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ملزمة (١)

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The Gama Function F (X)

ثانيه كهرباء

* Special Functions *

I) The Gamma Function $\Gamma(x)$:

Only when $x > 0$, we define

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad ; \quad x > 0$$

If $x \leq 0$, this integration diverges .

Examples:- Evaluate these integrals

1) $\int_0^\infty x^\pi e^{-x} \, dx$

2) $\int_0^\infty \sqrt[3]{x} \, e^{-2x} \, dx$

3) $\int_0^1 \sqrt{-x \ln x} \, dx$

Solution:- 1) $\int_0^\infty x^\pi e^{-x} \, dx = \Gamma(\pi+1)$

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2) $\int_0^\infty \sqrt[3]{x} \, e^{-2x} \, dx$

let $2x = t \Rightarrow dx = \frac{dt}{2}$

$$\int_0^\infty \sqrt[3]{\frac{t}{2}} \, e^{-t} \cdot \frac{dt}{2} = \frac{1}{2\sqrt[3]{2}} \int_0^\infty t^{1/3} e^{-t} \, dt$$

$$= \frac{1}{2\sqrt[3]{2}} \cdot \Gamma(4/3) .$$

3) $\int_0^1 \sqrt{-x \ln x}\, dx$

Let $\ln x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$x : 0 \to 1 \Rightarrow t : \infty \to 0$

$$\int_\infty^0 \sqrt{e^{-t} \cdot t} \cdot (-e^{-t} dt) = \int_0^\infty e^{-\frac{3}{2}t}\, t^{\frac{1}{2}}\, dt$$

Let $\frac{3}{2}t = u \Rightarrow dt = \frac{2}{3} du$

$$\int_0^\infty e^{-u} \left(\frac{2}{3}u\right)^{1/2} \cdot \frac{2}{3} du = \left(\frac{2}{3}\right)^{3/2} \int_0^\infty e^{-u} \cdot u^{1/2} du$$

$$= (2/3)^{3/2}\, \Gamma(3/2)$$

Note : $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

Properties of $\Gamma(x)$:

1) $\Gamma(1) = 1$

Proof :- $\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 0 + 1 = 1$

2) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof :- $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$

Let $t^{1/2} = u \Rightarrow t = u^2 \Rightarrow dt = 2u\,du$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} \cdot \frac{1}{u} \cdot 2u\, du = 2\int_0^\infty e^{-u^2} du$$

$$= 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

The Reccurrence Relation :-

$$\boxed{\Gamma(x+1) = x\,\Gamma(x)}$$

Proof :- $\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x \, dt$

using integration by parts

$u = t^x \qquad dv = e^{-t}\, dt$

$du = x\, t^{x-1}\, dt \qquad v = -e^{-t}$

$$\Gamma(x+1) = \underbrace{-t^x e^{-t}\Big|_0^\infty}_{\text{Zero}} + x \int_0^\infty e^{-t}\, t^{x-1}\, dt = x\,\Gamma(x)$$

Examples :-

* $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\,\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

* $\Gamma(6) = 5\,\Gamma(5) = 5.4.\Gamma(4) = 5.4.3.\Gamma(3)$

-- $= 5.4.3.2.1\,\Gamma(1) = 5!$

In general, $\Gamma(n+1) = n!$

So, the Gamma fn. is a generalization for the factorial

$\Rightarrow \left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

$\left(-\frac{1}{4}\right)! = \Gamma(3/4)$

Examples : Evaluate these integrals

1) $\int_0^1 \ln^7 t \, dt$

$\Rightarrow$ let $x = -\ln t \Rightarrow t = e^{-x} \Rightarrow dt = -e^{-x} dx$

$I = \int_0^1 \ln^7 t \, dt = \int_\infty^0 (-x)^7 \cdot (-e^{-x}) \, dx$

$= \int_\infty^0 x^7 e^{-x} dx = -\int_0^\infty e^{-x} x^7 dx = \Gamma(8) = 7!$

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2) $\int_0^1 \sqrt[3]{x} \ln^5 x \, dx$

$\Rightarrow$ let $-t = \ln x \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$I = \int_0^1 x^{1/3} \ln^5 x \, dx = \int_\infty^0 (e^{-t})^{1/3} (-t)^5 (-e^{-t}) \, dt$

$= \int_\infty^0 e^{-4/3 t} t^5 dt \quad \Rightarrow$ let $u = \frac{4}{3}t \Rightarrow t = \frac{3}{4}u$

$\Rightarrow dt = \frac{3}{4} du$

$\Rightarrow I = \int_\infty^0 e^{-u} (\frac{3}{4}u)^5 \cdot \frac{3}{4} du = (\frac{3}{4})^6 \int_\infty^0 e^{-u} u^5 du$

$= -(3/4)^6 \int_0^\infty e^{-u} u^5 du = -(\frac{3}{4})^6 \Gamma(6) = -(\frac{3}{4})^6 \cdot 5!$

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3) $\int_0^\infty x^3 e^{-2x^5} dx$

let $2x^5 = t \Rightarrow x = (t/2)^{1/5} = (\frac{1}{2})^{1/5} \cdot t^{1/5}$

$\Rightarrow dx = (\frac{1}{2})^{1/5} \cdot \frac{1}{5} t^{-4/5} dt$

$I = \int_0^\infty (\frac{1}{2})^{3/5} \cdot t^{3/5} e^{-t} \cdot (\frac{1}{2})^{1/5} \cdot \frac{1}{5} t^{-4/5} dt$

$$I = \frac{1}{5}\left(\frac{1}{2}\right)^{4/5} \int_0^\infty t^{-1/5} e^{-t}\, dt$$

$$= (1/5)(1/2)^{4/5}\, \Gamma(4/5)$$

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4) $\int_0^\infty 3^{-x^2}\, dx$

Let $3^{-x^2} = e^{-t} \Rightarrow -x^2 \ln 3 = -t$

$\Rightarrow x = \sqrt{\frac{t}{\ln 3}} \Rightarrow dx = \frac{1}{\sqrt{\ln 3}} \cdot \frac{1}{2\sqrt{t}}\, dt$

$x: 0 \to \infty \Rightarrow t: 0 \to \infty$

$$\int_0^\infty e^{-t} \cdot \frac{1}{\sqrt{\ln 3}} \cdot \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2\sqrt{\ln 3}} \int_0^\infty t^{-1/2} e^{-t}\, dt$$

$$= \frac{1}{2\sqrt{\ln 3}}\, \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{\ln 3}}.$$

We can extend the definition of the gamma fn. to include +ve & -ve values of x by using the reccurrence relation, But def. 1 is a def. for $\Gamma(x)$ only for $x > 0$.

Remark :-

- For $x > 0$ we define $\Gamma(x)$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

or

$$= (x-1)\,\Gamma(x-1)$$

- For $x \leq 0$ we define $\Gamma(x)$ only by the use of the reccurrence relation in the form

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \longrightarrow (2)$$

4) $\Gamma(-n) = \pm\infty$; for $n \geq 0$

Proof :-

to get $\Gamma(0)$ we have $\Gamma(0) = \frac{\Gamma(1)}{0} = \pm\infty$ (using 2)

So, $\Gamma(-1)$ by using 2 also $= \frac{\Gamma(0)}{-1} = \pm\infty$

$\Gamma(-2)$ " " " $= \frac{\Gamma(-1)}{-2} = \pm\infty$

$\vdots$

$\Rightarrow \Gamma(\text{nonpositive integer}) = \pm\infty$

Examples:- Evaluate 1) $\Gamma(-5/2)$

2) $\Gamma(-4.3)$

3) $L(t^x)$ & hence $L(1/\sqrt{t})$.

Solution :-

1) $\Gamma(-5/2) = \frac{\Gamma(-3/2)}{-5/2} = \frac{\Gamma(-1/2)}{-\frac{5}{2}(-3/2)}$

$= \frac{\Gamma(1/2)}{-\frac{5}{2}(-\frac{3}{2})(-\frac{1}{2})} = -\frac{\sqrt{\pi}}{15/8}$

$= -\frac{8\sqrt{\pi}}{15}$.

2) $\Gamma(-4.3) = \frac{\Gamma(-3.3)}{-4.3} = \frac{\Gamma(-2.3)}{(-4.3)(-3.3)}$

$= \frac{\Gamma(-1.3)}{(-4.3)(-3.3)(-2.3)} = \frac{\Gamma(-0.3)}{(-4.3)(-3.3)(-2.3)(-1.3)}$

$= \frac{\Gamma(0.7)}{(-4.3)(-3.3)(-2.3)(-1.3)(-0.3)}$

$= \frac{\Gamma(1.7)}{(-4.3)(-3.3)(-2.3)(-1.3)(-0.3)(0.7)} = \frac{0.909}{-8.91} = -0.102$

3) $L(t^x) = \int_0^\infty t^x e^{-st} dt$

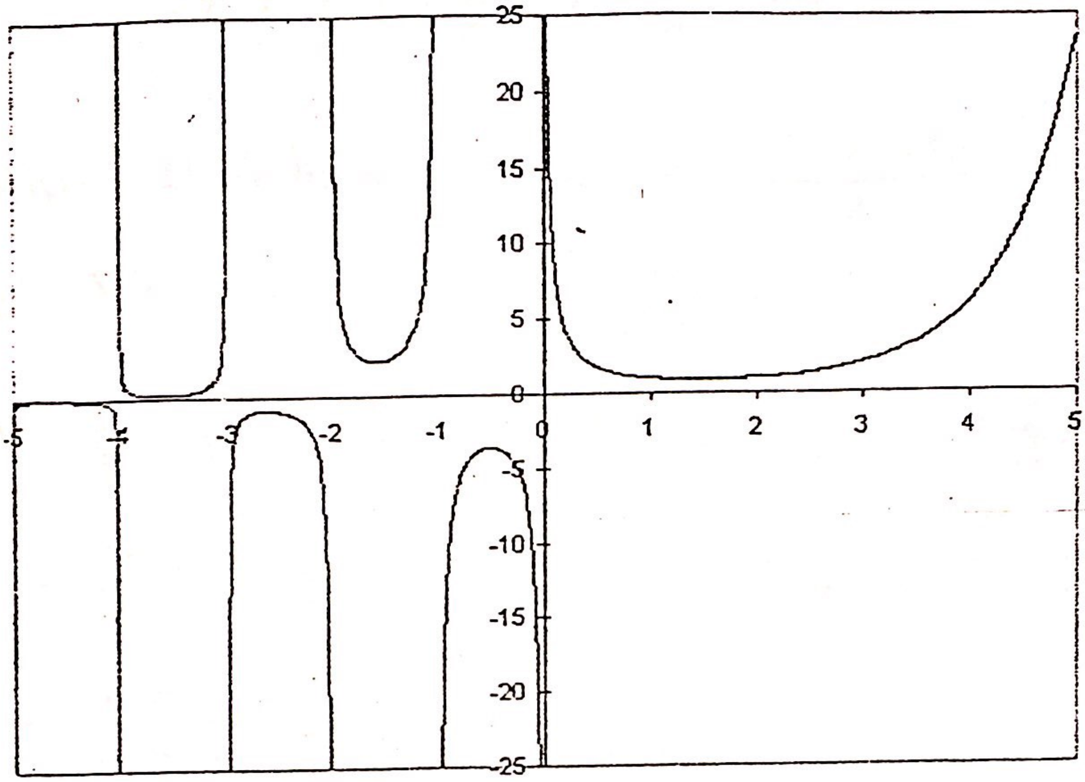
Let $st = u \Rightarrow t = \frac{u}{s} \Rightarrow dt = \frac{1}{s} du$

$L(t^x) = \int_0^\infty \frac{u^x}{s^x} e^{-u} \cdot \frac{1}{s} du = \frac{1}{s^{x+1}} \int_0^\infty u^x e^{-u} du$

$= \frac{\Gamma(x+1)}{s^{x+1}}$

$L(\frac{1}{\sqrt{t}}) = L(t^{-1/2}) \xrightarrow{Put\ x = -\frac{1}{2}} \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$.

5) The graph of the Gamma fn. $\Gamma(x)$:-



You can observe that $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(n+1) = n!$

& $\Gamma(-n) = \Gamma(0) = \pm\infty$.

6) The Multiplication formula :-

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Proof :- The proof is too ridiculous & we will not mention it , it starts by using the Weierstrass formula that yields a functional equation which ends by this required formula.

Example: Evaluate 1) $\Gamma(\frac{7}{3})\,\Gamma(-\frac{7}{3})$

2) $\Gamma(x+\frac{1}{2})\,\Gamma(\frac{1}{2}-x)$

Solution : 1) We have $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$

Then $\Gamma(-7/3) = \dfrac{\Gamma(-4/3)}{-7/3}$

$$\Rightarrow \Gamma(\tfrac{7}{3})\,\Gamma(-\tfrac{7}{3}) = -\frac{3}{7}\cdot\Gamma(\tfrac{7}{3})\,\Gamma(-\tfrac{4}{3})$$

$$= -\frac{3}{7}\cdot\Gamma(\tfrac{7}{3})\,\Gamma(1-\tfrac{7}{3}) = -\frac{3}{7}\cdot\frac{\pi}{\sin(\frac{7\pi}{3})}$$

$$= -\frac{3\pi}{7\sin(\frac{\pi}{3})} = -\frac{6\pi}{7\sqrt{3}}.$$

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2) $\Gamma(x+\frac{1}{2})\,\Gamma(\frac{1}{2}-x)$

$$= \Gamma(x+\tfrac{1}{2})\cdot\Gamma(1-(x+\tfrac{1}{2}))$$

$$= \frac{\pi}{\sin\pi(x+\frac{1}{2})} = \frac{\pi}{\sin(\pi x+\frac{\pi}{2})}$$

$$= \frac{\pi}{\cos\pi x}.$$

Examples

★ Find the Conditions under which the integral $\int_0^\infty x^a b^{-x}\,dx$ Converges & hence evaluate it interms of $\Gamma(x)$.

Solution :-

$$I = \int_0^\infty x^a b^{-x}\,dx$$

let $b^{-x} = e^{-t} \Rightarrow x \ln b = t \Rightarrow x = \frac{t}{\ln b}$

$$\Rightarrow dx = \frac{dt}{\ln b}$$

$$I = \int_0^\infty \frac{t^a}{(\ln b)^a} e^{-t} \cdot \frac{dt}{\ln b} \; ; \text{ for } \ln b > 0 \Rightarrow b > 1$$

$$= \frac{1}{(\ln b)^{a+1}} \int_0^\infty t^a e^{-t}\,dt$$

$$= \frac{\Gamma(a+1)}{(\ln b)^{a+1}} \; ; \text{ for } a+1 > 0 \Rightarrow a > -1$$

★ Use your result to evaluate i) $\int_0^\infty t^{3/2} 5^{-t}\,dt$

ii) $\int_0^\infty \frac{1}{\pi^x \, x\sqrt{x}}\,dx$.

Solution :-

i) We have $a = 3/2$ & $b = 5$

$$\Rightarrow I = \frac{\Gamma(3/2 + 1)}{(\ln 5)^{5/2}} = \frac{3/2\,\Gamma(3/2)}{(\ln 5)^{5/2}} = \frac{3/2 \cdot \frac{1}{2}\,\Gamma(\frac{1}{2})}{(\ln 5)^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4(\ln 5)^{5/2}}$$

ii) We have $I = \int_0^\infty \pi^{-x} \cdot x^{-3/2}\,dx \Rightarrow a = -\frac{3}{2}$ & $b = \pi$

$\Rightarrow$ Since $a = -3/2 < -1 \Rightarrow$ the integration diverges.

Example :- For what values of a does this integral

$$\int_0^{\sqrt{e}} (\ln x - \tfrac{1}{2})^a x^2 \, dx$$ has a value.

Solution :- Let $\ln x - \frac{1}{2} = u \Rightarrow \ln x = u + 1/2$

$x = e^{u+1/2} \Rightarrow x = \sqrt{e} \cdot e^u$

$\Rightarrow dx = \sqrt{e} \cdot e^u \, du$

$x : 0 \longrightarrow \sqrt{e} \Rightarrow u : -\infty \longrightarrow 0$

$$\int_{-\infty}^{0} u^a \cdot (\sqrt{e} \cdot e^u)^2 \cdot \sqrt{e} \cdot e^u \, du$$

$$= \int_{-\infty}^{0} u^a \, e\sqrt{e} \, e^{3u} \, du = e\sqrt{e} \int_{-\infty}^{0} u^a e^{3u} \, du$$

Let $3u = -t \Rightarrow du = -\frac{dt}{3}$

$$e\sqrt{e} \int_{\infty}^{0} (-\tfrac{t}{3})^a e^{-t} \cdot \frac{-dt}{3} = \frac{(-1)^a e\sqrt{e}}{3^{a+1}} \int_0^{\infty} t^a e^{-t} \, dt$$

$$= \frac{(-1)^a e\sqrt{e}}{3^{a+1}} \cdot \Gamma(a+1) \quad ; \quad a+1 > 0$$

The integration will converge if $a > -1$.

Example: Show that $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n} \cdot n!} \sqrt{\pi}$

Solution:-

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2})$$

$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2}) \cdots \tfrac{5}{2} \cdot \tfrac{3}{2} \cdot \tfrac{1}{2}\,\Gamma(1/2)$$

$$= \frac{1}{2^n}(2n-1)(2n-3) \cdots 5.3.1 \cdot \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)(2n-1)(2n-2)(2n-3) \cdots 6.5.4.3.2.1}{(2n) \quad (2n-2) \quad \cdots \quad 6 \cdot 4 \cdot 2}$$

$$= \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)!}{(2n)(2n-2) \cdots 6.4.2.}$$

$$= \frac{\sqrt{\pi}}{2^n \cdot 2^n} \cdot \frac{(2n)!}{n(n-1) \cdots 3.2.1} = \frac{\sqrt{\pi}}{2^{2n}} \cdot \frac{(2n)!}{n!}$$

Example: Show that $\Gamma(\nu) = \int_0^1 (\ln \frac{1}{x})^{\nu-1} dx \; ; \nu > 0$

Solution:- Let $\ln \frac{1}{x} = t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$$I = \int_{\infty}^{0} t^{\nu-1}(-e^{-t} dt)$$

$$= \int_0^{\infty} t^{\nu-1} e^{-t} dt = \Gamma(\nu), \text{ for } \nu > 0$$

Homework: Evaluate

1) $\int_0^{\infty} \sqrt[4]{x}\, e^{-\sqrt{x}} dx$

2) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$