Complex numbers

1 Rectangular form

$$z = \underbrace{x}_{Re(z)} + i\underbrace{y}_{Im(z)} \qquad \text{"}x, y \in R\text{"}$$

Ex: $z_1 = 2+3i$, $z_2 = 3-4i$

$Im(z_1) = 3$, $Im(z_2) = -4$

$z_1 + z_2 = (2+3) + (3-4)i = 5 - i$

$z_1 - z_2 = (2-3) + (3+4)i = -1 + 7i$

Re

$z_1 \cdot z_2 = (2+3i) \cdot (3-4i) = (6+12i^2) + i(9-8) = 18 + i$

Im

$$z_1 / z_2 = \frac{2+3i}{3-4i} * \frac{3+4i}{3+4i} = \frac{(6-12)+i(9+8)}{9-16i^2} = \frac{-6+17i}{25} = \frac{-6}{25} + \frac{17}{25}i$$

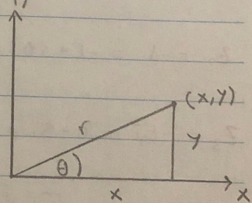
$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1} \frac{y}{x}$

$x = r\cos\theta$

$y = r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$



$\boxed{2}$ Polar form

$$z = r e^{i\theta}$$

$r \rightarrow |z| = \text{Mag.}(z)$, $\quad i\theta \rightarrow \text{Arg.}(z)$

$r \geq 0 \quad , \quad -\pi < \theta \leq \pi$

Ex: $\quad z_3 = 2e^{i(\pi/3)} \quad , \quad z_4 = 4e^{i(\pi/4)}$

$z_3 \cdot z_4 = (2e^{i\pi/3})(4e^{i\pi/4}) = 8e^{i\left(\frac{7\pi}{12}\right)}$ $\quad$ "$z_3 \cdot z_4 = r_4 r_3 e^{i(\theta_4+\theta_3)}$"

$z_3 / z_4 = \dfrac{2e^{i\pi/3}}{4e^{i\pi/4}} = \dfrac{1}{2} e^{i\left(\frac{\pi}{12}\right)}$ $\quad$ "$z_3/z_4 = \dfrac{r_3}{r_4} e^{i(\theta_3-\theta_4)}$"

$z_3 \pm z_4 = 2e^{i\pi/3} \pm 4e^{i\pi/4}$ $\quad$ "Convert to Rec. form first"

$= \left(2\cos\frac{\pi}{3} + 2i\sin\frac{\pi}{3}\right) \pm \left(4\cos\frac{\pi}{4} + 4i\sin\frac{\pi}{4}\right)$

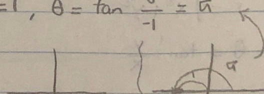
$= (1+\sqrt{3}i) \pm (2\sqrt{2} + 2\sqrt{2}i)$

Ex: write in the polar form

$z_1 = 1 = 1 + i0 = 1e^{i0}$ $\quad$ $r = 1;\ \theta = 0$

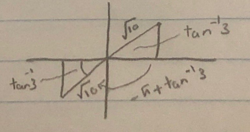
$z_2 = -1 = -1 + i0 = 1e^{i\pi}$ $\quad$ $r = \sqrt{1+0} = 1,\ \theta = \tan^{-1}\frac{0}{-1} = \pi$

$z_3 = -5i = 0 - 5i = 5e^{i\left(-\frac{\pi}{2}\right)}$



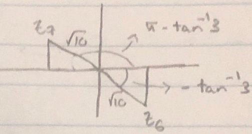
$z_4 = 1 + 3i = \sqrt{10}\, e^{i(\tan^{-1}3)}$ $\quad$ $r = \sqrt{1+9} = \sqrt{10},\ \theta = \tan^{-1}\left(\frac{3}{1}\right)$

$z_5 = -1 - 3i = \sqrt{10}\, e^{i(-\pi + \tan^{-1}3)}$



$z_6 = 1-3i = \sqrt{10}\, e^{i(-\tan^{-1}3)}$ $\qquad r = \sqrt{10}$, $\theta = \tan^{-1}\frac{-3}{1} = -\tan^{-1}3$

$z_7 = -1+3i = \sqrt{10}\, e^{i(\pi - \tan^{-1}3)}$



Bessel function

$y'' + xy = 0$ $\quad * x^2$ in terms of Bessel func.

$\Rightarrow x^2 y'' + x^3 y = 0$ ---- $(*)$ $\quad$ since Coeff $y' = 0$

let $y = x^\alpha u \Rightarrow y' = \alpha x^{\alpha-1} u + x u'$

$\therefore y'' = \alpha(\alpha-1) x^{\alpha-2} u + \alpha x^{\alpha-1} u' + x^\alpha u'' + \alpha x^{\alpha-1} u'$

$\therefore y'' = \alpha(\alpha-1) x^{\alpha-2} u + 2\alpha x^{\alpha-1} u' + x^\alpha u''$

substitute in $(*)$

$\Rightarrow x^2 [\alpha(\alpha-1) x^{\alpha-2} u + 2\alpha x^{\alpha-1} u' + x^\alpha u''] + x^3 (x^\alpha u) = 0$

$\Rightarrow x^{\alpha+2} u'' + 2\alpha x^{\alpha+1} u' + [x^{\alpha+3} + \alpha(\alpha-1) x^\alpha] u = 0 \qquad (* x^{-\alpha})$

$\Rightarrow x^2 u'' + 2\alpha x u' + [x^3 + \alpha(\alpha-1)] u = 0$

$\Rightarrow$ Coeff $(xu') = 1 \Rightarrow 2\alpha = 1 \Rightarrow \boxed{\alpha = \frac{1}{2}}$

$\therefore x^2 u'' + x u' + (x^{3} - \frac{1}{4}) u = 0$ $\quad$ should be $(\lambda^2 x^{2} - \mu^2)$

let $t^2 = x^3 \Rightarrow t = x^{3/2} \Rightarrow x = t^{2/3}$

$\Rightarrow u' = \dot{u}\, \frac{3}{2} x^{3/2} = \frac{3}{2} t^{1/3}\, \dot{u}$

$\Rightarrow u'' =$ ______ "Same as MT"

$\therefore t^2 \ddot{u} + t\dot{u} + \left(\frac{4}{9} t^2 - \frac{1}{9}\right) u = 0$

$\therefore u_{g.s} = C_1 J_{1/3}\left(\frac{2}{3} t\right) + C_2 J_{-1/3}\left(\frac{2}{3} t\right)$

$= C_1 J_{1/3}\left(\frac{2}{3} x^{3/2}\right) + C_2 J_{-1/3}\left(\frac{2}{3} x^{3/2}\right)$

$\therefore y_{g.s} = x^{1/2} u = x^{1/2}\left[C_1 J_{1/3}\left(\frac{2}{3} x^{3/2}\right) + C_2 J_{-1/3}\left(\frac{2}{3} x^{3/2}\right)\right]$

ترتيب الحل:
Coeff y'' ①
Coeff y' ②
Coeff y ③
مينفعش نغير الترتيب دا

Properties of Bessel function:

1. $J_{-n}(x) = (-1)^n J_n(x)$ → $n =$ integer $\Rightarrow J_n, J_{-n}$ are linearly dep.

2. $J_n(-x) = (-1)^n J_n(x) \rightarrow f(-x) = (-1)^n f(x)$
   - → n-even $\therefore J_n$ even funct.
   - → n-odd $\therefore J_n$ odd funct.

3. $\frac{2n}{x} J_n = J_{n-1}(x) + J_{n+1}(x)$ → Rec. Relation

4. $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \rightarrow J_0' = -J_1$

   ↳ $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2 J_n(x) \rightarrow \int J_1(x)\,dx = -J_0 + C$

5. $\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}$

6. $\frac{d}{dx}\left(x^{-n} J_n\right) = -x^{-n} J_{n+1}$

7. $\int x^n J_{n-1}\, dx = x^n J_n + C$

8. $\int x^{-n} J_{n+1}\, dx = -x^{-n} J_n + C$

9. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad , \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Closed form

$\hookrightarrow$ in terms of elementry function

Ex: write $J_{3/2}(x)$ in a closed form $\rightarrow$ "or in terms of elem funct."

$J_{3/2} \rightarrow J_{1/2}\, ,\, J_{-1/2} \rightarrow$ closed form

Ans: using Rec. Relation "property (3)"

$\frac{2n}{x} J_n = J_{n-1} + J_{n+1} \quad \Rightarrow n+1 = 3/2 \Rightarrow n = 1/2$

$\Rightarrow \frac{1}{x} J_{1/2} = J_{-1/2} + J_{3/2} \quad \Rightarrow J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2}$

$\therefore J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$

$J_{-5/2} \rightarrow J_{1/2}\, ,\, J_{-1/2} \rightarrow$ closed form $\Rightarrow n-1 = -5/2 \Rightarrow n = -3/2$

$-\frac{3}{x} J_{-3/2} = J_{-5/2} + J_{-1/2} \quad \Rightarrow J_{-5/2} = -\frac{3}{x} J_{-3/2} - J_{-1/2}$

$J_{-3/2} \Rightarrow n-1 = -3/2 \Rightarrow n = -\frac{1}{2}$ since $J_{-3/2}$ is not a closed form

$\Rightarrow -\frac{1}{x} J_{-1/2} = J_{-3/2} + J_{1/2} \Rightarrow J_{-3/2} = -\frac{1}{x} J_{-1/2} - J_{1/2}$

$\therefore J_{-5/2} = -\frac{3}{x}\left(-\frac{1}{x} J_{-1/2} - J_{1/2}\right) - J_{-1/2}$

$= \left(\frac{3}{x^2} - 1\right) J_{-1/2} + \frac{3}{x} J_{1/2} =$ ______

Ex: Find y'

① $y = x^3 J_3 \Rightarrow y' = x^3 J_2$

② $y = x^{-2} J_2 \Rightarrow y' = x^{-2} J_3$

③ $y = x^{-4} J_{-4} \Rightarrow y' = x^{-4} J_{-5}$ Since the power of x and the index of J are the same, we used property (5)

④ $y = x^3 J_{-3} \Rightarrow y' = -x^3 J_{-2}$

⑤ $y = x^5 J_3 \Rightarrow y = (x^2)(x^3 \cdot J_3) \Rightarrow y' = 2x(x^3 J_3) + x^2 (x^3 J_2)$

$= 2x^4 J_3 + x^5 J_2$

⑥ $y = x J_3 J_4$

$y' = \frac{d}{dx}(x J_3 J_4) * \frac{x^3}{x^3} = \frac{d}{dx}[x \cdot x^3 \cdot x^{-3} \cdot J_3 \cdot J_4] = \frac{d}{dx}[x^4 J_4 \cdot x^{-3} J_3]$

$\therefore y' = (x^4 J_3)(x^{-3} J_3) + (x^4 J_4)(-x^{-3} J_4) = x J_3^2 - x J_4^2 = x[J_3^2 - J_4^2]$