Numerical Analysis : Introduction

Why ?

- Exact Methods are limited

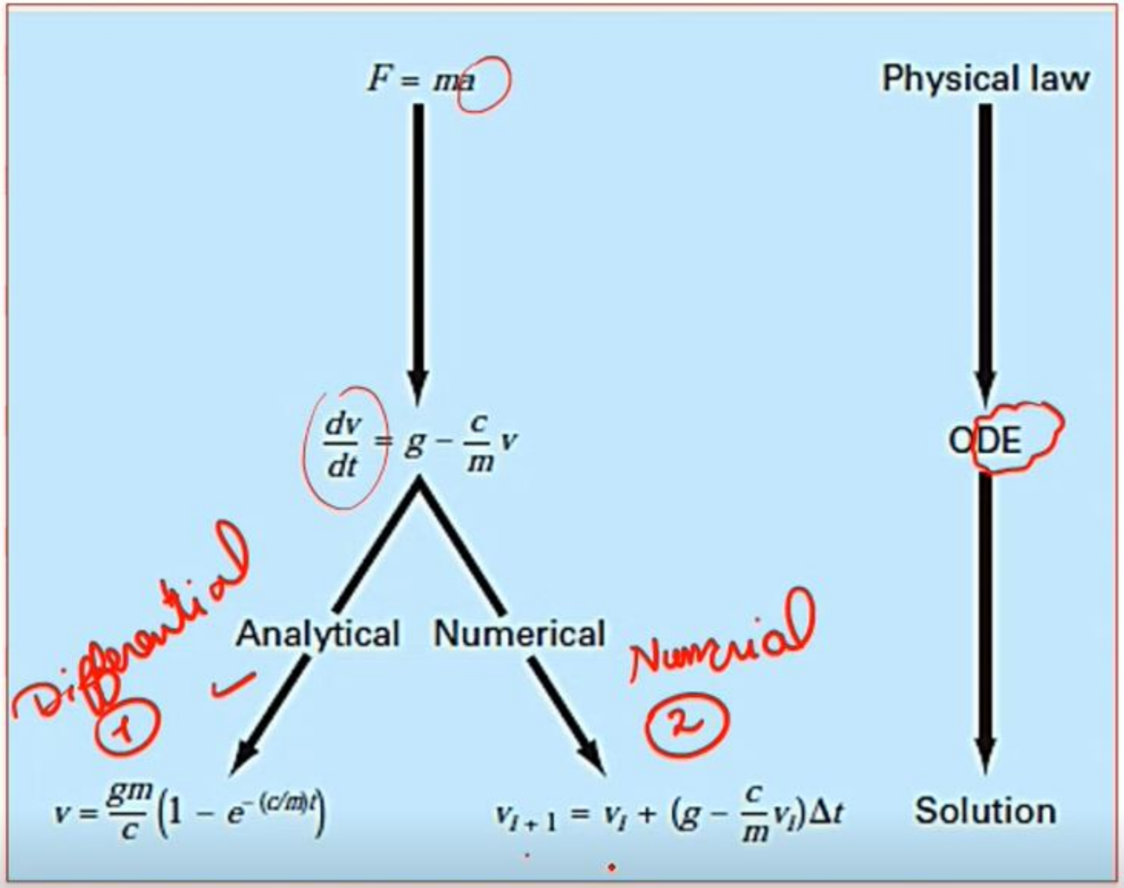
Sol: $x = y^2 + \sin(xy)$?? at $x = 1$ $y = ?$

- Solution of Exact Methods may be implicit functions

- Geometrical Methods are not accurate

Numerical Analysis : Introduction

How ?



Numerical Analysis : Introduction

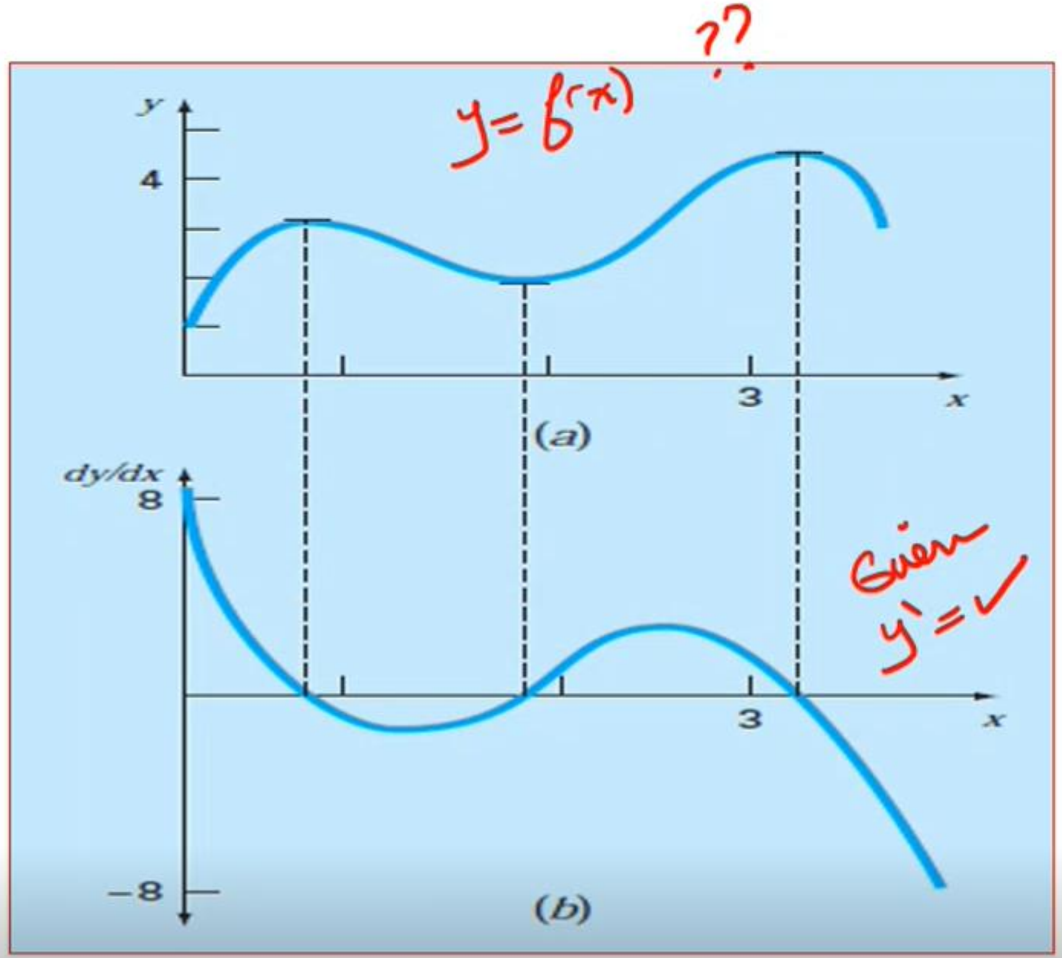
Idea ?

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

??

+c

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$



Numerical Analysis : Introduction

Idea ?

① D.E.

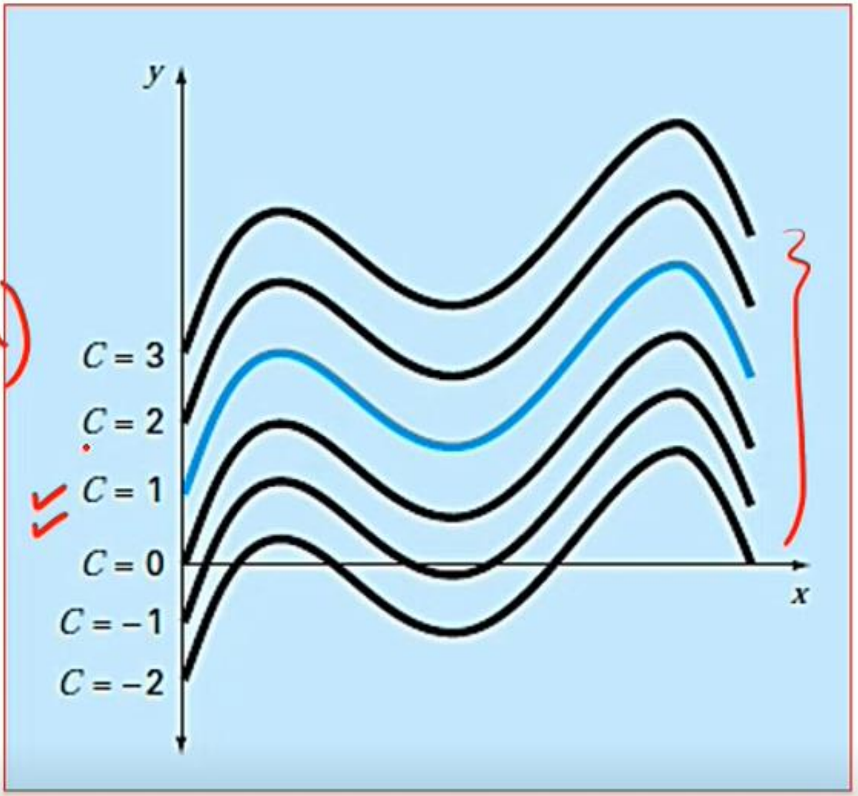
$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

② Point on y

at $x = 0,\ y = 1.$

IVP (Initial Value Problem)

system of 1 or more Differential Equations, together with 1 or more initial conditions.



Idea ?

Solution of IVP

Numerical methods gets the solution in a tabular form

n	x_n	y_n
0	0.0	0.000
1	0.2	0.000
2	0.4	0.04
3	0.6	0.128
4	0.8	0.274
5	1.0	0.488

$y = f(x)$ ✗

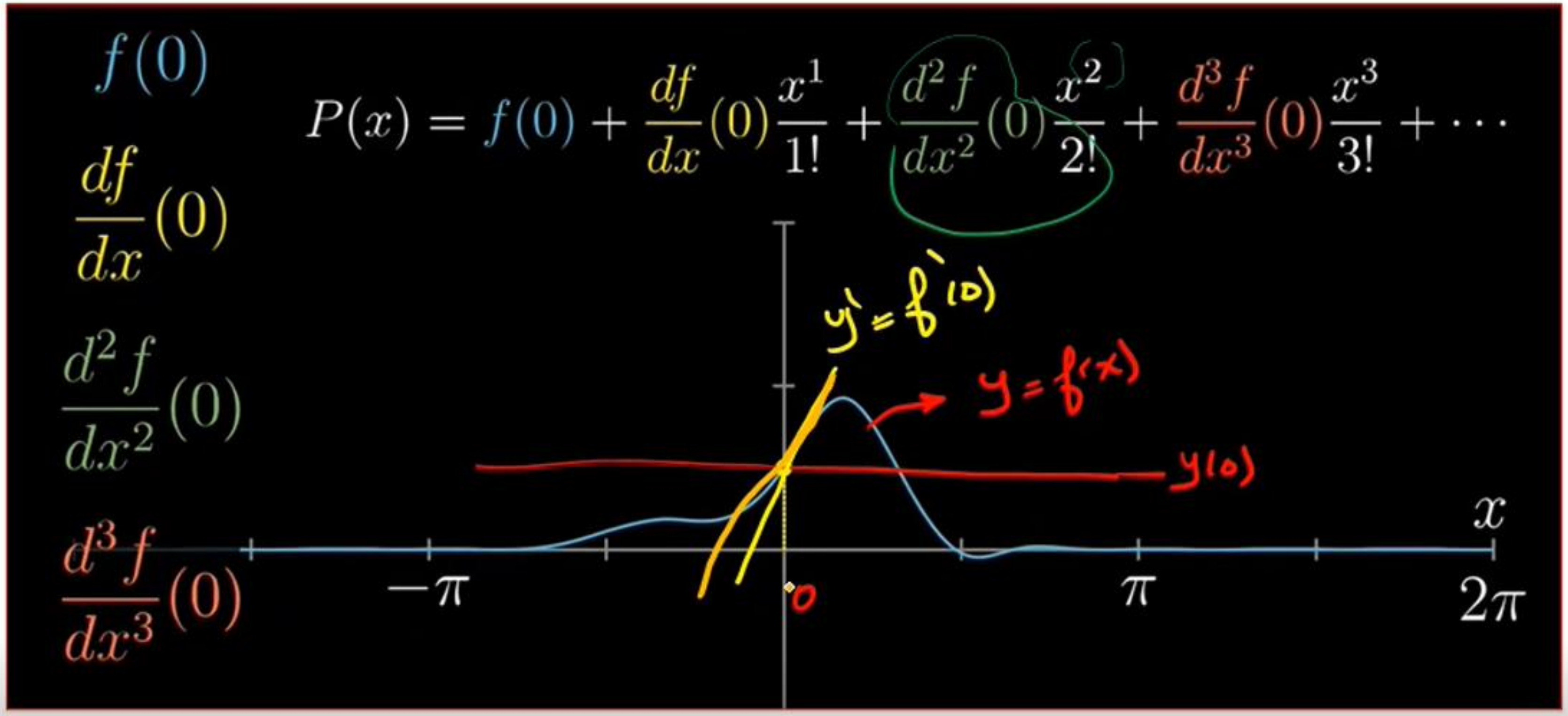
Numerical Analysis : Introduction

$y \simeq P(x)$

$P(x) = a + bx + cx^2 \dots$

dx^3

Series Approximation



$a = f(0)$

$b = f'(0)$

$c = ?$

$P''(x) = 0 + 0 + 2c = y''$

$2c = f''(0)$

$\boxed{c} = \frac{f''(0)}{2}$

$P'''(x) = 0 + 0 + 0 + d(3)(2)(1) = f'''(0)$

$\boxed{d} = \frac{f'''(0)}{3!}$

Numerical Analysis : Introduction

Maclaurin Series

$f(x) = e^x$ can be approximated near $x = 0$

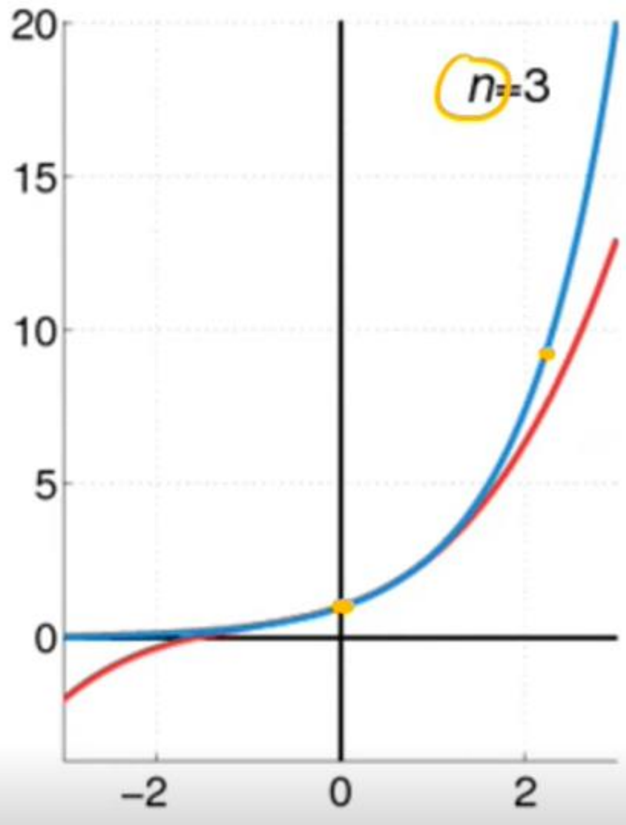
$$f(x) \cong e^0 = a = \boxed{1}$$

$$f(x) \cong a + bx = 1 + f'(x=0)\,x = \boxed{1 + x}$$

$$f(x) \cong a + bx + cx^2 = 1 + x + \frac{f''(x=0)}{2!}\,x^2$$

Near $x = 0$

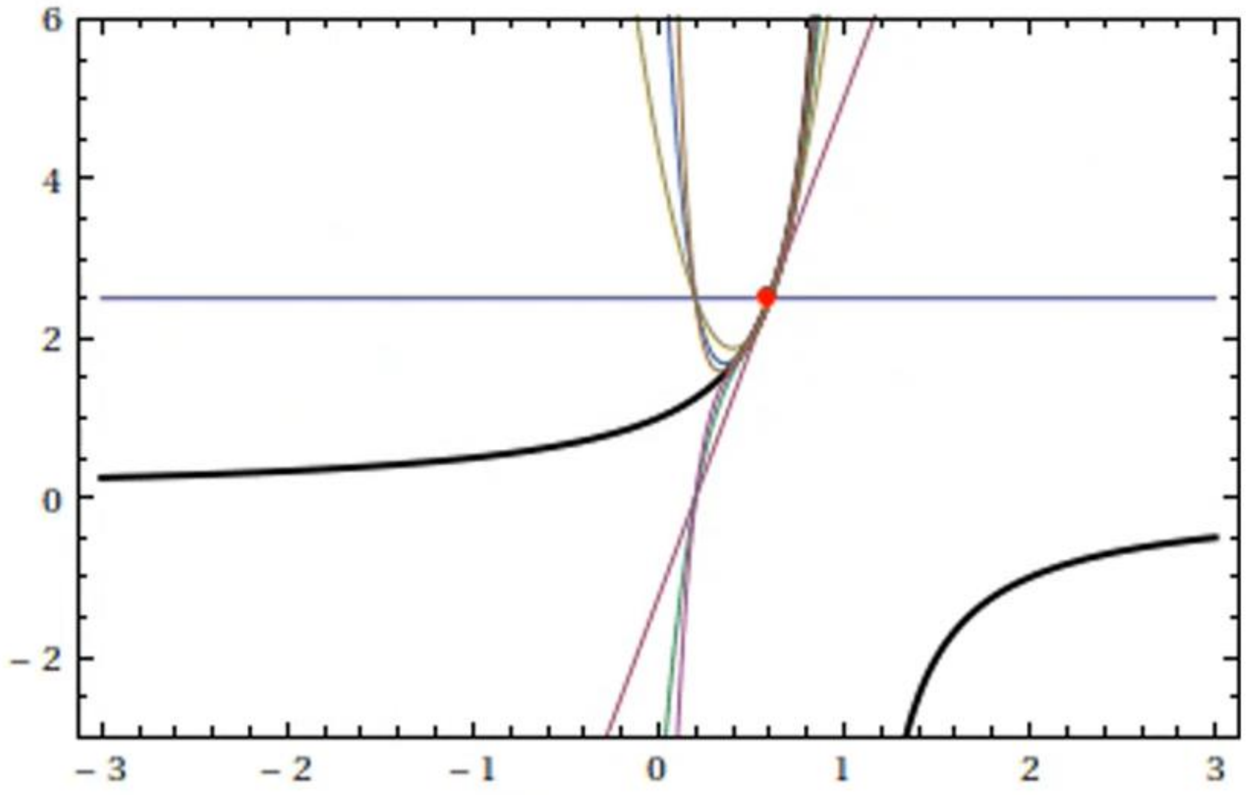
$$\mathrm{f}(x) \cong \mathrm{P}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x=0)}{k!}\,x^k$$



Taylor Series

Near $x = a$

$$f(x) \cong \sum_{k=0}^{\infty} \frac{f^{(k)}(x=a)}{k!}\,(x-a)^k$$



$$f(x) \cong f(a) + \frac{f'(a)}{1!}\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n$$

Taylor Series

The nth order Taylor polynomial of f centered at $x = a$ is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This degree n polynomial approximates $f(x)$ near $x = a$ and has the property that $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0 \ldots n$.

Numerical Analysis : Introduction

Taylor Series

Near $x = a$

$$f(x) \cong T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{n} c_k\,(x-a)^k$$

This approximation is valid as long as

Ratio Test

$$\lim_{k\to\infty}\left|\frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k}\right| = L$$

Exists & **Smaller than 1**

Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} - \sum_{n=0}^{K} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$K=n$

Error $E_{n,a}(x) = f(x) - \mathrm{T_n(x)} = 0 + 0 + \cdots + \frac{f^{(k+1)}(a)}{k+1!}(x-a)^{k+1} + \frac{f^{(k+2)}(a)}{k+2!}(x-a)^{k+2} \ldots$

$E_{n,a}(x)$ **is of order** $(x-a)^{k+1}$

ODE

ex $(y''')^4 + \sin xy = 0$

Order $= 3$
degree $= 4$

$$f\left(x, y, y', y'', \dots, y^{(m)}\right) = 0$$

Order => Highest derivative

Degree => Power of the highest derivative

Solution => all the functions y(x) that satisfy the ODE

$y = f(x)$ 1 Independent variable

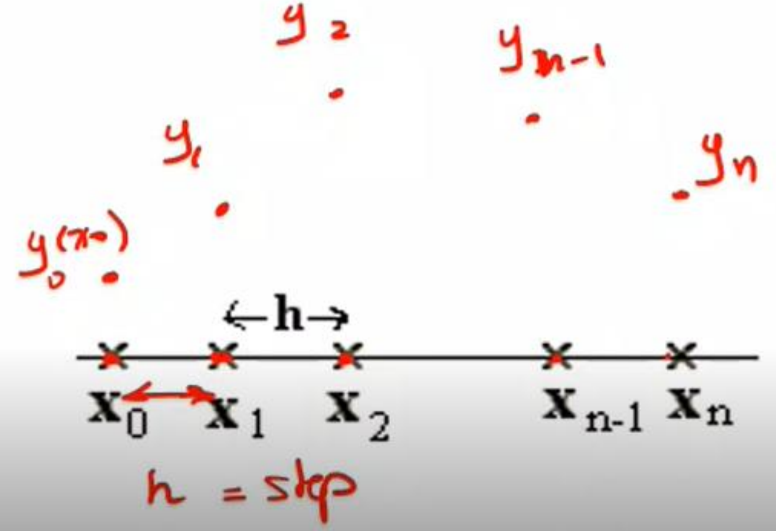
Agenda

- Introduction
- First Order ODE Numerical Methods $y' + f(x,y) = 0$
- Higher Order ODE Numerical Methods

ODE

To solve an ODE numerically over the interval [a,b]

Step 1 Let $x_0 = a, x_n = b$ and subdivide the interval into n equal parts such that $x_i = x_0 + ih \quad , i = 1, 2, 3, \dots n$



ODE

To solve an ODE numerically over the interval [a,b]

Step 2 Define y_i for each x_i according to the **ODE Method**

Equation

$$\rightarrow y_{i+1} = [\underset{i=0}{y_i} + \boxed{}]$$

Step 3 Solve the defined equation and get all y_i

$$\text{New value} = \text{old value} + [\text{slope} \times \text{step size}]$$

Depends on the Method: Euler-Caushy
Runge- Kutta (4)

ODE Methods

Single Step

compute new value y_{i+1} using only a single step (only previous y_i)

Multi- Step

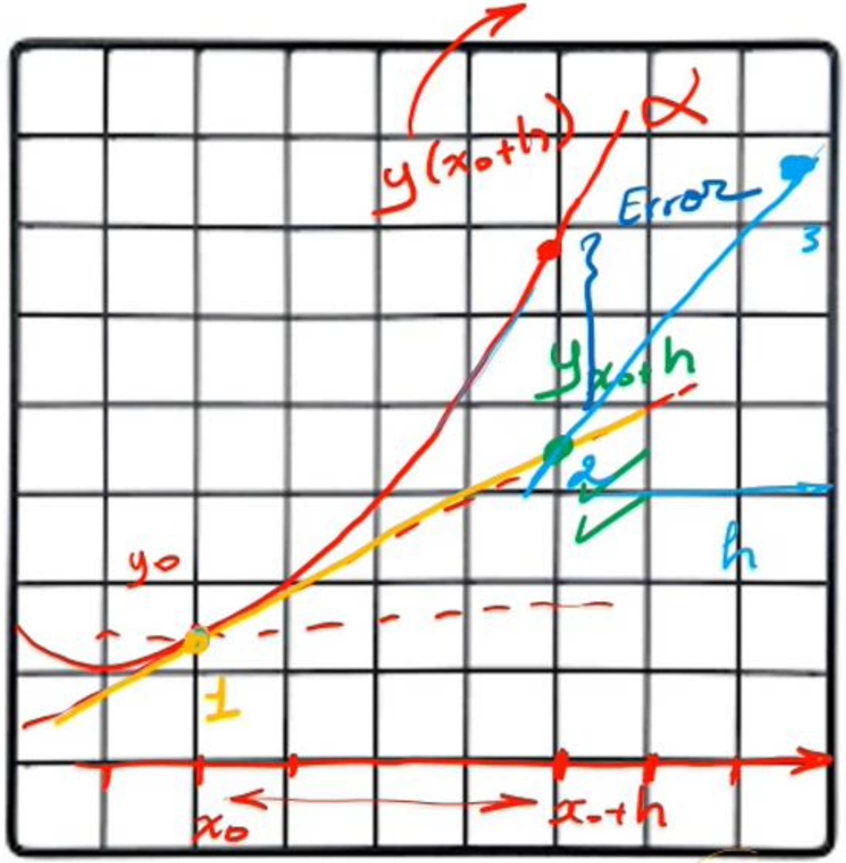
uses, in each step, values from two or more previous steps.

ODE : Euler-Cauchy Method

actual function

1- Method

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0,$$



Then!

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1),$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

New value = old value + slope × step size

$$y(x_0+h) = y(x_0) + f(x_0)\,h$$

$$(x_0+h-x_0)$$

2- Example 1

Apply Euler-Cauchy method with h = 0.1 to find the solution of the initial-value problem

$$y' = 1 - x + 4y,\ y(0) = 1$$

on the interval [0, 0.5] Given the exact solution below compute the error $E_n = y(x_n) - y_n$ in each step. Use 5 decimal points in your calculations.

$$y(x) = \frac{1}{4}x - \frac{3}{16} + \frac{19}{16}e^{4x}$$

ODE : Euler-Cauchy Method

2- Example 1

$$y' = 1 - x + 4y,\ y(0) = 1 \qquad y(x) = \frac{1}{4}x - \frac{3}{16} + \frac{19}{16}e^{4x}$$

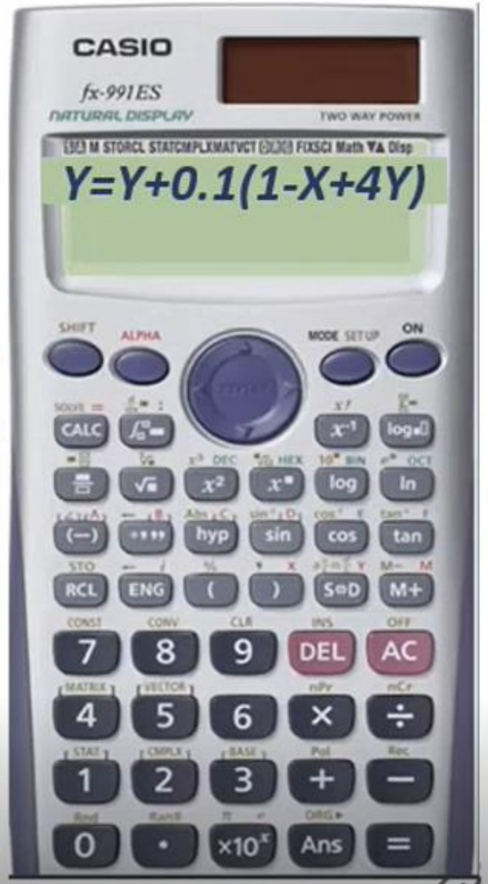
From Euler formula $y_{n+1} = y_n + h\, f(x_n, y_n)$, we have

$$y_{n+1} = y_n + 0.1\,(1 - x_n + 4\,y_n),\ n = 0, 1, 2, \ldots$$

The numerical solution is given in the following table:

① calculator ② calculator $y(0.1)$

n	x_n	y_n	f_n	$y_{n+1} = y_{approx}$	y_{exact}	Error = $y_{exact} - y_{approx}$
0	0	1.0	5.0	at $x=0.1$ 1.5	1.60904	0.10904
1	0.1	1.5	6.9	2.19	2.50533	0.31533
2	0.2	2.19	9.56	3.146	3.83014	0.68414
3	0.3	3.146	13.284	4.4744	5.79423	1.31923
4	0.4	4.4744	18.4976	6.32416	8.71200	$\boxed{2.38784}$ ↑↑



2- Example 2

Apply Euler-Cauchy method witl $h = 0.2$ to find the solution of the initial-value problem

$$y' = x + y \quad , y(0) = 0$$

on the interval [0, 1] Given the exact solution below compute the error $E_n = y(x_n) - y_n$ in each step. Use 3 decimal points in your calculations.

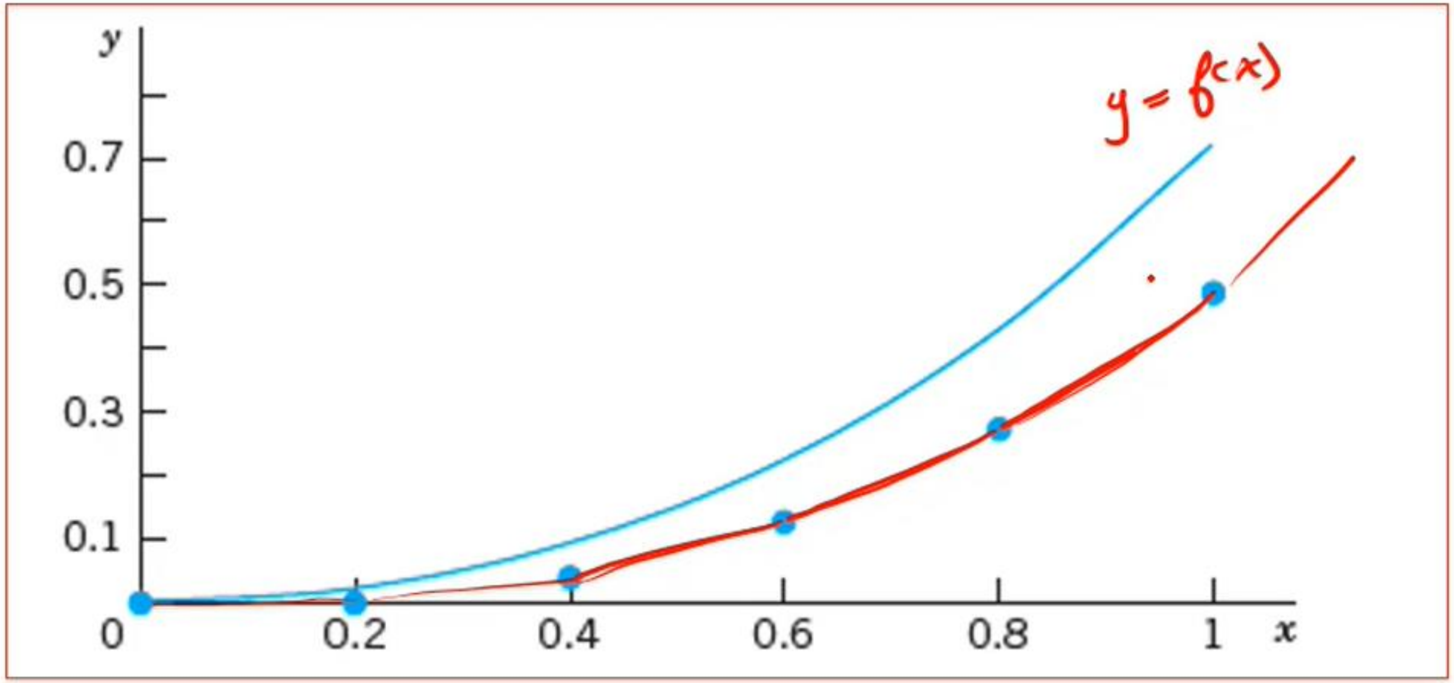
$$y = e^x - x - 1$$

2- Example 2

$$y' = x + y\ , \qquad y(0) = 0 \qquad y = e^{x} - x - 1$$

n	x_n	y_n	$y(x_n)$	Error
0	0.0	0.000	0.000	0.000
1	0.2	0.000	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230

2- Example 2



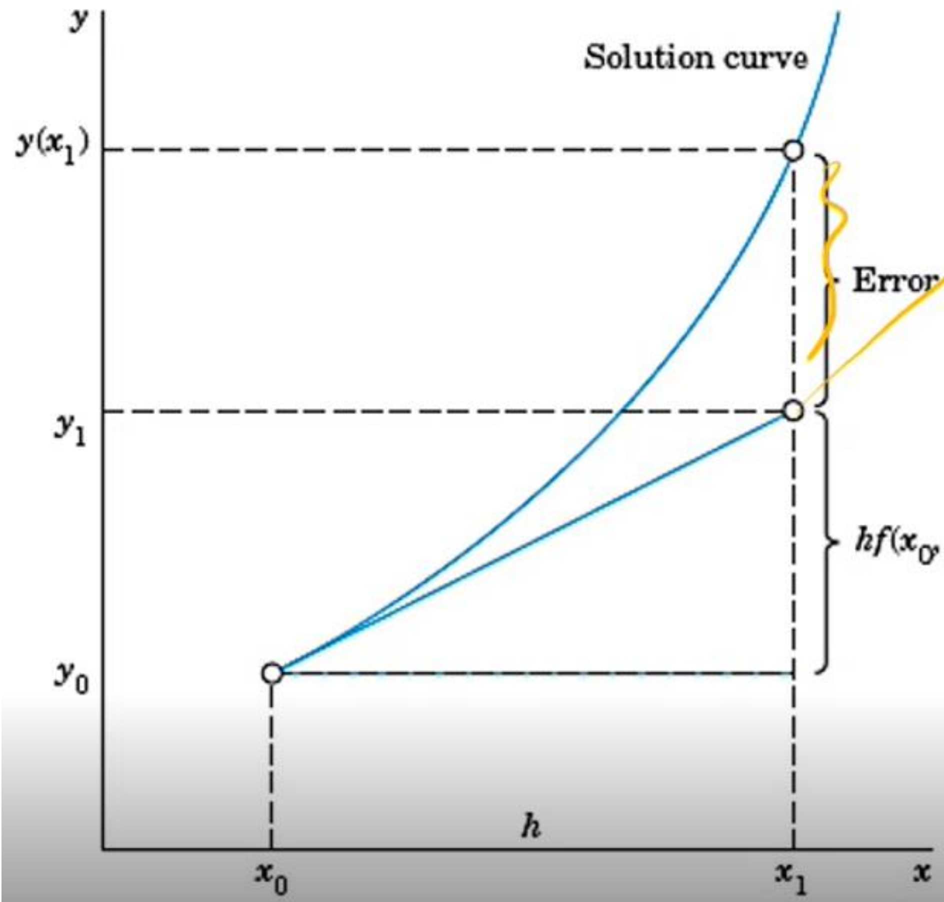
ODE : Euler-Cauchy Method

3- Error

Taylor

$$y(x+h) \simeq \underline{y(x)+\frac{h}{1!}y'(x)} + \frac{h^2}{2!}y''(x)+\ldots+\frac{h^n}{n!}y^{(n)}(x)+\frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi)$$

$\propto \quad \propto$



1- Round-off Error

2- Truncation Error

Local

Order of Error k

$n = \frac{k}{h}$

$n \propto \frac{1}{h}$

$$\frac{f'(x_i, y_i)}{2!}h^2 + \cdots + O(h^{n+1})$$

Global $E_{local} = O(h^2)$

constant

$n = \frac{[a,b]}{h}$

step

a h h h h b

[a, b]

$O(h)$ ← E_{global} of $[O(h^2) \times n] = [O(h^2) \times \frac{k}{\cancel{h}}]$

Remark ! Euler - Cauchy is a first order method

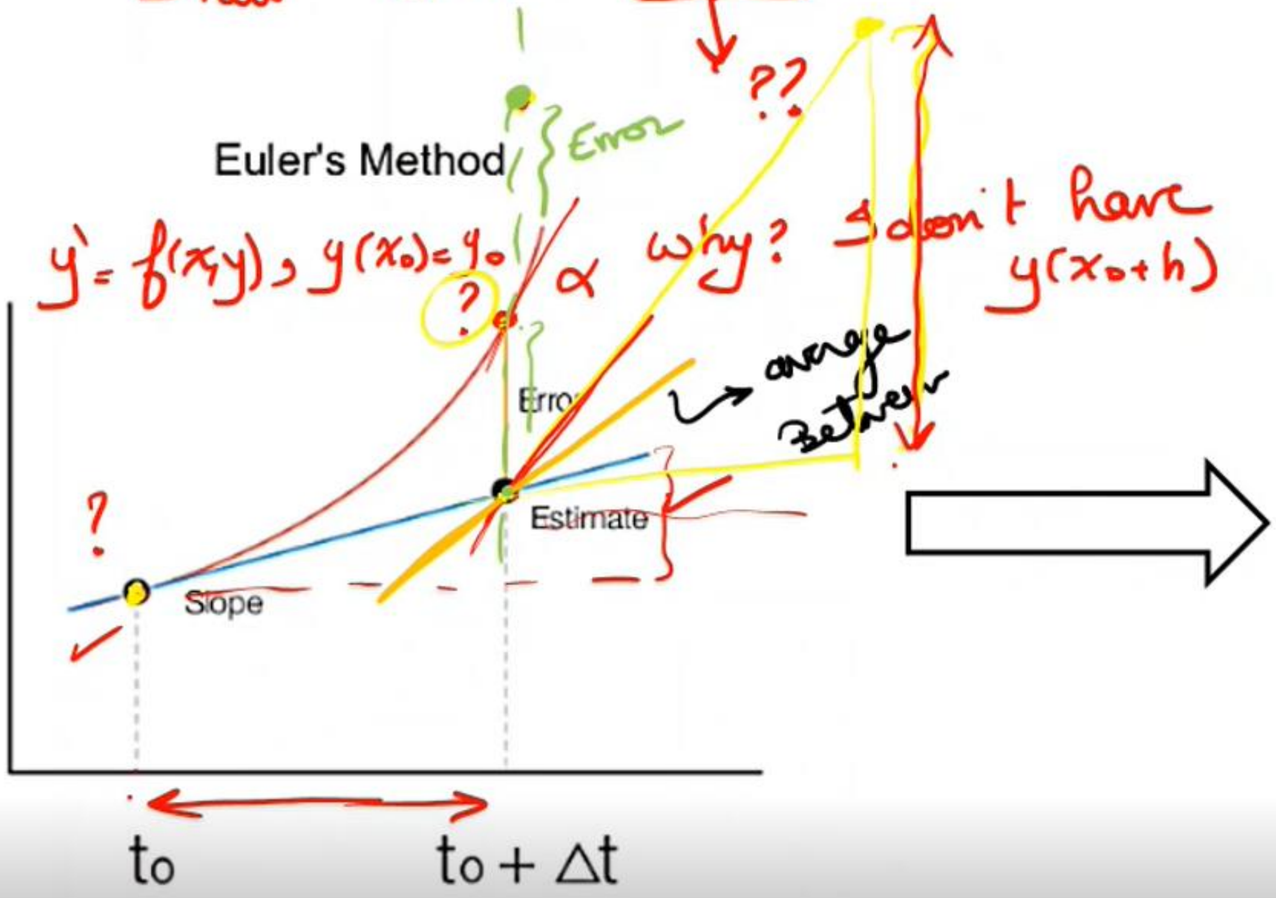
$O_E(h)$ → 1st order

The error between the y_{exact} and y_{euler} is considerably large
And decreasing the step size h n ↑↑
to reach an acceptable approximate y,
will make h very small leading to many operations

Improvements in Euler Method

Improvements in Euler Method

$y_{new} = y_{old} + \boxed{\text{Slope}}\, h$



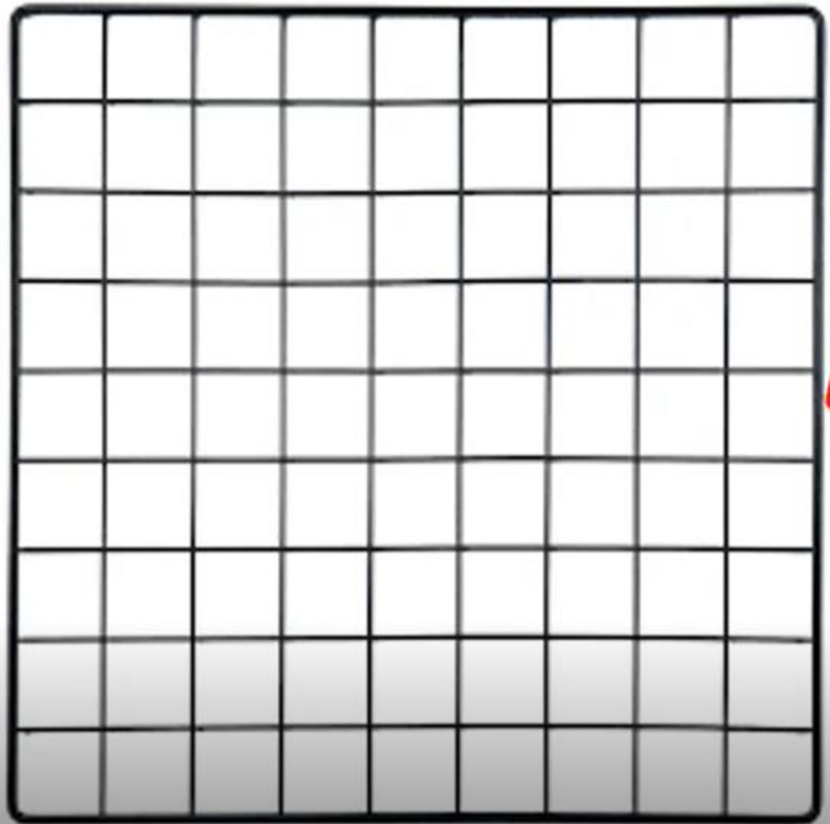
Improved Euler

$y_{new} = y_{old} + \frac{h}{2}\left[y'(x_0) + y'(x_0+h)\right]$

corrector

Local $O(h^3)$

Global $\boxed{O(h^2)}$

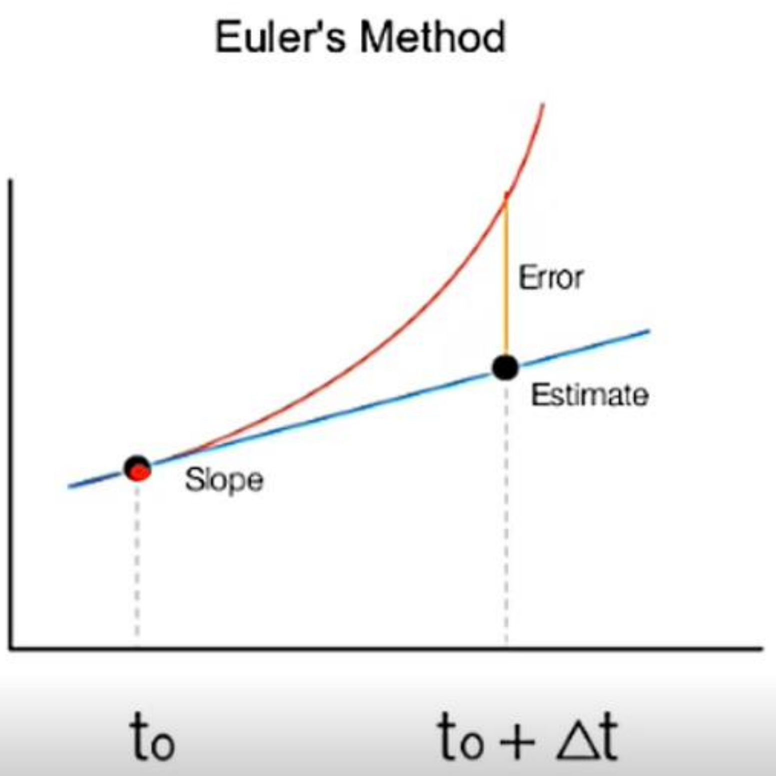


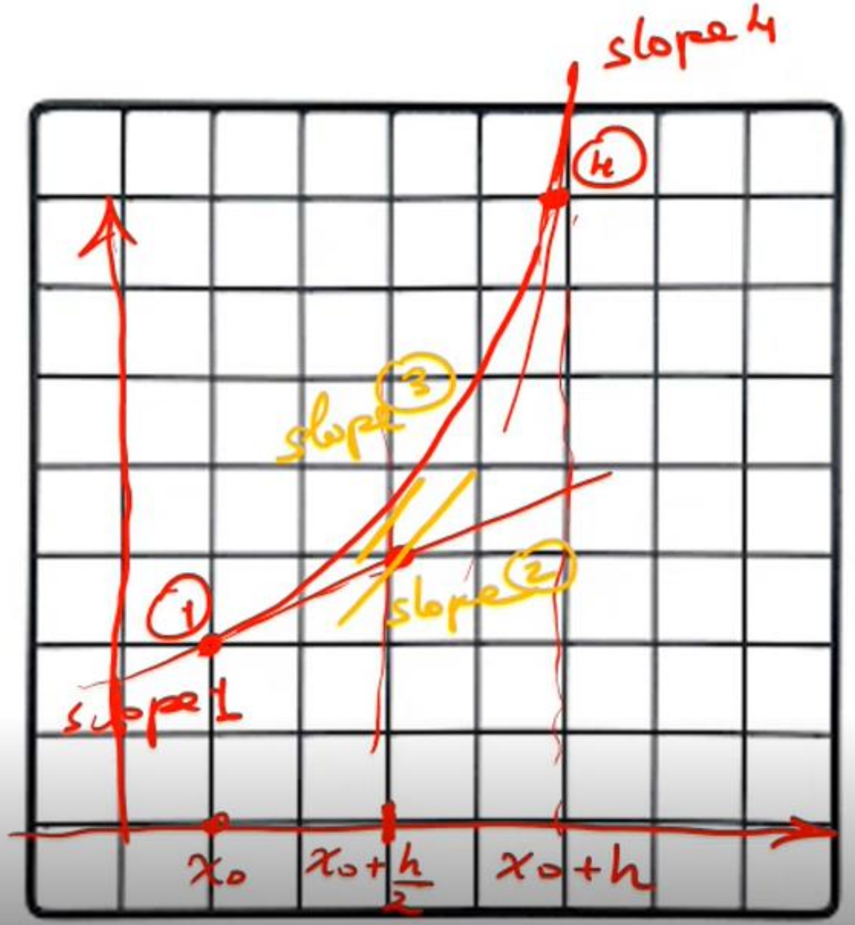
Improvements in Euler Method

$y_{new} = y_{old} + (\text{slope})\, h$

$(k_1 + k_2 + k_3 + k_4) h$

Runge-Kutta 4



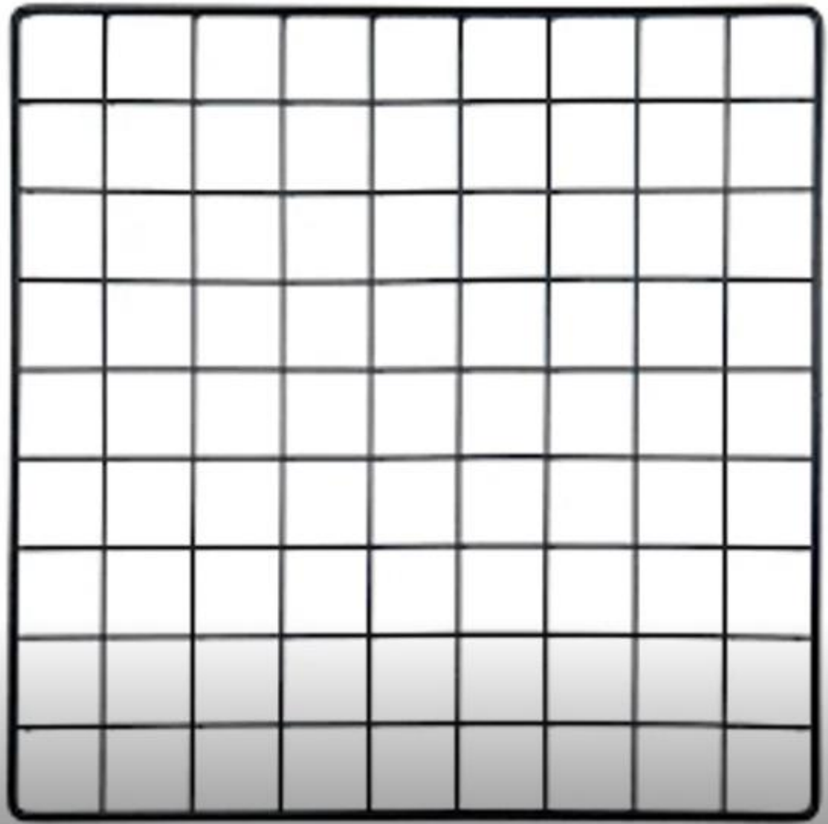


Improvements in Euler Method

Improved Euler

Runge-Kutta 4

$O_{Global}(h^4)$



Global $O(h^4)$

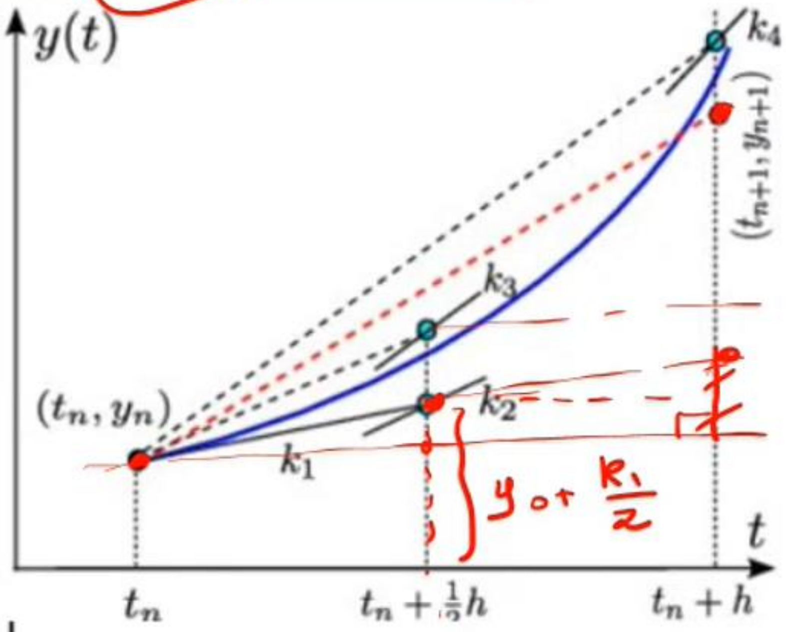
1- Method

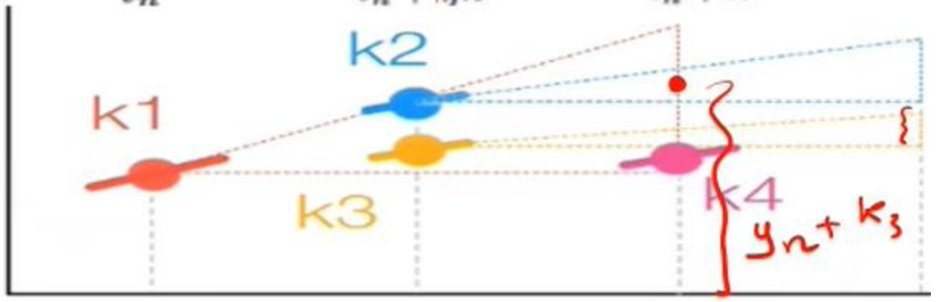
$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0,$$

Calculate

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$





Remark

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

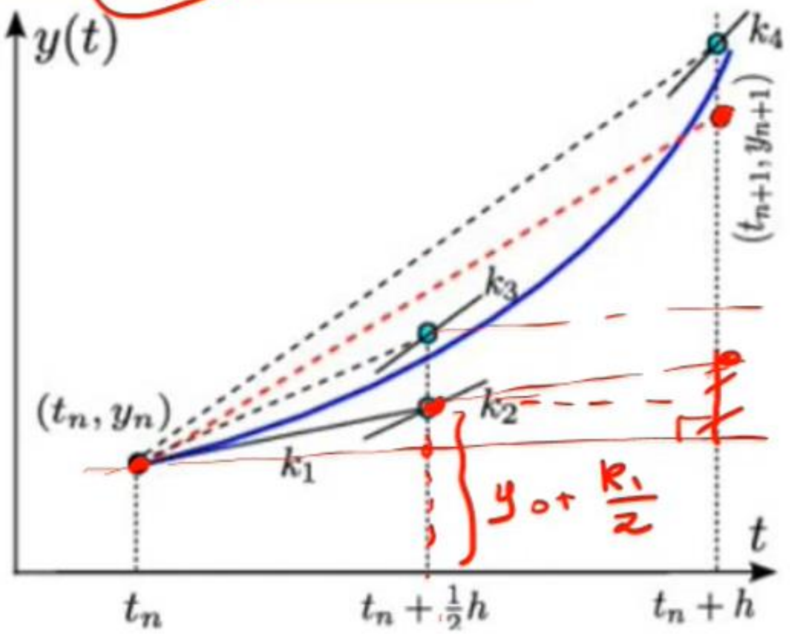
ODE : Runge-Kutta 4 Method

Global $O(h^4)$

1- Method

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0,$$



Calculate

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$y_n + k_3$

Remark

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

2- Example 1

Apply Runge-Kutta 4 method with h = 0.1 to find the solution of the initial-value problem

$$y' = 1 - x + 4y,\ y(0) = 1$$

Find y(0.2).

ODE : Runge-Kutta Method

2- Example 1

Take $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

Calculate $k_1 = h\, f\,(x_0, y_0)$

Calculate $k_2 = h\, f\,(x_0 + ½\, h,\ y_0 + ½\, k_1)$

Calculate $k_3 = h\, f\,(x_0 + ½\, h,\ y_0 + ½\, k_2)$

Calculate $k_4 = h\, f\,(x_0 + h,\ y_0 + k_3)$

Calculate $\Delta y = (k_1 + 2\, k_2 + 2\, k_3 + k_4)$

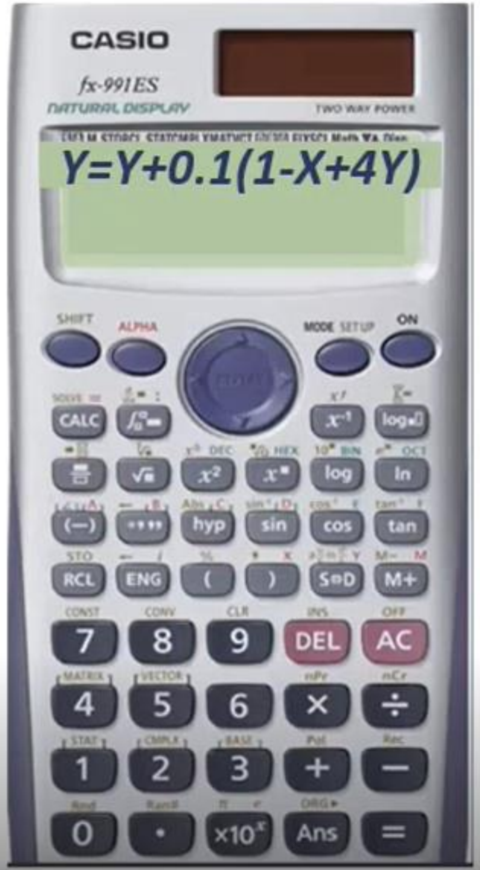
Calculate $y_1 = y\,(x_0 + h) = y\,(0 + 0.1) = y\,(0.1) = y(0) + 1/6\, \Delta y$

n	x	y	$k = h\,(1 - x + 4\,y)$	Δy $h\,f$
0	0	1	$0.1\,(1 - 0 + 4(1))$ 0.5 k_1	0.5
$+\frac{h}{2}$	0.05	$y + \frac{k_1}{2}$ 1.25	$0.1\,(1 - 0.05 + 4\,(1.25))$ 0.595 k_2	0.595 ×2
$+h/2$	0.05	$y + \frac{k_2}{2}$ 1.2975	$0.1\,(1 - (0.05) + 4\,(1.2975))$ 0.614	0.614 ×2
$+h$	0.1	$y + k_3$ 1.614	0.7356	0.7356
			sum	

÷2

÷2

$$y\,(0.1) = 1.00000 + (1/6)\,(3.65360) = \boxed{1.60893}$$



ODE : Runge-Kutta Method

2- Example 1

Take $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

Calculate $k_1 = h\, f(x_0, y_0)$

Calculate $k_2 = h\, f(x_0 + ½\, h,\ y_0 + ½\, w_1)$

Calculate $k_3 = h\, f(x_0 + ½\, h,\ y_0 + ½\, w_2)$

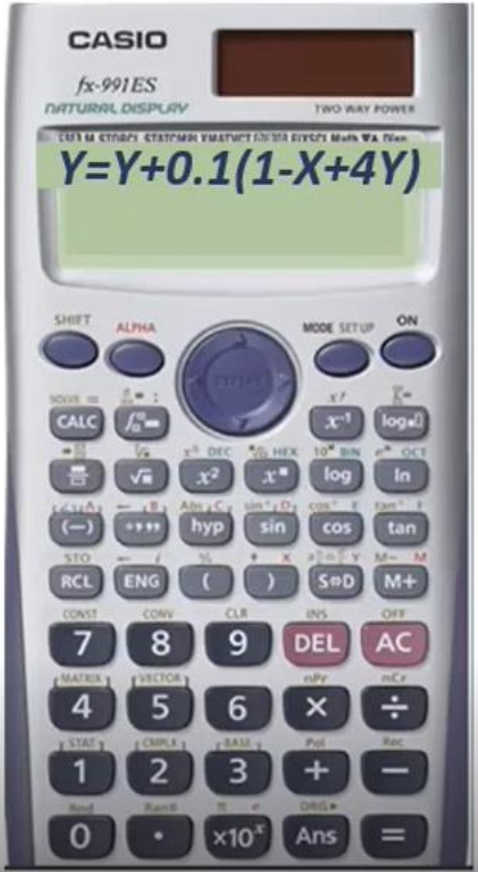
Calculate $k_4 = h\, f(x_0 + h,\ y_0 + w_3)$

Calculate $\Delta y = k_1 + 2\, k_2 + 2\, k_3 + k_4$

Calculate $y_1 = y(x_0 + h) = y(0 + 0.1) = y(0.1) = y(0) + 1/6\, \Delta y$

n	x	y	$k = h\,(1 - x + 4\,y)$	Δ y
0	0.00	1.00000	0.500000	0.50000
	0.05	1.25000	0.595000	1.19000
	0.05	1.29750	0.614000	1.22800
	0.10	1.61400	0.735600	0.73560
			sum	3.65360

$y(0.1) = 1.00000 + (1/6)\,(3.65360) = 1.60893$



ODE : Runge-Kutta Method

$y(0.1) = 1.60893$

$x_0 \quad y_0$

2- Example 1

Take $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

slope at start

slope at mid. y estimate 1 $k_1/2$

Calculate $k_1 = h\, f(x_0, y_0)$

Calculate $k_2 = h\, f(x_0 + ½\, h, y_0 + ½\, w_1)$

Calculate $k_3 = h\, f(x_0 + ½\, h, y_0 + ½\, w_2)$

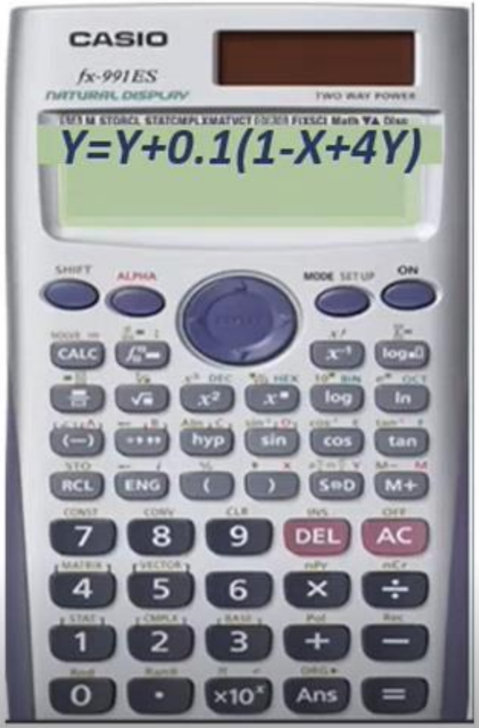
Calculate $k_4 = h\, f(x_0 + h, y_0 + w_3)$

Calculate $\Delta y = k_1 + 2\, k_2 + 2\, k_3 + k_4$

Calculate $y_1 = y(x_0 + h) = y(0 + 0.1) = y(0.1) = y(0) + 1/6\, \Delta y$

n	x	y	$k = h(1 - x + 4y)$	Δ y
1	0.10	1.60893	÷2 0.733572	0.733572
	0.15	1.97572	÷2 0.875288 ×2	1.75058
	0.15	2.04657	0.903628 ×2	1.80726
	0.20	y+k3 2.51250	1.08502	1.08502
			sum	5.37643

$y(0.2) = 1.60893 + (1/6)(5.37643) = \boxed{2.50500}$



ODE : Runge-Kutta Method

3- Error

Method	Function Evaluation per Step	Global Error $\left(\frac{1}{h}\right) \propto n \times$ local	Local Error at each step
Euler	1	$O(h)$	$O(h^2)$
Improved Euler	2	$O(h^2)$	$O(h^3)$
RK (fourth order)	4	$O(h^4)$	$O(h^5)$