# Exercise Sheet (Bessel Functions)

[1] Solve in terms of Bessel functions the following differential eq.

a) $y'' - \frac{1}{x}y' + \left(1 - \frac{3}{x^2}\right)y = 0$ ; Put $y = xu$

let $y = xu \Rightarrow y' = u + xu' \Rightarrow y'' = u' + u' + xu'' = 2u' + xu''$

$xu'' + 2u' - \frac{u}{x} - u' + xu - \frac{3}{x}u = xu'' + u' + \left(x - \frac{4}{x}\right)u = 0$

Multiply by $x \Rightarrow x^2u'' + xu' + (x^2 - 4)u = 0 \Rightarrow \boxed{\lambda = 1 \quad \nu = 2}$

$u_{g.s} = C_1 J_2(x) + C_2 Y_2(x) \Rightarrow y_{g.s} = xu = C_1 x J_2(x) + C_2 x Y_2(x)$

b) $xy'' - 3y' + xy = 0$ let $y = x^\alpha u \Rightarrow y' = \alpha x^{\alpha-1}u + x^\alpha u'$

$y'' = x^\alpha u'' + \alpha(\alpha-1)x^{\alpha-2}u + 2\alpha x^{\alpha-1}u'$

$x^{\alpha+1}u'' + \alpha(\alpha-1)x^{\alpha-1}u + 2\alpha x^\alpha u' - 3\alpha x^{\alpha-1}u - 3x^\alpha u' + x^{\alpha+1}u$

$= x^{\alpha+1}u'' + (2\alpha-3)x^\alpha u' + [\alpha(\alpha-1-3)x^{\alpha-1} + x^{\alpha+1}]u = 0$

* Multiply by $x^{1-\alpha} \Rightarrow$ to make $x^2u''$ $\therefore 2 - \alpha + 1 = 1 - \alpha$

$x^2u'' + (2\alpha-3)xu' + [\alpha(\alpha-4) + x^2]u = 0$

Coeff. of $xu' = 1$ $\therefore 2\alpha - 3 = 1 \Rightarrow \boxed{\alpha = 2}$

$x^2u'' + xu' + (x^2 - 4)u = 0 \Rightarrow \boxed{\lambda = 1 \quad \nu = \sqrt{4} = 2}$

$u_{g.s} = C_1 J_2(x) + C_2 Y_2(x)$

$y_{g.s} = x^2 u_{g.s} = C_1 x^2 J_2(x) + C_2 x^2 Y_2(x)$

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$x^{\alpha+1}u'' + (2\alpha-3)x^\alpha u' + [\alpha(\alpha-1-3)x^{\alpha-1} + x^{\alpha+1}]u = 0$

$2\alpha - 3 = 1 \Rightarrow \boxed{\alpha = 2}$ $\therefore x^3u'' + x^2u' + (x^3 - 4x)u = 0$

Divide by $x \Rightarrow x^2u'' + xu' + (x^2 - 4)u = 0$ وتكمل عادى يا معلم

c) $y'' + (3e^{2x} - 4)y = 0$ ; use $e^x = Z$

let $e^x = Z \Rightarrow \frac{dZ}{dx} = e^x = Z \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dZ} \cdot \frac{dZ}{dx} = Z \frac{dy}{dZ}$

$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dZ} \cdot \frac{dZ}{dx} = Z \frac{d}{dZ}\left(Z \frac{dy}{dZ}\right) = Z\left[\frac{dy}{dZ} + Z \frac{d^2y}{dZ^2}\right]$

$Z^2 \frac{d^2y}{dZ^2} + Z \frac{dy}{dZ} + (3Z^2 - 4)y = 0 \Rightarrow \boxed{\lambda = \sqrt{3} \quad , \nu = \sqrt{4} = 2}$

$y_{g.s} = C_1 J_2(\sqrt{3} Z) + C_2 Y_2(\sqrt{3} Z) = C_1 J_2(\sqrt{3} e^x) + C_2 Y_2(\sqrt{3} e^x)$

d) $xy'' + y = 0$ let $y = x^\alpha u \Rightarrow y' = x^\alpha u' + \alpha x^{\alpha-1} u$

$y'' = x^\alpha u'' + 2\alpha x^{\alpha-1} u' + \alpha(\alpha-1) x^{\alpha-2} u$

$x^{\alpha+1} u'' + 2\alpha x^\alpha u' + \alpha(\alpha-1) x^{\alpha-1} u + x^\alpha u = 0 \qquad * \; x^{1-\alpha}$

$x^2 u'' + 2\alpha x u' + [\alpha(\alpha-1) + x] u = 0$

Coeff. of $xu' = 1 \quad 2\alpha = 1 \Rightarrow \boxed{\alpha = \frac{1}{2}}$

$x^2 u'' + xu' + (x - \frac{1}{4})u = 0 \Rightarrow$ let $x = t^2 \Rightarrow \frac{dt}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} t^{-1}$

$u' = \frac{du}{dx} = \frac{du}{dt} \cdot \frac{dt}{dx} = \frac{1}{2} t^{-1} \dot{u}$ , $u'' = \frac{du'}{dt} \frac{dt}{dx} = \frac{1}{2} t^{-1}\left[\frac{-1}{2} t^{-2} \dot{u} + \frac{1}{2} t^{-1} \ddot{u}\right]$

$\therefore u'' = \frac{1}{4} t^{-2} \ddot{u} - \frac{1}{4} t^{-3} \dot{u}$

$t^4\left[\frac{1}{4} t^{-2} \ddot{u} - \frac{1}{4} t^{-3} \dot{u}\right] + t^2\left[\frac{1}{2} t^{-1} \dot{u}\right] + \left[t^2 - \frac{1}{4}\right] u = 0$

$\frac{1}{4} t^2 \ddot{u} + \frac{1}{4} t \dot{u} + (t^2 - \frac{1}{4}) u = 0 \qquad * \; 4$

$t^2 \ddot{u} + t \dot{u} + (4t^2 - 1) u = 0 \Rightarrow \boxed{\lambda = \sqrt{4} = 2 \quad , \nu = 1}$

$u_{g.s} = C_1 J_1(2t) + C_2 Y_1(2t) = C_1 J_1(2\sqrt{x}) + C_2 Y_1(2\sqrt{x})$

$y_{g.s} = \sqrt{x} u_{g.s} = C_1 \sqrt{x} J_1(2\sqrt{x}) + C_2 \sqrt{x} Y_1(2\sqrt{x})$

e) $xy'' + 5y' + xy = 0$ let $x^{\alpha}u = y \Rightarrow y' = \alpha x^{\alpha-1}u + x^{\alpha}u'$

$y'' = x^{\alpha}u'' + 2\alpha x^{\alpha-1}u' + \alpha(\alpha-1)x^{\alpha-2}u$

$x^{\alpha+1}u'' + 2\alpha x^{\alpha}u' + \alpha(\alpha-1)x^{\alpha-1}u + 5\alpha x^{\alpha-1}u + 5x^{\alpha}u' + x^{\alpha+1}u = 0$

$x^{\alpha+1}u'' + (2\alpha+5)x^{\alpha}u' + [\alpha(\alpha-1+5)x^{\alpha-1} + x^{\alpha+1}]u = 0 \quad * x^{1-\alpha}$

$x^2u'' + (2\alpha+5)xu' + (\alpha(\alpha+4) + x^2)u = 0$

Coeff. of $xu' = 1 \Rightarrow 2\alpha + 5 = 1 \Rightarrow \boxed{\alpha = -2}$

$\therefore x^2u'' + xu' + (x^2-4)u = 0 \Rightarrow \boxed{\lambda = 1 \quad , v = \sqrt{4} = 2}$

$u_{g.s} = C_1J_2(x) + C_2Y_2(x) \Rightarrow y_{g.s} = \frac{1}{x^2}u_{g.s} = \frac{1}{x^2}[C_1J_2(x) + C_2Y_2(x)]$

f) $xy'' - 7y' + xy = 0$ let $y = x^{\alpha}u \Rightarrow y' = \alpha x^{\alpha-1}u + x^{\alpha}u'$

$y'' = x^{\alpha}u'' + \alpha(\alpha-1)x^{\alpha-2}u + 2\alpha x^{\alpha-1}u'$

$x^{\alpha+1}u'' + \alpha(\alpha-1)x^{\alpha-1}u + 2\alpha x^{\alpha}u' - 7\alpha x^{\alpha-1}u - 7x^{\alpha}u' + x^{\alpha+1}u$

$= x^{\alpha+1}u'' + (2\alpha-7)x^{\alpha}u' + [\alpha(\alpha-1-7)x^{\alpha-1} + x^{\alpha+1}]u = 0 \quad * x^{1-\alpha}$

$x^2u'' + (2\alpha-7)xu' + [\alpha(\alpha-8) + x^2]u = 0$

Coeff. of $xu' = 1 \quad 2\alpha - 7 = 1 \Rightarrow \boxed{\alpha = 4}$

$xu'' + xu' + (x^2-16)u = 0 \Rightarrow \boxed{\lambda = 1 \quad , v = \sqrt{16} = 4}$

$u_{gs} = C_1J_4(x) + C_2Y_4(x) \Rightarrow y_{gs} = x^4u_{g.s} = C_1x^4J_4(x) + C_2x^4Y_4(x)$

g) $(x-1)^2y'' + (x-1)y' + (x^2-2x-3)y = 0$ let $x-1 = t \quad \therefore dx = dt$

$y' = \frac{dy}{dx} = \frac{dy}{dt} = \dot{y} \Rightarrow y'' = \ddot{y} \quad \therefore t^2\ddot{y} + t\dot{y} + (t^2-4)y = 0$

$\rightarrow$ By completing squared $x^2-2x-3 \quad x^2-2x-3+4-4 = (x^2-2x+1)-4$
$= (x-1)^2 - 4 = t^2 - 4$

$\rightarrow$ or By Sub. $x = t+1 \Rightarrow x^2-2x-3 = (t+1)^2 - 2(t+1) - 3 = t^2+1+\cancel{2t}-\cancel{2t}-2-3$
$= t^2 - 4$

$\boxed{\therefore \lambda = 1 \quad , v = 2}$ $\quad y_{g.s} = C_1J_2(t) + C_2Y_2(t) = C_1J_2(x-1) + C_2Y_2(x-1)$

[2] Evaluate

a) $\int_0^1 x^5 J_0(x)\,dx$ , in terms of $J_0$ and $J_1$

$\int x^n J_{n-1}\,dx = x^n J_n$

$$\boxed{\begin{array}{ccc} x^4 & & x J_0(x) \\ & \searrow & \\ 4x^3 & \xleftarrow[-\int]{} & x J_1(x) \end{array}}$$

$$= \left[x^5 J_1(x)\right]_0^1 - 4\int_0^1 x^4 J_1(x)$$

$$= \left[x^5 J_1(x) - 4x^4 J_2(x)\right]_0^1 + 8\int_0^1 x^3 J_2(x)$$

$$\boxed{\begin{array}{ccc} x^2 & & x^2 J_1(x) \\ & \searrow & \\ 2x & \xleftarrow[-\int]{} & x^2 J_2(x) \end{array}}$$

$$= \left[x^5 J_1 - 4x^4 J_2 + 8x^3 J_3\right]_0^1$$

$$\frac{2n}{x} J_n = J_{n-1} + J_{n+1} \Rightarrow J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$$

$$= \left[x^5 J_1 - 4x^4 J_2 + 8x^3 J_3\right]_0^1$$

$n+1=2$ $\quad$ $n+1=3$

$n=1$ $\quad$ $n=2$

$n-1=0$ $\quad$ $n-1=1$

$$= \left[x^5 J_1 - 4x^2\left(\frac{2}{x} J_1 - J_0\right) + 8x^3\left(\frac{4}{x} J_2 - J_1\right)\right]_0^1$$

$n+1=2$ $\quad$ $n=1$

$$= \left[x^5 J_1 - 8x J_1 + 4x^2 J_0 + 32x^2\left(\frac{2}{x} J_1 - J_0\right) - 8x^3 J_1\right]_0^1$$

$$= \left[(x^5 - 8x + 64x - 8x^3) J_1(x) + (4x^2 - 32x^2) J_0(x)\right]_0^1$$

$$= 49 J_1(1) - 28 J_0(1)$$

b) $\int x^4 J_1(x)\,dx$ , in terms of $J_0$ and $J_1$

$$\begin{array}{ccc} x^2 & & x^2 J_1(x) \\ 2x & \xleftarrow{-\int} & x^2 J_2(x) \end{array}$$

$x^4 J_2(x) - 2\int x^3 J_2(x) = x^4 J_2(x) - 2x^3 J_3(x)$

$x^4\left(\frac{2}{x}J_1 - J_0\right) - 2x^3\left[\frac{4}{x}\left(\frac{2}{x}J_1 - J_0\right) - J_1\right] + C$

$= 2x^3 J_1 - x^4 J_0 - 16x J_1 + 8x^2 J_0 + 2x^3 J_1 + C$

$= (4x^3 - 16x) J_1(x) - (x^4 - 8x^2) J_0(x) + C$

c) $\int x^{3/2} J_{-\frac{1}{2}}(x)\,dx$ By Parts

$$\begin{array}{ccc} x & & x^{\frac{1}{2}} J_{-\frac{1}{2}} \\ 1 & \xleftarrow{-\int} & x^{\frac{1}{2}} J_{\frac{1}{2}} \end{array}$$

$= x^{3/2} J_{\frac{1}{2}} - \int x^{\frac{1}{2}} J_{\frac{1}{2}}\,dx$ $\quad J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\sin x$

$= x^{\frac{3}{2}} J_{\frac{1}{2}} - \int \cancel{\sqrt{x}} \sqrt{\frac{2}{\pi \cancel{x}}}\sin x\,dx$

$J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\cos x$

$= x^{\frac{3}{2}} J_{\frac{1}{2}} - \sqrt{\frac{2}{\pi}}\int \sin x\,dx = x^{3/2} J_{\frac{1}{2}} - \sqrt{\frac{2}{\pi}}\cos x + C$

$= x^{3/2} J_{\frac{1}{2}} - \sqrt{x}\, J_{-\frac{1}{2}} + C$

d) $\int J_5(x)\,dx$ $\rightsquigarrow$ $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n$ , $J_0' = -J_1$

$n = 4$

$I = \int J_3(x)\,dx - 2J_4 = \int J_1(x)\,dx - 2J_2(x) - 2J_4(x)$

$= -J_0(x) - 2J_2(x) - 2J_4(x) + C$

[3] Show that

a) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ $\boxed{J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+v}}{n!\,\Gamma(n+v+1)}}$

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-\frac{1}{2}}}{n!\,\Gamma(n+\frac{1}{2})} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-\frac{1}{2}}}{2^{2n-\frac{1}{2}}\, n!\,\Gamma(n+\frac{1}{2})} \sqrt{\frac{\pi x}{2}}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cancel{\sqrt{2}}}{2^{2n} \cancel{\sqrt{x}}\, n!\,\Gamma(n+\frac{1}{2})} \sqrt{\frac{\cancel{\pi x}}{\cancel{2}}} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \sqrt{\pi}}{2^{2n}\, n!\,\Gamma(n+\frac{1}{2})}$$

By using Legendre's duplication formula $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$

$$\Rightarrow \Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}\Gamma(n)} = \frac{\sqrt{\pi}}{2^{2n-\cancel{1}}} \cdot \frac{\frac{\Gamma(2n+1)}{\cancel{2n}}}{\frac{\Gamma(n+1)}{\cancel{n}}} = \frac{\sqrt{\pi}}{2^{n}} \cdot \frac{\Gamma(2n+1)}{\Gamma(n+1)} = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,(n)!}$$

$$\therefore J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cancel{\sqrt{\pi}}\, \cancel{2^{2n}}\, \cancel{n!}}{\cancel{2^{2n}}\, \cancel{n!}\, \cancel{\sqrt{\pi}}\,(2n)!} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

b) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$ $J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}}{n!\,\Gamma(n+\frac{3}{2})}$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+\frac{1}{2}}}{2^{2n+\frac{1}{2}}\, n!\,\Gamma(n+\frac{3}{2})} \sqrt{\frac{\pi x}{2}} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sqrt{\pi}}{2^{2n+1}\, n!\,(n+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}$$

By using Legendre's duplication formula $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,(n)!}$

$$\therefore J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cancel{\sqrt{\pi}}\, \cancel{2^{2n}}\, \cancel{n!}}{2^{\cancel{2n}+1}\, \cancel{n!}\,(n+\frac{1}{2})\, \cancel{\sqrt{\pi}}\,(2n)!} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2\,(n+\frac{1}{2})\,(2n)!}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,(2n)!} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}} \sin x$$

c) i. $\frac{d}{dx}\left(x^{n} J_{n}(x)\right) = x^{n} J_{n-1}(x)$

$$\frac{d}{dx}\left[x^{n}\left(\sum_{K=0}^{\infty} \frac{(-1)^{K}\, x^{2K+n}}{2^{2K+n}\, K!\, \Gamma(K+n+1)}\right)\right] = \frac{d}{dx}\left(\sum_{K=0}^{\infty} \frac{(-1)^{K}\, x^{2K+2n}}{2^{2K+n}\, K!\, \Gamma(K+n+1)}\right)$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^{K}\, 2(K+n)\, x^{2K+2n-1}}{2^{2K+n}\, K!\, \Gamma(K+n+1)} = x^{n} \sum_{K=0}^{\infty} \frac{(-1)^{K}\, x^{2K+n-1}}{2^{2K+n-1}\, K!\, \underbrace{\frac{\Gamma(K+n+1)}{(K+n)}}_{\Gamma(K+n)}}$$

$$= x^{n} \sum_{K=0}^{\infty} \frac{(-1)^{K} \left(\frac{x}{2}\right)^{2K+(n-1)}}{K!\, \Gamma(K+(n-1)+1)} = x^{n} J_{n-1}(x)$$

ii. $\frac{d}{dx}\left(x^{-n} J_{n}(x)\right) = -x^{-n} J_{n+1}(x)$

$$\frac{d}{dx}\left[x^{-n} \sum_{K=0}^{\infty} \frac{(-1)^{K}\, x^{2K+n}}{2^{2K+n}\, K!\, \Gamma(K+n+1)}\right] = \frac{d}{dx}\left(\sum_{K=0}^{\infty} \frac{(-1)^{K}\, x^{2K}}{2^{2K+n}\, K!\, \Gamma(K+n+1)}\right)$$

@ $K=0$
the term $=0$
بعد كده
$K \to K+1$
$K+1=1$
$K=0$

$$\sum_{K=1}^{\infty} \frac{(-1)^{K}\, 2K\, x^{2K-1}}{2^{2K+n}\, K!\, \Gamma(K+n+1)} = \sum_{K=0}^{\infty} \frac{(-1)^{K+1}\, 2(K+1)\, x^{2(K+1)-1}}{2^{2K+n+2}\, (K+1)!\, \Gamma(K+n+1+1)}$$

$$= -x^{-n} \sum_{K=0}^{\infty} \frac{(-1)^{K}\, \cancel{(K+1)}\, x^{2K+n+1}}{2^{2K+n+1}\, \cancel{(K+1)}\, K!\, \Gamma(K+n+1+1)} = -x^{-n} \sum_{K=0}^{\infty} \frac{(-1)^{K} \left(\frac{x}{2}\right)^{2K+(n+1)}}{K!\, \Gamma(K+(n+1)+1)}$$

$$= -x^{-n} J_{n+1}(x)$$

iii. $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_{n}(x)$   from i & ii

$$\frac{d}{dx}\left(x^{n} J_{n}(x)\right) = x^{n} J_{n-1}(x) = n x^{n-1} J_{n}(x) + x^{n} J_{n}'(x) \quad \div x^{n}$$

$$\therefore J_{n-1}(x) = \frac{n}{x} J_{n}(x) + J_{n}'(x) \longrightarrow ①$$

$$\frac{d}{dx}\left(x^{-n} J_{n}(x)\right) = -x^{-n} J_{n+1}(x) = -n x^{-n-1} J_{n}(x) + x^{-n} J_{n}'(x) \quad * \; x^{n}$$

$$\therefore -J_{n+1}(x) = \frac{-n}{x} J_{n}(x) + J_{n}'(x) \longrightarrow ②$$

By Sub ①-②  $\therefore J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_{n}(x)$

d) $J_{n-1}(x) - J_{n+1}(x) = 2\, J_n'(x)$ from c) i & ii

$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x) = n\, x^{n-1} J_n(x) + x^n J_n'(x) \quad \div x^n$

$\therefore J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x) \rightarrow ①$

$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) = -n\, x^{-n-1} J_n(x) + x^{-n} J_n'(x) \quad * \; x^n$

$\therefore -J_{n+1}(x) = \frac{-n}{x} J_n(x) + J_n'(x) \rightarrow ②$

By add ① + ② $\quad J_{n-1}(x) - J_{n+1}(x) = 2\, J_n'(x)$

e) $\int J_{n+1}\, dx = \int J_{n-1}\, dx - 2 J_n$ from d) $J_{n-1} - J_{n+1} = 2 J_n'$

$\therefore J_{n+1} = J_{n-1} - 2J_n'$ By integration both sides by dx

$\therefore \int J_{n+1}(x)\, dx = \int J_{n-1}(x)\, dx - 2 J_n(x)$

f) $J_n(x)$ is an odd function when n is odd & it is an even function when n is even

The meaning of the question is show that $J_{-n}(x) = (-1)^n J_n(x)$

$J_n(x) = \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{x}{2}\right)^{2K+n}}{K!\, \Gamma(K+n+1)}$ , $J_{-n}(x) = \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{x}{2}\right)^{2K-n}}{K!\, \Gamma(K-n+1)}$

if $(K-n+1) = 0 \Rightarrow \Gamma(K-n+1) = \Gamma(0) = \infty \Rightarrow \frac{(-1)^K \left(\frac{x}{2}\right)^{2K-n}}{K!\, \Gamma(0)} = 0$

$\therefore K-n+1 \geq 1 \Rightarrow K-n \geq 0 \quad \boxed{\therefore K \geq n}$

$\therefore J_{-n}(x) = \sum_{K=n}^{\infty} \frac{(-1)^K \left(\frac{x}{2}\right)^{2K-n}}{K!\, \Gamma(K-n+1)}$ let $K-n = N \quad \therefore N \geq 0$

$J_{-n}(x) = \sum_{N=0}^{\infty} \frac{(-1)^{n+N} \left(\frac{x}{2}\right)^{2n+2N-n}}{(n+N)!\, \Gamma(N+1)} = \sum_{N=0}^{\infty} (-1)^n \frac{(-1)^N \left(\frac{x}{2}\right)^{2N+n}}{\Gamma(n+N+1)\, N!}$

$\therefore J_{-n}(x) = (-1)^n \sum_{N=0}^{\infty} \frac{(-1)^N \left(\frac{x}{2}\right)^{2N+n}}{N!\, \Gamma(N+n+1)} = (-1)^n J_n(x$

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