Math. 3 Midterm Exam.
2^{nd} Year Electrical Eng.
November 14^{th}, 2018. Allowed Time: 75 Minutes.

MODEL ANSWER	MODEL (A)

Question 1 (10 Marks)

(A) Evaluate in terms of the Gamma function $\displaystyle\int_0^{\sqrt{2}} \frac{dx}{\sqrt{4+x^4}}$.

[4 Marks]

$$let\ x^4 = 4\tan^2\theta \Rightarrow x = \sqrt{2}(\tan\theta)^{1/2} \Rightarrow dx = \frac{\sqrt{2}}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta \cdots\cdots \boxed{1}$$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\frac{\sqrt{2}}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta}{2\sec\theta} = \frac{\sqrt{2}}{4}\int_0^{\pi/4}\sin^{-1/2}\theta\cos^{-1/2}\theta d\theta = \frac{\sqrt{2}}{4}\int_0^{\pi/4}\left(\frac{1}{2}\sin 2\theta\right)^{-1/2} d\theta \cdots\cdots \boxed{1}$$

$$= \frac{\sqrt{2}*\sqrt{2}}{4}\int_0^{\pi/4}\sin^{-1/2} 2\theta d\theta$$

$$\rightarrow let\ \alpha = 2\theta \Rightarrow d\theta = \frac{1}{2}d\alpha \cdots\cdots \boxed{1}$$

$$\Rightarrow I = \frac{2}{8}\int_0^{\pi/2}\sin^{-1/2}\alpha d\alpha = \frac{2}{8}\left(\frac{1}{2}\right)\beta\left(\tfrac{1}{4},\tfrac{1}{2}\right) = \frac{1}{8}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \Rightarrow \boxed{I = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{8\Gamma\left(\frac{3}{4}\right)}} \cdots\cdots \boxed{1}$$

(B) Find and classify all the singularities of the following differential equation, hence, find its general solution in powers of x. $(1-x^2)\,y'' - 2\,x\,y' + 12\,y = 0$.

[6 Marks]

$$(1-x^2)y'' - 2xy' + 12y = 0 \cdots\cdots\cdots(1)$$

$$\Rightarrow p(x) = \frac{-2x}{(1-x^2)} = \frac{-2x}{(1-x)(1+x)}, q(x) = \frac{12}{(1-x^2)} = \frac{12}{(1-x)(1+x)}$$

$\rightarrow at\ x_0 = 1 \Rightarrow p(x) \& q(x)$ are not defined $\Rightarrow$ Singular Point

Let $P(x) = (x-1)p(x) = \dfrac{2x}{(1+x)}, Q(x) = (x-1)^2 q(x) = \dfrac{12(1-x)}{(1+x)}$

$\Rightarrow$ Both are defined at $x_0 = 1 \Rightarrow x_0 = 1$ is a Regular S.P.

$\rightarrow at\ x_0 = -1 \Rightarrow p(x) \& q(x)$ are not defined $\Rightarrow$ Singular Point

Let $P(x) = (x+1)p(x) = \dfrac{-2x}{(1-x)}, Q(x) = (x+1)^2 q(x) = \dfrac{12(1+x)}{(1-x)}$

$\Rightarrow$ Both are defined at $x_0 = -1 \Rightarrow x_0 = -1$ is a Regular S.P.

$\rightarrow at\ x_0 \in R - \{-1,1\} \Rightarrow$ Both are defined at $x_0 \Rightarrow x_0$ is an Ordinary Point $\cdots\cdots\cdots \boxed{1}$

$\Rightarrow x_0 = 0$ is an Ordinary Point.

$\rightarrow$ Let $y=\sum_{n=0}^{\infty} a_n x^n$ be a solution $\Rightarrow y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\rightarrow$ Substitute *in* (1) $\Rightarrow (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 12\sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 12 a_n\right] x^n = 0$$

$$\rightarrow \because coef.(x^n) = 0 \Rightarrow (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 12 a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{n(n-1)+2n-12}{(n+2)(n+1)} a_n = \frac{n^2+n-12}{(n+2)(n+1)} a_n = \frac{(n+4)(n-3)}{(n+2)(n+1)} a_n \;\; ; n \ge 0 \ldots \text{Recurrence Relation} \ldots \boxed{2}$$

$$\rightarrow n=0 \Rightarrow a_2 = \frac{(4)(-3)}{(2)(1)} a_0 , \rightarrow n=2 \Rightarrow a_4 = \frac{(6)(-1)}{(4)(3)} a_2 = \frac{(4*6)(-3*-1)}{(2*4)(1*3)} a_0 ,$$

$$n=4 \Rightarrow a_6 = \frac{(8)(1)}{(6)(5)} a_4 = \frac{(4*6*8)(-3*-1*1)}{(2*4*6)(1*3*5)} a_0$$

$$\Rightarrow a_{2k} = \frac{\left[4*6*\cdots*(2k+2)\right]\left[-3*-1*1*\cdots*(2k-5)\right]}{(2k)!} a_0 \cdots\cdots k \ge 1 \ldots\ldots\ldots \boxed{1}$$

$$\rightarrow n=1 \Rightarrow a_3 = \frac{(5)(-2)}{(3)(2)} a_1 \quad , n=3 \Rightarrow a_5 = 0 = a_7 = a_9 = \ldots\ldots\ldots\ldots \boxed{1}$$

$$\Rightarrow \boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0\left[1 + \sum_{k=1}^{\infty} \frac{\left[4*6*\cdots*(2k+2)\right]\left[-3*-1*1*\cdots*(2k-5)\right]}{(2k)!} x^{2k}\right] + a_1\left[x - \frac{5}{3}x^3\right]} \ldots\ldots\ldots \boxed{1}$$

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MODEL ANSWER	**MODEL (A)**

Question 2 (10 Marks)

Part A.

Two honest dice are rolled once. Let E = getting "odd outcome on the first die", F = getting "odd outcome on the second die", and G = getting "odd sum of both dice".Check independence of E, F, and G.

[2 Marks]

$$\Rightarrow P(E)=\frac{1}{2}, P(F)=\frac{1}{2}, P(G)=\frac{1}{2}, \ldots\ldots \boxed{0.5}$$

$$\to P(E\cap F)=\frac{1}{4}=P(E)*P(F)\Rightarrow E,F \text{ are independent}$$

$$\to P(E\cap G)=\frac{1}{4}=P(E)*P(G)\Rightarrow E,G \text{ are independent}$$

$$\to P(F\cap G)=\frac{1}{4}=P(F)*P(G)\Rightarrow F,G \text{ are independent}\ldots\ldots \boxed{0.5}$$

$$\to P(E\cap F\cap G)=0\neq P(E)*P(F)*P(G)\ldots\ldots \boxed{0.5}$$

$$\Rightarrow \boxed{E,F,G \text{ are Dependent but they are pairwise independent}}\ldots\ldots \boxed{0.5}$$

Part B.

Five balls are to be selected at random from a collection of 7 White, 6 Black, and 4 Green balls. Find the probability that the all selected balls are of the same color if we are:
(i) Sampling with replacement **(ii) sampling without replacement.**

[3 Marks]

$$P(I)=P(5W)+P(5B)+P(5G)=\left(\frac{7}{17}\right)^5+\left(\frac{6}{17}\right)^5+\left(\frac{4}{17}\right)^5=0.01803\ldots\ldots \boxed{1.5}$$

$$P(II)=P(5W)+P(5B)=\frac{C_5^7+C_5^6}{C_5^{17}}=\frac{27}{6188}=0.00436\ldots\ldots \boxed{1.5}$$

Part C.

(i) **Three machines A, B, and C produce respectively 35%, 25%, 40% of the total production of an item. The probabilities of producing a defective item on these machines are 0.03, 0.04, and 0.02, respectively. An item is chosen at random. Find the probability that it was found non-defective.**

[2 Marks]

(ii) **Three events E, F & G are defined on the sample space S, such that: $P(E)=0.3$, $P(F)=0.33$, $P(G)=0.54$, $P(E\cap F)=0.1$, $P(E\cap G)=0.11$, $P(F\cap G)=0.08$, $P(E\cap F\cap G)=0.03$. Find the probability that:**

(a) Exactly one event will occur. **(b) At least one event will occur.**

[3 Marks]

$$i)\quad P(ND)=P(ND|A)P(A)+P(ND|B)P(B)+P(ND|C)P(C)\ldots\ldots \boxed{1}$$

$$=(0.97)(0.35)+(0.96)(0.25)+(0.98)(0.4)=0.9715\ldots\ldots \boxed{1}$$

$ii)$

$$b)\ P(\text{At least one will occur})=P(E\cup F\cup G)$$

$$=P(E)+P(F)+P(G)-P(E\cap F)-P(E\cap G)-P(F\cap G)+P(E\cap F\cap G)=0.91\ldots\ldots \boxed{1.5}$$

$$a)\ P(\text{Exactly one will occur})=$$

$$=P(E\cup F\cup G)-P(E\cap F)-P(E\cap G)-P(F\cap G)+2P(E\cap F\cap G)=\frac{17}{25}=0.68\ldots\ldots \boxed{1.5}$$

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MODEL ANSWER	MODEL (B)

Question 1 (10 Marks)

(A) Evaluate in terms of the Gamma function $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{9+x^4}}$.

[4 Marks]

$$let\ x^4 = 9\tan^2\theta \Rightarrow x = \sqrt{3}(\tan\theta)^{1/2} \Rightarrow dx = \frac{\sqrt{3}}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta \cdots\cdots \boxed{1}$$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\frac{\sqrt{3}}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta}{3\sec\theta} = \frac{\sqrt{3}}{6}\int_0^{\pi/4} \sin^{-1/2}\theta\cos^{-1/2}\theta d\theta$$

$$= \frac{\sqrt{3}}{6}\int_0^{\pi/4}\left(\frac{1}{2}\sin 2\theta\right)^{-1/2} d\theta = \frac{\sqrt{3}*\sqrt{2}}{6}\int_0^{\pi/4} \sin^{-1/2} 2\theta d\theta \cdots\cdots \boxed{1}$$

$$\rightarrow let\ \alpha = 2\theta \Rightarrow d\theta = \frac{1}{2}d\alpha \cdots\cdots \boxed{1}$$

$$\Rightarrow I = \frac{\sqrt{6}}{12}\int_0^{\pi/2} \sin^{-1/2}\alpha d\alpha = \frac{\sqrt{6}}{12}\left(\frac{1}{2}\right)\beta\left(\tfrac{1}{4},\tfrac{1}{2}\right) = \frac{\sqrt{6}}{24}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \Rightarrow \boxed{I = \frac{\sqrt{6\pi}\,\Gamma\left(\frac{1}{4}\right)}{24\Gamma\left(\frac{3}{4}\right)}} \cdots\cdots \boxed{1}$$

(B) Find and classify all the singularities of the following differential equation, hence, find its general solution in powers of x. $(1-x^2)\,y'' - 2\,x\,y' + 6\,y = 0$.

[6 Marks]

$$\left(1-x^2\right)y'' - 2xy' + 6y = 0 \ldots\ldots\ldots(1)$$

$$\Rightarrow p(x) = \frac{-2x}{\left(1-x^2\right)} = \frac{-2x}{(1-x)(1+x)},\ q(x) = \frac{6}{\left(1-x^2\right)} = \frac{6}{(1-x)(1+x)}$$

$\rightarrow at\ x_0 = 1 \Rightarrow p(x)\ \&\ q(x)$ are not defined $\Rightarrow$ Singular Point

Let $P(x) = (x-1)\,p(x) = \dfrac{2x}{(1+x)},\ Q(x) = (x-1)^2 q(x) = \dfrac{6(1-x)}{(1+x)}$

$\Rightarrow$ Both are defined at $x_0 = 1 \Rightarrow x_0 = 1$ is a Regular S.P.

$\rightarrow at\ x_0 = -1 \Rightarrow p(x)\ \&\ q(x)$ are not defined $\Rightarrow$ Singular Point

Let $P(x) = (x+1)\,p(x) = \dfrac{-2x}{(1-x)},\ Q(x) = (x+1)^2 q(x) = \dfrac{6(1+x)}{(1-x)}$

$\Rightarrow$ Both are defined at $x_0 = -1 \Rightarrow x_0 = -1$ is a Regular S.P.

$\rightarrow at\ x_0 \in R - \{-1,1\} \Rightarrow$ Both are defined at $x_0 \Rightarrow x_0$ is an Ordinary Point ……… $\boxed{1}$

$\Rightarrow x_0 = 0$ is an Ordinary Point.

$$\rightarrow \text{Let } y=\sum_{n=0}^{\infty} a_n x^n \text{ be a solution } \Rightarrow y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\rightarrow \text{Substitute } in \ (1) \Rightarrow (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 6a_n\right] x^n = 0$$

$$\rightarrow \because coef.(x^n) = 0 \Rightarrow (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 6a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{n(n-1)+2n-6}{(n+2)(n+1)} a_n = \frac{n^2+n-6}{(n+2)(n+1)} a_n = \frac{(n+3)(n-2)}{(n+2)(n+1)} a_n \ ; n \geq 0 \ldots\ldots \text{Recurrence Relation} \ldots\ldots\ldots \boxed{2}$$

$$\rightarrow n=1 \Rightarrow a_3 = \frac{(4)(-1)}{(3)(2)} a_1 , \rightarrow n=3 \Rightarrow a_5 = \frac{(6)(1)}{(5)(4)} a_3 = \frac{(4*6)(-1*1)}{(2*4)(3*5)} a_1 ,$$

$$n=5 \Rightarrow a_7 = \frac{(8)(3)}{(7)(6)} a_5 = \frac{(4*6*8)(-1*1*3)}{(2*4*6)(3*5*7)} a_0$$

$$\Rightarrow a_{2k+1} = \frac{\left[4*6*\cdots*(2k+2)\right]\left[-1*1*3*\cdots*(2k-3)\right]}{(2k+1)!} a_1 \cdots\cdots k \geq 1 \ldots\ldots\ldots \boxed{1}$$

$$\rightarrow n=0 \Rightarrow a_2 = \frac{(3)(-2)}{(2)(1)} a_0 \quad , n=2 \Rightarrow a_4 = 0 = a_6 = a_8 = \ldots\ldots\ldots\ldots \boxed{1}$$

$$\Rightarrow \boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0\left[1-3x^2\right] + a_1\left[1 + \sum_{k=1}^{\infty} \frac{\left[4*6*\cdots*(2k+2)\right]\left[-1*1*3*\cdots*(2k-3)\right]}{(2k+1)!} x^{2k+1}\right]} \ldots\ldots\ldots \boxed{1}$$

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MODEL ANSWER	Model B

Question 2 (10 Marks)

Part A.

Two honest dice are rolled once. Let E = getting "odd outcome on the first die", F = getting "odd outcome on the second die", and G = getting "even sum of both dice".
Check independence of E, F, and G.

[2 Marks]

$$\Rightarrow P(E)=\frac{1}{2},\, P(F)=\frac{1}{2},\, P(G)=\frac{1}{2},\ldots\ldots \boxed{0.5}$$

$$\rightarrow P(E\cap F)=\frac{1}{4}=P(E)*P(F)\Rightarrow E,F \text{ are independent}$$

$$\rightarrow P(E\cap G)=\frac{1}{4}=P(E)*P(G)\Rightarrow E,G \text{ are independent}$$

$$\rightarrow P(F\cap G)=\frac{1}{4}=P(F)*P(G)\Rightarrow F,G \text{ are independent}\ldots\ldots \boxed{0.5}$$

$$\rightarrow P(E\cap F\cap G)=\frac{1}{4}\neq P(E)*P(F)*P(G)\ldots\ldots \boxed{0.5}$$

$$\Rightarrow \boxed{E,F,G \text{ are Dependent but they are pairwise independent}}\ldots\ldots \boxed{0.5}$$

Part B.

Five balls are to be selected at random from a collection of 4 yellow, 6 Black, and 8 Green balls. Find the probability that the all selected balls are of the same color if we are:
(i) Sampling with replacement (ii) sampling without replacement.

[3 Marks]

$$P(I)=P(5Y)+P(5B)+P(5G)=\left(\frac{4}{18}\right)^5+\left(\frac{6}{18}\right)^5+\left(\frac{8}{18}\right)^5=0.022\ldots\ldots \boxed{1.5}$$

$$P(II)=P(5B)+P(5G)=\frac{C_5^6+C_5^8}{C_5^{18}}=\frac{31}{4284}=0.00724\ldots\ldots \boxed{1.5}$$

Part C.

(i) **Three machines A, B, and C produce respectively 35%, 25%, 40% of the total production of an item. The probabilities of producing a defective item on these machines are 0.05, 0.03, and 0.04, respectively. An item is chosen at random. Find the probability that it was found non-defective.**

[2 Marks]

(ii) **Three events E , F & G are defined on the sample space S , such that: $P(E)=0.35, P(F)=0.42,$ $P(G)=0.55$, $P(E\cap F)=0.13$, $P(E\cap G)=0.15$, $P(F\cap G)=0.12$, $P(E\cap F\cap G)=0.05$. Find the probability that:**
(a) Exactly one event will occur. (b) At least one event will occur.

[3 Marks]

$$i)\quad P(ND)=P(ND|A)P(A)+P(ND|B)P(B)+P(ND|C)P(C)\ldots\ldots \boxed{1}$$

$$=(0.95)(0.35)+(0.97)(0.25)+(0.96)(0.4)=0.959\ldots\ldots \boxed{1}$$

$ii)$

$$b)\ P(\text{At least one will occur})=P(E\cup F\cup G)$$

$$=P(E)+P(F)+P(G)-P(E\cap F)-P(E\cap G)-P(F\cap G)+P(E\cap F\cap G)=0.97\ldots\ldots \boxed{1.5}$$

$$a)\ P(\text{Exactly one will occur})=$$

$$=P(E\cup F\cup G)-P(E\cap F)-P(E\cap G)-P(F\cap G)+2P(E\cap F\cap G)=0.67\ldots\ldots \boxed{1.5}$$