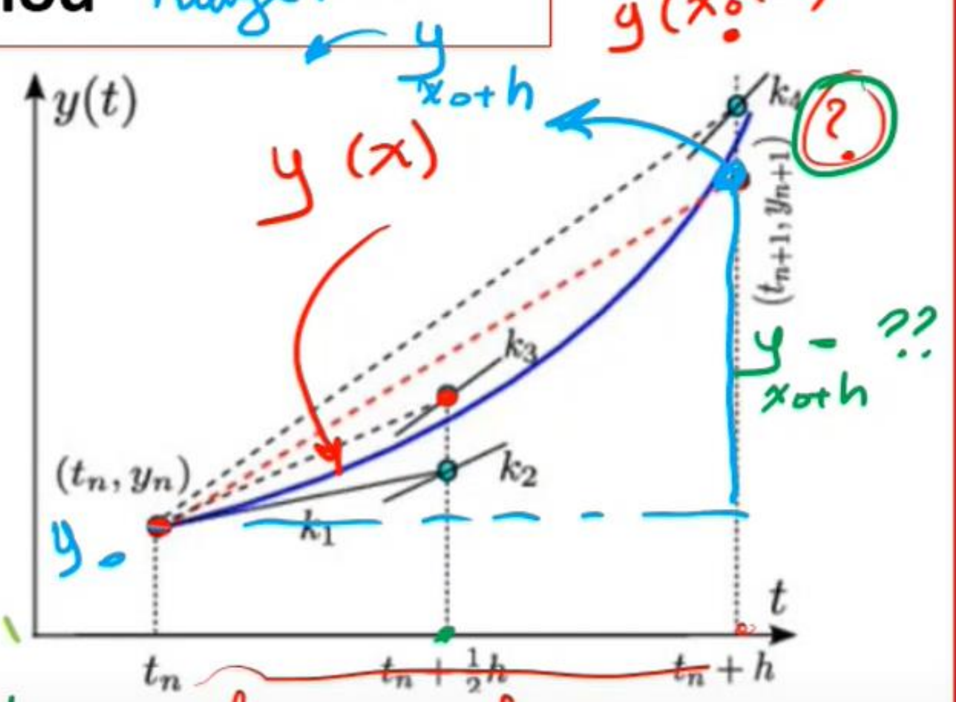
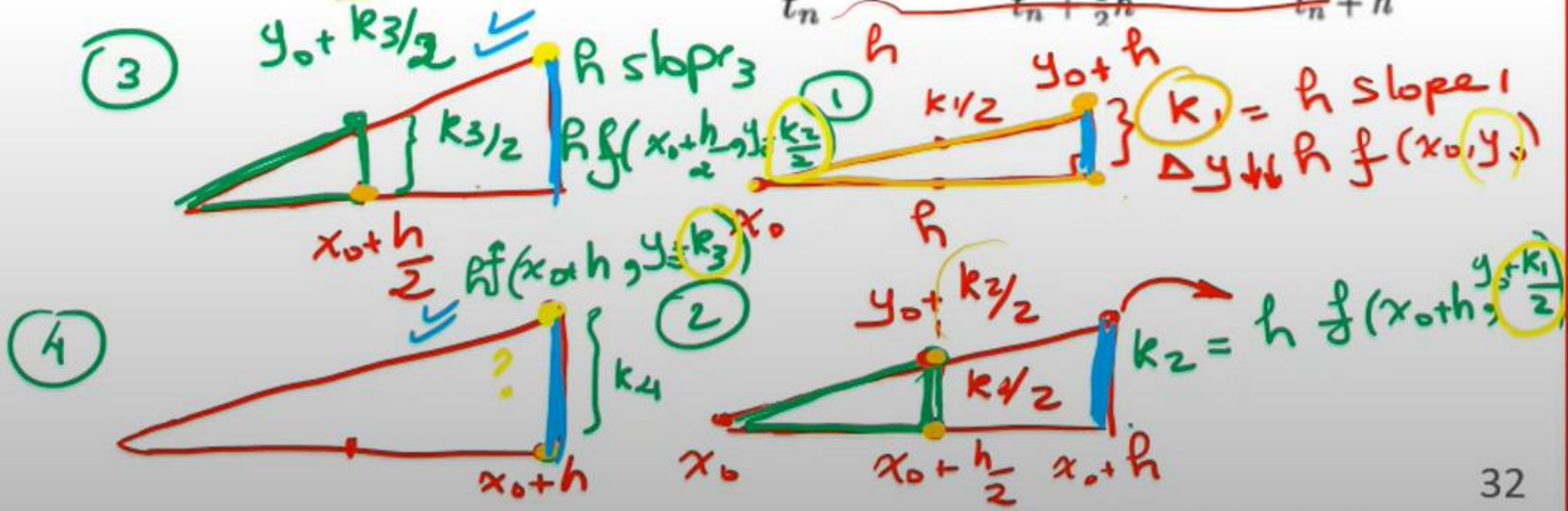
ODE : Runge-Kutta Method

Runge Kutta

$$y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

1- Method





Numerical Analysis: ODE

$\alpha\alpha\,(y')^2 + y'y + x^2 = 0$

slope $(y' =$

$y'' =$

1st degree

The second order ordinary differential equation

IVP

$$x'' = f(t, x, x') \qquad [\,x(t_0) = a,\ x'(t_0) = b\,]$$

Let $x' = y$ **Then.**

$$\begin{cases} x' = y & , x(t_0) = a \\ y' = f(t, x, y) & , y(t_0) = b \end{cases}$$

I can solve each By Runge-Kutta

2 First Order Differential Eq.

Numerical Analysis: ODE

2 First Order Differential Eq. solved using RK4

$x' = y \qquad , x(t_0) = a \qquad\qquad y' = f(t, x, y) \;,\; y(t_0) = b$

Euler

$$\begin{cases} v_1 = h\,f(t_n, x_n, y_n), \\ v_2 = h\,f(t_n + \frac{h}{2}, x_n + \frac{v_1}{2}, y_n + \frac{w_1}{2}), \\ v_3 = h\,f(t_n + \frac{h}{2}, x_n + \frac{v_2}{2}, y_n + \frac{w_2}{2}) \\ v_4 = h\,f(t_n + h, x_n + v_3, y_n + w_3) \end{cases}$$

$$\begin{cases} w_1 = h\,g(t_n, x_n, y_n) \\ w_2 = h\,g(t_n + \frac{h}{2}, x_n + \frac{v_1}{2}, y_n + \frac{w_1}{2}) \\ w_3 = h\,g(t_n + \frac{h}{2}, x_n + \frac{v_2}{2}, y_n + \frac{w_2}{2}) \\ w_4 = h\,g(t_n + h, x_n + v_3, y_n + w_3) \end{cases}$$

$$x_{n+1} = x_n + \tfrac{1}{6}\Delta x = x_n + \tfrac{1}{6}(v_1 + 2v_2 + 2v_3 + v_4) \qquad y_{n+1} = y_n + \tfrac{1}{6}\Delta y = y_n + \tfrac{1}{6}(w_1 + 2w_2 + 2w_3 + w_4)$$

Numerical Analysis: ODE

2 First Order Differential Eq. solved using RK4

n	t_n	x_n	y_n	$v = h\,f(x_n, y_n)$	$w = h\,g(x_n, y_n)$	Δx	Δy
0	t_0	x_0	y_0	v_1	w_1	v_1	w_1
1	$t_0 + h/2$	$x_0 + v_1/2$	$y_0 + w_1/2$	v_2	w_2	$2v_2$	$2w_2$
2	$t_0 + h/2$	$x_0 + v_2/2$	$y_0 + w_2/2$	v_3	w_3	$2v_3$	$2w_3$
3	$t_0 + h$	$x_0 + v_3$	$y_0 + w_3$	v_4	w_4	v_4	w_4
					sum	Sum (1)	Sum (2)

$h = 0.25 \quad 2h = 0.5$

Example (h^4) (4)

?

Use Runge – Kutta method with step size $\boxed{h = 0.25}$ to obtain **x (0.5)** with 5 decimal places for the IVP

$$x'' - t\,x' - x = 0,\ x(0) = 0,\ x'(0) = 1$$

$t_0 = 0 \quad x_0 = 0 \quad y_0 = 1 \hookrightarrow x'$

Solution

Assuming $x' = y$. Therefore $x'' = y'$. The equivalent system is

$\boxed{x' = y} = f(t, x, y)$, $\boxed{y' = x + t\,y} = g(t, x, y)$ such that $x(0) = 0$, $y(0) = 1$.

Numerical Analysis: ODE

$x' = y = f(t, x, y),\ y' = x + t\,y = g(t, x, y)$ such that $x(0) = 0, y(0) = 1.$

$(x+ty)$ $h = 0.25$

n	t	x	y	v $= 0.25y$	w $= 0.25$	Δx	Δy
0	0.000	0.00000	1.00000	0.25	0.00000	0.25	.000000
$t_0+\frac{h}{2}$	0.125	0.125	1.00000	0.25 ×2	0.0625 ×2	0.5	0.12500
$t_0+\frac{h}{2}$	0.125	0.125	1.0313	0.25783 ×2	0.063478 ×2	0.51566	0.12696
t_0+h	0.25	0.25783	1.0635	0.26588	0.13093	0.26588	0.13093
						0.25526	0.06382

÷2 ÷2 ÷2

$\boxed{x(0.25)} = x(0) + (1/6)(\Delta x) = \boxed{0} + (1/6)*(1.53154) = 0.25526$

$\boxed{y(0.25)} = y(0) + (1/6)(\Delta y) = \boxed{1} + (1/6)*(0.38289) = 1.0638$

Numerical Analysis: ODE

$x' = y = f(t, x, y)$, $y' = x + t\,y = g(t, x, y)$ such that $x(0) = 0$, $y(0) = 1$.

n	t	x	y	v	w	Δx	Δy
0	0.250	0.25526	1.0638	0.26595	0.1303	0.26595	0.1303
	0.375	0.38824	1.129	0.28225	0.2029	0.5645	0.4058
	0.375	0.39639	1.1653	0.29133	0.20848	0.58266	0.41696
	0.5	0.54659	1.2723	0.31808	0.29569	0.31808	0.29569
						0.25526	0.20813

$X(0.5) = x(0.250) + (1/6)(\Delta x) = 0.25526 + (1/6)*(1.73118) = 0.54379$

$y(0.5) = y(0.250) + (1/6)(\Delta y) = 1.0638 + (1/6)*(1.24878) = 1.2719$

Agenda

- Introduction
- Finite Difference Method
- Solution of PDE using Finite Difference Method
  - Steps of Solution
  - Laplace Equation Example
  - Poisson Equation Example

Numerical Analysis

PDE

$U(x,y) = ?$

Partial differential equations are used to characterize engineering systems where the behavior of a physical quantity is couched in terms of its rate of change with respect to two or more independent variables.

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x$$

$$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = x$$

$U(x,y) = \checkmark$

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0}$$

$\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $U(x,y)$

PDE: Introduction

Why ?

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

Many Applications in Engineering

steady state

$\boxed{T(x,y)}$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Laplace

$T(x, t)$

$$\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$$

Heat

$y(x, t)$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Wave

PDE: Introduction

Why ?

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

$$\left(B^2 - 4AC\right)$$

method of solution

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad < 0 \longrightarrow \textbf{Elliptic}$$

$$\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2} \qquad = 0 \longrightarrow \textbf{Parabolic}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \qquad > 0 \longrightarrow \textbf{Hyperbolic}$$

Idea ?

$$U_{xx} + U_{yy} = 0$$

$U(x,y)$ ✓

$\boxed{\text{BVP}}$

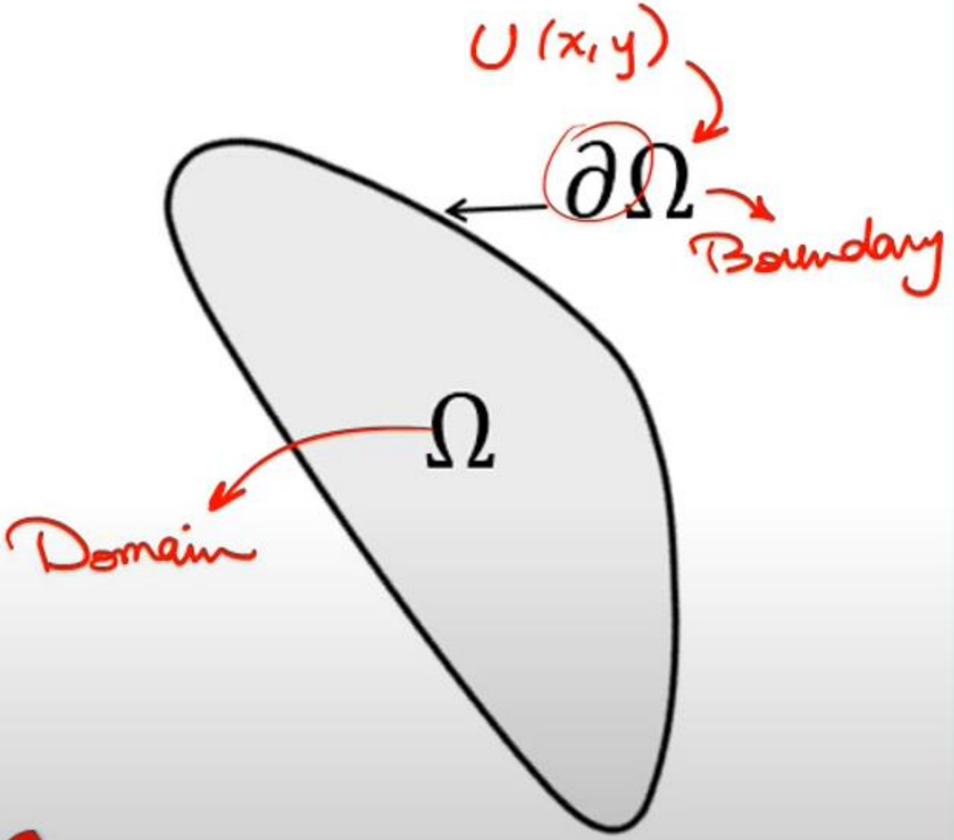
Partial Differential Equation, together with boundary conditions.

$$U_{xx} + U_{yy} = 0$$

$U(x,y) =$ ✓

at values of x, y on Boundary

Dirichlet Conditions



PDE: Finite Difference Method

Idea ?

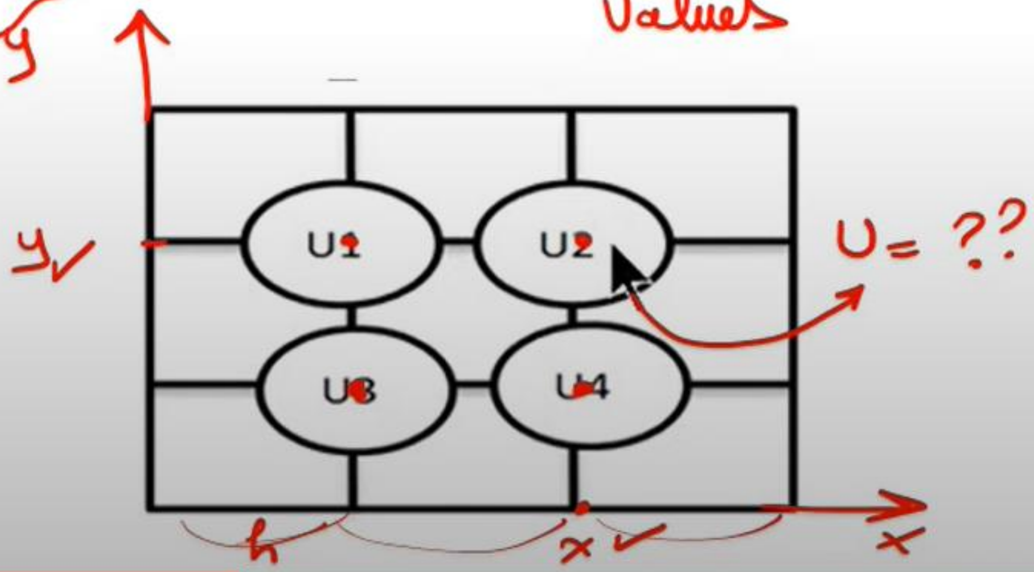
Find an approximate solution for PDE

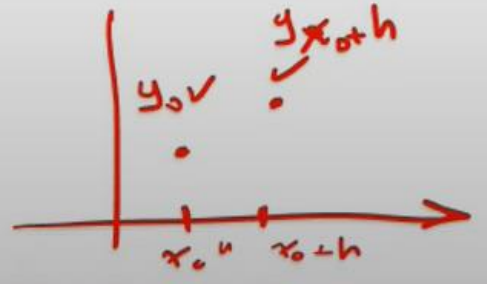
$$u_{xx} + u_{yy} = 0$$

$$u_{xx} + u_{yy} = f(x, y)$$

U (x, y)

→ in the form of values





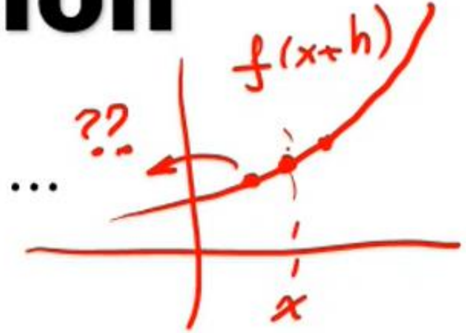
PDE: Finite Difference Method

Idea ?

Taylor Series Approximation

at x near a

$$\boxed{f(x)} = f(a) + \frac{f'(a)}{1!}\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \cdots$$



= different symbols

$a \rightarrow x$

near a $\rightarrow$ x+h

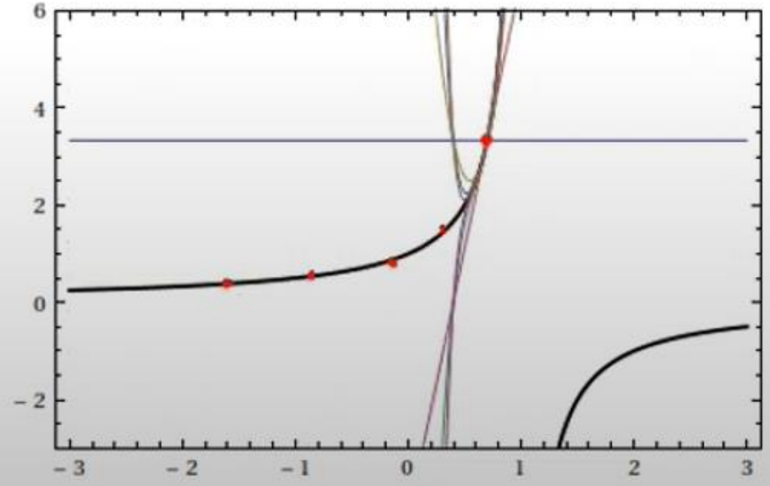
x $\rightarrow$ (x+h)

at (x+h) right

$$f(x+h) = f(x) + \frac{h}{1!}\,f'(x) + \frac{h^2}{2!}\,f''(x) + \frac{h^3}{3!}\,f'''(x)\ \ldots$$

left

$$f(x-h) = f(x) - \frac{h}{1!}\,f'(x) + \frac{h^2}{2!}\,f''(x) - \frac{h^3}{3!}\,f'''(x)\ \ldots$$



PDE: Finite Difference Method

P.D.E.

$U_{xx} + U_{yy} = 0$

↓

Alg. Equation

Idea ?

Taylor

I right

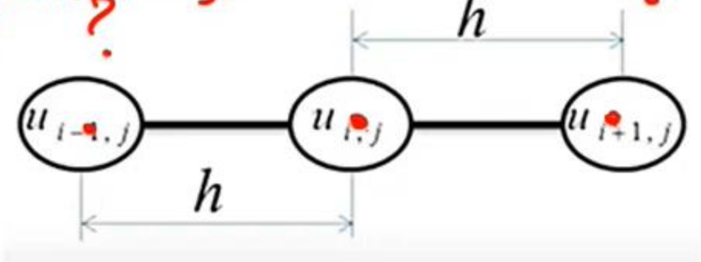
neglect

$$\rightarrow u(x+h,y) = u(x,y) + h\,u_x(x,y) + \frac{h^2}{2}\,u_{xx}(x,y) + \frac{h^3}{6}\,u_{xxx}(x,y)$$

① $\boxed{U_x(x,y)} = \dfrac{U(x+h,y) - U(x,y)}{h} - \dfrac{h^{1}}{2} U_{xx}(x,y) - \dfrac{h^2}{6} U_{xxx}(x,y)$?

Foward

?



II left

$$\rightarrow u(x-h,y) = u(x,y) - h\,u_x(x,y) + \frac{h^2}{2}\,u_{xx}(x,y) - \frac{h^3}{6}\,u_{xxx}(x,y)$$

② $\boxed{U_x(x,y)} = \dfrac{U(x,y) - U(x-h,y)}{h} + \dfrac{h^{1}}{2} U_{xx}(x,y)$ ----

Backward

central

③ $(I - II) \div 2h$ $\boxed{U_x(x,y)} = \dfrac{U(x+h,y) - U(x-h,y)}{2h} + \dfrac{h^{2}}{6} U_{xxx}(x,y)$ ....

PDE: Finite Difference Method

Idea ?

Approximation for $u_x(x, y)$	Formula
Forward – difference approximation	$u_x(x,y) \approx \dfrac{u(x+h,y)-u(x,y)}{h}$
Backward – difference approximation	$u_x(x,y) \approx \dfrac{u(x,y)-u(x-h,y)}{h}$
Central – difference approximation	$\boxed{u_x(x,y)} \approx \dfrac{u(x+h,y)-u(x-h,y)}{2h}$

more accurate

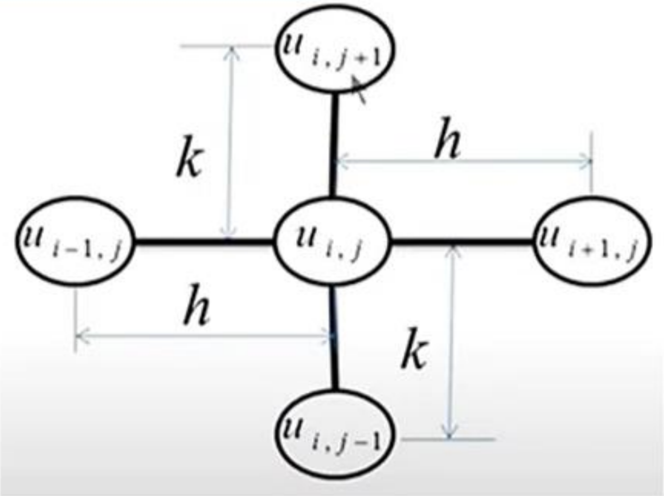
$(I + II) \div h^2 \rightarrow$

$$\boxed{u_{xx}(x,y)} \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$$

PDE: Finite Difference Method

Idea ?

$$u(x, y+k) = u(x, y) + k\, u_y(x, y) + \frac{k^2}{2} u_{yy}(x, y) + \cdots$$



$$u(x, y-k) = u(x, y) - k\, u_y(x, y) + \frac{k^2}{2} u_{yy}(x, y) + \cdots$$

PDE: Finite Difference Method

Idea ?

Approximation for $u_y(x, y)$	Formulas
Forward – difference approximation	$u_y(x,y) \approx \dfrac{u(x,y+k)-u(x,y)}{k}$
Backward – difference approximation	$u_y(x,y) \approx \dfrac{u(x,y)-u(x,y-k)}{k}$
Central – difference approximation	$\boxed{u_y(x,y)} \approx \dfrac{u(x,y+k)-u(x,y-k)}{2k}$

$$\boxed{u_{yy}(x,y)} \approx \frac{u(x,y+k)-2u(x,y)+u(x,y-k)}{k^2}$$

PDE: Finite Difference Method

Idea ?

Find an approximate solution for PDE

$U(x+h,y) - 2U(x,y) + U(x-h,y)/h^2$

$$u_{xx} + u_{yy} = 0 \xrightarrow{h^2} [u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)] = 0.$$

right up left down

$U(x,y+h) - 2U(x,y) + U(x,y-h)/h^2$

$$u_{xx} + u_{yy} = f(x,y) \rightarrow u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y) = h^2 f(x, y).$$

PDE: Steps of Solution

Idea ?

Step 1

Temperature

Define the BVP

Laplace

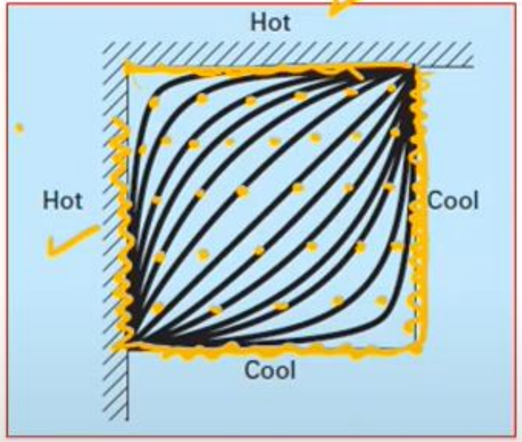
$$U_{xx} + U_{yy} = 0$$

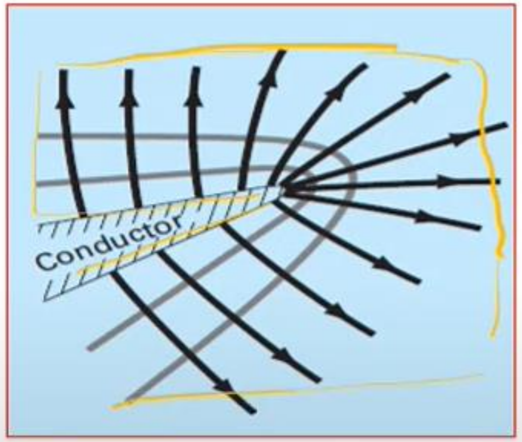
Poisson

$$U_{xx} + U_{yy} = f(x, y)$$

$$(x, y)|_{\partial\Omega} = u_0(x, y)$$

$$(x, y) \in \Omega = (a, b)x(c, d)$$





$T(x+h, y+h) = ✓$

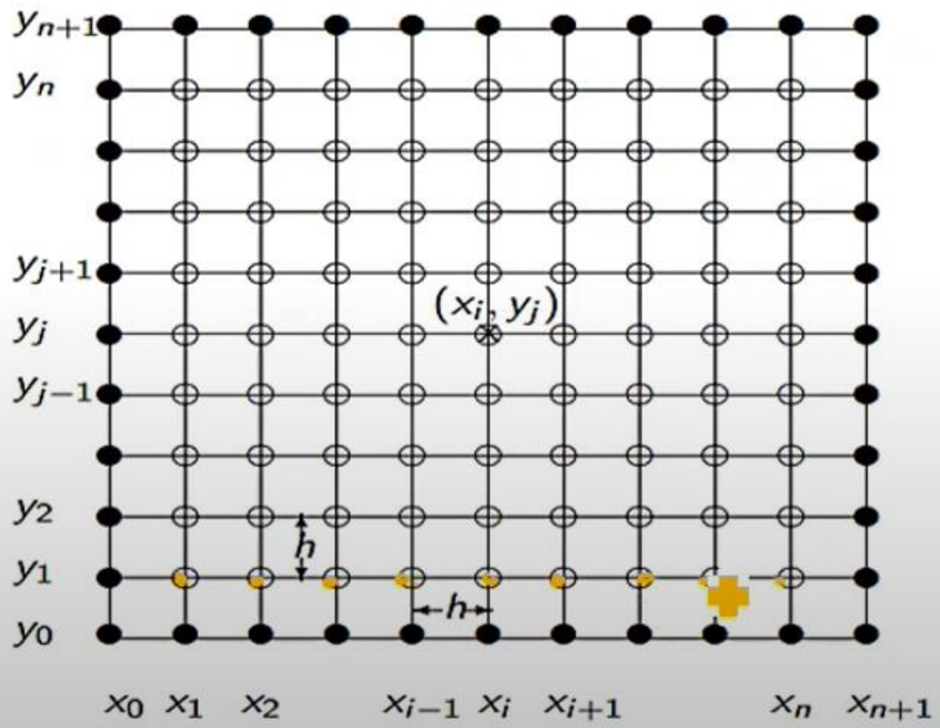
PDE: Steps of Solution

Idea ?

Step 2

Generate the grid
according to the mesh size h

equal mesh size in x, y



U (x, y)

PDE: Steps of Solution

Idea ?

Step 3

Use Finite Difference to convert PDE to Linear System of Algebraic Equations

stencil

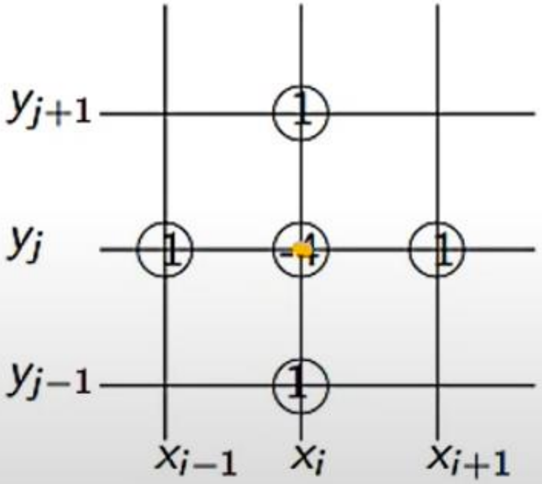
up

left

right

$$\nabla^2 u(x,y) \approx \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u(x,y)$$

down



Laplace Equation

$\rightarrow U_{xx}(x_i, y_j) + U_{yy}(x_i, y_j) = 0$

$\hookrightarrow U(x_i, y_{j+1}) + U(x_{i+1}, y_j) + U(x_{i-1}, j) + U(x_i, y_{j-1}) - 4U(x_i, y_j) = 0$

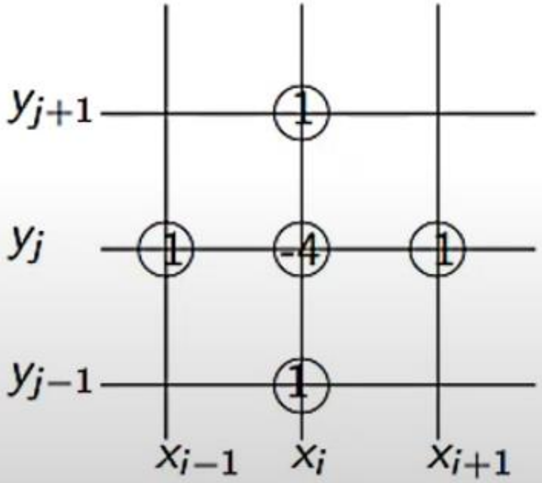
PDE: Steps of Solution

Idea ?

Step 3

Use Finite Difference to convert PDE to Linear System of Algebraic Equations

$$\nabla^2 u(x,y) \approx \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u(x,y) = f(x,y)$$



$$\left[U(x_{i+1}, j) + U(x_{i-1}, j) + U(x_i, y_{j+1}) + U(x_i, y_{j-1}) - 4\,U(x_i, y_j)\right] = h^2 f(x,y)$$

PDE: Steps of Solution

Idea ?

Step 4

Solve Linear System of Algebraic Equations

Gauss- Seidel Method

$$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

$|a_{11}| > |a_{12} + \cdots a_{1n}|$

$|a_{22}| >$

$|a_{33}| >$

PDE: Steps of Solution

Idea ?

Step 4

Solve Linear System of Algebraic Equations

Gauss-Seidel

Converges faster!! than Jacobi

initial value $x_1 = x_2 = \cdots x_n = \text{zero}$

$$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)}).$$

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$

$$x_3 = \cdots$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right).$$

PDE: Steps of Solution

Idea ?

Step 4

Solve Linear System of Algebraic Equations

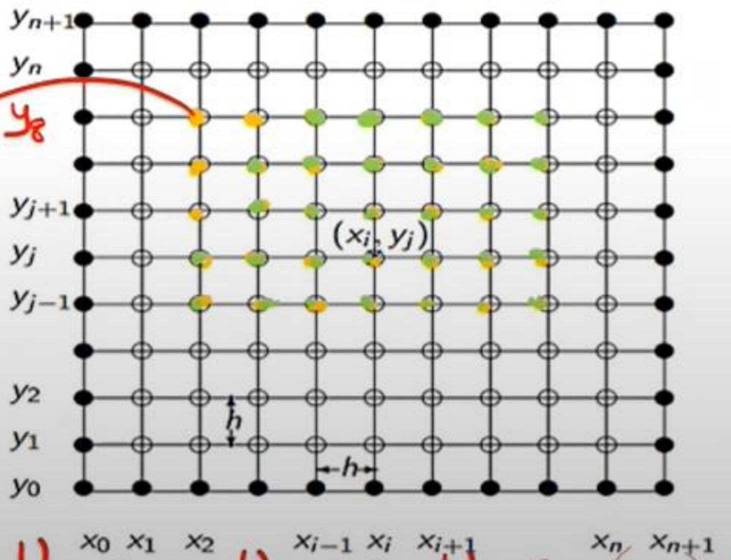
Why?

Gauss-Seidel

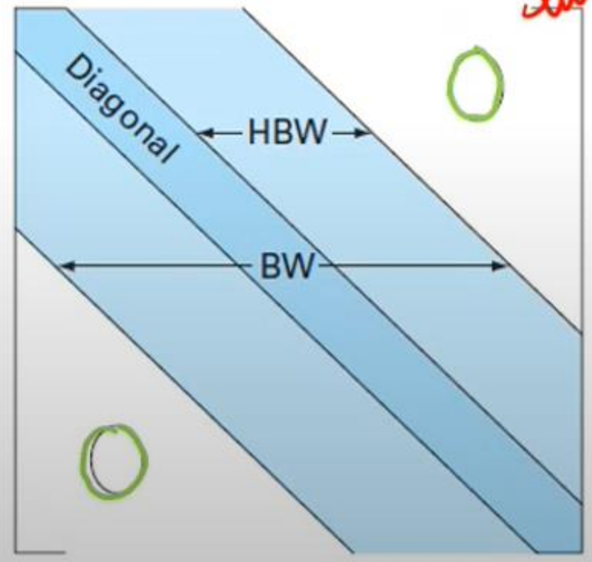
more efficient

diagonally dominant linear system

at $U_{2,8}$:



$$\left[U_{2,9} + U_{2,7} + U_{3,8} + U_{1,8} - 4\, U_{2,8} = 0\right]$$



PDE: Steps of Solution

Idea ?

Step 1 Define the PDE

Step 2 Decompose the domain

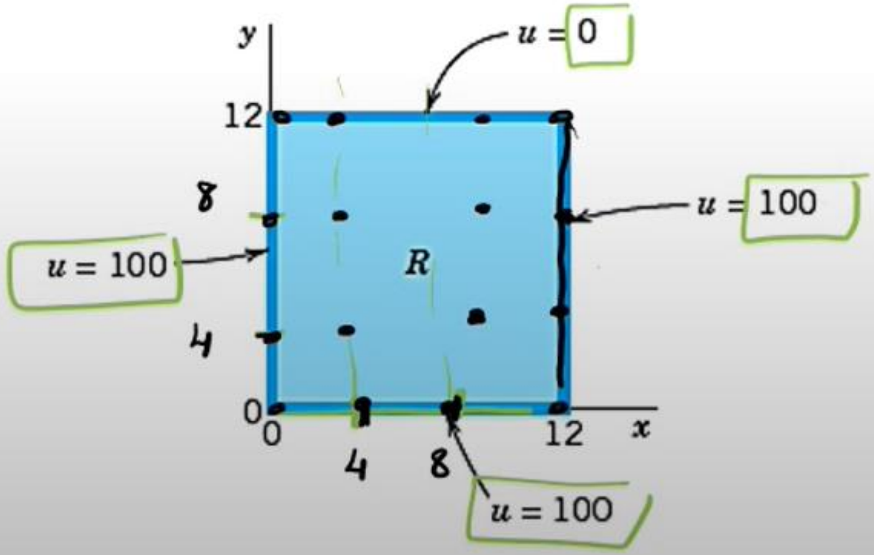
Step 3 Use Finite Difference to convert PDE to Linear System of Algebraic Equations

Step 4 Solve Linear System of Algebraic Equations

PDE: Example 1 Laplace Equation

Example 1:

The four sides of a square plate of side 12 cm, made of homogeneous material, are kept at constant temperature as shown in Figure. Using a (very wide) grid of mesh 4 cm and applying Gauss–Seidel (Liebmann) iteration method, find the steady-state temperature at the mesh points (which satisfies Laplace equation).



$U(x, y) = ?$ Dirishlet Problem

$$\left.\begin{aligned} &U_{xx} + U_{yy} = 0 \\ &U(x,0) = 100 \\ &U(0,y) = 100 \\ &U(12,y) = 100 \\ &U(x,12) = 0 \end{aligned}\right\}$$

PDE: Example 1 Laplace Equation

Solution

① **PDE**

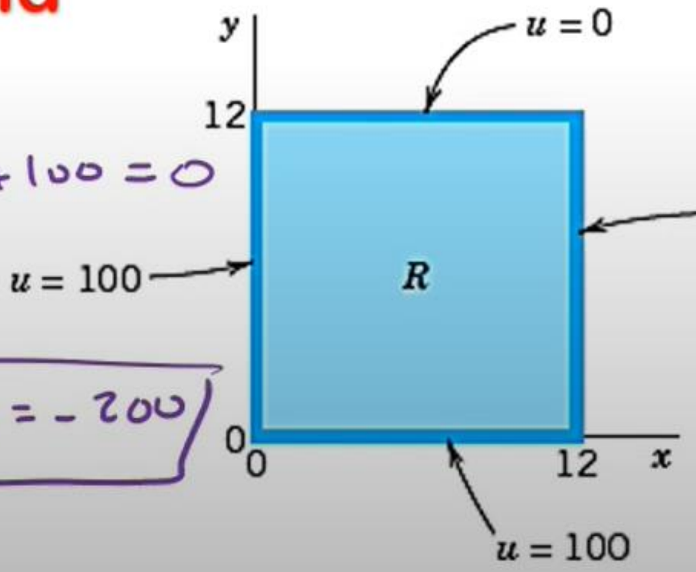
$$u_{xx} + u_{yy} = 0\,, \quad u(0,y) = u(x,0) = u(12,y) = 100 \text{ and } u(x,12) = 0,$$

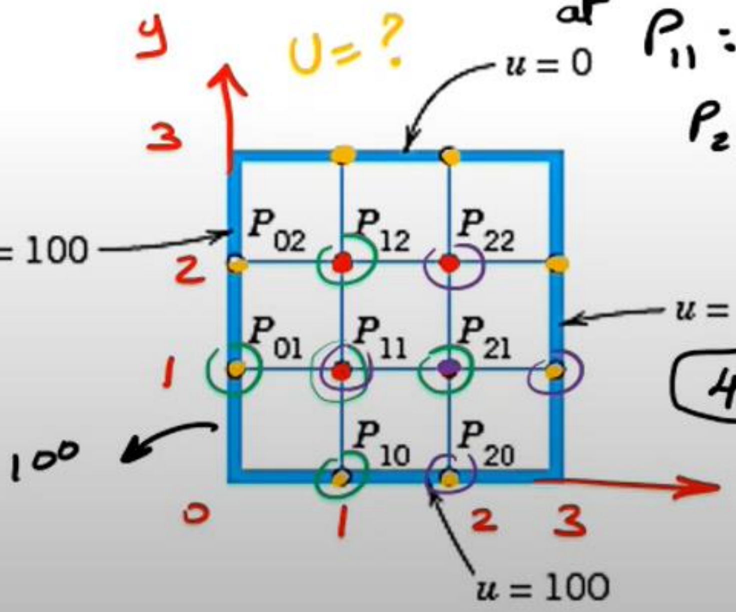
② **Mesh Grid**

at P_{21} :

$100 + U_{11} + U_{22} + 100 = 0$
$- 4\,U_{21}$

$$\boxed{4U_{21} - U_{11} - U_{22} = -200}$$





③ Finite difference

at P_{11} :

$P_{21} + 100 + P_{12} + 100 -$
$4\,P_{11} = 0$

$$\boxed{4\,U_{11} - U_{21} - U_{12} = -200}$$

\#

PDE: Example 1 Laplace Equation

Gauss - Seidel Method

$$\begin{aligned}
-4u_{11} + u_{21} + u_{12} \quad &= -200 \\
-u_{11} - 4u_{21} \quad + u_{22} &= -200 \\
u_{11} \quad - 4u_{12} + u_{22} &= -100 \\
u_{21} + u_{12} - 4u_{22} &= -100.
\end{aligned}$$

$$\begin{aligned}
u_{11} &= \quad 0.25\overset{100}{u_{21}} + 0.25\overset{50}{u_{12}} \quad + 50 \\
u_{21} &= 0.25u_{11} \quad + 0.25\overset{50}{u_{22}} + 50 \\
u_{12} &= 0.25u_{11} \quad + 0.25u_{22} + 25 \\
u_{22} &= \quad 0.25u_{21} + 0.25u_{12} \quad + 25.
\end{aligned}$$

at start

$U_{11}^{(0)} = 100$ $\qquad$ $U_{12}^{(0)} = 50$

$U_{21}^{(0)} = 100$ $\qquad$ $U_{22}^{(0)} = 50$

PDE: Example 1 Laplace Equation

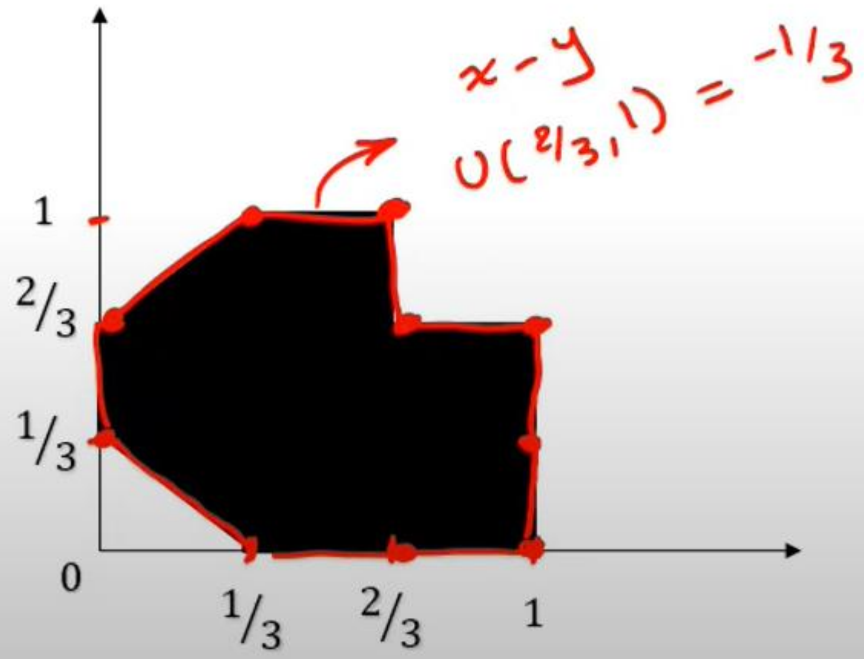
- For n = 0: $x_1^{(1)} = 87.50,\ x_2^{(1)} = 84.375,\ x_3^{(1)} = 53.375,\ x_4^{(1)} = 61.719$
- For n = 1: $x_1^{(2)} = 85.938,\ x_2^{(2)} = 86.914,\ x_3^{(2)} = 61.914,\ x_4^{(2)} = 62.207$
- For n = 2: $x_1^{(3)} = \boxed{87.207},\ x_2^{(3)} = \boxed{87.354},\ x_3^{(3)} = \boxed{62.354},\ x_4^{(3)} = \boxed{62.427}$
- Note that the exact solution of the given system is

$$x_1 = x_2 = \boxed{87.5} \text{ and } x_3 = x_4 = 62.5$$

PDE: Example 2 Poisson Equation

Example 2:

Consider the Dirichlet problem $\nabla^2 u(x,y) = \boxed{9\,(x^2+y^2)}$ in R , $u(x,y) = x - y$ in ∂R **where** ∂R **is the boundary of R and R is the region defined by the following figure with a mesh grid size h=1/3.**



PDE: Example 2 Poisson Equation

Solution

$U_{xx} + U_{yy} = h^2 f(x,y)$

$h = \frac{1}{3}$

at U_{11}:

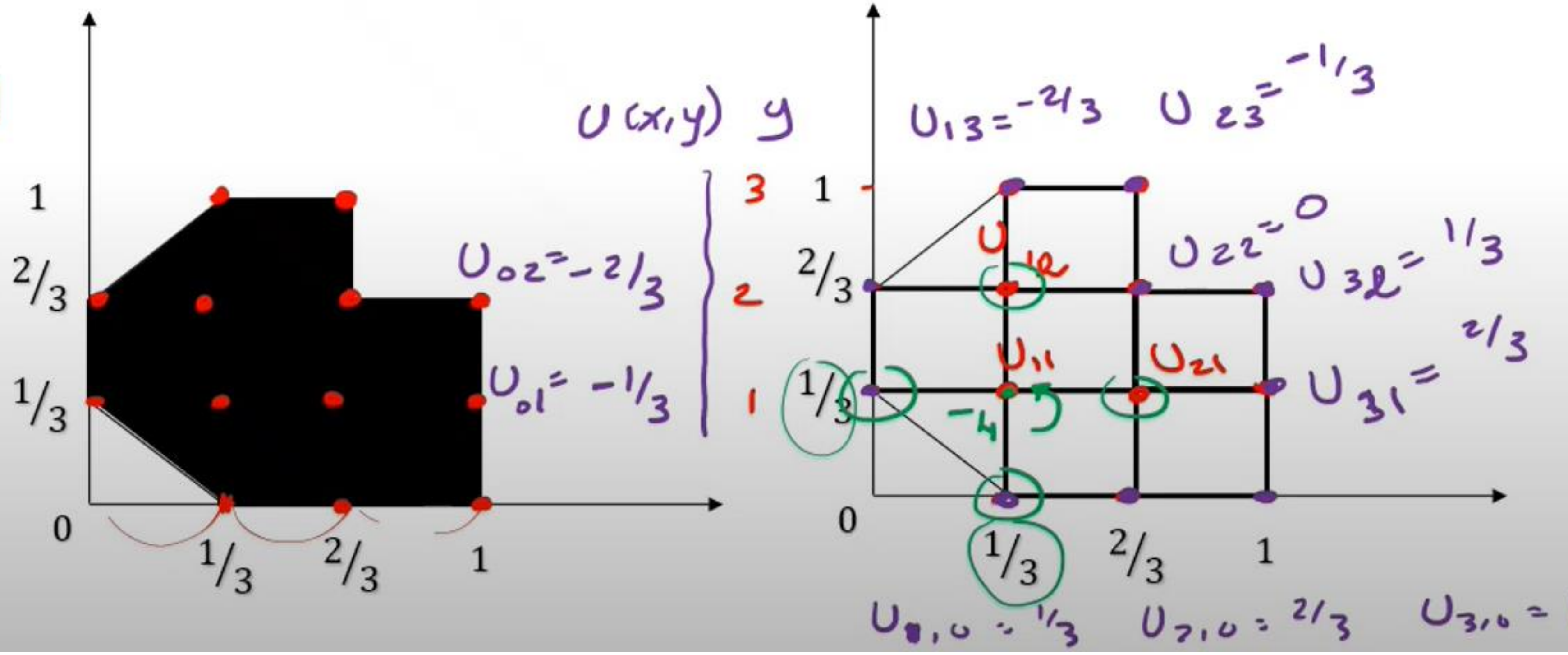
$\left(\frac{1}{3} + U_{21} - \frac{1}{3} + U_{12} - 4U_{11}\right) = \left(\frac{1}{3}\right)^2 9\left(\underbrace{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}_{2/9}\right)$

PDE

Poisson

$$u_{xx} + u_{yy} = 9\left(x^2 + y^2\right) \text{in R}, u(x,y) = x - y \text{ in } \partial\text{R}$$

Mesh Grid



PDE: Example 2 Poisson Equation

$$\left(-4u_{11}+u_{21}+u_{12}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}=\cancel{\frac{1}{9}}*\cancel{9}\left(\frac{1}{9}+\frac{1}{9}\right)\right)$$

$$\left(u_{11}-4u_{21}+\frac{2}{3}+0+\frac{2}{3}=\cancel{\frac{1}{9}}*\cancel{9}\left(\frac{4}{9}+\frac{1}{9}\right)\right)$$

$$\left(u_{11}-4u_{12}+0-\frac{2}{3}-\frac{2}{3}=\frac{1}{9}*9\left(\frac{1}{9}+\frac{4}{9}\right)\right)$$

Gauss Seidel

$$U_{11}^{(0)}=U_{21}^{(0)}=U_{12}^{(0)}=0$$

$$U_{11}=\left(\frac{2}{9}-U_{21}-U_{12}\right)\left(-\frac{1}{4}\right)$$

$$U_{21}=\left(\frac{7}{9}-U_{11}\right)\left(-\frac{1}{4}\right)$$

$$U_{12}=\left(\frac{14}{9}-U_{11}\right)\left(-\frac{1}{4}\right)$$

PDE: Example 2 Poisson Equation

$U_{11} = \checkmark$

then $U_{21} = \checkmark$

$U_{12} = \checkmark$

$$-4u_{11} + u_{21} + u_{12} = \frac{2}{9} = 0.222$$

$$u_{11} - 4u_{21} = -\frac{7}{9} = -0.778$$

$$u_{11} - 4u_{12} = \frac{17}{9} = 1.889$$

to get exact solution:

$$\boxed{u_{11}^{(3)} = -0.142,\ u_{21}^{(3)} = 0.159,\ u_{12}^{(3)} = -0.508}$$

Thank you ☺