Section 11: Suite: Numerical solution for ODE

Problem 2: $y' = y - y^2$, $y(0) = 0.2$ Find $y(1) = ?$

$F(y)$

Runge-Kutta (1 step): $n = 1$, $x_0 = 0$, $y_0 = 0.2$, $x_n = 1$, $y_n = ?$

$h = \frac{x_n - x_0}{n} = \frac{1-0}{1} = 1$

$$\boxed{\Rightarrow y_{n+1} = y_n + \Delta y_n}$$

this is the only thing that changes from a method to another

$$\Delta y_n = \frac{1}{6}(\omega_1 + 2\omega_2 + 2\omega_3 + \omega_4)$$

at $n=0$, $x_0 = 0$, $y_0 = 0.2$, $F = y - y^2$, $h = 1$

$\omega_1 = hF(x, y) = 1(0.2 - 0.2^2) = 0.16$

Δy of Euler method

$\omega_2 = hF(x + \frac{h}{2}, y + \frac{\omega_1}{2})$
$= F(0.5, 0.2 + \frac{0.16}{2})$
$= 1[0.28 - 0.28^2] = 0.2016$

$\omega_3 = hF(x + \frac{h}{2}, y + \frac{\omega_2}{2}) = hF(0.5, 0.2 + \frac{0.2016}{2}) = 1[( ) - ( )^2] = 0.2103$

$\omega_4 = hF(x + h, y + \omega_3) = hF(1, 0.2 + 0.2103) = 1[( ) - ( )^2] = 0.2420$

$\Rightarrow \Delta y = \frac{1}{6}(0.16 + 2(0.2016) + 2(0.2103) + 0.2420) = 0.2043$

$\Rightarrow y_1 = y_0 + \Delta y_0 = 0.2 + 0.2043 = 0.4043 \simeq y(1)$

	Euler	RK4
Error	$\propto h$	$\propto h^4$
	$O(h)$	$O(h^4)$

* 1 step in RK is tiresome but with higher accuracy

* each step is a table

b) RK4 (2 steps) $h = \frac{1-0}{2} = 0.5$

n	x_n	y_n	$\omega_i = hF$	Δy
0	0	0.2	ω_1	
	0.25		ω_2	
	0.25		ω_3	
	0.5		ω_4	$\Delta y =$

$y_1 = y_0 + \Delta y = y_0 + \Delta y = y(0.5)$

n	x_n	y_n	ω_1	Δy
1	0.5	0.2919	نفس الخطوات	$\Delta y = 0.1127$

$y_2 = y_1 + \Delta y \simeq y(1)$

* System of ODE : $y' = f(t, x, y)$
$x' = g(t, x, y)$

t_0, x_0, y_0, t_n $x_n = ?$ $y_n = ?$

$h = \frac{t_n - t_0}{n}$, $n = \frac{t_n - t_0}{h}$

$x_{n+1} = x_n + \Delta x_n$ $y_{n+1} = y_n + \Delta y_n$

$\Delta x = \frac{1}{6}(\omega_1 + 2\omega_2 + 3\omega_3 + \omega_4)$, $\Delta y = \frac{1}{6}(V_1 + 2V_2 + 2V_3 + V_4)$

$\omega_1 = hF(t_n, x_n, y_n)$, $V_1 = hg(t_n, x_n, y_n)$
$\omega_2 = hF(t_n + \frac{h}{2}, x_n + \frac{\omega_1}{2}, y_n + \frac{V_1}{2})$, $V_2 = hg(t_n + \frac{h}{2}, x_n + \frac{\omega_1}{2}, y_n + \frac{V_1}{2})$
$\omega_3 = hF(t_n + \frac{h}{2}, x_n + \frac{\omega_2}{2}, y_n + \frac{V_2}{2})$, $V_3 =$
$\omega_4 = hF(t_n + h, x_n + \omega_3, y_n + V_3)$, $V_4 = hg(t_n + h, x_n + \omega_3, y_n + V_3)$

$x'' - tx' - x = 0$ (higher order)
let $x' = y$ ① $x'' = y'$ $y' - ty - x = 0$
$\boxed{y' = ty + x}$ — ②

system. (you can even neglect calculating Δy because x is your target)

PDE : $\xrightarrow[\text{method}]{\text{Finite difference}}$ System of equations

we will solve using Gauss-Siedel method :

$u_1 = f_1(u_2, u_3, \dots, C)$

$u_2 = F_2(u_1, u_3, \dots, C)$

$u_3 = F_3(u_1, u_2, \dots, C)$

$a_{n1}x_1 + a_{12}x_2 + \dots = b_n$

* to take :

$x_1 = \frac{1}{a_{11}}(-a_{12}x_2 - a_{13}x_3 - \dots + b_n)$

* Cond :

$|a_{11}| > |a_{12}| + |a_{13}| + \dots$

to converge to the exact sol.

Exp: $2u_1 + u_2 - 4u_3 = 10$ → suitable for u_3

$3u_2 + u_3 - 4u_1 = 5$ ~~u1~~

$u_1 + 2u_3 - 4u_2 = -6$ → suitable for u_2

$^{(n+1)}u_2 = \frac{3}{2} + 0.25\,u_1^{(n+1)} + 0.5\,u_3^{(n)}$ (u_3 not yet updated)

$^{(n+1)}u_3 = -2.5 + 0.5\,u_1^{(n+1)} + 0.25\,u_2^{(n+1)}$

$^{(n+1)}u_1 = -1.25 + 0.75\,u_2^{(n)} + 0.25\,u_3^{(n)}$

reorder $u_1 =$
$u_2 =$
$u_3 =$

n	u_1	u_2	u_3
0	-1.25	1.5	-2.5
1	substitute	step	
2	by	step	

start your initial guess/approximation by using the constants from the equations unless given in the problem and we go on making successive approximation "Changing the first guess doesn't affect the outcome too much"

* When do we stop ?

perform n steps
$0 \to n$
(initial guess usually given)

Accurate to 2 Decimal places

اول لما اتنين 2D يثبتوا على
نفس الأرقام

□□.▨▨□□

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يعني لازم يثبتوا كل ال unknowns

* we work with 2 extra decimal places then our given accuracy

* Pros : making a mistake in one of the iteration, it will be self corrected after some iteration, the accuracy may be affected tho.

⚠ Don't get the equations wrong ⚠