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ملزمة (10)

رياضـــــة

Elementary function of complex variables

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Elementary functions of Complex Variables

For any Complex fn. $w = f(z)$ we have

$$w = f(z) = f(x+iy) = f(re^{i\theta}) = u+iv = Re^{i\phi}$$

where $z = x+iy = re^{i\theta}$ is a point from the domain

& $\omega = u+iv = Re^{i\phi}$ is a point in the Range.

1) Exponential function

$$W = f(z) = e^{z} = e^{x+iy} = e^{x} \cdot e^{iy} = e^{x} \operatorname{cis} y$$

$$= e^{x} (\cos y + i \sin y)$$

$\Rightarrow R = |f(z)| = e^{x}$

$\phi = \arg(f(z)) = y$

$u = \operatorname{Re}(f(z)) = e^{x} \cos y$

$v = \operatorname{Im}(f(z)) = e^{x} \sin y$

Example: $e^{1+i\sqrt{3}} = e^{1} \cdot e^{i\sqrt{3}} = e \cdot (\cos\sqrt{3} + i \sin\sqrt{3})$

$$= e\cos\sqrt{3} + i\, e\sin\sqrt{3}.$$

Examples :-

1) Evaluate $e^{2 \pm 3\pi i}$.

$$e^{2 \pm 3\pi i} = e^{2} \cdot e^{\pm 3\pi i} = e^{2}\left(\cos(\pm 3\pi) + i \sin(\pm 3\pi)\right)$$

$$= -e^{2}$$

2) $e^{\frac{1}{2} + \frac{\pi}{4} i}$

$$\Rightarrow e^{\frac{1}{2} + i\,\pi/4} = e^{1/2} \cdot e^{i\,\pi/4} = \sqrt{e}\left(\cos\frac{\pi}{4} + i \sin \pi/4\right)$$

$$= \sqrt{e}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{\frac{e}{2}}\,(1+i)$$

3) Find all roots of these equations :-

i) $e^{z} = -2$

Let $z = x + iy \Rightarrow e^{z} = e^{x} \cdot e^{iy} = -2 = 2e^{i(\pi + 2k\pi)}$

$\Rightarrow e^{x} = 2$ & $e^{iy} = e^{i(\pi + 2k\pi)}$

$\Rightarrow x = \ln 2$ & $y = \pi + 2k\pi$

Roots are $z = \ln 2 + i(\pi + 2k\pi)$; $k = 0, \pm 1, \ldots$

ii) $e^{z} = 1 + i\sqrt{3}$

Let $z = x + iy \Rightarrow e^{z} = e^{x} \cdot e^{iy} = 1 + i\sqrt{3}$

$$= 2\, e^{i\left(\frac{\pi}{3} + 2k\pi\right)}$$

$\Rightarrow e^{x} = 2 \quad \& \quad e^{iy} = e^{i(\frac{\pi}{3} + 2k\pi)}$

$\Rightarrow x = \ln 2 \quad \& \quad y = \pi/3 + 2k\pi$

Roots are $z = \ln 2 + i(\frac{\pi}{3} + 2k\pi)$

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iii) $e^{2z-1} = 1$

let $z = x + iy \Rightarrow e^{2x-1+i2y} = 1 \Rightarrow e^{2x-1} \cdot e^{i2y} = 1\, e^{i2k\pi}$

$\Rightarrow e^{2x-1} = 1 \quad \& \quad 2y = 2k\pi$

$\Rightarrow 2x-1 = \ln 1 \Rightarrow x = \frac{1}{2} \quad \& \quad y = k\pi$

Roots are $z = \frac{1}{2} + ik\pi \; ; \; k = 0, \pm 1, \ldots$

Logarithmic function :-

$$w = f(z) = \ln z = \ln(re^{i\theta}) = \ln r + i\theta$$

$$\Rightarrow u = \ln r$$
$$v = \theta$$

Note that

1) $\ln z$ is a multivalue fn. , i.e there exist infinite values for $\ln z$ for each value z.

2) If $w = \ln z \Rightarrow z = e^{w}$

3) $\operatorname{Ln} z =$ principle value of $\ln z$
or $\operatorname{Log} z$

$$\Rightarrow \begin{matrix}\operatorname{Ln} z \\ \operatorname{Log} z\end{matrix} = \ln r + i \text{ principle value of } \theta$$

$$= \ln r + i \operatorname{Arg}(z)$$

Examples:- Evaluate all values of :-

1) $\ln \sqrt{2}$

$$= \frac{1}{2}\ln 2 = \frac{1}{2}\ln(2e^{i2k\pi}) = \frac{1}{2}(\ln 2 + i\,2k\pi)$$

$$= \frac{1}{2}\ln 2 + i\,k\pi \quad ; k = 0, \pm 1, \ldots$$

$\ln i$

$= \ln(1 e^{i(\frac{\pi}{2}+2k\pi)}) = \ln 1 + i(\frac{\pi}{2}+2k\pi)$

$= i(\frac{\pi}{2}+2k\pi) \; ; k = 0, \pm 1, \ldots$

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3) $\ln(-2 + i\frac{2}{\sqrt{3}})$

$= \ln(\frac{4}{\sqrt{3}} e^{i(\pi - \pi/6 + 2k\pi)})$

$= \ln\frac{4}{\sqrt{3}} + i(\frac{5\pi}{6} + 2k\pi)$

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4) $\ln\sqrt[3]{i}$

$= \frac{1}{3}\ln i = \frac{1}{3}\ln(1e^{i(\pi/2+2k\pi)})$

$= \frac{1}{3}(\ln 1 + i(\pi/2 + 2k\pi)) = \frac{i}{3}(\frac{\pi}{2} + 2k\pi)$

$; k = 0, \pm 1, \ldots$

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5) $Ln(-ei)$

$= Ln(e e^{i(-\pi/2)}) = \ln e - i\pi/2 = 1 - i\pi/2$

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6) $Ln(1-i)$

$= Ln(\sqrt{2} e^{i(-\pi/4)}) = \ln\sqrt{2} - i\pi/4$

$$\mathrm{Ln}\,(-3 + i\sqrt{27})$$

$$= \mathrm{Ln}\,(6e^{i(\pi-\pi/3)}) = \ln 6 + i(\pi - \pi/3)$$

$$= \ln 6 + i\,\tfrac{2}{3}\pi$$

- Examples:- Solve these equations

i) $\ln z = \frac{\pi i}{2}$

let $z = re^{i\theta} \Rightarrow \ln r + i\theta = i\,\pi/2$

$\Rightarrow \ln r = 0 \quad \& \quad \theta = \pi/2$

$\Rightarrow r = 1 \quad \& \quad \theta = \pi/2$

$z = 1e^{i\pi/2} = i$

OR $\ln z = \frac{\pi}{2} i \Rightarrow z = e^{i\pi/2} = i$

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ii) $\ln z = 1 + \pi i$

$\Rightarrow$ let $z = re^{i\theta} \Rightarrow \ln r + i\theta = 1 + i\pi$

$\Rightarrow \ln r = 1 \quad \& \quad \theta = \pi$

$r = e \quad \& \quad \theta = \pi$

$\Rightarrow z = e\,e^{i\pi} = -e$

OR $\ln z = 1 + \pi i \Rightarrow z = e^{1+\pi i} = e.e^{i\pi} = -e$

Example:- Solve $\ln(z - i\sqrt{3}z) = -1-2i$.

Solution:-

$$\ln(z - i\sqrt{3}z) = \ln(z(1-i\sqrt{3}))$$

$$= \ln(re^{i\theta} * 2e^{i(-\pi/3)}) = \ln(2re^{i(\theta-\pi/3)})$$

$$= \ln 2r + i(\theta - \pi/3) = -1-2i$$

$$\Rightarrow \ln 2r = -1 \quad \Rightarrow \quad 2r = e^{-1} \Rightarrow r = \frac{1}{2e}$$

$$\& \quad \theta - \pi/3 = -2 \quad \Rightarrow \theta = \frac{\pi}{3} - 2$$

$$\Rightarrow z = \frac{1}{2e} e^{i(\pi/3 - 2)}$$

Complex exponent

To evaluate z^c where c & z are Complex numbers we use the fact that

$$z^c = e^{\ln z^c} = e^{c \ln z}$$

Note

1) z^c is multivalue due to $\ln z$
2) The principle value of z^c is $e^{c \operatorname{Ln} z}$

Examples:- Find all values of:-

1) $(1+i)^i$

$$= e^{\ln (1+i)^i} = e^{i \ln (1+i)}$$

but, $\ln(1+i) = \ln\left(\sqrt{2}\, e^{i(\pi/4 + 2k\pi)}\right)$

$$= \ln\sqrt{2} + i(\pi/4 + 2k\pi)$$

$$= \frac{1}{2}\ln 2 + i(\pi/4 + 2k\pi)$$

$$\Rightarrow (1+i)^i = e^{i\left(\frac{1}{2}\ln 2 + i(\pi/4 + 2k\pi)\right)}$$

$$= e^{-(\pi/4 + 2k\pi)} \cdot e^{\frac{i}{2}\ln 2}$$

$$= e^{-(\pi/4 + 2k\pi)}\left(\cos\frac{\ln 2}{2} + i\sin\frac{\ln 2}{2}\right) \; ; \; k = 0, \pm 1, \ldots$$

$(-1)^{1/\pi}$

$$= e^{\ln(-1)^{1/\pi}} = e^{\frac{1}{\pi}\ln(-1)} = e^{\frac{1}{\pi}(\ln(1e^{i(\pi+2k\pi)}))}$$

$$= e^{\frac{1}{\pi}(i(\pi+2k\pi))} = e^{i(2k+1)}$$

$$= \cos(2k+1) + i\sin(2k+1) \; ; \; k = 0, \pm 1, \dots$$

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3) $\left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i}$

$$= e^{\ln\left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i}} = e^{3\pi i \ln\left(\frac{e}{2}(-1-i\sqrt{3})\right)}$$

but, $\ln\left(\frac{e}{2}(-1-i\sqrt{3})\right) = \ln\left(\frac{e}{2} \cdot 2e^{i(-\frac{2\pi}{3}+2k\pi)}\right)$

$$= \ln\left(e \cdot e^{i(-\frac{2\pi}{3}+2k\pi)}\right)$$

$$= \underbrace{\ln e}_{1} + i\left(-\frac{2\pi}{3}+2k\pi\right)$$

$$\Rightarrow \left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i} = e^{3\pi i\left(1 + i\left(-\frac{2\pi}{3}+2k\pi\right)\right)}$$

$$= e^{i(3\pi)} \cdot e^{3\pi\left(\frac{2\pi}{3}-2k\pi\right)}$$

$$= e^{\pi(2\pi-6k\pi)} \cdot (\cos 3\pi + i\sin 3\pi)$$

$$= -e^{\pi(2\pi-6k\pi)} \quad ; \; k = 0, \pm 1, \dots$$

Note: the principle value of $\left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i}$ is $e^{2\pi^2}$

Trignometric & hyperbolic functions :-

Since $e^{iz} = \cos z + i \sin z$

& $e^{-iz} = \cos z - i \sin z$

$\Rightarrow$ $\boxed{\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}}$ & $\boxed{\cos z = \dfrac{e^{iz} + e^{-iz}}{2}}$

We also define

$\boxed{\operatorname{sh} z = \dfrac{e^{z} - e^{-z}}{2}}$ & $\boxed{\operatorname{ch} z = \dfrac{e^{z} + e^{-z}}{2}}$

So we have

$\sin iz = i \operatorname{sh} z$	$\cos iz = \operatorname{ch} z$
$\operatorname{sh} iz = i \sin z$	$\operatorname{ch} iz = \cos z$

Remember that :

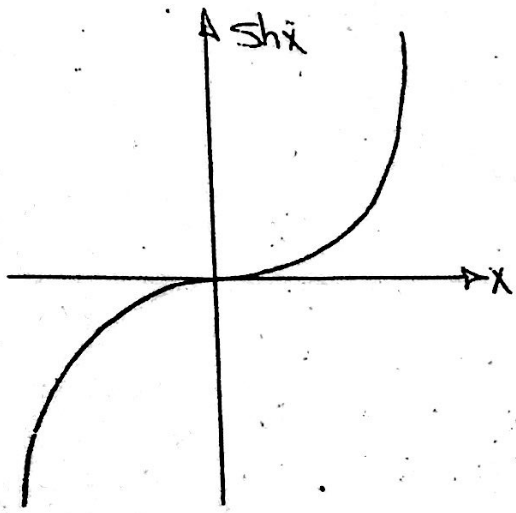
1) $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$

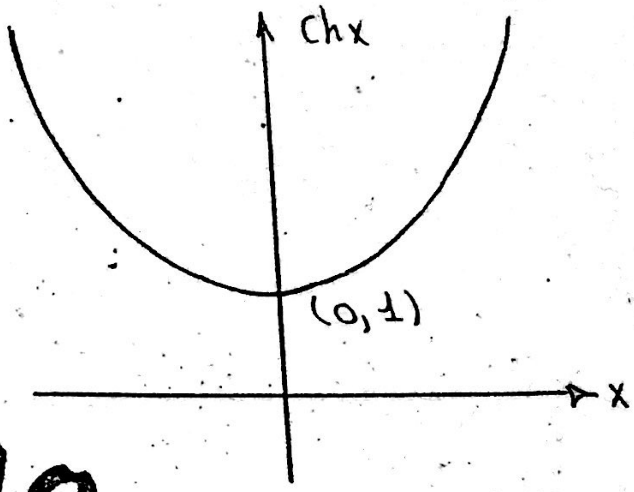
2) $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

3) $\operatorname{sh}(a \pm b) = \operatorname{sh} a \operatorname{ch} b \pm \operatorname{sh} b \operatorname{ch} a$

4) $\operatorname{ch}(a \pm b) = \operatorname{ch} a \operatorname{ch} b \pm \operatorname{sh} a \operatorname{sh} b$

5)





Example :- Show that e^z is an entire function & use this to show that $\frac{d}{dz}(\cos z) = -\sin z$, $\frac{d}{dz}(\sh z) = \ch z$

Solution :- $W = f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$$= e^x \cos y + i\, e^x \sin y$$

$\Rightarrow u = e^x \cos y$ $\qquad v = e^x \sin y$

$u_x = e^x \cos y$ $\qquad v_x = e^x \sin y$

$u_y = -\, e^x \sin y$ $\qquad v_y = e^x \cos y$

- Since $u_x = v_y$ & $u_y = -v_x$ for all values of x, y

$\Rightarrow f(z)$ is differentiable on all the z-plane $\Rightarrow$ $f(z)$ is analytic everywhere $\Rightarrow$ $f(z)$ is an entire fn.

$$\Rightarrow f'(z) = u_x + i v_x = e^x \cos y + i\, e^x \sin y = e^z.$$

But

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\Rightarrow \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{1}{2}(i e^{iz} - i e^{-iz}) = \frac{i}{2}(e^{iz} - e^{-iz})$$

$$= -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

Also $\quad \sh z = \dfrac{e^z - e^{-z}}{2} \Rightarrow \dfrac{d}{dz}(\sh z) = \dfrac{d}{dz}\left(\dfrac{e^z - e^{-z}}{2}\right)$

$$= \frac{1}{2}(e^z + e^{-z}) = \ch z.$$

Example :

1) Evaluate $sh(2-3i)$

we have $sh(2-3i) = sh\,2\,ch\,3i - sh\,3i\,ch\,2$

$= sh\,2\cos 3 - i\sin 3\,ch\,2$

2) Evaluate $\cos(2\,cis\,\pi/6)$

$\Rightarrow \cos(2\,cis\frac{\pi}{6}) = \cos(2\cos\frac{\pi}{6} + i\,2\sin\frac{\pi}{6})$

$= \cos(\sqrt{3}+i) = \cos\sqrt{3}\cos i - \sin\sqrt{3}\sin i$

$= \cos\sqrt{3}\,ch\,1 - i\sin\sqrt{3}\,sh\,1$

3) Show that $ch(z_1+z_2) = ch\,z_1\,ch\,z_2 + sh\,z_1\,sh\,z_2$

we have

$$L.H.S = ch(z_1+z_2) = \frac{e^{z_1+z_2}+e^{-z_1-z_2}}{2} \longrightarrow (1)$$

$R.H.S = ch\,z_1\,ch\,z_2 + sh\,z_1\,sh\,z_2$

$$= \frac{e^{z_1}+e^{-z_1}}{2} * \frac{e^{z_2}+e^{-z_2}}{2} + \frac{e^{z_1}-e^{-z_1}}{2} * \frac{e^{z_2}-e^{-z_2}}{2}$$

$$= \frac{1}{4}\left(e^{z_1+z_2} + \cancel{e^{z_1-z_2}} + \cancel{e^{-z_1+z_2}} + e^{-z_1-z_2}\right)$$

$$+ \frac{1}{4}\left(e^{z_1+z_2} - \cancel{e^{z_1-z_2}} - \cancel{e^{-z_1+z_2}} + e^{-z_1-z_2}\right)$$

$$= \frac{1}{2}\left(e^{z_1+z_2} + e^{-z_1-z_2}\right) = L.H.S$$

Example :- Show that $\sin(iz) = i\,\text{sh}\,z$ &

$\text{ch}(iz) = \cos z$.

Solution :- $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \Rightarrow \sin iz = \dfrac{e^{i(iz)} - e^{-i(iz)}}{2i}$

$\Rightarrow \sin iz = \dfrac{1}{i} \cdot \dfrac{e^{-z} - e^{z}}{2} = i\left(\dfrac{e^{z} - e^{-z}}{2}\right) = i\,\text{sh}\,z$

& $\text{ch}(iz) = \dfrac{e^{iz} + e^{-iz}}{2} = \cos z$.

Example :- Show that $\text{ch}\,z$ is an entire fn & find its derivatives.

Solution :- $f(z) = \text{ch}\,z = \text{ch}(x+iy)$

$= \text{ch}x\,\text{ch}iy + \text{sh}x\,\text{sh}iy$

$= \text{ch}x \cos y + i\,\text{sh}x \sin y$

$\Rightarrow u = \text{ch}x \cos y \qquad v = \text{sh}x \sin y$

$u_x = \text{sh}x \cos y \qquad v_x = \text{ch}x \sin y$

$u_y = -\text{ch}x \sin y \qquad v_y = \text{sh}x \cos y$

Cauchy - Reiman Eq. ($u_x = v_y$ & $v_x = -u_y$) are satisfied

For all x and y $\Rightarrow$ $f(z)$ is analytic on the whole

z plane $\Rightarrow$ entire fn.

& $f'(z) = u_x + i v_x = \text{sh}x \cos y + i\,\text{ch}x \sin y$

$= \text{sh}\,z$.

Show that $\cos^{-1} z = -i \ln(z \pm \sqrt{z^2-1})$

Let $w = \cos^{-1} z \Rightarrow z = \cos w = \dfrac{e^{iw} + e^{-iw}}{2}$

$$\Rightarrow 2z = e^{iw} + e^{-iw} \qquad * \, e^{iw}$$

$$e^{2iw} - 2z e^{iw} + 1 = 0$$

$$\Rightarrow e^{iw} = \frac{2z \pm \sqrt{4z^2-4}}{2} = z \pm \sqrt{z^2-1}$$

$$\Rightarrow iw = \ln(z \pm \sqrt{z^2-1})$$

$$\Rightarrow w = -i \ln(z \pm \sqrt{z^2-1})$$

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5) Show that $\sinh^{-1} z = \ln(z \pm \sqrt{z^2+1})$

Let $w = \sinh^{-1} z \Rightarrow z = \sinh w = \dfrac{e^{w} - e^{-w}}{2}$

$$\Rightarrow 2z = e^{w} - e^{-w} \qquad * \, e^{w}$$

$$e^{2w} - 2z e^{w} - 1 = 0$$

$$\Rightarrow e^{w} = \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1}$$

$$\Rightarrow w = \ln(z \pm \sqrt{z^2+1})$$

<u>You Can try proving</u>

1) $\sin^{-1} z = -i \ln(iz + \sqrt{1-z^2})$

2) $\cosh^{-1} z = \ln(z + \sqrt{z^2-1})$

Examples:- Find all roots of the following equations:-

1) $\operatorname{Ch} z = \frac{1}{2}$

let $z = x + iy \implies$

$$\operatorname{Ch} z = \operatorname{ch} x \cos y + i \operatorname{sh} x \sin y = \frac{1}{2} + i0$$

$\implies \operatorname{ch} x \cos y = \frac{1}{2}$ & $\operatorname{sh} x \sin y = 0$

$$\operatorname{Sh} x \sin y = 0$$

$\operatorname{Sh} x = 0$

$\implies x = 0$

$\implies \cos y = \frac{1}{2}$

$\implies y = \pm\frac{\pi}{3} + 2n\pi$

$\Downarrow$

$z = 0 + i\,(\pm\pi/3 + 2n\pi)$

$\sin y = 0$

$\Downarrow$

$y = n\pi$

$\Downarrow$

$\operatorname{ch} x \; (-1)^n = \frac{1}{2}$

$\Downarrow$

rejected.

Roots are $z = i\left(\pm\frac{\pi}{3} + 2n\pi\right)$; $n = 0, \pm 1, \ldots$

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2) $\cos z = 0$

let $z = x + iy$

$\implies \cos z = \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y = 0$

$\cos x \, ch y = 0$

$\cos x = 0$

$\Rightarrow x = (2n+1)\pi/2$

$\Rightarrow \sin x \, sh y = (-1)^n sh y = 0$

$\Rightarrow y = 0$

$ch y = 0$ rejected

Roots are $z = x + iy = (2n+1)\pi/2$ only real roots

3) $ch z = -2$

let $z = x + iy \Rightarrow ch x \cos y + i \, sh x \sin y = -2$

$\Rightarrow ch x \cos y = -2$ & $sh x \sin y = 0$

$sh x = 0$

$\Rightarrow x = 0$

$\Rightarrow \cos y = -2$ rejected

$\sin y = 0$

$\Rightarrow y = n\pi$

$ch x (-1)^n = -2$

$\Rightarrow ch x = (-1)^n (-2)$

$= (-1)^{n+1} 2$

n must be odd no. $\Rightarrow$ let $n = 2m+1$

$\Rightarrow ch x = 2$

$\Rightarrow x = ch^{-1} 2$

Roots are $ch^{-1}2 + i(2m+1)\pi$; $m = 0, \pm 1, \ldots$

$\cos z = 2i$

$\Rightarrow \cos x \, ch y - i \sin x \, sh y = 2i$

$\Rightarrow \cos x \, ch y = 0 \quad \& \quad \sin x \, sh y = -2$

$\cos x = 0$

$ch y = 0$

rejected

$\Rightarrow x = (2n+1)\pi/2$

$\Rightarrow \sin x \, sh y = (-1)^n \, sh y = -2$

$\Rightarrow sh y = 2(-1)^{n+1}$

$y = sh^{-1}\left(2(-1)^{n+1}\right)$

" may be written as $(-1)^{n+1} sh^{-1} 2$ "

Roots are $z = (2n+1)\pi/2 + i\,(-1)^{n+1} sh^{-1} 2 \quad ; \; n = 0, \pm 1, -$

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5) $\sin z = ch 4$

$\Rightarrow \sin x \, ch y + i \cos x \, sh y = ch 4$

$\Rightarrow \sin x \, ch y = ch 4 \quad \& \quad \cos x \, sh y = 0$

$\cos x = 0$

$sh y = 0$

$y = 0$

$\Downarrow$

$\sin x = ch 4$

$\Downarrow$

rejected

$\Rightarrow x = (2n+1)\pi/2$

$\Rightarrow \sin x \, ch y = (-1)^n ch y = ch 4$

$\Downarrow$

$\Rightarrow ch\,y = (-1)^n ch\,4$

n must be even $\Rightarrow$ let $n = 2m$

$ch\,y = ch\,4$

$\Rightarrow y = \pm 4$

$Z =$ roots $= (4m+1)\frac{\pi}{2} \pm i\,4 \quad ; m = 0, \pm 1, --$

<u>Example</u> : Evaluate $\tanh(\ln 3 + i\pi/4)$

<u>Solution</u> :-

$$sh(\ln 3 + i\pi/4) = sh\ln 3 \cos\pi/4 + i\, ch\ln 3 \sin\pi/4$$

$$= sh(\ln 3)\cdot\frac{1}{\sqrt{2}} + i\, ch(\ln 3)\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{4}{3} + i\,\frac{5}{3}\right)$$

Similarily, $ch(\ln 3 + i\pi/4) = ch\ln 3 \cos\frac{\pi}{4} + i\, sh\ln 3 \sin\pi/4$

$$= \frac{1}{\sqrt{2}}(5/3 + i\,4/3)$$

$$\Rightarrow \tanh(\ln 3 + i\,\pi/4) = \frac{sh(\ln 3 + i\pi/4)}{ch(\ln 3 + i\,\pi/4)}$$

$$= \frac{4/3 + i\,5/3}{5/3 + i\,4/3} = \frac{4 + i\,5}{5 + i\,4} * \frac{5 - i\,4}{5 - i\,4}$$

$$= \frac{1}{41}(20 + 20 + i(25 - 16))$$

$$= \frac{1}{41}(40 + i\,9)$$
