Gamma function

$$* \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad , \quad x > 0$$

Ex:

$$\int_0^\infty t^3 e^{-t}\, dt = \Gamma(4) = 3! = 6 \qquad \text{" } x-1=3 \Rightarrow x=4 \text{ "}$$

① $\displaystyle\int_0^\infty x^5 e^{-2x^3}\, dx$ $\qquad x=\infty \longrightarrow t=\infty, \quad x=0 \longrightarrow t=0$

Ans: let $t = 2x^3 \Rightarrow x^3 = \frac{t}{2}$

$\Rightarrow x = \left(\frac{1}{2}\right)^{1/3} t^{1/3}$

$\Rightarrow dx = \left(\frac{1}{2}\right)^{1/3} \cdot \frac{1}{3} t^{-2/3}\, dt$

$$I = \int_0^\infty \left(\left(\frac{1}{2}\right)^{1/3} t^{1/3}\right)^5 e^{-t} \left(\frac{1}{2}\right)^{1/3} \cdot \frac{1}{3} t^{-2/3}\, dt$$

$$= \left(\frac{1}{2}\right)^{5/3+\frac{1}{3}} \left(\frac{1}{3}\right) \int_0^\infty t^{5/3-2/3} e^{-t}\, dt$$

$$= \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right) \int_0^\infty t\, e^{-t}\, dt = \frac{1}{12} \int_0^\infty t e^{-t}\, dt = \frac{1}{12}\Gamma(2) = \frac{1}{12}$$

② $\int_0^\infty x^5 e^{1-x^2} dx$

Ans: let $-t = 1-x^2 \Rightarrow t = x^2-1$ $x=\infty \to t=\infty$, $x=0 \to t=-1$ ✗

$I = \int_0^\infty x^5 e^{1-x^2} dx = \int_0^\infty x^5 . e . e^{-x^2} dx$

$= e \int_0^\infty x^5 e^{-x^2} dx \Rightarrow$ let $t = x^2$ $x=\infty \to t=\infty$, $x=0 \to t=0$

$\Rightarrow dx = \frac{1}{2} t^{-1/2} dt$

$\therefore I = e \int_0^\infty (t^{1/2})^5 e^{-t} . \frac{1}{2} t^{-1/2} dt = \frac{e}{2} \int_0^\infty t^2 e^{-t} dt$

$= \frac{e}{2} \Gamma(3)$

③ $\int_1^\infty x^8 e^{1-x^3} dx$

Ans: let $-t = 1-x^3 \Rightarrow t = x^3-1$ $x=\infty \to t=\infty$, $x=1 \to t=0$

$\Rightarrow x = (t+1)^{1/3} \Rightarrow dx = \frac{1}{3}(t+1)^{-2/3} dt$

$I = \int_0^\infty ((t+1)^{1/3})^8 e^{-t} . \frac{1}{3}(t+1)^{-2/3} dt$

$= \frac{1}{3} \int_0^\infty (t+1)^2 e^{-t} dt = \frac{1}{3} \int_0^\infty (t^2+2t+1) e^{-t} dt$

$= \frac{1}{3} \int_0^\infty t^2 e^{-t} dt + \frac{2}{3} \int_0^\infty t e^{-t} dt + \frac{1}{3} \int_0^\infty e^{-t} dt$

$= \frac{1}{3} \Gamma(3) + \frac{2}{3} \Gamma(2) + \frac{1}{3} \Gamma(1)$

④ $\int_0^1 \sqrt[3]{x}\,(\ln x)^6\,dx$

Ans: let $-t = \ln x \Rightarrow x = e^{-t}$ $\quad x=1 \to t=0$, $x=0 \to t=\infty$

$\Rightarrow dx = -e^{-t}\,dt$

$$I = \int_\infty^0 (e^{-t})^{1/3}\,(-t)^6\,(-e^{-t}\,dt) = \int_0^\infty t^6\,e^{-\frac{4}{3}t}\,dt$$

let $u = \frac{4}{3}t \Rightarrow t = \frac{3}{4}u \Rightarrow dt = \frac{3}{4}du$

$$\therefore I = \int_0^\infty \left(\tfrac{3}{4}u\right)^6 e^{-u}\cdot\tfrac{3}{4}du = \left(\tfrac{3}{4}\right)^7 \int_0^\infty u^6 e^{-u}\,du$$

$$= \left(\tfrac{3}{4}\right)^7 \Gamma(7)$$

* $\int_0^\infty 3^{-x^2}\,dx$

Ans: let $3^{-x^2} = e^{-t} \Rightarrow -x^2\ln(3) = -t$

$\Rightarrow x^2 = \frac{t}{\ln(3)} = \left(\frac{1}{\ln(3)}\right)t$

* $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, $n > -1$

Ex:

① $L\{t^{5/2}\} = \frac{\Gamma(7/2)}{s^{7/2}}$

② $L\{\sqrt{t}\,e^{-3t}\} = e^{-3t}L\{t^{1/2}\}\Big|_{s\to s+3} = \frac{\Gamma(3/2)}{s^{3/2}}\Big|_{s\to s+3}$

$= \frac{\Gamma(3/2)}{(s+3)^{3/2}}$

Properties of $\Gamma(x)$:-

1) $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2) $\Gamma(x+1) = x\,\Gamma(x)$ $\rightsquigarrow$ Reccurrence Relation

3) $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$ $\rightsquigarrow$ " " for $-$ve values of x

4) $\Gamma(n+1) = n!$, n +ve integer

Examples :

① $\Gamma(3.5) = 2.5\,\Gamma(2.5) = (2.5)(1.5)\,\Gamma(1.5) = (2.5)(1.5)(0.5)\,\Gamma(0.5)$
$= (2.5)(1.5)(0.5)(\sqrt{\pi})$

② $\Gamma(4.3) = 3.3\,\Gamma(3.3) = (3.3)(2.3)\,\Gamma(2.3) = (3.3)(2.3)(1.3)\,\underbrace{\Gamma(1.3)}$

From table

③ $\Gamma(-1.5) = \dfrac{\Gamma(-0.5)}{(-1.5)} = \dfrac{\Gamma(0.5)}{(-1.5)(-0.5)} = \dfrac{\sqrt{\pi}}{(1.5)(0.5)}$