بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

PHM212s:

Complex, Special Functions and Numerical Analysis

Special Functions

﴿يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖ﴾

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* Gamma Function :

$$\rightarrow \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0)$$

the integral diverges for $x \leq 0$, so in order to define the gamma function in terms of the integral then x must be > 0

for other values of x, we extend the domain by using properties of gamma function

the gamma function is a generalization of the factorial function where the gamma function is defined for any non integer values while the factorial function defined for integers only

• the Gamma function for negative values of x :

→ since the integral only converges for positive values of x, so the definition of the gamma function can't be obtained from the integral in case of negative values of x

→ we agree to extend the domain of the function for negative values by using the following relation (also used to evaluate values of $x < 1$)

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \qquad x+1 > 0$$

(Example)

→ using this definition, the gamma function of discrete negative integers is equal to ∞ or $-\infty$

$\Gamma(-n) = \pm\infty$

• Properties of gamma function : # Proofs

1. $\Gamma(1) = 1$
2. $\Gamma(1/2) = \sqrt{\pi}$
3. $\Gamma(x+1) = x\,\Gamma(x)$ (Recurrence)
4. $\Gamma(n+1) = n!$ # for any positive integer n
5. $\Gamma(x)\,\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$ (Multiplication property)

   summation of arguments must be equal to one

6. Table of values of the gamma function for $x \in [1, 2]$

   → by using property no. 3, we can determine the value of gamma function for any x through successive steps (Example)

7. Legendre's Duplication Formula :

$$\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2})$$

* Beta Function :

$$\rightarrow \beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \qquad (x>0, y>0)$$

don't forget it

$$= 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta \qquad\qquad = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du$$

(Proof) (Proof)

→ the Beta function is a symmetric function $\boxed{\beta(x,y) = \beta(y,x)}$

$$\rightarrow \beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

this relation is valid for all values of x & y

Exercise Sheet

* How to Solve?

1. Determine whether it's Gamma Function or Beta function

- key for Gamma Function is the exponential term $[e^{-t}]$
- key for Beta Function is either
  - $(1-t)$ — or any other Constants
  - $(1+u)$ in the denominator
  - the presence of trigonometric Functions

2. After determining the Function, check for the key whether it's written properly or not → Ex: $[e^{-2x^3}, (1-t^5)^4, (1+u^4)]$

- if it's not, Use integration by substitution to get the well-known form
- Check for the limits of integration to make sure that the substitution is Correct
- you might have to take a Common factor to get the well-known form

3. check for the limits of the integral $\left(\int_0^\infty, \int_0^{\frac{\pi}{2}}, \int_0^1 \right)$

4. Use the gamma function properties to evaluate

* Tricky Tips:

→ if the limits of the integration goes to $\left[\int_{-a}^{a}\right]$ you might have to use even function property $\left[2\int_0^a\right]$: must be an even function

→ if you find $[\ln x]$ in the integration, the substitution might be let $\ln x = -t$

→ if you find a Constant that's raised to a Variable power like $[3^{-x^2}]$ the substitution might be → let $3^{-x^2} = e^{-t}$

→ Some integrations direct substitution will lead to wrong limits So you will have to use other integration techniques like:

trigonometric substitution (Example)

* Remember:

→ $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(S)$ (*Laplace Summary)

→ trigonometric Substitution: (Explanation)

- For $\sqrt{a^2 - x^2}$ → let $x = a\sin\theta$
- For $\sqrt{x^2 + a^2}$ → let $x = a\tan\theta$
- For $\sqrt{x^2 - a^2}$ → let $x = a\sec\theta$

→ Double Angle:

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\tan(2\theta) = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

$\cos(2\theta) =$
- $\cos^2(\theta) - \sin^2\theta$
- $2\cos^2(\theta) - 1$
- $1 - 2\sin^2(\theta)$

Product to Sum:

$\sin(\ )\cos(\ ) = \frac{1}{2}[\sin(+) + \sin(-)]$ — sincos $= \frac{1}{2}$sin $++-$

$\cos(\ )\sin(\ ) = \frac{1}{2}[\sin(+) - \sin(-)]$ — cossin $= \frac{1}{2}$sin $+--$

$\cos(\ )\cos(\ ) = \frac{1}{2}[\cos(+) + \cos(-)]$ — coscos $= \frac{1}{2}$cos $++-$

$\sin(\ )\sin(\ ) = \frac{1}{2}[\cos(-) - \cos(+)]$ — sinsin $= \frac{1}{2}$cos $--+$