Series Solutions Of Linear Differential Equations

Introduction: Up to now we have primarily solved linear differential equations of order two or higher when the equation had *constant coefficients*. And we found that, their solutions are ***elementary functions*** such as e^{ax}, $\sin ax$, $\cos ax$, $\ln ax$ etc. However, if those coefficients are not constant but depend on x, the situation is more complicated and the solutions may be nonelementary functions – *transcendental* (non-algebraic) *functions*, known as *special functions*. ***Airy's equation, Legendre's equation, Bessel's equation*** and other equations are of this type, which play an important role in applied engineering mathematics.

Throughout this chapter we shall consider only equations, which have at least one variable coefficient. Under Certain wide conditions, the best that we can usually expect from equations of this sort is an ***Infinite Series Solution***. One should note that the ***Cauchy-Euler Equation*** is an exception to this rule, see section I.1.3.

In essence, this chapter is devoted to two standard basic methods of solution and their applications: the ***Power Series Method***, which yields solutions in the form of ***power series***, and an extension of it, called the ***Frobenius Method***.

I. Basic Concepts and Definitions

Since our primary goal in the present chapter is to find general solution of Linear Higher-Order DEs with Variable Coefficients as an infinite power series, we first need to recall and examine some basic terminologies, definitions and theorems (without proofs) in concern.

1.1 Linear Higher-Order Differential Equations with Variable Coefficients

1.1.1 Preliminary Theory

The general *linear* differential equation of order n has the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)\,. \qquad (1)$$

A differential equation that cannot be written in this form is called ***non-linear***. If $g(x)=0$, the equation is called a ***homogeneous*** equation, otherwise it is called a ***non-homogeneous*** equation. If at least one of the coefficients $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ is not a constant, then equation (1) is said to have ***variable coefficients***.

Ex. 1: $xy''' + y' - 3x^4 y = \tan x$ is a third order *linear* equation.

Ex. 2: $yy''' + y' - 3x^4 y = \tan x$ is a third order *nonlinear* equation.

Theorem 1: ***"Existence of a Unique Solution"***

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are continuous on an interval I and $a_n(x)\neq 0$ for every x in this interval, then there exists one and only one solution to (1) on the interval which satisfies the conditions $y(x_0)=y_0, y'(x_0)=y_1, \ldots, y^{(n-1)}(x_0)=y_{n-1}$ where $x=x_0$ is any point in the interval I.

Definition 1: ***"Linear Dependence and Independence"***

A set of n functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ is said to be linearly dependent on an interval I, if there exist n constants $c_1, c_2, \ldots\ldots, c_n$, not all zero, such that $c_1y_1(x)+c_2y_2(x)+\ldots\ldots+c_ny_n(x)=0$ for every x in the interval. Otherwise, the set of functions is said to be linearly independent.

Ex. 3: The set of functions $y_1(x)=\sin 2x$ and $y_2(x)=\sin x\cos x$ is linearly dependent on $(-\infty,\infty)$ since $y_1(x)$ is a constant multiple of $y_2(x)$. Recall from the double angle formula that $\sin 2x = 2\sin x\cos x$.

<u>Ex. 4:</u> $4e^{2x}, 2e^{2x}, 8e^{-4x}$ are linearly dependent on $(-\infty,\infty)$ since we can find constants c_1, c_2, c_3 not all zero such that:

$c_1(4e^{2x})+c_2(2e^{2x})+c_3(8e^{-4x})=0$ identically; e.g. $c_1=1, c_2=-2, c_3=0$.

<u>Ex. 5:</u> e^x and xe^x are linearly independent since $c_1e^x+c_2xe^x=0$ identically if and only if $c_1=0$, $c_2=0$.

<u>Theorem 2:</u> "Linear Independence"

The set of n functions $y_1(x), y_2(x), \ldots, y_n(x)$ (assumed differentiable) is linearly independent on an interval I if and only if the determinant

$$W(y_1,y_2,\ldots,y_n)=\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix},$$

called the "Wronskian" of $y_1(x), y_2(x), \ldots, y_n(x)$, is different from zero on the interval I.

<u>Theorem 3:</u> "Superposition Principle – Homogeneous Equations"

If $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of the nth-order linear differential equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=0\,, \qquad (2)$$

on an interval I, it is called the fundamental set of solutions. Then the linear combination:

$$y(x)=c_1y_1(x)+c_2y_2(x)+\ldots+c_ny_n(x),$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants, is the general solution of (2) on the interval.

__Example 6:__ The functions $y_1(x)=e^x$ and $y_2(x)=e^{-x}$ are both solutions of the homogeneous linear equation $y''-y=0$ on the interval $(-\infty,\infty)$. By inspection, the solutions are linearly independent on the x-axis. This fact can be corroborated by observing that the ***Wronskian:***

$$W(e^x,e^{-x})=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-2\neq 0, \text{ for every } x.$$

Hence, we conclude that $y_1 \& y_2$ form a ***fundamental set of solutions*** and, consequently $y=c_1e^x+c_2e^{-x}$ is the ***general solution*** of $y''-y=0$ on the interval $(-\infty,\infty)$.

__Definition 2:__ "Particular Solution – Nonhomogeneous Equations"

Any function y_p, free of arbitrary parameters, that satisfies (1) is said to be a __particular solution or particular integral of the equation.__

__Ex.7:__ It is a straightforward task to show that the constant function $y_p=-6$ is a particular solution of the nonhomogeneous equation $y''-y=6$.

__Ex.8:__ *"Superposition – Nonhomogeneous DE"*

It is easy to verify that: $y_{p_1}=e^{2x}$ is a particular solution of $y''-y=3e^{2x}$, and $y_{p_2}=-6$ is a particular solution of $y''-y=6$. So, $y_p=y_{p_1}+y_{p_2}=e^{2x}-6$ is a particular solution of $y''-y=3e^{2x}+6$.

__Theorem 4:__ "General Solution – Nonhomogeneous Equations"

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (1) on an interval I, and let $y_1(x), y_2(x), \ldots, y_n(x)$ be a fundamental set of solutions of the associated homogeneous differential equation (2) on I. Then the general solution of equation (1) on this interval is given by

$$y(x)=c_1y_1(x)+c_2y_2(x)+\ldots\ldots+c_ny_n(x)+y_p \quad , \tag{3}$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants and the linear combination

$$c_1y_1(x)+c_2y_2(x)+\ldots\ldots+c_ny_n(x)=y_c(x)$$

__is called the complementary function for equation (1).__

__Ex.9:__ The general solution of the nonhomogeneous equation: $y''-y=6+3e^{2x}$ is given by $y=y_c+y_p=c_1e^x+c_2e^{-x}+e^{2x}-6$ (see Ex.6 & Ex.8).

I.1.2 Variation of Parameters Method

Generally, the ***particular solution*** y_p of equation (1) - see ***Theorem 4***- can be obtained by using the ***variation of parameters method***. This method is already studied before to solve linear differential equations ***with constant coefficients***. However, it should be adapted to solve nonhomogeneous linear differential equation ***with variable coefficients***.
For the sake of simplicity, we shall summarize this method to solving the nonhomogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x). \qquad (4)$$

The procedure: Under the assumption that the conditions of ***Theorem 1*** are verified, the general solution of (4) is given by $y = y_c + y_p$ ***(Theorem 4)***.

1. Find the complementary function $y_c(x) = c_1y_1(x) + c_2y_2(x)$, where $y_1(x)$ and $y_2(x)$ are the fundamental set of solutions of the ***associated homogeneous equation***

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

2. Put equation (4) into the standard form:

$$y'' + p(x)y' + q(x)y = f(x) \qquad (5),$$

by dividing it by the leading coefficient $a_2(x)$.

3. Find the particular solution y_p

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \qquad (6),$$

where $u_1(x)$ and $u_2(x)$ are two unknown functions satisfying the conditions:

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \qquad (7).$$

That is, solving these two equations simultaneously for determining the derivatives $u_1'(x)$ & $u_2'(x)$ and then, a direct integration gives u_1 & u_2. ∎
We shall illustrate this procedure in the next section, see ***Example 12***.

I.1.3 Cauchy-Euler Equation

It is one of the most important types of differential equations with variable coefficients whose general solution can always be written in terms of powers of x, sines, cosines, and logarithmic functions, i.e. there is no need to seek its solution as an infinite series. Moreover, its method of solution is quite similar to that for constant coefficient equations in that an *auxiliary equation* must be solved. Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots + a_1 x \frac{dy}{dx} + a_0 y = g(x) \qquad (8),$$

where $a_n, a_{n-1}, \dots, a_0$ are constants, is said to be a ***Cauchy-Euler equation.*** For the sake of simplicity, we shall confine our attention to solving the homogeneous second-order equation:

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0 \qquad (9).$$

And the solution of higher-order equations follows analogously. Moreover, we can also solve the nonhomogeneous equation

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = g(x)$$

by the ***variation of parameters method,*** once we have determined the complementary function $y_c(x)$.

Remark 1: ***One should note that*** the coefficient ax^2 of y'' is zero at $x = 0$. Hence in order to guarantee that the fundamental results of ***Theorem 1*** are applicable to the Cauchy-Euler equation, we confine our attention to finding the general solutions defined on the interval $(0, \infty)$. Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ into the differential equation. That is, the general solutions of Cauchy-Euler equation is not defined at $x = 0$ which is called ***Regular Singular point,*** as we shall define in the next section. Thereat, another expression is normally used in solving differential equations with variable coefficients namely "***solutions about/around a point***" as we shall discuss later.

Method of Solution:

We try a solution of the form $y = x^m$, where m is to be determined. Substituting $y = x^m$ and its derivatives into equation (9), we get:

$$\left[am(m-1)+bm+c\right]x^m = 0.$$

Omitting x^m, which is not zero if $x \neq 0$, we obtain the ***auxiliary equation:***

$$am^2+(b-a)m+c=0 \qquad (10)$$

Thus, $y = x^m$ is a solution of (9) whenever m is a solution of the auxiliary equation (10). Therefore, there are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots appear as a conjugate pair.

Case I: "Distinct Real Roots"

If $m_1 \neq m_2$, then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence, by ***Theorem 3***, the general solution of (9) is given by

$$\boxed{y = c_1x^{m_1} + c_2x^{m_2}}$$

Case II: "Repeated Real Roots"

If $m_1 = m_2 = -(b-a)/2a \equiv m$, then we obtain only one solution – namely, $y_1 = x^m$. According to Theorem 3, the second solution y_2 must be linearly independent with y_1 to construct the fundamental set of solutions. Therefore, applying the ***method of reduction of order***, that is substituting $y_2 = uy_1 = ux^m$ and its derivatives into (9) and solving the resultant DE, we get $y_2 = y_1 \ln x$. The general solution is then

$$\boxed{y = c_1x^m + c_2x^m \ln x}$$

Note For higher-order equations, if m is a root of multiplicity k, then it can be shown that

$$x^m, x^m \ln x, x^m(\ln x)^2, \ldots, x^m(\ln x)^{k-1}$$

are k linearly independent solutions.

Case III: "Conjugate Complex Roots"

If $m_{1,2} = \mu \pm i\nu$, then:

$$y_{1,2} = x^{\mu \pm i\nu} = x^{\mu}\left[\cos(\nu \ln x) \pm i \sin(\nu \ln x)\right].$$

And it is immediate to see that the general solution can be written as:

$$\boxed{y = x^{\mu}\left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]}$$

Remember that: $x^{\nu} = (e^{\ln x})^{\nu} = e^{\nu \ln x}$ and $e^{\pm i\theta} = [\cos\theta \pm i \sin\theta]$, which is the well-known ***Euler's formula***.

Cauchy-Euler equations occur in certain applications, and we illustrate this by the following simple example from electrostatics.

***Example 10:* "Electric potential field between two concentric spheres"**
Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 4cm$ & $r_2 = 8cm$ kept at potentials $v_1 = 110$ volts & $v_2 = 0$, respectively.
Physical information $v(r)$ is a solution of $rv'' + 2v' = 0$, where $v' = dv/dr$.

Solution We have $r(rv'' + 2v') = 0$, which is a Cauchy-Euler equation, setting $v = r^m \to v' = mr^{m-1}$ & $v'' = m(m-1)r^{m-2}$. The obtained auxiliary equation is $m^2 + m = 0$ which has the roots 0 and -1. This gives the general solution $v(r)=c_1+c_2/r$. From the given *'boundary conditions'* (the potentials on the sphere): $v(8)=c_1+c_2/8=0$ and $v(4)=c_1+c_2/4=110$. By subtraction, $c_2=880\to c_1=-110$. Thus, $v(r)=-110+880/r$ volts.

***Example 11:* "Third-Order Equation"**

Solve $\quad x^3y'''+5x^2y''+7xy'+8y=0.$

Solution $y=x^m\to y'=mx^{m-1}, y''=m(m-1)x^{m-2}, y'''=m(m-1)(m-2)x^{m-3}$

A direct substitution results in the following auxiliary equation

$$m^3+2m^2+4m+8=(m+2)(m^2+4)=0\Rightarrow m_1=-2, m_{2,3}=\pm 2i.$$

Hence, the general solution is: $y=c_1x^{-2}+c_2\cos(2\ln x)+c_3\sin(2\ln x)$.

***Example 12:* "Nonhomogeneous Equation – Variation of Parameters"**

Solve $\quad x^2y''-3xy'+3y=2x^4e^x.$

Solution Since the equation is nonhomogeneous, we first solve the associated homogeneous equation $x^2y''-3xy'+3y=0$ that has an auxiliary equation $(m-1)(m-3)=0\to\ y_c=c_1x+c_2x^3$. Now, by using the ***variation of parameters*** (see section I.1.2) $y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$, where $u_1(x)$ and $u_2(x)$ must satisfy conditions (7), that is:

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \Rightarrow \begin{cases} xu_1' + x^3u_2' = 0 \\ u_1' + 3x^2u_2' = \dfrac{2x^4e^x}{x^2} \end{cases}.$$

Solving this system results in $u_1'(x)=-x^2e^x$ and $u_2'(x)=e^x$, integrating these two functions yields $u_1=-x^2e^x+2xe^x-2e^x$ & $u_2=e^x$. Hence y_p is: $y_p=u_1y_1+u_2y_2=2x^2e^x-2xe^x$. Finally, $y=c_1x+c_2x^3+2x^2e^x-2xe^x$.

Remark 2:* *"Alternative Method of Solution"

Any Cauchy-Euler differential equation can be reduced to an equation with constant coefficients by means of the substitution $x=e^t$, which in turn requires the use of the ***Chain Rule*** of differentiation.

I.2 Power Series

Since the main aim of this chapter is to find a solution in the form of an infinite series which often turns out to be a power series, it is appropriate to list some of the more important facts about the power series. For a more in-depth review of this theory, the reader should encourage to consult a calculus text.

I.2.1 Background Material - Review of Power Series

In this section we review a few relevant facts on power series from calculus.

• **Definition 3:** ***"Power Series"***

A power series in $x-x_0$ is a functional infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + .. + a_n(x-x_0)^n + .. \qquad (11)$$

whose terms are the products of constant factors $a_0, a_1, a_2, \ldots,$ called the "coefficients of the series", by integral powers of the difference $x-x_0$. x_0 is a constant, called the "center of the series", and x is a variable.

If in particular $x_0 = 0$, we obtain a *power series* in powers of x: $\sum_{n=0}^{\infty} a_n x^n$.

•Convergence

A power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is convergent at a specified value of x, if its sequence of partial sums $\{S_N(x)\}$ converges – that is,

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n \text{ exist.}$$

If the limit does not exist at x, the series is said to be divergent.

•Interval of Convergence

Every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges.

- ***Radius of Convergence***

Every power series has a radius of convergence R. If $R > 0$, then a power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges for $|x-x_0| < R$, i.e. $x_0 - R < x < x_0 + R$; and diverges for $|x-x_0| > R$. A power series may or may not converge at the end points $x_0 - R$ and $x_0 + R$ of this interval. If $R = 0$, then the series converges only at its center x_0; when $R = \infty$, the series converges $\forall\, x$.

- ***Determination of 'R'***

The radius of convergence is usually determined by the *Ratio Test* (the *Root Test* is sometimes applicable):

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} <1 \rightarrow \text{the series converges absolutely,} \\ =1 \rightarrow \text{the test is inconclusive,} \\ >1 \rightarrow \text{the series diverges.} \end{cases}$$

Ex 13:

For the power series $\sum_{n=1}^{\infty} (x-6)^n / 2^n n$, centered at $x_0 = 6$, the ratio test gives:

$$\lim_{n\to\infty}\left[\frac{(x-6)^{n+1}}{2^{n+1}(n+1)}\right] \Big/ \left[\frac{(x-6)^n}{2^n n}\right] = |x-6| \lim_{n\to\infty}\left|\frac{n+1}{2n}\right| = \frac{1}{2}|x-6|.$$

Hence, the series converges for $\frac{1}{2}|x-6| < 1 \Rightarrow 4 < x < 8$, and it is easy to prove that the series converges at $x = 4$ and diverges at $x = 8$. Thus, the interval of convergence of the series is $[4,8)$, and the radius of convergence is $R = 2$.

- ***Identity Property***

If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0, R > 0$ for all numbers x in the interval of converges, then $a_n = 0$ for all values of n.

- ***A power series can be differentiated/integrated term-wise within its interval of convergence.***

• ***Shifting the Summation Index***

For the remainder of this chapter, it is important that you become adept at simplifying the sum of two or more power series to an expression with a single "*Sigma*" $\sum$ which is often requires a ***reindexing*** – that is, a ***shift in the index*** of summation. We illustrate that by the aide of the following example.

Example 14: ***Writing Two Added Power series as one Power Series***

$$\text{Write } \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} \text{ as one power series.}$$

Solution In order to write more than one added power series as only one series, it is necessary that all the summation indices start with the same number and the powers of x in each series be "in phase" – i.e. of same power at all values of the different indices. ***The procedure:***

1. ***Make all the series start with the same power of*** x.

By writing the first term of the first series outside the summation notation,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 2.1a_2 x^0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}. \qquad (*)$$

We see that both series on the right-hand side start with x^1.

2. ***Make all the series start with the same summation index.***

Now to get the same summation index, we are inspired by the exponents of x which are $n-2$ in the first series and $n+1$ in the second series – see (*). So, we let $k = n-2$ in the first series and at the same time let $k = n+1$ in the second one. The right-hand side of (*) becomes

$$2a_2 + \sum_{k=1}^{\infty} (k+2)(K+1)a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 2a_2 + \sum_{k=1}^{\infty} \left[(k+2)(K+1)a_{k+2} + a_{k-1}\right] x^k.$$

That is:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_{n-1}\right] x^n. \qquad (12)$$

N.B. We remind the reader that the summation index is a "dummy" variable. The fact that $k = n-2$ in one case and $k = n+1$ in the other should cause no confusion if you keep in mind that it is the ***value*** of the summation index that is important. If you are not convinced of the above result, then write out a few terms on both sides of the equality.

I.2.2 Expanding Functions Into Power Series

If a function $f(x)$ can be represented as the sum of a power series we say that it is ***expanded into the power series***. The existence of such an expansion is extremely important since it makes it possible to write/replace, approximately, the function (elementary/nonelementary) by the sum of the first several terms of the power series, i.e. by a polynomial.
In this section, we shall investigate the following question:

Given a function $f(x)$, what conditions guarantee that it can be represented as the sum of a power series?

Answer Under the assumption that $f(x)$ is infinitely differentiable in a neighborhood of a point x_0. Suppose that it is representable as the sum of a power series convergent within an interval of containing the point x_0 :

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\dots+a_n(x-x_0)^n+\dots \quad (13)$$

Thus, this representation is right if we can determine the coefficients $a_0,a_1,a_2,\dots$. By using the general properties of power series and the known values of the function $f(x)$ together with its derivatives at the point x_0, we can find the undetermined coefficients $a_0,a_1,a_2,\dots$ as follows:

♦ Setting $x=x_0$ in (13), yields $a_0=f(x_0)$. Now, differentiate the power series and substitute $x=x_0$ into the differentiated series. This results in $a_1=f'(x_0)$. Similarly, the repeated differentiation and substitution $x=x_0$ gives $a_2=f''(x_0)$. Proceeding in this way we receive

$$a_n=\frac{f^{(n)}(x_0)}{n!}.$$

♦ We thus consecutively determine the coefficients $a_0,a_1,a_2,\dots$ of the expansion (13). On substituting the expressions found into equality (13) we at the series:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\dots (14)$$

This is the well-known "Taylor series" of the function $f(x)$.

Thus, the result we have established is:

Lemma 1:

A function $f(x)$ is analytic at $x = x_0$, if it has a Taylor expansion about $x = x_0$ which is convergent in some interval $|x - x_0| < R$ with radius of convergence $R > 0$.

♦ Therefrom, the expression "*Series Solutions about/around a Point $x = x_0$*" is more appropriate to be used in solving differential equations by either the *Power Series Method or Frobenius Method.*

• If in particular $x_0 = 0$, we obtain Taylor series centered at $x = 0$ which is called "*Maclaurin series*".

Recall, for example, that:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots (-\infty < x < \infty),$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots (-\infty < x < \infty),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (-\infty < x < \infty),$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots \text{for } |x| < 1, \text{ where } m \text{ is any real number,}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{for } |x| < 1,$$

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) \quad \text{for } |x| < 1.$$

I.2.3 Idea of Power Series Method

In this section, we shall illustrate the practical procedure of the power series method for solving differential equations by describing it for a simple equation whose solutions we know; thereat we can see what is going on. The mathematical justification of the method follows in the coming sections.

The Procedure: For a given equation of the form:

$$y'' + p(x)y' + q(x)y = 0$$

1. Represent the functions $p(x)$ *and* $q(x)$ by power series (see *Lemma 1*) in powers of x (or of $(x-x_0)$ if solutions in powers of $(x-x_0)$ are wanted). Note that, if $p(x)$ & $q(x)$ are polynomials, then nothing needs to be done in this step.

2. Assume a solution in the form of a power series *with unknown coefficients,*

$$y = \sum_{n=0}^{\infty} a_n x^n \rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

3. Write the given differential equation as the sum of added power series.

4. Apply the identity property of the power series to get the unknown coefficients $a_0, a_1, a_2, \ldots$ *and consequently, the solution* $y = \sum_{n=0}^{\infty} a_n x^n$.

We illustrate this procedure for a very simple equation that can be solved elementary.

Example 15:

Solve: $y' - y = 0$.

Solution

Assume that the solution is $y = \sum_{n=0}^{\infty} a_n x^n$, so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Inserting these series in the given equation and collecting like powers of x, we get:

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots = 0.$$

Equating the coefficients of each power of x to zero and solving the obtained equations in which we can express $a_1, a_2, \ldots$ in terms of a_0, where a_0 remains arbitrary constant; we get:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \ldots\ldots$$

With these coefficients, one can see that, we have obtained the familiar general solution:

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) = a_0 e^x.$$

Of course we do not need the power series method for such kind of simple equations; this was just to explain the method. In fact, we need more details about the conditions and the theory of the power series method (and an extension of it – ***Frobenius*** Method) to solve many important equations of practical interest as mentioned in the introduction of this chapter. This will be the main aim of the rest of this chapter.

I.3 Essential Conditions for the Application of Series Solution Methods

The forgoing properties of power series just discussed construct the foundation of the power series method. It remains to answer the following important question:

Given $y'' + p(x)y' + q(x)y = f(x)$ what conditions guarantee that it has series solutions?

Answer

The answer is: If $p(x), q(x)$ and $f(x)$ have power series representations about $x = x_0$, then the considered equation has series solutions about the point x_0 and we can apply the ***Power Series Method*** to solve it. Otherwise, we must extend these conditions to meet the appliance requirements of the ***Frobenius Method***.

To formulate this answer in a precise way, we shall use the following important basic concepts and definitions (Sections 1.3.1 & 1.3.2), which are of great general interest.

1.3.1 Real Analytic Function

Definition 4: "Real Analytic Function"

A real function $f(x)$ is called analytic at a point $x = x_0$, if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

♦That is, $f(x)$ is ***analytic at*** $x = x_0$ if $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ such that the right-hand series converges to $f(x)$ for $|x - x_0| < R$, where $R > 0$.

♦Accordingly; all polynomials, e^{ax}, $\sin ax$ and $\cos ax$ (where a is any real constant) are ***analytic*** functions at any point of the real line. But, for example, the function $f(x) = 1/x$ is analytic at every point of the real line except at $x = 0$. Since, if $x_0 \neq 0$ then

$$\frac{1}{x} = \frac{1}{\left[x_0\left(1 + [x - x_0]/x_0\right)\right]} = x_0^{-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x - x_0}{x_0}\right)^n.$$

And this series converges to $1/x$ for $|x - x_0| < |x_0|$.

1.3.2 Classification of Solution Points: Ordinary and Regular/ Irregular Singular Points

This is the first step to be done in solving any linear differential equation with variable coefficients - the only exception is the Cauchy-Euler equation - whereby one can answer immediately the following questions:

1. Whether the given equation has series solutions about a point $x = x_0$ at all?

If it has. 2. What is the more appropriate applicable method – the Power Series Method or Frobenius Method?

If not. 3. How can we solve it?

In this section we shall answer these questions through the so-called *classification of solution points.* Henceforth, unless otherwise stated, we shall mainly consider the following homogeneous linear second order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (15).$$

Or equivalently, its *standard form:*

$$y'' + p(x)y' + q(x)y = 0 \qquad (16),$$

by dividing (15) by the leading term $a_2(x)$. And we have the following important definitions.

Definition 5: ***"Ordinary and Singular Points"***

A point $x = x_0$ is said to be an Ordinary Point of equation (15) if both $p(x)$ and $q(x)$ in the standard form (16) are analytic at the point $x = x_0$. If at least one of the functions $p(x)$ and $q(x)$ in (16) fails to be analytic at $x = x_0$, then the point $x = x_0$ is said to be a Singular point of the equation.

Ex.16:

The differential equation $y'' + (1-x)y' + e^x y = 0$ has no singular points since both $p(x) = 1 - x$ and $q(x) = e^x$ are analytic at every point of the real line.

Ex.17:

The equation $xy'' + (\sin x)y = 0$ has an ordinary point at $x = 0$ since it can be shown that $q(x) = (\sin x)/x$ possesses the power series expansion

$$q(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots,$$

which converges for all finite real values of x.

Ex.18:

The equation $(1-x^2)y'' - 2xy' + x(1-x^2)y = 0$ has two singular points at $x = \pm 1$, which are the solutions of $x^2 - 1 = 0$, since $p(x) = 2x/(1-x^2)$ is not analytic at these points.

Definition 6: ***"Regular and Irregular Singular Points"***

A singular point $x = x_0$ of equation (15) is said to be a "Regular Singular Point" if the functions

$$P(x) = (x - x_0)p(x) \text{ and } Q(x) = (x - x_0)^2 q(x)$$

are both analytic at x_0.

If at least one of the functions $P(x)$ and $Q(x)$ fails to be analytic at x_0, then the point x_0 is said to be an Irregular Singular point of the equation.

Example19: ***Find and classify the singular points of the following differential equation:***

(i) $ax^2 \dfrac{d^2y}{dx^2} + bx\dfrac{dy}{dx} + cy = 0$, where a, b & c are constants.

Solution

Dividing by x^2, we put the equation into its standard form (16) $ay'' + (b/x)y' + (c/x^2)y = 0$. That is, $p(x) = (b/x)$ and $q(x) = (c/x^2)$ which are both analytic everywhere except at $x = 0$. Hence, $x = 0$ is the only singular point of the given equation and we have to test $P(x) = xp(x) = b$ and $Q(x) = x^2 q(x) = c$ at this point. Both $P(x)$ & $Q(x)$ are analytic (polynomials) at $x = 0$, so $x = 0$ is a regular singular point of the equation.

We remind that this equation is a Cauchy-Euler equation - see Remark 1.

(ii) $(1 - x^2)y'' + y' - x(1 + x)y = 0$

Solution We have $p(x) = 1/(1 - x^2)$ and $q(x) = -x/(1 - x)$. Since $p(x)$ is not analytic at $x = \pm 1$, the equation has two singular points at $x = \pm 1$. To classify these singular points, we have to test $P(x)$ and $Q(x)$ at each singular point.

At $x = 1$: $P(x) = (x - 1)p(x) = -1/(1 + x)$ and $Q(x) = (x - 1)^2 q(x) = x(x - 1)$. $Q(x)$ is a polynomial which is analytic everywhere and it can be shown that $P(x)$ possesses the Taylor expansion:

$$P(x) = -[1/(1 + x)]_{x=1} = -\frac{1}{2}\left(1 \Big/ \left[1 + \left(\frac{x-1}{2}\right)\right]\right) = -\frac{1}{2}\left[1 - \left(\frac{x-1}{2}\right) + \left(\frac{x-1}{2}\right)^2 - \ldots\right],$$

which is convergent for $|x-1|/2<1 \rightarrow -1<x<3$. That is, $x=1$ is included in the interval of convergence $(-1,3)$, which implies that $P(x)$ is also analytic at $x=1$. Hence, the point $x=1$ is a regular singular point of the given equation. Similarly, it is easy to show that $x=-1$ is also a regular singular point.

(iii) $x^3y'' + xy' + x(2-x)y = 0$

<u>*Solution*</u> The point $x=0$ is an irregular singular point of this equation since $P(x) = xp(x) = 1/x$ is not analytic at $x=0$.

Finally, we end this section with the following essential remarks.

I.3.3 Essential Remarks and Comments

****<u>*Remark 3:*</u>**

1. In equation (15), if the coefficients $a_2(x), a_1(x)$, and $a_0(x)$ are ***polynomials*** with no common factors, then both rational functions $p(x) = a_1(x)/a_2(x)$ and $q(x) = a_0(x)/a_2(x)$ are analytic except where $a_2(x) = 0$. It follows, then, that a point $x = x_0$ is an Ordinary Point of (15) if $a_2(x)\big|_{x=x_0} \neq 0$ whereas $x = x_0$ is a Singular Point of (15) if $a_2(x_0) = 0$. Inspection of the Cauchy-Euler equation (9) shows that it has a singular point at $x = 0$.

2. Singular points need not be real numbers. The equation $(x^2+1)y'' + xy' - y = 0$ has singular points at $x = \pm i$ and all other values of x, real or complex, are ordinary points.

3. ***A differential equation has series solutions about a point $x = x_0$ if this point is either Ordinary or Regular Singular Point.***

♦That is, if x_0 is an ***Ordinary Point*** then every solution of equation (15) is analytic at $x = x_0$ and we can use the ***"Power Series Method"***, whereas if x_0 is a ***Regular Singular Point*** then an infinite series solutions of equation (15) can be obtained by applying the ***"Frobenius Method"***. Related theorems and procedures are presented in the coming sections.

♦Otherwise, if $x = x_0$ is an ***Irregular Singular Point***, then we ***may not be*** able to find a solution of equation (15) ***as an infinite series*** and we must apply one of the ***Numerical Methods – e.g. Euler's Method, Miline's Method, Runge-Kutta Methods***, ...etc. – which in turn require some initial conditions to get an approximated solution of the equation. These Numerical Methods will be presented in the last chapter titled "Numerical Methods for Ordinary DE's".

"Problems I"

1. Apply "Definition 1" to determine whether the given set of functions is linearly independent on the indicated interval.

a) $f_1(x)=1+x,\ f_2(x)=x\ \&\ f_3(x)=x^2;\ -\infty<x<\infty$. *{Ans. Independent}*

b) $f_1(x)=\cos 2x,\ f_2(x)=\sin^2 x\ \&\ f_3(x)=\cos^2 x;\ -\infty<x<\infty$. {Ans. Dep.}

c) $f_1(x)=5+\sqrt{x},\ f_2(x)=5x+\sqrt{x},\ f_3(x)=x-1\ \&\ f_4(x)=x^2;\ x\in[0,\infty)$.
{Hint: $f_2(x)=1.f_1(x)+5\,f_3(x)+0.f_4(x)$, on the interval $[0,\infty)$}

d) $f_1(x)=x\ \&\ f_2(x)=|x|;\ -\infty<x<\infty$. *{Ans. Independent}*

2. Use Theorem 2 to verify that the given functions form a fundamental set of solutions of the DE on the indicated interval, then form the general solution.

a) $y''-4y=0;\ \cosh 2x, \sinh 2x, (-\infty,\infty)$

b) $y''-y'-12y=0;\ e^{-3x}, e^{4x}, (-\infty,\infty)$ (c) $x^2y''-6xy'+12y=0;\ x^3, x^4, (0,\infty)$

3. Solve the given differential equations and write its interval of validity:

a) $3x^2y''+6xy'+y=0$ b) $x^3y'''-6y=0$

c) $x^2y''-xy'+y=\ln x$ d) $xy''-4y'=x^4$

4.* Find the general solution of $x^2y''+xy'-k^2y=0$ for any real number k^2.

5.* How would you use the method of Section I.1.3 to solve

$$(x+2)^2y''\ \cancel{(x+2)y''}+(x+2)y'+y=0\,?$$

Carry out your ideas. State an interval over which the solution is defined. {Hint: substitute $z=x+2$, then solve the new DE for $y(z)$ in powers of z, a back-substitution gives $y(x)$}.

6.* Solve the I.V.P.: $4x^2y''+y=0,\ y(-1)=2,\ y'(-1)=4$ on the interval $(-\infty,0)$.

7.* *Without solving the DE: $\left(x^2-kx\right)y''+\dfrac{2x^3}{(x-k)}y'-5y=0$, and for all values of the real number k; suggest the appropriate method (methods) to find its solution about any real point $x=x_0$. {Hint: firstly, classify the point $x=x_0$ as O.P./S.P. according to the different values of k}.*

II. Power Series Method & Frobenius Method

In the remaining material of this chapter, our goal is to find an infinite series solution about ***Ordinary/Regular Singular Points*** with the use of ***Power-Series/Frobenius Method*** for differential equations of type (15) in which the coefficients, unless otherwise stated, are polynomials. Significantly, we shall be concerned with the appreciable techniques of these two methods; therefore the basic used theorems will be stated without proof.

II.1 Series Solutions about an Ordinary Point: "The Power Series Method"

Theorem 5: ***"Existence of Power Series Solutions"***

If $x = x_0$ is an Ordinary Point of the differential equation (15), then we can always find two linearly independent solutions y_1, y_2 in the form of a power series centered at x_0 - that is, the general solution of (15) is :

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0y_1+a_1y_2 \qquad (17),$$

where a_0 and a_1 are two arbitrary constants.

The series solution (17) converges at least on some interval defined by $|x-x_0|<R$, where R is the distance from x_0 to the closest singular point. ■

♦A solution of the form $y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$ is said to be a "***Solution about the Ordinary Point x_0***", or equivalently "***Series Solution in powers of $(x-x_0)$***".

♦♦♦

♦**The Procedure:**

To find the series solution (17), we simply substitute series (17) together with its derivatives into equation (15); that is:

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n \to y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1} \to y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2} \quad (18).$$

Then we collect like powers of x and equate the sum of the coefficients of each occurring power of x to zero. This gives relations from which we can determine the unknown coefficients in (17) successively and consequently the required pre-assumed series solution. Of course, it is more practical to try to deduce some sort of a ***recurrence relation*** for a_n in terms of a_m where $m<n$. Now, we shall illustrate this procedure by means of the following selected examples which are of special practical importance.

II.1.1 Airy's Equation

The differential equation $y''+xy=0$ is called ***Airy's Equation*** and is encountered in the study of diffraction of light, diffraction of radio waves around the earth, aerodynamics, and the deflection of a uniform thin vertical column that bends under its own weight. Other common forms of ***Airy's Equation*** are $y''-xy=0$ and $y''+\alpha^2 xy=0$.

Example20: "Airy's Equation"

Find the series solution about $x=0$ of Airy's Equation $y''+xy=0$, and write its interval of validity.

Solution:

1. Classification of the solution point $x=0$:

Comparing the given DE with the standard form $y''+p(x)y'+q(x)y=0$, both $p(x)=0$ and $q(x)=x$ are analytic everywhere (polynomials). So, $x=0$ is an ***Ordinary Point*** and we can apply the ***Power Series Method - Theorem 5.***

2. The Series Solution in Powers of x:

♦ Since $x=0$ is an ordinary point and the equation has no finite singular points, ***Theorem 5*** guarantees that there exist two independent power series solutions y_1 & y_2 centered at 0 and a general solution

$$y=\sum_{n=0}^{\infty}a_n x^n=a_0y_1+a_1y_2\text{, convergent for } |x|<\infty.$$

That is, the radius of convergence of the series solution of Airy's equation is ∞, and hence this solution will be valid for $-\infty<x<\infty$.

♦ Substituting $y=\sum_{n=0}^{\infty}a_n x^n$ and $y''=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$ into the given DE gives:

$$y''+xy=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+x\sum_{n=0}^{\infty}a_n x^n=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty}a_n x^{n+1}. \quad (19)$$

In *Example 14* we already add the last two series on the right-hand side of equality (19) by shifting the summation index. From the result given in (12),

$$y'' + xy = 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right]x^n = 0 \qquad (20).$$

At this point we invoke the ***identity property*** of the power series (see Sec.I.2.1). Since (20) is identically zero, it is necessary that the coefficients of each power of x be set equal to zero. That is,

$$2a_2 = 0 \rightarrow a_2 = 0 \text{ and } (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$\rightarrow a_{n+2} = -a_{n-1}/\left[(n+1)(n+2)\right],\ n = 1,2,3,\ldots \qquad (21).$$

Equation (21) is called the ***"Recurrence Relation"***.
In (21), since a_{n+2} is given in terms of a_{n-1}, so it is obvious that the required coefficients will be determined in steps of three. That is, a_0 determines a_3 which in turn determines a_6,..; a_1 determines a_4 which in turn determines a_7,.. and a_2 determines a_5 which in turn determines a_8,.... etc. Setting $n = 1,2,3,\ldots$ in the recurrence relation (21) we get:

a_0	a_1	$a_2 = 0$
$a_3 = \dfrac{-a_0}{3.2}$	$a_4 = \dfrac{-a_1}{4.3}$	$a_5 = \dfrac{-a_2}{5.4} = 0$
$a_6 = \dfrac{-a_3}{6.5} = \dfrac{a_0}{(6.5)(3.2)}$	$a_7 = \dfrac{-a_4}{7.6} = \dfrac{a_1}{(7.6)(4.3)}$	$a_8 = \dfrac{-a_5}{8.7} = 0$
$a_9 = \dfrac{-a_6}{9.8} = \dfrac{-a_0}{(9.8)(6.5)(3.2)}$	$a_{10} = \dfrac{-a_7}{10.9} = \dfrac{-a_1}{(10.9)(7.6)(4.3)}$	$a_{11} = 0$
....	....	....
....	....	....
$a_{3n} = \dfrac{(-1)^n a_0}{(3n)(3n-1)(3n-3)(3n-4)\ldots.6.5.3.2}$	$a_{3n+1} = \dfrac{(-1)^n a_1}{(3n+1)(3n)(3n-2)(3n-3)\ldots.7.6.4.3}$	$a_{3n+2} = 0$

Hence the required general series solution of ***Airy's Equation*** is:

$$y = a_0\left(1-\frac{x^3}{3.2}+\frac{x^6}{6.5.3.2}-....\right)+a_1\left(x-\frac{x^4}{4.3}+\frac{x^7}{7.6.4.3}-....\right)=a_0y_1(x)+a_1y_2(x),$$

where

$$y_1(x)=1-\frac{x^3}{3.2}+\frac{x^6}{6.5.3.2}-......=1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{(3n)(3n-1)(3n-3)(3n-4)....6.5.3.2},$$

&

$$y_2(x)=x-\frac{x^4}{4.3}+\frac{x^7}{7.6.4.3}-....=x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{3n+1}}{(3n+1)(3n)(3n-2)(3n-3)....7.6.4.3}$$

are the two linearly independent solutions of *Airy's Equation*. And since the recursive use of (21) leaves a_0 & a_1 completely undetermined, they can be chosen arbitrarily, where $a_0 = y(0)$ and $a_1 = y'(0)$.

<u>***Note***</u> Although we know from ***Theorem 5*** that each series solution (y_1 & y_2) converges for $|x|<\infty$, this fact can be also verified by the ratio test – see ***Ex.13***.

II.1.2 Legendre's Equation

We now consider one of the most important differential equations, namely ***Legendre's Equation:***

$$(1-x^2)y''-2xy'+k(k+1)y=0 \tag{22}$$

It arises in numerous applied problems, mostly in those exhibiting spherical symmetry.

Solution of "Legendre's Equation" for all values of k – General Case:

Comparing (22) with the general form (15):

$$a_2(x)y''+a_1(x)y'+a_0(x)y=0.$$

We see that the coefficients of ***Legendre's Equation*** are analytic at $x=0$ (they are very special polynomials) and $a_2(x)=1-x^2\neq 0$ at $x=0$. Hence, $x=0$ is an ***Ordinary Point (see Remark 3)*** and we can apply the ***Power Series Method***.

♦ By ***Theorem 5***, every solution of (22) is some power series $y=\sum_{m=0}^{\infty}a_m x^m$ centered at 0 and the general solution is

$$y=\sum_{m=0}^{\infty}a_m x^m = a_0 y_1 + a_1 y_2 \text{ , convergent only for } |x|<1.$$

That is; since $x=\pm 1$ are the singular points of Legendre's equation, so the radius of convergence of its series solution is $|0-1|=1$, and hence this solution will be valid only on the interval $|x|<1$.

♦ Substitute: $y=\sum_{m=0}^{\infty}a_m x^m$, $y'=\sum_{m=0}^{\infty}m a_m x^{m-1}$ & $y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$ into (22), we get:

$$\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m-1)a_m x^m-\sum_{m=1}^{\infty}2m a_m x^m+\sum_{m=0}^{\infty}k(k+1)a_m x^m=0.$$

♦ Shift the summation index of the first series by 2 and combine series to get:

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m-\sum_{m=2}^{\infty}m(m-1)a_m x^m-\sum_{m=1}^{\infty}2m a_m x^m+\sum_{m=0}^{\infty}k(k+1)a_m x^m=0\rightarrow$$

$$[k(k+1)a_0+2a_2]+[(k-1)(k+2)a_1+6a_3]x+$$

$$\sum_{m=2}^{\infty}[(m+2)(m+1)a_{m+2}+(m-k)(m+k+1)a_m]x^m=0$$

Since this equality must be an identity in x if $y=\sum_{m=0}^{\infty}a_m x^m$ is to be a solution of (22), the sum of the coefficients of each power of x must be zero, this gives:

$$a_2=-k(k+1)a_0/2!\text{ , }a_2=-k(k+1)a_0/2!$$

&

$$a_{m+2}=-\frac{(k-m)(m+k+1)}{(m+2)(m+1)}a_m,\ m\geq 2.$$

In fact, the last relation is valid for all $m \geq 0$ as the reader may readily verify, that is, we have the following ***Recurrence Relation***:

$$a_{m+2} = -\frac{(k-m)(m+k+1)}{(m+2)(m+1)}a_m, \; m = 0,1,2,3,..$$

This relation gives each coefficient in terms of the second one preceding it with step two, except for a_0 and a_1, which are left as arbitrary constants. We find successively:

a_0	a_1
$a_2 = -\frac{k(k+1)a_0}{2.1}$	$a_3 = -\frac{(k-1)(k+2)}{3.2}a_1$
$a_4 = -\frac{(k-2)(k+3)}{4.3}a_2$	$a_5 = -\frac{(k-3)(k+4)}{5.4}a_3$
$= \frac{k(k-2)(k+1)(k+3)}{4!}a_0$	$= \frac{(k-3)(k-1)(k+2)(k+4)}{5!}a_1$
....	....

Thus, we have proved the following lemma.

Lemma 2: "Solution of Legendre's Equation in Ascending Powers of x"

The general solution of Legendre's Equation

$$(1-x^2)y'' - 2xy' + k(k+1)y = 0,$$

for all values of k – real or complex numbers, on the interval $-1 < x < 1$ is given by:

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \qquad (22\text{-}1),$$

where a_0 & a_1 are two arbitrary constants and

$$y_1(x) = 1 - \frac{k(k+1)}{2!}x^2 + \frac{(k-2)k(k+1)(k+3)}{4!}x^4 - ..+.. \qquad (22\text{-}1\text{-}a),$$

&

$$y_2(x) = x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-3)(k-1)(k+2)(k+4)}{5!}x^5 - ..+.. (22\text{-}1\text{-}b),$$

are two linearly independent power series solutions of (22).

Solution of "Legendre's Equation" for non-negative integer value of k (k=n) - "Case of Practical Appliance":

Evidently, in most applications the parameter k in "*Legendre's Equation*" is a non-negative integer number, moreover only one particular solution of Legendre's equation is needed. That is, the most popular applicable form of "*Legendre's Equation*" is:

$$\boxed{(1-x^2)y''-2xy'+n(n+1)y=0 \quad , n=0,1,2,3,... \qquad (23)}$$

And we have the following important remark.

Remark 4:

1. ***In Lemma 2:***
   If k=n, where n is a nonnegative integer, then one of the two infinite series solutions y_1 or y_2 (not both - depending on whether n is an even or odd positive integer) reduces to a polynomial (i.e. terminates) and the other becomes a polynomial multiplied by $\ln[(1+x)/(1-x)]$, as can be shown.

2. ***The polynomial solution, with a_0 or a_1 so chosen that the value of the polynomial becomes 1 at $x=1$, is called "Legendre Polynomial of Order n" - denoted by*** $P_n(x)$. The non-terminating (infinite series) solution with a_0 or a_1 suitably adjusted is called "***Legendre Function of Second Kind***" - denoted by $Q_n(x)$. That is, ***the general solution of "Legendre's Equation" (23) on the interval $-1<x<1$ is given by:***

$$\boxed{y(x)=c_1P_n(x)+c_2Q_n(x) \qquad (23\text{-}1)}$$

Example 21 - Demo:

Find the solution of Legendre's Equation

$$(1-x^2)y''-2xy'+n(n+1)y=0\,,$$

for the following values of n:

(i) $n=0$.

(ii) $n=1$.

(iii) $n=2$.

Solution By using *Lemma 2*, we have:

(i) **When $n=0$**, then (22-1-a) gives

$y_1(x)=1$, the simplest ***"Legendre Polynomial" denoted by $P_0(x)$***.

And (22-1-b) gives:

$$y_2(x)=x+\frac{2}{3!}x^3+\frac{(-3)(-1)2.4}{5!}x^5+...=x+\frac{x^3}{3}+\frac{x^5}{5}+...=\frac{1}{2}\ln\frac{1+x}{1-x}.$$

The student may verify this result by solving ***Legendre's Equation:*** $(1-x^2)u'-2xu=0$, where $u=y'$; by separating variables.

(ii) **When $n=1$**, then (22-1-a) gives

$y_2(x)=x$, the next ***"Legendre Polynomial" denoted by $P_1(x)=x$***.

And (22-1-b) gives:

$$y_1(x)=1-\frac{x^2}{1}-\frac{x^4}{3}-\frac{x^6}{5}-...=1-x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+..\right)=1-\frac{1}{2}x\ln\frac{1+x}{1-x}.$$

(iii) **When $n=2$**, then (22-1-a) gives

$$y_1(x)=1-\frac{2.3}{2!}x^2=1-3x^2,$$

which is a polynomial ***but not*** a ***"Legendre Polynomial"*** since $\left.(1-3x^2)\right|_{x=1}=-2\neq 1$, (see *Remark 4*).

It is obvious that: *A constant multiple of a solution of Legendre's Equation is also a solution, hence it is traditional to choose specific values for a_0 or a_1* (depending on the value of n). That is, setting $a_0=-c_0/2$ we get the ***"second Legendre Polynomial" denoted by***

$$P_2(x)=\frac{1}{2}(3x^2-1).$$

And (22-1-b) gives the second particular solution $y_2(x)=....$, which is an infinite series solution converges for $|x|<1$.

We end this section with the following result which is not difficult to prove.

Corollary 1: ***"Particular Solution of Legendre's Equation in Descending Powers of x - Case of Practical Appliance"***

Any Legendre's Equation:

$$(1-x^2)y''-2xy'+n(n+1)y=0,$$

with a non-negative integer 'n' has a basic particular solution $P_n(x)$ on the interval $-1<x<1$ given by:

$$P_n(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m},\ M=\begin{cases}n/2 & ,\text{if } n \text{ even}\\ (n-1)/2, & \text{if } n \text{ odd}\end{cases} \quad (23\text{-}2)$$

♦The principle value of ***Legendre Polynomials*** is not only in solving Legendre's Equation but also that they belong to the so-called ***"Special Functions"*** which are of great importance in applied mathematics.
From the practical point of view, we reject the study of the ***"Legendre Function of Second Kind-$Q_n(x)$"*** whereas the main properties and the so-called ***orthogonality*** of ***Legendre Polynomials*** will be discussed extensively in the next chapter.

II.2 Series Solutions about a Regular Singular Point

Theorem 6: ***"Frobenius Method"***

If $x = x_0$ is a Regular Singular Point of the differential equation (15), then there exists at least one solution of the form

$$y(x,s) = (x-x_0)^s \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+s} \qquad (24),$$

where the exponent s may be any (real or complex) number and s is chosen so that $a_0 \neq 0$.

That is, substituting series (24) together with its derivatives into equation (15) and then equating the coefficient of the lowest power of x (for $n=0$) to zero, we obtain a quadratic polynomial of the form $(As^2+Bs+c)a_0 = 0,\ a_0 \neq 0$. This gives the so-called the *Indicial Equation*:

$$As^2 + Bs + c = 0,$$

it has two roots s_1 and s_2 which are called *the exponents of the singularity* at $x = x_0$.

Let s_1 and s_2 be the roots of the indicial equation such that $s_1 \geq s_2$, then the general solution of (15) is :

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x)} \qquad (25),$$

where c_1 & c_2 are two arbitrary constants, and y_1 is the first solution given by:

$$y_1(x) = y(x,s)\big|_{s=s_1} = \sum_{n=0}^{\infty} a_n(s_1)(x-x_0)^{n+s_1} \qquad (25\text{-}1).$$

$y_2(x)$ is the second solution (such that y_1 & y_2 are linearly independent) that may be similar to y_1 (with $s = s_2$ and different coefficients) or may contain a logarithmic term. The form of $y_2(x)$ depends on the nature of the roots s_1 & s_2 and we have the following three cases:

Case I: *"Distinct roots not differing by an integer"* - $s_1 \neq s_2, s_1 - s_2 \neq$ *integer*

$$y_1(x) = y(x,s)\big|_{s=s_1} = \sum_{n=0}^{\infty} b_n(s_1)(x-x_0)^{n+s_1} \quad (25\text{-}2\text{-}I).$$

Case II: *"Double root"* - $s_1 = s_2 = s^*$

$$y_2(x) = \frac{\partial y(x,s)}{\partial s}\bigg|_{s=s^*} = y_1(x)\ln(x-x_0) + \sum_{n=1}^{\infty}\left[\frac{d}{ds}a_n(s)\bigg|_{s=s^*}\right](x-x_0)^{n+s^*} \quad (25\text{-}2\text{-}II).$$

Case III: *"Roots differing by an integer"* - $s_1 \neq s_2, s_1 - s_2 =$ *integer*

In this case there exist three possibilities/cases for $y_2(x)$:

Case III-a: *The second exponent s_2 makes some of the coefficients of $y(x,s)$ infinite, i.e. one of the a's equal to ∞. We define a new function:*

$$\bar{y}(x,s) = (s-s_2)y(x,s),$$

then y_2 is given by:

$$y_2(x) = \frac{\partial \bar{y}(x,s)}{\partial s}\bigg|_{s=s_2} \quad (25\text{-}2\text{-}III\text{-}a),$$

which will contain a logarithmic term.

Case III-b: *The second exponent s_2 makes some of the coefficients of $y(x,s)$ indeterminate, e.g. one of the a's equal to 0/0. We consider this coefficient arbitrary, so the series solution $y(x,s)\big|_{s=s_2}$ will contain two arbitrary constants. Hence, the required general solution of (15) is given by:*

$$\boxed{y_{g.s.}(x) = y(x,s) \equiv y_2(x) = y(x,s_2)} \quad (25\text{-}2\text{-}III\text{-}b).$$

In this case, there is no need to find $y_1(x) = y(x,s)\big|_{s=s_1}$ - the only exception.

Case III-c: *Neither (III-a) nor (III-b) is happen. (case similar to Case I)*

$$y_2(x) = y(x,s)\big|_{s=s_2} = \sum_{n=0}^{\infty} b_n(s_2)(x-x_0)^{n+s_2} \quad (25\text{-}2\text{-}III\text{-}c).$$

Lastly, *the series (25) will converge at least on some interval $0<(x-x_0)<R$, where R is the distance from x_0 to the nearest singular point.*

Now, to obtain skill in handling the ***method of Frobenius*** practically, we shall demonstrate this "big" theorem suggested by the German mathematician *F.G. Frobenius* (1849-1917) in 1873 by means of the following examples.

Example 22: "Case I -Distinct roots not differing by an integer"

Find the series solution in powers of x of the differential equation:

$$2x^2y''+3xy'-(x^2+1)y=0\,,$$

and write its interval of validity.

Solution:

1. Classification of the solution point:

Since we want to find series solution in powers of x, we must first determine whether the point $x=0$ is an ***ordinary point***, a **regular singular point**, or an ***irregular singular point***. Comparing the given DE with the general form (15):

$$a_2(x)y''+a_1(x)y'+a_0(x)y=0\,.$$

It is clear that, the coefficients are analytic everywhere (polynomials) and since $a_2(x)=2x^2=0$ at $x=0$, therefore, $x=0$ is a ***Singular Point (see Remark 3)***. To classify this singularity, we have to study the analyticity of the functions:

$$P(x)=xp(x)=x(3x/2x^2)=3/2$$

&

$$Q(x)=x^2q(x)=x^2\left[-(x^2+1)/2x^2\right]=-(x^2+1)/2\,.$$

Both $P(x)$ and $Q(x)$ are analytic everywhere (polynomials), hence $x=0$ is a ***Regular Singular Point*** of the given DE and we can use ***Frobenius Method - Theorem 6***.

2. The Series Solution in Powers of x:

♦ Since $x=0$ is a regular singular point and the equation has no other singular points, then ***Theorem 6*** guarantees that there exists ***at least one solution*** of the form (24):

$$y(x,s)=\sum_{n=0}^{\infty}a_nx^{n+s}\,,$$

centered at $x=0$ and a general solution:

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

which is convergent for $0<x<\infty$. That is, the radius of convergence R of the series solution of the given DE is ∞, and hence this solution will be valid, at least, for $0<x<\infty$.

♦ Substituting with the series expressions:

$$y=\sum_{n=0}^{\infty}a_nx^{n+s},\ y'=\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1}\ \&\ y''=\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2}$$

into the given differential equation, we get after some simplifications that

$$\sum_{n=0}^{\infty}[2(n+s)(n+s-1)+3(n+s)-1]a_nx^{n+s}-\sum_{n=2}^{\infty}a_{n-2}x^{n+s}=0$$

$$\rightarrow\sum_{n=0}^{\infty}(2n+2s-1)(n+s+1)a_nx^{n+s}-\sum_{n=2}^{\infty}a_{n-2}x^{n+s}=0,$$

which gives:

$$a_0(2s-1)(s+1)x^s+(2s+1)(s+2)a_1x^{s+1}+$$

$$\left[\sum_{n=2}^{\infty}(2n+2s-1)(n+s+1)a_n-a_{n-2}\right]x^{n+s}=0 \qquad (26)$$

♦***The Indicial Equation:***

Equating the coefficient of the lowest power of x (for $n=0$), i.e. x^s, to zero:

$$a_0(2s-1)(s+1)=0,\ a_0\neq0$$

which is an arbitrary constant. So, the *Indicial Equation* is:

$$(2s-1)(s+1)=0\rightarrow s_{1,2}=1/2,-1$$

That is, the exponents of the singularity at $x=0$ are $s_1=1/2$, $s_2=-1$ (such that $s_1\geq s_2$). Since $s_1-s_2=3/2\neq$ *integer number*, then, according to *Theorem 6*, the series solution about $x=0$ for the given DE is classified as *Case I*. That is the first solution $y_1(x)$ is given by (25-1) and the second solution $y_2(x)$ is of the form (25-2-I).

♦ **The Recurrence Relation:**

Since (26) is identically zero, it is necessary that the coefficients of each power of x be set equal to zero (the ***identity property*** of the power series ,see Sec.I.2.1). That is, $a_1 = 0$ and the ***Recurrence Relation*** is given by:

$$a_n(s) = \frac{a_{n-2}(s)}{(2n+2s-1)(n+s+1)}, n \geq 2 \qquad (27)$$

Since $a_n(s)$ is given in terms of $a_{n-2}(s)$, so the required coefficients will be determined in steps of two. That is, a_0 (arbitrary constant) determines a_2 which in turn determines a_4, etc. and a_1 determines $a_3, a_5, \ldots$.

Since $a_1(s) = 0$, then

$$a_3 = a_5 = a_7 = \ldots = a_{2m+1}(s) = \ldots = 0, \ m \geq 0 \quad (s = s_1, s = s_2).$$

♦ ***The first solution*** $y_1(x)$:

Substituting $s = s_1 = 1/2$ into (27), then the recurrence relation becomes:

$$a_n(\tfrac{1}{2}) = a_{n-2}(\tfrac{1}{2})/[2n(n+\tfrac{3}{2})] = a_{n-2}(\tfrac{1}{2})/[n(2n+3)], n \geq 2.$$

Setting $n = 2,4,6,..$ in this formula, we get

$$a_2 = \frac{a_0}{2.7}, \ a_4 = \frac{a_2}{4.11} = \frac{a_0}{2.4.7.11}, \ a_6 = \frac{a_0}{2.4.6.7.11.15}, \ \ldots.$$

$$\rightarrow \qquad a_{2m}(\tfrac{1}{2}) = \frac{a_0}{[2.4.6\ldots(2m)][7.11.15\ldots(4m+3)]}, \ m \geq 1 .$$

Hence, the first series solution is

$$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{2.7} + \frac{x^4}{2.4.7.11} + \ldots\right) = x^{1/2}\left[1 + \sum_{m=1} \frac{x^{2m}}{2.4\ldots(2m).7.11\ldots(4m+3)}\right]$$

♦ ***The second solution*** $y_2(x)$:

Similarly, substituting $s = s_2 = -1$ into (27), then the recurrence relation becomes: $a_n(-1) = a_{n-2}(-1)/[n(2n-3)], n \geq 2$, see (25-2-I). This gives the following second linearly independent series solution

$$y_2(x) = x^{-1}\left(1+\frac{x^2}{2.1}+\frac{x^4}{2.4.1.5}+....\right) = x^{-1}\left[1+\sum_{m=1}^{\infty}\frac{x^{2m}}{2.4....(2m).1.5.1...(4m-3)}\right]$$

◆ The general solution of the given equation is: $y(x)=c_1y_1(x)+c_2y_2(x)$, where c_1 & c_2 are two arbitrary constants, which is valid for $0<x<\infty$.

Example 23: ***"Case II - Double root"***

Find the series solution of the differential equation: $x^2y''+xy'+x^2y=0$, ***and write its interval of validity.***

Solution:

1. ***Classification of the solution point:***

It is obvious that, the point $x=0$ is a ***Regular Singular Point*** (the student should verify that) of the given DE, hence we can apply ***Frobenius Method.***

2. ***The Series Solution in Powers of x:***

◆ Since $x=0$ is a regular singular point and the equation has no other singular points, then ***Theorem 6*** guarantees that there exists ***at least one solution*** of the form (24): $y(x,s)=\sum_{n=0}^{\infty}a_nx^{n+s}$ centered at $x=0$ and a general solution: $y(x)=c_1y_1(x)+c_2y_2(x)$ which is convergent for $0<x<\infty$.

◆ Substituting with the series expressions:

$$y=\sum_{n=0}^{\infty}a_nx^{n+s},\quad y'=\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1}\quad\text{and}\quad y''=\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2}$$

into the given differential equation, we get

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s}+\sum_{n=0}^{\infty}(n+s)a_nx^{n+s}+\sum_{n=0}^{\infty}a_nx^{n+s+2}=0$$

$$\rightarrow\quad s^2a_0x^s+(s+1)^2a_1x^{1+s}+\sum_{n=2}^{\infty}\left[(n+s)^2a_n+a_{n-2}\right]x^{n+s}=0 \tag{28}$$

♦The *Indicial Equation* is: $s^2=0$, so the exponents are $s_1=s_2=s^*=0$; i.e. we have the case of equal exponents – *Case II* - *Theorem 6*. Equality (28) implies that the coefficient of x^{1+s} is equal to zero if and only if a_1 is identically zero, that is we must choose $a_1=0$. The *recurrence relation* is

$$a_n(s)=-\frac{a_{n-2}(s)}{(n+s)^2}, \qquad (n\ge 2) \tag{29}$$

Since $a_1(s)=0$, then

$$a_3=a_5=a_7=\ldots=a_{2m+1}(s)=0,\ m\ge 0 \quad (\forall\, s=s^*).$$

♦<u>The first solution</u> $y_1(x)$:

Setting $s=s_1=0$ in (29), we get: $a_n(0)=-a_{n-2}(0)/n^2$, $n=2,4,6,..$

Substituting $n=2,4,6,..$ into this formula, we get

$$a_2(0)=-\frac{a_0}{2^2},\ a_4(0)=-\frac{a_2(0)}{(2.2)^2}=\frac{a_0}{(2.2)^2 2^2}=\frac{a_0}{2^4 2^2},\ a_6(0)=-\frac{a_4}{(2.3)^2}=-\frac{a_0}{2^6(3.2)^2},\ldots$$

$$\rightarrow \quad a_{2m}(0)=\frac{(-1)^m a_0}{2^{2m}(m!)^2},\ m=1,2,3,\ldots \tag{30}$$

Hence, the first series solution is

$$y_1(x)=1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}.$$

♦<u>The second solution</u> $y_2(x)$:

From *Theorem 6 – Case II* (formula 25-2-II), we have:

$$y_2(x)=\frac{\partial y(x,s)}{\partial s}\bigg|_{s=s^*}=\frac{\partial}{\partial s}\sum_{n=0}^{\infty}a_n(s)x^{n+s}\bigg|_{s=s^*}=y_1(x)\ln x+\sum_{n=1}^{\infty}\left[\frac{d}{ds}a_n(s)\bigg|_{s=s^*}\right]x^{n+s^*}.$$

Since $a_{2m+1}(s)=0,\ m\ge 0$, so we only need to compute $\frac{d}{ds}a_{2m}(s)\Big|_{s=s^*=0}$, from (29) we have:

$$a_2(s) = -\frac{a_0}{(2+s)^2},\ a_4(s) = -\frac{a_2(s)}{(4+s)^2} = \frac{a_0}{(4+s)^2(2+s)^2},\ldots\ldots$$

$$\rightarrow a_{2m}(s) = \frac{(-1)^m a_0}{(2m+s)^2(2m-2+s)^2\ldots(4+s)^2(2+s)^2},\quad m = 1,2,3,\ldots\ldots$$

◆ To compute $a'_{2m}(0)$, it is more easy to use the ***logarithmic differentiation:***

$$\frac{a'_{2m}(s)}{a_{2m}(s)} = -2\left(\frac{1}{2m+s} + \frac{1}{2m+s} + \frac{1}{2m-2+s} + \ldots \frac{1}{2+s}\right)$$

Setting $s = s^* = 0$, we obtain

$$a'_{2m}(0) = -2\left[\frac{1}{2m} + \frac{1}{2(m-1)^2} + \frac{1}{2(m-2)} + \ldots\ldots \frac{1}{2}\right] a_{2m}(0)$$

Putting

$$H_m = \frac{1}{m} + \frac{1}{m-1} + \ldots + \frac{1}{2} + 1,$$

and substituting for $a_{2m}(0)$ from (30); we get:

$$a'_{2m}(0) = -H_m \frac{(-1)^m}{2^{2m}(m!)^2} a_0, \qquad (m \ge 1)$$

Hence, the second linearly independent series solution is

$$y_2(x) = y_1(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2^{2m}(m!)^2} H_m x^{2m}$$

◆ That is, the general solution of the given equation is: $y = c_1 y_1(x) + c_2 y_2(x)$, where c_1 & c_2 are two arbitrary constants, which is valid for $0 < x < \infty$.

Example 24: "Case III -Roots differing by an integer"

Obtain a series solution in powers of x for the DE:

$$xy'' + 2y' - y = 0 \text{ ,}$$

and write its interval of validity.

Solution:

1. Classification of the solution point:

The point $x = 0$ is a ***Regular Singular Point*** (the student should verify that) of the given DE, hence we can apply ***Frobenius Method.***

2. The Series Solution around $x = 0$:

♦Since $x = 0$ is a regular singular point, then ***Theorem 6*** guarantees that there exists at least one solution of the form (24):

$y(x,s) = \sum_{n=0}^{\infty} a_n x^{n+s}$ centered at $x = 0$ and the general solution is

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ which is convergent for $0 < x < \infty$ (since there are no singular points other than $x = 0$, then $R = \infty$).

♦ Setting

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}, \; y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \text{ and } y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

in the given DE, we get

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} 2(n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\rightarrow \sum_{n=-1}^{\infty} \left[(n+s+1)(n+s+2)\right] a_{n+1} x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0.$$

Thus, we have

$$s(s+1) a_0 x^{s-1} + \sum_{n=0}^{\infty} \left[(n+s+1)(n+s+2) a_{n+1} - a_n\right] x^{n+s} = 0 \qquad (31)$$

•Equation (31) yields that:

•The *Indicial Equation* is $s(s+1)=0$, so the exponents are $s_1=0$ & $s_2=-1$.

•And the *Recurrence Relation* is given by

$$a_{n+1}(s)=a_n(s)/(n+s+1)(n+s+2), \qquad (n\geq 0) \tag{32}$$

•Since $s_1\neq s_2, s_1-s_2=integer$ – *Case III* - *Theorem 6*, so it is more better to find first the general term $a_n(s)$. Setting $n=0,1,2,...$ in (32), we get

$$a_1(s)=\frac{a_0}{(s+1)(s+2)},\ a_2(s)=\frac{a_1(s)}{(s+2)(s+3)}=\frac{a_0}{(s+1)(s+2)^2(s+3)},...$$

And in general,

$$a_n(s)=\frac{a_0}{(s+1)(s+2)^2(s+3)^2...(s+n)^2(s+n+1)},\ n\geq 1 \tag{33}$$

Since at $s=s_2=-1$, formula (33) implies that $a_n(s)$ become infinite for $n\geq 1$; that is we are in *Case III-a* of *Theorem 6*, see formulas (25-1) and (25-2-III-a).

•<u>*The first series solution*</u> $y_1(x)$:

Setting $s=s_1=0$ in (33), we get

$$a_n(0)=\frac{a_0}{1.2^2.3^2....n^2.(n+1)}=\frac{a_0}{n!(n+1)!},\ n\geq 1.$$

Substituting in (25-1) gives,

$$y_1(x)=y(x,s)\big|_{s=s_1=0}=\sum_{n=0}^{\infty}a_n(s_1)x^{n+s_1}=\sum_{n=0}^{\infty}\frac{x^n}{n!(n+1)!}$$

• ***The second solution*** $y_2(x)$:

From formula (25-2-III-a), we have:

$$y_2(x)=\left.\frac{\partial\bar{y}(x,s)}{\partial s}\right|_{s=s_2},$$

which will contain a logarithmic term, where $\bar{y}(x,s)=(s-s_2)y(x,s)$. Thus,

$$\bar{y}(x,s)=(s+1)y(x,s)=\sum_{n=0}^{\infty}b_n(s)x^{n+s},$$

where

$b_0(s)=(s+1)a_0$, $b_1(s)=\dfrac{a_0}{(s+2)}$ and

$$b_n(s)=\frac{a_0}{(s+2)^2(s+3)^2...(s+n)^2(s+n+1)},\ n\geq 2.$$

Now, we must evaluate $\left[\dfrac{d}{ds}b_n(s)\right]_{s=s_2=-1}$, we have:

$$b_0'(s)=a_0\rightarrow b_0'(-1)=a_0,\ b_1'(s)=-a_0/(s+2)^2\rightarrow b_1'(-1)=-a_0,$$

and

$$b_n'(s)=-\left(\frac{2}{s+2}+\frac{2}{s+3}+...+\frac{2}{s+n}+\frac{1}{s+n+1}\right)b_n(s)$$

$$\Rightarrow b_n'(-1)=-\left(\frac{2}{1}+\frac{2}{2}+\frac{2}{3}+...+\frac{2}{n-1}+\frac{1}{n}\right)b_n(-1)=-\frac{H_{n-1}+H_n}{n!(n-1)!}a_0,\ (n\geq 2),$$

where

$$H_n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}.$$

Note that:

$$b_0(-1)=0,\ b_1(-1)=a_0,\text{ and } b_n(-1)=\frac{1}{n!(n-1)!}a_0 \text{ for } n\geq 2.$$

- That is, the second series solution is (we set $a_0 = 1$):

$$y_2(x) = \left.\frac{\partial \bar{y}(x,s)}{\partial s}\right|_{s=s_2=-1} = \sum_{n=0}^{\infty}\left[b_n(s)x^{n+s}\ln x\right]_{s=s_2=-1} + \sum_{n=0}^{\infty}\left[b_n'(s)x^{n+s}\right]_{s=s_2=-1}$$

$$= \ln x \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!(n-1)} + \sum_{n=0}^{\infty} b_n'(-1)x^{n-1}$$

$$= y_1(x)\ln x + \frac{1}{x} - 1 - \sum_{n=2}^{\infty}\frac{H_{n-1}+H_n}{n!(n-1)!}x^{n-1}.$$

- And the general solution of the given DE is:

$$y = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 & c_2 are two arbitrary constants, which is valid for $0 < x < \infty$.

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"Problems II"

In Problems 1-14 find the series solution about $x = 0$ of the following differential equations and write its interval of validity:

1. $y'' - 2xy' - 6y = 0$
2. $xy'' - y = 0$
3. $(1-x^2)y'' - 4xy' + 4y = 0$
4. $4xy'' + 2y' + y = 0$
5. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$
6. $xy'' - xy' - y = 0$
7. $(3-x^2)y'' - xy' + 25y = 0$
8. $xy'' + y' + y = 0$
9. $(x - x^2)y'' + (1-5x)y' - 4y = 0$
10. $(x^4 - x^2)y'' + 2x^3y' + 6y = 0$
11. $(x - x^3)y'' + 3(1-x^2)y' + 8xy = 0$
12. $xy'' + (1+x)y' + 2y = 0$
13.* $(x^2+1)y'' + xy' - y = 0$ {Ans. $|x| \le 1$}
14. $x(1-x)y'' - 3xy' - y = 0$

15.* $y'' + (\cos x)y = 0,\ y(0) = 2,\ y'(0) = 6$;*{Hint: use Maclaurin series for* $\cos x$, *Sec. 1.2.2.-Ans.* $y_{g.s.}(x) = c_1\left(1 - \frac{x^2}{2} + \frac{x^4}{12} - ..\right) + c_2\left(x - \frac{x^3}{6} + \frac{x^5}{30} - ..\right)$, *valid for* $|x| \le \infty$ *}*.

16.* Discuss how the power series method can be used to solve non-homogeneous equations such as $y'' - xy = 1$ and $y'' - 4xy' - 4y = e^x$. Carry out your ideas by solving both equations.

17.* We have seen that $x = 0$ is a Regular S.P. of any Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$. Are the Indicial and Auxiliary Equations for this equation related? Discuss. By using two different methods find the general solution of the DE: $x^2y'' - 2xy' + 2y = 0$.

18.* The DE $x^4y'' + \lambda y = 0$ has an Irregular S.P. at $x = 0$. Show that the substitution $t = 1/x$ yields a new DE which has a R.S.P. at $t = 0$ and solve it in powers of t. Then, express the series solution of the original equation in terms of elementary functions.

{Ans. $y(x) = x\left[c_1 \sin(\sqrt{\lambda}/x) + c_2 \cos(\sqrt{\lambda}/x)\right]$}

19.**Given $x^2y'' - xy' + (\lambda^2x^2 + k^2)y = 0$, where $\lambda \ne 0$ and k are any real constants. Apply Frobenius Method to find the range of values of k under which the equation has a real series solution near the origin. Hence, find these solutions for $k = 0, -1\ \&\ \pm\sqrt{3/4}$.

20.**Find the series solution of $x^2y'' + xy' + (\lambda^2x^2 - k^2)y = 0$, where λ and k are any real constants, about:

(a) $x_0 = 1$, for $\lambda = 0$ and all values of k .

{Hint: substitute $z = (x - x_0)\big|_{x_0=1}$, then solve the new DE for $y(z)$ in powers of z, where $z = 0$ is an O.P.. Finally, a back-substitution gives $y(x)$ }

(b) $x_0 = 0$, for $\lambda \ne 0$ and $2k = 0, 2k = 1\ \&\ 2k$ is any non-integer real number.