Singular point

* Find a series Solution in Power of x :

① $3xy'' + y' - y = 0$

1 → Compare $y'' + p(x)y' + q(x)y = 0$

$$\Rightarrow p(x) = \frac{1}{3x} \quad , \quad q(x) = \frac{-1}{3x}$$

at $x = 0$ ⇒ p, q both are not defined

⇒ $x = 0$ is a Singular point

$$P(x) = (x-0)\,p(x) = \frac{1}{3} \quad , \quad Q(x) = (x-0)^2 q(x) = -\frac{x}{3}$$

⇒ $P(x)$, $Q(x)$ are both defined at $x = 0$

∴ $x = 0$ is a regular Singular Point

2 → let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty} (n+s)\, a_n x^{n+s-1}$

$$y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n x^{n+s-2} \qquad a_n \neq 0$$

3 → Substitute in the differential equation

$$\Rightarrow 3\sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s)\, a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\Rightarrow 3\sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s)\, a_n x^{n+s-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+s-1} = 0$$

$$\Rightarrow \left[3(s(s-1)) + s\right] a_0 x^{s-1} + \sum_{n=1}^{\infty} \left[\begin{array}{l} 3(n+s)(n+s-1)\, a_n \\ + (n+s)\, a_n - a_{n-1} \end{array}\right] x^{n+s-1} = 0$$

$\xrightarrow{4}$ Coeff $(x^{s-1}) = 0$ $\Rightarrow$ $[3s(s-1) + s]a_0 = 0$

$\therefore s(3s-2) = 0$ $\therefore (3s-2) = 0$, $s = 0$

$\Rightarrow s_1 = \frac{2}{3}$ or $s_2 = 0$ $a_0 \neq 0$

$\therefore s_1 - s_2 \neq$ integer "Case 1"

$\Rightarrow$ Coeff $(x^{n+s-1}) = 0$

$\Rightarrow 3(n+s)(n+s-1)a_n + (n+s)a_n - a_{n-1} = 0$

$$\Rightarrow a_n = \frac{1}{3(n+s)(n+s-1) + (n+s)} a_{n-1}$$

$$\therefore a_n = \frac{1}{(n+s)[3(n+s)-2]} a_{n-1}$$

Rec. Relation

$\xrightarrow{5}$ at $n=1 \Rightarrow$ $a_1 = \frac{1}{(1+s)[3(1+s)-2]} a_0 = \frac{1}{(s+1)[3s+1]} a_0$

at $n=2 \Rightarrow$ $a_2 = \frac{1}{(s+2)[3(s+2)-2]} a_1 = \frac{1}{(s+1)(s+2)(3s+1)(3s+4)} a_0$

at $n=3 \Rightarrow$ $a_3 = \frac{1}{(s+3)[3(s+3)-2]} a_2 = \frac{1}{(s+1)(s+2)(s+3)(3s+1)(3s+4)(3s+7)} a_0$

$$\therefore a_n = \frac{1}{[(s+1)(s+2)\cdots(s+n)][(3s+1)(3s+4)\cdots(3s+3n-2)]} a_0 \quad n \geq 1$$

$\xrightarrow{6}$ $y_{GS} = c_1 y_1 + c_2 y_2$ let $a_0 = 1$

$$a_n\left(\tfrac{2}{3}\right) = \frac{1}{[\frac{5}{3}\cdot\frac{8}{3}\cdots(\frac{2}{3}+n)][3\cdot 6\cdots(3n)]}$$

$$y_1 = x^{2/3} + \sum_{n=1}^{\infty} a_n\left(\tfrac{2}{3}\right) x^{n+2/3} = x^{2/3}\left[1 + \sum_{n=1}^{\infty} \frac{1}{[\frac{5}{3}\cdot\frac{8}{3}\cdots(\frac{2}{3}+n)][3\cdot 6\cdots(3n)]} x^n\right]$$

$$a_n(0) = \frac{1}{[1.2.3\ldots n][1.4.7\ldots(3n-2)]} = \frac{1}{(n!)[1.4.7\ldots(3n-2)]}$$

$$y_2 = 1 + \sum_{n=1}^{\infty} a_n(0)\, x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{(n!)[1.4.7\ldots(3n-2)}$$

$$\therefore y_{GS} = c_1 y_1 + c_2 y_2$$

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② $x^2 y'' + x y' + (x^2 - \frac{1}{9}) y = 0$

$\xrightarrow{1}$ compare $y'' + P(x) y' + q(x) = 0$

$\Rightarrow P(x) = \frac{1}{x}$, $q(x) = \frac{x^2 - 1/9}{x^2}$

at $x = 0 \Rightarrow$ both p and q are not defined

$\Rightarrow x = 0$ is a singular point

$P(x) = (x-0)\frac{1}{x} = 1$, $Q(x) = (x-0)^2 \frac{x^2 - 1/9}{x^2} = x^2 - 1/9$

$\Rightarrow P(x), Q(x)$ are both defined at $x = 0$

$\therefore x = 0$ is a regular singular point

$\xrightarrow{2}$ let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$, $y'' = \sum_{n=0} (n+s)(n+s-1) a_n x^{n+s-2}$

$\xrightarrow{3}$ substitute in the differential equation

$$\Rightarrow x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + (x^2 - \tfrac{1}{9}) \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} - \frac{1}{9} \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\Rightarrow s(s-1)a_0x^s + (1+s)(s)a_1x^{s+1} + s a_0 x^s + (s+1)a_1 x^{s+1} - \frac{1}{9}a_0x^s - \frac{1}{9}a_1x^{s+1} + \sum_{n=2}^{\infty}\begin{pmatrix}(n+s)(n+s-1)a_n \\ +(n+s)a_n + a_{n-2} \\ -\frac{1}{9}a_n\end{pmatrix}x^{n+s} = 0$$

4 $\rightarrow$ coeff $(x^s) = 0 \Rightarrow (s^2 - \frac{1}{9})a_0 = 0$ $\qquad \because a_0 \neq 0$

$\therefore s_1 = \frac{1}{3}, \ s_2 = -\frac{1}{3} \qquad \Rightarrow s_1 - s_2 \neq$ integer $\qquad$ "Case 1"

coef $(x^{s+1}) = 0 \Rightarrow (s^2 + 2s + \frac{8}{9})a_1 = 0$

for $s_1 = \frac{1}{3} \Rightarrow (1 + \frac{2}{3})a_1 = 0$, $\ s_2 = -\frac{1}{3} \Rightarrow (1 - \frac{2}{3})a_1 = 0$

$\therefore a_1 = 0$

coef $(x^{s+n}) = 0 \Rightarrow \left[(n+s)(n+s-1) + (n+s) - \frac{1}{9}\right]a_n + a_{n-2} = 0$

Rec. Relation

$$a_n = \frac{-1}{(n+s-\frac{1}{3})(n+s+\frac{1}{3})}a_{n-2}, \quad n \geq 2$$

5 $\rightarrow$ $a_1 = 0 \Rightarrow a_{2n+1} = 0, \quad n \geq 0$

$$n=2 \Rightarrow a_2 = \frac{-1}{(s+\frac{5}{3})(s+\frac{7}{3})}a_0$$

$$n=4 \Rightarrow a_4 = \frac{-1}{(s+\frac{11}{3})(s+\frac{13}{3})}a_2 = \frac{(-1)^2}{\left[(s+\frac{5}{3})(s+\frac{11}{3})\right]\left[(s+\frac{7}{3})(s+\frac{13}{3})\right]}a_0$$

$$\therefore a_{2K}(s) = \frac{(-1)^K}{\left[(s+\frac{5}{3})(s+\frac{11}{3})\cdots(s+2K-\frac{1}{3})\right]\left[(s+\frac{7}{3})(s+\frac{13}{3})\cdots(s+2K+\frac{1}{3})\right]}$$

$K \geq 1$, let $a_0 = 1$

$$Y_{GS} = c_1 Y_1 + c_2 Y_2$$

$$Y_1 = x^{1/3} + \sum_{K=1}^{\infty} a_{2K}\left(\tfrac{1}{3}\right) x^{n+\frac{1}{3}}$$

$$Y_2 = x^{-1/3} + \sum_{K=1}^{\infty} a_{2K}\left(-\tfrac{1}{3}\right) x^{n-\frac{1}{3}}$$

* Note that $a_{2K}\left(\frac{1}{3}\right) \neq \frac{1}{3} a_{2K}$ but similar to $a_{2K}(s)$ as a_{2K} is a function of s