BESSEL FUNCTIONS

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Consider the following differential equation:

$$\boxed{t^2 \frac{d^2y}{dt^2} + t\,\frac{dy}{dt} + (\lambda^2 t^2 - \upsilon^2)\, y = 0}$$

Which is called ***Bessel differential equation*** of order υ and of parameter λ .

$$Let \quad \lambda t = x \quad \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \quad \Rightarrow \frac{dy}{dt} = \lambda \frac{dy}{dx} \quad \equiv \lambda\, y'$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) \times \frac{dx}{dt} = \frac{d}{dx}(\lambda\, y') \times \lambda \quad = \lambda^2\, y''$$

Substitute in the differential equation

$$t^2 \lambda^2\, y'' + t\lambda\, y' + (\lambda^2 t^2 - \upsilon^2)\, y = 0$$

$$\boxed{\Rightarrow x^2\, y'' + x\, y' + (x^2 - \upsilon^2)\, y = 0}$$

Which is also ***Bessel differential equation*** of order υ and of parameter 1 .

$x = 0$ is a regular singular point

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+s} \quad , \quad y' = \sum_{n=0}^{\infty} (n+s)\, a_n x^{n+s-1} \quad , \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n x^{n+s-2}$$

Substitute in the differential equation

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} - \upsilon^2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

Step 1

Shifting the index of the third summation so that the powers of x are the same in all summations.

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} - \upsilon^2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

Step 2

$$a_0\left(s(s-1)+s-\upsilon^2\right)x^s+a_1\left((s+1)s+(s+1)-\upsilon^2\right)x^{s+1}$$

$$+\sum_{n=2}^{\infty}\left(a_n[(n+s)(n+s-1)+(n+s)-\upsilon^2]+a_{n-2}\right)x^{n+s}=0$$

Coefficient of $x^s=0 \Rightarrow a_0\left(s(s-1)+s-\upsilon^2\right)=0$

$$a_0\left(s^2-\upsilon^2\right)=0 \qquad \Rightarrow s^2=\upsilon^2 \qquad \Rightarrow s_1=\upsilon \quad \& \quad s_2=-\upsilon$$

Step 3

Coefficient of $x^{s+n}=0$

$$a_n\left((n+s)(n+s-1)+(n+s)-\upsilon^2\right)+a_{n-2}=0 \qquad \Rightarrow a_n=\frac{-1}{(n+s)^2-\upsilon^2}a_{n-2}$$

$$\boxed{a_n=\frac{-1}{((n+s)-\upsilon)((n+s)+\upsilon)}a_{n-2}\quad , \quad n\geq 2}$$

Step 4

For $s_1 = \upsilon$

$$\boxed{\Rightarrow \quad a_n = \frac{-1}{n\,(2\upsilon+n)}\,a_{n-2} \quad , \quad n \geq 2}$$

$$a_2 = \frac{-1}{(2)(2\upsilon+2)}\,a_0$$

$$a_4 = \frac{-1}{(4)(2\upsilon+4)}\,a_2 \qquad = \frac{(-1)^2}{(2)(4)(2\upsilon+2)(2\upsilon+4)}\,a_0$$

$$a_6 = \frac{-1}{(6)(2\upsilon+6)}\,a_4 \qquad = \frac{(-1)^3}{(2)(4)(6)\times(2\upsilon+2)(2\upsilon+4)(2\upsilon+6)}\,a_0$$

$$a_{2n} = \frac{(-1)^n}{(2\times4\times6\times\ldots\times2n)[(2\upsilon+2)(2\upsilon+4)\ldots(2\upsilon+2n)]}\,a_0 \quad , \quad n \geq 1$$

$$a_{2n} = \frac{(-1)^n}{2^n\times n!\times 2^n\times(\upsilon+1)(\upsilon+2)\ldots(\upsilon+n)}\,a_0 \quad , \quad n \geq 1$$

$$\Rightarrow a_{2n} = \frac{(-1)^n\ \Gamma(\upsilon+1)}{2^{2n}\times n!\times\Gamma(n+\upsilon+1)}\,a_0 \quad , \quad n \geq 0$$

Step 5 - 1

$$\therefore\ y_1 = \sum_{n=0}^{\infty} a_n(\upsilon)\ x^{n+\upsilon} \qquad = \sum_{n=0}^{\infty} a_{2n}(\upsilon)\ x^{2n+\upsilon}$$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n\ \Gamma(\upsilon+1)\ x^{2n+\upsilon}}{2^{2n} \times n! \times \Gamma(n+\upsilon+1)}\, a_0$$

$$Take \quad 2^{\upsilon}\ \ \Gamma(\upsilon+1)\ \ a_0\ =\ 1 \qquad\qquad \Rightarrow y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n\ \Gamma(\upsilon+1)\ x^{2n+\upsilon}\ 2^{\upsilon}}{2^{2n+\upsilon} \times n! \times \Gamma(n+\upsilon+1)}\, a_0$$

$$\boxed{y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(n+\upsilon+1)} \left(\frac{x}{2}\right)^{2n+\upsilon} \equiv J_{\upsilon}(x)}$$

This is the first solution for Bessel Differential equation

Step 6 - 1

Solving the recurrence relation for $s_2 = -\upsilon$, we will find another solution

$$\boxed{y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(n-\upsilon+1)} \left(\frac{x}{2}\right)^{2n-\upsilon} \equiv J_{-\upsilon}(x)}$$

Step 6 - 2

Note that $J_{\upsilon}(x)$ *and* $J_{-\upsilon}(x)$ are called Bessel functions of the first kind

If υ is a fraction, the two solutions $J_{\upsilon}(x)$ *and* $J_{-\upsilon}(x)$ are linearly independent, hence,

$$\boxed{y_{gs}(x) = C_1 J_{\upsilon}(x) + C_2 J_{-\upsilon}(x)}$$

$$\boxed{\Rightarrow y_{gs}(t) = C_1 J_{\upsilon}(\lambda t) + C_2 J_{-\upsilon}(\lambda t)}$$

If $\upsilon = N$ is an integer, $\boxed{J_{-N}(x) = (-1)^N J_N(x)}$, hence, $C_1 J_N(x) + C_2 J_{-N}(x)$ doesn't represent the general solution but only one independent solution and we need another linearly independent solution

Consider $Y_\upsilon(x) = \dfrac{J_\upsilon(x)\cos\upsilon\pi - J_{-\upsilon}(x)}{\sin\upsilon\pi}$

The second linearly independent solution is

$$\boxed{Y_N(x) = \lim_{\upsilon \to N} \frac{J_\upsilon(x)\cos\upsilon\pi - J_{-\upsilon}(x)}{\sin\upsilon\pi}}$$

Which is called Bessel functions of the second kind

At which

$$\boxed{y_{gs}(x) = C_1 J_N(x) + C_2 Y_N(x)} \qquad \boxed{\Rightarrow y_{gs}(\lambda t) = C_1 J_N(\lambda t) + C_2 Y_N(\lambda t)}$$