermodynamic objects. They possess temperatures and have their own version of the zeroth and first law. This suggests that black holes have microstates that are averaged out once we look at them in a macroscopic scale. So, in order to understand the thermodynamic nature of black holes, we need to provide a quantum statistical description for them. Let us assume that we already know the quantum theory of black holes so we can write the generating function of this quantum theory as:

$$\mathcal{Z} = \int \mathcal{D}g e^{i\mathcal{S}[g]} \tag{3.34}$$

where $\mathcal{S}$ is the gravitational action. Now, Wick rotating time, we can pass to the Euclidean path intergral

$$\mathcal{Z} = \int \mathcal{D}g e^{-\mathcal{S}_E[g]} \tag{3.35}$$

where we take the Wick rotated time coordinate to be periodic in order to avoid conical singularities. This Euclidean path integral is simply the quantum statistical partition function of the black hole.

We can use the partition function 3.35 to get thermodynamic quantities and derive the thermodynamic relations but we don't know to define the partition function. However, we know that in the semi-classical limit, $\mathcal{Z}$ is dominated by the classical on-shell action. We can use the saddle point approximation to get the partition function in this limit to be:

$$\mathcal{Z} \approx e^{-\mathcal{S}_E^{on-shell}} \tag{3.36}$$

We can now identify the on-shell action as $\mathcal{S}_E^{on-shell} \equiv \beta F$ where F is the Helmholtz free energy. Using the Helmholtz free energy, it is now easy to compute thermodynamic relations. Particularly, we can verify that the Schwarzschild black hole entropy is given by

$$S = \frac{A}{4G} \tag{3.37}$$

where A is the area of the black hole and G is the gravitational constant. This formula is valid for a wide variety of black holes even in different number of dimensions. Interestingly, the black hole entropy is proportional to the area and not the volume. This behaviour is quite different as we usually expect the entropy of a particular system to scale with volume. This suggests that black holes are described by a microscopic theory that lives in one dimensional less than the gravitational theory. This is our second hint for the presence of a holographic duality.

4 "Proving" AdS/CFT

4.1 Overview of String Theory

String theory started as a theory of strong interaction in the sixties. It initially started in an attempt to explain the hadronic resonance of high spin observed experimentally. According to these experiments, the square of the mass of the hadronic particles is linearly proportional to the spin of these particles.

$$M^2 \sim J \tag{4.1}$$

This behavior is known as Regge Trajectories which string theory managed to reproduce accurately. It is not difficult to understand how string theory managed to reproduce this relation. In strong theory model of strong interaction, a meson is modeled by a quark-antiquark pair connected by a string with string tension T and length L. Since the string has tension, it would collapse to a point unless the string is oscillating or rotating in order to balance the tension. For now, let us assume that the string is rotating with angular momentum J around its center of mass. From dimensional considerations, we expect the mass to behave like $M \sim TL$ while the angular momentum to be $J \sim PL$ where P is the linear momentum. We also expect from a relevistic theory that $P \sim M$. From this, it is easy to see that $J \sim PL \sim ML \sim T^{-1}M^2$. Thus, we reproduce Regge trajectory

$$J \approx \frac{1}{T}M^2 + \hbar \tag{4.2}$$

where $\hbar$ is a quantum correction.

Phenomenologically, we have $J = \alpha' M^2 + \alpha(0)$. Here $\frac{1}{\alpha'}$ represents the string tension. It has the unit of length squared so we define the string length scale as

$$\alpha' = l_s^2$$

The string description of strong interaction manage to capture one more aspect of strong interaction which is the shape of flux lines. The color flux lines of the strong force don't spread like electric flux lines. The color flux line between a quark-antiquark pair for a thin flux tube connecting the pair. If we ignore the thickness of this tube, we can identify it with a string.

While string theory managed to capture some aspects of strong interaction, it has various problems that prevents it from being a true theory of string interaction, mainly these problems are:

- String theory needs more dimensions to be consistently quantized
- string theory fails to describe other aspects of strong theory such as confinement and asymptotic freedom.

Due to these shortcomings, string theory was abandoned as a theory of strong interaction but was later revived as a unified theory later by John Schwartz(see [23])

[13] This chapter follows the construction given in [25]

4.2 String interactions

String theory has two main objects, open strings and closed strings. Strings are the generalization to point-particles. The starting point of string theory is from the classical action of a relativistic string, the so-called Nambu-Goto action defined as:

$$\mathcal{S}_{NG} = -\frac{1}{2\pi l_s} \int \sqrt{-\det\left[g_{MN}\partial_\sigma x^M \partial_\tau x^N\right]} d\sigma d\tau \tag{4.3}$$

The classical equation of motion can be solved for flat spacetime and for both Neumann and Dirichlet boundary conditions. The coordinate $x(\tau, \sigma)$ can be expanded in Fourier series as an infinite superposition of plane waves. To quantize the string, we promote $x(\tau, sigma)$ and the conjugate momenta to operators and impose the canonical commutation relations. In this sense, the classical plane waves decomposition of $x(\tau, sigma)$ corresponds to an infinite set of particles with different mass spectrum characterized by the string length scale l_s. The modes of oscillations of an open string give rise to spin-1 particles while for closed strings they give rise to spin-2 particles.

Quantizing strings led to some curious results. One of them is that the mass spectrum of particles allow the existence of tachyonic particles(particles with $m^2 < 0$). This is usually a sign of instability in the system. In order to avoid this, the string had to have fermionic coordinates and supersymmetry had to be introduced. This give rise to the superstring theory. Another curious aspect of string quantization is the requirement that the spacetime has to have specific number of dimensions in order to quantize strings consistently. For superstring theory, spacetime has to be 10.

In order to quantize open strings, one need to give boundary conditions to fix the open string end points. One often fix the string end points to end on extended hypersurfaces known as Dp-branes. Dp-branes are extended objects that extend in p+1 dimensions. Dp-branes are important objects in the study of Gauge/Gravity duality. Beside their role as the hypersurfaces were string ends lie, they are also dynamical objects that can deform or rigidly move.

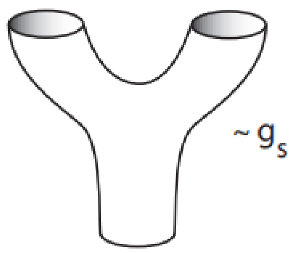
In order to understand the dynamics of Dp-branes, first let us consider the end points of an open string. The end points can be thought of as objects that are charged under U(1) gauge group. Since a charge can act as a source to a gauge field, we see that there is an Abelian gauge field A_μ living in the Dp-brane world volume. The action that takes into account the deformation and the presense of gauge field in the Dp-brane is called Dirac-Born-Infeld action, which can be written as:

$$\mathcal{S}_{DBI} = -\frac{1}{(2\pi)^p g_s l_s^{p+1}} \int d^{p+1}x \sqrt{-\det(g_{MN} + 2\pi l_s^2 F_{NM})} \tag{4.4}$$

where g_s is the string coupling constant.
Notice that if the gauge field isn't excited, the Dirac-Born-Infeld action reduces to a generalization to Nambu-Goto action.

Dp-branes and strings can interact in various ways. Let us consider simple interactions of strings. Strings propagate in spacetime forming a world sheet. Strings then interact with each other by splitting or by joining with other strings. The world sheet for these interactions is a two dimensional surface with holes and boundaries. These 2-dimensional surfaces are in fact the perturbative description of string interactions via Feynman diagrams.

Let us consider a closed string. The basic process a closed string can do is to split into two closed strings. The world sheet of this process corresponds to a Feynman diagram vertex as shown in the figure below.



We assign g_s to be the strength of this process. In other words, g_s is the coupling constant for closed strings. A world sheet with a hole in it corresponds to a 1-loop diagram and has two vertices so it is proportional to g_s^2. If h denotes the number of holes in a world sheet, we see that the string perturbative expansion takes the following form:

$$\ln \mathcal{Z}_{string} = \sum_{h=0} g_s^{2h-2} \tilde{f}_h(\alpha') = \sum_{h=0} g_s^{-\chi} \tilde{f}_h(\alpha') \tag{4.5}$$

In the weak coupling limit $g_s \ll 1$, the diagrams with the lowest number of holes becomes important. This gives a classical theory of supergravity in 10 dimensions with the following actions[14]

$$\mathcal{S}_{SUGRA_{10}} = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-g}(R + \alpha' R^2 + \dots) \tag{4.6}$$

where $G_{10} \simeq g_s^2 l_s^8$ is Newton's constant in 10 dimensions. The first term in this action corresponds to Einstein's gravity while higher order terms in α' are known as α' corrections. These are the terms that arise from string theory corrections to gravity. If $\alpha' R \ll 1$, we can ignore these corrections and so, we end up with Einstein's gravity in 10 dimensions. This corresponds to the lowest order diagram in perturbation series (the sphere world sheet).

[14] Actually, the most general form for the classical gravity approximation is

$$\mathcal{S} = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-g}e^{-2\phi}(R + 4(\nabla\phi)^2) + \dots$$

where ϕ is a scalar field called the dilaton. We will not consider the dilaton in this thesis.

4.3 AdS = CFT

As we discussed earlier, the closed string generating functional takes a very similar form to the SU(N) generating functional.

$$\ln \mathcal{Z}_{gauge} = \sum_h N^{\chi} f_h(\lambda) \tag{4.7}$$

$$\ln \mathcal{Z}_{string} = \sum_h (\frac{1}{g_s})^{\chi} \tilde{f}_h(\alpha') \tag{4.8}$$

It is natural to think that these two expressions are related. In fact, thanks to the duality between open and closed strings, we can relate the the coupling constants on both sides as we discussed in the last section. However, we have an inconsistency between the gravity side and the gauge theory side. Mainly,

- Large N gauge theory should be described by string theory
- but string theory is inconsistent in 4 dimensions!

To resolve this inconsistency, we note that string theory is a theory of both gravity and gauge theory, meaning that spacetime in string theory doesn't have to be flat. In other words, we can put the gauge theory on a curved background. Since gauge theory has to have $SO(1,3) \times \mathbb{R}^{1,3}$ invariance. To make this possible on a curved background, we need a curved spacetime of at least 5 dimensions with the same isometry group.[15] One such spacetime is:-

$$ds_5^2 = \Omega(w)^2(-dt^2 + d\vec{x}_3^2) + dw^2 \tag{4.9}$$

This metric has $SO(1,3) \times \mathbb{R}^{1,3}$ isometry in the $(t, \vec{x}_3)$ coordinates thus, the gague theory can live in these coordinates.

We now know how to fit the gauge theory into higher dimensional geometries however We still don't know which theory is described by the D3-branes. To determine the gauge theory, let's consider an open string. An open string attached to N coincident D-branes gives rise to U(N) gauge theory. In 10 dimensional spacetime, the string has 8 oscillatory degrees of freedom. Two of which are associated to the U(N) gauge field itself while the other six give rise to six scalars and because of supersymmetry, we require that we have 8 fermionic degrees of freedom. This gives rise to 4 Weyl fermions. Since everything originated from the same string, we expect that they belong to the same representation of U(N). In other words, they are in the adjoint representation. This means that the gauge theory is in fact the super Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry or $\mathcal{N} = 4$ SYM theory for short. This theory is scale invariant since it has vanishing β function.

$$\beta(g_{YM}) = \mu \frac{\partial g_{YM}}{\partial \mu} = -\frac{g_{YM}^3}{48\pi^2} N(11 - 2n_f - \frac{1}{2} n_s) \tag{4.10}$$

[15] Note that this is consistent with the fact that black hole entropy is proportional to the area rather than the volume.

where n_f is the number of Weyl fermions, n_s is the number of real scalars, and μ is the energy scale. The Lagragian of $\mathcal{N} = 4$ SYM thoery takes the form:

$$\mathcal{L}_{SYM} = \frac{1}{g_{YM}^2} \mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}^2 - (D_\mu\phi_i)^2 - i\bar{\lambda}_j\gamma^\mu D_\mu\lambda^j + O(\phi^4) + O(\bar{\lambda}\lambda\phi)\right] \tag{4.11}$$

Since the gauge theory has conformal symmetry, we expect that the space where the theory lives to have the conformal group as an isometry. Consider the metric:

$$ds_5^2 = \Omega(w)^2(-dt^2 + d\vec{x}_3) + dw^2 \tag{4.12}$$

Now consider the transformation

$$x^\mu \to ax^\mu \quad \mu = 0, 1, 2, 3; a \in \mathbb{R} \tag{4.13}$$

The metric 4.12 is invariant if we have the transformation law $\Omega^2(w) \to a^{-2}\Omega^2(w)$
To do that, we need w to transform in a certain way while keeping dw^2 invariant. The only possible transformation is $w \to w + \alpha$ where α has dimension of length.

It is easy to see that $\Omega^2(w)$ must be $e^{-2\frac{w}{L}}$ with $\alpha = L \ln a$

So:-

$$\begin{aligned} ds_5^2 &= e^{-2\frac{w}{L}}(-dt^2 + d\vec{x}_3^2) + dw^2 \\ &= (\frac{L}{z})^2(-dt^2 + d\vec{x}_3^2 + dz^2) \end{aligned} \tag{4.14}$$

with $z = Le^{\frac{w}{L}}$

This is the 5-dimensional Anti-de Sitter space with AdS radius L. It is the solution of to gravity with the following action action

$$\mathcal{S}_5 = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g_5}(R_5 + \frac{12}{L^2}) \tag{4.15}$$

where G_5 is Newton's constant in 5 dimensions.

As the $\mathcal{N} = 4$ SYM theory and the AdS metric share the same symmetries and because of open-closed string duality, we expect the following relation:

$$\mathcal{Z}_{CFT} = \mathcal{Z}_{AdS_5} \tag{4.16}$$

Where the left hand side is a gauge theory with conformal invariance while the right hand side is a string theory on AdS_5 spacetime. The dictionary to relate both sides of the relation is

$$N^2 = \frac{\pi}{2}\frac{L^3}{G_5} \qquad \lambda = (\frac{L}{l_s})^4 \tag{4.17}$$

where l_s is the string length scale, L is the AdS radius, In this thesis, we are only interested in the large N limit so classical gravity is enough. So,

$$\mathcal{Z}_{CFT,N\gg 1} = e^{-S_E^{on-shell}+O(l_s^2)} \tag{4.18}$$

Here, $S_E^{on-shell}$ is the Euclidean on-shell action action obtained by substituting the classical solution to the metric into the action. the $O(l_s^@)$ corresponds to the higher order corrections to General relativity due to string theory and can be ignored if the curvature is small. Small curvatures correspond to strongly coupled gauge theory according to the above dictionary. So in the strong coupling regime $\lambda \gg 1$, we can ignore the l_s^2 corrections to classical gravity. So we finally end up with the following relation:

$$\mathcal{Z}_{CFT,N\gg\lambda\gg 1} = e^{-S_E^{on-shell}} \tag{4.19}$$

Before we end this section, we note that the AdS space isn't the only solution to 4.15. The other allowed metric is the AdS black hole, a black hole geometry that is asymptotically Anti de-Sitter. Since black holes are thermodynamic systems, one generally make the following connection:

- gauge theory at zero temperature $\longleftrightarrow$ AdS_5 spacetime
- gauge theory at finite temperature $\longleftrightarrow$ AdS_5 black hole

and we identify the temperature of the black hole as the temperature of the field theory.

4.4 Compactifying the 5-sphere

We mentioned before that string theory needs 10 dimensions to be quantized consistently yet we only considered string theory on AdS_5. It is natural to ask how this is possible. To answer this question, let us revisit the super Yang-Mills theory once more.

We already know that $\mathcal{N} = 4$ SYM theory has conformal invariance. This conformal invariance is also present in the AdS space as we expect. What we don't know is that the gauge theory has another symmetry. If we look at the Lagrangian in eq 4.11, we will notice that the Lagrangian has a global SO(6) symmetry for the scalar fields ϕ_i. This symmetry is known as R-symmetry. This global symmetry isn't present in AdS_5. To resolve this, we consider the spacetime to be $AdS_5 \times \mathbb{S}^5$. This spacetime has the same global symmetries of the gauge theory as we required and is indeed 10-dimensional. The metric of this spacetime is given by

$$ds^2 = (\frac{L}{z})^2(-dt^2 + d\vec{x}_3^2 + dz^2) + +L^2 d\Omega_5^2 \tag{4.20}$$

The reason why the spherical part doesn't appear in the AdS_5/CFT_4 relation is because it is often compactified. This compactification results in various effects for example, let us consider a massless scalar field ϕ in $AdS_5 \times \mathbb{S}^5$.This field can be reduced to infinitely many particles with different masses by compactifying the sphere. To see this, let us express ϕ in terms of spherical harmonics as:

$$\phi(x, \Omega) = \sum_l \phi_l(x) Y_l(\Omega) \tag{4.21}$$

where x is the coordinate vector on AdS_5, Ω is the the coordinate vector on the $\mathbb{S}^5$, and $Y_l(\Omega)$ is the spherical harmonics on $\mathbb{S}^5$.

The scalar field obeys the Klein-Gordon equation:

$$\nabla^2\phi = 0 \tag{4.22}$$

where ∇^2 is the D'Alembertian operator in $AdS_5 \times \mathbb{S}^5$. Since the metric 4.20 factorizes into an AdS_5 part and a $\mathbb{S}^5$ part, the D'Alembertian is additive

$$\nabla^2\phi = \nabla^2_{AdS_5}\phi + \nabla^2_{\mathbb{S}^5}\phi \tag{4.23}$$

here, $\nabla^2_{\mathbb{S}^5}$ is the square of the angular momentum generators of SO(6). The eigenvalues of $\nabla^2_{\mathbb{S}^5}$ acting on the spherical harmonics are:

$$\nabla^2_{\mathbb{S}^5} Y_l = \frac{l(l+4)}{L^2} Y_l \tag{4.24}$$

The Klein-Gordon equation then becomes:

$$\nabla^2_{AdS_5}\phi_l(x) - m_l^2\phi, \qquad\qquad m_l^2 = \frac{l(l+4)}{L^2} \tag{4.25}$$

So, we see that compactifying the sphere gives us a tower of massive scalar fields of mass m_l. In the next section, we will show how to assign each field in gravity side to its dual operator in gauge theory side.

4.5 Remarks on the extra dimension in AdS_5

In order to understand the meaning of this extra dimension in AdS_5, we need start from the renormalization group. Consider a a classical system on a lattice with lattice spacing a and with the following hamiltonian:

$$H = \sum_{i,x} J_i(a,x)\mathcal{O}^i(x) \tag{4.26}$$

here, x denotes the different lattice sites and i denotes different operators. Here $J_i(x,a)$ is the coupling constants at different lattice sites x when the spacing is a. From the techniques of renormalization, we know that we can coarse grain the lattice by considering larger lattice spacings and grouping different lattice sites into a single site. This procedure will change the coupling constants the more we increase the lattice spacing. For example, we can have something like this:

$$J_i(x,a) \to J_i(x,2a) \to J_i(x,3a) \to \dots \tag{4.27}$$

This means that the coupling constants become dependent on the scale(lattice spacing). Let us denote the scale by μ, then, we can determine the evolution of the coupling constants by the renormalization flow equation:

$$\mu\frac{\partial J_i(x,\mu)}{\partial\mu} = \beta_i\big(J_i(x,\mu),\mu\big) \tag{4.28}$$

where β_i is the beta function of the i-th coupling constant. At weak coupling, we can determine the beta functions via perturbation series. At strong coupling, it is hard to determine the beta function but the AdS/CFT suggest that we can think of the scale u as another dimension. In this point of view, the succession of different scales is equivalent to consider different slices of a higher dimensional space.

In this context, we can think of J_i which source the operators O_i as classical fields in this higher dimensional space. So, we write

$$J_i(x, \mu) \equiv \phi_i(x, \mu) \tag{4.29}$$

The dynamics of ϕ_i is given by gravitational theory in the higher dimensional space. In this context, we can thing of AdS/CFT as a geometrization of the renormalization group. The sources of a quantum field theory in the UV can be identified with the values of the bulk fields evaluated at the boundary of this extra-dimensional space. In this sense, one can say that the quantum field theory lives on the boundary of AdS space.

In order to identify sources ϕ_i with their dual operators $\mathcal{O}_i$ in the field theory, we need to match the transformation laws of the sources and operators such that $\phi_i\mathcal{O}^i$ is a scalar in the group sense with respect to both the conformal and the Lorentz groups. For example, a scalar field *phi* is dual to a scalar operator $\mathcal{O}$, a a vector field A^μ is dual to a current J_μ and a spin 2 field such as the metric tensor $g_{\mu\nu}$ is dual to the energy-momentum-stress tensor $T^{\mu\nu}$. This will be more clear when we consider a scalar field example in the next section.

5 AdS/CFT Calculations

In this chapter, we finally start using AdS/CFT to compute relevant quantities in field theory, mainly, the correlation functions. This chapter follows closely the construction given in [24]

5.1 Scalars in AdS_5

In order to do calculations in AdS/CFT, one needs to identify the correspondence between classical fields in the gravity theory and field operators in the gauge theory. As we argued before, the basic idea is that we think of fields in AdS as sources to some dual operators in the gauge theory. In this section, we consider a massive scalar field in AdS space. For the sake of gerenality, we will first work in d+1 dimesnional AdS then, we specialize to the case of AdS_5. The metric of $AdS_d + 1$ in Euclidean sigis given by:

$$ds^2_{d+1} = \frac{L^2}{z^2}(dt^2 + d\vec{x}^2_{d-1} + dz^2) \tag{5.1}$$

A Scalar field in $AdS_d + 1$ has the following action:

$$\mathcal{S} = -\frac{1}{2}\int d^{d+1}x\sqrt{g}\big[g^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2\big] \tag{5.2}$$

It is easy to check that the equation of motion is:

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\partial_M(\sqrt{g}g^{MN}\partial_N\phi) = m^2\phi \tag{5.3}$$

Using the metric 5.1, we can write the equation of motion more explicitly as

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi) + z^2\delta_{\mu\nu}\partial^\mu\partial^\nu\phi - m^2L^2\phi = 0 \tag{5.4}$$

Now, we Fourier transform the field in the x^μ coordinates:

$$\phi(z, x^\mu) = \int \frac{d^dk}{(2\pi)^d}\phi_k(z) \tag{5.5}$$

So, the equation of motion becomes

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi_k) - z^2k^2\phi_k - m^2L^2\phi_k = 0 \tag{5.6}$$

In AdS/CFT, we are interested in the values of the fields near the boundary of AdS ($z \sim 0$). So, we make the ansatz $\phi_k \sim z^\beta$ for some exponent β and only consider the leading order terms of the equation of motion near the AdS boundary $z \sim 0$. It is now easy to see that β obeys the following equation:

$$\beta(\beta - d) - m^2L^2 = 0 \tag{5.7}$$

which has the following two solutions

$$\beta_- = d - \Delta, \qquad \beta_+ = \Delta \tag{5.8}$$

where $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2}$

We see that near $z \sim 0$, ϕ_k takes the form:

$$\phi_k(z) \approx A(k)z^{d-\Delta} + B(k)z^{\Delta} \tag{5.9}$$

By inverting the Fourier transform, we get

$$\phi(z, x^{\mu}) \approx A(x^{\mu})z^{d-\Delta} + B(x^{\mu})z^{\Delta} \tag{5.10}$$

Note that Δ has to be real in order to maintain stability of the AdS space with a scalar. This puts a lower bound on the mass of the scalar field

$$m^2 \geqslant -\frac{d^2}{4L^2} \tag{5.11}$$

This bound is known as the Breitenlohner-Freedman bound(see [19] and references within). Notice that m^2 can be negative which means that ϕ can be tachyonic even though it is satisfying the Breitenlohner-Freedman bound. Moreover, we see that when the mass satisfies this bound, we have

$$d - \Delta \leqslant \Delta \tag{5.12}$$

Then, we see that near $z \sim 0$, the term behaving like $z^{d-\Delta}$ is dominant. Let us take $z = \epsilon$ where ϵ is an infinitesmal number and let us focus on the dominant term in ϕ. We have:

$$\phi(\epsilon, x^{\mu}) \approx \epsilon^{d-\Delta} A(x^{\mu}) \tag{5.13}$$

Note that if $m^2 > 0$, the leading term in ϕ is divergent. In order to avoid this divergence in our AdS/CFT, we define the source φ of a field operator in the CFT from the value of the dual field ϕ in AdS by:

$$\varphi(x^{\mu}) = \lim_{z \to 0} z^{\Delta - d}\phi(z, x^{\mu}) \tag{5.14}$$

In order to understand the meaning of Δ, let us focus on the boundary theory action. Let the boundary be at $z = \epsilon$. If $\mathcal{O}$ is the operator dual to the field ϕ then the boundary action looks like

$$S_{boundary} \sim \int d^dx \sqrt{\gamma_{\epsilon}}\phi(\epsilon, x^{\mu})\mathcal{O}(\epsilon, x^{\mu}) \tag{5.15}$$

where $\gamma_{\epsilon} = \frac{L^{2d}}{\epsilon^{2d}}$ is the determinant of the induced metric on the boundary. Substituting $\phi(\epsilon, x^{\mu}) = z^{d-\Delta}\varphi(x^{\mu})$ to the leading order, we get

$$S_{boundary} \sim L^d \int d^dx \varphi(x^{\mu})\epsilon^{-\Delta}\mathcal{O}(\epsilon, x^{\mu}) \tag{5.16}$$

Now, we see that in order to make the boundary action finite and independent of ϵ as we take the limit $\epsilon \to 0$, we require that

$$\mathcal{O}(\epsilon, x^{\mu}) = \epsilon^{\Delta}\mathcal{O}(x^{\mu}) \tag{5.17}$$

Now, in section 4.5, we argued that the extra dimension of AdS corresponds to the scale of QFT so passing from $z = 0$ to $z = \epsilon$ is a scale transformation. In other words, $\mathcal{O}(\epsilon, x^\mu) = \epsilon^\Delta \mathcal{O}(x^\mu)$ is the wavefunction renormalization of the operator $\mathcal{O}$ as we change the scale. This implies that Δ is the mass scaling dimension of the operator $\mathcal{O}$. Similarly, $d - \Delta$ is the mass scaling dimension of the source φ. Using this fact, we can identify the dual operator to a field in AdS.

Remember when we discussed a massless scalar field in $AdS_5 \times \mathbb{S}^5$, we should that compactifying the sphere leads to a tower of scalar fields with mass $m^2 = \frac{l(l+4)}{L^2}$. Let us check the mass scaling dimension of these scalar fields and try to find their dual operators. In 5 dimensional AdS, we have

$$\Delta = 2 + \sqrt{4 + m^2 L^2} \tag{5.18}$$

First, let us consider $l = 0$ case which gives the massless scalar ϕ_0. The scaling dimension in this case is $\Delta = 4$ so we expect that the dual operator in the gauge theory to have a scaling dimension of 4. Since the massless scalar has $l = 0$, it transforms as a singlet under SO(6). In other words, the dual operator shouldn't contain the scalar operators of the gauge theory. The only candidate in the gauge theory that has the correct scaling dimension and is a singlet under SO(6) is the so-called glueball operator:

$$\mathcal{O}_0 = \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \tag{5.19}$$

In general, the scaling dimension for any l is given by

$$\Delta_l = 2 + \sqrt{4 + l(l+4)} = l + 4 \tag{5.20}$$

The dual operator should be in the l representation of SO(6) group. We can construct the dual operator in such a way: we know that scalar field operators of $\mathcal{N} = 4$ SYM theory transform in the $l = 1$ representation of SO(6). (i.e. they transform like vectors with respect to SO(6)). The dual operator for any given l is then:

$$\mathcal{O}_{i_1,\ldots,i_l} = \mathrm{Tr}[\phi_{(i_1,\ldots,i_l)} F_{\mu\nu} F^{\mu\nu}] \tag{5.21}$$

where $\phi_{(i_1,\ldots,i_l)}$ is the traceless symmetric product of i scalar field operators of $\mathcal{N} = 4$ SYM theory.

5.2 Correlation functions

Now that we know how to assign a a field in the gravitational theory to its dual operator in the gauge theory, we are ready to discuss how to get correlation functions in AdS/CFT.

In section 4, we established that

$$\mathcal{Z}_{CFT, N \gg \lambda \gg 1} = e^{-S_E^{on-shell}} \tag{5.22}$$

But in order to get correlation functions between operators, we are interested in the case where there is a source perturbing the gauge theory. So we need to define the AdS/CFT relation for the generating functional defined as:

$$\mathcal{Z}[\varphi] = \int \mathcal{D}\chi e^{-\mathcal{S}_{SYM}[\chi]+\int d^d x\varphi(x)\mathcal{O}} \tag{5.23}$$

where χ is a collection of all independent operators of the gauge theory.

The claim of AdS/CFT is that

$$\mathcal{Z}[\varphi] = e^{-\mathcal{S}^{on-shell}_{gravity}[\phi]} \tag{5.24}$$

where, we require that $\phi(z,x^\mu) = z^{d-\Delta}\varphi(x^\mu)$ to leading order at the boundary $z\sim 0$. We note that $\mathcal{S}^{on-shell}[\phi]_{gravity}$ often diverges and care should be given when evaluating it. In order to evaluate it, we follow the procedures of holographic renormalization (see [20]) and the on-shell action is replaced by a renormalizable one $\mathcal{S}^{ren}_{gravity}$. We will illustrate this by considering scalar field example. To conclude, the n-point correlation function is defined by:

$$\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = \frac{\delta^{(n)}S^{ren}_{gravity}}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)}\bigg|_{\varphi=0} \tag{5.25}$$

This is known as the WKP-Witten relation [21, 22]

One-point function

Let us try to compute the one-point function of an operator $\mathcal{O}$ that is dual to a scalar field ϕ in the presence of a source φ. According to 5.25, the one point function is

$$\langle\mathcal{O}(x)\rangle = \frac{\delta\mathcal{S}^{ren}_{gravity}}{\delta\varphi(x)}\bigg|_{\varphi=0} \tag{5.26}$$

Using the definition of φ, we can rewrite this as

$$\langle\mathcal{O}(x)\rangle = \lim_{z\to 0} z^{d-\Delta}\frac{\delta\mathcal{S}^{ren}_{gravity}}{\delta\phi(z,x)}\bigg|_{\varphi=0} \tag{5.27}$$

Let us try to compute the action in a closed form. Let the gravitational action be given by

$$\mathcal{S}_{gravity} = \iint dz d^d x\mathcal{L}[\phi,\partial\phi] \tag{5.28}$$

Under an infinitesimal change $\phi\to\phi+\delta\phi$, the change in the action is:

$$\delta\mathcal{S}_{gravity} = \iint dz d^d x\left[\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_M\Big(\frac{\partial\mathcal{L}}{\partial(\partial_M\phi)}\Big)\right)\delta\phi + \partial_M\left(\frac{\partial\mathcal{L}}{\partial(\partial_M\phi)}\delta\phi\right)\right] \tag{5.29}$$

The first term vanishes on-shell because of the equations of motion. Now because this expression diverges at the boundary of spacetime $z\sim 0$, we let the boundary to be at $z=\epsilon$ and we later take $\epsilon\to 0$. We get

$$\delta\mathcal{S}^{on-shell}_{gravity} = \int_\epsilon^\infty dz d^d x\partial_z\left(\frac{\partial\mathcal{L}}{\partial(\partial_z\phi)}\delta\phi\right) = -\int_{z=\epsilon} d^d x\frac{\partial\mathcal{L}}{\partial(\partial_z\phi)}\delta\phi \tag{5.30}$$

We define Π as

$$\Pi = -\frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} \tag{5.31}$$

Here, Π takes the form of canonical momentum. We see that

$$\frac{\delta \mathcal{S}^{on-shell}_{gravity}}{\delta \phi(\epsilon, x)} = \Pi(\epsilon, x) \tag{5.32}$$

This expression diverges as $\epsilon \to 0$. To avoid this, we define the renormalized action by

$$\mathcal{S}^{ren}_{gravity} = \mathcal{S}^{on-shell}_{gravity} + \mathcal{S}_{ct} \tag{5.33}$$

where $\mathcal{S}_{ct}$ is the action of counter terms to the divergent part of $\mathcal{S}^{on-shell}_{gravity}$ defined at $z = \epsilon$. The renormalized momentum is then defined as

$$\Pi^{ren}(\epsilon, x) = \frac{\delta \mathcal{S}^{ren}_{gravity}}{\delta \phi(\epsilon, x)} \tag{5.34}$$

We then conclude that the one-point function is given by

$$\langle \mathcal{O}(x) \rangle_\phi = \lim_{z \to 0} z^{d-\Delta} \Pi^{ren}(z, x) \tag{5.35}$$

The two-point function can be obtained from the one point function easily. To see how, let us start from the path integral definition of the one point function

$$\langle \mathcal{O}(x) \rangle_\varphi = \int \mathcal{D}\chi \mathcal{O}(x) e^{-\mathcal{S}_{SYM}[\chi] + \int d^d x \varphi(x) \mathcal{O}} \tag{5.36}$$

We can expand it as a power series of the source φ and keep only linear order in φ. We get

$$\langle \mathcal{O}(x) \rangle_\varphi = \langle \mathcal{O}(x) \rangle_{\varphi=0} + \int d^d y \varphi(y) G_R(x-y) + \dots \tag{5.37}$$

where $G_R(x-y)$ is the retarded green function(two point function) defined as

$$G_R(x-y) = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle \tag{5.38}$$

The first term in 5.37 is the vacuum expectation value in the absence of a source. If the operator is normal-ordered then this term is zero. We assume it is normal-ordered so we get

$$\langle \mathcal{O}(x) \rangle_\varphi = \int d^d y \varphi(y) G_R(x-y) \tag{5.39}$$

Or, in momentum space, we have

$$\langle \mathcal{O}(k) \rangle_\varphi = \varphi(k) G_R(k) \tag{5.40}$$

So, we arrive at:

$$G_R(k) = \frac{\langle \mathcal{O}(k) \rangle_\varphi}{\varphi(k)} \tag{5.41}$$

We can use this formula in the framework of AdS/CFT to get the two point function. In AdS/CFT, we have $\varphi = \lim_{z \to 0} z^{\Delta - d} \phi$, so the above formula in AdS/CFT becomes:

$$G_R(k) = \lim_{z \to 0} z^{2(d-\Delta)} \frac{\Pi^{ren}(z, k)}{\phi(z, k)} \tag{5.42}$$

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