FUNCTIONS OF COMPLEX VARIABLES (2)

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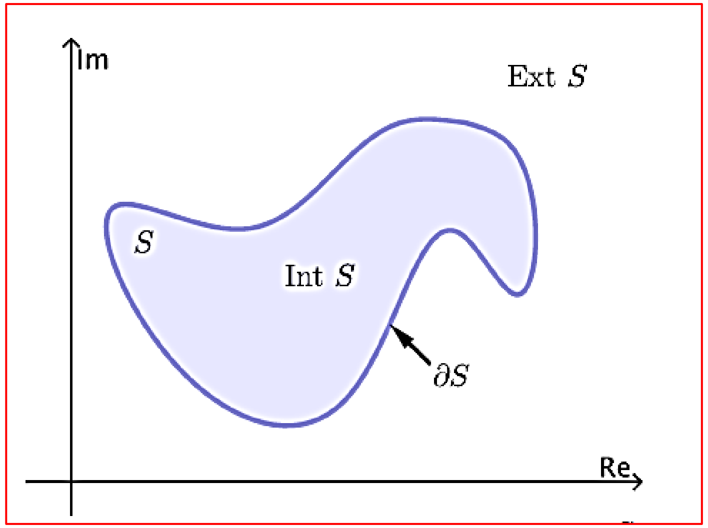
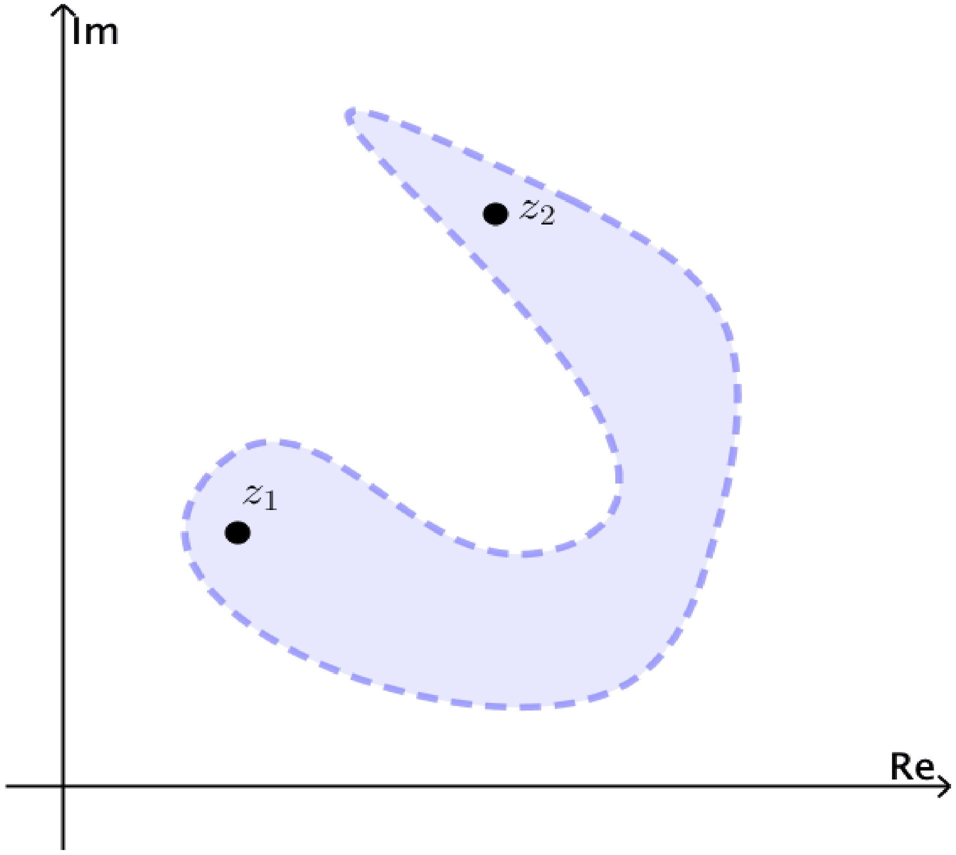
Functions of Complex Variables



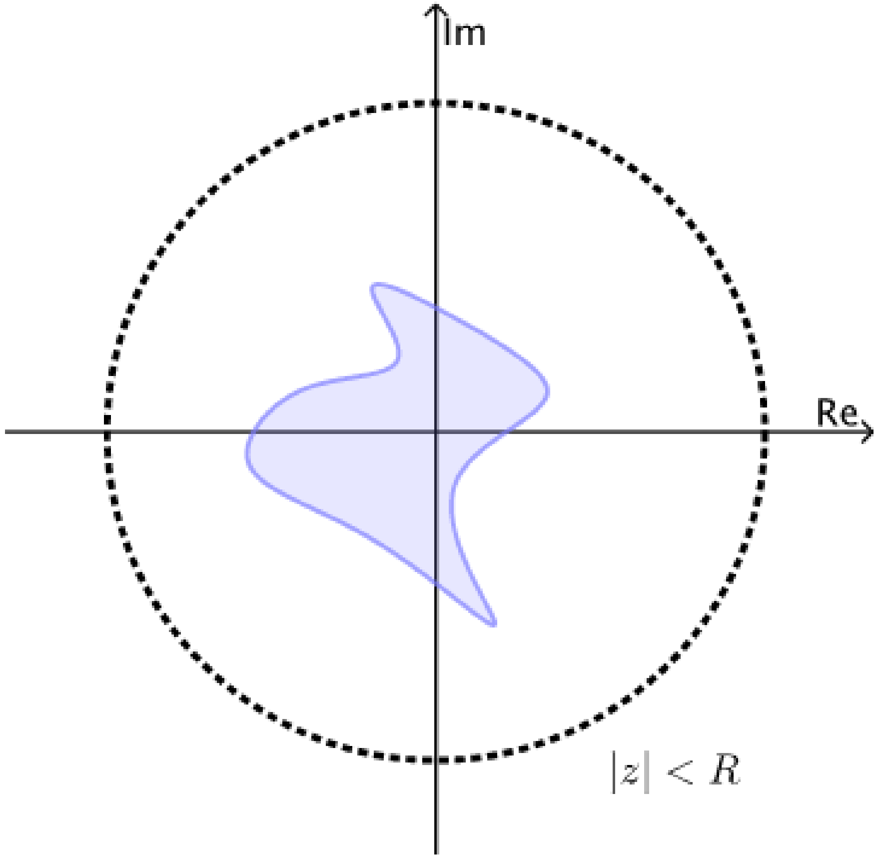
neighborhood open circular disk

open set **boundary** **closed set**

Derivatives of Functions of Complex Variables



connected set

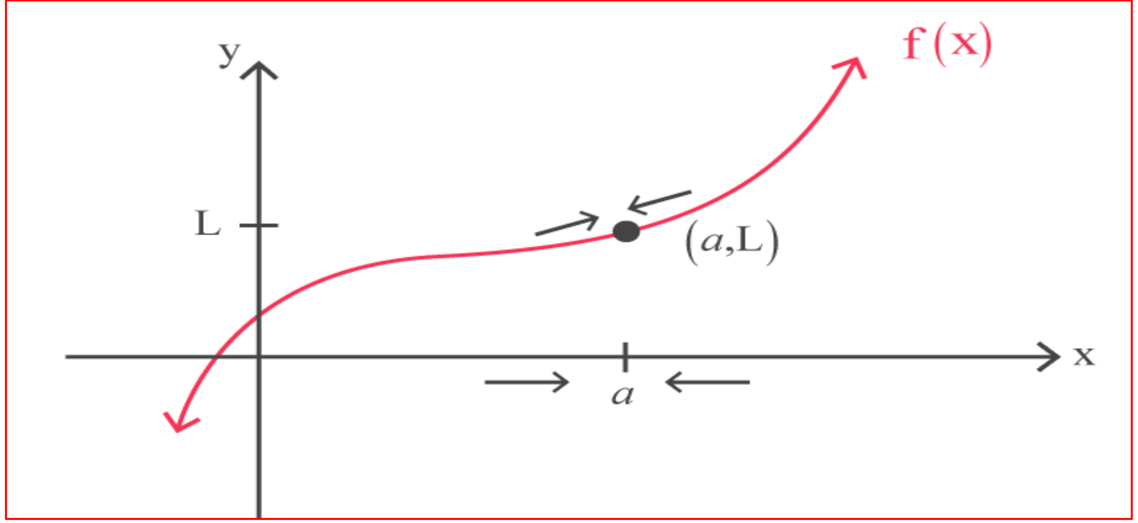


bounded set

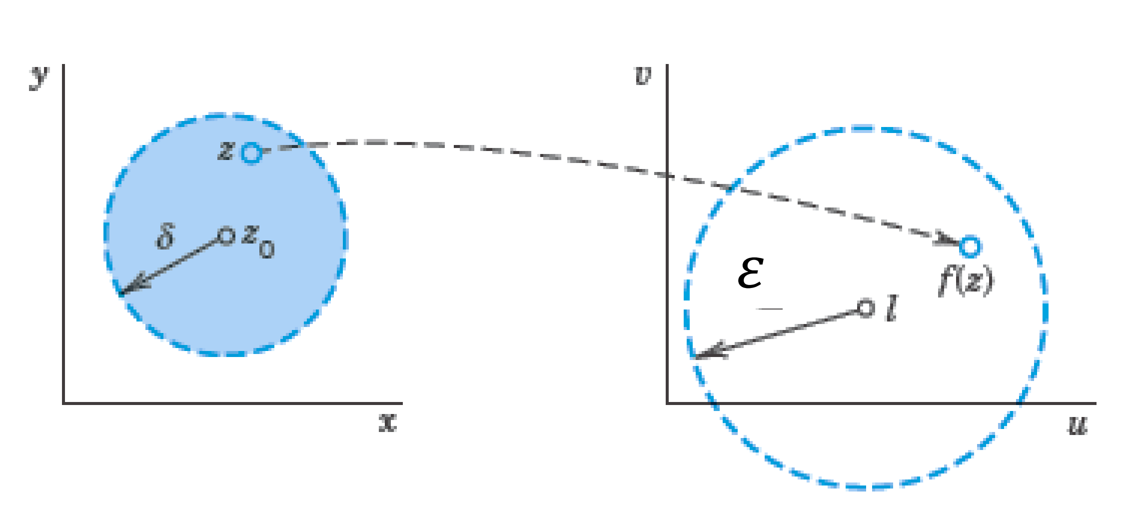
Derivatives of Functions of Complex Variables

Limits

$$\lim_{\underbrace{x\to a}_{\text{“As you approach } a \text{ along the x-axis”}}} \overbrace{f(x)}^{\text{function}} = \underbrace{L}_{\text{“What is the y-value getting closer to?”}}$$



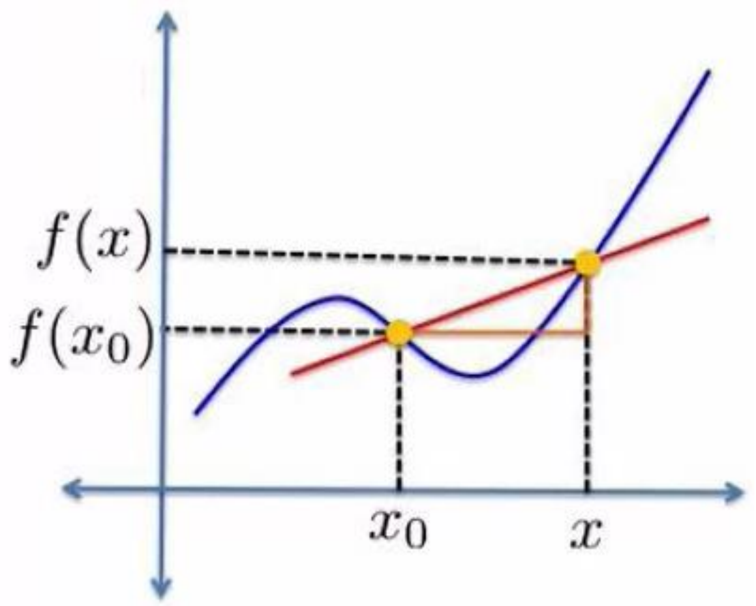
$$\lim_{z\to z_0} f(z) = l$$

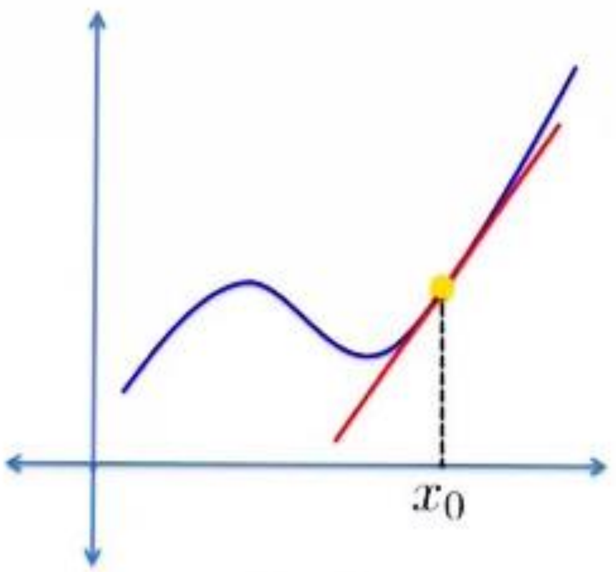


Derivatives of Functions of Complex Variables

Differentiable

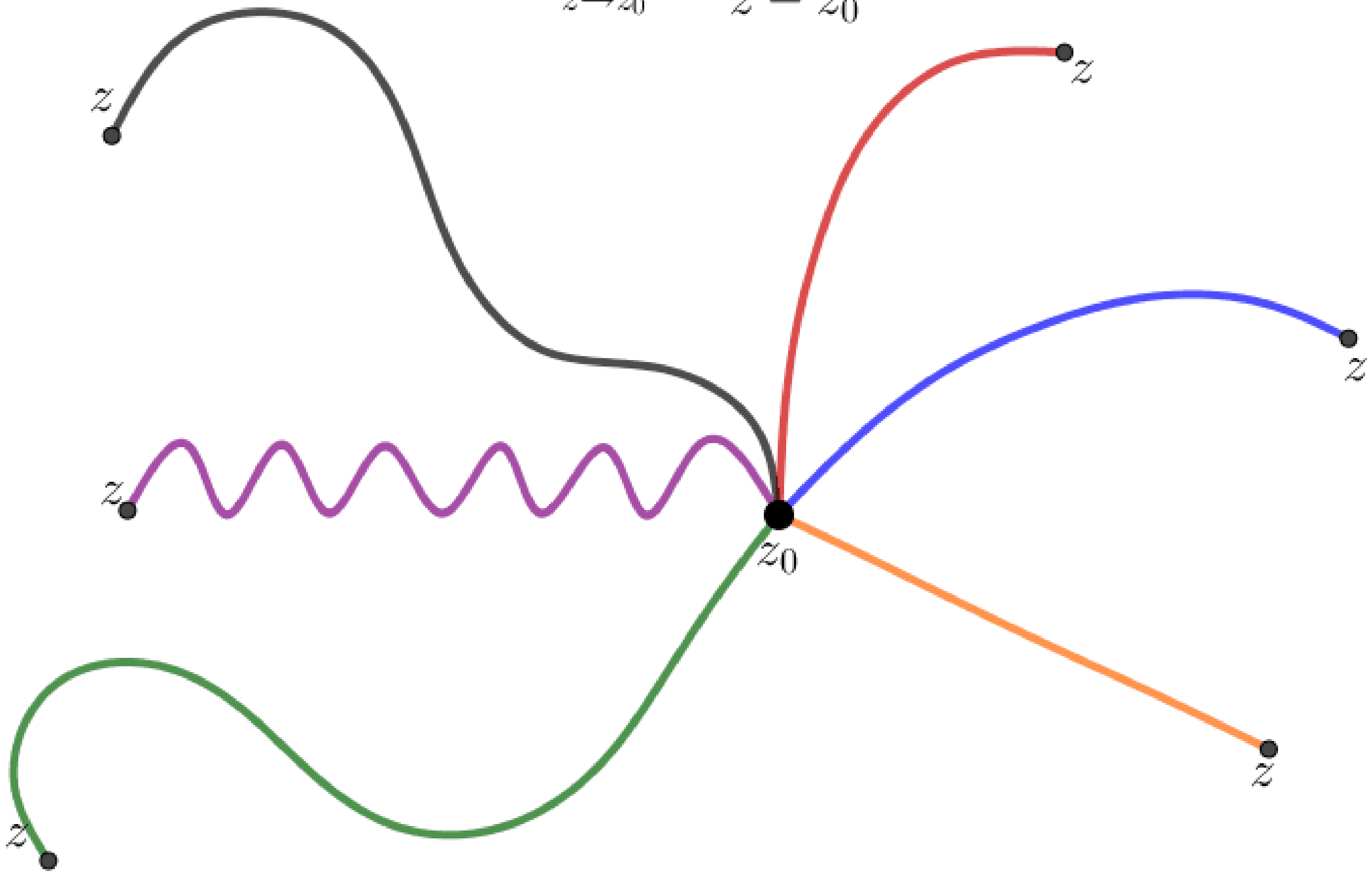
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$





Differentiable

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$



Derivatives of Functions of Complex Variables

Theorem:

The function $f(z)=u(x,y)+i\,v(x,y)$ is differentiable if and only if it satisfies Cauchy – Riemann equations $\boxed{u_x=v_y \quad \text{and} \quad u_y=-v_x}$ at which the derivative is

$$\boxed{f'(z)=u_x+iv_x=v_y-iu_y}$$

Note that the existence of the derivative thus implies the existence of the four partial derivatives

Example 1:

Show that $f(z)=z^2$ is differentiable everywhere and $f'(z)=2z$.

Solution:

$$f(z)=z^2=(x+iy)^2=(x^2-y^2)+i(2xy) \quad \Rightarrow u=(x^2-y^2) \;\; \& \;\; v=2xy$$

$$u_x=2x \;,\; v_y=2x \quad \Rightarrow u_x=v_y \quad \forall \; z$$

$$u_y=-2y \;,\; v_x=2y \quad \Rightarrow u_y=-v_x \quad \forall \; z$$

$$\boxed{\Rightarrow f(z) \text{ is differentiable everywhere.}}$$

$$\Rightarrow f'(z) = u_x + i v_x = 2x + i2y = 2(x+iy) = 2z$$

Example 2:

Show that $f(z) = \bar{z}$ is not differentiable anywhere.

Solution:

$$f(z) = x - iy \qquad \Rightarrow \quad u = x \quad \& \; v = -y$$

$$u_x = 1 \quad \& \; v_y = -1$$

$$\because \textit{It is impossible to equate } u_x \textit{ and } v_y$$

$$\Rightarrow f(z) = \bar{z} \textit{ is not differentiable anywhere.}$$

Example 3:

Show that $w = e^z$ is differentiable everywhere and $\frac{dw}{dz} = e^z$

Solution:

$$w = e^{x+iy} = e^x\ (\cos y + i \sin y)$$

$$u(x, y) = e^x \cos y \qquad v(x, y) = e^x \sin y$$

$$u_x = e^x \cos y\,, v_y = e^x \cos y \Rightarrow u_x = v_y \quad \forall z$$

$$u_y = -e^x \sin y\,, v_x = e^x \sin y \Rightarrow u_y = -v_x \quad \forall z$$

Similarly, we can find the derivatives of all the known functions

$$\frac{d}{dz}\left(z^n\right)=n\,z^{n-1}$$

$$\frac{d}{dz}\left((f(z))^n\right)=n(f(z))^{n-1}\times f'(z)$$

$$\frac{d}{dz}\left(\cos z\right)=-\sin z$$

$$\frac{d}{dz}\left(\tan z\right)=\sec^2 z$$

$$\frac{d}{dz}\left(\cot z\right)=-\csc^2 z$$

$$\frac{d}{dz}\left(\sec z\right)=\sec z\,\tan z$$

$$\frac{d}{dz}\left(\csc z\right)=-\csc z\,\cot z$$

$$\frac{d}{dz}\left(e^z\right)=e^z$$

$$\frac{d}{dz}\left(\ln z\right)=\frac{1}{z}$$

$$\frac{d}{dz}\left(f\,g\right)=f\,g'+f'\,g$$

$$\frac{d}{dz}\left(\frac{f}{g}\right)=\frac{g\,f'-f\,g'}{g^2}$$

Example 4:

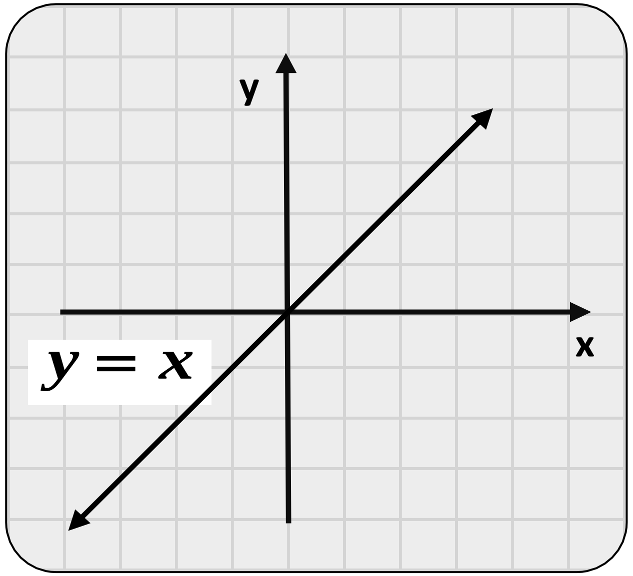
Show where the function $f(z)=(x^2+y)+i(y^2-x)$ is differentiable.

Solution:

$$u_x = 2x \;,\; v_y = 2y \qquad \text{for} \quad u_x = v_y \;\Rightarrow x = y$$

$$u_y = 1 \;,\; v_x = -1 \quad \Rightarrow u_y = -v_x \quad \forall \; z$$

This function is differentiable only on the line $y = x$



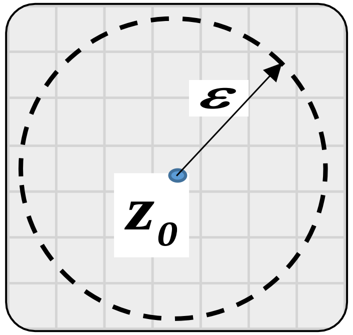
Analytic and Harmonic Functions

Definition:

A function $f(z)$ is called analytic at a point z_0 if it is differentiable at z_0 and on a neighborhood of z_0 .

Definition:

A neighborhood of a point Z_0 is the set of all points Z such that $|z - z_0| < \varepsilon \quad where \ \varepsilon > 0$



Definition:

A function is called Entire function if it is analytic everywhere and this happens if it is differentiable everywhere

We can show that the functions z^n , $\sin z$, $\cos z$, e^z are entire and their composite functions e^{z^n} , $\sin e^z$, $e^{\sin z}$, ...

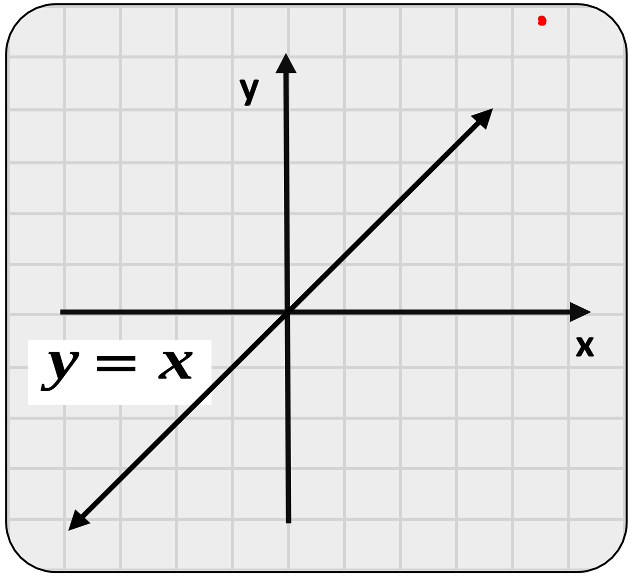
Example 5:

For the function $f(z) = (x^2 + y) + i(y^2 - x)$ which is given in example 4, state where the function is analytic.

Solution:

$$u_x = 2x \; , \; v_y = 2y \quad for \quad u_x = v_y \;\Rightarrow x = y$$

$$u_y = 1 \; , \; v_x = -1 \quad \Rightarrow u_y = -v_x \quad \forall \; z$$



This function is differentiable only on the line $y = x$

Hence, it is not analytic anywhere.

Harmonic Functions

Definition:

A function $u(x,y)$ is called harmonic on a certain domain $''D''$ if it satisfies Laplace's equation $\boxed{u_{xx} + u_{yy} = 0}$ on D

Theorem:

If $f(z) = u + iv$ is analytic on a certain domain "D" then both u and v are harmonic functions on the same domain "D" where u is called the harmonic conjugate of v and also v is called the harmonic conjugate of u.

Proof:

$$\because\ f(z)\ \textit{is analytic},\ \therefore\ \textit{it is differentiable} \Rightarrow\ u_x = v_y\ \ \&\ \ u_y = -v_x$$

$$\Rightarrow\ u_{xx} = v_{yx}\ \ \&\ \ u_{yy} = -v_{xy}\ \ \Rightarrow\ u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \qquad \therefore u\ \textit{is harmonic}$$

Similarly, we can prove that v is also harmonic.

Example 6:

Show that $u(x,y) = y^3 - 3x^2 y$ is harmonic and find its conjugate "v" hence, find the analytic function $f(z) = u + i\,v$ in terms of z.

Solution:

$$u_x = -6xy$$

$$u_{xx} = -6y$$

$$u_y = 3y^2 - 3x^2$$

$$u_{yy} = 6y$$

$$u_{xx} + u_{yy} = -6y + 6y = 0$$

$$\therefore\ u \text{ is harmonic}$$

$$u_x = v_y \quad \Rightarrow v_y = -6xy \qquad (1)$$

$$u_y = -v_x \quad \Rightarrow v_x = 3x^2 - 3y^2 \qquad (2)$$

Integrating (1) w. r. t. y $\Rightarrow v = -3xy^2 + h(x)$

Using (2) $\Rightarrow v_x = -3y^2 + h'(x) = 3x^2 - 3y^2$

$$h'(x) = 3x^2 \Rightarrow h(x) = x^3 + k$$

$$\boxed{v = x^3 - 3xy^2 + k}$$

$$\boxed{f(z) = (y^3 - 3x^2 y) + i(x^3 - 3xy^2 + k)}$$

Putting $y = 0$

$$f(x) = i(x^3 + k)$$

$$\boxed{\Rightarrow f(z) = i(z^3 + k)}$$

Differentiation in polar coordinates

Next time ☺

In polar coordinates $\boxed{f(z) = u(r,\theta) + iv(r,\theta)}$

Cauchy – Riemann equations are $\boxed{r\,u_r = v_\theta \quad \& \quad r\,v_r = -u_\theta}$

Differentiation in polar coordinates

Laplace's equation is $$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$r\, u_r = v_\theta \quad \& \quad r\, v_r = -u_\theta$$

$$f'(z) = \frac{r}{z}\left(u_r + i\, v_r\right) = \frac{1}{z}\left(v_\theta - i\, u_\theta\right)$$

Differentiation in polar coordinates

$$f'(z) = \frac{r}{z}\left(u_r + i\,v_r\right) = \frac{1}{z}\left(v_\theta - i\,u_\theta\right)$$

Example 7:

Show that $\dfrac{d}{dz}\left(\ln z\right) = \dfrac{1}{z}$

Solution:

$$\ln z = \ln(r\,e^{i\theta}) = \ln r + i\,\theta$$

$$u_r = \frac{1}{r} \quad , \quad v_\theta = 1 \quad \Rightarrow \quad r\,u_r = v_\theta$$

$$v_r = 0 \quad , \quad u_\theta = 0 \quad \Rightarrow \quad r\,v_r = -\,u_\theta$$

$\Rightarrow f(z) = \ln z$ is differentiable everywhere except at $z = 0$ or the negative real axis.

$$f'(z) = \frac{r}{z}\left(u_r + i\,v_r\right) = \frac{r}{z}\left(\frac{1}{r}\right) = \frac{1}{z}$$