Next time ☺

➢ **Horizontal and vertical lines are transformed into circles tangent the u and v axes respectively**

Example:

in w plane

Find the image of the line $\boxed{y = 1/2}$ **under the transformation** $w = 1/z$ **. Show the regions graphically.**

$x = f_1(u,v)$ $y = f_2(u,v)$

Solution:

step 1

get x, y in terms of u, v

$$w = 1/z \Rightarrow z = \frac{1}{w} = \frac{1}{(u+iv)} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2} \equiv x + iy$$

$x =$ $y =$

$$\boxed{\Rightarrow x = \frac{u}{u^2+v^2} \quad \& \quad y = \frac{-v}{u^2+v^2}}$$

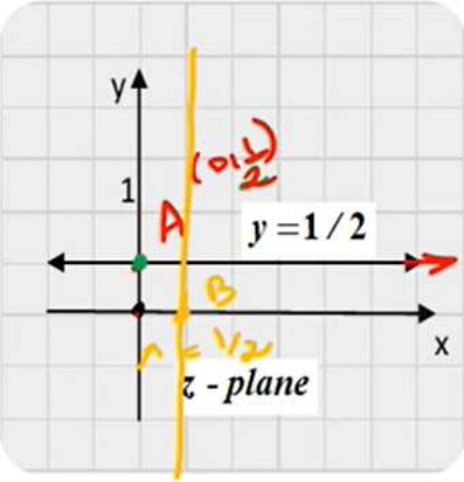
$$\Rightarrow y = 1/2 \Rightarrow \frac{-v}{u^2+v^2} = \frac{1}{2} \Rightarrow u^2+v^2+2v=0 \Rightarrow u^2+(v+1)^2=1$$

$u^2+v^2 = 2(-v)$

pre-image

$A = \frac{1}{2} e^{i\frac{\pi}{2}}$ $\quad 1/z \quad$ $A' = 2 e^{-i\frac{\pi}{2}}$

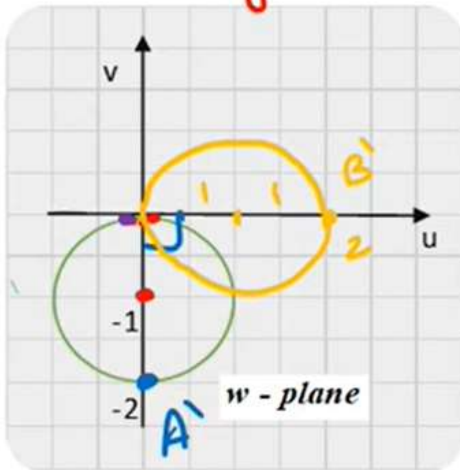
image



$-\infty$ $\quad$ ∞

$z \to 0$

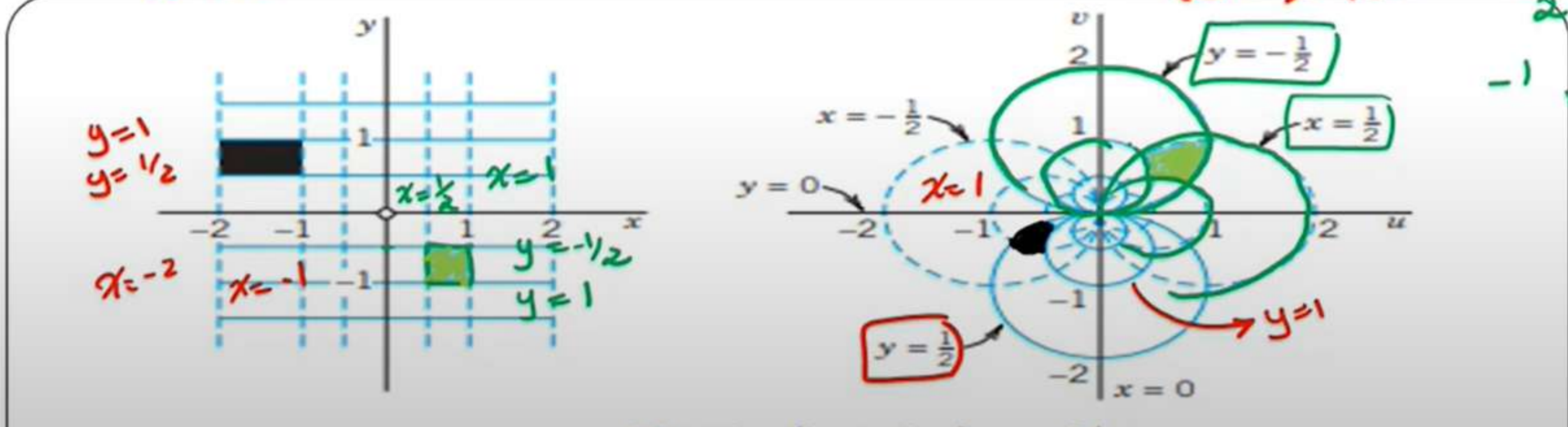
$$\Rightarrow u^2 + (v+1)^2 = 1$$



$w \to ??\pm\infty$

$\frac{1}{2} < x < 1$

$-1 < y < -\frac{1}{2}$



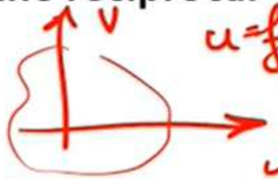
Mapping (Inversion) $w = 1/z$

general equation of st lines & circles

$ax + by + c = 0$
$x^2 + y^2 = r^2$) +

Example:

(Show that straight lines or circles are transformed into straight lines or circles under the reciprocal transformation.)



u=f(v)

Solution:

in Z-plane pre-image

Consider the following equation

all st lines and circles

$$\text{I} \leftarrow \boxed{A\,(x^2 + y^2) + B\,x + C\,y + D = 0}$$

$w = \frac{1}{z}$ $z = \frac{1}{w}$

Which represents straight lines or circles if A, B, C and D are real.

$x = ✓$ $y = ✓$

Under the reciprocal transformation, $\boxed{(x^2 + y^2) = \frac{1}{u^2 + v^2}}$, $\boxed{x = \frac{u}{u^2 + v^2}}$ & $\boxed{y = \frac{-v}{u^2 + v^2}}$

r_z^2 ?? $\frac{1}{r_w^2}$

$|z| = \frac{1}{|w|}$

substitute

$$\Rightarrow \frac{A}{u^2 + v^2} + \frac{B\,u}{u^2 + v^2} - \frac{C\,v}{u^2 + v^2} + D = 0$$

$(u^2 + v^2)$ *

$$\Rightarrow \boxed{D\,(u^2 + v^2) + B\,u - C\,v + A} = 0$$

Which represents straight lines or circles also.

$(x^3 + ax^2 + by^2 + cx + dy = 0)$ → any equation pass through origin because no constant term

preimage — image

$$\left[A(x^2+y^2)+Bx+Cy+D\right]=0 \qquad \Rightarrow \left(D(u^2+v^2)+Bu-Cv+A\right)=0$$

Remarks!!

ex

z-plane

$z \neq 0$ — $w \neq \pm\infty$

(a) a circle ($A \neq 0$) not passing through the origin ($D \neq 0$) in the z plane is transformed into a circle not passing through the origin in the w plane;

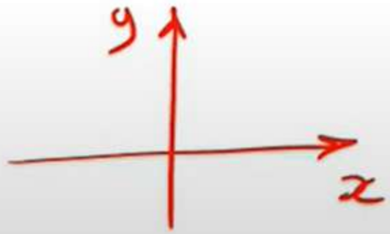
(b) a circle ($A \neq 0$) through the origin ($D = 0$) in the z plane is transformed into a line that does not pass through the origin in the w plane;

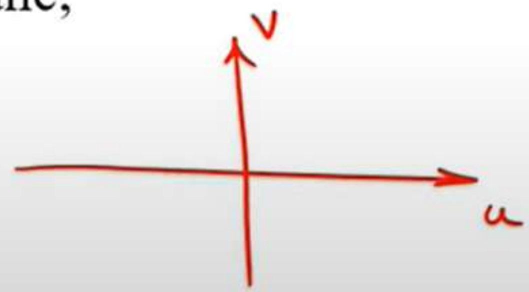
$z \neq \pm\infty$ — $w \neq 0$

$z = 0$ — $w \to \pm\infty$

$$\left[\underset{\alpha}{A}(x^2+y^2)+Bx+Cy+\underset{\alpha}{D}\right]=0 \qquad \Rightarrow D(u^2+v^2)+Bu-Cv+\underset{\alpha}{A}=0$$

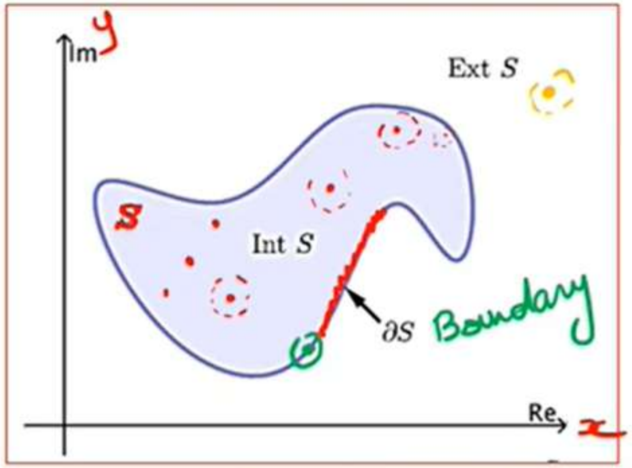
(*(c)* a line (A = 0) not passing through the origin (D $\neq$ 0) in the z plane is transformed into a circle through the origin in the w plane;

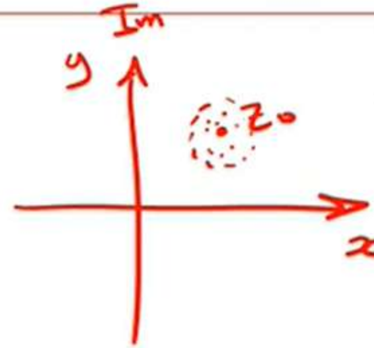




(d) a line (A = 0) through the origin (D = 0) in the z plane is transformed into a line through the origin in the w plane.

Functions of Complex Variables





$B(z_0)$ $|z-z_0|<\varepsilon$ → epsilon

$\varepsilon > 0$

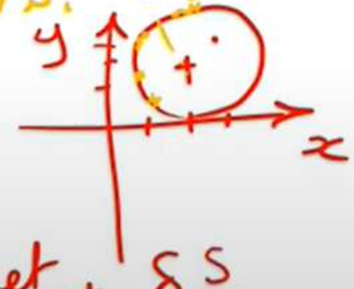
neighborhood open circular disk

Int S : $\forall z$, $B(z) \subset S$

∂S : $\forall z$, some $B(z) \subset S$

some $B(z) \not\subset S$

Ext S: $\forall z$, $B(z) \not\subset S$



set of all Int. S, $\therefore \forall z$

$B(z) \subset S$

open set $|z-3|<3$

set of all ∂S

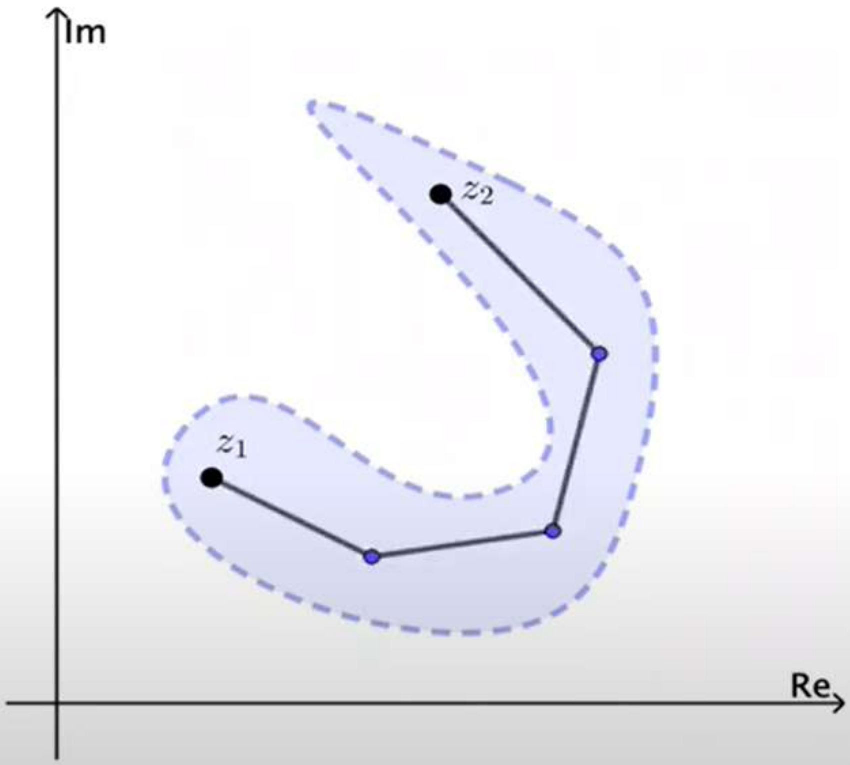
$|z-3|=3$

boundary

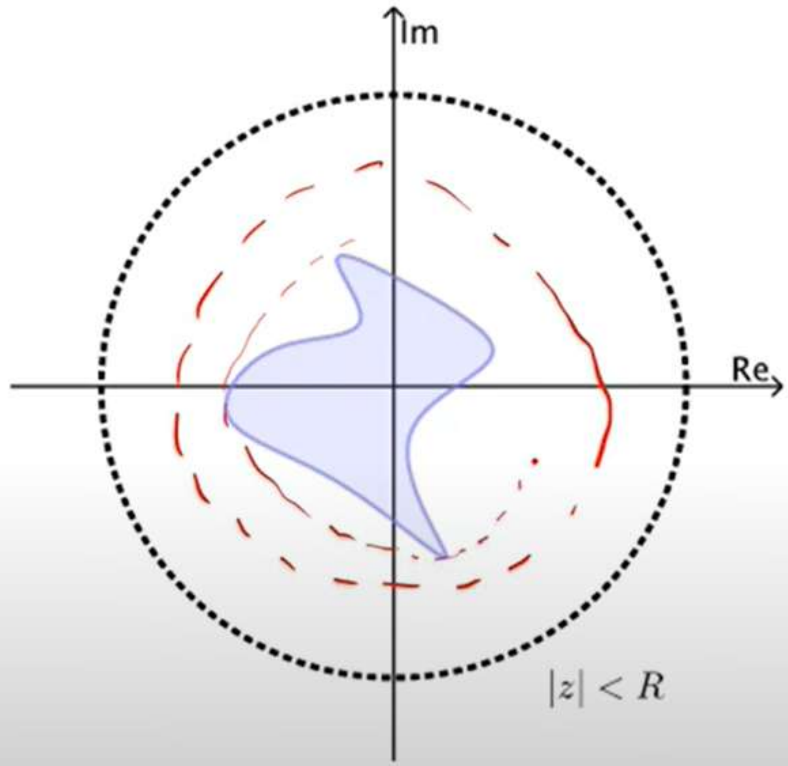
open set + δS

closed set

Derivatives of Functions of Complex Variables



connected set



bounded set

Derivatives of Functions of Complex Variables

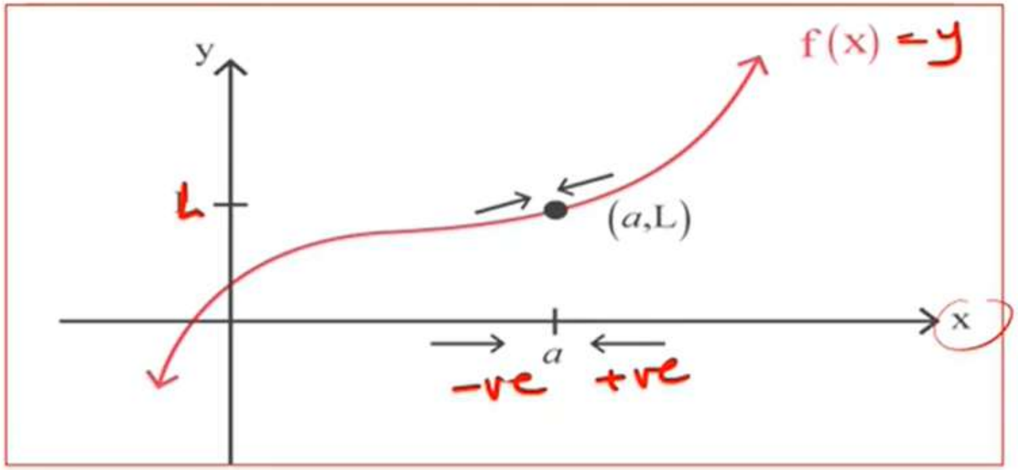
Limits

Real

$$\lim_{x \to a} \overbrace{f(x)}^{\text{function}} = \underbrace{L}_{\text{"What is the y-value getting closer to?"}}$$

function $\to y$

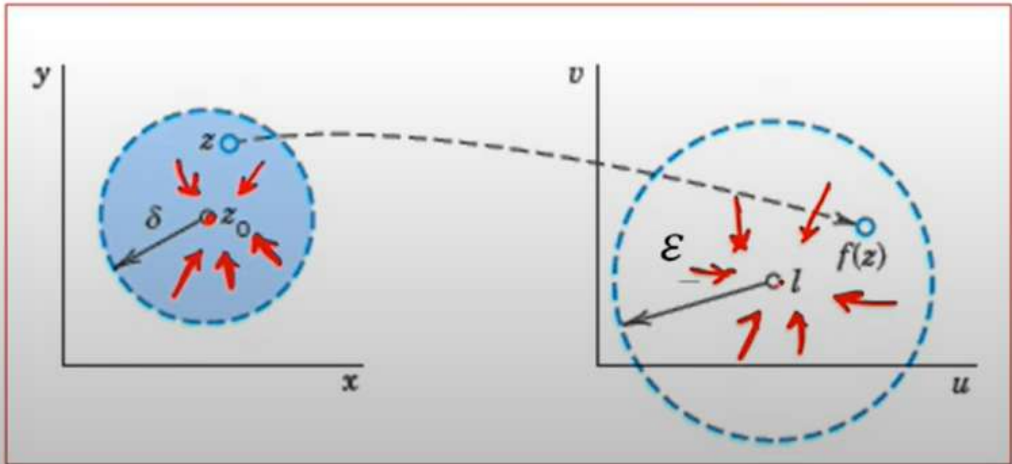
$x \to a$: "As you approach a along the x-axis"



Complex

$$\lim_{(z \to z_0)} f(z) = l$$

$\hookrightarrow |z - z_0|$

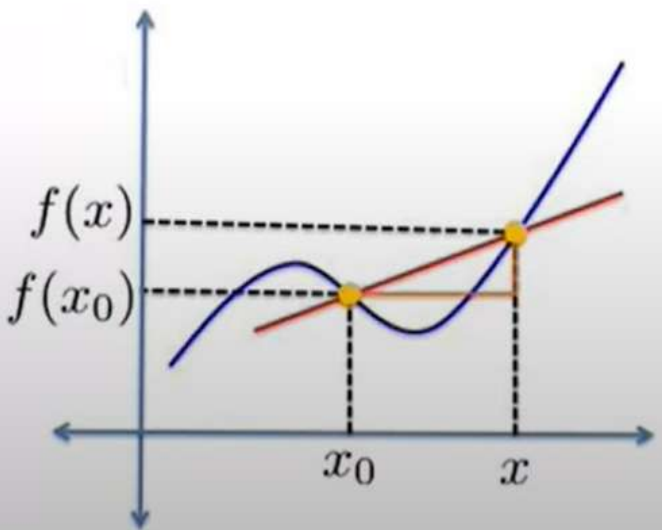


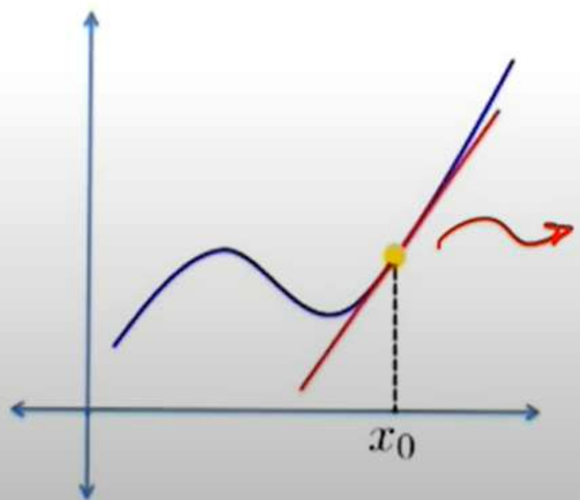
Derivatives of Functions of Complex Variables

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Differentiable

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$





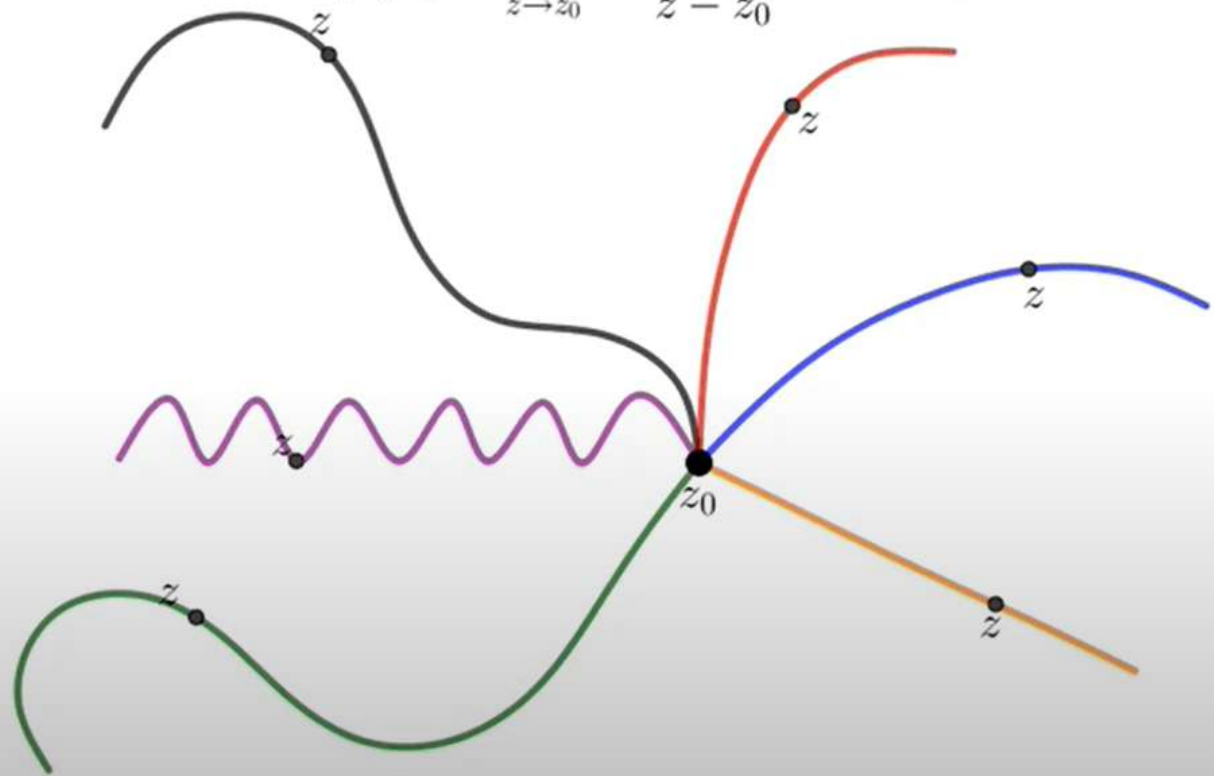
$f'(x) =$

Derivatives of Functions of Complex Variables

Differentiable

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



Derivatives of Functions of Complex Variables

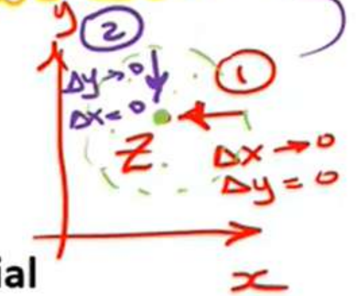
$\Delta x + i\Delta y$

$\lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$

Theorem:

The function $f(z) = u(x,y) + i\,v(x,y)$ is differentiable if and only if it satisfies Cauchy – Riemann equations $\boxed{u_x = v_y \quad and \quad u_y = -v_x}$ at which the derivative is

$$\boxed{f'(z) = (u_x + i v_x) = (v_y - i u_y)} \longrightarrow (\mathrm{I})$$



Note that the existence of the derivative thus implies the existence of the four partial derivatives U_x , U_y , V_x , V_y

Example 1:

Show that $f(z) = z^2$ is differentiable everywhere and $f'(z) = (2z.)$

Solution:

$u \quad +i \quad v$

$$f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) \quad \Rightarrow u = (x^2 - y^2) \;\; \& \;\; v = 2xy$$

$$u_x = 2x \;,\; v_y = 2x \quad \Rightarrow \boxed{u_x = v_y} \quad \forall \; z$$

$$u_y = -2y \;,\; v_x = 2y \quad \Rightarrow \boxed{u_y = -v_x} \quad \forall \; z$$

$$\boxed{\Rightarrow f(z) \text{ is differentiable everywhere.}}$$

$f'(z) = U_x + iV_x = 2x + i\,2y = 2z$

$2(x+iy) \to z$

$$\Rightarrow f'(z) = u_x + i\,v_x = 2x + i\,2y = 2(x+iy) = 2z$$

Example 2:

Show that $f(z) = \bar{z}$ is not differentiable anywhere.

Solution:

$$f(z) = x - i\,y \qquad \Rightarrow \qquad u = x \;\; \& \;\; v = -\,y$$

Ux = Vy not satisfied !!

$\boxed{u_x = 1}$ & $\boxed{v_y = -1}$ Uy = 0 ✓ Vx = 0 ✓

$\because$ *It is impossible to equate u_x and v_y*

$\Rightarrow f(z) = \bar{z}$ *is not differentiable anywhere.*

Example 3:

Show that $w = e^z$ is differentiable everywhere and $\frac{dw}{dz} = e^z$

Euler

Solution:

$$w = e^{x+iy} = e^x\ (\cos y + i \sin y)$$

$$u(x, y) = e^x \cos y \qquad v(x, y) = e^x \sin y$$

$$u_x = \underline{e^x \cos y},\ v_y = \underline{e^x \cos y} \Rightarrow u_x = v_y \quad \forall z$$

$$u_y = \underline{-e^x \sin y},\ v_x = \underline{e^x \sin y} \Rightarrow u_y = -v_x \quad \forall z$$

$$\frac{dw}{dz} = f'(z) = u_x + i v_x = (e^x \cos y + i e^x \sin y)$$

$$e^x e^{iy} = e^{x+iy} = e^z$$

$$\frac{d}{dz}(e^z) = e^z$$

Similarly, we can find the derivatives of all the known functions

$$\frac{d}{dz}\left(z^n\right) = n\,z^{n-1}$$

$$\frac{d}{dz}\left((f(z))^n\right) = n(f(z))^{n-1} \times f'(z)$$

$$\frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\tan z) = \sec^2 z$$

$$\frac{d}{dz}(\cot z) = -\csc^2 z$$

$$\frac{d}{dz}(\sec z) = \sec z\,\tan z$$

$$\frac{d}{dz}(\csc z) = -\csc z\,\cot z$$

$$\frac{d}{dz}\left(e^z\right) = e^z$$

$$\frac{d}{dz}(\ln z) = \frac{1}{z}$$

$$\frac{d}{dz}(f\,g) = f\,g' + f'\,g$$

$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{g\,f' - f\,g'}{g^2}$$

Example 4:

u + i v

Show where the function $f(z) = (x^2 + y) + i(y^2 - x)$ is differentiable.

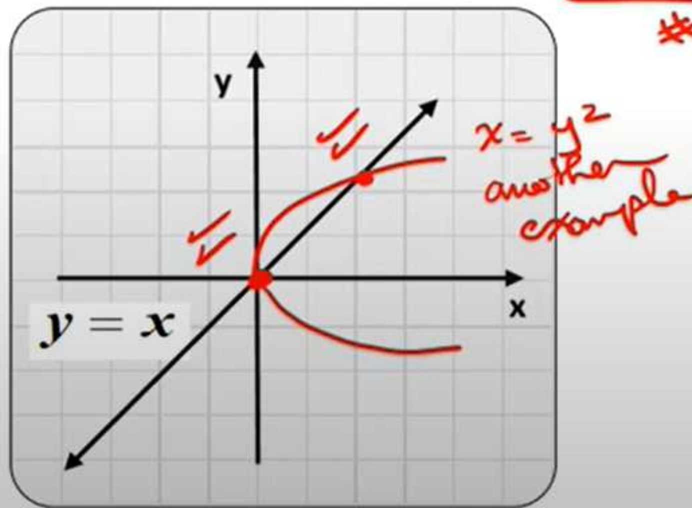
Solution:

① $u_x = 2x$, $v_y = 2y$ for $u_x = v_y \Rightarrow \boxed{x = y}$ *) &

② $u_y = 1$, $v_x = -1 \Rightarrow u_y = -v_x \quad \forall z$ ✓

another problem!!
$U_y = -x \quad V_x = y^2$
$U_y = -V_x$
$-x = -y^2 \quad \boxed{x = y^2}$ *

This function is differentiable only on the line $y = x$



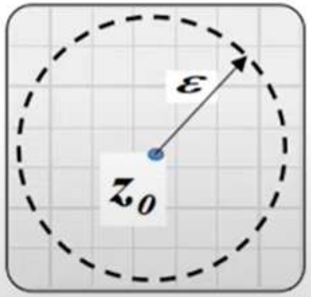
Analytic and Harmonic Functions

Definition:

A function $f(z)$ is called analytic at a point z_0 if it is differentiable at z_0 and on a neighborhood of z_0 .

Definition:

A neighborhood of a point z_0 is the set of all points z such that $|z - z_0| < \varepsilon \quad where \ \varepsilon > 0$



Definition:

A function is called Entire function if it is analytic everywhere and this happens if it is differentiable everywhere

We can show that the functions z^n , $\sin z$, $\cos z$, e^z are entire and their composite functions e^{z^n} , $\sin e^z$, $e^{\sin z}$, ...

Caushy - Rieman are satisfied $\forall z$
Diff → everywhere
analytic → everywhere
Entire

Example 5:

For the function $f(z) = (x^2 + y) + i(y^2 - x)$ which is given in example 4, state where the function is analytic.

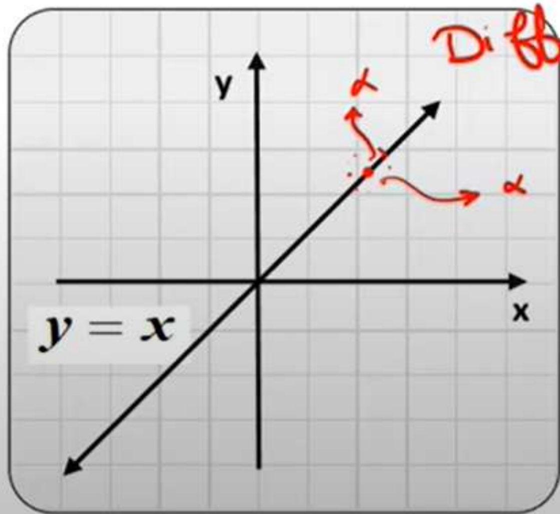
Solution:

$$u_x = 2x \;,\; v_y = 2y \quad for \quad u_x = v_y \;\Rightarrow \boxed{x = y}$$

$$u_y = 1 \;,\; v_x = -1 \;\Rightarrow u_y = -v_x \;\;\forall\; z$$

This function is differentiable only on the line $y = x$

Hence, it is not analytic anywhere.



Harmonic Functions

Definition:

A function $u(x,y)$ is called harmonic on a certain domain $''D''$ if it satisfies Laplace's equation $\boxed{u_{xx} + u_{yy} = 0}$ on D

satisfies
caushy - Reiman

$U_{xx} + U_{yy} = 0$
$V_{xx} + V_{yy} = 0$

Theorem:

If $f(z) = u + iv$ is analytic on a certain domain "D" then both u and v are harmonic functions on the same domain "D" where u is called the harmonic conjugate of v and also v is called the harmonic conjugate of u.

$V_{xy} = V_{yx}$

Proof:

$$\because f(z) \text{ is analytic}, \therefore \text{ it is differentiable} \Rightarrow (u_x = v_y) \;\&\; (u_y = -v_x)$$

$$\Rightarrow u_{xx} = v_{yx} \;\&\; u_{yy} = -v_{xy} \Rightarrow u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \qquad \therefore u \text{ is harmonic}$$

Similarly, we can prove that v is also harmonic.

$U_{xx} + U_{yy} = 0$

Example 6:

Show that $u(x,y) = y^3 - 3x^2 y$ is harmonic and find its conjugate "v" hence, find the analytic function $f(z) = u + i\,v$ in terms of z.

Solution:

$\boxed{u_x = -6xy} \rightarrow V_y$

$u_{xx} = -6y$

$\boxed{u_y = 3y^2 - 3x^2} \rightarrow -V_x$

$u_{yy} = 6y$

$u_{xx} + u_{yy} = -6y + 6y = 0$

Laplace Equ. is True

$\therefore$ u is harmonic

$u_x = v_y \quad \Rightarrow v_y = -6xy \qquad (1)$

$u_y = -v_x \quad \Rightarrow v_x = 3x^2 - 3y^2 \qquad (2)$

Integrating (1) w. r. t. y $\quad \Rightarrow v = -3xy^2 + \boxed{h(x)}$

$V_x = v$

Using (2) $\quad \Rightarrow v_x = \cancel{-3y^2} + h'(x) = 3x^2 \cancel{- 3y^2}$

$h'(x) = 3x^2 \Rightarrow h(x) = x^3 + k$

$\boxed{v = x^3 - 3xy^2 + k}$

$\boxed{f(z) = (y^3 - 3x^2 y) + i(x^3 - 3xy^2 + k)}$

very long!! Method 1

Method 2

u $\quad$ v

Putting $y = 0$

$x = \frac{z + \bar{z}}{2}$

$y = \frac{z - \bar{z}}{2i}$

$f(x) = i(x^3 + k) \quad x \rightarrow z$

$\boxed{\Rightarrow f(z) = i(z^3 + k)}$