FUNCTIONS OF COMPLEX VARIABLES

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Agenda

- Introduction
- Algebraic interpretation
- Geometric interpretation
- Mapping

Applications

Fundamental Theorem of Algebra

AC circuit analysis

Solution of Differential Equations

Fourier Transform

Motivation

The equation $\boldsymbol{x^2 + 1 = 0}$

a real problem but has no real solutions

Why ?

So we make up a new symbol for the roots and call it a **complex number**.

Definition. The symbols $\pm$ i will stand for the solutions to the equation $\boldsymbol{x^2 = -1}$

According to the defined quantity $\sqrt{-1}$ = i

$$\boldsymbol{i^2 = -1}, \qquad \boldsymbol{i^3 = -i}, \qquad \boldsymbol{i^4 = (-1)^2}$$

Motivation

This number $\pm i$ is called an **imaginary number**.
These are valid numbers that don't lie on the real number line.

If $z = a+bi$,
a is called the real part of z denoted by **Re**{z}
b is called the imaginary part of z denoted by **Im**{z}
The symbol z is called a complex variable.

We're going to show the algebra,
geometry of complex numbers

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- Geometry of Complex Numbers
- Mapping

Algebra of Complex Numbers

- **Addition** $(x_1 + i\,y_1) + (x_2 + i\,y_2) = (x_1 + x_2) + i(y_1 + y_2)$
  - Example: $(2 +\ 3i) + (1 + 2i)$
- **Subtraction** $(x_1 + i\,y_1) - (x_2 + i\,y_2) = (x_1 + x_2) - i\,(y_1 + y_2)$
  - Example: $(2 + 3i) - (1 + 2i)$
- **Multiplication** $(x_1 + i\,y_1)(x_2 + i\,y_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
  - Example: $(2 + 3i)(1 + 2i)$

Algebra of Complex Numbers

- **Complex Conjugation** $\overline{(x+yi)} = x - yi$
  - Example: $\overline{(2+3i)}$
- **Norm or Absolute Value** $|(x+yi)| = \sqrt{Z\bar{Z}} = \sqrt{x^2+y^2}$
  - Example: $|(2+3i)|$
- **Division** $\frac{(x_1+y_1 i)}{(x_2+y_2 i)} = \frac{(x_1+y_1 i)(x_2-y_2 i)}{(x_2+y_2 i)(x_2-y_2 i)} = \frac{(x_1 x_2+y_1 y_2)+(x_2 y_1+x_1 y_2)i}{({x_2}^2+{y_2}^2)}$
  - Example: $\frac{(2+3i)}{(1+2i)}$

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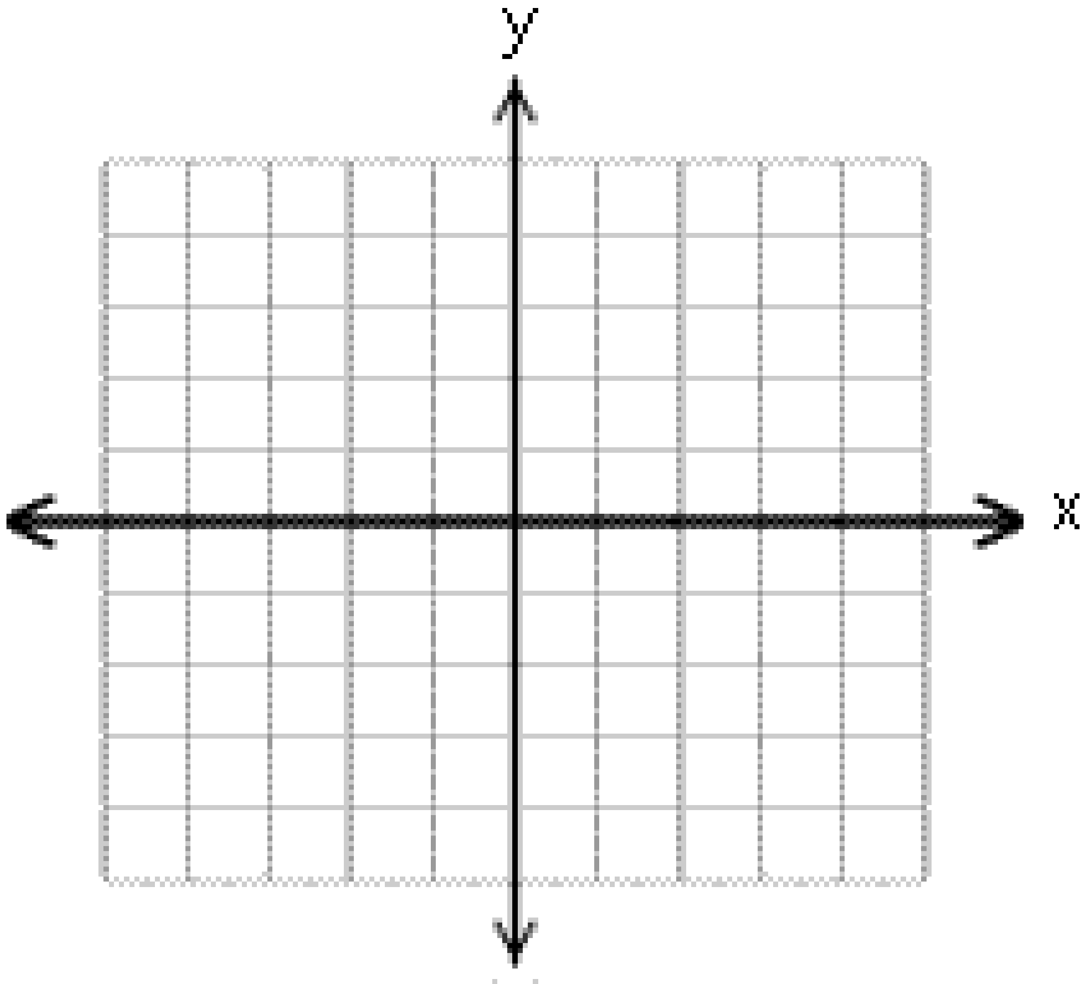
Geometry of Complex Numbers

$Z_1 = 3 + i \qquad Z_2 = 1 + 2i$

$Z_1 + \overline{Z_1} =$

$Z_1 + 2 =$

$Z_1 + Z_2 =$



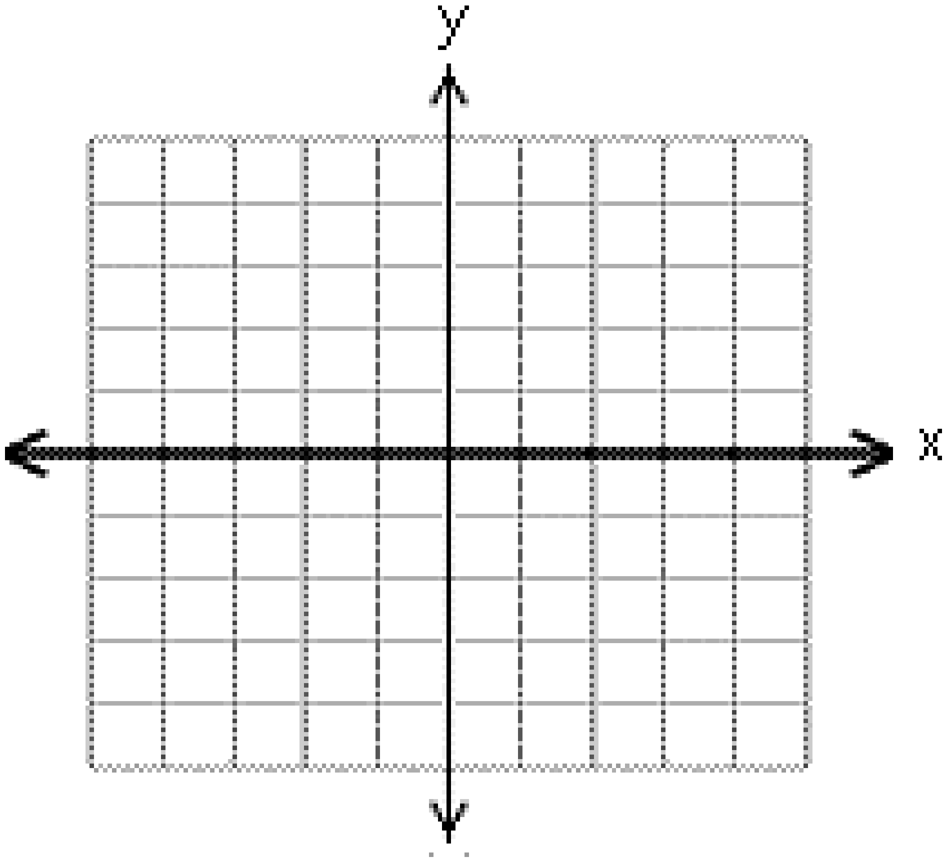
Complex Numbers in Polar form

If $z = re^{i\theta}$,
r is called the absolute value of z
θ is called argument of z

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\boldsymbol{z = re^{i\theta} = \mathrm{r}\cos\theta + i\, r\sin\theta}$$



Complex Numbers in Polar form

$$Z_1 = (\sqrt{3} + i)$$

$r = \sqrt{3+1} = 2$ $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Let $Z_2 = Z_1$ $r_{z_2} = r_{z_1} = 2$

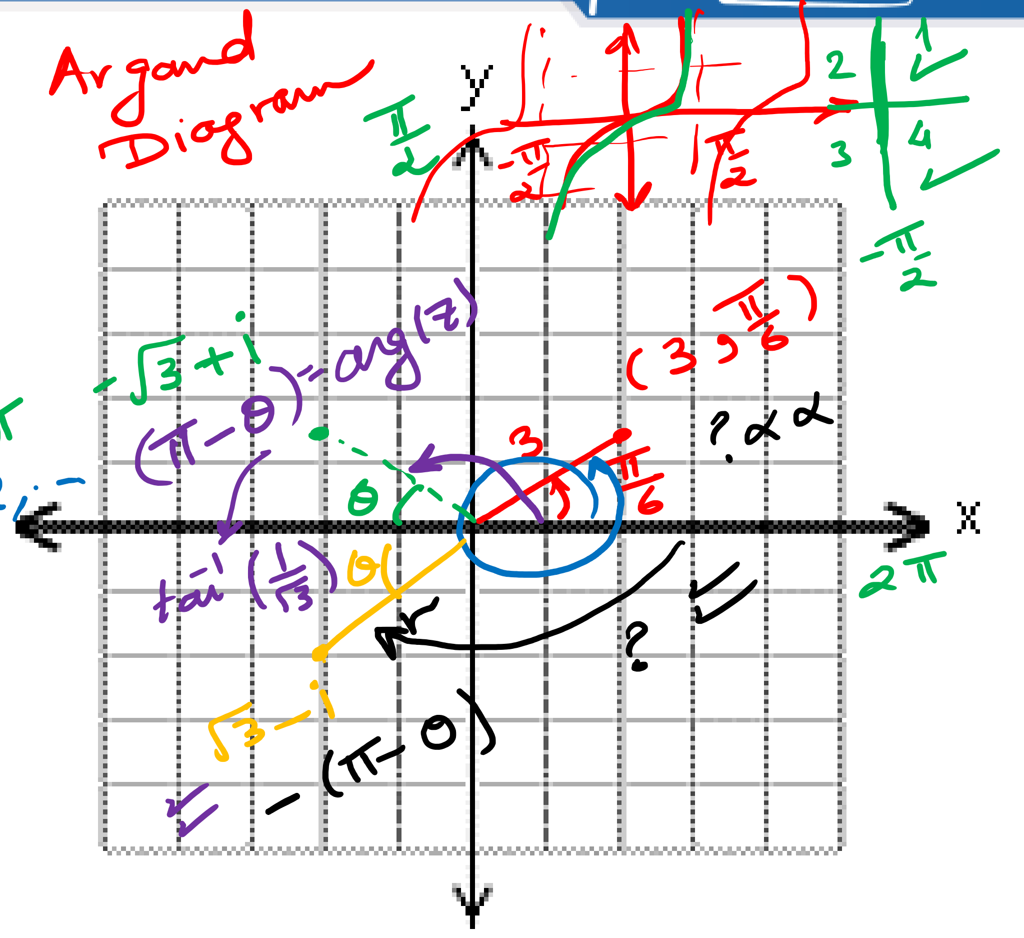
$\theta_{z_2} = \theta_{z_1} + 2n\pi$ $n = 0, \pm 1, \pm 2, \ldots$

The Principal Argument Arg (z)ϵ $(-\pi, \pi]$

closed

open why??

Argand Diagram



$\arctan \frac{y}{x}$	if $x > 0, y \in \mathbb{R}$
$\arctan \frac{y}{x} + \pi$	if $x < 0, y \geq 0$
$\arctan \frac{y}{x} - \pi$	if $x < 0, y < 0$
$\frac{\pi}{2}$	if $x = 0, y > 0$
$-\frac{\pi}{2}$	if $x = 0, y < 0$

Complex Numbers in Polar form

- **Multiplication**

$$(\boldsymbol{r_1},\boldsymbol{\theta_1})(\boldsymbol{r}_2,\boldsymbol{\theta_2}) = \boldsymbol{r_1}\,\boldsymbol{r_2}(\cos\theta_1 + i\sin\theta_1)\,(\cos\theta_2 + i\sin\theta_2)$$

$$\boldsymbol{r_1}\,\boldsymbol{r_2}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$= (\boldsymbol{r_1}\boldsymbol{r_2}, \boldsymbol{\theta_1} + \boldsymbol{\theta_2})$$

Results

$$\boxed{(\boldsymbol{r_1},\boldsymbol{\theta_1})\ldots.(\boldsymbol{r}_n,\boldsymbol{\theta_n}) = (\boldsymbol{r_1}\ldots.\boldsymbol{r_n},\boldsymbol{\theta_1}\ldots+\boldsymbol{\theta_n})}$$

$$\boxed{(\boldsymbol{r_1},\boldsymbol{\theta_1})^n = (\,\boldsymbol{r_1}^n, n\,\boldsymbol{\theta_1})}$$

$$\boxed{(\boldsymbol{Z})^n = \boldsymbol{r}\,(\cos\theta + i\sin\theta\,) = \boldsymbol{r}^{\,n}\,(\cos n\,\boldsymbol{\theta_1} + i\sin n\,\boldsymbol{\theta_1})}$$

De Moivre Theorem

Geometry of Complex Numbers

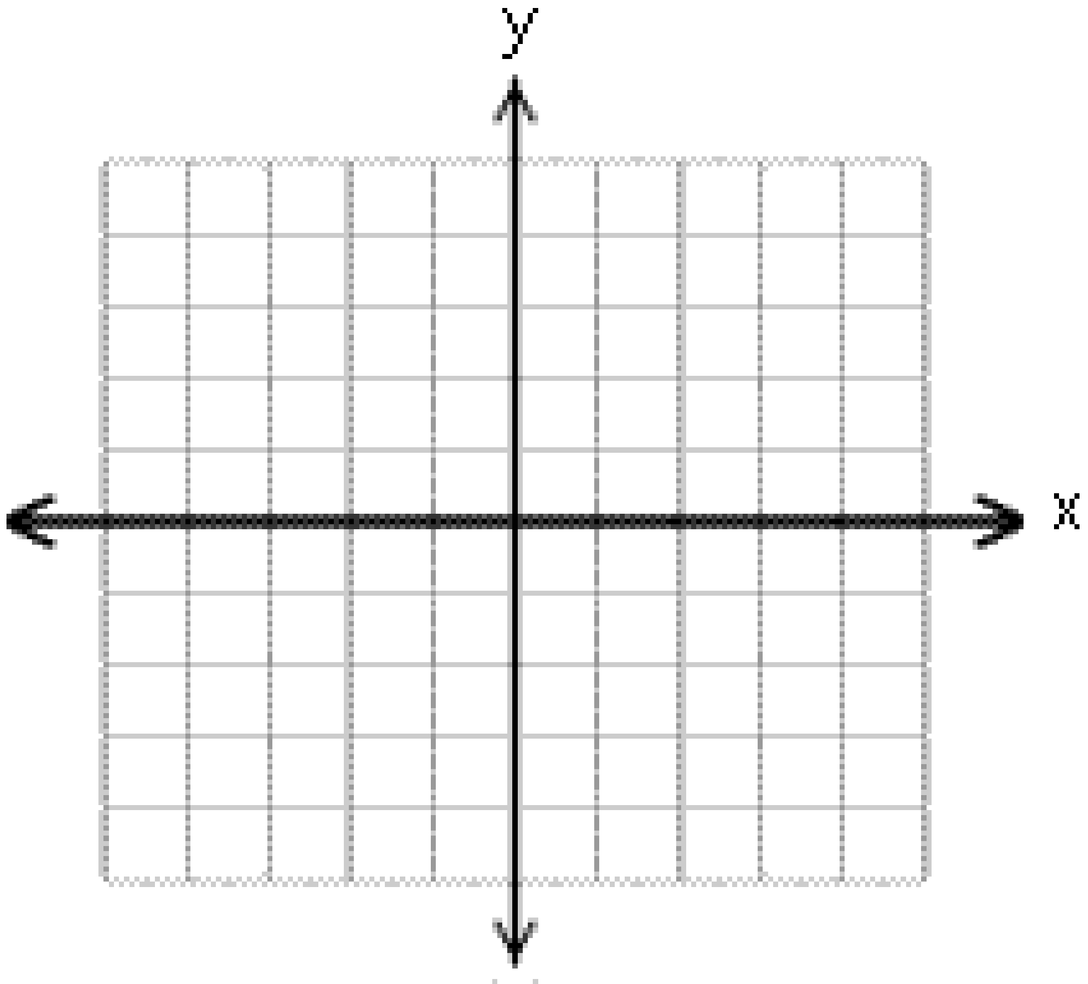
$Z_1 = 3e^{i\frac{\pi}{4}}$ $\quad Z_2 = 2e^{i\frac{\pi}{3}}$

$Z_1 . 2 =$

$Z_1 . i =$

$Z_1 . (e^{i\frac{4\pi}{3}}) =$

$Z_1 Z_2 =$



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Functions of Complex Variables as Transformations

Consider the function

$$\boxed{w = f(z) \quad , \quad z = x + i\,y} \qquad \boxed{\Rightarrow w = u(x,y) + i\,v(x,y)}$$

Example 1:

$$w = z^2 \quad = (x + iy)^2 \quad = (x^2 - y^2) + i\,(2\,x\,y)$$

$$\boxed{\Rightarrow u(x,y) = x^2 - y^2 \quad \& \quad v(x,y) = 2\,x\,y}$$

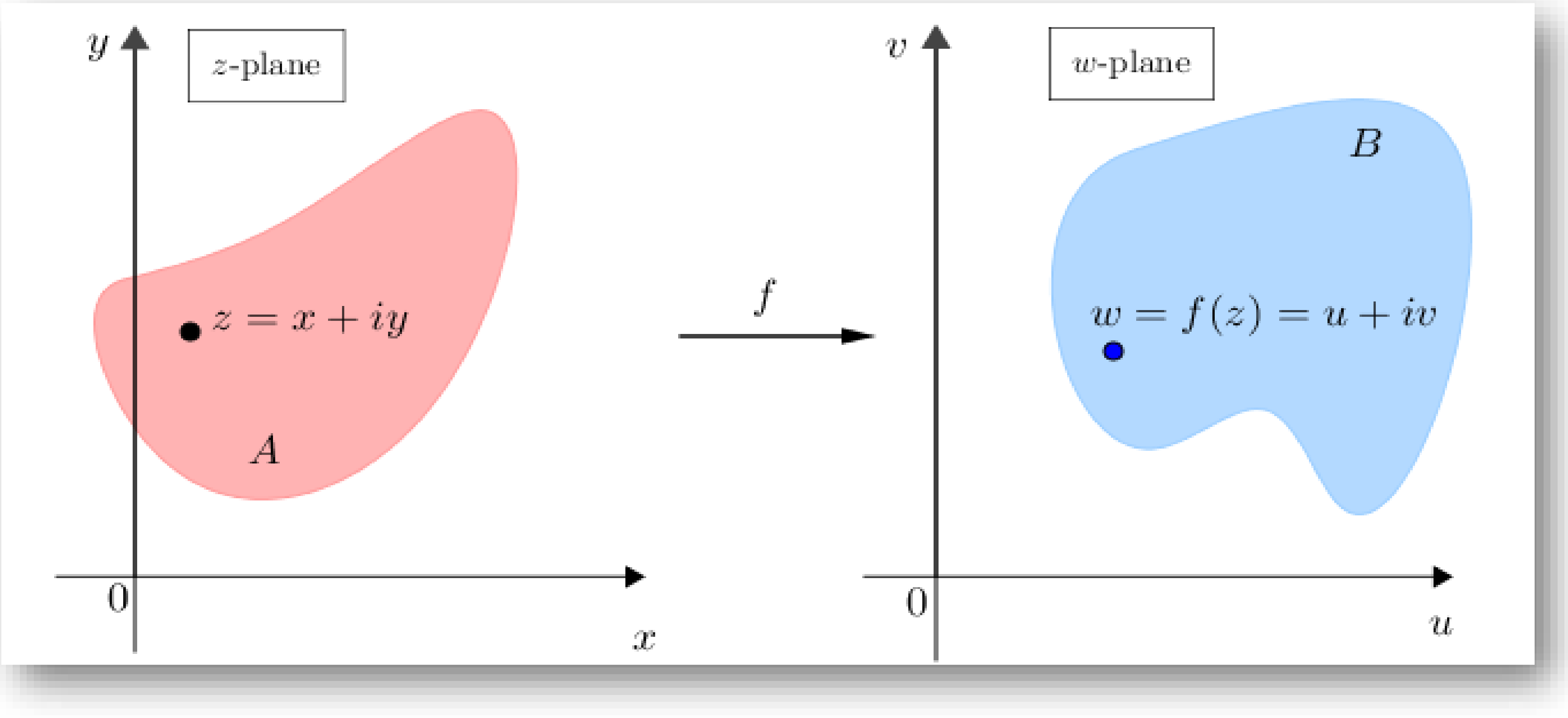
Example 2:

$$w = e^z \quad = e^{x+iy} \quad = e^x\,(\cos y + i \sin y)$$

$$\boxed{\Rightarrow u(x,y) = e^x \cos y \qquad v(x,y) = e^x \sin y}$$

$Z = f(x,y) \quad D \in R^2$

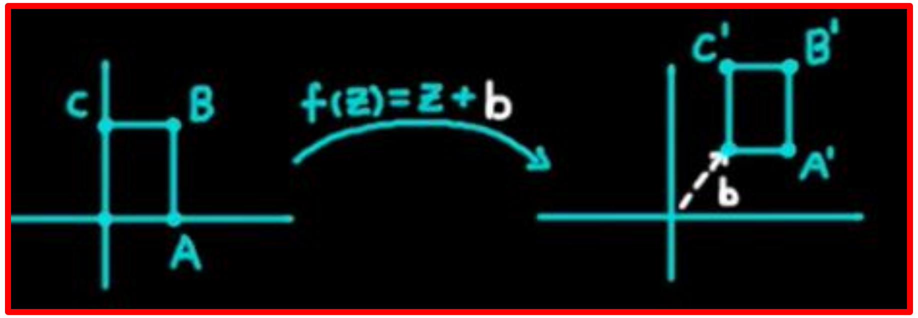
$w = u + iv \quad \text{Range} \in R^2$



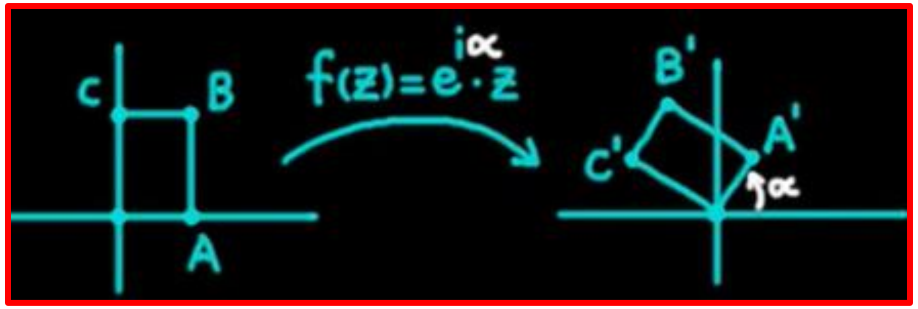
The region B is called the image od A and A is called the pre – image of B under the transformation $w = f(z)$ **.**

Transformations of Complex Numbers

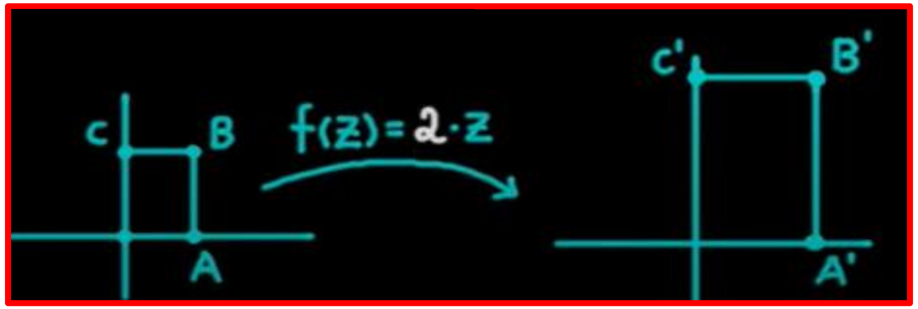
Translation



Rotation



Scaling



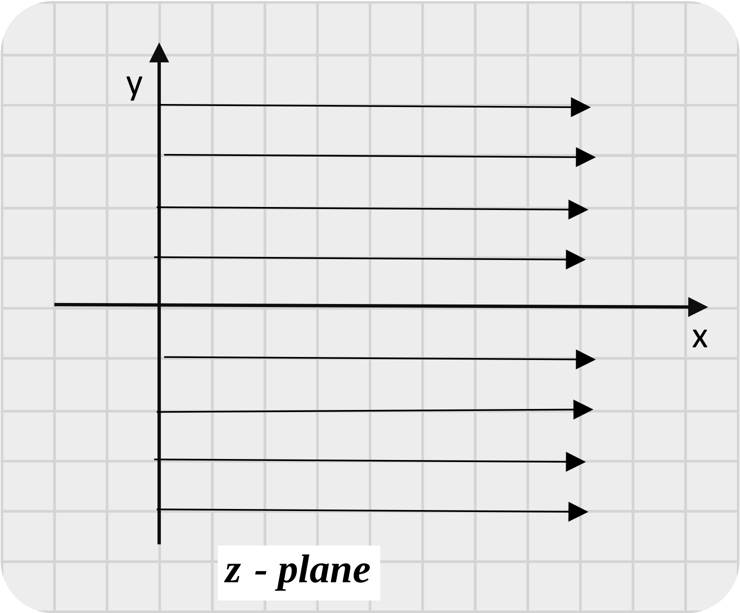
Example:

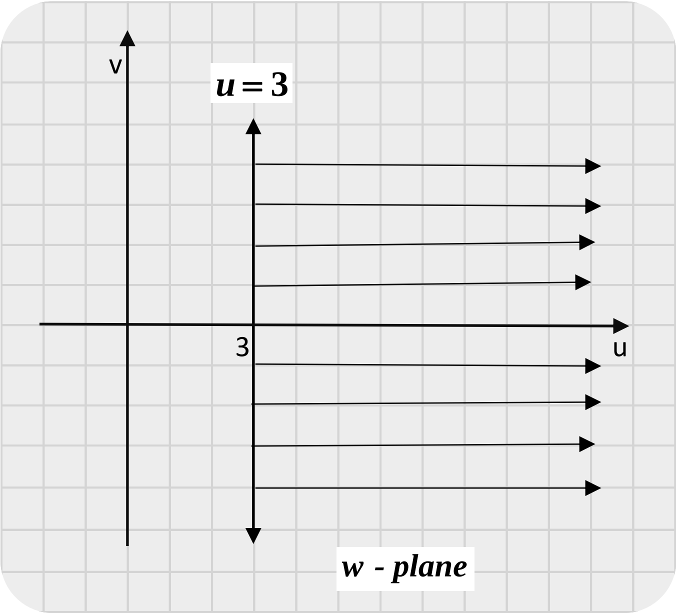
Find the image of the region $R(z) \geq 0$ under the transformation $w = z + 3$. Show the regions graphically.

Solution:

$$w = z + 3 = x + i y + 3 = (x + 3) + i y$$

$$\Rightarrow u = (x + 3) \quad \& \quad v = y \qquad \text{If} \quad x \geq 0 \Rightarrow u \geq 3$$





Example:

Find the image of the region $|z| \leq 2 \quad , \quad 0 < \arg(z) < \pi / 4$ under the transformation $w = z^2$. (Show the regions graphically).

Solution:

$$w = z^2 = (r\ e^{i\theta})^2 = r^2\ e^{i(2\theta)} \qquad \boxed{\Rightarrow |w| = |z|^2 \ ,} \qquad \boxed{\arg(w) = 2\arg(z)}$$

$$\because\ |w| = |z|^2 \ , \ |z| \leq 2 \qquad \Rightarrow |w| \leq 4$$

$$\because\ \arg(w) = 2\arg(z) \ , \ 0 < \arg(z) < \pi / 4 \qquad \Rightarrow 0 < \arg(w) < \pi / 2$$

The Linear Transformations $w = a\,z + b$

Where a and b are in general complex constants

- The linear transformation doesn't change figures shapes

- The linear transformation has three effects:

1: Shifting the origin to the point b

2: Scaling with $|a|$. (shrinking if $|a| < 1$ and enlarging if $|a| > 1$)

3: Rotation with $arg(a)$. (Anti clockwise if $arg(a) > 0$ and with clockwise if $arg(a) < 0$)

Example:

Find the image of the region bounded by the rectangle with vertices $(0,0),(1,0),(1,2)\&(0,2)$ under the transformation $w=(1+i)\,z+2$. (Show the regions graphically).

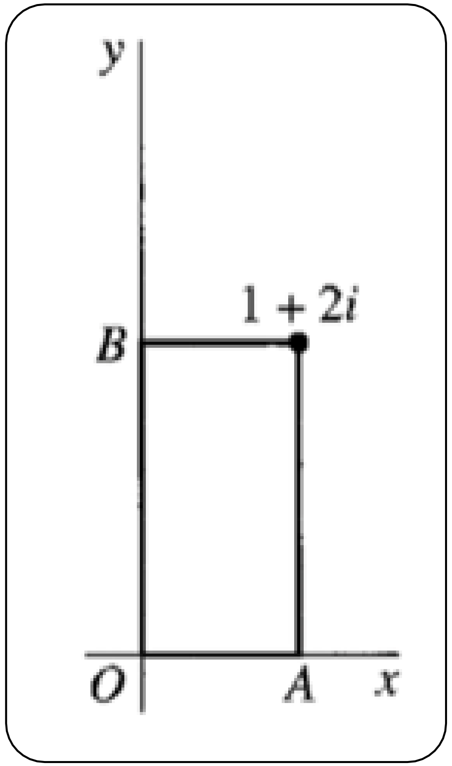
Solution:

$$w(0,0)=(1+i)\,(0)+2=2 \equiv (2,0)$$

$$w(1,0)=(1+i)\,(1)+2=3+i \equiv (3,1)$$

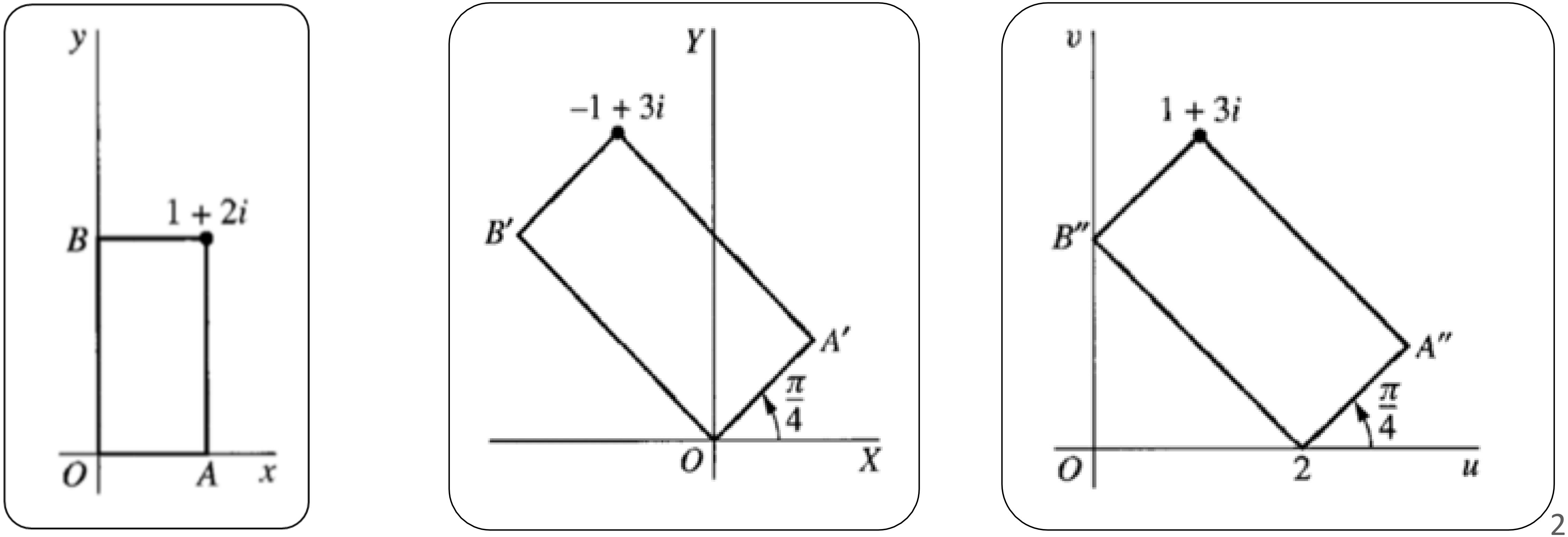
$$w(1,2)=(1+i)\,(1+2i)+2=1+3i \equiv (1,3)$$

$$w(0,2)=(1+i)\,(2i)+2=2i \equiv (0,2)$$



$$(0,0) \to (2,0)$$
$$(1,0) \to (3,1)$$
$$(1,2) \to (1,3)$$
$$(0,2) \to (0,2)$$

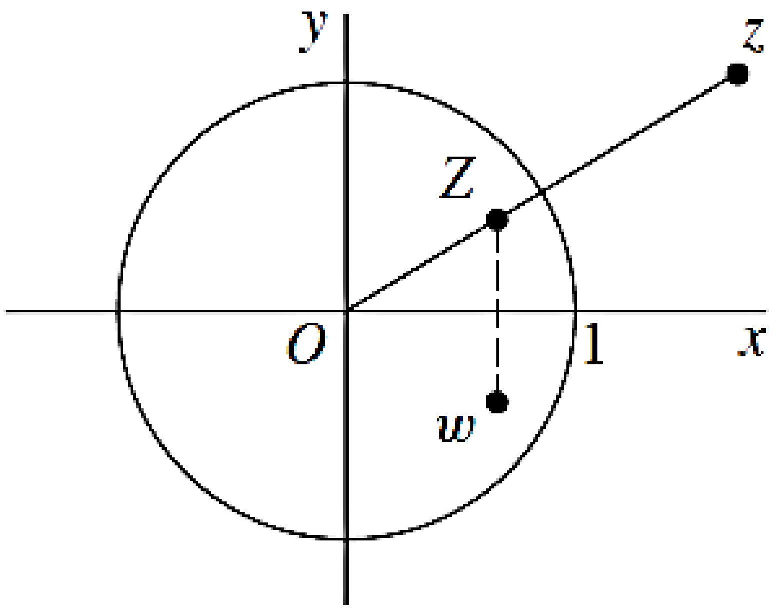
$$w = (1+i)\,z + 2$$



The Reciprocal Transformations $w = 1/z$

$$w = \frac{1}{z}$$

$$Z = \frac{z}{|z|^2}, \qquad w = \overline{Z}.$$



The Reciprocal Transformations $w = 1/z$

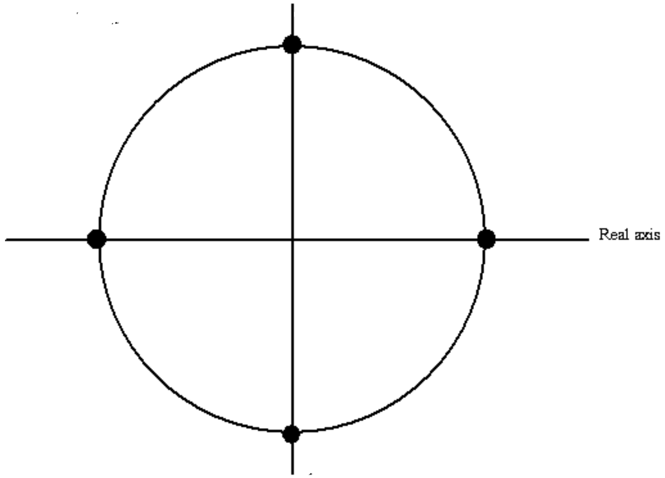
$$w = \frac{1}{z} = \frac{1}{r\,e^{i\theta}} = \frac{1}{r}\,e^{-i\theta}$$

$$\boxed{\Rightarrow |w| = \frac{1}{|z|}}$$

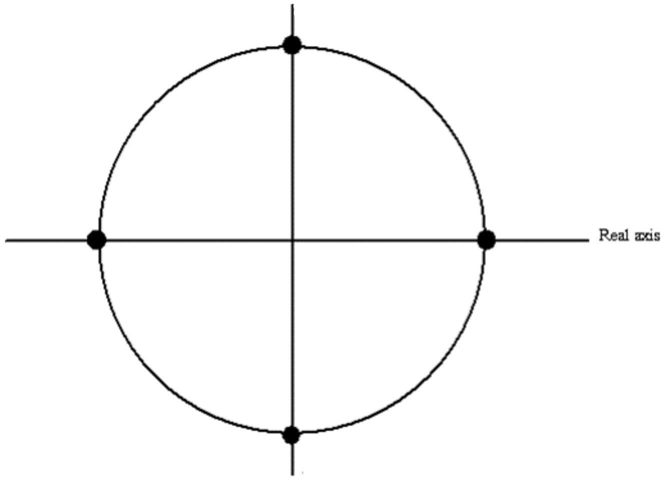
$$\boxed{\Rightarrow arg(w) = -arg(z)}$$

$$|z| = a \;\Rightarrow\; |w| = 1/a \qquad |z| < a \;\Rightarrow |w| > 1/a \quad \& \quad |z| > a \;\Rightarrow |w| < 1/a$$

$$arg(z) = \alpha \;\Rightarrow\; arg(w) = -\alpha$$



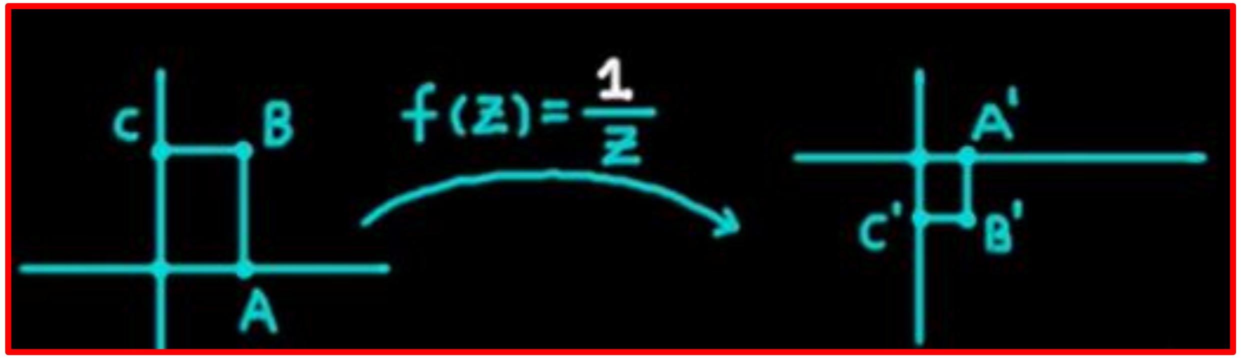
The z plane



The w plane

Transformations of Complex Numbers

Inversion



Next time ☺

- Horizontal and vertical lines are transformed into circles tangent the u and v axes respectively

Example:

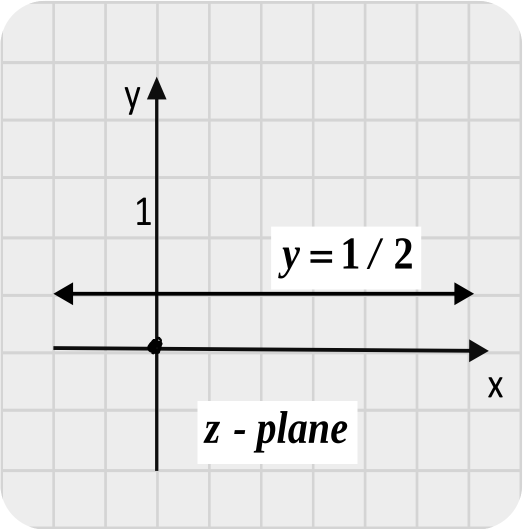
Find the image of the line $y = 1/2$ under the transformation $w = 1/z$. Show the regions graphically.

Solution:

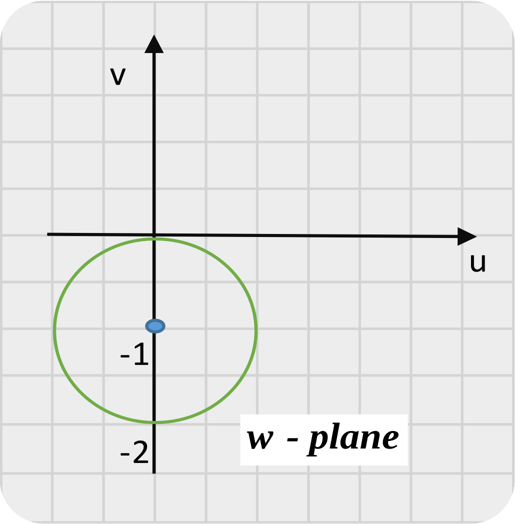
$$w = 1/z \quad \Rightarrow z = \frac{1}{w} = \frac{1}{u+iv} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2} \equiv x+iy$$

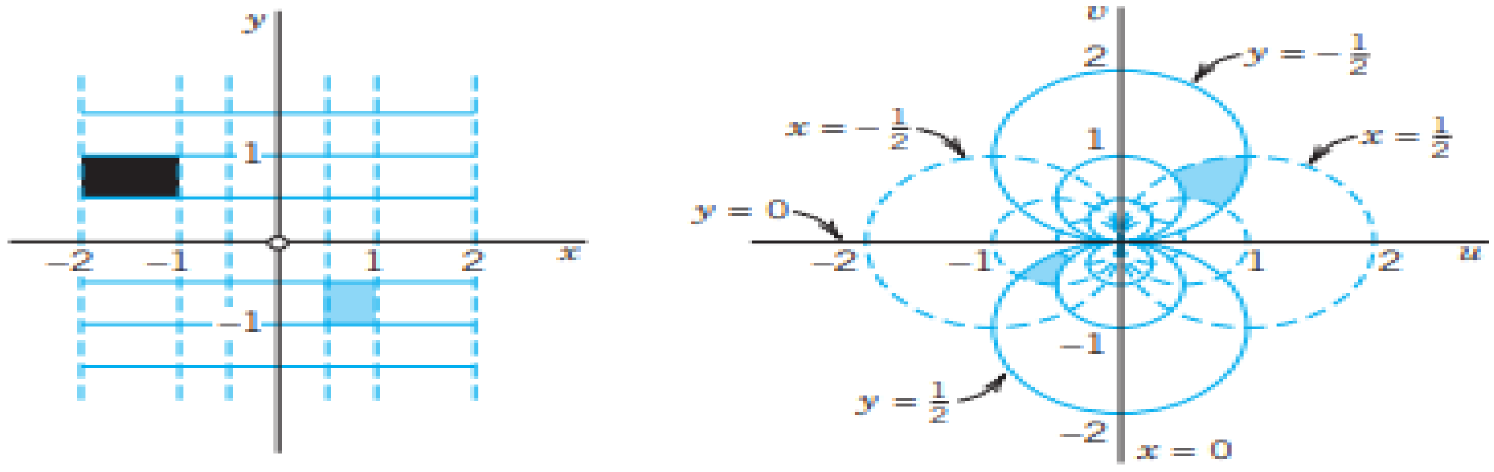
$$\boxed{\Rightarrow x = \frac{u}{u^2+v^2} \quad \& \quad y = \frac{-v}{u^2+v^2}}$$

$$\Rightarrow y = 1/2 \quad \Rightarrow \frac{-v}{u^2+v^2} = \frac{1}{2} \quad \Rightarrow \quad u^2+v^2+2v=0 \qquad \Rightarrow u^2+(v+1)^2=1$$



$$\Rightarrow u^2 + (v+1)^2 = 1$$





Mapping (Inversion) $w = 1/z$

Example:

Show that straight lines or circles are transformed into straight lines or circles under the reciprocal transformation.

Solution:

Consider the following equation

$$A\,(x^2+y^2)+B\,x+C\,y+D=0$$

Which represents straight lines or circles if A, B, C and D are real.

Under the reciprocal transformation, $(x^2+y^2)=\dfrac{1}{u^2+v^2}\ ,\ x=\dfrac{u}{u^2+v^2}\quad \&\quad y=\dfrac{-v}{u^2+v^2}$

$$\Rightarrow \frac{A}{u^2+v^2}+\frac{B\,u}{u^2+v^2}-\frac{C\,v}{u^2+v^2}+D=0$$

$$\Rightarrow D(u^2+v^2)+B\,u-C\,v+A=0$$

Which represents straight lines or circles also.

$$A(x^2+y^2)+Bx+Cy+D=0 \qquad \Rightarrow D(u^2+v^2)+Bu-Cv+A=0$$

(a) a circle ($A \neq 0$) not passing through the origin ($D \neq 0$) in the z plane is transformed into a circle not passing through the origin in the w plane;

(b) a circle ($A \neq 0$) through the origin ($D = 0$) in the z plane is transformed into a line that does not pass through the origin in the w plane;

$$A(x^2+y^2)+Bx+Cy+D=0 \qquad \Rightarrow D(u^2+v^2)+Bu-Cv+A=0$$

(*(c)* a line (A = 0) not passing through the origin (D $\neq$ 0) in the z plane is transformed into a circle through the origin in the w plane;

(d) a line (A = 0) through the origin (D = 0) in the z plane is transformed into a line through the origin in the w plane.