

The aim of this worksheet is to find an expression for the critical point where infected adults disappear in a disease model in which juvenile and adult consumers compete for a basic resource, while adult consumers are foraged on by a stage-specific predators. The disease can only infect juvenile consumers, which can mature into infected adult consumers as long as the resource density is sufficiently high.

Here are the model equations:

$$\frac{dR}{dt} = \delta(R_{\max} - R) - a \cdot q \cdot R \cdot (C_A + I_A) - \frac{a \cdot R}{q} \cdot (C_J + I_J)$$

$$\frac{dC_J}{dt} = \beta_C \cdot \max(\sigma_C \cdot a \cdot q \cdot R - T_C) \cdot C_A + \beta_C \cdot \max(\sigma_C \cdot (1 - e) \cdot a \cdot q \cdot R - (1 + d) \cdot T_C) \cdot I_A - \max\left(\frac{\sigma_C \cdot a \cdot R}{q} - T_C\right) \cdot C_J - \mu_C \cdot C_J - i \cdot C_J \cdot (I_J + I_A)$$

$$\frac{dI_J}{dt} = i \cdot C_J \cdot (I_J + I_A) - \max\left(\frac{\sigma_C \cdot (1 - e) \cdot a \cdot R}{q} - (1 + d) \cdot T_C\right) \cdot I_J - \mu_C \cdot I_J$$

$$\frac{dC_A}{dt} = \max\left(\frac{\sigma_C \cdot a \cdot R}{q} - T_C\right) \cdot C_J - \mu_C \cdot C_A - b \cdot C_A \cdot P$$

$$\frac{dI_A}{dt} = \max\left(\frac{\sigma_C \cdot (1 - e) \cdot a \cdot R}{q} - (1 + d) \cdot T_C\right) \cdot I_J - \mu_C \cdot I_A - b \cdot I_A \cdot P$$

$$\frac{dP}{dt} = \beta_P \cdot \max(\sigma_P \cdot b \cdot (C_A + I_A) - T_P) \cdot P - \mu_P \cdot P$$

The condition determining that infected adult consumers disappear from the model ($I_A = 0$) is given by the condition:

$$\frac{(1 - e) \cdot \sigma_C \cdot a \cdot R}{q} - (1 + d) \cdot T_C = 0$$

or

$$R = \frac{q \cdot (1 + d) \cdot T_C}{(1 - e) \cdot \sigma_C \cdot a}$$

In an equilibrium without any infected adults, the density of susceptible adult consumers is determined by the predator:

$$C_A := \frac{(T_P + \mu_P)}{\beta_P \cdot \sigma_P \cdot b}$$

$$C_A := \frac{T_P + \mu_P}{\beta_P \sigma_P b} \quad (1)$$

If maturation of infected juveniles does not occur and hence infected adults are not present, the density of infectde susceptible juveniles can be derived from the dynamic equation for the infected juveniles as:

$$C_J := \frac{\mu_C}{i}$$

$$C_J := \frac{\mu_C}{i} \quad (2)$$

The differential equation for C_J then yields an expression for I_J in equilibrium:

$$I_J := \frac{1}{i} \cdot \left(\frac{\beta_C \cdot (\sigma_C \cdot a \cdot q \cdot R - T_C) \cdot C_A}{C_J} - \left(\frac{\sigma_C \cdot a \cdot R}{q} - T_C \right) - \mu_C \right)$$

$$I_J := \frac{\frac{\beta_C (\sigma_C a q R - T_C) (T_P + \mu_P) i}{\beta_P \sigma_P b \mu_C} - \frac{\sigma_C a R}{q} + T_C - \mu_C}{i} \quad (3)$$

The sum $C_J + I_J$ hence can be written as:

$$C_J + I_J = \frac{1}{i} \cdot \left(\frac{\beta_C \cdot (\sigma_C \cdot a \cdot q \cdot R - T_C) \cdot C_A}{C_J} - \left(\frac{\sigma_C \cdot a \cdot R}{q} - T_C \right) \right)$$

The above expressions for C_J , C_A and I_J can be substituted in the differential equation for R to yield:

$$R_{cond} := \delta \cdot (R_{\max} - R) - a \cdot q \cdot R \cdot C_A - \frac{a \cdot R}{q} \cdot \frac{1}{i} \cdot \left(\frac{\beta_C \cdot (\sigma_C \cdot a \cdot q \cdot R - T_C) \cdot C_A}{C_J} - \left(\frac{\sigma_C \cdot a \cdot R}{q} - T_C \right) \right)$$

$$R_{cond} := \delta (R_{\max} - R) - \frac{a q R (T_P + \mu_P)}{\beta_P \sigma_P b} - \frac{a R \left(\frac{\beta_C (R a q \sigma_C - T_C) (T_P + \mu_P) i}{\beta_P \sigma_P b \mu_C} - \frac{\sigma_C a R}{q} + T_C \right)}{q i} \quad (4)$$

This is a quadratic equation in R :

$$collect(R_{cond}, R)$$

$$-\frac{a \left(\frac{\beta_C a q \sigma_C (T_P + \mu_P) i}{\beta_P \sigma_P b \mu_C} - \frac{\sigma_C a}{q} \right) R^2}{q i} + \left(-\delta - \frac{a q (T_P + \mu_P)}{\beta_P \sigma_P b} - \frac{a \left(-\frac{\beta_C T_C (T_P + \mu_P) i}{\beta_P \sigma_P b \mu_C} + T_C \right)}{q i} \right) R + \delta R_{\max} \quad (5)$$

with as unknowns:

$$indets(R_{cond})$$

$$\{R, R_{\max}, T_C, T_P, a, b, \beta_C, \beta_P, \delta, i, \mu_C, \mu_P, q, \sigma_C, \sigma_P\} \quad (6)$$

Substituting the expression $R = \frac{q \cdot (1 + d) \cdot T_C}{(1 - e) \cdot \sigma_C \cdot a}$ in this equation, which determines that the maturation rate of infected juveniles reaches 0, yields the expression in terms of all parameters determining the point where $I_A = 0$:

$$\begin{aligned}
 BPcond &:= \text{subs} \left(R = \frac{q \cdot (1 + d) \cdot T_C}{(1 - e) \cdot \sigma_C \cdot a}, Rcond \right) \\
 BPcond &:= \delta \left(R_{\max} - \frac{q (1 + d) T_C}{(1 - e) \sigma_C a} \right) - \frac{q^2 (1 + d) T_C (T_P + \mu_P)}{(1 - e) \sigma_C \beta_P \sigma_P b} \\
 &\quad - \frac{(1 + d) T_C \left(\frac{\beta_C \left(\frac{q^2 (1 + d) T_C}{1 - e} - T_C \right) (T_P + \mu_P) i}{\beta_P \sigma_P b \mu_C} - \frac{(1 + d) T_C}{1 - e} + T_C \right)}{(1 - e) \sigma_C i}
 \end{aligned} \tag{7}$$

Here are the default parameters:

$$\begin{aligned}
 pars &:= \{ \delta = 0.5, R_{\max} = 400, a = 0.2, \sigma_C = 0.5, T_C = 1.8, \mu_C = 0.1, \beta_C = 1.5, b = 0.1, \sigma_P = 0.5, T_P = 1, \mu_P = 0.05, \beta_P = 1, d = 0.5, q = 0.5, i = 0.1 \} \\
 pars &:= \{ R_{\max} = 400, T_C = 1.8, T_P = 1, a = 0.2, b = 0.1, \beta_C = 1.5, \beta_P = 1, d = 0.5, \delta = 0.5, i = 0.1, \mu_C = 0.1, \mu_P = 0.05, q = 0.5, \sigma_C = 0.5, \sigma_P \\
 &= 0.5 \}
 \end{aligned} \tag{8}$$

This leads to the prediction that the branching point where $I_A = 0$ occurs at a value of e equal to:

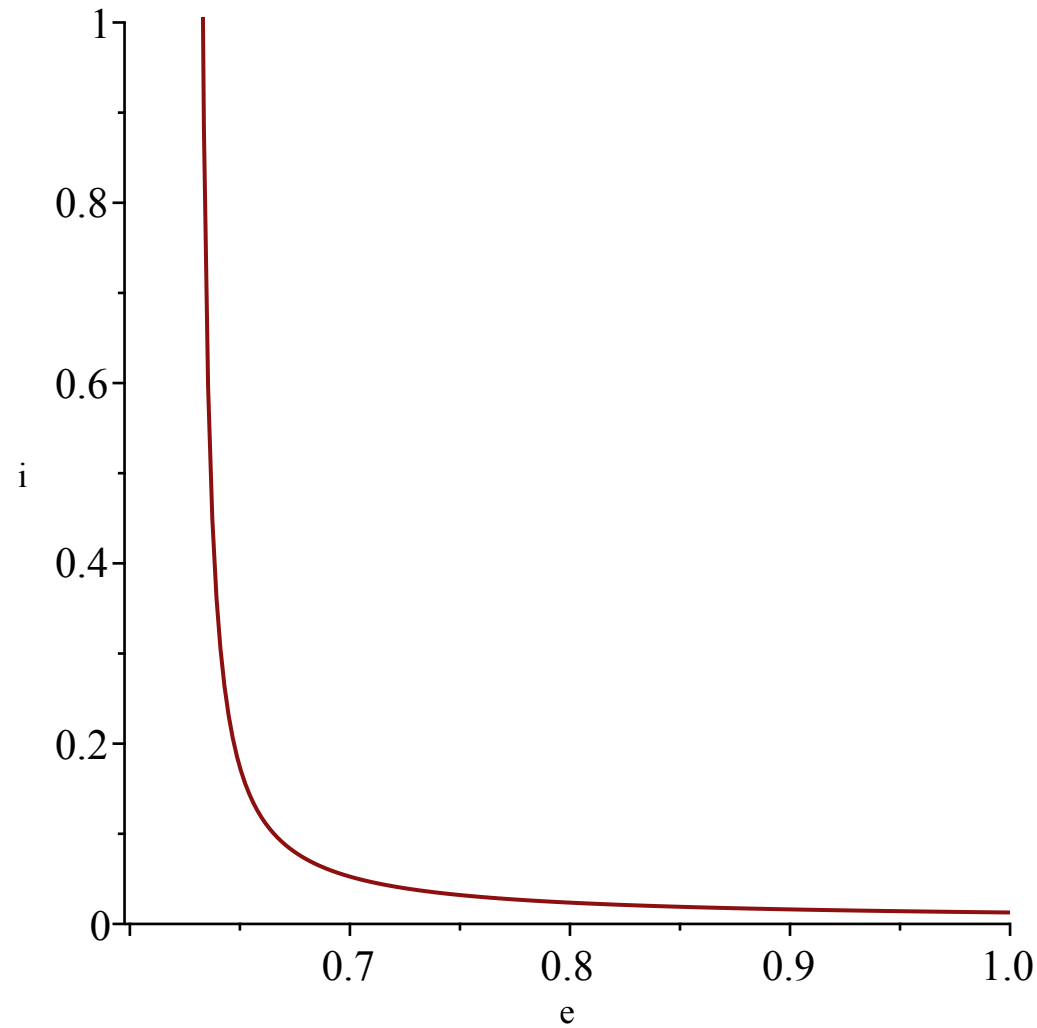
$$\begin{aligned}
 &fsolve(\text{subs}(pars, BPcond), e = 0.7) \\
 &0.6654741456
 \end{aligned} \tag{9}$$

The expression for the branching point can be most easily solved for i :

$i_{limit} := \text{simplify}(\text{solve}(BPcond, i))$ assuming *positive*

$$i_{limit} := \left(T_C^2 a b \mu_C \beta_P \sigma_P (1+d) (d+e) \right) / \left((1+d) \beta_C (q^2 (1+d) - 1 + e) (T_P + \mu_P) a T_C^2 - (1+d) q \mu_C (q (T_P + \mu_P) a + \beta_P \sigma_P b \delta) (-1 + e) T_C - a R_{\max} \beta_P \sigma_C \sigma_P b \delta \mu_C (-1 + e)^2 \right) \quad (10)$$

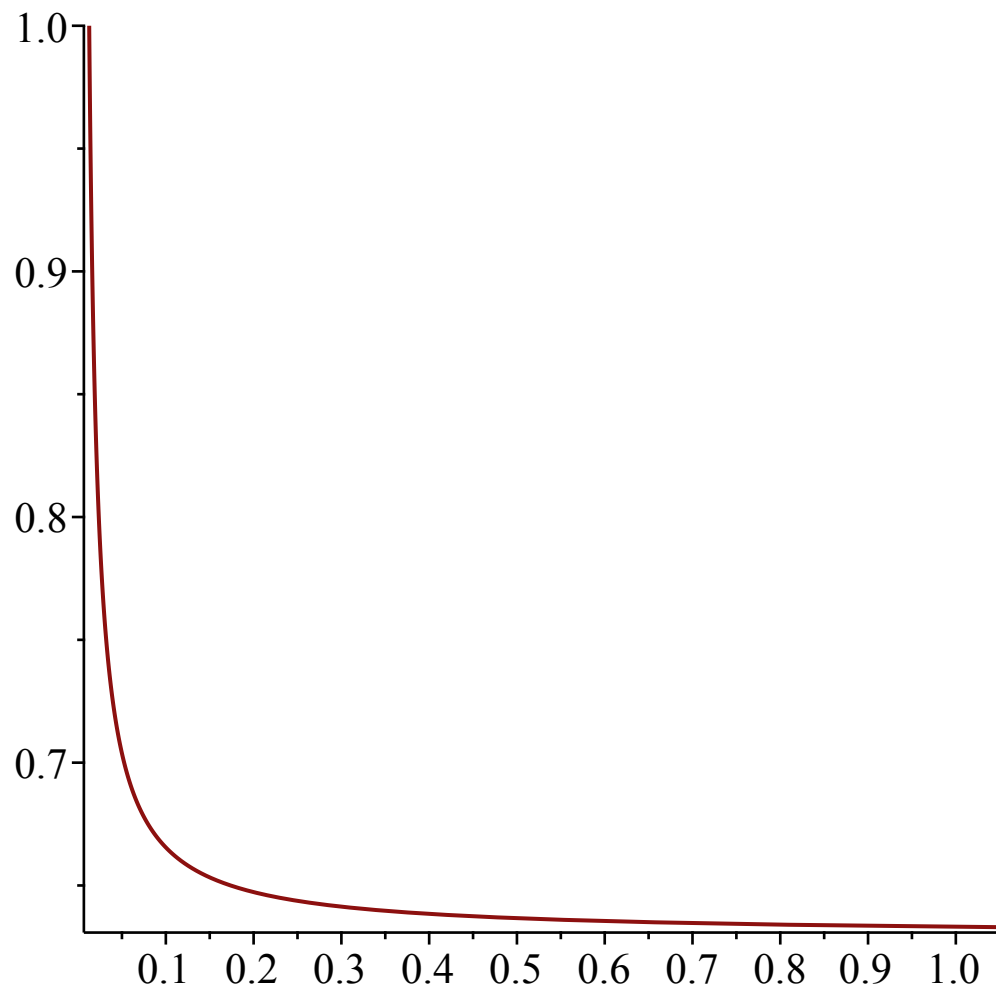
$\text{plot}(\text{subs}(\text{pars}, i_{limit}), e = 0.63 \dots 1, \text{view} = [0.6 \dots 1, 0 \dots 1], \text{labels} = ["e", "i"])$



$$fsolve(subs(pars, ilimit - 1), e = 0.7) \quad 0.6331815760 \quad (11)$$

$$evalf(subs(e = 0.633, pars, ilimit)) \quad 1.055033866 \quad (12)$$

```
IEcurve := [ ]:
for x from 0.633 by 0.001 to 1.0 do IEcurve := [op(IEcurve), [evalf(subs(e = x, pars, ilimit)), x]] end do:
plot(IEcurve)
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writedata("IEcurve.dat", IEcurve);
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