

## **Tutorial**

### **Numerical Solutions for ODE**

- 1)  $y' = x\sqrt{y}$  ,  $y(2) = 1$ . Find  $y(3)$  using the following:
  - a) Exact Method
  - b) Euler Method with  $h = 0.2$
  - c) Euler Method with  $n = 10$
  - d) Runge-Kutta Method with  $h = 1$
  
- 2)  $y' = y - y^2$  ,  $y(0) = 0.2$ . Find  $y(1)$  using
  - a) Runge-Kutta method with 1 step.
  - b) Runge-Kutta method with 2 steps.
  
- 3)  $x'' - tx' - x = 0$  ,  $x(0) = 1$ ,  $x'(0) = 1$ . Find  $x(1)$  using
  - a) Runge-Kutta method with 1 step.
  - b) Runge-Kutta method with 2 steps.

## Solution

1) a. Exact Method

$$\because y' = x\sqrt{y} \Rightarrow \frac{dy}{dx} = x\sqrt{y} \Rightarrow y^{-1/2} dy = x dx$$

$$\Rightarrow \int y^{-1/2} dy = \int x dx \Rightarrow 2y^{1/2} = \frac{1}{2}x^2 + c$$

$$\because y(2) = 1 \Rightarrow 2 = 2 + c \Rightarrow c = 0$$

$$\Rightarrow 2y^{1/2} = \frac{1}{2}x^2 \Rightarrow y = \frac{1}{16}x^4 \Rightarrow y(3) = 5.0625$$

b, c & d

$$\rightarrow y' = x\sqrt{y} = f(x, y) \quad \rightarrow y(2) = 1 \Rightarrow x_0 = 2, y_0 = 1$$

$$\rightarrow h = \frac{x_n - x_0}{n}, \quad \text{Required : at } x_n = 3, y_n = ?$$

$$\text{For Euler : } y_{n+1} = y_n + \Delta y \quad \text{where } \Delta y = h f(x_n, y_n)$$

$$\text{For Runge - Kutta : } y_{n+1} = y_n + \Delta y$$

$$\text{where } \Delta y = \frac{1}{6}(w_1 + 2w_2 + 2w_3 + w_4)$$

$$w_1 = h f(x_n, y_n)$$

$$w_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{w_1}{2}\right)$$

$$w_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{w_2}{2}\right)$$

$$w_4 = h f(x_n + h, y_n + w_3)$$

		b. Euler with h=0.2			
n	Xn	Yn	$\Delta y = 0.2 * Xn * \sqrt{Yn}$	Yn+1	Error
0	2	1	0.4	1.4	0
1	2.2	1.4	0.520615021	1.92062	0.0641
2	2.4	1.9206	0.665214026	2.58583	0.15298
3	2.6	2.5858	0.836186686	3.42202	0.27027
4	2.8	3.422	1.035926703	4.45794	0.41958
5	3	<b>4.4579</b>			0.60456

	c. Euler with ,n=10 → h=0.1				
n	Xn	Yn	$\Delta y = 0.1 * Xn * \sqrt{Yn}$	Yn+1	Error
0	2	1	0.2	1.2	0
1	2.1	1.2	0.230043474	1.43004	0.01551
2	2.2	1.43	0.263085735	1.69313	0.03406
3	2.3	1.6931	0.299276687	1.99241	0.05588
4	2.4	1.9924	0.338766261	2.33117	0.08119
5	2.5	2.3312	0.381704414	2.71288	0.11023
6	2.6	2.7129	0.428241119	3.14112	0.14322
7	2.7	3.1411	0.478526363	3.61964	0.18039
8	2.8	3.6196	0.53271014	4.15235	0.22196
9	2.9	4.1524	0.590942457	4.7433	0.26815
10	3	<b>4.7433</b>		4.7433	0.3192

		d. R-K with h=1			
n	Xn	Yn	w=(1)*Xn*sqrt(Yn)		Δy
0	2	1	2	w1	2
	2.5	2	3.535533906	w2	7.071068
	2.5	2.767767	4.159151771	w3	8.318304
	3	5.159152	6.814129874	w4	6.81413
				Δy=	4.033917
		Y1=	5.033916871		

2)

$$\rightarrow y' = y - y^2 = f(x, y) \quad \rightarrow y(0) = 0.2 \Rightarrow x_0 = 0, y_0 = 0.2$$

$$\rightarrow h = \frac{x_n - x_0}{n}, \quad \text{Required : at } x_n = 1, y_n = ?$$

		a. R-K with h=1 (1 step)			
n	Xn	Yn	w=(1)(Yn-Yn^2)		Δy
0	0	0.2	0.16	w1	0.16
	0.5	0.28	0.2016	w2	0.4032
	0.5	0.3008	0.21031936	w3	0.420639
	1	0.410319	0.241957383	w4	0.241957
				Δy=	0.204299
		Y1=	0.40429935		

		b. R-K with h=0.5 (2 steps)			
n	Xn	Yn	w=(0.5)(Yn-Yn^2)		Δy
0	0	0.2	0.08	w1	0.08
	0.25	0.24	0.0912	w2	0.1824
	0.25	0.2456	0.09264032	w3	0.185281
	0.5	0.29264	0.103500982	w4	0.103501
				Δy=	0.091864
		Y1=	0.291863604		
n	Xn	Yn	w=(0.5)(Yn-Yn^2)		Δy
1	0.5	0.291864	0.10333962	w1	0.10334
	0.75	0.343533	0.112759104	w2	0.225518
	0.75	0.348243	0.11348493	w3	0.22697
	1	0.405349	0.12052055	w4	0.120521
				Δy=	0.112725
		Y2=	0.40458831		

3)

$$\rightarrow x'' - tx' - x = 0, x(0) = 1, x'(0) = 1, \text{ Required } x(1) = ?$$

$$\text{let } x' = y = f(t, x, y) \Rightarrow x'' = y' = ty + x = g(t, x, y)$$

$$\Rightarrow t_0 = 0, x_0 = 1, y_0 = 1, \text{ Required at } t_n = 1, x_n = ?$$

$$x_{n+1} = x_n + \Delta x$$

$$y_{n+1} = y_n + \Delta y$$

$$\Delta x = \frac{1}{6}(w_1 + 2w_2 + 2w_3 + w_4)$$

$$\Delta y = \frac{1}{6}(v_1 + 2v_2 + 2v_3 + v_4)$$

$$w_1 = h f(t_n, x_n, y_n)$$

$$v_1 = h g(t_n, x_n, y_n)$$

$$w_2 = h f\left(t_n + \frac{h}{2}, x_n + \frac{w_1}{2}, y_n + \frac{v_1}{2}\right)$$

$$v_2 = h g\left(t_n + \frac{h}{2}, x_n + \frac{w_1}{2}, y_n + \frac{v_1}{2}\right)$$

$$w_3 = h f\left(t_n + \frac{h}{2}, x_n + \frac{w_2}{2}, y_n + \frac{v_2}{2}\right)$$

$$v_3 = h g\left(t_n + \frac{h}{2}, x_n + \frac{w_2}{2}, y_n + \frac{v_2}{2}\right)$$

$$w_4 = h f(t_n + h, x_n + w_3, y_n + v_3)$$

$$v_4 = h g(t_n + h, x_n + w_3, y_n + v_3)$$

a. R-K with h=1 (1 step)									
n	Tn	Xn	Yn	w = (1)Yn		$\Delta x$	v = (1)(Tn*Yn+Xn)		$\Delta y$
0	0	1	1	1	w1	1	1	v1	1
	0.5	1.5	1.5	1.5	w2	3	2.25	v2	4.5
	0.5	1.75	2.125	2.125	w3	4.25	2.8125	v3	5.625
	1	3.125	3.8125	3.8125	w4	3.8125	6.9375	v4	6.9375
					$\Delta x =$	2.010417		$\Delta y =$	3.010417
			X1=	3.010417		Y1=	4.010416667		

b. R-K with h=0.5 (2 steps)									
n	Tn	Xn	Yn	w = (0.5)Yn		$\Delta x$	$v = (0.5)(Tn*Yn+Xn)$		$\Delta y$
0	0	1	1	0.5	w1	0.5	0.5	v1	0.5
	0.25	1.25	1.25	0.625	w2	1.25	0.78125	v2	1.5625
	0.25	1.3125	1.390625	0.695313	w3	1.390625	0.830078125	v3	1.660156
	0.5	1.6953125	1.8300781	0.915039	w4	0.915039	1.305175781	v4	1.305176
					$\Delta x =$	0.675944		$\Delta y =$	0.837972
			X1=	1.675944		Y1=	1.837972005		
n	Tn	Xn	Yn	w = (0.5)Yn		$\Delta x$	$v = (0.5)(Tn*Yn+Xn)$		$\Delta y$
1	0.5	1.675944	1.837972	0.918986	w1	0.918986	1.297465007	v1	1.297465
	0.75	2.135437	2.4867045	1.243352	w2	2.486705	2.000232697	v2	4.000465
	0.75	2.2976201	2.8380884	1.419044	w3	2.838088	2.213093201	v3	4.426186
	1	3.0949882	4.0510652	2.025533	w4	2.025533	3.573026697	v4	3.573027
					$\Delta x =$	1.378219		$\Delta y =$	2.216191
			X2=	3.054163		Y2=	4.054162588		