AIN SHAMS UNIVERSITY
FACULTY OF ENGINEERING
ENG. PHYSICS & MATH. DEPARTMENT
Electronics and Communications Engineering Program
Junior Students.

Mid Term Examination

Spring 2021 *Exam Time: 60 minutes.*

PHM212s: Complex, Special Functions and Numerical Analysis

The Exam Consists of TWO Questions in THREE Pages. Answer All Questions *Total Marks: 20 Marks*

MODEL ANSWER		

General Instructions:

- ***Please read the examination paper carefully.***
- ***Be sure to solve each question in its paper (you can use the back).***
- ***Programmable & Graphical Calculators are NOT Allowed.***

Question no. 1 (12 marks)

a) By two different methods obtain a closed form for $\Gamma(n+3/2)$ where n is any positive integer. **[4 Marks]**

Method (1)

Using Legendre Duplication Formula

$$\because \sqrt{\pi}\,\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \quad \ldots\boxed{1}$$

Let $x=n+1$

$$\Rightarrow \sqrt{\pi}\,\Gamma(2n+2)=2^{2n+1}\Gamma(n+1)\Gamma\left(n+\frac{3}{2}\right)$$

$$\Rightarrow \Gamma\left(n+\frac{3}{2}\right)=\frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!} \quad \ldots\boxed{1}$$

Method (2)

$$\Gamma\left(n+\frac{3}{2}\right)=\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\ldots\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$

$$=\frac{1}{2^{n+1}}(2n+1)(2n-1)(2n-3)\ldots5.3.1\sqrt{\pi} \quad \ldots\boxed{1}$$

$$=\frac{(2n+1)!\sqrt{\pi}}{2^{n+1}\left[2n(2n-2)(2n-4)\ldots6.4.2\right]}=\frac{(2n+1)!\sqrt{\pi}}{2^{n+1}\left[2^n n!\right]}$$

$$\Rightarrow \Gamma\left(n+\frac{3}{2}\right)=\frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!} \quad \ldots\boxed{1}$$

b) Evaluate in terms of the Gamma function the integral $\int_0^\infty \frac{x^k}{k^x}\,dx$ and state the condition on k such that the integral converges. **[4 Marks]**

$$I=\int_0^\infty \frac{x^k}{k^x}dx=\int_0^\infty x^k k^{-x}dx$$

$$Let\ k^{-x}=e^{-t} \Rightarrow -x\ln k=-t \quad \ldots\boxed{1}$$

$$\Rightarrow x=\frac{t}{\ln k} \Rightarrow dx=\frac{1}{\ln k}dt$$

$$\Rightarrow I=\int_0^\infty \left(\frac{t}{\ln k}\right)^k e^{-t}\frac{1}{\ln k}dt \quad \ldots\boxed{1}$$

$$=\left(\frac{1}{\ln k}\right)^{k+1}\int_0^\infty t^k e^{-t}dt$$

$$=\boxed{\left(\frac{1}{\ln k}\right)^{k+1}\Gamma(k+1)} \quad \ldots\boxed{1}$$

$$\rightarrow k+1>0 \Rightarrow k>-1\left(Domain\ \ for\ \Gamma(x)\right)$$

$$\rightarrow k>0\left(Domain\ \ for\ \ln(x)\right)$$

$$\rightarrow \ln k>0 \Rightarrow k>1\left(\ for\ Integral\ Limits\right)$$

$$\Rightarrow \boxed{k>1} \quad \ldots\boxed{1}$$

C) Find the area enclosed by the curve

$$x^{2/5} + y^{2/5} = 1$$

[4 Marks]

$$Area = 4\int_0^1 y\,dx$$

$$= 4\int_0^1 \left(1 - x^{2/5}\right)^{5/2} dx \quad \ldots \boxed{1}$$

$$let\ x^{2/5} = t \Rightarrow x = t^{5/2} \Rightarrow dx = \frac{5}{2} t^{3/2} dt$$

$$\Rightarrow Area = 4\int_0^1 (1-t)^{5/2} \frac{5}{2} t^{3/2} dt \quad \ldots \boxed{1}$$

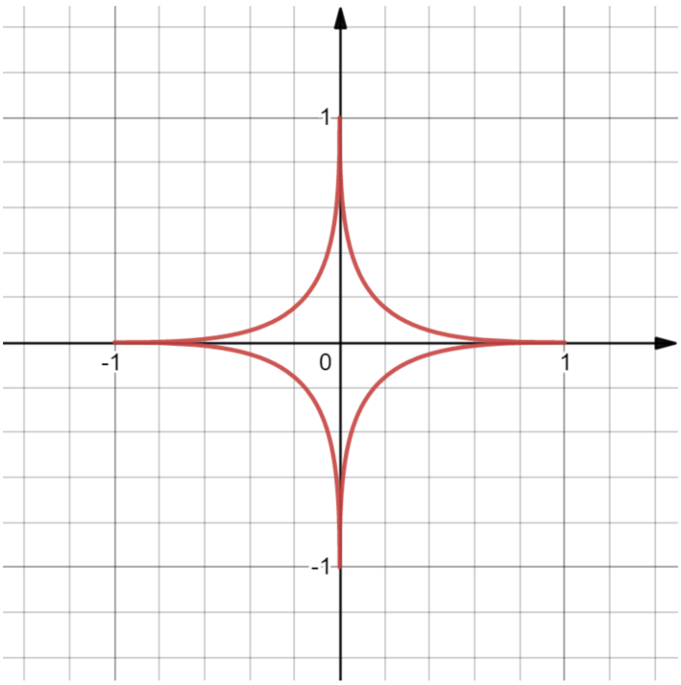
$$= 10\int_0^1 t^{3/2} (1-t)^{5/2}\, dt$$

$$= 10\beta\left(\frac{5}{2}, \frac{7}{2}\right) \quad \ldots \boxed{1}$$

$$= 10\frac{\Gamma\left(5/2\right)\Gamma\left(7/2\right)}{\Gamma(6)}$$

$$= 10\frac{3/2 . 1/2 . \sqrt{\pi}\ 5/2 . 3/2 . 1/2 . \sqrt{\pi}}{5!}$$

$$\Rightarrow Area = \frac{15\pi}{2^7}\ square\,units \quad \ldots \boxed{1}$$



Question no. 3 (8 marks)

Find two linearly independent solutions in powers of " x " for the following differential equations:

$$(1-x^2)\,y'' - 2\,x\,y' + 12\,y = 0$$

$\left(1-x^2\right)y''-2xy'+12y=0 \ldots\ldots\ldots equation[1]$

$1) \rightarrow p(x)=\dfrac{-2x}{\left(1-x^2\right)}, q(x)=\dfrac{12}{\left(1-x^2\right)}$

$at\ x_0=0 \Rightarrow p(x)\ \&\ q(x)\ are\ defined \Rightarrow x_0=0$ is an Ordinary Point ...$\boxed{1}$

$2) \rightarrow$ Let $y=\sum_{n=0}^{\infty} a_n x^n$ be a solution $\Rightarrow y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ ...$\boxed{1}$

$3) \rightarrow$ Substitute $in\ equation[1]$

$$\Rightarrow \left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 12\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 12 a_n\right]x^n = 0$$

$4) \rightarrow \because coef.\left(x^n\right)=0 \Rightarrow (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 12 a_n = 0$

$$\Rightarrow a_{n+2} = \frac{n(n-1)+2n-12}{(n+2)(n+1)}a_n = \frac{n^2+n-12}{(n+2)(n+1)}a_n$$

$\Rightarrow \boxed{a_{n+2}=\dfrac{(n+4)(n-3)}{(n+2)(n+1)}a_n}$ $; n \geq 0 \ldots$ Recurrence Relation ...$\boxed{3}$

$5) \rightarrow n=0 \Rightarrow a_2=\dfrac{(4)(-3)}{(2)(1)}a_0, \rightarrow n=2 \Rightarrow a_4=\dfrac{(6)(-1)}{(4)(3)}a_2=\dfrac{(4*6)(-3*-1)}{(2*4)(1*3)}a_0,$

$n=4 \Rightarrow a_6=\dfrac{(8)(1)}{(6)(5)}a_4=\dfrac{(4*6*8)(-3*-1*1)}{(2*4*6)(1*3*5)}a_0$

$\Rightarrow a_{2k}=\dfrac{\left[4*6*\cdots*(2k+2)\right]\left[-3*-1*1*\cdots*(2k-5)\right]}{(2k)!}a_0 \cdots\cdots k \geq 1$ ...$\boxed{1}$

$\rightarrow n=1 \Rightarrow a_3=\dfrac{(5)(-2)}{(3)(2)}a_1=\dfrac{-5}{3}a_1 \quad , n=3 \Rightarrow a_5=0=a_7=a_9=\ldots=a_{2k+1}$ ...$\boxed{1}$

$\Rightarrow \boxed{y=\sum_{n=0}^{\infty} a_n x^n = a_0\left[1+\sum_{k=1}^{\infty}\dfrac{\left[4*6*\cdots*(2k+2)\right]\left[-3*-1*1*\cdots*(2k-5)\right]}{(2k)!}x^{2k}\right]+a_1\left[x-\dfrac{5}{3}x^3\right]}$ ...$\boxed{1}$

Best Wishes,
Dr. Makram Roshdy Eskaros.