section 5

Find series solution in powers of x

ⓐ $xy'' + (1+x)y' + 2y = 0$ $\Rightarrow$ R.S.P (do the test)

P let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$; y' ... , y'' = ...

$x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + (1+x) \sum_{n=0} (n+s) a_n x^{n+s-1}$

$+ 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$

$\Rightarrow 0 + 0 + \sum_{n=1}^{\infty} (n+s-1) a_{n-1} x^{n+s-1} + \sum_{n=1} 2 a_{n-1} x^{n+s-1} = 0$

$\Rightarrow s(s-1) a_0 x^{s-1} + s a_0 x^{s-1} + \sum_{n=1}^{\infty} \left[\begin{matrix} (n+s)(n+s-1) a_n \\ + (n+s) a_n \\ + (n+s-1) a_{n-1} \\ + 2a_{n-1} \end{matrix} \right] x^{n+s-1} = 0$

Coef $(x^{s-1}) = 0$ $s^2 a_0 = 0$ $\because a_0 \neq 0$ $s^2 = 0$ $s_1 = s_2 = 0$

Case 2

Coef $(x^{n+s-1}) = 0$ $\left[(n+s)(n+s-1) + (n+s) \right] a_n + \left[(n+s-1) + 2 \right] a_{n-1} = 0$

$\Rightarrow a_n = - \dfrac{[n+s+1]}{(n+s)^2} a_{n-1}$ $n \geq 1$

$n = 1 \Rightarrow a_1 =$

$n = 2 \Rightarrow a_2$

$n = 3 \Rightarrow a_3$ get the general form in terms of s for case 2

$$\boxed{a_n(s) = \frac{(-1)^n (s+2)(s+3) \cdots \overset{(s+n)}{} (s+n+1)}{(s+1)^2 (s+2)^2 \cdots (s+n)^2}}$$ let $a_0 = 1$

Can be $a_n(s) = \dfrac{(-1)^n (s+n+1)}{(s+1)^2 \left[(s+2)(s+3) \cdots (s+n) \right]}$ will be better in in differentiation

Complex number but still a constant

$a_n'(s) = ?$ $\ln a_n = \ln(-1)^n + \ln(s+2) + \ln(s+3) + \dots + \ln(s+n+1)]$
$- 2[\ln(s+1) + \ln(s+2) + \dots + \ln(s+n)]$

$$\frac{a_n'}{a_n} = 0 + \left[\frac{1}{s+2} + \frac{1}{s+3} + \dots + \frac{1}{s+n+1}\right] - 2\left[\frac{1}{s+1} + \frac{1}{s+2} + \dots + \frac{1}{s+n}\right]$$

$$\Rightarrow a_n(0) = \frac{(-1)^n (2.3.4 \dots (n+1))}{1.2.3 \dots n} = \frac{(-1)^n (n+1)!}{(n!)^2}$$

$y_1 = y(x, 0)$

$$y_1 = 1 + \sum_{n=1}^{\infty} a_n(0) x^{n+0}$$

$a_0 x^s = a_0 \cdot 1 \quad s = 0$

$\therefore a_0 x^s = 1$

$\because s_1 = s_2 = 0$

$$\therefore y_2 = y_1 \ln|x - 0| + \sum_{n=1} a_n'(0) x^{n+0}$$

$\therefore$ ~~$a_n(s) =$~~

$$a_n'(0) = a_n(0)\left[\frac{1}{2} + \frac{1}{3} \dots + \frac{1}{n+1}\right] - 2\left[1 + \frac{1}{2} \dots + \frac{1}{n}\right]$$

$H_{n+1} - 1 \qquad H_n$

$$y_2 = y_1 \ln|x| + \sum_{n=1}^{\infty} [a_n'(0)] x^n$$

$\therefore y_{gs} = C_1 y_1 + C_2 y_2$ #

$$\Rightarrow y_2 = \left.\frac{\partial y}{\partial s}\right|_s = \frac{\partial}{\partial s}\left(\sum_{n=0} a_n(s) x^{(n+s)}\right) = \sum_{n=0} a_n(s) x^{n+s} \ln|x|$$
$$+ \sum_{n=1} a_n'(s) x^{n+s}$$

Find series solution in powers of $(x+1)$

$$xy'' + (1+x)y' + 2y = 0$$

let $t = x+1 \Rightarrow x = t-1$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(\dot{y}) = \frac{d}{dt}(\dot{y}) \cdot \frac{dt}{dx} = \ddot{y}$$

$$\therefore (t-1)\ddot{y} + t\dot{y} + 2y = 0 \Rightarrow y = \sum_{n=0}^{\infty} a_n t^n$$

Bessel function

$$\boxed{\begin{array}{l} x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0 \\ y_{gs} = C_1 J_\nu(\lambda x) + C_2 J_{-\nu}(\lambda x), \quad \nu \neq \text{integer} \\ y_{gs} = C_1 J_n(\lambda x) + C_2 Y_n(\lambda x), \quad \nu = n = \text{integer} \end{array}}$$

solve in terms of bessel function :-

① $x^2 y'' + xy' + (x^2 - \frac{1}{9}) y = 0$ $\qquad \lambda^2 = 1 \quad \lambda = 1$

$\nu^2 = \frac{1}{9} \quad \nu = \frac{1}{3} \neq \text{integer}$

$$y_{gs} = C_1 J_{\frac{1}{3}}(x) + C_2 J_{-\frac{1}{3}}(x)$$

2) $x^2 y'' + x y' + (2x^2 - 1) y = 0$

$\lambda^2 = 2 \quad \lambda = \sqrt{2} \qquad \nu^2 = 1 \quad \nu = 1$ (integer)

$y_{gs} = c_1 J_1(\sqrt{2}\,x) + c_2 Y_1(\sqrt{2}\,x)$

③ $x^2 y'' + x y' + (x-4) y = 0$

$x = t^2 \quad$ — ①

$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} t^{-1} \dot{y}$ — ②

$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{2} t^{-1} \dot{y}\right) = \frac{d}{dt}\left(\frac{1}{2} t^{-1} \dot{y}\right) \cdot \frac{dt}{dx}$

$= \frac{1}{2}\left[-t^{-2}\dot{y} + t^{-1}\ddot{y}\right]\left[\frac{1}{2} t^{-1}\right]$

$y'' = \frac{1}{4}\left(-t^{-3}\dot{y} + t^{-2}\ddot{y}\right)$ — ③

$\Rightarrow \frac{t^4}{4}\left(-t^{-3}\dot{y} + t^{-2}\ddot{y}\right) + t^2\left(\frac{1}{2} t^{-1} \dot{y}\right) + (t^2 - 4) y = 0$

$\Rightarrow \frac{t^2}{4}\ddot{y} + \left(\frac{t}{4}\right)\dot{y} + (t^2 - 4) y = 0 \qquad * 4$

$\Rightarrow t^2 \ddot{y} + t \dot{y} + (4t^2 - 16) y = 0$

$\therefore y_{gs} = c_1 J_4(2t) + c_2 Y_4(2t) = c_1 J_4(2x^{\frac{1}{2}}) + c_2 Y_4(2x^{\frac{1}{2}})$

(4) $x^2 y'' + xy' + 4(x^4 - n^2) y = 0$

let $x^4 = t^2 \Rightarrow t = x^2 \Rightarrow x = t^{1/2}$

$y' = \frac{d}{dx}(y) = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y}(2x) = \dot{y}(2t^{1/2})$

$y'' = \frac{d}{dx}(y') = \frac{d}{dt}[2t^{1/2}\dot{y}]\frac{dt}{dx} = 2\{\frac{1}{2}t^{-1/2}\dot{y} + t^{1/2}\ddot{y}\}(2t^{1/2})$

$y'' = 2\dot{y} + 4t\ddot{y}$

$\Rightarrow t[2\dot{y} + 4t\ddot{y}] + t^{1/2}[2t^{1/2}\dot{y}] + 4[t^2 - n^2]y = 0$

$\Rightarrow \cancel{4}t^2\ddot{y} + \cancel{4}t\dot{y} + \cancel{4}(t^2 - n^2)y = 0$

$y_{gs} = C_1 J_n(t) + C_2 Y_n(t)$

(assumin n integer

$= C_1 J_n(x^2) + C_2 Y_n(x^2)$