Lec6 Bessel Functions (2)

المرة اللى فاتت كان المشكلة فى تالت term اللى جنب ال y

$$x^2y'' + xy' + \boxed{(\lambda^2x^2 - v^2)}\,y = 0$$

المشكلة المرة دى فى أول أو تانى term $\quad xy'' + 3y' + xy = 0$

لو ضربنا x فى ال ترم الأول كدا هيبقى $\quad x^2y'' + 3xy' + \lambda^2x^2y = 0$

لسة فى مشكلة فى ال 3 جنب ال xy' فهنعوض عن ال y

$y = x^\alpha u$ و نجيب ال y' و ال y''

Example 4:

Solve in terms of Bessel functions the following differential equation:

$$x\,y'' + 3\,y' + x\,y = 0$$

Solution:

$\boxed{Let \quad y = x^\alpha u} \qquad \boxed{\Rightarrow \; y' = x^\alpha u' + \alpha x^{\alpha-1} u}$

ضرب داليه $\qquad \boxed{\Rightarrow \; y'' = x^\alpha u'' + 2\alpha x^{\alpha-1} u' + \alpha(\alpha-1)x^{\alpha-2} u}$

Substitute in the differential equation

$$x^{\alpha+1} u'' + 2\alpha x^\alpha u' + \alpha(\alpha-1)x^{\alpha-1} u$$
$$+ 3x^\alpha u' + 3\alpha x^{\alpha-1} u + x^{\alpha+1} u = 0$$

$$x^{\alpha+1} u'' + \underline{(2\alpha+3)}\, x^\alpha u' + \left([\alpha(\alpha-1)+3\alpha]\, x^{\alpha-1} + x^{\alpha+1}\right)u = 0$$

$$2\alpha+3 = 1 \;\Rightarrow\; \alpha = -1 \qquad \Rightarrow u'' + \frac{1}{x}u' + \left(\frac{-1}{x^2}+1\right)u = 0$$

عشان نكون على شكل Bessel لازم تكون $x^2u'' + xu' + \ldots$ فهناخد ال Coeff. اللى جنب ال $x^\alpha u'$

ونساويه ب 1 , ما ينفعش ناخد $x^\alpha = x$ أو $x^{\alpha+1} = x^2$ ونقول إن ال $\alpha = 1$

عشان دى معادلة Homogeneous أقدر أضرب فى x أس أى حاجة

$$\Rightarrow x^2 u'' + x u' + (x^2 - 1) u = 0$$

كده الـ مظبوط

$$\therefore \quad u_{gs} = C_1 J_1(x) + C_2 Y_1(x)$$

but $y = x^{\alpha} u$

$y = \frac{1}{x} u$

$$\boxed{\Rightarrow y_{gs} = \frac{1}{x}\left(C_1 J_1(x) + C_2 Y_1(x)\right)}$$

$\boxed{\textit{Prove that } J_{-N}(x) = (-1)^N J_N(x) \textit{ for any positive integer } N} \longrightarrow J_2 = J_{-2}, \; J_4 = J_{-4}, \dots$

$J_{-1} = -J_1, \; J_{-3} = -J_3, \dots$

$$* J_{\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}$$

Proof:

$$J_{-N}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{n!\,\Gamma(n-N+1)} \qquad = \sum_{n=N}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{n!\,\Gamma(n-N+1)}$$

Let $n - N = k \qquad \Rightarrow J_{-N}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+N} \left(\frac{x}{2}\right)^{2k+2N-N}}{(k+N)!\,\Gamma(k+1)}$

$$\Rightarrow J_{-N}(x) = (-1)^N \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+N}}{k!\,\Gamma(k+N+1)} \qquad = (-1)^N J_N(x)$$

$$J_{-N}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{n!\,\Gamma(n-N+1)} \longrightarrow \text{if } (n-N+1) = 0 \quad \therefore \Gamma(0) = \infty$$

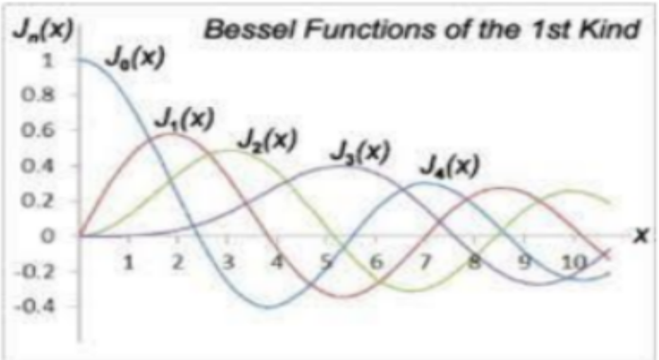
$$J_{-N} = \sum \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{\infty} \Rightarrow J_{-N} = 0$$

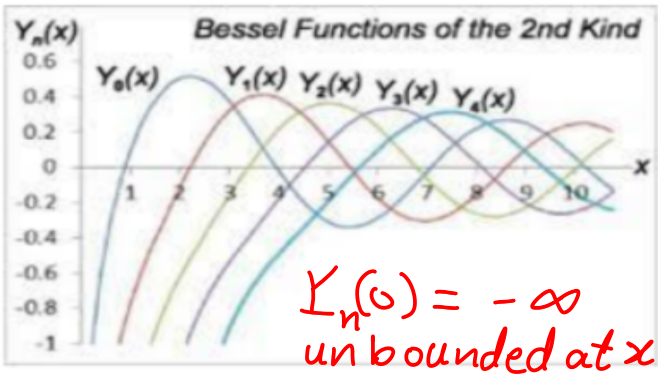
$\therefore n - N + 1 \geq 1 \quad \therefore \boxed{n \geq N}$

$$\therefore J_{-N}(x) = \sum_{n=N}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{n!\,\Gamma(n-N+1)} \Longrightarrow \text{Let } n - N = K$$

عشان نرجع العداد
يبدأ من صفر تاني

Graph of Bessel functions:





$Y_n(0) = -\infty$

unbounded at $x=0$

From the graphs we can conclude the following facts:

$J_0(0) = 1 \quad \& \quad J_n(0) = 0 \quad \forall n$ (for all n) $\quad$ and $J_{\infty}(x) = 0 \qquad y = \frac{A}{\sqrt{x}} \sin(x + \varphi)$

$J_n(x)$ is a bounded function but $Y_n(x)$ is unbounded at x=0 $\quad |J_n(x)| < 1$

Bessel functions can be represented as damped sine function for large x

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!\,\Gamma(k+n+1)} = \frac{\left(\frac{x}{2}\right)^n}{\Gamma(n+1)} - \frac{\left(\frac{x}{2}\right)^{n+2}}{1!\,\Gamma(n+2)} + \frac{\left(\frac{x}{2}\right)^{n+4}}{2!\,\Gamma(n+3)} \Rightarrow J_0 = 1 - \frac{\left(\frac{x}{2}\right)^2}{\Gamma(2)} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\Gamma(3)} \pm \dots$$

$$J_0(0) = 1$$

الدكتور قال
اللى عايز
يحله عادى
لكنه مش
Assignment

Extra problem. $x^2y''+xy'+(\lambda^2x^2-\nu^2)y=$

Show that Bessel Function for larg values of x can be approximated by $\frac{A}{\sqrt{x}}\sin(x+\varphi)$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Show that $\boxed{J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x}$ & $\boxed{J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x}$

Proof:

$$J_{1/2}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n+\frac{1}{2}}}{n!\ \Gamma(n+\frac{1}{2}+1)}=\sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n+\frac{1}{2}}}{n!\ \Gamma(n+\frac{1}{2}+1)}\sqrt{\frac{\pi x}{2}}$$

$$=\sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n\ x^{2n+1}\ \sqrt{\pi}}{2^{2n+1}n!\ \Gamma(n+\frac{3}{2})}$$

Using Legendre's duplication formula $\sqrt{\pi}\,\Gamma(2x)=2^{2x-1}\,\Gamma(x)\,\Gamma(x+\frac{1}{2})$

$$\Rightarrow\quad 2^{2n+1}\ n!\ \Gamma(x+\tfrac{3}{2})=\sqrt{\pi}\ (2n+1)!$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n\ x^{2n+1}}{(2n+1)!}=\sqrt{\frac{2}{\pi x}}\sin x$$

Similarly, we can prove that $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

Show that $\boxed{\frac{d}{dx}\left(x^n\ J_n(x)\right)=x^n\ J_{n-1}(x)\qquad (1)}$

For example: $\frac{d}{dx}\left(x^5J_5(x)\right)=x^5J_4(x)$

Proof:

$$\frac{d}{dx}\left(x^n\ J_n(x)\right)=\frac{d}{dx}\left(x^n\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+n}}{k!\,\Gamma(k+n+1)}\right)$$

$$=\frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+2n}}{2^{2k+n}k!\,\Gamma(k+n+1)}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k\,2(k+n)\ x^{2k+2n-1}}{2^{2k+n}k!\,\Gamma(k+n+1)}$$

$$=x^n\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+n-1}}{2^{2k+n-1}k!\,\Gamma(k+n)}=x^n\sum_{k=0}^{\infty}\frac{(-1)^k\ \left(\frac{x}{2}\right)^{2k+n-1}}{k!\,\Gamma(k+(n-1)+1)}=x^n\ J_{n-1}(x)$$

Similarly, we can prove that $\boxed{\dfrac{d}{dx}\left(x^{-n}\ J_n(x)\right) = -x^{-n}\ J_{n+1}(x) \qquad (II)}$

أي حاجة جوة
برواز مطالبين
نثبتها

Using I and II we can show that

$$\boxed{J_n(x) = \frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right) \qquad (III)}$$

$$\boxed{J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) \qquad (IV)}$$

We can use III to express $J_n(x)$ in terms of $J_0(x)$ and $J_1(x)$ only.

We can also use III to express $J_{\frac{2n+1}{2}}(x)$ sinx and cosx.

We can use IV to evaluate $\int J_n(x)$.

$$(1)\quad \underbrace{\frac{d}{dx}(x^n J_n)}= x^n J_{n-1} = x^n J_n' + n x^{n-1} J_n \quad \div x^n$$

$$(2)\quad \underbrace{\frac{d}{dx}\left(\frac{1}{x^n} J_n\right)} = -\frac{1}{x^n} J_{n+1} = \frac{1}{x^n} J_n' - \frac{n}{x^{n+1}} J_n \quad \times x^n$$

$$J_{n-1} = J_n' + \frac{n}{x} J_n \qquad (3)$$

$$-J_{n+1} = J_n' - \frac{n}{x} J_n \qquad (4)$$

لو طرحنا ③ - ④
هيطلع III

ولو جمعنا ③ + ④
هيطلع IV

$$\rightarrow \int J_{n+1}\,dx = \int J_{n-1}\,dx - 2J_n$$

$$J_{-1} = -J_1$$

$$\boxed{J_0' = -J_1}$$

$$\text{find } \int J_5\,dx \Rightarrow \int J_5\,dx = \int J_3\,dx - 2J_4 = -J_0 - 2J_2 - 2J_4$$

$$\int J_3\,dx = \int J_1 - 2J_2 = -J_0 - 2J_2$$

$$\int J_1\,dx = \int J_{-1}\,dx - 2J_0$$

$$= -\int J_1\,dx - 2J_0 \Rightarrow \cancel{2}\int J_1\,dx = -\cancel{2}J_0$$

We can Use I and II as follows

$\int x^5 J_4(x)dx = x^5 J_5(x)+c$ $\boxed{\int x^n\ J_{n-1}(x)\ dx\ =\ x^n\ J_n(x) + C}$

$\int x^{-5} J_6(x)\,dx = -x^{-5} J_5(x)+c$ $\boxed{\int x^{-n}\ J_{n+1}(x)\ dx\ =\ -x^{-n}\ J_n(x) + C}$

Example:

Evaluate $\int x^4\ J_1(x)\ dx$ *in terms of* $J_0(x)$ *and* $J_1(x)$ *only*

$I = x^4 J_2 - 2\int x^3 J_2 = x^4 J_2 - 2x^3 J_3 + C$

$u = x^2 \quad dv = x^2 J_1$

$du = 2x\,dx \quad v = x^2 J_2$

$-\int$

But $J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$

$I = x^4\left(\frac{2\times 1}{x} J_1 - J_0\right) - 2x^3\left(\frac{2\times 2}{x} J_2 - J_1\right) + C$

$= 2x^3 J_1 - x^4 J_0 - 8x^2\left(\frac{2\times 1}{x} J_1 - J_0\right) + 2x^3 J_1 + C$

$= (2x^3 - 16x + 2x^3) J_1 - (x^4 - 8x^2) J_0 + C$

$= (4x^3 - 16x) J_1 - (x^4 - 8x^2) J_0 + C$

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