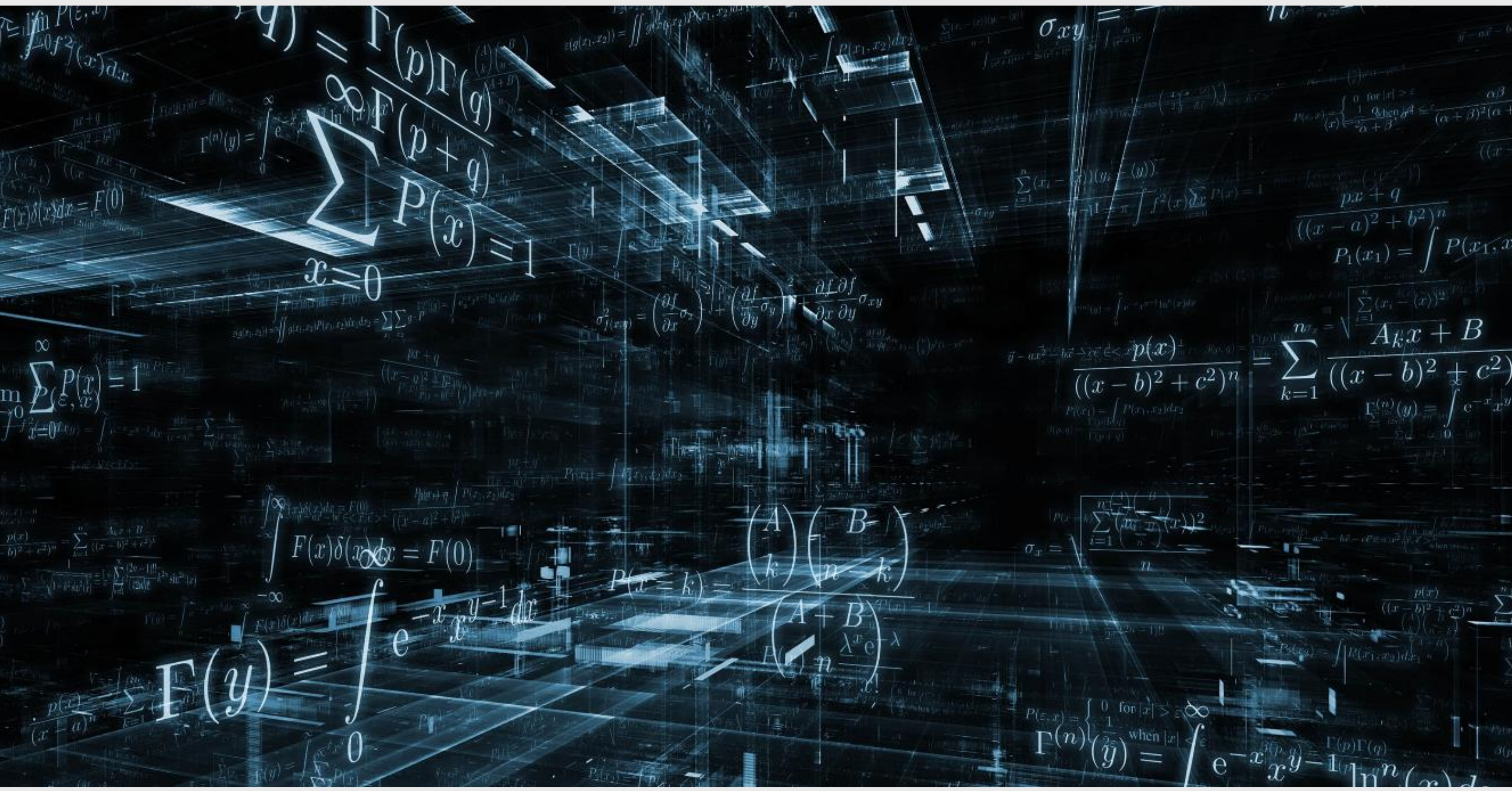
PHM212s: Complex, Special Functions and Numerical Analysis

Exercise Sheet

Functions of Complex Variables



“All the world's a differential equation, and the men and women are merely variables.”

-Ben Orlin

[1] Show that:

(a) $u = \dfrac{y}{x^2 + y^2}$ is harmonic and find its conjugate "**v**" and then **find** the analytic function $f(z) = u + iv$ in terms of "**z**". *[Fall 2015]*

(b) $v = \dfrac{x}{x^2 + y^2}$ is harmonic and find its conjugate "**u**" and then **find** the analytic function $f(z) = u + iv$ in terms of "**z**". *[Fall 2016]*

(c) $v = ln\sqrt{x^2 + y^2}$ is harmonic and find its conjugate "**u**" and then **find** the analytic function $f(z) = u + iv$ in terms of "**z**". *[Fall 2017]*

(d) $u = 2\,tan^{-1}\left(\dfrac{y}{x}\right)$ is harmonic and find its conjugate "**v**" and then **find** the analytic function $f(z) = u + iv$ in terms of "**z**". *[Fall 2018]*

(e) $f(z) = e^{(x^2 - y^2)}(cos\,2x\,y + i\,sin\,2xy)$ is an entire function. Hence, **find its derivative** in terms of z. *[Fall 2016]*

(f) $u = \dfrac{x}{x^2 + y^2}$ is harmonic and find its conjugate "**v**" and then **find** the analytic function $f(z) = u + iv$ in terms of "**z**". *[Summer 2015]*

[2] Find & Solve:

i. $e^{2z} = 1 + \sqrt{3}i$

ii. $cos\, z = cosh\, 5$ (all the roots)

iii. $cosh\, z = -2$ (all the roots)

iv. $\sin z = 4$ (all the roots)

v. $\left(\frac{e}{2}\left(1+\sqrt{3}i\right)\right)^{3\pi i}$ (hence find its principal value)

vi. $\left(\sqrt{3}+i\right)^{5\pi i}$ (hence find its principal value)

vii. $\left(1+\sqrt{3}i\right)^{2+i}$

viii. $e^{sin(z)} = i$ (all values of z)

ix. $sin\, z = cosh\, 4$ (all the roots)

x. $e^{3z-1} = 1 + i$

xi. $(1-i)^{i}$ (all the values & hence find its principle value and represent it in the complex plane)

[3] Find the image of the semi-infinite strip $0 \le y \le \frac{1}{4}\,, x \ge 0$ under the reciprocal transformation $w = \frac{1}{z}$. $(Show\ the\ regions\ graphically)$ [Fall 2015]

[4] Find the image of the semi-infinite strip $1 \le x \le 2\,, y \ge 0$ under the reciprocal transformation $w = \frac{1}{z}$. $(Show\ the\ regions\ graphically)$ [Fall 2017]

[5] Find the image of the triangular region bounded by $x = 0\,, y = 0\ \ \&\ \ x + y = 1$ under the reciprocal transformation $w = \frac{1}{z}$. $(Show\ the\ regions\ graphically)$ [Fall 2018]

[6] Find the image of the region $G = \left\{z\colon |z| \le 2{,}0 \le arg(z) \le \frac{\pi}{2}\right\}$ in the w-plane under mapping $w = (1-i)z + 1$. $(Sketch\ both\ regions)$ [Spring 2017]

[7] Find the image of the semi-infinite strip $0 \leq y \leq \frac{1}{2}, x \geq 0$ under the reciprocal transformation $w = \frac{1}{z}$. *(Show the regions graphically)* [Summer 2015]

[8] Starting with $f'(z) = (u_x + iv_x)$ **prove that** $f'(z) = \frac{r}{z}(u_r + iv_r)$ and then **use it** to **Find the derivative** of $ln\,z$ for $z \neq 0$ [Fall 2015]

[9] Starting with $f'(z) = (u_x + iv_x)$ **Find** the polar form of the derivative in terms of u_r and v_r only. [Fall 2018]

[10] Given $v = e^x \sin y + 2xy$, **show that** v is harmonic and find the corresponding analytic function: $f(z) = u_{(x,y)} + iv_{(x,y)}$ [Spring 2017]

[11] Use the transformation $w = (1+i)z + (2+i)$ to find the image of the rectangle bounded by the lines $x = 0, y = 0, x = 3$ & $y = 2$. *(Show the regions graphically and comment on your answer)* [Fall 2018]

[4] Show that if "v" is a harmonic conjugate of "u" is a domain ***D***, the "uv" is also harmonic in the same domain ***D***. [Fall 2016]

[12] Show that Laplace's equation in the polar form is given by $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

[Fall 2016] [Fall 2017] [Summer 2015]