

الإسـم: علاء

ملزمة (٩)

رياضة

**Cauchy- Reiman Equation &
Harmonic Functions**

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Cauchy-Reimanⁿ Eqs & Harmonic fn.

$f(z) = u + iv$ is differentiable at all the points on the z -plane that satisfies

- 1) Cauchy-Reiman Eqs.
- 2) Continuous Partial derivatives.

* Cauchy-Reiman Equations :-

- Rectangular form, when u & v are easier to be represented in terms of x & y

$$u_x = v_y \quad \& \quad u_y = -v_x$$

- Polar form, when u & v are easier to be written in terms of r & θ , we use

$$u_r = \frac{1}{r} v_\theta \quad \& \quad u_\theta = -r v_r$$

* At those points of the z -plane where Cauchy Reiman are satisfied & the partial derivatives u_x, u_y, v_x, v_y are continuous, we can find

$f(z)$, as $f(z)$ is differentiable, using

$$f'(z) = u_x + i v_x$$

* $f(z)$ is analytic at z , if: it is differentiable at z and differentiable at all the points around z .
Entire f_n = Analytic everywhere.

Example: Discuss the differentiability of

1) $f(z) = z^2$

2) $f(z) = \bar{z}$

Solution:- 1) $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i 2xy$

$$\Rightarrow u = x^2 - y^2$$

$$v = 2xy$$

$$u_x = 2x$$

$$v_x = 2y$$

$$u_y = -2y$$

$$v_y = 2x$$

Since, $u_x = v_y$ & $u_y = -v_x$, also they are all continuous (polynomials) $\Rightarrow f(z)$ is everywhere differentiable \Rightarrow Everywhere Analytic \Rightarrow Entire

$$\& f'(z) = u_x + i v_x = 2x + i 2y = 2z$$

We can use the polar form here as follow:-

$$f(z) = z^2 = r^2 e^{i2\theta} = r^2 (\cos 2\theta + i \sin 2\theta)$$

$$\Rightarrow u = r^2 \cos 2\theta$$

$$u_r = 2r \cos 2\theta$$

$$u_\theta = -2r^2 \sin 2\theta$$

$$v = r^2 \sin 2\theta$$

$$v_r = 2r \sin 2\theta$$

$$v_\theta = 2r^2 \cos 2\theta$$

Since $u_r = \frac{v_\theta}{r}$ & $\frac{u_\theta}{r} = -v_r$ also they are continuous fns $\Rightarrow f(z)$ is differentiable everywhere \Rightarrow analytic everywhere.

$$2) f(z) = \bar{z} = x - iy$$

$$\Rightarrow u = x$$

$$u_x = 1$$

$$u_y = 0$$

$$v = -y$$

$$v_x = 0$$

$$v_y = -1$$

we have $u_x \neq v_y \Rightarrow f(z)$ is nowhere diff. \Rightarrow nowhere analytic.

Also if we use the polar form $\Rightarrow f(z) = \bar{z} = r e^{-i\theta}$
 $= r \cos \theta - i r \sin \theta$

$$\Rightarrow u = r \cos \theta$$

$$u_r = \cos \theta$$

$$u_\theta = -r \sin \theta$$

$$v = -r \sin \theta$$

$$v_r = -\sin \theta$$

$$v_\theta = -r \cos \theta$$

$$\text{For } u_r = \frac{v_\theta}{r} \Rightarrow \cos \theta = -\cos \theta \Rightarrow \cos \theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$$

$$\text{for } \frac{u_\theta}{r} = -v_r \Rightarrow -\sin \theta = -\sin \theta \Rightarrow \sin \theta = 0 \Rightarrow \theta = n\pi$$

Both conditions can't be satisfied together \Rightarrow

$f(z)$ is nowhere diff & so nowhere analytic.

Example:- Determine where the functions below are differentiable & where they are analytic.

i) $f(z) = \text{Im}(z)$

ii) $f(z) = e^y (\cos x + i \sin x)$

iii) $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

iv) $f(z) = (1-i)x^3 + (1+i)y^3$

Solution:-

i) $f(z) = \text{Im}(z) = y$

$$u = y$$

$$v = 0$$

$$u_x = 0$$

$$v_x = 0$$

$$u_y = 1$$

$$v_y = 0$$

$\Rightarrow u_x = v_y$, but nowhere $u_y = -v_x \Rightarrow$

$f(z)$ is nowhere differentiable \Rightarrow nowhere $f(z)$ is analytic.

ii) $f(z) = e^y (\cos x + i \sin x)$

$$u = e^y \cos x$$

$$v = e^y \sin x$$

$$u_x = -e^y \sin x$$

$$v_x = e^y \cos x$$

$$u_y = e^y \cos x$$

$$v_y = e^y \sin x$$

suchy Reiman Eqs are satisfied if

$$1) U_x = V_y \Rightarrow -e^y \sin x = e^y \sin x$$

$$\Rightarrow \sin x = 0 \Rightarrow x = 0, \pm \pi, \pm 2\pi, \dots$$

$$2) U_y = -V_x \Rightarrow e^y \cos x = -e^y \cos x$$

$$\Rightarrow \cos x = 0 \Rightarrow x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$$

Both Conditions can't be satisfied together

$f(z)$ is no where differentiable \Rightarrow no where Analytic

$$\text{iii.) } f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$$

$$u = x^3 + 3xy^2 - 3x$$

$$v = y^3 + 3x^2y - 3y$$

$$u_x = 3x^2 + 3y^2 - 3$$

$$v_x = 6xy$$

$$u_y = 6xy$$

$$v_y = 3y^2 + 3x^2 - 3$$

as seen, anywhere $u_x = v_y$, but for $u_y = -v_x$

$$\Rightarrow 6xy = -6xy \Rightarrow xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

Since partial derivatives are continuous, then

$f(z)$ is differentiable only on the x -axis & y axis, with $f'(z) = u_x + i v_x$

$$\Rightarrow f'(z) = 3x^2 + 3y^2 - 3 + i \cdot 0$$

but, $f(z)$ is nowhere analytic.

Note that

$$\text{If } f(z) = x^3 - 3xy^2 - 3x + i(3x^2y - y^3 - 3y)$$

$$\Rightarrow u = x^3 - 3xy^2 - 3x \quad v = 3x^2y - y^3 - 3y$$

$$u_x = 3x^2 - 3y^2 - 3$$

$$v_x = 6xy$$

$$u_y = -6xy$$

$$v_y = 3x^2 - 3y^2 - 3$$

Cauchy-Riemann are satisfied everywhere

$f(z)$ is everywhere diff \Rightarrow everywhere Analytic

$\Rightarrow f(z)$ is entire f_n

$$f'(z) = u_x + i v_x$$

$$= 3x^2 - 3y^2 - 3 + i(6xy)$$

Note that to get $f(z)$ or $f'(z)$ in terms of z

for an entire f_n , set $y=0$ & replace x by z .

$$\text{So, we have } f(z) = z^3 - 3z$$

$$\& f'(z) = 3z^2 - 3$$

$$1) f(z) = (1-i)x^3 + (1+i)y^3$$

$$u = x^3 + y^3$$

$$v = y^3 - x^3$$

$$u_x = 3x^2$$

$$v_x = -3x^2$$

$$u_y = 3y^2$$

$$v_y = 3y^2$$

Cauchy Reiman are satisfied if:

$$1) u_x = v_y \Rightarrow 3x^2 = 3y^2 \Rightarrow y^2 = x^2$$

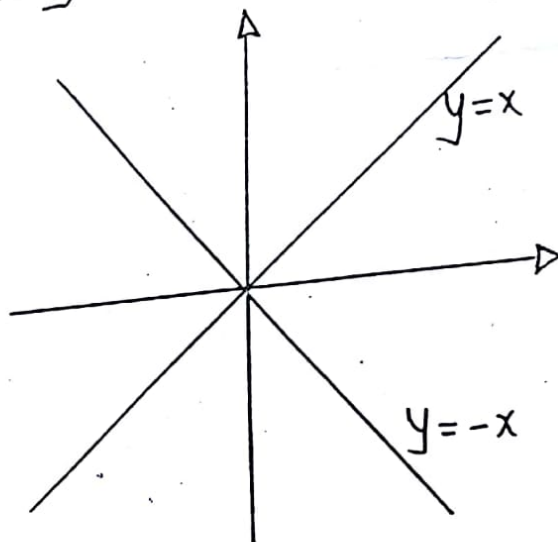
$$\Rightarrow y = \pm x$$

$$2) u_y = -v_x \Rightarrow 3y^2 = 3x^2 \Rightarrow y = \pm x$$

$f(z)$ is differentiable only on $y = \pm x$ (2 st. lines)

$$\begin{aligned} \text{with } f(z) &= u_x + i v_x \\ &= 3x^2 - i 3x^2 \\ &= 3(1-i)x^2 \end{aligned}$$

$f(z)$ is nowhere Analytic



$f(z) \begin{cases} \text{diff on } y = \pm x \\ \text{no where Analytic} \end{cases}$

Example :- Determine where

$$f(z) = (x^3 - xy^2) + i(x^2y + y^3)$$

is analytic & where it is differentiable.

Solution :-

We have

$$u = x^3 - xy^2$$

$$u_x = 3x^2 - y^2$$

$$u_y = -2xy$$

$$v = x^2y + y^3$$

$$v_x = 2xy$$

$$v_y = x^2 + 3y^2$$

The 2nd Cauchy - Reiman Eq. Condition is already satisfied, also all partial derivatives are continuous (Polynomials), for the 1st

Condition to be satisfied $\Rightarrow u_x = v_y$

$$\Rightarrow 3x^2 - y^2 = x^2 + 3y^2 \Rightarrow y^2 = \frac{x^2}{2} \Rightarrow y = \pm \frac{x}{\sqrt{2}}$$

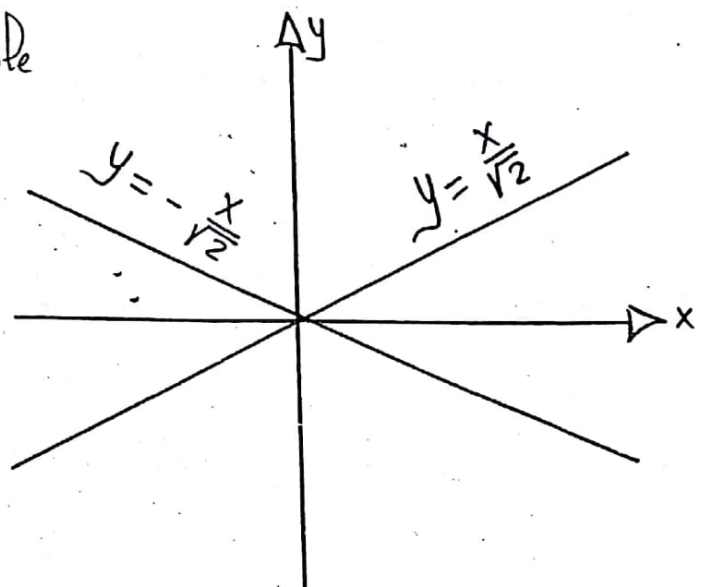
So, $f(z)$ is differentiable

on these 2 lines only &

$$f'(z) = (3x^2 - y^2) + i \cdot 2xy$$

But, $f(z)$ is nowhere

Analytic



harmonic function:-

A function ϕ of two variables $(x \& y)$ or $(r \& \theta)$ is said to be harmonic, if it satisfies Laplace Equation.

Laplace Eq. is

| |
|--|
| Rectangular form $\phi(x, y)$ |
| $\phi_{xx} + \phi_{yy} = 0$ |
| Polar form $\phi(r, \theta)$ |
| $r^2 \phi_{rr} + r \phi_r + \phi_{\theta\theta} = 0$ |

Theorem:-

If $f(z) = u + i v$ is analytic then u is harmonic & v is harmonic, they are called harmonic conjugate

Result: If he says that u & v are harmonic conjugate then we have:

- 1) u is harmonic $\Rightarrow u_{xx} + u_{yy} = 0$ OR $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$
- 2) v is harmonic $\Rightarrow v_{xx} + v_{yy} = 0$ OR $r^2 v_{rr} + r v_r + v_{\theta\theta} = 0$
- 3) $f(z) = u + i v$ is analytic \Rightarrow Cauchy - Reiman are satisfied.

Example :- Show that the following fns are harmonic & find their corresponding analytic fn

i) $u = x^3 - 3xy^2 + y$

ii) $u = \cos x \cosh y$

Solution:-

i) $u = x^3 - 3xy^2 + y$

$$\Rightarrow u_x = 3x^2 - 3y^2$$

$$u_{xx} = 6x$$

$$u_y = -6xy + 1$$

$$u_{yy} = -6x$$

Since $u_{xx} + u_{yy} = 6x - 6x = 0 \Rightarrow u$ is harmonic

To find its harmonic Conjugate v (so that $f(z) = u + iv$ is analytic) we must have

$$u_x = v_y = 3x^2 - 3y^2 \Rightarrow v = 3x^2y - y^3 + h(x)$$

$$\Rightarrow v_x = 6xy + h'(x) = -u_y = 6xy - 1$$

$$\Rightarrow h'(x) = -1 \Rightarrow h(x) = -x + C$$

$$\Rightarrow v = 3x^2y - y^3 - x + C$$

$$f(z) = u + iv = x^3 - 3xy^2 + y + i(3x^2y - y^3 - x) + C$$

$$= z^3 - i z + C$$

Also to find $f'(z) = u_x + i v_x = 3x^2 - 3y^2 + i(6xy - 1)$
 $= 3z^2 - i$

ii) $u = \cos x \cosh y$

$$\Rightarrow u_x = -\sin x \cosh y$$

$$u_{xx} = -\cos x \cosh y$$

$$u_y = \cos x \sinh y$$

$$u_{yy} = \cos x \cosh y$$

$$\Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow u \text{ is harmonic}$$

To find its harmonic conjugate v

$$\Rightarrow v_y = u_x = -\sin x \cosh y \Rightarrow v = -\sin x \sinh y + h(x)$$

$$u_x = -u_y \Rightarrow -\cos x \sinh y + h'(x) = -\cos x \sinh y$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = C$$

$$v = -\sin x \sinh y + C$$

$$f(z) = u + iv = \cos x \cosh y - i \sin x \sinh y + C \\ = \cos z$$

$$\text{To find } f'(z) = u_x + i v_x \\ = -\sin x \cosh y - i \cos x \sinh y \\ = -\sin z$$

Example :-

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Show that (uv) is harmonic if v is a harmonic conjugate of u .

* Solution *

It is required to prove that

$$(uv)_{xx} + (uv)_{yy} = 0$$

$$\frac{\partial}{\partial x} (uv) = u_x v + u v_x$$

$$\frac{\partial^2}{\partial x^2} (uv) = u_{xx} v + u_x v_x + u_x v_x + u v_{xx}$$

$$= u_{xx} v + 2 u_x v_x + u v_{xx}$$

$\therefore u$ & v are conjugate harmonic

$$\therefore \textcircled{1} \quad u_{xx} + u_{yy} = 0 \quad \Rightarrow \quad u_{xx} = -u_{yy}$$

$$v_{xx} + v_{yy} = 0 \quad \Rightarrow \quad v_{xx} = -v_{yy}$$

$$\textcircled{2} \quad f(z) = u + i v \text{ is analytic}$$

$$\Rightarrow \quad u_x = v_y \quad u_y = -v_x$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} (uv) = u_{xx} v + 2 u_x v_x + u v_{xx}$$

$$= (-u_{yy}) v + 2 (v_y) (-u_y) + u (-v_{yy})$$

$$= - [u_{yy} v + 2 u_y v_y + u v_{yy}]$$

$$= - \frac{\partial^2}{\partial y^2} (uv)$$

$$\Rightarrow (uv) \text{ is harmonic.}$$

sample:- If $f(z) = u + iv$ & $u = v^2$ is analytic, then show that $f(z) = \text{Constant}$

Solution:-

Since $f(z)$ is analytic then $\frac{\partial}{\partial x}(u) = v_y \Rightarrow \frac{\partial}{\partial x}(v^2) = v_y$

$$1) u_x = v_y \Rightarrow$$

$$\Rightarrow 2v v_x = v_y \rightarrow \textcircled{1}$$

$$2) u_y = -v_x \Rightarrow \frac{\partial}{\partial y}(u) = -v_x \Rightarrow \frac{\partial}{\partial y}(v^2) = -v_x$$

$$\Rightarrow 2v v_y = -v_x \rightarrow \textcircled{2}$$

Substituting with $\textcircled{2}$ in $\textcircled{1} \Rightarrow$

$$2v(-2v v_y) = v_y \Rightarrow -4v^2 v_y = v_y$$

$$\Rightarrow v_y(1 + 4v^2) = 0$$

OR

$$v_y = 0$$

from $\textcircled{2}$

$$v_x = 0$$

\Downarrow

$$v = \text{Constant}$$

\Downarrow

$$u = v^2 = \text{Constant}$$

\Downarrow

$$f(z) = u + iv = \text{Constant}$$

$$1 + 4v^2 = 0$$

\Downarrow

$$v^2 = -1/4$$

\Downarrow

$$v = \text{Constant}$$

\Downarrow

$$u = v^2 = \text{Constant}$$

\Downarrow

$$f(z) = u + iv = \text{Constant}$$

Example :-

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Show that $u = \ln^2 r - \theta^2$ is harmonic
& find $f(z) = u + iv$ to be analytic

* Solution *
 $u = \ln^2 r - \theta^2$

$$u_r = 2 \ln r \cdot \frac{1}{r}$$

$$= \frac{2 \ln r}{r}$$

$$u_\theta = -2\theta$$

$$u_{\theta\theta} = -2$$

$$u_{rr} = \frac{2/r \cdot r - 2 \ln r}{r^2}$$

$$\Rightarrow r^2 u_{rr} = 2 - 2 \ln r$$

$$r u_r = 2 \ln r$$

$$\Rightarrow r^2 u_{rr} + r u_r + u_{\theta\theta} = \cancel{2} - 2 \ln r + 2 \ln r - \cancel{2} = 0$$

$\Rightarrow u$ is harmonic

$\therefore f(z)$ is analytic $\therefore u_r = \frac{1}{r} v_\theta, u_\theta = -r v_r$

$$\Rightarrow v_\theta = r u_r = 2 \ln r$$

$$\Rightarrow v = 2\theta \ln r + h(r)$$

$$v_r = \frac{2\theta}{r} + h'(r) = -\frac{1}{r} u_\theta = \frac{2\theta}{r}$$

$$\Rightarrow h'(r) = 0 \Rightarrow h(r) = c$$

$$f(z) = u + iv = \ln^2 r - \theta^2 + i 2\theta \ln r$$

$$\Rightarrow f(z) = \ln^2 z$$

Example : Show that $v = \frac{y}{x^2 + y^2}$ is harmonic
& find its harmonic Conjugate.

Sol.

$$v = \frac{y}{x^2 + y^2} = \frac{r \sin \theta}{r^2} = \frac{\sin \theta}{r}$$

$$v_r = - \frac{\sin \theta}{r^2}$$

$$v_\theta = \frac{\cos \theta}{r}$$

$$v_{rr} = \frac{2 \sin \theta}{r^3}$$

$$v_{\theta\theta} = - \frac{\sin \theta}{r}$$

$$\Rightarrow r^2 v_{rr} + r v_r + v_{\theta\theta} = 2 \frac{\sin \theta}{r} - \frac{\sin \theta}{r} - \frac{\sin \theta}{r} = 0$$

$\Rightarrow v$ is harmonic

To get its harmonic Conjugate u

$$u_r = \frac{1}{r} v_\theta \Rightarrow u_r = \frac{\cos \theta}{r^2} \Rightarrow u = - \frac{\cos \theta}{r} + h(\theta)$$

$$\frac{1}{r} u_\theta = -v_r \Rightarrow \frac{\sin \theta}{r} + h'(\theta) = \frac{\sin \theta}{r}$$

$$\Rightarrow h'(\theta) = 0 \Rightarrow h(\theta) = c$$

$$\Rightarrow u = - \frac{\cos \theta}{r} + c$$

$$f(z) = - \frac{\cos \theta}{r} + i \frac{\sin \theta}{r} + c$$

$$= - \frac{1}{z} + c$$

Example Determine where $f(z) = z^2 \bar{z}$ is analytic & find $f'(z)$.

Solution

$$f(z) = z^2 \bar{z} = (x+iy)^2(x-iy)$$

$$f(z) = (x+iy)(x^2+y^2) = x^3+xy^2+i(x^2y+y^3)$$

$$\Rightarrow u = x^3+xy^2 \quad \& \quad v = x^2y+y^3$$

$$u_x = 3x^2+y^2$$

$$v_x = 2xy$$

$$u_y = 2xy$$

$$v_y = x^2+3y^2$$

$$① \quad u_x = v_y \Rightarrow 3x^2+y^2 = x^2+3y^2 \Rightarrow x^2=y^2$$

$$② \quad u_y = -v_x \Rightarrow 2xy = -2xy \Rightarrow xy=0$$

For Cauchy Reiman to be satisfied we must have
 $y = \pm x$ & $(x=0 \text{ or } y=0)$

This is satisfied for $x=y=0 \Rightarrow$ the origin

$f(z)$ is diff. only at $z=0$ & $f'(z) = u_x + i v_x$

$$\Rightarrow f'(z) = (3x^2+y^2) + i \cdot 2xy \Rightarrow f'(0) = 0$$

The function is not analytic at any Point.

Note that: we can use the Polar Coordinates
by setting $z = re^{i\theta}$

$$f(z) = z^2 \bar{z} = r^2 e^{i2\theta} \cdot r e^{-i\theta} \\ = r^3 e^{i\theta} = r^3 \cos \theta + i r^3 \sin \theta$$

$$\Rightarrow u = r^3 \cos \theta \quad \& \quad v = r^3 \sin \theta$$

$$u_r = 3r^2 \cos \theta$$

$$v_r = 3r^2 \sin \theta$$

$$u_\theta = -r^3 \sin \theta$$

$$v_\theta = r^3 \cos \theta$$

Satisfying Cauchy-Reiman

$$\textcircled{1} \quad u_r = \frac{v_\theta}{r} \Rightarrow 3r^2 \cos \theta = r^2 \cos \theta$$

$$\Rightarrow 2r^2 \cos \theta = 0 \quad \begin{cases} \Rightarrow r=0 \\ \Rightarrow \cos \theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2} \end{cases}$$

$$\textcircled{2} \quad v_r = -\frac{u_\theta}{r} \Rightarrow 3r^2 \sin \theta = r^2 \sin \theta$$

$$\Rightarrow 2r^2 \sin \theta = 0 \Rightarrow r=0 \\ \downarrow \\ \sin \theta = 0 \Rightarrow \theta = n\pi$$

Both Equations will not satisfied except at $r=0$

$\Rightarrow z=0 \Rightarrow f(z)$ is diff only at $z=0$

\Rightarrow is nowhere analytic.

Q.10:- Show that $v = 3x + y \cosh \cos(x+1) - x \sinh \sin(x+1)$ is harmonic. Find the analytic fn $f(z) = u + iv$. Then find $f'(z)$.

Solution:- $v = 3x + y \cosh \cos(x+1) - x \sinh \sin(x+1)$

$$v_x = 3 - y \cosh \sin(x+1) - \sinh \sin(x+1) - x \sinh \cos(x+1)$$

$$v_{xx} = -y \cosh \cos(x+1) - \sinh \cos(x+1) - \sinh \cos(x+1) + x \sinh \sin(x+1) \rightarrow (1)$$

$$v_y = \cosh \cos(x+1) + y \sinh \cos(x+1) - x \cosh \sin(x+1)$$

$$v_{yy} = \sinh \cos(x+1) + \sinh \cos(x+1) + y \cosh \cos(x+1) - x \sinh \sin(x+1) \rightarrow (2)$$

From (1) & (2) $\Rightarrow v_{xx} + v_{yy} = \text{Zero} \Rightarrow \text{harmonic}$

To find the harmonic Conjugate u :-

Apply Cauchy - Reiman

$$1) u_x = v_y = \cosh \cos(x+1) + y \sinh \cos(x+1) - x \cosh \sin(x+1)$$

Integrate w.r.t. x to get u .

$$chy \sin(x+1) + y shy \sin(x+1) - chy'(-x \cos(x+1) + \sin(x+1)) + h(y) \cdot$$

$$1 = y shy \sin(x+1) + x chy \cos(x+1) + h(y) \cdot$$

$$?) \quad U_y = -V_x \Rightarrow \text{to get } h(y)$$

$$\begin{aligned} & \cancel{shy \sin(x+1)} + y \cancel{chy \sin(x+1)} + x shy \cos(x+1) + h'(y) \\ &= -3 + y \cancel{chy \sin(x+1)} + \cancel{shy \sin(x+1)} + x \cancel{shy \cos(x+1)} \\ &\Rightarrow h'(y) = -3 \Rightarrow h(y) = -3y + C \end{aligned}$$

$$\text{Now, } f(z) = u + i \cdot v$$

$$\begin{aligned} &= y shy \sin(x+1) + x chy \cos(x+1) - 3y + C \\ &+ i \cdot (3x + y chy \cos(x+1) - x shy \sin(x+1)) \end{aligned}$$

to get it in terms of $z \Rightarrow$ Put $y=0$ & $x=z$

$$f(z) = z \cos(z+1) + i \cdot 3z + C$$

$$\text{For, } f'(z) = U_x + i \cdot V_x$$

$$\begin{aligned} &= chy \cos(x+1) + y shy \cos(x+1) - x chy \sin(x+1) \\ &+ i \cdot (3 - y chy \sin(x+1) - shy \sin(x+1) - x shy \cos(x+1)) \\ &\Rightarrow f'(z) = \cos(z+1) - z \sin(z+1) + 3i \end{aligned}$$

Example:-

Prove that an analytic fn $f(z)$ must be Constant if

1) $\overline{f(z)}$ is analytic

or

2) $|f(z)| = \text{constant}$.

Solution

Let $f(z) = u + iv \Rightarrow$ Since it is analytic then
 $u_x = v_y \rightarrow \textcircled{1}$ & $u_y = -v_x \rightarrow \textcircled{2}$

1) If $\overline{f(z)} = u - iv$ is analytic then

$$u_x = (-v)_y \Rightarrow u_x = -v_y \rightarrow \textcircled{3}$$

$$\& u_y = -(-v)_x \Rightarrow u_y = v_x \rightarrow \textcircled{4}$$

$$\text{add } \textcircled{1} \& \textcircled{3} \Rightarrow u_x = 0 \Rightarrow v_y = 0$$

$$\text{add } \textcircled{2} \& \textcircled{4} \Rightarrow u_y = 0 \Rightarrow v_x = 0$$

$$\Rightarrow u_x = u_y = 0 \Rightarrow u = C_1 \& v_x = v_y = 0 \Rightarrow v = C_2$$

$$\Rightarrow f(z) = C_1 + iC_2 = \text{Constant}.$$

$$i) |f(z)| = \sqrt{u^2 + v^2} = k \Rightarrow u^2 + v^2 = k^2$$

$$\text{Diff. w.r.t. } x \Rightarrow 2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \rightarrow \textcircled{5}$$

$$\text{" " } y \Rightarrow 2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \rightarrow \textcircled{6}$$

$$\text{From } \textcircled{1} \& \textcircled{5} \Rightarrow uv_y + vv_x = 0 \rightarrow \textcircled{7}$$

$$\text{From } \textcircled{2} \& \textcircled{6} \Rightarrow -uv_x + vv_y = 0 \rightarrow \textcircled{8}$$

$$\textcircled{7} * u + \textcircled{8} * v \Rightarrow (u^2 + v^2) v_y = 0 \Rightarrow v_y = 0 \xrightarrow{\textcircled{1}} u_x = 0$$

$$\textcircled{7} * v - \textcircled{8} * u \Rightarrow (u^2 + v^2) v_x = 0 \Rightarrow v_x = 0 \xrightarrow{\textcircled{2}} u_y = 0$$

Example :-

Find the image of $x^2 + y^2 = a^2$ under $w = \frac{1}{z}$

Solution :-

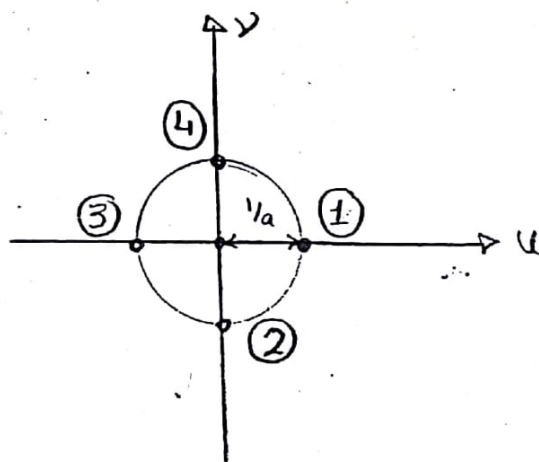
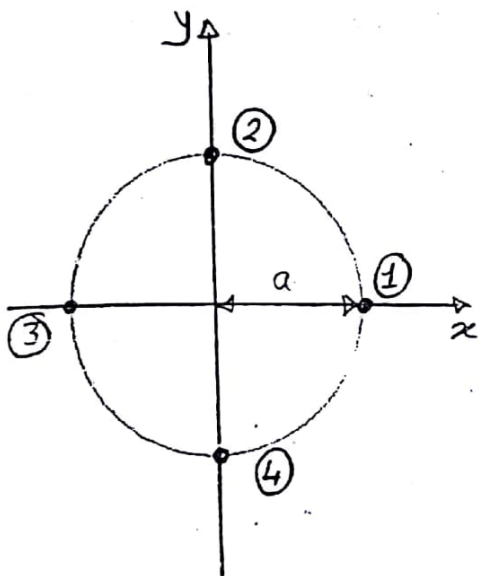
$$x^2 + y^2 = a^2 \rightarrow \text{Set } x = \frac{u}{u^2 + v^2}$$
$$y = -\frac{v}{u^2 + v^2}$$

$$\Rightarrow \frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = a^2$$

$$\Rightarrow \frac{1}{u^2 + v^2} = a^2$$

$$\Rightarrow u^2 + v^2 = \frac{1}{a^2}$$

is the equation of
circle of center (0,0)
& radius $\frac{1}{a}$.



From above, we can say that the Transformation of the inside of the circle $x^2 + y^2 = 4$ (i.e. $x^2 + y^2 \leq 4$) is the outside of the circle $u^2 + v^2 = \frac{1}{4}$ (i.e. $u^2 + v^2 \geq \frac{1}{4}$).