FUNCTIONS OF COMPLEX VARIABLES (3)

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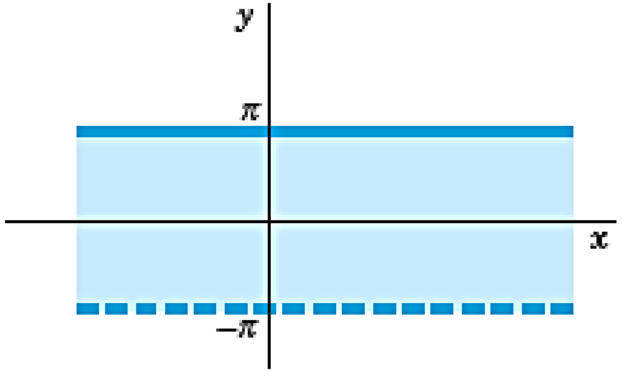
Functions of Complex Variables

Exponential Function

$$w = e^{x+iy} = e^{x}\,(\cos y + i\sin y)$$

$$u(x,y) = e^{x}\cos y \qquad v(x,y) = e^{x}\sin y$$

Entire Function

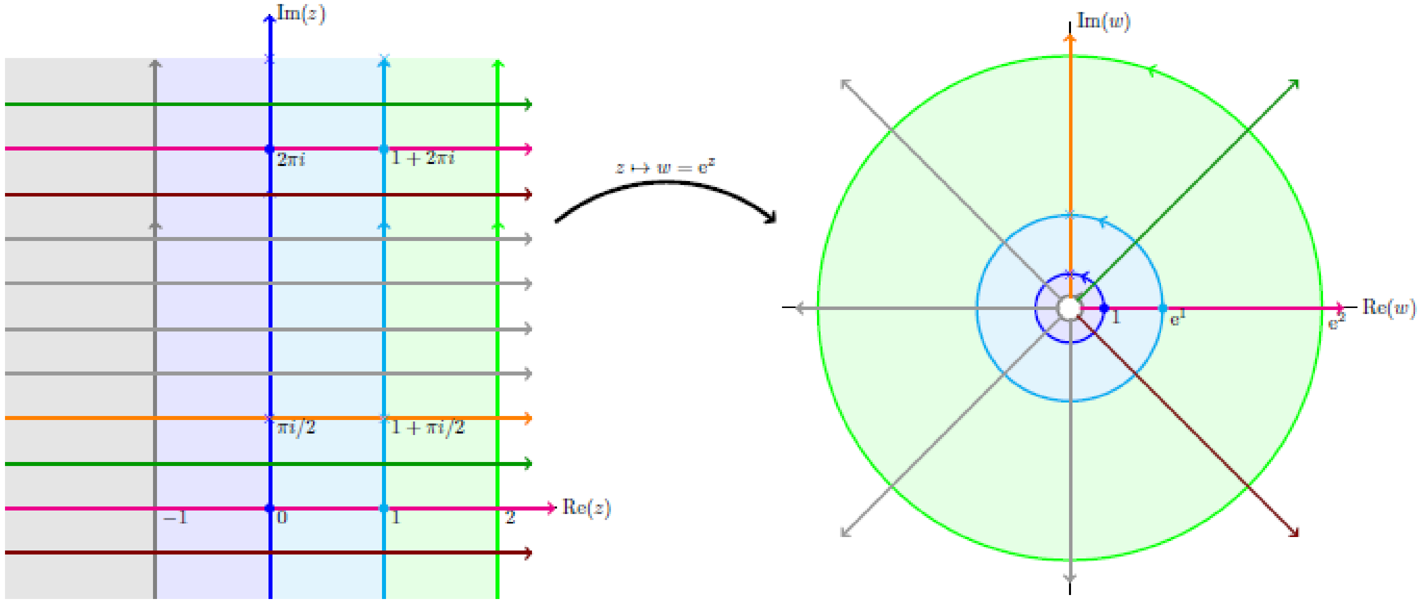


Periodic Function

Functions of Complex Variables

Exponential Function

$$w = e^{x+iy} = e^{x}\,(\cos y + i \sin y)$$



Example 1:

Find all the values of z such that $e^{4z} = 1+i$.

Solution:

Take ln to both sides

$$4z = \ln(1+i) = \ln(\sqrt{2}\; e^{i(\frac{\pi}{4}+2\pi k)}) = \ln\sqrt{2} + i(\tfrac{\pi}{4}+2\pi k)$$

$$\Rightarrow z = \frac{1}{4}\left(\ln\sqrt{2} + i\left(\tfrac{\pi}{4}+2\pi k\right)\right)$$

Functions of Complex Variables

Trigonometric and Hyperbolic Function

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} =$$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right),$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

$$\sin iz = i \sinh z, \qquad \cos iz = \cosh z, \qquad \tan iz = i \tanh z$$

$$\sinh iz = i \sin z, \qquad \cosh iz = \cos z, \qquad \tanh iz = i \tan z$$

Example 2:

Show that :

a- $\cos z = \cos x \cosh y - i \sin x \sinh y$

b- $|\cos z|^2 = cos^2 x + sinh^2 y$

Solution:

a-

$$\cos z = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$$

$$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$$

$$= \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{1}{2}i(e^{y} - e^{-y})\sin x.$$

$$\cosh y = \frac{1}{2}(e^{y} + e^{-y}), \qquad \sinh y = \frac{1}{2}(e^{y} - e^{-y});$$

b-

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y.$$

Example 3:

Show that $f(z) = \sin z$ is differentiable everywhere and $\dfrac{d}{dz}\left(\sin z\right) = \cos z$.

Solution:

$$w = \sin z = \sin(x + i\,y) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$u_x = \cos x \cosh y \quad , \quad v_y = \cos x \cosh y \qquad \Rightarrow u_x = v_y \;\; everywhere$$

$$u_y = \sin x \sinh y \quad , \quad v_x = -\sin x \sinh y \qquad \Rightarrow u_y = -v_x \;\; everywhere$$

$$\Rightarrow f(z) = \sin z \;\; \text{is differentiable everywhere.}$$

$$\Rightarrow f'(z) = u_x + i\,v_x = \cos x \cosh y - i \sin x \sinh y$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos(x + i\,y)$$

$$\boxed{\therefore \frac{d}{dz}\left(\sin z\right) = \cos z.}$$

Example 4:

Find all the values of z such that $\sin z = \cosh 2$

Solution:

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$\Rightarrow \sin x \cosh y + i \cos x \sinh y = \cosh 2$$

$$\cos x \sinh y = 0 \qquad \text{and} \qquad \sin x \cosh y = \cosh 2$$

$$\cos x \sinh y = 0 \quad \Rightarrow y = 0 \quad \text{or} \quad x = (2n+1)\tfrac{\pi}{2}$$

$$y = 0 \quad \Rightarrow \quad \sin x = \cosh 2 \quad \text{(Refused)}$$

$$x = (2n+1)\tfrac{\pi}{2} \Rightarrow \sin(2n+1)\tfrac{\pi}{2} \cosh y = \cosh 2$$

$$(-1)^n \cosh y = \cosh 2$$

$$\text{If } n \text{ is odd}, \quad \cosh y = -\cosh 2 \quad \text{(Refused)}$$

$$\text{If } n \text{ is even}, \quad \cosh y = \cosh 2 \quad \Rightarrow \quad y = \pm 2$$

$$\boxed{\Rightarrow z = (4k+1)\tfrac{\pi}{2} \pm 2i}$$

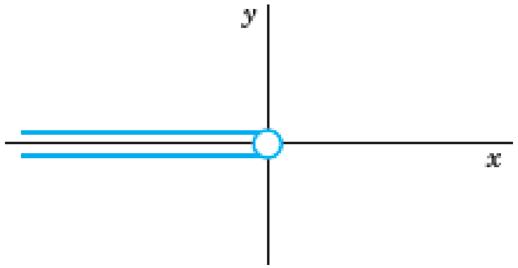
Functions of Complex Variables

Logarithmic Function

$$\ln z = \ln r + i\theta$$

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z$$

$$\ln z = \operatorname{Ln} z \pm 2n\pi i$$



$$e^{\ln z} = z \qquad \ln (e^{z}) = z \pm 2n\pi i,$$

Functions of Complex Variables

Logarithmic Function

Remarks.

1. Since $\arg(z)$ has infinitely many possible values, so does $\log(z)$.
2. $\log(0)$ is not defined. (Both because $\arg(0)$ is not defined and $\log(|0|)$ is not defined.)
3. Choosing a branch for $\arg(z)$ makes $\log(z)$ single valued. The usual terminology is to say we have chosen a branch of the log function.
4. The principal branch of log comes from the principal branch of arg. That is,

$$\log(z) = \log(|z|) + i\arg(z), \quad \text{where} \quad -\pi < \arg(z) \leq \pi \quad \text{(principal branch)}.$$

Logarithmic Function

Example 6:

Find $\log(1)$

$$\log(1) = 2n\pi i, \quad \text{where } n \text{ is any integer}$$

Compute all the values of $\log(i)$.

$$\log(i) = \log(1) + i\frac{\pi}{2} + i2n\pi = i\frac{\pi}{2} + i2n\pi, \ \text{where } n \text{ is any integer.}$$

Functions of Complex Variables

Logarithmic Function

$$z^a = e^{a \log(z)}.$$

Example 5:

Compute all the values of $\sqrt{2i}$.

$$\sqrt{2i} = (2i)^{1/2} = e^{\frac{\log(2i)}{2}} = e^{\frac{\log(2)}{2} + \frac{i\pi}{4} + in\pi} = \sqrt{2}e^{\frac{i\pi}{4} + in\pi}.$$

Compute all the cube roots of i.

$$i^{1/3} = e^{\frac{\log(i)}{3}} = e^{i\frac{\pi}{6} + i\frac{2n\pi}{3}}$$