* Cauchy Rieman in polar:

$U_x = V_y \longrightarrow rU_r = V_\theta$

$U_y = -V_x \qquad rV_r = -U_\theta$

* Laplace:

$U_{xx} + U_{yy} = 0 \longrightarrow r^2U_{rr} + rU_r + U_{\theta\theta} = 0$

* Discuss differentiability:

$f(z) = \frac{1}{z}$ (in rectangular) $= \frac{1}{z} = \frac{1}{x+iy} * \frac{x-iy}{x-iy}$

$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$

↳ finding U_x & U_y & V_x & V_y is hard because you'll have differentiate a quotient + you'll have to simplify.

in Polar Form: $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{i(-\theta)}$

$= \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$

$= \frac{1}{r}[\underbrace{\cos\theta}_{u} - i\underbrace{\sin\theta}_{v}]$

$u = \frac{1}{r}\cos\theta \qquad v = \frac{-1}{r}\sin\theta$

u, v are continuous on z-plane except $r=0$ $\Rightarrow (0,0)$

$U_r = \frac{-1}{r^2}\cos\theta \qquad V_r = \frac{1}{r^2}\sin\theta$

$U_\theta = \frac{-1}{r}\sin\theta \qquad V_\theta = \frac{-1}{r}\cos\theta$

$\therefore f(z)$ is diff on $z$plane

$rU_r = -\frac{1}{r}\cos\theta = V_\theta \qquad rV_r = \frac{1}{r}\sin\theta = -U_\theta$

$f'(z) = u_x + i v_x = \frac{r}{z}[u_r + i v_r]$

exp. $z^2 \bar{z} = (x+iy)^2 (x-iy)$

$= (re^{i\theta})^2 (re^{-i\theta})$

$= r^3 e^{i\theta} = \underbrace{r^3 \cos\theta}_{u} + \underbrace{i r^3 \sin\theta}_{v}$

easier to find u_r, u_θ
v_r, v_θ

exp = $\frac{1}{z^2} = \frac{1}{r^2 e^{i\theta 2}} = \frac{1}{r^2} e^{i(-2\theta)}$

$= \frac{1}{r^2}\cos(-2\theta) + i\frac{1}{r^2}\sin(-2\theta)$

even fn

exp: $v = \frac{y}{x^2+y^2}$ (show it's harmonic & find a corresponding analytics function $f(z) = u+iv$)

sol: $v = \frac{r\sin\theta}{r^2} = \frac{1}{r}\sin\theta$

$(r^2 = x^2+y^2)$ # $v_r = -\frac{1}{r^2}\sin\theta \quad v_{rr} = \frac{2}{r^3}\sin\theta$

$v_\theta = \frac{1}{r}\cos\theta \quad v_{\theta\theta} = \frac{-1}{r}\sin\theta$

$r^2 v_{rr} + r v_r + v_{\theta\theta} =$

$\cancel{r^2}\frac{2}{\cancel{r^3}}\sin\theta + \cancel{r}\frac{-1}{\cancel{r^2}}\sin\theta + \frac{-1}{r}\sin\theta =$

$\frac{2\sin\theta}{r} - \frac{1}{r}\sin\theta - \frac{1}{r}\sin\theta = 0$ # $\therefore$ harmonic

exp = $u = \arg(z) = \cancel{\tan^{-1}\left(\frac{y}{x}\right)}$ $\Rightarrow u = \theta$ //
very hard way easy

$z = x + iy$ $|z| = \sqrt{x^2 + y^2}$ $\arg(z) = \theta = \tan^{-1}\frac{x}{y}$

$f(z) = W = u + iV$ $|W| = \sqrt{u^2 + v^2}$

* Prove that $F(z)$ is Const if : $|F(z)|$ is analytic

$W = |f(z)| + i0$ analytic = Complex function that is always real.

$= \sqrt{u^2 + v^2} + i(\text{zero})$ is analytic

$U \quad + \quad iV \qquad V = 0$

$U_x = V_y = 0 \Rightarrow U_x = \frac{2uu_x}{2\sqrt{u^2+v^2}} = 0$

$2uu_x = 0$ —— (1)

$U = 0$ (trivial

$-V_x = 0 = U_y = \frac{2uu_y}{2\sqrt{u^2+v^2}} = 0$ $\boxed{u_x = 0}$

$2u\,u_y = 0$ —— (2)

$u = 0$

$\boxed{u_y = 0}$

Imp

Section 10 : Prove that $f(z)$ must be constant if

$\Rightarrow |f(z)| = \text{Const}$ $\qquad f(z) = u + iv$

sol: for this to be true we need the derivatives to equal zero (derivative of a constant = 0)

$f(z) = C_1 = u + iv = C_2 + iC_3$

where $u = C_2$ $\qquad v = C_3$

$u_x = 0 = u_y$ $\qquad v_x = 0 = v_y$

$f(z) = u + iv$ is analytic $\therefore$ $u_x = v_y$ — ②

& $u_y = -v_x$ — ③

$\boxed{\begin{array}{l}\text{harmonic} \\ u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0\end{array}}$

& $\sqrt{u^2 + v^2} = c$

$u^2 + v^2 = c^2$ — ①

diffe w.r.t $x \Rightarrow$ $2uu_x + 2vv_x = 0$

$uu_x + vv_x = 0 \rightarrow$ ④

diff w.r.t $y \Rightarrow$ $uu_y + vv_y = 0 \rightarrow$ ⑤

From CR in ④ $-uv_x + vu_x = 0$ — ⑥

④ $* u + v *$ ⑥ :

$u^2 u_x + uvv_x - uvv_x + v^2 u_x = 0$

$u^2 u_x + v^2 u_x = 0 \Rightarrow (u^2 + v^2) u_x = 0$

$u^2 + v^2 = 0$ $\qquad u_x = v_y = 0$

trivial solution $\qquad u_y = -v_x = 0$

"special case" $\qquad \therefore u = C_2 \qquad v = C_3$

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$\therefore f(z) = C_2 + iC_3 = C_1$