AIN SHAMS UNIVERSITY
FACULTY OF ENGINEERING
SPECIALIZED ENGINEERING PROGRAMS
JUNIOR COMMUNICATION ENGINEERING PROGRAM

Juba

SPRING 2022	Assignment #1	Total: 5 marks

PHM212s: Special Functions, Complex Analysis & Numerical Analysis

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Name:	ID:	Deadline: Week 3

Please, Solve each problem in its assigned place ONLY (the empty space below it)

Gamma and Beta Functions

1. Prove that $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$ for any real number $n > -1$.

$* \; L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$

let $u = st \longrightarrow t = \frac{u}{s}$

$dt = \frac{1}{s}\,du$

@ $t = 0 \rightarrow u = 0$

$t = \infty \rightarrow u = \infty$

$\therefore L\{t^n\} = \int_0^\infty \left(\frac{u}{s}\right)^n e^{-u} \left(\frac{1}{s}\right) du$

$= \left(\frac{1}{s}\right)^{n+1} \int_0^\infty u^n e^{-u}\,du$

$x - 1 = n$

$\therefore x = n+1$

$= \left(\frac{1}{s}\right)^{n+1} * \Gamma(n+1)$ #

Hence, find Laplace transform for each of the following functions:

a) $t^{5/2}$ $\quad n = 5/2: \quad \dfrac{\Gamma(7/2)}{s^{7/2}} = \dfrac{(5/2)(3/2)(1/2)\Gamma(1/2)}{s^{7/2}} = \dfrac{15\sqrt{\pi}}{8\,s^{7/2}}$

b) $t^{-1/3}$ $\quad n = -1/3: \quad \dfrac{\Gamma(2/3)}{s^{2/3}} = \dfrac{\Gamma(5/3)}{2/3 \; s^{2/3}} = \dfrac{(3)(0.903335)}{(2)\; s^{2/3}} = \dfrac{1.355}{s^{2/3}}$

c) $\sqrt{t}\; e^{-3t}$ $\quad n = 1/2:$ but shifted $\quad \dfrac{\Gamma(3/2)}{(s+3)^{3/2}} = \dfrac{\frac{1}{2}\Gamma(\frac{1}{2})}{(s+3)^{3/2}} = \dfrac{\sqrt{\pi}}{2\,(s+3)^{3/2}}$

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2. Given that n is a positive integer and x is a real number, show that

$\beta(x, n) = \dfrac{(n-1)!}{x(x+1)(x+2)\ldots(x+n-1)}$. Hence, evaluate $\beta(0.1, 3)$

$\rightarrow \beta(x,n) = \dfrac{\Gamma(x)\,\Gamma(n)}{\Gamma(x+n)}$, $\because$ n is a positive integer $\quad \therefore \Gamma(n) = (n-1)! \longrightarrow (1)$

• by using reccurrence : $\Gamma(x+n) = (x+n-1) \ldots (x+2)(x+1)(x)\Gamma(x) \longrightarrow (2)$

$\therefore \beta(x,n) = \dfrac{\cancel{\Gamma(x)}\,(n-1)!}{(x+n-1)\ldots(x+2)(x+1)(x)\,\cancel{\Gamma(x)}} = \dfrac{(n-1)!}{(x)(x+1)(x+2) - (x+n-1)}$

$\therefore \beta(0.1, 3) = \dfrac{2!}{(0.1)(1.1)(2.1)} = 8.658$

3. For any non-negative integer 'n', show that

$$\Gamma(n+\frac{1}{2})=\frac{(2n)!}{2^{2n}\ .\ n!}\sqrt{\pi}$$

* legendre's duplication formula

$$\boxed{\sqrt{\pi}\,\Gamma(2x)=2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2})}$$

a) Using Legendre duplication formula

$\rightarrow$ let $x=n$

$$\therefore \Gamma(n+\tfrac{1}{2})=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}\,\Gamma(n)}=\frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi}=\frac{(\frac{1}{\cancel{2n}})(2n)!}{2^{2n-1}(\frac{1}{\cancel{n}})(n)!}\sqrt{\pi}=\frac{(2n)!}{(2)(2^{2n-1})(n)!}\sqrt{\pi}=\frac{(2n)!}{2^{2n}\ .\ n!}\sqrt{\pi}$$

b) Without using Legendre duplication formula $\rightarrow$ using reccurrence

$$\therefore \Gamma(n+\tfrac{1}{2})=(n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\dots(\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})$$

$$=\left(\frac{1}{2^n}\right)\left[(2n-1)(2n-3)(2n-5)\dots(5)(3)(1)\right]\sqrt{\pi}$$

$$=\frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\dots(5)(4)(3)(2)(1)}{[2^n][(2n)(2n-2)(2n-4)\dots(4)(2)]}\sqrt{\pi}$$

$$=\frac{(2n)!\ \sqrt{\pi}}{[2^n][2^n](n)(n-1)(n-2)\dots(2)(1)}=\frac{(2n)!}{2^{2n}\,(n)!}\sqrt{\pi}$$

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4. Show that $\int_0^\infty x^a\ b^{-x}\ dx=\frac{\Gamma(a+1)}{(\ln b)^{a+1}}$, where $a>-1$ and $b>1$

* let $b^x=e^t$

$x\ln(b)=t\,\ln(e)$ ($\ln(e)\to 1$)

$x=\frac{1}{\ln b}t\rightarrow dx=\frac{1}{\ln b}\,dt$

@ $x=0\rightarrow t=0$

@ $x=\infty\rightarrow t=\infty$

$$\int_0^\infty\left(\frac{1}{\ln b}t\right)^a e^{-t}\left(\frac{1}{\ln b}\right)dt=\left[\frac{1}{\ln b}\right]^{a+1}\int_0^\infty t^a\,e^{-t}\,dt$$

$$=\frac{\Gamma(a+1)}{(\ln b)^{a+1}}$$

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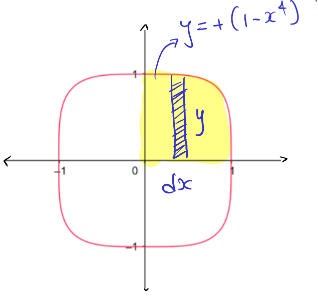
5. Show that the area enclosed by the curve $x^4+y^4=1$ is $\Gamma^2(\frac{1}{4})/(2\sqrt{\pi})$



$\rightarrow y=\pm(1-x^4)^{1/4}$

$$*\ A=4\int_0^1 y\,dx=4\int_0^1(1-x^4)^{1/4}dx$$

$\rightarrow$ let $x^4=t$

$x=t^{1/4}\rightarrow dx=\frac{1}{4}t^{-3/4}\,dt$

@ $x=0\rightarrow t=0$

@ $x=1\rightarrow t=1$

$$A=\cancel{(4)}\int_0^1(1-t)^{1/4}(\tfrac{1}{\cancel{4}})(t^{-3/4})\,dt$$

$$=\int_0^1(t)^{-3/4}(1-t)^{1/4}\,dt$$

$\therefore x-1=-3/4\rightarrow x=1/4$

$\therefore y-1=1/4\rightarrow y=5/4$

$$\therefore A=\frac{\Gamma(1/4)\,\Gamma(5/4)}{\Gamma(3/2)}=\frac{\frac{1}{4}\Gamma(1/4)\,\Gamma(1/4)}{\frac{1}{2}\Gamma(1/2)}$$

$$\therefore A=\frac{\Gamma^2(1/4)}{2\sqrt{\pi}}$$

6. Use the Gamma and the Beta functions to evaluate the following integrals:

a) $\int_0^\infty x^3 e^{-2x^5} dx$

@ $x=0 \to t=0$
@ $x=\infty \to t=\infty$

$\to$ let $2x^5 = t$

then $x = (\frac{1}{2}t)^{\frac{1}{5}} \to dx = (\frac{1}{5})(\frac{1}{2})(\frac{1}{2}t)^{-\frac{4}{5}} dt$

$\therefore \int_0^\infty (\frac{1}{2}t)^{\frac{3}{5}} e^{-t} (\frac{1}{2})(\frac{1}{5})(\frac{1}{2}t)^{-\frac{4}{5}} dt =$

$(\frac{1}{2})^{\frac{3}{5}-\frac{4}{5}+1} (\frac{1}{5}) \int_0^\infty t^{-1/5} e^{-t} dt =$

$\frac{\Gamma(4/5)}{(2)^{4/5}(5)} = \frac{\Gamma(9/5)}{(2)^{4/5}(\cancel{5})(4/\cancel{5})} = \frac{0.931384}{(2)^{4/5}(4)}$

$= 0.133735$

b) $\int_0^\infty 3^{-x^2} dx \longrightarrow$ let $3^{x^2} = e^t$

$\therefore x^2 \ln 3 = t \longrightarrow x = (\frac{1}{\ln 3})^{1/2} t^{1/2}$

$dx = (\frac{1}{\ln 3})^{1/2} (\frac{1}{2} t^{-1/2}) dt$

@ $x=0 \to t=0$
@ $x=\infty \to t=\infty$

$\therefore \int_0^\infty (\frac{1}{\ln 3})^{1/2} (\frac{1}{2}) t^{-1/2} e^{-t} dt =$

$\frac{1}{2\sqrt{\ln 3}} \int_0^\infty t^{-1/2} e^{-t} dt =$

$\frac{1}{2\sqrt{\ln 3}} \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2\sqrt{\ln 3}} = 0.845518$

c) $\int_0^1 \sqrt[3]{x} \ln^5 x \, dx$

@ $x=0 \to t=\infty$
@ $x=1 \to t=0$

$\to$ let $\ln x = -t$

then $x = e^{-t} \longrightarrow dx = -e^{-t} dt$

$\therefore \int_\infty^0 (e^{-t})^{1/3} (-t)^5 (-e^{-t}) dt = -\int_0^\infty (e^{-\frac{4}{3}t})(t^5) dt$

$\to$ let $u = \frac{4}{3}t$

then $t = \frac{3}{4}u \longrightarrow dt = \frac{3}{4} du$

$\therefore -\int_0^\infty (\frac{3}{4})^5 u^5 e^{-u} (\frac{3}{4}) du = -(\frac{3}{4})^6 \int_0^\infty u^5 e^{-u} du$

$-(\frac{3}{4})^6 \Gamma(6) = -(\frac{3}{4})^6 (5)!$

$= -21.3574$

d) $\int_{-\infty}^\infty \frac{dx}{1+x^4} = 2\int_0^\infty \frac{dx}{1+x^4}$ (even function)

$\to$ let $x^4 = u$

@ $x=0 \to u=0$
@ $x=\infty \to u=\infty$

then $x = u^{1/4}$

$dx = \frac{1}{4} u^{-3/4} du$

$\to x-1 = -3/4$
$x = 1/4$
$\to x+y = 1$
$y = 3/4$

$\therefore (2) \int_0^\infty \frac{\frac{1}{4} u^{-3/4}}{(1+u)} du =$

$\frac{1}{2} \int_0^\infty \frac{u^{-3/4}}{1+u} du = \frac{1}{2} \beta(1/4, 3/4)$

$= \frac{\Gamma(1/4)\Gamma(3/4)}{2\Gamma(1)} = \frac{\pi}{2\sin(\frac{\pi}{4})} = 2.22144$

e) $\int_0^{\pi/2} \sin^{3.04} x \, dx = \frac{1}{2}\beta(2.02, 0.5)$

$= \frac{\Gamma(2.02)\Gamma(1/2)}{2\Gamma(2.52)} = \frac{(1.02)\Gamma(1.02)\Gamma(1/2)}{2(1.52)\Gamma(1.52)}$

$= \frac{(1.02)(0.988844)(\sqrt{\pi})}{(1.52)(0.887039)(2)} = 0.662959$

f) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$

$= \frac{1}{2}\beta(3/4, 1/4) = \frac{\Gamma(3/4)\Gamma(1/4)}{2\Gamma(1)} = \frac{\pi}{2\sin(\frac{\pi}{4})}$

$= 2.22144$

g) $\int_0^\infty \frac{x \; dx}{1+x^6}$ $\rightarrow$ let $x^6 = u$

then $x = u^{1/6}$

$dx = \left(\frac{1}{6}\right) u^{\frac{-5}{6}} du$

$\therefore \int_0^\infty \frac{(u^{1/6})(\frac{1}{6})(u^{-5/6})}{1+u} du =$

$\frac{1}{6}\int_0^\infty \frac{u^{\frac{-2}{3}}}{1+u} du = \frac{1}{6}\beta(1/3, 2/3) = \frac{\Gamma(1/3)\Gamma(2/3)}{6\,\Gamma(1)}$

$= \frac{\pi}{6\sin(\frac{\pi}{3})} = 0.604599$

h) $\int_0^2 x\,(8-x^3)^{1/3}dx = (8)^{1/3}\int_0^2 x\left(1-\frac{x^3}{8}\right)^{1/3} dx$

$\rightarrow$ let $\frac{x^3}{8} = t$ then $x = 2t^{1/3}$ $\rightarrow$ $dx = \left(\frac{2}{3}\right)t^{-2/3} dt$

$\therefore (8)^{1/3}\int_0^1 (2)(t)^{1/3}(1-t)^{1/3}\left(\frac{2}{3}\right)(t)^{-2/3} dt =$

@ $x=0 \rightarrow t=0$
@ $x=2 \rightarrow t=1$

$\frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3} dt = \frac{8}{3}\beta(2/3, 4/3) = \frac{8\,\Gamma(2/3)\Gamma(4/3)}{3\,\Gamma(2)}$

$\frac{8(1/3)\,\Gamma(1/3)\Gamma(2/3)}{3} = \frac{8\pi}{9\sin(\frac{\pi}{3})} = 3.22453$

7. Show that $\beta(n, n+1) = \frac{\Gamma^2(n)}{2\,\Gamma(2n)}$.

$\beta(n,n+1) = \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} = \frac{\cancel{n}\Gamma(n)\Gamma(n)}{2\cancel{n}\Gamma(2n)} = \frac{\Gamma^2(n)}{2\,\Gamma(2n)} \rightarrow 1$

Hence, deduce that $\int_0^{\pi/2}(\sin^{-3}\theta - \sin^{-2}\theta)^{1/4}\cos\theta\, d\theta = \frac{\Gamma^2(1/4)}{2\sqrt{\pi}}$

$\rightarrow \int_0^{\pi/2}[\sin^{-3/4}\theta](1-\sin\theta)^{1/4}\cos\theta\, d\theta$ $\rightarrow$ let $\sin\theta = t$

then $\cos\theta\, d\theta = dt$

@ $\theta = 0 \rightarrow t = 0$
@ $\theta = \pi/2 \rightarrow t = 1$

$\therefore \int_0^1 [t^{-3/4}](1-t)^{1/4}\,\cancel{\cos\theta}\,\frac{dt}{\cancel{\cos\theta}} = \int_0^1 (t)^{\frac{-3}{4}}(1-t)^{\frac{1}{4}} dt = \beta(1/4, 5/4)$

* From 1 $\longrightarrow \beta(1/4, 5/4) = \frac{\Gamma^2(1/4)}{2\,\Gamma(1/2)} = \frac{\Gamma^2(1/4)}{2\sqrt{\pi}}$

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8. Show that $\beta(x, y) = \frac{y-1}{x}\beta(x+1, y-1)$

$\rightarrow$ R.H.S $= \frac{y-1}{x}\beta(x+1, y-1) = \frac{y-1}{x}\cdot\frac{\Gamma(x+1)\,\Gamma(y-1)}{\Gamma(x+y)} = \frac{(y-1)\Gamma(y-1)\cdot\frac{\Gamma(x+1)}{x}}{\Gamma(x+y)}$

$\frac{\Gamma(y)\,\Gamma(x)}{\Gamma(x+y)} = \beta(x, y) = $ L.H.S #

Best wishes,
Dr. Makram Roshdy