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Odd-frequency Pairing in Floquet Topological Superconductors

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Abstract:

Time-periodic (Floquet) Hamiltonians offer a unique and tunable way to engineer topological systems with intriguing edge modes. In particular, Floquet superconductors can possess multi-Majorana edge modes at energies $E = 0$ and $E = \pi$.

It is well-established that there is a direct relationship between odd-frequency Cooper pairing amplitudes and the topological invariants in the static superconductors. In our study, we discuss this relationship in the time-periodic regime.

We consider a Kitaev chain alternating in time between two different values for chemical potential. By tuning the time-periodicity of the alternating chemical potential, we show that the chain admits multiple zero and π energy Majorana modes at the edge of the chain. Furthermore, We show that odd-frequency Cooper pairing amplitude at the surface of the chain is correlated to the presence of Zero and π Majorana modes.

Contents

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1 Introduction

Fermi statistics implies that Cooper pairs have to be antisymmetric under the exchange of all quantum numbers of the two bounded electrons. Conventional wisdom dictates that Cooper pair symmetry can be classified into two groups: spin-singlet, even-parity Cooper pairs (found in s and d wave superconductors), and spin-triplet odd-parity Cooper pairs (p-wave). However, if we allow Cooper pairs to form between electrons at different times, the symmetry classification of superconductors broadens to include odd and even time (or frequency) pairs[16, 25]. The existence of odd-frequency pairing is exotic and is often associated with the existence of topologically non-trivial order[20–22].

So far, odd-frequency pairing has never been observed in a homogeneous uniform system. However, various models were proposed for odd-frequency pairing in heterostructures[7, 24, 30] and multi-band systems with non-trivial inter-band interactions[29]. A recent paper by Cayao, et. al. offers a new approach to this problem via floquet engineering[4].

On the other hand, time-periodic (Floquet) systems offer a platform to realize novel hamiltonians that are difficult -and sometimes impossible- to realize under static conditions[3, 10]. Furthermore, Floquet systems host unique topological phenomena that have no static counterparts such as the emergence of Majorana fermions at energy $E = \pi$ in periodically driven topological superconductors[12].

Motivated by the new phenomena allowed by Floquet engineering, we ask whether the intricate topologies in Floquet systems can have an impact on the generation of odd-frequency Cooper pairing. Previous studies have already shown that odd-frequency pairing at the edge of a static semi-infinite superconductor is correlated with the winding number defined in the bulk[20–22]. To our knowledge, no one has considered the relationship between odd-frequency pairing and topology in periodically-driven systems. In this thesis, we investigate this relationship by studying odd-frequency pairing at the edge of a periodically driven Kitaev chain.

Organization of the thesis

Section 2 highlights the main concepts of odd-frequency pairing and establishes the relationship between odd-frequency pairing in topological superconductors and their non-trivial topology. We also introduce the Kitaev chain, a p-wave topological superconductor, and illustrates how topology and odd-frequency pairing correlate to each other in the Kitaev chain.

Section 3 explains the basic tools to deal with time-periodic systems. We start this section by reviewing Floquet theorem for linear systems of differential equations. We follow it by explaining the theorem in the context of quantum mechanics. We, then, introduce two equivalent frameworks to deal with time periodic systems, mainly, the rotating frame formalism and the the extended Hilbert space formalism.

Before delving into our work, we give a quick overview on the previous work on time periodic superconductors in section 4. We start by introducing a recent results on odd-frequency pairing in time-periodic systems. We follow it by introducing time-periodic topological superconductivity and illustrates the presence of the exotic π Majorana fermions.

Section 5 is dedicated entirely to our work. We introduce our model which is a time-periodic variant of Kitaev chain that we introduced previously in section 2.3. We derive the effective hamiltonian in the rotating frame formalism and discuss some analytical properties of our model. Then, we obtain the quasienergy spectrum numerically and show the existence of both zero and π Majorana states under various conditions. We also introduce the Green's function using different formalisms and discuss their analytical properties. We end this section by numerically obtaining the odd-frequency pairing amplitudes and discuss its relationship to the topology of our model.

Throughout this thesis, we adopt the unit system in which $\hbar = c = 1$.

2 Odd-Frequency Superconductivity and topology

2.1 odd-frequency pairing

In superconductors, Cooper pairs form as a bound state between two electrons. The symmetry classification of Cooper pairing is of vital interest to the research on superconductivity. The symmetry of Cooper pairing imposes non-trivial relationship between the internal degrees of freedom of the two electrons within a Cooper pair. For example, in conventional superconductors, a spin singlet Cooper pairing of electrons necessarily have even-parity wavefunction. On the other hand, a spin-triplet pairing leads to an odd-parity wavefunction. However, this behaviour assumes that the pairing forms between electrons located at the same time. If we allow Cooper pairing between electrons at different time coordinates to form, an enlarged symmetry classification emerges. [16, 25]

For the bulk majority of the research on superconductivity, authors assumed that Cooper pairing form between electrons with the same time coordinates, thus, the symmetry under the exchange of the two time coordinates is necessarily even. However, in 1974, Berezinskii realized that if we allow Cooper pairs to form between electrons with non-zero relative time coordinate t, an odd-time Cooper pair amplitude can form. [1]

In this section, we provide a full characterization of the possible symmetry classes for Cooper pairing. We start by considering the anomalous component of the Nanbu Green's function:

$$F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) = \langle \mathcal{T} c_{\sigma_1,\eta_1}(t_1,x_1) c_{\sigma_2,\eta_2}(t_2,x_2)\rangle \tag{2.1}$$

The anomalous Green's function is given by the correlation function between two electrons. Here, $c_{\sigma,\eta}(t,x)$ is the annihilation operator of an electron at position x, time t, spin σ, and has other internal degrees of freedom summarized in η. The anomalous Green's function provides us with information about the strength of Cooper pairing between two electrons. Thus, we can study the symmetry classification of Cooper pairing by studying the anomalous Green's function.

For any fermionic system, Fermi statistics implies that the correlation function 2.1 must satisfy the total anti-symmetry criterion given below.

$$F_{\sigma_2,\sigma_1;\eta_2,\eta_1}(t_2,x_2;t_1,x_1) = -F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) \tag{2.2}$$

We can break down the anti-symmetry criterion into the combined action of four discrete transformation. spin permutation $\mathcal{S}$, position permutation $\mathcal{P}$, time permutation $\mathcal{T}$, and orbital index permutation $\mathcal{O}$. The action of each transformation is summarized below:

$$F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) \xrightarrow[\mathcal{P}]{\text{position permutation}} F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_2;t_2,x_1) \tag{2.3a}$$

$$F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) \xrightarrow[\mathcal{T}]{\text{time permutation}} F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_2,x_1;t_1,x_2) \tag{2.3b}$$

$$F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) \xrightarrow[\mathcal{S}]{\text{spin permutation}} F_{\sigma_2,\sigma_1;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) \tag{2.3c}$$

$$F_{\sigma_1,\sigma_2;\eta_1,\eta_2}(t_1,x_1;t_2,x_2) \xrightarrow[\mathcal{O}]{\text{orbital index permutation}} F_{\sigma_1,\sigma_2;\eta_2,\eta_1}(t_1,x_1;t_2,x_2) \tag{2.3d}$$

Equation 2.2 implies that the combined action of these four transformations should satisfy

$$\mathcal{SPOT} = -1 \tag{2.4}$$

We note that the Cooper pair correlation can be even or odd under each of the four transformations separately as long as the combined action of the four transformations satisfies equation 2.4. Thus, we can construct a symmetry classification of Cooper pairing based on its behavior under these transformations separately. The symmetry classes are summarized in the table below

class	$\mathcal{P}$ $(x_1, x_2) \to (x_2, x_1)$	$\mathcal{T}$ $(t_1, t_2) \to (t_2, t_1)$	$\mathcal{S}$ $(\sigma_1, \sigma_2) \to (\sigma_2, \sigma_1)$	$\mathcal{O}$ $(\eta_1, \eta_2) \to (\eta_2, \eta_1)$
1	Even	Even	Singlet	Even
2	Even	Even	Triplet	Odd
3	Odd	Even	Singlet	Odd
4	Odd	Even	Triplet	Even
5	Even	Odd	Triplet	Even
6	Even	Odd	Singlet	Odd
7	Odd	Odd	Triplet	Odd
8	Odd	Odd	Singlet	Even

Table 1. All possible pairing symmetries in multi-orbital superconductors allowed by Fermi statistics

So far, odd-frequency pairing has never been confirmed in bulk. In fact, it was shown that odd-frequency pairing is energetically unstable in the bulk. However, it is possible to generate odd-frequency pairing if one or more of the bulk symmetries are broken. [16, 25]

Examples:

1. space translation symmetry: In superconducting junctions, parity is not a good quantum number since space translation symmetry is broken. This implies that the anomalous Green's function has even-parity and odd-parity components. For a long chain, space translation is almost a symmetry in the bulk; however, as we get close to the boundary, the effect of spatial translation symmetry breaking is strongest. Thus, we expect that the parity near the boundary/surface of the material is different from that in the bulk. For example, if we start with an even-frequency spin-triplet odd-parity pair potential in the bulk, an odd-frequency spin-triplet even-parity pair amplitude is induced on the interface of superconducting junctions. [23, 24]

2. spin rotation symmetry : In superconductor/ferromagnet junctions, spin rotation symmetry is broken and the anomalous Green's function has spin-triplet and spin-singlet components mixing together, thus, an odd-frequency pairing can be generated. [7, 30]

2.2 Topological Superconductivity and Odd-Frequency Pairing

We, now, discuss topological superconductivity and its relationship to odd-frequency pairing. Topological condensed matter physics is a relatively new and active field in physics. It started with the discovery of the integer quantum hall effect[9]. In 1980, Klitzing discovered that the conductance of 2d electron systems at low temperature and strong magnetic field is quantized and undergoes phase transitions to take these quantized values[14]. This was a shocking discovery since phases of matter were characterized by broken symmetries according to the Landau paradigm, however, the quantum hall phase transition was different. It was later discovered that each phase is characterized by different values of a topological invariant known as Chern number. Phase transition happens as the Chern number of the system changes by varying the magnetic field.

Topological materials are characterized by the presence of protected boundary states. These boundary states are a direct result of the non-trivial topology in the system's wavefunction which can be measured in terms of topological invariants such as the Chern number or the winding number [15, 26]. We note that these topological numbers remain invariant under adiabatic change of the system's hamiltonian as long as the spectrum of the hamiltonian remains gaped.

In this section, we are interested in topological superconductors and their boundary states. When Cooper pairs form, electrons and holes are hybridized, opening an energy gap on the Fermi surface. This is known as the superconducting phase, and many metals are known to undergo a phase transition to a superconductor at low temperatures. In ground state of a superconductor, electrons completely fill up the states below the Fermi energy and a finite energy is required to create a fermion excitation, thus realizing a state similar to an insulator. Due to this similarity, it is possible to realize a superconducting ground state with non trivial topological invariant as is the case for topological insulators[2, 28]. Such superconductors with non trivial topological invariants are fittingly called topological superconductors. These topological superconductors are characterized by hosting Majorana fermions on the boundary[19].

Given electron creation and annihilation operators $c^\dagger$ and c, one can construct Majorana operators[17] as follows:

$$\gamma_1 = c + c^\dagger \qquad \gamma_2 = -i(c - c^\dagger) \tag{2.5}$$

We recall that the electron creation and annihilation operators satisfy the following properties

$$\left\{c, c^\dagger\right\} = 1 \qquad \{c, c\} = \left\{c^\dagger, c^\dagger\right\} = 0 \tag{2.6}$$

This implies that the Majorana operators γ_1, γ_2 satisfy

$$\{\gamma_i, \gamma_j\} = \delta_{ij} \qquad \gamma_i^\dagger = \gamma_i \tag{2.7}$$

In superconductors, the quasiparticle excitation is given by a superposition of both particle and hole states. Thus, it is theoretically possible to have a quasiparticle excitation that behaves like a Majorana fermion. We recall that a BdG hamiltonian H satisfies

$$\mathcal{C} H \mathcal{C}^{-1} = -H \tag{2.8}$$

where $\mathcal{C}$ is the particle-hole symmetry operator[5]. This implies that for a quasiparticle excitation with energy E, there exists a quasihole excitation with energy $-E$. Since Majorana fermions are their own anti-particles, they can only exist as an excitation in a superconductor if and only if they are located at zero energy. One can show that if Majorana fermions exist, they have to be located at the boundary of a superconductor[19].

It is shown that one can establish a bulk-boundary correspondence between the odd-frequency pairing at the boundary of a topological superconductor and the non-trivial topology of the bulk of the topological superconductor[16, 21, 22, 25]. We highlight the basics of this bulk-boundary correspondence below.

We start by defining chiral symmetric systems. A chiral symmetric system is system whose hamiltonian H anticommutes with a chiral operator Γ

$$\Gamma H \Gamma = -H \tag{2.9}$$

We note that Γ is both unitary and hermitian. Thus, the chiral operator satisfies

$$\Gamma^2 = 1 \tag{2.10}$$

It is easy to see that it implies that the chiral operator commutes with the square of the hamiltonian.

$$H^2\Gamma - \Gamma H^2 = 0 \tag{2.11}$$

This implies that there is a common basis that diagonalizes both Γ and H^2. However, this basis does not necessarily diagonalize H as well. This can be easily seen as follows: let $|n\rangle$ be an eigenstate of the hamiltonian with eigenvalue E_n. We know from the particle-hole symmetry that there exist a hole state with opposite energy. We denote that state with $|-n\rangle$. This implies that all eigenvalues of the operator H^2 are at least doubly degenerate. Equation 2.11 and 2.9 imply that all eigenstates of the chiral operator are necessarily a linear superposition of particle and hole states with energy E and $-E$ respectively. This means that the chiral operator Γ and the hamiltonian H can have a common eigenstate if and only if this eigenstate has energy eigenvalue equal to zero.

We can show that these chiral zero energy states are absent in the bulk. Thus, we need to work with a finite or semi-infinite system. In such a system, the chiral zero energy states will be located at the boundary. We interested in counting the number of chiral zero energy states. In order to do so, let us define the following quantity

$$\lim_{z\to 0} z\,\mathrm{Tr}\{\Gamma G(z)\} = \lim_{z\to 0} \mathrm{Tr}\left\{\Gamma \frac{z}{z-H}\right\} \tag{2.12}$$

We can show that this quantity computes the difference in number between zero energy modes with positive chirality and zero energy modes with negative chirality. Let N_+ be the number of zero modes with positive chirality and let N_- be the number of zero modes with negative chirality. We have

$$\begin{aligned}\lim_{z\to 0} z\,\mathrm{Tr}\{\Gamma G(z)\} &= \lim_{z\to 0} \sum_{E_n=0} \langle n|\Gamma \frac{z}{z-H}|n\rangle + \lim_{z\to 0} \sum_{E_n\neq 0} \langle n|\Gamma \frac{z}{z-H}|n\rangle \\ &= \sum_{E_n=0} \langle n|\Gamma|n\rangle = N_+ - N_- \end{aligned} \tag{2.13}$$

Now, let us consider the Bulk hamiltonian $h(k)$. Assume that we have periodic boundary condition so that the momentum space hamiltonian satisfies

$$h(k+2\pi) = h(k) \tag{2.14}$$

As we adiabatically change the momentum, the eigenstates of hamiltonian wind in the Hilbert space giving rise to a winding number[20]. The winding number is given by

$$W = \frac{i}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}k\, \mathrm{Tr}\big\{\Gamma h^{-1}(k)\partial_k h(k)\big\} \in \mathbb{Z} \tag{2.15}$$

The Bulk-boundary correspondence states that we can relate the number of states at the boundary of a system with open-boundary condition to the winding number defining in the bulk of the system with periodic boundary condition[20]. Thus, we have

$$W = \lim_{z\to 0} z\,\mathrm{Tr}\{\Gamma G(z)\} \tag{2.16}$$

We note that an extension of the bulk-boundary correspondence for nonzero frequencies z was conjectured by Tamura et. al[21]. It was dubbed "spectral bulk boundary correspondence(SBBC)".

It was used to show that the odd-frequency pairing amplitude at the boundary is related to the topology in the bulk. SBBC was later proven by Daido and Yanase[6]. The basic idea of SBBC is to replace the bulk hamiltonian $h(k)$ in equation 2.15 by the bulk Green's function $g(z,k)$. Thus, the winding number can be extended to a frequency dependent quantity $W(z)$ defined as

$$W(z) = \frac{i}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}k \,\mathrm{Tr}\{\Gamma g(z,k)\partial_k g(z,k)\} \tag{2.17}$$

For the semi-infinite system, we define the odd-frequency pairing amplitude as

$$F(z) = \mathrm{Tr}\{\Gamma G(z)\} \tag{2.18}$$

The SBBC is given by

$$W(z) = zF(z) \tag{2.19}$$

We notice that in the limit $z \to 0$, the spectral bulk boundary correspondence reduces to the bulk boundary correspondence.

2.3 Kitaev Chain

The archetypal example of a topological superconductor is a p-wave superconductor. The minimal model possessing p-wave pairing covering both the topological and trivial regime is the Kitaev chain[13]. It is a one-dimensional chain of spinless electrons. Electrons are allowed to hop to the first neighboring sites and electrons are paired by a spin-triplet p-wave superconducting pairing. The hamiltonian is given by

$$H = \sum_{i=1}^{N-1}\Big(-wc_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + h.c.\Big) - \sum_{i=1}^{N}\mu c_i^\dagger c_i \tag{2.20}$$

where $c_i, c_i^\dagger$ are the annihilation and creation operators of electron at position i, μ is the chemical potential, w is the nearest neighbor hopping, and Δ is the superconducting gap.

In order to see Majorana states, it is more convenient to work in the Majorana representation. We define Majorana operators as follows

$$\gamma_{i1} = c_i + c_i^\dagger \qquad\qquad \gamma_{i2} = -i(c_i - c_i^\dagger) \tag{2.21}$$

In terms of the Majorana representation, the Kitaev hamiltonian becomes

$$H = \frac{i}{2}\sum_{i=1}^{N-1}\Big((w+\Delta)\gamma_{i2}\gamma_{(i+1)1-(w-\Delta)\gamma_{i1}\gamma_{(i+1)2}}\Big) - \frac{\mu}{2}\sum_{i=1}^{N}(i\gamma_{i1}\gamma_{i2}+1) \tag{2.22}$$

Now, let us focus on two extreme limits that illustrates the topological and trivial regime. First, let us focus on the trivial regime. Let $\Delta = w = 0, \mu < 0$. In this limit, the hamiltonian is given by

$$H = -\frac{\mu}{2}\sum_{i=1}^{N}(i\gamma_{i1}\gamma_{i2}+1) \tag{2.23}$$

In this limit, Kitaev chain consists of uncoupled sites. The Majorana operators at each site couple to each other and there is no coupling between different sites. Adding a fermion will cost a nonzero energy μ. Thus, the system is fully gaped and no zero energy states are present. This is the trivial case.

Let us consider the other extreme. We take the following limit: $w = \Delta, \mu = 0$. In this limit, the hamiltonian becomes

$$H = \sum_{i=1}^{N-1} iw\gamma_{i2}\gamma_{(i+1)1} \tag{2.24}$$

In this limit, Majorana operators at different sites are coupled to each other. We notice that two Majorana operators at the edge γ_{11} and γ_{N2} are decoupled from the chain. This implies that these two Majorana states can be occupied with zero energy cost. Thus, we have shown that Kitaev chain hosts two Majorana zero energy modes at the edge.

Now, we discuss the relationship between the topology of Kitaev chain and odd frequency pairing. It is convenient to work with BdG hamiltonians thus we rewrite the Kitaev hamiltonians in the BdG form as follows:

$$H = \sum_i \psi_i^\dagger(-\mu\tau_z)\psi_i + \psi_i^\dagger(-w\tau_z + i\Delta\tau_y)\psi_{i+1} \tag{2.25}$$

where $\psi_i = (c_i, c_i^\dagger)^T$ is the Nambu spinor,and, $\tau_i(i = x, y, z)$ are Pauli matrices in the particle-hole space.

We study the odd-frequency pairing at the edge of the Kitaev chain by means of the bulk-boundary correspondence. First, we investigate the bulk properties. The bulk hamiltonian in momentum space is given by

$$H = \int \mathrm{d}k\psi^\dagger(k)h(k)\psi(k) = \int \mathrm{d}k\psi^\dagger(k)\Big(-(\mu + 2w\cos k)\tau_z + 2\Delta\sin k\tau_y\Big)\psi(k) \tag{2.26}$$

We can easily verify that $\Gamma = \tau_x$ anticommutes with $h(k)$. Thus, Kitaev chain is a chiral symmetric system. It is convenient to consider the limit $w = \Delta, \mu = 0$. In this limit, we find that the winding number is given by

$$\begin{aligned} W(z) &= \frac{i}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}k\, \mathrm{Tr}\{\Gamma g(z,k)\partial_k g(z,k)\} \in \mathbb{Z} \\ &= \frac{-w^2}{z^2 - w^2} \end{aligned} \tag{2.27}$$

Now, we focus on the semi-infinite Kitaev chain. Obtaining the right hand side of the SBBC 2.19 is quite tricky. However, using the recursive Green's function method[27], one can construct the Green's function for semi-infinite chain. After applying the recursive method, we find that the anomalous Green's function is given by

$$\begin{aligned} F(z) &= \mathrm{Tr}\{\Gamma G(z)\} \\ &= \frac{1}{z}\frac{-w^2}{z^2 - w^2} \end{aligned} \tag{2.28}$$

We conclude that the Kitaev chain obeys the spectral bulk boundary correspondence

$$W(z) = zF(z) \tag{2.29}$$

3 Floquet Theory

In the theory of linear differential equations, Floquet theorem is a theorem that describes the solutions of a time periodic linear differential equation[8].

Let us consider a very general linear system of differential equations. Assume that we have a linear system of n differntial equations of the following form:

$$\partial_t \vec{X}(t) = A(t)\vec{X}(t) \qquad X(0) = \vec{X}_0 \tag{3.1}$$

where $\vec{X}(t)$ is a vector field and $A(t)$ is a linear $n \times n$ coefficient matrix that depends on time.

The general solution is given in terms of the following time-ordered product

$$\vec{X}(t) = \mathcal{T}\exp\left(\int_0^t A(s)\mathrm{d}s\right)\vec{X}_0 \tag{3.2}$$

Generally speaking, there is no closed-form solution to the differential equation; however, Floquet theorem indicates that a closed-form solution can be found provided that the coefficient matrix $A(t)$ is time-periodic.

Theorem
Consider a linear differential equation of the form

$$\partial_t \vec{X}(t) = A(t)\vec{X}(t) \tag{3.3a}$$

and let $A(t+T) = A(t)$ for some non-zero positive T, then, there exist a solution to the differential equation of the following form

$$\vec{X}_\alpha(t) = e^{\epsilon_\alpha t}\vec{Y}_\alpha(t) \tag{3.3b}$$

such that ϵ_α is defined modulo $\frac{2\pi}{T}$ and $\vec{Y}(t)$ satisfies $\vec{Y}(t+T) = \vec{Y}(t)$

3.1 Floquet theory in quantum mechanics

Central equation of quantum mechanics is the Schrodinger equation. Since Schrodinger equation is a linear equation in time, we can apply Floquet theorem to quantum systems with time-periodic Hamiltonians. For a reference on Floquet theorem in the context of quantum mechanics, we recommend the following article [11].

Let $H(t)$ be a given hamiltonian satisfying the following condition.

$$H(t+T) = H(t) \tag{3.4}$$

Consider the associated time-dependent Schrodinger equation

$$i\partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle \qquad \psi(t_0) = |\psi_0\rangle \tag{3.5}$$

We can integrate the Schrodinger equation to obtain the general solution

$$|\psi(t)\rangle = \mathcal{T}\exp\left(-i\int_{t_0}^t H(s)\mathrm{d}s\right)|\psi(t_0)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{3.6}$$

However, we need to deal with time-ordering in the above equation. The conventional method to deal with the above expression is by writing the propagator $U(t, t_0)$ in terms of Dyson series and applying Wick theorems to express the time ordered quantities in terms of normal-ordered quantities which are easier to evaluate(cf. chapter 4 in [18]).

The problem with the conventional approach is that it only works for small perturbations and cannot capture topological effects. Floquet theorem provide an alternative formulation for time-periodic hamiltonians that captures topological properties.

Floquet theorem states that the general solutions to Schrodinger's equation for the time-periodic hamiltonian are given by

$$\begin{aligned} |\psi_j(t)\rangle &= e^{-i\epsilon_j(t-t_0)}\,|u_j(t)\rangle \\ \text{with}\quad |u_j(t+T)\rangle &= |u_j(t)\rangle\,, \qquad \epsilon_j \in \Big\{-\frac{\Omega}{2}, \frac{\Omega}{2}\Big\} \end{aligned} \tag{3.7}$$

where $\Omega = \frac{2\pi}{T}$, the states $|u_j(t)\rangle$ are called Floquet modes, and ϵ_j is called quasienergy. The quasienergy plays the role of crystal momentum in Bloch theorem and it is conserved modulo Ω

Given any initial state $|\psi_0\rangle$, one can expand it in terms of the states $\{|u_j(t_0)\rangle$ since they form a complete orthogonal basis. Let $|\psi_0\rangle$ be given by

$$|\psi_0\rangle = \sum_j C_j\,|u_j(t_0)\rangle \tag{3.8}$$

It follows that the solution to Schrodinger equation with the boundary condition $|\psi(t)\rangle = |\psi_0\rangle$ is

$$|\psi(t)\rangle = \sum_j C_j e^{-i\epsilon_j(t-t_0)}\,|u_j(t)\rangle \tag{3.9}$$

Proof of Floquet Theorem:
We now show the proof of Floquet theorem.

Let $H(t+T) = H(t)$. The propagator is given by

$$U(t, t_0) = \mathcal{T}\exp\Big\{-i\int_{t_0}^{t} H(t')\mathrm{d}t'\Big\} \tag{3.10}$$

In the language of group theory, the propagator is the time-translation operator and it is an element of $U(1)$ Lie group. Thus, it satisfies the following two properties:

$$U(t_0, t_0) = \mathbb{I} \tag{3.11a}$$

$$U(t', t)U(t, t_0) = U(t', t_0) \tag{3.11b}$$

It also obeys Schrodinger equation

$$i\partial_t U(t, t_0) = H(t)U(t, t_0) \tag{3.11c}$$

From Schrodinger equation,we can show that the following.

$$i\partial_t U(t+nT, t_0+nT) = H(t+nT)U(t+nT, t_0+nT) = H(t)U(t+nT, t_0+nT) \tag{3.12}$$

Where $n \in \mathbb{Z}$. Since the operators $U(t+nT, t_0+nT)$ and $U(t, t_0)$ satisfy the same equation with the same initial condition, it follows that they are the same operator

$$U(t+nT, t_0+nT) = U(t, t_0) \tag{3.13}$$

Next, we use the $U(1)$ group properties. We see that for $t \in [t_0, t_0+T]$, the following property is satisfied

$$U(t+nT, t_0) = U(t+nT, t_0+nT)U(t_0+nT, t_0) \tag{3.14}$$

It is easy to show that $U(t_0+nT, t_0) = [U(t_0, t_0)]^n$. We define the one-period propagator $U_T[t_0]$ as follows that

$$U_T[t_0] = U(t_0+T, t_0) \tag{3.15}$$

Since $U_T[t_0]$ is unitary, we can diagonalize it. Let $|u_j(t_0)\rangle$ be the eigenvector associated with the eigenvalue $\exp(-iT\epsilon_j)$

$$U_T[t_0]\,|u_j(t_0)\rangle = \exp(-iT\epsilon_j)\,|u_j(t_0)\rangle \tag{3.16}$$

Since all eigenvalues of a unitary operators have modulous $= 1$, it follows that ϵ_j is only defined modulo $\frac{2\pi}{T}$

From the one-period propagator $U_T[t_0]$, it is possible to define a hermitian operator $H_f[t_0]$ by taking the logarithm of $U_T[t_0]$. we define

$$H_f[t_0] = \frac{i}{T}\log(U_T[t_0]) \tag{3.17}$$

$H_f[t_0]$ is known as the Floquet hamiltonian

Now, let us restrict t to the region $[t_0, t_0+T]$. In this region, we define the following unitary operator

$$P(t, t_0) = U(t, t_0)\exp(iH_f[t_0](t-t_0)) \tag{3.18}$$

It is easy to show that the above operator satisfies the following properties

$$P(t_0, t_0) = \mathbb{I} \tag{3.19a}$$

$$P(t+T, t_0) = P(t, t_0) \tag{3.19b}$$

This implies that $P(t, t_0)$ is periodic in the first argument.

The propagator can be rewritten in terms of $H_f[t_0]$ and $P(t, t_0)$ as

$$U(t, t_0) = U(t, t_0)\exp(iH_f[t_0](t-t_0))\exp(-iH_f[t_0](t-t_0)) = P(t, t_0)\exp(-iH_f[t_0](t-t_0)) \tag{3.20}$$

It follows that the solution to Schrodinger equation is

$$|\psi_j(t)\rangle = e^{-i\epsilon_j(t-t_0)}\,|u_j(t)\rangle$$
$$\text{with} \quad |u_j(t)\rangle = P(t, t_0)\,|u_j(t_0)\rangle\,, \qquad \epsilon_j \in \Big\{-\frac{\Omega}{2}, \frac{\Omega}{2}\Big\}$$

where $\Omega = \frac{2\pi}{T}$

3.2 Rotating Frame Method

Floquet theory provides two equivalent frameworks to deal with time-periodic quantum systems. These two systems are known as: (1)Rotating Frame Formalism, and (2)Extended Hilbert space formalism. In the next two sections, we will briefly introduce the two formalisms.

In quantum mechanics, it is often convenient to change the frame of reference in order to simplify the calculations. For example, the interaction picture is often employed when dealing with perturbation theory. Now, one can ask the following question: can we find a rotating frame of reference where the time-periodic hamiltonian is static? The answer is yes.

Assume that we know the unitary operator $P(t, t_0)$. It is possible to define a rotating frame using the unitary operator $P(t, t_0)$. First, let us define the solution to Schrodinger equation as $|\psi(t)\rangle = P(t, t_0)\left|\tilde{\psi}(t)\right\rangle$. We substitute into Schrodinger equation:

$$i\partial_t\Big(P(t,t_0)\left|\tilde{\psi}\right\rangle\Big) = iP(t,t_0)\left|\partial_t\tilde{\psi}(t)\right\rangle + \Big(i\partial_t P(t,t_0)\Big)\left|\tilde{\psi}(t)\right\rangle = H(t)P(t,t_0)\left|\tilde{\psi}(t)\right\rangle \tag{3.21}$$

The Schrodinger equation implies that:

$$i\partial_t\left|\tilde{\psi}(t)\right\rangle = \Big[P^\dagger(t,t_0)H(t)P(t,t_0) - iP^\dagger(t,t_0)\partial_t P(t,t_0)\Big]\left|\tilde{\psi}(t)\right\rangle \tag{3.22}$$

In other words, the state $\left|\tilde{\psi}(t)\right\rangle$ satisfies Schrodinger equation For the following hamiltonian:

$$\tilde{H}(t) = P^\dagger(t,t_0)H(t)P(t,t_0) - iP^\dagger(t,t_0)\partial_t P(t,t_0) \tag{3.23}$$

We notice that the hamiltonian $\tilde{H}(t)$ is not related to the original hamiltonian $H(t)$ by a simple unitary transformation. Rather, it contains an additional term. We note that the additional term can be thought of as a gauge transformation. This gauge term is necessary since the unitary transformation $P(t, t_0)$ is local in time.

Now, recall the definition of $P(t, t_0)$

$$P(t,t_0) = U(t,t_0)\exp(iH_f[t_0](t-t_0))$$

By differentiating the above equation, we find that:

$$\begin{aligned}\partial_t P(t,t_0) =& \Big(\partial_t U(t,t_0)\Big)e^{iH_f[t_0](t-t_0)} + iU(t,t_0)e^{iH_f[t_0](t-t_0)}H_f[t_0] \\ =& -iH(t)P(t,t_0) + iP(t,t_0)H_f[t_0]\end{aligned}$$

Substituting into equation (3.23), we find that

$$\tilde{H}(t) = H_f[t_0] \tag{3.24}$$

This implies that the hamiltonian in the rotating frame is nothing but the Floquet hamiltonian.

3.3 Extended Hilbert Space Formalism

The extended Hilbert space formalism is a different approach to deal with time-periodic systems. The basic idea of the extended Hilbert space formalism stems from a simple observation.
Consider the following operator

$$K(t') = H(t') - i\partial_{t'} \tag{3.25}$$

Let us apply the operator $K(t')$ to the Floquet mode $|u_j(t')\rangle$

$$\begin{aligned} K(t')\,|u_j(t')\rangle &= \Big(H(t') - i\partial_{t'}\Big)e^{+i\epsilon_j(t'-t_0)}e^{-i\epsilon_j(t'-t_0)}\,|u_j(t')\rangle \\ &= e^{+i\epsilon_j(t'-t_0)}\Big(H(t') - i\partial_{t'} + \epsilon_j\Big)e^{-i\epsilon_j(t'-t_0)}\,|u_j(t')\rangle \\ &= \epsilon_j\,|u_j(t')\rangle \end{aligned} \tag{3.26}$$

In the last step, we used the fact that $e^{-i\epsilon_j(t'-t_0)}\,|u_j(t')\rangle$ satisfy the time-dependent Schrodinger equation for the hamiltonian $H(t')$

From the above equation, we see that the Floquet modes $|u_j(t')\rangle$ are eigenstates of the operator $K(t')$ with eigenvalues ϵ_j

Furthermore, we notice that multiplying by $\varphi_n(t') = e^{-i\Omega n t'}$ gives us the following result:

$$K(t')\,|u_{jn}(t')\rangle = K(t')\varphi_n(t')\,|u_j(t')\rangle = (\epsilon_j - \Omega n)\varphi_n(t')\,|u_j(t')\rangle = \epsilon_{jn}\,|u_{jn}(t')\rangle \tag{3.27}$$

where we defined $\epsilon_{jn} = \epsilon_j - \Omega n$

Since we can define the eigenvalue equation (3.26) at any $t' \in [t_0, T + t_0]$, it is possible to construct an extended Hilbert space where the operator $K(t')$ is treated as the Hamiltonian. This construction was developed by J. Howland in 1974. We give a summary below.

Let $\mathcal{H}$ be the Hilbert space for the quantum system of interest. We extend the Hilbert space $\mathcal{H}$ to the extended Hilbert space $\mathcal{H}_{ext} = L^2[0,T] \otimes \mathcal{H}$ where $L^2[0,T]$ is the set of all T-periodic normalizable functions. In this extended Hilbert space, we treat the operator $K(t')$ as the hamiltonian operator.

In the extended Hilbert space, the temporal part is spanned by Fourier vectors $\Big\{\varphi_m(t') = e^{-im\Omega t'}\Big\}_{m\in\mathbb{Z}}$

The inner product is defined as:

$$\langle\langle u(t')|v(t')\rangle\rangle_T = \frac{1}{T}\int_0^T \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x u^*(x,t)v(x,t) \tag{3.28}$$

Let $\Big\{\phi_\alpha\Big\}_{\alpha\in\mathbb{Z}}$ be a complete orthonormal basis of the original Hilbert space $\mathcal{H}$. Then, any state in the external Hilbert space $\mathcal{H}_{ext}$ is given by

$$\begin{aligned} |u(t')\rangle &= \sum_\alpha \sum_{m=-\infty}^{\infty} u_{m,\alpha}\varphi_m(t')\,|\phi_\alpha\rangle \\ &= \sum_\alpha C_\alpha(t')\,|\phi_\alpha\rangle \qquad \text{with} \quad C_\alpha = \sum_{m=-\infty}^{\infty} u_{m,\alpha}\varphi_m(t') \\ &= \sum_m \varphi_m(t')\,|\Phi_m\rangle \qquad \text{with} \quad |\Phi_m\rangle = \sum_\alpha u_{m,\alpha}\,|\phi_\alpha\rangle \end{aligned}$$

We see that the operator $K(t')$ is naturally defined as a linear operator acting on the extended Hilbert space $\mathcal{H}_{ext}$.

Let the operator $K(t')$ be the hamiltonian operator in the extended Hilbert space $\mathcal{H}_{ext}$. The solutions to Schrodinger equation in the extended space $\mathcal{H}_{ext}$ can be easily obtained. They are given by:

$$|\psi_{jn}(t,t')\rangle = e^{-i\epsilon_{jn}t}\,|u_{jn}(t')\rangle \tag{3.29}$$

We notice that the solutions to the Schrodinger equation in the extended Hilbert space reduce to the solutions for Schrodinger equation in the original Hilbert space $\mathcal{H}$ when $t = t'$

$$|\psi_{jn}(t,t'=t)\rangle = |\psi_j(t)\rangle \tag{3.30}$$

It is often convenient to express the extended Hilbert space hamiltonian $K(t')$ in terms of its Fourier modes. Consider the eigenvalue problem in the extended Hilbert space:

$$K(t')\,|u_j(t')\rangle = \Big[H(t') - i\partial_{t'}\Big]\,|u_j(t')\rangle = \epsilon_j\,|u_j(t')\rangle \tag{3.31}$$

We can expand the above equation in the Fourier basis $\left\{\varphi_m(t') = e^{-im\Omega t'}\right\}_{m\in\mathbb{Z}}$

The eignestate is given by

$$|u_j(t')\rangle = \sum_m \varphi_m(t')\,\left|u_j^m\right\rangle \tag{3.32}$$

Meanwhile, the extended Hilbert space hamiltonian becomes

$$K_{mn} = \frac{1}{T}\int_0^T \varphi_m^*(s)K(s)\varphi_n(s)\mathrm{d}s = H_{mn} - n\Omega\delta_{mn} \tag{3.33}$$

The eigenvalue equation becomes

$$\sum_n K_{mn}\left|u_j^n\right\rangle = \sum_n H_{mn}\left|u_j^n\right\rangle - m\Omega\left|u_j^m\right\rangle = \epsilon_j\left|u_j^m\right\rangle \tag{3.34}$$

To end this chapter, we note that the hamiltonians $H(t), H_f[t_0], K(t')$ are related to each other non-trivially. We emphasis that $H(t)$ is NOT unitarily equivalent to the effective hamiltonians $H_f[t_0]$ and $K(t')$. Meanwhile, $H_f[t_0]$ and $K(t')$ are unitarily equivalent.
Proof:

$$\begin{aligned}
K(t') &= H(t)\Big|_{t=t'} - i\partial_{t'}\\
&= \sum_j \epsilon_j\,|u_j(t')\rangle\langle u_j(t')|\\
&= \sum_j \epsilon_j P(t',t_0)\,|u_j\rangle\langle u_j|\,P^\dagger(t',t_0)\\
&= P(t',t_0)H_f[t_0]P^\dagger(t',t_0)
\end{aligned}$$

4 Floquet Superconductivity

In this chapter, we highlight some of the previous results in the area of Floquet engineering and superconductivity that motivated our research.

4.1 odd-frequency pairing in Floquet superconductors

While floquet engineering of superconductors is not a new concept, odd frequency pairing induced by time-periodic drives is a very novel idea. In this section, we highlight the main points in reference[4] by Cayao et. al.. The interesting find of Cayao's paper is that the addition of the Floquet index can enlarge the symmetry classification of superconductors. Particularly, in the case of a floquet s-wave superconductor, the paper show that an emergent odd-frequency p-wave order can arise even though the initial symmetry is an s-wave.

Let us start our discussion by summarizing the main concept of odd-frequency pairing in floquet superconductors. Consider a superconducting system described by the hamiltonian H_{sc}. Now, we assume that a time-periodic perturbation is applied to the system so that the overall hamiltonian is given by

$$H(t) = H_{sc} + V(t) \tag{4.1}$$

We assume that the perturbation $V(t)$ is periodic in time with a period T. This perturbation can take any form and we will not assume any particular form at the moment.

The periodicity of the perturbation will introduce Floquet bands degrees of freedom to the superconducting system. These Floquet bands are nothing but the Fourier bands in the extended Hilbert space hamiltonian described in the previous chapter.

Generally speaking, for a generic superconductor (static or driven), the pair amplitude is given by the off-diagonal elements of the Green's function in Nambu space. Due to the floquet bands, the anomalous Green's function can be expanded as follows:

$$F_{\sigma_1,\sigma_2}(k_1, k_2, t_1, t_2) = \sum_{m,n} \int \mathrm{d}\omega F^{n,m}_{\sigma_1,\sigma_2}(k_1, k_2, \omega) e^{-i(\omega+n\Omega)t_1 + i(\omega+m\Omega)t_2} \tag{4.2}$$

where $n, m \in \mathbf{Z}$ are the Floquet band indices. Here, $F^{n,m}_{\sigma_1,\sigma_2}(k_1, k_2, \omega)$ is called the Floquet anomalous Green's function. Applying Fermi statistics to the above quantity, it flows that

$$F^{n,m}_{\sigma_1,\sigma_2}(k_1, k_2, \omega) = -F^{-m,-n}_{\sigma_2,\sigma_1}(k_2, k_1, -\omega) \tag{4.3}$$

One may naively think that the Floquet index acts as new band index; however, we note that this is not a simple generalization of multiband superconductor with infinitely many bands. In fact, the floquet index expands the symmetry classification of Cooper pairs significantly. The classification is summarized in the table below:

Now, we apply the above discussion to a concrete example. Consider an s-wave superconductor driven by circularly polarized electric field. The electric field can be written as $E = \partial_t A(t)$ with $A(t) = A_0(\sin \Omega t, \cos \Omega t, 0)$. The system can be modeled in Nambu space by the following Hamiltonian:

$$H(t) = H_{sc} + V(t) \tag{4.4}$$

class	**Floquet index** $(n,m) \to (-m,-n)$	**Frequency** $\omega \to -\omega$	**Spin** $s = \pm 1$	**Momentum** $\boldsymbol{k}_1 \to \boldsymbol{k}_2$
1	Even	Even	Singlet	Even
2	Even	Even	Triplet	Odd
3	Odd	Even	Singlet	Odd
4	Odd	Even	Triplet	Even
5	Even	Odd	Triplet	Even
6	Even	Odd	Singlet	Odd
7	Odd	Odd	Triplet	Odd
8	Odd	Odd	Singlet	Even

Table 2. All possible pairing symmetries in single-orbital Floquet superconductors allowed by Fermi statistics

where

$$H_{sc} = \zeta_k \tau_z + \Delta \tau_x \tag{4.5}$$

$$V(t) = \frac{e}{m} A(t) \cdot k \tau_z \tag{4.6}$$

To find Namub Green's function, we have to solve the following equation:

$$\sum_{m'} \Big[(\omega + m'\Omega - H_{sc})\delta_{n,m'} - U_k \delta_{n+1,m'} - U_k^* \delta_{n-1,m'} \Big] \mathcal{G}_{m',m}(k,\omega) = \delta_{nm} \tag{4.7}$$

where $\omega \in [-\Omega/2, \Omega/2]$ and $U_k = \frac{eA_0}{2m}(k_y - ik_x)\tau_z$. The term U_k is the Fourier transform of the time-periodic perturbation. It couples nearest-neighbors Floquet bands.

Solving the above equation is generally unfeasible so we need to use an approximation. We can simplify this equation by only considering the Floquet bands $1, 0, -1$. This truncation of the summation above approximates the exact answer very well. The off-diagonal elements of the Nambu Green's function are the Floquet pair amplitudes which we are interested in. We can obtain the solution to the above equation by employing Dyson series as follows:

$$\mathcal{G}_{nm} = \langle n|G|m\rangle \approx \langle n|g|m\rangle + \langle n|gVg|m\rangle + \langle n|gVgVg|m\rangle + \dots \tag{4.8}$$

Here, G is the dressed Green's function while g is the bare Green's function. The state $|n\rangle$ is the state of the nth Floquet band.

The bare Green's function $\langle n|g|m\rangle$ is only non-zero when $n = m$ while the potential V only couples nearest-neighbors bands. Using this information, we arrive at

$$\mathcal{G}_{00} \approx g_{00} + g_{00}V_{01}g_{11}V_{10}g_{00} + g_{00}V_{0,-1}g_{-1,-1}V_{-1,0}g_{00} \tag{4.9}$$

$$\mathcal{G}_{01} \approx g_{00}V_{01}g_{11} \tag{4.10}$$

$$\mathcal{G}_{0,-1} \approx g_{00}V_{0,-1}g_{-1,-1} \tag{4.11}$$

$$\mathcal{G}_{10} \approx g_{11}V_{10}g_{00} \tag{4.12}$$

$$\mathcal{G}_{-1,0} \approx g_{-1,-1}V_{-1,0}g_{00} \tag{4.13}$$

$$\mathcal{G}_{1,-1} \approx g_{11}V_{10}g_{00}V_{0,-1}g_{-1,-1} \tag{4.14}$$

$$\mathcal{G}_{-1,1} \approx g_{-1,-1}V_{-1,0}g_{00}V_{01}g_{11} \tag{4.15}$$

$$\mathcal{G}_{11} \approx g_{11} + g_{11}V_{10}g_{00}V_{01}g_{11} \tag{4.16}$$

$$\mathcal{G}_{-1,-1} \approx g_{-1,-1} + g_{-1,-1}V_{-1.0}g_{00}V_{0,-1}g_{-1,-1} \tag{4.17}$$

From the above equations, we can get the floquet pair amplitudes. Now, let us define $F^{\pm}_{nm} = \frac{F_{nm} \pm F_{-m,-n}}{2}$. The none-zero amplitudes are summarized below:

$$
\begin{aligned}
F^{+}_{0,0}(\boldsymbol{k},\omega) &\approx \frac{\Delta}{D} + \frac{\Delta |U_{\boldsymbol{k}}|^2 A^{\Omega}_{\omega}}{D^2 D_{-1} D_1}\,, \\
F^{+}_{1,1}(\boldsymbol{k},\omega) &\approx \frac{\Delta[D+\Omega^2]}{D_1 D_{-1}} + \frac{\Delta |U_{\boldsymbol{k}}|^2 B^{\Omega}_{\omega}}{2D(D_{-1}D_1)^2}\,, \\
F^{-}_{1,1}(\boldsymbol{k},\omega) &\approx \frac{-2\omega\Omega\Delta}{D_1 D_{-1}} + \frac{2\omega\Omega\Delta |U_{\boldsymbol{k}}|^2 C^{\Omega}_{\omega}}{D(D_{-1}D_1)^2}\,, \\
F^{+}_{1,-1}(\boldsymbol{k},\omega) &\approx -\frac{\Delta [U^{*}_{\boldsymbol{k}}]^2 E^{\Omega}_{\omega}}{D D_1 D_{-1}}\,, \\
F^{+}_{0,1}(\boldsymbol{k},\omega) &\approx -\frac{2\omega\Delta U_{\boldsymbol{k}}}{D_1 D_{-1}}\,, \\
F^{-}_{0,1}(\boldsymbol{k},\omega) &\approx -\frac{\Omega\Delta U_{\boldsymbol{k}} E^{\Omega}_{\omega}}{D D_1 D_{-1}}\,,
\end{aligned}
\tag{4.18}
$$

where $D(\omega) = \omega^2 - (\Delta^2 + \xi^2_{\boldsymbol{k}})$, and $D_n(\omega) = D(\omega + n\Omega)$. Here, A^{Ω}_{ω}, B^{Ω}_{ω}, C^{Ω}_{ω}, E^{Ω}_{ω}, $D_1 D_{-1}$ are even functions of $\boldsymbol{k}$ and ω and their analytic expression is not relevant for our discussion. The first three equations describe Cooper pair formation within each band (intraband Cooper pairs) while the last three equations describe Cooper pair formation between different bands (interband Cooper pairs). Using the above analytic expressions, one can determine the symmetry groups of each component. Surprisingly, we find that the even floquet index interband pair amplitude F^{+}_{01} is odd in frequency while being odd in momentum. Thus, this implies that it is a p-wave even though the superconducting system is an s-wave. This can be understood as a result from the linear coupling between the driven electric field and the momentum.

4.2 Topological Floquet superconductors

Since the discovery of the quantum hall effect, topology has played a central role in condensed matter physics. However, materials that realize non-trivial topological characteristics are scarce in nature. In particular, topological p-wave superconductors are yet to be confirmed in experiment.

Several proposals to engineer topological superconductors were published. Of particular interest to us is the Floquet engineering of topological phases. Kitagawa et. al. proposed that Majorana fermions can be engineered in cold atom experiments using optical traps[12]. In this section, we present a summary of this proposal, highlighting the elusive Floquet Majorana π modes.

Let us consider a one dimensional chain of fermionic atoms trapped inside a 3D Bose-Einstein condensate(BEC) of molecules. We assume that the atoms have two internal degrees of freedom that we denote by $\sigma =\uparrow$ and $\sigma =\downarrow$. The hamiltonian is given by

$$H = \sum_{p,\sigma} a^\dagger_\sigma(p)\big(\epsilon(p) + V\big)a(p) + \sum_p \big(\Delta a^\dagger_\uparrow(p)a^\dagger_\downarrow(-p) + h.c.\big) \tag{4.19}$$

Here, $a_\sigma(p)$ is the annihilation operator of an atom with internal degree of freedom σ. The kinetic energy is given by $\epsilon(p) = \frac{p^2}{2m}$. V is the optical trap potential. The atoms can move between the BEC and the optical trap. This movement creates a fictitious s-wave superconducting order characterized by Δ.

Let us shine two laser beams on the trapped atoms with recoil momentum k in the direction of the chain. This will induce a coupling between atoms. Let the coupling strength be B. The hamiltonian now becomes

$$H \to H = \sum_{p,\sigma} a^\dagger_\sigma(p)\big(\epsilon(p) + V\big)a(p) + \sum_p \big(\Delta a^\dagger_\uparrow(p)a^\dagger_\downarrow(-p) + Ba^\dagger_\uparrow(p+k)a_\downarrow(p-k) + h.c.\big) \tag{4.20}$$

After applying a unitary transformation to the above hamiltonian, we can show that this system is unitarily equivalent to the Rashba nanowire hamiltonian[**?**]. The Rashba nanowire hamiltonian is given by

$$H = \sum_p \psi^\dagger(p)\Big(\big(\epsilon(p) - \mu\big)\tau_z + \lambda p\sigma_z + B\tau_z\sigma_x + \Delta\tau_y\sigma_y\Big)\psi(p) \tag{4.21}$$

where $\psi(p) = \big(a_\uparrow(p), a_\downarrow(p), a^\dagger_\uparrow(p), a^\dagger_\downarrow(p)\big)^T$ is the Nanbu spinor. τ_i, σ_i is the i-th pauli matrix in the particle-hole and spin space respectively. The chemical potential is given by $\mu = -(V + \frac{k^2}{2m})$ and the Rashba spin-orbit coupling strength is given by $\lambda = k/m$.

By adjusting the optical trap potential or the incident laser beams frequency, it is possible to engineer a time-periodic chemical potential. For example, let us consider the Rashba nanowire hamiltonian with a time-periodic chemical potential given by

$$\mu(t) = \begin{cases} \mu_1 & nT < t < (n+\frac{1}{2})T \\ \mu_2 & (n+\frac{1}{2})T < t < (n+1)T \end{cases} \qquad n \in \mathbb{Z} \tag{4.22}$$

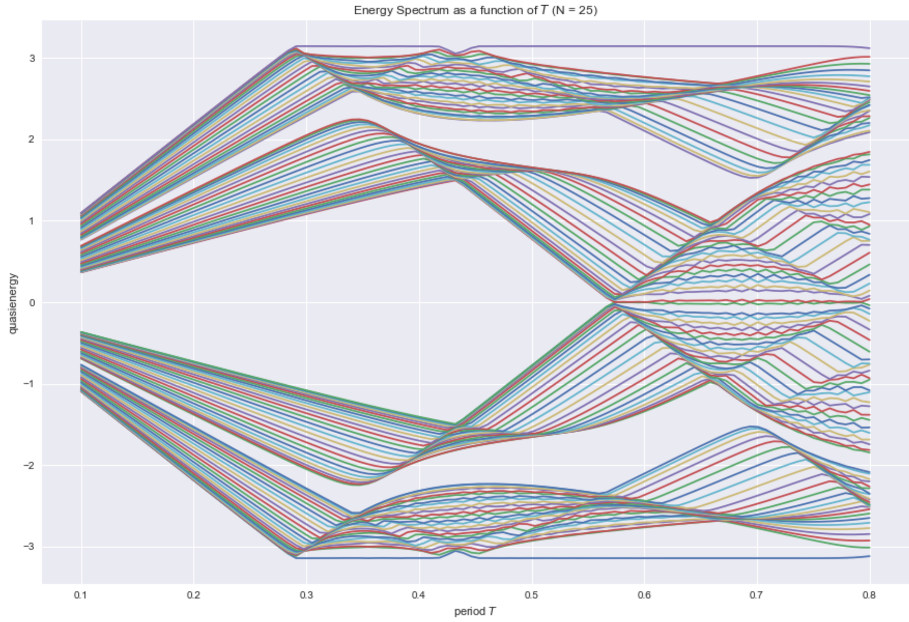

Figure 1. Quasienergy spectrum of Rashba nanowire with 25 sites. The parameters are given in terms of the hoping integral w as follows: $B = 2w, \Delta = 4w, \lambda = 2w, \mu_1 = -2w, \mu_2 = -6w$

Kitagawa et. al. studied this system within the Floquet theory. They found that in addition to Majorana zero modes, the system also possesses unique Majorana edge modes at π energy even in the static trivial regime(see figure). By altering the period, the system's topology and the number of Majorana π and zero modes can be tuned.

We emphasis that the Floquet Majorana edge modes are unique to time-periodic systems and are drastically different from the conventional Majorana edge modes in the static case.One obvious difference is that Floquet Majorana edge modes are not restricted to lie at zero energy. This is because time-periodicity of the hamiltonian implies that the quasienergy spectrum is $\frac{2\pi}{T}$-periodic. Thus, quasiparticle excitations at energy E and $E + \frac{2\pi}{T}$ correspond to the same state. Meanwhile, particle-hole symmetry implies that for a quasiparticle excitation at energy E, there exist a quasi-hole excitation at energy $-E$. The periodicity of the spectrum and particle-hole symmetry imply that excitations at energy $E = \frac{\pi}{2}$ are there own antiparticles. Thus, Majorana states can be found at energy $E = \frac{\pi}{2}$

5 Odd-frequency pairing in topological floquet superconductors

As we have seen in the previous chapters, Floquet engineering allows us to generate very interesting quantum systems. Periodically-driving a conventional s-wave superconductor can produces p-wave Cooper pairs in the bulk which is inconceivable in a static system without a more complicated setup. Moreover, we have already seen that driving a topological superconductor leads to new type of topological edge states known as Floquet Majorana fermions. We have seen that these Floquet Majorana can be present at either zero or π energies.

Motivated by the new phenomena allowed by Floquet engineering, we ask whether the intricate topologies in Floquet systems can have an impact on the generation of Cooper pairs. In this chapter, we introduce our model and analysis methods, and main results of the thesis.

5.1 Model

Our main goal in the thesis is to study the effect of Floquet Majorana bound states on the generation of odd-frequency pairing in superconductors. The simplest static superconducting model that admits Majorana bound states at the edge is the Kitaev chain. In our study, we start with a static Kitaev chain given by the following BdG Hamiltonian:

$$H_{Kitaev} = \sum_i \psi_i^\dagger(-\mu\tau_z)\psi_i + \psi_i^\dagger(-w\tau_z + i\Delta\tau_y)\psi_{i+1} \tag{5.1}$$

where $\psi_i = (c_i, c_i^\dagger)^T$ is the Nambu spinor, $\tau_i (i = x, y, z)$ are Pauli matrices in the particle-hole space, μ is the chemical potential, w is the nearest neighbor hopping, and Δ is the superconducting gap.

We drive the system by introducing a time-dependant chemical potential $\mu(t)$. Generally speaking, the choice of the function $\mu(t)$ can drastically change the dynamics of the model. Furthermore, different techniques are more suitable to treat certain forms than others. In our study, we consider a simple step-like periodic chemical potential. Thus, the model we investigate in this thesis is given below

$$H_{model}(t) = \sum_i \psi_i^\dagger(-\mu(t)\tau_z)\psi_i + \psi_i^\dagger(-w\tau_z + i\Delta\tau_y)\psi_{i+1} \tag{5.2}$$

with

$$\mu(t) = \begin{cases} \mu_1 & nT < t < (n+\frac{1}{2})T \\ \mu_2 & (n+\frac{1}{2})T < t < (n+1)T \end{cases} \qquad n \in \mathbb{Z} \tag{5.3}$$

Here, our system effectively alternates in time between two different Hamiltonians H_1 and H_2 where H_i is the static Kitaev chain with chemical potential μ_i.

We chose this particular hamiltonian since it is more tractable in numerical and analytical calculation. To analyze this system, we chose to work in the rotating frame formalism as it is the simplest and more straightforward formalism. Furthermore, we wanted to avoid dealing with the extended Hilbert space formalism since it adds infinite degrees of freedom in the form of Floquet bands. Thus, the extended Hilbert space formalism is costly both in analytical analysis as well as the computational analysis even after truncation of the Floquet bands.

From here onward, we refer to our hamiltonian as H. Now, we start our analysis by constructing the propagator of the system. To construct the propagator, we write down the Schrodinger equation

$$\Big(i\partial_t - H(t)\Big)\,|\psi(t)\rangle = 0 \tag{5.4}$$

By integrating the Schrodinger equation, we obtain the propagator $U(t, t_0)$ given by

$$U(t, t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} H(s)\mathrm{d}s} \tag{5.5}$$

We are interested in the one-period propagator $U(T + t_0, t_0) \equiv U_T(t_0)$. The one-period propagator gives us information about the behaviour of the system in the long timescale with time steps equal to the periodic time T. Generally speaking, finding a closed-form formula for the one-period propagator is not feasible. However, It is possible to obtain the one-period propagator in a closed-form due to the simplicity of the model hamiltonian. The one-period propagator is given by

$$\begin{aligned} U_T(t_0) =& U(t_0 + T, t_0) = \\ & \mathcal{T} e^{-i\int_{t_0}^{t_0+T} H(s)\mathrm{d}s} = \\ & \begin{cases} e^{-iH_1 t_0} e^{-iH_2 T/2} e^{-iH_1(T/2 - t_0)} & 0 < t_0 < \frac{T}{2} \\ e^{-iH_2 t_0} e^{-iH_1 T/2} e^{-iH_2(T - t_0)} & \frac{T}{2} < t_0 < T \end{cases} \end{aligned} \tag{5.6}$$

We see that the explicit form of the one-period propagator depends heavily on the initial time t_0. However, we can easily show that the choice of initial time t_0 is equivalent to choosing a frame of reference for the time coordinate. From elementary quantum mechanics, we know that choosing a frame of reference only amounts to a unitary transformation and it does not affect the eigenvalues of the operators. The proof to this statement is a straightforward application of the composition property of the time-translation operator. Below, we will show the proof.

First, we consider a different initial time t_0'. We can assume that without loss of generality that $t_0' > t_0$. Now, we write down the one-period propagator as a function of initial time t_0'

$$U_T(t_0') = U(t_0' + T, t_0') = \mathcal{T} e^{-i\int_{t_0'}^{t_0'+T} H(s)\mathrm{d}s} \tag{5.7}$$

Since we have $t_0' > t_0$, we can decompose $U(T + t_0', t_0')$ as follows:

$$\begin{aligned} U(T + t_0', t_0') &= U(T + t_0', T + t_0)U(T + t_0, t_0') \\ &= U(T + t_0', T + t_0)U(T + t_0, t_0')U(t_0', t_0)U^{-1}(t_0', t_0) \\ &= U(t_0', t_0)U(T + t_0, t_0)U^{-1}(t_0', t_0) \end{aligned} \tag{5.8}$$

In the last line, we used the time-periodicity in the propagator.
As we can see, choosing a different initial time only applies a unitary transformation. For the sake of simplicity, we take the initial time to be $t_0 = 0$ and denote the one period propagator as $U_T(0) \equiv U_T$. With this choice, we have

$$U_T = e^{-iH_2 T/2} e^{-iH_1 T/2} \tag{5.9}$$

Now, we construct the Floquet Hamiltonian H_f from the one period operator U_T.

$$H_f = \frac{i}{T} \log U_T \tag{5.10}$$

We note that the Floquet hamiltonian is generally ill-defined because the logarithm depends on the choice of the branch cut. Generally speaking, we can define different Floquet hamiltonians by changing the branch cut. In this thesis, we choose the branch cut such that the spectrum of H_f lies in the region $[-\frac{\pi}{T}, \frac{\pi}{T})$. In other words, for any arbitrary state $|\psi\rangle$ the Floquet hamiltonian needs to satisfy the following relation

$$|\langle\psi|H_f|\psi\rangle| \leq \frac{\pi}{T} \tag{5.11}$$

The Floquet hamiltonian can be obtained using the Baker-Campbell-Hausdorff (BCH) formula. For any two operators A, B, we have

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \tag{5.12}$$

Let $A = \frac{iT}{2}H_2$ and $B = \frac{iT}{2}H_1$. It follows that the Floquet hamiltonian is given by

$$H_f = \frac{(H_1 + H_2)}{2} - \frac{iT}{8}[H_2, H_1] + \dots \tag{5.13}$$

The above equation is easily computed in the momentum space. In momentum space, we have

$$H_i = \sum_k \psi^\dagger(k) h_i(k) \psi(k) \tag{5.14}$$

where $h_i(k)$ given by

$$h_i(k) = (-\mu_i - 2w\cos k)\tau_z + 2\Delta \sin k\tau_y \tag{5.15}$$

It follows that the Floquet hamiltonian H_f at first order of T is given by

$$H_f = \sum_k \psi^\dagger(k) h_f(k) \psi(k) \tag{5.16}$$

where $h_i(k)$ given by

$$h_f(k) = \Big(-\frac{\mu_1 + \mu_2}{2} - 2w\cos k\Big)\tau_z + 2\Delta \sin k\tau_y + \frac{T\Delta(\mu_2 - \mu_1)}{4}\sin k\tau_x + \dots \tag{5.17}$$

From the above expression, we can already see that the Floquet hamiltonian depends heavily on T. As the periodic time increases, higher terms in the BCH formula become more dominant. Furthermore, while the original hamiltonian has short-range interaction, we can already see that higher terms in the expansion can produce long range hopping and pairing between electrons. This behaviour can lead to higher number of Majorana fermions under certain conditions. This is indeed what we observe in the spectrum which we will discuss in the next section.

5.2 Quasienergy spectrum and Floquet Majorana

Before showing the quasienergy spectrum of the Floquet Kitaev chain, we would like to provide a meaning to quasienergies. From the Floquet theory, we can already see that quasienergy plays a similar role to crystal momentum in space-periodic quantum systems. In this section, we give a more mathematicall explanation to quasienergy in terms of geometry. To do so, we discuss the adiabatic theorem in the context of time-periodic quantum systems.

Consider a system prepared at an initial time t_0. We construct the one-period propagator at this time as

$$U_T(t_0) = \mathcal{T} e^{-i\int_{t_0}^{t_0+T} H(s)\mathrm{d}s} \tag{5.18}$$

We have already discussed that under a change of initial time t_0, the one-period propagator transforms according to the following transformation law

$$U_T(t_0') = U(t_0', t_0)U_T(t_0)U^\dagger(t_0', t_0) \tag{5.19}$$

We know that we can write the propagator $U(t, t')$ in terms of a periodic part $P(t, t')$ and a non-periodic part $U_T(t')^{(t-t')/T}$. This implies that we can rewrite the transformation law (equation (5.19)) as

$$\begin{aligned} U_T(t_0') =& U(t_0', t_0)U_T(t_0)U^\dagger(t_0', t_0) \\ =& P(t_0', t_0)U_T(t_0)P^\dagger(t_0', t_0) \end{aligned} \tag{5.20}$$

In other words, the transformation operator is the periodic operator $P(t, t')$. Since H_f is defined as the logarithm of the one-period propagator, it also transforms similarly.

Now, at initial time t, let us consider a system prepared in the Floquet mode $|\psi_j(t_0)\rangle = |u_j(t_0)\rangle$. We adiabatically progress in time so that the final state of the system stays in the same eigenstate of the Floquet Hamiltonian at time t.

$$|\psi_j(t_0)\rangle \to |\psi_j(t)\rangle = e^{-i\theta(t)}\,|u_j(t)\rangle \tag{5.21}$$

where $\theta(t)$ is an overall phase factor. Now, let us evolve the system from t_0 to $t_0 + T$. we know that from the adiabatic theorem that the phase factor is a summation of the average energy and a geometric phase. Thus, we find that the phase factor $\theta(t_0 + T)$ is given by

$$\begin{aligned} \theta(t_0+T) &= \int_{t_0}^{t_0+T} \langle u_j(t)|H(t) - i\partial_t|u_j(t)\rangle\,\mathrm{d}t \\ &= \int_{t_0}^{t_0+T} \langle u_j(t_0)|P^\dagger(t,t_0)\big(H(t) - i\partial_t\big)P(t,t_0)|u_j(t_0)\rangle\,\mathrm{d}t \\ &= \int_{t_0}^{t_0+T} \langle u_j(t_0)|H_f[t_0]|u_j(t_0)\rangle\,\mathrm{d}t \\ &= \int_{t_0}^{t_0+T} \epsilon_j \mathrm{d}t = T\epsilon_j \end{aligned} \tag{5.22}$$

The above equation implies that the quasienergies can be interpreted as a summation of two parts: (1)the average energy per one period, and (2)a geometric phase. We note that in our calculation, we were not able to identify the physical interpretation of this geometric phase, However, one of our future goals is to identify this geometric phase in our model and give it a physical interpretation.

Our next goal is to compute the quasienergy spectrum of the Floquet Hamiltonian. For the initial time, we used $t_0 = 0$. This simplifies the Floquet hamiltonian to the following form

$$H_f = \frac{i}{T} \log\Big(e^{-\frac{iT}{2}H_2} e^{-\frac{-iT}{2}H_1}\Big) \tag{5.23}$$

Our goal is to confirm the existence for Floquet Majorana states in our model and the necessary condition for their production. In order to do so, we diagonalized the Floquet hamiltonian H_f numerically in the lattice space and momentum space(open and periodic boundary conditions). While analytical study of the Floquet hamiltonian can offer a deeper understanding of the system's behaviour, it is tremendously complicated to determine the form of the Floquet hamiltonian analytically. While the momentum-space Floquet hamiltonian can be obtained from the BCH formula, one needs higher terms in the BCH formula in order to observe gap closing and opening which is a sign of topological phase transitions. Thus, we opted for numerical approach when dealing with most of the computations in this thesis.

We wrote a numerical code in python programming language to diagonalize the hamiltonian. We used numpy library for matrix manipulations and linear algebra operations. For data visualization and graphing, we used matplotlib library. The eigenvalues of the Floquet hamiltonian were calculated for different values of the periodic time T. In order to compare the eigenvalues at different periods, we measured the eigenvalues in units of $\frac{\pi}{T}$.

To investigate the presense of Floquet Majorana fermions, we considered different scenarios. First, we considered the case when our model hamiltonian $H(t)$ is always in the topological regime. what we mean is that we chose μ_1 and μ_2 such that the condition (5.24) is always satisfied. The resulting quasienergy spectrum is summarized in figure (2).

$$\mu(t) < 2w \qquad \forall t \in \mathbb{R} \tag{5.24}$$

Figure (2) shows the spectrum of the Floquet hamiltonian H_f with open boundary conditions as a function of the period T. we observe that the zero energy Majorana states persists for all values of T. Interestingly, as we increase the period T, we begin to observe clear well-defined lines at $\epsilon = \pm\frac{\pi}{T}$. These lines correspond to Floquet π Majorana states. This confirms that our hamiltonian can be engineered to host π Majorana fermions.

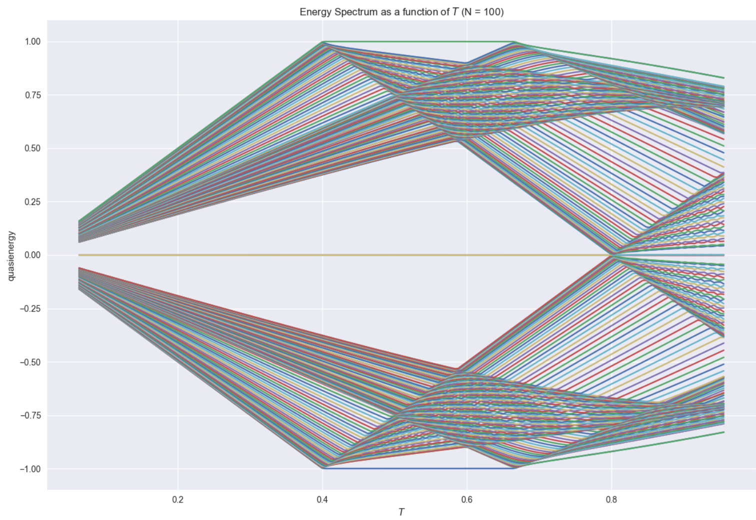

Figure 2. The graph shows quasienergy spectrum as a function of period. Here, we chose a chain of size 100 sites. The chemical potential is given by $\mu_1 = 0$ and $\mu_2 = w$. The superconducting gap is given by $\Delta = 0.5w$.

Next, we checked whether the π and zero Majorana fermions can persist when the condition (5.24) is not satisfied. We considered two scenarios. The first scenario is when the system is at the trivial regime at every moment in time. The second scenario is when the system alternates between the trivial regime and topological regime every half period.

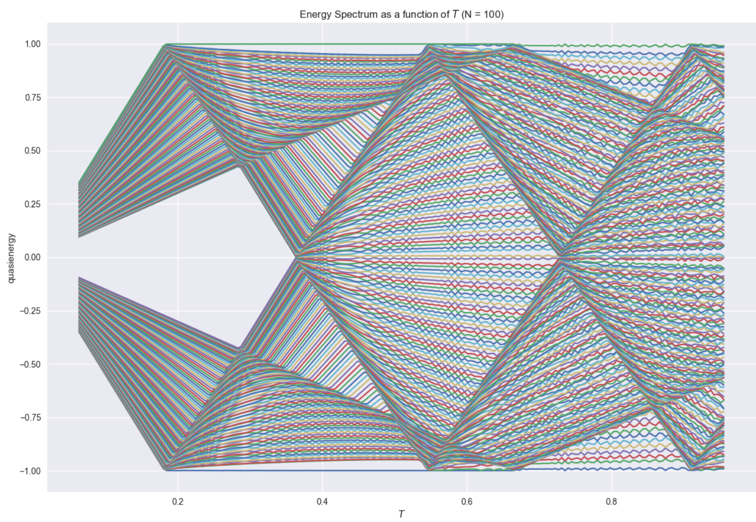

Figure 3. The graph shows quasienergy spectrum as a function of period. Here, we chose a chain of size 100 sites. The chemical potential is given by $\mu_1 = 3w$ and $\mu_2 = 4w$. The superconducting gap is given by $\Delta = 0.5w$.

Figure 3 shows the quasienergy spectrum for the first scenario. Here, the chemical potential is always larger than $2w$. We observe that for small values of the period T, Majorana states are absent. This is expected since for small T, we can approximate the Floquet hamiltonian as

$$H_f = \frac{i}{T}\log(U_T) \xrightarrow{T\to 0} H_f = (H_1 + H_2)/2 \tag{5.25}$$

This implies that for small T, the Floquet hamiltonian behaves closely to a static Kitaev chain with chemical potential $\mu = (\mu_1 + \mu_2)/2$. However, as we increase the period T, we notice that not only Majorana π modes start to appear in the spectrum but also we observe Majorana zero modes at higher values of period. This happens because higher terms in the BCH formula become more dominant and this affects the behavior and topology of the system.

We observe a similar behaviour in the second scenario as well. Figure 4 illustrates the spectrum for the second scenario with chemical potentials $\mu_1 = w, \mu_2 = 3w$. Since the average chemical potential lies exactly at the topological transition point, we see that for small values of T, we have states clustered close to zero energy but not exactly at zero. As the period increases, we observe both Majorana zero modes and Majorana π modes.

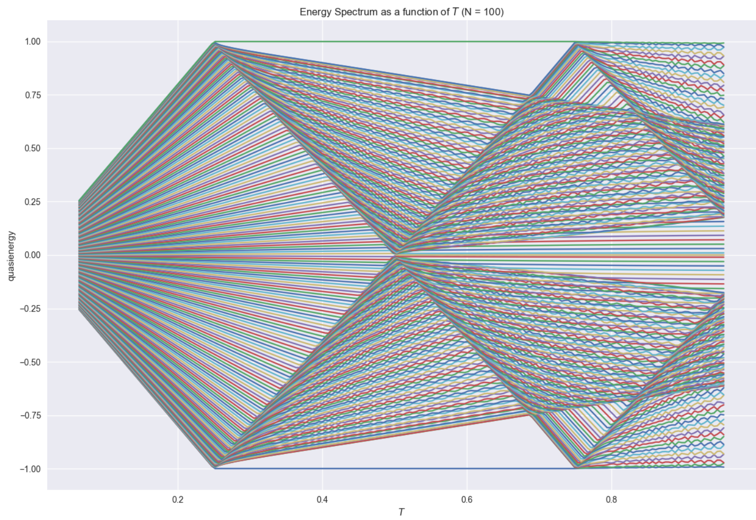

Figure 4. The graph shows quasienergy spectrum as a function of period. Here, we chose a chain of size 100 sites. The chemical potential is given by $\mu_1 = 1w$ and $\mu_2 = 3w$. The superconducting gap is given by $\Delta = 0.5w$.

Finally, we consider the effect of tuning the superconducting gap. We only present one example in the thesis. We set the superconducting gap to be $\Delta = w$ while keeping the chemical potential in the topological regime so we choose $\mu_1 = 0, \mu_2 = w$. For this set of parameters, we observe that the spectrum in figure 5 possesses many interesting gap-closing and gap-opening points. We notice that while the system is always in the topological regime, there is a gap-opening in the spectrum near zero energy. Thus, Majorana zero modes disappeared.

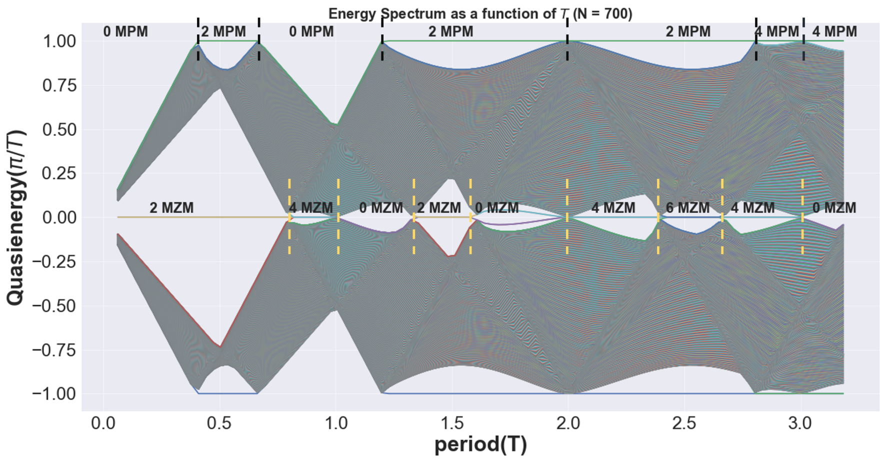

Figure 5. The graph shows quasienergy spectrum as a function of period. Here, we chose a chain of size 700 sites. The chemical potential is given by $\mu_1 = 0$ and $\mu_2 = w$. The superconducting gap is given by $\Delta = w$.

Another interesting point about the spectrum is that we can observe multiple Majorana states. The regions with different number of Majorana states are illustrated in figure 5. We note that the number of both Majorana zero and π modes can be tuned by changing the period T. The graph below shows the probability distribution of Majorana π modes at $T = 3.1$. We see that we have 4 states localized at the two edges of the Floquet Kitaev chain.

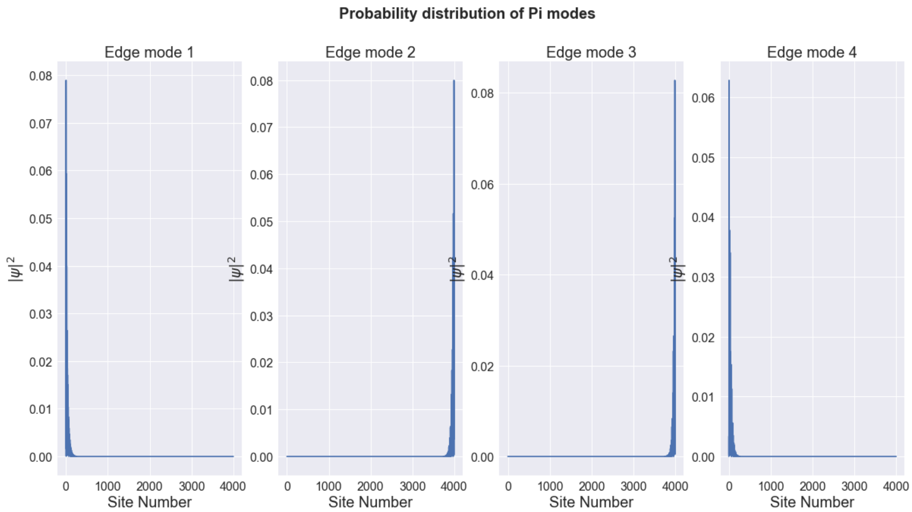

Figure 6. At $T = 3.1$, we observe 4 π-modes localized at the edge. Here, we chose a chain of size 2000 sites. The chemical potential is given by $\mu_1 = 0$ and $\mu_2 = w$. The superconducting gap is given by $\Delta = w$.

5.3 Effective Green's Function

In this section, we investigate the Nanbu Green's function of the Floquet Kitaev chain and study its properties. We recall that the Nanbu Green's function is given by the following equation:

$$\big(i\partial_t - H(t)\big)\mathcal{G}(t,t') = \delta(t-t') \tag{5.26}$$

Generally speaking, the Green's function depends on both time coordinates t, t'. Only when time translation is a symmetry of the system can we set the Green's function as a function of the relative time $\tau = t - t'$. In Floquet systems, time translation symmetry is broken and thus, the calculation of the Green's function is more involved. In this section, we provide an explanation of the Green's function method in both the extendend Hilbert space formalism and the rotating frame formalism.

We begin our discussion by recalling that the Green's function can be defined in terms of the time-translation operator (propagator) as follows:

$$\mathcal{G}(t,t') = \Theta(t-t')U(t,t') = \Theta(t-t')\mathcal{T}e^{-i\int_{t'}^{t} H(s)\mathrm{d}s} \tag{5.27}$$

where $\Theta(t-t')$ is the step function. This definition can be easily checked by using the fact that the derivative of the step function is the Dirac delta function and by recalling that the propagator satisfies Schrodinger's equation.
Given the above definition of the Green's function, we can easily manipulate the Green's function using the properties of the propagator. First, let us introduce the Green's function for the Floquet Hamiltonian H_f. According to the definition 5.27, we see that the Floquet Green's function is given by

$$\begin{aligned}
\mathcal{G}_f(t,t') &= \Theta(t-t')e^{-i(t-t')H_f} \\
&= \Theta(t-t')e^{-itH_f}e^{+it'H_f} \\
&= \Theta(t-t')e^{-itH_f}U^\dagger(t,0)U(t,0)U^\dagger(t',0)U(t',0)e^{+it'H_f} \\
&= \Theta(t-t')P^\dagger(t,0)U(t,0)U(0,t')P(t',0) \\
&= \Theta(t-t')P^\dagger(t,0)U(t,t')P(t',0) \\
&= P^\dagger(t,0)\mathcal{G}(t,t')P(t',0)
\end{aligned} \tag{5.28}$$

Or, in other words, we have:

$$\mathcal{G}(t,t') = P(t,0)\mathcal{G}_f(t,t')P^\dagger(t',0) \tag{5.29}$$

This implies that both the Floquet Green's function and the true Green's function are related to each other. Indeed, we can see that equations 5.26 and 5.29 imply the following:

$$\begin{aligned}
(i\partial_t - H(t))\mathcal{G}(t,t') &= \Big(i\partial_t - H(t)\Big)\Big(P(t,0)\mathcal{G}_f(t,t')P^\dagger(t',0)\Big) \\
&= \Big(P(t,0)i\partial_t + i\dot{P}(t,0) - H(t)P(t,0)\Big)\mathcal{G}_f(t,t')P^\dagger(t',0) \\
&= \Big(P(t,0)i\partial_t + H(t)P(t,0) - P(t,0)H_f - H(t)P(t,0)\Big)\mathcal{G}_f(t,t')P^\dagger(t',0) \\
&= P(t,0)\Big(i\partial_t - H_f\Big)\mathcal{G}_f(t,t')P^\dagger(t',0) \\
&= P(t,0)\delta(t-t')P^\dagger(t',0) \\
&= \delta(t-t')
\end{aligned} \tag{5.30}$$

where in the the last line, we used the fact that the delta function is zero everywhere except when $t = t'$. Thus, we can safely replace $P^\dagger(t',0)$ to $P\dagger(t,0)$ in the second to last line.

Now, we show the relationship between the extended Hilbert space Green's function and the true Green's function. We start from the defining equation 5.26 and use the substitution 5.29. Thus, we have

$$(i\partial_t - H(t))\mathcal{G}(t,t') = \Big(i\partial_t - H(t)\Big)\Big(P(t,0)\mathcal{G}_f(t,t')P^\dagger(t',0)\Big) \tag{5.31}$$

Now, we recall that the operator $P(t,0)$ is periodic in t with period $T = \frac{2\pi}{\Omega}$. Thus, one can use the following expansion

$$P(t,0) = \sum_{n=-\infty}^{\infty} P_n e^{-i\Omega n t} \tag{5.32}$$

Next, we recall that the Floquet Green's function $\mathcal{G}_f(t,t')$ has time-translation symmetry. Thus, we have $\mathcal{G}_f(t,t') = \mathcal{G}_f(t-t')$. Now, we expand the Floquet Green's function in terms of its Fourier components. We recall that the Floquet hamiltonian's spectrum is restricted to lie in the region $[-\frac{\Omega}{2}, \frac{\Omega}{2})$. Thus, the Fourier(frequency) expansion should be given by

$$\mathcal{G}_f(t-t') = \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{-i\omega(t-t')}\mathcal{G}_f(\omega)\mathrm{d}\omega \tag{5.33}$$

After We insert these two substitutions into equation 5.31, we arrive at:

$$\sum_{n,m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega e^{-it(\omega+n\Omega)+it'(\omega+m\Omega)}\Big(\omega + n\Omega - H(t)\Big)\Big(P_n\mathcal{G}_f(\omega)P_m^\dagger\Big) = \delta(t-t') \tag{5.34}$$

Now, we use the fact that the hamiltonian is periodic to so that we can write it as

$$H(t) = \sum_{m'=-\infty}^{\infty} H_{m'}e^{-i\Omega m' t} = \sum_{m'=-\infty}^{\infty} H_{m'-n}e^{-i\Omega(m'-n)t} \tag{5.35}$$

Now, we insert the expansion 5.35 into equation 5.34. We obtain the following:

$$\sum_{n,m,m'=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega e^{-it(\omega+m'\Omega)+it'(\omega+m\Omega)}\Big((\omega+n\Omega)\delta_{m',n} - H_{m'-n}\Big)\Big(P_n\mathcal{G}_f(\omega)P_m^\dagger\Big) = \delta(t-t') \tag{5.36}$$

The next step is to write the delta function $\delta(t-t')$ in a more convenient form. We can write the delta function as follows:

$$\begin{aligned}
\delta(t-t') &= \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t')} \\
&= \sum_{m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}+m\Omega}^{\frac{\Omega}{2}+m\Omega} \mathrm{d}\omega e^{-i\omega(t-t')} \\
&= \sum_{m,m'=-\infty}^{\infty} \int_{-\frac{\Omega}{2}+m\Omega}^{\frac{\Omega}{2}+m\Omega} \mathrm{d}\omega e^{-i\omega(t-t')}\delta_{m,m'} \\
&= \sum_{m,m'=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega e^{-i(\omega+m\Omega)(t-t')}\delta_{m,m'} \\
&= \sum_{m,m'=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega e^{-it(\omega+m\Omega)+it'(\omega+m\Omega))}\delta_{m,m'} \\
&= \sum_{m,m'=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega e^{-it(\omega+m'\Omega)+it'(\omega+m\Omega))}\delta_{m,m'}
\end{aligned} \tag{5.37}$$

In the last line, we replaced mt with $m't$ since the Kronecker delta is nonzero only when $m' = m$.

Now, we can use equation 5.37 and equation 5.36 to obtain the following equation

$$\sum_n \Big((\omega + n\Omega)\delta_{m',n} - H_{m'-n}\Big)\Big(P_n \mathcal{G}_f(\omega) P_m^\dagger\Big) = \delta_{m',m} \tag{5.38}$$

The above equation is remarkable. We can observe that the above equation can be rewriten as

$$\sum_n \Big((\omega\delta_{m',n} - K_{m',n}\Big)\mathcal{G}_{n,m} = \delta_{m',m} \tag{5.39}$$

where $K_{m',n} = n\Omega\delta_{m',n} - H_{m'-n}$ is the matrix element of the extended Hilbert space hamiltonian which we defined in equation 3.33. The operator $\mathcal{G}_{n,m}$ is the matrix element of the extended Hilbert space Green's function. It follows that the extended Hilbert space Green's function is given by

$$\mathcal{G}_{n,m} = P_n \mathcal{G}_f(\omega) P_m^\dagger \tag{5.40}$$

Equations 5.29 and 5.40 illustrates the relationship between the true Green's function $\mathcal{G}(t,t')$, the Floquet Green's function $\mathcal{G}_f(t-t')$ and the extended Hilbert space Green's function $\mathcal{G}_{n,m}(\omega)$.

Before we move on to the next section, we would like to establish the direct relationship between the true Green's function and the extended Hilbert space Green's function. First, we recall that the propagator $U(t,t')$ is invariant under the transformation that shifts both time coordinates by a multiple of the period T. It follows that the true Green's function is also invariant under discrete translation of both t and t'. Now, we Fourier transform the true Green's function by integrating the variable t'.

$$\mathcal{G}(t,\omega) = \int_{-\infty}^{\infty} \mathrm{d}t' e^{i\omega(t-t')}\mathcal{G}(t,t') \tag{5.41}$$

Since the true Green's function is periodic under shifting both time coordinates, it follows that $\mathcal{G}(t,\omega)$ is periodic in the time coordinate t. Thus, we apply a discrete Fourier transform.

$$\mathcal{G}(t,\omega) = \sum_{n=-\infty}^{\infty} \mathcal{G}_n(\omega) e^{-in\Omega t} \tag{5.42}$$

It follows that the true Green's function is given by

$$\begin{aligned}
\mathcal{G}(t,t') &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \mathcal{G}_n(\omega) e^{-in\Omega t - i\omega(t-t')} \\
&= \sum_{n,m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}+m\Omega}^{\frac{\Omega}{2}+m\Omega} \mathrm{d}\omega \mathcal{G}_n(\omega) e^{-in\Omega t - i\omega(t-t')} \\
&= \sum_{n,m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega \mathcal{G}_n(\omega + m\Omega) e^{-in\Omega t - i(\omega+m\Omega)(t-t')} \\
&= \sum_{n,m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega \mathcal{G}_{n-m}(\omega + m\Omega) e^{-i(n-m)\Omega t - i(\omega+m\Omega)(t-t')} \\
&= \sum_{n,m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega \mathcal{G}_n(\omega + m\Omega) e^{-it(\omega+n\Omega) + i(\omega+m\Omega)t'}
\end{aligned} \tag{5.43}$$

We can identify $\mathcal{G}_{nm}(\omega) = \mathcal{G}_n(\omega + m\Omega)$. Thus, we now establish the relationship between the true Green's function and the extended Hilbert space Green's function as follows:

$$\mathcal{G}(t,t') = \sum_{n,m=-\infty}^{\infty} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathrm{d}\omega \mathcal{G}_{n,m}(\omega) e^{-it(\omega+n\Omega) + i(\omega+m\Omega)t'} \tag{5.44}$$

5.4 Density of states and Odd-frequency pairing

In the previous section, we managed to show that the Green's function of the true hamiltonian and the Green's functions of the extended Hilbert space formalism and the rotating frame formalism are related to each other non-trivially. Since we established the relationships between these three formalisms, one can choose to work in any formalism. For numerical convenience, we choose to work in the rotating frame formalism. In the rotated space formalism, the Floquet Green's function is defined according to the equation

$$(i\partial_t - H_f)\mathcal{G}_f(t-t') = \delta(t-t') \tag{5.45}$$

where $H_f = \frac{i}{T}\log U_T$ is the Floquet hamiltonian. In the frequency domain, this equation leads to

$$\mathcal{G}_f^{R(A)}(\omega) = (\omega \pm i\delta - H_f)^{-1}$$

Here $\mathcal{G}_f^{R(A)}(\omega)$ refers to the retarded(advanced) Floquet Green's function in Nanbu space. We can explicitly express $\mathcal{G}_f^{R(A)}(\omega)$ in Nanbu space as follows:

$$\mathcal{G}_f^{R(A)}(\omega) = \begin{pmatrix} G_f^{R(A)}(\omega) & F_f^{R(A)}(\omega) \\ -F_f^{R(A)\dagger}(\omega) & -G_f^{R(A)\dagger}(\omega) \end{pmatrix} \tag{5.46}$$

We recall that in section 5.2, we showed that the number of Majorana π and zero modes in the Floquet Kitaev chain can be tuned by changing the period T. Here, we would like to investigate the validity of this statement using the Green's function method.

First, we introduce the density of states (LDOS) of the Floquet Kitaev chain. The LDOS can be obtained from the following expression:

$$\rho_f(\omega, j) = -\frac{1}{\pi}\,\mathrm{Im}\big\{G^R_{fjj}(\omega)\big\} \tag{5.47}$$

where j is the lattice index. Equivelently, we can define the LDOS in terms of the Nanbu Floquet Green's function as follows

$$\rho_f(\omega, j) = -\frac{1}{2\pi}\,\mathrm{Im}\big\{\mathrm{Tr}\big\{\mathcal{G}^R_{fjj}(\omega)\big\}\big\} \tag{5.48}$$

Where the trace is taken over the particle-hole degrees of freedom.

Now, we confirm the quasienergy spectrum in figure 5. For this purpose, it is convenient to construct the full density of states (DOS) from the LDOS by summing over the lattice index j.

$$\rho(\omega) = \sum_j \rho(\omega, j) = \sum_j -\frac{1}{2\pi}\,\mathrm{Im}\big\{\mathrm{Tr}\big\{\mathcal{G}^{jj}(\omega)\big\}\big\} = -\frac{1}{2\pi}\,\mathrm{Im}\{\mathrm{Tr}\{\mathcal{G}(\omega)\}\} \tag{5.49}$$

where the trace is now taken over both the particle-hole and lattice degrees of freedom. From the definition of the DOS, we expect the DOS $\rho(\omega)$ to diverge when the frequency approaches one of the quasienergies. Thus, we plot the DOS as a function of both the frequency and period.

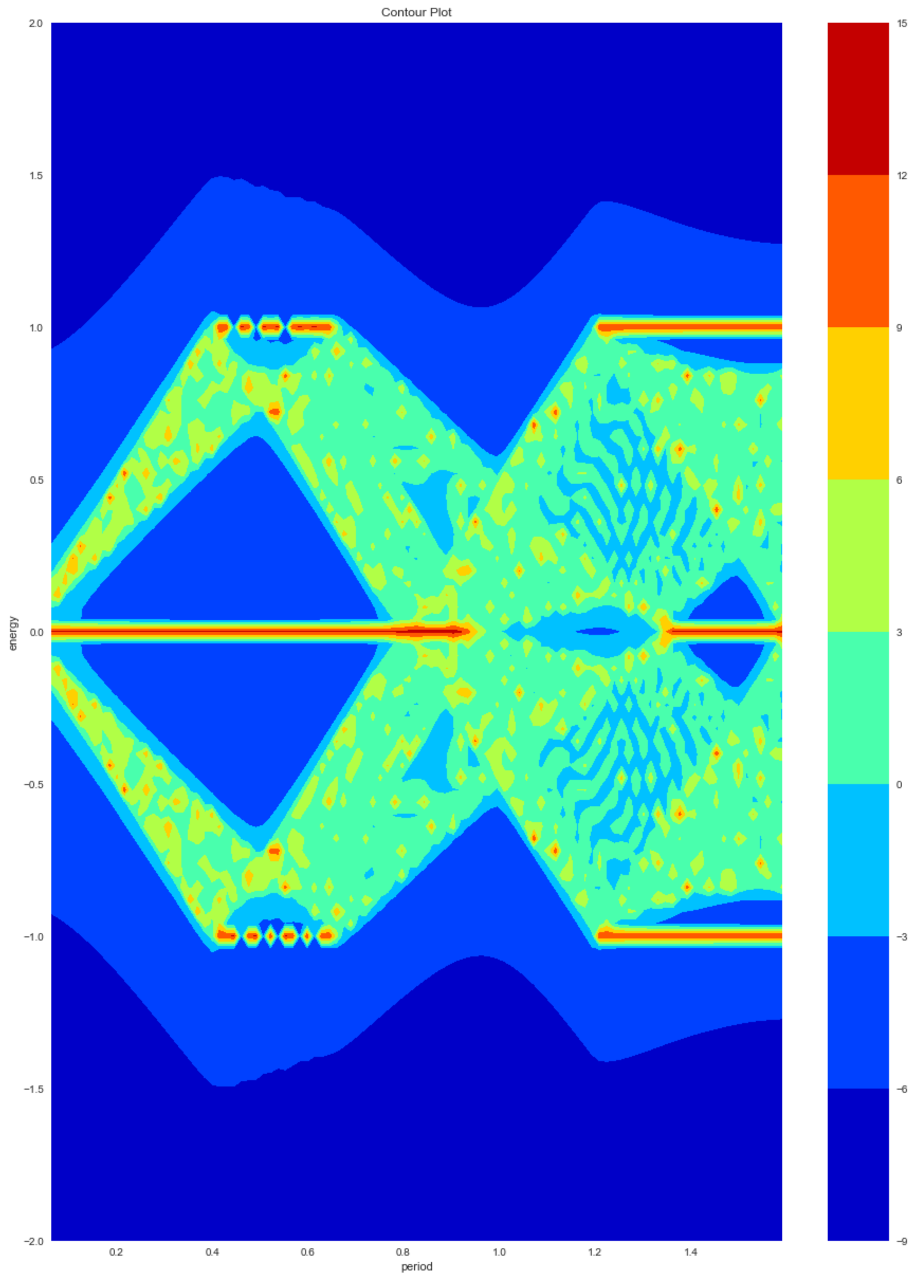

Figure 7. A contour graph of the logarithm of the DOS as a function of frequency ω and period T. The chemical potentials are $\mu_1 = 0, \mu_2 = w$ and the superconducting gap is $\Delta = w$

Figure 7 shows the logarithm of density of states as a function of both the frequency and period. As we can see, the graph is very reminiscent of the quasienergy spectrum in figure 5. This confirms that the Green's function method agrees with the results obtained in section 5.2 by brute force diagonalization.

Moreover, we can confirm the existense of multi-Majorana states by calculating the DOS evaluated at $\omega = 0, \pm\frac{\Omega}{2}$ as a function of the period T. Figure 8 illustrates how the value of the DOS changes as the number of Majorana zero modes changes. Initially, two zero modes are present but as T increases, the number of zero modes double to 4 zero modes. The number drops to zero and goes back to 2 zero modes as T increases further. We notice that the transition points in figure 8 agrees precisely with the gap closing and opening points in the spectrum in figure 5.

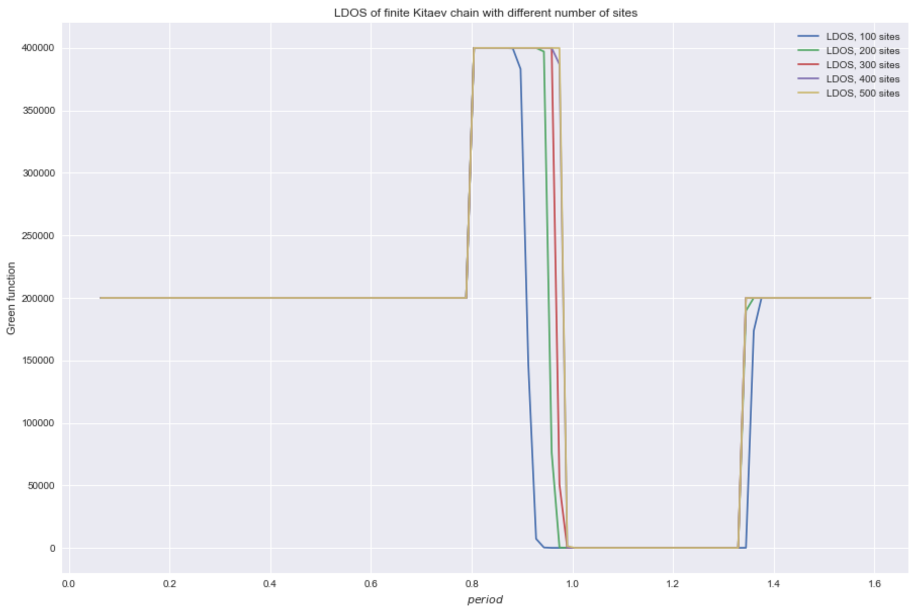

Figure 8. The graph shows the DOS at $\omega = 0$ as a function of the period T. Multiple Majorana zero modes and their hybridization can be observed in the DOS of the Floquet hamiltonian H_f. The chemical potentials are $\mu_1 = 0, \mu_2 = w$ and the superconducting gap is $\Delta = w$

Next, we discuss the anomalous part of the Green's function. The Floquet anomalous Green's function $F_f^{R(A)}(t,t')$ describes the correlation function between electrons in the rotating frame. Thus, it is given by

$$F_f^{R(A)}(t,t') = -i \langle BCS_f | \mathcal{T}^{R(A)} \{c_f(t) c_f(t')\} | BCS_f \rangle \tag{5.50}$$

where $\mathcal{T}^{R(A)}$ is the retarded(advanced) time-ordering operator which orders operators on the retarded(advanced) contour. The operator $c_f(t)$ is the annihilation operator evolving in time according to the Floquet hamiltonian H_f. Thus $c_f(t)$ is an operator in the rotating frame. The state $|BCS_f\rangle$ is the BCS ground state in the rotating frame. We will not discuss the existence of this state here but assume that it exists and describes the ground state in the rotating frame.

The Floquet anomalous Green's function is computed in the rotating frame and thus, it doesn't measure the correlation function between real electrons but rather, it gives us the correlations between "rotating" electrons described by the annihilation operator $C_f(t)$. Nevertheless, we can still give a physical meaning to the Floquet anomalous Green's function. We claim that the Floquet anomalous Green's function is equivalent to the true anomalous Green's function when the difference between the two time coordinates is a multiple of the period T.

The proof of our claim is straightforward. First, we note that we can write equation 5.50 in the following form:

$$F_f^{R(A)}(t,t') = -i \langle BCS_f | \mathcal{T}^{R(A)} \{U_f(0,t) c(0) U_f(t,0) U_f(0,t') c(0) U_f(t',0)\} | BCS_f \rangle \tag{5.51}$$

here, $c(0)$ is just the true electron annihilation operator, and $U_f(t_1, t_2)$ is the propagator in the rotating frame. The rotating frame propagator is given by

$$
\begin{aligned}
U_f(t_1, t_2) &= e^{-i(t_1 - t_2)H_f} \\
&= e^{-it_1 H_f} e^{+it_2 H_f} = U_f(t_1, 0) U_f(0, t_2)
\end{aligned}
\tag{5.52}
$$

Now, we set the first time coordinate to be $t = t' + nT$ where $n \in \mathbb{Z}$. Thus, the relative time between the two electrons is a multiple of the period. We know that the floquet propagator $U_f(t_1, t_2)$ and the true propagator $U(t_1, t_2)$ coincide when $t_1 = nT, t_2 = t_0 = 0$. Thus, we have

$$
\begin{aligned}
F_f^{R(A)}(t' + nT, t') &= -i \langle BCS_f | \mathcal{T}^{R(A)} \{ U_f(0, t' + nT) c(0) U_f(t' + nT, 0) U_f(0, t') c(0) U_f(t', 0) \} | BCS_f \rangle \\
&= -i \langle BCS_f | \mathcal{T}^{R(A)} \{ U_f(0, t') U(0, nT) c(0) U(nT, 0) U_f(t', 0) U_f(0, t') c(0) U_f(t', 0) \} | BCS_f \rangle \\
&= -i \langle BCS_f | \mathcal{T}^{R(A)} \{ U_f(0, t') U(0, nT) c(0) U(nT, 0) c(0) U_f(t', 0) \} | BCS_f \rangle \\
&= -i \langle BCS_f | \mathcal{T}^{R(A)} \{ U(0, nT) c(0) U(nT, 0) c(0) \} | BCS_f \rangle
\end{aligned}
\tag{5.53}
$$

where in the last line,we used the fact that the state $|BCS_f\rangle$ is an eigenstate of the Floquet propagator. Now, we recall that

$$
U(0, nT) c(0) U(nT, 0) = c(nT) \tag{5.54}
$$

Thus, we have

$$
F_f^{R(A)}(t' + nT, t') = -i \langle BCS_f | \mathcal{T}^{R(A)} \{ c(nT) c(0) \} | BCS_f \rangle \tag{5.55}
$$

Now, we argue that the Floquet BCS state and the true BCS state have a huge overlap. For small values of the period, this statement is true since the BCH expansion of Floquet hamiltonian H_f is dominated by $\frac{H_1 + H_2}{2}$. Thus, it follows that the Floquet BCS state will have a huge overlap with the the true BCS state of the hamiltonian $\frac{H_1 + H_2}{2}$ for small values of the period T. If T grows larger, we can still argue that the Floquet BCS state and the true BCS state have a huge overlap if the difference in chemical potential is small. We can easily inspect that higher terms in the BCH expansion of the Floquet hamiltonian should be proportional to higher powers of $(\mu_1 - \mu_2)$. If we assume that $(\mu_1 - \mu_2) << w$, we can conclude that the overlap between the Floquet BCS state and the true BCS state is large. Thus, we arrive at the following expression

$$
\begin{aligned}
F_f^{R(A)}(t' + nT, t') &= -i \langle BCS_f | \mathcal{T}^{R(A)} \{ c(nT) c(0) \} | BCS_f \rangle \\
&\approx -i \langle BCS | \mathcal{T}^{R(A)} \{ c(nT) c(0) \} | BCS \rangle = F^{R(A)}(nT, 0)
\end{aligned}
\tag{5.56}
$$

We can show that shifting both time coordinates in the right hand side by an equal time shift δt is equivalent to a unitary transformation on the left hand side by the unitary operator $P(\delta t, 0)$. Thus, we conclude that the Floquet anomalous Green's function tells us the correlation between true electrons with relative time equal to a multiple of the period T.

We are interested in describing the odd-time component of the Floquet anomalous Green's function. For this purpose, we conveniently work in the frequency domain and define the odd and even frequency components of the Floquet anomalous Green's function as follows:

$$
F_f^{\pm}(\omega) = \frac{1}{2} \Big(F_f^R(\omega) \pm F_f^A(-\omega) \Big) \tag{5.57}
$$

In static superconductors, the value of the odd-frequency component of the anomalous Green's function at the edge is correlated to the topology of the system. Here, we numerically investigate the relationship between topology and odd-frequency pairing in superconductors. First, we check the odd-frequency component near $\omega = 0$

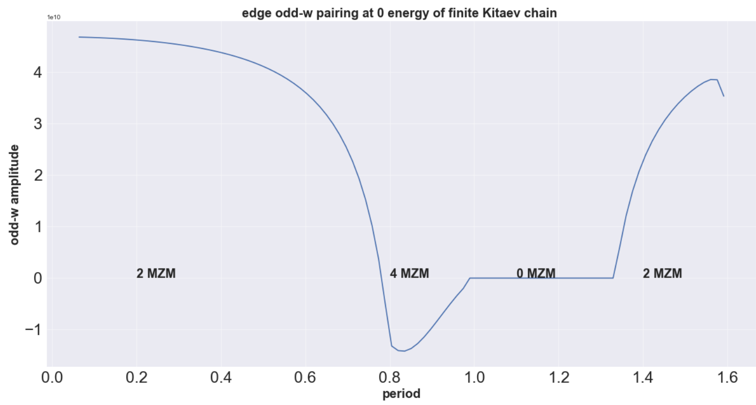

Figure 9. Odd-frequency pairing amplitude near $\omega = 0$ as a function of period. The chemical potentials are $\mu_1 = 0, \mu_2 = w$, the superconducting gap is $\Delta = w$, and the total number of lattice sites is 1000 sites

Figure 9 shows the odd-frequency component of the Floquet Green's function near zero as a function of period T. We notice that the odd-frequency pairing is maximum for small values of the period T. As the period increases, the odd-frequency pairing starts to decrease until it reaches zero at roughly $T \approx 0.8$. After that, it keeps decreasing for a while and then it starts increasing until it reaches zero again at $T \approx 1$. From $T \approx 1$ until $T \approx 1.35$, the value of the odd-frequency pairing stays at zero. After $T \approx 1.35$, it starts to increase again. If we compare figure 9 with figure 5, We notice that the odd-frequency pairing crosses zero whenever we have a gap-closing/gap-opening point at zero in the spectrum. This indicates that value of odd-frequency pairing near $\omega = 0$ is indeed correlated with the number of Floquet Majorana zero modes as we expected.

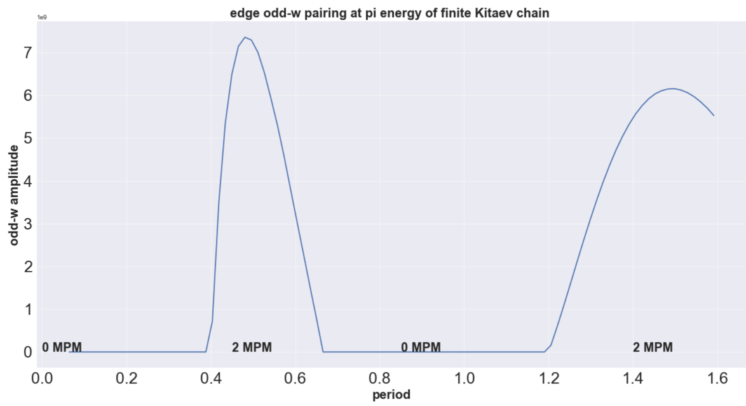

Figure 10. Odd-frequency pairing amplitude near $\omega = \frac{\Omega}{2}$ as a function of period. The chemical potentials are $\mu_1 = 0, \mu_2 = w$, the superconducting gap is $\Delta = w$, and the total number of lattice sites is 1000 sites

We notice a similar behavior in figure 10. Figure 10 shows the odd-frequency Floquet anomalous Green's function. We notice that it is also non-zero whenever Floquet π Majorana modes are present. This indicates that the value of odd-frequency pairing near $\omega = \frac{\Omega}{2}$ is correlated with the presence of Majorana π modes. Overall, we conclude that odd-frequency pairing amplitude is correlated with the topology of the Floquet Kitaev chain.

6 Conclusion

In this thesis, we studied Kitaev chain with time dependent chemical potential. The chemical potential alternates between two values μ_1 and μ_2. We conclude that the time-periodic Kitaev chain non-trivial Majorana edge states even when both values of chemical potential are bigger than the hoping integral. Furthermore, the Majorana edge modes are located not only at energy $E = 0$, but also at $E = \pi$. These exotic Majorana π modes are unique to time-periodic topological superconductors. The number of Majorana zero modes and Majorana π modes can be controlled by tuning the driving period.

In the thesis, we obtained odd-frequency pairing amplitude in the rotating frame formalism. We found that the odd-frequency Pairing amplitude at the edge crosses zero whenever we have a gap-closing/gap-opening point in the spectrum. Moreover, the edge odd-frequency Pairing near $\omega = 0(\pi/T)$ is non-zero only when Majorana zero (π) modes are present. Thus, we conclude that odd-frequency pairing at the edge is correlated with the presence of Majorana fermions in periodically-driven topological superconductors.

Outlook on future research: This thesis main message is that odd-frequency pairing is correlated to Majorana fermions. This result was shown numerically only. An analytical proof of our work is yet to be found. Furthermore, We would like to investigate whether a version of the spectral bulk-boundary correspondence [21] exist for the time-periodic regime or not. We also plan to apply the numerical tools that we developed here to other more involved models with different periodic driving protocols, different interactions, and different topological classes. We are specially interested in studying the effect of strong interaction, spin-orbit coupling, and random impurities on our results.

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