SPECIAL FUNCTIONS

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1. THE GAMMA FUNCTION

The Gamma function is defined as

$$\boxed{\Gamma(x)=\int_0^{\infty} t^{x-1}\; e^{-t}\; dt \qquad , \quad x>0}$$

This integral diverges for $x \le 0$

Properties of the Gamma function:

$$\boxed{1)\quad \Gamma(1)=1}$$

Proof:

$$Set \quad x=1 \qquad \Rightarrow \Gamma(1)=\int_0^{\infty} e^{-t}\,dt \;=-\,e^{-t}\Big|_0^{\infty}=1$$

$$\boxed{2)\quad \Gamma(1/2)=\sqrt{\pi}}$$

Proof:

$$\Gamma(1/2)=\int_0^\infty t^{-\frac{1}{2}}\; e^{-t}\; dt \;=\int_0^\infty y^{-1}e^{-y^2}\,.\,2y\,dy \;=2\int_0^\infty e^{-y^2}\,dy\;=\sqrt{\pi}$$

Note that we’ve used the well known formula $\int_0^\infty e^{-x^2}\; dx=\dfrac{\sqrt{\pi}}{2}$

$$\boxed{3)\;\; \Gamma(x+1)=x\,\Gamma(x)}$$

Proof:

$$\Gamma(x+1)=\int_0^\infty t^x\; e^{-t}\; dt$$

Integrating by parts we find:

$$\Gamma(x+1)=-\,t^x\, e^{-t}\Big|_0^\infty \;+\; x\int_0^\infty t^{x-1}\, e^{-t}\, dt \;=\; x\Gamma(x)$$

Note that $\lim\limits_{t\to\infty} t^x / e^t = 0$

$$\boxed{4)\quad \Gamma(n+1)=n! \quad \text{For any positive integer } n}$$

Note: Because of property 4, the Gamma function is a generalization of the factorial function where the Gamma function is defined for any non integer values while the factorial function is defined for integers only.

Example:

Show that $L\{t^{-1/2}\} = \sqrt{\dfrac{\pi}{s}}$

Solution:

$$L\{t^{-1/2}\} = \int_0^\infty t^{-1/2}\ e^{-st}\ dt \qquad , \quad s > 0$$

$$Set\ \ u = st \ \Rightarrow\ dt = \tfrac{1}{s}\, du$$

$$\Rightarrow L\{t^{-1/2}\} = \int_0^\infty \left(\frac{u}{s}\right)^{-1/2} e^{-u}\ \frac{1}{s} du \ = \frac{1}{\sqrt{s}} \int_0^\infty u^{-\frac{1}{2}}\ e^{-u} du \ = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

In what follows we will give a table of values of the Gamma function for $x \in [1,2]$ in steps of 0.02. For values of x not cited in the table, we can use linear interpolation to find $\Gamma(x)$.

x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$
1.00	1.000000	1.34	0.892216	1.68	0.905001
1.02	0.988844	1.36	0.890185	1.70	0.908639
1.04	0.978438	1.38	0.888537	1.72	0.912581
1.06	0.968744	1.40	0.887264	1.74	0.916826
1.08	0.959725	1.42	0.866356	1.76	0.921275
1.10	0.951351	1.44	0.885805	1.78	0.926227
1.12	0.943590	1.46	0.885604	1.80	0.931384
1.14	0.936416	1.48	0.885747	1.82	0.936845
1.16	0.929803	1.50	0.886227	1.84	0.942612
1.18	0.923728	152	0.887039	1.86	0.948667
1.20	0.918169	1.54	0.888178	1.88	0.955017
1.22	0.913106	1.56	0.889390	1.90	0.961766
1.24	0.908521	1.58	0.891420	1.92	0.968774
1.26	0.904397	1.60	0.893515	1.94	0.976099
1.28	0.900718	1.62	0.895924	1.96	0.983743
1.30	0.897471	1.64	0.898642	1.98	0.991708
1.32	0.894640	1.66	0.901668	2.00	1.000000

Definition of the Gamma function for negative values of x.

The definition of the gamma function can't be obtained from the integral because it converges only for positive values of x. We agree to extend the domain of definition of the Gamma function for negative values using the following relation

$$\boxed{\Gamma(x)=\frac{\Gamma(x+1)}{x} \quad , \quad x+1>0}$$

For example,

$$\Gamma(-1/2)=\frac{\Gamma(1/2)}{-1/2}=-2\sqrt{\pi}$$

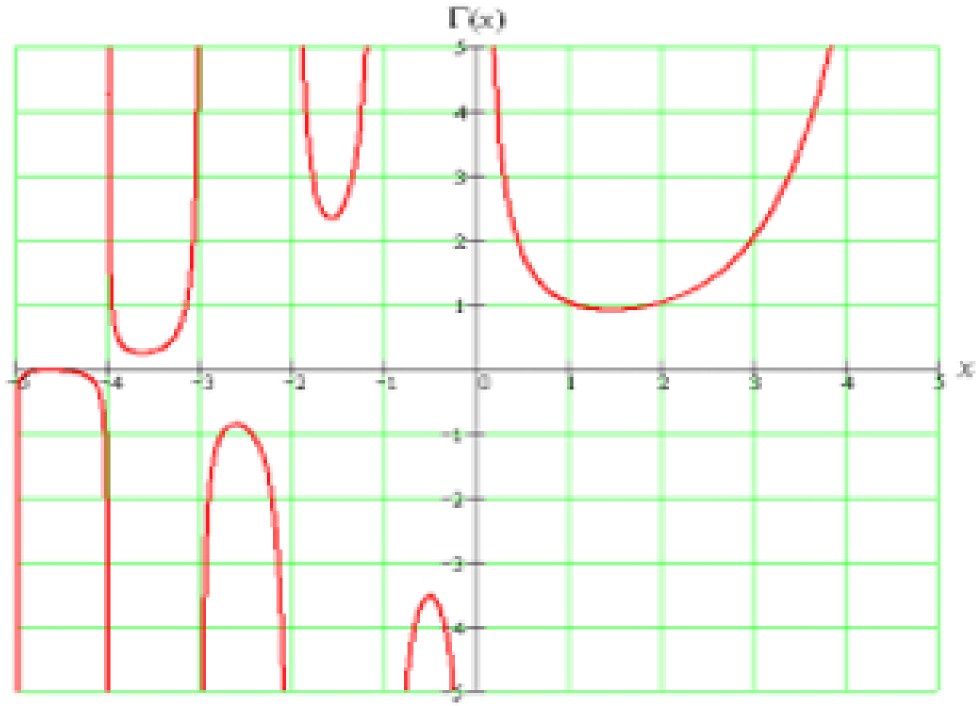
Similarly,

$$\Gamma(-3.4)=\frac{\Gamma(1.6)}{(-3.4)(-2.4)(-1.4)(-0.4)(0.6)}$$

We can use the tables to evaluate $\Gamma(1.6)$.

Using this definition we can show that $\Gamma(-n)=\pm\infty$ **where n is a positive integer.**

Using these information we can sketch the curve of the Gamma function for both positive and negative values.



5) The multiplication property:

$$\boxed{\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}}$$

Example:

Evaluate $\displaystyle\int_0^\infty x^{3/2}\ 5^{-x}\ dx$

Solution:

$$Set\ \ 5^x=e^t \quad\Rightarrow\quad x\ln 5=t \quad\Rightarrow\quad dx=\frac{dt}{\ln 5}$$

$$\Rightarrow\ I\ =\int_0^\infty\left(\frac{t}{\ln 5}\right)^{\frac{3}{2}}\,.\,e^{-t}\,.\,\frac{dt}{\ln 5}\qquad=\frac{1}{(\ln 5)^{5/2}}\int_0^\infty t^{\frac{3}{2}}\,.\,e^{-t}\,.\,dt$$

$$\Rightarrow I=\frac{1}{(\ln 5)^{5/2}}\Gamma\left(\frac{5}{2}\right)\qquad=\frac{1}{(\ln 5)^{5/2}}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\qquad=\frac{3\sqrt{\pi}}{4\,(\ln 5)^{5/2}}$$

2. THE BETA FUNCTION

The Beta function is defined as

$$\beta(x\,,\ y)=\int_0^1 t^{x-1}\ (1-t)^{y-1}\ dt \qquad ;\quad x>0\ ,\quad y>0 \qquad (1)$$

Putting $t\ =\ \sin^2\theta$ we find another form for the Beta function.

$$\beta(x\,,\ y)=2\int_0^{\pi/2} \sin^{2x-1}\theta\ \cos^{2y-1}\theta\quad d\theta \qquad (2)$$

Putting $t\ =\ \dfrac{u}{1+u}$ we find the third form for the Beta function.

$$\beta(x\,,\ y)=\int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\ du \qquad (3)$$

Relation between the Beta and Gamma functions

We can prove that

$$\boxed{\beta(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}}$$

Simply we can show that the Beta function is a symmetric function $\beta(x\,,\;y)=\;\beta(y\,,\;x)$

Examples:

$$i)\quad \beta(2,4)=\frac{\Gamma(2)\Gamma(4)}{\Gamma(6)}\quad =\frac{1!\;.\;3!}{5!}=\frac{1}{20}$$

$$ii)\quad \beta(\frac{1}{4},\frac{7}{4})=\frac{\Gamma(1/4)\Gamma(7/4)}{\Gamma(2)}\quad =\frac{\Gamma(1/4).\,\frac{3}{4}\Gamma(3/4)}{1!}\quad =\frac{3}{4}.\frac{\pi}{\sin\pi/4}\quad =\frac{3\sqrt{2}}{4}\pi$$

Example:

Evaluate in terms of the Gamma function $I = \int_0^2 \sqrt{x}\;(16 - x^4)^{5/8}\; dx$.

Solution:

$$Set\;\; x^4 = 16\,t \quad \Rightarrow \quad x = 2t^{1/4} \quad \Rightarrow \quad dx = \frac{1}{2}\,t^{-3/4}\; dt$$

$$\Rightarrow I = \int_0^1 \sqrt{2}\;\; t^{1/8}\;\; (16 - 16\,t)^{5/8}\;.\frac{1}{2}\,t^{-3/4}\,dx \quad = 4\int_0^1 t^{-5/8}\;(1 - t)^{5/8}\;\, dt$$

$$\Rightarrow I \;=4\;\; \beta(3/8\,,13/8) \quad = 4\,\frac{\Gamma(3/8)\Gamma(13/8)}{\Gamma(2)} \quad = \;4.\frac{5}{8}\Gamma(\frac{5}{8})\Gamma(\frac{3}{8})$$

$$= \frac{5}{2}\;.\;\frac{\pi}{\sin(3\pi/8)} \quad \approx \;8.501088$$

Legendre's Duplication Formula

We can prove that

$$\boxed{\sqrt{\pi}\ \Gamma(2x) = 2^{2x-1}\ \Gamma(x)\ \Gamma(x+\frac{1}{2})}$$

Extra Exercise:

By two different methods find a closed form for $\Gamma(n+\frac{1}{2})$ where n is a positive integer.

Example:

Find in terms of the Gamma function the area enclosed by the Astroid $x^{2/3} + y^{2/3} = 1$

Solution:

We have to sketch the Astroid

$$For\ \ x=0\ \Rightarrow y=\pm 1 \qquad For\ \ y=0\ \Rightarrow x=\pm 1 \qquad \&\ y=\left(1-x^{2/3}\right)^{3/2}$$

$$A = 4\int_0^1 y\ dx = 4\int_0^1 \left(1 - x^{2/3}\right)^{3/2} dx$$

$$Set\ \ x^{2/3} = t \qquad \Rightarrow x = t^{3/2} \qquad \Rightarrow dx = \frac{3}{2}\sqrt{t}\ dt$$

$$\Rightarrow\ A = 4\int_0^1 (1-t)^{3/2} \cdot \frac{3}{2}\sqrt{t}\ dt$$

$$= 6\int_0^1 t^{1/2}\ (1-t)^{3/2} dt$$

$$= 6\ \beta\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{3}{8}\pi$$