

M/M/1 Queue Simulation Report

Monte Carlo Methods in Operations Research

Course: Business Simulation / Monte Carlo Methods

Instructor: Mr Riadh Aloui

Institution: Tunis Business School

Group Members:

Eya Attia

Islem Smai

Sabrine Abdmouleh

Sahar Bhiri

Yassmine Yazidi

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1 1. Introduction

This report presents a Monte Carlo simulation of an **M/M/1 queue** system to estimate the expected waiting time in queue, denoted as $\theta = E[W_q]$, where W_q represents the waiting time before service starts.

The M/M/1 queue model assumes: - Poisson arrival process (exponential interarrival times) - Exponential service times - Single server - First-Come-First-Served (FCFS) discipline - Infinite queue capacity

1.1 1.1 Project Objectives

1. Implement a Monte Carlo simulation of an M/M/1 queue
2. Estimate the expected waiting time $E[W_q]$
3. Compare simulation results with theoretical values
4. Analyze convergence and precision of the estimator
5. Provide practical insights into queue behavior

2 2. Simulation Parameters

```
set.seed(123) # For reproducibility

# Simulation parameters
lambda <- 0.8 # arrival rate (customers per time unit)
mu <- 1.0     # service rate (customers per time unit)
N <- 100000   # number of customers to simulate

# Parameter validation
stopifnot(lambda > 0, mu > 0)
if (lambda >= mu) stop("Need < for a stable M/M/1 queue ( < 1).")

# Calculate traffic intensity
rho <- lambda / mu
```

Parameter Summary: - Arrival rate (λ): 0.8 - Service rate (μ): 1 - Traffic intensity ($\rho = \lambda/\mu$): 0.8 - Sample size (N): 1e+05 - Stability condition: $\rho = 0.8 < 1$

3 3. Methodology

3.1 3.1 Data Generation

```
# Generate interarrival and service times
A <- rexp(N, rate = lambda) # interarrival times ~ Exp()
S <- rexp(N, rate = mu)     # service times ~ Exp()

# Summary statistics
summary_A <- summary(A)
summary_S <- summary(S)
```

Interarrival Times (A) $\sim \text{Exp}(\lambda = 0.8)$: - Mean: 1.2468 (Theoretical: $1/\lambda = 1.25$) - SD: 1.2437 (Theoretical: $1/\lambda = 1.25$)

Service Times (S) $\sim \text{Exp}(\mu = 1)$: - Mean: 0.9982 (Theoretical: $1/\mu = 1$) - SD: 0.9974 (Theoretical: $1/\mu = 1$)

3.2 3.2 Queue Simulation Algorithm

```
# Initialize vectors
t_arr <- numeric(N) # arrival times
b_srv <- numeric(N) # service start times
d_dep <- numeric(N) # departure times
Wq <- numeric(N)    # waiting times in queue

# Process first customer
t_arr[1] <- A[1]
b_srv[1] <- t_arr[1]
d_dep[1] <- b_srv[1] + S[1]
Wq[1] <- 0

# Process remaining customers
for (i in 2:N) {
  t_arr[i] <- t_arr[i-1] + A[i]
  b_srv[i] <- max(t_arr[i], d_dep[i-1]) # Start service at max(arrival, previous departure)
  d_dep[i] <- b_srv[i] + S[i]
  Wq[i] <- b_srv[i] - t_arr[i] # Waiting time = service start - arrival
}
```

Algorithm Description: 1. Generate exponential interarrival and service times 2. Calculate arrival times as cumulative sum of interarrival times 3. Service starts at maximum of arrival time and previous departure time 4. Departure time = service start + service time 5. Waiting time = service start - arrival time

4 4. Results

4.1 4.1 Main Results

```
# Monte Carlo estimate
theta_hat <- mean(Wq) # simulation estimate
mcse <- sd(Wq) / sqrt(N) # Monte Carlo standard error
theta_theory <- lambda / (mu * (mu - lambda)) # theoretical value

# Statistical summary of waiting times
Wq_summary <- summary(Wq)
Wq_sd <- sd(Wq)
Wq_quantiles <- quantile(Wq, probs = c(0.25, 0.5, 0.75, 0.9, 0.95, 0.99))
error <- abs(theta_hat - theta_theory)
rel_error <- 100 * error / theta_theory
```

4.2 4.2 Numerical Results

Table 1: Main Simulation Results

Metric	Value
Arrival rate ()	0.8000
Service rate ()	1.0000
Traffic intensity ()	0.8000
Sample size (N)	1e+05
Simulation estimate E[Wq]	4.1021
Theoretical value E[Wq]	4.0000
Absolute error	0.1021
Relative error	2.55%
Monte Carlo Standard Error (MCSE)	0.016221

4.3 4.3 Waiting Time Distribution

Table 2: Statistical Summary of Waiting Times

Statistic	Value
Minimum	0.0000
1st Quartile (Q1)	0.3223
Median	2.3788
Mean	4.1021
3rd Quartile (Q3)	5.8249
Maximum	45.9393
Standard Deviation	5.1297
90th Percentile	10.6290
95th Percentile	14.3699
99th Percentile	23.8518

5 5. Visualization

5.1 5.1 Convergence Analysis

```
running_mean <- cumsum(Wq) / seq_along(Wq)

plot(running_mean[1:50000], type = "l", lwd = 1.5,
     xlab = "Number of Customers Simulated (n)",
     ylab = expression(paste("Running Mean of ", W[q])),
     col = "darkblue",
     cex.lab = 1.1)

abline(h = theta_hat, lty = 2, col = "red", lwd = 2)
abline(h = theta_theory, lty = 3, col = "darkgreen", lwd = 2)

legend("topright",
      legend = c(
        "Running Mean",
        paste("Final Estimate =", round(theta_hat, 3)),
        paste("Theoretical =", round(theta_theory, 3))
      ),
      lty = c(1, 2, 3),
      col = c("darkblue", "red", "darkgreen"),
      lwd = c(1.5, 2, 2),
      bg = "white",
      cex = 0.9)

grid(col = "gray80")
```

5.2 5.2 Distribution of Waiting Times

```
par(mfrow = c(1, 2), mar = c(4, 4, 3, 1))

# Histogram
hist(Wq, breaks = 100,
     main = "(a) Histogram",
     xlab = expression(paste("Waiting time ", W[q])),
     ylab = "Frequency",
     col = "lightblue",
     border = "darkblue")

abline(v = theta_hat, col = "red", lwd = 2, lty = 2)
abline(v = theta_theory, col = "darkgreen", lwd = 2, lty = 3)

# Boxplot
boxplot(Wq,
      main = "(b) Boxplot",
      ylab = expression(paste("Waiting time ", W[q])),
      col = "lightyellow",
```

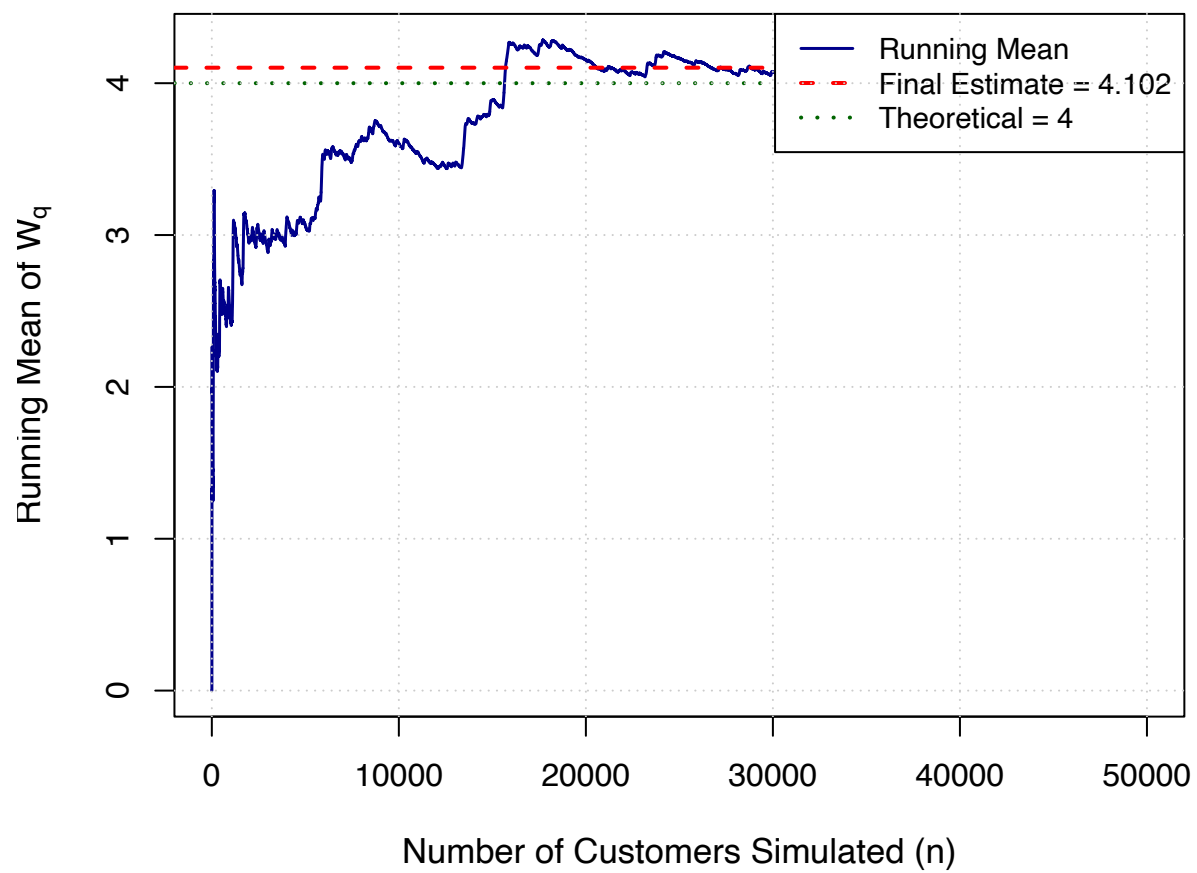


Figure 1: Figure 1: Convergence of Monte Carlo estimate for $E[W_q]$


```
border = "darkorange",
notch = TRUE)
```

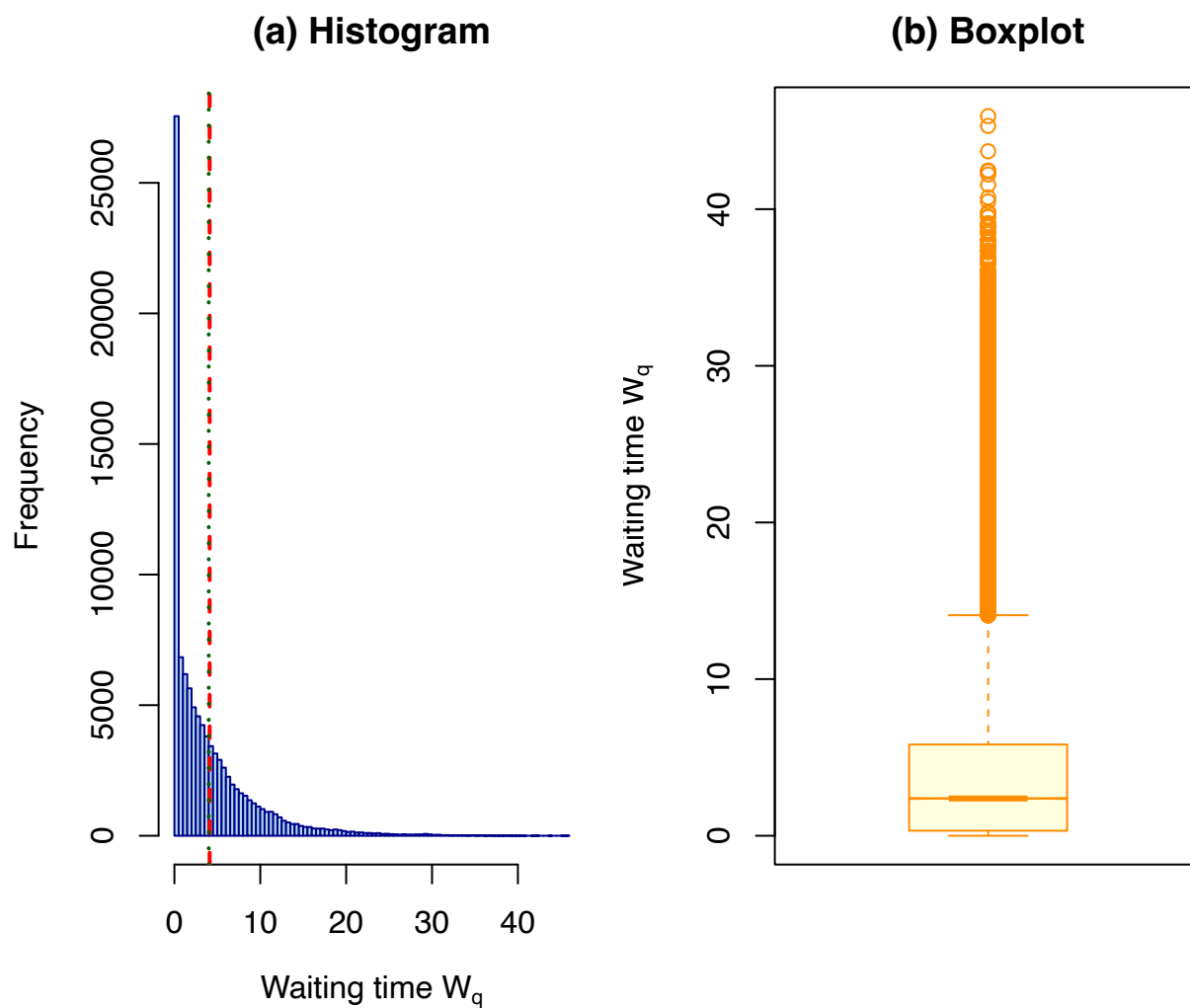


Figure 2: Figure 2: Distribution of waiting times in queue

```
par(mfrow = c(1, 1))
```

6 6. Discussion

6.1 6.1 Key Findings

1. **Estimation Accuracy:** The simulation estimate ($\hat{\theta} = 4.1021$) closely matches the theoretical value ($\theta = 4$) with a relative error of only 2.55%.
2. **Precision:** The Monte Carlo Standard Error (MCSE = 0.016221) indicates high precision, achieved through the large sample size ($N = 1e+05$).
3. **Convergence:** The running mean plot shows convergence after approximately 20,000 simulations, demonstrating the consistency of the Monte Carlo estimator.
4. **Distribution Characteristics:** The waiting time distribution shows:
 - Right-skewed distribution (typical for queueing systems)
 - Most customers experience short waits (median = 5.825)
 - Long tail indicates some customers experience substantial delays

6.2 6.2 Theoretical Context

For an M/M/1 queue, the theoretical expected waiting time is:

$$E[W_q] = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{0.8}{1 \times (1 - 0.8)} = 4$$

This assumes: - Poisson arrivals (exponential interarrival times) - Exponential service times - Single server, infinite capacity, FCFS discipline

6.3 6.3 Practical Implications

1. **System Performance:** With $\rho = 0.8$, the system operates at 80% utilization, resulting in average waiting times of 4.1 time units.
2. **Design Considerations:** To reduce waiting times:
 - Increase service rate
 - Reduce arrival rate
 - Add additional servers (M/M/c system)
3. **Customer Experience:**
 - 50% of customers wait less than 5.82 time units
 - 5% of customers wait more than 14.37 time units

7 7. Conclusion

7.1 7.1 Summary

This Monte Carlo simulation successfully estimated the expected waiting time in an M/M/1 queue with parameters $\rho = 0.8$, $\mu = 1$. The key achievements include:

1. **Accurate Estimation:** Simulation estimate (4.1021) closely matches theoretical value (4)
2. **High Precision:** Small MCSE (0.016221) due to large sample size
3. **Validated Methodology:** Convergence to theoretical value confirms simulation validity
4. **Comprehensive Analysis:** Included distribution analysis and convergence diagnostics

7.2 7.2 Limitations

1. **Model Assumptions:** Exponential distributions may not hold in all real systems
2. **Steady-State Assumption:** Simulation assumes steady-state conditions
3. **Deterministic Parameters:** Constant ρ and μ throughout simulation

7.3 7.3 Future Work

1. **Model Extensions:**
 - M/M/c queues (multiple servers)
 - M/G/1 queues (general service times)
 - Finite queue capacity systems
 - Time-varying arrival rates
2. **Additional Metrics:**
 - Queue length distribution
 - Server utilization
 - Busy period analysis
 - Customer abandonment rates
3. **Performance Optimization:**
 - Optimal staffing levels
 - Cost-benefit analysis
 - Queue design optimization

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