Math. 3 Midterm Exam.

2nd Year Electrical Eng. November 6th , 2017. Allowed Time: 1 Hour.

Model Answer	**Class:**	**Model A**

Question 1 (12 Marks)

(A) Evaluate in terms of the Gamma function

$$\int_0^{\pi/2} \left(\csc^3\theta - \csc^2\theta\right)^{1/5} \cos\theta \; d\theta .$$

[4 Marks]

$$\to t = \sin\theta \Rightarrow dt = \cos\theta d\theta \qquad \to \boxed{1}$$

$$\Rightarrow \csc\theta = 1/t = t^{-1}$$

$$\Rightarrow I = \int_0^1 \left(t^{-3} - t^{-2}\right)^{1/5} dt \qquad \to \boxed{1}$$

$$= \int_0^1 t^{-3/5} (1-t)^{1/5} dt$$

$$\to x-1 = -3/5, \; y-1 = 1/5$$

$$= \beta\left(\frac{2}{5}, \frac{6}{5}\right) \qquad \to \boxed{1}$$

$$= \frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{6}{5}\right)}{\Gamma\left(\frac{8}{5}\right)} \qquad \to \boxed{1}$$

(B) Find the general solution in powers of x for

$$(2 - x^2)\, y'' - x\, y' + 25\, y = 0 .$$

[8 Marks]

$$p(x) = \frac{-x}{2-x^2}, q(x) = \frac{25}{2-x^2}$$

$\Rightarrow p(x)\,\&\, q(x)$ are both analytic at $x_0 = 0$

$\Rightarrow x_0 = 0$ is an ordinary point $\Rightarrow$ Power series Method .... $\boxed{1}$

$$\text{let } y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \ldots\ldots\ldots\ldots \boxed{2}$$

$$\Rightarrow (2-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 25\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 25 a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 25 a_n x^n = 0$$

$$coef\,(x^n) = 0$$

$$\Rightarrow 2(n+2)(n+1)a_{n+2} - n(n-1)a_n - n a_n + 25 a_n = 0$$

$$\Rightarrow 2(n+2)(n+1)a_{n+2} = \left[n(n-1) + n - 25\right] a_n$$

$$\Rightarrow a_{n+2} = \frac{(n-5)(n+5)}{2(n+2)(n+1)} a_n, n \ge 0 \ldots\ldots\ldots\ldots \boxed{2}$$

$$n = 0 \Rightarrow a_2 = \frac{-5.5}{2.2.1} a_0,$$

$$n = 2 \Rightarrow a_4 = \frac{-3.7}{2.4.3} a_2 = \frac{(-5.-3)(5.7)}{2^2(4.3.2.1)} a_0,$$

$$\Rightarrow a_{2m} = \frac{\left[-5.-3.-1....(2m-7)\right]\left[5.7.9...(2m+3)\right]}{2^m m!} a_0 \ldots\ldots\ldots\ldots \boxed{1}$$

$$n = 1 \Rightarrow a_3 = \frac{-4.6}{2.3.2} a_1 = -2a_1,$$

$$n = 3 \Rightarrow a_5 = \frac{-2.8}{2.5.4} a_3 = \frac{(-4.-2)(6.8)}{2^2(5.4.3.2)} a_1 = \frac{4}{5} a_1,$$

$$n = 5 \Rightarrow a_7 = 0 = a_9 = a_{11} = ... = a_{2m+1} \ldots\ldots\ldots\ldots \boxed{1}$$

$$\Rightarrow y = a_0\left(1 + \sum_{m=1}^{\infty} a_{2m} x^{2m}\right) + a_1\left(x - 2x^3 + \frac{4}{5}x^5\right) \ldots\ldots\ldots\ldots \boxed{1}$$

Question 2 (10 Marks)

_(A)_

Let A, B, and C be defined on S, such that P (A) = 0.35, P (B) = 0.25, and P (A ∪ B ∪ C) = 0.78.
Find P (C) when A, B, and C are
(1) mutually exclusive
(2) statistically independent

1) Mutually Exclusive

$$\Rightarrow P(A\cup B\cup C)=P(A)+P(B)+P(C)$$

$$\Rightarrow 0.78=0.35+0.25+P(C)\Rightarrow \boxed{P(C)=0.18}\text{..................}\boxed{1}$$

2) Independent

$$\Rightarrow P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A)P(B)-P(A)P(C)-P(B)P(C)+P(A)P(B)P(C)\text{..............................}\boxed{1}$$

$$\Rightarrow 0.78=0.35+0.25+P(C)-(0.35)(0.25)-(0.35)P(C)-(0.25)P(C)+(0.35)(0.25)P(C)$$

$$\Rightarrow 0.2675=0.4875P(C)\Rightarrow \boxed{P(C)=0.5487}\text{..................}\boxed{1}$$

_(C)_

Three machines produce respectively 25 %, 55%, and 20% of the total production of an item in a certain factory. The probabilities of producing a defective item on these machines are 0.02, 0.05, and 0.01 respectively. An item is selected at random and tested.
(i) Find the probability that it was found defective
(ii) Find the probability that the selected defective item was produced by the 2nd machine

$$P(D)=P(D\,|\,M_1)P(M_1)+P(D\,|\,M_2)P(M_2)+P(D\,|\,M_3)P(M_3)$$
$$=(0.02)(0.25)+(0.05)(0.55)+(0.01)(0.2)$$

$$\Rightarrow \boxed{P(D)=0.0345}\text{................}\boxed{2}$$

$$P(M_2\,|\,D)=\frac{P(D\,|\,M_2)P(M_2)}{P(D)}$$

$$\Rightarrow \boxed{P(M_2\,|\,D)=0.797}\text{................}\boxed{1}$$

_<u>(B)</u>_

Four balls are selected at random from a collection of 4 White, 3 Red, and 5 Black balls. Find the probability that all selected balls have the same color when:
(i) sampling with replacement
(ii) sampling without replacement

1) With Replacement

$$P(SameColor)=P(4W)+P(4R)+P(4B)\text{.......}\boxed{1}$$

$$=\left(\frac{4}{12}\right)^4+\left(\frac{3}{12}\right)^4+\left(\frac{5}{12}\right)^4=0.04639\text{.................}\boxed{1}$$

1) Without Replacement

$$P(SameColor)=P(4W)+P(4B)\text{....................}\boxed{1}$$

$$=\frac{{}^4C_4\,{}^3C_0\,{}^5C_0+{}^4C_0\,{}^3C_0\,{}^5C_4}{{}^{12}C_4}=\frac{6}{495}=0.01212\text{......}\boxed{1}$$

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Model Answer	**Class:**	**Model B**

Question 1 (12 Marks)

(A) Evaluate in terms of the Gamma function

$$\int_0^{\pi/2}\left(\sec^3\theta-\sec^2\theta\right)^{1/4}\sin\theta\ d\theta.$$

[4 Marks]

$$\rightarrow t=\cos\theta\Rightarrow dt=-\sin\theta d\theta \qquad \rightarrow \boxed{1}$$

$$\Rightarrow \sec\theta=\frac{1}{t}=t^{-1}$$

$$\Rightarrow I=-\int_1^0\left(t^{-3}-t^{-2}\right)^{1/4}dt \qquad \rightarrow \boxed{1}$$

$$=\int_0^1 t^{-3/4}\left(1-t\right)^{1/4}dt$$

$$\rightarrow x-1=-3/4,\ y-1=1/4$$

$$=\beta\left(\frac{1}{4},\frac{5}{4}\right) \qquad \rightarrow \boxed{1}$$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{6}{4}\right)} \qquad \rightarrow \boxed{1}$$

(B) Find the general solution in powers of x for

$$(3-x^2)\,y''-x\,y'+16\,y=0.$$

[8 Marks]

$$p(x)=\frac{-x}{3-x^2},\ q(x)=\frac{16}{3-x^2}$$

$\Rightarrow p(x)\,\&\,q(x)$ are both analytic at $x_0=0$

$\Rightarrow x_0=0$ is an ordinary point $\Rightarrow$ Power series Method .... $\boxed{1}$

$$let\ y=\sum_{n=0}^{\infty}a_nx^n\Rightarrow y'=\sum_{n=1}^{\infty}na_nx^{n-1}\Rightarrow y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}\ldots\ldots\ldots\ldots\boxed{2}$$

$$\Rightarrow(3-x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-x\sum_{n=1}^{\infty}na_nx^{n-1}+16\sum_{n=0}^{\infty}a_nx^n=0$$

$$\Rightarrow\sum_{n=2}^{\infty}3n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}16a_nx^n=0$$

$$\Rightarrow\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_nx^n-\sum_{n=0}^{\infty}na_nx^n+\sum_{n=0}^{\infty}16a_nx^n=0$$

$$coef\,(x^n)=0$$

$$\Rightarrow 3(n+2)(n+1)a_{n+2}-n(n-1)a_n-na_n+16a_n=0$$

$$\Rightarrow 3(n+2)(n+1)a_{n+2}=\left[n(n-1)+n-16\right]a_n$$

$$\Rightarrow a_{n+2}=\frac{(n-4)(n+4)}{3(n+2)(n+1)}a_n,\ n\ge 0\ldots\ldots\ldots\ldots\boxed{2}$$

$$n=0\Rightarrow a_2=\frac{-4.4}{3.2.1}a_0=\frac{-8}{3}a_0,$$

$$n=2\Rightarrow a_4=\frac{-2.6}{3.4.3}a_2=\frac{(-4.-2)(4.6)}{3^2(4.3.2.1)}a_0=\frac{8}{9}a_0,$$

$$n=4\Rightarrow a_6=0=a_8=a_{10}=\ldots=a_{2m}\ \ldots\ldots\ldots\ldots\boxed{1}$$

$$n=1\Rightarrow a_3=\frac{-3.5}{3.3.2}a_1,$$

$$n=3\Rightarrow a_5=\frac{-1.7}{3.5.4}a_3=\frac{(-3.-1)(5.7)}{3^2(5.4.3.2)}a_1,$$

$$\Rightarrow a_{2m+1}=\frac{\left[-3.-1.1\ldots.(2m-5)\right]\left[5.7.9\ldots(2m+3)\right]}{3^m(2m+1)!}a_0\ \ldots\ldots\ldots\ldots\boxed{1}$$

$$\Rightarrow y=a_0\left(1-\frac{8}{3}x^2+\frac{8}{9}x^4\right)+a_1\left(x+\sum_{m=1}^{\infty}a_{2m+1}x^{2m+1}\right)\ldots\ldots\ldots\ldots\boxed{1}$$

Question 2 (10 Marks)

(A)

Let A, B, and C be defined on S, such that P (A) = 0.35, P (B) = 0.18, and P (A ∪ B ∪ C) = 0.78. Find P (C) when A, B, and C are
(1) mutually exclusive
(2) statistically independent

1) Mutually Exclusive

$$\Rightarrow P(A\cup B\cup C)=P(A)+P(B)+P(C)$$

$$\Rightarrow 0.78=0.35+0.18+P(C)\Rightarrow \boxed{P(C)=0.25}\ldots\ldots\ldots\ldots\boxed{1}$$

2) Independent

$$\Rightarrow P(A\cup B\cup C)=P(A)+P(B)+P(C)$$
$$-P(A)P(B)-P(A)P(C)-P(B)P(C)$$
$$+P(A)P(B)P(C)\ldots\ldots\ldots\ldots\boxed{1}$$

$$\Rightarrow 0.78=0.35+0.18+P(C)-(0.35)(0.18)$$
$$-(0.35)P(C)-(0.18)P(C)+(0.35)(0.18)P(C)$$

$$\Rightarrow 0.313=0.533P(C)\Rightarrow \boxed{P(C)=0.5872}\ldots\ldots\ldots\ldots\boxed{1}$$

(C)

Three machines produce respectively 25 %, 55%, and 20% of the total production of an item in a certain factory. The probabilities of producing a defective item on these machines are 0.03, 0.05, and 0.01 respectively. An item is selected at random and tested.
(i) Find the probability that it was found defective
(ii) Find the probability that the selected defective item was produced by the 3rd machine

$$P(D)=P(D\mid M_1)P(M_1)+P(D\mid M_2)P(M_2)$$
$$+P(D\mid M_3)P(M_3)$$
$$=(0.03)(0.25)+(0.05)(0.55)+(0.01)(0.2)$$

$$\Rightarrow \boxed{P(D)=0.037}\ldots\ldots\ldots\ldots\boxed{2}$$

$$P(M_3\mid D)=\frac{P(D\mid M_3)P(M_3)}{P(D)}$$

$$\Rightarrow \boxed{P(M_3\mid D)=\frac{2}{37}=0.05405}\ldots\ldots\boxed{1}$$

(B)

Four balls are selected at random from a collection of 4 White, 5 Red, and 3 Black balls. Find the probability that all selected balls have the same color when:
(i) sampling with replacement
(ii) sampling without replacement

1) With Replacement

$$P(SameColor)=P(4W)+P(4R)+P(4B)\ldots\ldots\boxed{1}$$

$$=\left(\frac{4}{12}\right)^4+\left(\frac{5}{12}\right)^4+\left(\frac{3}{12}\right)^4=0.04639\ldots\ldots\ldots\ldots\boxed{1}$$

1) Without Replacement

$$P(SameColor)=P(4W)+P(4R)\ldots\ldots\ldots\ldots\boxed{1}$$

$$=\frac{{}^4C_4\,{}^5C_0\,{}^3C_0+{}^4C_0\,{}^5C_4\,{}^3C_0}{{}^{12}C_4}=\frac{6}{495}=0.01212\ldots\ldots\boxed{1}$$