FUNCTIONS OF COMPLEX VARIABLES

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Agenda

- Introduction
- Algebraic interpretation
- Geometric interpretation
- Mapping

Applications

complex
number of roots = degree

$x^2 + x + 1 = 0$
$2 \to$

Fundamental Theorem of Algebra

(i) ✓

current, Z

AC circuit analysis

Solution of Differential Equations

$y'' + y' + y = 0$
$m^2 + m + 1 = 0$
aux. equ.
Roots!!

Fourier Transform

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-i2\pi f t}\, dt$$

Motivation

N Natural 1, 2, 3

Q $\frac{3}{2}$?? $\frac{a}{b}$

The equation $x^2 + 1 = 0$

a real problem but has no real solutions

Why ?

So we make up a new symbol for the roots and call it a **complex number**.

Definition. The symbols $\pm$ i will stand for the solutions to the equation $x^2 = -1$ $(\sqrt{-1}) = i$

According to the defined quantity $\sqrt{-1}$ = i

$$i^2 = -1\,, \qquad i^3 = -i, \qquad i^4 = (-1)^2$$

Motivation

This number ±i is called an **imaginary number**.
These are valid numbers that don't lie on the real number line.

If z = a+bi,

a is called the real part of z denoted by **Re**{z}

b is called the imaginary part of z denoted by **Im**{z}

The symbol z is called a complex variable.

a

b

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Algebra of Complex Numbers

- **Addition** $(x_1 + i\, y_1) + (x_2 + i\, y_2) = (x_1 + x_2) + i(y_1 + y_2)$
  - **Example:** $(2 + \ 3i) + (1 + 2i)$ $= 3 + 5i$
- **Subtraction** $(x_1 + i\, y_1) - (x_2 + i\, y_2) = (x_1 + x_2) - i\,(y_1 + y_2)$
  - **Example:** $(2 + 3i) - (1 + 2i)$ $= 1 + i$
- **Multiplication** $(x_1 + i\, y_1)(x_2 + i\, y_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
  - **Example:** $(2 + 3i)(1 + 2i)$ $= 2 + 4i + 3i + 6(-1)$

    $= -4 + 7i$

Algebra of Complex Numbers

$\frac{3}{2} \notin Z$

Q

$\sqrt{Z \cdot \overline{Z}}$

$\sqrt{(x+yi)(x-yi)}$

- **Complex Conjugation** $\overline{(x+yi)} = x - yi$
  - **Example:** $\overline{(2+3i)}$ $= 2+3i$
- **Norm or Absolute Value** $|(x+yi)| = \sqrt{Z\overline{Z}} = \sqrt{x^2+y^2}$ real
  - **Example:** $|(2+3i)|$ $= \sqrt{4+9} = \sqrt{13}$
- **Division** $\frac{(x_1+y_1 i)}{(x_2+y_2 i)} = \frac{(x_1+y_1 i)(x_2-y_2 i)}{(x_2+y_2 i)(x_2-y_2 i)} = \frac{(x_1x_2+y_1y_2)+(x_2y_1+x_1y_2)i}{(x_2{}^2+y_2{}^2)}$ real

  get rid of i from dinominator

  - **Example:** $\frac{(2+3i)}{(1+2i)} \frac{(1-2i)}{(1-2i)} = \frac{2-4i+3i+6}{(1+4)} = \frac{1}{5}(8-i)$ Complex

    $\in C$

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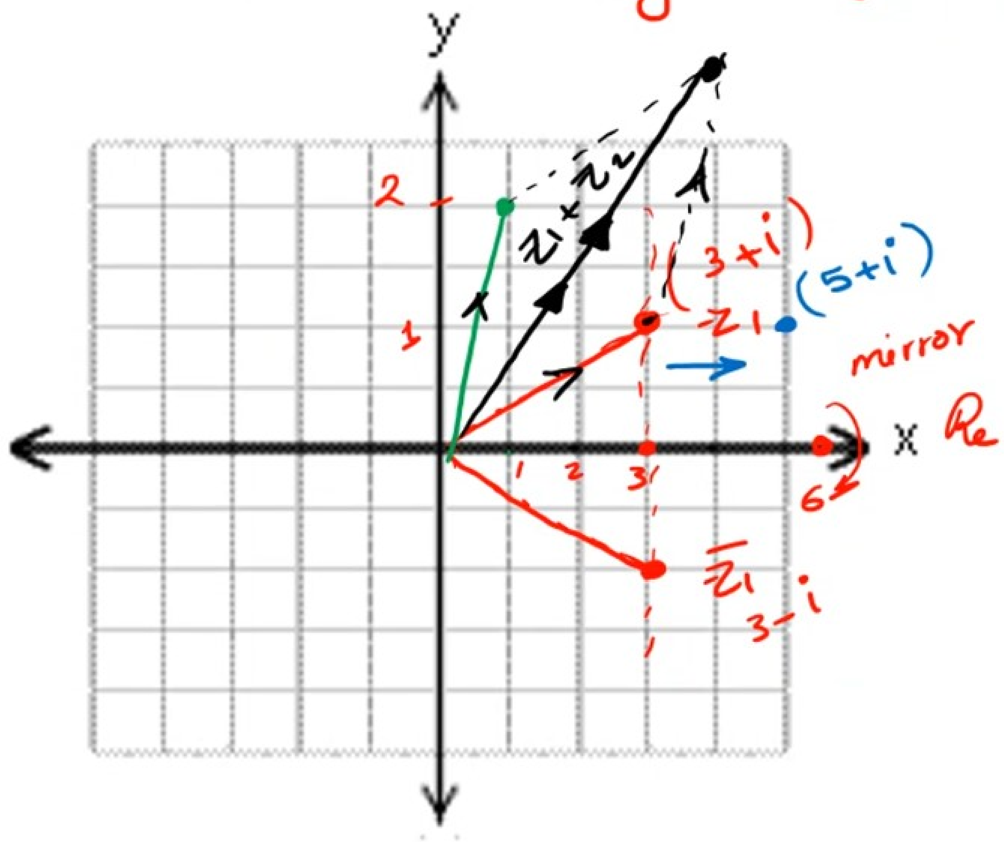
Geometry of Complex Numbers

$x+yi$

Im Argand diagram



$Z_1 = 3+i \qquad Z_2 = 1+2i$

$Z_1 + \overline{Z_1} = 2\,Re\{z\} = 6$

shift right

$Z_1 + 2 = 5+i$

$Z_1 + Z_2 = 4+3i$

Complex Numbers in Polar form

If $z = re^{i\theta}$,
r is called the absolute value of z
θ is called argument of z

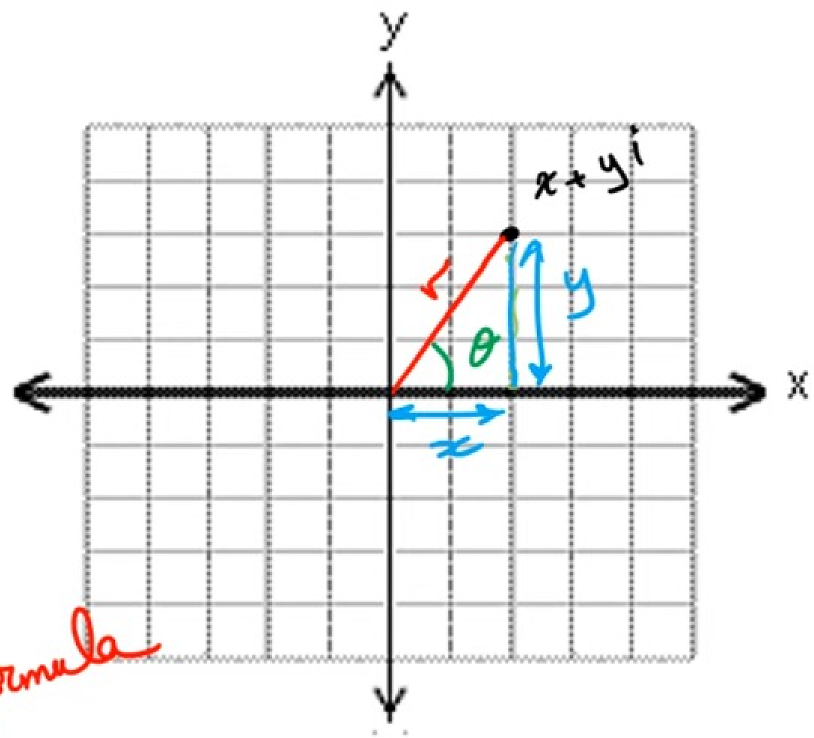
$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\mathbf{z = re^{i\theta} = \overbrace{rcos\theta + i\,rsin\theta}^{x \,+\, iy}}$$

$re^{i\theta} = (r\cos\theta + i\,r\sin\theta)$

$re^{i\theta} = r(\cos\theta + i\sin\theta)$ Euler formula



Complex Numbers in Polar form

$Z_1 = (\sqrt{3} + i)$ $\quad r = \sqrt{3+1} = 2 \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

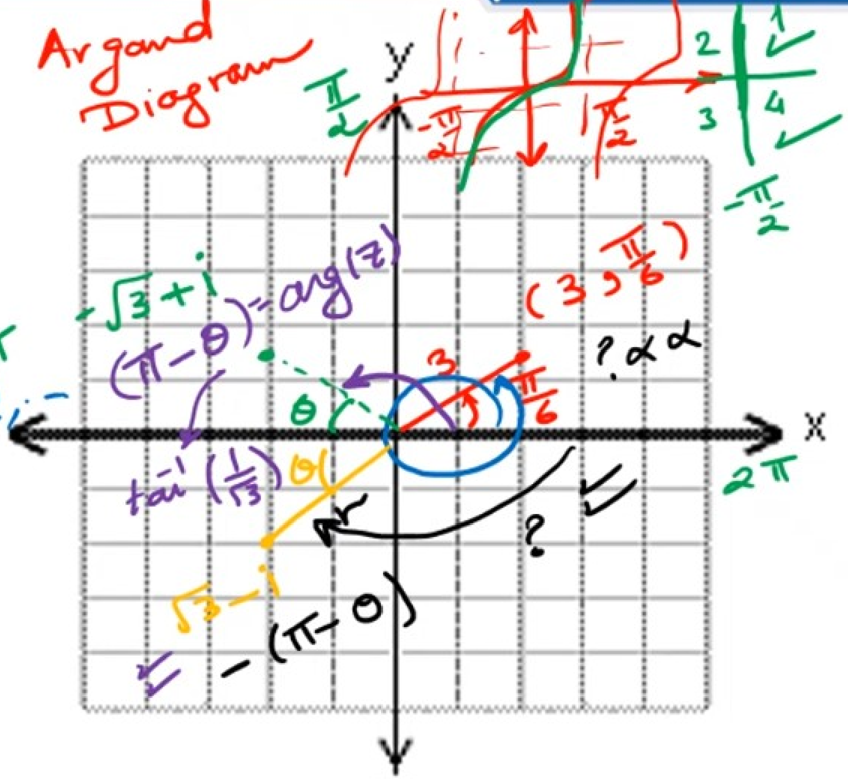
let $Z_2 = Z_1 \quad r_{Z_2} = r_{Z_1} = 2$

$\theta_{Z_2} = \theta_{Z_1} + 2n\pi \quad n = 0, \pm 1, \pm 2, \ldots$

The Principal Argument Arg $(z) \epsilon\ (-\pi, \pi]$

open why??

closed



$$
\begin{array}{ll}
\arctan \frac{y}{x} & \text{if } x > 0, y \in \mathbb{R} \\
\arctan \frac{y}{x} + \pi & \text{if } x < 0, y \geq 0 \\
\arctan \frac{y}{x} - \pi & \text{if } x < 0, y < 0 \\
\frac{\pi}{2} & \text{if } x = 0, y > 0 \\
-\frac{\pi}{2} & \text{if } x = 0, y < 0
\end{array}
$$

Complex Numbers in Polar form

- **Multiplication**

$$(\boldsymbol{r_1}, \boldsymbol{\theta_1})(\boldsymbol{r_2}, \boldsymbol{\theta_2}) = \boldsymbol{r_1}\,\boldsymbol{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$\boldsymbol{r_1}\,\boldsymbol{r_2}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\underbrace{\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2}_{\sin(\theta_1+\theta_2)})$$

$$= (\boldsymbol{r_1 r_2}, \boldsymbol{\theta_1} + \boldsymbol{\theta_2})$$

$$r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$r_1 r_2\, c\, i(\theta_1+\theta_2)$

Results

$$\boxed{(\boldsymbol{r_1}, \boldsymbol{\theta_1}) \ldots (\boldsymbol{r_n}, \boldsymbol{\theta_n}) = (\boldsymbol{r_1} \ldots \boldsymbol{r_n}, \boldsymbol{\theta_1} \ldots + \boldsymbol{\theta_n})}$$

$$\boxed{(\boldsymbol{r_1}, \boldsymbol{\theta_1})^n = (\boldsymbol{r_1}^n, n\,\boldsymbol{\theta_1})}$$

$$\boxed{(\boldsymbol{Z})^n = (\boldsymbol{r}(\cos\theta + i\sin\theta))^n = \boldsymbol{r}^n(\cos n\,\boldsymbol{\theta_1} + i\sin n\,\boldsymbol{\theta_1})}$$

Roots of complex numbers

complex number

De Moivre Theorem

Geometry of Complex Numbers

$Z_1 = 3e^{i\frac{\pi}{4}}$ $\quad Z_2 = 2e^{i\frac{\pi}{3}}$

① $Z_1 . 2 =$ $z_1\, 2\, e^{i0}$ Same argument of z

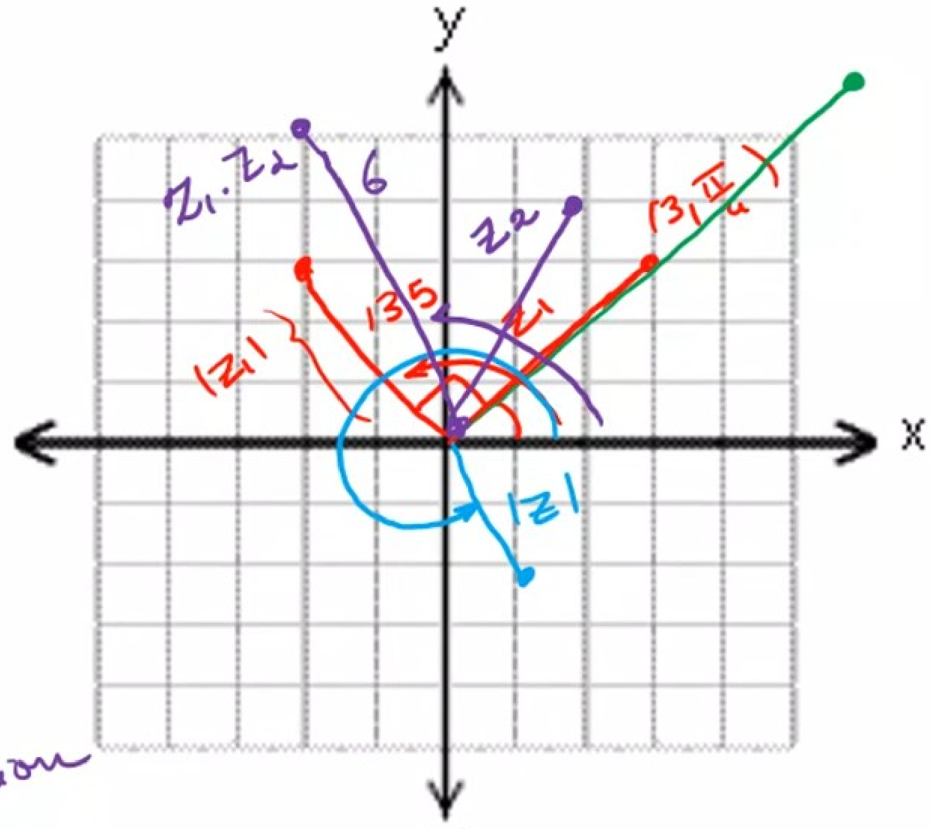
scaling

② $Z_1 . i =$ $z_1\, e^{i\frac{\pi}{2}}$ rotation

③ $Z_1 . (e^{i\frac{4\pi}{3}}) =$ $\xrightarrow{} 240$

④ $Z_1 Z_2 =$ $(3.2)\, e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = 6\, e^{i\frac{7\pi}{12}}$

scaling and rotation



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Functions of Complex Variables as Transformations

Consider the function

$x+iy$ complex variable

Re{w} Im(w)

$$w = f(z) \quad , \quad z = x + i\,y \qquad \Rightarrow w = u(x,y) + i\,v(x,y)$$

Example 1:

$$w = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)}_{u} + i\,\underbrace{(2\,x\,y)}_{v}$$

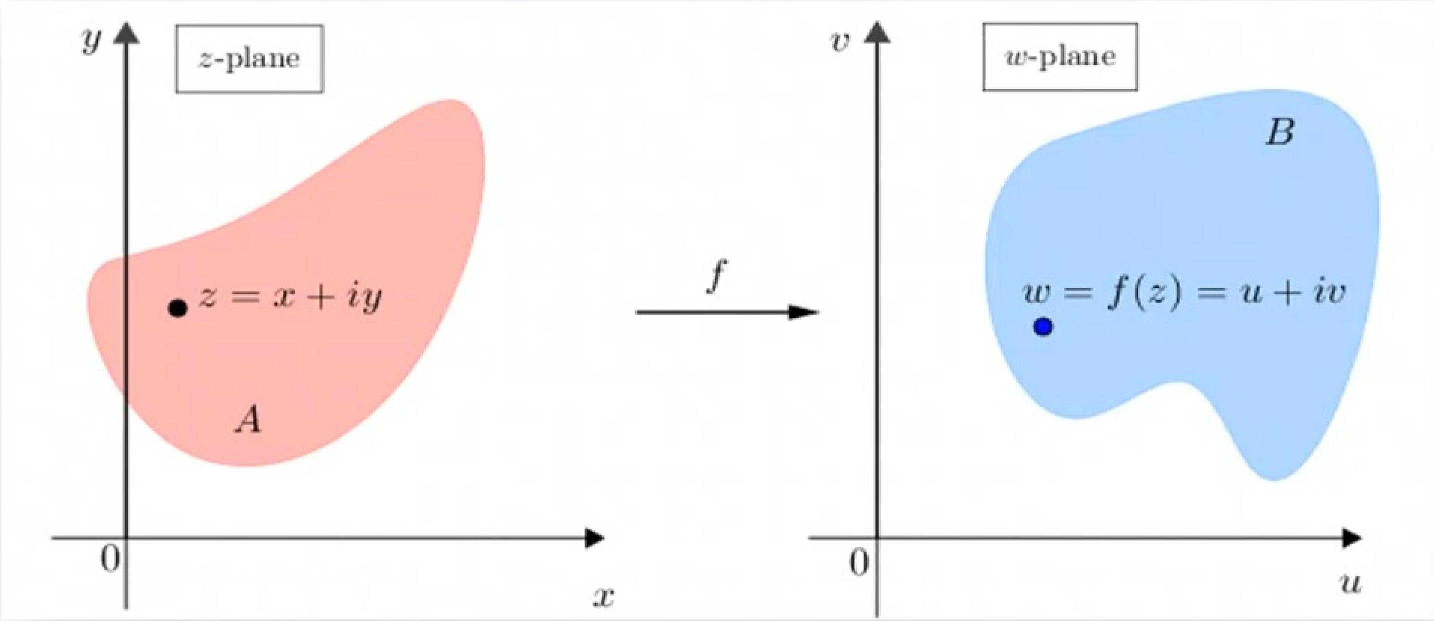
$$\boxed{\Rightarrow u(x,y) = x^2 - y^2 \quad \& \quad v(x,y) = 2\,x\,y}$$

Example 2:

$$w = e^z = e^{x+iy} = \underbrace{e^x}_{u}\,(\cos y + i\,\underbrace{\sin y}_{v})$$

$$\boxed{\Rightarrow u(x,y) = e^x \cos y \qquad v(x,y) = e^x \sin y}$$

$Z = f(x, y)$ $D \in R^2$ $\alpha\alpha$ $W = u + iv$ Range $\in R^2$

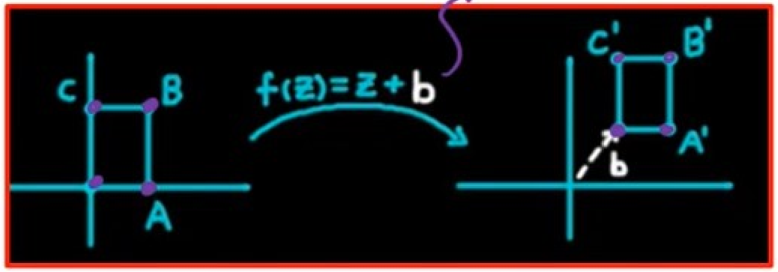


The region B is called the image od A and A is called the pre – image of B under the transformation $w = f(z)$.

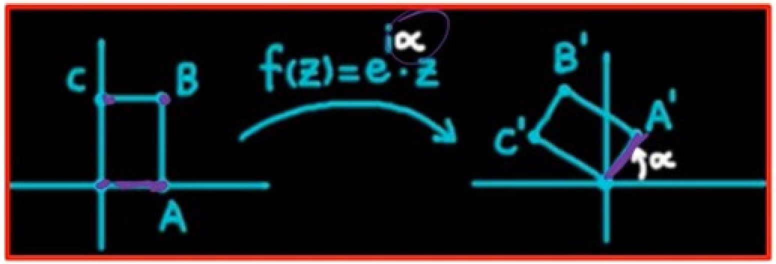
Transformations of Complex Numbers

Constant complex number

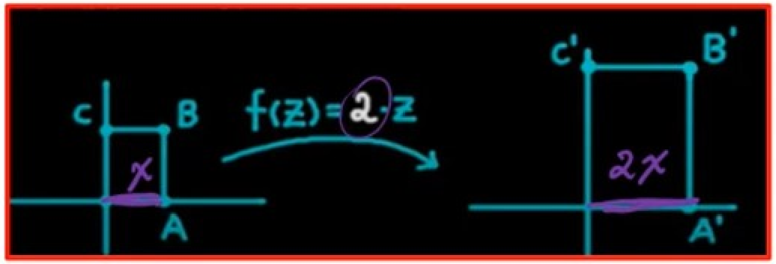
1. Translation



2. Rotation



3. Scaling



Example:

$x \geq 0$

$3i$

Find the image of the region $R(z) \geq 0$ under the transformation $w = (z + 3)$. Show the regions graphically.

addition cartezian easier

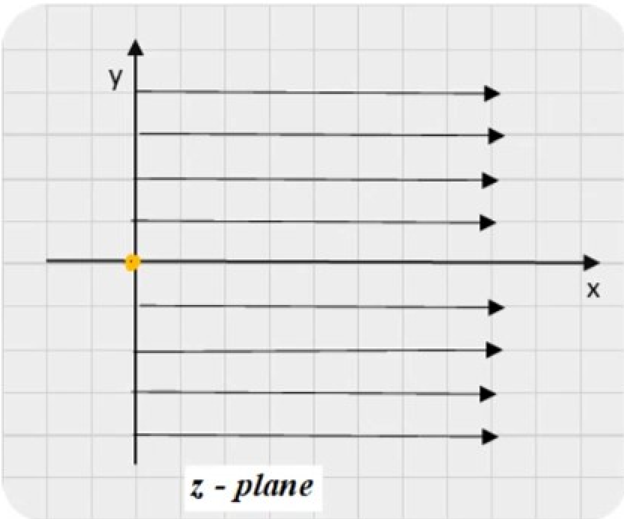
Solution:

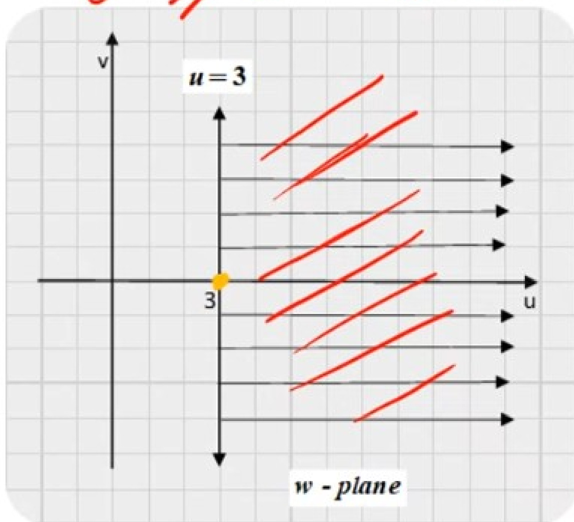
$$w = (z) + 3 = (x + i\,y) + 3 = \underbrace{(x+3)}_{u} + i\,\underbrace{y}_{v}$$

$x \geq 0$

$$\Rightarrow u = (x+3) \quad \& \quad v = y \qquad\qquad If \quad x \geq 0 \;\Rightarrow\; u \geq 3$$

$U \geq 3 \qquad U = 3$





Example:

Find the image of the region $|z| \le 2$, $0 < arg(z) < \pi/4$ under the transformation $w = z^2$. (Show the regions graphically).

$r \le 2$ $r = 2$

Solution:

polar form why multiplication in f(z) $z^2 = z \cdot z$

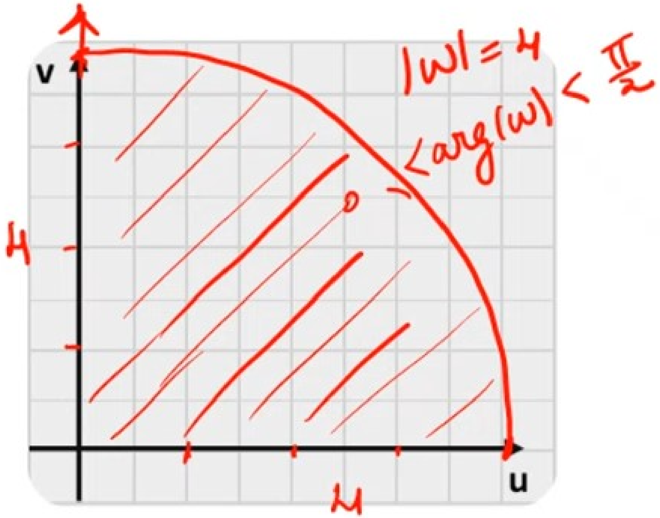
$$w = z^2 = (r\ e^{i\theta})^2 = r^2\ e^{i(2\theta)} \boxed{\Rightarrow |w| = |z|^2 ,} \quad \boxed{arg(w) = 2\,arg(z)}$$

$$\because\ |w| = |z|^2 ,\ |z| \le 2 \xrightarrow{(\ )^2} \Rightarrow |w| \le 4$$

$2\,arg(z)$

$$\because\ arg(w) = 2\,arg(z) ,\ 0 < arg(z) < \pi/4 \xrightarrow{2*} \Rightarrow 0 < arg(w) < \pi/2$$

$\xrightarrow{(\ )^2}$



Famous Transformation

1- **The Linear Transformations** $w = a\,z + b$

Where a and b are in general complex constants

- The linear transformation doesn't change figures shapes
- The linear transformation has three effects:

1: Shifting the origin to the point b

2: Scaling with $|a|$. (shrinking if $|a| < 1$ and enlarging if $|a| > 1$)

3: Rotation with $arg(a)$. (Anti clockwise if $arg(a) > 0$ and with clockwise if $arg(a) < 0$)

Example:

Find the image of the region bounded by the rectangle with vertices $(0,0);(1,0),(1,2) \& (0,2)$ under the transformation $w = (1+i)z + 2$. (Show the regions graphically).

Solution:

$0+0i$

$$w(0,0) = (1+i)(0) + 2 = 2 \equiv (2,0)$$

$1+0i$

$$w(1,0) = (1+i)(1) + 2 = 3+i \equiv (3,1)$$

$$w(1,2) = (1+i)(1+2i) + 2 = 1+3i \equiv (1,3)$$

$$w(0,2) = (1+i)(2i) + 2 = 2i \equiv (0,2)$$

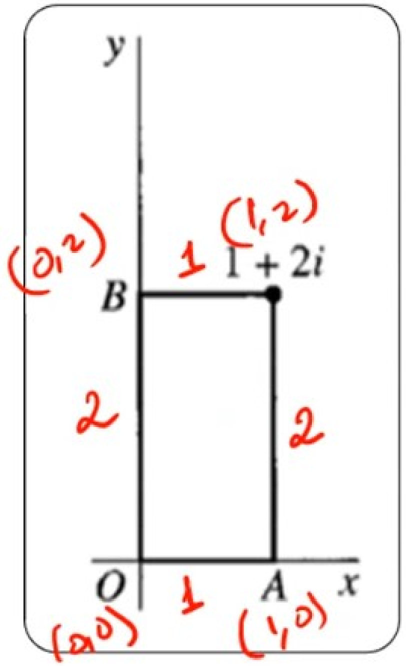


image in w-plane

$(0,0) \rightarrow (2,0)$

$(1,0) \rightarrow (3,1)$

$(1,2) \rightarrow (1,3)$

$(0,2) \rightarrow (0,2)$

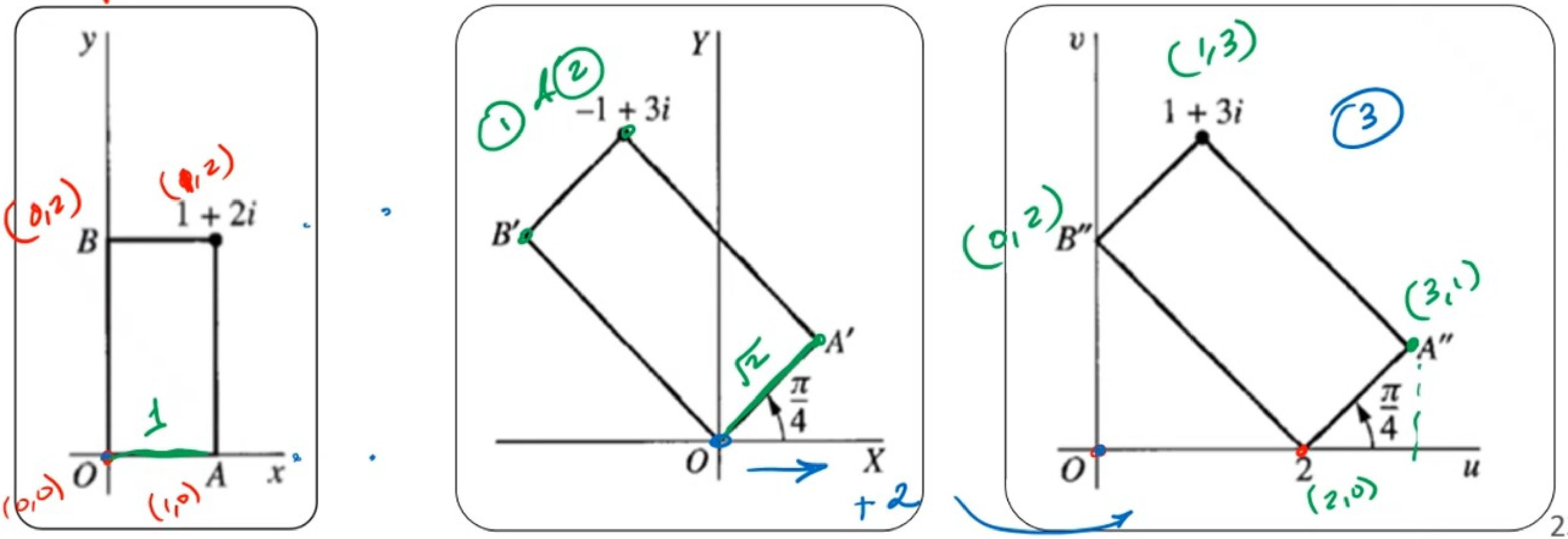
Pre-image

$$w = \left((1+i)\,z\right) + 2$$

① rotation $\frac{\pi}{4} \leftarrow \tan^{-1}(1)$

② & scaling $\sqrt{2} = \sqrt{1^2+1^2} > 1$

③



Famous Transformation

(2) The Reciprocal Transformations $w = 1/z$

$w = \frac{1}{z} =$

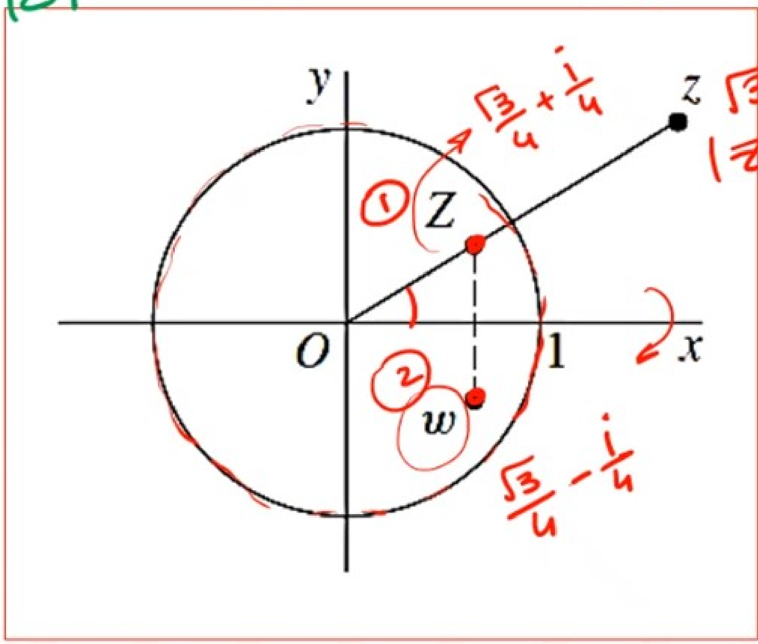
$z\bar{z} = |z|^2$ ① $\frac{1}{\bar{z}} = \frac{z}{|z|^2}$ ② $\overline{\left(\frac{1}{\bar{z}}\right)} = \frac{1}{z}$

z plane

① $Z = \frac{z}{|z|^2}$, ② $w = \overline{Z}.$

Scaling
shrinking
or magnifying.

minor
about x-axis



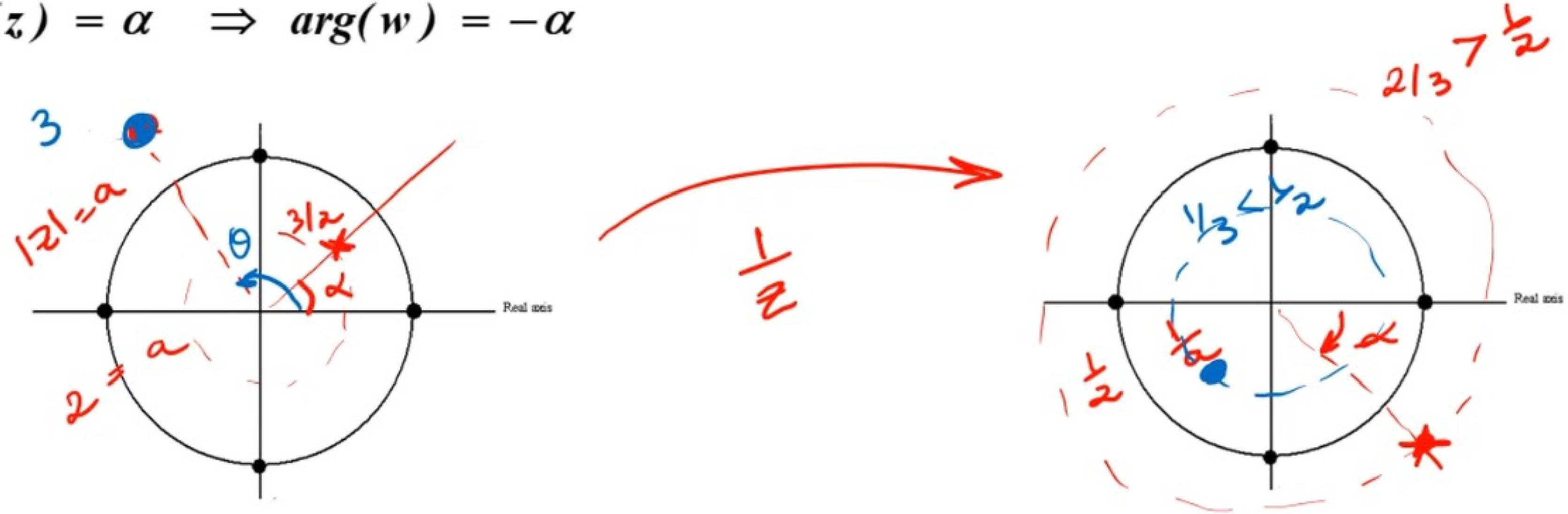
The Reciprocal Transformations $w = 1/z$

$$w = \frac{1}{z} = \frac{1}{r\,e^{i\theta}} = \frac{1}{r}\,e^{-i\theta}$$

$$\boxed{\Rightarrow |w| = \frac{1}{|z|}} \qquad \boxed{\Rightarrow arg(w) = -arg(z)}$$

1 $|z| = a \Rightarrow |w| = 1/a$ 2 $|z| < a \;;\; \Rightarrow |w| > 1/a$ & 3 $|z| > a \Rightarrow |w| < 1/a$

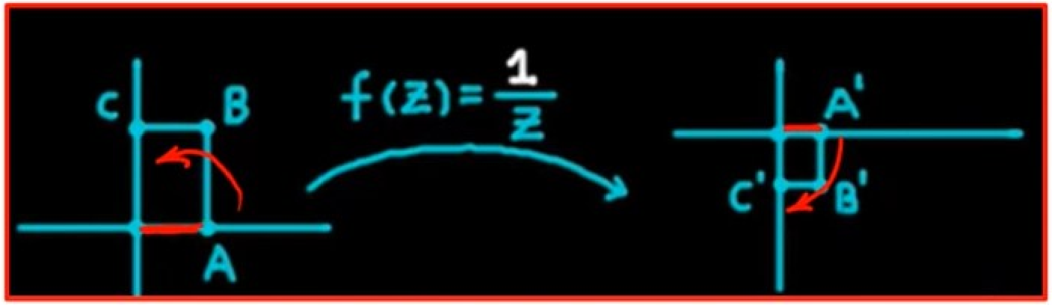
$$arg(z) = \alpha \quad \Rightarrow \quad arg(w) = -\alpha$$



The z plane *The w plane*

Transformations of Complex Numbers

4 Inversion



scaling
rotation counter clock wise
mirror about x axis