The George Washington Bridge connects Manhattan, New York, and Fort Lee, New Jersey. This suspension bridge carries traffic on two levels over roadways that are supported by a system of beams. Trusses are used both to connect these roadways to complete the overall bridge span as well as to form the towers. The bridge span itself is supported by the cable system.

CHAPTER 7

Forces in Beams and Cables

Chapter 7 Forces in Beams and Cables

7.1 Introduction
7.2 Internal Forces in Members
7.3 Various Types of Loading and Support
7.4 Shear and Bending Moment in a Beam
7.5 Shear and Bending-Moment Diagrams
7.6 Relations Among Load, Shear, and Bending Moment
7.7 Cables with Concentrated Loads
7.8 Cables with Distributed Loads
7.9 Parabolic Cable
7.10 Catenary

*7.1 INTRODUCTION

In preceding chapters, two basic problems involving structures were considered: (1) determining the external forces acting on a structure (Chap. 4) and (2) determining the forces which hold together the various members forming a structure (Chap. 6). The problem of determining the internal forces which hold together the various parts of a given member will now be considered.

We will first analyze the internal forces in the members of a frame, such as the crane considered in Secs. 6.1 and 6.10, noting that whereas the internal forces in a straight two-force member can produce only *tension* or *compression* in that member, the internal forces in any other type of member usually produce *shear* and *bending* as well.

Most of this chapter will be devoted to the analysis of the internal forces in two important types of engineering structures, namely,

1. *Beams*, which are usually long, straight prismatic members designed to support loads applied at various points along the member.
2. *Cables*, which are flexible members capable of withstanding only tension, designed to support either concentrated or distributed loads. Cables are used in many engineering applications, such as suspension bridges and transmission lines.

*7.2 INTERNAL FORCES IN MEMBERS

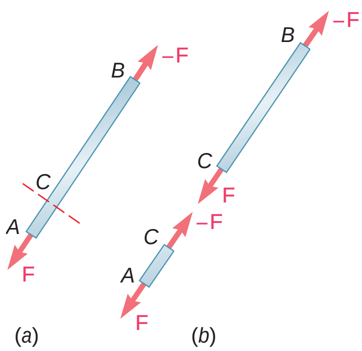


Fig. 7.1

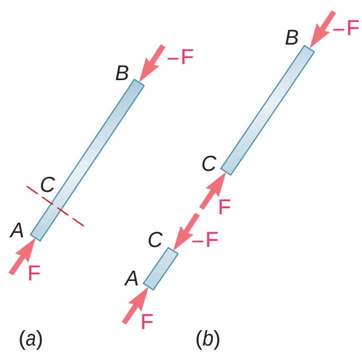


Fig. 7.2

Let us first consider a *straight two-force member AB* (Fig. 7.1*a*). From Sec. 4.6, we know that the forces **F** and −**F** acting at *A* and *B*, respectively, must be directed along *AB* in opposite sense and have the same magnitude *F*. Now, let us cut the member at *C*. To maintain the equilibrium of the free bodies *AC* and *CB* thus obtained, we must apply to *AC* a force −**F** equal and opposite to **F**, and to *CB* a force **F** equal and opposite to −**F** (Fig. 7.1*b*). These new forces are directed along *AB* in opposite sense and have the same magnitude *F*. Since the two parts *AC* and *CB* were in equilibrium before the member was cut, *internal forces* equivalent to these new forces must have existed in the member itself. We conclude that in the case of a straight two-force member, the internal forces that the two portions of the member exert on each other are equivalent to *axial forces*. The common magnitude *F* of these forces does not depend upon the location of the section *C* and is referred to as the *force in member AB*. In the case considered, the member is in tension and will elongate under the action of the internal forces. In the case represented in Fig. 7.2, the member is in compression and will decrease in length under the action of the internal forces.

Next, let us consider a *multiforce member*. Take, for instance, member *AD* of the crane analyzed in Sec. 6.10. This crane is shown again in Fig. 7.3*a*, and the free-body diagram of member *AD* is drawn in Fig. 7.3*b*. We now cut member *AD* at *J* and draw a free-body diagram for each of the portions *JD* and *AJ* of the member

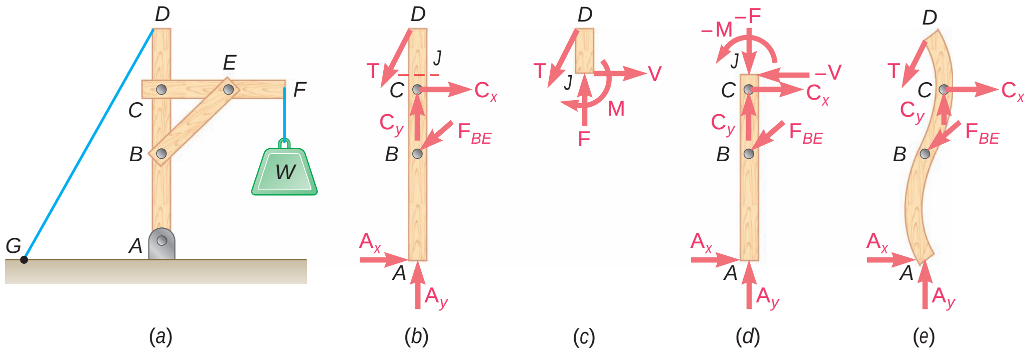


Fig. 7.3

(Fig. 7.3*c* and *d*). Considering the free body *JD*, we find that its equilibrium will be maintained if we apply at *J* a force **F** to balance the vertical component of **T**, a force **V** to balance the horizontal component of **T**, and a couple **M** to balance the moment of **T** about *J*. Again we conclude that internal forces must have existed at *J* before member *AD* was cut. The internal forces acting on the portion *JD* of member *AD* are equivalent to the force-couple system shown in Fig. 7.3*c*. According to Newton's third law, the internal forces acting on *AJ* must be equivalent to an equal and opposite force-couple system, as shown in Fig. 7.3*d*. It is clear that the action of the internal forces in member *AD is not limited to producing tension or compression* as in the case of straight two-force members; the internal forces *also produce shear and bending*. The force **F** is an *axial force;* the force **V** is called a *shearing force;* and the moment **M** of the couple is known as the *bending moment at J*. We note that when determining internal forces in a member, we should clearly indicate on which portion of the member the forces are supposed to act. The deformation which will occur in member *AD* is sketched in Fig. 7.3*e*. The actual analysis of such a deformation is part of the study of mechanics of materials.

It should be noted that in a *two-force member which is not straight,* the internal forces are also equivalent to a force-couple system. This is shown in Fig. 7.4, where the two-force member *ABC* has been cut at *D*.

Photo 7.1 The design of the shaft of a circular saw must account for the internal forces resulting from the forces applied to the teeth of the blade. At a given point in the shaft, these internal forces are equivalent to a force-couple system consisting of axial and shearing forces and a couple representing the bending and torsional moments.

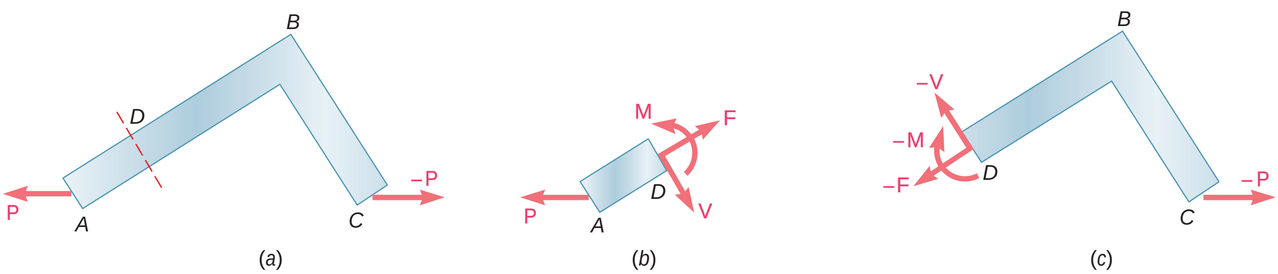
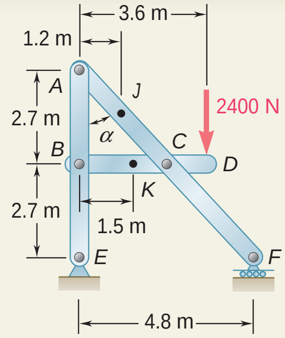
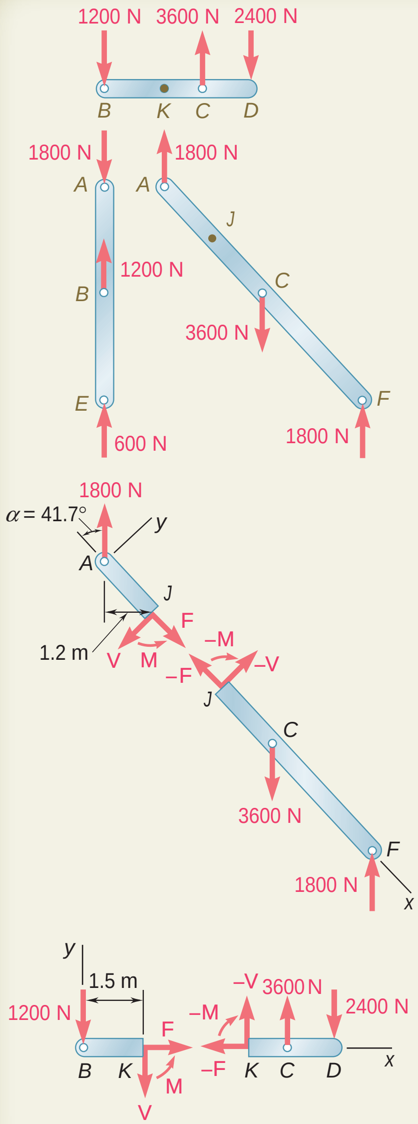


Fig. 7.4



SAMPLE PROBLEM 7.1

In the frame shown, determine the internal forces (*a*) in member *ACF* at point *J*, (*b*) in member *BCD* at point *K*. This frame has been previously considered in Sample Prob. 6.5.



SOLUTION

Reactions and Forces at Connections. The reactions and the forces acting on each member of the frame are determined; this has been previously done in Sample Prob. 6.5, and the results are repeated here.

a. Internal Forces at *J*. Member *ACF* is cut at point *J*, and the two parts shown are obtained. The internal forces at *J* are represented by an equivalent force-couple system and can be determined by considering the equilibrium of either part. Considering the *free body AJ*, we write

$$+\circlearrowleft \Sigma M_J = 0: \quad -(1800 \text{ N})(1.2 \text{ m}) + M = 0$$
$$M = +2160 \text{ N} \cdot \text{m} \qquad \mathbf{M} = 2160 \text{ N} \cdot \text{m} \circlearrowleft \blacktriangleleft$$
$$+\searrow \Sigma F_x = 0: \quad F - (1800 \text{ N}) \cos 41.7° = 0$$
$$F = +1344 \text{ N} \qquad \mathbf{F} = 1344 \text{ N} \searrow \blacktriangleleft$$
$$+\nearrow \Sigma F_y = 0: \quad -V + (1800 \text{ N}) \sin 41.7° = 0$$
$$V = +1197 \text{ N} \qquad \mathbf{V} = 1197 \text{ N} \swarrow \blacktriangleleft$$

The internal forces at *J* are therefore equivalent to a couple **M**, an axial force **F**, and a shearing force **V**. The internal force-couple system acting on part *JCF* is equal and opposite.

b. Internal Forces at *K*. We cut member *BCD* at *K* and obtain the two parts shown. Considering the *free body BK*, we write

$$+\circlearrowleft \Sigma M_K = 0: \quad (1200 \text{ N})(1.5 \text{ m}) + M = 0$$
$$M = -1800 \text{ N} \cdot \text{m} \qquad \mathbf{M} = 1800 \text{ N} \cdot \text{m} \circlearrowright \blacktriangleleft$$
$$\xrightarrow{+} \Sigma F_x = 0: \quad F = 0 \qquad \mathbf{F} = 0 \blacktriangleleft$$
$$+\uparrow \Sigma F_y = 0: \quad -1200 \text{ N} - V = 0$$
$$V = -1200 \text{ N} \qquad \mathbf{V} = 1200 \text{ N} \uparrow \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to determine the internal forces in the member of a frame. The internal forces at a given point in a *straight two-force member* reduce to an axial force, but in all other cases, they are equivalent to a *force-couple system* consisting of an *axial force* **F**, a *shearing force* **V**, and a couple **M** representing the *bending moment* at that point.

To determine the internal forces at a given point *J* of the member of a frame, you should take the following steps.

1. Draw a free-body diagram of the entire frame, and use it to determine as many of the reactions at the supports as you can.

2. Dismember the frame, and draw a free-body diagram of each of its members. Write as many equilibrium equations as are necessary to find all the forces acting on the member on which point *J* is located.

3. Cut the member at point *J*, and draw a free-body diagram of each of the two portions of the member that you have obtained, applying to each portion at point *J* the force components and couple representing the internal forces exerted by the other portion. Note that these force components and couples are equal in magnitude and opposite in sense.

4. Select one of the two free-body diagrams you have drawn and use it to write three equilibrium equations for the corresponding portion of member.

a. Summing moments about *J* and equating them to zero will yield the bending moment at point *J*.

b. Summing components in directions parallel and perpendicular to the member at *J* and equating them to zero will yield, respectively, the axial and shearing force.

5. When recording your answers, be sure to specify the portion of the member you have used, since the forces and couples acting on the two portions have opposite senses.

Since the solutions of the problems in this lesson require the determination of the forces exerted on each other by the various members of a frame, be sure to review the methods used in Chap. 6 to solve this type of problem. When frames involve pulleys and cables, for instance, remember that the forces exerted by a pulley on the member of the frame to which it is attached have the same magnitude and direction as the forces exerted by the cable on the pulley [Prob. 6.90].

PROBLEMS

7.1 and 7.2 Determine the internal forces (axial force, shearing force, and bending moment) at point J of the structure indicated.

7.1 Frame and loading of Prob. 6.75
7.2 Frame and loading of Prob. 6.78

7.3 Determine the internal forces at point J when a = 90°.

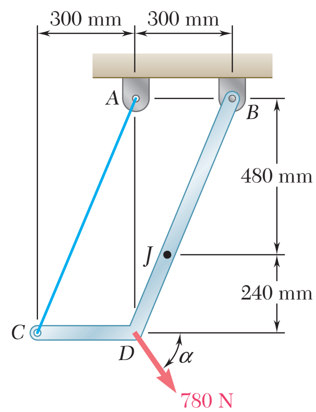


Fig. P7.3 and P7.4

7.4 Determine the internal forces at point J when a = 0.

7.5 and 7.6 Knowing that the turnbuckle has been tightened until the tension in wire AD is 850 N, determine the internal forces at the point indicated:

7.5 Point J
7.6 Point K

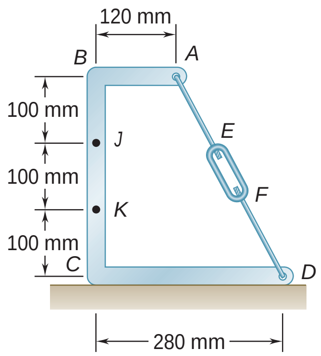


Fig. P7.5 and P7.6

7.7 Two members, each consisting of a straight and a quarter-circular portion of rod, are connected as shown and support a 75-lb load at A. Determine the internal forces at point J.

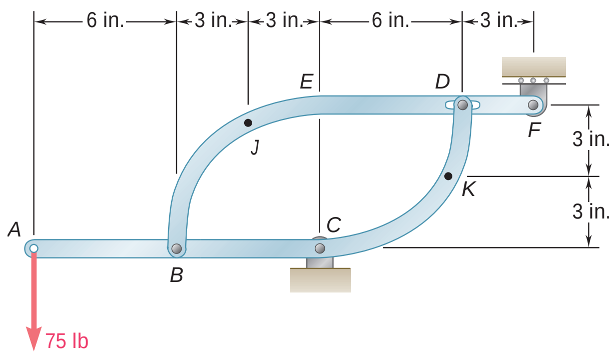


Fig. P7.7 and P7.8

7.8 Two members, each consisting of a straight and a quarter-circular portion of rod, are connected as shown and support a 75-lb load at A. Determine the internal forces at point K.

7.9 A semicircular rod is loaded as shown. Determine the internal forces at point *J*.

7.10 A semicircular rod is loaded as shown. Determine the internal forces at point *K*.

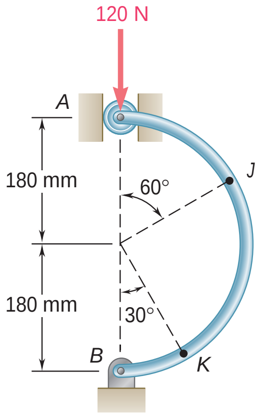


Fig. P7.9 and P7.10

7.11 A semicircular rod is loaded as shown. Determine the internal forces at point *J* knowing that $u = 30°$.

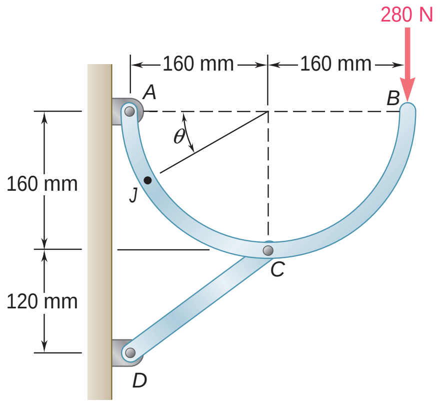


Fig. P7.11 and P7.12

7.12 A semicircular rod is loaded as shown. Determine the magnitude and location of the maximum bending moment in the rod.

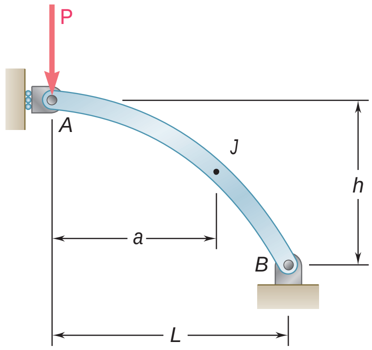


Fig. *P7.13* and *P7.14*

7.13 The axis of the curved member *AB* is a parabola with vertex at *A*. If a vertical load **P** of magnitude 450 lb is applied at *A*, determine the internal forces at *J* when $h = 12$ in., $L = 40$ in., and $a = 24$ in.

7.14 Knowing that the axis of the curved member *AB* is a parabola with vertex at *A*, determine the magnitude and location of the maximum bending moment.

7.15 Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at point *J* of the frame shown.

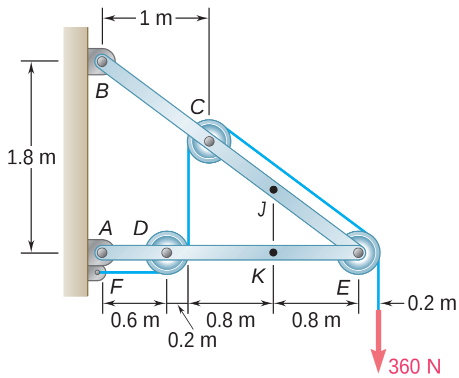


Fig. P7.15 and P7.16

7.16 Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at point *K* of the frame shown.

7.17 A 5-in.-diameter pipe is supported every 9 ft by a small frame consisting of two members as shown. Knowing that the combined weight of the pipe and its contents is 10 lb/ft and neglecting the effect of friction, determine the magnitude and location of the maximum bending moment in member *AC*.

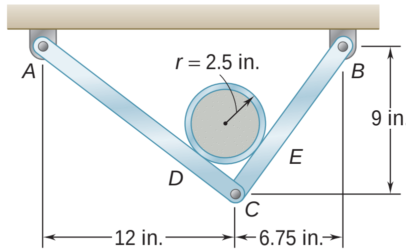


Fig. P7.17

7.18 For the frame of Prob. 7.17, determine the magnitude and location of the maximum bending moment in member *BC*.

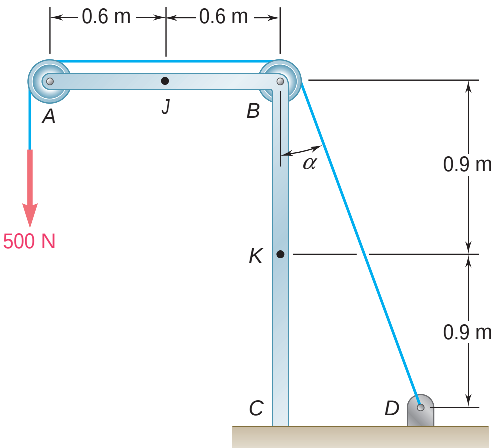


Fig. P7.19 and P7.20

7.19 Knowing that the radius of each pulley is 150 mm, that $a = 20°$, and neglecting friction, determine the internal forces at (*a*) point *J*, (*b*) point *K*.

7.20 Knowing that the radius of each pulley is 150 mm, that $a = 30°$, and neglecting friction, determine the internal forces at (*a*) point *J*, (*b*) point *K*.

7.21* and *7.22 A force **P** is applied to a bent rod that is supported by a roller and a pin and bracket. For each of the three cases shown, determine the internal forces at point *J*.

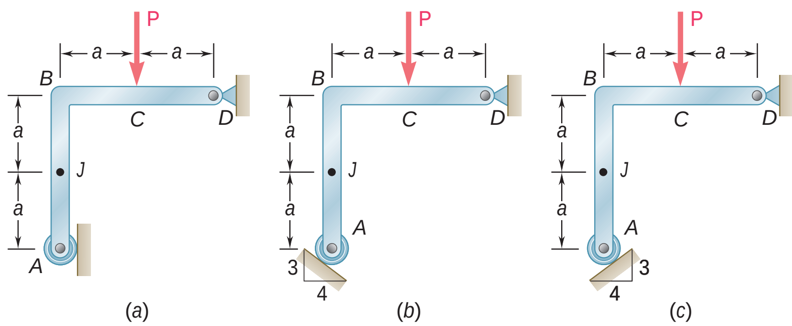


Fig. *P7.21*

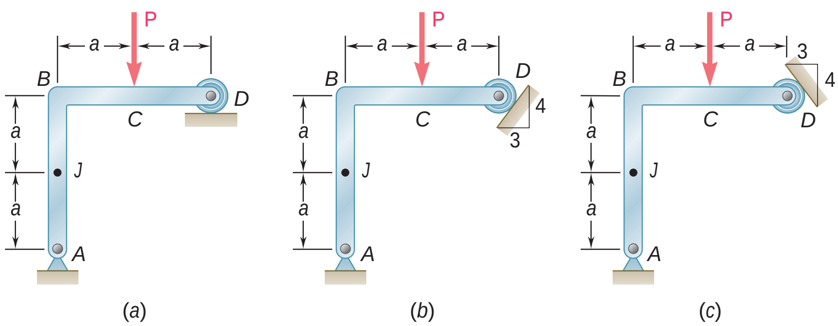


Fig. *P7.22*

7.23 and 7.24 A quarter-circular rod of weight W and uniform cross section is supported as shown. Determine the bending moment at point J when u = 30°.

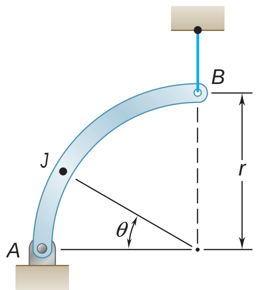


Fig. P7.23

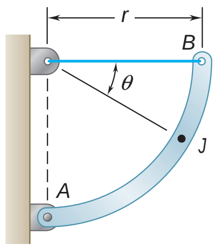


Fig. P7.24

7.25 For the rod of Prob. 7.23, determine the magnitude and location of the maximum bending moment.

7.26 For the rod of Prob. 7.24, determine the magnitude and location of the maximum bending moment.

7.27* and *7.28 A half section of pipe rests on a frictionless horizontal surface as shown. If the half section of pipe has a mass of 9 kg and a diameter of 300 mm, determine the bending moment at point J when u = 90°.

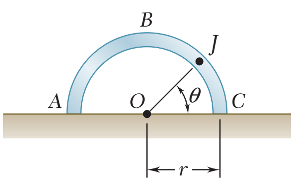


Fig. *P7.27*

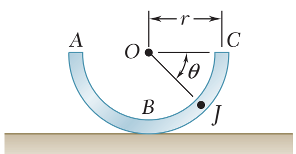


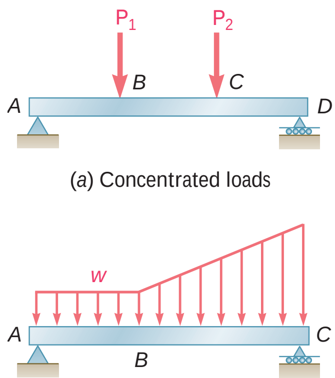
Fig. *P7.28*

BEAMS

*7.3 VARIOUS TYPES OF LOADING AND SUPPORT

A structural member designed to support loads applied at various points along the member is known as a *beam*. In most cases, the loads are perpendicular to the axis of the beam and will cause only shear and bending in the beam. When the loads are not at a right angle to the beam, they will also produce axial forces in the beam.

Beams are usually long, straight prismatic bars. Designing a beam for the most effective support of the applied loads is a two-part



(a) Concentrated loads

(b) Distributed load

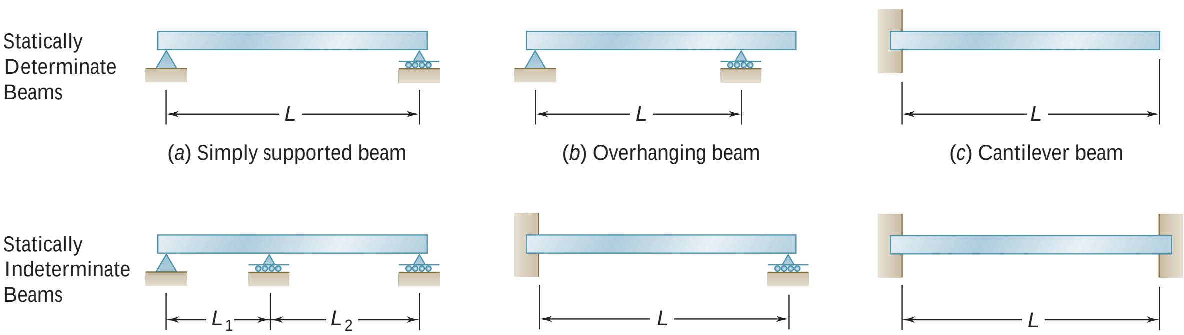
Fig. 7.5

process: (1) determining the shearing forces and bending moments produced by the loads and (2) selecting the cross section best suited to resist the shearing forces and bending moments determined in the first part. Here we are concerned with the first part of the problem of beam design. The second part belongs to the study of mechanics of materials.

A beam can be subjected to *concentrated loads* $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., expressed in newtons, pounds, or their multiples kilonewtons and kips (Fig. 7.5*a*), to a *distributed load* w, expressed in N/m, kN/m, lb/ft, or kips/ft (Fig. 7.5*b*), or to a combination of both. When the load w per unit length has a constant value over part of the beam (as between A and B in Fig. 7.5*b*), the load is said to be *uniformly distributed* over that part of the beam. The determination of the reactions at the supports is considerably simplified if distributed loads are replaced by equivalent concentrated loads, as explained in Sec. 5.8. This substitution, however, should not be performed, or at least should be performed with care, when internal forces are being computed (see Sample Prob. 7.3).

Beams are classified according to the way in which they are supported. Several types of beams frequently used are shown in Fig. 7.6. The distance L between supports is called the *span*. It should be noted that the reactions will be determinate if the supports involve only three unknowns. If more unknowns are involved, the reactions will be statically indeterminate and the methods of statics will not be sufficient to determine the reactions; the properties of the beam with regard to its resistance to bending must then be taken into consideration. Beams supported by two rollers are not shown here; they are only partially constrained and will move under certain loadings.

Sometimes two or more beams are connected by hinges to form a single continuous structure. Two examples of beams hinged at a point H are shown in Fig. 7.7. It will be noted that the reactions at the supports involve four unknowns and cannot be



(a) Simply supported beam

(b) Overhanging beam

(c) Cantilever beam

(d) Continuous beam

(e) Beam fixed at one end and simply supported at the other end

(f) Fixed beam

Fig. 7.6

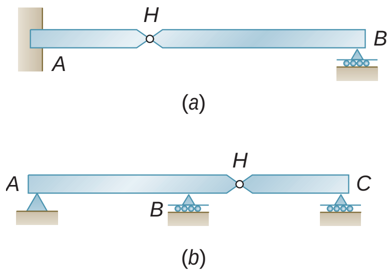


Fig. 7.7

determined from the free-body diagram of the two-beam system. They can be determined, however, by considering the free-body diagram of each beam separately; six unknowns are involved (including two force components at the hinge), and six equations are available.

*7.4 SHEAR AND BENDING MOMENT IN A BEAM

Consider a beam AB subjected to various concentrated and distributed loads (Fig. 7.8a). We propose to determine the shearing force and bending moment at any point of the beam. In the example considered here, the beam is simply supported, but the method used could be applied to any type of statically determinate beam.

First we determine the reactions at A and B by choosing the entire beam as a free body (Fig. 7.8b); writing $\Sigma M_A = 0$ and $\Sigma M_B = 0$, we obtain, respectively, $\mathbf{R}_B$ and $\mathbf{R}_A$.

Photo 7.2 The internal forces in the beams of the overpass shown vary as the truck crosses the overpass.

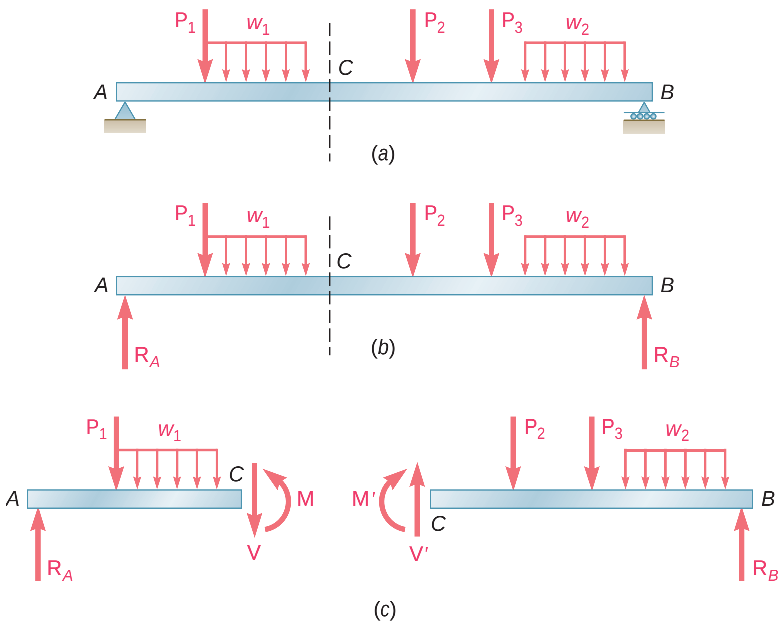


Fig. 7.8

To determine the internal forces at C, we cut the beam at C and draw the free-body diagrams of the portions AC and CB of the beam (Fig. 7.8*c*). Using the free-body diagram of AC, we can determine the shearing force $\mathbf{V}$ at C by equating to zero the sum of the vertical components of all forces acting on AC. Similarly, the bending moment $\mathbf{M}$ at C can be found by equating to zero the sum of the moments about C of all forces and couples acting on AC. Alternatively, we could use the free-body diagram of CB† and determine the shearing force $\mathbf{V}'$ and the bending moment $\mathbf{M}'$ by equating to zero the sum of the vertical components and the sum of the moments about C of all forces and couples acting on CB. While this choice of free bodies may facilitate the computation of the numerical values of the shearing force and bending moment, it makes it necessary to indicate on which portion of the beam the internal forces considered are acting. If the shearing force and bending moment are to be computed at every point of the beam and efficiently recorded, we must find a way to avoid having to specify every time which portion of the beam is used as a free body. We shall adopt, therefore, the following conventions:

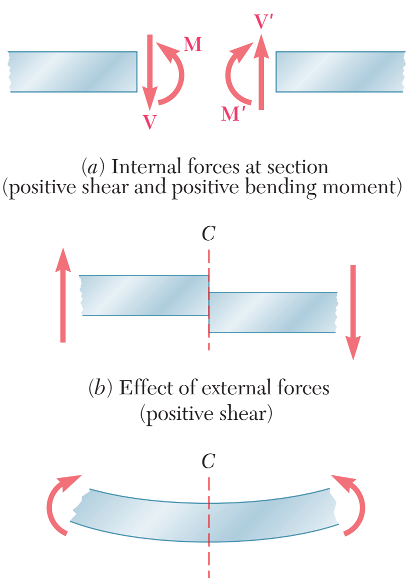
In determining the shearing force in a beam, *it will always be assumed* that the internal forces $\mathbf{V}$ and $\mathbf{V}'$ are directed as shown in Fig. 7.8*c*. A positive value obtained for their common magnitude V will indicate that this assumption was correct and that the shearing forces are actually directed as shown. A negative value obtained for V will indicate that the assumption was wrong and that the shearing forces are directed in the opposite way. Thus, only the magnitude V, together with a plus or minus sign, needs to be recorded to define completely the shearing forces at a given point of the beam. The scalar V is commonly referred to as the *shear* at the given point of the beam.

Similarly, *it will always be assumed* that the internal couples $\mathbf{M}$ and $\mathbf{M}'$ are directed as shown in Fig. 7.8*c*. A positive value obtained for their magnitude M, commonly referred to as the bending moment, will indicate that this assumption was correct, and a negative value will indicate that it was wrong. Summarizing the sign conventions we have presented, we state:

The shear V and the bending moment M at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. 7.9a.

These conventions can be more easily remembered if we note that:

1. *The shear at C is positive when the* **external** *forces (loads and reactions) acting on the beam tend to shear off the beam at C as indicated in Fig. 7.9b.*
2. *The bending moment at C is positive when the* **external** *forces acting on the beam tend to bend the beam at C as indicated in Fig. 7.9c.*



(*a*) Internal forces at section (positive shear and positive bending moment)

(*b*) Effect of external forces (positive shear)

(*c*) Effect of external forces (positive bending moment)

Fig. 7.9

†The force and couple representing the internal forces acting on CB will now be denoted by $\mathbf{V}'$ and $\mathbf{M}'$, rather than by $-\mathbf{V}$ and $-\mathbf{M}$ as done earlier, in order to avoid confusion when applying the sign convention which we are about to introduce.

It may also help to note that the situation described in Fig. 7.9, in which the values of the shear and of the bending moment are positive, is precisely the situation which occurs in the left half of a simply supported beam carrying a single concentrated load at its midpoint. This particular example is fully discussed in the following section.

*7.5 SHEAR AND BENDING-MOMENT DIAGRAMS

Now that shear and bending moment have been clearly defined in sense as well as in magnitude, we can easily record their values at any point of a beam by plotting these values against the distance x measured from one end of the beam. The graphs obtained in this way are called, respectively, the *shear diagram* and the *bending-moment diagram*. As an example, consider a simply supported beam AB of span L subjected to a single concentrated load $\mathbf{P}$ applied at its midpoint D (Fig. 7.10*a*). We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 7.10*b*); we find that the magnitude of each reaction is equal to $P/2$.

Next we cut the beam at a point C between A and D and draw the free-body diagrams of AC and CB (Fig. 7.10*c*). *Assuming that shear and bending moment are positive*, we direct the internal forces $\mathbf{V}$ and $\mathbf{V}'$ and the internal couples $\mathbf{M}$ and $\mathbf{M}'$ as indicated in Fig. 7.9*a*. Considering the free body AC and writing that the sum of the vertical components and the sum of the moments about C of the forces acting on the free body are zero, we find $V = +P/2$ and $M = +Px/2$. Both shear and bending moment are therefore positive; this can be checked by observing that the reaction at A tends to shear off and to bend the beam at C as indicated in Fig. 7.9*b* and *c*. We can plot V and M between A and D (Fig. 7.10*e* and *f*); the shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point E between D and B and considering the free body EB (Fig. 7.10*d*), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain $V = -P/2$ and $M = P(L - x)/2$. The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. 7.9*c* but tends to shear it off in a manner opposite to that shown in Fig. 7.9*b*. We can complete, now, the shear and bending-moment diagrams of Fig. 7.10*e* and *f*; the shear has a constant value $V = -P/2$ between D and B, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$.

It should be noted that when a beam is subjected to concentrated loads only, the shear is of constant value between loads and the bending moment varies linearly between loads, but when a beam is subjected to distributed loads, the shear and bending moment vary quite differently (see Sample Prob. 7.3).

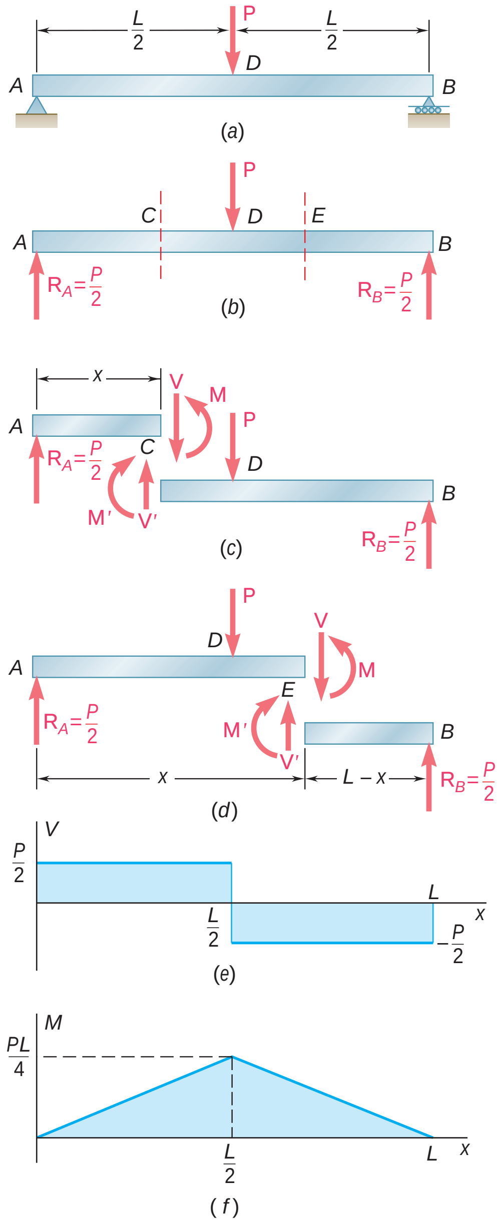
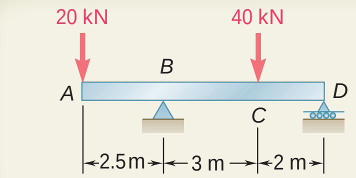
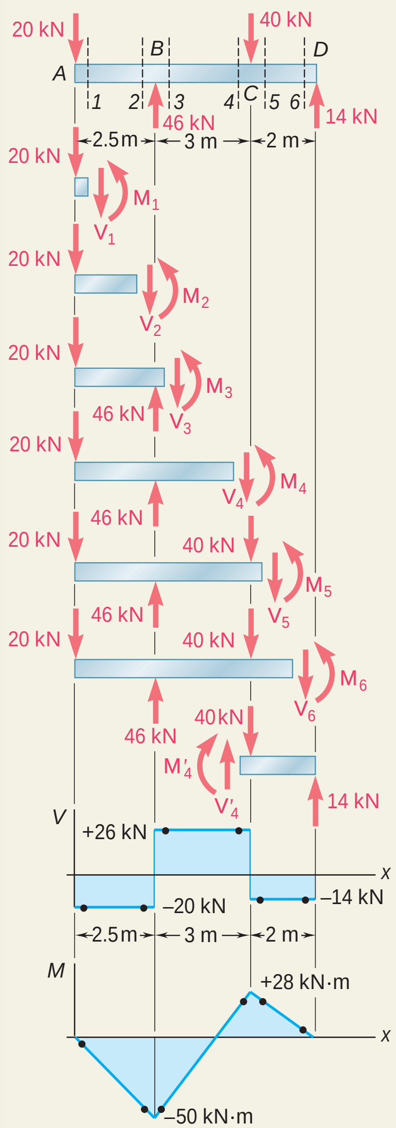


Fig. 7.10



SAMPLE PROBLEM 7.2

Draw the shear and bending-moment diagrams for the beam and loading shown.



SOLUTION

Free-Body: Entire Beam. From the free-body diagram of the entire beam, we find the reactions at B and D:

$$\mathbf{R}_B = 46 \text{ kN x} \qquad \mathbf{R}_D = 14 \text{ kN x}$$

Shear and Bending Moment. We first determine the internal forces just to the right of the 20-kN load at A. Considering the stub of beam to the left of section *1* as a free body and assuming V and M to be positive (according to the standard convention), we write

$$+\text{x}\Sigma F_y = 0: \quad -20 \text{ kN} - V_1 = 0 \qquad V_1 = -20 \text{ kN}$$
$$+\text{l}\ \Sigma M_1 = 0: \quad (20 \text{ kN})(0 \text{ m}) + M_1 = 0 \qquad M_1 = 0$$

We next consider as a free body the portion of the beam to the left of section 2 and write

$$+\text{x}\Sigma F_y = 0: \quad -20 \text{ kN} - V_2 = 0 \qquad V_2 = -20 \text{ kN}$$
$$+\text{l}\ \Sigma M_2 = 0: \quad (20 \text{ kN})(2.5 \text{ m}) + M_2 = 0 \qquad M_2 = -50 \text{ kN} \cdot \text{m}$$

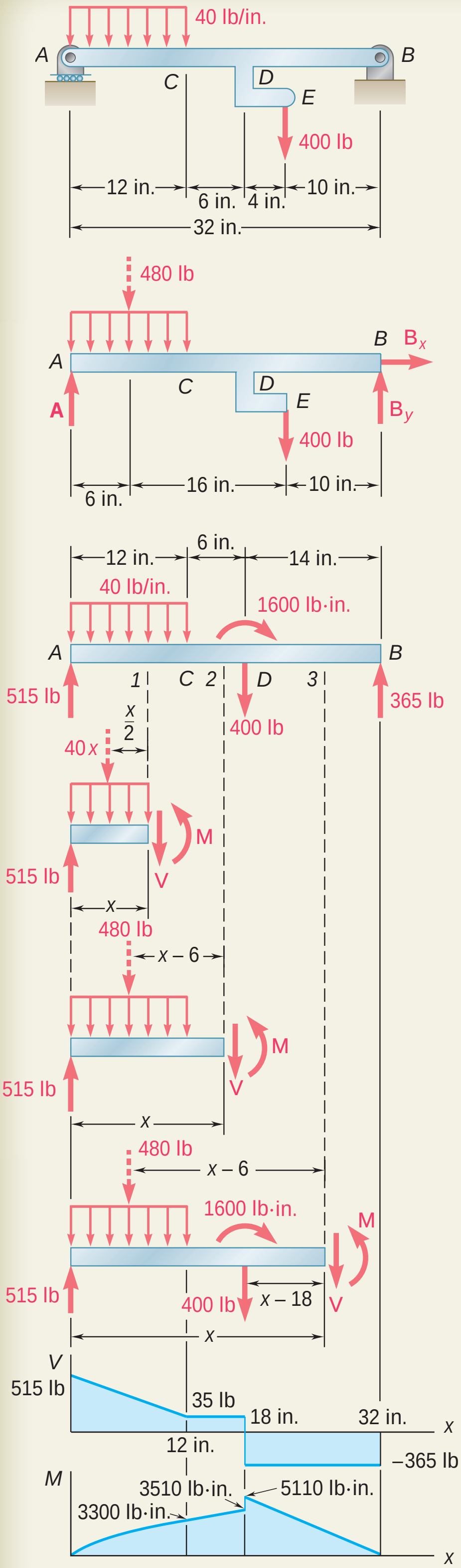
The shear and bending moment at sections *3*, *4*, *5*, and *6* are determined in a similar way from the free-body diagrams shown. We obtain

$$V_3 = +26 \text{ kN} \qquad M_3 = -50 \text{ kN} \cdot \text{m}$$
$$V_4 = +26 \text{ kN} \qquad M_4 = +28 \text{ kN} \cdot \text{m}$$
$$V_5 = -14 \text{ kN} \qquad M_5 = +28 \text{ kN} \cdot \text{m}$$
$$V_6 = -14 \text{ kN} \qquad M_6 = 0$$

For several of the latter sections, the results are more easily obtained by considering as a free body the portion of the beam to the right of the section. For example, considering the portion of the beam to the right of section *4*, we write

$$+\text{x}\Sigma F_y = 0: \quad V_4 - 40 \text{ kN} + 14 \text{ kN} = 0 \qquad V_4 = +26 \text{ kN}$$
$$+\text{l}\ \Sigma M_4 = 0: \quad -M_4 + (14 \text{ kN})(2 \text{ m}) = 0 \qquad M_4 = +28 \text{ kN} \cdot \text{m}$$

Shear and Bending-Moment Diagrams. We can now plot the six points shown on the shear and bending-moment diagrams. As indicated in Sec. 7.5, the shear is of constant value between concentrated loads, and the bending moment varies linearly; we therefore obtain the shear and bending-moment diagrams shown.



SAMPLE PROBLEM 7.3

Draw the shear and bending-moment diagrams for the beam *AB*. The distributed load of 40 lb/in. extends over 12 in. of the beam, from *A* to *C*, and the 400-lb load is applied at *E*.

SOLUTION

Free-Body: Entire Beam. The reactions are determined by considering the entire beam as a free body.

$+l\ \Sigma M_A = 0$: $\quad B_y(32 \text{ in.}) - (480 \text{ lb})(6 \text{ in.}) - (400 \text{ lb})(22 \text{ in.}) = 0$

$B_y = +365 \text{ lb} \qquad \mathbf{B}_y = 365 \text{ lb x}$

$+l\ \Sigma M_B = 0$: $\quad (480 \text{ lb})(26 \text{ in.}) + (400 \text{ lb})(10 \text{ in.}) - A(32 \text{ in.}) = 0$

$A = +515 \text{ lb} \qquad \mathbf{A} = 515 \text{ lb x}$

$\overset{+}{y}\ \Sigma F_x = 0$: $\quad B_x = 0 \qquad \mathbf{B}_x = 0$

The 400-lb load is now replaced by an equivalent force-couple system acting on the beam at point *D*.

Shear and Bending Moment. ***From A to C.*** We determine the internal forces at a distance *x* from point *A* by considering the portion of the beam to the left of section *1*. That part of the distributed load acting on the free body is replaced by its resultant, and we write

$+x\Sigma F_y = 0$: $\quad 515 - 40x - V = 0 \qquad V = 515 - 40x$

$+l\ \Sigma M_1 = 0$: $\quad -515x + 40x(\tfrac{1}{2}x) + M = 0 \qquad M = 515x - 20x^2$

Since the free-body diagram shown can be used for all values of *x* smaller than 12 in., the expressions obtained for *V* and *M* are valid throughout the region $0 < x < 12$ in.

From C to D. Considering the portion of the beam to the left of section 2 and again replacing the distributed load by its resultant, we obtain

$+x\Sigma F_y = 0$: $\quad 515 - 480 - V = 0 V = 35 \text{ lb}$

$+l\ \Sigma M_2 = 0$: $\quad -515x + 480(x - 6) + M = 0 \qquad M = (2880 + 35x) \text{ lb} \cdot \text{in.}$

These expressions are valid in the region 12 in. $< x <$ 18 in.

From D to B. Using the portion of the beam to the left of section 3, we obtain for the region 18 in. $< x <$ 32 in.

$+x\Sigma F_y = 0$: $\quad 515 - 480 - 400 - V = 0 \qquad V = -365 \text{ lb}$

$+l\ \Sigma M_3 = 0$: $\quad -515x + 480(x - 6) - 1600 + 400(x - 18) + M = 0$

$M = (11{,}680 - 365x) \text{ lb} \cdot \text{in.}$

Shear and Bending-Moment Diagrams. The shear and bending-moment diagrams for the entire beam can now be plotted. We note that the couple of moment 1600 lb · in. applied at point *D* introduces a discontinuity into the bending-moment diagram.

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to determine the shear V and the *bending moment* M at any point in a beam. You also learned to draw the *shear diagram* and the *bending-moment diagram* for the beam by plotting, respectively, V and M against the distance x measured along the beam.

A. Determining the shear and bending moment in a beam. To determine the shear V and the bending moment M at a given point C of a beam, you should take the following steps.

1. Draw a free-body diagram of the entire beam, and use it to determine the reactions at the beam supports.

2. Cut the beam at point C, and, using the original loading, select one of the two portions of the beam you have obtained.

3. Draw the free-body diagram of the portion of the beam you have selected, showing:

a. The loads and the reaction exerted on that portion of the beam, replacing each distributed load by an equivalent concentrated load as explained earlier in Sec. 5.8.

b. The shearing force and the bending couple representing the internal forces at C. To facilitate recording the shear V and the bending moment M after they have been determined, follow the convention indicated in Figs. 7.8 and 7.9. Thus, if you are using the portion of the beam located to the *left of C*, apply at C a *shearing force* $\mathbf{V}$ *directed downward* and a *bending couple* $\mathbf{M}$ *directed counterclockwise*. If you are using the portion of the beam located to the *right of C*, apply at C a *shearing force* $\mathbf{V}'$ *directed upward* and a *bending couple* $\mathbf{M}'$ *directed clockwise* [Sample Prob. 7.2].

4. Write the equilibrium equations for the portion of the beam you have selected. Solve the equation $\Sigma F_y = 0$ for V and the equation $\Sigma M_C = 0$ for M.

5. Record the values of V and M with the sign obtained for each of them. A positive sign for V means that the shearing forces exerted at C on each of the two portions of the beam are directed as shown in Figs. 7.8 and 7.9; a negative sign means that they have the opposite sense. Similarly, a positive sign for M means that the bending couples at C are directed as shown in these figures, and a negative sign means that they have the opposite sense. In addition, a positive sign for M means that the concavity of the beam at C is directed upward, and a negative sign means that it is directed downward.

B. Drawing the shear and bending-moment diagrams for a beam. These diagrams are obtained by plotting, respectively, V and M against the distance x measured along the beam. However, in most cases the values of V and M need to be computed only at a few points.

1. For a beam supporting only concentrated loads, we note [Sample Prob. 7.2] that

a. The shear diagram consists of segments of horizontal lines. Thus, to draw the shear diagram of the beam you will need to compute V only just to the left or just to the right of the points where the loads or the reactions are applied.

b. The bending-moment diagram consists of segments of oblique straight lines. Thus, to draw the bending-moment diagram of the beam you will need to compute M only at the points where the loads or the reactions are applied.

2. For a beam supporting uniformly distributed loads, we note [Sample Prob. 7.3] that under each of the distributed loads:

a. The shear diagram consists of a segment of an oblique straight line. Thus, you will need to compute V only where the distributed load begins and where it ends.

b. The bending-moment diagram consists of an arc of parabola. In most cases you will need to compute M only where the distributed load begins and where it ends.

3. For a beam with a more complicated loading, it is necessary to consider the free-body diagram of a portion of the beam of arbitrary length x and determine V and M as functions of x. This procedure may have to be repeated several times, since V and M are often represented by different functions in various parts of the beam [Sample Prob. 7.3].

4. When a couple is applied to a beam, the shear has the same value on both sides of the point of application of the couple, but the bending-moment diagram will show a discontinuity at that point, rising or falling by an amount equal to the magnitude of the couple. Note that a couple can either be applied directly to the beam, or result from the application of a load on a curved member rigidly attached to the beam [Sample Prob. 7.3].

PROBLEMS

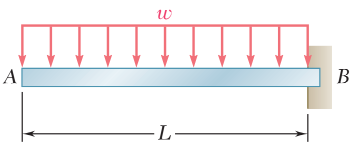


Fig. P7.29

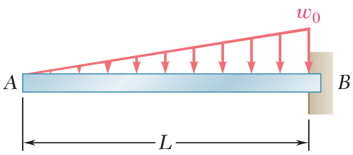


Fig. P7.30

7.29 through 7.32 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

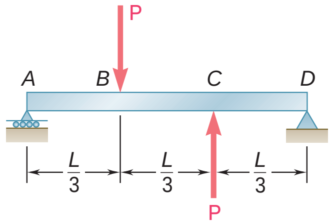


Fig. P7.31

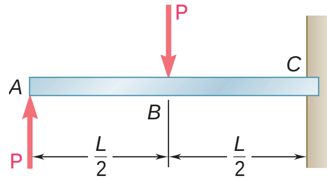


Fig. P7.32

7.33 and 7.34 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

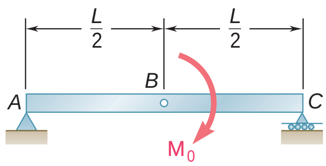


Fig. *P7.33*

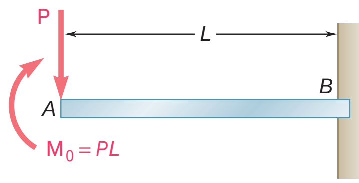


Fig. *P7.34*

7.35 and 7.36 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

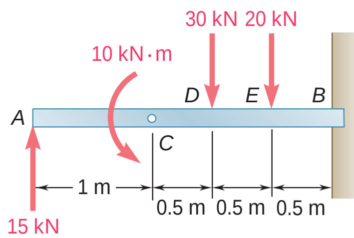


Fig. P7.35

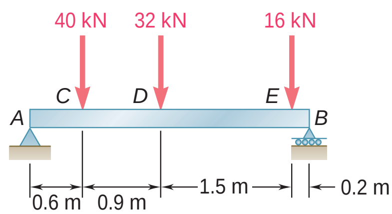


Fig. P7.36

7.37 and 7.38 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

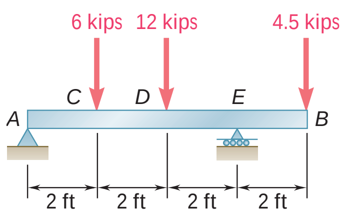


Fig. *P7.37*

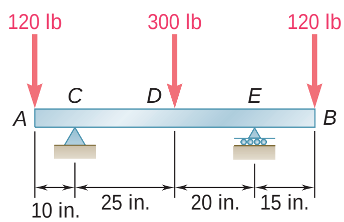


Fig. *P7.38*

7.39 through 7.42 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

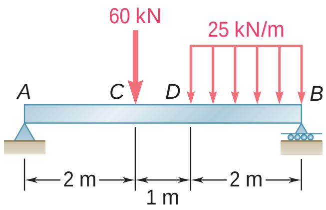


Fig. P7.39

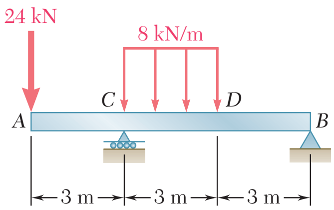


Fig. P7.40

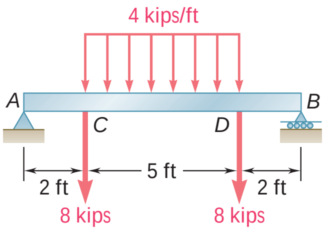


Fig. P7.41

7.43 Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed and knowing that $P = wa$, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

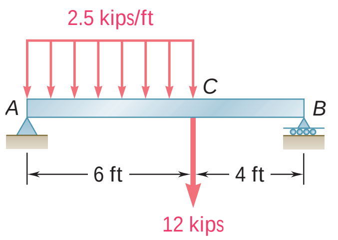


Fig. P7.42

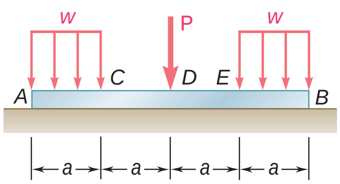


Fig. *P7.43*

7.44 Solve Prob. 7.43 knowing that $P = 3wa$.

7.45 Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed and knowing that $a = 0.3$ m, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

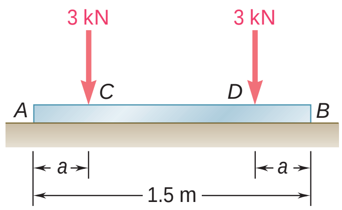


Fig. P7.45

7.46 Solve Prob. 7.45 knowing that $a = 0.5$ m.

7.47 and 7.48 Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

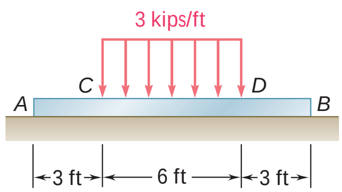


Fig. P7.47

7.49 and 7.50 Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

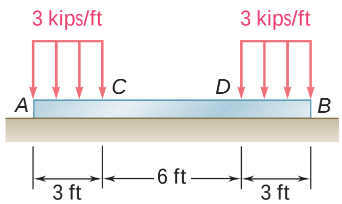


Fig. P7.48

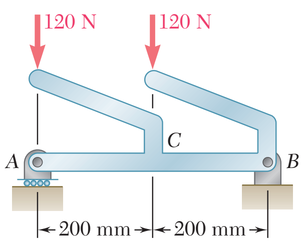


Fig. P7.49

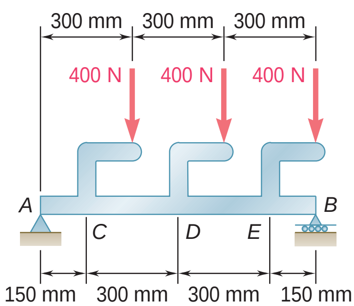


Fig. P7.50

7.51 and 7.52 Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

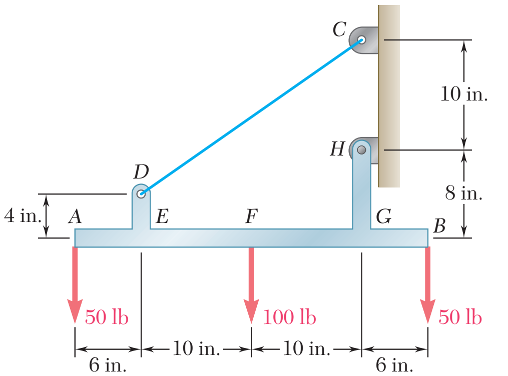


Fig. P7.51

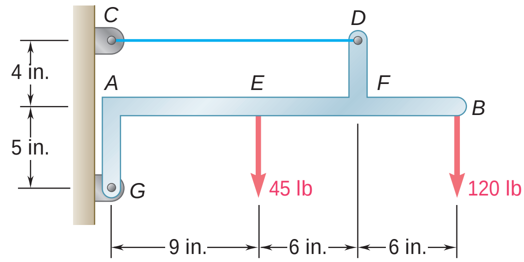


Fig. P7.52

7.53 Two small channel sections *DF* and *EH* have been welded to the uniform beam *AB* of weight $W = 3$ kN to form the rigid structural member shown. This member is being lifted by two cables attached at *D* and *E*. Knowing that u = 30° and neglecting the weight of the channel sections, (*a*) draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the maximum absolute values of the shear and bending moment in the beam.

7.54 Solve Prob. 7.53 when u = 60°.

7.55 For the structural member of Prob. 7.53, determine (*a*) the angle u for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (*b*) the corresponding value of $|M|_{max}$. (*Hint:* Draw the bending-moment diagram and then equate the absolute values of the largest positive and negative bending moments obtained.)

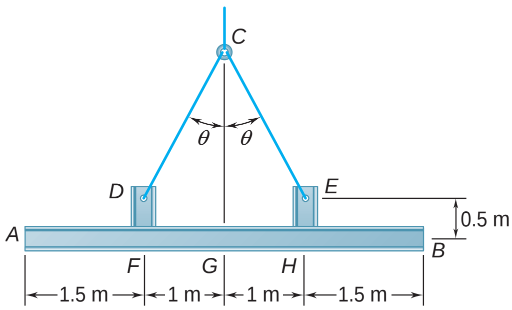


Fig. *P7.53*

7.56 For the beam of Prob. 7.43, determine (*a*) the ratio $k = P/wa$ for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{max}$. (See hint for Prob. 7.55.)

7.57 For the beam of Prob. 7.45, determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{max}$. (See hint for Prob. 7.55.)

7.58 For the beam and loading shown, determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{max}$. (See hint for Prob. 7.55.)

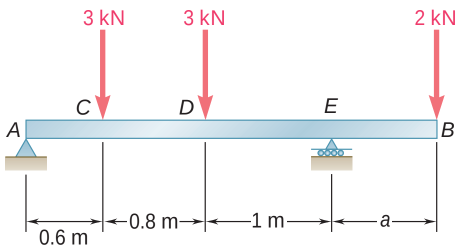


Fig. P7.58

7.59 A uniform beam is to be picked up by crane cables attached at *A* and *B*. Determine the distance *a* from the ends of the beam to the points where the cables should be attached if the maximum absolute value of the bending moment in the beam is to be as small as possible. (*Hint:* Draw the bending-moment diagram in terms of *a*, *L*, and the weight per unit length *w*, and then equate the absolute values of the largest positive and negative bending moments obtained.)

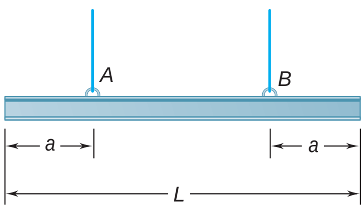


Fig. P7.59

7.60 Knowing that $P = Q = 150$ lb, determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (*b*) the corresponding value of $|M|_{max}$. (See hint for Prob. 7.55.)

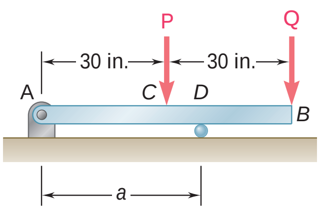


Fig. *P7.60*

7.61 Solve Prob. 7.60 assuming that $P = 300$ lb and $Q = 150$ lb.

***7.62** In order to reduce the bending moment in the cantilever beam AB, a cable and counterweight are permanently attached at end B. Determine the magnitude of the counterweight for which the maximum absolute value of the bending moment in the beam is as small as possible and the corresponding value of $|M|_{max}$. Consider (*a*) the case when the distributed load is permanently applied to the beam, (*b*) the more general case when the distributed load may either be applied or removed.

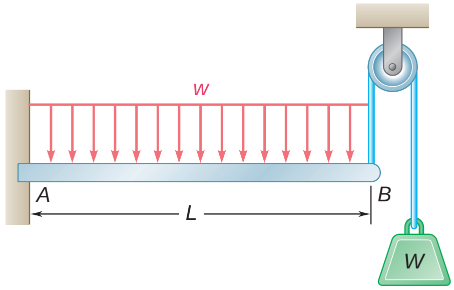


Fig. P7.62

*7.6 RELATIONS AMONG LOAD, SHEAR, AND BENDING MOMENT

When a beam carries more than two or three concentrated loads, or when it carries distributed loads, the method outlined in Sec. 7.5 for plotting shear and bending moment is likely to be quite cumbersome. The construction of the shear diagram and, especially, of the bending-moment diagram will be greatly facilitated if certain relations existing among load, shear, and bending moment are taken into consideration.

Let us consider a simply supported beam AB carrying a distributed load w per unit length (Fig. 7.11*a*), and let C and C' be two points of the beam at a distance Δx from each other. The shear and bending moment at C will be denoted by V and M, respectively, and will be assumed positive; the shear and bending moment at C' will be denoted by $V + \Delta V$ and $M + \Delta M$.

Let us now detach the portion of beam CC' and draw its free-body diagram (Fig. 7.11*b*). The forces exerted on the free body include a load of magnitude $w\,\Delta x$ and internal forces and couples at C and C'. Since shear and bending moment have been assumed positive, the forces and couples will be directed as shown in the figure.

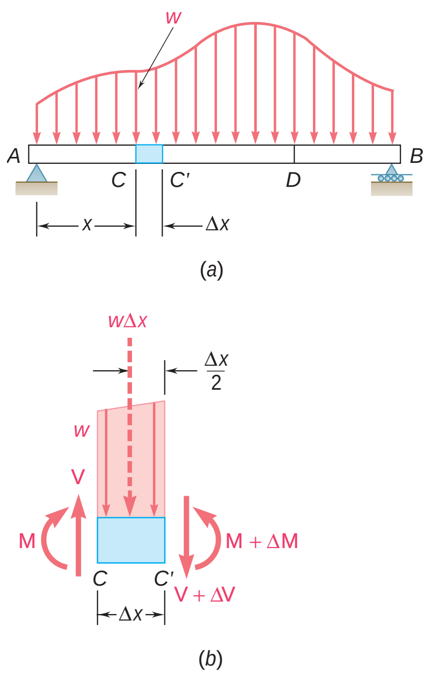


Fig. 7.11

Relations Between Load and Shear. We write that the sum of the vertical components of the forces acting on the free body CC' is zero:

$$V - (V + \Delta V) - w\,\Delta x = 0$$
$$\Delta V = -w\,\Delta x$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\boxed{\frac{dV}{dx} = -w} \qquad (7.1)$$

Formula (7.1) indicates that for a beam loaded as shown in Fig. 7.11*a*, the slope dV/dx of the shear curve is negative; the numerical value of the slope at any point is equal to the load per unit length at that point.

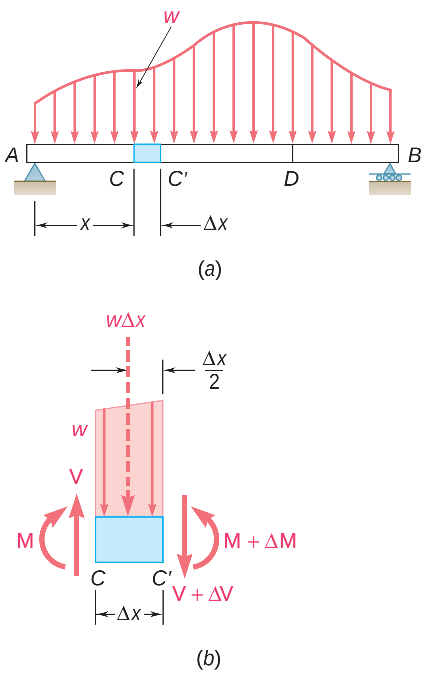


Fig. 7.11 *(repeated)*

Integrating (7.1) between points C and D, we obtain

$$V_D - V_C = -\int_{x_C}^{x_D} w\,dx \tag{7.2}$$

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \tag{7.2′}$$

Note that this result could also have been obtained by considering the equilibrium of the portion of beam CD, since the area under the load curve represents the total load applied between C and D.

It should be observed that formula (7.1) *is not valid* at a point where a concentrated load is applied; the shear curve is discontinuous at such a point, as seen in Sec. 7.5. Similarly, formulas (7.2) and (7.2′) cease to be valid when concentrated loads are applied between C and D, since they do not take into account the sudden change in shear caused by a concentrated load. Formulas (7.2) and (7.2′), therefore, should be applied only between successive concentrated loads.

Relations Between Shear and Bending Moment. Returning to the free-body diagram of Fig. 7.11*b*, and writing now that the sum of the moments about C' is zero, we obtain

$$(M + \Delta M) - M - V\,\Delta x + w\Delta x\frac{\Delta x}{2} = 0$$

$$\Delta M = V\,\Delta x - \tfrac{1}{2}w(\Delta x)^2$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\frac{dM}{dx} = V \tag{7.3}$$

Formula (7.3) indicates that the slope dM/dx of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value, i.e., at any point where no concentrated load is applied. Formula (7.3) also shows that the shear is zero at points where the bending moment is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending.

Integrating (7.3) between points C and D, we obtain

$$M_D - M_C = \int_{x_C}^{x_D} V\,dx \tag{7.4}$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \tag{7.4′}$$

Note that the area under the shear curve should be considered positive where the shear is positive and negative where the shear is negative. Formulas (7.4) and (7.4′) are valid even when concentrated loads are applied between C and D, as long as the shear curve has been correctly drawn. The formulas cease to be valid, however, if a *couple* is applied at a point between C and D, since they do not take into account the sudden change in bending moment caused by a couple (see Sample Prob. 7.7).

EXAMPLE Let us consider a simply supported beam AB of span L carrying a uniformly distributed load w (Fig. 7.12a). From the free-body diagram of the entire beam we determine the magnitude of the reactions at the supports: $R_A = R_B = wL/2$ (Fig. 7.12b). Next, we draw the shear diagram. Close to the end A of the beam, the shear is equal to R_A, that is, to $wL/2$, as we can check by considering a very small portion of the beam as a free body. Using formula (7.2), we can then determine the shear V at any distance x from A. We write

$$V - V_A = -\int_0^x w\,dx = -wx$$

$$V = V_A - wx = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right)$$

The shear curve is thus an oblique straight line which crosses the x axis at $x = L/2$ (Fig. 7.12c). Considering, now, the bending moment, we first observe that $M_A = 0$. The value M of the bending moment at any distance x from A can then be obtained from formula (7.4); we have

$$M - M_A = \int_0^x V\,dx$$

$$M = \int_0^x w\left(\frac{L}{2} - x\right)dx = \frac{w}{2}(Lx - x^2)$$

The bending-moment curve is a parabola. The maximum value of the bending moment occurs when $x = L/2$, since V (and thus dM/dx) is zero for that value of x. Substituting $x = L/2$ in the last equation, we obtain $M_{\max} = wL^2/8$. ■

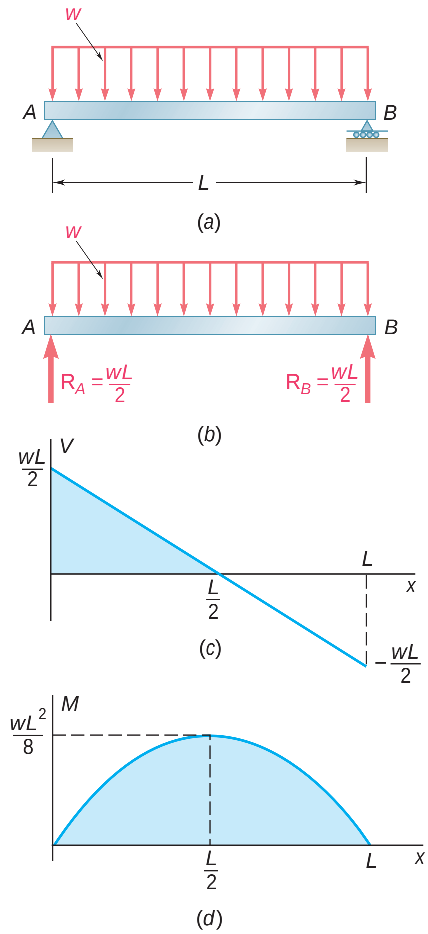
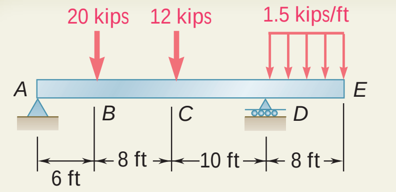


Fig. 7.12

In most engineering applications, the value of the bending moment needs to be known only at a few specific points. Once the shear diagram has been drawn, and after M has been determined at one of the ends of the beam, the value of the bending moment can then be obtained at any given point by computing the area under the shear curve and using formula (7.4′). For instance, since $M_A = 0$ for the beam of Fig. 7.12, the maximum value of the bending moment for that beam can be obtained simply by measuring the area of the shaded triangle in the shear diagram:

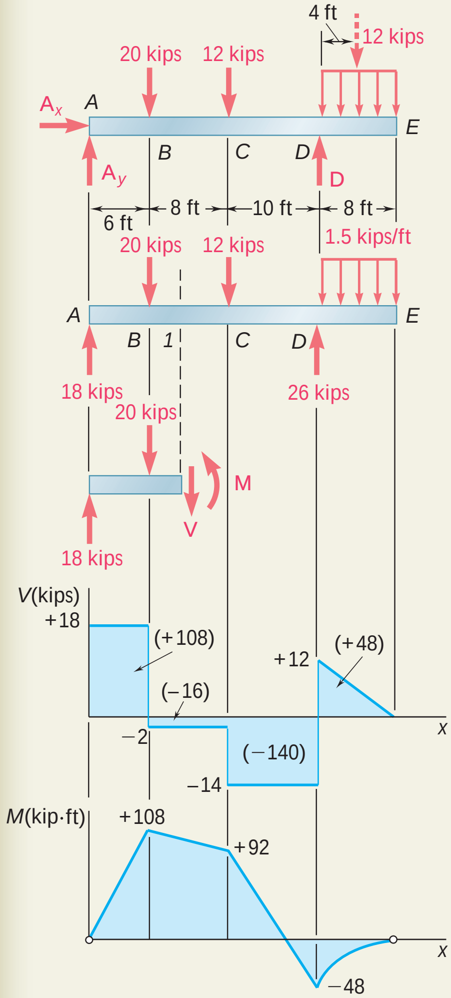
$$M_{\max} = \frac{1}{2}\frac{L}{2}\frac{wL}{2} = \frac{wL^2}{8}$$

In this example, the load curve is a horizontal straight line, the shear curve is an oblique straight line, and the bending-moment curve is a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree), and the bending-moment curve would have been a cubic (third degree). The shear and bending-moment curves will always be, respectively, one and two degrees higher than the load curve. Thus, once a few values of the shear and bending moment have been computed, we should be able to sketch the shear and bending-moment diagrams without actually determining the functions $V(x)$ and $M(x)$. The sketches obtained will be more accurate if we make use of the fact that at any point where the curves are continuous, the slope of the shear curve is equal to $-w$ and the slope of the bending-moment curve is equal to V.



SAMPLE PROBLEM 7.4

Draw the shear and bending-moment diagrams for the beam and loading shown.



SOLUTION

Free-Body: Entire Beam. Considering the entire beam as a free body, we determine the reactions:

$+\Sigma l\, M_A = 0$:

$$D(24 \text{ ft}) - (20 \text{ kips})(6 \text{ ft}) - (12 \text{ kips})(14 \text{ ft}) - (12 \text{ kips})(28 \text{ ft}) = 0$$

$$D = +26 \text{ kips} \qquad \mathbf{D} = 26 \text{ kips x}$$

$+\text{x}\,\Sigma F_y = 0$: $\quad A_y - 20 \text{ kips} - 12 \text{ kips} + 26 \text{ kips} - 12 \text{ kips} = 0$

$$A_y = +18 \text{ kips} \qquad \mathbf{A}_y = 18 \text{ kips x}$$

$\overset{+}{\text{y}}\,\Sigma F_x = 0$: $\quad A_x = 0 \qquad \mathbf{A}_x = 0$

We also note that at both A and E the bending moment is zero; thus two points (indicated by small circles) are obtained on the bending-moment diagram.

Shear Diagram. Since $dV/dx = -w$, we find that between concentrated loads and reactions the slope of the shear diagram is zero (i.e., the shear is constant). The shear at any point is determined by dividing the beam into two parts and considering either part as a free body. For example, using the portion of beam to the left of section *1*, we obtain the shear between B and C:

$+\text{x}\,\Sigma F_y = 0$: $\quad +18 \text{ kips} - 20 \text{ kips} - V = 0 \qquad V = -2 \text{ kips}$

We also find that the shear is $+12$ kips just to the right of D and zero at end E. Since the slope $dV/dx = -w$ is constant between D and E, the shear diagram between these two points is a straight line.

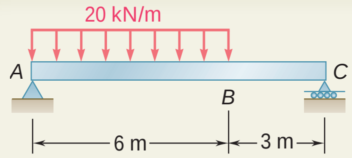
Bending-Moment Diagram. We recall that the area under the shear curve between two points is equal to the change in bending moment between the same two points. For convenience, the area of each portion of the shear diagram is computed and is indicated on the diagram. Since the bending moment M_A at the left end is known to be zero, we write

$$\begin{aligned} M_B - M_A &= +108 & M_B &= +108 \text{ kip} \cdot \text{ft} \\ M_C - M_B &= -\ 16 & M_C &= +\ 92 \text{ kip} \cdot \text{ft} \\ M_D - M_C &= -140 & M_D &= -\ 48 \text{ kip} \cdot \text{ft} \\ M_E - M_D &= +\ 48 & M_E &= 0 \end{aligned}$$

Since M_E is known to be zero, a check of the computations is obtained.

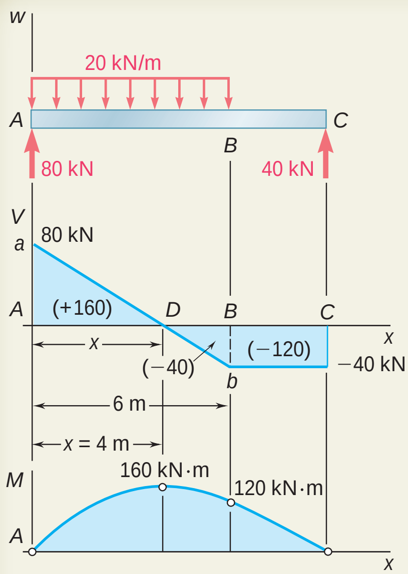
Between the concentrated loads and reactions the shear is constant; thus the slope dM/dx is constant, and the bending-moment diagram is drawn by connecting the known points with straight lines. Between D and E, where the shear diagram is an oblique straight line, the bending-moment diagram is a parabola.

From the V and M diagrams we note that $V_{max} = 18$ kips and $M_{max} = 108$ kip · ft.



SAMPLE PROBLEM 7.5

Draw the shear and bending-moment diagrams for the beam and loading shown and determine the location and magnitude of the maximum bending moment.



SOLUTION

Free-Body: Entire Beam. Considering the entire beam as a free body, we obtain the reactions

$$\mathbf{R}_A = 80 \text{ kN} \uparrow \qquad \mathbf{R}_C = 40 \text{ kN} \uparrow$$

Shear Diagram. The shear just to the right of A is $V_A = +80$ kN. Since the change in shear between two points is equal to *minus* the area under the load curve between the same two points, we obtain V_B by writing

$$V_B - V_A = -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN}$$
$$V_B = -120 + V_A = -120 + 80 = -40 \text{ kN}$$

Since the slope $dV/dx = -w$ is constant between A and B, the shear diagram between these two points is represented by a straight line. Between B and C, the area under the load curve is zero; therefore,

$$V_C - V_B = 0 \qquad V_C = V_B = -40 \text{ kN}$$

and the shear is constant between B and C.

Bending-Moment Diagram. We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section D of the beam where $V = 0$. We write

$$V_D - V_A = -wx$$
$$0 - 80 \text{ kN} = -(20 \text{ kN/m})x$$

and, solving for x: $x = 4$ m ◀

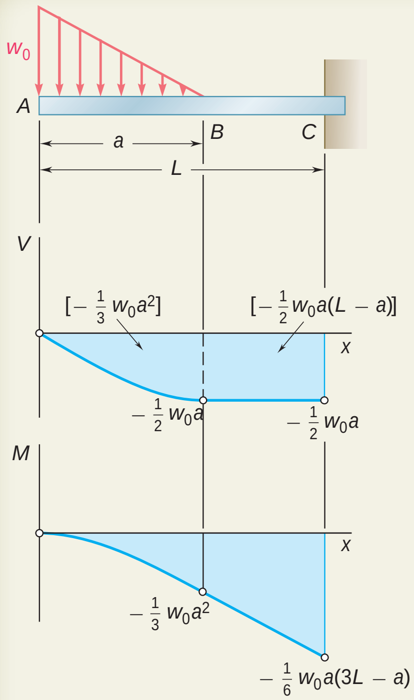
The maximum bending moment occurs at point D, where we have $dM/dx = V = 0$. The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

$$M_D - M_A = +160 \text{ kN}\cdot\text{m} \qquad M_D = +160 \text{ kN}\cdot\text{m}$$
$$M_B - M_D = -\ 40 \text{ kN}\cdot\text{m} \qquad M_B = +120 \text{ kN}\cdot\text{m}$$
$$M_C - M_B = -120 \text{ kN}\cdot\text{m} \qquad M_C = 0$$

The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at A is equal to the value of V at that point.

The maximum bending moment is

$$M_{max} = M_D = +160 \text{ kN}\cdot\text{m}$$ ◀



SAMPLE PROBLEM 7.6

Sketch the shear and bending-moment diagrams for the cantilever beam shown.

SOLUTION

Shear Diagram. At the free end of the beam, we find $V_A = 0$. Between A and B, the area under the load curve is $\frac{1}{2}w_0a$; we find V_B by writing

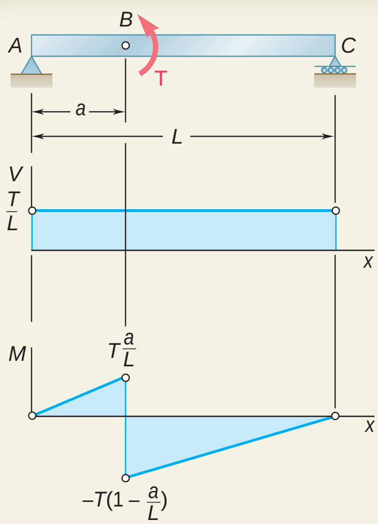
$$V_B - V_A = -\tfrac{1}{2}w_0a \qquad V_B = -\tfrac{1}{2}w_0a$$

Between B and C, the beam is not loaded; thus $V_C = V_B$. At A, we have $w = w_0$, and, according to Eq. (7.1), the slope of the shear curve is $dV/dx = -w_0$, while at B the slope is $dV/dx = 0$. Between A and B, the loading decreases linearly, and the shear diagram is parabolic. Between B and C, $w = 0$, and the shear diagram is a horizontal line.

Bending-Moment Diagram. We note that $M_A = 0$ at the free end of the beam. We compute the area under the shear curve and write

$$M_B - M_A = -\tfrac{1}{3}w_0a^2 \qquad M_B = -\tfrac{1}{3}w_0a^2$$
$$M_C - M_B = -\tfrac{1}{2}w_0a(L-a)$$
$$M_C = -\tfrac{1}{6}w_0a(3L-a)$$

The sketch of the bending-moment diagram is completed by recalling that $dM/dx = V$. We find that between A and B the diagram is represented by a cubic curve with zero slope at A, and between B and C the diagram is represented by a straight line.



SAMPLE PROBLEM 7.7

The simple beam AC is loaded by a couple of magnitude T applied at point B. Draw the shear and bending-moment diagrams for the beam.

SOLUTION

Free-Body: Entire Beam. The entire beam is taken as a free body, and we obtain

$$\mathbf{R}_A = \frac{T}{L}\,\mathrm{x} \qquad \mathbf{R}_C = \frac{T}{L}\,\mathrm{w}$$

Shear and Bending-Moment Diagrams. The shear at any section is constant and equal to T/L. Since a couple is applied at B, the bending-moment diagram is discontinuous at B; the bending moment decreases suddenly by an amount equal to T.

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned how to use the relations existing among load, shear, and bending moment to simplify the drawing of the shear and bending-moment diagrams. These relations are

$$\frac{dV}{dx} = -w \tag{7.1}$$

$$\frac{dM}{dx} = V \tag{7.3}$$

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \qquad (7.2')$$

$$M_D - M_C = (\text{area under shear curve between } C \text{ and } D) \qquad (7.4')$$

Taking into account these relations, you can use the following procedure to draw the shear and bending-moment diagrams for a beam.

1. Draw a free-body diagram of the entire beam, and use it to determine the reactions at the beam supports.

2. Draw the shear diagram. This can be done as in the preceding lesson by cutting the beam at various points and considering the free-body diagram of one of the two portions of the beam that you have obtained [Sample Prob. 7.3]. You can, however, consider one of the following alternative procedures.

a. The shear V at any point of the beam is the sum of the reactions and loads to the left of that point; an upward force is counted as positive, and a downward force is counted as negative.

b. For a beam carrying a distributed load, you can start from a point where you know V and use Eq. (7.2′) repeatedly to find V at all the other points of interest.

3. Draw the bending-moment diagram, using the following procedure.

a. Compute the area under each portion of the shear curve, assigning a positive sign to areas located above the x axis and a negative sign to areas located below the x axis.

b. Apply Eq. (7.4′) repeatedly [Sample Probs. 7.4 and 7.5], starting from the left end of the beam, where $M = 0$ (except if a couple is applied at that end, or if the beam is a cantilever beam with a fixed left end).

c. Where a couple is applied to the beam, be careful to show a discontinuity in the bending-moment diagram by *increasing* the value of M at that point by an amount equal to the magnitude of the couple if the couple is *clockwise,* or *decreasing* the value of M by that amount if the couple is *counterclockwise* [Sample Prob. 7.7].

(*continued*)

4. Determine the location and magnitude of $|M|_{max}$. The maximum absolute value of the bending moment occurs at one of the points where $dM/dx = 0$, that is, according to Eq. (7.3), at a point where V is equal to zero or changes sign. You should, therefore:

a. Determine from the shear diagram the value of $|M|$ where V changes sign; this will occur under the concentrated loads [Sample Prob. 7.4].

b. Determine the points where $V = 0$ and the corresponding values of $|M|$; this will occur under a distributed load. To find the distance x between point C, where the distributed load starts, and point D, where the shear is zero, use Eq. (7.2′); for V_C use the known value of the shear at point C, for V_D use zero, and express the area under the load curve as a function of x [Sample Prob. 7.5].

5. You can improve the quality of your drawings by keeping in mind that at any given point, according to Eqs. (7.1) and (7.3), the slope of the V curve is equal to $-w$ and the slope of the M curve is equal to V.

6. Finally, for beams supporting a distributed load expressed as a function $w(x)$, remember that the shear V can be obtained by integrating the function $-w(x)$, and the bending moment M can be obtained by integrating $V(x)$ [Eqs. (7.3) and (7.4)].

PROBLEMS

7.63 Using the method of Sec. 7.6, solve Prob. 7.29.

7.64 Using the method of Sec. 7.6, solve Prob. 7.30.

7.65 Using the method of Sec. 7.6, solve Prob. 7.31.

7.66 Using the method of Sec. 7.6, solve Prob. 7.32.

7.67 Using the method of Sec. 7.6, solve Prob. 7.33.

7.68 Using the method of Sec. 7.6, solve Prob. 7.34.

7.69 and 7.70 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

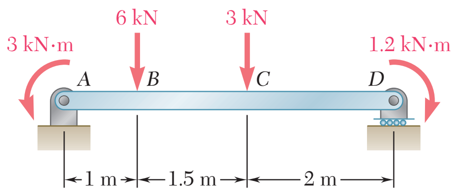


Fig. P7.69

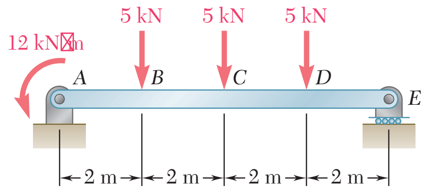


Fig. P7.70

7.71 Using the method of Sec. 7.6, solve Prob. 7.39.

7.72 Using the method of Sec. 7.6, solve Prob. 7.40.

7.73 Using the method of Sec. 7.6, solve Prob. 7.41.

7.74 Using the method of Sec. 7.6, solve Prob. 7.42.

7.75* and *7.76 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

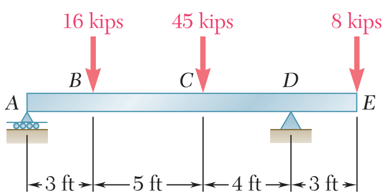


Fig. *P7.75*

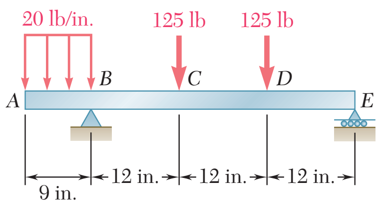


Fig. *P7.76*

7.77 through 7.79 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

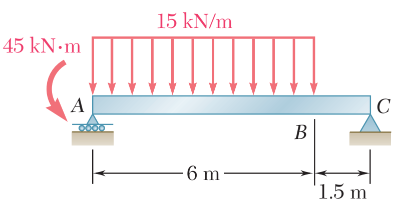


Fig. P7.77

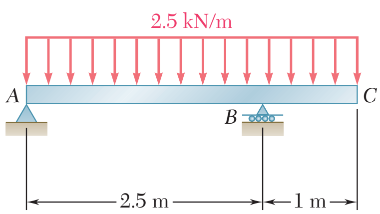


Fig. P7.78

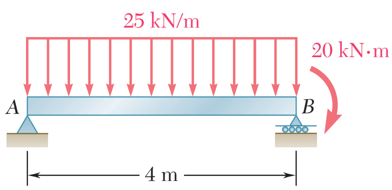


Fig. P7.79

7.80 Solve Prob. 7.79 assuming that the 20-kN · m couple applied at *B* is counterclockwise.

7.81 For the beam shown, draw the shear and bending-moment diagrams, and determine the magnitude and location of the maximum absolute value of the bending moment, knowing that (*a*) $M = 0$, (*b*) $M = 24$ kip · ft.

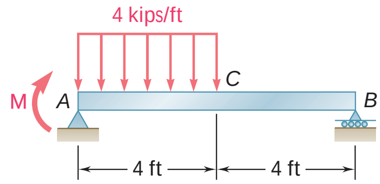


Fig. P7.81

7.82 For the beam shown, draw the shear and bending-moment diagrams, and determine the magnitude and location of the maximum absolute value of the bending moment, knowing that (*a*) $P = 6$ kips, (*b*) $P = 3$ kips.

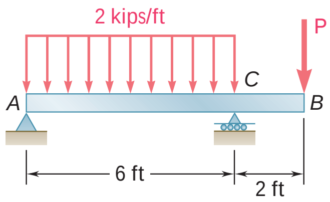


Fig. P7.82

7.83 (*a*) Draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

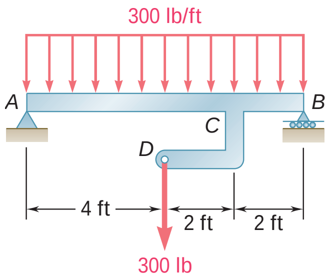


Fig. *P7.83*

7.84 Solve Prob. 7.83 assuming that the 300-lb force applied at *D* is directed upward.

7.85 through 7.87 For the beam and loading shown, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the magnitude and location of the maximum bending moment.

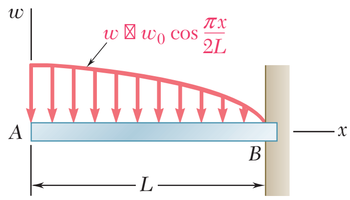


Fig. *P7.85*

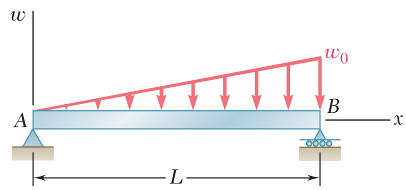


Fig. P7.86

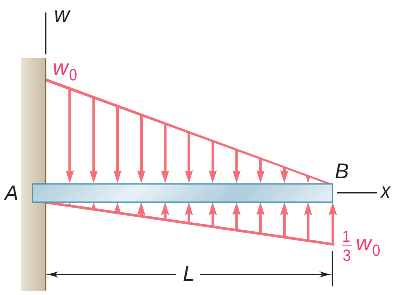


Fig. P7.87

7.88 For the beam and loading shown, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the maximum bending moment.

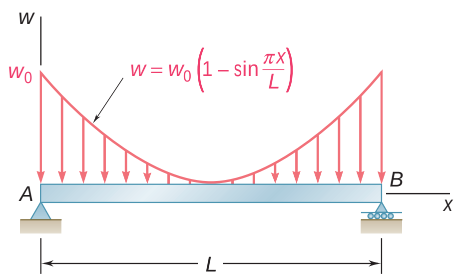


Fig. *P7.88*

7.89 The beam *AB* is subjected to the uniformly distributed load shown and to two unknown forces **P** and **Q**. Knowing that it has been experimentally determined that the bending moment is +800 N · m at *D* and +1300 N · m at *E*, (*a*) determine **P** and **Q**, (*b*) draw the shear and bending-moment diagrams for the beam.

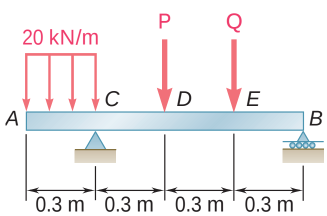


Fig. P7.89

7.90 Solve Prob. 7.89 assuming that the bending moment was found to be +650 N · m at *D* and +1450 N · m at *E*.

***7.91** The beam AB is subjected to the uniformly distributed load shown and to two unknown forces **P** and **Q**. Knowing that it has been experimentally determined that the bending moment is $+6.10$ kip · ft at D and $+5.50$ kip · ft at E, (*a*) determine **P** and **Q**, (*b*) draw the shear and bending-moment diagrams for the beam.

***7.92** Solve Prob. 7.91 assuming that the bending moment was found to be $+5.96$ kip · ft at D and $+6.84$ kip · ft at E.

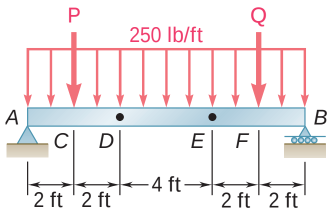


Fig. P7.91

CABLES

*7.7 CABLES WITH CONCENTRATED LOADS

Cables are used in many engineering applications, such as suspension bridges, transmission lines, aerial tramways, guy wires for high towers, etc. Cables may be divided into two categories, according to their loading: (1) cables supporting concentrated loads, (2) cables supporting distributed loads. In this section, cables of the first category are examined.

Consider a cable attached to two fixed points A and B and supporting n vertical concentrated loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ (Fig. 7.13*a*). We assume that the cable is *flexible*, i.e., that its resistance to bending is small and can be neglected. We further assume that the *weight of the cable is negligible* compared with the loads supported by the cable. Any portion of cable between successive loads can therefore be considered as a two-force member, and the internal forces at any point in the cable reduce to a *force of tension directed along the cable*.

We assume that each of the loads lies in a given vertical line, i.e., that the horizontal distance from support A to each of the loads is known; we also assume that the horizontal and vertical distances between the supports are known. We propose to determine the shape of the cable, i.e., the vertical distance from support A to each of the points $C_1, C_2, \ldots, C_n$, and also the tension T in each portion of the cable.

Photo 7.3 Since the weight of the cable of the chairlift shown is negligible compared to the weights of the chairs and skiers, the methods of this section can be used to determine the force at any point in the cable.

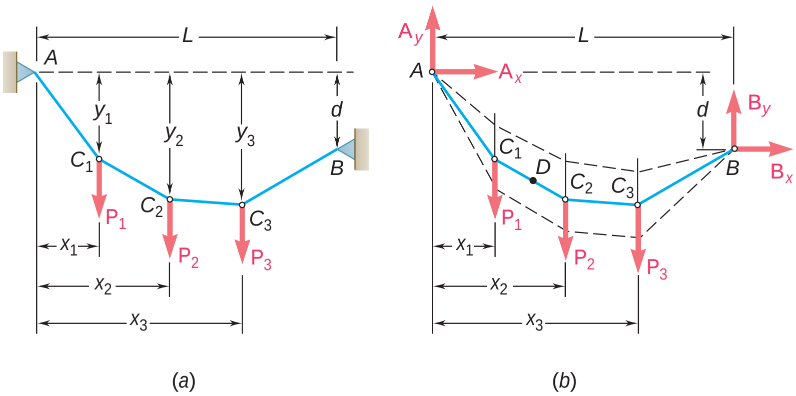


Fig. 7.13

We first draw the free-body diagram of the entire cable (Fig. 7.13*b*). Since the slope of the portions of cable attached at A and B is not known, the reactions at A and B must be represented by two components each. Thus, four unknowns are involved, and the three equations of equilibrium are not sufficient to determine the reactions at A and B.† We must

†Clearly, the cable is not a rigid body; the equilibrium equations represent, therefore, *necessary but not sufficient conditions* (see Sec. 6.11).

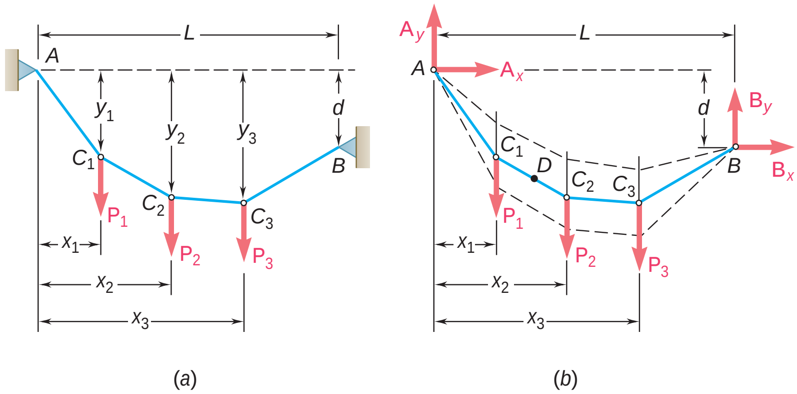


Fig. 7.13 (*repeated*)

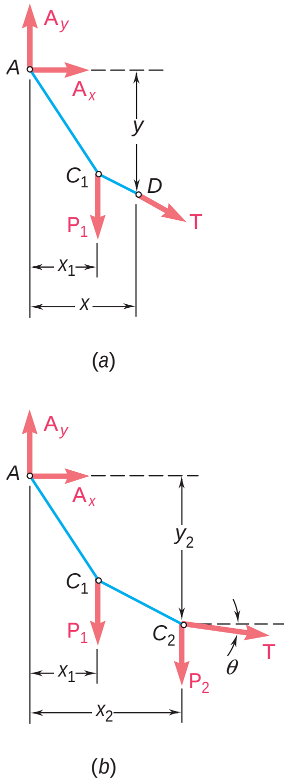


Fig. 7.14

therefore obtain an additional equation by considering the equilibrium of a portion of the cable. This is possible if we know the coordinates x and y of a point D of the cable. Drawing the free-body diagram of the portion of cable AD (Fig. 7.14*a*) and writing $\Sigma M_D = 0$, we obtain an additional relation between the scalar components A_x and A_y and can determine the reactions at A and B. The problem would remain indeterminate, however, if we did not know the coordinates of D, unless some other relation between A_x and A_y (or between B_x and B_y) were given. The cable might hang in any of various possible ways, as indicated by the dashed lines in Fig. 7.13*b*.

Once A_x and A_y have been determined, the vertical distance from A to any point of the cable can easily be found. Considering point C_2, for example, we draw the free-body diagram of the portion of cable AC_2 (Fig. 7.14*b*). Writing $\Sigma M_{C_2} = 0$, we obtain an equation which can be solved for y_2. Writing $\Sigma F_x = 0$ and $\Sigma F_y = 0$, we obtain the components of the force **T** representing the tension in the portion of cable to the right of C_2. We observe that $T \cos u = -A_x$; *the horizontal component of the tension force is the same at any point of the cable*. It follows that the tension T is maximum when cos u is minimum, i.e., in the portion of cable which has the largest angle of inclination u. Clearly, this portion of cable must be adjacent to one of the two supports of the cable.

*7.8 CABLES WITH DISTRIBUTED LOADS

Consider a cable attached to two fixed points A and B and carrying a *distributed load* (Fig. 7.15*a*). We saw in the preceding section that for a cable supporting concentrated loads, the internal force at any point is a force of tension directed along the cable. In the case of a cable carrying a distributed load, the cable hangs in the shape of a curve, and the internal force at a point D is a force of tension **T** *directed along the tangent to the curve*. In this section, you will learn to determine the tension at any point of a cable supporting a given distributed load. In the following sections, the shape of the cable will be determined for two particular types of distributed loads.

Considering the most general case of distributed load, we draw the free-body diagram of the portion of cable extending from the lowest point C to a given point D of the cable (Fig. 7.15*b*). The

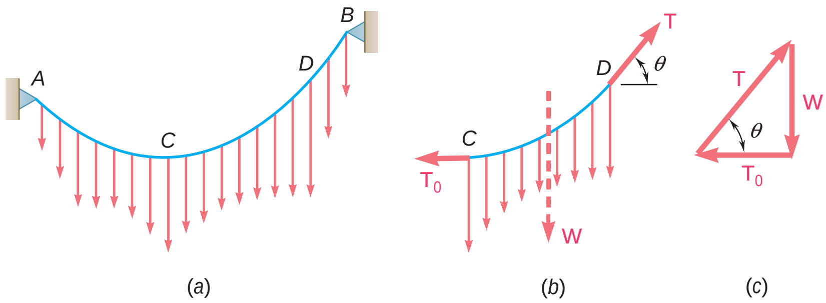


Fig. 7.15

forces acting on the free body are the tension force $\mathbf{T}_0$ at C, which is horizontal, the tension force $\mathbf{T}$ at D, directed along the tangent to the cable at D, and the resultant $\mathbf{W}$ of the distributed load supported by the portion of cable CD. Drawing the corresponding force triangle (Fig. 7.15c), we obtain the following relations:

$$T \cos \theta = T_0 \qquad T \sin \theta = W \tag{7.5}$$

$$T = \sqrt{T_0^2 + W^2} \qquad \tan \theta = \frac{W}{T_0} \tag{7.6}$$

From the relations (7.5), it appears that the horizontal component of the tension force $\mathbf{T}$ is the same at any point and that the vertical component of $\mathbf{T}$ is equal to the magnitude W of the load measured from the lowest point. Relations (7.6) show that the tension T is minimum at the lowest point and maximum at one of the two points of support.

*7.9 PARABOLIC CABLE

Let us assume, now, that the cable AB carries a load *uniformly distributed along the horizontal* (Fig. 7.16a). Cables of suspension bridges may be assumed loaded in this way, since the weight of the cables is small compared with the weight of the roadway. We denote by w the load per unit length (*measured horizontally*) and express it in N/m or in lb/ft. Choosing coordinate axes with origin at the lowest point C of the cable, we find that the magnitude W of the total load carried by the portion of cable extending from C to the point D of coordinates x and y is $W = wx$. The relations (7.6) defining the magnitude and direction of the tension force at D become

$$T = \sqrt{T_0^2 + w^2x^2} \qquad \tan \theta = \frac{wx}{T_0} \tag{7.7}$$

Moreover, the distance from D to the line of action of the resultant $\mathbf{W}$ is equal to half the horizontal distance from C to D (Fig. 7.16b). Summing moments about D, we write

$$+\circlearrowleft \Sigma M_D = 0: \qquad wx\frac{x}{2} - T_0 y = 0$$

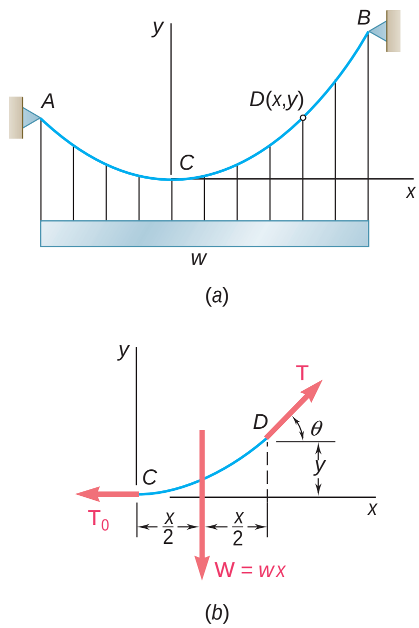


Fig. 7.16

and, solving for y,

$$y = \frac{wx^2}{2T_0} \tag{7.8}$$

This is the equation of a *parabola* with a vertical axis and its vertex at the origin of coordinates. The curve formed by cables loaded uniformly along the horizontal is thus a parabola.†

When the supports A and B of the cable have the same elevation, the distance L between the supports is called the *span* of the cable and the vertical distance h from the supports to the lowest point is called the *sag* of the cable (Fig. 7.17*a*). If the span and sag of a cable are known, and if the load w per unit horizontal length is given, the minimum tension T_0 may be found by substituting $x = L/2$ and $y = h$ in Eq. (7.8). Equations (7.7) will then yield the tension and the slope at any point of the cable and Eq. (7.8) will define the shape of the cable.

When the supports have different elevations, the position of the lowest point of the cable is not known and the coordinates x_A, y_A and x_B, y_B of the supports must be determined. To this effect, we express that the coordinates of A and B satisfy Eq. (7.8) and that $x_B - x_A = L$ and $y_B - y_A = d$, where L and d denote, respectively, the horizontal and vertical distances between the two supports (Fig. 7.17*b* and *c*).

The length of the cable from its lowest point C to its support B can be obtained from the formula

$$s_B = \int_0^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{7.9}$$

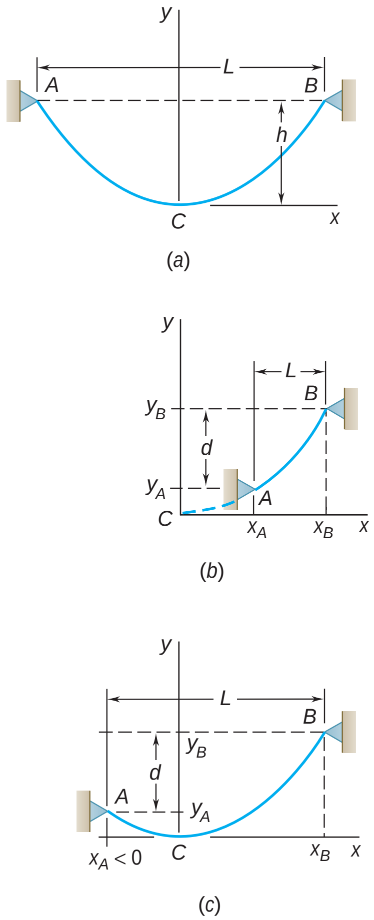


Fig. 7.17

Differentiating (7.8), we obtain the derivative $dy/dx = wx/T_0$; substituting into (7.9) and using the binomial theorem to expand the radical in an infinite series, we have

$$s_B = \int_0^{x_B} \sqrt{1 + \frac{w^2x^2}{T_0^2}}\, dx = \int_0^{x_B} \left(1 + \frac{w^2x^2}{2T_0^2} - \frac{w^4x^4}{8T_0^4} + \cdots\right) dx$$

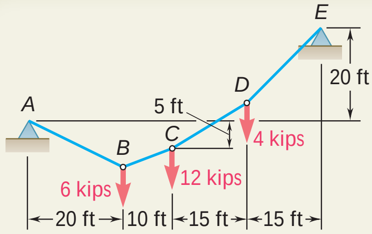
$$s_B = x_B\left(1 + \frac{w^2x_B^2}{6T_0^2} - \frac{w^4x_B^4}{40T_0^4} + \cdots\right)$$

and, since $wx_B^2/2T_0 = y_B$,

$$s_B = x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4 + \cdots\right] \tag{7.10}$$

The series converges for values of the ratio y_B/x_B less than 0.5; in most cases, this ratio is much smaller, and only the first two terms of the series need be computed.

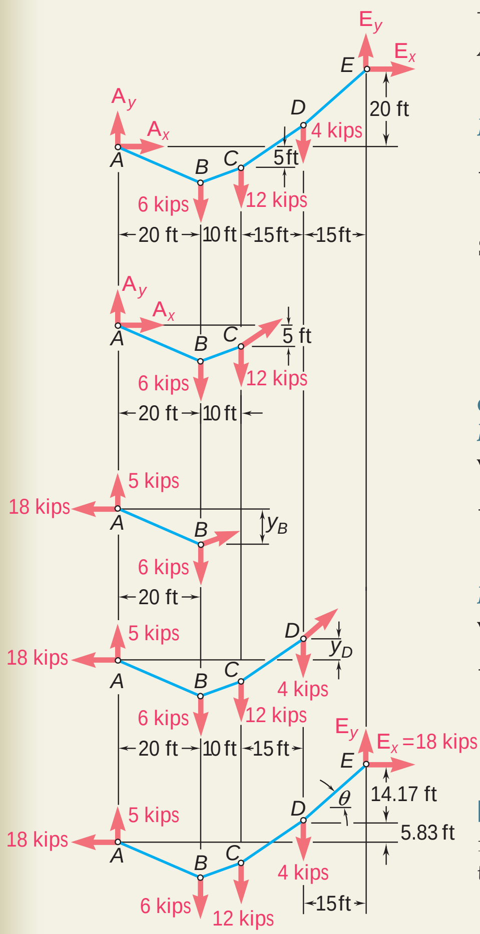
†Cables hanging under their own weight are not loaded uniformly along the horizontal, and they do not form a parabola. The error introduced by assuming a parabolic shape for cables hanging under their weight, however, is small when the cable is sufficiently taut. A complete discussion of cables hanging under their own weight is given in the next section.



SAMPLE PROBLEM 7.8

The cable AE supports three vertical loads from the points indicated. If point C is 5 ft below the left support, determine (*a*) the elevation of points B and D, (*b*) the maximum slope and the maximum tension in the cable.

SOLUTION



Reactions at Supports. The reaction components $\mathbf{A}_x$ and $\mathbf{A}_y$ are determined as follows:

Free Body: Entire Cable

$+1\ \Sigma M_E = 0$:

$$A_x(20\text{ ft}) - A_y(60\text{ ft}) + (6\text{ kips})(40\text{ ft}) + (12\text{ kips})(30\text{ ft}) + (4\text{ kips})(15\text{ ft}) = 0$$
$$20A_x - 60A_y + 660 = 0$$

Free Body: ABC

$$+1\ \Sigma M_C = 0: \qquad -A_x(5\text{ ft}) - A_y(30\text{ ft}) + (6\text{ kips})(10\text{ ft}) = 0$$
$$-5A_x - 30A_y + 60 = 0$$

Solving the two equations simultaneously, we obtain

$$A_x = -18\text{ kips} \qquad \mathbf{A}_x = 18\text{ kips z}$$
$$A_y = +5\text{ kips} \qquad \mathbf{A}_y = 5\text{ kips x}$$

a. Elevation of Points *B* and *D*.

Free Body: AB Considering the portion of cable AB as a free body, we write

$$+1\ \Sigma M_B = 0: \qquad (18\text{ kips})y_B - (5\text{ kips})(20\text{ ft}) = 0$$

$y_B = 5.56$ ft below A ◀

Free Body: ABCD Using the portion of cable $ABCD$ as a free body, we write

$+1\ \Sigma M_D = 0$:

$$-(18\text{ kips})y_D - (5\text{ kips})(45\text{ ft}) + (6\text{ kips})(25\text{ ft}) + (12\text{ kips})(15\text{ ft}) = 0$$

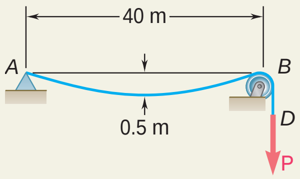
$y_D = 5.83$ ft above A ◀

b. Maximum Slope and Maximum Tension. We observe that the maximum slope occurs in portion DE. Since the horizontal component of the tension is constant and equal to 18 kips, we write

$$\tan u = \frac{14.17}{15\text{ ft}} \qquad u = 43.4° ◀$$

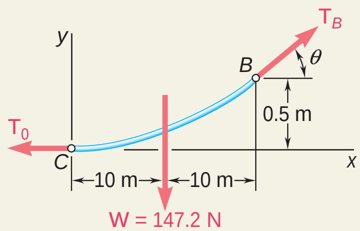
$$T_{\max} = \frac{18\text{ kips}}{\cos u} \qquad T_{\max} = 24.8\text{ kips} ◀$$

SAMPLE PROBLEM 7.9



A light cable is attached to a support at A, passes over a small pulley at B, and supports a load $\mathbf{P}$. Knowing that the sag of the cable is 0.5 m and that the mass per unit length of the cable is 0.75 kg/m, determine (*a*) the magnitude of the load $\mathbf{P}$, (*b*) the slope of the cable at B, (*c*) the total length of the cable from A to B. Since the ratio of the sag to the span is small, assume the cable to be parabolic. Also, neglect the weight of the portion of cable from B to D.

SOLUTION



a. Load P. We denote by C the lowest point of the cable and draw the free-body diagram of the portion CB of cable. Assuming the load to be uniformly distributed along the horizontal, we write

$$w = (0.75 \text{ kg/m})(9.81 \text{ m/s}^2) = 7.36 \text{ N/m}$$

The total load for the portion CB of cable is

$$W = wx_B = (7.36 \text{ N/m})(20 \text{ m}) = 147.2 \text{ N}$$

and is applied halfway between C and B. Summing moments about B, we write

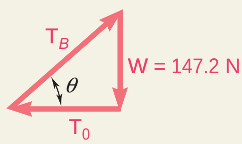
$$+\circlearrowleft \Sigma M_B = 0: \qquad (147.2 \text{ N})(10 \text{ m}) - T_0(0.5 \text{ m}) = 0 \qquad\qquad T_0 = 2944 \text{ N}$$

From the force triangle we obtain

$$\begin{aligned} T_B &= \sqrt{T_0^2 + W^2} \\ &= \sqrt{(2944 \text{ N})^2 + (147.2 \text{ N})^2} = 2948 \text{ N} \end{aligned}$$

Since the tension on each side of the pulley is the same, we find

$$P = T_B = 2948 \text{ N} \quad ◀$$



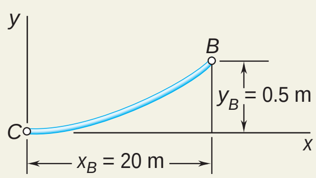
b. Slope of Cable at *B*. We also obtain from the force triangle

$$\tan \theta = \frac{W}{T_0} = \frac{147.2 \text{ N}}{2944 \text{ N}} = 0.05$$

$$\theta = 2.9° \quad ◀$$

c. Length of Cable. Applying Eq. (7.10) between C and B, we write

$$\begin{aligned} s_B &= x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 + \cdots\right] \\ &= (20 \text{ m})\left[1 + \frac{2}{3}\left(\frac{0.5 \text{ m}}{20 \text{ m}}\right)^2 + \cdots\right] = 20.00833 \text{ m} \end{aligned}$$



The total length of the cable between A and B is twice this value,

$$\text{Length} = 2s_B = 40.0167 \text{ m} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In the problems of this section you will apply the equations of equilibrium to *cables that lie in a vertical plane*. We assume that a cable cannot resist bending, so that the force of tension in the cable is always directed along the cable.

A. In the first part of this lesson we considered cables subjected to concentrated loads. Since the weight of the cable is neglected, the cable is straight between loads.

Your solution will consist of the following steps:

1. Draw a free-body diagram of the entire cable showing the loads and the horizontal and vertical components of the reaction at each support. Use this free-body diagram to write the corresponding equilibrium equations.

2. You will be confronted with four unknown components and only three equations of equilibrium (see Fig. 7.13). You must therefore find an additional piece of information, such as the *position* of a point on the cable or the *slope* of the cable at a given point.

3. After you have identified the point of the cable where the additional information exists, cut the cable at that point, and draw a free-body diagram of one of the two portions of the cable you have obtained.

a. If you know the position of the point where you have cut the cable, writing $\Sigma M = 0$ about that point for the new free body will yield the additional equation required to solve for the four unknown components of the reactions [Sample Prob. 7.8].

b. If you know the slope of the portion of the cable you have cut, writing $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for the new free body will yield two equilibrium equations which, together with the original three, can be solved for the four reaction components and for the tension in the cable where it has been cut.

4. To find the elevation of a given point of the cable and the slope and tension at that point once the reactions at the supports have been found, you should cut the cable at that point and draw a free-body diagram of one of the two portions of the cable you have obtained. Writing $\Sigma M = 0$ about the given point yields its elevation. Writing $\Sigma F_x = 0$ and $\Sigma F_y = 0$ yields the components of the tension force, from which its magnitude and direction can easily be found.

(continued)

5. For a cable supporting vertical loads only, you will observe that *the horizontal component of the tension force is the same at any point*. It follows that, for such a cable, the *maximum tension occurs in the steepest portion of the cable*.

B. In the second portion of this lesson we considered cables carrying a load uniformly distributed along the horizontal. The shape of the cable is then parabolic.

Your solution will use one or more of the following concepts:

1. Placing the origin of coordinates at the lowest point of the cable and directing the x and y axes to the right and upward, respectively, we find that *the equation of the parabola* is

$$y = \frac{wx^2}{2T_0} \tag{7.8}$$

The minimum cable tension occurs at the origin, where the cable is horizontal, and the maximum tension is at the support where the slope is maximum.

2. If the supports of the cable have the same elevation, the sag h of the cable is the vertical distance from the lowest point of the cable to the horizontal line joining the supports. To solve a problem involving such a parabolic cable, you should write Eq. (7.8) for one of the supports; this equation can be solved for one unknown.

3. If the supports of the cable have different elevations, you will have to write Eq. (7.8) for each of the supports (see Fig. 7.17).

4. To find the length of the cable from the lowest point to one of the supports, you can use Eq. (7.10). In most cases, you will need to compute only the first two terms of the series.

PROBLEMS

7.93 Three loads are suspended as shown from the cable *ABCDE*. Knowing that $d_C = 3$ m, determine (*a*) the components of the reaction at *E*, (*b*) the maximum tension in the cable.

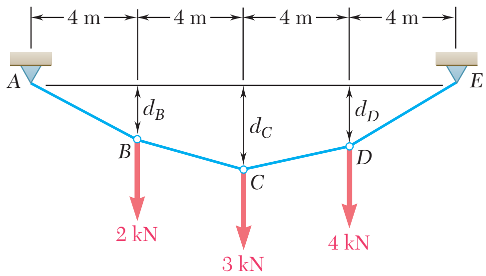


Fig. P7.93 and P7.94

7.94 Knowing that the maximum tension in cable *ABCDE* is 13 kN, determine the distance d_C.

7.95 If $d_C = 8$ ft, determine (*a*) the reaction at *A*, (*b*) the reaction at *E*.

7.96 If $d_C = 4.5$ ft, determine (*a*) the reaction at *A*, (*b*) the reaction at *E*.

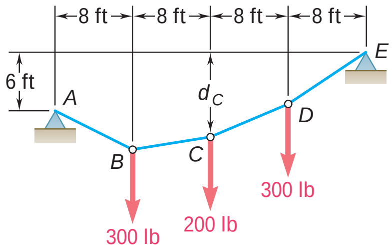


Fig. P7.95 and P7.96

7.97 Knowing that $d_C = 3$ m, determine (*a*) the distances d_B and d_D, (*b*) the reaction at *E*.

7.98 Determine (*a*) distance d_C for which portion *DE* of the cable is horizontal, (*b*) the corresponding reactions at *A* and *E*.

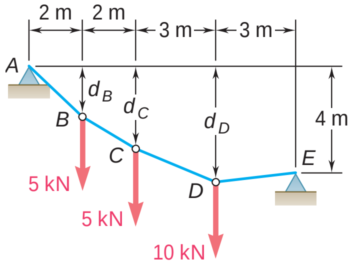


Fig. P7.97 and P7.98

7.99 An oil pipeline is supported at 6-ft intervals by vertical hangers attached to the cable shown. Due to the combined weight of the pipe and its contents, the tension in each hanger is 400 lb. Knowing that $d_C = 12$ ft, determine (*a*) the maximum tension in the cable, (*b*) the distance d_D.

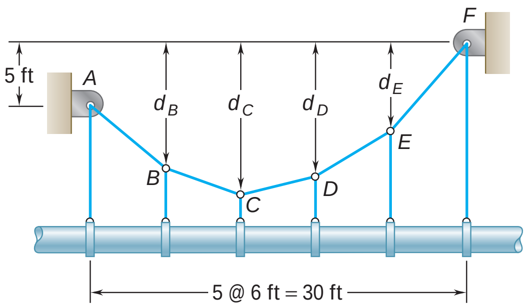


Fig. *P7.99* and *P7.100*

7.100 Solve Prob. 7.99 assuming that $d_C = 9$ ft.

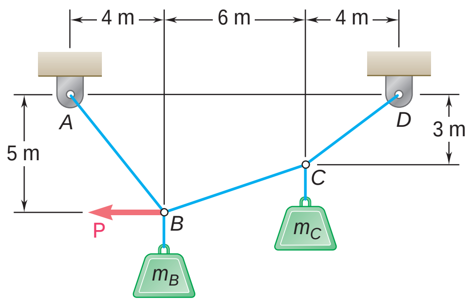


Fig. P7.101 and P7.102

7.101 Knowing that $m_B = 70$ kg and $m_C = 25$ kg, determine the magnitude of the force **P** required to maintain equilibrium.

7.102 Knowing that $m_B = 18$ kg and $m_C = 10$ kg, determine the magnitude of the force **P** required to maintain equilibrium.

7.103 Cable *ABC* supports two loads as shown. Knowing that $b = 21$ ft, determine (*a*) the required magnitude of the horizontal force **P**, (*b*) the corresponding distance *a*.

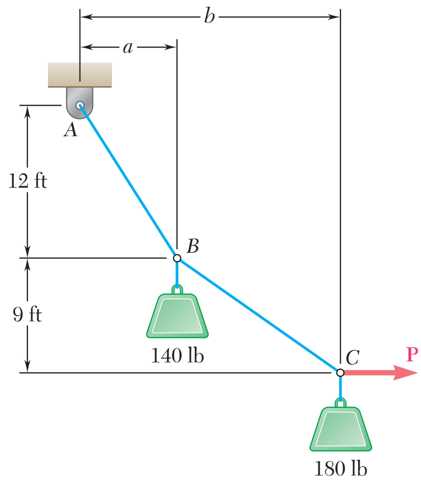


Fig. P7.103 and P7.104

7.104 Cable *ABC* supports two loads as shown. Determine the distances *a* and *b* when a horizontal force **P** of magnitude 200 lb is applied at *C*.

7.105 If $a = 3$ m, determine the magnitudes of **P** and **Q** required to maintain the cable in the shape shown.

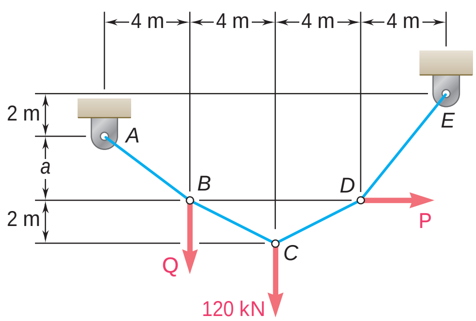


Fig. *P7.105* and *P7.106*

7.106 If $a = 4$ m, determine the magnitudes of **P** and **Q** required to maintain the cable in the shape shown.

7.107 A transmission cable having a mass per unit length of 0.8 kg/m is strung between two insulators at the same elevation that are 75 m apart. Knowing that the sag of the cable is 2 m, determine (*a*) the maximum tension in the cable, (*b*) the length of the cable.

7.108 The total mass of cable ACB is 20 kg. Assuming that the mass of the cable is distributed uniformly along the horizontal, determine (*a*) the sag h, (*b*) the slope of the cable at A.

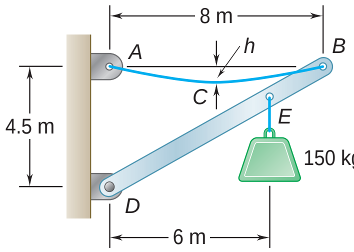


Fig. *P7.108*

7.109 The center span of the Verrazano-Narrows Bridge consists of two uniform roadways suspended from four cables. The uniform load supported by each cable is w = 10.8 kips/ft along the horizontal. Knowing that the span L is 4260 ft and that the sag h is 390 ft, determine (*a*) the maximum tension in each cable, (*b*) the length of each cable.

7.110 The center span of the Verrazano-Narrows Bridge consists of two uniform roadways suspended from four cables. The design of the bridge allows for the effect of extreme temperature changes that cause the sag of the center span to vary from h_w = 386 ft in winter to h_s = 394 ft in summer. Knowing that the span is L = 4260 ft, determine the change in length of the cables due to extreme temperature changes.

7.111 Each cable of the Golden Gate Bridge supports a load w = 11.1 kips/ft along the horizontal. Knowing that the span L is 4150 ft and that the sag h is 464 ft, determine (*a*) the maximum tension in each cable, (*b*) the length of each cable.

7.112 Two cables of the same gauge are attached to a transmission tower at B. Since the tower is slender, the horizontal component of the resultant of the forces exerted by the cables at B is to be zero. Knowing that the mass per unit length of the cables is 0.4 kg/m, determine (*a*) the required sag h, (*b*) the maximum tension in each cable.

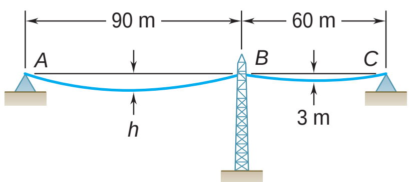


Fig. P7.112

7.113 A 50.5-m length of wire having a mass per unit length of 0.75 kg/m is used to span a horizontal distance of 50 m. Determine (*a*) the approximate sag of the wire, (*b*) the maximum tension in the wire. [*Hint:* Use only the first two terms of Eq. (7.10).]

7.114 A cable of length $L + \Delta$ is suspended between two points that are at the same elevation and a distance L apart. (*a*) Assuming that Δ is small compared to L and that the cable is parabolic, determine the approximate sag in terms of L and Δ. (*b*) If L = 100 ft and Δ = 4 ft, determine the approximate sag. [*Hint:* Use only the first two terms of Eq. (7.10).]

7.115 The total mass of cable AC is 25 kg. Assuming that the mass of the cable is distributed uniformly along the horizontal, determine the sag h and the slope of the cable at A and C.

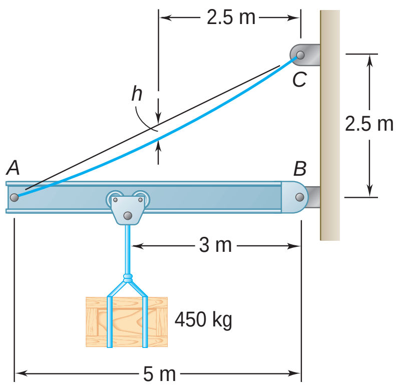


Fig. P7.115

7.116 Cable *ACB* supports a load uniformly distributed along the horizontal as shown. The lowest point *C* is located 9 m to the right of *A*. Determine (*a*) the vertical distance *a*, (*b*) the length of the cable, (*c*) the components of the reaction at *A*.

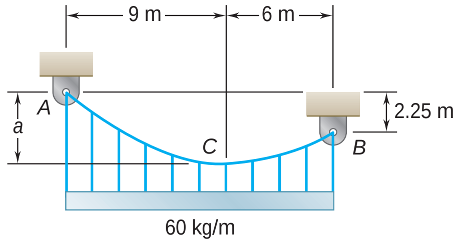


Fig. P7.116

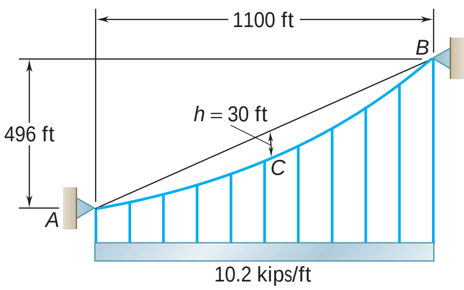


Fig. P7.117

7.117 Each cable of the side spans of the Golden Gate Bridge supports a load $w = 10.2$ kips/ft along the horizontal. Knowing that for the side spans the maximum vertical distance h from each cable to the chord *AB* is 30 ft and occurs at midspan, determine (*a*) the maximum tension in each cable, (*b*) the slope at *B*.

7.118 A steam pipe weighing 45 lb/ft that passes between two buildings 40 ft apart is supported by a system of cables as shown. Assuming that the weight of the cable system is equivalent to a uniformly distributed loading of 5 lb/ft, determine (*a*) the location of the lowest point *C* of the cable, (*b*) the maximum tension in the cable.

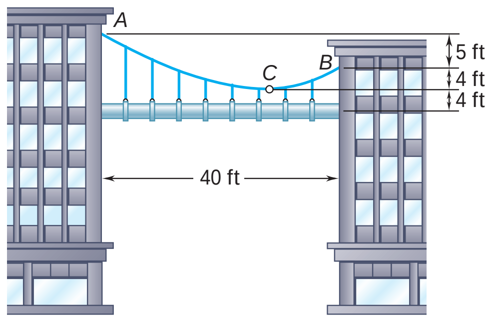


Fig. P7.118

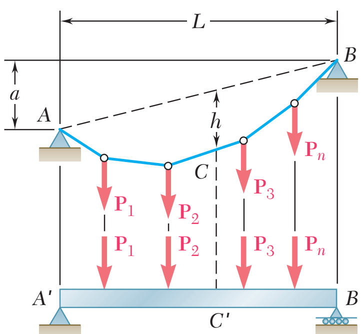


Fig. P7.119

***7.119** A cable *AB* of span *L* and a simple beam *A′B′* of the same span are subjected to identical vertical loadings as shown. Show that the magnitude of the bending moment at a point *C′* in the beam is equal to the product T_0h, where T_0 is the magnitude of the horizontal component of the tension force in the cable and h is the vertical distance between point *C* and the chord joining the points of support *A* and *B*.

7.120 through 7.123 Making use of the property established in Prob. 7.119, solve the problem indicated by first solving the corresponding beam problem.

7.120 Prob. 7.94.
7.121 Prob. 7.97*a*.
7.122 Prob. 7.99*b*.
7.123 Prob. 7.100*b*.

***7.124** Show that the curve assumed by a cable that carries a distributed load $w(x)$ is defined by the differential equation $d^2y/dx^2 = w(x)/T_0$, where T_0 is the tension at the lowest point.

***7.125** Using the property indicated in Prob. 7.124, determine the curve assumed by a cable of span L and sag h carrying a distributed load $w = w_0 \cos (\pi x/L)$, where x is measured from mid-span. Also determine the maximum and minimum values of the tension in the cable.

***7.126** If the weight per unit length of the cable AB is $w_0/\cos^2 \theta$, prove that the curve formed by the cable is a circular arc. (*Hint:* Use the property indicated in Prob. 7.124.)

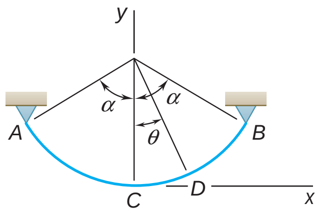


Fig. P7.126

*7.10 CATENARY

Let us now consider a cable AB carrying a load *uniformly distributed along the cable itself* (Fig. 7.18*a*). Cables hanging under their own weight are loaded in this way. We denote by w the load per unit length (*measured along the cable*) and express it in N/m or in lb/ft. The magnitude W of the total load carried by a portion of cable of length s extending from the lowest point C to a point D is $W = ws$. Substituting this value for W in formula (7.6), we obtain the tension at D:

$$T = \sqrt{T_0^2 + w^2 s^2}$$

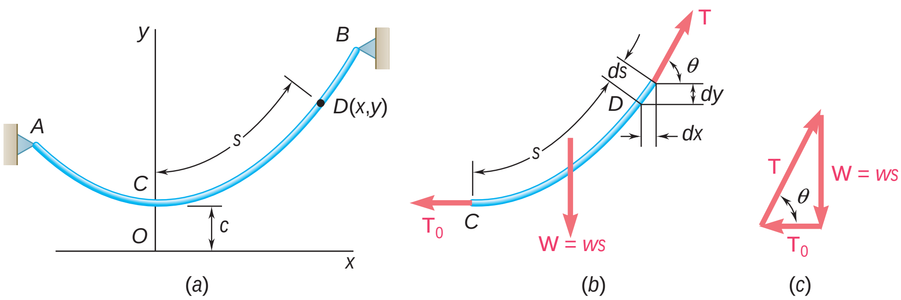


Fig. 7.18

Photo 7.4 The forces on the supports and the internal forces in the cables of the power line shown are discussed in this section.

In order to simplify the subsequent computations, we introduce the constant $c = T_0/w$. We thus write

$$T_0 = wc \qquad W = ws \qquad T = w\sqrt{c^2 + s^2} \qquad (7.11)$$

The free-body diagram of the portion of cable CD is shown in Fig. 7.18*b*. This diagram, however, cannot be used to obtain directly the equation of the curve assumed by the cable, since we do not know the horizontal distance from D to the line of action of the resultant **W** of the load. To obtain this equation, we first write that the horizontal projection of a small element of cable of length ds is

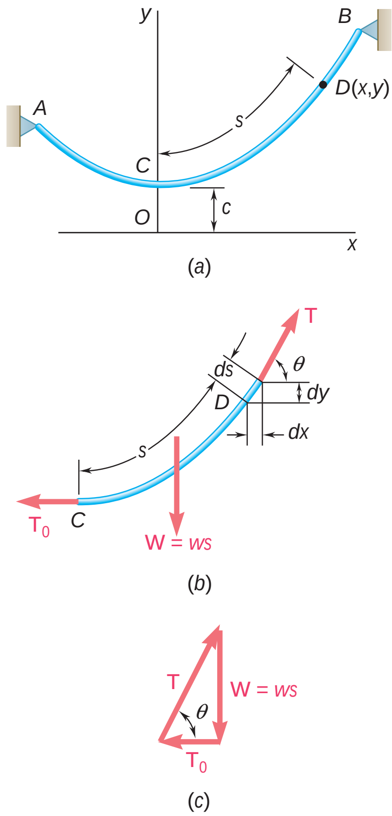


Fig. 7.18 *(continued)*

$dx = ds \cos \theta$. Observing from Fig. 7.18*c* that $\cos \theta = T_0/T$ and using (7.11), we write

$$dx = ds \cos \theta = \frac{T_0}{T}\,ds = \frac{wc\,ds}{w\sqrt{c^2 + s^2}} = \frac{ds}{\sqrt{1 + s^2/c^2}}$$

Selecting the origin O of the coordinates at a distance c directly below C (Fig. 7.18*a*) and integrating from $C(0, c)$ to $D(x, y)$, we obtain†

$$x = \int_0^s \frac{ds}{\sqrt{1 + s^2/c^2}} = c\left[\sinh^{-1}\frac{s}{c}\right]_0^s = c \sinh^{-1}\frac{s}{c}$$

This equation, which relates the length s of the portion of cable CD and the horizontal distance x, can be written in the form

$$s = c \sinh \frac{x}{c} \tag{7.15}$$

The relation between the coordinates x and y can now be obtained by writing $dy = dx \tan \theta$. Observing from Fig. 7.18*c* that $\tan \theta = W/T_0$ and using (7.11) and (7.15), we write

$$dy = dx \tan \theta = \frac{W}{T_0}\,dx = \frac{s}{c}\,dx = \sinh \frac{x}{c}\,dx$$

Integrating from $C(0, c)$ to $D(x, y)$ and using (7.12) and (7.13), we obtain

$$y - c = \int_0^x \sinh \frac{x}{c}\,dx = c\left[\cosh \frac{x}{c}\right]_0^x = c\left(\cosh \frac{x}{c} - 1\right)$$

$$y - c = c \cosh \frac{x}{c} - c$$

†This integral can be found in all standard integral tables. The function

$$z = \sinh^{-1} u$$

(read "arc hyperbolic sine u") is the *inverse* of the function $u = \sinh z$ (read "hyperbolic sine z"). This function and the function $v = \cosh z$ (read "hyperbolic cosine z") are defined as follows:

$$u = \sinh z = \tfrac{1}{2}(e^z - e^{-z}) \qquad v = \cosh z = \tfrac{1}{2}(e^z + e^{-z})$$

Numerical values of the functions sinh z and cosh z are found in *tables of hyperbolic functions*. They may also be computed on most calculators either directly or from the above definitions. The student is referred to any calculus text for a complete description of the properties of these functions. In this section, we use only the following properties, which are easily derived from the above definitions:

$$\frac{d \sinh z}{dz} = \cosh z \qquad \frac{d \cosh z}{dz} = \sinh z \tag{7.12}$$

$$\sinh 0 = 0 \qquad \cosh 0 = 1 \tag{7.13}$$

$$\cosh^2 z - \sinh^2 z = 1 \tag{7.14}$$

which reduces to

$$y = c \cosh \frac{x}{c} \tag{7.16}$$

This is the equation of a *catenary* with vertical axis. The ordinate c of the lowest point C is called the *parameter* of the catenary. Squaring both sides of Eqs. (7.15) and (7.16), subtracting, and taking (7.14) into account, we obtain the following relation between y and s:

$$y^2 - s^2 = c^2 \tag{7.17}$$

Solving (7.17) for s^2 and carrying into the last of the relations (7.11), we write these relations as follows:

$$T_0 = wc \qquad W = ws \qquad T = wy \tag{7.18}$$

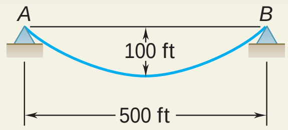
The last relation indicates that the tension at any point D of the cable is proportional to the vertical distance from D to the horizontal line representing the x axis.

When the supports A and B of the cable have the same elevation, the distance L between the supports is called the *span* of the cable and the vertical distance h from the supports to the lowest point C is called the *sag* of the cable. These definitions are the same as those given in the case of parabolic cables, but it should be noted that because of our choice of coordinate axes, the sag h is now

$$h = y_A - c \tag{7.19}$$

It should also be observed that certain catenary problems involve transcendental equations which must be solved by successive approximations (see Sample Prob. 7.10). When the cable is fairly taut, however, the load can be assumed uniformly distributed *along the horizontal* and the catenary can be replaced by a parabola. This greatly simplifies the solution of the problem, and the error introduced is small.

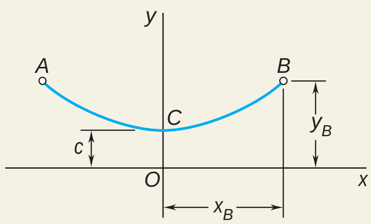
When the supports A and B have different elevations, the position of the lowest point of the cable is not known. The problem can then be solved in a manner similar to that indicated for parabolic cables, by expressing that the cable must pass through the supports and that $x_B - x_A = L$ and $y_B - y_A = d$, where L and d denote, respectively, the horizontal and vertical distances between the two supports.



SAMPLE PROBLEM 7.10

A uniform cable weighing 3 lb/ft is suspended between two points A and B as shown. Determine (a) the maximum and minimum values of the tension in the cable, (b) the length of the cable.

SOLUTION



Equation of Cable. The origin of coordinates is placed at a distance c below the lowest point of the cable. The equation of the cable is given by Eq. (7.16),

$$y = c \cosh \frac{x}{c}$$

The coordinates of point B are

$$x_B = 250 \text{ ft} \qquad y_B = 100 + c$$

Substituting these coordinates into the equation of the cable, we obtain

$$100 + c = c \cosh \frac{250}{c}$$
$$\frac{100}{c} + 1 = \cosh \frac{250}{c}$$

The value of c is determined by assuming successive trial values, as shown in the following table:

c	$\frac{250}{c}$	$\frac{100}{c}$	$\frac{100}{c} + 1$	$\cosh \frac{250}{c}$
300	0.833	0.333	1.333	1.367
350	0.714	0.286	1.286	1.266
330	0.758	0.303	1.303	1.301
328	0.762	0.305	1.305	1.305

Taking $c = 328$, we have

$$y_B = 100 + c = 428 \text{ ft}$$

a. Maximum and Minimum Values of the Tension. Using Eqs. (7.18), we obtain

$$T_{\min} = T_0 = wc = (3 \text{ lb/ft})(328 \text{ ft}) \qquad T_{\min} = 984 \text{ lb} \blacktriangleleft$$
$$T_{\max} = T_B = wy_B = (3 \text{ lb/ft})(428 \text{ ft}) \qquad T_{\max} = 1284 \text{ lb} \blacktriangleleft$$

b. Length of Cable. One-half the length of the cable is found by solving Eq. (7.17):

$$y_B^2 - s_{CB}^2 = c^2 \qquad s_{CB}^2 = y_B^2 - c^2 = (428)^2 - (328)^2 \qquad s_{CB} = 275 \text{ ft}$$

The total length of the cable is therefore

$$s_{AB} = 2s_{CB} = 2(275 \text{ ft}) \qquad s_{AB} = 550 \text{ ft} \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In the last section of this chapter you learned to solve problems involving a *cable carrying a load uniformly distributed along the cable*. The shape assumed by the cable is a catenary and is defined by the equation:

$$y = c \cosh \frac{x}{c} \qquad (7.16)$$

1. You should keep in mind that the origin of coordinates for a catenary is located at a distance *c* directly below the lowest point of the catenary. The length of the cable from the origin to any point is expressed as

$$s = c \sinh \frac{x}{c} \qquad (7.15)$$

2. You should first identify all of the known and unknown quantities. Then consider each of the equations listed in the text (Eqs. 7.15 through 7.19), and solve an equation that contains only one unknown. Substitute the value found into another equation, and solve that equation for another unknown.

3. If the sag *h* is given, use Eq. (7.19) to replace y by $h + c$ in Eq. (7.16) if x is known [Sample Prob. 7.10], or in Eq. (7.17) if s is known, and solve the equation obtained for the constant c.

4. Many of the problems that you will encounter will involve the solution by trial and error of an equation involving a hyperbolic sine or cosine. You can make your work easier by keeping track of your calculations in a table, as in Sample Prob. 7.10, or by applying a numerical methods approach using a computer or calculator.

PROBLEMS

7.127 A 20-m chain of mass 12 kg is suspended between two points at the same elevation. Knowing that the sag is 8 m, determine (*a*) the distance between the supports, (*b*) the maximum tension in the chain.

7.128 A 600-ft-long aerial tramway cable having a weight per unit length of 3.0 lb/ft is suspended between two points at the same elevation. Knowing that the sag is 150 ft, find (*a*) the horizontal distance between the supports, (*b*) the maximum tension in the cable.

7.129 A 40-m cable is strung as shown between two buildings. The maximum tension is found to be 350 N, and the lowest point of the cable is observed to be 6 m above the ground. Determine (*a*) the horizontal distance between the buildings, (*b*) the total mass of the cable.

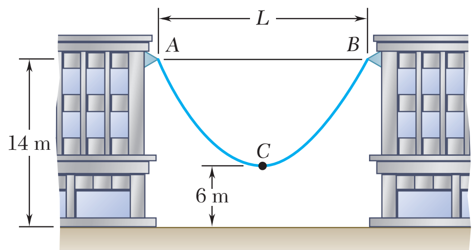


Fig. P7.129

7.130 A 200-ft steel surveying tape weighs 4 lb. If the tape is stretched between two points at the same elevation and pulled until the tension at each end is 16 lb, determine the horizontal distance between the ends of the tape. Neglect the elongation of the tape due to the tension.

7.131 A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at *A* and to a collar at *B*. Neglecting the effect of friction, determine (*a*) the force **P** for which $h = 8$ m, (*b*) the corresponding span *L*.

7.132 A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at *A* and to a collar at *B*. Knowing that the magnitude of the horizontal force applied to the collar is $P = 20$ N, determine (*a*) the sag *h*, (*b*) the span *L*.

7.133 A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at *A* and to a collar at *B*. Neglecting the effect of friction, determine (*a*) the sag *h* for which $L = 15$ m, (*b*) the corresponding force **P**.

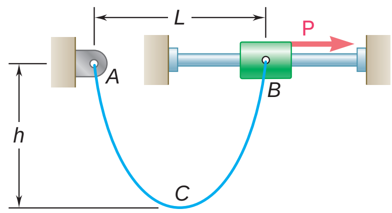


Fig. *P7.131, P7.132,* and P7.133

7.134 Determine the sag of a 30-ft chain that is attached to two points at the same elevation that are 20 ft apart.

7.135 A 10-ft rope is attached to two supports A and B as shown. Determine (*a*) the span of the rope for which the span is equal to the sag, (*b*) the corresponding angle θ_B.

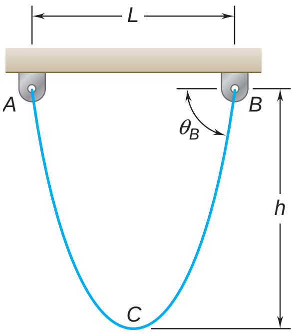


Fig. P7.135

7.136 A 90-m wire is suspended between two points at the same elevation that are 60 m apart. Knowing that the maximum tension is 300 N, determine (*a*) the sag of the wire, (*b*) the total mass of the wire.

7.137 A cable weighing 2 lb/ft is suspended between two points at the same elevation that are 160 ft apart. Determine the smallest allowable sag of the cable if the maximum tension is not to exceed 400 lb.

7.138 A uniform cord 50 in. long passes over a pulley at B and is attached to a pin support at A. Knowing that $L = 20$ in. and neglecting the effect of friction, determine the smaller of the two values of h for which the cord is in equilibrium.

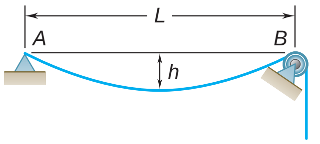


Fig. *P7.138*

7.139 A motor M is used to slowly reel in the cable shown. Knowing that the mass per unit length of the cable is 0.4 kg/m, determine the maximum tension in the cable when $h = 5$ m.

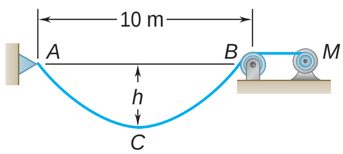


Fig. P7.139 and P7.140

7.140 A motor M is used to slowly reel in the cable shown. Knowing that the mass per unit length of the cable is 0.4 kg/m, determine the maximum tension in the cable when $h = 3$ m.

7.141 The cable ACB has a mass per unit length of 0.45 kg/m. Knowing that the lowest point of the cable is located at a distance $a = 0.6$ m below the support A, determine (*a*) the location of the lowest point C, (*b*) the maximum tension in the cable.

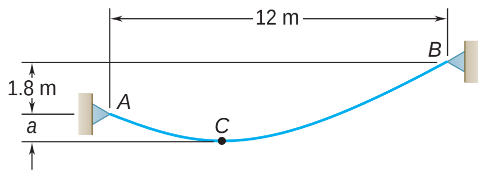


Fig. *P7.141* and *P7.142*

7.142 The cable ACB has a mass per unit length of 0.45 kg/m. Knowing that the lowest point of the cable is located at a distance $a = 2$ m below the support A, determine (*a*) the location of the lowest point C, (*b*) the maximum tension in the cable.

7.143 A uniform cable weighing 3 lb/ft is held in the position shown by a horizontal force **P** applied at B. Knowing that $P = 180$ lb and $\theta_A = 60°$, determine (*a*) the location of point B, (*b*) the length of the cable.

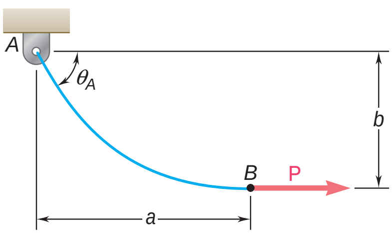


Fig. P7.143 and P7.144

7.144 A uniform cable weighing 3 lb/ft is held in the position shown by a horizontal force **P** applied at B. Knowing that $P = 150$ lb and $\theta_A = 60°$, determine (*a*) the location of point B, (*b*) the length of the cable.

7.145 To the left of point B the long cable $ABDE$ rests on the rough horizontal surface shown. Knowing that the mass per unit length of the cable is 2 kg/m, determine the force $\mathbf{F}$ when $a = 3.6$ m.

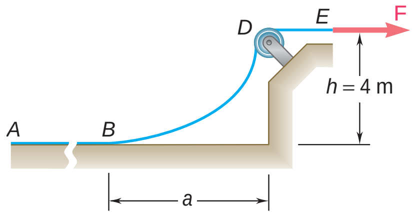


Fig. P7.145 and P7.146

7.146 To the left of point B the long cable $ABDE$ rests on the rough horizontal surface shown. Knowing that the mass per unit length of the cable is 2 kg/m, determine the force $\mathbf{F}$ when $a = 6$ m.

***7.147** The 10-ft cable AB is attached to two collars as shown. The collar at A can slide freely along the rod; a stop attached to the rod prevents the collar at B from moving on the rod. Neglecting the effect of friction and the weight of the collars, determine the distance a.

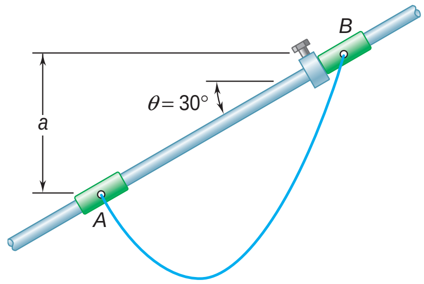


Fig. P7.147

***7.148** Solve Prob. 7.147 assuming that the angle u formed by the rod and the horizontal is 45°.

7.149 Denoting by u the angle formed by a uniform cable and the horizontal, show that at any point (*a*) $s = c$ tan u, (*b*) $y = c$ sec u.

***7.150** (*a*) Determine the maximum allowable horizontal span for a uniform cable of weight per unit length w if the tension in the cable is not to exceed a given value T_m. (*b*) Using the result of part *a*, determine the maximum span of a steel wire for which $w = 0.25$ lb/ft and $T_m = 8000$ lb.

***7.151** A cable has a mass per unit length of 3 kg/m and is supported as shown. Knowing that the span L is 6 m, determine the *two* values of the sag h for which the maximum tension is 350 N.

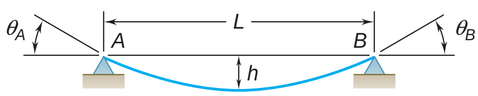


Fig. P7.151, P7.152, and P7.153

***7.152** Determine the sag-to-span ratio for which the maximum tension in the cable is equal to the total weight of the entire cable AB.

***7.153** A cable of weight per unit length w is suspended between two points at the same elevation that are a distance L apart. Determine (*a*) the sag-to-span ratio for which the maximum tension is as small as possible, (*b*) the corresponding values of u_B and T_m.

REVIEW AND SUMMARY

In this chapter you learned to determine the internal forces which hold together the various parts of a given member in a structure.

Forces in straight two-force members

Considering first a *straight two-force member AB* [Sec. 7.2], we recall that such a member is subjected at A and B to equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ directed along AB (Fig. 7.19*a*). Cutting member AB at C and drawing the free-body diagram of portion AC, we conclude that the internal forces which existed at C in member AB are equivalent to an *axial force* $-\mathbf{F}$ equal and opposite to $\mathbf{F}$ (Fig. 7.19*b*). We note that in the case of a two-force member which is not straight, the internal forces reduce to a force-couple system and not to a single force.

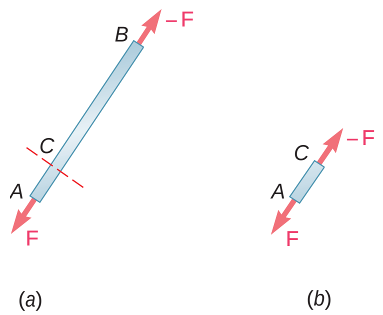


Fig. 7.19

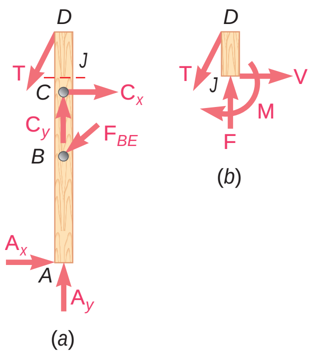


Fig. 7.20

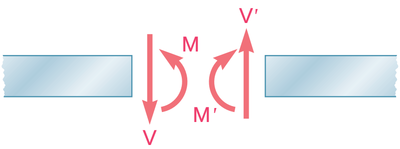
Forces in multiforce members

Considering next a *multiforce member AD* (Fig. 7.20*a*), cutting it at J, and drawing the free-body diagram of portion JD, we conclude that the internal forces at J are equivalent to a force-couple system consisting of the *axial force* $\mathbf{F}$, the *shearing force* $\mathbf{V}$, and a couple $\mathbf{M}$ (Fig. 7.20*b*). The magnitude of the shearing force measures the *shear* at point J, and the moment of the couple is referred to as the *bending moment* at J. Since an equal and opposite force-couple system would have been obtained by considering the free-body diagram of portion AJ, it is necessary to specify which portion of member AD was used when recording the answers [Sample Prob. 7.1].

Forces in beams

Most of the chapter was devoted to the analysis of the internal forces in two important types of engineering structures: *beams* and *cables*. *Beams* are usually long, straight prismatic members designed to support loads applied at various points along the member. In general the loads are perpendicular to the axis of the beam and produce only *shear and bending* in the beam. The loads may be either *concentrated* at specific points, or *distributed* along the entire length or a portion of the beam. The beam itself may be supported in various ways; since only statically determinate beams are considered in this text, we limited our analysis to that of *simply supported beams, overhanging beams,* and *cantilever beams* [Sec. 7.3].

Shear and bending moment in a beam



Internal forces at section (positive shear and positive bending moment)

Fig. 7.21

To obtain the *shear* V and *bending moment* M at a given point C of a beam, we first determine the reactions at the supports by considering the entire beam as a free body. We then cut the beam at C and use the free-body diagram of one of the two portions obtained in this fashion to determine V and M. In order to avoid any confusion regarding the sense of the shearing force $\mathbf{V}$ and couple $\mathbf{M}$ (which act in opposite directions on the two portions of the beam), the sign convention illustrated in Fig. 7.21 was adopted [Sec. 7.4]. Once the values of the shear and bending moment have been determined at a few selected points of the beam, it is usually possible to draw a *shear diagram* and a *bending-moment diagram* representing, respectively, the shear and bending moment at any point of the beam [Sec. 7.5]. When a beam is subjected to concentrated loads only, the shear is of constant value between loads and the bending moment varies linearly between loads [Sample Prob. 7.2]. On the other hand, when a beam is subjected to distributed loads, the shear and bending moment vary quite differently [Sample Prob. 7.3].

Relations among load, shear, and bending moment

The construction of the shear and bending-moment diagrams is facilitated if the following relations are taken into account. Denoting by w the distributed load per unit length (assumed positive if directed downward), we have [Sec. 7.5]:

$$\frac{dV}{dx} = -w \qquad (7.1)$$

$$\frac{dM}{dx} = V \qquad (7.3)$$

or, in integrated form,

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \qquad (7.2')$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \qquad (7.4')$$

Equation (7.2′) makes it possible to draw the shear diagram of a beam from the curve representing the distributed load on that beam and the value of V at one end of the beam. Similarly, Eq. (7.4′) makes it possible to draw the bending-moment diagram from the shear diagram and the value of M at one end of the beam. However, concentrated loads introduce discontinuities in the shear diagram and concentrated couples in the bending-moment diagram, none of which are accounted for in these equations [Sample Probs. 7.4 and 7.7]. Finally, we note from Eq. (7.3) that the points of the beam where the bending moment is maximum or minimum are also the points where the shear is zero [Sample Prob. 7.5].

Cables with concentrated loads

The second half of the chapter was devoted to the analysis of *flexible cables*. We first considered a cable of negligible weight supporting *concentrated loads* [Sec. 7.7]. Using the entire cable AB as a free body (Fig. 7.22), we noted that the three available equilibrium equations were not sufficient to determine the four unknowns representing the reactions at the supports A and B. However, if the coordinates of a point D of the cable are known, an additional equation can be obtained by considering the free-body diagram of the portion AD or DB of the cable. Once the reactions at the supports have been determined, the elevation of any point of the cable and the tension in any portion of the cable can be found from the appropriate free-body diagram [Sample Prob. 7.8]. It was noted that the horizontal component of the force **T** representing the tension is the same at any point of the cable.

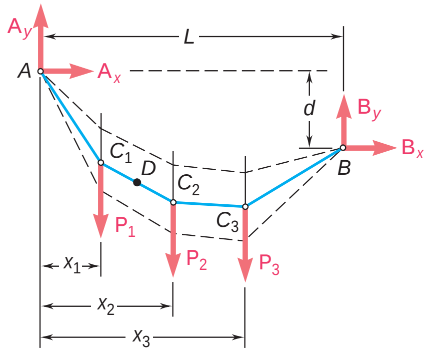


Fig. 7.22

Cables with distributed loads

We next considered cables carrying *distributed loads* [Sec. 7.8]. Using as a free body a portion of cable CD extending from the lowest point C to an arbitrary point D of the cable (Fig. 7.23), we observed that the horizontal component of the tension force **T** at D is constant and equal to the tension T_0 at C, while its vertical component is equal to the weight W of the portion of cable CD. The magnitude and direction of **T** were obtained from the force triangle:

$$T = \sqrt{T_0^2 + W^2} \qquad \tan \theta = \frac{W}{T_0} \tag{7.6}$$

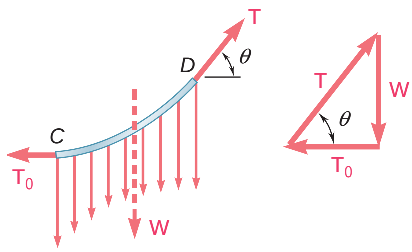


Fig. 7.23

Parabolic cable

In the case of a load *uniformly distributed along the horizontal*—as in a suspension bridge (Fig. 7.24)—the load supported by portion CD is $W = wx$, where w is the constant load per unit horizontal length [Sec. 7.9]. We also found that the curve formed by the cable is a *parabola* of equation

$$y = \frac{wx^2}{2T_0} \tag{7.8}$$

and that the length of the cable can be found by using the expansion in series given in Eq. (7.10) [Sample Prob. 7.9].

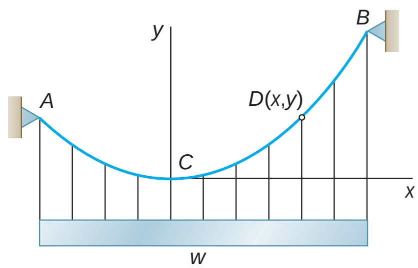


Fig. 7.24

Catenary

In the case of a load *uniformly distributed along the cable itself*—e.g., a cable hanging under its own weight (Fig. 7.25)—the load supported by portion CD is $W = ws$, where s is the length measured along the cable and w is the constant load per unit length [Sec. 7.10]. Choosing the origin O of the coordinate axes at a distance $c = T_0/w$ below C, we derived the relations

$$s = c \sinh \frac{x}{c} \tag{7.15}$$

$$y = c \cosh \frac{x}{c} \tag{7.16}$$

$$y^2 - s^2 = c^2 \tag{7.17}$$

$$T_0 = wc \qquad W = ws \qquad T = wy \tag{7.18}$$

which can be used to solve problems involving cables hanging under their own weight [Sample Prob. 7.10]. Equation (7.16), which defines the shape of the cable, is the equation of a *catenary*.

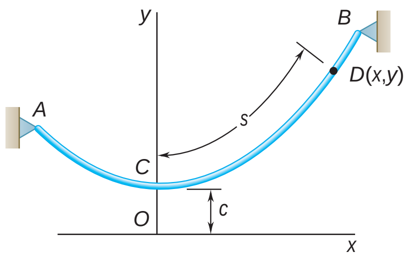


Fig. 7.25

REVIEW PROBLEMS

7.154 Determine the internal forces at point J of the structure shown.

7.155 Determine the internal forces at point K of the structure shown.

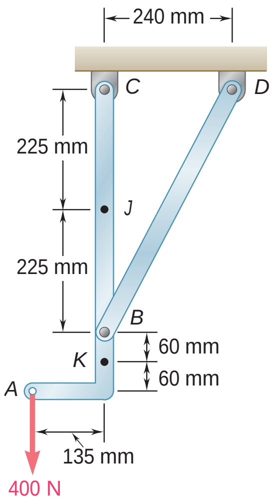


Fig. P7.154 and *P7.155*

7.156 An archer aiming at a target is pulling with a 45-lb force on the bowstring. Assuming that the shape of the bow can be approximated by a parabola, determine the internal forces at point J.

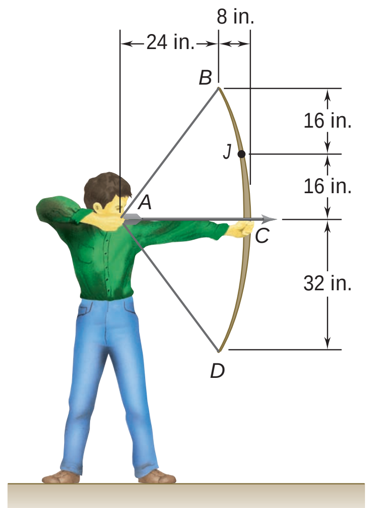


Fig. P7.156

7.157 Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at point J of the frame shown.

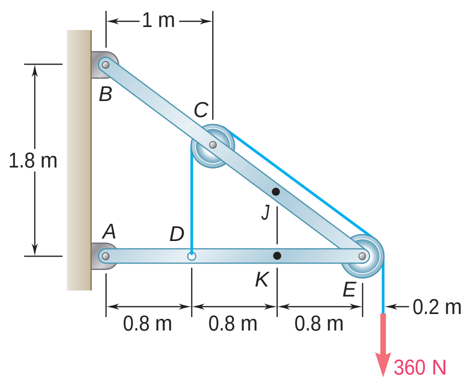


Fig. P7.157

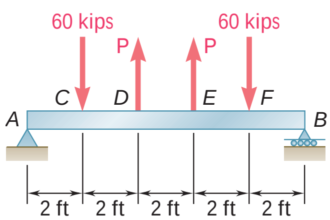


Fig. P7.158

7.158 For the beam shown, determine (*a*) the magnitude P of the two upward forces for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{max}$.

7.159 and 7.160 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

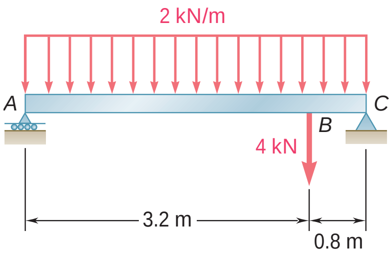


Fig. P7.159

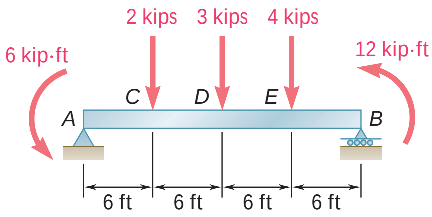


Fig. P7.160

7.161 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

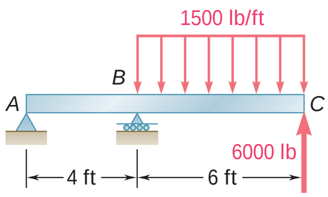


Fig. P7.161

7.162 The beam *AB*, which lies on the ground, supports the parabolic load shown. Assuming the upward reaction of the ground to be uniformly distributed, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the maximum bending moment.

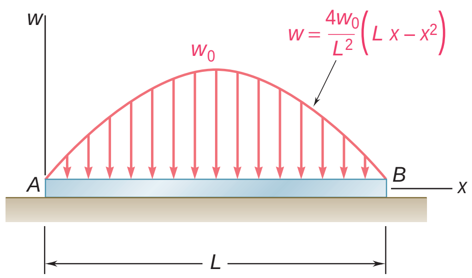


Fig. P7.162

7.163 Two loads are suspended as shown from the cable *ABCD*. Knowing that $d_B = 1.8$ m, determine (*a*) the distance d_C, (*b*) the components of the reaction at *D*, (*c*) the maximum tension in the cable.

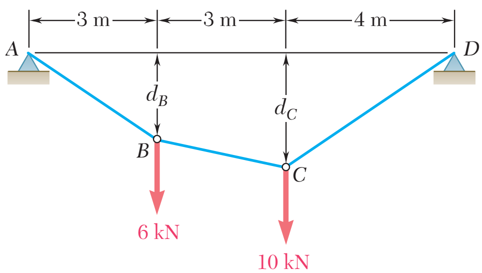


Fig. P7.163

7.164 A wire having a mass per unit length of 0.65 kg/m is suspended from two supports at the same elevation that are 120 m apart. If the sag is 30 m, determine (*a*) the total length of the wire, (*b*) the maximum tension in the wire.

7.165 A counterweight *D* is attached to a cable that passes over a small pulley at *A* and is attached to a support at *B*. Knowing that $L = 45$ ft and $h = 15$ ft, determine (*a*) the length of the cable from *A* to *B*, (*b*) the weight per unit length of the cable. Neglect the weight of the cable from *A* to *D*.

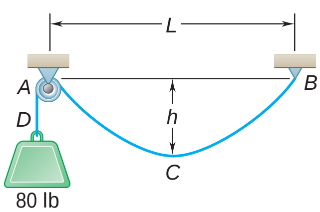


Fig. P7.165

COMPUTER PROBLEMS

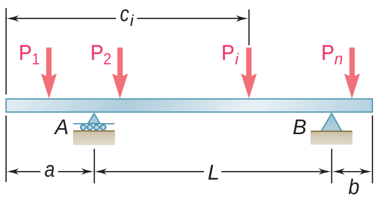


Fig. P7.C1

7.C1 An overhanging beam is to be designed to support several concentrated loads. One of the first steps in the design of the beam is to determine the values of the bending moment that can be expected at the supports *A* and *B* and under each of the concentrated loads. Write a computer program that can be used to calculate those values for the arbitrary beam and loading shown. Use this program for the beam and loading of (*a*) Prob. 7.36, (*b*) Prob. 7.37, (*c*) Prob. 7.38.

7.C2 Several concentrated loads and a uniformly distributed load are to be applied to a simply supported beam *AB*. As a first step in the design of the beam, write a computer program that can be used to calculate the shear and bending moment in the beam for the arbitrary loading shown using given increments Δx. Use this program for the beam of (*a*) Prob. 7.39, with $\Delta x = 0.25$ m; (*b*) Prob. 7.41, with $\Delta x = 0.5$ ft; (*c*) Prob. 7.42, with $\Delta x = 0.5$ ft.

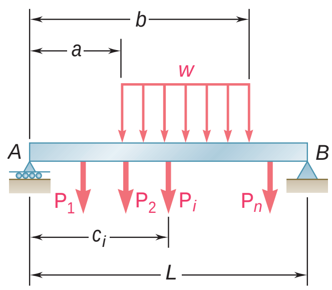


Fig. P7.C2

7.C3 A beam *AB* hinged at *B* and supported by a roller at *D* is to be designed to carry a load uniformly distributed from its end *A* to its midpoint *C* with maximum efficiency. As part of the design process, write a computer program that can be used to determine the distance *a* from end *A* to the point *D* where the roller should be placed to minimize the absolute value of the bending moment *M* in the beam. (*Note:* A short preliminary analysis will show that the roller should be placed under the load and that the largest negative value of *M* will occur at *D*, while its largest positive value will occur somewhere between *D* and *C*. Also see the hint for Prob. 7.55.)

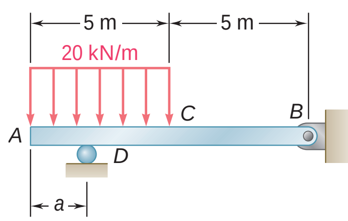


Fig. P7.C3

7.C4 The floor of a bridge will consist of narrow planks resting on two simply supported beams, one of which is shown in the figure. As part of the design of the bridge, it is desired to simulate the effect that driving a 3000-lb truck over the bridge will have on this beam. The distance between the truck's axles is 6 ft, and it is assumed that the weight of the truck is equally distributed over its four wheels. (*a*) Write a computer program that can be used to calculate the magnitude and location of the maximum bending moment in the beam for values of x from -3 ft to 10 ft using 0.5-ft increments. (*b*) Using smaller increments if necessary, determine the largest value of the bending moment that occurs in the beam as the truck is driven over the bridge and determine the corresponding value of x.

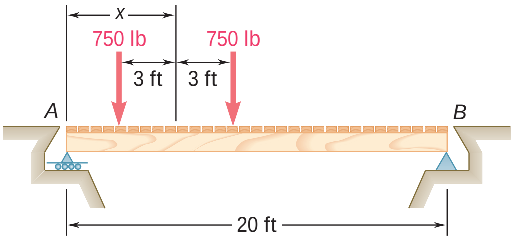


Fig. P7.C4

***7.C5** Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam of Prob. 7.C1. Using this program and a plotting increment $\Delta x \le L/100$, plot the V and M diagrams for the beam and loading of (*a*) Prob. 7.36, (*b*) Prob. 7.37, (*c*) Prob. 7.38.

***7.C6** Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam of Prob. 7.C2. Using this program and a plotting increment $\Delta x \le L/100$, plot the V and M diagrams for the beam and loading of (*a*) Prob. 7.39, (*b*) Prob. 7.41, (*c*) Prob. 7.42.

7.C7 Write a computer program that can be used in the design of cable supports to calculate the horizontal and vertical components of the reaction at the support A_n from values of the loads $P_1, P_2, \ldots, P_{n-1}$, the horizontal distances $d_1, d_2, \ldots, d_n$, and the two vertical distances h_0 and h_k. Use this program to solve Probs. 7.95*b*, 7.96*b*, and 7.97*b*.

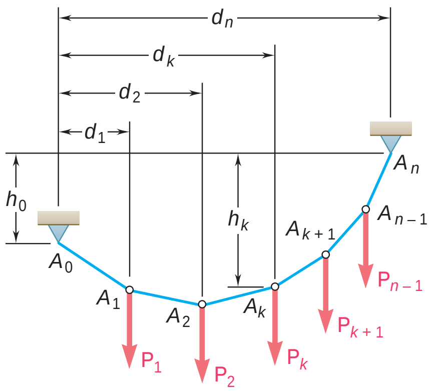


Fig. P7.C7

7.C8 A typical transmission-line installation consists of a cable of length s_{AB} and weight w per unit length suspended as shown between two points at the same elevation. Write a computer program and use it to develop a table that can be used in the design of future installations. The table should present the dimensionless quantities h/L, s_{AB}/L, T_0/wL, and T_{max}/wL for values of c/L from 0.2 to 0.5 using 0.025 increments and from 0.5 to 4 using 0.5 increments.

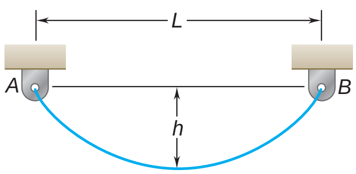


Fig. P7.C8

7.C9 Write a computer program and use it to solve Prob. 7.132 for values of P from 0 to 50 N using 5-N increments.