١٢٥

الإســـــراء

ملزمة (2)

رياضـــــة

Special Function

ثانية كهرباء

* Special Functions *

I) The Gamma Function $\Gamma(x)$:

Only when $x > 0$, we define

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad ; \quad x > 0$$

If $x \leq 0$, this integration diverges.

Examples:- Evaluate these integrals

1) $\int_0^\infty x^\pi e^{-x} \, dx$

2) $\int_0^\infty \sqrt[3]{x} \, e^{-2x} \, dx$

3) $\int_0^1 \sqrt{-x \ln x} \, dx$

Solution:- 1) $\int_0^\infty x^\pi e^{-x} \, dx = \Gamma(\pi+1)$

2) $\int_0^\infty \sqrt[3]{x} \, e^{-2x} \, dx$

let $2x = t \Rightarrow dx = \frac{dt}{2}$

$$\int_0^\infty \sqrt[3]{\frac{t}{2}} \, e^{-t} \cdot \frac{dt}{2} = \frac{1}{2\sqrt[3]{2}} \int_0^\infty t^{1/3} e^{-t} \, dt$$

$$= \frac{1}{2\sqrt[3]{2}} \cdot \Gamma(4/3).$$

3) $\int_0^1 \sqrt{-x \ln x}\ dx$

Let $\ln x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$x : 0 \to 1 \Rightarrow t : \infty \to 0$

$$\int_\infty^0 \sqrt{e^{-t} \cdot t} \cdot (-e^{-t} dt) = \int_0^\infty e^{-\frac{3}{2}t}\, t^{\frac{1}{2}}\, dt$$

Let $\frac{3}{2}t = u \Rightarrow dt = \frac{2}{3} du$

$$\int_0^\infty e^{-u} \left(\frac{2}{3}u\right)^{1/2} \cdot \frac{2}{3} du = \left(\frac{2}{3}\right)^{3/2} \int_0^\infty e^{-u} \cdot u^{1/2} du$$

$$= (2/3)^{3/2}\ \Gamma(3/2)$$

Note : $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

Properties of $\Gamma(x)$:

1) $\Gamma(1) = 1$

Proof :- $\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 0 + 1 = 1$

2) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof :- $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$

Let $t^{1/2} = u \Rightarrow t = u^2 \Rightarrow dt = 2u\,du$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} \cdot \frac{1}{u} \cdot 2u\, du = 2\int_0^\infty e^{-u^2} du$$

$$= 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

3) The Reccurrence Relation :-

$$\Gamma(x+1) = x\,\Gamma(x)$$

Proof :- $\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x \, dt$

using integration by parts.

$u = t^x \qquad dv = e^{-t} dt$

$du = x\, t^{x-1} dt \qquad v = -e^{-t}$

$$\Gamma(x+1) = \underbrace{-t^x e^{-t}\Big|_0^{\infty}}_{\text{Zero}} + x \int_0^{\infty} e^{-t} t^{x-1} dt = x\,\Gamma(x)$$

Examples :-

$* \ \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

$* \ \Gamma(6) = 5\Gamma(5) = 5.4.\Gamma(4) = 5.4.3.\Gamma(3)$

$-- = 5.4.3.2.1\,\Gamma(1) = 5!$

Ingeneral, $\Gamma(n+1) = n!$

So, the Gamma fn. is a generalization for the factorial

$\Rightarrow \left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

$\left(-\frac{1}{4}\right)! = \Gamma(3/4)$

Examples : Evaluate these integrals

1) $\int_0^1 \ln^7 t \, dt$

$\Rightarrow$ let $x = -\ln t \Rightarrow t = e^{-x} \Rightarrow dt = -e^{-x} dx$

$$I = \int_0^1 \ln^7 t \, dt = \int_\infty^0 (-x)^7 \cdot (-e^{-x}) \, dx$$

$$= \int_\infty^0 x^7 e^{-x} dx = -\int_0^\infty e^{-x} x^7 dx = \Gamma(8) = 7!$$

2) $\int_0^1 \sqrt[3]{x} \ln^5 x \, dx$

$\Rightarrow$ let $-t = \ln x \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$$I = \int_0^1 x^{1/3} \ln^5 x \, dx = \int_\infty^0 (e^{-t})^{1/3} (-t)^5 (-e^{-t}) \, dt$$

$$= \int_\infty^0 e^{-4/3 t} t^5 dt \Rightarrow \text{let } u = \frac{4}{3}t \Rightarrow t = \frac{3}{4}u$$

$$\Rightarrow dt = \frac{3}{4} du$$

$$\Rightarrow I = \int_\infty^0 e^{-u} (\tfrac{3}{4}u)^5 \cdot \tfrac{3}{4} du = (\tfrac{3}{4})^6 \int_\infty^0 e^{-u} u^5 du$$

$$= -(3/4)^6 \int_0^\infty e^{-u} u^5 du = -(\tfrac{3}{4})^6 \Gamma(6) = -(\tfrac{3}{4})^6 \cdot 5!$$

3) $\int_0^\infty x^3 e^{-2x^5} dx$

let $2x^5 = t \Rightarrow x = (t/2)^{1/5} = (\frac{1}{2})^{1/5} \cdot t^{1/5}$

$$\Rightarrow dx = (\tfrac{1}{2})^{1/5} \tfrac{1}{5} t^{-4/5} dt$$

$$I = \int_0^\infty (\tfrac{1}{2})^{3/5} \cdot t^{3/5} e^{-t} \cdot (\tfrac{1}{2})^{1/5} \cdot \tfrac{1}{5} t^{-4/5} dt$$

$$I = \frac{1}{5}\left(\frac{1}{2}\right)^{4/5} \int_0^\infty t^{-1/5} e^{-t}\, dt$$

$$= (1/5)(1/2)^{4/5}\, \Gamma(4/5)$$

---

4) $\displaystyle\int_0^\infty 3^{-x^2}\, dx$

Let $3^{-x^2} = e^{-t} \Rightarrow -x^2 \ln 3 = -t$

$$\Rightarrow x = \sqrt{\frac{t}{\ln 3}} \Rightarrow dx = \frac{1}{\sqrt{\ln 3}} \cdot \frac{1}{2\sqrt{t}}\, dt$$

$x: 0 \rightarrow \infty \Rightarrow t: 0 \rightarrow \infty$

$$\int_0^\infty e^{-t} \cdot \frac{1}{\sqrt{\ln 3}} \cdot \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2\sqrt{\ln 3}} \int_0^\infty t^{-1/2} e^{-t}\, dt$$

$$= \frac{1}{2\sqrt{\ln 3}}\, \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{\ln 3}}.$$

- We can extend the definition of the gamma fn. to include +ve & -ve values of x by using the reccurrence relation, But def. (1) is a def. for $\Gamma(x)$ only for $x > 0$.

- Remark :-

- For $x > 0$ we define $\Gamma(x)$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\text{or} \quad = (x-1)\,\Gamma(x-1)$$

- For $x \leq 0$ we define $\Gamma(x)$ only by the use of the reccurrence relation in the form

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \longrightarrow (2)$$

4) $\Gamma(-n) = \pm\infty \quad ; \text{ for } n \geq 0$

Proof :-

to get $\Gamma(0)$ we have $\Gamma(0) = \dfrac{\Gamma(1)}{0} = \pm\infty$ (using (2))

So, $\Gamma(-1)$ by using (2) also $= \dfrac{\Gamma(0)}{-1} = \pm\infty$

$\Gamma(-2)$ " " " $= \dfrac{\Gamma(-1)}{-2} = \pm\infty$

⋮

$\Rightarrow \Gamma(\text{nonpositive integer}) = \pm\infty$

Examples:- Evaluate 1) $\Gamma(-5/2)$

2) $\Gamma(-4.3)$

3) $L(t^x)$ & hence $L(1/\sqrt{t})$.

Solution :-

1) $\Gamma(-5/2) = \dfrac{\Gamma(-3/2)}{-5/2} = \dfrac{\Gamma(-1/2)}{-\frac{5}{2}(-3/2)}$

$= \dfrac{\Gamma(1/2)}{-\frac{5}{2}(-\frac{3}{2})(-\frac{1}{2})} = -\dfrac{\sqrt{\pi}}{15/8}$

$= -\dfrac{8\sqrt{\pi}}{15}$.

2) $\Gamma(-4.3) = \dfrac{\Gamma(-3.3)}{-4.3} = \dfrac{\Gamma(-2.3)}{(-4.3)(-3.3)}$

$= \dfrac{\Gamma(-1.3)}{(-4.3)(-3.3)(-2.3)} = \dfrac{\Gamma(-0.3)}{(-4.3)(-3.3)(-2.3)(-1.3)}$

$= \dfrac{\Gamma(0.7)}{(-4.3)(-3.3)(-2.3)(-1.3)(-0.3)}$

$= \dfrac{\Gamma(1.7)}{(-4.3)(-3.3)(-2.3)(-1.3)(-0.3)(0.7)} = \dfrac{0.909}{-8.91} = -0.102$

3) $L(t^x) = \int_0^\infty t^x e^{-st}\, dt$

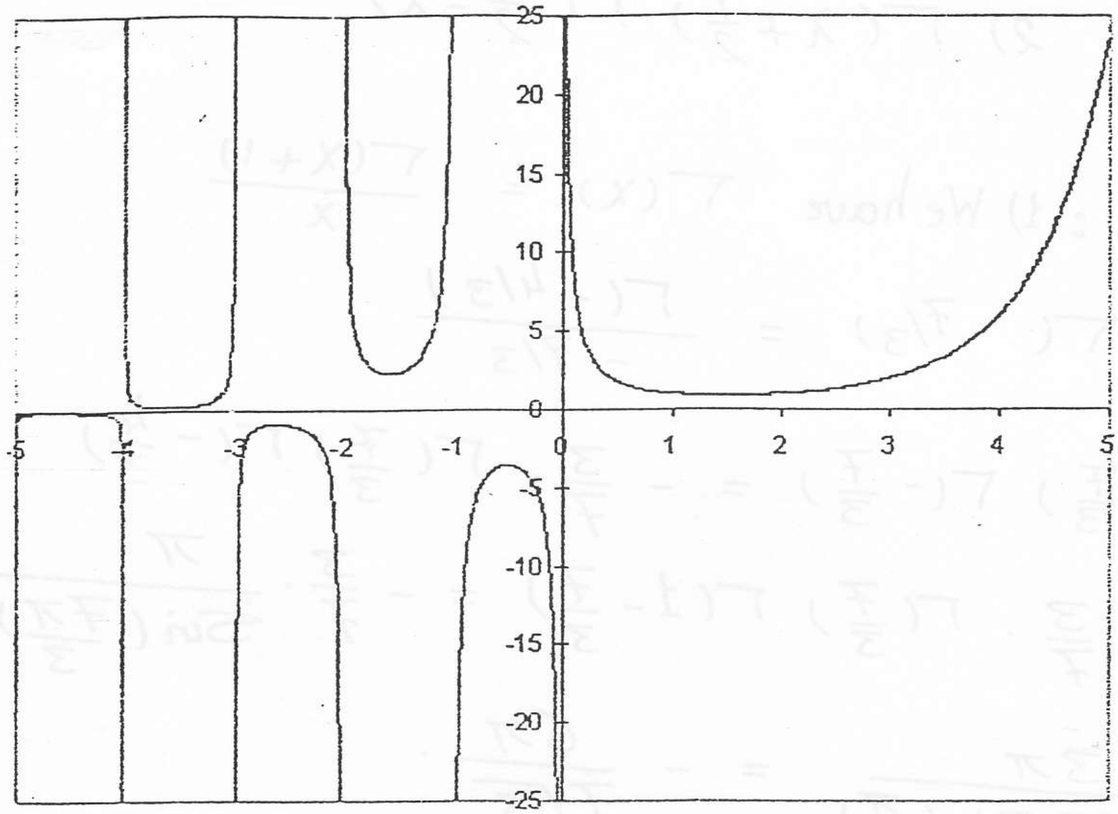
Let $st = u \Rightarrow t = \dfrac{u}{s} \Rightarrow dt = \dfrac{1}{s} du$

$L(t^x) = \int_0^\infty \dfrac{u^x}{s^x} e^{-u} \cdot \dfrac{1}{s} du = \dfrac{1}{s^{x+1}} \int_0^\infty u^x e^{-u} du$

$= \dfrac{\Gamma(x+1)}{s^{x+1}}$

$L(\dfrac{1}{\sqrt{t}}) = L(t^{-1/2}) \xrightarrow{\text{Put } x = -\frac{1}{2}} \dfrac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\dfrac{\pi}{s}}$.

5) The graph of the Gamma fn. $\Gamma(x)$:-



You can observe that $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(n+1) = n!$

& $\Gamma(-n) = \Gamma(0) = \pm\infty$.

6) The Multiplication formula :-

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Proof :-

The proof is too ridiculous & we will not mention it , it starts by using the Weierstrass formula that yields a functional equation which ends by this required formula.

<u>**Example:-**</u> Evaluate $\quad$ 1) $\Gamma(\frac{7}{3})\,\Gamma(-\frac{7}{3})$

2) $\Gamma(x+\frac{1}{2})\,\Gamma(\frac{1}{2}-x)$

<u>Solution</u> : 1) We have $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$

Then $\Gamma(-7/3) = \dfrac{\Gamma(-4/3)}{-7/3}$

$$\Rightarrow \Gamma(\tfrac{7}{3})\,\Gamma(-\tfrac{7}{3}) = -\frac{3}{7}\cdot\Gamma(\tfrac{7}{3})\,\Gamma(-\tfrac{4}{3})$$

$$= -\frac{3}{7}\cdot\Gamma(\tfrac{7}{3})\,\Gamma(1-\tfrac{7}{3}) = -\frac{3}{7}\cdot\frac{\pi}{\sin(\frac{7\pi}{3})}$$

$$= -\frac{3\pi}{7\sin(\frac{\pi}{3})} = -\frac{6\pi}{7\sqrt{3}}.$$

---

2) $\Gamma(x+\frac{1}{2})\,\Gamma(\frac{1}{2}-x)$

$$= \Gamma(x+\tfrac{1}{2})\cdot\Gamma(1-(x+\tfrac{1}{2}))$$

$$= \frac{\pi}{\sin\pi(x+\frac{1}{2})} = \frac{\pi}{\sin(\pi x+\frac{\pi}{2})}$$

$$= \frac{\pi}{\cos\pi x}.$$

Examples

★ Find the Conditions under which the integral $\int_0^\infty x^a b^{-x}\,dx$ Converges & hence evaluate it interms of $\Gamma(x)$.

Solution:-

$$I = \int_0^\infty x^a b^{-x}\,dx$$

let $b^{-x} = e^{-t} \Rightarrow x \ln b = t \Rightarrow x = \dfrac{t}{\ln b}$

$$\Rightarrow dx = \frac{dt}{\ln b}$$

$$I = \int_0^\infty \frac{t^a}{(\ln b)^a} e^{-t} \cdot \frac{dt}{\ln b} \quad ; \text{ for } \ln b > 0 \Rightarrow b > 1$$

$$= \frac{1}{(\ln b)^{a+1}} \int_0^\infty t^a e^{-t}\,dt$$

$$= \frac{\Gamma(a+1)}{(\ln b)^{a+1}} \quad ; \text{ for } a+1 > 0 \Rightarrow a > -1$$

★ Use your result to evaluate i) $\int_0^\infty t^{3/2} 5^{-t}\,dt$

ii) $\int_0^\infty \frac{1}{\pi^x\, x\sqrt{x}}\,dx$.

Solution:-

i) We have $a = 3/2$ & $b = 5$

$$\Rightarrow I = \frac{\Gamma(3/2+1)}{(\ln 5)^{5/2}} = \frac{3/2\,\Gamma(3/2)}{(\ln 5)^{5/2}} = \frac{3/2 \cdot \frac{1}{2}\,\Gamma(\frac{1}{2})}{(\ln 5)^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4\,(\ln 5)^{5/2}}$$

ii) we have $I = \int_0^\infty \pi^{-x} \cdot x^{-3/2}\,dx \Rightarrow a = -\frac{3}{2}$ & $b = \pi$

$\Rightarrow$ Since $a = -3/2 < -1 \Rightarrow$ the integration diverges.

Example:- For what values of a does this integral
$$\int_0^{\sqrt{e}} (\ln x - \tfrac{1}{2})^a x^2 \, dx$$ has a value.

Solution:- Let $\ln x - \frac{1}{2} = u \Rightarrow \ln x = u + \frac{1}{2}$

$x = e^{u+1/2} \Rightarrow x = \sqrt{e}\,.\,e^u$

$\Rightarrow dx = \sqrt{e}\,.\,e^u \, du$

$x : 0 \longrightarrow \sqrt{e} \Rightarrow u : -\infty \longrightarrow 0$

$$\int_{-\infty}^{0} u^a . (\sqrt{e}\,.\,e^u)^2 . \sqrt{e}\,.\,e^u \, du$$

$$= \int_{-\infty}^{0} u^a \, e\sqrt{e} \, e^{3u} \, du = e\sqrt{e} \int_{-\infty}^{0} u^a e^{3u} \, du$$

Let $3u = -t \Rightarrow du = -\frac{dt}{3}$

$$e\sqrt{e} \int_{\infty}^{0} (-\tfrac{t}{3})^a e^{-t} . \frac{-dt}{3} = \frac{(-1)^a e\sqrt{e}}{3^{a+1}} \int_0^{\infty} t^a e^{-t} \, dt$$

$$= \frac{(-1)^a e\sqrt{e}}{3^{a+1}} . \Gamma(a+1) \quad ; \quad a+1 > 0$$

The integration will converge if $a > -1$.

Example: Show that $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n} \cdot n!} \sqrt{\pi}$

Solution:-

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2})$$

$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2}) \cdots \tfrac{5}{2} \cdot \tfrac{3}{2} \cdot \tfrac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$= \frac{1}{2^n} (2n-1)(2n-3) \cdots 5.3.1 \,.\, \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)(2n-1)(2n-2)(2n-3)\cdots 6.5.4.3.2.1}{(2n)\quad(2n-2)\quad\cdots\quad 6\cdot 4\cdot 2}$$

$$= \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)!}{(2n)(2n-2)\cdots 6.4.2.}$$

$$= \frac{\sqrt{\pi}}{2^n . 2^n} \cdot \frac{(2n)!}{n(n-1)\cdots 3.2.1} = \frac{\sqrt{\pi}}{2^{2n}} \cdot \frac{(2n)!}{n!}$$

Example: Show that $\Gamma(\nu) = \int_0^1 (\ln \frac{1}{x})^{\nu-1} dx \;;\; \nu > 0$

Solution:- Let $\ln\frac{1}{x} = t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$

$$I = \int_\infty^0 t^{\nu-1} (-e^{-t}dt)$$

$$= \int_0^\infty t^{\nu-1} e^{-t}\, dt = \Gamma(\nu), \text{ for } \nu > 0$$

Home work: Evaluate

1) $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

2) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$