Laplace Trasform

1- Laplace transform of functions

***Definition:** The Laplace transform of the function $f(t)$ is denoted by $F(s)$ or $L[f(t)]$ and can be defined as:

$$\boxed{F(s) = L[f(t)] = \int_0^{\infty} f(t)\, e^{-st}\, dt}$$ هام

*** Note that:**

$$\boxed{L[c_1 f_1(t) \pm c_2 f_2(t)] = c_1 L[f_1(t)] \pm c_2 L[f_2(t)]}$$

where c_1 and c_2 are constants.

*** Laplace Transforms of elementary functions:** هام

① If $f(t) = a$ (constant)

$$\Rightarrow \boxed{L[a] = \frac{a}{s}}, \quad s > 0.$$

② If $f(t) = e^{kt}$, k is constant

$$\Rightarrow \boxed{L[e^{kt}] = \frac{1}{s-k}}, \quad s > k$$

③ If $f(t) = sh(kt)$, k is constant

$$\Rightarrow \boxed{L[sh(kt)] = \frac{k}{s^2 - k^2}}$$

④ If $f(t) = ch(kt)$, k is constant

$$\Rightarrow \boxed{L[ch(kt)] = \frac{s}{s^2-k^2}}$$

⑤ If $f(t) = \cos(kt)$, k is constant

$$\Rightarrow \boxed{L[\cos(kt)] = \frac{s}{s^2+k^2}}$$

⑥ If $f(t) = \sin(kt)$, k is constant

$$\Rightarrow \boxed{L[\sin(kt)] = \frac{k}{s^2+k^2}}$$

⑦ If $f(t) = t^n$, n is positive integer.

$$\Rightarrow \boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

ملخص القوانين السابقة:

$f(t)$	$F(s)=L[f(t)]$	$f(t)$	$F(s)=L[f(t)]$
① a (Constant)	$\frac{a}{s}$	⑤ $\cos(kt)$	$\frac{s}{s^2+k^2}$
② e^{kt}	$\frac{1}{s-k}$	⑥ $\sin(kt)$	$\frac{k}{s^2+k^2}$
③ $\sinh(kt)$	$\frac{k}{s^2-k^2}$	⑦ t^n (n is +ve integer)	$\frac{n!}{s^{n+1}}$
④ $\cosh(kt)$	$\frac{s}{s^2-k^2}$		

Example: (i) $L[t^3 + e^{4t}] = L[t^3] + L[e^{4t}]$

$$= \frac{3!}{s^4} + \frac{1}{s-4} = \frac{6}{s^4} + \frac{1}{s-4} \quad \#$$

(ii) $L[4\sin 3t + 3e^{-2t} + 5]$

$$= 4L[\sin 3t] + 3L[e^{-2t}] + L[5]$$

$$= 4 \cdot \frac{3}{s^2+9} + 3 \cdot \frac{1}{s+2} + \frac{5}{s} \quad \#$$

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* **Theorem ①:** (The first shift theorem):

مهم

if $L[f(t)] = F(s)$, then $\boxed{L[e^{-at} f(t)] = F(s+a)}$

, a is constant.

EX: Evaluate:

(i) $L[t\,e^{-3t}]$. (ii) $L[e^{-t}\cos 4t]$. (iii) $L[e^{2t}(t^3+2t-$

solution

(i) $\because L[t] = \frac{1!}{s^2} = \frac{1}{s^2}$

$$\Rightarrow L[t\,e^{-3t}] = \frac{1}{(s+3)^2} \quad \#$$

(ii) $\because L[\cos 4t] = \frac{s}{s^2+16}$

$$\Rightarrow L[e^{-t}\cos 4t] = \frac{s+1}{(s+1)^2+16} = \frac{s+1}{s^2+2s+17} \quad \#$$

(iii) $\because L[t^3+2t-1] = \frac{3!}{s^4} + 2 \cdot \frac{1!}{s^2} - \frac{1}{s} = \frac{6}{s^4} + \frac{2}{s^2} - \frac{1}{s}$

$$= \frac{6+2s^2-s^3}{s^4}$$

$$\Rightarrow L\left[e^{2t}(t^3+2t-1)\right] = \frac{6+2(s-2)^2-(s-2)^3}{(s-2)^4} \quad \#$$

*** Theorem 2:** If $L[f(t)] = F(s)$, then $\boxed{L[t\,f(t)] = -\frac{d}{ds}F(s)}$

Generally, $\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)}$

EX: i) Evaluate: $L[t \sin 3t]$

ii) Evaluate: $L[t^2\, sh\, 2t]$.

Solution

i) $\because L[\sin 3t] = \dfrac{3}{s^2+9}$

$$\Rightarrow L[t \sin 3t] = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right] = -3\left[\frac{-(2s)}{(s^2+9)^2}\right]$$

$$= \boxed{\frac{6s}{(s^2+9)^2}} \quad \#$$

ii) $\because L[sh\, 2t] = \dfrac{2}{s^2-4}$

$$\Rightarrow L[t^2\, sh 2t] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{2}{s^2-4}\right]$$

$$= 2\frac{d}{ds}\left[\frac{-(2s)}{(s^2-4)^2}\right] = -4\frac{d}{ds}\left[\frac{s}{(s^2-4)^2}\right]$$

$$= -4 \cdot \frac{(s^2-4)^{\cancel{2}} - 2\cancel{(s^2-4)}(2s)(s)}{(s^2-4)^{\cancel{4}\,3}}$$

$$= -4 \cdot \frac{s^2-4-4s^2}{(s^2-4)^3} = \boxed{4\,\frac{(3s^2+4)}{(s^2-4)^3}} \quad \#$$

Theorem 3: if $L[f(t)] = F(s)$, $\lim\limits_{t\to 0} \dfrac{f(t)}{t}$ exists,

Then $$\boxed{L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds.}$$

EX: Find $L\left[\dfrac{\sin t}{t}\right]$

Solution

$\because L[\sin t] = \dfrac{1}{s^2+1}$

$,\because \lim\limits_{t\to 0} \dfrac{\sin t}{t} = 1 \Rightarrow$ the limit exists.

$$\therefore L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1} s \Big|_s^\infty = \boxed{\frac{\pi}{2} - \tan^{-1} s}$$

#

EX: Find $L\left[\dfrac{1-\operatorname{ch} t}{t}\right]$

Solution

$\because L[1-\operatorname{ch} t] = \dfrac{1}{s} - \dfrac{s}{s^2-1}$

$$,\because \lim_{t\to 0} \frac{1-\operatorname{ch} t}{t} \overset{\frac{0}{0}}{=} \lim_{\substack{t\to 0 \\ \text{لوبيتال}}} \frac{-\operatorname{sh} t}{1} = 0$$

$\therefore$ The limit exists.

$$\Rightarrow L\left[\frac{1-\operatorname{ch} t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2-1}\right) ds$$

$$= \ln s - \frac{1}{2}\ln(s^2-1)\Big|_s^\infty = \ln\left(\frac{s}{\sqrt{s^2-1}}\right)\Big|_s^\infty$$

$$\Rightarrow L\left[\frac{1-\operatorname{ch} t}{t}\right] = \lim_{s\to\infty} \ln\left(\frac{s}{\sqrt{s^2-1}}\right) - \ln\frac{s}{\sqrt{s^2-1}}$$

$$= 0 - \ln\frac{s}{\sqrt{s^2-1}} = \boxed{\ln\left(\frac{\sqrt{s^2-1}}{s}\right)} \quad \#$$

Problems 1: حل بعض مسائل من

انظر حل المسائل رقم 8 و 11 ، 13 ، و 14 ، و 16
و 17 في الحلول.

مسائل إضافية:

1) $L[t^2+at+b] = \frac{2!}{s^3} + a\cdot\frac{1!}{s^2} + \frac{b}{s}$

$$= \frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s} \quad \#$$

2) $L\left[\sin\left(\frac{2n\pi}{T}t\right)\right] = \frac{\left(\frac{2n\pi}{T}\right)}{s^2+\left(\frac{2n\pi}{T}\right)^2} \quad \#$

$\searrow k$

3) $L[\cos^2 t] = L\left[\frac{1}{2}(1+\cos 2t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] \#$

4) $L[e^{at+b}] = L[e^{at}\cdot e^{b}] = e^{b}\cdot\frac{1}{s-a} \quad \#$

5) $L[\cos(\omega t+b)] = L[\cos\omega t\cos b - \sin\omega t\sin b]$

$$= (\cos b)L[\cos\omega t] - (\sin b)L[\sin\omega t]$$

$$= (\cos b)\cdot\frac{s}{s^2+\omega^2} - (\sin b)\frac{\omega}{s^2+\omega^2} \quad \#$$

② Problems ① حلول التمارين

① $L[8] = \boxed{\frac{8}{s}}$ #

② $L[4e^{5t}] = 4L[e^{5t}] = 4\cdot\frac{1}{s-5} = \boxed{\frac{4}{s-5}}$ #

③ $L[2t+t^2] = 2L(t) + L(t^2) = 2\cdot\frac{1!}{s^2} + \frac{2!}{s^3} = \frac{2}{s^2} + \frac{2}{s^3}$ #

④ $L[t\sin 6t] = ??$

$\because L[\sin 6t] = \frac{6}{s^2+36}$

$\therefore L[t\sin 6t] = -\frac{d}{ds}\left[\frac{6}{s^2+36}\right] = -6\left[-\frac{(2s)}{(s^2+36)^2}\right]$

$= \frac{12s}{(s^2+36)^2}$ #

⑤ $L[e^{-5t}(ch2t + sh2t)] = ??$

$\because L[ch2t + sh2t] = \frac{s}{s^2-4} + \frac{2}{s^2-4} = \frac{s+2}{s^2-4} = \frac{1}{s-2}$

$\therefore L[e^{-5t}(ch2t + sh2t)] = \frac{1}{s+5-2} = \boxed{\frac{1}{s+3}}$ #

⑥ $L[e^{-t}\sin^2 t]$

$\because L[\sin^2 t] = L\left[\frac{1}{2}(1-\cos 2t)\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right] = \frac{2}{s(s^2+4)}$

$\therefore L[e^{-t}\sin^2 t] = \frac{2}{(s+1)[(s+1)^2+4]} = \frac{2}{(s+1)(s^2+2s+5)}$

⑦ $L[e^{4t}(t^4+3t^2-2)]$

$\because L(t^4+3t^2-2) = \frac{4!}{s^5} + 3\cdot\frac{2!}{s^3} - \frac{2}{s}$

$= \frac{24}{s^5} + \frac{6}{s^3} - \frac{2}{s}$

$\therefore L[e^{4t}(t^4+3t^2-2)] = \frac{24}{(s-4)^5} + \frac{6}{(s-4)^3} - \frac{2}{(s-4)}$ #

⑧ $L[t^n e^{-5t}]$

$\because L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L[t^n e^{-5t}] = \frac{n!}{(s+5)^{n+1}}$ #

⑨ $L[\frac{sht}{t}]$

$\because L[sht] = \frac{1}{s^2-1}$. $\because \lim_{t\to 0}\frac{sht}{t} = \lim_{t\to 0}\frac{cht}{1} = 1$

$\therefore$ the limit exists.

$\therefore L[\frac{sht}{t}] = \int_s^\infty \frac{1}{s^2-1}\,ds = -\int_s^\infty \frac{ds}{1-s^2}$

$= -\int_s^\infty \frac{ds}{(1-s)(1+s)}$

$\because \frac{1}{(1-s)(1+s)} = \frac{A}{1-s} + \frac{B}{1+s}$

$\Rightarrow 1 = A(1+s) + B(1-s)$

let $s=-1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

let $s=1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$\therefore L[\frac{sht}{t}] = -\int_s^\infty (\frac{1/2}{1-s} + \frac{1/2}{1+s})\,ds$

$= -\frac{1}{2}[-\ln(1-s) + \ln(1+s)]_s^\infty$

$= \frac{1}{2}[\ln(1-s) - \ln(1+s)]_s^\infty = \frac{1}{2}\ln\left|\frac{1-s}{1+s}\right|_s^\infty$

$= \frac{1}{...}$

$$L\left[\frac{\sinh t}{t}\right] = \int_s^\infty \frac{1}{s^2-1}\,ds = \int_s^\infty \frac{1}{(s-1)(s+1)}\,ds$$

$$\because \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + B(s-1)$$

let $s=1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$.

, let $s=-1 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$.

$$\therefore L\left[\frac{\sinh t}{t}\right] = \int_s^\infty \left(\frac{1/2}{s-1} - \frac{1/2}{s+1}\right) ds = \frac{1}{2}\Big[\ln(s-1) - \ln(s+1)\Big]_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{s-1}{s+1}\right)\right]_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty} \ln\left(\frac{s-1}{s+1}\right) - \ln\frac{s-1}{s+1}\right]$$

$$= \frac{1}{2}\left[0 - \ln\frac{s-1}{s+1}\right] = \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right) \quad \#$$

⑩ $L[e^{-4t}\cos 3t]$

$$\because L[\cos 3t] = \frac{s}{s^2+9} \Rightarrow L[e^{-4t}\cos 3t] = \frac{(s+4)}{(s+4)^2+9} \quad \#$$

⑪ $L\left[\frac{e^{-2t} - e^{-3t}}{t}\right] = ?$

$$\because L[e^{-2t} - e^{-3t}] = \frac{1}{s+2} - \frac{1}{s+3}$$

$$\because \lim_{t\to 0} \frac{e^{-2t} - e^{-3t}}{t} = \lim_{t\to 0} \frac{-2e^{-2t} + 3e^{-3t}}{1} = 1 \Rightarrow \text{The limit exist}$$

$$\therefore L\left[\frac{e^{-2t} - e^{-3t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+2} - \frac{1}{s+3}\right) ds$$

$$= \ln(s+2) - \ln(s+3)\Big|_s^\infty = \ln\left(\frac{s+2}{s+3}\right)\Big|_s^\infty$$

$$= \lim_{s\to\infty} \ln\frac{s+2}{s+3} - \ln\left(\frac{s+2}{s+3}\right) = -\ln\left(\frac{s+2}{s+3}\right)$$

$$= \ln\left(\frac{s+3}{s+2}\right) \quad \#$$

(12) $L\left[\dfrac{\cos t - \cos 3t}{t}\right]$

$\because L[\cos t - \cos 3t] = \dfrac{s}{s^2+1} - \dfrac{s}{s^2+9}$

$, \because \lim\limits_{t\to 0}\left(\dfrac{\cos t - \cos 3t}{t}\right) = \lim\limits_{t\to 0}\left(\dfrac{-\sin t + 3\sin 3t}{1}\right) = 0.$

$\Rightarrow$ The limit exists.

$$\therefore L\left[\frac{\cos t - \cos 3t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right) ds.$$

$$= \frac{1}{2}\ln(s^2+1) - \frac{1}{2}\ln(s^2+9)\Big|_s^\infty = \frac{1}{2}\ln\left(\frac{s^2+1}{s^2+9}\right)\Big|_s^\infty$$

$$= \frac{1}{2}\left[0 - \ln\left(\frac{s^2+1}{s^2+9}\right)\right] = \frac{1}{2}\ln\left(\frac{s^2+9}{s^2+1}\right) \ \#$$

(13) $L[e^{-t}(\sin t - \cos t)^2] = L[e^{-t}(\sin^2 t - 2\sin t\cos t + \cos^2 t)]$

$= L[e^{-t}(1 - \sin 2t)]$

$\because L[1 - \sin 2t] = \dfrac{1}{s} - \dfrac{2}{s^2+4}$

$\Rightarrow L[e^{-t}(1-\sin 2t)] = \dfrac{1}{(s+1)} - \dfrac{2}{(s+1)^2+4} \ \#$

(14) $L[t e^{-2t}\sin at]$

$\because L[\sin at] = \dfrac{a}{s^2+a^2} \Rightarrow L[e^{-2t}\sin at] = \dfrac{a}{(s+2)^2+a^2} = \dfrac{a}{s^2+4s+a^2}$

$\Rightarrow L[t e^{-2t}\sin at] = -\dfrac{d}{ds}\left[\dfrac{a}{s^2+4s+a^2+4}\right]$

$$= -a\left[-\frac{(2s+4)}{(s^2+4s+a^2+4)^2}\right] \ \#$$

(15) $L[e^{2t}(t-1)^2] = L[e^{2t}(t^2-2t+1)]$

$\because L[t^2-2t+1] = \dfrac{2!}{s^3} - 2\dfrac{1!}{s^2} + \dfrac{1}{s} = \dfrac{2}{s^3} - \dfrac{2}{s^2} + \dfrac{1}{s} = \dfrac{2-2s+}{s^3}$

$\Rightarrow L[e^{2t}(t-1)^2] = \dfrac{2 - 2(s-2) + (s-2)^2}{(s-2)^3} \ \#$

6) $L[t^2\cos^2 t]$ $\quad \because L[\cos^2 t] = L[\frac{1}{2}(1+\cos 2t)]$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$\therefore L[t^2\cos^2 t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]\right)$$

$$= \frac{1}{2}\frac{d}{ds}\left[\frac{-1}{s^2} + \frac{s^2+4-2s^2}{(s^2+4)^2}\right] = \frac{1}{2}\frac{d}{ds}\left[\frac{-1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{(-2s)(s^2+4)^2 - 2(s^2+4)(2s)(4-s^2)}{(s^2+4)^{\cancel{4}\,3}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s^3-24s}{(s^2+4)^3}\right] = \frac{1}{s^3} + \frac{s^3-12s}{(s^2+4)^3}$$ #

17) $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt = \cancel{\int_0^3 (0)e^{-st}dt}^{\,0} + \int_3^\infty (2)e^{-st}dt$$

$$= 2\frac{e^{-st}}{-s}\Big|_3^\infty = -\frac{2}{s}[0-e^{-3s}] = \frac{2e^{-3s}}{s}$$ #

18) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t > 1 \end{cases}$

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^1 \overset{u}{t}\,\overset{v}{e^{-st}}dt + \int_1^\infty (1)e^{-st}dt$$

$$= t\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\Big|_0^1 + \frac{e^{-st}}{-s}\Big|_1^\infty$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{1}{s}(0-e^{-s})$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$ #

(c) [redacted] (3)

Inverse Laplace Transform

$$\text{if } L[f(t)] = F(s) \Rightarrow f(t) = L^{-1}[F(s)]$$

L^{-1}: inverse Laplace transform

ونستخدم فى ذلك الجدول التالى

احفظ

	$f(t)$	$F(s) = L(f(t))$
1	a	$\frac{a}{s}$
2	e^{kt}	$\frac{1}{s-k}$
3	$\sin kt$	$\frac{k}{s^2+k^2}$
4	$\cos kt$	$\frac{s}{s^2+k^2}$
5	$\sinh kt$	$\frac{k}{s^2-k^2}$
6	$\cosh kt$	$\frac{s}{s^2-k^2}$
7	t^n	$\frac{n!}{s^{n+1}}$
8	$e^{-at} \sin kt$	$\frac{k}{(s+a)^2+k^2}$
9	$e^{-at} \cos kt$	$\frac{(s+a)}{(s+a)^2+k^2}$
10	$e^{-at} \sinh kt$	$\frac{k}{(s+a)^2-k^2}$
11	$e^{-at} \cosh kt$	$\frac{(s+a)}{(s+a)^2-k^2}$
12	$e^{-at} t^n$	$\frac{n!}{(s+a)^{n+1}}$

فمثلاً : $L^{-1}\left[\frac{5}{s}\right]=5$, $L^{-1}\left[\frac{1}{s+7}\right]=e^{-7t}$

$L^{-1}\left[\frac{s}{s^2+9}\right]=\cos 3t$, $L^{-1}\left[\frac{5}{s^3}\right]=\frac{5}{2!}L^{-1}\left[\frac{2!}{s^3}\right]=\boxed{\frac{5}{2!}t^2}$

$9 \to k^2$, $3 \to 2+1$

وهكذا...

EX (1): Find the inverse transform of:

$$\frac{4}{s^2+8s+25}$$

solution

بإكمال المربع

$$F(s)=\frac{4}{s^2+8s+25}=\frac{4}{(s+4)^2+25-16}=\frac{4}{(s+4)^2+9}$$

$\div 2$

$$\Rightarrow F(s)=\frac{4}{(s+4)^2+(3)^2}=\frac{4}{3}\cdot\frac{3}{(s+4)^2+(3)^2}$$

$$f(t) \Rightarrow L^{-1}(F(s))=\frac{4}{3}e^{-4t}\sin 3t \quad \#$$

EX (2): Find $L^{-1}\left[\frac{s-2}{s^3+s^2-2s}\right]$

solution

$$F(s)=\frac{s-2}{s^3+s^2-2s}=\frac{s-2}{s(s^2+s-2)}=\frac{s-2}{s(s+2)(s-1)}$$

(درجة البسط < درجة المقام والمقام يمكن تحليله) by using partial fractions

$$\frac{s-2}{s(s+2)(s-1)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-1}$$

$$\Rightarrow s-2=A(s+2)(s-1)+Bs(s-1)+Cs(s+2)$$

let $s=0 \Rightarrow -2=-2A \Rightarrow \boxed{A=1}$

let $s=-2 \Rightarrow -4=6B \Rightarrow \boxed{B=\frac{-2}{3}}$

let $s=1 \Rightarrow -1=3C \Rightarrow \boxed{C=-1/3}$

$$\therefore F(s) = \frac{1}{s} + \frac{-2/3}{s+2} + \frac{-1/3}{s-1}$$

$$\Rightarrow L^{-1}\left[\frac{s-2}{s^3+s^2-2s}\right] = L^{-1}\left[\frac{1}{s} + \frac{-2/3}{s+2} + \frac{-1/3}{s-1}\right]$$

$$= 1 - \frac{2}{3} e^{-2t} - \frac{1}{3} e^{t} \quad \#$$

ملحوظة: يمكن حل المسألة السابقة باستخدام طريقة heaviside التالى ذكرها

EX ③ : Find: $L^{-1}\left[\int_s^{\infty} \frac{ds}{s(s+1)}\right]$

solution

$$\because L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s)\, ds \Rightarrow L^{-1}\left[\int_s^{\infty} F(s)\, ds\right] = \frac{f(t)}{t}$$

$$\because F(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 1 = A(s+1) + Bs$$

let $s=0 \Rightarrow 1 = A$, let $s=-1 \Rightarrow 1 = -B \Rightarrow B=-1$

$$\therefore F(s) = \frac{1}{s} - \frac{1}{s+1} \Rightarrow f(t) = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\Rightarrow f(t) = 1 - e^{-t}$$

$$\Rightarrow L^{-1}\left[\int_s^{\infty} \frac{ds}{s(s+1)}\right] = \boxed{\frac{1-e^{-t}}{t}} \quad \#$$

ملحوظة: فالمثال السابق يمكن إيجاد $f(t)$ بطريقة أخرى:

$$f(t) = L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s^2+s}\right] = L^{-1}\left[\frac{1}{(s+\frac{1}{2})^2 - \frac{1}{4}}\right]$$

$$= 2L^{-1}\left[\frac{\frac{1}{2}}{(s+\frac{1}{2})^2 - (\frac{1}{2})^2}\right] = 2e^{-\frac{1}{2}t} \sinh \tfrac{1}{2}t$$

$$\left(\frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{2}\right)$$

$$\Rightarrow f(t) = \boxed{1 - e^{-t}}$$

* Heaviside's Expansion Theorem:

Let $F(s) = \dfrac{P(s)}{Q(s)}$ where,

(i) $P(s)$ & $Q(s)$ are polynomials without common factors

(ii) the degree of $P(s)$ is lower than that of $Q(s)$

Then,

Case (1): if $(s-a)$ be unrepeated factor of $Q(s)$

$\Longrightarrow$ Then, the term of $f(t) = L^{-1}[F(s)]$ corresponding to the factor $(s-a)$ is:

$$\boxed{\frac{P(a)}{Q'(a)}\, e^{at}}$$

حفظ

Note that: If $Q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$ where $a_1, a_2, \ldots, a_n$ are different numbers

then: $f(t) = L^{-1}\left[\dfrac{P(s)}{Q(s)}\right] =$

$$\boxed{\frac{P(a_1)}{Q'(a_1)}\, e^{a_1 t} + \frac{P(a_2)}{Q'(a_2)}\, e^{a_2 t} + \cdots + \frac{P(a_n)}{Q'(a_n)}\, e^{a_n t}}$$

حفظ

Ex. Find $L^{-1}\left[\dfrac{s^3+s}{(s^2+4)^2}\right]$

$$F(s) = \frac{s^3+s}{(s^2+4)^2} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s^2+4)^2}$$

$$\Rightarrow s^3+s = (As+B)(s^2+4) + (Cs+D).$$

Coeff. of s^3: $1 = A \Rightarrow \boxed{A=1}$

Coeff. of s^2: $0 = B \Rightarrow \boxed{B=0}$

Coeff. of s: $1 = 4\underset{1}{A} + C \Rightarrow 1 = 4 + C \Rightarrow \boxed{C=-3}$

Coeff. of s^0: $0 = 4\underset{0}{B} + D \Rightarrow \boxed{D=0}$

$$\therefore F(s) = \frac{s}{s^2+4} - \frac{3s}{(s^2+4)^2}$$

$\because$ لإيجاد $L^{-1}\left[\dfrac{-3s}{(s^2+4)^2}\right]$ نلاحظ أن :

$$\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = -\frac{2s}{(s^2+4)^2}$$

$$\Rightarrow F(s) = \frac{s}{s^2+4} + \frac{3}{2}\,\frac{-2s}{(s^2+4)^2}$$

$$F(s) = \frac{s}{s^2+4} + \frac{3}{2}\,\frac{d}{ds}\left(\frac{1}{s^2+4}\right)$$

$$\because L[t f(t)] = -\frac{d}{ds}F(s) \Rightarrow L^{-1}\left[\frac{d}{ds}F(s)\right] = -t f(t)$$

$$\Rightarrow F(s) = \frac{s}{s^2+4} + \frac{3}{2}\,\frac{d}{ds}\left[\frac{1}{2}\cdot\frac{2}{s^2+\underset{(2)^2}{4}}\right]$$

$$F(s) = \frac{s}{s^2+4} + \frac{3}{4}\,\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$\Rightarrow f(t) = L^{-1}[F(s)] = \cos 2t + \frac{3}{4}\cdot(-t\sin 2t) \ \#$$

EX (4): Find $L^{-1}\left[\dfrac{2s-1}{s^3-s^2-2s}\right]$

solution

$$\because F(s)=\frac{2s-1}{s^3-s^2-2s}=\frac{2s-1}{s(s^2-s-2)}=\frac{\overbrace{2s-1}^{P(s)}}{\underbrace{s(s-2)(s+1)}_{Q(s)}}$$

$$a_1=0,\ a_2=2,\ a_3=-1$$

$$\therefore f(t)=L^{-1}\left[\frac{2s-1}{s(s-2)(s+1)}\right]=\frac{P(0)}{Q'(0)}e^{0t}+\frac{P(2)}{Q'(2)}e^{2t}+\frac{P(-1)}{Q'(-1)}e^{-t}$$

$$=\frac{-1}{(-2)(1)}e^{0t}+\frac{3}{(2)(3)}e^{2t}+\frac{-3}{(-1)(-3)}e^{-t}$$

$$=\frac{1}{2}+\frac{1}{2}e^{2t}-e^{-t}\ \#$$

لاحظ أن: في المثال السابق: عند ايجاد $Q'(0)$ نضع $s=0$ في $(s-2)(s+1)$
وعند ايجاد $Q'(2)$ نضع $s=2$ في $s(s+1)$
وعند ايجاد $Q'(-1)$ نضع $s=-1$ في $s(s-2)$

Case (2): if $(s-a)$ be a factor of $Q(s)$ which is repeated twice (مكرر مرتين)

let $\boxed{\Phi(s)=(s-a)^2F(s)}$ حفظ

$\Rightarrow$ the term in $f(t)$ corresponding to the factor $(s-a)^2$ is:

$$\boxed{\left[\Phi'(a)+\Phi(a)t\right]e^{at}}$$ حفظ

Case 3 : The term in $f(t)$ corresponding to $(s-a)^3$ is:

$$\boxed{\frac{1}{2!}\left[\Phi''(a) + 2\Phi'(a)\,t + \Phi(a)\,t^2\right] e^{at}}$$

where, $\Phi(s) = (s-a)^3 F(s)$.

EX 5 : Find: $\mathcal{L}^{-1}\left[\dfrac{s}{(s+1)(s-2)^2}\right]$

solution

$$F(s) = \frac{\overbrace{s}^{=P(s)}}{\underbrace{(s+1)(s-2)^2}_{Q(s)}}$$

$\therefore$ The term in $f(t)$ corresponding to the factor $(s+1)$:

$$\frac{P(-1)}{Q'(-1)}\, e^{-t} = \frac{-1}{(-3)^2}\, e^{-t} = \boxed{\frac{-1}{9}\, e^{-t}}$$

$\therefore$ For the factor $(s-2)^2$:

$$\Phi(s) = (s-2)^2 F(s) = \frac{s}{s+1}$$

$$\therefore \Phi'(s) = \frac{s+1-s}{(s+1)^2} = \frac{1}{(s+1)^2}$$

$\therefore$ the term in $f(t)$ corresponding to $(s-2)^2$:

$$\left[\Phi'(2) + \Phi(2)\,t\right] e^{2t} = \boxed{\left[\frac{1}{9} + \frac{2}{3}t\right] e^{2t}}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s+1)(s-2)^2}\right] = \frac{-1}{9}e^{-t} + \left(\frac{1}{9} + \frac{2}{3}t\right)e^{2t} \quad \#$$

Case (4): if $(s-\alpha)^2+\beta^2$

be unrepeated factor of second degree of $Q(s)$.

, Let $G(s) = [(s-\alpha)^2+\beta^2] F(s)$.

and let $G(\alpha+i\beta) = R + i I$.

Then, the term in $f(t)$ Corresponding to the factor $[(s-\alpha)^2+\beta^2]$ is: $\boxed{\frac{1}{\beta} e^{\alpha t}[I \cos\beta t + R \sin\beta t]}$

EX (7): Find $L^{-1}\left[\frac{s}{(s-2)(s^2-4s+13)}\right]$

Solution

$$F(s) = \frac{\overset{P(s)}{s}}{(s-2)\underbrace{(s^2-4s+13)}_{Q(s)}}$$

For the factor $(s-2)$:

the term in $f(t)$ corresponding to $(s-2)$ is:

$$\frac{P(2)}{Q'(2)} e^{2t} = \frac{2}{(4-8+13)} e^{2t} = \boxed{\frac{2}{9} e^{2t}}$$

For the factor $(s^2-4s+13)$: $s^2-4s+13 = (s-2)^2+13-4$
$= (s-2)^2+9$

$\therefore \alpha = 2, \beta = 3$. $\boxed{\frac{1}{i} = -i}$

$$\therefore G(s) = [(s-2)^2+9] F(s) = \frac{s}{s-2}$$

$$\therefore G(2+i3) = \frac{2+3i}{2+3i-2} = \frac{2+3i}{3i} = 1 + \frac{2}{3i} = \overset{R}{1} - \overset{I}{\frac{2}{3}}i$$

$\therefore R = 1, \quad I = -\frac{2}{3}$.

$\therefore$ the term in $f(t)$ corresponding to $(s^2 - 4s + 13)$ is

$$\boxed{\frac{1}{3} e^{2t} \left[-\frac{2}{3} \cos 3t + \sin 3t \right]}$$

$$\therefore f(t) = L^{-1}[F(s)] = \frac{2}{9} e^{2t} + \frac{1}{3} e^{2t} \left[-\frac{2}{3} \cos 3t + \sin 3t \right]$$

#

نلحظ أن:

$i^2 = -1 \quad , \quad i^3 = -i \quad , \quad i^4 = 1$

$, \quad \frac{1}{i} = -i$

ارى كهرباء

التفاضل تحت علامة التكامل

قاعدة:

$$\text{if} \quad f(t) = \int_a^b f(x,t)\,dx$$

$$\Rightarrow \boxed{\frac{df}{dt} = \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx}$$

تستخدم هذه القاعدة في حساب كثير من التكاملات المحددة التي من الصعب أو المستحيل حسابها مباشرة.

مثال (1): باستخدام التفاضل تحت علامة التكامل أوجد قيمة:

$$\int_0^\infty \frac{1-e^{-ax}}{xe^x}\,dx \quad , \quad a>-1$$

solution

$$\text{let} \quad f(a) = \int_0^\infty \frac{1-e^{-ax}}{xe^x}\,dx$$

بتفاضل الطرفين بالنسبة إلى a

$$\Rightarrow f'(a) = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{1-e^{-ax}}{xe^x}\right)dx$$

$$= \int_0^\infty \frac{\cancel{x}\,e^{-ax}}{\cancel{x}e^x}\,dx = \int_0^\infty e^{-(a+1)x}\,dx$$

$$\Rightarrow f'(a) = \left.\frac{e^{-(a+1)x}}{-(a+1)}\right|_0^\infty = -\frac{1}{a+1}(0-1)$$

$$\Rightarrow f'(a) = \frac{1}{a+1}$$

ثم بتكامل الطرفين بالنسبة إلى a

$$\Rightarrow f(a) = \ln(a+1) + C$$

ولحساب الثابت C نضع $a=0$ في الطرفين

(نختار قيمة a بحيث تعطي $f(0)=0$)

$$\Rightarrow f(0) = \ln(1) + C \Rightarrow 0 = 0 + C$$

$$f(0) = \int_0^\infty \frac{1-1}{xe^x}\,dx = 0$$

$$\Rightarrow \boxed{C = 0}$$

$$\Rightarrow f(a) = \ln(a+1) \Rightarrow \boxed{\int_0^\infty \frac{1-e^{-ax}}{xe^x}\,dx = \ln(a+1)}$$

#

فمثلاً : اذا كانت $a=4$

$$\Rightarrow \int_0^\infty \frac{1-e^{-4x}}{xe^x}\,dx = \ln(5)$$

وهكذا .

مثال ②

أوجد قيمة $\int_0^1 \frac{x^4-x}{\ln x}\,dx$ باستخدام التفاضل تحت علامة التكامل .

solution

let $f(a) = \int_0^1 \frac{x^a-x}{\ln x}\,dx$

Diffrintiate w.r.t. a

$$\Rightarrow f'(a) = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-x}{\ln x}\right)dx$$

$$= \int_0^1 \frac{x^a \cancel{\ln x}}{\cancel{\ln x}}\,dx = \int_0^1 x^a\,dx$$

$$= \frac{x^{a+1}}{a+1}\Big|_0^1 = \frac{1}{a+1}$$

$$\therefore f'(a) = \frac{1}{a+1}$$

Integrate w.r.t. a

$$\Longrightarrow f(a) = \ln(a+1) + C$$

let $a = 1$ in bot sides

$$\Longrightarrow f(1) = \ln 2 + C \Rightarrow C = -\ln 2$$

$$\int_0^1 \frac{x-x}{\ln x}\,dx = 0$$

$$\Longrightarrow f(a) = \ln(a+1) - \ln 2$$

$$= \boxed{\ln \frac{a+1}{2}}$$

let $a = 4$

$$\Longrightarrow f(4) = \int_0^1 \frac{x^4 - x}{\ln x}\,dx = \boxed{\ln\left(\frac{5}{2}\right)}$$

#

ملحوظة: يمكن باستخدام قاعدة التفاضل تحت علامة التكامل اثبات Theorem 2 في الملزمة السابقة كما يلي:

if $L[f(t)] = F(s) \Rightarrow L[t f(t)] = -\frac{d}{ds}F(s)$

Proof

$$\because F(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

بالتفاضل بالنسبة لـ s

$$\Longrightarrow \frac{dF(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s} f(t) \cdot e^{-st}\, dt = -\int_0^\infty t f(t)\, e^{-st}\, dt$$

$$= -L[t f(t)] \Longrightarrow \boxed{L[t f(t)] = -\frac{dF(s)}{ds}}$$ #

* The Convolution Theorem :

نستخدم هذه النظرية لإيجاد L^{-1} لحاصل ضرب دالتين $[F(s) \cdot G(s)]$

فعند حساب : $L^{-1}[F(s) \cdot G(s)]$

نوجد أولاً : $L^{-1}[F(s)] = f(t)$, و

$L^{-1}[G(s)] = g(t)$

فيكون :

$$L^{-1}[F(s) \cdot G(s)] = \int_0^t f(t-u)\, g(u)\, du = \underbrace{f * g}_{\text{the convolution of } f \text{ and } g.}$$

OR $$L^{-1}[F(s) \cdot G(s)] = \int_0^t g(t-u)\, f(u)\, du = g * f$$

Note that: $f * g = g * f$

EX 1 Find $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$ by using the convolution theorem.

Solution

$$\because L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\underset{\substack{\uparrow \\ F(s)}}{\frac{1}{s^2}} \cdot \underset{\substack{\uparrow \\ G(s)}}{\frac{1}{s^2+1}}\right]$$

$\therefore f(t) = L^{-1}\left[\frac{1}{s^2}\right] = \boxed{t}$

$\therefore g(t) = L^{-1}\left[\frac{1}{s^2+1}\right] = \boxed{\sin t}$

$\therefore L^{-1}\left[\frac{1}{s^2} \cdot \frac{1}{s^2+1}\right] = t * \sin t = \int_0^t \underbrace{(t-u)\sin u}_{\text{by parts (بالتجزئ)}} \, du$

$= (t-u)(-\cos u) - (-1)(-\sin u)\Big|_0^t$

$= \boxed{t - \sin t}$ #

لاحظ أنه : يمكن حل المثال السابق باستخدام Heaviside's theorem

EX (2) : Find $L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$

solution

$\because L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\underbrace{\frac{1}{s^2+1}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+1}}_{G(s)}\right]$

$\therefore f(t) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$, $g(t) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$\therefore L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \sin t * \sin t$

$= \int_0^t \sin(t-u)\sin u \, du$

$= \frac{1}{2}\int_0^t \left[\cos(t-2u) - \underbrace{\cos(t)}_{\text{ثابت}}\right] du$

$= \frac{1}{2}\left[\frac{\sin(t-2u)}{-2} - u\cos t\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin(-t) - \sin t}{-2} - t\cos t\right]$

$= \frac{1}{2}\left[\frac{-2\sin t}{-2} - t\cos t\right] = \frac{1}{2}\left[\sin t - t\cos t\right]$ #

③ ملحوظة هامة: في المثال السابق لا ينفع استخدام ال Heaviside's th نظراً لوجود $(s^2+1)^2$ ولذلك يجب في هذه الحالة استخدام ال Convolution Th.

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*حل بعض المسائل من تمارين ②:

④ $L^{-1}\left[\dfrac{1}{(s+4)(s^2+4s+1)}\right]$

نلاحظ أنه: $s^2+4s+1=(s+2)^2-3$

ليس على صورة $(s-\alpha)^2+\beta^2$

ولذا يجب استخدام ال Convolution Th.

$F(s)=\dfrac{1}{s+4}$, $G(s)=\dfrac{1}{s^2+4s+1}$

$\therefore f(t)=L^{-1}\left[\dfrac{1}{s+4}\right]=e^{-4t}$

$\therefore g(t)=L^{-1}\left[\dfrac{1}{s^2+4s+1}\right]=L^{-1}\left[\dfrac{1}{(s+2)^2-3}\right]$; $3 \to (\sqrt{3})^2$

$=\dfrac{1}{\sqrt{3}}L^{-1}\left[\dfrac{\sqrt{3}}{(s+2)^2-3}\right]$

$=\dfrac{1}{\sqrt{3}}e^{-2t}\sinh\sqrt{3}\,t$

$\therefore L^{-1}\left[\dfrac{1}{(s+4)(s^2+4s+1)}\right]=e^{-4t}*\dfrac{1}{\sqrt{3}}e^{-2t}\sinh\sqrt{3}\,t$

$=\dfrac{1}{\sqrt{3}}\int_0^t e^{-4(t-u)}e^{-2u}\sinh\sqrt{3}\,u\,du$

$=\dfrac{1}{\sqrt{3}}\int_0^t \underbrace{e^{-4t+2u}}_{e^{-4t}e^{2u}}\sinh\sqrt{3}\,u\,du$

$=\dfrac{1}{\sqrt{3}}e^{-4t}\int_0^t e^{2u}\underbrace{\sinh\sqrt{3}\,u}_{\left(\frac{e^{\sqrt{3}u}-e^{-\sqrt{3}u}}{2}\right)}\,du$

$$L^{-1} = \frac{e^{-4t}}{2\sqrt{3}} \int_0^t [e^{(\sqrt{3}+2)u} - e^{(2-\sqrt{3})u}]\,du$$ ④

$$= \frac{e^{-4t}}{2\sqrt{3}} \left[\frac{e^{(\sqrt{3}+2)u}}{(\sqrt{3}+2)} - \frac{e^{(2-\sqrt{3})u}}{(2-\sqrt{3})} \right]_0^t$$

$$= \frac{e^{-4t}}{2\sqrt{3}} \left[\frac{e^{(\sqrt{3}+2)t} - 1}{(\sqrt{3}+2)} - \frac{e^{(2-\sqrt{3})t} - 1}{(2-\sqrt{3})} \right] \#$$

⑦ $L^{-1}\left[\frac{1}{s^4 - 2s^3}\right] = L^{-1}\left[\frac{1}{s^3(s-2)}\right]$

<u>for the factor $(s-2)$:</u> $P(s) = 1$, $Q(s) = s^3(s-2)$

$\therefore$ the term in $f(t)$ corresponding to $(s-2)$ is:

$$\frac{P(2)}{Q'(2)} e^{2t} = \boxed{\frac{1}{8} e^{2t}}$$

<u>for the factor s^3:</u> $a = 0$

$$\therefore \Phi(s) = s^3 F(s) = \frac{1}{s-2}$$

$$\therefore \Phi'(s) = -\frac{1}{(s-2)^2} \Rightarrow \Phi''(s) = \frac{2}{(s-2)^3}$$

$\therefore$ the term in $f(t)$ corresponding to s^3 is:

$$\frac{1}{2!}\left[\Phi''(0) + 2\Phi'(0)t + \Phi(0)t^2\right] e^{0t}$$

$$= \frac{1}{2}\left[-\frac{1}{4} + 2\left(-\frac{1}{4}\right)t - \frac{1}{2}t^2\right] e^{0t}$$

$$= \frac{1}{2}\left[-\frac{1}{4} - \frac{1}{2}t - \frac{1}{2}t^2\right]$$

$$\therefore f(t) = L^{-1}\left[\frac{1}{s^3(s-2)}\right] = \frac{1}{8}e^{2t} + \frac{1}{2}\left[-\frac{1}{4} - \frac{1}{2}t - \frac{1}{2}t^2\right] \#$$

11 $$L^{-1}\left[\frac{s}{(s^2+1)^3}\right] = L^{-1}\left[\frac{s}{s^2+1}\cdot\frac{1}{(s^2+1)^2}\right]$$

$$\because L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

و $$\because L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right]$$

و $$\because L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \sin t * \sin t$$

$$= \int_0^t \sin(t-u)\sin u \, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(t-2u) - \cos(t)\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(t-2u)}{-2} - u\cos t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(-t) - \sin t}{-2} - t\cos t\right]$$

$$= \frac{1}{2}\left[\sin t - t\cos t\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+1)^3}\right] = \cos t * \frac{1}{2}(\sin t - t\cos t)$$

$$= \frac{1}{2}\int_0^t \cos(t-u)(\sin u - u\cos u)\, du$$

$$= \frac{1}{2}\int_0^t \left[\sin u\cos(t-u) - u\cos(t-u)\cos u\right] du$$

$$= \frac{1}{2}\int_0^t \left[\frac{1}{2}(\sin t + \sin(2u-t)) - \frac{u}{2}(\cos t + \cos(t-2u$$

$$= \frac{1}{4}\int_0^t \left[\sin t + \sin(2u-t) - u\cos t - \underbrace{u\cos(t-2u)}_{\text{تجزيء}}\right]$$

$$= \frac{1}{4}\Big[u \sin t - \frac{\cos(2u-t)}{2} - \frac{u^2}{2}\cos t - \Big(u\,\frac{\sin(t-2u)}{-2} - (1)\Big(\frac{-\cos(t-2u)}{4}\Big)\Big)\Big]$$

$$= \frac{1}{4}\Big[t \sin t - \frac{1}{2}\big(\cancel{\cos t - \cos(-t)}\big) - \frac{t^2}{2}\cos t + \frac{1}{2}t\sin(-t) - \frac{1}{4}\big(\cancel{\cos(-t) - \cos t}\big)\Big]$$

$\cos(-t) \to \cos t$

$$= \frac{1}{4}\Big[\frac{1}{2}t \sin t - \frac{t^2}{2}\cos t\Big] \quad \#$$

---

* مسائل إضافية : هام

(1) Find $L^{-1}\Big[\ln\Big(1+\frac{1}{s^2}\Big)\Big]$

$\because F(s) = \ln\Big(1+\frac{1}{s^2}\Big)$

$\therefore L[t f(t)] = (-1)\frac{dF(s)}{ds} = -\frac{d}{ds}\Big[\ln\Big(1+\frac{1}{s^2}\Big)\Big]$

$$= -\frac{\frac{-2}{s^3}}{1+\frac{1}{s^2}} = \frac{2}{s^3+s}$$

$$\Rightarrow L[t f(t)] = \frac{2}{s(s^2+1)} \Rightarrow t f(t) = L^{-1}\Big[\frac{2}{s(s^2+1)}\Big]$$

For the factor s: $P(s) = 2$, $Q(s) = s(s^2+1)$

$\therefore$ the term corresponding to s is:

$$\frac{P(0)}{Q'(0)}e^{0t} = \frac{2}{1}e^{0t} = \boxed{2}$$

, for the factor s^2+1: $s^2+1 = (s-0)^2+1$

$\therefore \alpha = 0,\ \beta = 1$

$$\therefore G(s) = (s^2+1)\cdot\frac{2}{s(s^2+1)} = \boxed{\frac{2}{s}}$$

$$\therefore G(0+i) = G(i) = \frac{2}{i} = \boxed{-2i}$$

$\therefore R = 0,\ I = -2$

$\therefore$ the term corresponding to (s^2+1) is:

$$\frac{1}{1} e^{0t}\left[-2\cos t + 0\right] = \boxed{-2\cos t}$$

$$\therefore t f(t) = 2 - 2\cos t$$

$$\Rightarrow f(t) = \frac{2-2\cos t}{t} = L^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]$$

\#

2) Find the inverse L.T. of:

$$\tan^{-1}\left(\frac{1}{s}\right)$$

Solution

$$F(s) = \tan^{-1}\left(\frac{1}{s}\right)$$

$$\Rightarrow L(t f(t)) = (-1)\frac{d F(s)}{ds} = -\frac{-\frac{1}{s^2}}{1+\frac{1}{s^2}} = \frac{1}{s^2+1}$$

$$\therefore t f(t) = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$\Rightarrow f(t) = \frac{\sin t}{t} \quad \# $$

$$= L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$$

⑧ حل آخر للمثال في صفحة ③ في الملزمة السابقة :

Find $L^{-1}\left[\frac{s^3+s}{(s^2+4)^2}\right]$

$$F(s)=\frac{s^3+s}{(s^2+4)^2}=\frac{As+B}{s^2+4}+\frac{Cs+D}{(s^2+4)^2}$$

كما سبق بالكسور الجزئية

$$\Rightarrow F(s)=\frac{s}{s^2+4}-\frac{3s}{(s^2+4)^2}\Rightarrow f(t)=\cos 2t-3\underbrace{L^{-1}\left[\frac{s}{(s^2+4)^2}\right]}_{①}$$

ثم نوجد $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$ كما يلي .

$$L^{-1}\left[\frac{s}{(s^2+4)^2}\right]=L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{1}{s^2+4}\right]$$

$$\because L^{-1}\left[\frac{s}{s^2+4}\right]=\cos 2t \text{ و } L^{-1}\left[\frac{1}{s^2+4}\right]=\frac{1}{2}\sin 2t$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+4)^2}\right]=\cos 2t*\frac{1}{2}\sin 2t$$

$$=\frac{1}{2}\int_0^t \sin 2(t-u)\cos 2u\,du$$

$$=\frac{1}{4}\int_0^t\left[\sin 2t+\sin(2t-4u)\right]du$$

$$=\frac{1}{4}\left[u\sin 2t-\frac{\cos(2t-4u)}{-4}\right]_0^t$$

$$=\frac{1}{4}\left[t\sin 2t+\frac{1}{4}\left(\cancel{\cos 2t-\cos 2t}\right)\right]$$

$$=\frac{1}{4}t\sin 2t$$

in ①

$$\Rightarrow f(t)=\boxed{\cos 2t-\frac{3}{4}t\sin 2t}\ \#$$

5

Problems 2

① $L^{-1}\left[\frac{1}{2s-7}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s-7/2}\right] = \frac{1}{2} e^{7/2 t}$ #

② $L^{-1}\left[\frac{1}{(s-3)^3}\right] = \frac{1}{2} L^{-1}\left[\frac{2!}{(\underset{\downarrow}{s-3})^{\overset{3}{\downarrow}}}\right]$ $\quad a=-3 \quad n+1$

$= \frac{1}{2} e^{3t} t^2.$ #

③ $L^{-1}\left[\frac{1}{s^2+4s+8}\right] = L^{-1}\left[\frac{1}{(s+2)^2 + \underset{\downarrow (2)^2}{4}}\right]$

$= \frac{1}{2} L^{-1}\left[\frac{2}{(s+2)^2+(2)^2}\right]$

$= \frac{1}{2} e^{-2t} \sin 2t$ #.

④ $L^{-1}\left[\frac{1}{(s+4)(s^2+4s+1)}\right] \longrightarrow$ يمكن حلها ب Convolution theorem

$\therefore \frac{1}{(s+4)(s^2+4s+1)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+4s+1}$

$\Rightarrow 1 = A(s^2+4s+1) + (Bs+C)(s+4).$

Coeff of s^2: $0 = A + B \longrightarrow ①$

Coeff of s: $0 = 4A+4B+C \longrightarrow ②$

coeff of s^0: $1 = A + 4C \longrightarrow 3$

$$3 \Rightarrow C = \frac{1-A}{4}.$$

in $2 \Rightarrow 0 = 4A + 4B + \frac{1-A}{4} \longrightarrow 4$

from $1 \Rightarrow B = -A \xrightarrow{\text{in } 4} 0 = \frac{1-A}{4}$

$$\Rightarrow \boxed{A = 1} \Rightarrow B = \boxed{-1}$$

$$\therefore C = \boxed{0}.$$

$$\therefore F(s) = \frac{1}{s+4} - \frac{s}{s^2+4s+1}$$

$$\therefore L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+4}\right] - L^{-1}\left[\frac{s}{s^2+4s+1}\right]$$

$$= e^{-4t} - L^{-1}\left[\frac{s}{(s+2)^2 - 3}\right]$$

$$= e^{-4t} - L^{-1}\left[\frac{(s+2) - 2}{(s+2)^2 - 3}\right]$$

$$= e^{-4t} - L^{-1}\left[\frac{s+2}{(s+2)^2 - \underset{(\sqrt{3})^2}{3}}\right] + \frac{2}{\sqrt{3}} L^{-1}\left[\frac{\sqrt{3}}{(s+2)^2 - \underset{(\sqrt{3})}{3}}\right]$$

$$= e^{-4t} - e^{-2t}\cosh\sqrt{3}t + \frac{2}{\sqrt{3}} e^{-2t}\sinh\sqrt{3}t$$

#

(5) $L^{-1}\left[\dfrac{1}{(s-1)(s-2)^2}\right]$

For the factor $(s-1)$:

$P(s) = 1$, $Q(s) = (s-1)(s-2)^2$

$Q'(s) = (s-2)^2 + 2(s-1)(s-2)$

$\therefore$ The term in $f(t)$ corresponding to the factor $(s-1)$ is:

$$\frac{P(1)}{Q'(1)} e^{t} = \frac{1}{1} e^{t} = \boxed{e^{t}}$$

For the factor $(s-2)^2$:

$$\Phi(s) = (s-2)^2 F(s) = \frac{1}{s-1}$$

$$\therefore \Phi'(s) = \frac{-1}{(s-1)^2}$$

$\therefore$ the term in $f(t)$ corresponding to $(s-2)^2$ is:

$$\left[\Phi'(2) + \Phi(2)\, t\right] e^{2t}$$

$$= \boxed{[-1 + t]\, e^{2t}}.$$

$$\therefore L^{-1}[F(s)] = e^{t} + (t-1)e^{2t} \quad \#$$

(6) $$L^{-1}\left[\frac{s^2-10s-25}{s^3-25s}\right]=L^{-1}\left[\frac{s^2-10s-25}{s(s^2-25)}\right]$$

$$=L^{-1}\left[\frac{s^2-10s-25}{s(s-5)(s+5)}\right]$$

$$\because P(s)=s^2-10s-25,\ Q(s)=s^3-25s$$

$$\Rightarrow Q'(s)=3s^2-25.$$

$$\therefore L^{-1}[F(s)]=\frac{P(0)}{Q'(0)}e^{0t}+\frac{P(5)}{Q'(5)}e^{5t}+\frac{P(-5)}{Q'(-5)}e^{-5t}$$

$$=\frac{-25}{-25}+\frac{-50}{50}e^{5t}+\frac{50}{50}e^{-5t}$$

$$=1-e^{5t}+e^{-5t}\ \#$$

---

(7) $$L^{-1}\left[\frac{1}{s^4-2s^3}\right]=L^{-1}\left[\frac{1}{s^3(s-2)}\right]$$

For the factor s^3:

$$\Phi(s)=s^3F(s)=\frac{1}{s-2}$$

$$\Rightarrow\Phi'(s)=\frac{-1}{(s-2)^2}\Rightarrow\Phi''(s)=\frac{2}{(s-2)^3}$$

$\therefore$ The term in $f(t)$ corresponding to s^3:

$$\frac{1}{2!}\left[\Phi''(0)+2\Phi'(0)t+\Phi(0)t^2\right]e^{0t}$$

$$=\frac{1}{2}\left[-\frac{1}{4}+2\left(-\frac{1}{4}\right)t-\frac{1}{2}t^2\right]=\cancel{\frac{1}{2}\left(-\frac{3}{4}\ldots-\frac{1}{2}t^2\right)}$$

$$=\frac{1}{2}\left[-\frac{1}{4}-\frac{1}{2}t-\frac{1}{2}t^2\right]\ \#$$

For the factor $(s-2)$:

$$P(s) = 1, \quad Q(s) = s^4 - 2s^3$$

$$\Rightarrow Q'(s) = 4s^3 - 6s^2$$

$\therefore$ The term in $f(t)$ corresponding to $(s-2)$:

$$\frac{P(2)}{Q'(2)} e^{2t} = \boxed{\frac{1}{8} e^{2t}}$$

$$\therefore L^{-1}[F(s)] = \frac{1}{2}\left(-\frac{1}{4} - \frac{1}{2}t - \frac{1}{2}t^2\right) + \frac{1}{8} e^{2t}$$

\#

8 $L^{-1}\left[\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right]$

For the factors $(s+2)$, $(s-3)$:

$$P(s) = s^2 - 3, \quad Q(s) = (s+2)(s-3)(s^2+2s+5)$$

$$\Rightarrow Q'(s) = (s-3)(s^2+2s+5) + (s+2)(s^2+2s+5) + (s+2)(s-3)(2s+2).$$

$\therefore$ The term in $f(t)$ corresponding to $(s+2)$, $(s-3)$:

$$\frac{P(-2)}{Q'(-2)} e^{-2t} + \frac{P(3)}{Q'(3)} e^{3t}.$$

$$= \frac{1}{-25} e^{-2t} + \frac{6}{100} e^{3t} = \boxed{-\frac{1}{25} e^{-2t} + \frac{3}{50} e^{3t}}$$

For the factor (s^2+2s+5):

$$s^2+2s+5 = (s+1)^2 + 4.$$

$$\alpha = -1, \quad \beta = 2$$

$$\therefore G(s) = \left[(s+1)^2+4\right] F(s) = \frac{s^2-3}{(s+2)(s-3)}$$

$$\therefore G(-1+2i) = \frac{(-1+2i)^2-3}{(-1+2i+2)(-1+2i-3)}$$

$$= \frac{1-4i-4-3}{(1+2i)(-4+2i)} = \frac{-6-4i}{-8-6i}$$

$$= \frac{(-6-4i)(-8+6i)}{(-8-6i)(-8+6i)} = \frac{48+24-4i}{64+36}$$

$$= \frac{72-4i}{100} = \frac{18-i}{25} = \frac{18}{25} - \frac{1}{25}i$$

R $\quad$ I

$\therefore$ The term in $f(t)$:

$$\frac{1}{2} e^{-t}\left[-\frac{1}{25}\cos 2t + \frac{18}{25}\sin 2t\right].$$

$$\therefore L^{-1}[F(s)] = -\frac{1}{25}e^{-2t} + \frac{3}{50}e^{3t} + \frac{e^{-t}}{2}\left[-\frac{\cos 2t}{25} + \frac{18s}{2}\right.$$

#

9 $L^{-1}\left[\dfrac{1}{(s+2)^2(s-2)}\right]$.

$F(s)=\dfrac{1}{(s+2)^2}$, $G(s)=\dfrac{1}{s-2}$

$f(t)=L^{-1}\left[\dfrac{1}{(s+2)^2}\right]=e^{-2t}t$

, $g(t)=L^{-1}\left[\dfrac{1}{s-2}\right]=e^{2t}$.

$\therefore L^{-1}[F(s)\cdot G(s)]=f*g$

$$=e^{-2t}t*e^{2t}=\int_0^t e^{2(t-u)}\cdot e^{-2u}u\,du.$$

$$=e^{2t}\int_0^t e^{-4u}\cdot u\,du.$$

$$=e^{2t}\left[u\left(\frac{e^{-4u}}{-4}\right)-(1)\left(\frac{e^{-4u}}{16}\right)\right]_0^t$$

$$=e^{2t}\left[\frac{t\,e^{-4t}}{-4}-\frac{1}{16}\left(e^{-4t}-1\right)\right]$$ #

---

10 $L^{-1}\left[\dfrac{2}{s^3(s^2+1)}\right]$

$F(s)=\dfrac{2}{s^3}$, $G(s)=\dfrac{1}{s^2+1}$

$\therefore f(t)=L^{-1}\left(\dfrac{2}{s^3}\right)=L^{-1}\left(\dfrac{2!}{s^3}\right)=t^2$

, $g(t)=L^{-1}\left(\dfrac{1}{s^2+1}\right)=\sin t$

$$\therefore L^{-1}[F(s)\,G(s)] = t^2 * \sin t$$

$$= \int_0^t \sin(t-u)\cdot u^2\,du.$$

$$= u^2\left(-\frac{\cos(t-u)}{-1}\right) - (2u)\left(\frac{-\sin(t-u)}{1}\right) + (2)\left(\frac{\cos(t-u)}{-1}\right)\Bigg|_0^t$$

$$= t^2 + 2[0-0] - 2[1-\cos t]$$

$$= t^2 - 2 + 2\cos t \quad \#.$$

---

(11) $L^{-1}\left[\frac{s}{(s^2+1)^3}\right] = L^{-1}\left[\frac{s}{s^2+1}\cdot\frac{1}{(s^2+1)^2}\right]$

~~F(s)~~ $\because L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$

$$\text{と } L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{s^2+1}\right]$$

~~#~~ $\text{て} \because L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \sin t * \sin t$$

$$= \int_0^t \sin(t-u)\sin u\,du$$

$$= \frac{1}{2}\int_0^t [\cos(t-2u) - \cos t]\,du.$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \frac{1}{2}(\sin t - t\cos t)$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+1)^3}\right] = L^{-1}\left[\frac{s}{s^2+1}\cdot\frac{1}{(s^2+1)^2}\right]$$

$$= \cos t * \frac{1}{2}(\sin t - t\cos t)$$

$$= \int_0^t \cos(t-u)\cdot\frac{1}{2}(\sin u - u\cos u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\sin u\cos(t-u) - u\cos u\cos(t-u)\right]du$$

$$= \frac{1}{4}\int_0^t \underbrace{\sin t}_{\text{ثابت}} + \sin(2u-t) - u\left(\cos t + \cos(2u-t)\right)du$$

$$= \frac{1}{4}\left[u\sin t - \frac{\cos(2u-t)}{2} - \frac{u^2}{2}\cos t\Big|_0^t - \int_0^t \underbrace{u\cos(2u-t)\,du}_{\text{بالتجزئ}}\right]$$

$$= \frac{1}{8}t\sin t - \frac{1}{8}t^2\cos t \;\#$$

1.0

(12) $L^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right]$

$\because L^{-1}\left[\dfrac{1}{s^2}\right] = t \;,\quad L^{-1}\left[\dfrac{1}{(s+1)^2}\right] = e^{-t}\,t$

$\therefore L^{-1}\left[\dfrac{1}{s^2}\cdot\dfrac{1}{(s+1)^2}\right] = t * e^{-t}t$

$= \displaystyle\int_0^t (t-u)\, e^{-u}\cdot u \, du$

$= \displaystyle\int_0^t (tu-u^2)\, e^{-u} \, du$

$= (tu-u^2)\left(\dfrac{e^{-u}}{-1}\right) - (t-2u)\left(e^{-u}\right)$
$\quad + (-2)\left(\dfrac{e^{-u}}{-1}\right)\Big|_0^t$

$= (t+2)e^{-t} + t - 2$ #

(13) $L^{-1}\left[\dfrac{1}{(s^2+1)^3}\right] = L^{-1}\left[\dfrac{1}{s^2+1}\cdot\dfrac{1}{(s^2+1)^2}\right]$

$L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t \,,\; L^{-1}\left[\dfrac{1}{(s^2+1)^2}\right] = L^{-1}\left[\dfrac{1}{(s^2+1)}\cdot\dfrac{1}{(s^2+1)}\right]$
$= \sin t * \sin t$

(11) ثم أكمل كما 12 أعلاه

(14) $$L^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+4s+13)}\right]$$ (11)

$$\because L^{-1}\left[\frac{1}{s^2+2s+2}\right]=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\sin t$$

$$\& \because L^{-1}\left[\frac{1}{s^2+4s+13}\right]=L^{-1}\left[\frac{1}{(s+2)^2+\underset{(3)^2}{9}}\right]$$

$$=\frac{1}{3}L^{-1}\left[\frac{3}{(s+2)^2+(3)^2}\right]$$

$$=\frac{1}{3}e^{-2t}\sin 3t.$$

$$\therefore L^{-1}\left[\frac{1}{s^2+2s+2}\cdot\frac{1}{s^2+4s+13}\right]$$

$$=e^{-t}\sin t * \frac{1}{3}e^{-2t}\sin 3t.$$

$$=\int_0^t e^{-(t-u)}\sin(t-u)\,\frac{1}{3}e^{-2u}\sin 3u\,du.$$

مع تمنياتي