Special Functions

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* Prove that $\beta(x,y) = \beta(y,x)$

$\Rightarrow \beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt \quad ; x>0 \ \& \ y>0$

put $u = 1-t \Rightarrow t = 1-u \Rightarrow dt = -du \Rightarrow \begin{matrix} t=0 \rightarrow u=1 \\ t=1 \rightarrow u=0 \end{matrix}$

$\Rightarrow \beta(x,y) = \int_1^0 (1-u)^{x-1} u^{y-1} (-du)$

$= \int_0^1 u^{y-1} (1-u)^{x-1}\, du = \beta(y,x)$ #

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* Prove that $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$

$\Rightarrow \beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt$

put $t = \sin^2\theta \Rightarrow dt = 2\sin\theta\cos\theta\, d\theta \Rightarrow \begin{matrix} t=0 \rightarrow \theta=0 \\ t=1 \rightarrow \theta=\frac{\pi}{2} \end{matrix}$

$\Rightarrow 1-t = \cos^2\theta$

$\Rightarrow \beta(x,y) = \int_0^{\frac{\pi}{2}} \sin^{2x-2}\theta \cdot \cos^{2y-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cdot \cos^{2y-1}\theta\, d\theta$ #

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* Prove that $\beta(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du$

$\Rightarrow \beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt$

put $t = \frac{u}{1+u} \Rightarrow 1-t = \frac{1}{1+u} \Rightarrow dt = \frac{(1+u-u)du}{(1+u)^2} \Rightarrow \begin{matrix} t=0 \rightarrow u=0 \\ t=1 \rightarrow u=\infty \end{matrix}$

$\Rightarrow \beta(x,y) = \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1} \cdot \left(\frac{1}{1+u}\right)^{y-1} \cdot \left(\frac{1}{(1+u)^2}\right) du$

$= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y-\cancel{2}+\cancel{2}}}\, du = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du$ #

Prove that $\beta(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$\Rightarrow \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad \Gamma(y) = \int_0^\infty u^{y-1} e^{-u}\, du$

$\Rightarrow \Gamma(x) \cdot \Gamma(y) = \int_0^\infty \int_0^\infty t^{x-1} u^{y-1} e^{-(t+u)}\, dt \cdot du$

to change variables $\Rightarrow$

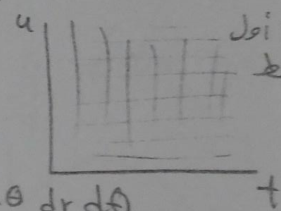
put $t = r\sin^2\theta \qquad u = r\cos^2\theta$

$$dt \cdot du = \left|J\left(\frac{t,u}{r,\theta}\right)\right| dr\, d\theta = \begin{vmatrix} \frac{\partial t}{\partial r} & \frac{\partial t}{\partial \theta} \\ \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \end{vmatrix} dr\, d\theta$$

$$= \begin{vmatrix} \sin^2\theta & 2r\sin\theta\cos\theta \\ \cos^2\theta & -2r\cos\theta\sin\theta \end{vmatrix} dr\, d\theta = \left|-2r\cos\theta\sin\theta\right| dr\, d\theta = 2r\cos\theta\sin\theta\, dr\, d\theta$$

$r \quad 0 \longrightarrow \infty$

$\theta \quad 0 \longrightarrow \frac{\pi}{2}$



$$\Rightarrow \Gamma(x) \cdot \Gamma(y) = \int_0^{\frac{\pi}{2}} \int_0^\infty (r\sin^2\theta)^{x-1} \cdot (r\cos^2\theta)^{y-1} e^{-r} \cdot 2r\cos\theta\sin\theta\, dr\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_0^\infty r^{x+y-1} \cdot \sin^{2x-1}\theta \cdot \cos^{2y-1}\theta \cdot e^{-r}\, dr\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta \cdot \int_0^\infty r^{x+y-1} e^{-r}\, dr$$

$$= \beta(x,y) \cdot \Gamma(x+y)$$

$$\Rightarrow \beta(x,y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)} \qquad \#$$

...ing β & Γ relation, prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Rightarrow$ put $x = \frac{1}{2}$ & $y = \frac{1}{2}$ $\Rightarrow \dfrac{\Gamma^2(\frac{1}{2})}{\Gamma(1)} = \beta(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \sin^0\theta \cos^0\theta \, d\theta$

$$\frac{\Gamma^2(\frac{1}{2})}{\underbrace{\Gamma(1)}_{1}} = 2\int_0^{\frac{\pi}{2}} d\theta = 2\theta\Big|_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2}\right) = \pi$$

$\Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$ #

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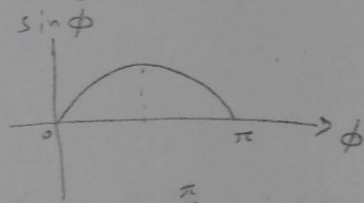
* Prove that $\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma(x+\frac{1}{2})$

$$\beta(x,x) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2x-1}\theta \, d\theta = 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2x-1} d\theta$$

$$\frac{\Gamma^2(x)}{\Gamma(2x)} = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2x-1} d\theta = \frac{2}{2^{2x-1}}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2x-1} d\theta$$

put $2\theta = \phi$ $\Rightarrow$ @ $\theta = 0 \Rightarrow \phi = 0$ & @ $\theta = \frac{\pi}{2} \Rightarrow \phi = \pi$

$\Rightarrow d\phi = 2\,d\theta$ or $d\theta = \frac{1}{2}\,d\phi$



$$\Rightarrow \frac{\Gamma^2(x)}{\Gamma(2x)} = \frac{2}{2^{2x-1} \cdot 2}\int_0^{\pi} (\sin\phi)^{2x-1} d\phi$$

$$= \frac{2 \cdot \cancel{2}}{2^{2x-1} \cdot \cancel{2}}\int_0^{\frac{\pi}{2}} (\sin\phi)^{2x-1} d\phi$$

but $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$

$\therefore y = \frac{1}{2}$

$$\frac{\Gamma^2(x)}{\Gamma(2x)} = \frac{1}{2^{2x-1}}\,\beta(x, \tfrac{1}{2}) = \frac{1}{2^{2x-1}}\,\frac{\Gamma(x)\,\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})} \quad \Gamma(\tfrac{1}{2}) \to \sqrt{\pi}$$

$\Rightarrow \sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma(x+\frac{1}{2})$ #

$\int_0^\infty x^a b^{-x} dx = I$

$\Rightarrow \Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt \quad \Rightarrow$ put $b^{-x} = e^{-t}$

$-x \ln b = -t \ln e \Rightarrow x = \frac{t}{\ln b}$

$dx = \frac{1}{\ln b} dt$

$\Rightarrow I = \int_0^\infty \left(\frac{t}{\ln b}\right)^a e^{-t} \cdot \frac{dt}{\ln b} = \frac{1}{(\ln b)^{a+1}} \int_0^\infty t^a e^{-t} dt$

if $u-1 = a \Rightarrow u = a+1$

$$\Rightarrow \boxed{I = \frac{1}{(\ln b)^{a+1}} \Gamma(a+1)} \quad \#$$

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* $\int_0^1 \sqrt[3]{x} \ln^5(x) dx = I$ put $u = \ln x \Rightarrow I = \int e^{\frac{1}{3}u} u^5 e^u du$

$\Rightarrow x = e^u$

$dx = e^u du$

$= \int u^5 e^{\frac{4}{3}u} du$

put $v = \frac{u}{3} \Rightarrow u = 3v \Rightarrow du = 3dv \Rightarrow I = \int (3v)^5 e^{4v} \cdot 3dv$

$\Rightarrow I = 729 \int v^5 e^{4v} dv$

$f = v^5 \qquad dg = e^{4v} dv$

$df = 5v^4 dv \qquad g = \frac{1}{4} e^{4v}$

$\Rightarrow I = 729\left[\frac{1}{4} v^5 e^{4v} - \frac{5}{4} \int v^4 e^{4v} dv\right]$ but $\int v^4 e^{4v} = \frac{1}{4} v^4 e^{4v} - \frac{4}{4} \int v^3 e^{4v} dv$

& $\int v^3 e^{4v} dv = \frac{1}{4} v^3 e^{4v} - \frac{3}{4} \int v^2 e^{4v} dv$

& $\int v^2 e^{4v} dv = \frac{1}{4} v^2 e^{4v} - \frac{2}{4} \int v e^{4v} dv$

& $\int v e^{4v} dv = \frac{1}{4} v e^{4v} - \frac{1}{4} \int e^{4v} dv$

& $\int e^{4v} dv = \frac{1}{4} e^{4v}$

$$= 729\left[\frac{1}{4} v^5 e^{4v} - \frac{5}{4}\left[\frac{1}{4} v^4 e^{4v} - \left[\frac{1}{4} v^3 e^{4v} - \frac{3}{4}\left[\frac{1}{4} v^2 e^{4v} - \frac{1}{2}\left[\frac{1}{4} v e^{4v} - \frac{1}{4}\left[\frac{1}{4} e^{4v}\right]\right]\right]\right]\right]\right]$$

$\because v = \frac{u}{3} \Rightarrow e^{4v} = e^{\frac{4}{3}u} = (e^u)^{\frac{4}{3}}$ but $e^u = x \Rightarrow \boxed{e^{4v} = x^{\frac{4}{3}}}$

$v^5 = \frac{\ln^5 x}{3^5}$, $v^4 = \frac{\ln^4 x}{3^4}$, $v^3 = \frac{\ln^3 x}{3^3}$, $v^2 = \frac{\ln^2 x}{3^2}$, $v = \frac{\ln x}{3}$

$$\Rightarrow I = 729\left[\frac{1}{4}\frac{\ln^5 x}{3^5}\cdot x^{4/3} - \frac{5}{4}\left[\frac{1}{4}\frac{\ln^4 x}{3^4}\cdot x^{4/3} - \left[\frac{1}{4}\frac{\ln^3 x}{3^3}\cdot x^{4/3} - \frac{3}{4}\left[\frac{1}{4}\frac{\ln^2 x}{3^2}\cdot x^{4/3} - \frac{1}{2}\left[\frac{1}{4}\frac{\ln x}{3} x^{4/3} - \frac{1}{16} x^{4/3}\right]\right]\right]\right]\right]$$

$; x = 0 \to x = 1$

if $x = 1 \Rightarrow \ln x = \ln 1 = 0$

if $x = 0 \Rightarrow x^{4/3} = 0$

$$\Rightarrow I = 729\left[-\frac{5}{4}\left[-1\left[-\frac{3}{4}\left[-\frac{1}{2}\left[-\frac{1}{16}(1)^{4/3}\right]\right]\right]\right]\right] = \boxed{-21.357}$$

$$* \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{2}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta\, d\theta = I$$

But $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta$

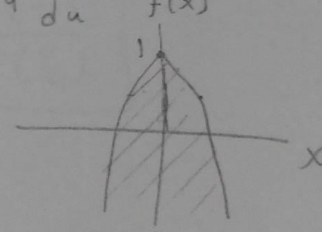
$\Rightarrow 2x - 1 = \frac{1}{2} \Rightarrow x = \frac{3}{4}$ $\quad 2y - 1 = -\frac{1}{2} \Rightarrow y = \frac{1}{4}$

$$I = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)} = \frac{1}{2}\frac{\pi}{\sin(\frac{3}{4}\pi)} = \boxed{\frac{\pi}{\sqrt{2}}}$$

$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = I$ put $x^4 = u \Rightarrow x = u^{\frac{1}{4}} \Rightarrow dx = \frac{1}{4} u^{-\frac{3}{4}} du$

$x = 0 \to u = 0$

$x\,\infty \to u = \infty$



$I = 2\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{2}{4}\int_0^{\infty} \frac{u^{-\frac{3}{4}}}{1 + u} du$ But $\beta(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} du$

$\therefore x - 1 = -\frac{3}{4} \Rightarrow x = \frac{1}{4}$ & $x + y = 1 \Rightarrow y = 1 - x = \frac{3}{4}$

$$\Rightarrow I = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)} = \frac{1}{2}\frac{\pi}{\sin(\frac{3}{4}\pi)} = \boxed{\frac{\pi}{\sqrt{2}}}$$

The function is symetric about y axis $\Rightarrow I_{-\infty \to \infty} = 2I'_{0 \to \infty}$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\Rightarrow \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{4}-\frac{1}{4}\cos 2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\frac{3}{4}-\frac{1}{4}\cos 2\theta}} = 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{3-\cos 2\theta}} = I$$

put $\cos 2\theta = 1 - 2\sqrt{u} \Rightarrow \theta = \frac{1}{2}\cos^{-1}(1-2\sqrt{u})$

$$\Rightarrow d\theta = \frac{1}{2}\frac{-du}{\sqrt{1-(1-2\sqrt{u})^2}} \cdot \left(\frac{-2}{2}u^{-\frac{1}{2}}\right) = \frac{1}{2}\frac{du}{\sqrt{u}\sqrt{1-(1-4\sqrt{u}+4u)}}$$

$$d\theta = \frac{1}{2}\frac{du}{\sqrt{u}\sqrt{4\sqrt{u}-4u}} = \frac{1}{4}\frac{du}{\sqrt{u}\sqrt{\sqrt{u}-u}}$$

@ $\theta = 0 \Rightarrow u = 0$

$\theta = \frac{\pi}{2} \Rightarrow u = 1$

$$\Rightarrow I = \frac{2}{2\cdot 2}\int_0^1 \frac{1}{\sqrt{3-1+2\sqrt{u}}} \cdot \frac{du}{\sqrt{u}\sqrt{\sqrt{u}-u}} = \int_0^1 \frac{du}{\sqrt{2+2\sqrt{u}}\sqrt{\sqrt{u}-u}\sqrt{u}}$$

$$= \frac{1}{\sqrt{2}}\int_0^1 \frac{du}{\sqrt{1+\sqrt{u}}\sqrt{1-\sqrt{u}}\sqrt{u}\sqrt{\sqrt{u}}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{du}{\sqrt{1-u}\; u^{3/4}}$$

$$= \frac{1}{\sqrt{2}}\int_0^1 u^{-3/4}(1-u)^{-\frac{1}{2}}\,du \quad \text{but } \beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$\therefore x - 1 = \frac{-3}{4} \Rightarrow x = \frac{1}{4} \qquad y - 1 = -\frac{1}{2} \Rightarrow y = \frac{1}{2}$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} = \boxed{\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}}$$

Use duplicate formula to show that $\beta(m,m) = 2^{1-2m}\beta(m,\frac{1}{2})$

$$\beta(m,m) = \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} \quad \text{but } \Gamma(2m) = \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma(m)\,\Gamma(m+\tfrac{1}{2})$$

$$= \frac{\Gamma(m)\,\cancel{\Gamma(m)}\,\sqrt{\pi}}{2^{2m-1}\,\cancel{\Gamma(m)}\,\Gamma(m+\frac{1}{2})} = 2^{1-2m}\frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2^{1-2m}\beta(m,\tfrac{1}{2})$$

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Laplace Transform

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* Prove that $L[\cos(kt)] = \frac{s}{s^2+k^2}$

$$L[\;] = \int_0^\infty e^{-st}\left(\frac{e^{ikt}+e^{-ikt}}{2}\right)dt = \frac{1}{2}\int_0^\infty \left(e^{(-s+ik)t} + e^{(-s-ik)t}\right)dt$$

$$= \frac{1}{2}\left(\frac{e^{-(s-ik)t}}{-s+ik} + \frac{e^{-(s+ik)t}}{-s-ik}\right)\Bigg|_0^\infty$$

$$= \frac{1}{2}\left[\frac{-1}{-s+ik} - \frac{1}{-s-ik}\right] = \frac{1}{2}\left[\frac{s+ik+s-ik}{s^2+k^2}\right] = \boxed{\frac{s}{s^2+k^2}}$$

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* $L^{-1}\left[\frac{6s}{(s^2+9)^2}\right]$

if $f(t) = \sin 3t$

$F(s) = \frac{3}{s^2+9}$

$F'(s) = \frac{-3(2s)}{(s^2+9)^2} = \frac{-6s}{(s^2+9)^2}$

but $L[f(t)\,t] = -F'(s) = \frac{6s}{(s^2+9)^2}$

$\Rightarrow L^{-1}[\;] = \boxed{t \sin 3t}$

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* Prove that $L^{-1}\left[\ln\sqrt{\frac{s^2-25}{s^2+9}}\right] = \frac{\cos 3t - \operatorname{ch} 5t}{t}$

$L[\cos 3t] = \frac{s}{s^2+9}$

$L[\operatorname{ch} 5t] = \frac{s}{s^2-25}$

$$L[R.H.S] = L\left[\frac{\cos 3t}{t}\right] - L\left[\frac{\operatorname{ch} 5t}{t}\right]$$

$$= \frac{1}{2}\int_s^\infty \frac{2s\,ds}{s^2+9} - \frac{1}{2}\int_s^\infty \frac{2s\,ds}{s^2-25}$$

$$= \frac{1}{2}\ln(s^2+9) - \frac{1}{2}\ln(s^2-25)\Big|_s^\infty = \ln\sqrt{\frac{s^2+9}{s^2-25}}\Bigg|_s^\infty$$

$$= \lim_{s\to\infty} \ln\sqrt{\frac{s^2+9}{s^2-25}} - \ln\sqrt{\frac{s^2+9}{s^2-25}} = \lim_{s\to\infty} \frac{1}{2}\cancel{\ln\frac{s^2}{s^2}} - \ln\sqrt{\frac{s^2+9}{s^2-25}} = \boxed{\ln\sqrt{\frac{s^2-25}{s^2+9}}}$$

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$L[\sin^2 t] = L[\frac{1}{2} - \frac{1}{2}\cos 2t] = \boxed{\frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4}}$

$* L[\frac{e^{-3t} - e^{-4t}}{t}]$ $\quad L[e^{-3t} - e^{-4t}] = \frac{1}{s+3} - \frac{1}{s+4} = F(s)$

$\Rightarrow L[\ ] = \int_s^\infty F(s)\,ds = \int_s^\infty \frac{1}{s+3} - \frac{1}{s+4}\cdot ds = \ln(s+3) - \ln(s+4)\Big|_s^\infty$

$= \ln\frac{s+3}{s+4}\Big|_s^\infty = \lim_{s\to\infty} \underbrace{\ln\frac{s+3}{s+4}}_{0} + \ln\frac{s+4}{s+3} = \boxed{\ln\frac{s+4}{s+3}}$ #

$* L[t\cos^2(2t)]$ $\quad \cos^2 2t = \frac{\cos 4t + 1}{2} = \frac{1}{2}\cos 4t + \frac{1}{2}$

$L[\ ] = -\frac{d}{ds}L[\frac{1}{2}\cos 4t + \frac{1}{2}] = \frac{-d}{ds}\left[\frac{s}{2(s^2+16)} + \frac{1}{2s}\right]$

$= \frac{-1}{2}\left[\frac{s^2+16-2s^2}{(s^2+16)^2} + \frac{-1}{s^2}\right] = \boxed{\frac{1}{2}\left[\frac{s^2-16}{(s^2+16)^2} - \frac{1}{s^2}\right]}$ #

$L^{-1}\left(\frac{2s+3}{(s-2)^2(s+1)^2}\right) = f_1(t) + f_2(t)$

$\phi_1(s) = (s-2)^2 F(s) = \frac{2s+3}{(s+1)^2}$ $\quad \phi_1'(s) = \frac{2(s+1)^2 - (2s+3).2(s+1)}{(s+1)^4}$

$\phi_1(2) = \frac{7}{9}$ $\quad \phi_1'(2) = -\frac{8}{27}$

$f_1(t) = [\phi_1'(2) + t\phi(2)]e^{2t} = \frac{-8}{27}e^{2t} + \frac{7}{9}te^{2t}$

$\phi_2(s) = (s+1)^2 F(s) = \frac{2s+3}{(s-2)^2}$ $\quad \phi_2'(s) = \frac{2(s-2)^2 - (2s+3).2(s-2)}{(s-2)^4}$

$\phi_2(-1) = \frac{1}{9}$ $\quad \phi_2'(-1) = \frac{4}{27}$

$f_2(t) = [\phi_2'(-1) + t\phi(-1)]e^{-t} = \frac{4}{27}e^{-t} + \frac{1}{9}te^{-t}$

$\Rightarrow \boxed{L^{-1}[\ ] = \frac{-8}{27}e^{2t} + \frac{7}{9}te^{2t} + \frac{4}{27}e^{-t} + \frac{1}{9}te^{-t}}$ #

$$\left[\frac{5s-3}{(s+4)(s-3)^3}\right]$$

□ By partial Fractions

$$\frac{5s-3}{(s+4)(s-3)^3} = \frac{A}{s+4} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3}$$

$$\Rightarrow 5s-3 = A(s-3)^3 + B(s+4)(s-3)^2 + C(s+4)(s-3) + D(s+4)$$

if $s=3 \Rightarrow 12 = 7D \Rightarrow D = \frac{12}{7}$

if $s=-4 \Rightarrow -23 = -343A \Rightarrow A = \frac{23}{343}$

coeff. $s^3 = 0 \Rightarrow A + B = 0 \Rightarrow B = -A = \frac{-23}{343}$

coeff. $s^0 = -3 \Rightarrow -27A + 36B - 12C + 4D = -3 \Rightarrow C = \frac{23}{49}$

$$\Rightarrow L^{-1}[\;] = L^{-1}\left[\frac{23}{343}\cdot\frac{1}{s+4} - \frac{23}{343}\,\frac{1}{s-3} + \frac{23}{49}\cdot\frac{1}{(s-3)^2} + \frac{12}{7}\cdot\frac{1}{(s-3)^3}\right]$$

$$\boxed{L^{-1}[\;] = \frac{23}{343}e^{-4t} - \frac{23}{343}e^{3t} + \frac{23}{49}e^{3t}t + \frac{12}{7}e^{3t}\frac{t^2}{2}}$$ #

□ By Heaviside Method:

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right] = f_1(t) + f_2(t)$$

$$f_1(t) = \frac{P(-4)e^{-4t}}{Q'(-4)} = \frac{-23}{-343}e^{-4t} = \frac{23}{343}e^{-4t}$$

$$\phi(s) = (s-3)^3F(s) = \frac{5s-3}{s+4} \qquad \phi'(s) = \frac{5(s+4)-(5s-3)}{(s+4)^2} = \frac{23}{(s+4)^2}$$

$$\phi''(s) = \frac{-46}{(s+4)^3} \qquad \Rightarrow \phi(3) = \frac{12}{7} \qquad \phi'(3) = \frac{23}{49} \qquad \phi''(3) = \frac{-46}{343}$$

$$\Rightarrow f_2(t) = \frac{1}{2}\left[\phi''(3) + 2t\phi'(3) + t^2\phi(3)\right]e^{3t} = \frac{-23}{343}e^{3t} + \frac{23}{49}te^{3t} - \frac{12}{14}t^2e^{3t}$$

$$\Rightarrow \boxed{L^{-1}[\;] = \frac{23}{343}e^{-4t} - \frac{23}{343}e^{3t} + \frac{23}{49}te^{3t} - \frac{12}{14}t^2e^{3t}}$$ #

$$\left[\frac{s-1}{(s+3)(s^2+2s+2)}\right] = f_1(t) + f_2(t)$$

$$f_1(t) = \frac{P(-3)}{Q'(-3)} e^{-3t} = \frac{-4}{5} e^{-3t}$$

$$\phi(s) = (s^2+2s+2)\,F(s) = \left[(s+1)^2+1\right] F(s) \Rightarrow \alpha = -1 \quad \beta = 1$$

$$\phi(s) = \frac{s-1}{s+3}$$

$$\phi(\alpha+i\beta) = \frac{-1+i-1}{-1+i+3} = \frac{-2+i}{2+i} \cdot \frac{2-i}{2-i} = \frac{-(4-4i-1)}{5} = \frac{-3+4i}{5}$$

$$\phi(\alpha+i\beta) = \frac{-3}{5} + \frac{4}{5} i = R + I i \Rightarrow R = \frac{-3}{5} \quad I = \frac{4}{5}$$

$$\Rightarrow f_2(t) = \frac{1}{\beta} e^{\alpha t}\left[I \cos\beta t + R \sin\beta t\right] = e^{-t}\left[\frac{4}{5}\cos t - \frac{3}{5}\sin t\right]$$

$$\Rightarrow \boxed{L^{-1}[\;] = \frac{-4}{5} e^{-3t} + \frac{4}{5} e^{-t}\cos t - \frac{3}{5} e^{-t}\sin t}$$

#

Power series

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* $y' - y = 0$ near $x = 0$

$P_{(x)} = 1 \qquad q(x) = -1 \Rightarrow$ analytic $\Rightarrow x_0 = 0$ is O.P.

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow y' - y = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

or $$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - a_n\right] x^n = 0$$

$$(n+1) a_{n+1} = a_n \qquad ; n \geq 0$$

$n=0 \Rightarrow a_0 = a_1 \qquad \Rightarrow a_1 = a_0 = \frac{a_0}{1!}$

$n=1 \Rightarrow a_1 = 2a_2 = a_0 \qquad \Rightarrow a_2 = \frac{1}{2} a_0 = \frac{a_0}{2!}$

$n=2 \Rightarrow a_2 = 3a_3 = \frac{1}{2} a_0 \qquad \Rightarrow a_3 = \frac{1}{6} a_0 = \frac{a_0}{3!}$

$n=3 \Rightarrow a_3 = 4a_4 = \frac{1}{6} a_0 \qquad \Rightarrow a_4 = \frac{1}{24} a_0 = \frac{a_0}{4!}$

$\vdots$

$$\Rightarrow y = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

$$= a_0 \left[1 + x^1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}\right]$$

$$\Rightarrow \boxed{y = a_0 e^x}$$

$y'' + y' = 0$

$p(x) = 1 \quad q(x) = 0 \Rightarrow$ analytic for all $x \Rightarrow x_0 = 0$ is o.p.

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$y'' + y' = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$

or $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0$

$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} \right] x^n = 0$

$\Rightarrow (n+2)\cancel{(n+1)} a_{n+2} = -\cancel{(n+1)} a_{n+1} \quad \Rightarrow a_{n+1} = -(n+2) a_{n+2}$

$; n \geq 0$

$n=0 \Rightarrow a_1 = -2a_2 \quad \Rightarrow a_2 = \frac{-1}{2} a_1 = \frac{-1}{2!} a_1$

$n=1 \Rightarrow a_2 = -3a_3 = \frac{-1}{2} a_1 \Rightarrow a_3 = \frac{1}{6} a_1 = \frac{1}{3!} a_1$

$n=2 \Rightarrow a_3 = -4a_4 = \frac{1}{6} a_1 \Rightarrow a_4 = \frac{-1}{24} a_1 = \frac{-1}{4!} a_1$

$\vdots$

$y = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$

$= a_0 + a_1 \underbrace{\left[x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots \right]}_{y_1}$

$y = a_0 + a_1 y_1$

$y'' + (x-1)^2 y' - 4(x-1)y = 0$ near $x = 1$

$p(x) = (x-1)^2 \quad q(x) = -(x-1) \Rightarrow$ analytic for all x

$\Rightarrow x_0 = 1 \Rightarrow$ O.P

$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$

$y'' + (x-1)^2 y' - 4(x-1) y = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n+1} - 4 \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = 0$

or $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} (x-1)^n - 4 \sum_{n=1}^{\infty} a_{n-1} (x-1)^n = 0$

$2a_2 + 6a_3(x-1) - 4a_0(x-1) + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n-1) a_{n-1} - 4a_{n-1} \right] (x-1)^n = 0$

$\Rightarrow 2a_2 - 6a_3 + 4a_0 = 0 \to (1)$

$6a_3 - 4a_0 = 0 \to (2) \Rightarrow \boxed{a_3 = \frac{2}{3} a_0} \xrightarrow{\text{in } (1)} 2a_2 - 4a_0 + 4a_0 = 0 \Rightarrow \boxed{a_2 = 0}$

$(n+2)(n+1) a_{n+2} + (n-5) a_{n-1} = 0 \quad ; n \geq 2$

$n = 2 \Rightarrow 12 a_4 - 3a_1 = 0 \Longrightarrow a_4 = \frac{1}{4} a_1$

$n = 3 \Rightarrow 20 a_5 - 2a_2 = 0 \Longrightarrow 20 a_5 = 0 \Rightarrow a_5 = 0$

$n = 4 \to 30 a_6 - a_3 = 0 \Longrightarrow a_6 = \frac{a_3}{30} = \frac{2}{90} a_0 = \frac{1}{45} a_0$

$n = 5 \Rightarrow 42 a_7 = 0 \Rightarrow a_7 = 0$

$n = 6 \Rightarrow 56 a_8 + a_5 = 0 \Rightarrow a_8 = \frac{-a_5}{56} = 0$

$n = 7 \Rightarrow 72 a_9 + 2a_6 = 0 \Rightarrow a_9 = \frac{-1}{36} a_6 = \frac{-1}{45 \cdot 36} a_0$

$n = 8 \Rightarrow 90 a_{10} + 3 a_7 = 0 \Rightarrow a_{10} = 0$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0\left[1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{45\cdot 36}(x-1)^9 + a_{12}(x-1)^{12}\cdots\right]$$

$$+ a_1\left[(x-1) + \frac{1}{4}(x-1)^4\right]$$

$$\boxed{y = a_0 y_1 + a_1 y_2}$$

---

* $(1+4x^2)y'' - 8y = 0$ near $x=0$

$$y'' - \frac{8}{1+4x^2} y = 0$$

$p(x) = 0 \qquad q(x) = \frac{-8}{1+4x^2}$ @ $x_0 = 0 \Rightarrow p \,\&\, q$ are analytic

$\Rightarrow x_0 = 0$ is o.p

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} * (1+4x^2) - 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 4\sum_{n=2}^{\infty} n(n-1) a_n x^n - 8\sum_{n=0}^{\infty} a_n x^n = 0$$

or $$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 4\sum_{n=2}^{\infty} n(n-1) a_n x^n - 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 x - 8[a_0 + a_1 x] + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + 4n(n-1)a_n - 8a_n\right]x^n = 0$$

$$2a_2 - 8a_0 = 0 \Rightarrow a_2 = 4a_0$$

$$6a_3 - 8a_1 = 0 \Rightarrow a_3 = \frac{4}{3}a_1$$

continued...

$(n+2)(n+1)a_{n+2} + 4n(n-1)a_n - 8a_n = 0 \quad ; n \geqslant 2$

$n=2 \Rightarrow 12a_4 + \underbrace{8a_2 - 8a_2}_{} = 0 \quad \Rightarrow a_4 = 0$

$n=3 \Rightarrow 20a_5 + 16a_3 = 0 \quad \Rightarrow a_5 = \frac{-16}{20}a_3 = \frac{-4 * 16}{3 * 20}a_1$

$n=4 \Rightarrow 30a_6 + 40a_4 = 0 \quad \Rightarrow a_6 = 0$

$n=5 \Rightarrow 42a_7 + 72a_5 = 0 \quad \Rightarrow a_7 = \frac{-72}{42}a_5 = \frac{72 * 4 * 16}{42 * 3 * 20}a_1$

$\Rightarrow y = a_0[1 + 4x^2] + a_1\left[x^1 + \frac{4}{3}x^3 - \frac{4 * 16}{3 * 20}x^5 + \frac{72 * 4 * 16}{42 * 3 * 20}x^7 - - -\right]$

$\boxed{y = a_0 y_1 + a_1 y_2}$

Exercise (2) – Complex

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[1] a) $F'(z) = \lim_{\Delta z \to 0} \dfrac{F(z+\Delta z) - F(z)}{\Delta z}$

---

b) A function is said to be analytic if it's defined & differentiable at all points in a domain or at the neighborhod of a point.

---

c) A Harmonic function is a function that has continous second order partial derivatives and satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in a domain D.

If $w = u + iv$ is analytic, then u & v are called Conjugate harmonic functions.

---

[4] a) $w = \bar{z} = x - iy = u + iv \Rightarrow u = x \quad v = -y$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$\Rightarrow \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y} \Rightarrow$ The function is neither differentiable nor analytic.

---

b) $w = \operatorname{Im} z = y = u + iv \Rightarrow u = y \quad v = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = 0$$

$\Rightarrow \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$

& $\dfrac{\partial v}{\partial x} \neq -\dfrac{\partial u}{\partial y}$ $\Rightarrow$ The function is neither differentiable nor analytic.

$w = 2y - ix = u + iv \Rightarrow u = 2y \quad v = -x$

$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = -1$

$\frac{\partial u}{\partial y} = 2 \qquad \frac{\partial v}{\partial y} = 0$

$\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$ ⇒ The function is neither differentiable nor analytic.

---

d) $W = e^x(\cos y + i\sin y) = u + iv \Rightarrow u = e^x \cos y \quad v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

& $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ⇒ The function is differentiable & analytic for all values of x & y

$\Rightarrow W = e^{x+iy} = e^z$

$$\boxed{\frac{dW}{dz} = e^z = e^x(\cos y + i\sin y)}$$

---

e) $W = e^y(\cos x + i\sin x) = u + iv \Rightarrow u = e^y \cos x \quad v = e^y \sin x$

$\frac{\partial u}{\partial x} = -e^y \sin x \qquad \frac{\partial v}{\partial x} = e^y \cos x$

$\frac{\partial u}{\partial y} = e^y \cos x \qquad \frac{\partial v}{\partial y} = e^y \sin x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ if $\sin x = 0 \Rightarrow x = n\pi \quad ; n = 0, \pm 1, \pm 2, \ldots$

& $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ if $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + k\pi \quad ; k = 0, \pm 1, \pm 2, \ldots$

⇒ The solution sets don't intersect ⇒ The function is not differentiable

$W = z^2\bar{z} = (x+iy)(x+iy)(x-iy)$
$= (x+iy)(x^2-y^2)$
$= x(x^2-y^2) + iy(x^2-y^2) = u+iv$

$\Rightarrow u = x^3 - xy^2 \qquad V = x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - y^2 \qquad \frac{\partial V}{\partial x} = 2xy$

$\frac{\partial u}{\partial y} = -2xy \qquad \frac{\partial V}{\partial y} = x^2 - 3y^2$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}$ if $3x^2 - y^2 = x^2 - 3y^2$

$2x^2 = -2y^2 \Rightarrow x^2 = -y^2$

$\Rightarrow$ Valid for point $(0,0)$ only

$\Rightarrow W = 0 \quad \Rightarrow \boxed{W' = 0}$

$\Rightarrow$ differentiable @ $(0,0)$ only & not analytic

---

g) $W = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{x - iy}{x^2+y^2} = u + iV$

$\Rightarrow u = \frac{x}{x^2+y^2} \qquad V = \frac{-y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial V}{\partial x} = \frac{+y(2x)}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{-x(2y)}{(x^2+y^2)^2} \qquad \frac{\partial V}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}$
& $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

$\Rightarrow$ The function is differentiable and analytic for all values of x & y except @ point $(0,0)$

$W = z^{-1} \Rightarrow \boxed{W' = -1 \cdot z^{-2} = \frac{-1}{z^2}}$

$w = x^3+3xy^2-3x+i(y^3+3x^2y-3y) = u+iv$

$\Rightarrow u = x^3+3xy^2-3x \qquad v = y^3+3x^2y-3y$

$\frac{\partial u}{\partial x} = 3x^2+3y^2-3 \qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = 6xy \qquad \frac{\partial v}{\partial y} = 3y^2+3x^2-3$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ but $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ if $x=0$ or $y=0$

if $x=0$ $\Rightarrow$ The function is differentiable for all values of y but not analytic as the solution set is line.

$$\Rightarrow w' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{x=0} = \boxed{3y^2-3}$$

---

[5] $f(z) = 3x^2+2x-3y^2-1+i(6xy+2y) = u+iv$

$\Rightarrow u = 3x^2+2x-3y^2-1 \qquad v = 6xy+2y$

$\frac{\partial u}{\partial x} = 6x+2 \qquad \frac{\partial v}{\partial y} = 6x+2$

$\frac{\partial u}{\partial y} = -6y \qquad \frac{\partial v}{\partial x} = 6y$

$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\Rightarrow$ This function is differentiable and analytic for all values of x & y $\Rightarrow$ It's an entire function

$\because x = \frac{1}{2}(z+\bar{z})$ & $y = \frac{1}{2}(z-\bar{z})$

$$\Rightarrow \boxed{f(z) = 3\left(\frac{z+\bar{z}}{2}\right)^2 + 2\left(\frac{z+\bar{z}}{2}\right) - 3\left(\frac{z-\bar{z}}{2}\right)^2 - 1 + i\left(\frac{6}{4}(z^2-\bar{z}^2) + z - \bar{z}\right)}$$

a) $\mathrm{Re}\, f(z) = \text{const}$

$\Rightarrow f(z) = c + i\, g(z) = u + iv$

$\Rightarrow u = c \qquad v = g(z)$

$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = \frac{\partial g(z)}{\partial y}$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = \frac{\partial g(z)}{\partial x}$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ if } \frac{\partial g(z)}{\partial y} = 0 \\ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ if } \frac{\partial g(z)}{\partial x} = 0 \end{array}\right\} \Rightarrow g(z) = \text{const}$$

$\Rightarrow F(z) = C_1 + i C_2 \Rightarrow$ Analytic for any value of x & y.

---

b) $\mathrm{Im}\, f(z) = \text{const}.$

$\Rightarrow F(z) = g(z) + ic = u + iv$

$u = g(z) \qquad v = c$

$\frac{\partial u}{\partial x} = \frac{\partial g(z)}{\partial x} \qquad \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = \frac{\partial g(z)}{\partial y} \qquad \frac{\partial v}{\partial x} = 0$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ if } \frac{\partial g(z)}{\partial x} = 0 \\ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ if } \frac{\partial g(z)}{\partial y} = 0 \end{array}\right\} \Rightarrow g(z) = \text{const}.$$

$\Rightarrow F(z) = C_1 + i C_2 \Rightarrow$ Analytic for any value of x & y.

$F(z) = u + iv$ & $|F(z)| = \text{const.}$

$\Rightarrow \sqrt{u^2+v^2} = \text{const.} \Rightarrow$ valid if u & v were constants only

$\Rightarrow F(z) = C_1 + iC_2 \Rightarrow$ Analytic for any value of x & y.

---

d) $\overline{F(z)}$ is analytic

$F(z) = u + iv$
$= g(z) + i\,h(z)$

$\overline{F(z)} = u - iv$
$= g(z) - i\,h(z)$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial g(z)}{\partial x} \qquad \dfrac{\partial v}{\partial y} = \dfrac{\partial h(z)}{\partial y}$

$\dfrac{\partial u}{\partial y} = \dfrac{\partial g(z)}{\partial y} \qquad \dfrac{\partial v}{\partial x} = \dfrac{\partial h(z)}{\partial x}$

$\because \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow \dfrac{\partial g(z)}{\partial x} = \dfrac{\partial h(z)}{\partial y} = 0 \Rightarrow g(z) = C_1$

$h(z) = C_2$

& $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\Rightarrow F(z) = C_1 + iC_2 \Rightarrow$ Analytic for any value of x & y.

---

e) $|F(z)|$ is analytic ??

$= 2x(1-y)$

$\frac{\partial u}{\partial x} = 2(1-y)$ $\quad \frac{\partial^2 u}{\partial x^2} = 0$ $\quad \frac{\partial u}{\partial y} = -2x$ $\quad \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$ The function is harmonic

$\therefore \frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} = 2(1-y) \Rightarrow V = 2y - y^2 + g(x)$

$\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y} = 2x$ $\quad \frac{\partial V}{\partial x} = \frac{\partial g(x)}{\partial x} = 2x \Rightarrow g(x) = x^2 + c$

$\Rightarrow \boxed{F(z) = 2x(1-y) + i(2y - y^2 + x^2 + c)}$

---

b) $u = \cos x \, ch y$

$\frac{\partial u}{\partial x} = -\sin x \, ch y$ $\quad \frac{\partial^2 u}{\partial x^2} = -\cos x \, ch y$

$\frac{\partial u}{\partial y} = \cos x \, sh y$ $\quad \frac{\partial^2 u}{\partial y^2} = \cos x \, ch y$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$ The function is harmonic

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} = -\sin x \, ch y \Rightarrow V = -\sin x \, sh y + g(x)$

$\Downarrow$

$\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y} = -\cos x \, sh y$ $\quad \frac{\partial V}{\partial x} = -\cos x \, sh y + \underbrace{\frac{\partial g(x)}{\partial x}}_{0} = -\cos x \, sh$

$\Rightarrow V = -\sin x \, sh y + c$

$\Rightarrow \boxed{F(z) = \cos x \, ch y - i(\sin x \, sh y + c)}$

$$v = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{-y(2x)}{(x^2+y^2)^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{-2y(x^2+y^2)^2 + 2(2x)(x^2+y^2).2xy}{(x^2+y^2)^4}$$

$$\frac{\partial v}{\partial y} = \frac{x^2+y^2 - y(2y)}{(x^2+y^2)^2} \xrightarrow{} x^2-y^2 \qquad \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)(2)(2y)(x^2+y^2)}{(x^2+y^2)^4}$$

$$x^4+2x^2y^2+y^4$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{-2y(x^2+y^2)^2 + 8x^2y(x^2+y^2) - 2y(x^2+y^2)^2 - 4y(x^4-y^4)}{(x^2+y^2)^4}$$

nominator $= -2yx^4 - 4x^2y^3 - 2y^5 + 8x^4y + 8x^2y^3 - 2yx^4 - 4x^2y^3 - 2y^5 - 4x^4y + 4y^5 = 0$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow \text{harmonic function}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow u = \frac{-x}{x^2+y^2} + g(y)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = \frac{x(2y)}{(x^2+y^2)^2} + \frac{\partial g(y)}{\partial y} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial g(y)}{\partial y} = 0$$

$$\Rightarrow u = \frac{-x}{x^2+y^2} + c$$

$$\Rightarrow \boxed{F(z) = \frac{-x}{x^2+y^2} + c + i\,\frac{y}{x^2+y^2}}$$

---

d) $u = x^3 - 3xy^2 + y$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6yx + 1 \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{Harmonic function}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \Rightarrow v = 3x^2y - y^3 + g(x) \Rightarrow \frac{\partial v}{\partial x} = 6xy + \frac{\partial g(x)}{\partial x} = 6xy -$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy + 1 \qquad \Rightarrow \frac{\partial g(x)}{\partial x} = -1 \Rightarrow g(x) = -x + c$$

$$\rightarrow \boxed{F(z) = x^3 - 3xy^2 + y + i(3x^2y - y^3 - x + c)}$$

$\text{Arg } z = \theta$

$\frac{\partial u}{\partial r} = 0 \qquad \frac{\partial^2 u}{\partial r^2} = 0 \qquad \frac{\partial u}{\partial \theta} = 1 \qquad \frac{\partial^2 u}{\partial \theta^2} = 0$

$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = 0 \Rightarrow$ harmonic function

$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta} = 0 \Rightarrow V = g(r) + C$

$\Downarrow$

$\frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r} \qquad \frac{\partial V}{\partial r} = \frac{\partial g(r)}{\partial r} = \frac{-1}{r} \Rightarrow g(r) = -\ln r$

$\Rightarrow V = \ln\left(\frac{1}{r}\right) + C$

$\Rightarrow F(z) = \theta + i\left(\ln\left(\frac{1}{r}\right) + C\right)$

$\boxed{9}$ ∵ V is harmonic $\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

u is harmonic $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}$

$\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

∴ $\frac{\partial (uV)}{\partial x} = V\frac{\partial u}{\partial x} + \frac{\partial V}{\partial x}u \qquad \frac{\partial^2 (uV)}{\partial x^2} = \frac{\partial V}{\partial x}\cdot\overbrace{\frac{\partial u}{\partial x}}^{\frac{\partial V}{\partial y}} + V\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 V}{\partial x^2}u + \frac{\partial V}{\partial x}\cdot\overbrace{\frac{\partial u}{\partial x}}^{\frac{\partial V}{\partial y}}$

$\frac{\partial (uV)}{\partial y} = V\frac{\partial u}{\partial y} + \frac{\partial V}{\partial y}u \qquad \frac{\partial^2 (uV)}{\partial y^2} = \frac{\partial V}{\partial y}\cdot\overbrace{\frac{\partial u}{\partial y}}^{-\frac{\partial V}{\partial x}} + V\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 V}{\partial y^2}u + \frac{\partial V}{\partial y}\cdot\overbrace{\frac{\partial u}{\partial y}}^{-\frac{\partial V}{\partial x}}$

$\frac{\partial^2 (uV)}{\partial x^2} + \frac{\partial^2 (uV)}{\partial y^2} = \cancel{2\frac{\partial V}{\partial x}\cdot\frac{\partial V}{\partial y}} - \cancel{2\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}} + V\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{0} + u\underbrace{\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)}_{0} = 0$

$\Rightarrow \boxed{uV \text{ is harmonic}}$ #

Exercise (3) - complex

محمد طلعت ماهر
ميكانيكا قوى (ربيع ٢٠١٨)

1 i) $e^{2\pm 3\pi i} = -e^2$

$L.H.S = e^{2\pm 3\pi i} = e^2 . e^{\pm 3\pi i} = e^2 [\cos(\pm 3\pi) + i \underbrace{\sin(\pm 3\pi)}_{0}]$

$= e^2 \cos(3\pi) = -e^2$

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ii) $e^{\frac{\pi}{2} i} = i$

$L.H.S = \underbrace{\cos\frac{\pi}{2}}_{0} + i \sin\frac{\pi}{2} = i$

---

iii) $e^{\frac{1}{2} + \frac{\pi}{4} i} = \sqrt{e} \cdot \frac{1+i}{\sqrt{2}}$

$L.H.S = e^{\frac{1}{2}} e^{\frac{\pi}{4} i} = \sqrt{e}\ [\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}] = \sqrt{e}\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \sqrt{e}\,\frac{1+i}{\sqrt{2}}$

---

iv) $e^{\ln r + i\theta} = Z = r e^{i\theta}$

$L.H.S = e^{\ln r} \cdot e^{i\theta}$

$= r \cdot e^{i\theta}$

if $a = e^{\ln r}$

$\ln a = \ln r \underbrace{\ln e}_{1}$

$\Rightarrow a = r$

---

2 i) $e^z = -2 \Rightarrow e^z = e^{x+iy} = e^x e^{iy} = e^x [\cos y + i \sin y] = -2$

$e^x \cos y = -2 \qquad e^x \sin y = 0 \Rightarrow y = 0, \pm n\pi \ ; n = 0, 1, 2, 3 \ldots$

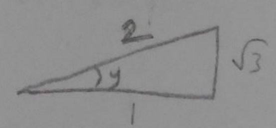
$\Rightarrow e^x = -2$

$\Rightarrow e^{2x} = 4 \Rightarrow 2x = \ln 4 \Rightarrow x = \frac{\ln 4}{2} = \boxed{\ln 2}$

---

ii) $e^z = 1 + i\sqrt{3} \Rightarrow e^x [\cos y + i \sin y] = 1 + i\sqrt{3}$

$\Rightarrow e^x \cos y = 1 \rightarrow ① \quad e^x \sin y = \sqrt{3} \rightarrow ②$



$\frac{②}{①} = \tan y = \sqrt{3} \Rightarrow \boxed{y = \frac{\pi}{3} \pm n\pi \ ; n = 0, 1, 2, \ldots}$

if $y = \frac{\pi}{3} \overset{①}{\Rightarrow} e^x \cdot \frac{1}{2} = 1 \Rightarrow e^x = 2 \quad \boxed{x = \ln 2}$

$e^{2z-1} = 1$

$\Rightarrow e^{2x-1+2yi} = e^{2x-1}\left[\cos(2y) + i\sin(2y)\right] = 1$

$\Rightarrow e^{2x-1}\cos(2y) = 1$ $\quad$ $e^{2x-1}\sin(2y) = 0$

$\sin(2y) = 0 \Rightarrow 2y = 0 \pm n\pi$; $n = 0, 1, 2 ..$

$\Rightarrow \boxed{y = 0 \pm n\frac{\pi}{2}}$

if $2y = 0$

$\Rightarrow e^{2x-1} = 1 = e^{0}$

$2x - 1 = 0$

$\Rightarrow \boxed{x = \frac{1}{2}}$

---

iv) $e^{i\bar{z}} = \overline{(e^{iz})}$

L.H.S $= e^{i(x-iy)} = e^{y+ix}$

$= e^{y}\left[\cos x + i\sin x\right]$

R.H.S $= \overline{(e^{iz})} = e^{\overline{i(x+iy)}} = e^{\overline{-y+ix}}$

$= e^{-y}\left[\cos x - i\sin x\right]$

$\because$ L.H.S = R.H.S

$\Rightarrow e^{y}\cos x = e^{-y}\cos x$ $\quad$ $e^{y}\sin x = -e^{-y}\sin x$

$\cos x.\left[e^{y} - e^{-y}\right] = 0$ $\quad$ $\sin x.\left[e^{y} - e^{-y}\right] = 0$

$\cos x = 0$ or $y = 0$ $\quad$ $\sin x = 0$ or $y = 0$

as $\cos x$ & $\sin x$ can't be zero at the same time $\Rightarrow \boxed{y = 0}$

$\Rightarrow \cancel{\cos x} + i\sin x = \cancel{\cos x} - i\sin x$

$2i\sin x = 0 \Rightarrow \boxed{x = 0 \pm n\pi}$; $n = 0, 1, 2, 3 ...$

---

4 i) $\cos z = 2$

$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \, ch y - i \sin x \, sh y = 2$

$\Rightarrow \cos x \, ch y = 2 \rightarrow ①$ $\quad$ $\sin x \, sh y = 0 \rightarrow ②$ $\Rightarrow \sin x = 0$ or $sh y = 0$

$\Downarrow$ $\quad$ $\Downarrow$

$x = n\pi$; $n = 0, \pm 1, \pm 2, ..$ $\quad$ $y = 0$ (refused)

f $y = 0 \overset{①}{\Rightarrow} \cos x = 2$ (refused as $\cos \leq 1$)

f $x = n\pi \overset{①}{\Rightarrow} \pm ch y = 2$

but $ch \geq 1 \Rightarrow$ -ve values of $\cos x$ are refused

$\Rightarrow \boxed{x = k\pi}$; $k = 0, \pm 2, \pm 4, ...$ $\quad$ $\Rightarrow y = ch^{-1}(2) = \boxed{1.317}$

$\sin Z = ch\,4$

$\sin Z = \sin(x+iy) = \sin x \cos iy + \sin iy \cos x = \sin x\, ch\, y + i\, sh\, y \cos x = ch\,4$

$\Rightarrow \sin x\, ch\, y = ch\,4 \rightarrow ①$ $\quad sh\, y \cos x = 0 \rightarrow ②$ $\Rightarrow \cos x = 0$ or $sh\, y = 0$

$\cos x = 0 \Downarrow x = \frac{\pi}{2} + n\pi \;;n = 0, \pm1, \pm2, \dots$

$sh\, y = 0 \Downarrow y = 0$ (refused)

if $y = 0 \overset{①}{\Rightarrow} \sin x = ch\,4 > 1$ (refused)

f $x = \frac{\pi}{2} + n\pi \overset{①}{\Rightarrow} \pm\, ch\, y = ch\,4$

as ch is +ve $\Rightarrow$ −ve values are refused $\Rightarrow \boxed{x = \frac{\pi}{2} + k\pi}$ $;k = 0, \pm2, \pm4, \dots$

$ch\, y = ch\,4 \Rightarrow \boxed{y = 4}$

---

iii) $\cos Z = 0$

$\cos(x+iy) = \cos x\, ch\, y - i \sin x\, sh\, y = 0 \Rightarrow \cos x\, ch\, y = 0 \quad \sin x\, sh\, y = 0$

$ch\, y = 0$ (refused) $\quad \cos x = 0 \Rightarrow \boxed{x = \frac{\pi}{2} + n\pi}$ $;n = 0, \pm1, \pm2, \dots$

$\sin x = 0$ (refused) $\quad sh\, y = 0 \Downarrow \boxed{y = 0}$

---

6 i) $sh\, Z = 0$

$sh\, Z = sh\, x \cdot \cos y + i\, ch\, x \sin y = 0$

$sh\, x \cdot \cos y = 0 \Downarrow sh\, x = 0 \Rightarrow \boxed{x = 0}$

$ch\, x \sin y = 0 \Downarrow \sin y = 0 \Downarrow \boxed{y = 0 + n\pi}$ $;n = 0, \pm1, \pm2, \dots$

ii) $ch\, Z = 0$

$ch\, Z = ch\, x \cos y + i\, sh\, x \sin y = 0$

$ch\, x \cos y = 0$ $\quad \cos y = 0 \Downarrow \boxed{y = \frac{\pi}{2} + n\pi}$ $;n = 0, \pm1, \pm2 \dots$

$sh\, x \sin y = 0 \Downarrow sh\, x = 0 \Rightarrow \boxed{x = 0}$

i) $ch\, z = \frac{1}{2}$

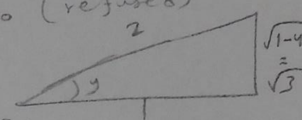
$ch\, z = ch\, x \cos y + i\, sh\, x \sin y = \frac{1}{2}$

$ch\, x \cos y = \frac{1}{2}$ $\qquad$ $sh\, x \sin y = 0$

$sh\, x = 0$ $\qquad$ $\sin y = 0$

$x = 0$ $\qquad$ $y = 0$ (refused)

if $y = 0 \Rightarrow ch\, x = \frac{1}{2}$ (refused)



f $\boxed{x = 0} \Rightarrow \cos y = \frac{1}{2} \Rightarrow \boxed{y \begin{cases} \frac{\pi}{3} + 2n\pi \\ \frac{-\pi}{3} + 2n\pi \end{cases}}$; $n = 0, \pm 1, \pm 2 \ldots$

---

ii) $sh\, z = i$

$sh\, z = sh\, x \cos y + i\, ch\, x \sin y = i$

$sh\, x \cos y = 0 \rightarrow (1)$ $\qquad$ $ch\, x \sin y = 1 \longrightarrow (2)$

$sh\, x = 0 \Rightarrow x = 0$ $\qquad$ $\cos y = 0 \Rightarrow y = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

if $x = 0 \overset{(2)}{\Rightarrow} \sin y = 1 \Rightarrow y = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$

if $y = \frac{\pi}{2} \overset{(2)}{\Rightarrow} \pm ch\, x = 1$

but $ch\, x \geq 1$

-ve values are refused

$ch\, x = 1$

$x = 0$

$$\Rightarrow \boxed{x = 0 \qquad y = \frac{\pi}{2} + n\pi \quad ; n = 0, \pm 2, \pm 4, \ldots}$$

---

8 i) $\ln 1 = W = u + iv \Rightarrow 1 = e^{u+iv} = e^{u}[\cos v + i \sin v]$

$\Rightarrow e^{u} \cos v = 1 \rightarrow (1)$ $\qquad$ $e^{u} \sin v = 0 \Rightarrow v = 0, \pi, 2\pi \ldots$

in (1) $\Rightarrow \pm e^{u} = 1$

but $e^{u} > 0 \Rightarrow$ -ve values of $\cos v$ are refused

$\Rightarrow u = \ln 1 = 0 \Rightarrow \boxed{W = i 2n\pi}$

; $n = 0, \pm 1, \pm 2, \ldots$

$\ln i = W = u + iV$

$\Rightarrow i = e^u[\cos V + i\sin V] \Rightarrow e^u\cos V = 0 \qquad e^u \sin V = 1$

$\cos V = 0$

$V = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \qquad \Rightarrow \pm e^u = 1$

but $e^u > 0$

$\Rightarrow$ -ve values are refused

$\Rightarrow e^u = 1 \Rightarrow u = 0$

$\Rightarrow \boxed{W = i\left(\frac{\pi}{2} + 2n\pi\right)}$

$; n = 0, \pm 1, \pm 2, \dots$

---

iii) $\ln(-1) = W = u + iV$

$\Rightarrow -1 = e^u[\cos V + i\sin V] \Rightarrow e^u\cos V = -1 \qquad e^u\sin V = 0$

$\Rightarrow \pm e^u = -1 \qquad \sin V = 0$

but as $e^u > 0 \qquad V = 0, \pi, 2\pi \dots$

$\Rightarrow$ +ve values are refused

$\Rightarrow u = 0$

$\Rightarrow \boxed{W = i(\pi + 2n\pi) \quad ; n = 0, \pm 1, \pm 2, \dots}$

---

iv) $\ln\sqrt{i} = \ln i^{\frac{1}{2}} = \frac{1}{2}\ln i = \frac{1}{2} i\left(\frac{\pi}{2} + 2n\pi\right) = \boxed{i\left(\frac{\pi}{4} + n\pi\right) \quad ; n = 0, \pm 1, \pm 2, \dots}$

see Pb ii)

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v) $\mathrm{Ln}(-ei) = \mathrm{Ln}(e) \cdot \mathrm{Ln}\, i^3 = 1 \cdot 3\,\mathrm{Ln}\, i = \boxed{3\frac{\pi}{2} i}$ as $\pi > \theta > -\pi$

---

vi) $\mathrm{Ln}(1-i) = u + iV = W$

$\Rightarrow 1 - i = e^u[\cos V + i\sin V] \Rightarrow e^u \cos V = 1 \rightarrow ① \qquad e^u\sin V = -1 \rightarrow ②$

$\frac{②}{①} = \tan V = \frac{-1}{1} \Rightarrow V = \frac{-\pi}{4} \overset{①}{\Rightarrow} e^u = \sqrt{2}$

$\Rightarrow u = \ln\sqrt{2} = \frac{1}{2}\ln 2$

$\Rightarrow \boxed{W = \frac{1}{2}\ln(2) - \frac{\pi}{4} i}$

$\ln(-3+i\sqrt{27}) = u+iv \Rightarrow e^u[\cos v + i\sin v] = -3+i\sqrt{27}$

$\Rightarrow e^u\cos v = -3 \rightarrow ①$ $\quad e^u \sin v = \sqrt{27} \rightarrow ②$

$\frac{②}{①} = \tan v = \frac{\sqrt{3}}{-1} \Rightarrow v = \frac{2\pi}{3} \overset{①}{\Longrightarrow} e^u = 6 \Rightarrow u = \ln 6$

$\Rightarrow \boxed{W = \ln(6) + i\frac{2\pi}{3}}$

---

viii) $(1+i)^i = W = Z^c \quad ; Z = 1+i \quad c = i$

$c\ln Z = \ln W$

$\ln Z = \ln(1+i) = u+iv \Rightarrow 1+i = e^u[\cos v + i\sin v]$

$e^u\cos v = 1 \qquad e^u \sin v = 1$

$\tan v = 1 \Rightarrow v = \frac{\pi}{4} + 2n\pi \quad ; n = 0, \pm1, \pm2, \ldots$

$\Rightarrow e^u = \sqrt{2} \Rightarrow u = \frac{1}{2}\ln 2$

$\Rightarrow \ln Z = \frac{1}{2}\ln(2) + i\left(\frac{\pi}{4} + 2n\pi\right)$

$i\ln Z = -\left(\frac{\pi}{4} + 2n\pi\right) + i\frac{1}{2}\ln(2)$

$\Rightarrow W = e^{c\ln Z} = e^{-(\frac{\pi}{4}+2n\pi) + i(\frac{1}{2}\ln(2))} = \boxed{e^{-(\frac{\pi}{4}+2n\pi)}\left[\cos \ln\sqrt{2} + i\sin\ln\sqrt{2}\right]}$

---

x) $(-1)^{\frac{1}{z}} = W \Rightarrow \frac{1}{z}\ln(-1) = \ln W \Rightarrow W = e^{\frac{1}{z}\ln(-1)}$

$\ln(-1) = i(\pi + 2n\pi) \quad ; n = 0, \pm1, \pm2, \ldots$

from Pb $\boxed{8}$ iii)

$\Rightarrow \frac{1}{z}\ln(-1) = \frac{i(\pi+2n\pi)}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{y(\pi+2n\pi) + ix(\pi+2n\pi)}{x^2+y^2}$

$\Rightarrow e^{\frac{1}{z}\ln(-1)} = e^{\frac{y}{x^2+y^2}(\pi+2n\pi)} \cdot e^{\frac{ix}{x^2+y^2}(\pi+2n\pi)} = W$

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