5) The multiplication Property:

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Ex: $\Gamma(\tfrac{1}{3})\,\Gamma(\tfrac{2}{3}) = \dfrac{\pi}{\sin\frac{\pi}{3}} = \dfrac{2\pi}{\sqrt{3}}$

Example Evaluate $\int_0^\infty x^{3/2}\, 5^{-x}\, dx$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

let $5^x = e^t$

$x \ln 5 = t$

$dx = \dfrac{dt}{\ln 5}$

$$I = \int_0^\infty x^{3/2}\, 5^{-x}\, dx = \int_0^\infty \left(\frac{t}{\ln 5}\right)^{\frac{3}{2}} e^{-t} \frac{dt}{\ln 5}$$

$$= \frac{1}{(\ln 5)^{5/2}} \int_0^\infty t^{(\frac{5}{2}-1)} e^{-t}\, dt = \frac{\Gamma(\frac{5}{2})}{(\ln 5)^{5/2}} = \frac{\frac{3}{2} * \frac{1}{2} * \Gamma(\frac{1}{2})}{(\ln 5)^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4(\ln 5)^{5/2}}$$

[2] The Beta Function

$$\boxed{\beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt \;;\; x>0,\; y>0}$$

$$\boxed{\beta\Big|_{t=\sin^2\theta} = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \,\cos^{2y-1}\theta \, d\theta}$$

$t = \sin^2\theta$

$dt = 2\sin\theta\cos\theta\, d\theta$

$t \quad 0 \to 1$

$\theta \quad 0 \to \frac{\pi}{2}$

$$\boxed{\beta\Big|_{t=\frac{u}{1+u}} = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du}$$

$t = \dfrac{u}{1+u}$

$dt = \dfrac{du}{(1+u)^2}$

$t \; 0 \to 1, \quad u \; 0 \to \infty$

* Relation between the Beta and Gamma functions

$$\boxed{\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

Ex: $\beta(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3!\,4!}{8!}$

we can show that the Beta function is a Symmetric function $\beta(x,y) = \beta(y,x)$ $\quad \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)}$

Ex $\beta(2,4) = \frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{1!\,3!}{5!} = \frac{1}{20}$

$$\beta(\tfrac{1}{4},\tfrac{1}{7}) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{7}{4})}{\Gamma(2)} = \frac{\Gamma(\frac{1}{4})\frac{3}{4}\Gamma(\frac{3}{4})}{1!} = \frac{3}{4}\Gamma(\tfrac{1}{4})\Gamma(1-\tfrac{1}{4})$$

$$= \frac{3}{4}\frac{\pi}{\sin(\frac{\pi}{4})} = \frac{3\sqrt{2}}{4}\pi$$

Ex Evaluate in terms of the Gamma function

$$I = \int_0^2 \sqrt{x}\,(16-x^4)^{5/8}\,dx$$

let $x^4 = 16t$
$x = 2t^{1/4}$
$dx = \frac{1}{2}t^{-3/4}dt$
$x=2 \longrightarrow t = \frac{x^4}{16} = \frac{16}{16} = 1$

$$I = \int_0^1 \sqrt{2}\,t^{1/8}(16-16t)^{5/8} * \frac{1}{2}t^{-\frac{3}{4}}dt$$

$$= \frac{16^{5/8}}{\sqrt{2}}\int_0^1 t^{-\frac{5}{8}}(1-t)^{5/8}dt$$

$$= \frac{16^{5/8}}{\sqrt{2}}\beta(\tfrac{3}{8},\tfrac{13}{8}) = \frac{16^{5/8}}{\sqrt{2}}\frac{\Gamma(\frac{3}{8})\Gamma(\frac{13}{8})}{\Gamma(2)} = \frac{16^{5/8}\,\frac{5}{8}}{\sqrt{2}}\frac{\Gamma(\frac{3}{8})\Gamma(1-\frac{3}{8})}{1!}$$

$$= \frac{5}{2}\frac{\pi}{\sin(\frac{3}{8}\pi)}$$

GHARIB

Subject: .... / /

* Legendre's Duplication Formula

$$\boxed{\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2})}$$

Ex By two different methods find a closed form for $\Gamma(n+\frac{1}{2})$ where n is a positive integer

① with Legen.: $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{\Gamma(2n)}{\Gamma(n)} = \frac{\sqrt{\pi}\,\Gamma(2n+1) \cdot n}{2^{2n-1} \cdot 2n\,\Gamma(n+1)}$

$$= \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)} = \frac{\sqrt{\pi}}{2^{2n}} \frac{(2n)!}{n!} \quad \#$$

② without: $\Gamma(n+\frac{1}{2}) = (n+\frac{1}{2})\,\Gamma(n-\frac{1}{2}) = (n+\frac{1}{2})(n-\frac{1}{2})\,\Gamma(n-\frac{3}{2})$

$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2}) \cdots \tfrac{5}{2} \cdot \tfrac{3}{2} \cdot \tfrac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$= \frac{1}{2^n}(2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1 \cdot \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2^n} \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4) \cdots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2n)(2n-2)(2n-4) \cdots 6 \cdot 4 \cdot 2}$$

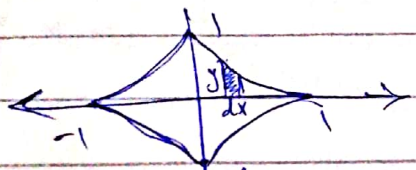
$$= \frac{\sqrt{\pi}}{2^n \cdot 2^n} \frac{(2n)!}{(n)(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1} = \frac{\sqrt{\pi}}{2^{2n}} \frac{(2n)!}{n!} \quad \#$$

Example Find in terms of the Gamma function the area enclosed by the Astroid $x^{2/3} + y^{2/3} = 1$

we have to sketch Astroid

$x = 0 \rightarrow y = \pm 1$

$y = 0 \rightarrow x = \pm 1$



$$\text{Area} = 4\int_0^1 y\,dx = 4\int_0^1 (1 - x^{\frac{2}{3}})^{\frac{3}{2}}\,dx$$

let $x^{2/3} = t$

$x = t^{3/2}$

$dx = \frac{3}{2} t^{\frac{1}{2}}\,dt$

$$A = 6\int_0^1 t^{\frac{1}{2}} (1-t)^{\frac{3}{2}}\,dx = 6\,\beta(\tfrac{3}{2}, \tfrac{5}{2}) = \frac{3}{8}\pi$$

$$6\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{5}{2})}{\Gamma(4)} = 6\,\frac{\frac{1}{2}\Gamma(\frac{1}{2})\,\frac{3}{2} \times \frac{1}{2} \times \Gamma(\frac{1}{2})}{3!} = \frac{3}{8}\pi$$

Subject: Series Solutions for linear differential equations

Basic Principles

- $\boxed{y'' + P(x)\,y' + q(x)\,y = 0}$ — 2^{nd} order, 1^{st} degree Linear, homog., with variable coeff.
- It's required to find a solution in the form of infinite series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \Rightarrow \text{Power series}$$

Note that the solution is Convergent on the interval $|x - x_0| < R$ where $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$x_0 - R$ ——R—— x_0 ——R—— $x_0 + R$: called Solution around the Point x_0 or a Series solution in Power of $(x - x_0)$

Classification:

[1] If both $P(x_0)$ and $q(x_0)$ are well defined (ليها قيمة), x_0 is Called ordinary Point, otherwise, x_0 is Called a Singular Point (singularity of the differential equation). مش ليها قيمة ex: $\frac{1}{0}$

[2] If x_0 is a singularity but both $P(x) = (x - x_0)\,P(x_0)$ and $Q(x) = (x - x_0)^2 q(x_0)$ are well defined, x_0 is Called a regular Singular Point (Regular singularity), otherwise is Called Irregular sigularity

Ex $x^2 y'' + x y' + x^3 y = 0$ Find the singular points

$y'' + \frac{1}{x} y' + x y = 0 \qquad \therefore P(x) = \frac{1}{x} \qquad q(x) = x$

$\therefore x = 0$ is the only singularity $\qquad P = x \cdot \frac{1}{x} = 1$

$\therefore x = 0$ is Regular Singularity $\qquad Q = x^2 \cdot x = x^3$ Defined

Theorem

[1] If x_0 is an ordinary Point,

$$\boxed{y=\sum_{n=0}^{\infty} a_n (x-x_0)^n}$$ is a solution

[2] If x_0 is a regular singularity,

$$\boxed{y=\sum_{n=0}^{\infty} a_n (x-x_0)^{n+s}}$$ is a solution where

s is a constant called the singularity exponent

[3] If x_0 is an irregular singularity, No series solution can be obtained.

Solution around an ordinary Point

Illustrative examples:

Find a series solution in power of x for the differential equation $y''+y=0$ $\hookrightarrow x_0 = \text{zero}$

Ans. $y''+0y'+1y=0 \quad \therefore P(x)=0, \; q(x)=1$

$x=0$ is an ordinary Point $\longrightarrow y=\sum_{n=0}^{\infty} a_n x^n$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\hookrightarrow y'\big|_{n=0}=0 \qquad \hookrightarrow y''\big|_{n=0}=0, \; y''\big|_{n=1}=0$$

$$\therefore \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \longrightarrow \text{step 1 substitute}$$

Subject : ....   / /

Step 2 shifting the index of the first summation so that the powers of x are the same in the two summations

$$\sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}\,x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$=\sum_{n=0}^{\infty}\big((n+2)(n+1)\,a_{n+2}+a_n\big)x^n=0$$

* All the coefficients of x^n must vanish

$\hookrightarrow (n+2)(n+1)\,a_{n+2}+a_n=0$

$$\therefore a_{n+2}=\frac{-1}{(n+2)(n+1)}\,a_n\ ,\ n\geq 0 \rightarrow$$ step 3 Recurrence relation.

Step 4 Solve the Recurrence relation twice $\begin{cases}\text{even}\\ \text{odd}\end{cases}$

@ $n=0,2,4,\ldots$ even

$$a_2=\frac{-1}{(2)(1)}a_0$$

$$a_4=\frac{-1}{(4)(3)}a_2$$

$$=\frac{(-1)(-1)}{(4)(3)(2)(1)}a_0=\frac{(-1)^2}{4!}a_0$$

$$a_6=\frac{-1}{(6)(5)}a_4=\frac{(-1)^3}{6!}a_0$$

@ $n=1,3,5,\ldots$ odd

$$a_3=\frac{-1}{(3)(2)}a_1=\frac{-1}{3!}a_1$$

$$a_5=\frac{-1}{(5)(4)}a_3=\frac{(-1)^2}{5!}a_1$$

$$a_7=\frac{-1}{(7)(6)}a_5=\frac{(-1)^3}{7!}a_1$$

$$\boxed{\therefore a_{2n}=\frac{(-1)^n}{(2n)!}a_0\ ,\ n\geq 0}$$

$$\boxed{\therefore a_{2n+1}=\frac{(-1)^n}{(2n+1)!}a_1\ ,\ n\geq 0}$$

Subject:

even odd

Step 5 $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$

$$\therefore y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\therefore y = a_0 \cos x + a_1 \sin x$$

Remember: (Maclaurin series of some functions)

① $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

② $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

③ $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

④ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

⑤ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

⑥ $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

for $|x| \le 1$, $x \ne \pm i$

GHARIB