الإسراء

ملزمة (٩)

رياضة

Cauchy- Reiman Equation& Harmonic Functions

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Cauchy - Reiman Eqs & Harmonic fn.

$f(z) = u + iv$ is differentiable at all the Points on the z-plane that satisfies

1) Cauchy - Reiman Eqs.
2) Continous Partial derivatives.

* **Cauchy - Reiman Equations :-**

- Rectangular form, when u & v are easier to be represented interms of x & y

$$\boxed{U_x = V_y \quad \& \quad U_y = -V_x}$$

- Polar form, when u & v are easier to be written interms of r & θ, we use

$$\boxed{U_r = \frac{1}{r} V_\theta \quad \& \quad U_\theta = -r V_r}$$

* At those points of the z-plane where Cauchy Reiman are satisfied & the partial derivatives U_x, U_y, V_x, V_y are Continous, we can find

(z), as $f(z)$ is differentiable, using

$$\boxed{f'(z) = u_x + i\,v_x}$$

* $f(z)$ is analytic at z, if: it is differentiable at z and differentiable at all the points around z.

* Entire fn = Analytic everywhere.

Example: Discuss the differentiability of

1) $f(z) = z^2$ 2) $f(z) = \bar{z}$

Solution:- 1) $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy$

$\Rightarrow u = x^2 - y^2$ $v = 2xy$

$u_x = 2x$ $v_x = 2y$

$u_y = -2y$ $v_y = 2x$

Since, $u_x = v_y$ & $u_y = -v_x$, also they are all continous (polynomials) $\Rightarrow$ $f(z)$ is every where differentiable $\Rightarrow$ Every where Analytic $\Rightarrow$ Entire

& $f'(z) = u_x + i v_x = 2x + i\,2y = 2z$

We can use the polar form here as follow:-

3.

$f(z) = z^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + i\, r^2 \sin 2\theta$

$\Rightarrow \quad u = r^2 \cos 2\theta \qquad\qquad v = r^2 \sin 2\theta$

$u_r = 2r \cos 2\theta \qquad\qquad v_r = 2r \sin 2\theta$

$u_\theta = -2r^2 \sin 2\theta \qquad\qquad v_\theta = 2r^2 \cos 2\theta$

Since $u_r = \frac{v_\theta}{r}$ & $\frac{u_\theta}{r} = -v_r$ also they are Continous $f\underline{n}s$ $\Rightarrow$ $f(z)$ is differentiable every-where $\Rightarrow$ analytic everywhere.

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2) $f(z) = \bar{z} = x - iy$

$\Rightarrow \quad u = x \qquad\qquad v = -y$

$u_x = 1 \qquad\qquad v_x = 0$

$u_y = 0 \qquad\qquad v_y = -1$

we have $u_x \neq v_y \Rightarrow$ $f(z)$ is nowhere diff. $\Rightarrow$ nowhere analytic.

Also if we use the Polar form $\Rightarrow f(z) = \bar{z} = re^{-i\theta}$
$= r\cos\theta - i\, r \sin\theta$

$\Rightarrow \quad u = r\cos\theta \qquad\qquad v = -r\sin\theta$

$u_r = \cos\theta \qquad\qquad v_r = -\sin\theta$

$u_\theta = -r\sin\theta \qquad\qquad v_\theta = -r\cos\theta$

for $u_r = \frac{v_\theta}{r} \Rightarrow \cos\theta = -\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$

for $\frac{u_\theta}{r} = -v_r \Rightarrow -\sin\theta = -\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi$

Both Conditions Can't be satisfied together $\Rightarrow$
$f(z)$ is nowhere diff & so nowhere analytic.

Example:- Determine where the functions below are differentiable & where they are analytic

i) $f(z) = Im(z)$

ii) $f(z) = e^{y}(\cos x + i \sin x)$

iii) $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

iv) $f(z) = (1-i)x^3 + (1+i)y^3$

Solution:- i) $f(z) = Im(Z) = y$

$u = y$ $\quad\quad\quad\quad$ $v = 0$

$u_x = 0$ $\quad\quad\quad\quad$ $v_x = 0$

$u_y = 1$ $\quad\quad\quad\quad$ $v_y = 0$

$\Rightarrow u_x = v_y$, but no where $u_y = -v_x \Rightarrow$

$f(z)$ is no where differentiable $\Rightarrow$ Nowhere $f(z)$ is analytic.

ii) $f(z) = e^{y}(\cos x + i \sin x)$

$u = e^{y}\cos x$ $\quad\quad\quad\quad$ $v = e^{y}\sin x$

$u_x = -e^{y}\sin x$ $\quad\quad\quad\quad$ $v_x = e^{y}\cos x$

$u_y = e^{y}\cos x$ $\quad\quad\quad\quad$ $v_y = e^{y}\sin x$

...uchy Reiman Eqs are satisfied if

1) $U_x = V_y \Rightarrow -\cancel{e^y}\sin x = \cancel{e^y}\sin x$

$\Rightarrow \sin x = 0 \Rightarrow x = 0, \pm\pi, \pm 2\pi, ----$

2) $U_y = -V_x \Rightarrow \cancel{e^y}\cos x = -\cancel{e^y}\cos x$

$\Rightarrow \cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ---$

Both Conditions can't be satisfied together

f(z) is no where differentiable $\Rightarrow$ no where Analytic

iii) $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

$u = x^3 + 3xy^2 - 3x$ $\qquad$ $v = y^3 + 3x^2y - 3y$

$U_x = 3x^2 + 3y^2 - 3$ $\qquad$ $V_x = 6xy$

$U_y = 6xy$ $\qquad$ $V_y = 3y^2 + 3x^2 - 3$

us seen, anywhere $U_x = V_y$, but for $U_y = -V_x$

$\Rightarrow 6xy = -6xy \Rightarrow xy = 0 \Rightarrow x = 0$ or $y = 0$

Since partial derivatives are Continous, then

f(z) is differentiable only on the x-axis & y axis, with $f'(z) = U_x + i V_x$

$\Rightarrow f'(z) = 3x^2 + 3y^2 - 3 + i\cdot 0$

Out, $f(z)$ is nowhere analytic.

Note that

If $f(z) = x^3 - 3xy^2 - 3x + i(3x^2y - y^3 - 3y)$

$\Rightarrow u = x^3 - 3xy^2 - 3x \qquad v = 3x^2y - y^3 - 3y$

$u_x = 3x^2 - 3y^2 - 3 \qquad v_x = 6xy$

$u_y = -6xy \qquad v_y = 3x^2 - 3y^2 - 3$

Cauchy-Reiman are satisfied everywhere

$f(z)$ is everywhere diff $\Rightarrow$ everywhere Analytic

$\Rightarrow$ $f(z)$ is entire fn

$f'(z) = u_x + i\,v_x$

$= 3x^2 - 3y^2 - 3 + i(6xy)$

Note that to get $f(z)$ or $f'(z)$ interms of z

for an entire fn, set $y = 0$ & replace x by z.

So, we have $f(z) = z^3 - 3z$

& $f'(z) = 3z^2 - 3$

) $f(z) = (1-i)x^3 + (1+i)y^3$

$u = x^3 + y^3 \qquad\qquad v = y^3 - x^3$

$u_x = 3x^2 \qquad\qquad v_x = -3x^2$

$u_y = 3y^2 \qquad\qquad v_y = 3y^2$

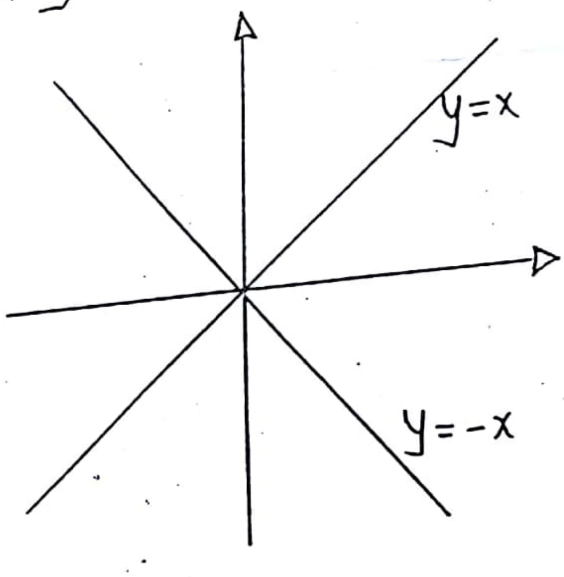
Cauchy Reiman are satisfied if:

1) $u_x = v_y \Rightarrow 3x^2 = 3y^2 \Rightarrow y^2 = x^2$

$\Rightarrow y = \pm x$

2) $u_y = -v_x \Rightarrow 3y^2 = 3x^2 \Rightarrow y = \pm x$

$f(z)$ is differentiable only on $y = \pm x$ (2 st. lines)



with $f'(z) = u_x + i v_x$

$= 3x^2 - i\,3x^2$

$= 3(1-i)x^2$

$f(z)$ is no where Analytic

$f(z) \begin{cases} \text{diff on } y = \pm x \\ \text{No where Analytic} \end{cases}$

Example :- Determine where

$$f(z) = (x^3 - xy^2) + i\,(x^2y + y^3)$$

is analytic & where it is differentiable.

Solution :- We have

$u = x^3 - xy^2$ $\qquad\qquad$ $v = x^2y + y^3$

$u_x = 3x^2 - y^2$ $\qquad\qquad$ $v_x = 2xy$

$u_y = -2xy$ $\qquad\qquad$ $v_y = x^2 + 3y^2$

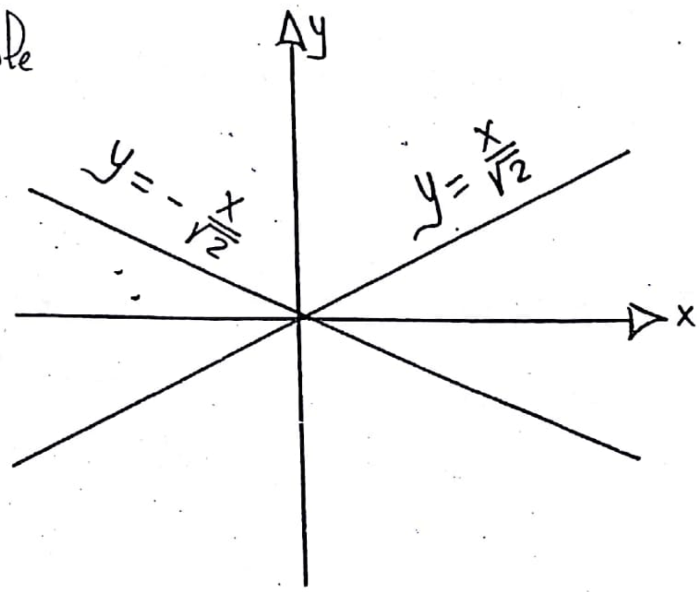
The 2nd Cauchy - Reiman Eq. Condition is already satisfied, also all partial derivatives are continous (Polynomials), for the 1st condition to be satisfied $\Rightarrow u_x = v_y$

$$\Rightarrow 3x^2 - y^2 = x^2 + 3y^2 \Rightarrow y^2 = \frac{x^2}{2} \Rightarrow y = \pm\frac{x}{\sqrt{2}}$$

So, $f(z)$ is differentiable on these 2 lines only &

$$f'(z) = (3x^2 - y^2) + i\,2xy$$

But, $f(z)$ is nowhere Analytic



Harmonic Function :-

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A function of two variables (x & y) or (r & θ) is said to be harmonic, if it satisfies Laplace Equation.

Laplace Eq. is
- Rectangular form $\phi(x,y)$: $\phi_{xx} + \phi_{yy} = 0$
- Polar form $\phi(r,\theta)$: $r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0$

Theorm :-

If $f(z) = u + iv$ is analytic then u is harmonic & v is harmonic, they are called harmonic Conjugate

Result : If he says that u & v are harmonic Conjugate then we have :

1) U is harmonic $\Rightarrow U_{xx} + U_{yy} = 0$ OR $r^2U_{rr} + rU_r + U_{\theta\theta} =$
2) V is harmonic $\Rightarrow V_{xx} + V_{yy} = 0$ OR $r^2V_{rr} + rV_r + V_{\theta\theta} = ($
3) $f(z) = u + iv$ is analytic $\Rightarrow$ Cauchy - Reiman are Satisfied.

Example :- Show that the following fns are harmonic & find their Corresponding analytic fn

i) $u = x^3 - 3xy^2 + y$

ii) $u = \cos x \, \text{Ch} y.$

Solution :-

i) $u = x^3 - 3xy^2 + y$

$\Rightarrow u_x = 3x^2 - 3y^2$ $\qquad$ $u_y = -6xy + 1$

$u_{xx} = 6x$ $\qquad$ $u_{yy} = -6x$

Since $u_{xx} + u_{yy} = 6x - 6x = 0 \Rightarrow u$ is harmonic

To find its harmonic Conjugate v (So that $f(z) = u + iv$ is analytic) we must have

$u_x = v_y = 3x^2 - 3y^2 \Rightarrow v = 3x^2y - y^3 + h(x)$

$\Rightarrow v_x = 6xy + h'(x) = -u_y = 6xy - 1$

$\Rightarrow h'(x) = -1 \Rightarrow h(x) = -x + C$

$\Rightarrow v = 3x^2y - y^3 - x + C$

$f(z) = u + iv = x^3 - 3xy^2 + y + i(3x^2y - y^3 - x) + C$

$= z^3 - iz + C$

Also to find $f'(z) = u_x + i v_x = 3x^2 - 3y^2 + i(6xy - 1)$

$= 3z^2 - i$

ii) $u = \cos x \, \text{ch} y$

$\Rightarrow u_x = -\sin x \, \text{ch} y \qquad u_y = \cos x \, \text{sh} y$

$u_{xx} = -\cos x \, \text{ch} y \qquad u_{yy} = \cos x \, \text{ch} y$

$\Rightarrow u_{xx} + u_{yy} = 0 \quad \Rightarrow u$ is harmonic

To find it's harmonic Conjugate v

$\Rightarrow v_y = u_x = -\sin x \, \text{ch} y \quad \Rightarrow v = -\sin x \, \text{sh} y + h(x)$

$v_x = -u_y \quad \Rightarrow -\cos x \, \text{sh} y + h'(x) = -\cos x \, \text{sh} y$

$\Rightarrow h'(x) = 0 \quad \Rightarrow h(x) = C$

$v = -\sin x \, \text{sh} y + C$

$f(z) = u + iv = \cos x \, \text{ch} y - i \sin x \, \text{sh} y + C$

$= \cos z$

To find $f'(z) = u_x + i v_x$

$= -\sin x \, \text{ch} y - i \cos x \, \text{sh} y$

$= -\sin z$

Example :-

Show that (uv) is harmonic if v is a harmonic Conjugate of u.

* Solution *

It is requirred to prove that

$$(uv)_{xx} + (uv)_{yy} = 0$$

$$\frac{\partial}{\partial x}(uv) = u_x v + u v_x$$

$$\frac{\partial^2}{\partial x^2}(uv) = u_{xx} v + u_x v_x + u_x v_x + u v_{xx}$$

$$= u_{xx} v + 2 u_x v_x + u v_{xx}$$

$\because$ u & v are Conjugate harmonic

$\therefore$ ① $u_{xx} + u_{yy} = 0 \Rightarrow u_{xx} = -u_{yy}$

$v_{xx} + v_{yy} = 0 \Rightarrow v_{xx} = -v_{yy}$

② $f(z) = u + iv$ is analytic

$\Rightarrow u_x = v_y \qquad u_y = -v_x$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(uv) = u_{xx} v + 2 u_x v_x + u v_{xx}$$

$$= (-u_{yy})v + 2(v_y)(-u_y) + u(-v_{yy})$$

$$= -\left[u_{yy} v + 2 u_y v_y + u v_{yy} \right]$$

$$= -\frac{\partial^2}{\partial y^2}(uv)$$

$\Rightarrow$ (uv) is harmonic.

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Example :- If $f(z) = u + iv$ & $u = v^2$ is analytic, then show that $f(z) =$ Constant

Solution :-

Since $f(z)$ is analytic then

1) $u_x = v_y \Rightarrow \frac{\partial}{\partial x}(u) = v_y \Rightarrow \frac{\partial}{\partial x}(v^2) = v_y$

$\Rightarrow 2v v_x = v_y \longrightarrow (1)$

2) $u_y = -v_x \Rightarrow \frac{\partial}{\partial y}(u) = -v_x \Rightarrow \frac{\partial}{\partial y}(v^2) = -v_x$

$\Rightarrow 2v v_y = -v_x \longrightarrow (2)$

Subsituting with (2) in (1) $\Rightarrow$

$2v(-2v v_y) = v_y \Rightarrow -4v^2 v_y = v_y$

$\Rightarrow v_y(1 + 4v^2) = 0$

OR

$v_y = 0$	$1 + 4v^2 = 0$
$\downarrow$ from (2)	$\Downarrow$
$v_x = 0$	$v^2 = -1/4$
$\Downarrow$	$\Downarrow$
$v =$ Constant	$v =$ Constant
$\Downarrow$	$\Downarrow$
$u = v^2 =$ Constant	$u = v^2 =$ Constant
$\Downarrow$	$\Downarrow$
$f(z) = u + iv =$ Constant	$f(z) = u + iv$ $=$ Constant

Example :-

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Show that $u = \ln^2 r - \theta^2$ is harmonic & find $f(z) = u + iv$ to be analytic

*** Solution ***

$u = \ln^2 r - \theta^2$

$u_r = 2\ln r \cdot \frac{1}{r} = \frac{2\ln r}{r}$ $\qquad u_\theta = -2\theta$

$\qquad u_{\theta\theta} = -2$

$$u_{rr} = \frac{\frac{2}{r} \cdot r - 2\ln r}{r^2}$$

$\Rightarrow r^2 u_{rr} = 2 - 2\ln r$

$r\, u_r = 2\ln r$

$\Rightarrow r^2 u_{rr} + r u_r + u_{\theta\theta} = \cancel{2} - \cancel{2\ln r} + \cancel{2\ln r} - \cancel{2} = 0$

$\Rightarrow u$ is harmonic

$\because f(z)$ is analytic $\quad \therefore u_r = \frac{1}{r} v_\theta \;,\; u_\theta = -r v_r$

$\Rightarrow v_\theta = r\, u_r = 2\ln r$

$\Rightarrow v = 2\theta \ln r + h(r)$

$v_r = \frac{2\theta}{r} + h'(r) = -\frac{1}{r} u_\theta = \frac{2\theta}{r}$

$\Rightarrow h'(r) = 0 \quad \Rightarrow \quad h(r) = c$

$f(z) = u + iv = \ln^2 r - \theta^2 + i\, 2\theta \ln r$

$\Rightarrow f(z) = \ln^2 z$

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Example: Show that $v = \dfrac{y}{x^2+y^2}$ is harmonic & find its harmonic Conjugate.

Sol.

$$v = \frac{y}{x^2+y^2} = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$$

$$v_r = -\frac{\sin\theta}{r^2} \qquad v_\theta = \frac{\cos\theta}{r}$$

$$v_{rr} = \frac{2\sin\theta}{r^3} \qquad v_{\theta\theta} = -\frac{\sin\theta}{r}$$

$$\Rightarrow r^2 v_{rr} + r v_r + v_{\theta\theta} = 2\frac{\sin\theta}{r} - \frac{\sin\theta}{r} - \frac{\sin\theta}{r} = 0$$

$\Rightarrow$ v is harmonic

To get its harmonic Conjugate u

$$u_r = \frac{1}{r}v_\theta \Rightarrow u_r = \frac{\cos\theta}{r^2} \Rightarrow u = -\frac{\cos\theta}{r} + h(\theta)$$

$$\frac{1}{r}u_\theta = -v_r \Rightarrow \frac{\sin\theta}{r} + h'(\theta) = \frac{\sin\theta}{r}$$

$$\Rightarrow h'(\theta) = 0 \Rightarrow h(\theta) = c$$

$$\Rightarrow u = -\frac{\cos\theta}{r} + c$$

$$f(z) = -\frac{\cos\theta}{r} + i\frac{\sin\theta}{r} + c$$

$$= -\frac{1}{z} + c$$

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xample: Determine where $f(z) = z^2\bar{z}$ is analytic & find $f'(z)$.

Solution

$$f(z) = z^2\bar{z} = (x+iy)^2(x-iy)$$

$$f(z) = (x+iy)(x^2+y^2) = x^3+xy^2+i(x^2y+y^3)$$

$$\Rightarrow u = x^3+xy^2 \quad \& \quad v = x^2y+y^3$$

$u_x = 3x^2+y^2$ $\qquad$ $v_x = 2xy$

$u_y = 2xy$ $\qquad$ $v_y = x^2+3y^2$

① $u_x = v_y \Rightarrow 3x^2+y^2 = x^2+3y^2 \Rightarrow x^2 = y^2$

② $u_y = -v_x \Rightarrow 2xy = -2xy \Rightarrow xy = 0$

For Cauchy Reiman to be satisfied we must have
$y = \pm x$ & ($x=0$ or $y=0$)

This is satisfied for $x = y = 0 \Rightarrow$ the origin

$f(z)$ is diff. only at $z=0$ & $f'(z) = u_x + i v_x$

$$\Rightarrow f'(z) = (3x^2+y^2) + i\,2xy \Rightarrow f'(0) = 0$$

<u>The function is not analytic at any Point.</u>

<u>Note that:</u> we can use the Polar Coordinates by setting $z = re^{i\theta}$

$$f(z) = z^2 \bar{z} = r^2 e^{i2\theta} \cdot r e^{-i\theta}$$

$$= r^3 e^{i\theta} = r^3 \cos\theta + i\, r^3 \sin\theta$$

$\Rightarrow u = r^3 \cos\theta$ & $v = r^3 \sin\theta$

$u_r = 3r^2 \cos\theta$ $\quad$ $v_r = 3r^2 \sin\theta$

$u_\theta = - r^3 \sin\theta$ $\quad$ $v_\theta = r^3 \cos\theta$

<u>Satisfying</u> <u>Cauchy - Reiman</u>

(1) $u_r = \dfrac{v_\theta}{r} \Rightarrow 3r^2 \cos\theta = r^2 \cos\theta$

$\Rightarrow 2r^2 \cos\theta = 0 \Rightarrow r = 0$

$\quad \Rightarrow \cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2}$

(2) $v_r = -\dfrac{u_\theta}{r} \Rightarrow 3r^2 \sin\theta = r^2 \sin\theta$

$\Rightarrow 2r^2 \sin\theta = 0 \Rightarrow r = 0$

$\Downarrow$

$\sin\theta = 0 \Rightarrow \theta = n\pi$

Both Equations will not satisfied except at $r = 0$

$\Rightarrow z = 0 \Rightarrow f(z)$ is diff only at $z = 0$

$\Rightarrow$ is nowhere analytic.

Ple :- Show that $v = 3x + y\,chy\,Cos(x+1) - x\,shy\,Sin(x+1)$ is harmonic. Find the analytic fn $f(z) = u + iv$ Then find $f'(z)$.

Solution :-

$$v = 3x + y\,chy\,Cos(x+1) - x\,shy\,Sin(x+1)$$

$$v_x = 3 - y\,chy\,Sin(x+1) - shy\,Sin(x+1) - x\,shy\,Cos(x+1)$$

$$v_{xx} = -y\,chy\,Cos(x+1) - shy\,Cos(x+1) - shy\,Cos(x+1) + x\,shy\,Sin(x+1) \rightarrow (1)$$

$$v_y = chy\,Cos(x+1) + y\,shy\,Cos(x+1) - x\,chy\,Sin(x+1)$$

$$v_{yy} = shy\,Cos(x+1) + shy\,Cos(x+1) + y\,chy\,Cos(x+1) - x\,shy\,Sin(x+1) \rightarrow (2)$$

From (1) & (2) $\Rightarrow v_{xx} + v_{yy} = $ zero $\Rightarrow$ harmonic

To find the harmonic Conjugate u :-

Apply Cauchy - Reiman

1) $u_x = v_y = chy\,Cos(x+1) + y\,shy\,Cos(x+1) - x\,chy\,Sin(x+1)$

Integrate w.r.t. x to get u

$\cosh y \sin(x+1) + y \sinh y \sin(x+1) - \cosh y \big(-x\cos(x+1)$

$\quad + \sin(x+1)\big) + h(y).$

$v = y \sinh y \sin(x+1) + x \cosh y \cos(x+1) + h(y).$

ii) $u_y = -v_x \Rightarrow$ to get $h(y)$

$\cancel{\sinh y \sin(x+1)} + \cancel{y \cosh y \sin(x+1)} + \cancel{x \sinh y \cos(x+1)} + h'(y)$

$= -3 + \cancel{y \cosh y \sin(x+1)} + \cancel{\sinh y \sin(x+1)} + \cancel{x \sinh y \cos(x+1)}$

$\Rightarrow h'(y) = -3 \Rightarrow h(y) = -3y + C$

Now, $f(z) = u + i\, v$

$= y \sinh y \sin(x+1) + x \cosh y \cos(x+1) - 3y + C$

$\quad + i\,\big(3x + y\cosh y \cos(x+1) - x \sinh y \sin(x+1)\big)$

to get it interms of $z \Rightarrow$ put $y = 0$ & $x = z$

$f(z) = z\cos(z+1) + i\, 3z + C$

for, $f'(z) = u_x + i\, v_x$

$= \cosh y \cos(x+1) + y \sinh y \cos(x+1) - x \cosh y \sin(x+1)$

$\quad + i\,\big(3 - y\cosh y \sin(x+1) - \sinh y \sin(x+1) - x \sinh y \cos(x+1)\big)$

$\Rightarrow f'(z) = \cos(z+1) - z\sin(z+1) + 3i$

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Example:-

Prove that an analytic fn $f(z)$ must be Constant if

1) $\overline{f(z)}$ is analytic

or

2) $|f(z)| = $ constant.

Solution

Let $f(z) = u + iv \Rightarrow$ Since it is analytic then

$u_x = v_y \rightarrow (1)$ & $u_y = -v_x \rightarrow (2)$

1) if $\overline{f(z)} = u - iv$ is analytic then

$u_x = (-v)_y \Rightarrow u_x = -v_y \rightarrow (3)$

& $u_y = -(-v)_x \Rightarrow u_y = v_x \rightarrow (4)$

add (1) & (3) $\Rightarrow u_x = 0 \Rightarrow v_y = 0$

add (2) & (4) $\Rightarrow u_y = 0 \Rightarrow v_x = 0$

$\Rightarrow u_x = u_y = 0 \Rightarrow u = c_1$ & $v_y = v_x = 0 \Rightarrow v = c_2$

$\Rightarrow f(z) = c_1 + i c_2 = $ Constant.

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2) $|f(z)| = \sqrt{u^2 + v^2} = k \Rightarrow u^2 + v^2 = k^2$

Diff. w.r.t. x $\Rightarrow 2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \rightarrow (5)$

" " y $\Rightarrow 2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \rightarrow (6)$

From (1) & (5) $\Rightarrow uv_y + vv_x = 0 \rightarrow (7)$

From (2) & (6) $\Rightarrow -uv_x + vv_y = 0 \rightarrow (8)$

(7) $* u$ + (8) $* v \Rightarrow (u^2 + v^2) v_y = 0 \Rightarrow v_y = 0 \overset{(1)}{\Rightarrow} u_x = 0$

(7) $* v$ - (8) $* u \Rightarrow (u^2 + v^2) v_x = 0 \Rightarrow v_x = 0 \overset{(2)}{\Rightarrow} u_y = 0$

Example :-

Find the image of $x^2+y^2=a^2$ under $w=\frac{1}{z}$

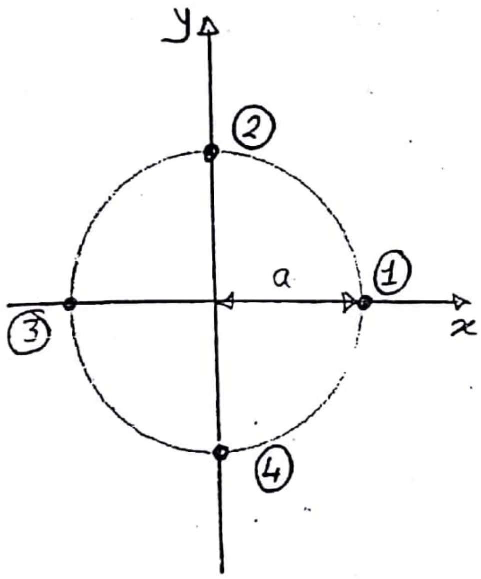
Solution :-

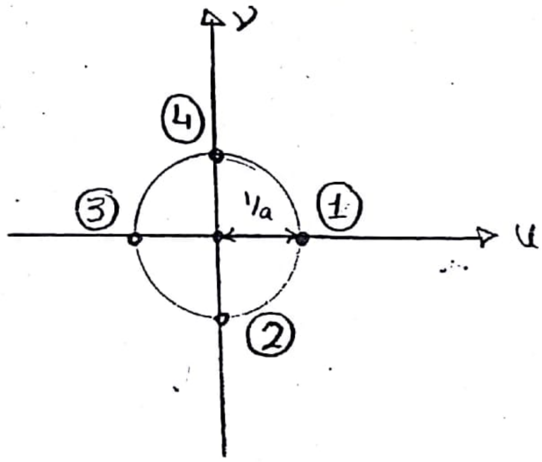
$$x^2+y^2=a^2 \longrightarrow \text{set } x=\frac{u}{u^2+v^2}$$
$$y=-\frac{v}{u^2+v^2}$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}=a^2$$

$$\Rightarrow \frac{1}{u^2+v^2}=a^2$$

$\Rightarrow u^2+v^2=\frac{1}{a^2}$ is the equation of circle of center $(0,0)$ & radius $\frac{1}{a}$.





From above, we can say that the Transformation of the <u>inside</u> of the circle $x^2+y^2=4$ (i.e $x^2+y^2\leq 4$) is the <u>outside</u> of the circle $u^2+v^2=\frac{1}{4}$ (i.e. $u^2+v^2\geq 1/4$).