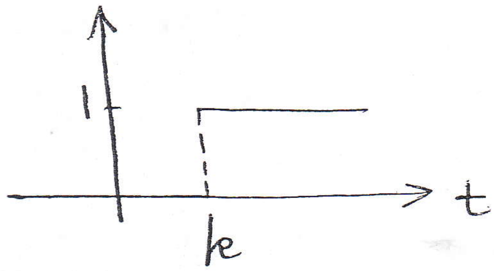
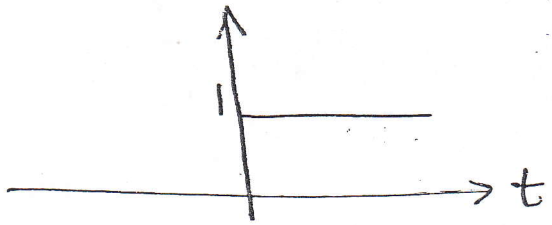
Heaviside unit step function

Heaviside unit step function is defined as follows:

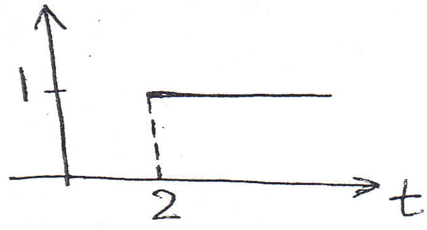
$$H(t-k)=\begin{cases}0, & t<k\\ 1, & t\geq k\end{cases}$$



فمثلاً :



$H(t)$



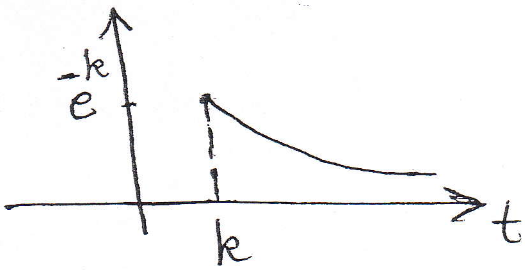
$H(t-2)$

<u>EX</u>: sketch the graph of the functions:

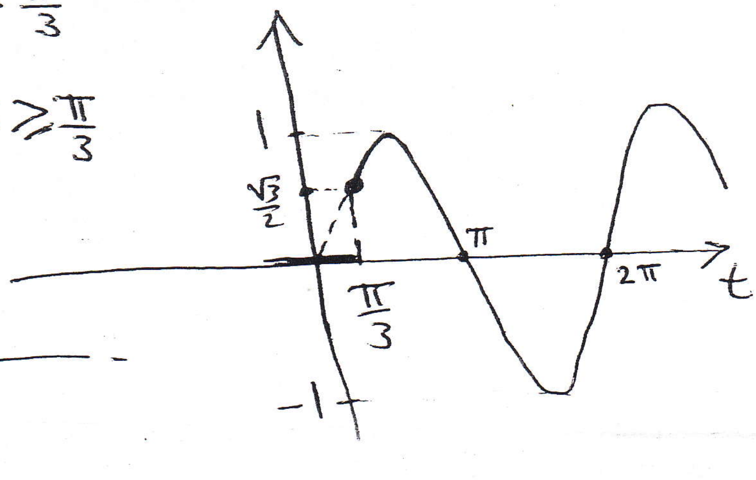
$H(t-k)e^{-t}$ and $H(t-\frac{\pi}{3})\sin t$.

<u>Solution</u>

$$\because H(t-k)e^{-t}=\begin{cases}0, & t<k\\ e^{-t}, & t\geq k\end{cases}$$



$$, H(t-\tfrac{\pi}{3})\sin t=\begin{cases}0, & t<\frac{\pi}{3}\\ \sin t, & t\geq\frac{\pi}{3}\end{cases}$$



* Laplace Transform of $H(t-k)$:

$$\boxed{L[H(t-k)] = \frac{e^{-ks}}{s}}$$ حفظ

* Second shift theorem:

if $L[f(t)] = F(s)$

Then, $$\boxed{L[H(t-k)\,f(t-k)] = e^{-ks}F(s)}$$ حفظ

For example,

$$L[H(t-k)\sin(t-k)] = e^{-ks}.L[\sin t] = e^{-ks}.\frac{1}{s^2+1} \quad \#$$

$$, L[H(t-\tfrac{\pi}{3})\cos(t-\pi/3)] = e^{-\frac{\pi}{3}s}L[\cos t] = e^{-\frac{\pi}{3}s}.\frac{s}{s^2+1} \quad \#$$

$$, L[H(t-3)\,sh\,2(t-3)] = e^{-3s}L[sh2t] = e^{-3s}.\frac{2}{s^2-4} \quad \#$$

* The Second shift theorem may be formulated as:

$$\boxed{L^{-1}[e^{-ks}F(s)] = H(t-k)\,f(t-k)}$$ حفظ

For example, $L^{-1}\left[\frac{s}{s^2+4}.e^{-3s}\right] = ??$

$$\because L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t \Rightarrow L^{-1}\left[\frac{s}{s^2+4}e^{-3s}\right] = \boxed{H(t-3)\cos 2(t-3)}$$

#

$$, L^{-1}\left[\frac{3}{s^3}e^{-4s}\right] = ??$$

$$\because L^{-1}\left[\frac{3}{s^3}\right] = \frac{3}{2}L^{-1}\left[\frac{\overset{2!}{2}}{s^3}\right] = \frac{3}{2}t^2$$

$$\therefore L^{-1}\left[\frac{3}{s^3}e^{-4s}\right] = \frac{3}{2}H(t-4)\cdot(t-4)^2 \ \#$$

<u>**EX:**</u> Find: $f(t) = L^{-1}\left[\overset{\frac{3}{s}e^{0s}}{\frac{3}{s}} + \frac{2e^{-2s}}{s^2} - \frac{2}{s^2}e^{-5s}\right]$

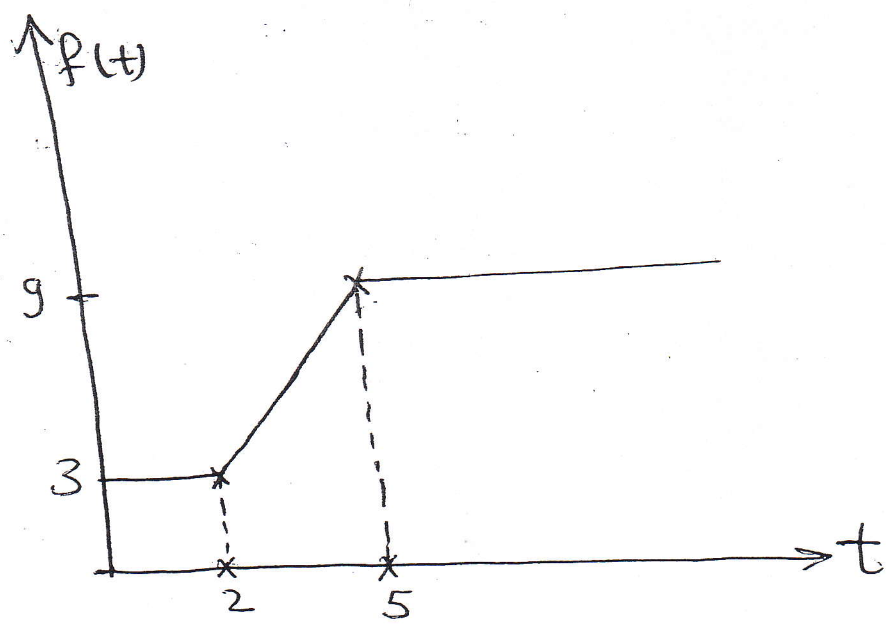
and sketch the graph of this function.

<u>Solution</u>

$$f(t) = 3H(t) + H(t-2)\cdot 2(t-2) - H(t-5)\cdot 2(t-5)$$

$$\therefore f(t) = \begin{cases} 3 & 0<t<2 \\ 3+2(t-2) & 2<t<5 \\ 3+2(t-2)-2(t-5) & 5<t \end{cases} = \begin{cases} 3 & 0<t<2 \\ 2t-1 & 2<t<5 \\ 9 & 5<t \end{cases}$$

Then the graph of $f(t)$ is:



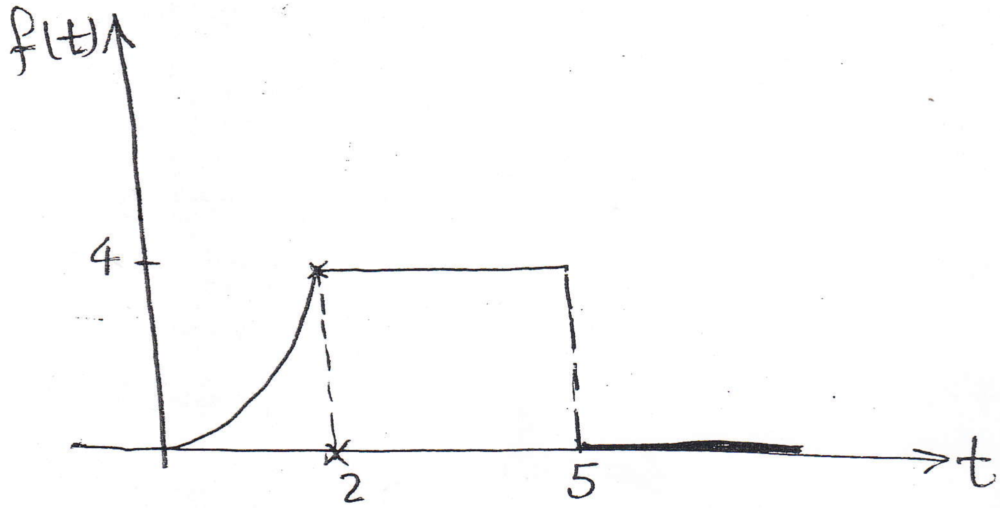
④

EX: Sketch the graph of the function:

$$f(t)=\begin{cases} t^2 , & 0<t<2 \\ 4 , & 2<t<5 \\ 0 , & 5<t \end{cases}$$

and find its transform

Solution



$$\therefore f(t) = t^2\left[H(t-0)-H(t-2)\right]+4\left[H(t-2)-H(t-5)\right] + (0)\,H(t-5)$$

$$= t^2 H(t) - (t^2-4)\,H(t-2) - 4\,H(t-5)$$

$$= t^2 H(t) - \left[(t-2)^2+4t-8\right] H(t-2) - 4H(t-5)$$

$$= t^2 H(t) - \left[(t-2)^2+4(t-2)+8-8\right] H(t-2) - 4H(t-5)$$

$$f(t) = t^2 H(t) - \left[(t-2)^2+4(t-2)\right] H(t-2) - 4H(t-5)$$

$$\therefore F(s) = \frac{2}{s^3}\,e^{0s} - \left[\frac{2}{s^3}+4\cdot\frac{1}{s^2}\right] e^{-2s} - 4\,\frac{e^{-5s}}{s}$$

$$= \frac{2}{s^3} - \left(\frac{2}{s^3}+\frac{4}{s^2}\right) e^{-2s} - 4\,\frac{e^{-5s}}{s} \quad \#$$

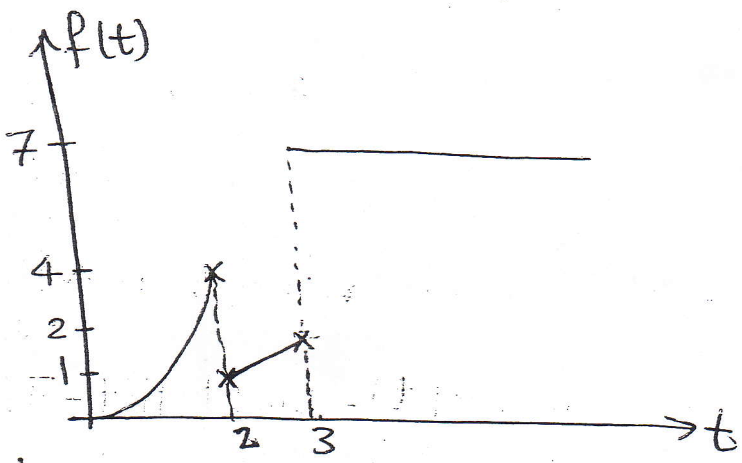
* حل بعض مسائل من تمارين (٤):

(3) sketch the graph of f(t) and find its transform:

$$f(t) = \begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & 3<t \end{cases}$$

Solution

$\therefore f(t) = t^2\left[H(t) - H(t-2)\right]$
$+(t-1)\left[H(t-2) - H(t-3)\right]$
$+7\,H(t-3).$

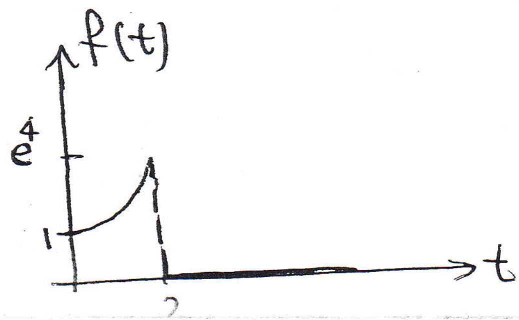


$$\therefore f(t) = t^2 H(t) - (t^2 - t + 1)\,H(t-2) - (t-8)\,H(t-3).$$
$$= t^2 H(t) - \left[(t-2)^2 + 4t - 4 - t + 1\right] H(t-2) - \left[(t-3) + 3 - 8\right] H(t-3)$$
$$= t^2 H(t) - \left[(t-2)^2 + 3t - 3\right] H(t-2) - \left[(t-3) - 5\right] H(t-3)$$
$$\therefore f(t) = t^2 H(t) - \left[(t-2)^2 + 3(t-2) + 3\right] H(t-2) - \left[(t-3) - 5\right] H(t-3)$$
$$\Rightarrow F(s) = \frac{2}{s^3} - \left[\frac{2}{s^3} + 3\cdot\frac{1}{s^2} + \frac{3}{s}\right] e^{-2s} - \left[\frac{1}{s^2} - \frac{5}{s}\right] e^{-3s} \quad \#$$

(4) $$f(t) = \begin{cases} e^{2t}, & 0<t<2 \\ 0, & 2<t \end{cases}$$

Solution

$\therefore f(t) = e^{2t}\left[H(t) - H(t-2)\right]$



$$\therefore f(t) = e^{2t}\, H(t) - e^{2t}\, H(t-2)$$

$$= e^{2t}\, H(t) - e^{[2(t-2)+4]} \cdot H(t-2)$$

$$= e^{2t}\, H(t) - e^{4} \cdot e^{2(t-2)} \cdot H(t-2)$$

$$\therefore F(s) = \frac{1}{s-2} - e^{4} \cdot \frac{1}{s-2}\, e^{-2s} \quad \#$$

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(5)
$$f(t) = \begin{cases} \sin 2t , & 0 < t < \pi \\ 0 , & \pi < t \end{cases}$$

Solution

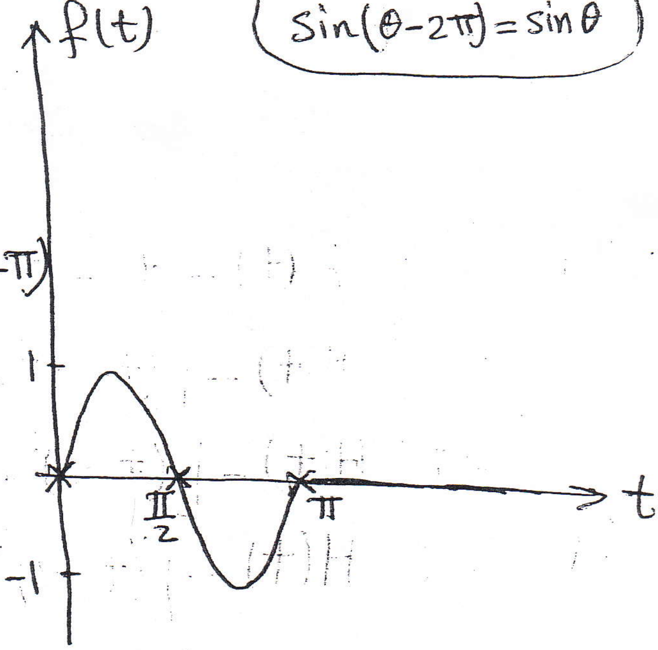
لاحظ أن:
$\sin(\theta - 2\pi) = \sin\theta$

$$\therefore f(t) = \sin 2t \left[H(t) - H(t-\pi) \right]$$

$$= \sin 2t\, H(t) - \sin 2t\, H(t-\pi)$$

$$= \sin 2t\, H(t) - \sin[2(t-\pi)]\, H(t-\pi)$$

$$\Rightarrow F(s) = \frac{2}{s^2+4} - \frac{2}{s^2+4} \cdot e^{-\pi s} \quad \#$$



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(7) Find and sketch the graph of the inverse transform of:

$$F(s) = \frac{(1-e^{-2s})(1+2e^{-3s})}{s^2}$$

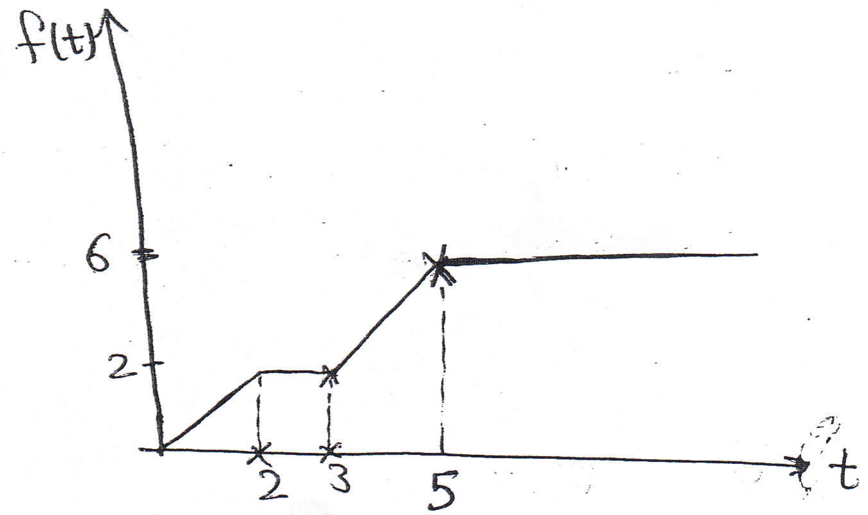
Solution

$$\because F(s) = \frac{1 - e^{-2s} + 2e^{-3s} - 2e^{-5s}}{s^2} = \frac{1}{s^2}\overset{e^{0s}}{} - \frac{1}{s^2}e^{-2s} + \frac{2}{s^2}e^{-3s} - \frac{2}{s^2}e^{-5s}$$

$$\therefore f(t) = t\,H(t) - (t-2)H(t-2) + 2(t-3)H(t-3) - 2(t-5)H(t-5)$$

$$\therefore f(t) = \begin{cases} t & , 0<t<2 \\ t-(t-2) & , 2<t<3 \\ t-(t-2)+2(t-3) & , 3<t<5 \\ t-(t-2)+2(t-3)-2(t-5) & , 5<t \end{cases} = \begin{cases} t & , 0<t<2 \\ 2 & , 2<t<3 \\ 2t-4 & , 3<t<5 \\ 6 & , 5<t \end{cases}$$

The graph:



9 Find : $L^{-1}\left[\frac{e^{-2s}}{s^2-6s+13}\right]$

Solution

$$\because L^{-1}\left[\frac{1}{s^2-6s+13}\right] = L^{-1}\left[\frac{1}{(s-3)^2+4}\right] \quad , (2)^2$$

$$= \frac{1}{2}L^{-1}\left[\frac{2}{(s-3)^2+4}\right] = \frac{1}{2}\cdot e^{3t}\sin 2t$$

$$\therefore L^{-1}\left[\frac{e^{-2s}}{s^2-6s+13}\right] = \frac{1}{2}e^{3(t-2)}\sin 2(t-2)\cdot H(t-2) \;\#$$

11 Find and sketch the inverse of:

$$\frac{s}{(s^2+1)(1-e^{-\pi s})}$$

<u>solution</u>

$$\because F(s) = \frac{s}{s^2+1} \cdot \frac{1}{1-e^{-\pi s}} = \frac{s}{s^2+1} \cdot [1+e^{-\pi s}+e^{-2\pi s}+\cdots]$$

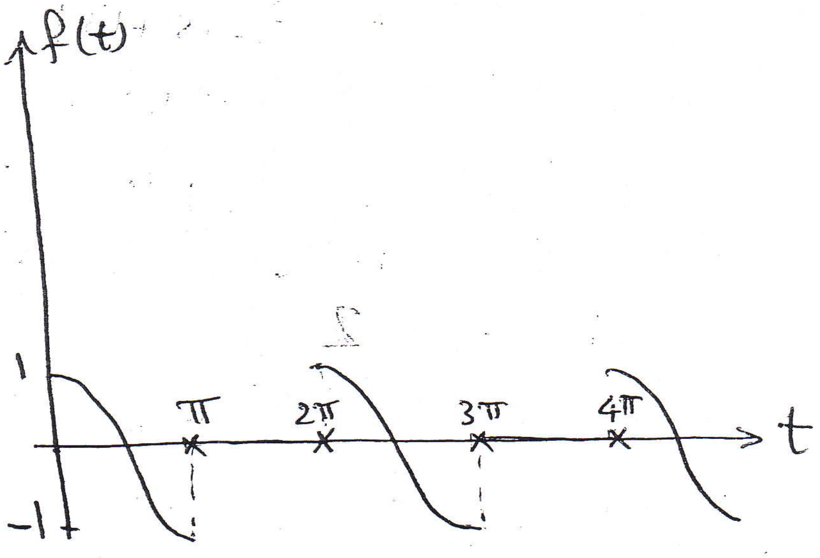
$$= \frac{s}{s^2+1} \cdot \overset{e^{0s}}{1} + \frac{s}{s^2+1} \cdot e^{-\pi s} + \frac{s}{s^2+1} e^{-2\pi s} + \cdots$$

$$\therefore f(t) = \cos t . H(t) + \cos(t-\pi) H(t-\pi) + \cos(t-2\pi) H(t-2\pi) + \cdots$$

$$\therefore f(t) = \begin{cases} \cos t & , 0 < t < \pi \\ \cos t + \overset{\to -\cos t}{\cos(t-\pi)} & , \pi < t < 2\pi \\ \cos t + \cos(t-\pi) + \overset{\to \cos t}{\cos(t-2\pi)} & , 2\pi < t < 3\pi \\ \vdots & \end{cases}$$

$$\therefore f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \cos t & 2\pi < t < 3\pi \\ \vdots & \end{cases}$$

$\therefore$ The graph:



13 solve the initial value problem;

$$y'' + y = \begin{cases} 3, & 0<t<4 \\ 2t-5, & 4<t \end{cases} \quad ; y(0)=1,\ y'(0)=0$$

solution

$$y''+y = 3\left[H(t) - H(t-4)\right] + (2t-5)\left[H(t-4)\right]$$

$$= 3H(t) + \left[2t-8\right] H(t-4).$$

$$\therefore y''+y = 3H(t) + 2(t-4)\, H(t-4).$$

by taking L.T. for both sides

$$\Rightarrow s^2 Y(s) - s(1) - 0 + Y(s) = \frac{3}{s} + 2\frac{1}{s^2} e^{-4s}$$

$$\Rightarrow (s^2+1)\, Y(s) = \frac{3}{s} + s + \frac{2}{s^2} e^{-4s}$$

$$= \frac{s^2+3}{s} + \frac{2}{s^2} e^{-4s}$$

$$\Rightarrow Y(s) = \frac{s^2+3}{s(s^2+1)} + \frac{2}{s^2(s^2+1)} e^{-4s}$$

$$\therefore y(t) = L^{-1}\left[\frac{s^2+3}{s(s^2+1)}\right] + 2L^{-1}\left[\frac{1}{s^2(s^2+1)} e^{-4s}\right]$$

$L^{-1}\left[\frac{s^2+3}{s(s^2+1)}\right]$: for the factor s:

$$\frac{P(0)}{Q'(0)} e^{0t} = \frac{3}{1} e^{0t} = \boxed{3}$$

for the factor (s^2+1):

$$\alpha = 0,\ \beta = 1$$

$$\therefore G(s) = \frac{s^2+3}{s} \Rightarrow G(i) = \frac{-1+3}{i} = \frac{2}{i} = -2i$$

$$\therefore R = 0,\ I = -2 \Rightarrow \frac{1}{1} e^{0t}(-2\cos t + 0) = \boxed{-2\cos t}$$

$$\therefore L^{-1}\left[\frac{s^2+3}{s(s^2+1)}\right] = \boxed{3-2\cos t}$$

$$\text{6)}\ L^{-1}\left[\frac{1}{s^2(s^2+1)}e^{-4s}\right]:$$

We find $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$

for the factor s^2: $\Phi(s) = \dfrac{1}{s^2+1}$

$$\therefore \Phi'(s) = \frac{-2s}{(s^2+1)^2}$$

$$\Rightarrow [\Phi'(0) + \Phi(0)t]e^{0t} = (0+t)e^{0t} = \boxed{t}$$

for the factor (s^2+1): $\alpha=0,\ \beta=1$

$$G(s) = \frac{1}{s^2} \Rightarrow G(i) = \frac{1}{i^2} = -1$$

$$\Rightarrow \frac{1}{1}e^{0\alpha}[0+(-1)\sin t] = \boxed{-\sin t}$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \boxed{t-\sin t}$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+1)}e^{-4s}\right] = [(t-4)-\sin(t-4)]\,H(t-4).$$

$$\therefore y(t) = 3-2\cos t + 2[(t-4)-\sin(t-4)]\,H(t-4).$$

#

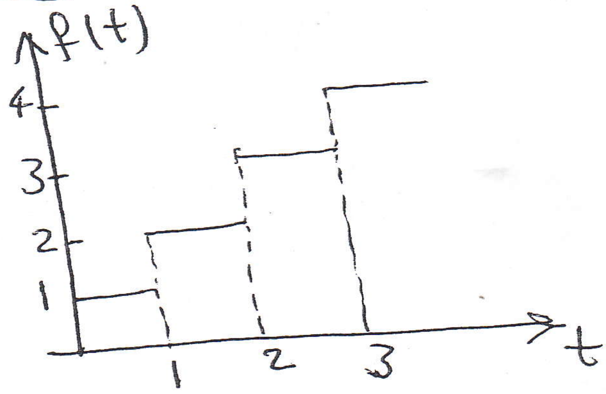
(14) $f(t) = n+1,\ n<t<n+1$ with $n=0,1,2,\cdots$

sketch $f(t)$ and solve :

$$y'' + y = f(t)\ ;\ y(0) = y'(0) = 0 .$$

solution

$$\therefore f(t) = \begin{cases} 1, & 0<t<1 \\ 2, & 1<t<2 \\ 3, & 2<t<3 \\ \vdots \end{cases}$$



$$\therefore y'' + y = (1)[H(t) - H(t-1)] + 2[H(t-1) - H(t-2)] + 3[H(t-2) - H(t-3)] + \cdots$$

$$\Rightarrow y'' + y = H(t) + H(t-1) + H(t-2) + \cdots$$

by taking L.T. for both sides

$$\Rightarrow s^2 Y(s) - \cancel{s(0)} - \cancel{0} + Y(s) = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots$$

$$\Rightarrow (s^2+1) Y(s) = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots$$

$$\Rightarrow Y(s) = \frac{1}{s(s^2+1)} + \frac{1}{s(s^2+1)} e^{-s} + \frac{1}{s(s^2+1)} e^{-2s} + \cdots$$

$\because L^{-1}\left[\frac{1}{s(s^2+1)}\right]$: The factor s : $\frac{P(0)}{Q'(0)} e^{0t} = \frac{1}{1} e^{0t} = \boxed{1}$

The factor (s^2+1) : $\alpha = 0,\ \beta = 1$

$$G(s) = \frac{1}{s} \Rightarrow G(i) = \frac{1}{i} = \boxed{-i}$$

$$\Rightarrow \frac{1}{1} e^{0t} [-\cos t + 0] = \boxed{-\cos t}$$

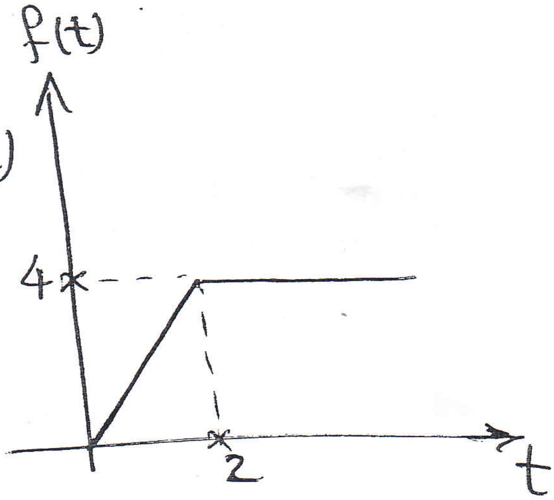
$$\Rightarrow L^{-1}\left[\frac{1}{s(s^2+1)}\right] = \boxed{1 - \cos t}$$

$$\therefore y(t) = 1 - \cos t + (1 - \cos(t-1)) H(t-1) + (1 - \cos(t-2)) H(t-2) + \cdots$$

#

problems ④ (Laplace) ①

① $f(t) = \begin{cases} 2t & 0<t<2 \\ 4 & 2<t \end{cases}$



$f(t) = 2t\left[H(t) - H(t-2)\right] + 4H(t-2)$

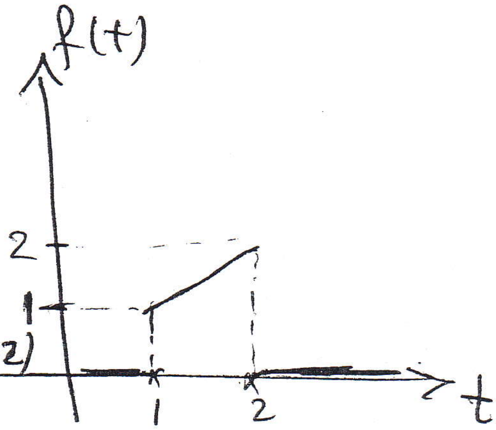
$= 2tH(t) + (4-2t)H(t-2)$

$= 2tH(t) + \left[(-2)(t-2)\right]H(t-2)$

$f(t) = 2t\,H(t) - 2(t-2)H(t-2)$

$\Rightarrow F(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-2s}$ #

② $f(t) = \begin{cases} 0, & 0<t<1 \\ t, & 1<t<2 \\ 0, & 2<t \end{cases}$



$f(t) = t\left[H(t-1) - H(t-2)\right]$

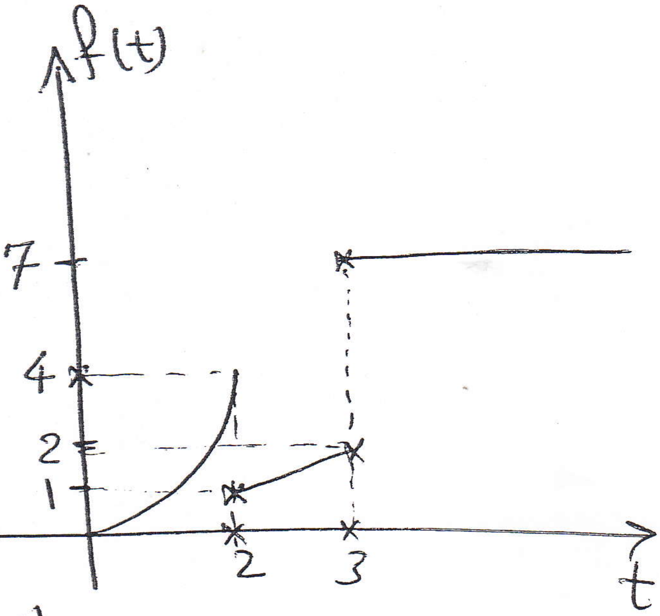
$= tH(t-1) - tH(t-2)$

$= (t-1+1)H(t-1) - (t-2+2)H(t-2)$

$f(t) = (t-1)H(t-1) + H(t-1) - (t-2)H(t-2) - 2H(t-2)$

$\Rightarrow F(s) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s}$ #

$$③\ f(t) = \begin{cases} t^2 & , 0<t<2 \\ t-1 & , 2<t<3 \\ 7 & , 3<t \end{cases}$$



$$f(t) = t^2\left[H(t) - H(t-2)\right] + (t-1)\left[H(t-2) - H(t-3)\right] + 7\,H(t-3).$$

$$\Rightarrow f(t) = t^2 H(t) + (-t^2 + t - 1) H(t-2) + \left[-t + 1 + 7\right] H(t-3)$$

$$= t^2 H(t) - (t^2 - t + 1) H(t-2) - (t-8)\ H(t-3)$$

$$= t^2\ H(t) - \left[(t-2)^2 + 3t - 3\right] H(t-2) - \left[t-3-5\right] H(t-$$

$$= t^2 H(t) - \left[(t-2)^2 + 3(t-2) + 3\right] H(t-2) - [t-3) H(t-$$

$$+ 5\,H(t-3).$$

$$\Rightarrow f(t) = t^2 H(t) - (t-2)^2 H(t-2) - 3(t-2) H(t-2) - 3H(t-2) - (t-3) H(t-3) + 5\ H(t-3).$$

$$\Rightarrow F(s) = \frac{2}{s^3} - \frac{2}{s^3} e^{-2s} - \frac{3}{s^2} e^{-2s} - 3\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} + 5\frac{e^{-3s}}{s}$$

\#

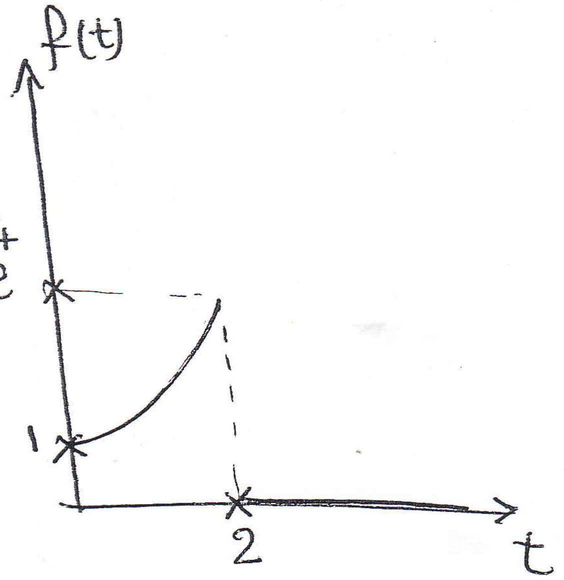
4) $$f(t) = \begin{cases} e^{2t}, & 0 < t < 2 \\ 0, & 2 < t \end{cases}$$

$$f(t) = e^{2t}\left[H(t) - H(t-2)\right]$$

$$= e^{2t} H(t) - e^{2t} H(t-2)$$

$$= e^{2t} H(t) - e^{2(t-2+2)} H(t-2)$$

$$= e^{2t} H(t) - e^{2(t-2)} e^{4} H(t-2)$$



$$\Rightarrow F(s) = \frac{1}{s-2} - e^{4} \cdot \frac{e^{-2s}}{s-2} \quad \#$$

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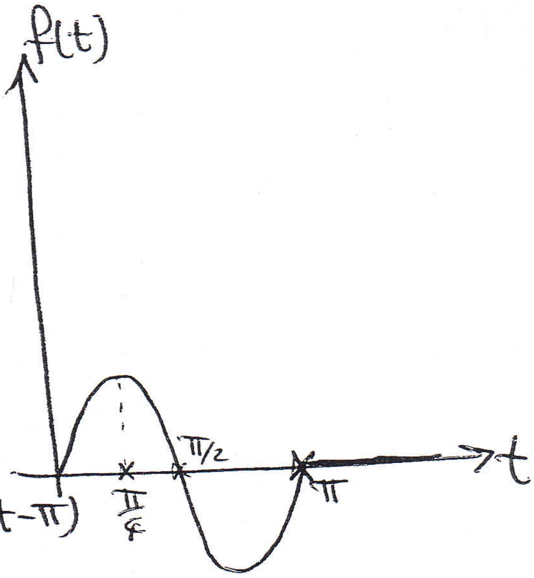
5) $$f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & \pi < t \end{cases}$$

$$\therefore f(t) = \sin 2t\left[H(t) - H(t-\pi)\right]$$

$$= \sin 2t\, H(t) - \sin 2t \cdot H(t-\pi)$$

$$= \sin 2t\, H(t) - \sin\left[2(t-\pi) + 2\pi\right] * H(t-\pi).$$

$$f(t) = \sin 2t \cdot H(t) - \sin\left[2(t-\pi)\right] H(t-\pi)$$

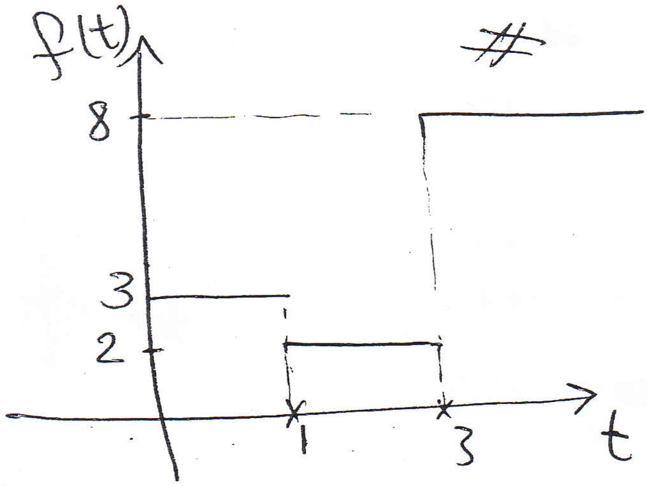


$$\Rightarrow F(s) = \frac{2}{s^2+4} - \frac{2}{s^2+4} e^{-\pi s} \quad \#$$

$\boxed{6}$ $F(s) = \dfrac{3}{s} - \dfrac{e^{-s}}{s} + \dfrac{6e^{-3s}}{s}$.

$\Rightarrow f(t) = 3H(t) - H(t-1) + 6H(t-3)$.

$$= \begin{cases} 3 & 0<t<1 \\ 3-1=2 & 1<t<3 \\ 2+6=8 & 3<t \end{cases} = \begin{cases} 3 & , 0<t<1 \\ 2 & , 1<t<3 \\ 8 & , 3<t \end{cases}$$



#

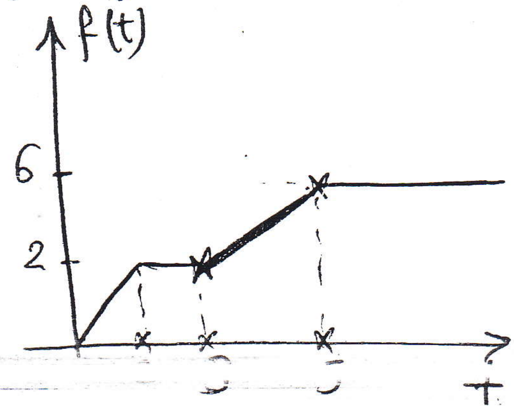
⑦ $F(s) = \dfrac{(1-e^{-2s})(1+2e^{-3s})}{s^2} = \dfrac{1 - e^{-2s} + 2e^{-3s} - 2e^{-5s}}{s^2}$

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s^2}e^{-3s} - \frac{2}{s^2}e^{-5s}.$$

$\therefore f(t) = tH(t)\ \cancel{2tH(t} - (t-2)H(t-2)$

$\quad + 2(t-3)H(t-3) - 2(t-5)H(t-5)$.

$$= \begin{cases} t & 0<t<2 \\ 2 & 2<t<3 \\ 2t-4 & 3<t<5 \\ 6 & 5<t \end{cases}$$



8 $F(s) = \dfrac{e^{-2s}}{(s+3)^2}$:

$\because L^{-1}\left(\dfrac{1}{(s+3)^2}\right) = t e^{-3t}$.

$\therefore L^{-1}\left[\dfrac{e^{-2s}}{(s+3)^2}\right] = (t-2) e^{-3(t-2)} \; H(t-2)$

#

9 $L^{-1}\left[\dfrac{e^{-2s}}{s^2-6s+13}\right]$

$\because L^{-1}\left[\dfrac{1}{s^2-6s+13}\right] = \dfrac{1}{2} L^{-1}\left[\dfrac{2}{(s-3)^2+4}\right]$

$= \dfrac{1}{2} e^{3t} \sin 2t$

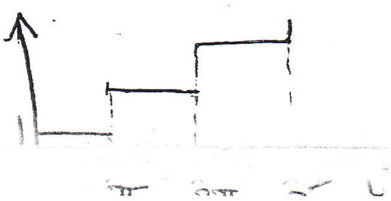
$\Rightarrow L^{-1}\left[\dfrac{e^{-2s}}{s^2-6s+13}\right] = \dfrac{1}{2} e^{3(t-2)} \sin 2(t-2) \cdot H(t-2)$

#

10 $L^{-1}\left[\dfrac{1}{s(1-e^{-\pi s})}\right] = L^{-1}\left[\dfrac{1}{s}\left(1+e^{-\pi s}+e^{-2\pi s}+\cdots\right)\right]$

$= L^{-1}\left[\dfrac{1}{s} + \dfrac{e^{-\pi s}}{s} + \dfrac{e^{-2\pi s}}{s} + \cdots\right]$

$= H(t) + H(t-\pi) + H(t-2\pi) + \cdots$

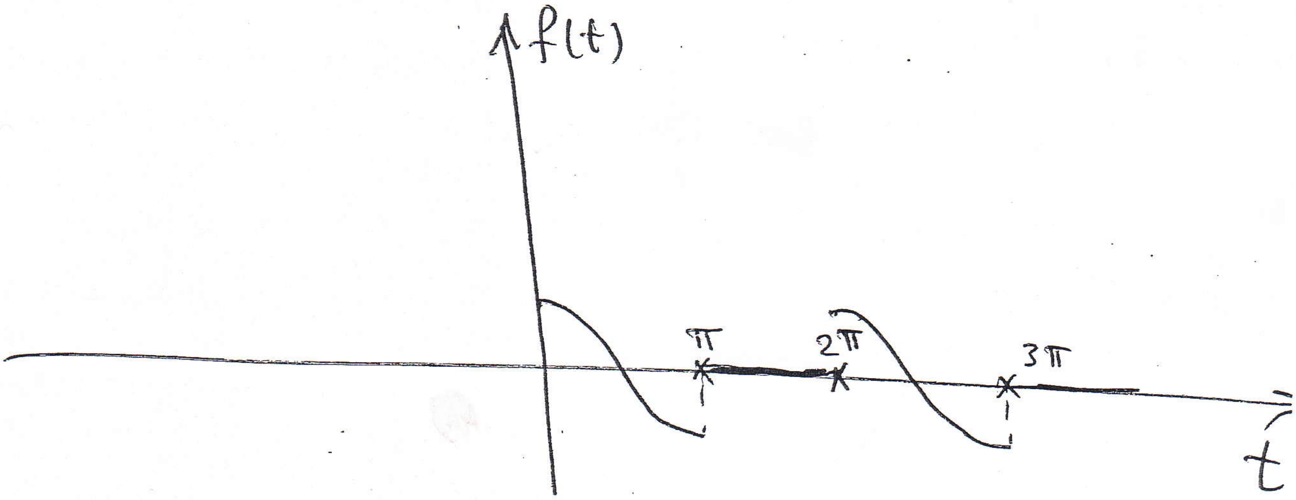


11) $$L^{-1}\left[\frac{s}{(s^2+1)(1-e^{-\pi s})}\right] = L^{-1}\left[\frac{s}{s^2+1}\left(1+e^{-\pi s}+e^{-2\pi s}+\cdots\right)\right]$$

$$= L^{-1}\left[\frac{s}{s^2+1} + \frac{s}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-2\pi s} + \cdots\right]$$

$$= \cos t \cdot H(t) + \cos(t-\pi)\, H(t-\pi) + \cos(t-2\pi) * H(t-2\pi) + \cdots$$

$$= \cos t \cdot H(t) - \cos t\, H(t-\pi) + \cos t\, H(t-2\pi) - \cdots$$



12) $$\ddot{I} - 4\dot{I} - 5I = 30\,H(t-2).$$

at $t=0$, $I=0$, $\dot{I}=0$.

by L.T. for both sides

$$\cancel{s^2 I(s) - s(0) - 0 - 4(sI(s) - 0) = 30\frac{e^{-2s}}{s}}$$

$$\cancel{(s^2 - 4s)\, I(s) = 30\frac{e^{-2s}}{s} \Rightarrow I(s) = 30\frac{e^{-2s}}{s^2(s-4)}}$$

$$\Rightarrow s^2 I(s) - s(0) - 0 - 4\left(sI(s) - 0\right) - 5I(s) = 30\frac{e^{-2s}}{s}.$$

$$\Rightarrow \left(s^2 - 4s - 5\right) I(s) = \frac{30}{s} e^{-2s}.$$

$$\Rightarrow I(s) = \frac{30}{s(s^2-4s-5)} e^{-2s} = \frac{30}{s(s-5)(s+1)} e^{-2s}$$

$$\therefore L^{-1}\left[\frac{30}{s(s-5)(s+1)}\right] = \frac{P(0)}{Q'(0)} + \frac{P(5)}{Q'(5)} e^{5t} + \frac{P(-1)}{Q'(-1)} e^{-t}$$

$$= \frac{30}{-5} + \frac{30}{30} e^{5t} + \frac{30}{6} e^{-t}.$$

$$= -6 + e^{5t} + 5e^{-t}.$$

$$\therefore I(t) = \left[-6 + e^{5(t-2)} + 5e^{-(t-2)}\right] H(t-2)$$

\#

---

13 $y'' + y = \begin{cases} 3 & , 0<t<4 \\ 2t-5 & 4<t \end{cases}$; $y(0)=1$, $y'(0)=0$.

$$\Rightarrow y'' + y = 3\left[H(t) - H(t-4)\right] + (2t-5)H(t-4)$$

$$= 3H(t) + (2t-8)H(t-4)$$

$$\Rightarrow y'' + y = 3H(t) + 2(t-4)H(t-4)$$

$$\Rightarrow s^2 Y(s) - s(1) - 0 + Y(s) = \frac{3}{s} + \frac{2}{s^2} e^{-4s}$$

$$(s^2+1)Y(s) = \frac{3}{s} + \frac{2}{s^2} e^{-4s} + s$$

8

$$\Rightarrow Y(s) = \frac{3}{s(s^2+1)} + \frac{s}{s^2+1} + \frac{2}{s^2(s^2+1)} e^{-4s}.$$

$$L^{-1}\left[\frac{3}{s(s^2+1)}\right] = 3L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+1}\right]$$

$$= 3 \quad (1) * \sin t$$

$$= 3\int_0^t \sin u\, du = 3\left(-\cos u\right)_0^t$$

$$= -3[\cos t - 1].$$

$$\& \; L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t.$$

$$\& \; L^{-1}\left[\frac{2}{s^2(s^2+1)} e^{-4s}\right]$$

$$\because L^{-1}\left[\frac{2}{s^2(s^2+1)}\right] = 2\,L^{-1}\left[\frac{1}{s^2} \cdot \frac{1}{s^2+1}\right]$$

$$= 2 \; t * \sin t = 2\int_0^t (t-u)\sin u\, du$$

$$= 2\left[(t-u)(-\cos u) - (-1)(-\sin u)\right]_0^t$$

$$= 2[t - \sin t]$$

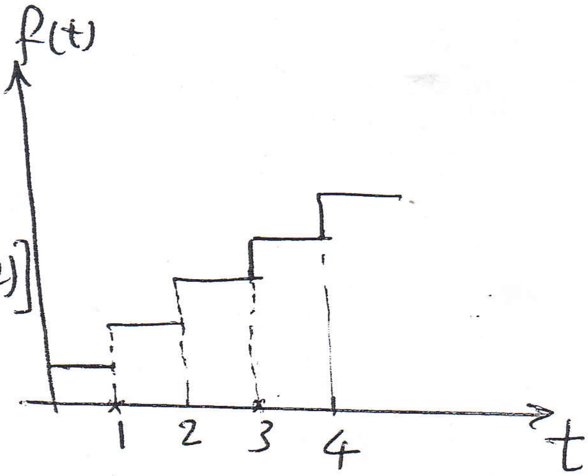
$$\therefore L^{-1}\left[\frac{2}{s^2(s^2+1)} e^{-4s}\right] = 2[(t-4) - \sin(t-4)] * H(t-4).$$

$$\Rightarrow y(t) = 3(1-\cos t) + \cos t + 2[t-4-\sin(t-4)]H(t-4)$$

#

14) $f(t) = n+1 \quad , \quad n < t < n+1 \quad , \quad n = 0, 1, 2, \cdots$

$$f(t) = \begin{cases} 1 & , \; 0<t<1 \\ 2 & , \; 1<t<2 \\ 3 & , \; 2<t<3 \\ \vdots & \end{cases}$$



$y'' + y' = f(t) \; ; \; y(0) = y'(0) = 0.$

$\because f(t) = H(t) - H(t-1) + 2[H(t-1) - H(t-2)]$
$+ 3[H(t-2) - H(t-3)]$
$+ \cdots$

$f(t) = H(t) + H(t-1) + H(t-2) + \cdots$

$\Rightarrow F(s) = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots$

$\Rightarrow s^2 Y(s) + s Y(s) = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots$

$Y(s)[s^2 + s] = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots$

$\Rightarrow Y(s) = \frac{1}{s^2(s+1)} + \frac{1}{s^2(s+1)} e^{-s} + \frac{1}{s^2(s+1)} e^{-2s} + \cdots$

$\because L^{-1}\left(\frac{1}{s^2(s+1)}\right) = L^{-1}\left(\frac{1}{s^2} \cdot \frac{1}{s+1}\right)$

$= t * e^{-t} = \int_0^t (t-u) e^{-u} \, du$

$= (t-u)(-e^{-u}) - (-1)(e^{-u}) \Big|_0^t$

$= t + e^{-t} - 1$

(10

$$\Rightarrow y(t) = t-1+e^{-t} + \left[(t-1)-1+e^{-(t-1)}\right]H(t-1)$$
$$+\left[(t-2)-1+e^{-(t-2)}\right]H(t-2)+\cdots$$

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