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ملزمة (٤)

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The power series method

ثانية كهرباء

The Power Series Method -

* To solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ about the point x_0 (or in powers of $(x-x_0)$), we write the Equation in the standard form

$$y'' + P(x)y' + q(x)y = 0.$$

* If both $P(x)$ & $q(x)$ are analytic at $x = x_0$

$\Rightarrow$ x_0 is an ordinary point & we have to use the Power Series method $\Rightarrow$ let $y = \sum_{n=0} a_n(x-x_0)^n$

$\Rightarrow y' = \sum_{n=1} n a_n (x-x_0)^{n-1}$, $y'' = \sum_{n=2} n(n-1)a_n(x-x_0)^{n-2}$

$\Rightarrow$ Sub. in the diff. equation

* Obtain a reccurrence relation in the form

$$a_n = \boxed{\frac{(\;)(\;)}{(\;)(\;)}}\, a_{n-2} \text{ or } a_{n-3} \ldots$$

* Get the Coff. a_n all interms of a_0 & a_1, then Sub. by them in $y(x) = \sum_{n=0} a_n(x-x_0)^n$

* Obtain the solution as $y(x) = a_0 y_1 + a_1 y_2$.

* The Interval of validity of the solution is $|x-x_0| < R$, where R is the distance from x_0 to the nearst S.P.

Note that :-

① we have to classify the D.E according to all points x_0.

② We get the interval of the solution when solving about the given x_0 by getting Taylor expansion about x_0 for $p(x)$ & $q(x)$

③ We start solving using $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

Examples :-

① Solve Airy's Eq. about $x_0 = 0$

② Solve $(1-x^2)y'' - 4xy' + 4y = 0$ about $x = 0$

③ Solve $(3-x^2)y'' - xy' + 25y = 0$ about $x = 0$

④ Solve $y'' - 2xy' - 6y = 0$ around $x_0 = 0$

⑤ Solve the Legendre Diff Eq.

Solution:- We will show first step one, classify all x_0 for the given D.E, & step two, then start solving. (step three)

① Airy's Eq. $\Rightarrow$ $y'' + xy = 0$

$p(x) = 0$ & $q(x) = x$

<u>step one</u>: For any point x_0, $p(x)$ & $q(x)$ are analytic

$\Rightarrow$ x_0 is ordinary point $\Rightarrow$ Use Power Series

<u>step two</u> :-

$$\left.\begin{aligned} p(x) &= 0 \\ q(x) &= x \end{aligned}\right\} \text{Taylor about } x_0 = 0 \text{ for all } x$$

Then the solution interval is all values of x

i.e. $|x| < \infty$

<u>Step Three</u> :- See Page number

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② $(1-x^2)y'' - 4xy' + 4y = 0$

<u>step one</u> :- $p(x) = \frac{-4x}{1-x^2}$ $q(x) = \frac{4}{1-x^2}$

$\Rightarrow$ Any point $x_0 \neq \pm 1$ is an ordinary point

For $x_0 = \pm 1$ we have

$P(x) = \frac{-4x}{1-x^2}(x \mp 1)$ $Q(x) = \frac{4}{1-x^2}(x \mp 1)^2$

So, $x_0 = \pm 1$ is a Regular Singular Point

<u>Step two</u>:-

$p(x) = \frac{-4x}{1-x^2} = -4x(1 + x^2 + x^4 \dots)$; $|x| < 1$

& $q(x) = \frac{4}{1-x^2} = 4(1 + x^2 + x^4 \dots)$; $|x| < 1$

$\Rightarrow$ The solution interval is $|x| < 1$

Step Three :- See page number

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③ $(3-x^2)y'' - xy' + 25y = 0$

$p(x) = -\frac{x}{3-x^2}$ & $q(x) = \frac{25}{3-x^2}$

Step One :- all $x_0 \neq \pm\sqrt{3}$ are ordinary points

$x_0 = \pm\sqrt{3}$ are Regular Sugular Points

Step Two :- $p(x) = \frac{-x}{3-x^2}$ has Taylor expansion around $x_0 = 0$ as

$$-\frac{x}{3}\left(\frac{1}{1-\frac{x^2}{3}}\right) = -\frac{x}{3}\left(1 + \frac{x^2}{3} + \frac{x^4}{9} \cdots\right)$$

For $\left|\frac{x^2}{3}\right| < 1 \Rightarrow |x| < \sqrt{3}$

Also, $q(x) = \frac{25}{3-x^2} = \frac{25}{3}\left(\frac{1}{1-x^2/3}\right)$

$$= \frac{25}{3}\left(1 + \frac{x^2}{3} + \frac{x^4}{9} \cdots\right) ; \quad |x| < \sqrt{3}$$

$\Rightarrow$ The solution interval is $|x| < \sqrt{3}$

Step Three : Do the same steps as number ②

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$y'' - 2xy' - 6y = 0$

Step one :- $p(x) = -2x$ $q(x) = -6$

Any point x_0 is an ordinary point, because $p(x)$ & $q(x)$ are everywhere analytic

Step Two :-

$$\left.\begin{array}{l} p(x) = -2x \\ q(x) = -6 \end{array}\right\}$$ Taylor exp. around $x_0 = 0$ for all values of x

$\Rightarrow$ The solution interval is $|x| < \infty$ i.e for all x

Step Three :- See page number

* We will show now how to apply the power series method (step Three) for all the above problems.

Example :- Solve Airy's Equation $y'' + xy = 0$

about $x = 0$

Solution :- we have $x_0 = 0$, $p(x) = 0$ & $q(x) = x$

$\Rightarrow$ $p(x)$ & $q(x)$ are analytic at $x_0 = 0$ $\Rightarrow$

x_0 is an ordinary Point $\Rightarrow$ Use the Power Series

method $\Rightarrow$ let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow y' = \sum_{n=1} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$$

Subsitute in the diff equation

$$\sum_{n=2} n(n-1) a_n x^{n-2} + \sum_{n=0} a_n x^{n+1} = 0$$

$\Downarrow$ replace n by n-3

$$\Rightarrow \sum_{n=2} n(n-1) a_n x^{n-2} + \sum_{n=3} a_{n-3} x^{n-2} = 0$$

$$\Rightarrow 2a_2 + \sum_{n=3} n(n-1) a_n x^{n-2} + \sum_{n=3} a_{n-3} x^{n-2} = 0$$

$$\Rightarrow 2a_2 + \sum_{n=3} (n(n-1) a_n + a_{n-3}) x^{n-2} = 0$$

$$\Rightarrow 2a_2 = 0 \quad \& \quad n(n-1) a_n + a_{n-3} = 0$$

$$\Rightarrow a_2 = 0 \quad \& \quad a_n = -\frac{a_{n-3}}{n(n-1)} \quad \text{for } n \geq 3$$

Using the recurrence relation to get $a_3, a_4, \ldots$

Since $a_2 = 0 \Rightarrow a_5 = a_8 = a_{11} \cdots = \text{zero}$

$$a_3 = -\frac{a_0}{3(2)} \quad , \quad a_4 = -\frac{a_1}{4(3)}$$

$$a_6 = -\frac{a_3}{6(5)} = +\frac{a_0}{(2.5)(3.6)} \quad , \quad a_7 = -\frac{a_4}{7(6)} = \frac{a_1}{(4.7)(3.6)}$$

$$a_9 = -\frac{a_0}{(2.5.8)(3.6.9)} \quad , \quad a_{10} = -\frac{a_1}{(4.7.10)(3.6.9)}$$

$$\vdots \qquad\qquad \vdots$$

$$\Rightarrow y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2.3} x^3 - \frac{a_1}{3.4} x^4 + \frac{a_0}{(2.5)(3.6)} x^6 + \frac{a_1}{(4.7)(3.6)} x^7$$

$$- \frac{a_0}{(2.5.8)(3.6.9)} x^9 - \frac{a_1}{(4.7.10)(3.6.9)} x^{10} \cdots$$

$$= a_0 \left(1 - \frac{1}{2.3} x^3 + \frac{1}{(2.5)(3.6)} x^6 - \frac{1}{(2.5.8)(3.6.9)} x^9 \cdots \right)$$

$$+ a_1 \left(x - \frac{1}{3.4} x^4 + \frac{1}{(4.7)(3.6)} x^7 - \frac{1}{(4.7.10)(3.6.9)} x^{10} \cdots \right)$$

$$= a_0 \sum_{n=0} \frac{(-1)^n}{[2.3.\cdots(3n-1)][3.6\cdots(3n)]} x^{3n}$$

$$+ a_1 \sum_{n=0} \frac{(-1)^n}{[4.7.10.\cdots(3n+1)][3.6.9\cdots(3n)]} x^{3n+1}$$

$$= a_0 y_1 + a_1 y_2 .$$

The solution is valid for all x i.e. $|x| < \infty$

Example :- find the series solution about $x=0$ for

$$(1-x^2)y'' - 4xy' + 4y = 0$$

Solution : the standard form of the diff. Eq. is

$$y'' - \frac{4x}{1-x^2}y' + \frac{4}{1-x^2}y = 0$$

$\Rightarrow P(x) = -\frac{4x}{1-x^2}$, $q(x) = \frac{4}{1-x^2}$ both are analytic

at $x_0 = 0$ $\Rightarrow$ x_0 is an ordinary Point $\Rightarrow$ Use the Power Series Method $\Rightarrow$ let $y = \sum_{n=0} a_n x^n$

$y' = \sum_{n=1} n a_n x^{n-1}$ and $y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$

$\Rightarrow$ Subsitute in the diff. Equation

$$(1-x^2)\sum_{n=2} n(n-1)a_n x^{n-2} - 4x\sum_{n=1} n a_n x^{n-1} + 4\sum_{n=0} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2} n(n-1)a_n x^{n-2} - \underbrace{\sum_{n=2} n(n-1)a_n x^n}_{\text{replace } n \text{ by } n-2} - 4\underbrace{\sum_{n=1} n a_n x^n}_{\text{replace } n \text{ by } n-2} + 4\underbrace{\sum_{n=0} a_n x^n}_{\text{replace } n \text{ by } n-2} = 0$$

$$\Rightarrow \sum_{n=2} n(n-1)a_n x^{n-2} - \sum_{n=4}(n-2)(n-3)a_{n-2}x^{n-2}$$

$$-4\sum_{n=3}(n-2)a_{n-2}x^{n-2} + 4\sum_{n=2} a_{n-2}x^{n-2} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=4} n(n-1)a_n x^{n-2} - \sum_{n=4} (n-2)(n-3)a_{n-2} x^{n-2}$$

$$\cancel{-4a_1 x} - 4\sum_{n=4} (n-2)a_{n-2} x^{n-2} + 4a_0 + \cancel{4a_1 x}$$

$$+4\sum_{n=4} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow (2a_2 + 4a_0) + 6a_3 x + \sum_{n=4} \big(n(n-1)a_n - (n-2)(n-3)a_{n-2}$$

$$- 4(n-2)a_{n-2} + 4a_{n-2} \big) x^{n-2} = 0$$

$$\Rightarrow 2a_2 + 4a_0 = 0 \Rightarrow a_2 = -2a_0$$

$$6a_3 = 0 \Rightarrow a_3 = 0$$

$$n(n-1)a_n - (n-2)(n-3)a_{n-2} - 4(n-2)a_{n-2} + 4a_{n-2} = 0$$

$$n(n-1)a_n - \big((n-2)(n-3) + 4(n-2) - 4\big)a_{n-2} = 0$$

$$n(n-1)a_n = (n^2 - 5n + 6 + 4n - 8 - 4)a_{n-2}$$

$$n(n-1)a_n = (n^2 - n - 6)a_{n-2}$$

$$n(n-1)a_n = (n+2)(n-3)a_{n-2}$$

$$a_n = \frac{(n+2)(n-3)}{n(n-1)} a_{n-2} \; ; \text{ for } n \geq 4$$

Since $a_3 = 0 \Rightarrow a_5 = a_7 = a_9 \cdots = \text{Zero}$

$$a_4 = \frac{6(1)}{4(3)} a_2 = \frac{6(1)}{4(3)}(-2)a_0$$

$$a_6 = \frac{(8)(3)}{6(5)} a_4 = \frac{(6.8)(1.3)}{(4.6)(3.5)}(-2)a_0$$

$$a_8 = \frac{(6.8.10)(1.3.5)}{(4.6.8)(3.5.7)}(-2a_0)$$

$\vdots$

$$\Rightarrow a_2 = -2a_0 \quad , \quad a_4 = \frac{6(1)}{4(3)}(-2a_0) \quad , \quad a_6 = \frac{8(1)}{4(5)}(-2a_0)$$

$$, \quad a_8 = \frac{(10)(1)}{(4)(7)}(-2a_0) \quad , \quad \ldots$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_1 x + a_0 \left(1 - 2x^2 - 2\left(\frac{6}{4.3}\right)x^4 - 2\left(\frac{8}{4.5}\right)x^6 - 2\left(\frac{10}{4.7}\right)x^8 \ldots \right)$$

$$= a_1 x + a_0 \left(1 - 2 \sum_{n=1}^{\infty} \frac{(2n+2)}{4(2n-1)} x^{2n} \right)$$

The interval of the validity of the solution is

$|x| < 1$.

Example: Solve $(3-x^2)y'' - xy' + 25y = 0$ around $x_0 = 0$

Solution :- Try it by yourself & note that the interval of validity will be $|x| < \sqrt{3}$.

Example :- Solve $y'' - 2xy' - 6y = 0$ around $x_0 = 0$.

Solution :- The Eq. is in its standard form $\Rightarrow$

$p(x) = -2x$ & $q(x) = -6$ $\Rightarrow$ both are analytic

at $x_0 = 0$ $\Rightarrow$ x_0 is an ordinary Point $\Rightarrow$ Use the Power

Series $\Rightarrow$ let $y = \sum_{n=0} a_n x^n$ $\Rightarrow$ $y' = \sum_{n=1} n a_n x^{n-1}$

$\Rightarrow y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$ $\Rightarrow$ Sub. in the D.E.

$$\sum_{n=2} n(n-1) a_n x^{n-2} - 2\sum_{n=1} n a_n x^n - 6\sum_{n=0} a_n x^n = 0$$

Replace in the last two Summations n by $n-2$.

$$\sum_{n=2} n(n-1) a_n x^{n-2} - 2\sum_{n=3} (n-2) a_{n-2} x^{n-2} - 6\sum_{n=2} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow 2a_2 + \sum_{n=3} n(n-1) a_n x^{n-2} - 2\sum_{n=3} (n-2) a_{n-2} x^{n-2}$$

$$- 6a_0 - 6\sum_{n=3} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow (2a_2 - 6a_0) + \sum_{n=3} \left(n(n-1) a_n - [2(n-2) + 6] a_{n-2}\right) x^{n-2} = 0$$

$$\Rightarrow 2a_2 - 6a_0 = 0 \Rightarrow a_2 = 3a_0$$

$$n(n-1) a_n - (2(n-2) + 6) a_{n-2} = 0$$

$$\Rightarrow a_n = \frac{2(n-2)+6}{n(n-1)} a_{n-2}.$$

$$\Rightarrow a_n = \frac{2(n+1)}{[illegible]} a_{n-2} \quad \text{for } n \geq 3$$

$$a_3 = \frac{2(4)}{3(2)}\, a_1 \quad , \quad a_4 = \frac{2(5)}{4(3)}\, a_2 = \frac{2(5)}{4(3)}\,(3a_0)$$

$$a_5 = \frac{2(6)}{5(4)}\, a_3 = \frac{2^2(4\cdot 6)}{(3\cdot 5)(2\cdot 4)}\, a_1 \quad , \quad a_6 = \frac{2(7)}{6(5)}\, a_4 = \frac{2^2(5\cdot 7)}{(4\cdot 6)(3\cdot 5)}\,(3a_0)$$

$$a_7 = \frac{2^3(4\cdot 6\cdot 8)}{(3\cdot 5\cdot 7)(2\cdot 4\cdot 6)}\, a_1 \quad , \quad a_8 = \frac{2^3(5\cdot 7\cdot 9)}{(4\cdot 6\cdot 8)(3\cdot 5\cdot 7)}\,(3a_0)$$

$$\vdots$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0\left(1 + 3x^2 + \frac{2(5)}{4(3)}(3)x^4 + \frac{2^2(5\cdot 7)}{(4\cdot 6)(3\cdot 5)}(3)x^6 + \cdots\right)$$

$$+ a_1\left(x + \frac{2(4)}{3(2)}x^3 + \frac{2^2(4\cdot 6)}{(3\cdot 5)(2\cdot 4)}x^5 + \cdots\right)$$

$$= a_0\left(1 + 3x^2 + \sum_{n=1} \frac{2^n(5\cdot 7\cdot 9\cdots(2n+3))}{[4\cdot 6\cdots(2n+2)][3\cdot 5\cdots(2n+1)]}\,x^{2n+2}\right)$$

$$+ a_1\left(x + \sum_{n=1} \frac{2^n(4\cdot 6\cdot 8\cdots(2n+2))}{[3\cdot 5\cdots(2n+1)][2\cdot 4\cdots(2n)]}\,x^{2n+1}\right)$$

This solution is valid for all $x \Rightarrow |x| < \infty$

The Legendre Differential Equation :-

Ex: Solve the Legendre Diff. Eq. around $x_0 = 0$

Solution :- The Legendre diff. eq. is

$$(1-x^2)\, y'' - 2x y' + k(k+1)\, y = 0 \quad ; k \text{ is real no.}$$

Its standard form is

$$y'' - \frac{2x}{1-x^2}\, y' + \frac{k(k+1)}{1-x^2}\, y = 0$$

$$\Rightarrow p(x) = -\frac{2x}{1-x^2} \quad \& \quad q(x) = \frac{k(k+1)}{1-x^2}$$

Both $p(x)$ & $q(x)$ are analytic at $x_0 = 0$, to show this mathematically we have

$$p(x) = -\frac{2x}{1-x^2} = -2x\,(1 + x^2 + x^4 + x^6 + \cdots)$$

$$= -2x - 2x^3 - 2x^5 \cdots \Rightarrow \text{Taylor about } x_0 = 0$$

Also $q(x) = \dfrac{k(k+1)}{1-x^2} = k(k+1) \cdot \dfrac{1}{1-x^2}$

$$= k(k+1)\,(1 + x^2 + x^4 + x^6 + \cdots) \Rightarrow \text{Taylor}$$

So $x_0 = 0$ is an ordinary Point & we will use the Power Series Method

Let $y = \sum\limits_{m=0} a_m x^m \Rightarrow y' = \sum\limits_{m=1} m\, a_m x^{m-1}$

$$y'' = \sum_{m=2} m(m-1)\, a_m x^{m-2}$$

Subsitute in the Leg. D. E.

$$(1-x^2)\sum_{m=2} m(m-1)a_m x^{m-2} - 2x\sum_{m=1} m a_m x^{m-1} + k(k+1)\sum_{m=0} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2} m(m-1)\,a_m\,x^{m-2} - \sum_{m=2} m(m-1)\,a_m\,x^m$$

$$-2\sum_{m=1} m\,a_m\,x^m + k(k+1)\sum_{m=0} a_m\,x^m = 0.$$

replace m by m+2 in the first $\sum$ $\Rightarrow$

$$\sum_{m=0}(m+2)(m+1)\,a_{m+2}\,x^m - \sum_{m=2} m(m-1)\,a_m\,x^m$$

$$-2\sum_{m=1} m\,a_m\,x^m + k(k+1)\sum_{m=0} a_m\,x^m = 0$$

$$\Rightarrow 2a_2 + 3(2)a_3x - 2a_1x + k(k+1)a_0 + k(k+1)a_1x$$

$$+\sum_{m=2}\Big((m+2)(m+1)\,a_{m+2} - m(m-1)\,a_m - 2m\,a_m + k(k+1)\,a_m\Big)\,x^m = 0.$$

$$\Rightarrow (2a_2 + k(k+1)a_0) + (6a_3 + (k(k+1)-2)a_1)\,x$$

$$+\sum_{m=2}\Big((m+2)(m+1)\,a_{m+2} - [m(m-1)+2m-k(k+1)]\,a_m\Big)x^m = 0$$

$$\Rightarrow 2a_2 + k(k+1)a_0 \longrightarrow a_2 = -\frac{k(k+1)}{2}a_0$$

$$6a_3 + [k(k+1)-2]\,a_1 = 0$$

$$\Rightarrow 6a_3 + (k^2+k-2)\,a_1 = 0$$

$6a_3 + (k-1)(k+2)\,a_1 = 0$

$$\Rightarrow \boxed{a_3 = \frac{(1-k)(k+2)}{6}\,a_1}$$

$$(m+2)(m+1)\,a_{m+2} = \big(m(m-1) + 2m - k(k+1)\big)\,a_m$$

$$= \big(m^2 + m - k(k+1)\big)\,a_m$$

$$= (m-k)\big(m+(k+1)\big)\,a_m$$

$$\Rightarrow \boxed{a_{m+2} = \frac{(m-k)(m+k+1)}{(m+2)(m+1)}\,a_m \;; \text{ for } m \geq 2}$$

$$a_4 = \frac{(2-k)(k+3)}{4(3)}a_2 = \frac{[(-k)(2-k)][(k+1)(k+3)]}{(4.2)\quad(3.1)}\,a_0$$

$$a_5 = \frac{(3-k)(k+4)}{5(4)}\,a_3 = \frac{[(1-k)(3-k)][(k+2)(k+4)]}{(5.3)\quad(4.2)}\,a_1$$

$$a_6 = \frac{[(-k)(2-k)(4-k)][(k+1)(k+3)(k+5)]}{(6.4.2)(5.3.1)}\,a_0$$

$$a_7 = \frac{[(1-k)(3-k)(5-k)][(k+2)(k+4)(k+6)]}{(7.5.3)\quad(6.4.2)}\,a_1$$

The Solution $y(x) = \sum\limits_{m=0} a_m x^m$

$= a_0 + a_1 x + a_2 x^2 + \cdots$

$= a_0 y_1 + a_1 y_2$, where

$$= 1 + \frac{(-k)(k+1)}{2} x^2 + \frac{(-k)(2-k)\,(k+1)(k+3)}{(4.2)\,(3.1)} x^4 + \frac{(-k)(2-k)(4-k)\,(k+1)(k+3)(k+5)}{(6.4.2)\,(5.3.1)} x^6 + \cdots$$

$$y_1 = x + \frac{(1-k)(k+2)}{6} x^3 + \frac{(1-k)(3-k)\,(k+2)(k+4)}{(5.3)\,(4.2)} x^5 + \frac{(1-k)(3-k)(5-k)\,(k+2)(k+4)(k+6)}{(7.5.3)\,(6.4.2)} x^7 + \cdots$$

- Observe that y_1 is an even fn., while y_2 is an odd fn.

- The interval of the validity is $|x| < 1$.

- Special Case :- In the special case of $k = n$ = a positive integer, one of the two functions (y_1 or y_2) will terminate i.e. become a polynomial of order $= n$

If $k=0 \Rightarrow y_1 = 1$

If $k=1 \Rightarrow y_2 = x$

If $k=2 \Rightarrow y_1 = 1 - 3x^2$

If $k=3 \Rightarrow y_2 = x - \frac{5}{3}x^3$

⋮

} Legendre Polynomials

Note: For $k=0$, The D.E. is $(1-x^2)\,y'' - 2xy' = 0 \Rightarrow$
if we use $u = y' \Rightarrow (1-x^2)\cdot u' - 2xu = 0 \Rightarrow$
$\frac{u'}{u} = \frac{2x}{1-x^2} \Rightarrow \ln u = -\ln(1-x^2) + \ln c_1$

$\Rightarrow u = \frac{c_1}{1-x^2} \Rightarrow y' = \frac{c_1}{1-x^2} \Rightarrow y = c_1 \tanh^{-1} x + c_2$.