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الإسراء

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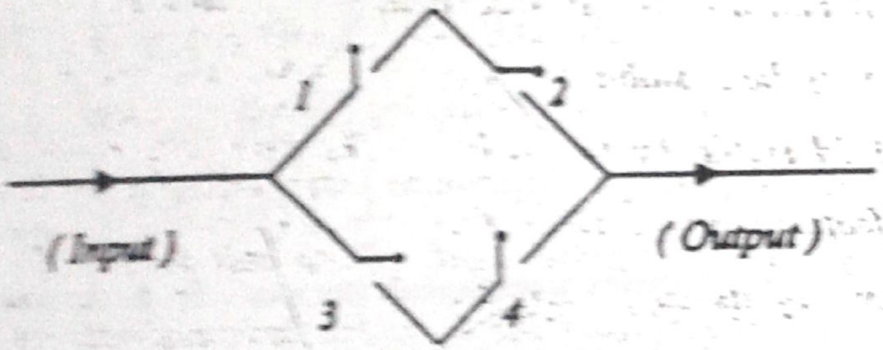
PROBABILITY : BASIC CONCEPTS

EXERCISES (1)

1)



Electric Circuit

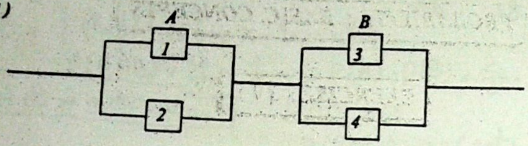
The Figure shows an electric circuit in which each of the switch i is independently closed with probability p_i and open with probability $1-p_i$, $i=1,2,3,4$. If a signal is fed to the input. What is the probability that it is transmitted to the output ?

$$P(12 \cup 34) = P(12) + P(34) - P(1234)$$

$$\boxed{= P_1P_2 + P_3P_4 - P_1P_2P_3P_4}$$

(Answer : $p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4$)

(2)



A machine has two components A and B. The first component A is held stable by two bolts while the second component is also held stable by two bolts. The bolts are redundant safety features, in the sense that the component stays stable if at least one of its supporting bolts stays tight. Each bolt fails with probability p_i, $i=1,2,3,4$, and the bolts fail independently of one another. Calculate the probability that the machine suffers instability.

$$P(A_1A_2 \cup A_3A_4) = P(A_1A_2) + P(A_3A_4) - P(A_1A_2 \cap A_3A_4)$$

$$\boxed{= P_1P_2 + P_3P_4 - P_1P_2P_3P_4}$$

(Answer : $p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4$)

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3) *Two fair coins are tossed. Let,*

A = event that the tail will appear on the second throw,

B = event that the tail will appear on both throws,

Are the two events A and B independent ?

$S = \{HH, Ht, tH, tt\}$

$A = \{Ht, tt\} \Rightarrow P(A) = \frac{2}{4} = \frac{1}{2}$

$B = \{tt\} \Rightarrow P(B) = \frac{1}{4}$;

$A \cap B = \{tt\} \Rightarrow P(A \cap B) = \frac{1}{4}$; $P(A) \cdot P(B) = \frac{1}{8}$

(Answer : A & B are not independent events.) ∴ not indep

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(4) *A pair of dice I, II is thrown. Let,*

A = event that the outcome on die I is less than 5,

B = event that the outcome on die II is more than 5,

C = event that the sum of the outcome is more than 11.

Are the events A & B and B & C independent ?

$S = \{(x,y) \mid x, y = 1, 2, 3, \dots 6\}$

$A = \{(x,y) \mid x = 1, 2, 3, 4 \,;\, y = 1, 2, \dots 6\}$

$P(A) = \frac{24}{36} = \frac{2}{3}$

$B = \{(x,y) \mid y = 6 \,;\, x = 1, 2, 3 \dots 6\}$

$P(B) = \frac{6}{36} = \frac{1}{6}$

$C = \{(5,6),(6,5)\} \Rightarrow P(C) = \frac{2}{36} = \frac{1}{18}$

$A \cap B = \{(x,y) \mid y = 6 ; x = 1, \dots 4\}$

$P(A \cap B) = \frac{4}{36} = \frac{1}{9}$; $P(A) \cdot P(B) = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$

$\therefore$ $\boxed{A, B \Rightarrow \text{independant}}$ $\quad B \cap C = \emptyset$

$\therefore P(B \cap C) = 0$ (not indep)

(Answer : A & B are independent events.

B & C are not independent events.)

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(5) *A lot contains 20 items of which 6 are defective. Three items are drawn at random from the lot one after the other. Find the probability that all the three items are defective.*

$$P(D) = \frac{6}{20} \cdot \frac{5}{19} \cdot \frac{4}{18} = \frac{1}{57}$$

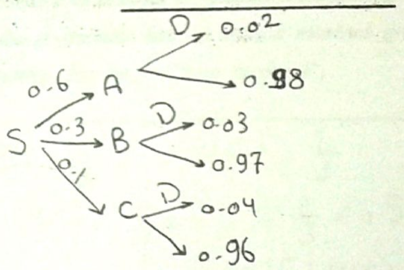
(Answer : $\frac{1}{57}$)

****

TREE - DIAGRAM

EXERCISES (2)

(1) *Three machines A , B and C produce respectively 60 % , 30 % and 10 % of the total number of items of a factory. The percentage of defective output of these machines are 2 % , 3 % and 4 %. If an item is selected at random. Find the probability that the selected item is defective.*



$P(D) = 0.02 * 0.6 + 0.03 * 0.3 + 0.04 * 0.1$

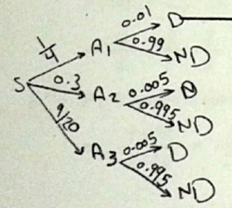
$\boxed{P(D) = 0.025}$

(Answer : 0.025)

(2) *In a certain factory, there are three machines. Given the following table.*

Machine A_i	*No. of items produced / day*	*% Defective items*
1	*1000*	*1.0*
2	*1200*	*0.5*
3	*1800*	*0.5*

An item is selected at random from the daily production of the factory. Find the probability that the selected item is defective. What is the probability that this defective item is of the production of machine A_3 ?



$$P(D) = 0.01 * \frac{1}{4} + 0.005 * 0.3 + 0.005 * \frac{9}{20} = \frac{1}{160}$$

$$P(A_3|D) = \frac{P(D|A_3)\,P(A_3)}{P(D)} = \frac{(0.005)(\frac{9}{20})}{\frac{1}{160}} = 0.36$$

(Answer : 0.00625 – 0.36)

****

The grade distribution of mathematics course of 3000 students for two faculties F_1 and F_2 of Ain Shams University is given by the following table :-

	% Grade			
	A	*B*	*C*	*F*
Ain Shams F_1	*20*	*30*	*40*	*10*
University F_2	*15*	*25*	*50*	*10*

where, F_1 has 1000 students and F_2 has 2000 students., If a student is selected at random from among the 3000 students, find the probability that the student received grade C and the Probability that he was from faculty F_1.

$$P(F_1) = \frac{1}{3} \quad ; \quad P(F_2) = \frac{2}{3}$$

$$P(C) = \frac{1}{3} * 0.4 + \frac{2}{3} * 0.5 = \frac{7}{15}$$

$$P(F_1|C) = \frac{P(C|F_1)\,P(F_1)}{P(C)} = \frac{0.4 * \frac{1}{3}}{\frac{7}{15}} = \frac{2}{7}$$

(Answer : $\frac{7}{15}$; $\frac{2}{7}$)

****

(4) *A box contains 8 radio tubes of which 2 are defective. The tubes are tested one after the other until the two defective tubes are discovered. What is the probability that the process stopped on :*

(i) second test *(ii) third test.*

If the process stopped on the third test, what is the probability that the first tube is non-defective ?

(i) P (process stopped on the ~~third~~ Second test) =

$$P(D_1 \cap D_2) = P(D_1) \cdot P(D_2) = \frac{2}{8} \cdot \frac{1}{7} = \frac{1}{28}$$

(ii) • P (process stopped on the third test) =

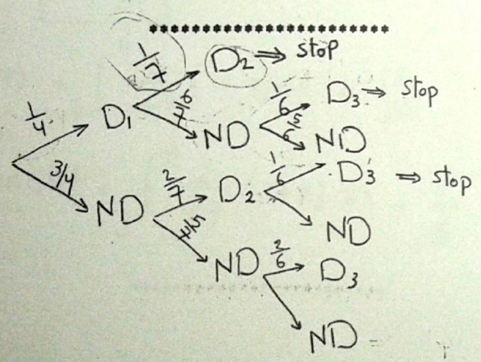
$$P(ND_1 \cap D_2 \cap D_3) + P(D_1 \cap ND_2 \cap D_3)$$

$$= \frac{6}{8} \cdot \frac{2}{7} \cdot \frac{1}{6} + \frac{2}{8} \cdot \frac{6}{7} \cdot \frac{1}{6} = \frac{1}{14}$$

• P (first tube is non-defective | process stopped on the third test)

$$= \frac{1/28}{1/14} = \frac{1}{2}$$

****



RANDOM VARIABLES AND DISTRIBUTIONS

EXERCISES (3)

1) *Three students have interviews scheduled for summer employment at an institute. In each case, the result of the interview will either be that a position is offered or not offered.*

(i) List the experimental outcomes.

(ii) Define a r. v. that represents the number of offers made.

(iii) Show what value the r. v. will assume for each of the experimental outcomes.

****

2) *A continuous r. v. x has a density function given by :-*

$$f(x) = \begin{cases} C & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) C

(ii) $P(2 < x < 2.5)$ *(iii) $P(x < 1.6)$*

$\int_1^3 C\,dx = 1 \Rightarrow Cx\Big|_1^3 = 1$

$C(2) = 1 \Rightarrow \boxed{C = \frac{1}{2}}$

$P(2<x<2.5) = \int_2^{2.5} 0.5\,dx = 0.5x\Big|_2^{2.5} = 0.25$

$P(x<1.6) = \int_1^{1.6} 0.5\,dx = 0.5x\Big|_1^{1.6} = 0.03$

****

(3) *A continuous r. v. x has a density function given by :-*

$$f(x) = \begin{cases} C(1+x) & 2<x<5 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) C

(ii) $P(x<4)$ (iii) $P(3<x<4)$

$\int_2^5 C(1+x)\,dx = 1 \Rightarrow C\left(x+\frac{x^2}{2}\right)\Big|_2^5 = 1$

$\therefore \boxed{C = \frac{2}{27}}$

$P(x<4) = \int_2^4 \frac{2}{27}(1+x)\,dx = \frac{16}{27}$

$P(3<x<4) = \int_3^4 \frac{2}{27}(1+x)\,dx = \frac{1}{3}$

****

(4) *A continuous r. v. x has a density function given by :-*

$$f(x) = \begin{cases} k\sqrt{x} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k

(ii) $P(0.3<x<0.6)$

$\int_0^1 k\sqrt{x}\,dx = 1$

$k\left(\frac{x^{3/2}}{3/2}\right)\Big|_0^1 = 1 \Rightarrow \boxed{K = 1.5}$

$P(0.3<x<0.6) = \int_{0.3}^{0.6} \frac{3}{2}\sqrt{x}\,dx$

$= 0.3$

****

(5) *Suppose that X & Y have the joint probability distribution as follows :-*

Y \ X	1	2	3	
1	0	1/6	1/12	$\frac{1}{4}$
2	c	1/9	0	?
3	2/15	1/4	1/18	$\frac{79}{180}$

Calculate (i) c $\left(\frac{1}{5}\right)$ (ii) Marginal probability distributions

(iii) Check independence of X & Y.

$$\sum_{\text{all } x} \sum_{\text{all } y} f(x,y) = 1 \Rightarrow \boxed{C = \frac{1}{5}}$$

y \ x	1	2	3	$f_2(y)$
1	0	1/6	1/12	1/4
2	1/5	1/9	0	14/45
3	2/15	1/4	1/18	79/180
$f_1(x)$	1/3	19/36	5/36	1

$$F(1,1) = 0 \neq f_1(1) \cdot f_2(1) = \frac{1}{4} \cdot \frac{1}{3}$$

$\therefore$ x, y are not indep

****

(6) *Suppose that X & Y have the joint probability distribution as follows :-*

Y \ X	2	4
1	0.10	0.15
2	0.20	C
3	0.10	0.15

Calculate (i) c (ii) Marginal probability distributions (iii) Check independence of X & Y.

$$\sum_{\text{all } x} \sum_{\text{all } y} f(x,y) = 1 \Rightarrow C = 0.3$$

y \ x	2	4	$F_2(y)$
1	0.1	0.15	0.25
2	0.2	0.3	0.5
3	0.1	0.15	0.25
$f_1(x)$	0.4	0.6	1

$F(2,1) = 0.1$

$f_1(2) = 0.4$

$f_2(1) = 0.25$

$f_1(2) \cdot f_2(1) = F(2,1)$

$\therefore$ x, y are indep

****

(7) *Two dice are thrown. The random variable X represents the minimum of the two faces and the random variable Y represents the number of sixes. Find*

(i) The probability distribution of the two random variables X & Y.

(ii) The marginal probability distribution of X and Y.

(iii) The probability that X will exceed 5 given that Y = 1.

y \ x	1	2	3	4	5	6	$F_2(y)$
0	9/36	7/36	5/36	3/36	1/36	0	25/36
1	2/36	2/36	2/36	2/36	2/36	0	10/36
2	0	0	0	0	0	1/36	1/36
$F_1(x)$	11/36	9/36	7/36	5/36	3/36	1/36	1

$$P(x>5 \mid y=1) = \frac{P(x>5 \cap y=1)}{P(y=1)} = \frac{0}{10/36} = 0$$

****

(8) Two dice are thrown. The random variable X represents the square difference between the two faces and the random variable Y represents the number of sixes. Find

(i) The probability distribution of the two random variables X & Y.

(ii) The marginal probability distribution of X and Y.

(iii) The probability that X will exceed 9 given that $Y = 2$.

$X = \{25, 16, 9, 4, 1, 0\}$

$y = \{0, 1, 2\}$

y \ x	0	1	4	9	16	25	$f_2(y)$
0	5/36	8/36	6/36	4/36	2/36	0	25/36
1	0	2/36	2/36	2/36	2/36	2/36	10/36
2	1/36	0	0	0	0	0	1/36
$f_1(x)$	6/36	10/36	8/36	6/36	4/36	2/36	1

$P(x > 9 \mid Y = 2) = 0$

(9) The continuous r. v. x & y have a density function given by :-

$$f(x,y) = \begin{cases} C(6-x-y) & 0<x<2 \;\&\; 2<y<4 \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

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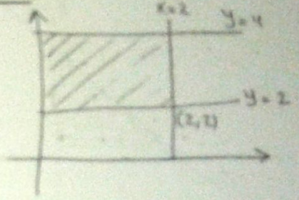
i] $\int_2^4 \int_0^2 C(6-x-y)\, dx\, dy = 1$



$C \int_2^4 \left(6x - \frac{x^2}{2} - xy\right)\Big|_0^2 dy = 1$

$C \int_2^4 (10-2y)\, dy = 1 \Rightarrow \boxed{C = \frac{1}{8}}$

ii] $f_1(x) = \int_2^4 f(x,y)\, dy$

$= \int_2^4 \frac{1}{8}(6-x-y)\, dy = \boxed{\frac{3-x}{4}}$

$f_2(y) = \int_0^2 \frac{1}{8}(6-x-y)\, dx = \boxed{\frac{5-y}{4}}$

$f(x,y) \neq f_1(x) \cdot f_2(y)$

$\boxed{\therefore x \,\&\, y \text{ are dep}}$

(10) The continuous r. v. x & y have a density function given by :-

$$f(x,y) = \begin{cases} C & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

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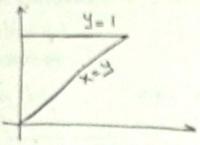
① $\int_0^1 \int_0^y C \, dx \, dy = 1$



$C * \frac{1}{2} = 1 \rightarrow \boxed{C = 2}$

② $f_1(x) = \int_x^1 f(x,y)\, dy = \int_x^1 2\, dy = 2(1-x)$

$f_2(y) = \int_0^y 2\, dx = 2y$

③ $f_1(x) \cdot f_2(x) = 4y(1-x) \neq f(x,y) = 2$

$\therefore$ $\boxed{\text{X \& y are not indep}}$

11) The continuous r. v. x & y have a density function given by :-

$$f(x,y) = \begin{cases} Cx & 0<x<1 \ \& \ 0<y<1-x \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

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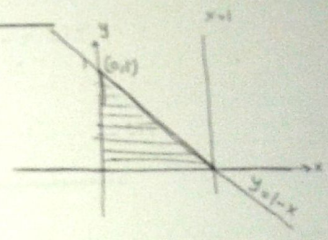
① $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$



$\int_0^1 \int_0^{1-y} Cx \, dx\, dy = 1$

$\frac{C}{2} \int_0^1 (1-2y+y^2)\, dy = 1$

$\boxed{C = 6}$

② $f_1(x) = \int_0^{1-x} f(x,y)\, dy = \int_0^{1-x} 6x\, dy = 6x(1-x)$

$f_2(y) = \int_0^{1-y} 6x\, dx = 3(1-y)^2$

③ $f_1(x)\, f_2(y) = 18x(1-x)(1-y)^2 \neq f(x,y) = 6x$

$\therefore$ X & y are not indep

STANDARD DEVIATION & CORRELATION COEFFICIENT

EXERCISES (4)

(1) *The probability distribution of the continuous r. v. x is :-*

$$f(x) = \begin{cases} c(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) $Var.(x)$ (iii) σ_x

(i) $\int_0^1 c(1-x)\,dx = 1 \Rightarrow c\left(x - \frac{x^2}{2}\right)\Big|_0^1 = 1 \Rightarrow \boxed{c=2}$

(ii) $E(x) = 2\int_0^1 (x - x^2)\,dx = \frac{1}{3}$

$E(x^2) = 2\int_0^1 (x^2 - x^3)\,dx = \frac{1}{6}$

$Var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{1}{18}$

(iii) $\sigma_x = \sqrt{Var(x)} = \frac{\sqrt{2}}{6}$

****

(2) *The probability distribution of the continuous r. v. x is :-*

$$f(x) = \begin{cases} k/(1+x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k (ii) $Var.(x)$ (iii) σ_x

$\int_0^1 \frac{k}{1+x^2}\,dx = k\tan^{-1}x\Big|_0^1 = 1$

$\therefore k = \frac{4}{\pi}$

$Var(x) = E(x^2) - (E(x))^2$

$\because E(x) = \int_0^1 \frac{4x}{\pi(1+x^2)}\,dx = 0.441$

$E(x^2) = \int_0^1 \frac{4x^2}{\pi(1+x^2)}\,dx = 0.273$

$\therefore Var(x) = 0.0788$

$\sigma_x = \sqrt{Var(x)} = 0.281$

****

(3) *Let x represent the outcome when a balanced die is tossed.*

Find (i) $E(x)$ (ii) $E(2x^2-5)$ (iii) σ_x

$x = \{1, 2, 3, 4, 5, 6\}$

i) $E(x) = \sum_{all\ x} x f(x)$

$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + 1 = \frac{21}{6}$

(ii) $E(2x^2-5) = 2E(x^2) - 5 = 2\left[\frac{91}{6}\right] - 5 = \frac{76}{3}$

(iii) $\sigma_x = \sqrt{Var(x)} = \sqrt{E(x^2) - (E(x))^2} = \sqrt{\frac{91}{6} - \left(\frac{21}{6}\right)^2}$

$\sigma_x = \sqrt{105}/6$

****

(4) *A continuous r. v. x has a density function given by :-*

$$f(x) = \begin{cases} C\,(1+x) & 2<x<5 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) $E(2x)$ (iii) σ_{2x}

(i) $\int_2^5 C(1+x)\,dx = 1 \Rightarrow C = \frac{2}{27}$

(ii) $E(x^2) = \int_2^5 \frac{2}{27}(x^2+x^3)\,dx = \frac{85}{6}$

$E(x) = \int_2^5 \frac{2}{27}(x+x^2)\,dx = \frac{11}{3}$

$E(2x) = 2E(x) = \frac{22}{3}$ #

(iii) $\sigma_{2x} = \sqrt{Var(2x)} = \sqrt{4\,Var(x)} = 2\sqrt{Var(x)}$

$= 2\sqrt{E(x^2) - (E(x))^2} = \frac{\sqrt{26}}{3}$ #

****

(5) *Show that* $Cov.(ax, by) = ab\, Cov.(x,y)$

$Cov(ax,by) = E(ax\,by) - E(ax)\cdot E(by)$

$= ab\,E(xy) - a\,E(x)\cdot b\,E(y)$

$= ab\,[E(xy) - E(x)E(y)] = ab\,Cov(x,y)$

****

(6) *Let x & y be independent random variables with variances* $\sigma_x^2 = 5$ *and* $\sigma_y^2 = 3$. *Find the variance of the random variable* $Z = -2x + 4y - 3$.

$Var(Z) = Var(-2x + 4y - 3)$

$= Var(2x) + Var(4y) + Var(-3)$

$= 4\,Var(x) + 16\,Var(y)$

$= 4\times 5 + 16\times 3 = 68$

****

(7) *Let x represent the number that occurs when a green die is tossed, and y represent the number occurs when a red die is tossed. Find the variance of the random variables :* $2x - y$ *and* $x + 3y - 5$.

$Var(2x-y) = 4\,Var(x) + Var(y)$

****

(8) *A continuous r. v. x has a density function given by :-*

$$f(x,y) = \begin{cases} c\,xy & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) Var.(2x) (iii) $\rho_{x,y}$

$$\int_0^1\int_0^1 cxy\,dx\,dy = 1 \rightarrow \frac{1}{2} * \frac{1}{2} c = 1 \rightarrow \boxed{c = 4}$$

$$f_1(x) = \int_0^1 4xy\,dy = 2x$$

$$f_2(y) = \int_0^1 4xy\,dy = 2y$$

$$Var(2x) = 4Var(x) =$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = 0.056$$

$$\boxed{Var(2x) = \frac{2}{9}}$$

$$\rho_{x,y} = \frac{\delta_{xy}}{\sigma_x \sigma_y}$$

$$\delta_{x,y} = E(xy) - E(x)E(y)$$

$$= \frac{4}{9} - \frac{2}{3} * \frac{2}{3} = 0$$

$$\therefore \boxed{\rho_{x,y} = 0}$$

****

(9) *A continuous r. v. x has a density function given by :-*

$$f(x) = \begin{cases} k\sqrt{x} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k (ii) E(2x) (iii) σ_{2x}

$$\int_0^1 k\sqrt{x}\,dx = 1 \Rightarrow \boxed{k = \frac{3}{2}}$$

$$E(2x) = 2E(x) = 2\int_0^1 \frac{3}{2} x\sqrt{x}\,dx = \frac{6}{5}$$

$$\sigma_{2x} = \sqrt{Var(2x)} = 2\sqrt{Var(x)}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \frac{3}{7}$$

$$\therefore Var(x) = \frac{3}{7} - \left(\frac{3}{5}\right)^2 = 0.0686$$

$$\therefore \sigma_{2x} = 2\sqrt{Var(x)} = 0.524$$

****

(10) *Suppose that X & Y have the joint probability distribution as follows :-*

Y \ X	1	2	3
1	0	1/6	1/12
2	1/9	c	0
3	2/15	1/4	1/18

Calculate (i) c (ii) $\rho_{x,y}$

$$\sum_{\text{all } x}\sum_{\text{all } y} f(x,y) = 1$$

$$\therefore \frac{1}{9} + \frac{2}{15} + \frac{1}{6} + c + \frac{1}{4} + \frac{1}{12} + 0 + \frac{1}{18} = 1$$

$$\therefore \boxed{c = \frac{1}{5}}$$

$$\rho_{x,y} = \frac{\delta_{x,y}}{\sigma_x \sigma_y}$$

$$\delta_{x,y} = E(xy) - E(x)E(y)$$

$$= \frac{721}{[illegible]} - \frac{341}{[illegible]} * \frac{197}{[illegible]} = -0.1412$$

$\sigma_x = \sqrt{Var x} = \sqrt{E(x^2) - (E(x))^2}$

$= \sqrt{\frac{713}{180} - \left(\frac{314}{180}\right)^2} = 0.958$

$\sigma_y = \sqrt{Var(y)} = \sqrt{E(y^2) - (E(y))^2}$

$= \sqrt{\frac{49}{9} - \left(\frac{197}{90}\right)^2} = 0.808$

$\therefore \boxed{\rho_{x,y} = -0.182}$

****

(11) *Suppose that X & Y have the joint probability distribution as follows :-*

Y \ X	2	4	
1	c	0.15	0.45
2	0.20	0.10	0.3
3	0.10	0.15	0.25
	0.6	0.4	

Calculate (i) c (ii) $\rho_{x,y}$

$\sum_{all\ x} \sum_{all\ y} f(x,y) = 1$

$\therefore c + 0.15 + 0.20 + 0.1 + 0.1 + 0.15 = 1 \rightarrow \boxed{c = 0.30}$

$\rho_{x,y} = \frac{S_{x,y}}{\sigma_x \sigma_y}$

$S_{x,y} = E(xy) - E(x)E(y)$

$= \frac{26}{5} - \frac{14}{5} * \frac{9}{5}$

$\sigma_x = \sqrt{Var(x)} = \sqrt{E(x^2) - (E(x))^2} = \sqrt{8.8 - (2.8)^2} = 0.$

$\sigma_y = \sqrt{Var(y)} = \sqrt{E(y^2) - (E(y))^2} = \sqrt{3.9 - (1.8)^2} = 0.8$

$\therefore \boxed{\rho_{x,y} = 0.201}$

****

(12) *Two dice are thrown. The random variable X represents the maximum of the two faces and the random variable Y represents the number of sixes. Find*

(i) The probability distribution of the two random variables X & Y

(ii) $\rho_{x,y}$

$X = \{1, 2, 3, 4, 5, 6\}$ $Y = \{0, 1, 2\}$

Y \ X	1	2	3	4	5	6	$f_2(y)$
0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	0	$\frac{25}{36}$
1	0	0	0	0	0	$\frac{10}{36}$	$\frac{10}{36}$
2	0	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$
$f_1(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

$\rho_{x,y} = \frac{S_{x,y}}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sqrt{Var(x)}\sqrt{Var(y)}}$

$E(x) = \frac{161}{36}$; $E(x^2) = \frac{791}{36}$; $E(y) = \frac{1}{3}$; $E(y^2) = \frac{7}{18}$

****

$\therefore Var(x) = 1.97$; $Var(y) = \frac{5}{18}$; $E(xy) = 2$

$\therefore \rho_{x,y} = \frac{0.509}{} = 0.6881$

SPECIAL PROBABILITY DISTRIBUTIONS

EXERCISES (5)

(1) In a single toss of 6 fair coins, find the probability of :
(i) 2 Or more heads (ii) fewer than 4 heads.

Hint : Use Binomial distribution.

$n = 6$ Prob of Head $= P = \frac{1}{2} \Rightarrow q = \frac{1}{2}$

- $F(x) = P(X = x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

- $x \geq 2 \Rightarrow F(x) = P(X \geq 2) = 1 - P(X<2) =$
$1 - P(X=1) - P(X=0) = 1 - \frac{3}{32} - \frac{1}{64} = \frac{57}{64}$

- $X<4 \Rightarrow F(x) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$
$= \frac{21}{32}$

****

(2) What is the probability of getting 9 exactly once in 3 throws of a pair of dice ?

Hint : Use Binomial distribution.

$n = 3 \quad P = \frac{4}{36} = \frac{1}{9} \Rightarrow q = \frac{8}{9}$

$F(x) = P(X=1) = {}^{3}C_{1} \left(\frac{1}{9}\right)^{1} \left(\frac{8}{9}\right)^{2} = 0.2634$

****

(3) Find the probability of guessing at least 6 of the 10 answers on a true false examination.

Hint : Use Binomial distribution.

$n = 10 \quad P = 0.5 \rightarrow q = 0.5$

$(X \geq 6) \Rightarrow$

$F(x) = P(X \geq 6) = P(X=10) + P(X=9)$
$+ P(X=8) + P(X=7) + P(X=6)$

$P(x) = \frac{193}{512} = 0.377$

****

(4) An insurance salesman sells policies to 5 men, all of identical age and in good health. According to the actuarial tables the probability that a man of this particular age will be alive 30 years hence is $\frac{2}{3}$. Find the probability that in 30 years
(i) all 5 men (ii) at least 1 man
will be alive.

Hint : Use Binomial distribution.

$n=5 \quad p=\frac{2}{3} \rightarrow q=\frac{1}{3}$

*① $x=5$

$F(x)=P(x=5)={}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0=\left(\frac{2}{3}\right)^5$

*② $x\geq 1$

$F(x)=P(x\geq 1)=1-P(x<1)=1-P(x=0)$

$=1-{}^5C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5=\frac{242}{243}$

....

(5) *If 3% of the electric bulbs manufactured by a company are defectives. Find the probability that in a sample of 100 bulbs:*

(i) more than 5 *(ii) between 1 and 3*

(iii) less than or equal to 2 *bulbs will be defective.*

Hint: Use Poisson distribution.

$n=100 \quad P=0.03 \rightarrow \lambda=np=3$

① $x>5$

$F(x)=P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4)+P(x=5)$

$=\frac{3^0e^{-3}}{0!}+\frac{3e^{-3}}{1}+\frac{3^2e^{-3}}{2!}+\frac{3^3e^{-3}}{3!}+\frac{3^4e^{-3}}{4!}+\frac{3^5e^{-3}}{5!}$

$=0.916$

② $1<x<3$

$F(x)=P(x=2)=\frac{3^2e^{-3}}{2!}=0.224$

③ $x\leq 2$

$F(x)=P(x=2)+P(x=1)+P(x=0)$

$=0.423$

(6) *If Z has standard normal distribution. Find*

(i) $P(Z\geq -1.64)$ *(ii) $P(-1.96\leq Z\leq 1.96)$*

(iii) $P(|Z|\geq 1)$.

① $P(Z\geq -1.64)=P(Z\leq 1.64)=\Phi(1.64)$

$=0.9495$

② $P(-1.96\leq Z\leq 1.96)=\Phi(1.96)-\Phi(-1.96)$

$=\Phi(1.96)-[1-\Phi(1.96)]=2\Phi(1.96)-1$

$=2*0.9750-1=0.95$

③ $P(|Z|\geq 1)$

....

(7) *Find the values of Z such that:*

(i) The area to the right of Z is 0.2266.

(ii) The area to the left of Z is 0.0314.

(iii) The area between -0.23 and Z is 0.5722.

(iv) The area between 1.15 and Z is 0.0.0730.

(v) The area between -Z and Z is 0.9000.

① $P(Z>z)=0.2266 \rightarrow 1-\Phi(z)=0.2266$

$\therefore \Phi(z)=0.7734 \Rightarrow \boxed{z=0.75}$

② $P(Z<-z)=0.0314$ $\rightarrow P(Z>z)=0.0314$

③ $\Phi(z) + \Phi(0.23) - 1 = 0.5722$

$\Phi(z) = 0.9812 \rightarrow \boxed{z = 2.08}$

④ $\Phi(z) - \Phi(1.15) = 0.0730$

$\Phi(z) = 0.9479 \rightarrow \boxed{z = 1.625}$

⑤ $\Phi(z) - \Phi(-z) = 0.9$

$2\Phi(z) - 1 = 0.9 \rightarrow \Phi(z) = 0.95 \rightarrow$

$\boxed{z \doteq 1.645}$

....

(8) *If the height of 300 students are normally distributed with mean 172 cm., and standard deviation 8 cm. How many students have heights :*

(i) Greater than 184 cm.

(ii) Less than or equal to 160 cm.

(iii) Between 164 and 180 cm.

$n = 300 \quad \mu = 172 \quad \sigma = 8$

① $x > 184$

$F(x) = P(x > 184) = P(Z > \frac{184-172}{8}) = 1 - \Phi(1.5)$

$= 0.0668$; $\boxed{\text{no of studs} = 21 \text{ Student}}$

② $x \leq 160$

$F(x) = P(x \leq 160) = P(Z < \frac{160-172}{8}) = \Phi(-1.5) = 1 - \Phi(1.5)$

....

$= 0.0668 \rightarrow \boxed{\text{no of sts} = 21 \text{ student}}$

③ $164 < x < 180$

$P(-1 < Z < 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1$

(9) *The marks of 300 students of Ain Shams University in mathematics examination are normally distributed with mean 172 marks, and standard deviation 8 marks. Determine how many students have marks :*

(I) Between 164 and 180 marks.

(II) Less than or equal to 164 marks.

(Given that $\Phi(-1) = 0.1587$)

Hint : $\Phi(1) + \Phi(-1) = 1$

$\mu = 172 \quad \sigma = 8 \quad n = 300$

① $164 < x < 180$

$P(\frac{164-172}{8} < Z < \frac{180-172}{8}) = P(-1 < Z < 1)$

$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826$

$\boxed{\text{no of students} = 205 \text{ students}}$

② $x \leq 164$

$P(Z \leq \frac{164-172}{8}) = P(Z \leq -1) = \Phi(-1)$

$= 1 - \Phi(1) = 0.1587$

$\boxed{\text{no. of students} = 48 \text{ student}}$

....

QUALITY CONTROL

EXERCISES (6)

(1) ***The lengths of items produced on a machine are distributed normally with mean 4.05 cm and a standard deviation of 0.04 cm. What is the probability that the mean of a sample of 16 items taken at random differs from the over all mean by more than 0.01 cm ?***

$\mu = 4.05$ $\quad \sigma = 0.04$ $\quad n = 16$

let x = mean of a random sample of size 16 $(n=16)$

$x \sim N(\mu , \frac{\sigma}{\sqrt{n}}) \Rightarrow x \sim N(4.05 , 0.01)$

req Prob $= P(|x-\mu| > 0.01)$

$= P(x-\mu > 0.01) + P(x-\mu < -0.01)$

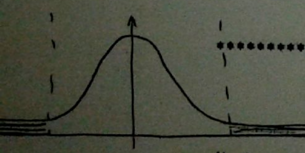
$= P(x > 4.06) + P(x < 4.04)$

$z = \frac{x-\mu}{\frac{\sigma}{\sqrt{n}}} \Rightarrow$ ∴ req prob $= P(z>1) + P(z<-1)$

$= 2(1-\Phi(1)) = 0.3174$

(Answer: $P(-1 < z < 1) = 0.6826$)

غلط



4.04 4.05 4.06

(2) ***A process produces items whose lengths are distributed normally with mean 1.5 units and standard deviation of 0.04 units. Samples of 16 items are taken at random. Between what limits do 90 % of the samples has a mean value μ = 1.5 units and standard error = 0.01 units.***

$\mu = 1.5$ $\quad \sigma = 0.04$ $\quad n = 16$

X = mean of a random sample of 16 item $(n=16)$

$X \sim N(1.5 , 0.01)$

$z = \frac{X-\mu}{\frac{\sigma}{\sqrt{2}}} = \frac{X-1.5}{0.01}$

let 90% of the samples are around the mean value by a value (a)

$P(\mu - a < X < \mu + a) \rightarrow P(\frac{-a}{0.01} < z < \frac{a}{0.01}) = 0.9$

(Answer : 1.484 & 1.516)

$2\Phi(\frac{a}{0.01}) - 1 = 0.9 \rightarrow \frac{a}{0.01} = 1.645 \rightarrow a = 0.01645$

∴ $\boxed{\text{limits are } 1.5 \pm 0.01645}$

(3) ***A process produces items whose lengths are distributed normally with mean 0.25 units and standard deviation of 0.05 units.***
(i) Calculate control limits for sample of size 25
(ii) Prepare a control chart.

$\mu = 0.25$, $\sigma = 0.05$

① $n = 25$

- Action limits $= \mu \pm 3.09 \frac{\sigma}{\sqrt{n}} = 0.2809$ or 0.

- Warning limits $= \mu \pm 1.96 \frac{\sigma}{\sqrt{n}} = 0.2696$ or 0.2

UAL = 0.2809

UWL = 0.2696

μ = 0.25

LWL = 0.2304

LAL = 0.2191

(Answer : 0.2809 , 0.2696 , 0.25 , 0.2304 , 0.2191)

****

(4) ***The lengths of items produced on a machine are distributed normally with standard deviation of 0.01 cm. What is the minimum size of a sample so that the standard error does not exceed 0.001 cm ?***

$\sigma = 0.01$. Standard error $\equiv$ standard div of a sample $= \frac{\sigma}{\sqrt{n}}$

$\frac{\sigma}{\sqrt{n}} \leqslant 0.001 \Rightarrow \frac{0.01}{\sqrt{n}} \leqslant 0.001$

$\sqrt{n} \geqslant 10 \rightarrow \boxed{n \geqslant 100}$

(Answer : $n > 100$)

****

MOMENT GENERATING FUNCTION

EXERCISES (7)

(1) *Let Y be a r. v. for the distribution whose probability density function $f(Y)$ is given by :-*

$$f(Y) = \frac{(\frac{1}{\beta})^{Y} e^{-\frac{1}{\beta}}}{Y!} \quad , \quad \beta > 0 \ , \ Y = 0,1,2,\dots$$

Find (i) $m_Y(t)$ (ii) $E(Y)$ (iii) $Var.(Y)$

Hint : Poisson distribution with parameter $\frac{1}{\beta}$.

****

Third :

Evaluate each of the following integrals by using gamma and beta functions :

1) $\int_0^1 (\ln x)^{5/3}\, dx$

let $x = e^{-t}$, $dx = -e^{-t}dt$, $dt = -\frac{dx}{x}$

$\ln x = -t$

$x : 0 \longrightarrow 1$

$t : \infty \longrightarrow 0$

$$I = \int_\infty^0 (-t)^{5/3} \cdot (-e^{-t}dt)$$

$$= -\int_0^\infty t^{5/3} e^{-t} dt$$

$x - 1 = \frac{5}{3}$

$x = \frac{8}{3}$

$$= -\Gamma(\tfrac{8}{3})$$

2) $\int_0^\infty x^2 e^{-x^2} dx$

let $x^2 = t$, $2x\,dx = dt$

$x = t^{1/2}$, $dx = \frac{1}{2} t^{-1/2} dt$

$x : 0 \rightarrow \infty$

$t : 0 \rightarrow \infty$

$$I = \int_0^\infty t\, e^{-t} \cdot (\tfrac{1}{2} t^{-1/2} dt)$$

$$= \frac{1}{2}\int_0^\infty t^{1/2} e^{-t} dt$$

$x - 1 = \frac{1}{2}$

$x = \frac{3}{2}$

$$= \frac{1}{2}\Gamma(\tfrac{3}{2}) = \frac{1}{2} \cdot \frac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \frac{\sqrt{\pi}}{4}$$

3) $\int_0^{\pi/2} \sin^{3.04} x \, dx$

$$\beta(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cdot \cos^{2y-1}\theta \, d$$

$$I = \frac{\beta(2.02, 0.5)}{2}$$

$2x - 1 = 3.04$

$x = 2.02$

$2y - 1 = \text{zero}$

$y = 1/2$

$$= \frac{\Gamma(2.02)\cdot\Gamma(0.5)}{2\,\Gamma(2.52)}$$

$$= \frac{\sqrt{\pi}}{2} \; \frac{1.01\,\Gamma(1.01)}{1.52\,\Gamma(1.52)}$$

4) $\int_0^2 x(8 - x^3)^{1/3} dx$

let $x^3 = 8t$, $t = \frac{x^3}{8}$

$x = 2t^{1/3}$ $\quad dt = \frac{3}{8}x^2 dx$

$x: 0 \to 2$ $\quad dx = \frac{2}{3} t^{-2/3} dt$

$t: 0 \to 1$

$$I = \int_0^1 (2t^{1/3})\,(8 - 8t)^{1/3} \cdot \left(\frac{2}{3} t^{-2/3} dt\right)$$

$$\frac{2 * 2 * 2}{3}\int_0^1 t^{-1/3}(1-t)^{1/3} dt = \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3} dt$$

$x - 1 = -1/3$

$x = \frac{2}{3}$

$y - 1 = 1/3$

$y = 4/3$

$$I = \frac{8}{3}\,\beta\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$= \frac{8}{3}\,\frac{\Gamma(\frac{2}{3})\cdot\Gamma(\frac{4}{3})}{\Gamma(2)} = \frac{8}{9}\,\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{1}{3}\right)$$

$$= \frac{8}{9}\,\frac{\pi}{\sin(\frac{\pi}{3})} = \frac{16\pi}{9\sqrt{3}}$$

5) $\int_0^1 \sqrt[n]{1-x^n}\, dx$

let $x^n = t$,

$x = t^{1/n}$, $dx = \frac{1}{n}\, t^{1/n - 1} dt$

$x : 0 \to 1$, $t : 0 \to 1$

$I = \int_0^1 (1-t)^{1/n} \cdot \frac{1}{n} (t)^{\frac{1}{n}-1} \cdot dt$

$= \frac{1}{n} \int_0^1 (t)^{\frac{1}{n}-1} \cdot (1-t)^{\frac{1}{n}} dt$

$\frac{1}{n} - 1 = x - 1$

$x = \frac{1}{n}$

$\frac{1}{n} = y - 1$

$y = \frac{1}{n} + 1$

$= \frac{1}{n}\, \beta(\frac{1}{n}, \frac{1}{n}+1)$

$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\Gamma(\frac{1}{n}+1)}{\Gamma(\frac{2}{n}+1)}$

$= \frac{1}{n} (\frac{1}{n}) (\frac{n}{2}) \frac{\Gamma(\frac{1}{n})\, \Gamma(\frac{1}{n})}{\Gamma(\frac{2}{n})} = \frac{1}{2n} \frac{\Gamma^2(\frac{1}{n})}{\Gamma(\frac{2}{n})}$

6) $\int_0^\infty \frac{dx}{\sqrt{1+x^4}}$

$\beta(x, y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du$

let $u = x^4$, $x = u^{1/4}$, $dx = \frac{1}{4} u^{-3/4} du$

$I = \int_0^\infty \frac{1}{(1+u)^{1/2}} \cdot \frac{1}{4} u^{-3/4} du$

$x - 1 = -\frac{3}{4}$ | $x + y = \frac{1}{2}$

$x = \frac{1}{4}$ | $y = \frac{1}{4}$

$= \frac{1}{4} \int_0^\infty \frac{u^{-3/4}}{(1+u)^{1/2}} du = \frac{1}{4} \beta(\frac{1}{4}, \frac{1}{4})$

$= \frac{1}{4} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})} = \frac{1}{4\sqrt{\pi}} (16\, \Gamma^2(\frac{5}{4}))$

$= \frac{4}{\sqrt{\pi}} \Gamma^2(\frac{5}{4})$

$$\beta(x,y) = \frac{1}{2}\int_0 \sin^{2x-1}\theta \cdot \cos$$

7)* $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

let $x^4 = \tan^2\theta$

$x^2 = \tan\theta$, $2x\,dx = \sec^2\theta\, d\theta$

$x = \tan^{1/2}\theta \Rightarrow dx = \frac{1}{2}\tan^{-1/2}\theta \cdot \sec^2\theta \cdot d\theta$

$x: 0 \to \theta = 0$, $x: 1 \to \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{2}\tan^{-1/2}\theta \cdot \sec^2\theta \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \cos\theta \cdot \left(\frac{\sin\theta}{\cos\theta}\right)^{-1/2} \cdot \left(\frac{1}{\cos\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (\sin\theta)^{-1/2} \cdot (\cos\theta)^{-1/2}\, d\theta$$

$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{\sin 2\theta}{2}\right)^{-1/2} d\theta = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}(2\theta) \cdot d(2\theta)$$

$2x-1 = \frac{-1}{2}$, $x = \frac{3}{4}$ | $2y-1 = 0$, $y = \frac{1}{2}$

$$I = \frac{1}{2\sqrt{2}} \frac{\beta(\frac{3}{4}, \frac{1}{2})}{2} = \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{\Gamma(\frac{7}{4})}{(3/4)} \cdot \frac{1}{\Gamma(\frac{5}{4})}$$

$$= 3\sqrt{\frac{\pi}{2}} \cdot \frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}$$

8) Find $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \cdot \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$$I = \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta - \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$$

$2x-1 = \frac{-1}{2}$, $2y-1 = 0$ | $2x-1 = \frac{1}{2}$, $2y-1 = 0$

$x = \frac{1}{4}$, $y = \frac{1}{2}$ | $x = \frac{3}{4}$, $y = \frac{1}{2}$

$$I = \frac{\beta(\frac{1}{4}, \frac{1}{2})}{2} \cdot \frac{\beta(\frac{3}{4}, \frac{1}{2})}{2}$$

$$= \frac{1}{4}\left(\frac{\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{1}{2})}{\cancel{\Gamma(\frac{3}{4})}} \cdot \frac{\cancel{\Gamma(\frac{3}{4})} \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})}\right)$$

$$= \frac{\pi}{4}\left(\frac{\cancel{\Gamma(\frac{1}{4})}}{(\frac{1}{4})\cancel{\Gamma(\frac{1}{4})}}\right) = \boxed{\pi}$$

Final Answers:

(1) $-\Gamma\left(\frac{8}{3}\right)$ (2) $\frac{\sqrt{\pi}}{4}$ (3) $\frac{1}{2}B\left(2.02, \frac{1}{2}\right) \cong 0.663$

(4) $\frac{16\pi}{9\sqrt{3}}$ (5) $\frac{1}{2n} \frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}$ (6) $\frac{1}{4\sqrt{\pi}} \Gamma^2\left(\frac{1}{4}\right)$

(7) $\frac{\sqrt{2\pi}}{8} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ (8) π

* * * *

Functions of a Complex Variable

Chapter I

Analytic Functions

EXERCISES I

pp. 12, 13

First: 1, 3-b, 4-b, 5-b

Final Answers:

3-b) The domain is the entire z-plane except $z=1$.

$$f(z) = \frac{1-x}{(1-x)^2+y^2} - \frac{y}{(1-x)^2+y^2}\, i$$

4-b) $|\text{Arg } w| \leq \frac{3\pi}{4}$, $|w| \leq 8$

5-b) 2

* * *

Second: 2, 3-c, 4-a, 5-c

Complete Answers:

2) Since $\lim_{n \to \infty} \frac{1}{n} = 0$, the boundary of E is consisting of only one point : $z = 0$.

Note that every neighbourhood of the point $z=0$ contains points in E as well as points not in E.

3 – c) $f(z) = \dfrac{z+i}{(z+i)(z-i)}$

The function is not defined at $z = \pm i$.
When $z \neq \pm i$:

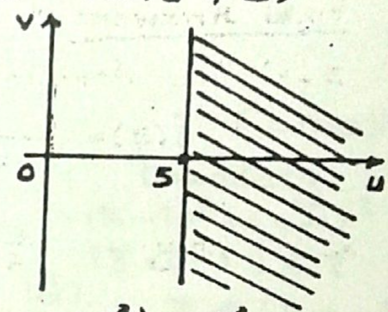
$$f(z) = \frac{1}{z-i} = \frac{1}{x+i(y-1)} \cdot \frac{x-i(y-1)}{x-i(y-1)}$$

$$= \frac{x}{x^2+(y-1)^2} - \frac{y-1}{x^2+(y-1)^2}\, i .$$

4 – a) Let $w = z + 5 \Rightarrow \operatorname{Re} w = \operatorname{Re}(z+5)$

$\Rightarrow \operatorname{Re} w = 5 + \operatorname{Re} z > 5$

The range is $\operatorname{Re} w > 5$.



5 – c) $\lim = \lim\limits_{\Delta z \to 0} \dfrac{(z_0^2 + 2z_0\Delta z + \Delta z^2) - z_0^2}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} \dfrac{2z_0\Delta z + \Delta z^2}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$

-o-o-o-

Third:

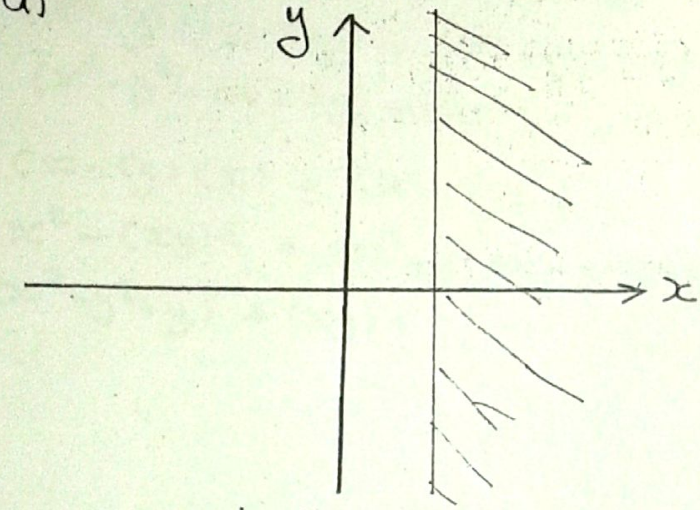
1 – Sketch each of the following sets and determine which of them is open, connected or closed. What is the boundary of each set:

(a) $\operatorname{Re} z > 1$ (b) $|z-2-i| \leq 2$

(c) $|z|<1$ or $|z-4|<1$ (d) $|z+i|>2$

(e) $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}$

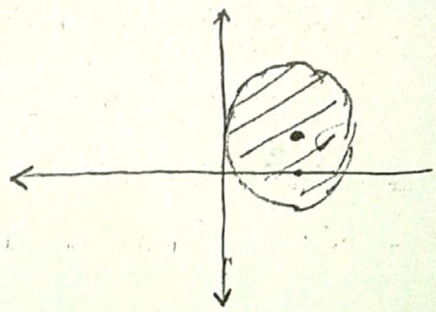
(a)



connected, open
boundary, $\cancel{z} = 1$

(b) $z-2-i = (x-2) + (y-1)i$

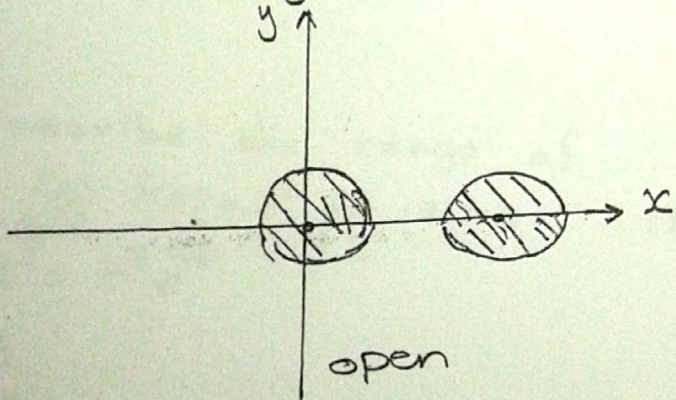
$(x-2)^2 + (y-1)^2 \leq 4$



Connected, closed
boundary
$(x-2)^2 + (y-1)^2 = 4$
$|z-2-i| \leq 2$

(c) $z-4 = (x-4) + iy$

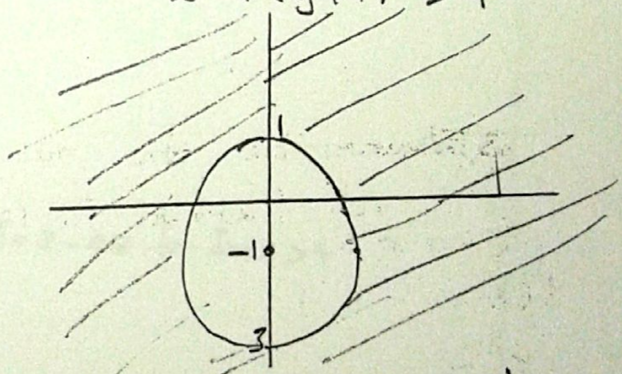
$(x-4)^2 + y^2 < 1$



open

(d) $\cancel{x} + (y+1)i$

$x^2 + (y+1)^2 > 4$



open, Connected

(e) $z = x + iy$

$y = 0, \quad x = \frac{n}{n+1} \quad n = 1, 2, 3, \dots$

$\lim_{n\to\infty} \frac{n}{n+1} = \boxed{1}$

boundary $\boxed{z = 1}$

2 - Express each function in the form $u+iv$:

(a) $f(z) = \bar{z}^2 + (2-3i)z$

(b) $f(z) = \bar{z}\,\text{Re}\,z + z^2 + \text{Im}\,z$

$z = x + iy \quad ; \quad \bar{z} = x - iy$

(a) $f(z) = (x-iy)^2 + (2-3i)(x+iy)$

$= (x^2-y^2) + i(-2xy) + (2x+3y) + (2y-3x)i$

$= (x^2-y^2+2x+3y) + (2y-3x-2xy)i$

(b) $f(z) = (x-iy)(x) + (x+iy)^2 + y$

$= x^2 - (xy)i + (x^2-y^2+y) + (2xy)i$

$= (2x^2-y^2+y) + (xy)i$

3) Describe the range of

(a) $f(z) = z^2$ for $|z|>3$ (b) $f(z) = 3-2z$ for $\text{Im}\,z > 4$

(a)

$z = re^{i\theta} \quad r > 3$

$f(z) = z^2 = r^2 e^{2i\theta}$

$r^2 > 9$

$|w| > 9$

(b) $f(z) = 3 - 2(x+iy) \quad y > 4$

$= (3-2x) - 2iy$

$v = -2y$

$\because y > 4$

$-2y < -8$

$\text{Im}\,w < -8$

(1+4i)

$(-1-i)+$

$-1+i$

43 $-(1+i) - \dfrac{\quad}{(-2-i)} = 1$

4) Find the following limits:

a) $\lim\limits_{z \to i} \dfrac{z^2+4z+2}{z+1}$ (b) $\lim\limits_{z \to i} \dfrac{z^4-1}{z-i}$

c) $\lim\limits_{z \to 1+i} \dfrac{z^2+z-1-3i}{z^2-2z+2}$ (Factorize!!)

(a) $\lim\limits_{z \to i} \dfrac{z^2+4z+2}{z+1} = \dfrac{-1+4i+2}{1+i} * \dfrac{1-i}{1-i} = \dfrac{5+3i}{2}$

(b) $\lim\limits_{z \to i} \dfrac{z^4-1}{z-i} = \dfrac{(z)^4-(i)^4}{(z)-(i)} = \dfrac{4}{1} * z^3 = 4i^3 = -4i$

(c) $\lim\limits_{z \to 1+i} \dfrac{z^2+z-1-3i}{z^2-2z+2} = \dfrac{(1+i)^2+(1+i)-1-3i}{(1+i)^2-2(1+i)+2} = \dfrac{0}{0}$

$= \lim\limits_{z \to 1+i} \dfrac{\cancel{(z-(1+i))}(z-(-2-i))}{\cancel{(z-(1+i))}(z+(-1+i))} = \dfrac{(1+i)-(-2-i)}{(1+i)+(-1+i)}$

$= \dfrac{3+2i}{2i} * \dfrac{-2i}{-2i}$

$= \dfrac{4-6i}{4} = 1-\dfrac{3}{2}i$

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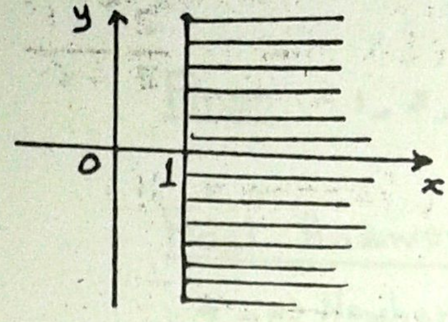
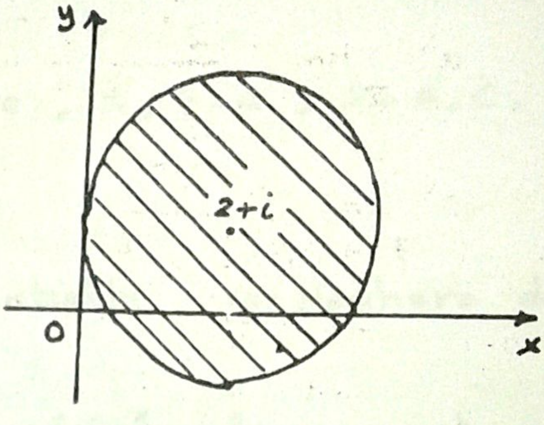
Final Answers:

1 – a) open and connected . The boundary is $Re\, z = 1$



(a)

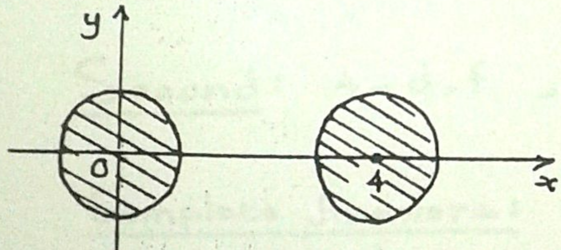


(b)

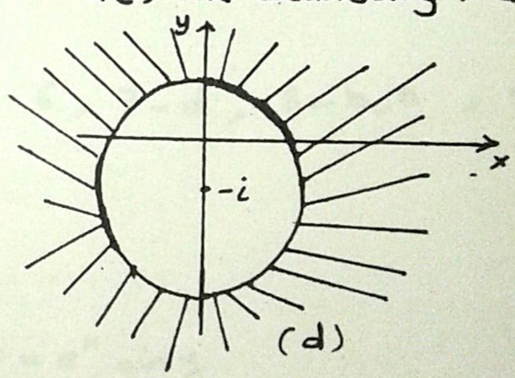
b) Closed and connected : $|z-(2+i)| = 2$

c) open : $|z| = 1$ or $|z-4| = 1$

d) open and connected (e) The boundary : $z = 1$



(c)



(d)

2 – a) $(x^2 + 2x + 3y - y^2) + i(-3x - 2xy + 2y)$

b) $(2x^2 - y^2 + y) + xyi$

3 – a) $|w| > 9$ (b) $\text{Im}\, w < -8$

4 – a) $\frac{1}{2}(5+3i)$ (b) $-4i$ (c) $1 - \frac{3}{2}i$

* * * *

EXERCISES II

pp. 30, 31

First : 1, 2, 3, 4 – a, e , 5 , 7 – a , 8 – a, c.

Final Answers:

4 – a) Nowhere differentiable (e) Nowhere diff.

5) $f(z) = 3z^2 + 2z - 1$

8 – a) $f(z) = 2x(1-y) + i(x^2 - y^2 + 2y + c)$

$= 2z + i(z^2 + c)$

c) $f(z) = \left(\dfrac{-x}{x^2+y^2} + c\right) + \dfrac{iy}{x^2+y^2} = -\dfrac{1}{z} + c$

* * *

Second : 4 – d, f , 6 , 7 – d , 8 – b, e , 9

Complete Answers:

4 – d) $u = e^x \cos y$, $v = e^x \sin y$

$\Rightarrow u_x = e^x \cos y$, $u_y = -e^x \sin y$

$v_x = e^x \sin y$, $v_y = e^x \cos y$

$\Rightarrow u_x = v_y$, $u_y = -v_x$

Cauchy – Riemann equations are satisfied everywhere, and hence the given function is entire.

$$f'(z) = u_x + iv_x = e^x(\cos y + i\sin y)$$

f) $W = (x+iy)^2(x-iy) = (x^2+y^2)(x+iy)$
$\quad = (x^3 + xy^2) + i(x^2y + y^3)$

$$\Rightarrow u = x^3 + xy^2 \;,\quad v = x^2y + y^3$$

$$u_x = 3x^2 + y^2 \;,\quad u_y = 2xy$$
$$v_x = 2xy \;,\quad v_y = x^2 + 3y^2$$

C–R eqns take the form:
$u_x = v_y \Rightarrow 3x^2 + y^2 = x^2 + 3y^2 \Rightarrow x^2 = y^2$

This eqn is satisfied only when $y = \pm x$. (1)

$u_y = -v_x \Rightarrow 2xy = -2xy \Rightarrow xy = 0$
$\Rightarrow x = 0$ or $y = 0$ (2)

From (1), (2) we see that C–R eqns are satisfied only at $z = 0$.
The fn is differentiable at $z = 0$ and nowhere analytic:

$$f'(0) = u_x(0,0) + i\,v_x(0,0) = 0.$$

—o-o-o—

6) $u = e^{x^2-y^2}\cos 2xy \;,\quad v = e^{x^2-y^2}\sin 2xy$

$\Rightarrow u_x = 2x\,e^{x^2-y^2}\cos 2xy - 2y e^{x^2-y^2}\sin 2xy$
$\quad v_y = -2y e^{x^2-y^2}\sin 2xy + 2x e^{x^2-y^2}\cos 2xy$

Hence $\quad u_x = v_y \quad$ for all $x, y \qquad (1)$

$$u_y = -2y e^{x^2-y^2} \cos 2xy - 2x e^{x^2-y^2} \sin 2xy$$

$$v_x = 2x e^{x^2-y^2} \sin 2xy + 2y e^{x^2-y^2} \cos 2xy$$

$\Rightarrow \quad u_y = -v_x \quad$ for all $x, y \qquad (2)$

From (1), (2) we that $f(z)$ is differentiable in the whole z-plane and consequently it is an entire fn.

$$f'(z) = u_x + i v_x$$

$$= 2e^{x^2-y^2}\left[(x \cos 2xy - y \sin 2xy) + i(x \sin 2xy + y \cos 2xy)\right]$$

7-d) Let $\quad f(z) = u(x,y) + i\, v(x,y)$. Then

$$\overline{f(z)} = u - iv = u + i(-v)$$

Since $\overline{f(z)}$ is analytic, it must satisfy C-R eqns:

$u_x = (-v)_y \Rightarrow u_x = -v_y \qquad (1)$

$u_y = -(-v)_x \Rightarrow u_y = v_x \qquad (2)$

$f(z)$ is also analytic. Hence

$u_x = v_y \qquad (3)$

$u_y = -v_x \qquad (4)$

Adding (1), (3) we get $u_x = 0$
Adding (2), (4) we get $u_y = 0$

$$\Rightarrow u = \text{const} = c_1$$

Substituting in (1) and ( ), we find

$$v_y = 0,\ v_x = 0 \Longrightarrow v = \text{const} = c_2$$

Hence $f(z) = c_1 + ic_2 = \text{const}$ ※

-o-o-o-

8-b) $u_x = -\sin x \operatorname{ch} y$, $u_y = \cos x \operatorname{sh} y$.

From C-R eqns:

$$v_y = -\sin x \operatorname{ch} y \Rightarrow v = \int -\sin x \operatorname{ch} y \, dy + h(x)$$

$$\Rightarrow v = -\sin x \operatorname{sh} y + h(x)$$

$$v_x = -u_y \Rightarrow -\cos x \operatorname{sh} y + h'(x) = -\cos x \operatorname{sh} y$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = c.$$

The corresponding analytic fn is

$$w = f(z) = \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y + ic.$$

e) $u = \operatorname{Arg} z = \theta \Rightarrow u_r = 0,\ u_\theta = 1,\ u_{rr} = 0,\ u_{\theta\theta} = 0$

$\Rightarrow u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \Rightarrow u = \theta$ is harmonic.

Using C-R eqns in polar form, we have

$$v_\theta = r u_r \Rightarrow v_\theta = 0 \Rightarrow v = h(r)$$

$$v_r = -\frac{1}{r} u_\theta \Rightarrow h'(r) = -\frac{1}{r} \Rightarrow h(r) = -\ln r + C$$
$$= \ln \frac{1}{r} + C.$$

$$\Rightarrow f(z) = \theta + i\left(\ln \frac{1}{r} + c\right). \qquad (r>0)$$

—o-o-o—

9) Since u and v are conjugate harmonic fns, they satisfy C-R eqns:

$$u_x = v_y \quad , \quad u_y = -v_x \qquad (1)$$

Also, we have

$$u_{xx} + u_{yy} = 0 \quad , \quad v_{xx} + v_{yy} = 0 \qquad (2)$$

$$(uv)_x = uv_x + vu_x \Rightarrow (uv)_{xx} = uv_{xx} + 2u_x v_x + vu_{xx}$$

$$(uv)_y = uv_y + vu_y \Rightarrow (uv)_{yy} = uv_{yy} + 2u_y v_y + vu_{yy}$$

Use (2):

$$(uv)_{xx} + (uv)_{yy} = u(\underbrace{v_{xx} + v_{yy}}_{0}) + 2(u_x v_x + u_y v_y)$$
$$+ v(\underbrace{u_{xx} + u_{yy}}_{0})$$

Use (1):

$$(uv)_{xx} + (uv)_{yy} = 2\left(v_y v_x + (-v_x)v_y\right) = 0$$

Hence uv is a harmonic fn ※

* * *

Third:

1) Use C-R eqns to determine where each fn is differentiable, and where it is analytic. Find $f'(z)$ if it exists:

$z = x + iy$

(a) $w = \text{Im}\, z$

$w = y$

$u = y \; ; \; v = 0$

$u_x = 0 \quad | \quad v_x = 0$

$u_y = 1 \quad | \quad v_y = 0$

$u_x = v_y = 0$

$u_y \neq -v_x$

w is nowhere diffrentiable and not analytic

(b) $w = \dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2}$

$u = \dfrac{x}{x^2+y^2} = \dfrac{r\cos\theta}{r^2} = \dfrac{\cos\theta}{r} \; ; \; v = \dfrac{-y}{x^2+y^2} = \dfrac{-r\sin\theta}{r^2} = \dfrac{-\sin\theta}{r}$

$u_r = \dfrac{-\cos\theta}{r^2} \quad | \quad v_r = \dfrac{\sin\theta}{r^2}$

$u_\theta = \dfrac{-\sin\theta}{r} \quad | \quad v_\theta = \dfrac{-\cos\theta}{r}$

$$\left.\begin{aligned} u_r &= \frac{-\cos\theta}{r^2} = \frac{1}{r} v_\theta \\ u_\theta &= \frac{-\sin\theta}{r} = -r v_r \end{aligned}\right\}$$

w is diffrentiable and analytic

$f'(z) = u_r + i v_r$

$= \dfrac{-\cos\theta}{r^2} + \dfrac{\sin\theta}{r^2} i$

$= \dfrac{-1}{z^2}$

(c) $w = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

$u = x^3 + 3xy^2 - 3x \qquad v = y^3 + 3x^2y - 3y$

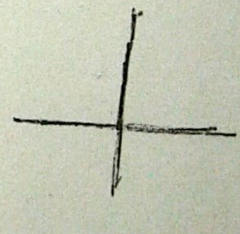
$u_x = 3x^2 + 3y^2 - 3 \quad | \quad v_x = 6xy$

$u_y = 6xy \quad | \quad v_y = 3y^2 + 3x^2 - 3$

$u_x = v_y$ (✓) $\qquad u_y = -v_x$

$6xy = -6xy \qquad xy = 0$

$x = 0 \; , \; y = 0$



Diffrentiable at x Axis or y Axis only and not analytic

$f'(z) = u_x + i v_x$

$= 3x^2 + 3y^2 - 3 + (0 \cdot x y) i$

$= 3z^2 - 3$

d) $f(z) = r^{1/2}\cos\frac{\theta}{2} + i\,r^{1/2}\sin\frac{\theta}{2}$ $\quad (r > 0,\ -\pi < \theta < \pi)$

$u = r^{1/2}\cos\frac{\theta}{2}$ | $v = r^{1/2}\sin\frac{\theta}{2}$

$u_r = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2}$ | $v_r = \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2}$

$u_\theta = \frac{-1}{2}r^{1/2}\sin\frac{\theta}{2}$ | $v_\theta = \frac{1}{2}r^{1/2}\cos(\frac{\theta}{2})$

$$\left.\begin{array}{l} u_r = \frac{1}{2}r^{-1/2}\cos(\frac{\theta}{2}) = \frac{1}{r}v_\theta \\ u_\theta = \frac{-1}{2}r^{1/2}\sin(\frac{\theta}{2}) = -r\,v_r \end{array}\right\}$$ differentiable and analytic fn

$f'(z) = u_r + i v_r$

$= \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2} + (\frac{1}{2}r^{-1/2}\sin\frac{\theta}{2})i$

2). Show that each given fn is harmonic and find the corresponding analytic fn $w = u + iv$:

(a) $u = x^3 - 3xy^2 + y$

$u_x = 3x^2 - 3y^2$ | $u_y = -6xy + 1$

$u_{xx} = 6x$ | $u_{yy} = -6x$

$u_{xx} + u_{yy} = 6x - 6x = 0$ (analytic function)

$v_x = -u_y = 6xy - 1$

$v = 3x^2y - x + h(y)$

$v_y = 3x^2 + h'(y) = u_x = 3x^2 - 3y^2$

$h'(y) = -3y^2$

$h(y) = -y^3$

$v = 3x^2y - x - y^3$

Corresponding analytic fn

$w = (x^3 - 3xy^2 + y) + i(3x^2y - x - y^3)$

(b) $u = \ln^2 r - \theta^2$; $r > 0, -\pi < \theta < \pi$

$u_r = 2\ln(r) * \frac{1}{r}$

$u_{rr} = \frac{2}{r}\left(\frac{1}{r}\right) + 2\ln(r)\frac{-1}{r^2}$

$= \frac{-2}{r^2}(\ln(r) - 1)$

$u_\theta = -2\theta$

$u_{\theta\theta} = -2$

$r^2 u_{rr} + r u_r + u_{\theta\theta} = -2(\ln(r) - 1) + 2\ln(r) - 2 = 0$

(Analytic fn)

$u_\theta = -r v_r$ / $u_r = \frac{1}{r} v_\theta$

$v_r = \frac{-1}{r} u_\theta = + \frac{2\theta}{r}$

$v = 2\theta \ln r + h(\theta)$

$v_\theta = 2\ln r + h'(\theta) = r u_r = 2\ln(r)$

$h'(\theta) = 0$, $h(\theta) = C$

$v = 2\theta \ln(r)$

Corresponding fn.

$w = (\ln^2 r - \theta^2) + i(2\theta \ln r) + C$

(c) $v = e^y \sin x$

$v_x = e^y \cos x$

$v_{xx} = -e^y \sin x$

$v_y = e^y \sin x$

$v_{yy} = e^y \sin x$

$v_{xx} + v_{yy} = e^y(\sin x - \sin x) = 0$

$u_x = v_y = e^y \sin x$

$u = -e^y \cos(x) + h(y)$

$u_y = -e^y \cos(x) + h'(y) = -v_x$

$= -e^y \cos x$

$h'(y) = 0$, $h(y) = C$

$w = (-e^y \cos x + C) + i(e^y \sin x)$

3) Let v be the harmonic conjugate of u. Prove that $h = u^2 - v^2$ is a harmonic function.

$h_x = 2uu_x - 2vv_x$

$h_{xx} = \cancel{2u_x^2} + 2uu_{xx} - \cancel{2v_x^2} - 2vv_{xx}$

$= 2uu_{xx} - 2vv_{xx}$

$h_{yy} = 2uu_{yy} - 2vv_{yy}$

we have u analytic

$u_{xx} + u_{yy} = 0$

$u_{yy} = -u_{xx}$

v analytic

$v_{xx} + v_{yy} = 0$

$v_{xx} = -v_{yy}$

$h_{xx} = \ldots$

$= -(2u u_{yy} - 2v v_{yy})$

$= -h_{yy}$

$h_{xx} + h_{yy} = 0$

h is analytic

4) Let $f(z)$ be analytic and nonzero in a domain $\mathcal{D}$. Prove that $\ln|f(z)|$ is harmonic in $\mathcal{D}$.

$f(z) = u + iv$

$W = \ln(u^2+v^2)^{1/2} = \frac{1}{2}\ln(u^2+v^2)$

$W_x = \frac{1}{2}\left(\frac{2uu_x + 2vv_x}{(u^2+v^2)}\right)$

$W_{xx} = \frac{1}{2}\left(\frac{2(u^2+v^2)(u_x^2+v_x^2+uu_{xx}+vv_{xx}) - (2uu_x+2vv_x)^2}{(u^2+v^2)^2}\right)$

$W_{xx}+W_{yy} = \frac{1}{2}\left[\frac{2(u^2+v^2)(u_x^2+v_x^2+u_y^2+v_y^2+u\cancel{(u_{xx}+u_{yy})}+v\cancel{(v_{xx}+v_{yy})})}{(u^2+v^2)^2}\right.$

$= -4\frac{(uu_x+vv_x)^2+(uu_y+vv_y)^2}{(u^2+v^2)^2}$

$u_x = v_y$, $u_y = -v_x$

$u_{xx} = v_{yx} = v_{xy}$, $u_{yy} = -v_{xy}$

$u_{xy} = v_{yy} = u_{yx}$, $u_{yx} = -v_{xx}$

$W_{xx}+W_{yy} = \frac{1}{2}\left[\frac{4(u^2+v^2)(u_x^2+v_x^2) - 4(u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2))}{(u^2+v^2)^2}\right.$

$= 2\left[\frac{(u^2+v^2)(u_x^2+v_x^2) - \cancel{(u^2+v^2)(u_x^2+v_x^2)}}{(u^2+v^2)^2}\right.$

$=$ zero

$\therefore$ W is harmonic

الاجابة فى ورقة الأسئلة　　　　　　　　　　الزمن : 1 ساعة

الفرقة الثانية كهرباء　　رياضيات هندسية (٣)　　< ١ >　　الامتحان ثلاثة أسئلة فى صفحة واحده

الاسم (باللغة العربية):	الفصل:	رقم البنش:

1. **Evaluate** i) $\int_0^\infty 3^{5-x^2}\,dx$　　ii) $\int_0^{\pi/2} (\tan\theta)^{3/2}\,d\theta$　　$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$

2. **Show that that the function: $u(x, y) = \cos 2x \cosh 2y$ is harmonic. and find its harmonic conjugate. If $f(Z) = u + iv$ Find $f'(Z)$.**

3. **Two dice are thrown. The random variable x represents the maximum of the two faces and the random variable y represents the number of sixes. Find**
   (i) The probability distribution of the two random variables x & y.
   (ii) The marginal probability distribution of x and y, then check independence.

1) i) $\int_0^\infty 3^{5-x^2}\,dx$

let $u = 3^{-x^2}$

$-u = -x^2 \cdot \ln(3)$

$u = x^2 \ln(3)$

$x = \frac{u^{1/2}}{\ln(3)}$, $dx = \frac{u^{-1/2}}{2\ln(3)}\,du$

$x = 0 \longrightarrow u = 0$

$x = \infty \longrightarrow u = \infty$

$I = \int_0^\infty 3^5 * e^{-u} \cdot \frac{u^{-1/2}}{2\ln(3)}\,du$

$= \frac{3^5}{2\ln(3)} \int_0^\infty u^{-1/2} e^{-u}\,du$

$= \frac{3^5}{2\ln(3)} \Gamma(\frac{1}{2}) = \frac{3^5\sqrt{\pi}}{2\ln(3)}$

(ii) $\int_0^{\pi/2} \sin^{3/2}\theta \cos^{-3/2}\theta\,d\theta$

$2x - 1 = \frac{3}{2}$, $2y - 1 = \frac{-3}{2}$

$x = \frac{5}{4}$　　$y = \frac{-1}{4}$

$= 2\beta(\frac{5}{4}, \frac{-1}{4})$

$= 2\,\frac{\Gamma(\frac{5}{4}) \cdot \Gamma(\frac{-1}{4})}{\Gamma(1)} =$

$\frac{2\pi}{\sin(\frac{5\pi}{4})} = -2\pi\sqrt{2}$

$u_x = -2\sin 2x \cosh 2y$

$u_{xx} = -4\cos 2x \cosh 2y$

$u_y = 2\cos 2x \sinh 2y$

$u_{yy} = 4\cos 2x \cosh 2y$

$u_{xx} + u_{yy} = 0$　∴ u is harmonics

$u_x = v_y$, $u_y = -v_x$

$v_x = -2\cos 2x \sinh 2y$

$v = -\sin(2x) \sinh 2y + h(y)$

$v_y = -2\sin(2x)\cosh 2y + h'(y)$

$= u_x = -2\sin 2x \cosh 2y$

$h'(y) = 0$, $h(y) = C$

$v = -\sin(2x) \sinh(2y) + C$

$f'(z) = u_x + i v_x$

$= -4\cos 2x \cosh 2y + i(-2\cos 2x \sinh 2y)$