Lecture 3

Solution around an ordinary point

Illustrative example.

* we want the solution to be in the form of series.

$y'' + y = 0$ in powers of x (means $x_0 = 0$)

1) $\therefore x_0 = 0$ is an O.P ($p(x), q(x)$ are defined)

$p = 0 \quad q = 1$

2) $y = \sum_{n=0}^{\infty} a_n x^n$

remember: general solution was $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

↳ because at $n = 0$ the coeff $= 0$ $\therefore$ it really starts at 1

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

3) Sub in the original equation

* we need to find the coefficients.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

* For this to be true the coeff. need to be equal zero

exp: $a + bx = 0$

$a + b = 0$

$\left.a + 2b = 0\right\}$ $a = 0, b = 0$

Valid for values of x

$\forall x$

* The solution will have a radius of convergence or an interval at which it'll converge

$-R \longleftarrow 0 \longrightarrow R$

↳ the valid values of x which is infinite amount of values within the interval

* shift the index of the first summation

$\sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n$ ↳ power of x

* If your solution is infinite i.e infinite number of solution for some function, means the coefficients should vanish.

4) $\Rightarrow \sum_{n=0}^{\infty} \left((n+2)(n+1)\, a_{n+2} + a_n\right) x^n = 0$

$\therefore$ the coefficients must be equal to zero.

5) $(n+2)(n+1)\, a_{n+2} + a_n = 0$

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n \quad , n \geq 0$$

Reccurence relation.

you can get a_2 function of a_0 and all the even a

all the odd coeff can be found function of a_1

6) Solve the Recc. relation twice

$$a_2 = \frac{-1}{(2)(1)} a_0$$

$$a_4 = \frac{-1}{(4)(3)} a_2 = \frac{(-1)^2}{(4)(3)(2)(1)} a_0 = \frac{(-1)^2}{4!} a_0$$

$$a_6 = \frac{-1}{(6)(5)} = \frac{(-1)^3}{6!} a_0$$

$$a_3 = \frac{-1}{(3)(2)} a_1 = \frac{-1}{3!} a_1$$

$$a_5 = \frac{-1}{(5)(4)} a_3 = \frac{(-1)^2}{5!} a_1$$

$$a_7 = \frac{-1}{(7)(6)} a_5 = \frac{(-1)^3}{7!} a_1$$

$2(3)+1=7 \rightarrow 2n+1$

$$\therefore\ \boxed{a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \ , n \geq 0} \qquad \boxed{a_{2n+1} = \frac{(-1)^n}{(2n+1)!} \quad n \geq 0}$$

* we don't know a_0, a_1 but we'll find our solution as function of them.

7) find y

$a_{2n} = \frac{(-1)^n}{(2n)!} a_0, \; n \geq 0$

$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1, \; n \geq 0$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

If n hadn't start from zero, we would've wrote the expression as:

$$a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n}$$

our general solution.

* This solution is valid considering a_0, a_1 our arbitrary constants, exp: for $y'' + y = 0$

$y_{cf} = c_1 y_1 + c_2 y_2$

c_1, c_2 can be found thro boundary conditions.

Notice: in the previous solution, the first term represents $\cos x$ and the second therm $\sin x$ (from taylor series)

$y = a_0 \cos x + a_1 \sin x$ (8th step)

→ summary of steps:

1. define X_0
2. Assume the solution based on the nature of X_0
3. Substitute in the original equation
4. Find the reccurence relation
5. sub with even and odd values of the index
6. Find the general form (solve the reccurence twice)
7. Find y, function in the general form you derived

* The question can be : Find a series of soln.

or " " general "

in powers of X.

i.e. ($X_0 = 0$)

exp : $(1-x^2)y'' - 4xy' + 4y = 0$

1) $X = 0$ is OP — p, q at 0 is defined

2) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\Rightarrow y'' - x^2 y'' - 4xy' + 4y = 0$

3) sub in it

دخلناها جوا الـ X

$X \cdot X^{n-1} = X^n$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 4\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

4) Shift index :

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - 4\sum_{n=0}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - \underbrace{(n(n-1) + 4n - 4)}_{= n^2+3n-4} a_n \right) x^n = 0$$

5) Coeff must vanish

$\Rightarrow (n+2)(n+1) a_{n+2} - (n^2 + 3n - 4) a_n = 0$, $n \geq 0$

$$a_{n+2} = \frac{(n-1)(n+4)}{(n+2)(n+1)} a_n , \quad n \geq 0$$

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يستحسن متحفظش الارقام عشان تعرف تستنتج القاعده وتعملها في ال general form

8) Solving the recurrence relation twice

$$a_2 = \frac{(-1)(4)}{(2)(1)} a_0$$

$$a_4 = \frac{(1)(6)}{(4)(3)} a_2 = \frac{(-1\times1)(4\times6)}{4!} a_0$$

$$a_6 = \frac{(3)(8)}{(6)(5)} a_4 = \frac{(-1\times1\times3)(4\times6\times8)}{6!} a_0$$

$$a_8 = \frac{(-1\times1\times3\times5)(4\times6\times8\times10)}{8!} a_0$$

لازم اجيب اخر رقم

$$a_{2n} = \frac{(-1\times1\times\dots\times(2n-3))(4\times6\times\dots\times(2n+2))}{(2n)!} a_0 \quad n \geq 1$$

Infinite series

2.2
2.3
2.4
$2. \vdots$
$2(n+1)$

$a_3 = 0$

$a_5 = 0$

$\vdots$

$a_{2n+1} = 0$

Terminated solution
→ Finite series
(Polynomial solution)
because odd indices only have a_1 now.

7) $$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1\times1\times\dots\times 2n-3)\times 2^n (n+1)!}{(2n)!} x^{2n} \right] + a_1 x$$

→ we call them general solution / 2 linearly independent solution

→ If we have $x_0 = 1$: $\sum_{n=0}^{\infty} a_n (x-1)^n$ or let $x-1 = t$

$$y = \sum_{n=0}^{\infty} a_n t^n$$

and in the end sub back x

→ intro to Series solutions around singular points

exp: Find a series solution in powers of x.

$$2x^2y'' + 3xy' - (x^2+1)y = 0$$

$P(x) = \frac{3}{2x}$ at $x=0$ $P(x) \to \infty$

1) ∴ $x_0 = 0$ is singular but RS regular singularity $q(x) = \frac{(x^2+1)}{-2x^2}$

(not regular aren't solved with series solution)

$P = 3/2$

$Q = \frac{x^2+1}{-2}$

2) $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ → singularity exponent

$$y' = \sum_{\boxed{n=0}}^{\infty} (n+s) a_n x^{n+s-1}$$

$$y'' = \sum_{\boxed{n=0}}^{\infty} (n+s)(s+n-1) a_n x^{n+s-2}$$

3) Sub in diffe equation

the only different one

$$\sum_{n=0}^{\infty} 2(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s)a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

shifting need

$$\Rightarrow \sum_{n=2}^{\infty} a_{n-2} x^{n+s}$$

it can't start at zero

* So I make them all start from two and I take out the terms that are at $n=0$, $n=1$

$a_0(2s(s-1)+3s-1)x^s + a_1(2(s+1)s+3(s+1)-1)x^{s+1} +$

$$\sum_{n=2}^{\infty}(a_n[2(n+s)(n+s-1)+3(n+s)-1]-a_{n-2})x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}2(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}3(n+s)a_n x^{n+s} - \sum_{n=0}^{\infty}a_n x^{n+s} - \sum_{n=2}^{\infty}a_{n-2}x^{n+s} = 0$$

$\Rightarrow$ extra step is to get s (Knowing all coeff must vanish.

$\Rightarrow$ For the least power of $x \Rightarrow x^s$, the coeff is called the indicial equation which should be quadratic and with two roots

$\hookrightarrow a_0(2s(s-1)+3s-1)=0$

$a_0(2s^2-2s+3s-1)=0$

$a_0(2s^2+s-1)=0 \longrightarrow a_0=0$

$\longrightarrow (2s-1)(s+1)=0$

$s_1=\frac{1}{2} \quad s_2=-1$

s: singularity exponents

$\hookrightarrow$ So what are the restrictions?