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AIN SHAMS UNIVERSITY
FACULTY OF ENGINEERING
SPECIALIZED ENGINEERING PROGRAMS
JUNIOR COMMUNICATION ENGINEERING PROGRAM

SPRING 2022	Assignment #4	Total: 5 marks

PHM212s: Special Functions, Complex Analysis & Numerical Analysis

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Name: **ID:** **Deadline: Week 11**

Please, Solve each problem in its assigned place ONLY (the empty space below it)

Functions of the complex variable

1. Find the real and imaginary parts of each of the following functions and state its domain

a) $f(z)=z^2 = (x+iy)^2 = (x^2-y^2)+i(2xy) \Rightarrow D \in R^2$

b) $f(z)=e^{z^2} = e^{(x+iy)^2} = e^{(x^2-y^2)}\, e^{i(2xy)} = e^{x^2-y^2}\left[\cos(2xy)+i\sin(2xy)\right]$

$= \left[e^{x^2-y^2}\cos(2xy)\right] + i\left[e^{x^2-y^2}\sin(2xy)\right] \Rightarrow D \in R^2$

c) $f(z)=\sin z = \sin(x+iy) = \sin(x)\cos(iy)+\cos(x)\sin(iy)$

$= \left[\sin(x)\cosh(y)\right] + i\left[\cos(x)\sinh(y)\right] \Rightarrow D \in R^2$

d) $f(z)= 2z^3-3z = 2(x+iy)^3-3(x+iy) = 2\left(x^3+i3x^2y-3xy^2-iy^3\right)-3(x+iy)$

$= \left[2x^3-6xy^2-3x\right] + i\left[6x^2y-2y^3-3y\right] \Rightarrow D \in R^2$

e) $f(z)=\dfrac{(z+i)}{(z^2+1)} = \dfrac{x+iy+i}{(x^2-y^2)+i(2xy)+1} = \dfrac{x+i(1+y)}{(x^2-y^2+1)+i(2xy)} \ast \dfrac{(x^2-y^2+1)-i(2xy)}{(x^2-y^2+1)-i(2xy)}$

$= \dfrac{\left[x(x^2-y^2+1)-(1+y)(2xy)\right]+i\left[(1+y)(x^2-y^2+1)-x(2xy)\right]}{(x^2-y^2+1)^2+(2xy)^2}$

$= \left[\dfrac{x(x^2-y^2+1)-(1+y)(2xy)}{(x^2-y^2+1)^2+(2xy)^2}\right] + i\left[\dfrac{(1+y)(x^2-y^2+1)-x(2xy)}{(x^2-y^2+1)^2+(2xy)^2}\right] \Rightarrow D \in R^2$

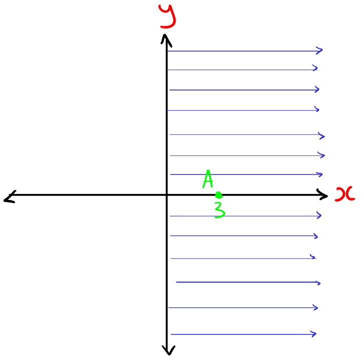
2. Describe the image of the following regions under the specified function (Show the regions graphically)

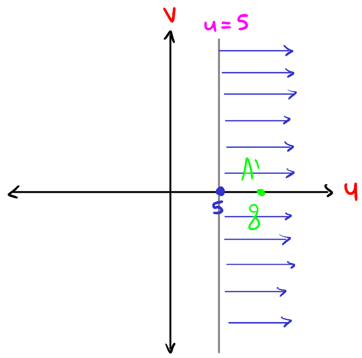
a) $\text{Re}(z) \geq 0$ for $f(z) = z + 5$

$x \geqslant 0$

• $f(z) = (x+5) + iy$

$\therefore u = x+5$ & $v = y \longrightarrow$ if $x \geqslant 0 \therefore u \geqslant 5$





b) $|\arg(z)| \leq \pi/4$, $|z| \leq 2$ for $f(z) = z^3$

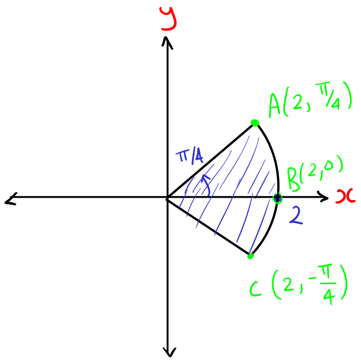
$\theta \leq \pi/4$ $r \leq 2$

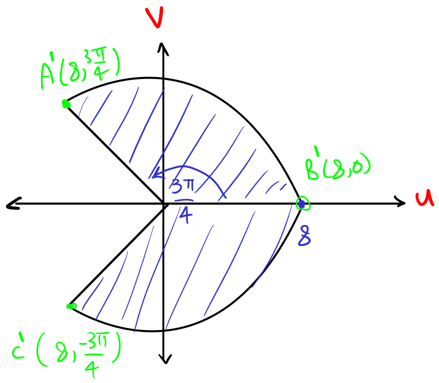
• $f(z) = (re^{i\theta})^3 = r^3 e^{i3\theta}$

$\therefore R = r^3$ & $\Phi = 3\theta$

if $\rightarrow r \leq 2$ $\therefore |f(z)| \leqslant 8$

$-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$ $\therefore \frac{-3\pi}{4} \leqslant \Phi \leqslant \frac{3\pi}{4}$



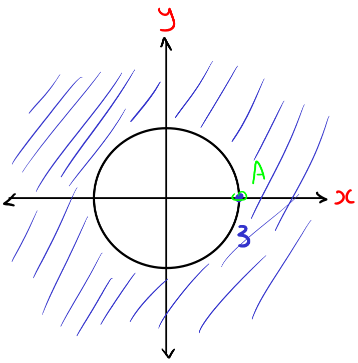


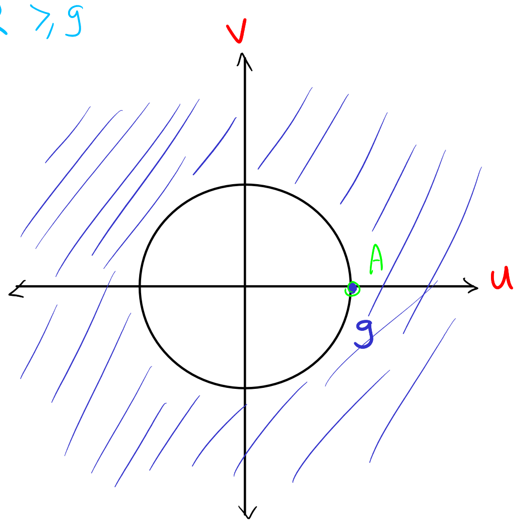
c) $|z| \geq 3$ for $f(z) = z^2$

• $f(z) = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$\therefore R = r^2$ & $\Phi = 2\theta$

if $\rightarrow r \geqslant 3$ $\therefore R \geqslant 9$

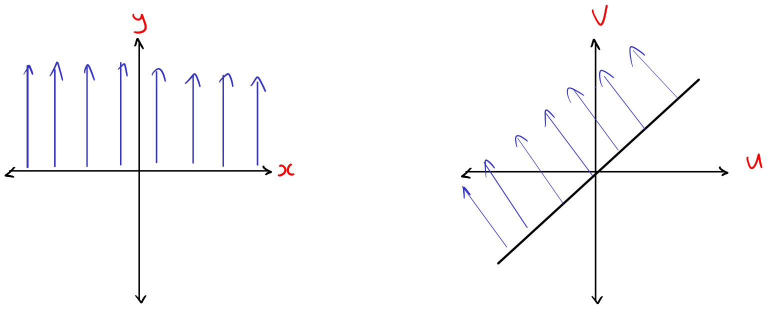




3. Find the region into which the half plane $y > 0$ is mapped by the function $w = (1+i)\,z$. Show the regions graphically.

- $z = \frac{1}{1+i}(w) = \left(\frac{1}{2} - \frac{1}{2}i\right)(u+iv) = \left(\frac{1}{2}u + \frac{1}{2}v\right) + i\left(-\frac{1}{2}u + \frac{1}{2}v\right)$

$\therefore x = \frac{1}{2}u + \frac{1}{2}v$ & $y = -\frac{1}{2}u + \frac{1}{2}v \longrightarrow$ if $y \geq 0$ $\therefore -\frac{1}{2}u + \frac{1}{2}v \geq 0$ $\therefore u \leq v$



4. Find the image of the semi – infinite strip $x > 0$, $0 < y < 2$ under the function $w = i\,z + 1$. Show the regions graphically.

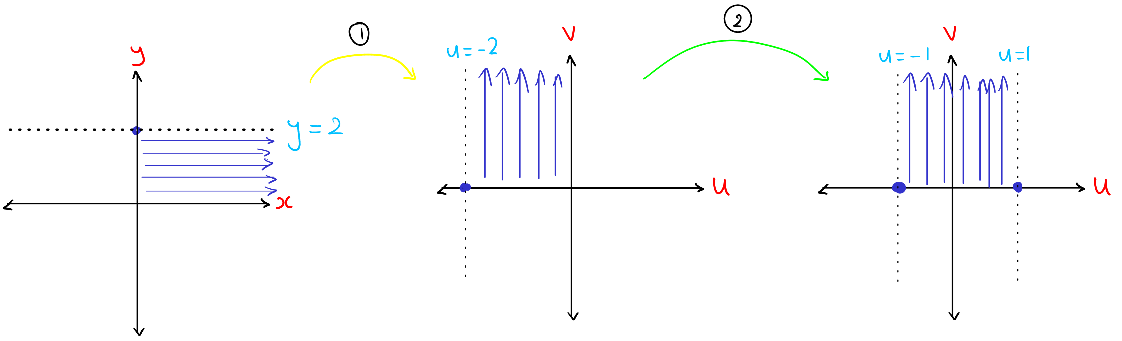
① rotating with $\frac{\pi}{2}$ Counter clock wise → ② Shifting with 1 in positive u

- $w = i(x+iy)+1$
  $= ix - y + 1$
  $= (1-y) + i(x)$

$\therefore u = (1-y)$ & $v = x$

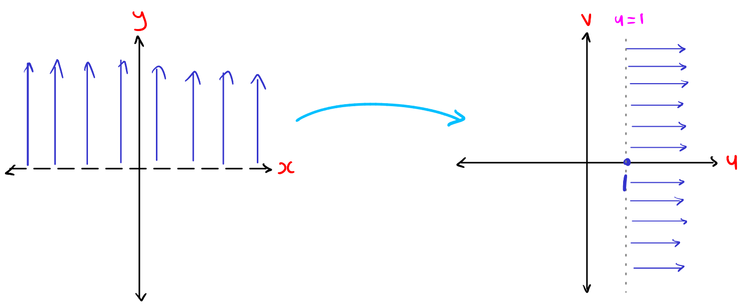
if: $0<y<2 \rightarrow -1<u<1$

: $x > 0 \longrightarrow v > 0$



5. Find a linear transformation that maps the half plane $\text{Im}(z) > 0$ into the region $\text{Re}(w) > 1$.

$u > 1$ $y > 0$



$\Rightarrow$ the region rotated by $\frac{\pi}{2}$ clockwise & shifted with 1 in positive u

$\therefore w = az + b = (\underbrace{r_a}_{=1} e^{i\overbrace{\theta_a}^{=-\frac{\pi}{2}}})z + \underbrace{b}_{=1}$

$\therefore w = e^{i(-\frac{\pi}{2})} z + 1 = -iz + 1$

6. Show that under the transformation $w = \frac{1}{z}$ circles or straight lines are mapped into circles or straight lines. → For $w = \frac{1}{z}$ ∴ $x = \frac{u}{u^2+v^2}$ & $y = \frac{-v}{u^2+v^2}$ ⇝ $x^2+y^2 = \frac{1}{u^2+v^2}$

• Consider the following equation: $A(x^2+y^2) + Bx + Cy + D = 0$

↳ which represents straight lines or circles depending on the values of constants

• by substituting: $A\left(\frac{1}{u^2+v^2}\right) + B\left(\frac{u}{u^2+v^2}\right) + C\left(\frac{-v}{u^2+v^2}\right) + D = 0$ $*(u^2+v^2)$

∴ $D(u^2+v^2) + Bu - Cv + A = 0$

A	D	pre image	image
$=0$	$=0$	Straight line passing through origin	Straight line passing through origin
$=0$	$\neq 0$	Straight line not passing through origin	Circle passing through origin
$\neq 0$	$=0$	Circle passing through origin	Straight line not passing through origin
$\neq 0$	$\neq 0$	Circle not passing through origin	Circle not passing through origin

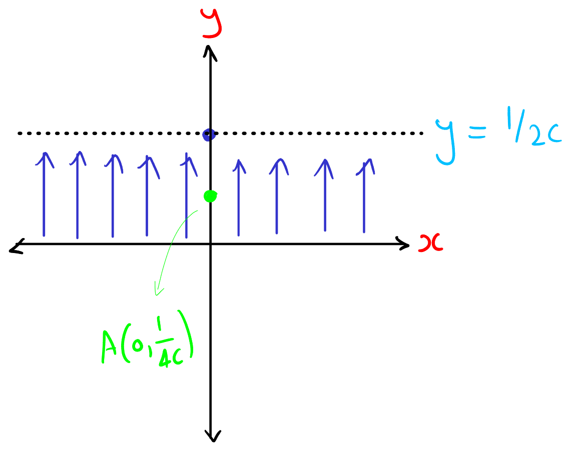
7. Under the Reciprocal transformation, find the image of the following regions & show the regions (The pre-image & The image) graphically. → $z = x + iy = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2}$

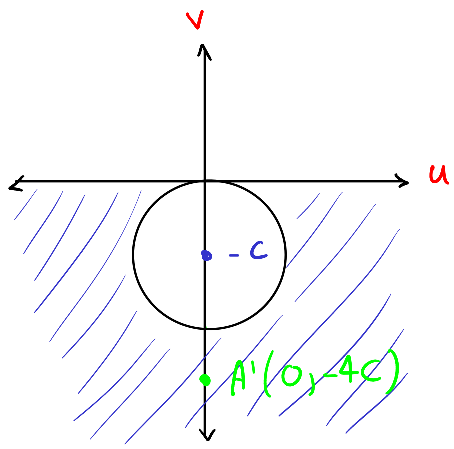
a) The infinite strip $0 < y < 1/2c$

∴ $x = \frac{u}{u^2+v^2}$ & $y = \frac{-v}{u^2+v^2}$

if $0 < y < 1/2c$ ∴ $0 < \frac{-v}{u^2+v^2} < 1/2c$ $\begin{cases} v < 0 \\ -v < \frac{1}{2c}(u^2+v^2) \rightarrow u^2+v^2+2cv > 0 \end{cases}$

$u^2 + (v+c)^2 > c^2$





b) The circle $|z-2|=1$

$\therefore \; |(x+iy)-2|=1,$

$|(x-2)+(iy)| = 1,$

$(x-2)^2+y^2 = 1^2,$

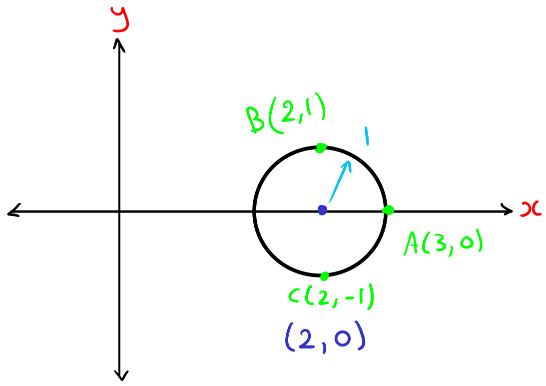
$\underbrace{x^2+y^2} - 4x + 3 = 0$

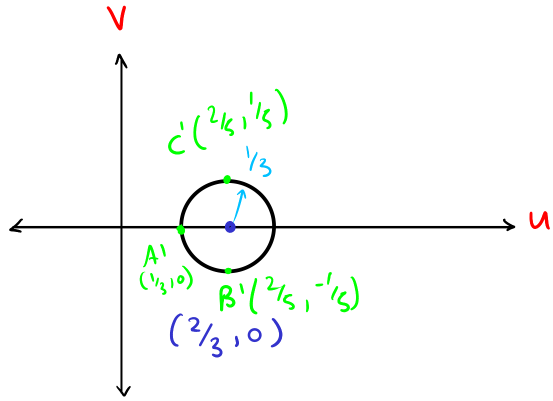
$\frac{1}{u^2+v^2} - \frac{4u}{u^2+v^2} + 3 = 0 \qquad * (u^2+v^2)$

$3(u^2+v^2) - 4u + 1 = 0$

$(u^2 - \frac{4}{3}u) + v^2 + \frac{1}{3} = 0$

$(u-\frac{2}{3})^2 + v^2 = 1/9$





c) The circle $|z-2|=2$

$\therefore \; |(x+iy)-2| = 2,$

$|(x-2)+(iy)| = 2,$

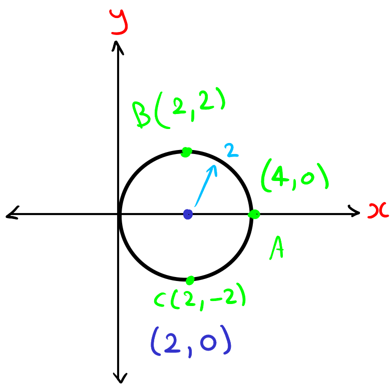
$(x-2)^2+y^2 = 4,$

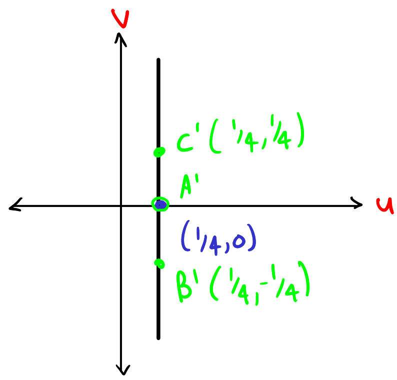
$\underbrace{x^2+y^2} - 4x = 0$

$\frac{1}{u^2+v^2} - \frac{4u}{u^2+v^2} = 0 \qquad * (u^2+v^2)$

$4u = 1$

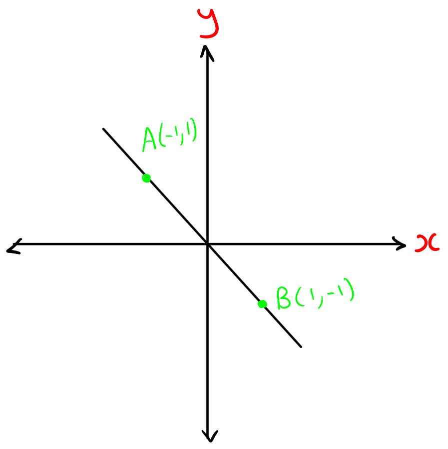
$u = 1/4$

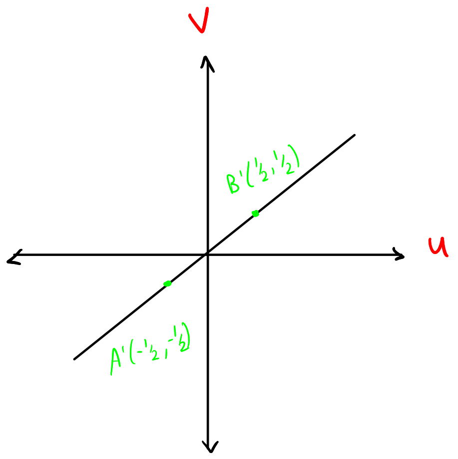




d) The line $x + y = 0$

- $$\frac{u}{u^2+v^2} - \frac{v}{u^2+v^2} = 0 \longrightarrow u = v$$



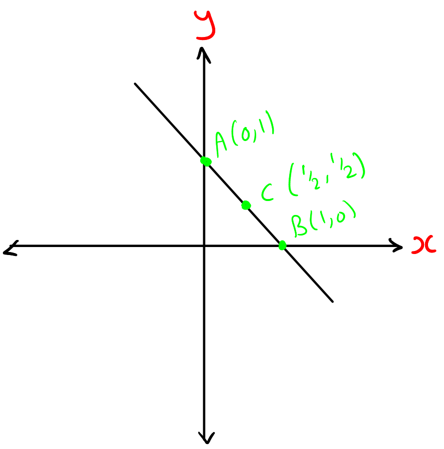


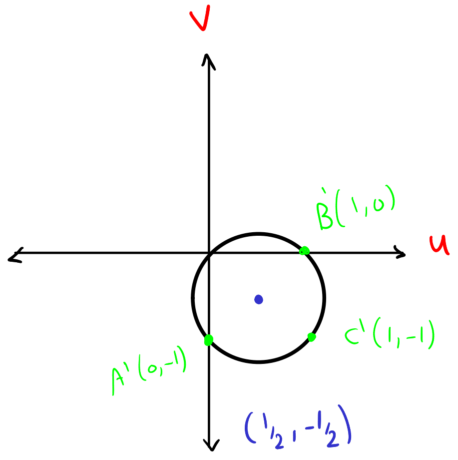
e) The line $x + y = 1$

- $$\frac{u}{u^2+v^2} - \frac{v}{u^2+v^2} = 1 \longrightarrow u - v = u^2 + v^2$$

$$u^2 - u + v^2 + v = 0$$

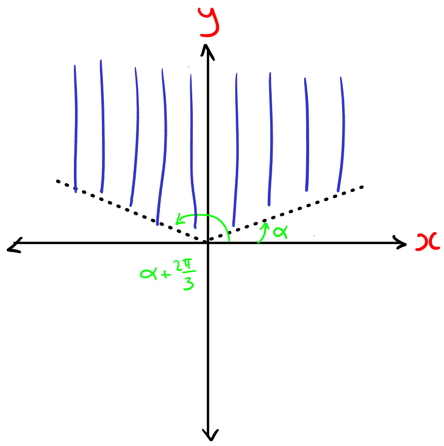
$$(u - \tfrac{1}{2})^2 + (v + \tfrac{1}{2})^2 = \tfrac{1}{2}$$

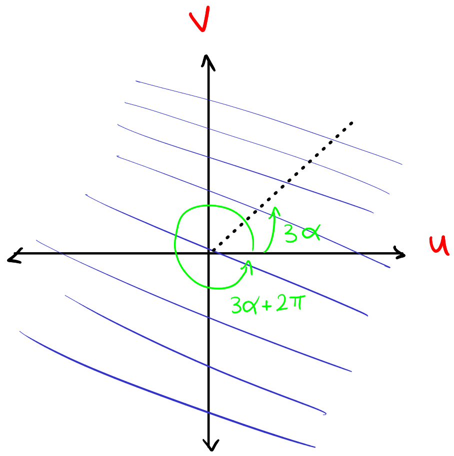




8. Find the image of the region $\alpha < \arg(z) < \alpha + 2\pi/3$ under the function $w = z^3$.

- $w = (re^{i\theta})^3 = r^3 e^{i3\theta}$ ∴ $\arg(w) = 3\arg(z)$ # Assume $\alpha < \frac{\pi}{2}$





9. Show that the imaginary axis $x = 0$ is mapped into the imaginary axis $u = 0$ under the mapping $w = \sin z$.

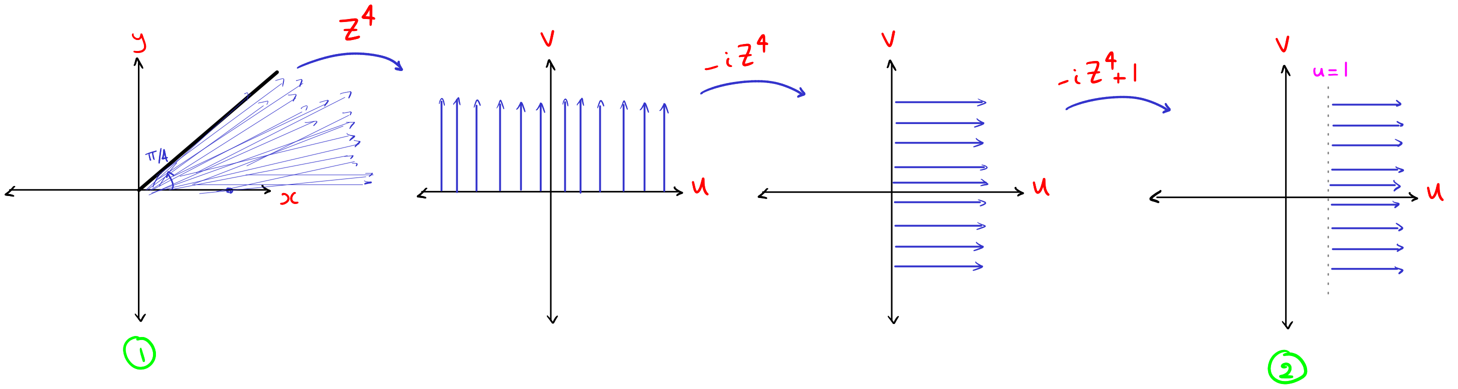
∴ $w = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$

$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

∴ $u = \sin(x)\cosh(y)$ & $v = \cos(x)\sinh(y)$

if $x = 0$ ∴ $u = 0$

10. Find a transformation that maps the region ① $0 < \arg(z) < \pi/4$ into the region ② $\mathrm{Re}(w) > 1$.



$\Longrightarrow$ $w = -iz^4 + 1$

11. Using Cauchy – Riemann's equations, determine where the following functions are differentiable and where they are analytic? If the function is differentiable, find its derivatives.

a) $w = \bar{z}$ • $w = x - iy$ ∴ $u = x$ & $v = -y$

$\rightarrow u_x = 1$ $\rightarrow v_y = -1$

$\rightarrow u_y = 0$ $\rightarrow v_x = 0$

∴ it's impossible to equate u_x & v_y

∴ w is not differentiable anywhere

∴ w is not analytic

b) $w = \cos z$ ∴ $w = \cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

∴ $u = \cos(x)\cosh(y)$ & $v = -\sin(x)\sinh(y)$

$\rightarrow u_x = -\sin(x)\cosh(y)$ $\rightarrow v_y = -\sin(x)\cosh(y)$

$\rightarrow u_y = \cos(x)\sinh(y)$ $\rightarrow v_x = -\cos(x)\sinh(y)$

∴ $u_x = v_y$ & $u_y = -v_x$

∴ w is differentiable everywhere

∴ w is analytic everywhere

$\Longrightarrow \dfrac{dw}{dz} = -\sin z$

c) $w = \ln z$

• $w = \ln(re^{i\theta}) = \ln r + i\theta$

∴ $u = \ln r$ & $v = \theta$

$\rightarrow u_r = 1/r$ $\rightarrow v_\theta = 1$

$\rightarrow u_\theta = 0$ $\rightarrow v_r = 0$

∴ $ru_r = v_\theta = 1$ & $rv_r = -u_\theta = 0$

∴ w is differentiable everywhere

∴ w is analytic everywhere

Except at $z = 0$ or the negative real axis

$\Longrightarrow \dfrac{dw}{dz} = \dfrac{1}{z}$

d) $w = 2y - i\,x$

∴ $u = 2y$ & $v = -x$

→ $u_x = 0$ → $v_y = 0$

→ $u_y = 2$ → $v_x = -1$

∵ it's impossible to equate u_y & $-v_x$

∴ w is not differentiable anywhere

∴ w is not analytic

e) $w = e^x(\cos y + i\sin y)$

∴ $u = e^x\cos y$ & $v = e^x\sin y$

→ $u_x = e^x\cos y$ → $v_y = e^x\cos y$

→ $u_y = -e^x\sin y$ → $v_x = e^x\sin y$

∵ $u_x = v_y$ & $u_y = -v_x$

∴ w is differentiable everywhere

∴ w is analytic everywhere

→ $w = e^x e^{iy} = e^{x+iy} = e^z$

$\Rightarrow \dfrac{dw}{dz} = e^z$

f) $w = z^2\,\bar{z}$

• $w = (x+iy)^2(x-iy) = (x^2-y^2+i2xy)(x-iy)$

$= x^3 - y^2x + i2x^2y - ix^2y + iy^3 + 2xy^2$

$= x^3 + xy^2 + i(x^2y + y^3)$

∴ $u = x^3 + xy^2$ & $v = x^2y + y^3$

→ $u_x = 3x^2+y^2$ → $v_y = x^2+3y^2$

→ $u_y = 2xy$ → $v_x = 2xy$

→ to equate u_x & v_y, also u_y & $-v_x$

• $3x^2+y^2 = x^2+3y^2$

$2x^2 = 2y^2 \rightarrow x^2 = y^2$

∴ $x = \pm y \rightarrow ①$

• $2xy = -2xy$

$4xy = 0$

∴ $xy = 0 \rightarrow ②$

② in ① $\Rightarrow \pm y^2 = 0 \left\{ \begin{array}{l} x = 0 \\ x = 0 \end{array} \right.$

∴ w is differentiable only at $(0,0)$

∴ w is not analytic

$\Rightarrow \dfrac{dw}{dz} = u_x + i v_x = (3x^2+y^2) + i(2xy)$

g) $w = e^y(\cos x + i\sin x)$

∴ $u = e^y\cos x$ & $v = e^y\sin x$

→ $u_x = -e^y\sin x$ → $v_y = e^y\sin x$

→ $u_y = e^y\cos x$ → $v_x = e^y\cos x$

• to equate u_x & v_y, also u_y & $-v_x$

∴ $-e^y\sin x = e^y\sin x$

∴ $2e^y\sin x = 0$

$\Rightarrow x = 0 \rightarrow ①$

∴ $e^y\cos x = -e^y\cos x$

$2e^y\cos x = 0$

$\Rightarrow x = \pm\pi(n+\tfrac{1}{2}) \rightarrow ②$, $n = 0,1,2,3,\ldots$

• from ① & ②

∵ it's impossible to equate u_y & $-v_x$ & u_x & v_y

∴ w is not differentiable anywhere

∴ w is not analytic

h) $w = 1/z$

- $w = \frac{1}{r} e^{-i\theta} = \frac{1}{r}\left[\cos(-\theta) + i \sin(-\theta)\right]$
  $= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$

$\therefore u = \frac{1}{r}\cos\theta$ & $v = -\frac{1}{r}\sin\theta$

$\rightarrow u_r = -\frac{1}{r^2}\cos\theta$ $\rightarrow v_\theta = -\frac{1}{r}\cos\theta$

$\rightarrow u_\theta = -\frac{1}{r}\sin\theta$ $\rightarrow v_r = \frac{1}{r^2}\sin\theta$

$\because r u_r = v_\theta = -\frac{1}{r}\cos\theta$ & $r v_r = -u_\theta = \frac{1}{r}\sin\theta$

$\therefore$ w is differentiable everywhere except at $(0,0)$

$\therefore$ w is analytic everywhere except at $(0,0)$

$\Rightarrow \frac{dw}{dz} = -1/z^2$

i) $w = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

$\therefore u = x^3 + 3xy^2 - 3x$ & $v = y^3 + 3x^2y - 3y$

$\rightarrow u_x = 3x^2 + 3y^2 - 3$ $\rightarrow v_y = 3y^2 + 3x^2 - 3$

$\rightarrow u_y = 6xy$ $\rightarrow v_x = 6xy$

$\therefore u_x = v_y$

$\rightarrow$ to equate u_y & $-v_x$

- $6xy = -6xy$ $\therefore 12xy = 0$ — $y = 0$, $x = 0$

$\therefore$ w is differentiable only at $y = 0$ or $x = 0$

(only at the Cartesian axes in z plane)

$\therefore$ w is not analytic

$\Rightarrow \frac{dw}{dz} = u_x + i v_x = (3x^2 + 3y^2 - 3) + i(6xy)$

12. Prove that the function $f(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$ is entire. Hence, write this function in terms of z.

$\therefore u = 3x^2 + 2x - 3y^2 - 1$ & $v = 6xy + 2y$

$\rightarrow u_x = 6x + 2$ $\rightarrow v_y = 6x + 2$

$\rightarrow u_y = -6y$ $\rightarrow v_x = 6y$

$\because u_x = v_y$ & $u_y = -v_x$

$\therefore f(z)$ is differentiable everywhere

$\therefore f(z)$ is analytic everywhere (Entire)

to rewrite the function in terms of z — put $x = z$, put $y = 0$

$\Longrightarrow f(z) = 3z^2 + 2z - 1$

13. Prove that the function $f(z)=e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$ is entire and find its derivative in terms of z.

∴ $u = e^{x^2-y^2}\cos 2xy$ & $v = e^{x^2-y^2}\sin 2xy$

$\rightarrow u_x = -2y e^{x^2-y^2}\sin 2xy + 2x e^{x^2-y^2}\cos 2xy$

$\rightarrow v_y = 2x e^{x^2-y^2}\cos 2xy - 2y e^{x^2-y^2}\sin 2xy$

$\rightarrow u_y = -2x e^{x^2-y^2}\sin 2xy - 2y e^{x^2-y^2}\cos 2xy$

$\rightarrow v_x = 2y e^{x^2-y^2}\cos 2xy + 2x e^{x^2-y^2}\sin 2xy$

∴ $f(z)$ is differentiable everywhere $= e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$

∴ $f(z)$ is analytic everywhere (Entire)

$\Longrightarrow f(z) = e^{z^2}$ $\Longrightarrow f'(z) = 2z e^{z^2}$

14. Prove that an analytic function $f(z)$ must be constant if any one of the following conditions hold:

a) $\mathrm{Re}(f(z)) =$ Constant.

- $u =$ Constant Assume it's equal to a
- $u = a$

$\rightarrow u_x = 0$ $\rightarrow u_y = 0$

∵ $f(z)$ is analytic

∴ $f(z)$ satisfies C.R equation

∴ $v_x = 0$ ∴ $v_y = 0 \rightarrow ①$

integrate w.r.t x

∴ $v = h(y)$

∴ $v_y = h'(y) \rightarrow ②$

$① = ②$

∴ $h'(y) = 0 \longrightarrow h(y) = c$

∵ $u =$ Constant & $v =$ Constant

∴ $f(z)$ is Constant #

b) $\mathrm{Im}(f(z)) =$ Constant.

- $v =$ Constant Assume it's equal to b
- $v = b$

$\rightarrow v_x = 0$ $\rightarrow v_y = 0$

∵ $f(z)$ is analytic

∴ $f(z)$ satisfies C.R equation

∴ $u_y = 0$ ∴ $u_x = 0 \rightarrow ①$

integrate w.r.t y

∴ $u = g(x)$

∴ $u_x = g'(x) \rightarrow ②$

$① = ②$

∴ $g'(x) = 0 \longrightarrow g(x) = c$

∵ $u =$ Constant & $v =$ Constant

∴ $f(z)$ is Constant #

c) $|f(z)| =$ Constant.

$\therefore \sqrt{u^2+v^2} = a$

$\therefore u^2+v^2 = a^2$

d/dx

$\therefore 2u\,u_x + 2v\,v_x = 0$

$\therefore u_x = -\frac{v}{u} v_x \rightarrow ①$

d/dy

$\therefore 2u\,u_y + 2v\,v_y = 0$

$\therefore u_y = -\frac{v}{u} v_y \rightarrow ②$

$\because f(z)$ is analytic

$\therefore f(z)$ satisfies C.R equation

$\therefore u_x = v_y$ & $u_y = -v_x \longrightarrow$ subs in ① & ②

$\therefore v_y = -\frac{u}{v} v_x \rightarrow ③$ $\therefore v_x = \frac{v}{u} v_y \rightarrow ④$

putting ③ in ④

$\therefore \cancel{v_x} = \frac{v}{u}\left(-\frac{u}{v}\right)\cancel{v_x} \longrightarrow u^2 = -v^2 \longrightarrow u^2+v^2 = 0$

$\therefore |f(z)| = 0$ $\therefore f(z) = 0$ #

d) $\overline{f(z)}$ is analytic.

- $f(z) = u + iv$ & $u_x = v_y$ & $u_y = -v_x \rightarrow ①$
- $\overline{f(z)} = u - iv$ & $u_x = -v_y$ & $u_y = v_x \rightarrow ②$

From ① & ②

$\therefore v_y = -v_y$ & $v_x = -v_x$

$\therefore v_y = 0$ & $v_x = 0$

$\therefore u_x = 0$ & $u_y = 0$

$\therefore u =$ Constant & $v =$ Constant

$\therefore f(z)$ is Constant #

e) $|f(z)|$ is analytic.

- $|f(z)| = \underbrace{\sqrt{u^2+v^2}}_{U} + i\underbrace{0}_{V}$

$\therefore U = \sqrt{u^2+v^2} = (u^2+v^2)^{1/2}$ & $V = 0$

$\rightarrow U_x = \frac{1}{2}(u^2+v^2)^{-1/2}(2u u_x + 2v v_x) \rightarrow V_y = 0$

$\rightarrow U_y = \frac{1}{2}(u^2+v^2)^{-1/2}(2u u_y + 2v v_y) \rightarrow V_x = 0$

$\because |f(z)|$ is analytic $\therefore U_x = V_y$ & $U_y = -V_x$

$\therefore 2u u_x + 2v v_x = 0$ $\therefore 2u u_y + 2v v_y = 0$

$\because f(z)$ is analytic $\therefore u_x = v_y$ & $u_y = -v_x$

$\therefore u_x = 0$ & $u_y = 0$ & $v_x = 0$ & $v_y = 0$

$\therefore u =$ Constant & $v =$ Constant $\therefore f(z)$ is Constant #

15. Show that each of the following functions is harmonic and find a corresponding analytics function $f(z) = u + i\,v$:

a) $u = 2\,x\,(1 - y)$

- $u_x = 2(1-y)$ & • $u_y = -2x$
- $u_{xx} = 0$ & • $u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0 \longrightarrow$ "$f(z)$ is harmonic function"

$\because f(z)$ is analytic

① — $v_y = u_x = 2(1-y)$

$\therefore v = 2y - y^2 + g(x)$ $\therefore v_x = g'(x) \rightarrow ①$

② — $v_x = -u_y = 2x \rightarrow ②$

① = ② $\longrightarrow g'(x) = 2x$

$\therefore g(x) = x^2 + k$

$\therefore v = 2y - y^2 + x^2 + k$

$\therefore f(z) = 2x(1-y) + i(2y - y^2 + x^2 + k)$

b) $u = \cos x \cosh y$

- $u_x = -\sin x \cosh y$
- $u_y = \cos x \sinh y$
- $u_{xx} = -\cos x \cosh y$
- $u_{yy} = \cos x \cosh y$

$\therefore u_{xx} + u_{yy} = \cos x \cosh y - \cos x \cosh y = 0$

"$f(z)$ is harmonic function" #

$\because f(z)$ is analytic

① — $v_y = u_x = -\sin x \cosh y$

$\therefore v = -\sin x \sinh y + g(x)$

- $v_x = -\cos x \sinh y + g'(x)$

② — $v_x = -u_y = -\cos x \sinh y$

① = ② $\longrightarrow g'(x) = 0$ $\therefore g(x) = k$

$\therefore v = -\sin x \sinh y + k$

$\therefore f(z) = \cos x \cos y - i(\sin x \sinh y + k)$

c) $v = y/(x^2+y^2)$

→ put $x = r\cos\theta$ & $y = r\sin\theta$ $\therefore x^2+y^2=r^2$

$\therefore v = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$

→ $v_r = -\frac{\sin\theta}{r^2}$ → $v_\theta = \frac{\cos\theta}{r}$

→ $v_{rr} = \frac{\sin\theta}{r^3}$ → $v_{\theta\theta} = \frac{-\sin\theta}{r}$

$\because v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = \frac{2\sin\theta}{r^3} + \frac{1}{r}\left(\frac{-\sin\theta}{r^2}\right) + \frac{1}{r^2}\left(\frac{-\sin\theta}{r}\right)$

$= \frac{2\sin\theta}{r^3} - \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} = \frac{2\sin\theta - 2\sin\theta}{r^3} = 0$

$\therefore f(z)$ is harmonic

$\therefore r v_r = -u_\theta$

$-\frac{\sin\theta}{r} = -u_\theta$

$u_\theta = \frac{\sin\theta}{r}$

$u = \frac{-\cos\theta}{r} + h(r)$

$\Rightarrow u_r = \frac{\cos\theta}{r^2} + h'(r)$ ①

$\therefore r u_r = v_\theta$

$u_r = \frac{1}{r}\left(\frac{\cos\theta}{r}\right)$

$u_r = \frac{\cos\theta}{r^2}$ ②

① = ②

$\therefore h'(r) = 0$

$h(r) = k$ $\therefore u = -\frac{\cos\theta}{r} + k$

$\therefore f(z) = \left(\frac{-x}{x^2+y^2} + k\right) + i\left(\frac{y}{x^2+y^2}\right)$

Another Solution in the last page

d) $v = x^3 - 3xy^2 + y$

- $v_x = 3x^2 - 3y^2$
- $v_{xx} = 6x$
- $v_y = -6xy + 1$
- $v_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 6x - 6x = 0$

"$f(z)$ is harmonic function" #

$\because f(z)$ is analytic

① — $u_x = v_y = -6xy + 1$

$\therefore u = -3x^2y + x + h(y)$

$\therefore u_y = -3x^2 + h'(y)$ → ①

② — $u_y = -v_x = -3x^2 + 3y^2$ → ②

① = ② $\longrightarrow h'(y) = 3y^2$ $\therefore h(y) = y^3 + k$

$\therefore u = y^3 - 3x^2y + x$

$\therefore f(z) = (y^3 - 3x^2y + x) + i(x^3 - 3xy^2 + y)$

e) $u = \arg(z) \longrightarrow u = \theta$

$\rightarrow u_r = 0$ $\rightarrow u_\theta = 1$

$\rightarrow u_{rr} = 0$ $\rightarrow u_{\theta\theta} = 0$

$\therefore u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$\therefore$ $F(Z)$ is harmonic

$\therefore r u_r = V_\theta = 0$ & $r V_r = -u_\theta = -1$

$\therefore V_r = -1/r \longrightarrow V = -\ln(r) + h(\theta)$

$\therefore V_\theta = 0 \longrightarrow h'(\theta) = 0 \longrightarrow h(\theta) = k$

$\Rightarrow V = -\ln(r) + k$

$\therefore F(Z) = \theta - i\left[\ln(r) + k\right]$

$= \tan^{-1}(y/x) - i\left[\frac{1}{2}\ln(x^2+y^2) + k\right]$

\# Another Solution on the last page

16. Prove that if $f(z) = u + i\,v$ is analytic in some domain **D**, then both ***u*** and ***v*** are harmonic functions in the same domain **D**.

$\therefore$ $F(Z)$ is analytic in some domain D

$\therefore u_x = V_y$ & $u_y = -V_x$

① $u_{xx} = V_{yx}$ ② $V_{yy} = u_{xy}$ ③ $u_{yy} = -V_{xy}$ ④ $V_{xx} = -u_{yx}$

$\therefore u_{xy} = u_{yx}$ & $V_{xy} = V_{yx}$ "Clairaut's theorem"

$\longrightarrow$ from ① & ③

$\therefore u_{xx} = -u_{yy}$

$u_{xx} + u_{yy} = 0$

$\longrightarrow$ from ② & ④

$\therefore V_{xx} = -V_{yy}$

$V_{xx} + V_{yy} = 0$

$\therefore$ u & V are harmonic function in D

17. Prove that if ***v*** is a harmonic conjugate of ***u*** in some domain **D**, then ***uv*** is harmonic in the same domain **D**.

$\therefore$ V is a harmonic conjugate of u

$\therefore u_x = V_y$ & $u_y = -V_x$

$\therefore u_x V_x = -u_y V_y$ $\therefore u_x V_x + u_y V_y = 0 \longrightarrow ①$

$\therefore u_{xx} + u_{yy} = 0$ & $V_{xx} + V_{yy} = 0 \longrightarrow ②$

* $\frac{d}{dx}(uv) = uV_x + u_x V$

* $\frac{d^2}{dx^2}(uv) = uV_{xx} + u_x V_x + u_x V_x + u_{xx} V$

$= u V_{xx} + u_{xx} V + 2u_x V_x \longrightarrow ③$

* $\frac{d}{dy}(uv) = uV_y + u_y V$

* $\frac{d^2}{dy^2}(uv) = uV_{yy} + u_y V_y + u_y V_y + u_{yy} V$

$= u V_{yy} + u_{yy} V + 2u_y V_y \longrightarrow ④$

\# ③ + ④

$\therefore uV_{yy} + u_{yy}V + 2u_yV_y + uV_{xx} + u_{xx}V + 2u_xV_x =$

$u\left(V_{xx} + V_{yy}\right) + V\left(u_{xx} + u_{yy}\right) + 2\left(u_xV_x + u_yV_y\right)$

\# from ① & ②

$u(\underbrace{V_{xx} + V_{yy}}_{=0}) + V(\underbrace{u_{xx} + u_{yy}}_{=0}) + 2(\underbrace{u_xV_x + u_yV_y}_{=0}) = 0$

$\therefore \frac{d^2}{dx^2}(uv) + \frac{d^2}{dy^2}(uv) = 0$

$\therefore$ uV is harmonic

18. Let $f(z)$ be analytic and non – zero in a domain **D**. Prove that $\ln|f(z)|$ is harmonic in **D**.

• $f(z) = u + iv \rightarrow |f(z)| = \sqrt{u^2+v^2} \Rightarrow \ln|f(z)| = \frac{1}{2}\ln(u^2+v^2)$

∵ $f(z)$ is analytic in some domain D ∴ $u_x = v_y$ & $u_y = -v_x$ ∴ $u_x^2 = v_y^2$ & $u_y^2 = v_x^2$ ∴ $u_x v_x = -u_y v_y$

Also $\rightarrow u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

* $\frac{d}{dx}\left(\ln|f(z)|\right) = \frac{uu_x + vv_x}{u^2+v^2}$
* $\frac{d}{dy}\left(\ln|f(z)|\right) = \frac{uu_y + vv_y}{u^2+v^2}$
* $\frac{d^2}{dx^2}\left(\ln|f(z)|\right) = \frac{(u^2+v^2)(uu_{xx} + u_x^2 + vv_{xx} + v_x^2) - 2(uu_x + vv_x)^2}{(u^2+v^2)^2} \rightarrow ①$
* $\frac{d^2}{dy^2}\left(\ln|f(z)|\right) = \frac{(u^2+v^2)(uu_{yy} + u_y^2 + vv_{yy} + v_y^2) - 2(uu_y + vv_y)^2}{(u^2+v^2)^2} \rightarrow ②$

① + ②

$$\rightarrow \frac{d^2}{dx^2}\left(\ln|f(z)|\right) + \frac{d^2}{dy^2}\left(\ln|f(z)|\right) = \frac{(u^2+v^2)(uu_{xx} + u_x^2 + vv_{xx} + v_x^2) - 2(uu_x + vv_x)^2 + (u^2+v^2)(uu_{yy} + u_y^2 + vv_{yy} + v_y^2) - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)\left[u(\underbrace{u_{xx} + u_{yy}}_{=0}) + v(\underbrace{v_{xx} + v_{yy}}_{=0}) + u_x^2 + v_y^2 + u_y^2 + v_x^2\right] - 2\left(u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y\right)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)\left[2(u_x^2 + u_y^2)\right] - 2\left[(u^2+v^2)(u_x^2 + u_y^2) + 2uv(\underbrace{u_xv_x + u_yv_y}_{=0})\right]}{(u^2+v^2)^2} = \frac{2(u^2+v^2)\left[u_x^2 + u_y^2\right] - 2(u^2+v^2)\left[u_x^2 + u_y^2\right]}{(u^2+v^2)^2} = 0$$

∴ $\ln|f(z)|$ is harmonic in D

19. Find all the roots of the following equations:

a) $\cos z = 2$ ∴ $\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = 2$

① → $\sin x \sinh y = 0$ ② → $\cos x \cosh y = 2$

∴ $\sin x = 0$ or ∴ $\sinh y = 0$

∴ $x = n\pi$ ($n = 0, \pm1, \pm2, \ldots$) ∴ $y = 0$

• Subs. in ② • Subs. in ②

∴ $(-1)^n \cosh y = 2$ ∴ $\cos x = 2$ "refused"

∴ $\cosh y = \frac{2}{(-1)^n} \longrightarrow$ cosh y can't be negative. So, n must be even

∴ $\cosh y = 2$ ∴ $y = \pm\cosh^{-1}(2)$

$\Rightarrow z = n\pi \pm i\cosh^{-1}(2)$ # $n = 0, \pm2, \pm4, \pm6, \ldots$

b) $\sin z = \cosh 4$ $\therefore \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y = \cosh 4$

① → $\cos x \sinh y = 0$ ② → $\sin x \cosh y = \cosh 4$

$\therefore \cos x = 0$ or $\therefore \sinh y = 0$

$\therefore x = (n + \frac{1}{2})\pi$ $n = 0, \pm1, \pm2, \ldots$

$\therefore y = 0$

• Subs. in ②

$\therefore \sin x = \cosh 4$ "refused"

• Subs. in ②

$\therefore (-1)^n \cosh y = \cosh 4$

$\therefore \cosh y = \frac{\cosh 4}{(-1)^n}$ → $\cosh y$ Can't be negative. So, n must be even

$\Rightarrow z = \pi(n + \frac{1}{2}) \pm i4$ $\# n = 0, \pm2, \pm4, \pm6, \ldots$

$\therefore \cosh y = \cosh 4$ $\therefore y = \pm 4$

c) $\cosh z = 0$ $\therefore \cosh(x+iy) = \cosh x \cosh(iy) + \sinh x \sinh(iy) = \cosh x \cos y + i \sinh x \sin y = 0$

① → $\cosh x \cos y = 0$ ② → $\sinh x \sin y = 0$

$\therefore \cos y = 0$ or $\therefore \cosh x = 0$ "refused"

$\therefore y = (n + \frac{1}{2})\pi$ $n = 0, \pm1, \pm2, \ldots$

• Subs. in ②

$\therefore (-1)^n \sinh x = 0$

$\Rightarrow z = i(n + \frac{1}{2})\pi$ $n = 0, \pm1, \pm2, \ldots$

$\therefore x = 0$

20. Find all the values of

• $\ln(z) = \ln r + i\theta$

a) $\ln 1 = \ln 1 + i(2n\pi) = 2n\pi i$ (where n is any integer)

b) $\ln(-1) = \ln(1) + i[(2n+1)\pi] = (2n+1)\pi i$ (where n is any integer)

c) $\ln i = \ln(1) + i[(2n+\frac{1}{2})\pi] = (2n+\frac{1}{2})\pi i$ (where n is any integer)

d) $\ln(-e\,i) = \ln(e) + i\left[(2n-\frac{1}{2})\pi\right] = 1 + i(2n-\frac{1}{2})\pi$ (where n is any integer)

e) $\ln(1-i) = \ln(\sqrt{2}) + i\left[(2n-\frac{1}{4})\pi\right] = \frac{1}{2}\ln 2 + i(2n-\frac{1}{4})\pi$ (where n is any integer)

$= \sqrt{2}\, e^{i(2n\pi-\frac{\pi}{4})}$

f) $\ln(-3+i\sqrt{27}) = \ln(6) + i\left[(2n+\frac{2}{3})\pi\right] = \ln 6 + i(2n+\frac{2}{3})\pi$ (where n is any integer)

$= 6\, e^{i(2n\pi+\frac{2}{3}\pi)}$

g) $(1+i)^i$ → let $Z = (1+i)^i$ $\therefore Z = e^{\ln(1+i)^i}$

$\therefore Z = e^{i\ln(1+i)}$

$\ast\ \theta = \ln(1+i) = \ln(\sqrt{2}) + i(2n+\frac{1}{4})\pi$

$\therefore i\theta = i\ln(\sqrt{2}) - (2n+\frac{1}{4})\pi$

$\therefore Z = e^{-(2n+1/4)\pi + i\ln(\sqrt{2})} = e^{-(2n+1/4)\pi}\, e^{i\ln(\sqrt{2})}$

$|Z| = e^{-(2n+1/4)\pi}$ (where n is any integer)

$\arg(Z) = \ln(\sqrt{2})$

h) $(-1)^{1/\pi}$ → let $Z = (-1)^{1/\pi}$ $\therefore Z = e^{\ln(-1)^{1/\pi}}$

$\therefore Z = e^{\frac{1}{\pi}\ln(-1)}$

$\ast\ \ln(-1) = \cancel{\ln(1)} + i\left[(2n+1)\pi\right] = (2n+1)\pi i$ (with 0 written above the struck $\ln(1)$)

$\therefore \frac{1}{\pi}\ln(-1) = (2n+1)i$

$\therefore Z = e^{(2n+1)i}$

$|Z| = 1$

$\arg(Z) = 2n+1$ (where n is any integer)

i) $\left(\frac{e}{2}\left(-1-i\sqrt{3}\right)\right)^{3\pi i}$ $\longrightarrow$ let $Z = \left[\frac{e}{2}\left(-1-i\sqrt{3}\right)\right]^{3\pi i}$

$$\therefore Z = \left[\frac{e}{\cancel{2}}\left(\cancel{2}e^{i(2n-\frac{2}{3})\pi}\right)\right]^{3\pi i}$$

$$= \left(e^{i(2n-\frac{2}{3})\pi+1}\right)^{3\pi i}$$

\# $(3\pi i) * \left[1 + i\pi\left(2n-\frac{2}{3}\right)\right] = 3\pi i - \pi^2(6n-2)$

$$= e^{-\pi^2(6n-2)+3\pi i}$$

$$= e^{\pi^2(2-6n)}\, e^{i(3\pi)}$$

$$\therefore Z = -e^{\pi^2(2-6n)}$$

$\rightarrow |Z| = e^{\pi^2(2-6n)}$ (where n is any integer)

$\rightarrow \arg(Z) = \pi$

Best wishes,

Dr. Makram Roshdy, Dr. Betty Nagy

c) $v = y/(x^2+y^2)$

- $V_x = -2xy\,(x^2+y^2)^{-2}$
- $V_{xx} = (-2xy)\left[-2(2x)(x^2+y^2)^{-3}\right] - 2y\,(x^2+y^2)^{-2} = 8x^2y\,(x^2+y^2)^{-3} - 2y\,(x^2+y^2)^{-2}$
- $V_y = -2y^2(x^2+y^2)^{-2} + (x^2+y^2)^{-1} = (x^2-y^2)(x^2+y^2)^{-2}$
- $V_{yy} = 8y^3(x^2+y^2)^{-3} - 4y\,(x^2+y^2)^{-2} - 2y\,(x^2+y^2)^{-2} = 8y^3(x^2+y^2)^{-3} - 6y\,(x^2+y^2)^{-2}$

$\therefore V_{xx}+V_{yy} = \dfrac{8x^2y - 2y(x^2+y^2)}{(x^2+y^2)^3} + \dfrac{8y^3 - 6y\,(x^2+y^2)}{(x^2+y^2)^3} = \dfrac{\cancel{8x^2y} - \cancel{2x^2y} - \cancel{2y^3} + \cancel{8y^3} - \cancel{6x^2y} - \cancel{6y^3}}{(x^2+y^2)^3}$

$\therefore V_{xx}+V_{yy} = 0$

"$f(z)$ is harmonic Function" #

$\therefore$ $f(z)$ is analytic

① — $u_y = -V_x = x\,\dfrac{2y}{(x^2+y^2)^2}$ $\therefore u = \dfrac{-x}{(x^2+y^2)} + g(x)$

$\therefore u_x = \dfrac{x^2-y^2}{(x^2+y^2)^2} + g'(x) \longrightarrow ①$

② — $u_x = V_y = \dfrac{x^2-y^2}{(x^2+y^2)^2} \longrightarrow ②$

① = ② $\longrightarrow g'(x) = 0$ $\therefore g(x) = k$

$\therefore u = \dfrac{-x}{(x^2+y^2)} + k$

$\therefore f(z) = \left(\dfrac{-x}{x^2+y^2} + k\right) + i\,\dfrac{y}{x^2+y^2}$

---

e) $u = \arg(z) \longrightarrow u = \tan^{-1}(y/x)$

$\therefore u_x = \dfrac{-y/x^2}{(1+y^2/x^2)} = \dfrac{-y}{x^2+y^2}$ $\therefore u_{xx} = \dfrac{2xy}{(x^2+y^2)^2}$

$\therefore u_y = \dfrac{1/x}{(1+y^2/x^2)} = \dfrac{x}{x^2+y^2}$ $\therefore u_{yy} = \dfrac{-2xy}{(x^2+y^2)^2}$

$\therefore u_{xx}+u_{yy} = \dfrac{2xy}{(x^2+y^2)^2} + \dfrac{-2xy}{(x^2+y^2)^2} = 0$

"$f(z)$ is harmonic Function" #

$\therefore$ $f(z)$ is analytic

① — $V_y = u_x = \dfrac{-y}{x^2+y^2}$ $\therefore V = -\frac{1}{2}\ln(x^2+y^2) + g(x)$

$\therefore V_x = \dfrac{-x}{x^2+y^2} + g'(x) \longrightarrow ①$

② — $V_x = -u_y = \dfrac{-x}{x^2+y^2} \longrightarrow ②$

① = ② $\longrightarrow g'(x) = 0$ $\therefore g(x) = k$

$\therefore V = -\frac{1}{2}\ln(x^2+y^2) + k$

$\Longrightarrow f(z) = \tan^{-1}(y/x) - i\left(\frac{1}{2}\ln(x^2+y^2) + k\right)$
