Math. 3 Midterm

2nd Year Electrical Eng. November 14th , 2016. Allowed Time: 1 Hour.

Model Answer

Question 1 (12 Marks)

(A) ***Evaluate in terms of the Gamma function*** $\int_0^\infty x^a \; b^{-x} \; dx$. ***Hence State the conditions on the constants*** a **&** b ***such that the integral converges.***

(B) ***Find the general solution in powers of*** x ***for*** $(1-x^2)\,y'' - 2x\,y' + 2y = 0$.

$$I = \int_0^\infty x^a b^{-x} dx$$

$$\text{let } b^{-x} = e^{-t} \Rightarrow -x\ln b = -t \cdots\cdots\cdots\cdots \boxed{1}$$

$$\Rightarrow x = \frac{t}{\ln b} \Rightarrow dx = \frac{dt}{\ln b} \Rightarrow \ln b > 0 \Rightarrow \boxed{b>1} \cdots\cdots \boxed{1}$$

$$\Rightarrow I = \int_0^\infty \left(\frac{t}{\ln b}\right)^a e^{-t}\frac{dt}{\ln b} \cdots\cdots\cdots\cdots \boxed{1}$$

$$= \left(\frac{1}{\ln b}\right)^{a+1} \int_0^\infty t^a e^{-t} dt$$

$$= \left(\frac{1}{\ln b}\right)^{a+1} \Gamma(a+1) \cdots\cdots\cdots\cdots \boxed{1}$$

$$\Rightarrow a+1>0 \Rightarrow \boxed{a>-1} \cdots\cdots\cdots\cdots \boxed{1}$$

$$\rightarrow p(x) = \frac{-2x}{1-x^2},\ q(x) = \frac{2}{1-x^2}$$

$\Rightarrow$ p,q are both analytic at $x_0 = 0 \cdots\cdots\cdots\cdots \boxed{1}$

$\Rightarrow x_0$ is Ordinary point $\Rightarrow$ Power series method

$$\rightarrow y = \sum_0^\infty a_n\ x^n \Rightarrow y' = \sum_1^\infty n a_n\ x^{n-1} \Rightarrow y'' = \sum_2^\infty n(n-1)a_n\ x^{n-2} \cdots\cdots\cdots \boxed{1}$$

$$\Rightarrow (1-x^2)\sum_2^\infty n(n-1)a_n\ x^{n-2} - 2x\sum_1^\infty n a_n\ x^{n-1} + 2\sum_0^\infty a_n\ x^n = 0$$

$$\Rightarrow \sum_2^\infty n(n-1)\, a_n x^{n-2} - \sum_2^\infty n(n-1)a_n\ x^n - \sum_1^\infty 2\ n a_n\ x^n + \sum_0^\infty 2a_n\ x^n = 0$$

$$\Rightarrow \sum_0^\infty (n+1)(n+2)a_{n+2}\ x^n - \sum_2^\infty n(n-1)\, a_n x^n - \sum_1^\infty 2\ n a_n\ x^n + \sum_0^\infty 2a_n\ x^n = 0 \cdots \boxed{1}$$

$$\Rightarrow \sum_0^\infty \left[(n+1)(n+2)a_{n+2} - \{n(n-1)+2n-2\}a_n\right]x^n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{n^2+n-6}{(n+1)(n+2)}a_n = \frac{(n-1)(n+2)}{(n+1)(n+2)}a_n = \frac{(n-1)}{(n+1)}a_n \cdots\cdots n\ge 0} \cdots\cdots\cdots \boxed{1}$$

$\boxed{n=0}$

$$\Rightarrow a_2 = \frac{-1}{1}a_0$$

$\boxed{n=1}$

$$\Rightarrow a_3 = 0 = a_5 = a_7 = \cdots$$

$\boxed{n=2}$

$$\Rightarrow a_4 = \frac{1}{3}a_2 = \frac{-1.1}{1.3}a_0 = \frac{-1}{3}a_0$$

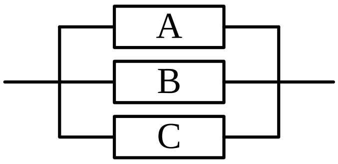
$\boxed{n=4}$

$$\Rightarrow a_6 = \frac{3}{5}a_4 = \frac{-1.1.3}{1.3.5}a_0 = \frac{-1}{5}a_0$$

$$\Rightarrow \boxed{a_{2n} = \frac{-1.1.3.........(2n-3)}{1.3.5\cdots\cdots\cdots(2n-1)}a_0 = \frac{-1}{2n-1}a_0 \cdots\cdots n\ge 1} \cdots\cdots\cdots \boxed{1}$$

$$\Rightarrow \boxed{y = a_0(1+\sum_1^\infty a_{2n}\ x^{2n}) + a_1(x)} \cdots\cdots\cdots\cdots \boxed{2}$$

3- Three switches A, B & C are connected in parallel.
Switch A is independently ON with a probability of 0.7.
When switch A is OFF, switch B is ON with a probability of 0.2.
Switch C is independently ON with a probability of 0.6.
<u>Find</u> the probability that the circuit is ON.



Given

$$P(A)=0.7, P(C)=0.6$$
$$P(B/A^c)=0.2$$
$$\Rightarrow P(A^c)=0.3, P(C^c)=0.4$$
$$P(B^c/A^c)=0.8$$

$$P(reqiured)=P\left(A\cup B\cup C\right)\cdots\cdots\cdots\cdots\boxed{1}$$
$$=1-P(A^c\cap B^c\cap C^c)\cdots\cdots\cdots\cdots\boxed{1}$$
$$=1-\left(P(A^c)*P(B^c/A^c)*P(C^c/A^c\cap B^c)\right)\cdots\cdots\boxed{1}$$
$$=1-((0.3)*(0.8)*(0.4))=0.904\cdots\cdots\cdots\boxed{1}$$

OR

We have A independently ON and C independently ON, then B is independently ON also.

$$\Rightarrow P(B/A^c)=0.2=P(B)$$
$$P(reqiured)=P\left(A\cup B\cup C\right)\cdots\cdots\boxed{1}$$
$$=P(A)+P(B)+P(C)$$
$$-P(A\cap B)-P(A\cap C)-P(B\cap C)\cdots\cdots\cdots\boxed{1}$$
$$+P(A\cap B\cap C)$$
$$=P(A)+P(B)+P(C)$$
$$-P(A)*P(B)-P(A)P(C)-P(B)*P(C)\cdots\cdots\cdots\boxed{1}$$
$$+P(A)*P(B)*P(C)$$
$$=0.7+0.2+0.6$$
$$-(0.7*0.2)-(0.7*0.6)-(0.2*0.6)\cdots\cdots\cdots\boxed{1}$$
$$+(0.7*0.2*0.6)=0.904$$

4- Six balls are drawn from a box containing 10 white and 20 black balls. Find the probability of having at least 5 black balls.

$$P(required)=P(5B,1W)+P(6B,0W)\cdots\cdots\boxed{2}$$

$$=\frac{{}^{20}C_5*{}^{10}C_1+{}^{20}C_6*{}^{10}C_0}{{}^{30}C_6}\cdots\cdots\cdots\cdots\boxed{1}$$

OR

$$=6*\left(\frac{20}{30}*\frac{19}{29}*\frac{18}{28}*\frac{17}{27}*\frac{16}{26}*\frac{10}{25}\right)$$
$$+\left(\frac{20}{30}*\frac{19}{29}*\frac{18}{28}*\frac{17}{27}*\frac{16}{26}*\frac{15}{25}\right)$$
$$=\mathbf{\frac{2584}{7917}=0.3264}\cdots\cdots\cdots\cdots\boxed{1}$$

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Model Answer

Question 1 (12 Marks)

(A) *Evaluate in terms of the Gamma function* $\int_0^\infty \frac{x^c}{c^x}\,dx$. *Hence State the conditions on the constant* C *such that the integral converges.*

(B) *Find the general solution in powers of* x *for* $(1-x^2)\,y'' - 2x\,y' + 6y = 0$.

$$I=\int_0^\infty x^c c^{-x}dx$$

$$\text{let } c^{-x}=e^{-t}\Rightarrow -x\ln c=-t\cdots\cdots\boxed{1}$$

$$\Rightarrow x=\frac{t}{\ln c}\Rightarrow dx=\frac{dt}{\ln c}\Rightarrow \ln c>0\Rightarrow c>1\cdots\cdots\boxed{1}$$

$$\Rightarrow I=\int_0^\infty\left(\frac{t}{\ln c}\right)^c e^{-t}\frac{dt}{\ln c}\cdots\cdots\boxed{1}$$

$$=\left(\frac{1}{\ln c}\right)^{c+1}\int_0^\infty t^c e^{-t}dt$$

$$=\left(\frac{1}{\ln c}\right)^{c+1}\Gamma(c+1)\cdots\cdots\boxed{1}$$

$$\Rightarrow c+1>0\Rightarrow c>-1\Rightarrow \boxed{c>1}\cdots\cdots\boxed{1}$$

$$\to p(x)=\frac{-2x}{1-x^2},\ q(x)=\frac{6}{1-x^2}$$

$\Rightarrow$ p,q are both analytic at $x_0=0\cdots\cdots\boxed{1}$

$\Rightarrow x_0$ is Ordinary point $\Rightarrow$ Power series method

$$\to y=\sum_0^\infty a_n\,x^n\Rightarrow y'=\sum_1^\infty na_n\,x^{n-1}\Rightarrow y''=\sum_2^\infty n(n-1)a_n\,x^{n-2}\cdots\cdots\boxed{1}$$

$$\Rightarrow (1-x^2)\sum_2^\infty n(n-1)a_n\,x^{n-2}-2x\sum_1^\infty na_n\,x^{n-1}+6\sum_0^\infty a_n\,x^n=0$$

$$\Rightarrow \sum_2^\infty n(n-1)\,a_nx^{n-2}-\sum_2^\infty n(n-1)a_n\,x^n-\sum_1^\infty 2\,na_n\,x^n+\sum_0^\infty 6a_n\,x^n=0$$

$$\Rightarrow \sum_0^\infty (n+1)(n+2)a_{n+2}\,x^n-\sum_2^\infty n(n-1)\,a_nx^n-\sum_1^\infty 2\,na_n\,x^n+\sum_0^\infty 6a_n\,x^n=0\cdots\boxed{1}$$

$$\Rightarrow \sum_0^\infty\left[(n+1)(n+2)a_{n+2}-\{n(n-1)+2n-6\}a_n\right]x^n=0$$

$$\Rightarrow \boxed{a_{n+2}=\frac{n^2+n-6}{(n+1)(n+2)}a_n=\frac{(n-2)(n+3)}{(n+1)(n+2)}a_n\cdots\cdots n\ge 0}\cdots\cdots\boxed{1}$$

$\boxed{n=0}$

$$\Rightarrow a_2=\frac{-2.3}{1.2}a_0$$

$\boxed{n=1}$

$$\Rightarrow a_3=\frac{-1.4}{2.3}a_1$$

$\boxed{n=2}$

$$\Rightarrow a_4=0=a_6=a_8=\ldots\ldots$$

$\boxed{n=3}$

$$\Rightarrow a_5=\frac{1.6}{4.5}a_3=\frac{-1.1.4.6}{5.4.3.2}a_1$$

$\boxed{n=5}$

$$\Rightarrow a_7=\frac{3.8}{6.7}a_5=\frac{-1.1.3.4.6.8}{7.6.5.4.3.2}a_1$$

$$\Rightarrow \boxed{a_{2n+1}=\frac{-1.1.3\ldots\ldots(2n-3)\ \ 4.6.8\ldots\ldots(2n+2)}{(2n+1)!}a_1\cdots\cdots n\ge 1}\cdots\cdots\boxed{1}$$

$$\Rightarrow \boxed{\begin{array}{l} y=a_0(1-3x^2)\\ +a_1\left(x+\sum_1^\infty a_{2n+1}\,x^{2n+1}\right)\end{array}}\cdots\cdots\boxed{2}$$

3- Three switches A, B & C are connected in parallel.
Switch A is independently ON with a probability of 0.7.
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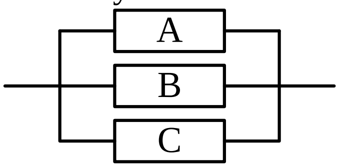
When switch A is OFF, switch C is ON with a probability of 0.2.
<u>Find</u> the probability that the circuit is ON.



Given

$$P(A) = 0.7, P(B) = 0.6$$

$$P(C/A^c) = 0.2$$

$$\Rightarrow P(A^c) = 0.3, P(B^c) = 0.4$$

$$P(C^c/A^c) = 0.8$$

$$P(reqiured) = P\left(A \cup B \cup C\right) \cdots\cdots\cdots \boxed{1}$$

$$= 1 - P(A^c \cap B^c \cap C^c) \cdots\cdots\cdots \boxed{1}$$

$$= 1 - \left(P(A^c) * P(C^c/A^c) * P(B^c/A^c \cap C^c)\right) \cdots\cdots\cdots \boxed{1}$$

$$= 1 - ((0.3)*(0.8)*(0.4)) = 0.904 \cdots\cdots\cdots \boxed{1}$$

OR

We have A independently ON and B independently ON, then C is independently ON also.

$$\Rightarrow P(C/A^c) = 0.2 = P(C)$$

$$P(reqiured) = P\left(A \cup B \cup C\right) \cdots\cdots\cdots \boxed{1}$$

$$= P(A) + P(B) + P(C)$$

$$-P(A \cap B) - P(A \cap C) - P(B \cap C) \cdots\cdots\cdots\cdots\cdots \boxed{1}$$

$$+P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C)$$

$$-P(A)*P(B) - P(A)P(C) - P(B)*P(C) \cdots\cdots\cdots\cdots\cdots\cdots \boxed{1}$$

$$+P(A)*P(B)*P(C)$$

$$= 0.7 + 0.6 + 0.2$$

$$-(0.7*0.6) - (0.7*0.2) - (0.6*0.2) \cdots\cdots\cdots\cdots\cdots\cdots \boxed{1}$$

$$+(0.7*0.6*0.2) = 0.904$$

4- Six balls are drawn from a box containing 20 white and 10 black balls. Find the probability of having at least 5 white balls.

$$P(required) = P(5W, 1B) + P(6W, 0B) \cdots\cdots\cdots \boxed{2}$$

$$= \frac{{}^{20}C_5 * {}^{10}C_1 + {}^{20}C_6 * {}^{10}C_0}{{}^{30}C_6} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \boxed{1}$$

OR

$$= 6 * \left(\frac{20}{30} * \frac{19}{29} * \frac{18}{28} * \frac{17}{27} * \frac{16}{26} * \frac{10}{25}\right)$$

$$+\left(\frac{20}{30} * \frac{19}{29} * \frac{18}{28} * \frac{17}{27} * \frac{16}{26} * \frac{15}{25}\right)$$

$$= \mathbf{\frac{2584}{7917} = 0.3264} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \boxed{1}$$