Subject: Lec 4 Solutions around sigularity

Example 1:

Find a series solution in powers of x for the diff. eq.

$$2x^2y'' + 3xy' - (x^2+1)y = 0$$

$$P(x) = \frac{3x}{2x^2} = \frac{3}{2x} \qquad \Rightarrow P(x) = xP(x) = \frac{3}{2}$$

$\longrightarrow x = 0$ is a sing $\longrightarrow$ regular

$$q(x) = \frac{-(x^2+1)}{2x^2} \qquad \Rightarrow Q(x) = x^2 q(x)\Big|_{x=0} = \frac{-1}{2}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

$$\sum_{n=0}^{\infty} 2(n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s) a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+2} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

we need shifting

Shifting the index of the third summation so that the powers of x are the same in all summations. $(n \longrightarrow n-2)$

$$\sum_{n=0}^{\infty} 2(n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s) a_n x^{n+s} - \sum_{n=2}^{\infty} a_{n-2} x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

هفك في كل ال Summations اللي بتبدأ من $n=0$ ال terms عند $n=0$ & $n=1$ عشان اخلي كله يبدأ من $n=2$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+s} \Longrightarrow$$ مش هينفع ابدء من $n=0$ زي ال ordinary

عشان مفيش a_{-2} & a_{-1}

ولا يوجد لهم معنى

$$2s(s-1)a_0x^s + 3s\,a_0x^s - a_0x^s - 2(s+1)s\,a_1x^{s+1}$$

$$+3(s+1)a_1x^{s+1} - a_1x^{s+1} + \sum_{n=2}^{\infty} 2(n+s)(n+s-1)a_nx^{n+s}$$

$$+\sum_{n=2}^{\infty} 3(n+s)a_nx^{n+s} - \sum_{n=2}^{\infty} a_{n-2}\,x^{n+s} - \sum_{n=2}^{\infty} a_nx^{n+s} = 0$$

$$= a_0\left[2s(s-1)+3s-1\right]x^s + a_1\left[2(s+1)s+3(s+1)-1\right]x^{s+1}$$

$$+\sum_{n=2}^{\infty}\left(a_n\left[2(n+s)(n+s-1)+3(n+s)-1\right]-a_{n-2}\right)x^{n+s} = 0$$

The coefficient of x to the least power equals zero (called the indicial equation) this equation is a quadratic equation in S and has two roots S_1 & S_2.

$(S_1 > S_2)$ مجرد اتفاق مع الدكتور مش أكثر

$$a_0(2s(s-1)+3s-1) = a_0(2s^2+s-1) = 0$$

$$a_0 \neq 0 \quad \therefore (2s-1)(s+1)=0 \quad \therefore S_1 = \frac{1}{2} \;\&\; S_2 = -1$$

• Classification:

Case ① If $S_1 - S_2$ is a fraction $\Rightarrow$ linear

$$y_{gs} = y_1(x,s_1) + y_2(x,s_2) = \sum_{n=0}^{\infty} a_n s_1\, x^{n+s_1} + \sum_{n=0}^{\infty} a_n s_2\, x^{n+s_2}$$

Case ② If $S_1 - S_2 = 0$

$$y_1(x) = \sum_{n=0}^{\infty} a_n s_1\, x^{n+s_1}, \quad y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} a'_n s_1\, x^{n+s_1}$$

نظرية اثبتها بس مش علينا

Subject :

Case ③ If $S_1 - S_2$ is a Positive integer → log. Cases / → non log cases / → as Case1

مش عليها في الامتحان

Return back to our example,

$S_1 = \frac{1}{2}$ & $S_2 = -1 \implies$ Case 1

Coefficient of $x^{s+1} = 0 \implies a_1(2s(s+1) + 3(s+1) - 1) = 0$

For $S_1 = \frac{1}{2} \to 5a_1 = 0 \therefore a_1 = 0$

$\boxed{\therefore a_{2n+1} = 0}$ only even

For $S_2 = -1 \to -a_1 = 0 \therefore a_1 = 0$

Coefficient of $x^{s+n} = 0$

$\therefore a_n(2(n+s)(n+s-1) + 3(n+s) - 1) - a_{n-2} = 0$

$n \geq 2$

$a_n(\boxed{2(n+s)}(n+s-1) + 3(n+s) - 1) = a_{n-2}$

$a_n(2(n+s)^2 - 2(n+s) + 3(n+s) - 1) = a_{n-2}$

$\therefore a_n = \frac{1}{2(n+s)^2 + (n+s) - 1} a_{n-2}$

$$\boxed{\therefore a_n = \frac{1}{(2(n+s)-1)((n+s)+1)} a_{n-2}, \quad n \geq 2}$$

For $S_1 = \frac{1}{2} \implies a_n = \frac{1}{n(2n+3)} a_{n-2}, \quad n \geq 2$

$$a_2 = \frac{1}{(2)(7)} a_0 \longrightarrow n = 2$$

$$n = 4 \longrightarrow a_4 = \frac{1}{(4)(11)} a_2 = \frac{1}{(2.4)(7.11)} a_0$$

$$n = 6 \longrightarrow a_6 = \frac{1}{(6)(15)} a_4 = \frac{1}{(2.4.6)(7.11.15)} a_0$$

$$\therefore a_{2n} = \frac{1}{\underbrace{[2.4.6 \cdots (2n)]}_{2^n (1.2.3 \cdots (n)) = 2^n n!}[7.11.15 \cdots (4n+3)]} a_0 \qquad n \geq 2$$

$$\therefore y_1 = \sum_{n=0}^{\infty} \underbrace{a_n(s_1)}_{\text{function of } s_1} x^{n+s_1} = \sqrt{x} \sum_{n=0}^{\infty} a_{2n}\left(\tfrac{1}{2}\right) x^{2n}$$

مش ضرب

زوجي بس

$$\Rightarrow y_1 = \sqrt{x}\left(a_0 + \sum_{n=1}^{\infty} a_{2n}\left(\tfrac{1}{2}\right) x^{2n}\right) = a_0\sqrt{x}\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n\, n!\, (7.11.15 \cdots (4n+3))}\right)$$

Repeat the same work for $s_2 = -1$

$$\Rightarrow a_n = \frac{1}{n(2n-3)} a_{n-2}, \qquad n \geq 2$$

$$a_2 = \frac{1}{(2)(1)} a_0$$

$$a_4 = \frac{1}{(4)(5)} a_2 = \frac{1}{(4)(2)(5)(1)} a_0$$

$$a_6 = \frac{1}{(6)(9)} a_4 = \frac{1}{(2.4.6)(1.5.9)} a_0$$

$$a_{2n} = \frac{1}{[2.4.6 --- (2n)][1.5.9 --- (4n-3)]} a_0 \quad , n \geq 1$$

$$y_2 = \sum_{n=0}^{\infty} a_n(S_2)\, x^{n+S_2} = \frac{1}{x}\sum_{n=0}^{\infty} a_n(-1)\, x^n$$

$$= \frac{1}{x}\sum_{n=0}^{\infty} a_{2n}(-1)\, x^{2n} = \frac{1}{x}\left(\sum_{n=1}^{\infty} a_{2n}(-1)\, x^{2n} + a_0\right)$$

عوضنا بـ n=0
في ال summation

$$y_2 = \frac{a_0}{x}\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! (1.5.9 --- (4n-3))}\right).$$

$$\underset{\text{general solution}}{y} = \text{Constant}_1\, y_1(x) + \text{Constant}_2\, y_2(x)$$

$$C_1 = \text{Constant}_1\, a_0$$
$$C_2 = \text{Constant}_2\, a_0$$

أقر ، أقول

$$\therefore y_{gs} = C_1\sqrt{x}\left(1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n! (7.11.15 --- (4n+3))}\right)$$

$$+ C_2 \frac{1}{x}\left(1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n! (1.5.9 --- (4n-3))}\right)$$

Subject : .... / /

Example 2: Find a two linearly independent solutions in Powers of x for the following differential equation:

$$x^2 y'' + x y' + x^2 y = 0$$

$P(x) = \frac{1}{x}$, $q(x) = 1$

$\bar{P}(x) = x P(x) = 1$

$\therefore x_0 = 0$ is regular sig. Point

$$\therefore y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \quad , \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

Sub.

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

shifting of the third summation

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0$$

$$a_0 \big(\underbrace{s(s-1) + s}_{s^2} \big) x^s + a_1 \big(\underbrace{(s+1)s + (s+1)}_{(s+1)(s+1) = (s+1)^2} \big) x^{1+s}$$

$$+ \sum_{n=2}^{\infty} \Big(a_n \big[\underbrace{(n+s)(n+s-1) + (n+s)}_{(n+s)^2 - (n+s) + (n+s) = (n+s)^2} \big] + a_{n-2} \Big) x^{n+s} = 0$$

Coeff. of $x^s = 0 \Rightarrow a_0 (s^2) = 0$ $\quad a_0 \neq 0$ $\quad \therefore s_1 = s_2 = 0$

Case 2

Coeff. of $x^{s+1} = 0 \Rightarrow a_1 = 0$ $\quad \boxed{a_{2n+1} = 0 \quad \text{odd}}$

Coeff. of $x^{n+s} = 0 \Rightarrow \boxed{a_n = \frac{-1}{(n+s)^2} a_{n-2} \quad , \quad n \geq 2}$

$$y_1(x) = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s_1} \rightarrow \text{for } s_1 = 0 \quad \boxed{a_n = \frac{-1}{n^2}\, a_{n-2}}$$

$$a_2 = \frac{-1}{4}\, a_0$$

$$a_4 = \frac{-1}{16}\, a_2 = \frac{(-1)^2}{(4)(16)}\, a_0$$

$$a_6 = \frac{-1}{36}\, a_4 = \frac{(-1)^3}{(4)(16)(36)}\, a_0$$

من أول اشتمام

بأنا مسم $a_{2n} = a_2$

$n = 1$

not valid at $n=0$

$$a_{2n} = \frac{(-1)^n}{[2.4.6 - \cdots - (2n)]^2}\, a_0 \quad , \quad n \geq 1$$

$$\rightarrow [2^n(1.2.3 - \cdots (n))]^2 = [2^n n!]^2 = 2^{2n} (n!)^2$$

valid at $n=0$

$$\therefore a_{2n} = \frac{(-1)^n}{2^{2n} (n!)^2}\, a_0 \quad , \quad n \geq 0 \qquad a_0 = \frac{(-1)^0 a_0}{2^0 (0!)^2} = a_0$$

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2}\, x^{2n}}$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} a_n'(s_1)\, x^{n+s_1} \quad , \quad a_n' = \left. \frac{da_n}{ds} \right|_{s=s_1}$$

To find $a_n'(s_1)$ we have to solve the recurrence relation in terms of s and differentiate it.

$$a_n = \frac{-1}{(n+s)^2}\, a_{n-2} \quad , \quad n \geq 2$$

$$a_2(s) = \frac{-1}{(2+s)^2}\, a_0$$

$$a_4(s) = \frac{-1}{(4+s)^2}\, a_2 = \frac{(-1)^2}{(2+s)^2(4+s)^2}\, a_0$$

$$a_6(s) = \frac{-1}{(6+s)^2}\, a_4 = \frac{(-1)^3}{(2+s)^2(4+s)^2(6+s)^2}\, a_0$$

$$a_{2n}(s) = \frac{(-1)^n}{\big((2+s)(4+s)(6+s)\text{---}(2n+s)\big)^2}\, a_0 \quad , n \geq 1.$$

$$\ln(a_{2n}(s)) = \overset{\text{costant}}{\ln(-1)^n} - 2\big(\ln(2+s) + \ln(4+s) \text{---} + \ln(2n+s)\big) + \overset{\text{Constat}}{\ln a_0}$$

Differentiate both sides with respect to s

$$\frac{a'_{2n}(s)}{a_{2n}(s)} = -2\left(\frac{1}{(2+s)} + \frac{1}{(4+s)} + \text{---} + \frac{1}{(2n+s)}\right)$$

$$a'_{2n}(s_1) = a'_{2n}(0) = -2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \text{---} + \frac{1}{2n}\right) a_{2n}(0)$$

$$= -\underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \text{---} + \frac{1}{n}\right)}_{H_n} \frac{(-1)^n}{2^{2n}(n!)^2}\, a_0$$

$$\Rightarrow a'_{2n}(0) = \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2}\, a_0$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2}\, a_0\, x^{2n}$$

$$y_{gs} = C_1 y_1(x) + C_2 y_2(x)$$

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