Functions of a Complex Variable

CHAPTER I

Analytic Functions

§1. Point Sets in the Complex Plane

In this section we shall give some notations and basic definitions, concerning point sets in the plane, which are of great importance and will be used in our further considerations.

As we know, the equation

$$|z - z_0| = \varepsilon$$

represents a circle K or radius ε with centre at z_0. Consequently, the inequality

$$|z - z_0| < \varepsilon \qquad \text{I.1}$$

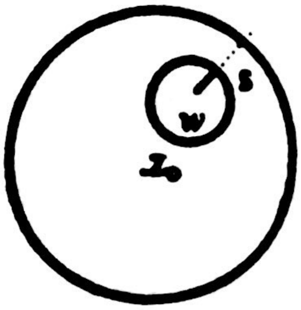
holds for any point z inside K.

<u>Definition</u>. The set of all points z which satisfy inequality I.1 is called an *open disk* or neighborhood of z_0.

It is clear that every point z_0 has infinitely many neighborhoods, each of which corresponds to a certain value of ε.

The neighborhood $|z| < 1$ of the origin is called the open unit disk.

Definition. Let E be a set of points in the z-plane. A point z of E is called an *interior point* of E if there exists a neighborhood of z, which is completely contained in E. If every point of E is an interior point of E, we say that E is an *open set*.



Lemma. Any neighborhood is an *open set*.

Proof. We must show that every point w of a neighborhood $|z - z_0| < \varepsilon$ is an interior point of this neighborhood. Let $s = \varepsilon - |w - z_0|$.

It is clear that the neighborhood of w

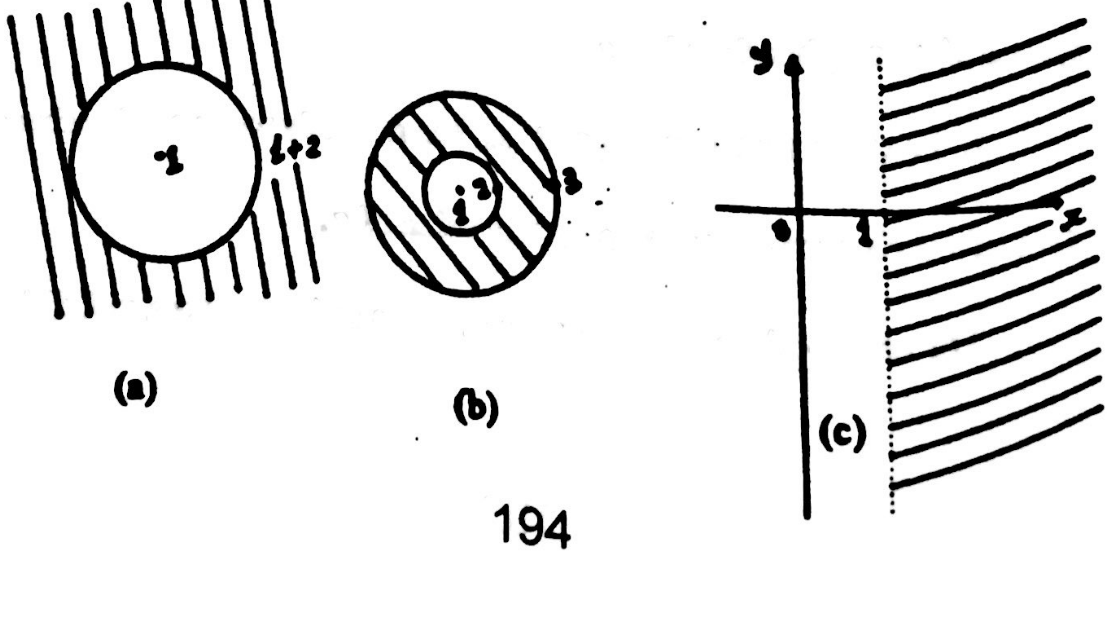
$$|z - w| < s/2$$

is completely contained in the considered neighborhood.

Each of the following inequalities describes an open set:

(a) $|z - 1| > 2$, (b) $1 < |z - 1| < 2$, (c) Re $z > 1$.

The sketches of these sets are as follows:



(a) (b) (c)

Note that the set described by the inequality

$$|z-2| \le 5$$

is not an open set since no point on the circle $|z-2|=5$ is an interior point of the considered set.

Definition. An open set E is said to be *connected* if every pair of points z_1 and z_2 in E can be joined by a polygonal line which lies entirely in E. An open connected set is called a *domain*.

In particular, the sets in (a), (b), and (c) are domains. The set of all points, for which $|z|<1$ or $|z|>2$ is open but not connected.

Definition. If each neighborhood of a point z contains points in E as well as points not in E, then z is called a *boundary point* of E. The set of all boundary points of E is called the *boundary* of E.

For example, the boundary of the set (a) is the circle $|z-1|=2$, the boundary of the set (b) is the two circles $|z-1|=1$ and $|z-1|=2$, and of (c) is the line is the line $x=1$.

Definition. A region is a domain with some, none, or all of its boundary points. If a region contains all its boundary points, we call it a *closed region*.

The region $0<|z|\le 1$ is not closed since it does not contain the boundary point O. The set $|z-z_0|\le r$ is a closed region, because it consists of the open disk $|z-z_0|<\varepsilon$ together with its boundary $|z-z_0|=\varepsilon$. This set is called a *closed disk.*

Definition. A set is called *bounded* if all its points are interior to a circle $|z|=C$ for some positive constant C. In other words, a set is bounded if it is contained in some neighborhood of the origin.

The set in (b) is bounded, while the set in (a) and (c) are unbounded.

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§2. Functions of a Complex Variable

By a function f defined on a set E of complex numbers we mean a rule which assigns to each z in E a unique complex number w. We then write

$$w=f(z).$$

The set E is called the *domain of definition of the function f*. The set of all values $f(z)$ corresponding to all z in E is called the *range* of $w=f(z)$.

For example, the domain of definition of the function $f(z)=|z|$ is the whole z-plane and its range is the non-negative half of the real axis. The domain of definition of the function

$$w=f(z)=\frac{z}{z^2+4}$$

is the entire z-plane except the two points $z = \pm 2i$ where the function is undefined.

Let u and v be the real and imaginary parts of w. Since w depends on $z = x + iy$, it is clear that in general u and v depend on x and y. Therefore, we may write

$$w = f(z) = u(x,y) + iv(x,y).$$

This shows that a complex function $f(z)$ is equivalent to two real functions $u(x,y)$ and $v(x,y)$ of the two real variables x and y.

Example 1.1. Write the function $w = f(z) = z^2 + 3iz$ in the form

$$w = u(x,y) + iv(x,y).$$

Solution. Setting $z = x+iy$, we obtain

$$w = f(z) = (x+iy)^2 + 3i(x+iy)$$

$$= x^2 - y^2 + 2xyi + 3xi - 3y$$

$$= (x^2 - y^2 - 3y) + i(2xy + 3x).$$

This is the desired form, where

$u = \text{Re } w = x^2 - y^2 - 3y$ and $v = \text{Im } w = 2xy + 3x$.

Example 1.2. Write the function $w = \left(x^2 - y^2 + 3x\right) + i(2xy - 3y)$ in the form $w = f(z)$.

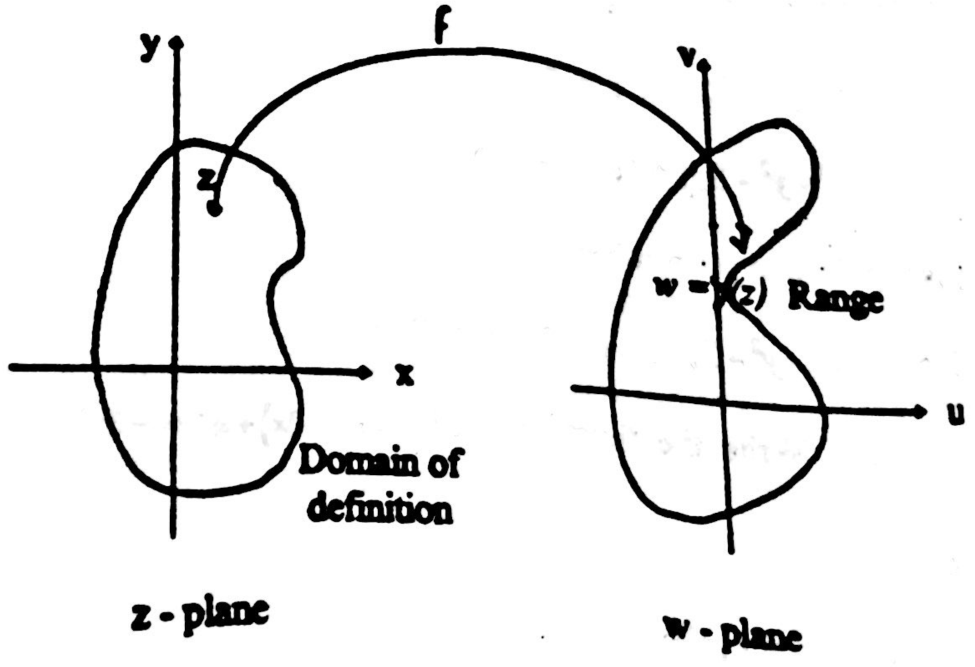
Solution. Using the relations

$$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z}),$$

we get

$$\begin{aligned} w = f(z) &= \left\{\left[\frac{1}{2}(z+\bar{z})\right]^2 - \left[\frac{1}{2i}(z-\bar{z})\right]^2 + \frac{3}{2}(z+\bar{z})\right\} + \\ &\quad i\left\{\left[(2)\frac{1}{2}(z+\bar{z})\right]\left[\frac{1}{2i}(z-\bar{z})\right] - \frac{3}{2i}(z-\bar{z})\right\} \\ &= z^2 + 3\bar{z}. \end{aligned}$$

It is impossible to draw the graph of a complex function, because it will need four dimensions. This suggests the use of two separate complex planes for the two variables z and w. We shall sketch the domain of definition of the function in the z-plane and its range in the w-plane. In this way the function $w = f(z)$ is a mapping of the domain of definition of f onto the range of f in the w-plane.



Example 1.3. Describe the function $f(z)$ in the semi - disk given by $|z| \leq 2$, $\text{Im } z \geq 0$, if

(a) $f(z) = z^2$, (b) $f(z) = z^3$.

Solution .

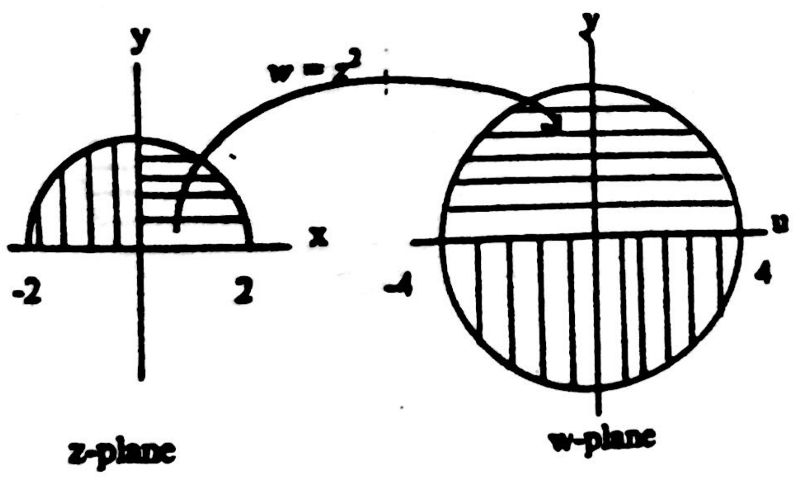
In this case the simplest procedure is the use of polar representation. We set

$z = r(\cos\theta + i \sin\theta)$ and $w = R(\cos\phi + i \sin\phi)$.

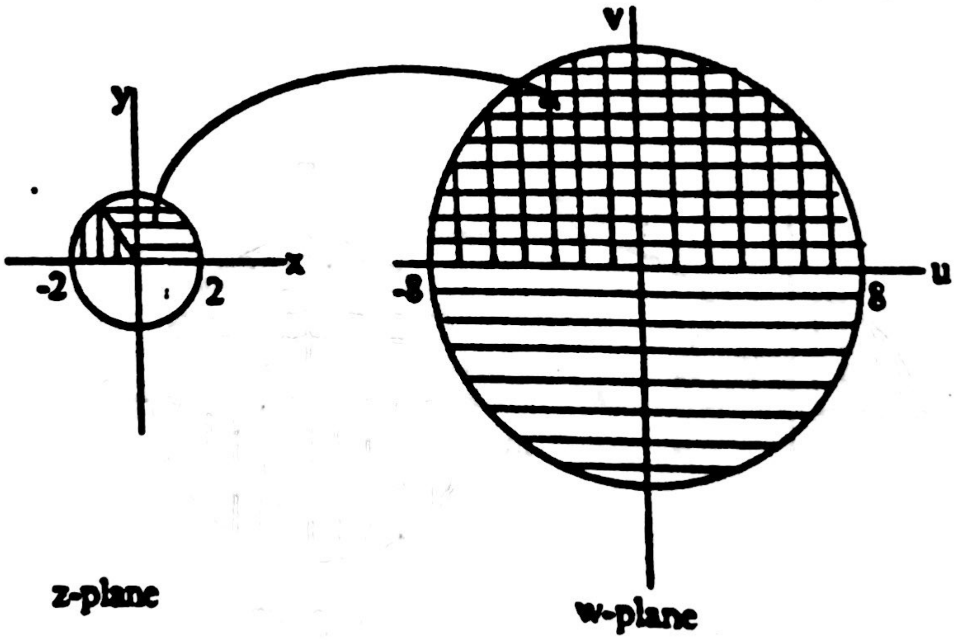
(a) It follows from $w = z^2$ that

$R = r^2$ and $\phi = 2\,\theta$.

Thus the points z in the given semidisk, when squared, cover the entire disk $|w| \leq 4$.



(b) If $w = z^3$, then $R = r^3$ and $\phi = 3\theta$. Thus, the points z in the sector of the semidisk from Arg $z = 0$ to Arg $z = 2\Pi/3$, when cubed, cover the entire disk $|w| \le 8$. The cubez of the remaining z-points also fall in this disk, overlapping it in the upper half - plane as illustrated in the figure.



z-plane

w-plane

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§3. Limits and Continuity

Definition. A function $f(z)$ is said to have the limit w_o as z approaches z_o if

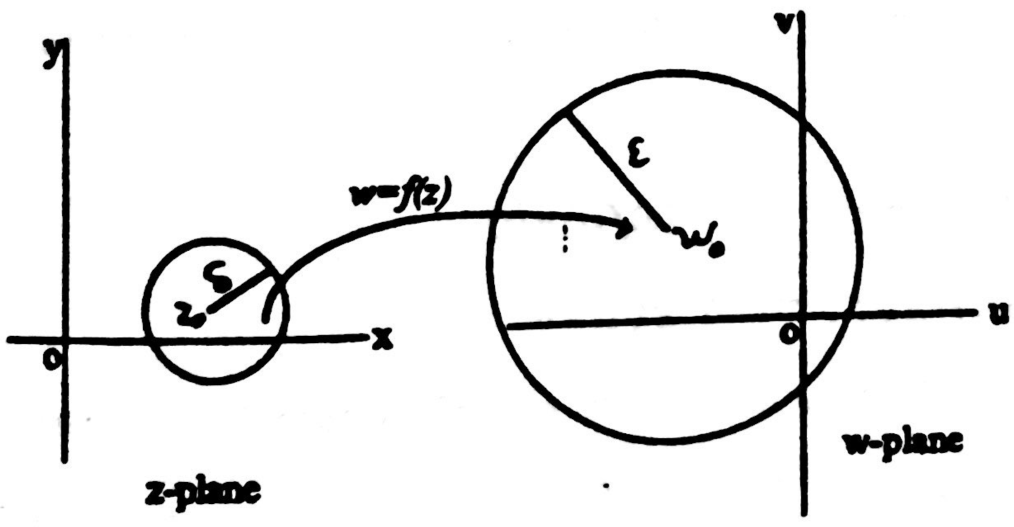
(i) $f(z)$ is defined in some neighborhood of z_o with the possible exception of the point z_o itself.

(ii) for every $\varepsilon > 0$ (no matter how small) there exists a positive number δ such that:

$$|f(z) - w_o| < \varepsilon \text{ whenever } 0 < |z - z_o| < \delta.$$

Geometrically, this definition requires that for every $\varepsilon > 0$, the neighborhood $|w - w_o| < \varepsilon$ contains all the values $f(z)$ for z in the full neighborhood $|z - z_o| < \delta$ for some $\delta > 0$, except possibly the value $f(z_o)$.

In this case we write $\lim_{z \to z_o} f(z) = w_o$.



Since z may be any point in the full neighborhood $0 < |z - z_o| < \delta$,

$f(z)$ approaches w_0 when z approaches z_0 from any direction, i.e. along any path.

Definition. A function $f(z)$ is said to be continuous at a point z_0 if $f(z_0)$ exists, $\lim_{z \to z_0} f(z)$ exists, and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

It is very important to note that this definition requires that $f(z)$ is defined in some neighborhood of z_0. Also, a function $f(z)$ is said to be continuous on a set E if it is continuous at such point of E.

As in the case of real calculus, it is easy to prove the following theorem:

Theorem. If the functions $f(z)$ and $g(z)$ are continuous at the point z_0, then so are

(a) $f(z) \pm g(z)$ (b) $f(z)\,g(z)$

(c) $f(z)/g(z)$ provided $g(z_0) \neq 0$.

Using the definition of continuity, we can show that the constant function and the function $f(z) = z$ are continuous on the whole complex plane. It follows from the above theorem that:

(i) the polynomial functions in z, i.e. functions of the form

$$a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n,$$

where the a_k are constants, are also continuous on the whole plane.

(ii) the rational functions in z, i.e. quotients of polynomials

$$\frac{a_0 + a_1 z + \dots\dots + a_n z^n}{b_0 + b_1 z + \dots\dots + b_m z^m}$$

are continuous at each point, where the denominator does not vanish.

These two facts give us a simple method for evaluating limits of polynomial and rational functions as continuous functions. To find $\lim_{z \to z_0} f(z)$, where $f(z)$ is continuous at z_0, we simply evaluate $f(z)$ at z_0.

Example 1.4. Find each of the following limits:

(a) $\lim_{z \to 2i} \frac{z^2 + 3}{iz}$ (b) $\lim_{z \to i} \frac{z^4 - 1}{z^2 + 1}$

Solution .

(a) Since the function $\frac{z^2 + 3}{iz}$ is continuous at $z = 2i$ we have

$$\lim_{z \to 2i} \frac{z^2 + 3}{iz} = \frac{(2i)^2 + 3}{i(2i)} = \frac{1}{2}.$$

(b) The function

$$f(z) = \frac{z^4 - 1}{z^2 + 1}$$

is not continuous at $z = i$, because it is not defined there. However, for $z \neq i$ and $z \neq -i$ we have:

$$f(z) = \frac{(z^2-1)(z^2+1)}{(z^2+1)} = z^2 - 1 = g(z).$$

Hence

$$\lim_{z \to i} f(z) = \lim_{z \to i} g(z) = g(i) = -2$$

EXERCISES 1

1. Write down the definition of each of the following concepts:

(a) neighborhood of a point.

(b) interior point of a set.

(c) open set (d) boundary of a set.

(e) connected set. (f) domain and region.

(g) limit of a function.

(h) continuity of a function at apoint and on a set.

2. Let E be the set consisting of the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. What is the boundary of E?

3. Find the domain of definition of the following functions and write each of them in the form $w = u(x,y) + iv(x,y)$:

(a) $f(z) = 2z^3 - 3z$

(b) $f(z) = \dfrac{1}{(1-z)}$

(c) $f(z)=\dfrac{(z+i)}{(z^2+1)}$.

4. Describe the range of each of the following functions:

(a) $f(z)=z+5$ for $\operatorname{Re} z>0$.

(b) $f(z)=z^3$ for $|Arg\, z|\le \pi/4, |z|\le 2$

(c) $f(z)=z^2$ for $|z|>3$.

5. Find each of the following limits:

(a) $\lim\limits_{z\to 1-2i}(2z+3i)^2$

(b) $\lim\limits_{z\to 3i}\dfrac{z^2+9}{z(z-3i)}$

(c) $\lim\limits_{\Delta z\to 0}\dfrac{(z_0+\Delta z)^2-z_0^2}{\Delta z}$.

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14. Derivatives and Analytic Functions

<u>Definition</u> . A function $f(z)$ is said to be *differentiable* at a point z_0 if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. This limit is called *the derivative* of $f(z)$ at z_0. Some times the symbol df/dz will be used instead of f'.

Note that this definition requires that $f(z)$ is defined in some neighborhood of z_0. Once more we recall that the notion of limit implies that $z = z_0 + \Delta z$ may approach z_0 from any direction. Thus, differentiability means that, along what's ever path $z = z_0 + \Delta z$ approaches z_0, the above difference quotient must tend to the same complex number.

Example 1.5. Discuss the differentiability of each of the following functions by using the definition:

(a) $f(z) = z^2$ (b) $f(z) = |z|^2$

Solution:

(a) We have at any point z that

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0}(2z + \Delta z) = 2z$$

Thus, the function $f(z) = z^2$ is differentiable for all z and has the derivative $f' = 2z$.

(b) Using that $|z|^2 = z\bar{z}$, we have

$$\frac{\Delta f}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} = \bar{z} + \Delta z + z\frac{\overline{\Delta z}}{\Delta z}.$$

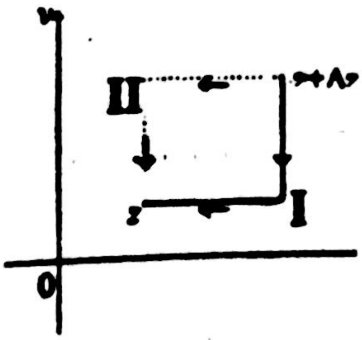
Note that $\Delta z = \Delta x + i\Delta y$, $\overline{\Delta z} = \Delta x - i\Delta y$.

Thus $\Delta z \to 0$ if and only if $\overline{\Delta z} \to 0$.

At $z = 0$ we have

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{z \to 0} \overline{\Delta z} = 0.$$

Hence the function is differentiable at $z = 0$ and $f'(0) = 0$.



Suppose $z \neq 0$. We show that the value of the limit of $\Delta f/\Delta z$ as $\Delta z \to 0$ depends on the manner in which Δz approaches 0. Along path I in the figure $\Delta y \to 0$ first, and then $\Delta x \to 0$. Thus, along this path

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0}\left(\lim_{\Delta y \to 0} \frac{\Delta f}{\Delta z}\right)$$

$$= \lim_{\Delta x \to 0}\left(\bar{z} + \Delta x + z\frac{\Delta x}{\Delta x}\right)$$

$$= \bar{z} + z.$$

Along path II

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y \to 0}\left(\bar{z} - i\Delta y + z\frac{-i\Delta y}{i\Delta y}\right) = \bar{z} - z.$$

Since $z \neq 0$, the value $\bar{z} + z$ and $\bar{z} - z$ are distict, so that the limit does not exist. Hence the function $f(z) = |z|^2$ is not differentiable for all $z \neq 0$.

As in the real case, differentiability implies continuity. Furthermore, all the familiar rules of real differential calclus continue to hold in complex case. For example, if f and g are differentiable, then

(a) $(fg)' = fg' + f'g$

(b) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

(c) $\frac{d}{dz} f(g(z)) = \frac{df}{dg}\frac{dg}{dz}$ (The chain rule)

(d) $\frac{d}{dz} z^n = nz^{n-1}$ (Prove this !)

Definition . A complex-valued function $f(z)$ is said to be *analytic* on an open set E if $f(z)$ is defined and differentiable at all points of E. The function $f(z)$ is said to be analytic at a point z_0 if it is analytic in some neighborhood of z_0 .

As we showed in example (1.5), the function $f(z)=|z|^2$ is differentiable only at the isolated point $z_0 = 0$

Thus the function is not analytic at $z=0$, because analyticity requires that $f(z)$ has a derivative at every point in some neighborhood of $z=0$, which is not the case.

The concept of analyticity is motivated by the fact that it is of no practical interest when a function is differentiable merely at a single point z_0 but not throughout some neighborhood of z_0 .

If $f(z)$ is analytic on the whole complex plane, then it is said to be entire. For example, all polynomial functions of z are entire. A rational function is analytic at every point for which its denominator is nonzero. In particular, the function

$$f(z) = \frac{z^3 + 3z + 5}{z(z - 2i)}$$

is analytic everywhere except at the two points $z = 0$ and $z = 2i$.

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§5.The Cauchy-Riemann Equations

The requirement of differentiability of a function of complex variable imposes very important conditions on the behavior of the real and imaginary parts of this function; known as the *Cauchy-Riemann equations* .These conditions can be formulated by means of the following two theorems:

Theorem 1. (Necessity of Cauchy-Riemann conditions)

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at z, the first order partial derivatives of u and v exist and satisfy the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \qquad \text{I.2}$$

called the *Cauchy-Riemann equations*.

Consequently, if $f(z)$ is analytic in an open set E then equations (I.2) must hold at every point of E.

Proof. Suppose $f\ (z)$ to be defined and continuous in some neighborhood of a fixed point z and differentiable at that z. Then the derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad \text{I.3}$$

at z exists. Here, Δz may approach 0 along any path in a neighborhood of z. Setting $\Delta z = \Delta x + i\Delta y$ and choosing path I in figure on page 15, we let $\Delta y \to 0$ first and then $\Delta x \to 0$ After Δy become zero, $\Delta z = \Delta x$, and by I.3

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - [u(x,y) + iv(x,y)]}{\Delta x}$$
$$= \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i \lim_{\Delta x \to 0}\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right].$$

Since $f'(z)$ exists, the last two real limits exist. They are the partial derivatives of u and v with respect to x. Hence the derivative $f'(z)$ can be written:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \qquad \text{I.4}$$

Similarly, choosing path II on page 15, we let $\Delta x \to 0$ first, after which $\Delta z = i\Delta y$, so that we now obtain

$$f'(z) = \lim_{\Delta y \to 0}\left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y}\right] + i \lim_{\Delta y \to 0}\left[\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}\right];$$

that is, since $1/i = -i$,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \qquad \text{I.5}$$

Thus, the existance of $f'(z)$ implies the existence of the four partial derivaties on the right-hand side of (1.4) and (1.5). Moreover, equating the real and imaginary parts of (1.4) and (1.5), we obtain the Cauchy-Riemann equations. The proof of theorem 1 is complete.

Theorem 2. (Sufficiency of the Caucy-Riemann equations)

Let $f(z) = u(x,y) + iv(x,y)$ be defined in some neighborhood of the point $z = x + iy$. If the first partial derivatives of u and v exist in this neighborhood, are continuous at z, and satisfy the *Cauchy - Riemann* equations at z, then $f(z)$ is differentiable at z.

Proof. Let $|\Delta z|$ be so small such that $z + \Delta z$ lies in the mentioned neighborhood. Because of our continuity assumptions, we may apply the mean-value theorem to obtain

$$\begin{aligned}\Delta u &= u(x+\Delta x, y+\Delta y) - u(x,y)\\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,\end{aligned}$$

$$\begin{aligned}\Delta v &= v(x+\Delta x, y+\Delta y) - v(x,y)\\ &= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y,\end{aligned}$$

where each partial derivative is evaluated at (x,y), i.e. at z, where each $\varepsilon_k \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, as $\Delta z \to 0$.

Therefore,

$$\Delta f = f(z+\Delta z) - f(z) = \Delta u + i\Delta v$$
$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y$$

Using the *Cauchy-Riemann* equations, we replace $\frac{\partial u}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$. Hence

$$\Delta f = \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + c_1\Delta x + c_2\Delta y,$$

where the complex numbers $c_1 = \varepsilon_1 + i\,\varepsilon_3$ and $c_2 = \varepsilon_2 + i\,\varepsilon_4$ approach zero as $\Delta z \to 0$. Dividing both sides by $\Delta z = \Delta x + i\Delta y$, we see that the difference quotient

$$\frac{\Delta f}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{c_1\Delta x + c_2\Delta y}{\Delta x + i\Delta y}$$

Since

$$\left|\frac{c_1\Delta x + c_2\Delta y}{\Delta x + i\Delta y}\right| = |c_1|\left|\frac{\Delta x}{\Delta x + i\Delta y}\right| + |c_2|\left|\frac{\Delta y}{\Delta x + i\Delta y}\right|$$

$$\le |c_1| + |c_2| \to 0 \text{ as } \Delta z \to 0,$$

we have

$$\lim_{\Delta z \to 0}\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Hence $f'(z)$ exists and the proof is complete.

Corollary. If the first partial derivitives of u and v are continuous and satisfy the *Cauchy-Riemann* equations at all points of an open E, then $f(z)$ is analytic in E.

Example I.6 . Discuss the analyticity of the function $f(z) = z^2$ by using the *Cauchy-Riemann* equations.

Solution. We have

$$u = x^2 - y^2 \text{ and } v = 2xy.$$

Thus,

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y, \frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x$$

Hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y,$$

so that the *Cauchy-Riemann* equations are satisfied at every point in the complex plane. The function $f(z) = z^2$ is entire and by (I.4)

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2yi = 2z$$

<u>Example I.7</u> . Discuss the differentialability and analyticity of the function $f(z) = (x^2 + y) + i(y^2 - x)$.

<u>Solution</u>. Since $u = x^2 + y$ and $v = y^2 - x$, we have

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial x} = -1, \frac{\partial v}{\partial y} = 2y,$$

The *Cauchy-Riemann* equations in this case have form

$$2x = 2y \ , \ 1 = 1$$

Hence they are simultaneously satisfied only on the line $x = y$. The given function is differentiable on the line $x = y$ and by (I.4)

$$f'(z) = 2x - i \, .$$

Since $f(z)$ is differentiable in no open disk, it is no where analytic.

If u and v are expressed in terms of polar coordinates (r, θ) then the *Cauchy-Riemann* equations can be written in the form

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{1.6}$$

In fact, let the *Cauchy-Riemann* equations (1.2) be satisfied at a point $z \neq 0$. We show that equations (1.6) are satisfied at that point. Setting $x=r\cos\theta$ *and* $x=r\sin\theta$, and using the chain rule, we get

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta, \tag{i}$$

$$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial v}{\partial x}(-r\sin\theta)+\frac{\partial v}{\partial y}(r\cos\theta).$$

Replacing $\frac{\partial v}{\partial x}$ by $-\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ in the last relation, we obtain

$$\begin{aligned}\frac{\partial v}{\partial \theta}&=\frac{\partial u}{\partial y}r\sin\theta+\frac{\partial u}{\partial x}r\cos\theta\\ &=r\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right).\end{aligned} \tag{ii}$$

From (i) and (ii) it follows that $\frac{\partial u}{\partial r}=r^{-1}\frac{\partial v}{\partial \theta}$. Similarly, we can show that the second of equations (1.6) is satisfied. It is left for the student to prove that if equations (1.6) are satisfied at a point, then equations (1.2) are satisfied at this point.

<u>*Example 1.8*</u> . Show that the function $f(z)=\text{Arg}\, z$ is nowhere analytic.

Solution. Since $u = \theta$ and $v = 0$, we have

$$\frac{\partial u}{\partial r} = 0, \frac{\partial u}{\partial \theta} = 1, \frac{\partial v}{\partial r} = 0, \frac{\partial v}{\partial \theta} = 0.$$

Since $\frac{\partial v}{\partial r} \neq -r^{-1}\frac{\partial u}{\partial \theta}$ at all points in the plane, the given function is not differentiable at any point, and consequently is nowhere analytic.

Example I.9. If $f(z)$ is analytic in a domain D and if $v = u^2$, prove that $f(z) = \text{constant}$.

Solution. Since $f(z)$ is analytic in D, it satisfies the *Cauchy-Riemann* equations there. Thus,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u^2}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = 2u\frac{\partial u}{\partial y}. \qquad \text{(i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial u^2}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = -2u\frac{\partial u}{\partial x}. \qquad \text{(ii)}$$

Substituting from (ii) into (I), we obtain

$$\frac{\partial u}{\partial x} = -4u^2\frac{\partial u}{\partial x} \Rightarrow \left(1 + 4u^2\right)\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0,$$

Fom (ii) it follows that $\frac{\partial u}{\partial y} = 0$. Thus, u is a real constant. Since $v = u^2$, then v also is a real constant, and, consequently, $f(z)$ is a constant function.

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§6 Harmonic Functions

The second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called *Laplace's equation* in the two real independent variables x and y.

<u>**Definition**</u>. A real-valued function $\phi(x,y)$ is called *harmonic* in domain D if it has continuous second order partial derivatives and satisfies *Laplace's equation* in D.

In particular, $\phi = x^2 - y^2$ and $\phi = e^x \cos y$ are harmonic functions.

We shall discover a practically important relation between complex analysis and the *Laplace's equation* in two variables as follows.

<u>***Theorem***</u>. If the function $f(z) = u + iv$ is analytic in some domain D, then both real and imaginary parts u, v of $f(z)$ are harmonic functions in D.

Proof. We shall see later that the derivatives of all orders of an analytic function are themselves analytic. Thus, u and v have continuous partial derivatives of all orders.

Since $f(z)$ is analytic in D, its real and imaginary parts satisfy the *Cauchy-Riemann* equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \text{ for all } (x,y)\in D$$

Differentiating the first of these equations with respect to x, and the second with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y},\ \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x} \text{ in D.} \qquad (*)$$

Since v has continuous partial derivatives of all orders, then the mixed partial derivatives can be taken in any order ; i.e.

$$\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}.$$

Adding equations (*), we find that u satisfies *Laplace's equation* in D, and hence u is harmonic in D.

Similarly, we may show that v is harmonic in D. The proof is complete.

Definition. If $w = u + iv$ is an analytic function in some domain D, then u and v are called *conjugate harmonic function* in D.

We can find the conjugate of a given harmonic function using the *Cauchy-Riemann* equations . This procedure will be illustrated by means of examples.

<u>Example 1.10</u> . Show that the function $u = y^3 - 3x^2 y$ is harmonic and find its conjugate v. Write the analytic function $u + iv$ in the form $f(z)$.

<u>Solution</u>. We have

$$u_x = -6xy, u_{xx} = -6y, u_y = 3y^2 - 3x^2, u_{yy} = 6y.$$

Hence

$$u_{xx} + u_{yy} = -6y + 6y = 0,$$

so that u is harmonic . To find the conjugate v of u , we have from the first of the *Cauchy-Riemann* equations that

$$v_y = u_x = -6xy.$$

Integrating both sides with respect to y , we find

$$v = -3xy^2 + h(x),$$

where $h(x)$ is a function of x only. But since $v_x = -u_y$, it follows that

$$-3y^2 + h'(x) = -\left(3y^2 - 3x^2\right).$$

We find that $h'(x) = 3x^2$, so that $h(x) = x^3 + c$, where c is an arbitrary real constant. Hence

$$v = -3xy^2 + x^3 + c.$$

The corresponding analytic function is

$w = f(z) = y^3 - 3x^2 y + i(x^3 - 3xy^2 + c).$ (**)

Setting $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$, we find

$$w = f(z) = i(z^3 + c).$$

Note. There is another way of obtaining $f(z)$. Setting $y = 0$ in both sides of (**), we get

$$f(x) = i(x^3 + c).$$

Replacing x by z, we obtain the same form as above for $f(z)$.

By using the chain rule for Differentiation, the student may show that *Laplace's equation* in polar coordinates has the form

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

Example 1.11. Show that the function $u = \ln r$ is harmonic in the domain $r>0$, $0 < \theta < 2\pi$. Find v such that the function $u + iv$ is analytic in that domain.

Solution. We have

$$u_r = r^{-1}, u_{rr} = -r^{-2}, u_\theta = u_{\theta\theta} = 0.$$

Hence

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = -r^{-2} + r^{-2} = 0.$$

Thus u is harmonic in the mentioned domain. To find the conjugate harmonic function , we use the *Cauchy-Riemann* equations in polar form as follows:

$$v_\theta = ru_r = rr^{-1} = 1 \qquad \text{(note that } r > 0\text{)}$$

Integrating both sides with respect to θ, we obtain

$$v = \theta + h(r).$$

From the equation $v_r = -ru_\theta$, we have

$$h'(r) = 0,$$

so that $h(r) = c$, where c is a real constant. Hence

$$v = \theta + c.$$

The required analytic function is

$$w = f(z) = \ln r + i(\theta + c).$$

EXERCISES II

1. Write down the definitions of

   (a) the derivative of a complex-valued function $f(z)$ at a point,

   (b) the analytic function in a domain and at a point,

   (c) harmonic and conjugate harmonic functions.

2. State and prove theorems about the necessity and sufficiency of the *Cauchy-Riemann* equations for the differentiability and analyticity of the function $f(z) = u + iv$.

3. Prove that if $u + iv$ is analytic in some domain D, then both u and v are harmonic functions in D.

4. Using the *Cauchy-Riemann* equations, determine where the following functions are differentiable and where they are analytic. If the function is differentiable find $f'(z)$:

(a) $w = \bar{z}$

(b) $w = \operatorname{Im} z$

(c) $w = 2y - ix$

(d) $w = e^{x}(\cos y + i \sin y)$

(e) $w = e^{y}(\cos x + i \sin x)$

(f) $w = z^2 \bar{z}$

(g) $w = 1/z$

(h) $w = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2 y - 3y)$.

5. Prove that the function

$$f(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$$

is entire. Write this function in terms of z.

6. Prove that the function

$$f(z) = e^{x^2 - y^2}(\cos 2xy + i \sin 2xy)$$

is entire, and find its derivative.

7. Prove that an analytic function $f(z)$ must be constant if any one of the following conditions hold:

(a) $\text{Re} f(z) =$ const
(b) $\text{Im} f(z) =$ const
(c) $|f(z)| =$ const
(d) $\overline{f(z)}$ is analytic
(e) $|f(z)|$ is analytic.

8. Show that each of the following functions is harmonic , and find a corresponding analytic function $w = u + iv$:

(a) $u = 2x(1-y)$
(b) $u = \cos x \cosh y$
(c) $v = y/(x^2+y^2)$
(d) $u = x^3 - 3xy^2 + y$
(e) $u = \text{Arg}\, z$

9. Show that if v is a harmonic conjugate of u in a domain D, then uv is harmonic in D.

10. Let $f(z)$ be analytic and nonzero in a domain D. Prove that $\ln|f(z)|$ is harmonic in D.

CHAPTER II

Elementary Functions of a Complex Variable

§1. Rational Functions. Root

In this chapter we study the most important elementary functions of a complex variable. We shall see that these function can easily be defined in such a way that, for real values of z the functions become identical with the familiar real functions.

The simplest elementary complex functions are

1) the powers $w = z^n$, where n is a positive integer. This function is analytic in the entire z-plane and $w' = n z^{n-1}$.

2) the functions of the form

$$w = P_n(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_n, \qquad (c_0 \neq 0)$$

where c_0, c_1, ..., c_n are complex or real constants. Such functions are called *polynomials* of the nth degree.

3) the quotients of two polynomials

$$w = \frac{P_n(z)}{Q_m(z)}$$

are called *rational functions*. Such a function is analytic everywhere, except at those z, where $Q_n(z) = 0$.

A rational function of the form $\dfrac{c}{(z - z_0)^k}$, where c and z_0 are complex numbers and k is a positive integer, is called a *partial fraction*. In algebra it is proved that any rational function can be represented as the sum of a polynomial and partial fractions. For example,

$$\frac{z^3}{z^2 + 4} = z - \frac{4z}{(z + 2i)(z - 2i)} = z - \frac{2}{z + 2i} - \frac{2}{z - 2i}.$$

4) The nth root of z

$$w = \sqrt[n]{z} = z^{1/n}$$

This function is multi-valued: to each value of z there correspond n distinct values of w.

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§2. Exponential Function

The exponential function e^z is defined in terms of real-valued functions by the equation

$$e^z = e^x(\cos y + i \sin y) \tag{2.1}$$

This definition is suggested by the following facts:-

1) When $z = x$ is real, i.e. $y = 0$, then $e^z = e^x$.

2) From *Cauchy-Riemann* equations it follows that e^z is analytic for all z.

From formula (1.4) we see that

$$\frac{d}{dz}e^z = \frac{\partial}{\partial x}\left(e^x \cos y\right) + i\frac{\partial}{\partial x}\left(e^x \sin y\right) = e^z.$$

3) Setting $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we find that

$$\begin{aligned} e^{z_1+z_2} &= e^{(x_1+x_2)+i(y_1+y_2)} \\ &= e^{x_1+x_2}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right] \\ &= e^{x_1}\operatorname{cis} y_1 . e^{x_2}\operatorname{cis} y_2 = e^{z_1}e^{z_2} \end{aligned}$$

Thus $e^{z_1}e^{z_2} = e^{z_1+z_2}$ (2.2)

The facts (1) - (3) are similar to those for the real exponential function e^x.

Applying formula (2.2) with $z_1 = x$ and $z_2 = iy$, we get

$$e^z = e^x e^{iy}$$

From here and (2.1) we obtain the so-called *Euler* formula

$$e^{iy} = \cos y + i\sin y. \tag{2.3}$$

Now any complex number $z = x + iy$ can be written in the following exponential form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{2.4}$$

From (2.3) we see that $\left|e^{iy}\right| = 1$. Consequently,

$$\left|e^z\right| = e^x, \arg e^z = y.$$

Thus $|e^z|>0$ for every value of z, which means that $e^z \neq 0$ for any number z.

It is easy to see that, $f(z)=e^z$ is periodic function with the complex period $2\pi i$, because

$$f(z+2\pi i)=e^{z+2\pi i}=e^z e^{2\pi i}=e^z \text{ cis } 2\pi$$
$$=e^z=f(z).$$

Example 2.1. Solve $e^z=-1$

Since $-1=\cos\pi+i\sin\pi$, it follows that

$$e^x(\cos y+i\sin y)=\cos\pi+i\sin\pi$$

Consequently, $e^x=1$, so that $x=0$, and $y=\pi\pm 2n\pi$ $(n=0,1,2,\ldots)$. The solution of $e^z=-1$ is $z=i(\pi\pm 2n\pi)$.

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§3. Trigonometric and Hyperbolic Functions

From the formulas

$$e^{iy}=\cos y+i\sin y,\quad e^{-iy}=\cos y-i\sin y$$

it follows that

$$\frac{e^{iy}+e^{-iy}}{2}=\cos y,\quad \frac{e^{iy}-e^{-iy}}{2i}=\sin y \tag{2.5}$$

for every real y. It is therefore natural to define the *cosine* and *sine* functions of z by the equations

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{2.6}$$

Since e^{iz} and e^{-iz} are analytic in the entire z-plane, it follows that $\cos z$ and $\sin z$ are analytic for all values of z. Differentiating equations (2.6), we get

$$\frac{d}{dz}\cos z = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

and $\dfrac{d}{dz}\sin z = \cos z.$

The other trigonometric functions are defined by the usual relations

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}$$
$$\sec z = \frac{1}{\cos x}, \csc z = \frac{1}{\sin z}$$

Thus $\tan z$ and $\sec z$ are analytic in any domain where $\cos z \neq 0$, and $\csc z$ are analytic in any domain where $\sin z \neq 0$. Differentiation of the last relations gives

$$\frac{d}{dz}\tan z = \sec^2 z, \frac{d}{dz}\cot z = -\csc^2 z,$$
$$\frac{d}{dz}\sec z = \sec z \tan z, \frac{d}{dz}\csc z = -\csc z \cot z.$$

To find the real and imaginary parts of $\cos z$ we behave as follows:

$$\cos z = \cos(x+iy) = \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$$
$$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$$
$$= \frac{e^{y}+e^{-y}}{2}\cos x - i\frac{e^{y}-e^{-y}}{2}\sin x$$

Thus

$$\cos z = \cos(x+iy) = \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y \tag{2.7}$$

Similarly, we can find that

$$\sin z = \sin(x+iy) = \sin x \operatorname{ch} y + i \cos x \operatorname{sh} y \tag{2.8}$$

It is clear from (2.7) and (2.8) that

$$\cos iy = \operatorname{ch} y, \sin iy = i\operatorname{sh} y, \text{ and}$$
$$\overline{\cos z} = \cos \bar{z}, \overline{\sin z} = \sin \bar{z}$$

Example 2.2. Find all values of z such that $\sin z = 0$.

From (2.3) we deduce that x and y must satisfy the simultaneous equation

$$\sin x \operatorname{ch} y = 0, \cos x \operatorname{sh} y = 0$$

Since $\operatorname{ch} y \geq 1$, it follow from the first equation that $\sin x = 0$, i.e. $x = n\pi$ ($n=0, \pm1, \pm2, \ldots$).

Since $\cos n\pi = (-1)^n \neq 0$, we get from the second equation that $\operatorname{sh} y = 0$. Thus $y = 0$. Therefore,

$$\sin z = 0 \text{ implies } z = n\pi \quad (n=0, \pm1, \pm2, \ldots).$$

The Hyperbolic functions of z are defined as in real calculus; that is,

$$\operatorname{sh} z = \frac{e^z - e^{-z}}{2}, \operatorname{ch} z = \frac{e^z + e^{-z}}{2},$$
$$\operatorname{th} z = \frac{\operatorname{sh} z}{\operatorname{ch} z}, \operatorname{coth} z = \frac{\operatorname{ch} z}{\operatorname{sh} z}, \tag{2.9}$$
$$\operatorname{sech} z = \frac{1}{\operatorname{ch} z}, \operatorname{csch} z = \frac{1}{\operatorname{sh} z}$$

It follows from (2.9) that $\operatorname{sh} z$ and $\operatorname{ch} z$ are analytic everywhere, $\operatorname{th} z$ is analytic in any domain where $\operatorname{ch} z \neq 0$ etc.

$$\frac{d}{dz}\operatorname{sh}z = \operatorname{ch}z, \frac{d}{dz}\operatorname{ch}z = \operatorname{sh}z, \frac{d}{dz}\operatorname{th}z = \operatorname{sech}^2 z, \ldots$$

From (2.6) and (2.8) we obtain the following relations between the hyperbolic and trigonometric functions :

$$\operatorname{sh} iz = i \sin z, \operatorname{ch} iz = \cos z,$$

$$\sin iz = i \operatorname{sh} z, \cos iz = \operatorname{ch} z.$$

To find real and imaginary parts of $\operatorname{ch} z$ we have by using (2.7) that

$$\operatorname{ch} z = \cos iz = \cos(-y + ix)$$
$$= \operatorname{ch} x \cos y + i \operatorname{sh} x \sin y \tag{2.10}$$

Similarly, $\operatorname{sh} z = \operatorname{sh} x \cos y + i \operatorname{ch} x \sin y. \quad (2.11)$

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§6. Logarithm. Complex Exponents

The natural logarithm of z is denoted by $\ln z$ (or by $\log z$) and defined as the inverse of the exponential function ; that is, $w = \ln z$ is the function which satisfies the relation

$$e^{w} = z$$

Let $w = u + iv$ and $z = re^{i\theta}$. We have

$$e^{u}e^{iv} = re^{i\theta}$$

It follows that

$$e^{u} = r \quad \text{or} \quad u = \ln r = \ln\sqrt{x^2 + y^2}$$

and $v = \theta = \arg z = \tan^{-1}\dfrac{y}{x}$.

Therefore,

$$\ln z = \ln r + i\theta = \ln\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x} \tag{2.12}$$

Since θ is determined only up to multiples of 2π, the function $\ln z$ is infinitely many-valued.

The value of $\ln \theta$ corresponding to the principal value of $\arg z$ is called the principal value of $\ln z$ and is often denoted by $\operatorname{Ln} z$ or $\operatorname{Log} z$, e.g.

$$\ln(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right), \quad (n = 0, \pm1, \pm2, \ldots)$$

$$\operatorname{Ln}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}.$$

The function $\operatorname{Ln} z = \ln r + i\theta$ $(-\pi < \theta \leq \pi)$ is single-valued with , $u = \ln r$ and $v = \theta$, which satisfy *Cauchy-Riemann* equations in polar form. Thus $w = \ln z$ is analytic everywhere, except on the negative real axis $\theta = \pi$ and the origin, and

$$\frac{d}{dz}\operatorname{Ln} z = \frac{1}{z} \tag{2.13}$$

We define z^c , where the exponent c is any complex number, by the equation:

$$z^c = e^{c \ln z} \tag{2.14}$$

This function is multi-valued, because $\ln z$ is multi-valued. The value $e^{c \operatorname{Ln} z}$ is called the principal value of z^c .

It is single-valued and analytic in the domain $-\pi < \theta < \pi$. We have

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \operatorname{Ln} z} = \frac{c}{z} e^{c \operatorname{Ln} z} = \frac{c}{z} z^c = c z^{c-1}$$

Note that z^{c-1} is the principal value of $e^{(c-1)\ln z}$.

<u>*Example 2.3*</u>. Find all values of i^{-2i} .

$$\begin{aligned} i^{-2i} &= e^{-2i \ln i} = e^{-2i\left(\frac{\pi}{2} + 2n\pi\right)i} \\ &= e^{\pi + 4n\pi} ,\left(n = 0, \pm 1, \pm 2, \ldots\right) \end{aligned}$$

The principal value of i^{-2i} is e^{π} .

Finally, according to (3.14), we define the exponential function with a complex constant as its base by

$$c^z = e^{z\ \ln c}\ .$$

EXERCISES III

1) Show that

i) $e^{2\pm 3\pi i} = -e^2$ (ii) $e^{\frac{\pi}{2}i} = 1$

iii) $e^{\frac{1}{2}+\frac{\pi}{4}i} = \sqrt{e}\,\dfrac{1+i}{\sqrt{2}}$ (iv) $\exp(\ln r + i\theta) = z$, where $z = re^{i\theta}$

2) Find all roots of the equations

i) $e^z = -2$ (ii) $e^z = 1 + i\sqrt{3}$

iii) $e^{2z-1} = 1$ (iv) $e^{\bar{z}} = \overline{(e^z)}$.

3) Prove that

i) $|\sin z|^2 = \sin^2 x + \mathrm{sh}^2 y, |\cos z|^2 = \cos^2 x + \mathrm{ch}^2 y$

ii) $\sin^2 z + \cos^2 z = 1$

iii) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

iv) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

v) $\sin(-z) = -\sin z, \cos(-z) = \cos z$

vi) $\sin\left(\dfrac{\pi}{2} - z\right) = \cos z$

vii) $\sin(z + 2\pi) = \sin z$, i.e. $\sin z$ is a periodic function with period 2π.

4) Find all roots of each of the following equations:

i) $\cos z = 2$ ii) $\sin z = \text{ch}\, 4$ iii) $\cos z = 0$

5) Prove that

i) $\text{ch}^2 z - \text{sh}^2 z = 1$

ii) $\text{sh}\, 2z = 2\, \text{sh}\, z\, \text{ch}\, z$

iii) $\text{ch}\left(z_1 + z_2\right) = \text{ch}\, z_1\, \text{ch}\, z_2 + \text{sh}\, z_1\, \text{sh}\, z_2$.

6) Find all zeros of $\text{sh}\, z$ and $\text{ch}\, z$.

7) Find all roots of the equations:

i) $\text{ch}\, z = \dfrac{1}{2}$ ii) $\text{sh}\, z = i$ iii) $\text{ch}\, z = -2$.

8) Find all values of

i) $\ln 1$ (ii) $\ln i$ (iii) $\ln(-1)$ (iv) $\ln\sqrt{i}$

v) $\text{Ln}(-ei)$ (vi) $\text{Ln}(1-i)$ (vii) $\text{Ln}(-3+i\sqrt{27})$

viii) $(1+i)^i$ (ix) $(-1)^{1/\pi}$ (x) $\left[\dfrac{e}{2}\left(-1-i\sqrt{3}\right)\right]^{3\pi i}$

9) Solve the following equations for z:

i) $\ln z = +\dfrac{\pi i}{2}$ (ii) $\ln z = 1 + \pi i$

10) Prove that

i) $\sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$

ii) $\cos^{-1} z = -i \ln\left(z + \sqrt{z^2 - 1}\right)$

iii) $\text{sh}^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$

iv) $\text{ch}^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$

11) Using formulas of problem (10), solve for z

i) $\sin z = 2$ (ii) $\cos z = \sqrt{2}$

12) Find all values of

i) $\tan^{-1}(2i)$ (ii) $\tan^{-1}(1+i)$ (iii) $\tan^{-1} 0$.

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§5. Elementary Transformations

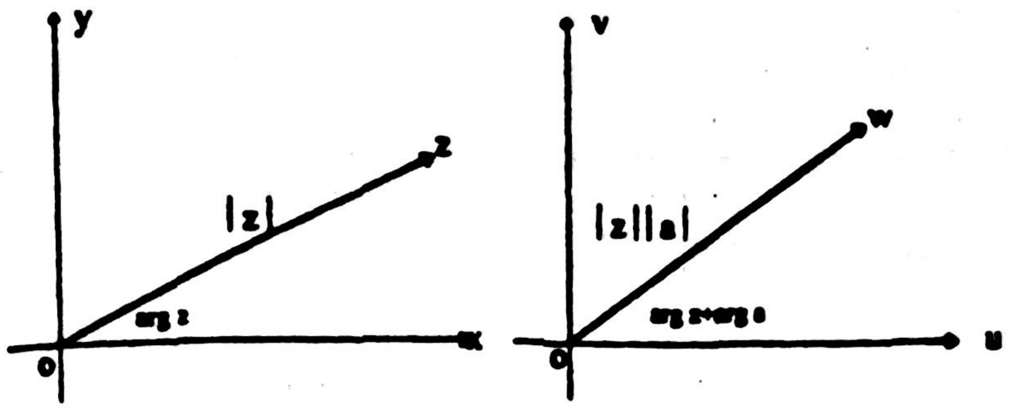
1. Linear Transformation :

A function of the form $f(z) = a\,z + b$ where a, b are complex constants, is called a linear function . If $a \neq 0$, then f is a one-one function . The inverse relation $z = \frac{w}{a} - \frac{b}{a}$ is also a linear function which maps the w-plane back onto z-plane. Let

$$z' = az \qquad \text{then} \qquad w = z' + b$$

The mapping $z' = a\,z$ is known as a rotation stretching because $|z'| = |a|.|z|$ and $\arg z' = \arg a + \arg z$, i.e. the modulus $|z|$ is stretched by a factor $|\,a\,|$ and the $\arg z$ is rotated by an angle $\arg a$.

The transformation $w = z' + b$ is called translation, has the property that it shifts or translates every point .



z-plane w-plane

Now, $w = a\,z + b$ has the effect of rotation $z' = e^{i\alpha} z$, $|a|=1$ or rotation stretching $z'=az$, $|a|>1$ or rotation shrinking $z'=a\,z$, $|a|>1$ followed by translation $w = z' + b$.

Example 2.4. Determine the region in the w-plane into which the region bounded by $x = 0$, $y = 0$, $x = 2$, and $y=1$ is mapped by the function

$w = (1+i\,)\,z + (1+2i\,)$

Solution: Let

$$z = x + iy \quad , \; w = u + iv$$

Since

$$w = (1+i)\,(x+iy) + (1+2i\,)$$

$$w = (x - y + 1) + i\,(x + y + 2)$$

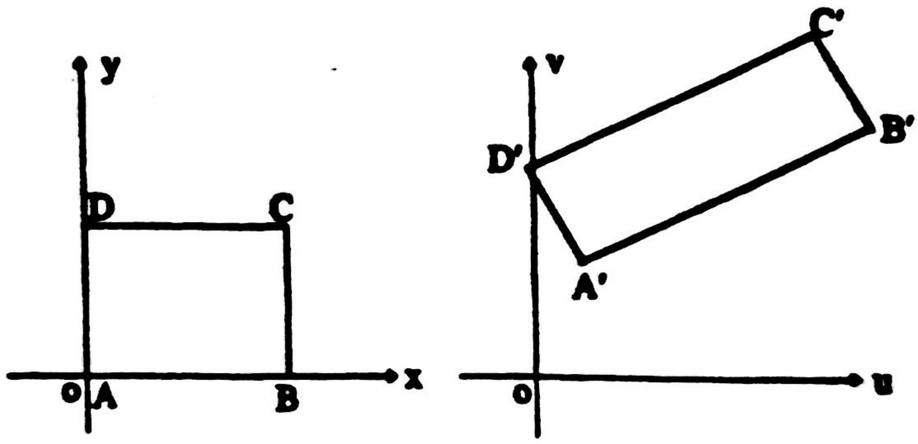
Hence,

$$u = x - y + 1 \quad , \; v = x + y + 2$$

The given region is a rectangluar region ABCD where A = (0,0), B = (2,0), C= (2,1), D = (0,1)

The points A,B,C, and D are mapped respectively into the points A′, B′, C′, and D′ in the w-plane where

A′ = (1,2), B′ = (3,4), C′ = (2,5), D′ =(0,3)



z-plane w-plane

The line $x = 0$ in z-plane is mapped into the line $v = -u + 3$ in w-plane

The line $y = 0$ in z-plane is mapped into the line $v = u + 1$ in w-plane

The line $x = 2$ in z-plane is mapped into the line $v = -u + 7$ in w-plane

The line $y = 1$ in z-plane is mapped into the line $v = u + 3$ in w-plane

The image of ABCD is again a rectangle A′B′C′D′.

2. The Reciprocal Transformation :

This mapping as we shall see transforms circles or straight lines into circles or straight lines.

Since $w = \frac{1}{z}$ we can write then

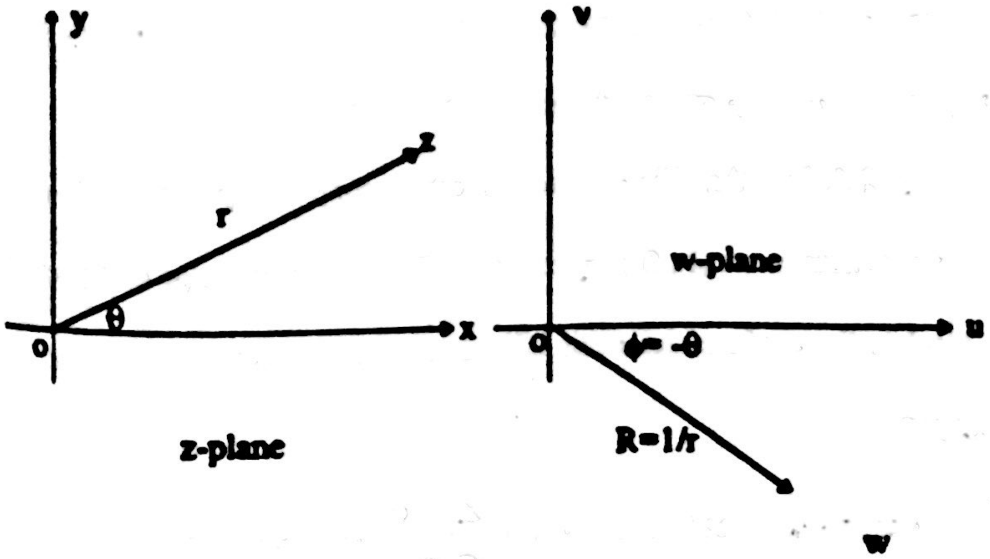
$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}$$

Also, $z = \frac{1}{w}$, i.e

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$$

Moreover, if $w = Re^{i\phi}$, and $z = re^{i\theta}$ then

$$R = \frac{1}{r}, \quad \phi = -\theta$$



3. Bilinear transformation :

The bilinear mapping is defined as:

$$w=\frac{az+b}{cz+d}, \text{ where } c\neq 0 \text{ and } (ad-bc)\neq 0$$

This transformation has C-$\frac{-d}{c}$ for its domain, and C-$\frac{a}{c}$ for its range since

$$w=\frac{az+b}{cz+d}=\frac{az+b}{c\left(z+\frac{d}{c}\right)}=\frac{a\left(z+\frac{d}{c}\right)+b-\frac{ad}{c}}{c\left(z+\frac{d}{c}\right)}$$

$$w=\frac{a}{c}+\left(\frac{bc-ad}{c}\right)\frac{1}{cz+d}$$

If we set

$$z'=cz+d \quad \text{and} \quad z''=\frac{1}{z}$$

Then

$$w=\frac{a}{c}+\left(\frac{bc-ad}{c}\right)z''$$

This implies that the bilinear transformation maps circles or straight lines in z-plane into circles or straight lines in w-plane.

Example 2.5. Find the bilinear function which maps the points $z=0, -i, -1$ into the points $w=i, 1, 0$ respectively. Then find the image of the circle $|z-1|=1$ in the w-plane under this bilinear transformation .

Solution.

Assume that the function is $w=\frac{az+b}{cz+d}$ then we have

$$i = \frac{b}{d} \quad (1)$$

$$1 = \frac{-ai + b}{-ci + d} \quad (2)$$

$$0 = \frac{-a + b}{-c + d} \quad (3)$$

It is easy to see that $b = a$, $d = -ia$, and $c=ia$. Hence the required function is then

$$w = \frac{az + a}{iaz - ia} = \frac{z+1}{i(z-1)} = (-i)\frac{z+1}{z-1}$$

If we solve for z , then

$$z = \frac{w-i}{w+i}$$

Now to find the image of $|z - 1|=1$ we have

$$\left| \frac{w-i}{w+i} - 1 \right| = 1$$

i.e. $\left|\frac{-2i}{w+i} - 1\right| = 1$ and hence, $|w+i|=2$, which is circle of center $-i = (0,-1)$ and radius 2.

EXERCISES VI

1. Find the region into which the half plane $y > 0$ is mapped by the function $w = (1+i)z$. Show the regions graphically.

2. Find the image of the semi-infinite strip $x > 0$, $0 < y < 2$. Under the function $w = iz + 1$. Show the regions graphically.

3. Find a linear transformation that maps the half plane $\mathrm{Im}(z) > 0$ into the region $R(w) > 1$.

4. Show that under the transformation $w = \dfrac{1}{z}$ circles or straight lines are mapped into circles or straight lines.

5. Find the image of the infinite strip $0 < y < \dfrac{1}{2c}$ under the function $w = \dfrac{1}{z}$. Show the regions graphically.

6. Find the image of the circle $|z-2|=1$ under the function $w = \dfrac{1}{z}$. Show the regions graphically.

7. Find the image of the circle $|z-2|=2$ under the function $w = \dfrac{1}{z}$. Show the regions graphically.

8. Find the image of the line $x + y = 0$ under the function $w = \dfrac{1}{z}$. Show the regions graphically.

9. Find the image of the line $x + y = 1$ under the function $w = \dfrac{1}{z}$. Show the regions graphically.

10. Find the bilinear mapping which maps the points $-1, 0, 1$ into the points $w = -1, i, 1$ respectively. Into what curve must this function map the y - axis?

11. Show that $w = \dfrac{z-1}{z+1}$ transforms the half plane $R(z) > 0$ into the unit disk $|w| < 1$.

12. Find the image of the region $\alpha \le \arg z < \alpha + \dfrac{2\pi}{3}$ under the function $w = z^3$.

13. Show that the imaginary axis $x = 0$ is mapped into the imaginary axis $u = 0$ under the mapping $w = \sin z$.

14. Find a linear transformation that maps the region $0 < \arg z < \pi/4$ into the region $R(w) > 1$.

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