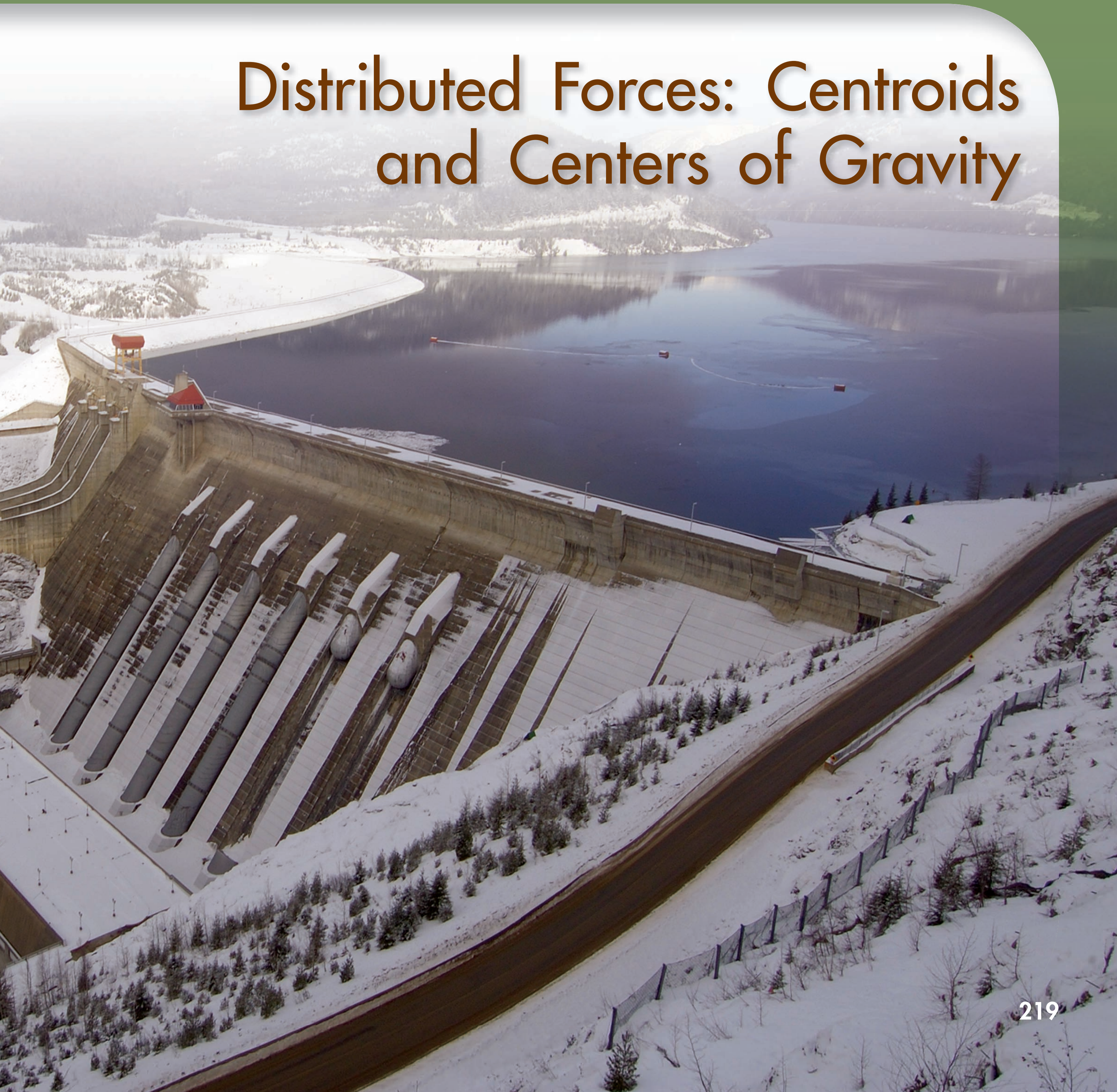
The Revelstoke Dam, located on the Columbia River in British Columbia, is subjected to three different kinds of distributed forces: the weights of its constituent elements, the pressure forces exerted by the water of its submerged face, and the pressure forces exerted by the ground on its base.

CHAPTER 5

Distributed Forces: Centroids and Centers of Gravity



Chapter 5 Distributed Forces: Centroids and Centers of Gravity

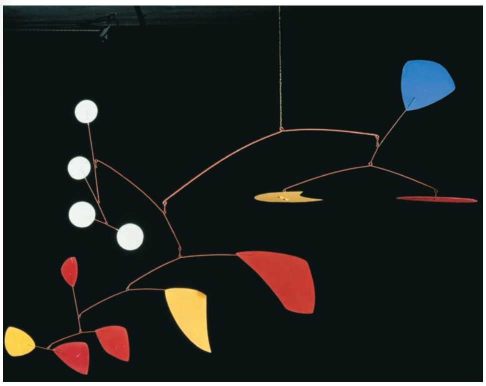


Photo 5.1 The precise balancing of the components of a mobile requires an understanding of centers of gravity and centroids, the main topics of this chapter.

5.1 INTRODUCTION

We have assumed so far that the attraction exerted by the earth on a rigid body could be represented by a single force **W**. This force, called the force of gravity or the weight of the body, was to be applied at the *center of gravity* of the body (Sec. 3.2). Actually, the earth exerts a force on each of the particles forming the body. The action of the earth on a rigid body should thus be represented by a large number of small forces distributed over the entire body. You will learn in this chapter, however, that all of these small forces can be replaced by a single equivalent force **W**. You will also learn how to determine the center of gravity, i.e., the point of application of the resultant **W**, for bodies of various shapes.

In the first part of the chapter, two-dimensional bodies, such as flat plates and wires contained in a given plane, are considered. Two concepts closely associated with the determination of the center of gravity of a plate or a wire are introduced: the concept of the *centroid* of an area or a line and the concept of the *first moment* of an area or a line with respect to a given axis.

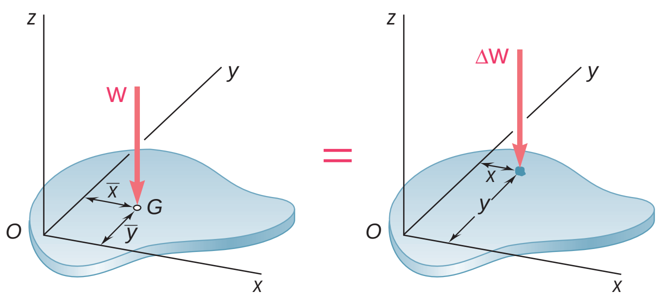
You will also learn that the computation of the area of a surface of revolution or of the volume of a body of revolution is directly related to the determination of the centroid of the line or area used to generate that surface or body of revolution (theorems of Pappus-Guldinus). And, as is shown in Secs. 5.8 and 5.9, the determination of the centroid of an area simplifies the analysis of beams subjected to distributed loads and the computation of the forces exerted on submerged rectangular surfaces, such as hydraulic gates and portions of dams.

In the last part of the chapter, you will learn how to determine the center of gravity of a three-dimensional body as well as the centroid of a volume and the first moments of that volume with respect to the coordinate planes.

AREAS AND LINES

5.2 CENTER OF GRAVITY OF A TWO-DIMENSIONAL BODY

Let us first consider a flat horizontal plate (Fig. 5.1). We can divide the plate into n small elements. The coordinates of the first element



$$\Sigma M_y: \quad \bar{x}W = \Sigma x\,\Delta W$$
$$\Sigma M_x: \quad \bar{y}W = \Sigma y\,\Delta W$$

Fig. 5.1 Center of gravity of a plate.

are denoted by x_1 and y_1, those of the second element by x_2 and y_2, etc. The forces exerted by the earth on the elements of the plate will be denoted, respectively, by $\Delta\mathbf{W}_1$, $\Delta\mathbf{W}_2$, ... , $\Delta\mathbf{W}_n$. These forces or weights are directed toward the center of the earth; however, for all practical purposes they can be assumed to be parallel. Their resultant is therefore a single force in the same direction. The magnitude W of this force is obtained by adding the magnitudes of the elemental weights.

$$\Sigma F_z: \qquad W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n$$

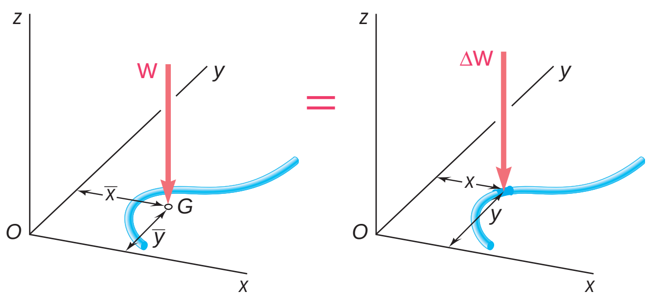
To obtain the coordinates $\bar{x}$ and $\bar{y}$ of the point G where the resultant $\mathbf{W}$ should be applied, we write that the moments of $\mathbf{W}$ about the y and x axes are equal to the sum of the corresponding moments of the elemental weights,

$$\begin{aligned} \Sigma M_y: \qquad \bar{x}W &= x_1\,\Delta W_1 + x_2\,\Delta W_2 + \cdots + x_n\,\Delta W_n \\ \Sigma M_x: \qquad \bar{y}W &= y_1\,\Delta W_1 + y_2\,\Delta W_2 + \cdots + y_n\,\Delta W_n \end{aligned} \tag{5.1}$$

If we now increase the number of elements into which the plate is divided and simultaneously decrease the size of each element, we obtain in the limit the following expressions:

$$W = \int dW \qquad \bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \tag{5.2}$$

These equations define the weight $\mathbf{W}$ and the coordinates $\bar{x}$ and $\bar{y}$ of the center of gravity G of a flat plate. The same equations can be derived for a wire lying in the xy plane (Fig. 5.2). We note that the center of gravity G of a wire is usually not located on the wire.



$\Sigma M_y: \quad \bar{x}W = \Sigma x\,\Delta W$

$\Sigma M_x: \quad \bar{y}W = \Sigma y\,\Delta W$

Fig. 5.2 Center of gravity of a wire.

5.3 CENTROIDS OF AREAS AND LINES

In the case of a flat homogeneous plate of uniform thickness, the magnitude ΔW of the weight of an element of the plate can be expressed as

$$\Delta W = \mathsf{g}t\,\Delta A$$

where g = specific weight (weight per unit volume) of the material
t = thickness of the plate
ΔA = area of the element

Similarly, we can express the magnitude W of the weight of the entire plate as

$$W = \mathsf{g}tA$$

where A is the total area of the plate.

If U.S. customary units are used, the specific weight g should be expressed in $\mathrm{lb/ft^3}$, the thickness t in feet, and the areas ΔA and A in square feet. We observe that ΔW and W will then be expressed in pounds. If SI units are used, g should be expressed in $\mathrm{N/m^3}$, t in meters, and the areas ΔA and A in square meters; the weights ΔW and W will then be expressed in newtons.†

Substituting for ΔW and W in the moment equations (5.1) and dividing throughout by $\mathsf{g}t$, we obtain

$$\Sigma M_y: \quad \bar{x}A = x_1\,\Delta A_1 + x_2\,\Delta A_2 + \cdots + x_n\,\Delta A_n$$
$$\Sigma M_x: \quad \bar{y}A = y_1\,\Delta A_1 + y_2\,\Delta A_2 + \cdots + y_n\,\Delta A_n$$

If we increase the number of elements into which the area A is divided and simultaneously decrease the size of each element, we obtain in the limit

$$\bar{x}A = \int x\,dA \qquad \bar{y}A = \int y\,dA \tag{5.3}$$

These equations define the coordinates $\bar{x}$ and $\bar{y}$ of the center of gravity of a homogeneous plate. The point whose coordinates are $\bar{x}$ and $\bar{y}$ is also known as the *centroid C of the area A* of the plate (Fig. 5.3). If the plate is not homogeneous, these equations cannot be used to determine the center of gravity of the plate; they still define, however, the centroid of the area.

In the case of a homogeneous wire of uniform cross section, the magnitude ΔW of the weight of an element of wire can be expressed as

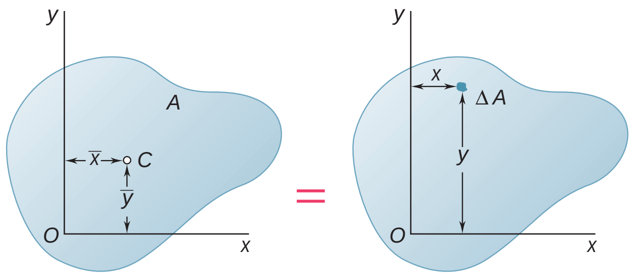
$$\Delta W = \mathsf{g}a\,\Delta L$$

where g = specific weight of the material
a = cross-sectional area of the wire
ΔL = length of the element

†It should be noted that in the SI system of units a given material is generally characterized by its density r (mass per unit volume) rather than by its specific weight g. The specific weight of the material can then be obtained from the relation

$$\mathsf{g} = \mathsf{r}g$$

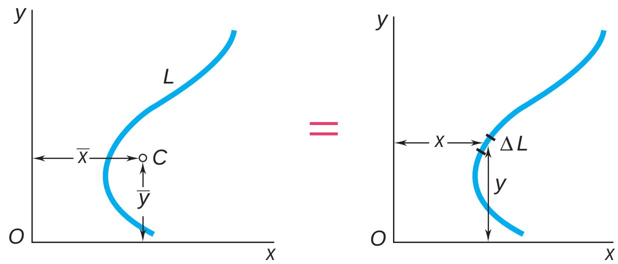
where $g = 9.81\ \mathrm{m/s^2}$. Since r is expressed in $\mathrm{kg/m^3}$, we observe that g will be expressed in $(\mathrm{kg/m^3})(\mathrm{m/s^2})$, that is, in $\mathrm{N/m^3}$.



ΣM_y: $\bar{x}A = \Sigma x\,\Delta A$

ΣM_x: $\bar{y}A = \Sigma y\,\Delta A$

Fig. 5.3 Centroid of an area.



ΣM_y: $\bar{x}L = \Sigma x\,\Delta L$

ΣM_x: $\bar{y}L = \Sigma y\,\Delta L$

Fig. 5.4 Centroid of a line.

The center of gravity of the wire then coincides with the *centroid C of the line L* defining the shape of the wire (Fig. 5.4). The coordinates $\bar{x}$ and $\bar{y}$ of the centroid of the line L are obtained from the equations

$$\bar{x}L = \int x\,dL \qquad \bar{y}L = \int y\,dL \tag{5.4}$$

5.4 FIRST MOMENTS OF AREAS AND LINES

The integral $\int x\,dA$ in Eqs. (5.3) of the preceding section is known as the *first moment of the area A with respect to the y axis* and is denoted by Q_y. Similarly, the integral $\int y\,dA$ defines the *first moment of A with respect to the x axis* and is denoted by Q_x. We write

$$Q_y = \int x\,dA \qquad Q_x = \int y\,dA \tag{5.5}$$

Comparing Eqs. (5.3) with Eqs. (5.5), we note that the first moments of the area A can be expressed as the products of the area and the coordinates of its centroid:

$$Q_y = \bar{x}A \qquad Q_x = \bar{y}A \tag{5.6}$$

It follows from Eqs. (5.6) that the coordinates of the centroid of an area can be obtained by dividing the first moments of that area by the area itself. The first moments of the area are also useful in mechanics of materials for determining the shearing stresses in beams under transverse loadings. Finally, we observe from Eqs. (5.6) that if the centroid of an area is located on a coordinate axis, the first moment of the area with respect to that axis is zero. Conversely, if the first moment of an area with respect to a coordinate axis is zero, then the centroid of the area is located on that axis.

Relations similar to Eqs. (5.5) and (5.6) can be used to define the first moments of a line with respect to the coordinate axes and

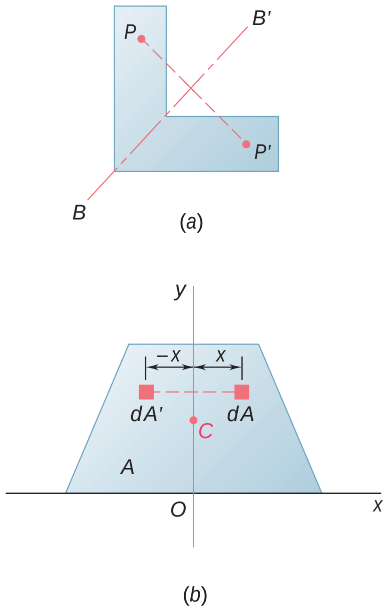


Fig. 5.5

to express these moments as the products of the length L of the line and the coordinates $\bar{x}$ and $\bar{y}$ of its centroid.

An area A is said to be *symmetric with respect to an axis* BB' if for every point P of the area there exists a point P' of the same area such that the line PP' is perpendicular to BB' and is divided into two equal parts by that axis (Fig. 5.5*a*). A line L is said to be symmetric with respect to an axis BB' if it satisfies similar conditions. When an area A or a line L possesses an axis of symmetry BB', its first moment with respect to BB' is zero, and its centroid is located on that axis. For example, in the case of the area A of Fig. 5.5*b*, which is symmetric with respect to the y axis, we observe that for every element of area dA of abscissa x there exists an element dA' of equal area and with abscissa $-x$. It follows that the integral in the first of Eqs. (5.5) is zero and, thus, that $Q_y = 0$. It also follows from the first of the relations (5.3) that $\bar{x} = 0$. Thus, if an area A or a line L possesses an axis of symmetry, its centroid C is located on that axis.

We further note that if an area or line possesses two axes of symmetry, its centroid C must be located at the intersection of the two axes (Fig. 5.6). This property enables us to determine immediately the centroid of areas such as circles, ellipses, squares, rectangles, equilateral triangles, or other symmetric figures as well as the centroid of lines in the shape of the circumference of a circle, the perimeter of a square, etc.

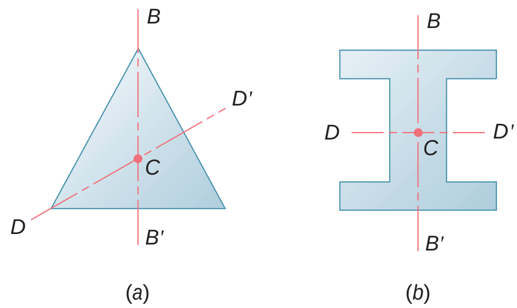


Fig. 5.6

An area A is said to be *symmetric with respect to a center* O if for every element of area dA of coordinates x and y there exists an element dA' of equal area with coordinates $-x$ and $-y$ (Fig. 5.7). It then follows that the integrals in Eqs. (5.5) are both zero and that $Q_x = Q_y = 0$. It also follows from Eqs. (5.3) that $\bar{x} = \bar{y} = 0$, that is, that the centroid of the area coincides with its center of symmetry O. Similarly, if a line possesses a center of symmetry O, the centroid of the line will coincide with the center O.

It should be noted that a figure possessing a center of symmetry does not necessarily possess an axis of symmetry (Fig. 5.7), while a figure possessing two axes of symmetry does not necessarily possess a center of symmetry (Fig. 5.6*a*). However, if a figure possesses two axes of symmetry at a right angle to each other, the point of intersection of these axes is a center of symmetry (Fig. 5.6*b*).

Determining the centroids of unsymmetrical areas and lines and of areas and lines possessing only one axis of symmetry will be discussed in Secs. 5.6 and 5.7. Centroids of common shapes of areas and lines are shown in Fig. 5.8A and B.

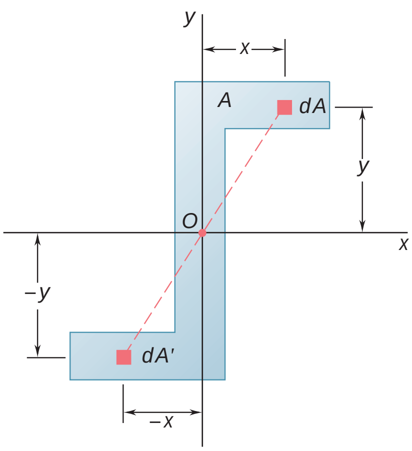


Fig. 5.7

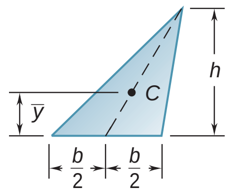
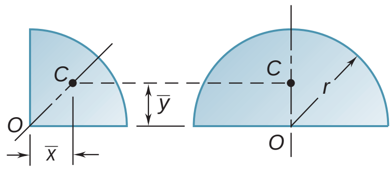
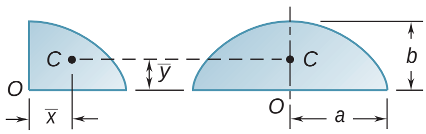
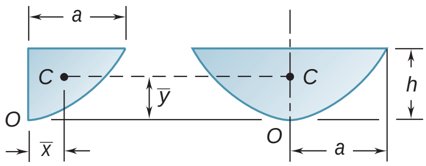
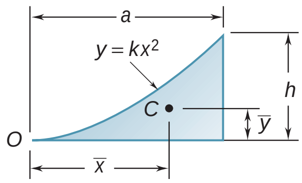
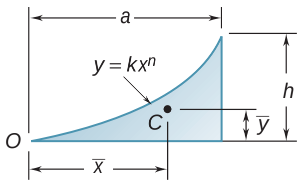
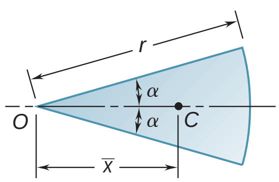
Shape		$\overline{x}$	$\overline{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Fig. 5.8A Centroids of common shapes of areas.

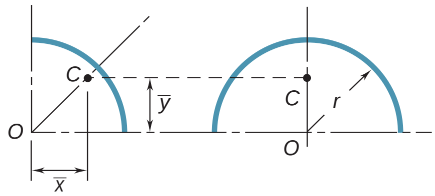
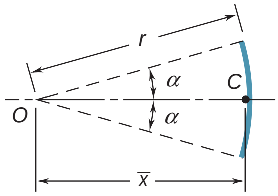
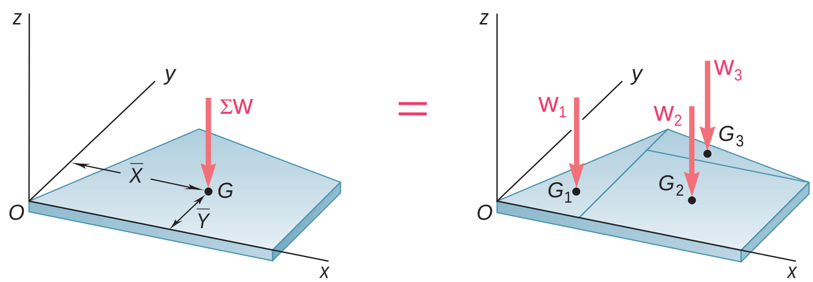
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Fig. 5.8B Centroids of common shapes of lines.

5.5 COMPOSITE PLATES AND WIRES

In many instances, a flat plate can be divided into rectangles, triangles, or the other common shapes shown in Fig. 5.8A. The abscissa $\bar{X}$ of its center of gravity G can be determined from the abscissas $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ of the centers of gravity of the various parts by expressing that the moment of the weight of the whole plate about the y axis is equal to the sum of the moments of the weights of the various parts about the same axis (Fig. 5.9). The ordinate $\bar{Y}$ of the center of gravity of the plate is found in a similar way by equating moments about the x axis. We write

$$\Sigma M_y: \quad \bar{X}(W_1 + W_2 + \cdots + W_n) = \bar{x}_1 W_1 + \bar{x}_2 W_2 + \cdots + \bar{x}_n W_n$$
$$\Sigma M_x: \quad \bar{Y}(W_1 + W_2 + \cdots + W_n) = \bar{y}_1 W_1 + \bar{y}_2 W_2 + \cdots + \bar{y}_n W_n$$



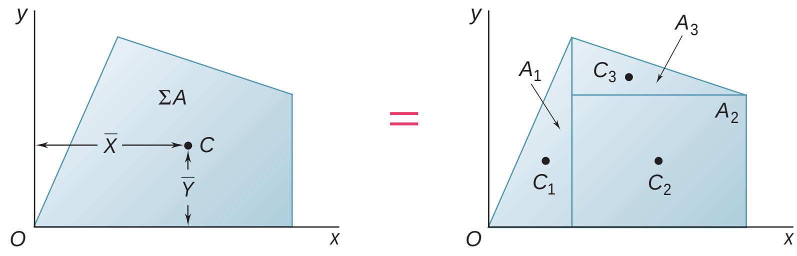
$$\Sigma M_y: \quad \bar{X}\,\Sigma W = \Sigma \bar{x} W$$
$$\Sigma M_x: \quad \bar{Y}\,\Sigma W = \Sigma \bar{y} W$$

Fig. 5.9 Center of gravity of a composite plate.

or, for short,

$$\bar{X}\Sigma W = \Sigma \bar{x} W \qquad \bar{Y}\Sigma W = \Sigma \bar{y} W \tag{5.7}$$

These equations can be solved for the coordinates $\bar{X}$ and $\bar{Y}$ of the center of gravity of the plate.



$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x} A$$
$$Q_x = \bar{Y}\Sigma A = \Sigma \bar{y} A$$

Fig. 5.10 Centroid of a composite area.

If the plate is homogeneous and of uniform thickness, the center of gravity coincides with the centroid C of its area. The abscissa $\bar{X}$ of the centroid of the area can be determined by noting that the first moment Q_y of the composite area with respect to the y axis can be expressed both as the product of $\bar{X}$ and the total area and as the sum of the first moments of the elementary areas with respect to the y axis (Fig. 5.10). The ordinate $\bar{Y}$ of the centroid is found in a similar way by considering the first moment Q_x of the composite area. We have

$$Q_y = \bar{X}(A_1 + A_2 + \cdots + A_n) = \bar{x}_1 A_1 + \bar{x}_2 A_2 + \cdots + \bar{x}_n A_n$$
$$Q_x = \bar{Y}(A_1 + A_2 + \cdots + A_n) = \bar{y}_1 A_1 + \bar{y}_2 A_2 + \cdots + \bar{y}_n A_n$$

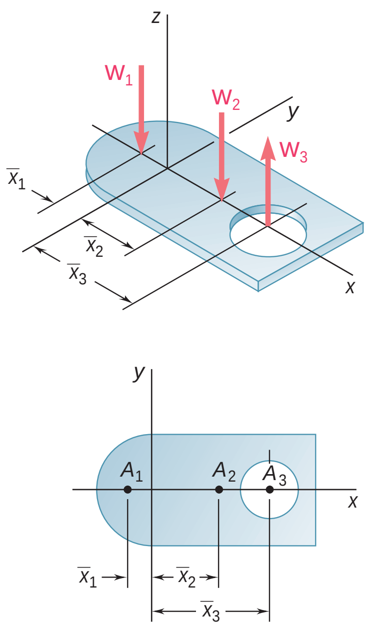
or, for short,

$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x} A \qquad Q_x = \bar{Y}\Sigma A = \Sigma \bar{y} A \tag{5.8}$$

These equations yield the first moments of the composite area, or they can be used to obtain the coordinates $\bar{X}$ and $\bar{Y}$ of its centroid.

Care should be taken to assign the appropriate sign to the moment of each area. First moments of areas, like moments of forces, can be positive or negative. For example, an area whose centroid is located to the left of the y axis will have a negative first moment with respect to that axis. Also, the area of a hole should be assigned a negative sign (Fig. 5.11).

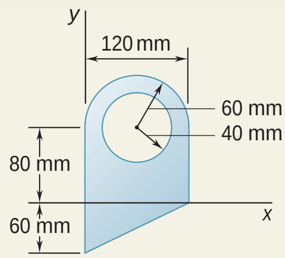
Similarly, it is possible in many cases to determine the center of gravity of a composite wire or the centroid of a composite line by dividing the wire or line into simpler elements (see Sample Prob. 5.2).



	$\bar{x}$	A	$\bar{x}A$
A_1 Semicircle	−	+	−
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	−	−

Fig. 5.11

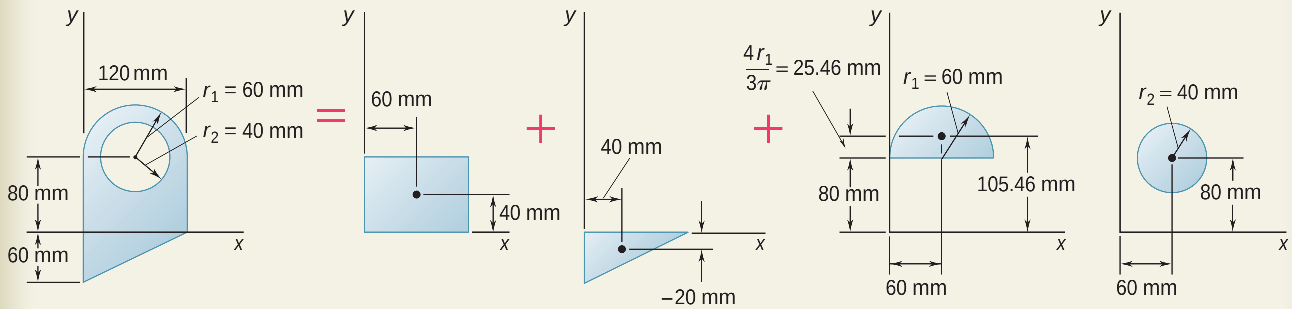
SAMPLE PROBLEM 5.1



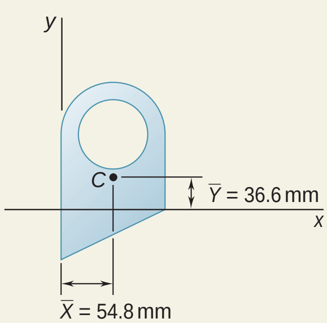
For the plane area shown, determine (*a*) the first moments with respect to the x and y axes, (*b*) the location of the centroid.

SOLUTION

Components of Area. The area is obtained by adding a rectangle, a triangle, and a semicircle and by then subtracting a circle. Using the coordinate axes shown, the area and the coordinates of the centroid of each of the component areas are determined and entered in the table below. The area of the circle is indicated as negative, since it is to be subtracted from the other areas. We note that the coordinate $\bar{y}$ of the centroid of the triangle is negative for the axes shown. The first moments of the component areas with respect to the coordinate axes are computed and entered in the table.



Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$



a. First Moments of the Area. Using Eqs. (5.8), we write

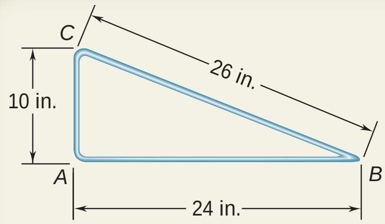
$$Q_x = \Sigma\bar{y}A = 506.2 \times 10^3 \text{ mm}^3 \qquad Q_x = 506 \times 10^3 \text{ mm}^3 \blacktriangleleft$$
$$Q_y = \Sigma\bar{x}A = 757.7 \times 10^3 \text{ mm}^3 \qquad Q_y = 758 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

b. Location of Centroid. Substituting the values given in the table into the equations defining the centroid of a composite area, we obtain

$$\bar{X}\Sigma A = \Sigma\bar{x}A: \qquad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$$
$$\bar{X} = 54.8 \text{ mm} \blacktriangleleft$$

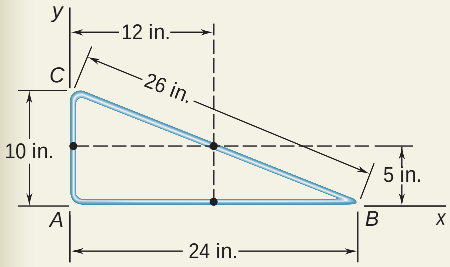
$$\bar{Y}\Sigma A = \Sigma\bar{y}A: \qquad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$$
$$\bar{Y} = 36.6 \text{ mm} \blacktriangleleft$$

SAMPLE PROBLEM 5.2



The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.

SOLUTION



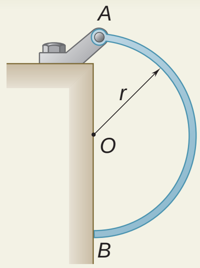
Since the figure is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. Therefore, that centroid will be determined. Choosing the coordinate axes shown, with origin at A, we determine the coordinates of the centroid of each line segment and compute the first moments with respect to the coordinate axes.

Segment	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
	$\Sigma L = 60$			$\Sigma \bar{x}L = 600$	$\Sigma \bar{y}L = 180$

Substituting the values obtained from the table into the equations defining the centroid of a composite line, we obtain

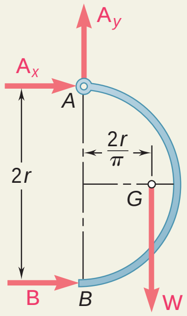
$\bar{X}\Sigma L = \Sigma \bar{x}L$: $\quad \bar{X}(60 \text{ in.}) = 600 \text{ in}^2 \qquad \bar{X} = 10 \text{ in.}$ ◀

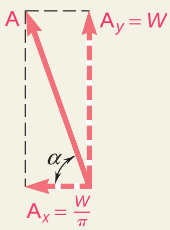
$\bar{Y}\Sigma L = \Sigma \bar{y}L$: $\quad \bar{Y}(60 \text{ in.}) = 180 \text{ in}^2 \qquad \bar{Y} = 3 \text{ in.}$ ◀



SAMPLE PROBLEM 5.3

A uniform semicircular rod of weight W and radius r is attached to a pin at A and rests against a frictionless surface at B. Determine the reactions at A and B.





SOLUTION

Free-Body Diagram. A free-body diagram of the rod is drawn. The forces acting on the rod are its weight **W**, which is applied at the center of gravity G (whose position is obtained from Fig. 5.8B); a reaction at A, represented by its components $\mathbf{A}_x$ and $\mathbf{A}_y$; and a horizontal reaction at B.

Equilibrium Equations

$+1\ \Sigma M_A = 0$: $\quad B(2r) - W\left(\frac{2r}{p}\right) = 0$

$$B = +\frac{W}{p} \qquad \mathbf{B} = \frac{W}{p}\ \text{y} \quad ◀$$

$\overset{+}{\text{y}}\ \Sigma F_x = 0$: $\quad A_x + B = 0$

$$A_x = -B = -\frac{W}{p} \qquad \mathbf{A}_x = \frac{W}{p}\ \text{z}$$

$+\text{x}\Sigma F_y = 0$: $\quad A_y - W = 0 \qquad \mathbf{A}_y = W\ \text{x}$

Adding the two components of the reaction at A:

$$A = \left[W^2 + \left(\frac{W}{p}\right)^2\right]^{1/2} \qquad A = W\left(1 + \frac{1}{p^2}\right)^{1/2} \quad ◀$$

$$\tan a = \frac{W}{W/p} = p \qquad a = \tan^{-1} p \quad ◀$$

The answers can also be expressed as follows:

$$\mathbf{A} = 1.049W\ \text{b}\ 72.3° \qquad \mathbf{B} = 0.318W\ \text{y} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we developed the general equations for locating the centers of gravity of two-dimensional bodies and wires [Eqs. (5.2)] and the centroids of plane areas [Eqs. (5.3)] and lines [Eqs. (5.4)]. In the following problems, you will have to locate the centroids of composite areas and lines or determine the first moments of the area for composite plates [Eqs. (5.8)].

1. Locating the centroids of composite areas and lines. Sample Problems 5.1 and 5.2 illustrate the procedure you should follow when solving problems of this type. There are, however, several points that should be emphasized.

a. The first step in your solution should be to decide how to construct the given area or line from the common shapes of Fig. 5.8. You should recognize that for plane areas it is often possible to construct a particular shape in more than one way. Also, showing the different components (as is done in Sample Prob. 5.1) will help you to correctly establish their centroids and areas or lengths. Do not forget that you can subtract areas as well as add them to obtain a desired shape.

b. We strongly recommend that for each problem you construct a table containing the areas or lengths and the respective coordinates of the centroids. It is essential for you to remember that areas which are "removed" (for example, holes) are treated as negative. Also, the sign of negative coordinates must be included. Therefore, you should always carefully note the location of the origin of the coordinate axes.

c. When possible, use symmetry [Sec. 5.4] to help you determine the location of a centroid.

d. In the formulas for the circular sector and for the arc of a circle in Fig. 5.8, the angle α must always be expressed in radians.

2. Calculating the first moments of an area. The procedures for locating the centroid of an area and for determining the first moments of an area are similar; however, for the latter it is not necessary to compute the total area. Also, as noted in Sec. 5.4, you should recognize that the first moment of an area relative to a centroidal axis is zero.

3. Solving problems involving the center of gravity. The bodies considered in the following problems are homogeneous; thus, their centers of gravity and centroids coincide. In addition, when a body that is suspended from a single pin is in equilibrium, the pin and the body's center of gravity must lie on the same vertical line.

It may appear that many of the problems in this lesson have little to do with the study of mechanics. However, being able to locate the centroid of composite shapes will be essential in several topics that you will soon encounter.

PROBLEMS

5.1 through 5.9 Locate the centroid of the plane area shown.

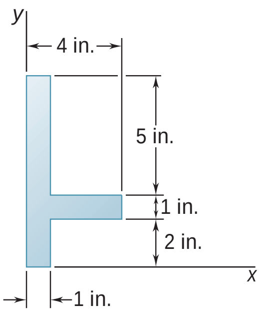


Fig. P5.1

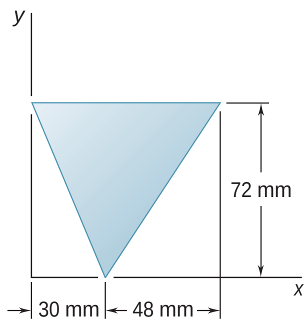


Fig. P5.2

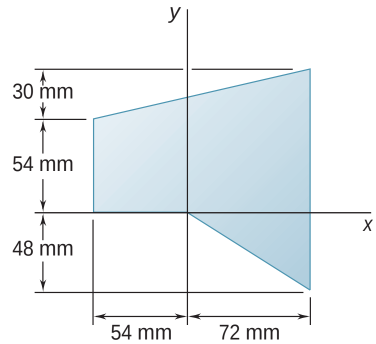


Fig. P5.3

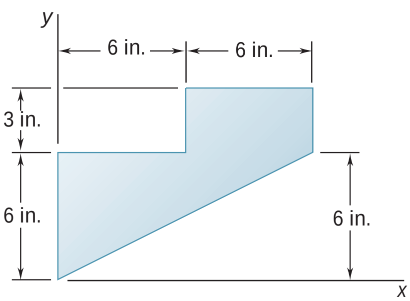


Fig. P5.4

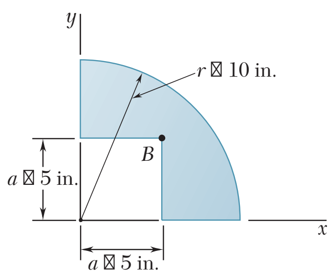


Fig. *P5.5*

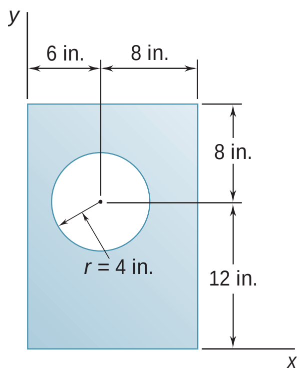


Fig. P5.6

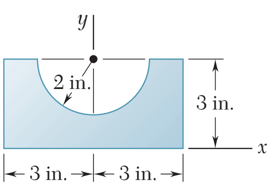


Fig. *P5.7*

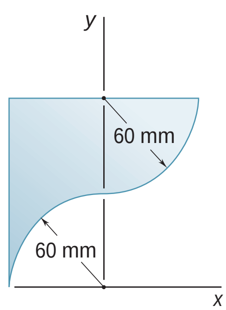


Fig. P5.8

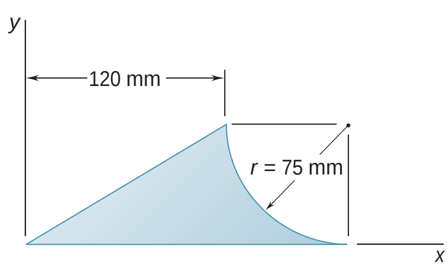


Fig. P5.9

5.10 through *5.15* Locate the centroid of the plane area shown.

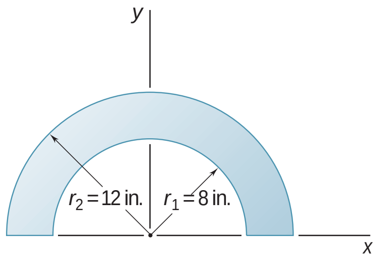


Fig. P5.10

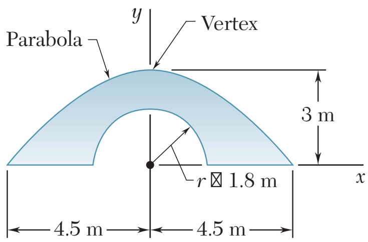


Fig. P5.11

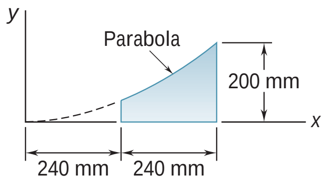


Fig. *P5.12*

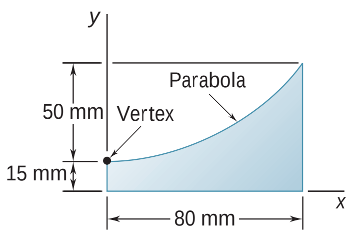


Fig. P5.13

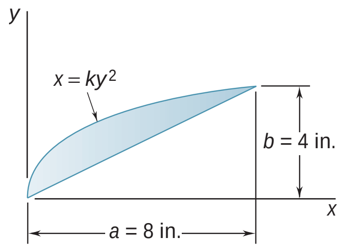


Fig. P5.14

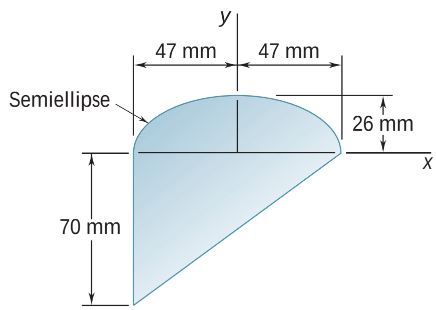


Fig. *P5.15*

5.16 Determine the x coordinate of the centroid of the trapezoid shown in terms of h_1, h_2, and a.

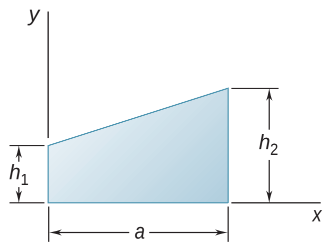


Fig. *P5.16*

5.17 For the plane area of Prob. 5.5, determine the ratio a/r so that the centroid of the area is located at point B.

5.18 Determine the y coordinate of the centroid of the shaded area in terms of r_1, r_2, and a.

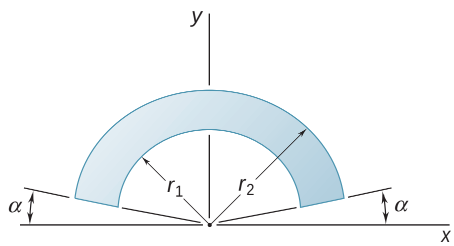


Fig. P5.18 and P5.19

5.19 Show that as r_1 approaches r_2, the location of the centroid approaches that for an arc of circle of radius $(r_1 + r_2)/2$.

5.20 and *5.21* The horizontal x axis is drawn through the centroid C of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x axis, and explain the results obtained.

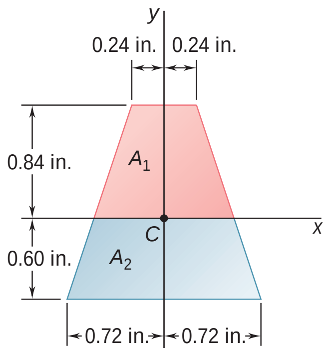


Fig. P5.20

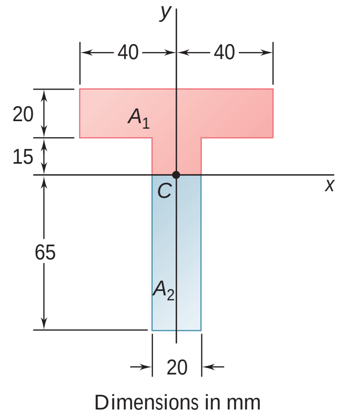


Fig. *P5.21*

5.22 A composite beam is constructed by bolting four plates to four $60 \times 60 \times 12$-mm angles as shown. The bolts are equally spaced along the beam, and the beam supports a vertical load. As proved in mechanics of materials, the shearing forces exerted on the bolts at A and B are proportional to the first moments with respect to the centroidal x axis of the red shaded areas shown, respectively, in parts a and b of the figure. Knowing that the force exerted on the bolt at A is 280 N, determine the force exerted on the bolt at B.

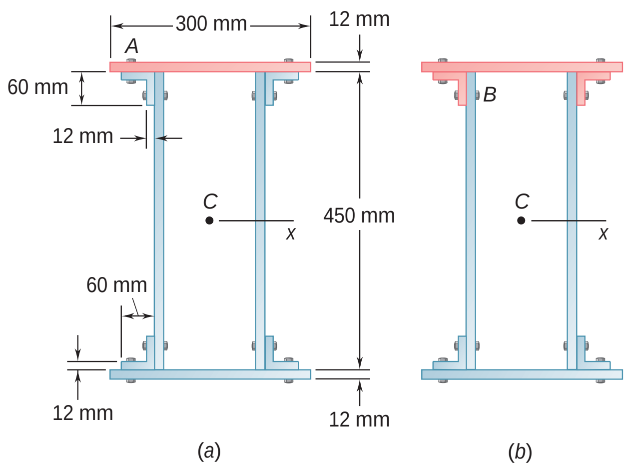


Fig. P5.22

5.23 The first moment of the shaded area with respect to the x axis is denoted by Q_x. (*a*) Express Q_x in terms of b, c, and the distance y from the base of the shaded area to the x axis. (*b*) For what value of y is Q_x maximum, and what is that maximum value?

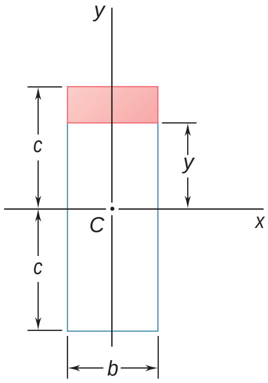


Fig. P5.23

5.24 through *5.27* A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

5.24 Fig. P5.2.
5.25 Fig. P5.3.
5.26 Fig. P5.4.
5.27 Fig. P5.5.

5.28 The homogeneous wire $ABCD$ is bent as shown and is attached to a hinge at C. Determine the length L for which portion BCD of the wire is horizontal.

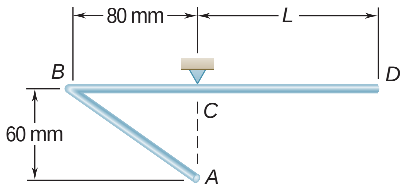


Fig. P5.28 and P5.29

5.29 The homogeneous wire $ABCD$ is bent as shown and is attached to a hinge at C. Determine the length L for which portion AB of the wire is horizontal.

5.30 The homogeneous wire ABC is bent into a semicircular arc and a straight section as shown and is attached to a hinge at A. Determine the value of u for which the wire is in equilibrium for the indicated position.

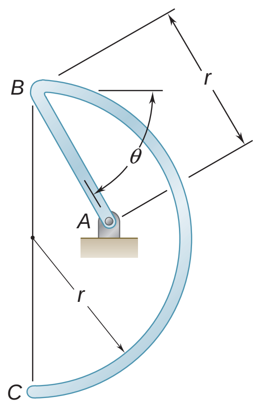


Fig. *P5.30*

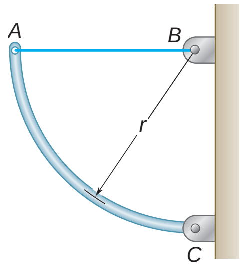


Fig. P5.31

5.31 A uniform circular rod of weight 8 lb and radius 10 in. is attached to a pin at C and to the cable AB. Determine (*a*) the tension in the cable, (*b*) the reaction at C.

5.32 Determine the distance h for which the centroid of the shaded area is as far above line BB' as possible when (*a*) $k = 0.10$, (*b*) $k = 0.80$.

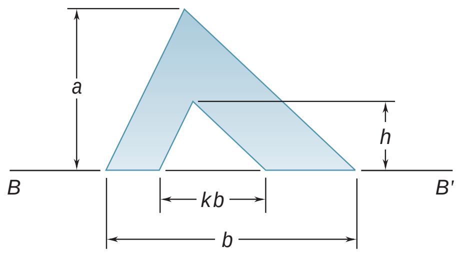


Fig. P5.32 and P5.33

5.33 Knowing that the distance h has been selected to maximize the distance $\bar{y}$ from line BB' to the centroid of the shaded area, show that $\bar{y} = 2h/3$.

5.6 DETERMINATION OF CENTROIDS BY INTEGRATION

The centroid of an area bounded by analytical curves (i.e., curves defined by algebraic equations) is usually determined by evaluating the integrals in Eqs. (5.3) of Sec. 5.3:

$$\bar{x}A = \int x\,dA \qquad \bar{y}A = \int y\,dA \tag{5.3}$$

If the element of area dA is a small rectangle of sides dx and dy, the evaluation of each of these integrals requires a *double integration* with respect to x and y. A double integration is also necessary if polar coordinates are used for which dA is a small element of sides dr and $r\,d\theta$.

In most cases, however, it is possible to determine the coordinates of the centroid of an area by performing a single integration. This is achieved by choosing dA to be a thin rectangle or strip or a thin sector or pie-shaped element (Fig. 5.12); the centroid of the thin rectangle is located at its center, and the centroid of the thin sector is located at a distance $\frac{2}{3}r$ from its vertex (as it is for a triangle). The coordinates of the centroid of the area under consideration are then obtained by expressing that the first moment of the entire area with respect to each of the coordinate axes is equal to the sum (or integral) of the corresponding moments of the elements of area.

Denoting by $\bar{x}_{el}$ and $\bar{y}_{el}$ the coordinates of the centroid of the element dA, we write

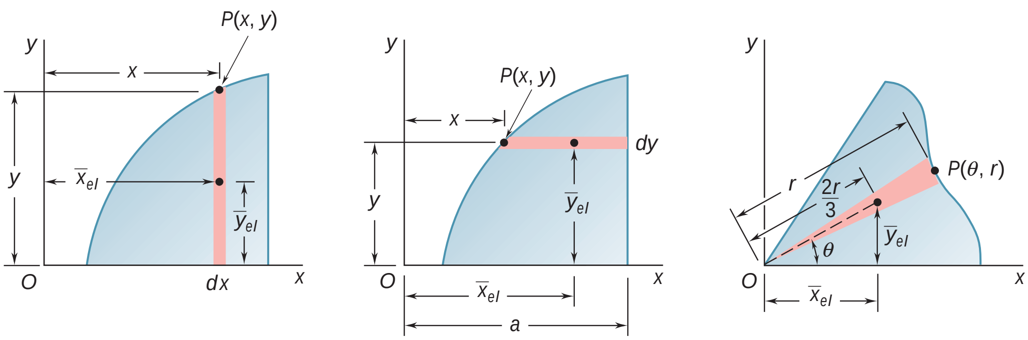
$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA$$
$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA \qquad (5.9)$$

If the area A is not already known, it can also be computed from these elements.

The coordinates $\bar{x}_{el}$ and $\bar{y}_{el}$ of the centroid of the element of area dA should be expressed in terms of the coordinates of a point located on the curve bounding the area under consideration. Also, the area of the element dA should be expressed in terms of the coordinates of that point and the appropriate differentials. This has been done in Fig. 5.12 for three common types of elements; the pie-shaped element of part c should be used when the equation of the curve bounding the area is given in polar coordinates. The appropriate expressions should be substituted into formulas (5.9), and the equation of the bounding curve should be used to express one of the coordinates in terms of the other. The integration is thus reduced to a single integration. Once the area has been determined and the integrals in Eqs. (5.9) have been evaluated, these equations can be solved for the coordinates $\bar{x}$ and $\bar{y}$ of the centroid of the area.

When a line is defined by an algebraic equation, its centroid can be determined by evaluating the integrals in Eqs. (5.4) of Sec. 5.3:

$$\bar{x}L = \int x\, dL \qquad \bar{y}L = \int y\, dL \qquad (5.4)$$



$\bar{x}_{el} = x$ | $\bar{x}_{el} = \dfrac{a+x}{2}$ | $\bar{x}_{el} = \dfrac{2r}{3}\cos\theta$

$\bar{y}_{el} = y/2$ | $\bar{y}_{el} = y$ | $\bar{y}_{el} = \dfrac{2r}{3}\sin\theta$

$dA = y\,dx$ | $dA = (a-x)\,dy$ | $dA = \dfrac{1}{2}r^2\,d\theta$

(a) | (b) | (c)

Fig. 5.12 Centroids and areas of differential elements.

The differential length dL should be replaced by one of the following expressions, depending upon which coordinate, x, y, or θ, is chosen as the independent variable in the equation used to define the line (these expressions can be derived using the Pythagorean theorem):

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

After the equation of the line has been used to express one of the coordinates in terms of the other, the integration can be performed, and Eqs. (5.4) can be solved for the coordinates $\bar{x}$ and $\bar{y}$ of the centroid of the line.

Photo 5.2 The storage tanks shown are all bodies of revolution. Thus, their surface areas and volumes can be determined using the theorems of Pappus-Guldinus.

5.7 THEOREMS OF PAPPUS-GULDINUS

These theorems, which were first formulated by the Greek geometer Pappus during the third century A.D. and later restated by the Swiss mathematician Guldinus, or Guldin, (1577–1643) deal with surfaces and bodies of revolution.

A *surface of revolution* is a surface which can be generated by rotating a plane curve about a fixed axis. For example (Fig. 5.13), the

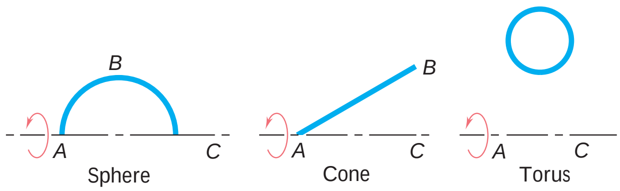


Fig. 5.13

surface of a sphere can be obtained by rotating a semicircular arc ABC about the diameter AC, the surface of a cone can be produced by rotating a straight line AB about an axis AC, and the surface of a torus or ring can be generated by rotating the circumference of a circle about a nonintersecting axis. A *body of revolution* is a body which can be generated by rotating a plane area about a fixed axis. As shown in Fig. 5.14, a sphere, a cone, and a torus can each be generated by rotating the appropriate shape about the indicated axis.

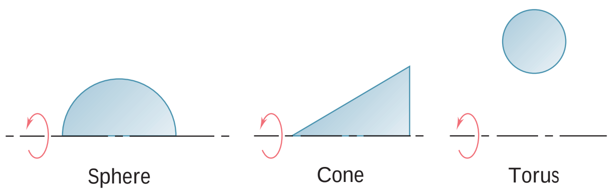


Fig. 5.14

THEOREM I. *The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated.*

Proof. Consider an element dL of the line L (Fig. 5.15), which is revolved about the x axis. The area dA generated by the element

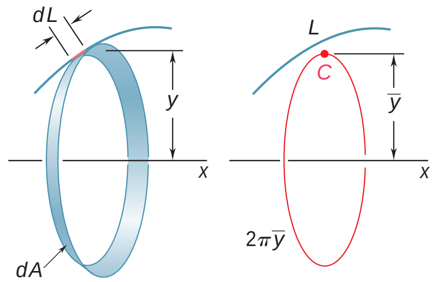


Fig. 5.15

dL is equal to $2\pi y\, dL$. Thus, the entire area generated by L is $A = \int 2\pi y\, dL$. Recalling that we found in Sec. 5.3 that the integral $\int y\, dL$ is equal to $\bar{y}L$, we therefore have

$$A = 2\pi\bar{y}L \tag{5.10}$$

where $2\pi\bar{y}$ is the distance traveled by the centroid of L (Fig. 5.15). It should be noted that the generating curve must not cross the axis about which it is rotated; if it did, the two sections on either side of the axis would generate areas having opposite signs, and the theorem would not apply.

THEOREM II. *The volume of a body of revolution is equal to the generating area times the distance traveled by the centroid of the area while the body is being generated.*

Proof. Consider an element dA of the area A which is revolved about the x axis (Fig. 5.16). The volume dV generated by the element dA is equal to $2\pi y\, dA$. Thus, the entire volume generated by A is $V = \int 2\pi y\, dA$, and since the integral $\int y\, dA$ is equal to $\bar{y}A$ (Sec. 5.3), we have

$$V = 2\pi\bar{y}A \tag{5.11}$$

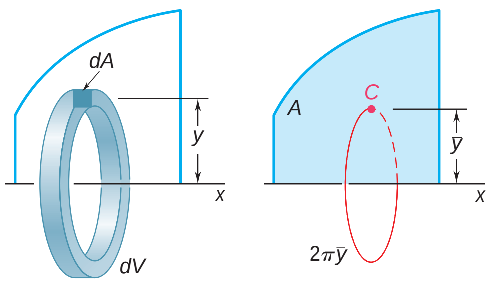
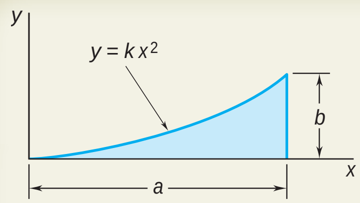


Fig. 5.16

where $2\pi\bar{y}$ is the distance traveled by the centroid of A. Again, it should be noted that the theorem does not apply if the axis of rotation intersects the generating area.

The theorems of Pappus-Guldinus offer a simple way to compute the areas of surfaces of revolution and the volumes of bodies of revolution. Conversely, they can also be used to determine the centroid of a plane curve when the area of the surface generated by the curve is known or to determine the centroid of a plane area when the volume of the body generated by the area is known (see Sample Prob. 5.8).



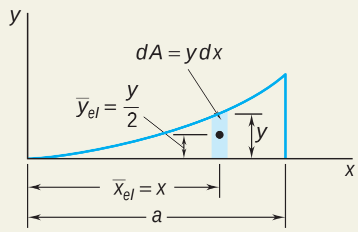
SAMPLE PROBLEM 5.4

Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION

Determination of the Constant k. The value of k is determined by substituting $x = a$ and $y = b$ into the given equation. We have $b = ka^2$ or $k = b/a^2$. The equation of the curve is thus

$$y = \frac{b}{a^2}x^2 \qquad \text{or} \qquad x = \frac{a}{b^{1/2}}y^{1/2}$$



Vertical Differential Element. We choose the differential element shown and find the total area of the figure.

$$A = \int dA = \int y\,dx = \int_0^a \frac{b}{a^2}x^2\,dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a = \frac{ab}{3}$$

The first moment of the differential element with respect to the y axis is $\bar{x}_{el}\,dA$; hence, the first moment of the entire area with respect to this axis is

$$Q_y = \int \bar{x}_{el}\,dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx = \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2b}{4}$$

Since $Q_y = \bar{x}A$, we have

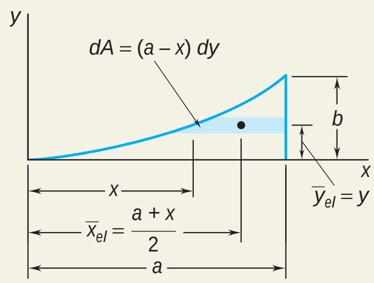
$$\bar{x}A = \int \bar{x}_{el}\,dA \qquad \bar{x}\frac{ab}{3} = \frac{a^2b}{4} \qquad \bar{x} = \tfrac{3}{4}a \quad \blacktriangleleft$$

Likewise, the first moment of the differential element with respect to the x axis is $\bar{y}_{el}\,dA$, and the first moment of the entire area is

$$Q_x = \int \bar{y}_{el}\,dA = \int \frac{y}{2}y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx = \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Since $Q_x = \bar{y}A$, we have

$$\bar{y}A = \int \bar{y}_{el}\,dA \qquad \bar{y}\frac{ab}{3} = \frac{ab^2}{10} \qquad \bar{y} = \tfrac{3}{10}b \quad \blacktriangleleft$$



Horizontal Differential Element. The same results can be obtained by considering a horizontal element. The first moments of the area are

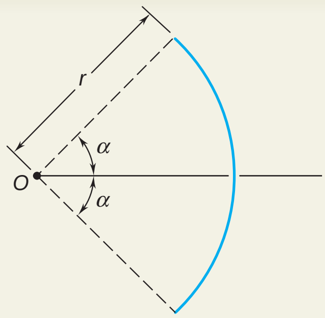
$$Q_y = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}(a-x)\,dy = \int_0^b \frac{a^2-x^2}{2}dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right)dy = \frac{a^2b}{4}$$

$$Q_x = \int \bar{y}_{el}\,dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right)dy$$

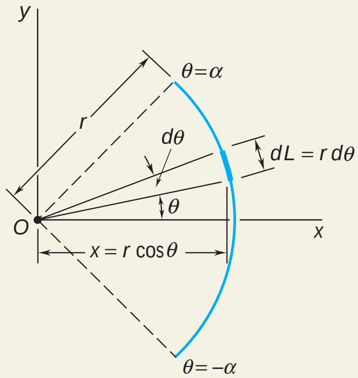
$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right)dy = \frac{ab^2}{10}$$

To determine $\bar{x}$ and $\bar{y}$, the expressions obtained are again substituted into the equations defining the centroid of the area.



SAMPLE PROBLEM 5.5

Determine the location of the centroid of the arc of circle shown.



SOLUTION

Since the arc is symmetrical with respect to the x axis, $\bar{y} = 0$. A differential element is chosen as shown, and the length of the arc is determined by integration.

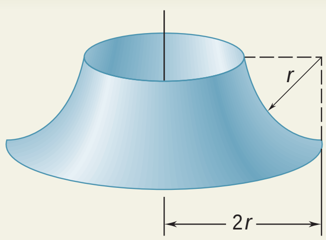
$$L = \int dL = \int_{-\alpha}^{\alpha} r\,d\theta = r\int_{-\alpha}^{\alpha} d\theta = 2r\alpha$$

The first moment of the arc with respect to the y axis is

$$Q_y = \int x\,dL = \int_{-\alpha}^{\alpha} (r\cos\theta)(r\,d\theta) = r^2\int_{-\alpha}^{\alpha} \cos\theta\,d\theta$$
$$= r^2[\sin\theta]_{-\alpha}^{\alpha} = 2r^2\sin\alpha$$

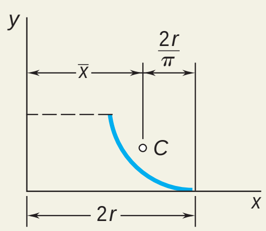
Since $Q_y = \bar{x}L$, we write

$$\bar{x}(2r\alpha) = 2r^2\sin\alpha \qquad \bar{x} = \frac{r\sin\alpha}{\alpha} \blacktriangleleft$$



SAMPLE PROBLEM 5.6

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.



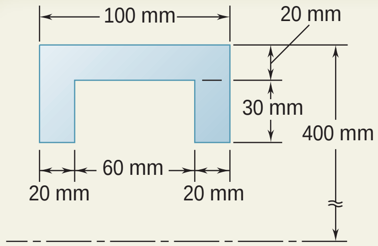
SOLUTION

According to Theorem I of Pappus-Guldinus, the area generated is equal to the product of the length of the arc and the distance traveled by its centroid. Referring to Fig. 5.8B, we have

$$\bar{x} = 2r - \frac{2r}{\pi} = 2r\left(1 - \frac{1}{\pi}\right)$$

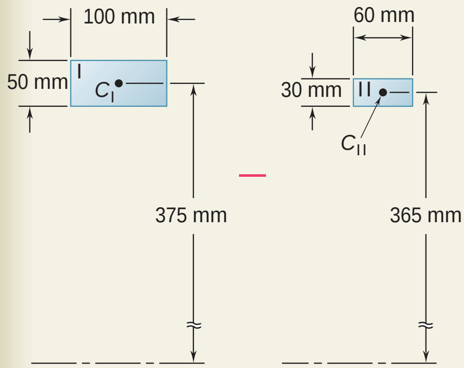
$$A = 2\pi\bar{x}L = 2\pi\left[2r\left(1 - \frac{1}{\pi}\right)\right]\left(\frac{\pi r}{2}\right)$$

$$A = 2\pi r^2(\pi - 1) \blacktriangleleft$$



SAMPLE PROBLEM 5.7

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85 \times 10^3$ kg/m^3, determine the mass and the weight of the rim.



SOLUTION

The volume of the rim can be found by applying Theorem II of Pappus-Guldinus, which states that the volume equals the product of the given cross-sectional area and the distance traveled by its centroid in one complete revolution. However, the volume can be more easily determined if we observe that the cross section can be formed from rectangle I, whose area is positive, and rectangle II, whose area is negative.

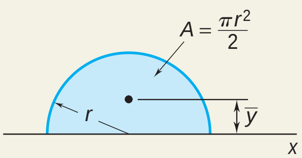
	Area, mm^2	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm^3
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

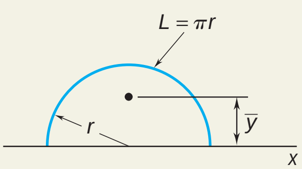
Since 1 mm $= 10^{-3}$ m, we have 1 mm$^3 = (10^{-3}$ m$)^3 = 10^{-9}$ m^3, and we obtain $V = 7.65 \times 10^6$ mm$^3 = (7.65 \times 10^6)(10^{-9}$ m$^3) = 7.65 \times 10^{-3}$ m^3.

$$m = \rho V = (7.85 \times 10^3 \text{ kg/m}^3)(7.65 \times 10^{-3} \text{ m}^3) \qquad m = 60.0 \text{ kg} ◀$$
$$W = mg = (60.0 \text{ kg})(9.81 \text{ m/s}^2) = 589 \text{ kg} \cdot \text{m/s}^2 \qquad W = 589 \text{ N} ◀$$

SAMPLE PROBLEM 5.8

Using the theorems of Pappus-Guldinus, determine (*a*) the centroid of a semicircular area, (*b*) the centroid of a semicircular arc. We recall that the volume and the surface area of a sphere are $\frac{4}{3}\pi r^3$ and $4\pi r^2$, respectively.





SOLUTION

The volume of a sphere is equal to the product of the area of a semicircle and the distance traveled by the centroid of the semicircle in one revolution about the x axis.

$$V = 2\pi\bar{y}A \qquad \tfrac{4}{3}\pi r^3 = 2\pi\bar{y}(\tfrac{1}{2}\pi r^2) \qquad \bar{y} = \frac{4r}{3\pi} ◀$$

Likewise, the area of a sphere is equal to the product of the length of the generating semicircle and the distance traveled by its centroid in one revolution.

$$A = 2\pi\bar{y}L \qquad 4\pi r^2 = 2\pi\bar{y}(\pi r) \qquad \bar{y} = \frac{2r}{\pi} ◀$$

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will use the equations

$$\bar{x}A = \int x\, dA \qquad \bar{y}A = \int y\, dA \tag{5.3}$$

$$\bar{x}L = \int x\, dL \qquad \bar{y}L = \int y\, dL \tag{5.4}$$

to locate the centroids of plane areas and lines, respectively. You will also apply the theorems of Pappus-Guldinus (Sec. 5.7) to determine the areas of surfaces of revolution and the volumes of bodies of revolution.

1. Determining by direct integration the centroids of areas and lines. When solving problems of this type, you should follow the method of solution shown in Sample Probs. 5.4 and 5.5: compute A or L, determine the first moments of the area or the line, and solve Eqs. (5.3) or (5.4) for the coordinates of the centroid. In addition, you should pay particular attention to the following points.

a. Begin your solution by carefully defining or determining each term in the applicable integral formulas. We strongly encourage you to show on your sketch of the given area or line your choice for dA or dL and the distances to its centroid.

b. As explained in Sec. 5.6, the x and the y in the above equations represent the *coordinates of the centroid* of the differential elements dA and dL. It is important to recognize that the coordinates of the centroid of dA are not equal to the coordinates of a point located on the curve bounding the area under consideration. You should carefully study Fig. 5.12 until you fully understand this important point.

c. To possibly simplify or minimize your computations, always examine the shape of the given area or line before defining the differential element that you will use. For example, sometimes it may be preferable to use horizontal rectangular elements instead of vertical ones. Also, it will usually be advantageous to use polar coordinates when a line or an area has circular symmetry.

d. Although most of the integrations in this lesson are straightforward, at times it may be necessary to use more advanced techniques, such as trigonometric substitution or integration by parts. Of course, using a table of integrals is the fastest method to evaluate difficult integrals.

2. Applying the theorems of Pappus-Guldinus. As shown in Sample Probs. 5.6 through 5.8, these simple, yet very useful theorems allow you to apply your knowledge of centroids to the computation of areas and volumes. Although the theorems refer to the distance traveled by the centroid and to the length of the generating curve or to the generating area, the resulting equations [Eqs. (5.10) and (5.11)] contain the products of these quantities, which are simply the first moments of a line ($\bar{y}L$) and an area ($\bar{y}A$), respectively. Thus, for those problems for which the generating line or area consists of more than one common shape, you need only determine $\bar{y}L$ or $\bar{y}A$; you do not have to calculate the length of the generating curve or the generating area.

PROBLEMS

5.34 through 5.36 Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

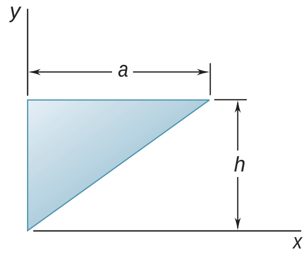


Fig. P5.34

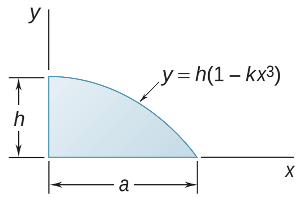


Fig. *P5.35*

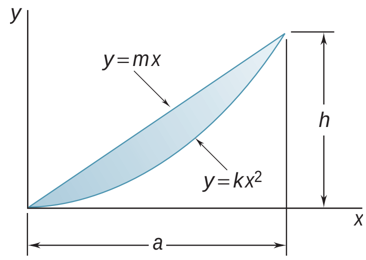


Fig. P5.36

5.37 through *5.39* Determine by direct integration the centroid of the area shown.

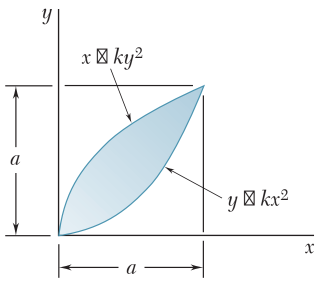


Fig. P5.37

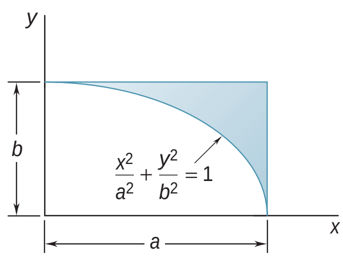


Fig. P5.38

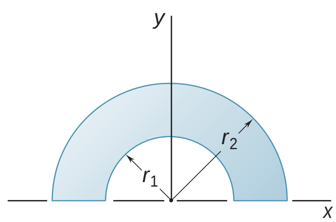


Fig. *P5.39*

5.40 and 5.41 Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

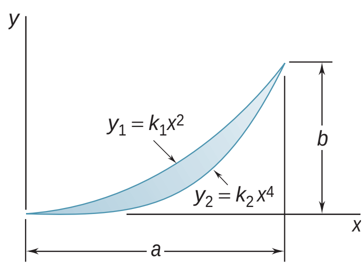


Fig. P5.40

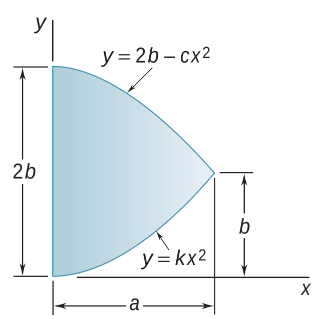


Fig. P5.41

5.42 Determine by direct integration the centroid of the area shown.

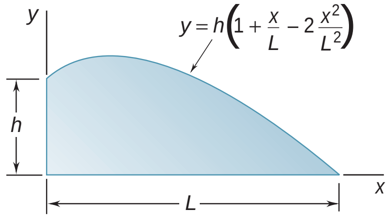


Fig. P5.42

5.43 and 5.44 Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

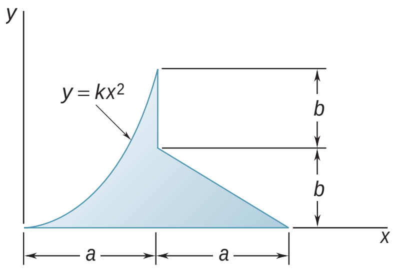


Fig. P5.43

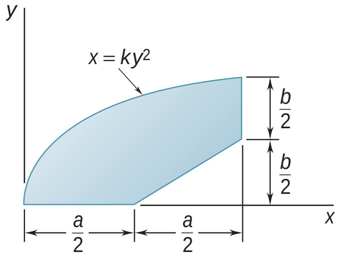


Fig. P5.44

5.45 and 5.46 A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid.

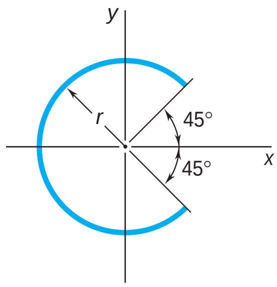


Fig. P5.45

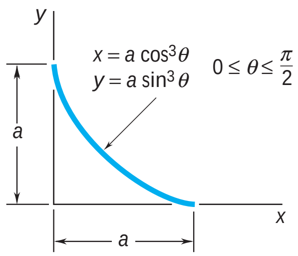


Fig. P5.46

***5.47** A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid. Express your answer in terms of a.

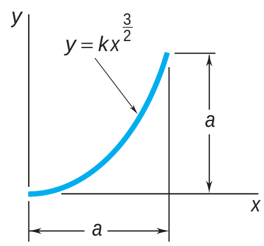


Fig. P5.47

***5.48 and *5.49** Determine by direct integration the centroid of the area shown.

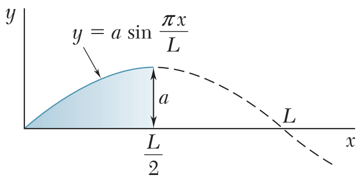


Fig. P5.48

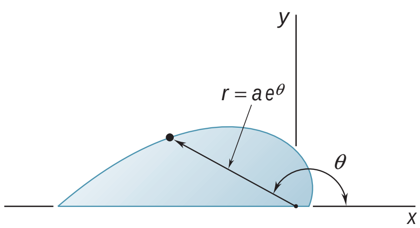


Fig. P5.49

5.50 Determine the centroid of the area shown when $a = 2$ in.

5.51 Determine the value of a for which the ratio $\bar{x}/\bar{y}$ is 9.

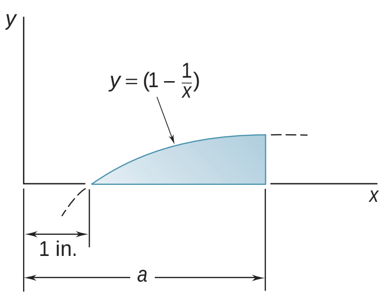


Fig. P5.50 and P5.51

5.52 Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.1 about (a) the x axis, (b) the y axis.

5.53 Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.2 about (*a*) the line $y = 72$ mm, (*b*) the x axis.

5.54 Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.8 about (*a*) the line $x = -60$ mm, (*b*) the line $y = 120$ mm.

5.55 Determine the volume of the solid generated by rotating the parabolic area shown about (*a*) the x axis, (*b*) the axis AA'.

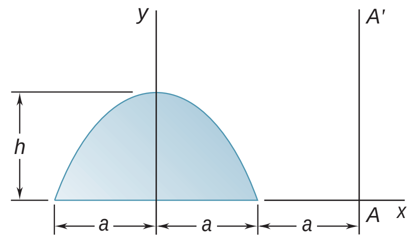


Fig. ***P5.55***

5.56 Determine the volume and the surface area of the chain link shown, which is made from a 6-mm-diameter bar, if $R = 10$ mm and $L = 30$ mm.

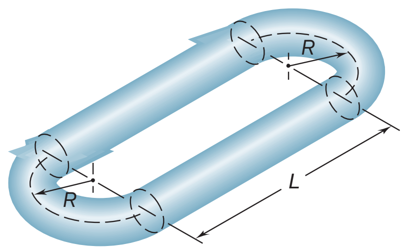


Fig. P5.56

5.57 Verify that the expressions for the volumes of the first four shapes in Fig. 5.21 on page 260 are correct.

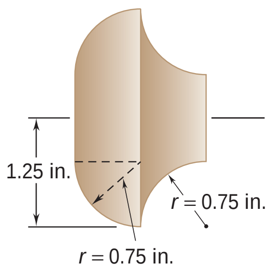


Fig. P5.58 and ***P5.59***

5.58 Determine the volume and weight of the solid brass knob shown, knowing that the specific weight of brass is 0.306 lb/in^3.

5.59 Determine the total surface area of the solid brass knob shown.

5.60 The aluminum shade for the small high-intensity lamp shown has a uniform thickness of 1 mm. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass of the shade.

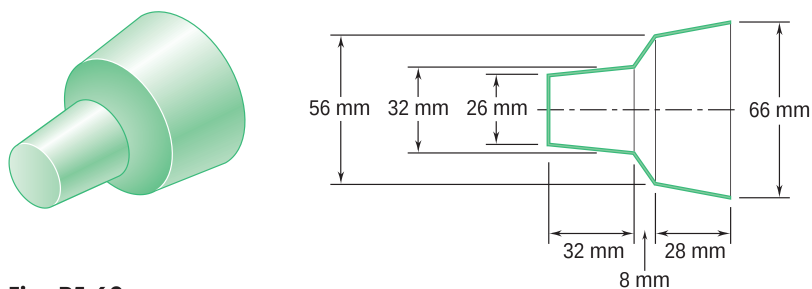


Fig. P5.60

5.61 The escutcheon (a decorative plate placed on a pipe where the pipe exits from a wall) shown is cast from brass. Knowing that the density of brass is 8470 kg/m^3, determine the mass of the escutcheon.

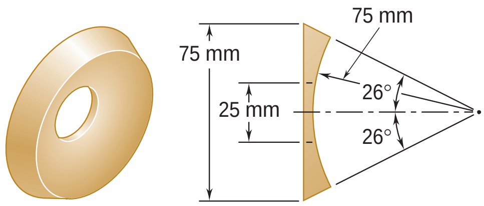


Fig. *P5.61*

5.62 A $\frac{3}{4}$-in.-diameter hole is drilled in a piece of 1-in.-thick steel; the hole is then countersunk as shown. Determine the volume of steel removed during the countersinking process.

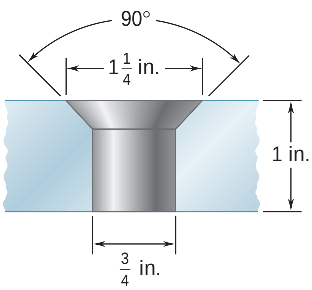


Fig. P5.62

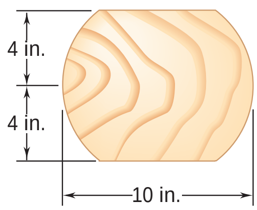


Fig. P5.63

5.63 Knowing that two equal caps have been removed from a 10-in.-diameter wooden sphere, determine the total surface area of the remaining portion.

5.64 Determine the capacity, in liters, of the punch bowl shown if $R = 250$ mm.

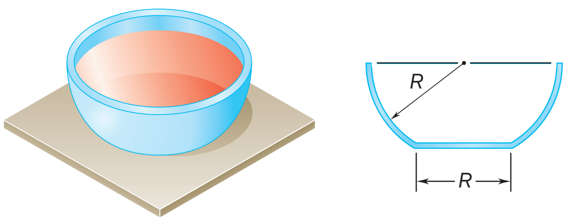


Fig. P5.64

***5.65** The shade for a wall-mounted light is formed from a thin sheet of translucent plastic. Determine the surface area of the outside of the shade, knowing that it has the parabolic cross section shown.

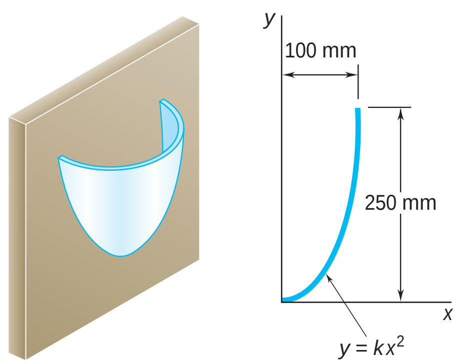


Fig. *P5.65*

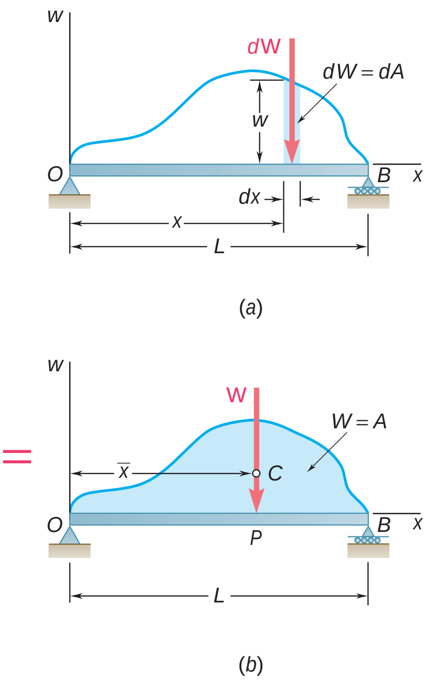


Fig. 5.17

*5.8 DISTRIBUTED LOADS ON BEAMS

The concept of the centroid of an area can be used to solve other problems besides those dealing with the weights of flat plates. Consider, for example, a beam supporting a *distributed load;* this load may consist of the weight of materials supported directly or indirectly by the beam, or it may be caused by wind or hydrostatic pressure. The distributed load can be represented by plotting the load w supported per unit length (Fig. 5.17); this load is expressed in N/m or in lb/ft. The magnitude of the force exerted on an element of beam of length dx is $dW = w\,dx$, and the total load supported by the beam is

$$W = \int_0^L w\,dx$$

We observe that the product $w\,dx$ is equal in magnitude to the element of area dA shown in Fig. 5.17*a*. The load W is thus equal in magnitude to the total area A under the load curve:

$$W = \int dA = A$$

We now determine where a *single concentrated load* **W**, of the same magnitude W as the total distributed load, should be applied on the beam if it is to produce the same reactions at the supports (Fig. 5.17*b*). However, this concentrated load **W**, which represents the resultant of the given distributed loading, is equivalent to the loading only when considering the free-body diagram of the entire beam. The point of application P of the equivalent concentrated load **W** is obtained by expressing that the moment of **W** about point O is equal to the sum of the moments of the elemental loads $d\mathbf{W}$ about O:

$$(OP)W = \int x\,dW$$

or, since $dW = w\,dx = dA$ and $W = A$,

$$(OP)A = \int_0^L x\,dA \tag{5.12}$$

Since the integral represents the first moment with respect to the w axis of the area under the load curve, it can be replaced by the product $\bar{x}A$. We therefore have $OP = \bar{x}$, where $\bar{x}$ is the distance from the w axis to the centroid C of the area A (this is *not* the centroid of the beam).

A distributed load on a beam can thus be replaced by a concentrated load; the magnitude of this single load is equal to the area under the load curve, and its line of action passes through the centroid of that area. It should be noted, however, that the concentrated load is equivalent to the given loading only as far as external forces are concerned. It can be used to determine reactions but should not be used to compute internal forces and deflections.

Photo 5.3 The roofs of the buildings shown must be able to support not only the total weight of the snow but also the nonsymmetric distributed loads resulting from drifting of the snow.

*5.9 FORCES ON SUBMERGED SURFACES

The approach used in the preceding section can be used to determine the resultant of the hydrostatic pressure forces exerted on a *rectangular surface* submerged in a liquid. Consider the rectangular plate shown in Fig. 5.18, which is of length L and width b, where b is measured perpendicular to the plane of the figure. As noted in Sec. 5.8, the load exerted on an element of the plate of length dx is $w\,dx$, where w is the load per unit length. However, this load can also be expressed as $p\,dA = pb\,dx$, where p is the gage pressure in the liquid† and b is the width of the plate; thus, $w = bp$. Since the gage pressure in a liquid is $p = \gamma h$, where γ is the specific weight of the liquid and h is the vertical distance from the free surface, it follows that

$$w = bp = b\gamma h \tag{5.13}$$

which shows that the load per unit length w is proportional to h and, thus, varies linearly with x.

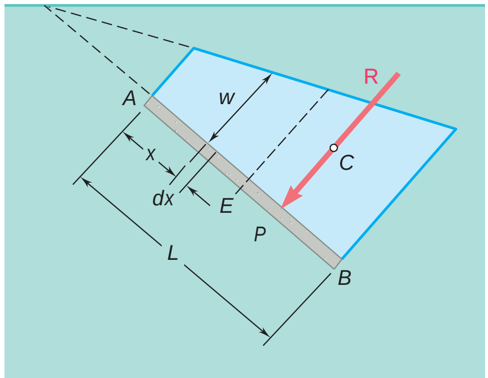
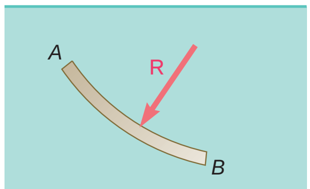


Fig. 5.18

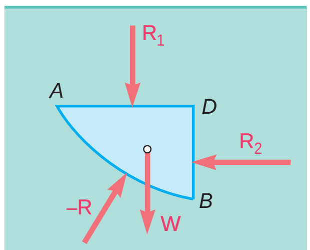
Recalling the results of Sec. 5.8, we observe that the resultant **R** of the hydrostatic forces exerted on one side of the plate is equal in magnitude to the trapezoidal area under the load curve and that its line of action passes through the centroid C of that area. The point P of the plate where **R** is applied is known as the *center of pressure*.‡

Next, we consider the forces exerted by a liquid on a curved surface of constant width (Fig. 5.19*a*). Since the determination of the resultant **R** of these forces by direct integration would not be easy, we consider the free body obtained by detaching the volume of liquid ABD bounded by the curved surface AB and by the two plane surfaces AD and DB shown in Fig. 5.19*b*. The forces acting on the free body ABD are the weight **W** of the detached volume of liquid, the resultant $\mathbf{R}_1$ of the forces exerted on AD, the resultant $\mathbf{R}_2$ of the forces exerted on BD, and the resultant $-\mathbf{R}$ of the forces exerted *by the curved surface on the liquid*. The resultant $-\mathbf{R}$ is equal and opposite to, and has the same line of action as, the resultant **R** of the forces exerted *by the liquid on the curved surface*. The forces **W**, $\mathbf{R}_1$, and $\mathbf{R}_2$ can be determined by standard methods; after their values have been found, the force $-\mathbf{R}$ is obtained by solving the equations of equilibrium for the free body of Fig. 5.19*b*. The resultant **R** of the hydrostatic forces exerted on the curved surface is then obtained by reversing the sense of $-\mathbf{R}$.

The methods outlined in this section can be used to determine the resultant of the hydrostatic forces exerted on the surfaces of dams and rectangular gates and vanes. The resultants of forces on submerged surfaces of variable width will be determined in Chap. 9.



(*a*)



(*b*)

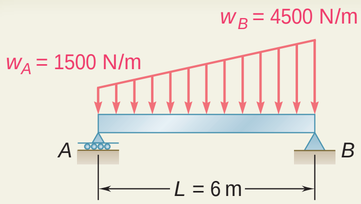
Fig. 5.19

†The pressure p, which represents a load per unit area, is expressed in N/m^2 or in lb/ft^2. The derived SI unit N/m^2 is called a *pascal* (Pa).

‡Noting that the area under the load curve is equal to $w_E L$, where w_E is the load per unit length at the center E of the plate, and recalling Eq. (5.13), we can write

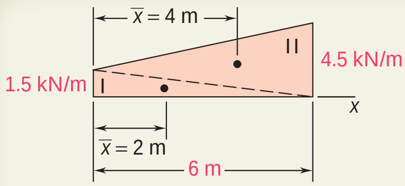
$$R = w_E L = (bp_E)L = p_E(bL) = p_E A$$

where A denotes the area of the *plate*. Thus, the magnitude of **R** can be obtained by multiplying the area of the plate by the pressure at its center E. The resultant **R**, however, *should be applied at P, not at E*.



SAMPLE PROBLEM 5.9

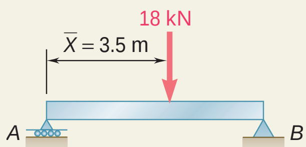
A beam supports a distributed load as shown. (*a*) Determine the equivalent concentrated load. (*b*) Determine the reactions at the supports.



SOLUTION

a. Equivalent Concentrated Load. The magnitude of the resultant of the load is equal to the area under the load curve, and the line of action of the resultant passes through the centroid of the same area. We divide the area under the load curve into two triangles and construct the table below. To simplify the computations and tabulation, the given loads per unit length have been converted into kN/m.

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$



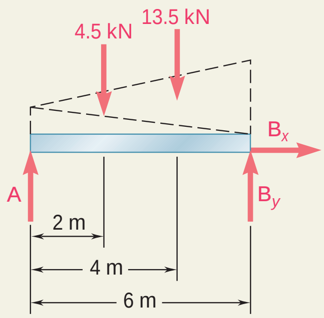
Thus, $\bar{X}\Sigma A = \Sigma \bar{x}A$: $\quad \bar{X}(18 \text{ kN}) = 63 \text{ kN} \cdot \text{m} \qquad \bar{X} = 3.5 \text{ m}$

The equivalent concentrated load is

$$\mathbf{W} = 18 \text{ kN w} \quad ◀$$

and its line of action is located at a distance

$$\bar{X} = 3.5 \text{ m to the right of } A \quad ◀$$



b. Reactions. The reaction at A is vertical and is denoted by $\mathbf{A}$; the reaction at B is represented by its components $\mathbf{B}_x$ and $\mathbf{B}_y$. The given load can be considered to be the sum of two triangular loads as shown. The resultant of each triangular load is equal to the area of the triangle and acts at its centroid. We write the following equilibrium equations for the free body shown:

$\overset{+}{\text{y}}\ \Sigma F_x = 0$: $\qquad \mathbf{B}_x = 0 \quad ◀$

$+1\ \Sigma M_A = 0$: $\qquad -(4.5 \text{ kN})(2 \text{ m}) - (13.5 \text{ kN})(4 \text{ m}) + B_y(6 \text{ m}) = 0$

$$\mathbf{B}_y = 10.5 \text{ kN x} \quad ◀$$

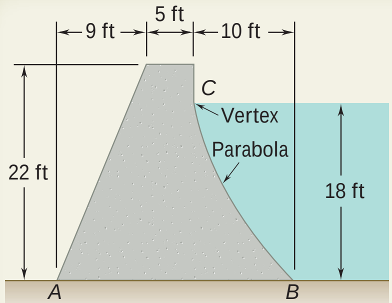
$+1\ \Sigma M_B = 0$: $\qquad +(4.5 \text{ kN})(4 \text{ m}) + (13.5 \text{ kN})(2 \text{ m}) - A(6 \text{ m}) = 0$

$$\mathbf{A} = 7.5 \text{ kN x} \quad ◀$$

Alternative Solution. The given distributed load can be replaced by its resultant, which was found in part *a*. The reactions can be determined by writing the equilibrium equations $\Sigma F_x = 0$, $\Sigma M_A = 0$, and $\Sigma M_B = 0$. We again obtain

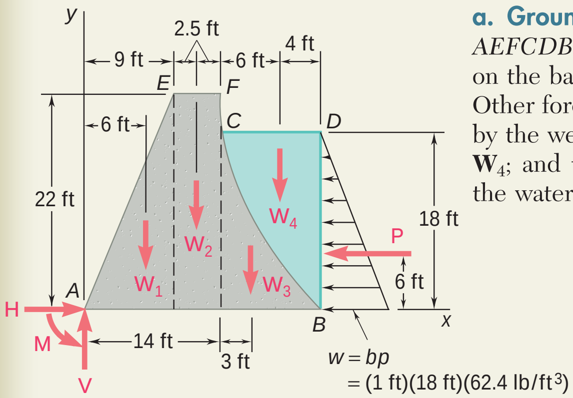
$$\mathbf{B}_x = 0 \qquad \mathbf{B}_y = 10.5 \text{ kN x} \qquad \mathbf{A} = 7.5 \text{ kN x} \quad ◀$$

SAMPLE PROBLEM 5.10



The cross section of a concrete dam is as shown. Consider a 1-ft-thick section of the dam, and determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam. The specific weights of concrete and water are 150 lb/ft^3 and 62.4 lb/ft^3, respectively.

SOLUTION



a. Ground Reaction. We choose as a free body the 1-ft-thick section *AEFCDB* of the dam and water. The reaction forces exerted by the ground on the base *AB* are represented by an equivalent force-couple system at *A*. Other forces acting on the free body are the weight of the dam, represented by the weights of its components $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_3$; the weight of the water $\mathbf{W}_4$; and the resultant **P** of the pressure forces exerted on section *BD* by the water to the right of section *BD*. We have

$$W_1 = \tfrac{1}{2}(9\text{ ft})(22\text{ ft})(1\text{ ft})(150\text{ lb/ft}^3) = 14{,}850\text{ lb}$$
$$W_2 = (5\text{ ft})(22\text{ ft})(1\text{ ft})(150\text{ lb/ft}^3) = 16{,}500\text{ lb}$$
$$W_3 = \tfrac{1}{3}(10\text{ ft})(18\text{ ft})(1\text{ ft})(150\text{ lb/ft}^3) = 9000\text{ lb}$$
$$W_4 = \tfrac{2}{3}(10\text{ ft})(18\text{ ft})(1\text{ ft})(62.4\text{ lb/ft}^3) = 7488\text{ lb}$$
$$P = \tfrac{1}{2}(18\text{ ft})(1\text{ ft})(18\text{ ft})(62.4\text{ lb/ft}^3) = 10{,}109\text{ lb}$$

Equilibrium Equations

$\overset{+}{\text{y}}\ \Sigma F_x = 0$: $\quad H - 10{,}109\text{ lb} = 0 \qquad$ **H** $= 10{,}110$ lb y ◀

$+\text{x}\ \Sigma F_y = 0$: $\quad V - 14{,}850\text{ lb} - 16{,}500\text{ lb} - 9000\text{ lb} - 7488\text{ lb} = 0$

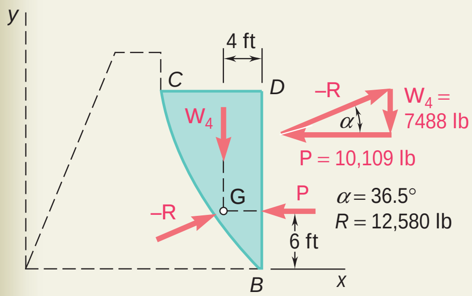
V $= 47{,}840$ lb x ◀

$+\text{l}\ \Sigma M_A = 0$: $\quad -(14{,}850\text{ lb})(6\text{ ft}) - (16{,}500\text{ lb})(11.5\text{ ft})$
$- (9000\text{ lb})(17\text{ ft}) - (7488\text{ lb})(20\text{ ft}) + (10{,}109\text{ lb})(6\text{ ft}) + M = 0$

M $= 520{,}960$ lb · ft l ◀

We can replace the force-couple system obtained by a single force acting at a distance *d* to the right of *A*, where

$$d = \frac{520{,}960\text{ lb}\cdot\text{ft}}{47{,}840\text{ lb}} = 10.89\text{ ft}$$



b. Resultant R of Water Forces. The parabolic section of water *BCD* is chosen as a free body. The forces involved are the resultant $-\mathbf{R}$ of the forces exerted by the dam on the water, the weight $\mathbf{W}_4$, and the force **P**. Since these forces must be concurrent, $-\mathbf{R}$ passes through the point of intersection *G* of $\mathbf{W}_4$ and **P**. A force triangle is drawn from which the magnitude and direction of $-\mathbf{R}$ are determined. The resultant **R** of the forces exerted by the water on the face *BC* is equal and opposite:

R $= 12{,}580$ lb d $36.5°$ ◀

SOLVING PROBLEMS ON YOUR OWN

The problems in this lesson involve two common and very important types of loading: distributed loads on beams and forces on submerged surfaces of constant width. As we discussed in Secs. 5.8 and 5.9 and illustrated in Sample Probs. 5.9 and 5.10, determining the single equivalent force for each of these loadings requires a knowledge of centroids.

1. Analyzing beams subjected to distributed loads. In Sec. 5.8, we showed that a distributed load on a beam can be replaced by a single equivalent force. The magnitude of this force is equal to the area under the distributed load curve and its line of action passes through the centroid of that area. Thus, you should begin your solution by replacing the various distributed loads on a given beam by their respective single equivalent forces. The reactions at the supports of the beam can then be determined by using the methods of Chap. 4.

When possible, complex distributed loads should be divided into the common-shape areas shown in Fig. 5.8A [Sample Prob. 5.9]. Each of these areas can then be replaced by a single equivalent force. If required, the system of equivalent forces can be reduced further to a single equivalent force. As you study Sample Prob. 5.9, note how we have used the analogy between force and area and the techniques for locating the centroid of a composite area to analyze a beam subjected to a distributed load.

2. Solving problems involving forces on submerged bodies. The following points and techniques should be remembered when solving problems of this type.

a. The pressure p at a depth h below the free surface of a liquid is equal to $\mathsf{g}h$ or $\mathsf{r}gh$, where g and r are the specific weight and the density of the liquid, respectively. The load per unit length w acting on a submerged surface of constant width b is then

$$w = bp = b\mathsf{g}h = b\mathsf{r}gh$$

b. The line of action of the resultant force $\mathbf{R}$ acting on a submerged plane surface is perpendicular to the surface.

c. For a vertical or inclined plane rectangular surface of width b, the loading on the surface can be represented by a linearly distributed load which is trapezoidal in shape (Fig. 5.18). Further, the magnitude of $\mathbf{R}$ is given by

$$R = \mathsf{g}h_E A$$

where h_E is the vertical distance to the center of the surface and A is the area of the surface.

d. The load curve will be triangular (rather than trapezoidal) when the top edge of a plane rectangular surface coincides with the free surface of the liquid, since the pressure of the liquid at the free surface is zero. For this case, the line of action of **R** is easily determined, for it passes through the centroid of a *triangular* distributed load.

e. For the general case, rather than analyzing a trapezoid, we suggest that you use the method indicated in part *b* of Sample Prob. 5.9. First divide the trapezoidal distributed load into two triangles, and then compute the magnitude of the resultant of each triangular load. (The magnitude is equal to the area of the triangle times the width of the plate.) Note that the line of action of each resultant force passes through the centroid of the corresponding triangle and that the sum of these forces is equivalent to **R**. Thus, rather than using **R**, you can use the two equivalent resultant forces, whose points of application are easily calculated. Of course, the equation given for R in paragraph *c* should be used when only the magnitude of **R** is needed.

f. When the submerged surface of constant width is curved, the resultant force acting on the surface is obtained by considering the equilibrium of the volume of liquid bounded by the curved surface and by horizontal and vertical planes (Fig. 5.19). Observe that the force $\mathbf{R}_1$ of Fig. 5.19 is equal to the weight of the liquid lying above the plane AD. The method of solution for problems involving curved surfaces is shown in part *b* of Sample Prob. 5.10.

In subsequent mechanics courses (in particular, mechanics of materials and fluid mechanics), you will have ample opportunity to use the ideas introduced in this lesson.

PROBLEMS

5.66 and 5.67 For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

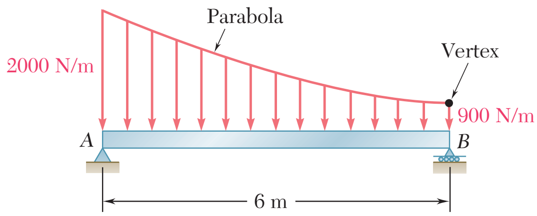


Fig. P5.66

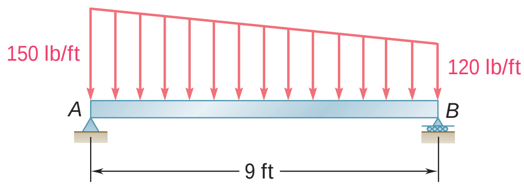


Fig. P5.67

5.68 through 5.73 Determine the reactions at the beam supports for the given loading.

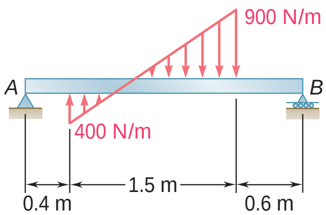


Fig. P5.68

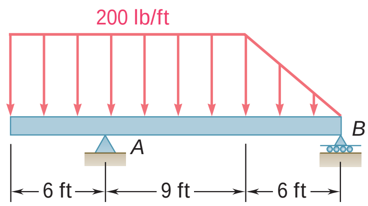


Fig. P5.69

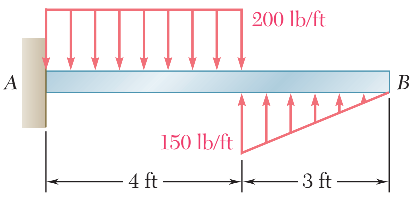


Fig. *P5.70*

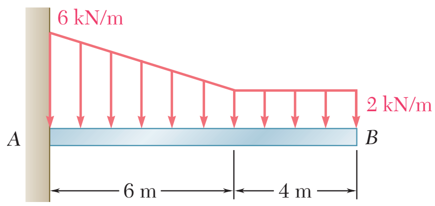


Fig. P5.71

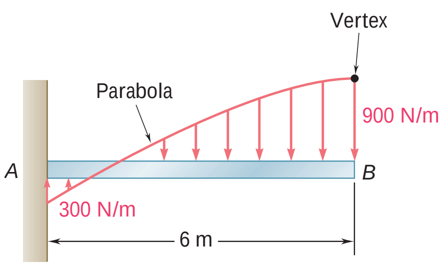


Fig. P5.72

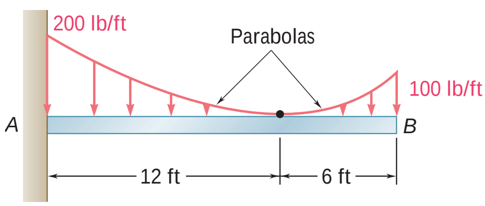


Fig. *P5.73*

5.74 Determine the reactions at the beam supports for the given loading when $w_0 = 400$ lb/ft.

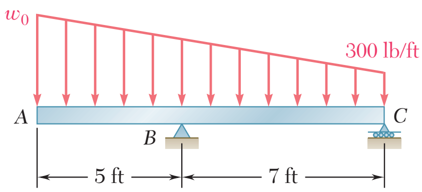


Fig. P5.74 and P5.75

5.75 Determine (*a*) the distributed load w_0 at the end A of the beam ABC for which the reaction at C is zero, (*b*) the corresponding reaction at B.

5.76 Determine (*a*) the distance a so that the vertical reactions at supports A and B are equal, (*b*) the corresponding reactions at the supports.

5.77 Determine (*a*) the distance a so that the reaction at support B is minimum, (*b*) the corresponding reactions at the supports.

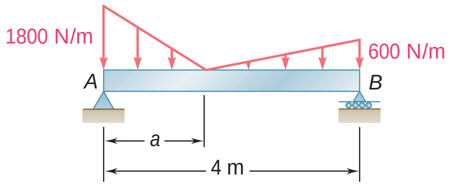


Fig. P5.76 and *P5.77*

5.78 A beam is subjected to a linearly distributed downward load and rests on two wide supports BC and DE, which exert uniformly distributed upward loads as shown. Determine the values of w_{BC} and w_{DE} corresponding to equilibrium when $w_A = 600$ N/m.

5.79 A beam is subjected to a linearly distributed downward load and rests on two wide supports BC and DE, which exert uniformly distributed upward loads as shown. Determine (*a*) the value of w_A so that $w_{BC} = w_{DE}$, (*b*) the corresponding values of w_{BC} and w_{DE}.

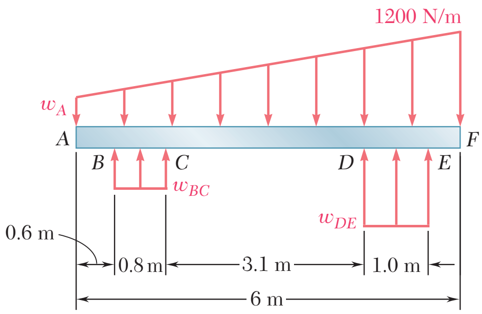


Fig. P5.78 and *P5.79*

In the following problems, use g = 62.4 lb/ft³ for the specific weight of fresh water and g_c = 150 lb/ft³ for the specific weight of concrete if U.S. customary units are used. With SI units, use r = 10^3 kg/m³ for the density of fresh water and r_c = 2.40×10^3 kg/m³ for the density of concrete. (See the footnote on page 222 for how to determine the specific weight of a material given its density.)

5.80 and 5.81 The cross section of a concrete dam is as shown. For a 1-m-wide dam section determine (*a*) the resultant of the reaction forces exerted by the ground on the base AB of the dam, (*b*) the point of application of the resultant of part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face BC of the dam.

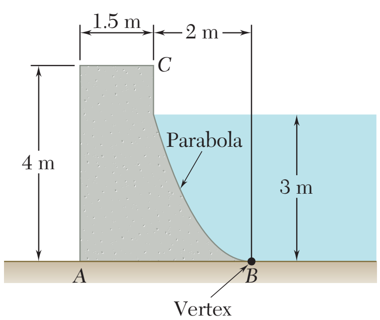


Fig. P5.80

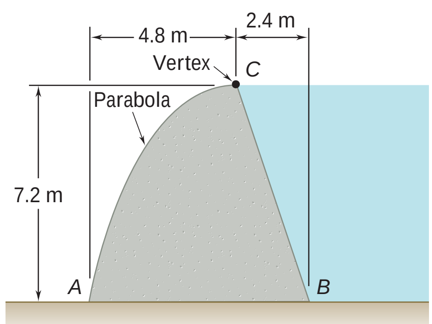


Fig. P5.81

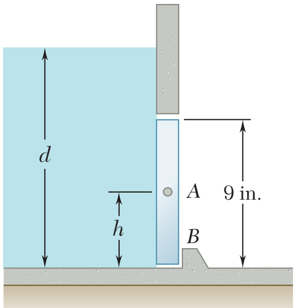


Fig. P5.82 and P5.83

5.82 An automatic valve consists of a 9 × 9-in. square plate that is pivoted about a horizontal axis through A located at a distance $h = 3.6$ in. above the lower edge. Determine the depth of water d for which the valve will open.

5.83 An automatic valve consists of a 9 × 9-in. square plate that is pivoted about a horizontal axis through A. If the valve is to open when the depth of water is $d = 18$ in., determine the distance h from the bottom of the valve to the pivot A.

5.84 The 3 × 4-m side AB of a tank is hinged at its bottom A and is held in place by a thin rod BC. The maximum tensile force the rod can withstand without breaking is 200 kN, and the design specifications require the force in the rod not to exceed 20 percent of this value. If the tank is slowly filled with water, determine the maximum allowable depth of water d in the tank.

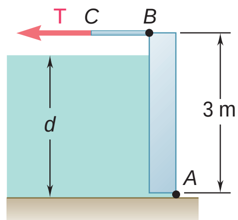


Fig. *P5.84* and P5.85

5.85 The 3 × 4-m side of an open tank is hinged at its bottom A and is held in place by a thin rod BC. The tank is to be filled with glycerine, whose density is 1263 kg/m^3. Determine the force **T** in the rod and the reactions at the hinge after the tank is filled to a depth of 2.9 m.

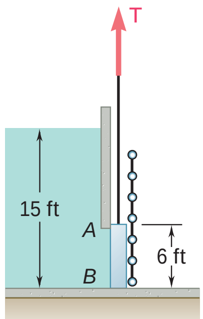


Fig. P5.86

5.86 The friction force between a 6 × 6-ft square sluice gate AB and its guides is equal to 10 percent of the resultant of the pressure forces exerted by the water on the face of the gate. Determine the initial force needed to lift the gate if it weighs 1000 lb.

5.87 A tank is divided into two sections by a 1 × 1-m square gate that is hinged at A. A couple of magnitude 490 N · m is required for the gate to rotate. If one side of the tank is filled with water at the rate of 0.1 m^3/min and the other side is filled simultaneously with methyl alcohol (density r_{ma} = 789 kg/m^3) at the rate of 0.2 m^3/min, determine at what time and in which direction the gate will rotate.

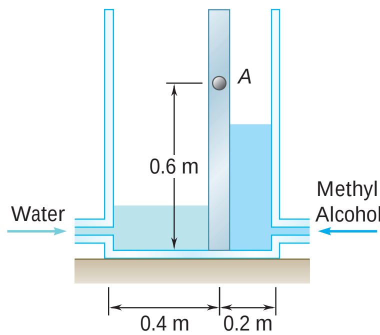


Fig. *P5.87*

5.88 A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at A and rests on a frictionless support at B. The pin is located at a distance $h = 0.10$ m below the center of gravity C of the gate. Determine the depth of water d for which the gate will open.

5.89 A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at A and rests on a frictionless support at B. The pin is located at a distance h below the center of gravity C of the gate. Determine the distance h if the gate is to open when $d = 0.75$ m.

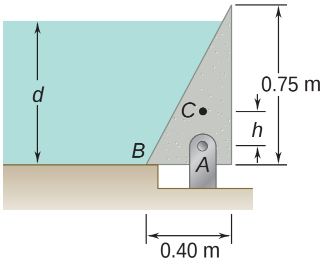


Fig. P5.88 and P5.89

5.90 The square gate AB is held in the position shown by hinges along its top edge A and by a shear pin at B. For a depth of water $d = 3.5$ ft, determine the force exerted on the gate by the shear pin.

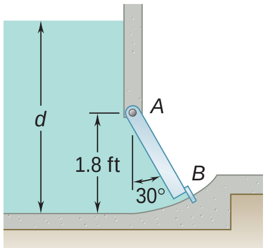


Fig. *P5.90*

5.91 A long trough is supported by a continuous hinge along its lower edge and by a series of horizontal cables attached to its upper edge. Determine the tension in each of the cables, at a time when the trough is completely full of water.

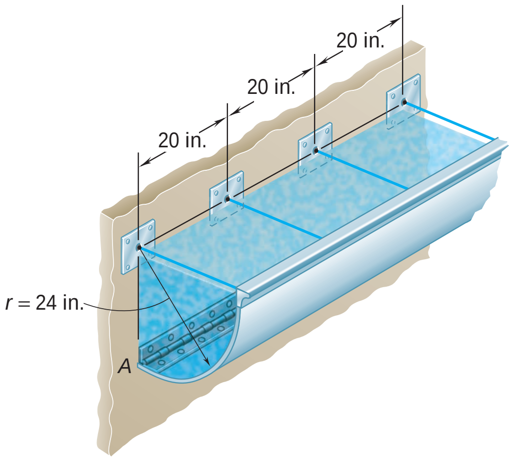


Fig. P5.91

5.92 A 0.5×0.8-m gate AB is located at the bottom of a tank filled with water. The gate is hinged along its top edge A and rests on a frictionless stop at B. Determine the reactions at A and B when cable BCD is slack.

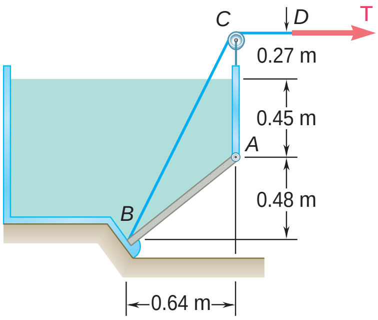


Fig. P5.92 and P5.93

5.93 A 0.5×0.8-m gate AB is located at the bottom of a tank filled with water. The gate is hinged along its top edge A and rests on a frictionless stop at B. Determine the minimum tension required in cable BCD to open the gate.

5.94 A 4×2-ft gate is hinged at A and is held in position by rod CD. End D rests against a spring whose constant is 828 lb/ft. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod CD on the gate remains horizontal, determine the minimum depth of water d for which the bottom B of the gate will move to the end of the cylindrical portion of the floor.

5.95 Solve Prob. 5.94 if the gate weighs 1000 lb.

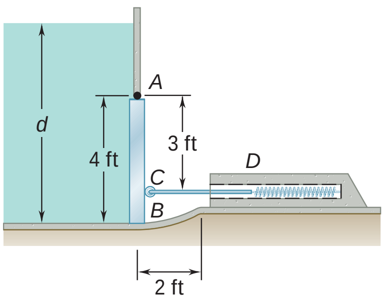


Fig. P5.94

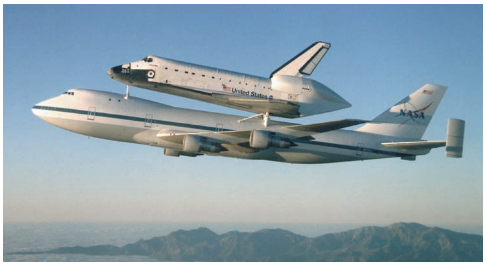


Photo 5.4 To predict the flight characteristics of the modified Boeing 747 when used to transport a space shuttle, the center of gravity of each craft had to be determined.

VOLUMES

5.10 CENTER OF GRAVITY OF A THREE-DIMENSIONAL BODY. CENTROID OF A VOLUME

The *center of gravity* G of a three-dimensional body is obtained by dividing the body into small elements and by then expressing that the weight $\mathbf{W}$ of the body acting at G is equivalent to the system of distributed forces $\Delta\mathbf{W}$ representing the weights of the small elements. Choosing the y axis to be vertical with positive sense upward (Fig. 5.20) and denoting by $\bar{\mathbf{r}}$ the position vector of G, we write that

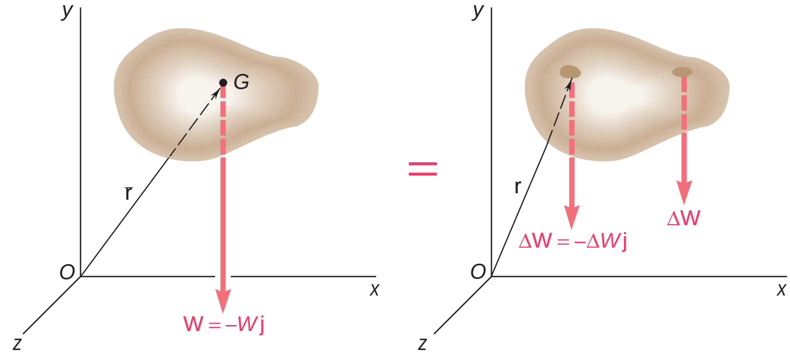


Fig. 5.20

$\mathbf{W}$ is equal to the sum of the elemental weights $\Delta\mathbf{W}$ and that its moment about O is equal to the sum of the moments about O of the elemental weights:

$$\Sigma\mathbf{F}: \qquad -W\mathbf{j} = \Sigma(-\Delta W\mathbf{j})$$
$$\Sigma\mathbf{M}_O: \qquad \bar{\mathbf{r}} \times (-W\mathbf{j}) = \Sigma[\mathbf{r} \times (-\Delta W\mathbf{j})] \tag{5.14}$$

Rewriting the last equation in the form

$$\bar{\mathbf{r}}W \times (-\mathbf{j}) = (\Sigma\mathbf{r}\,\Delta W) \times (-\mathbf{j}) \tag{5.15}$$

we observe that the weight $\mathbf{W}$ of the body is equivalent to the system of the elemental weights $\Delta\mathbf{W}$ if the following conditions are satisfied:

$$W = \Sigma\,\Delta W \qquad \bar{\mathbf{r}}W = \Sigma\mathbf{r}\,\Delta W$$

Increasing the number of elements and simultaneously decreasing the size of each element, we obtain in the limit

$$W = \int dW \qquad \bar{\mathbf{r}}W = \int \mathbf{r}\,dW \tag{5.16}$$

We note that the relations obtained are independent of the orientation of the body. For example, if the body and the coordinate axes were rotated so that the z axis pointed upward, the unit vector $-\mathbf{j}$ would be replaced by $-\mathbf{k}$ in Eqs. (5.14) and (5.15), but the relations (5.16) would remain unchanged. Resolving the vectors $\bar{\mathbf{r}}$ and $\mathbf{r}$ into rectangular components, we note that the second of the relations (5.16) is equivalent to the three scalar equations

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW \tag{5.17}$$

If the body is made of a homogeneous material of specific weight γ, the magnitude dW of the weight of an infinitesimal element can be expressed in terms of the volume dV of the element, and the magnitude W of the total weight can be expressed in terms of the total volume V. We write

$$dW = \gamma\, dV \qquad W = \gamma V$$

Substituting for dW and W in the second of the relations (5.16), we write

$$\bar{\mathbf{r}}V = \int \mathbf{r}\, dV \tag{5.18}$$

or, in scalar form,

$$\bar{x}V = \int x\, dV \qquad \bar{y}V = \int y\, dV \qquad \bar{z}V = \int z\, dV \tag{5.19}$$

The point whose coordinates are $\bar{x}, \bar{y}, \bar{z}$ is also known as the *centroid C of the volume V* of the body. If the body is not homogeneous, Eqs. (5.19) cannot be used to determine the center of gravity of the body; however, Eqs. (5.19) still define the centroid of the volume.

The integral $\int x\, dV$ is known as the *first moment of the volume with respect to the yz plane*. Similarly, the integrals $\int y\, dV$ and $\int z\, dV$ define the first moments of the volume with respect to the zx plane and the xy plane, respectively. It is seen from Eqs. (5.19) that if the centroid of a volume is located in a coordinate plane, the first moment of the volume with respect to that plane is zero.

A volume is said to be symmetrical with respect to a given plane if for every point P of the volume there exists a point P' of the same volume, such that the line PP' is perpendicular to the given plane and is bisected by that plane. The plane is said to be a *plane of symmetry* for the given volume. When a volume V possesses a plane of symmetry, the first moment of V with respect to that plane is zero, and the centroid of the volume is located in the plane of symmetry. When a volume possesses two planes of symmetry, the centroid of the volume is located on the line of intersection of the two planes. Finally, when a volume possesses three planes of symmetry which intersect at a well-defined point (i.e., not along a common line), the point of intersection of the three planes coincides with the centroid of the volume. This property enables us to determine immediately the locations of the centroids of spheres, ellipsoids, cubes, rectangular parallelepipeds, etc.

The centroids of unsymmetrical volumes or of volumes possessing only one or two planes of symmetry should be determined by integration (Sec. 5.12). The centroids of several common volumes are shown in Fig. 5.21. It should be observed that in general the centroid of a volume of revolution *does not coincide* with the centroid of its cross section. Thus, the centroid of a hemisphere is different from that of a semicircular area, and the centroid of a cone is different from that of a triangle.

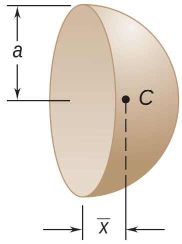
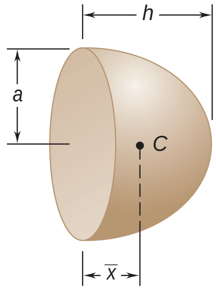
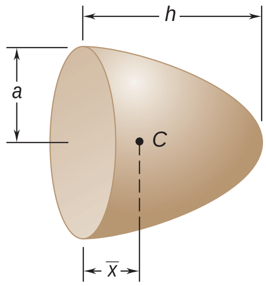
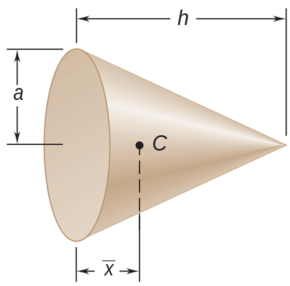
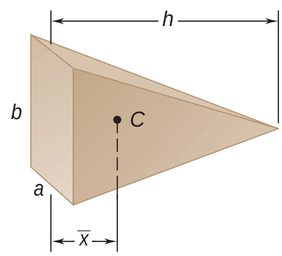
Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3} abh$

Fig. 5.21 Centroids of common shapes and volumes.

5.11 COMPOSITE BODIES

If a body can be divided into several of the common shapes shown in Fig. 5.21, its center of gravity G can be determined by expressing that the moment about O of its total weight is equal to the sum of the moments about O of the weights of the various component parts. Proceeding as in Sec. 5.10, we obtain the following equations defining the coordinates $\overline{X}$, $\overline{Y}$, $\overline{Z}$ of the center of gravity G.

$$\overline{X}\Sigma W = \Sigma\bar{x}W \qquad \overline{Y}\Sigma W = \Sigma\bar{y}W \qquad \overline{Z}\Sigma W = \Sigma\bar{z}W \tag{5.20}$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid of its volume, and we obtain:

$$\overline{X}\Sigma V = \Sigma\bar{x}V \qquad \overline{Y}\Sigma V = \Sigma\bar{y}V \qquad \overline{Z}\Sigma V = \Sigma\bar{z}V \tag{5.21}$$

5.12 DETERMINATION OF CENTROIDS OF VOLUMES BY INTEGRATION

The centroid of a volume bounded by analytical surfaces can be determined by evaluating the integrals given in Sec. 5.10:

$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV \tag{5.22}$$

If the element of volume dV is chosen to be equal to a small cube of sides dx, dy, and dz, the evaluation of each of these integrals requires a *triple integration*. However, it is possible to determine the coordinates of the centroid of most volumes by *double integration* if dV is chosen to be equal to the volume of a thin filament (Fig. 5.22). The coordinates of the centroid of the volume are then obtained by rewriting Eqs. (5.22) as

$$\bar{x}V = \int \bar{x}_{el}\,dV \qquad \bar{y}V = \int \bar{y}_{el}\,dV \qquad \bar{z}V = \int \bar{z}_{el}\,dV \tag{5.23}$$

and by then substituting the expressions given in Fig. 5.22 for the volume dV and the coordinates $\bar{x}_{el}$, $\bar{y}_{el}$, $\bar{z}_{el}$. By using the equation of the surface to express z in terms of x and y, the integration is reduced to a double integration in x and y.

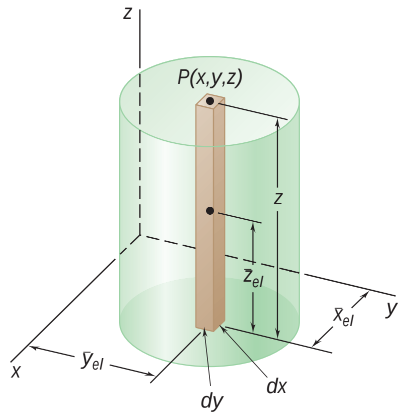
If the volume under consideration possesses *two planes of symmetry*, its centroid must be located on the line of intersection of the two planes. Choosing the x axis to lie along this line, we have

$$\bar{y} = \bar{z} = 0$$

and the only coordinate to determine is $\bar{x}$. This can be done with a *single integration* by dividing the given volume into thin slabs parallel to the yz plane and expressing dV in terms of x and dx in the equation

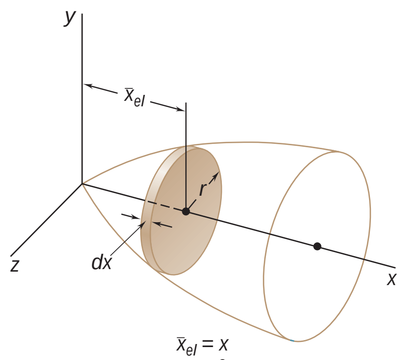
$$\bar{x}V = \int \bar{x}_{el}\,dV \tag{5.24}$$

For a body of revolution, the slabs are circular and their volume is given in Fig. 5.23.



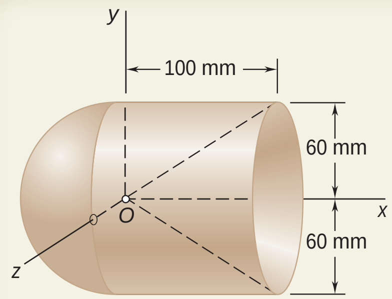
$\bar{x}_{el} = x,\ \bar{y}_{el} = y,\ \bar{z}_{el} = \frac{z}{2}$
$dV = z\,dx\,dy$

Fig. 5.22 Determination of the centroid of a volume by double integration.



$\bar{x}_{el} = x$
$dV = \pi r^2\,dx$

Fig. 5.23 Determination of the centroid of a body of revolution.

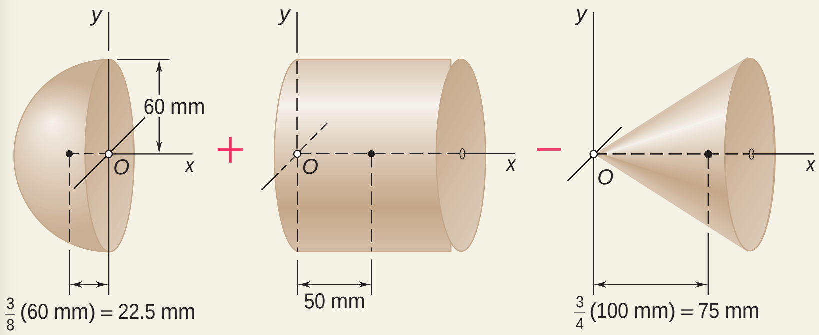


SAMPLE PROBLEM 5.11

Determine the location of the center of gravity of the homogeneous body of revolution shown, which was obtained by joining a hemisphere and a cylinder and carving out a cone.

SOLUTION

Because of symmetry, the center of gravity lies on the x axis. As shown in the figure below, the body can be obtained by adding a hemisphere to a cylinder and then subtracting a cone. The volume and the abscissa of the centroid of each of these components are obtained from Fig. 5.21 and are entered in the table below. The total volume of the body and the first moment of its volume with respect to the yz plane are then determined.



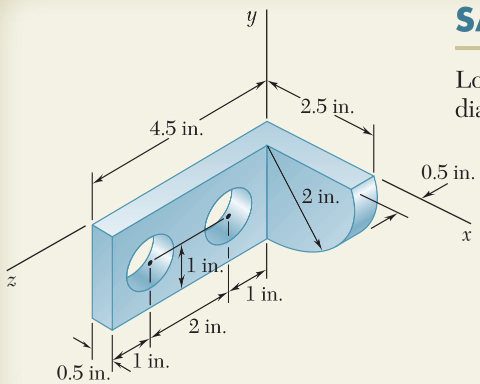
Component	Volume, mm^3	$\bar{x}$, mm	$\bar{x}V$, mm^4
Hemisphere	$\frac{1}{2}\frac{4\pi}{3}(60)^3 = 0.4524 \times 10^6$	-22.5	-10.18×10^6
Cylinder	$\pi(60)^2(100) = 1.1310 \times 10^6$	$+50$	$+56.55 \times 10^6$
Cone	$-\frac{\pi}{3}(60)^2(100) = -0.3770 \times 10^6$	$+75$	-28.28×10^6
	$\Sigma V = 1.206 \times 10^6$		$\Sigma\bar{x}V = +18.09 \times 10^6$

Thus,

$$\bar{X}\Sigma V = \Sigma\bar{x}V: \qquad \bar{X}(1.206 \times 10^6 \text{ mm}^3) = 18.09 \times 10^6 \text{ mm}^4$$

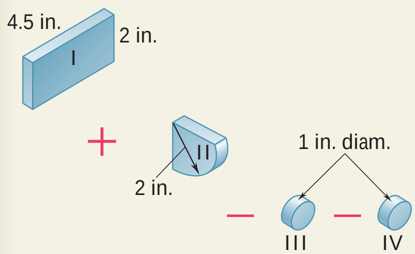
$$\bar{X} = 15 \text{ mm}$$

SAMPLE PROBLEM 5.12

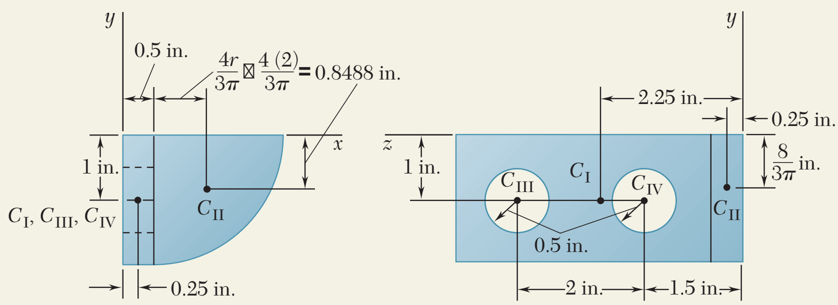


Locate the center of gravity of the steel machine element shown. The diameter of each hole is 1 in.

SOLUTION



The machine element can be obtained by adding a rectangular parallelepiped (I) to a quarter cylinder (II) and then subtracting two 1-in.-diameter cylinders (III and IV). The volume and the coordinates of the centroid of each component are determined and are entered in the table below. Using the data in the table, we then determine the total volume and the moments of the volume with respect to each of the coordinate planes.



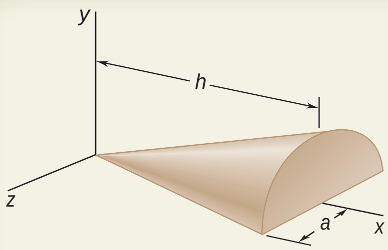
	V, in^3	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in^4	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

Thus,

$\bar{X}\Sigma V = \Sigma\bar{x}V$: $\quad \bar{X}(5.286 \text{ in}^3) = 3.048 \text{ in}^4 \qquad \bar{X} = 0.577$ in. ◀

$\bar{Y}\Sigma V = \Sigma\bar{y}V$: $\quad \bar{Y}(5.286 \text{ in}^3) = -5.047 \text{ in}^4 \qquad \bar{Y} = -0.955$ in. ◀

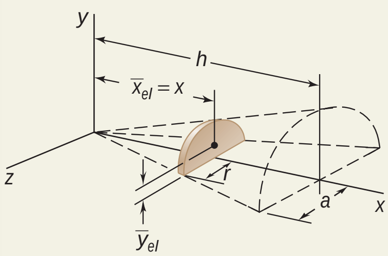
$\bar{Z}\Sigma V = \Sigma\bar{z}V$: $\quad \bar{Z}(5.286 \text{ in}^3) = 8.555 \text{ in}^4 \qquad \bar{Z} = 1.618$ in. ◀



SAMPLE PROBLEM 5.13

Determine the location of the centroid of the half right circular cone shown.

SOLUTION



Since the xy plane is a plane of symmetry, the centroid lies in this plane and $\bar{z} = 0$. A slab of thickness dx is chosen as a differential element. The volume of this element is

$$dV = \tfrac{1}{2}\pi r^2\,dx$$

The coordinates $\bar{x}_{el}$ and $\bar{y}_{el}$ of the centroid of the element are obtained from Fig. 5.8 (semicircular area).

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \frac{4r}{3\pi}$$

We observe that r is proportional to x and write

$$\frac{r}{x} = \frac{a}{h} \qquad r = \frac{a}{h}x$$

The volume of the body is

$$V = \int dV = \int_0^h \tfrac{1}{2}\pi r^2\,dx = \int_0^h \tfrac{1}{2}\pi\left(\frac{a}{h}x\right)^2 dx = \frac{\pi a^2 h}{6}$$

The moment of the differential element with respect to the yz plane is $\bar{x}_{el}\,dV$; the total moment of the body with respect to this plane is

$$\int \bar{x}_{el}\,dV = \int_0^h x(\tfrac{1}{2}\pi r^2)\,dx = \int_0^h x(\tfrac{1}{2}\pi)\left(\frac{a}{h}x\right)^2 dx = \frac{\pi a^2 h^2}{8}$$

Thus,

$$\bar{x}V = \int \bar{x}_{el}\,dV \qquad \bar{x}\frac{\pi a^2 h}{6} = \frac{\pi a^2 h^2}{8} \qquad \bar{x} = \tfrac{3}{4}h \quad \blacktriangleleft$$

Likewise, the moment of the differential element with respect to the zx plane is $\bar{y}_{el}\,dV$; the total moment is

$$\int \bar{y}_{el}\,dV = \int_0^h \frac{4r}{3\pi}(\tfrac{1}{2}\pi r^2)dx = \frac{2}{3}\int_0^h \left(\frac{a}{h}x\right)^3 dx = \frac{a^3 h}{6}$$

Thus,

$$\bar{y}V = \int \bar{y}_{el}\,dV \qquad \bar{y}\frac{\pi a^2 h}{6} = \frac{a^3 h}{6} \qquad \bar{y} = \frac{a}{\pi} \quad \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will be asked to locate the centers of gravity of three-dimensional bodies or the centroids of their volumes. All of the techniques we previously discussed for two-dimensional bodies—using symmetry, dividing the body into common shapes, choosing the most efficient differential element, etc.—may also be applied to the general three-dimensional case.

1. Locating the centers of gravity of composite bodies. In general, Eqs. (5.20) must be used:

$$\bar{X}\Sigma W = \Sigma\bar{x}W \qquad \bar{Y}\Sigma W = \Sigma\bar{y}W \qquad \bar{Z}\Sigma W = \Sigma\bar{z}W \tag{5.20}$$

However, for the case of a *homogeneous body*, the center of gravity of the body coincides with the *centroid of its volume*. Therefore, for this special case, the center of gravity of the body can also be located using Eqs. (5.21):

$$\bar{X}\Sigma V = \Sigma\bar{x}V \qquad \bar{Y}\Sigma V = \Sigma\bar{y}V \qquad \bar{Z}\Sigma V = \Sigma\bar{z}V \tag{5.21}$$

You should realize that these equations are simply an extension of the equations used for the two-dimensional problems considered earlier in the chapter. As the solutions of Sample Probs. 5.11 and 5.12 illustrate, the methods of solution for two- and three-dimensional problems are identical. Thus, we once again strongly encourage you to construct appropriate diagrams and tables when analyzing composite bodies. Also, as you study Sample Prob. 5.12, observe how the x and y coordinates of the centroid of the quarter cylinder were obtained using the equations for the centroid of a quarter circle.

We note that *two special cases* of interest occur when the given body consists of either uniform wires or uniform plates made of the same material.

a. For a body made of *several wire elements* of the *same uniform cross section*, the cross-sectional area A of the wire elements will factor out of Eqs. (5.21) when V is replaced with the product AL, where L is the length of a given element. Equations (5.21) thus reduce in this case to

$$\bar{X}\Sigma L = \Sigma\bar{x}L \qquad \bar{Y}\Sigma L = \Sigma\bar{y}L \qquad \bar{Z}\Sigma L = \Sigma\bar{z}L$$

b. For a body made of *several plates* of the *same uniform thickness*, the thickness t of the plates will factor out of Eqs. (5.21) when V is replaced with the product tA, where A is the area of a given plate. Equations (5.21) thus reduce in this case to

$$\bar{X}\Sigma A = \Sigma\bar{x}A \qquad \bar{Y}\Sigma A = \Sigma\bar{y}A \qquad \bar{Z}\Sigma A = \Sigma\bar{z}A$$

2. Locating the centroids of volumes by direct integration. As explained in Sec. 5.12, evaluating the integrals of Eqs. (5.22) can be simplified by choosing either a thin filament (Fig. 5.22) or a thin slab (Fig. 5.23) for the element of volume dV. Thus, you should begin your solution by identifying, if possible, the dV which produces the single or double integrals that are the easiest to compute. For bodies of revolution, this may be a thin slab (as in Sample Prob. 5.13) or a thin cylindrical shell. However, it is important to remember that the relationship that you establish among the variables (like the relationship between r and x in Sample Prob. 5.13) will directly affect the complexity of the integrals you will have to compute. Finally, we again remind you that $\bar{x}_{el}$, $\bar{y}_{el}$, and $\bar{z}_{el}$ in Eqs. (5.23) are the coordinates of the centroid of dV.

PROBLEMS

5.96 A hemisphere and a cone are attached as shown. Determine the location of the centroid of the composite body when (*a*) $h = 1.5a$, (*b*) $h = 2a$.

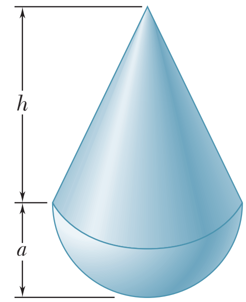


Fig. P5.96

5.97 Consider the composite body shown. Determine (*a*) the value of $\bar{x}$ when $h = L/2$, (*b*) the ratio h/L for which $\bar{x} = L$.

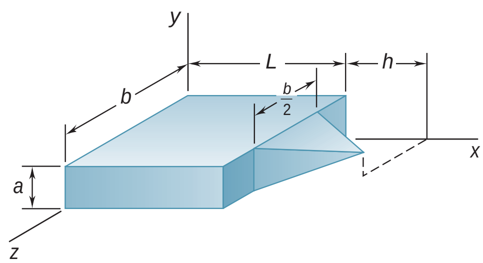


Fig. P5.97

5.98 Determine the y coordinate of the centroid of the body shown.

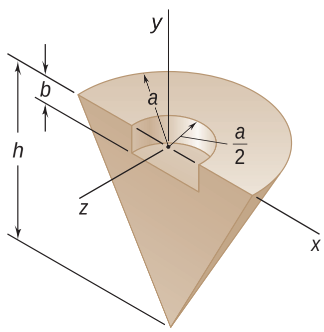


Fig. P5.98 and P5.99

5.99 Determine the z coordinate of the centroid of the body shown. (*Hint:* Use the result of Sample Prob. 5.13.)

5.100 and 5.101 For the machine element shown, locate the y coordinate of the center of gravity.

5.102 For the machine element shown, locate the x coordinate of the center of gravity.

5.103 For the machine element shown, locate the z coordinate of the center of gravity.

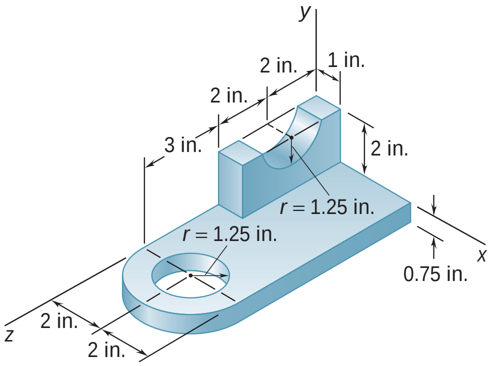


Fig. P5.100 and P5.103

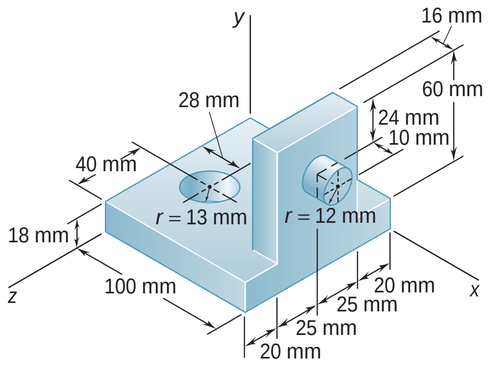


Fig. P5.101 and *P5.102*

5.104 For the machine element shown, locate the x coordinate of the center of gravity.

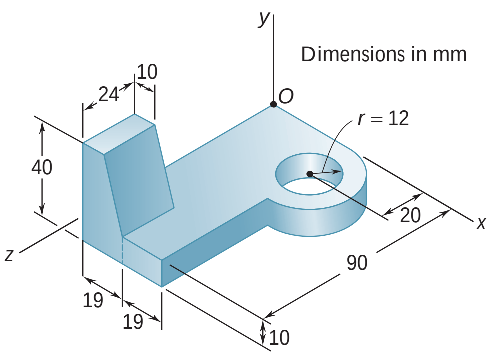


Fig. P5.104 and *P5.105*

5.105 For the machine element shown, locate the z coordinate of the center of gravity.

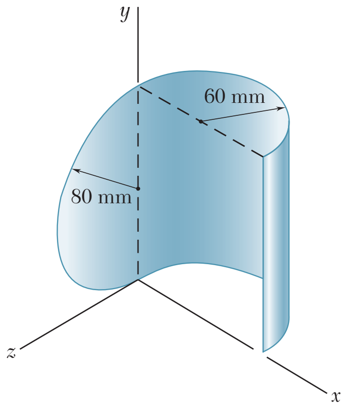


Fig. P5.106

5.106 and 5.107 Locate the center of gravity of the sheet-metal form shown.

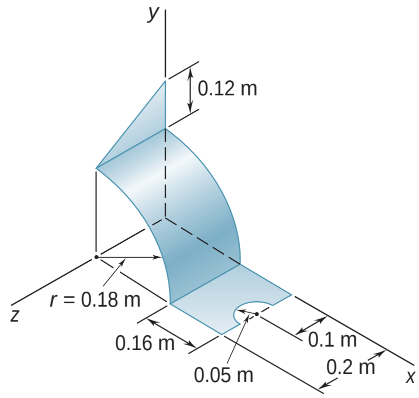


Fig. P5.107

5.108 A window awning is fabricated from sheet metal of uniform thickness. Locate the center of gravity of the awning.

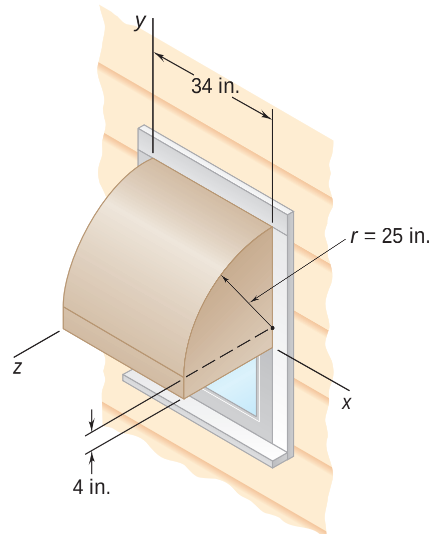


Fig. P5.108

5.109 A thin sheet of plastic of uniform thickness is bent to form a desk organizer. Locate the center of gravity of the organizer.

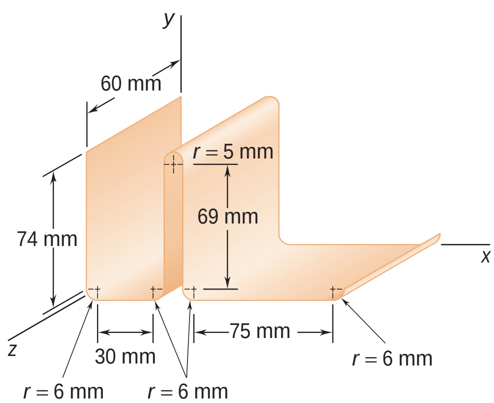


Fig. P5.109

5.110 A wastebasket, designed to fit in the corner of a room, is 16 in. high and has a base in the shape of a quarter circle of radius 10 in. Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.

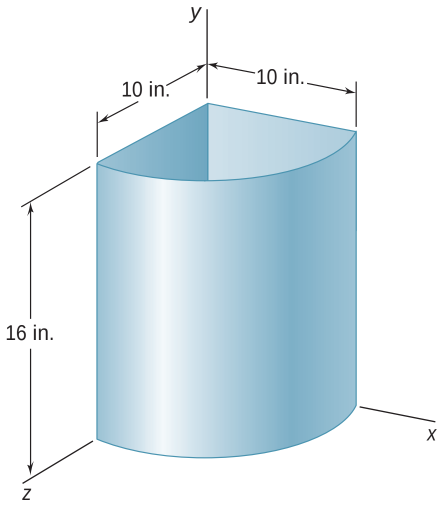


Fig. P5.110

5.111 A mounting bracket for electronic components is formed from sheet metal of uniform thickness. Locate the center of gravity of the bracket.

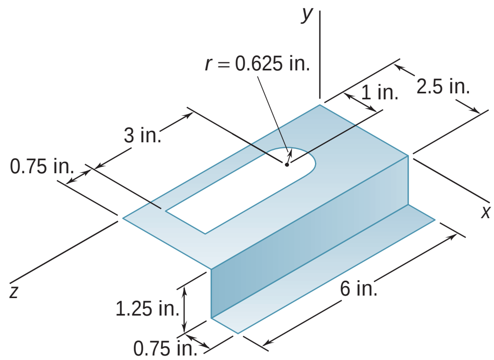


Fig. *P5.111*

5.112 An 8-in.-diameter cylindrical duct and a 4 × 8-in. rectangular duct are to be joined as indicated. Knowing that the ducts were fabricated from the same sheet metal, which is of uniform thickness, locate the center of gravity of the assembly.

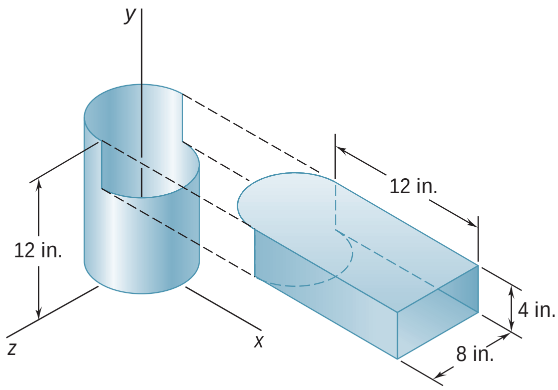


Fig. *P5.112*

5.113 An elbow for the duct of a ventilating system is made of sheet metal of uniform thickness. Locate the center of gravity of the elbow.

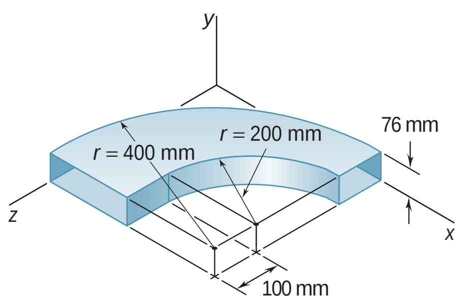


Fig. P5.113

5.114 and 5.115 Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

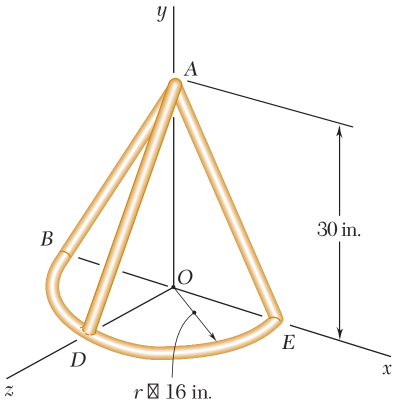


Fig. P5.114

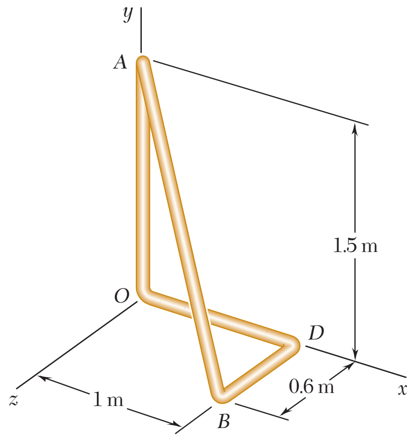


Fig. P5.115

5.116 A thin steel wire of uniform cross section is bent into the shape shown. Locate its center of gravity.

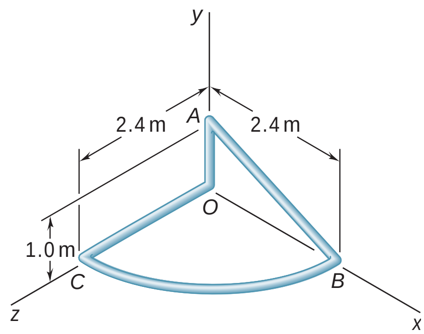


Fig. P5.116

5.117 The frame of a greenhouse is constructed from uniform aluminum channels. Locate the center of gravity of the portion of the frame shown.

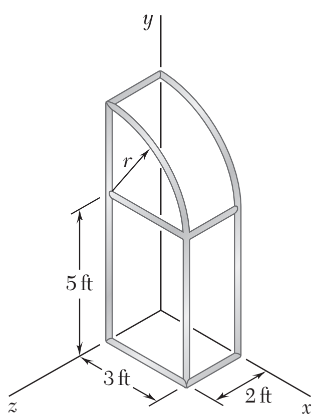


Fig. *P5.117*

5.118 Three brass plates are brazed to a steel pipe to form the flagpole base shown. Knowing that the pipe has a wall thickness of 8 mm and that each plate is 6 mm thick, determine the location of the center of gravity of the base. (Densities: brass = 8470 kg/m^3; steel = 7860 kg/m^3.)

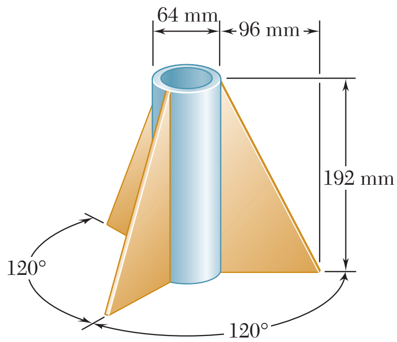


Fig. P5.118

5.119 A brass collar, of length 2.5 in., is mounted on an aluminum rod of length 4 in. Locate the center of gravity of the composite body. (Specific weights: brass = 0.306 lb/in^3, aluminum = 0.101 lb/in^3.)

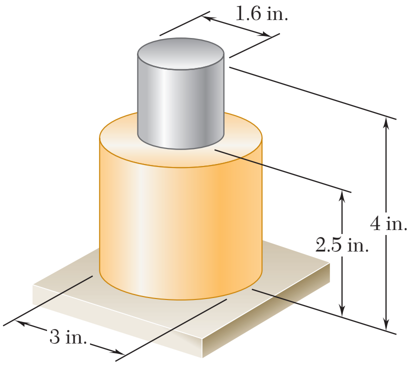


Fig. *P5.119*

5.120 A bronze bushing is mounted inside a steel sleeve. Knowing that the specific weight of bronze is 0.318 lb/in^3 and of steel is 0.284 lb/in^3, determine the location of the center of gravity of the assembly.

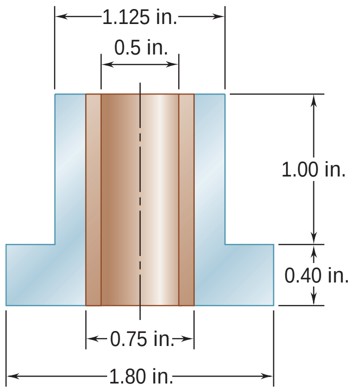


Fig. P5.120

5.121 A scratch awl has a plastic handle and a steel blade and shank. Knowing that the density of plastic is 1030 kg/m^3 and of steel is 7860 kg/m^3, locate the center of gravity of the awl.

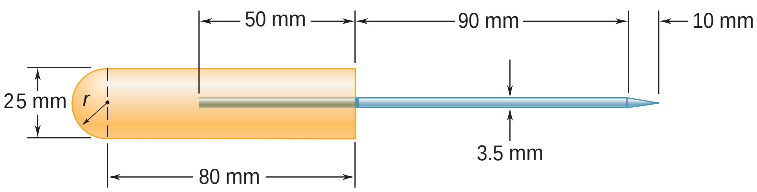


Fig. P5.121

5.122 through 5.124 Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Fig. 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

5.122 A hemisphere.
5.123 A semiellipsoid of revolution.
5.124 A paraboloid of revolution.

5.125 and 5.126 Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

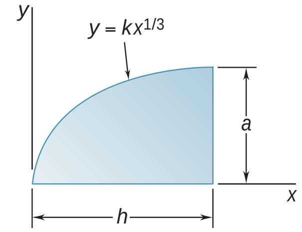


Fig. *P5.125*

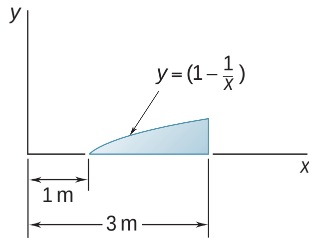


Fig. P5.126

5.127 Locate the centroid of the volume obtained by rotating the shaded area about the line $x = h$.

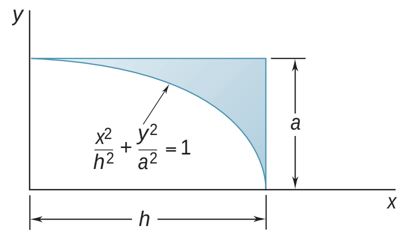


Fig. *P5.127*

***5.128** Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the x axis.

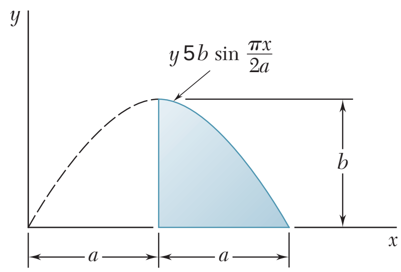


Fig. P5.128 and P5.129

***5.129** Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the y axis. (*Hint:* Use a thin cylindrical shell of radius r and thickness dr as the element of volume.)

***5.130** Show that for a regular pyramid of height h and n sides ($n = 3, 4, \ldots$) the centroid of the volume of the pyramid is located at a distance $h/4$ above the base.

5.131 Determine by direct integration the location of the centroid of one-half of a thin, uniform hemispherical shell of radius R.

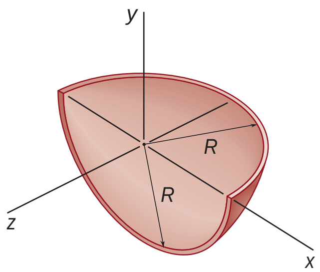


Fig. *P5.131*

5.132 The sides and the base of a punch bowl are of uniform thickness t. If $t \ll R$ and $R = 250$ mm, determine the location of the center of gravity of (*a*) the bowl, (*b*) the punch.

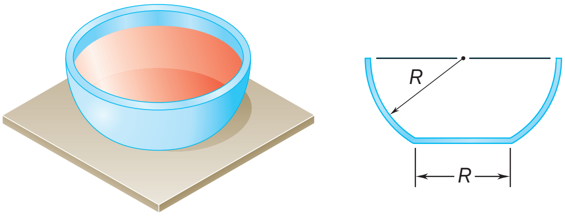


Fig. P5.132

5.133 Locate the centroid of the section shown, which was cut from a thin circular pipe by two oblique planes.

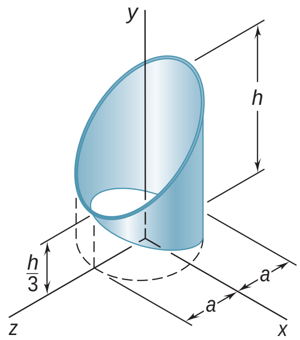


Fig. *P5.133*

***5.134** Locate the centroid of the section shown, which was cut from an elliptical cylinder by an oblique plane.

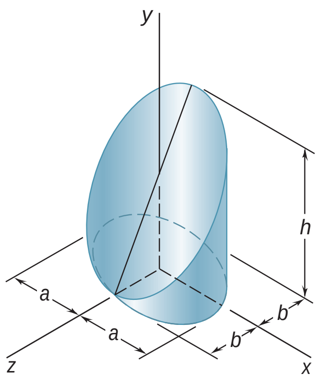


Fig. P5.134

5.135 After grading a lot, a builder places four stakes to designate the corners of the slab for a house. To provide a firm, level base for the slab, the builder places a minimum of 3 in. of gravel beneath the slab. Determine the volume of gravel needed and the x coordinate of the centroid of the volume of the gravel. (*Hint:* The bottom surface of the gravel is an oblique plane, which can be represented by the equation $y = a + bx + cz$.)

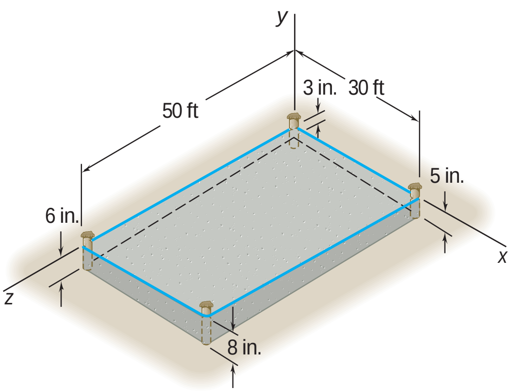


Fig. P5.135

5.136 Determine by direct integration the location of the centroid of the volume between the xz plane and the portion shown of the surface $y = 16h(ax - x^2)(bz - z^2)/a^2b^2$.

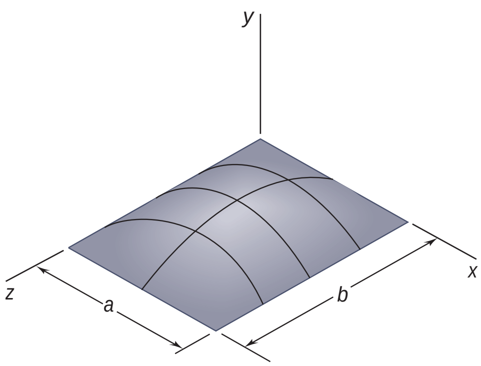


Fig. P5.136

REVIEW AND SUMMARY

This chapter was devoted chiefly to the determination of the *center of gravity* of a rigid body, i.e., to the determination of the point G where a single force $\mathbf{W}$, called the *weight* of the body, can be applied to represent the effect of the earth's attraction on the body.

Center of gravity of a two-dimensional body

In the first part of the chapter, we considered *two-dimensional bodies*, such as flat plates and wires contained in the xy plane. By adding force components in the vertical z direction and moments about the horizontal y and x axes [Sec. 5.2], we derived the relations

$$W = \int dW \qquad \bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \tag{5.2}$$

which define the weight of the body and the coordinates $\bar{x}$ and $\bar{y}$ of its center of gravity.

Centroid of an area or line

In the case of a *homogeneous flat plate of uniform thickness* [Sec. 5.3], the center of gravity G of the plate coincides with the *centroid C of the area A* of the plate, the coordinates of which are defined by the relations

$$\bar{x}A = \int x\,dA \qquad \bar{y}A = \int y\,dA \tag{5.3}$$

Similarly, the determination of the center of gravity of a *homogeneous wire of uniform cross section* contained in a plane reduces to the determination of the *centroid C of the line L* representing the wire; we have

$$\bar{x}L = \int x\,dL \qquad \bar{y}L = \int y\,dL \tag{5.4}$$

First moments

The integrals in Eqs. (5.3) are referred to as the *first moments* of the area A with respect to the y and x axes and are denoted by Q_y and Q_x, respectively [Sec. 5.4]. We have

$$Q_y = \bar{x}A \qquad Q_x = \bar{y}A \tag{5.6}$$

The first moments of a line can be defined in a similar way.

Properties of symmetry

The determination of the centroid C of an area or line is simplified when the area or line possesses certain *properties of symmetry*. If the area or line is symmetric with respect to an axis, its centroid C

lies on that axis; if it is symmetric with respect to two axes, C is located at the intersection of the two axes; if it is symmetric with respect to a center O, C coincides with O.

Center of gravity of a composite body

The *areas and the centroids of various common shapes* are tabulated in Fig. 5.8. When a flat plate can be divided into several of these shapes, the coordinates $\overline{X}$ and $\overline{Y}$ of its center of gravity G can be determined from the coordinates $\overline{x}_1, \overline{x}_2, \ldots$ and $\overline{y}_1, \overline{y}_2, \ldots$ of the centers of gravity $G_1, G_2, \ldots$ of the various parts [Sec. 5.5]. Equating moments about the y and x axes, respectively (Fig. 5.24), we have

$$\overline{X}\Sigma W = \Sigma\overline{x}W \qquad \overline{Y}\Sigma W = \Sigma\overline{y}W \tag{5.7}$$

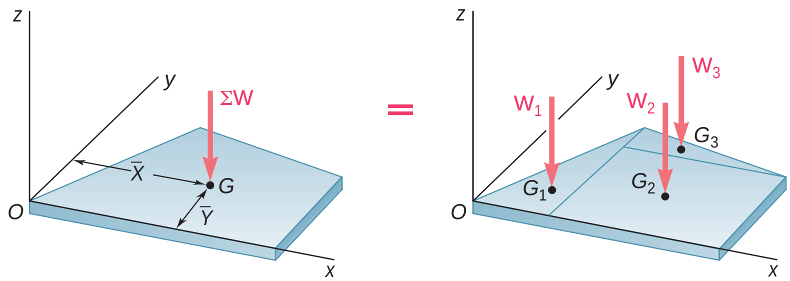


Fig. 5.24

If the plate is homogeneous and of uniform thickness, its center of gravity coincides with the centroid C of the area of the plate, and Eqs. (5.7) reduce to

$$Q_y = \overline{X}\Sigma A = \Sigma\overline{x}A \qquad Q_x = \overline{Y}\Sigma A = \Sigma\overline{y}A \tag{5.8}$$

These equations yield the first moments of the composite area, or they can be solved for the coordinates $\overline{X}$ and $\overline{Y}$ of its centroid [Sample Prob. 5.1]. The determination of the center of gravity of a composite wire is carried out in a similar fashion [Sample Prob. 5.2].

Determination of centroid by integration

When an area is bounded by analytical curves, the coordinates of its centroid can be determined by *integration* [Sec. 5.6]. This can be done by evaluating either the double integrals in Eqs. (5.3) or a *single integral* which uses one of the thin rectangular or pie-shaped elements of area shown in Fig. 5.12. Denoting by $\overline{x}_{el}$ and $\overline{y}_{el}$ the coordinates of the centroid of the element dA, we have

$$Q_y = \overline{x}A = \int \overline{x}_{el}\, dA \qquad Q_x = \overline{y}A = \int \overline{y}_{el}\, dA \tag{5.9}$$

It is advantageous to use the same element of area to compute both of the first moments Q_y and Q_x; the same element can also be used to determine the area A [Sample Prob. 5.4].

Theorems of Pappus-Guldinus

The *theorems of Pappus-Guldinus* relate the determination of the area of a surface of revolution or the volume of a body of revolution to the determination of the centroid of the generating curve or area [Sec. 5.7]. The area A of the surface generated by rotating a curve of length L about a fixed axis (Fig. 5.25*a*) is

$$A = 2\pi \bar{y} L \tag{5.10}$$

where $\bar{y}$ represents the distance from the centroid C of the curve to the fixed axis. Similarly, the volume V of the body generated by rotating an area A about a fixed axis (Fig. 5.25*b*) is

$$V = 2\pi \bar{y} A \tag{5.11}$$

where $\bar{y}$ represents the distance from the centroid C of the area to the fixed axis.

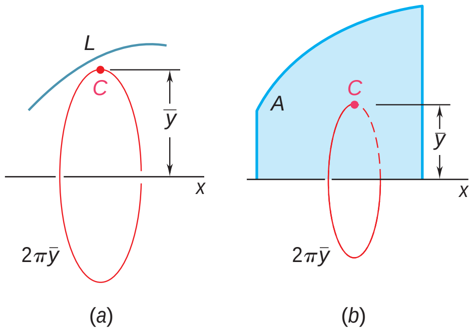


Fig. 5.25

Distributed loads

The concept of centroid of an area can also be used to solve problems other than those dealing with the weight of flat plates. For example, to determine the reactions at the supports of a beam [Sec. 5.8], we can replace a *distributed load* w by a concentrated load $\mathbf{W}$ equal in magnitude to the area A under the load curve and passing through the centroid C of that area (Fig. 5.26). The same approach can be used to determine the resultant of the hydrostatic forces exerted on a *rectangular plate submerged in a liquid* [Sec. 5.9].

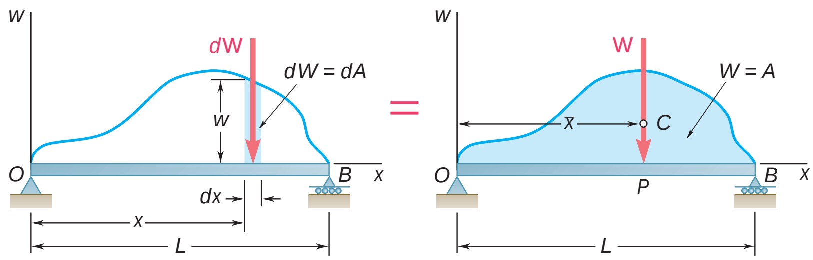


Fig. 5.26

Center of gravity of a three-dimensional body

The last part of the chapter was devoted to the determination of the *center of gravity G of a three-dimensional body*. The coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of G were defined by the relations

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW \tag{5.17}$$

Centroid of a volume

In the case of a *homogeneous body*, the center of gravity G coincides with the *centroid C of the volume V* of the body; the coordinates of C are defined by the relations

$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV \tag{5.19}$$

If the volume possesses a *plane of symmetry*, its centroid C will lie in that plane; if it possesses two planes of symmetry, C will be located on the line of intersection of the two planes; if it possesses three planes of symmetry which intersect at only one point, C will coincide with that point [Sec. 5.10].

The *volumes and centroids of various common three-dimensional shapes* are tabulated in Fig. 5.21. When a body can be divided into several of these shapes, the coordinates $\overline{X}, \overline{Y}, \overline{Z}$ of its center of gravity G can be determined from the corresponding coordinates of the centers of gravity of its various parts [Sec. 5.11]. We have

Center of gravity of a composite body

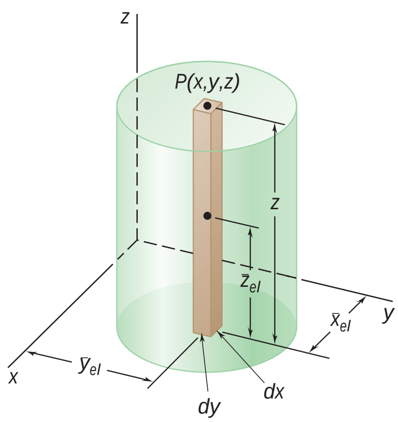
$$\overline{X}\Sigma W = \Sigma\bar{x}\,W \qquad \overline{Y}\Sigma W = \Sigma\bar{y}\,W \qquad \overline{Z}\Sigma W = \Sigma\bar{z}\,W \tag{5.20}$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid C of its volume, and we write [Sample Probs. 5.11 and 5.12]

$$\overline{X}\Sigma V = \Sigma\bar{x}\,V \qquad \overline{Y}\Sigma V = \Sigma\bar{y}\,V \qquad \overline{Z}\Sigma V = \Sigma\bar{z}\,V \tag{5.21}$$

Determination of centroid by integration

When a volume is bounded by analytical surfaces, the coordinates of its centroid can be determined by *integration* [Sec. 5.12]. To avoid the computation of the triple integrals in Eqs. (5.19), we can use elements of volume in the shape of thin filaments, as shown in Fig. 5.27.



$\bar{x}_{el} = x,\ \bar{y}_{el} = y,\ \bar{z}_{el} = \frac{z}{2}$
$dV = z\,dx\,dy$

Fig. 5.27

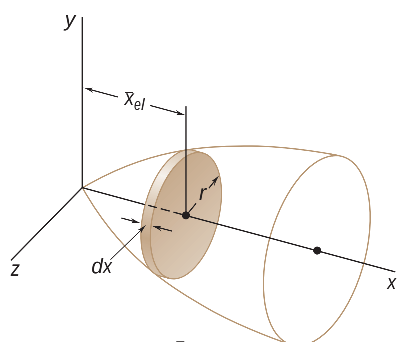
Denoting by $\bar{x}_{el}$, $\bar{y}_{el}$, and $\bar{z}_{el}$ the coordinates of the centroid of the element dV, we rewrite Eqs. (5.19) as

$$\bar{x}V = \int \bar{x}_{el}\,dV \qquad \bar{y}V = \int \bar{y}_{el}\,dV \qquad \bar{z}V = \int \bar{z}_{el}\,dV \tag{5.23}$$

which involve only double integrals. If the volume possesses *two planes of symmetry*, its centroid C is located on their line of intersection. Choosing the x axis to lie along that line and dividing the volume into thin slabs parallel to the yz plane, we can determine C from the relation

$$\bar{x}V = \int \bar{x}_{el}\,dV \tag{5.24}$$

with a *single integration* [Sample Prob. 5.13]. For a body of revolution, these slabs are circular and their volume is given in Fig. 5.28.



$\bar{x}_{el} = x$
$dV = \pi r^2\,dx$

Fig. 5.28

REVIEW PROBLEMS

5.137 and 5.138 Locate the centroid of the plane area shown.

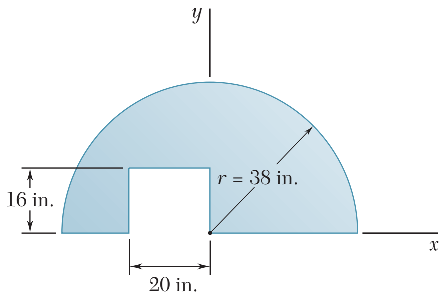


Fig. P5.137

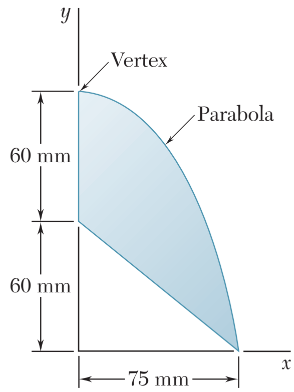


Fig. P5.138

5.139 The frame for a sign is fabricated from thin, flat steel bar stock of mass per unit length 4.73 kg/m. The frame is supported by a pin at C and by a cable AB. Determine (*a*) the tension in the cable, (*b*) the reaction at C.

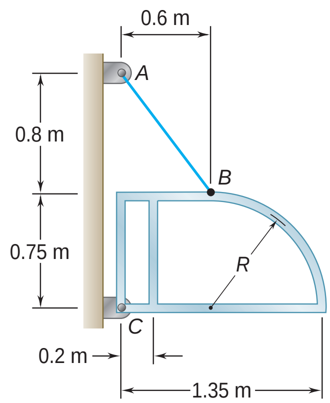


Fig. *P5.139*

5.140 Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

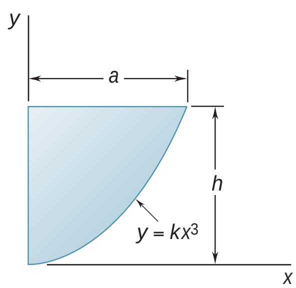


Fig. P5.140

5.141 Determine by direct integration the centroid of the area shown.

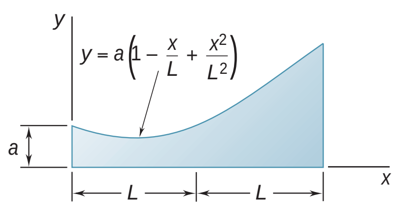


Fig. P5.141

5.142 Three different drive belt profiles are to be studied. If at any given time each belt makes contact with one-half of the circumference of its pulley, determine the *contact area* between the belt and the pulley for each design.

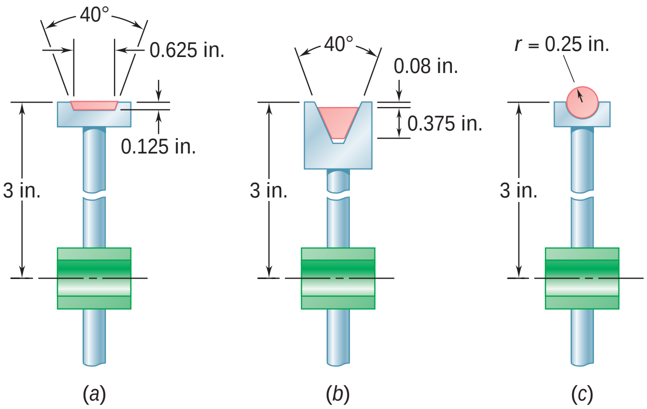


Fig. *P5.142*

5.143 Determine the reactions at the beam supports for the given loading.

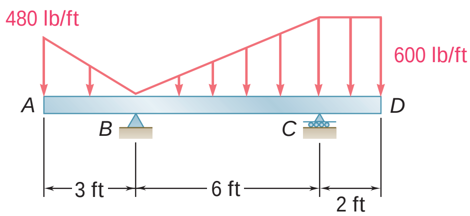


Fig. P5.143

5.144 The beam AB supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of w_A and w_B corresponding to equilibrium.

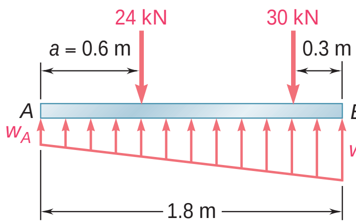


Fig. P5.144

5.145 The base of a dam for a lake is designed to resist up to 120 percent of the horizontal force of the water. After construction, it is found that silt (that is equivalent to a liquid of density $r_s = 1.76 \times 10^3$ kg/m^3) is settling on the lake bottom at the rate of 12 mm/year. Considering a 1-m-wide section of dam, determine the number of years until the dam becomes unsafe.

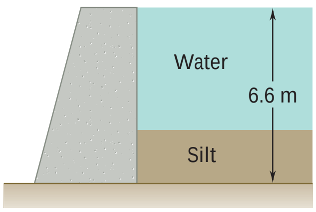


Fig. *P5.145*

5.146 Determine the location of the centroid of the composite body shown when (*a*) $h = 2b$, (*b*) $h = 2.5b$.

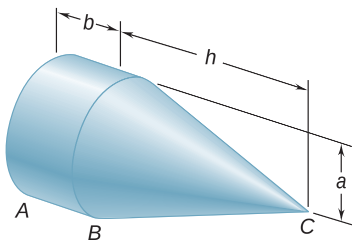


Fig. P5.146

5.147 Locate the center of gravity of the sheet-metal form shown.

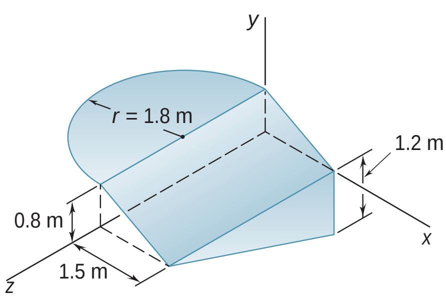


Fig. P5.147

5.148 Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

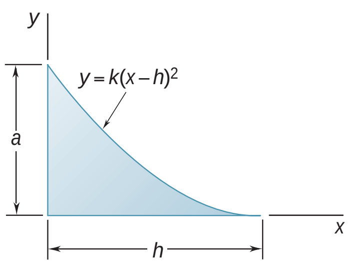


Fig. *P5.148*

COMPUTER PROBLEMS

5.C1 A beam is to carry a series of uniform and uniformly varying distributed loads as shown in part *a* of the figure. Divide the area under each portion of the load curve into two triangles (see Sample Prob. 5.9), and then write a computer program that can be used to calculate the reactions at *A* and *B*. Use this program to calculate the reactions at the supports for the beams shown in parts *b* and *c* of the figure.

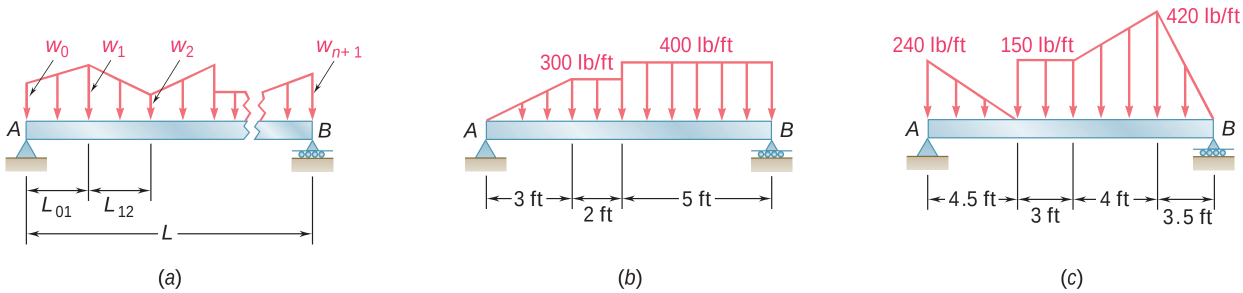


Fig. P5.C1

5.C2 The three-dimensional structure shown is fabricated from five thin steel rods of equal diameter. Write a computer program that can be used to calculate the coordinates of the center of gravity of the structure. Use this program to locate the center of gravity when (*a*) $h = 12$ m, $R = 5$ m, a $= 90°$; (*b*) $h = 570$ mm, $R = 760$ mm, a $= 30°$; (*c*) $h = 21$ m, $R = 20$ m, a $= 135°$.

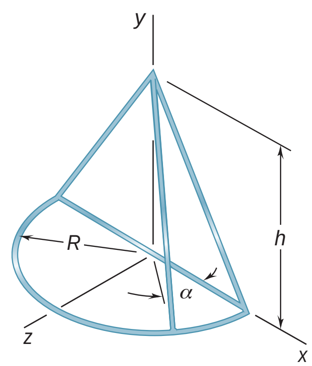


Fig. P5.C2

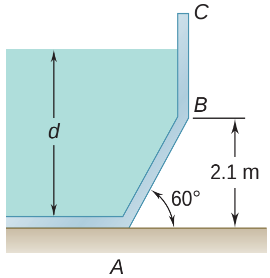


Fig. P5.C3

5.C3 An open tank is to be slowly filled with water. (The density of water is 10^3 kg/m^3.) Write a computer program that can be used to determine the resultant of the pressure forces exerted by the water on a 1-m-wide section of side *ABC* of the tank. Determine the resultant of the pressure forces for values of *d* from 0 to 3 m using 0.25-m increments.

5.C4 Approximate the curve shown using 10 straight-line segments, and then write a computer program that can be used to determine the location of the centroid of the curve. Use this program to determine the location of the centroid when (*a*) $a = 1$ in., $L = 11$ in., $h = 2$ in.; (*b*) $a = 2$ in., $L = 17$ in., $h = 4$ in.; (*c*) $a = 5$ in., $L = 12$ in., $h = 1$ in.

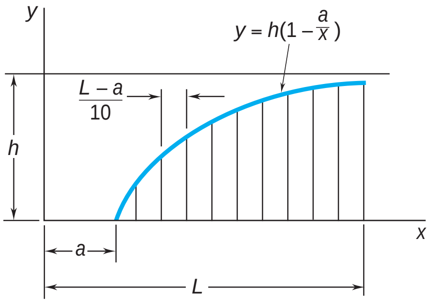


Fig. P5.C4

5.C5 Approximate the general spandrel shown using a series of n rectangles, each of width Δa and of the form $bcc'b'$, and then write a computer program that can be used to calculate the coordinates of the centroid of the area. Use this program to locate the centroid when (*a*) $m = 2$, $a = 80$ mm, $h = 80$ mm; (*b*) $m = 2$, $a = 80$ mm, $h = 500$ mm; (*c*) $m = 5$, $a = 80$ mm, $h = 80$ mm; (*d*) $m = 5$, $a = 80$ mm, $h = 500$ mm. In each case, compare the answers obtained to the exact values of $\bar{x}$ and $\bar{y}$ computed from the formulas given in Fig. 5.8A and determine the percentage error.

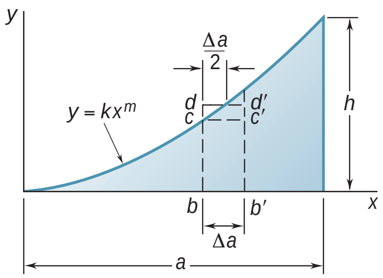


Fig. P5.C5

5.C6 Solve Prob. 5.C5, using rectangles of the form $bdd'b'$.

***5.C7** A farmer asks a group of engineering students to determine the volume of water in a small pond. Using cord, the students first establish a 2×2-ft grid across the pond and then record the depth of the water, in feet, at each intersection point of the grid (see the accompanying table). Write a computer program that can be used to determine (*a*) the volume of water in the pond, (*b*) the location of the center of gravity of the water. Approximate the depth of each 2×2-ft element of water using the average of the water depths at the four corners of the element.

		Cord									
		1	2	3	4	5	6	7	8	9	10
Cord	1	...	...	...	...	0	0	0	...	...	...
	2	...	...	0	0	0	1	0	0	0	...
	3	...	0	0	1	3	3	3	1	0	0
	4	0	0	1	3	6	6	6	3	1	0
	5	0	1	3	6	8	8	6	3	1	0
	6	0	1	3	6	8	7	7	3	0	0
	7	0	3	4	6	6	6	4	1	0	...
	8	0	3	3	3	3	3	1	0	0	...
	9	0	0	0	1	1	0	0	0	...	...
	10	...	...	0	0	0	0	...	...	...	...