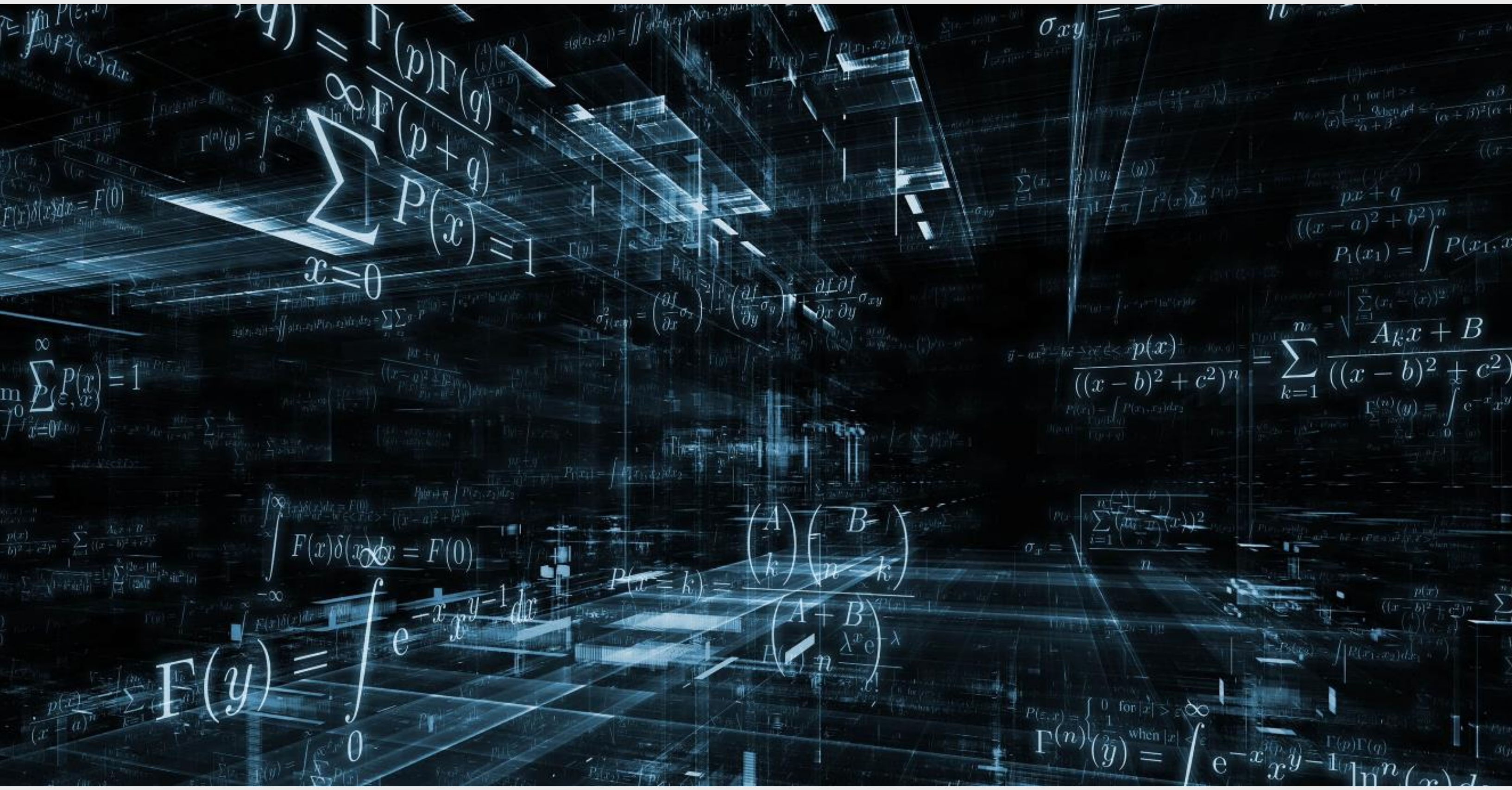
PHM212s: Complex, Special Functions and Numerical Analysis

Exercise Sheet

Bessel Functions



"All the world's a differential equation, and

the men and women are merely variables."

-Ben Orlin

[1] Solve in terms of Bessel Functions the following differential equation:

(a) $y'' - \frac{1}{x}y' + \left(1 - \frac{3}{x^2}\right)y = 0$; put $y = xu$ *[final 2013]*

(b) $xy'' - 3y' + xy = 0$ *[final 2016]*

(c) $y'' + (3e^{2x} - 4)y = 0$; use $e^x = z$ *[final 2017]*

(d) $xy'' + y = 0$ *[final 2018]*

(e) $xy'' + 5y' + xy = 0$ *[final Spring 2019]*

(f) $xy'' - 7y' + xy = 0$ *[final Spring 2021]*

(g) $(x-1)^2y'' + (x-1)y' + (x^2 - 2x - 3)y = 0$ *[Midterm Credit Spring 2022]*

[2] Evaluate:

(a) $\int_0^1 x^5 J_0(x)\,dx$ "in terms of J_0 and J_1" *[Final Fall 2015]*

(b) $\int x^4 J_1(x)\,dx$ "in terms of J_0 and J_1" *[Final 2018]*

(c) $\int x^{3/2} J_{-1/2}(x)\,dx$ *[Final 2016]*

(d) $\int J_5(x)\;dx$ *[Final 2016]*

[3] Show That:

-By using the formula: $$J_\theta(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\theta}}{2^{2m+\theta} m!\,\Gamma(m+\theta+1)}$$

(a) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ [Final Spring 2021]

(b) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ [Final 2015]

(c) [Final 2015] [Final 2017] [Final 2018]

i. $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$

ii. $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

iii. $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

(d) $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$ [Midterm Spring 2017] [Final 2013]

(e) $\int J_{n+1}\, dx = \int J_{n-1}\, dx - 2J_n$ [Final Spring 2021]

(f) $J_n(x)$ is an **Odd Function** when n is odd & it is an **Even Function** when n is even.