بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

PHM212s:

Complex, Special Functions and Numerical Analysis

Bessel Functions

﴿يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖ﴾

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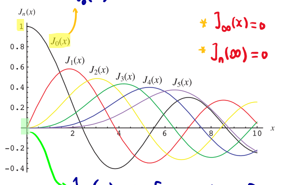
* Bessel Functions :

→ They are Canonical solutions of Bessel's differential equation of order ν, parameter λ

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(\lambda^2t^2-\nu^2)y=0$$

① Bessel Functions of First kind $(J_\nu(x))$:

$J_0(0)=1$



$J_\infty(x)=0$

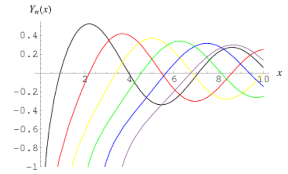
$J_n(\infty)=0$

$J_n(0)=0$: For all values of n except zero

• $J_\nu(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m+\alpha}}{m!\,\Gamma(m+\alpha+1)}$

* $J_\nu(x)$ is a bounded Function

② Bessel Functions of Second kind $(Y_\nu(x))$:



• $Y_\nu(x)=\frac{J_\nu(x)\cos(\nu\pi)-J_{-\nu}(x)}{\sin(\nu\pi)}$

→ in Case of integer order N : $Y_N(x)=\lim_{\nu\to N}Y_\nu(x)$

* $Y_\nu(x)$ is unbounded Function at $x=0$

The DE must be in the Standard Form : $x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0$

① the Coefficients of y'' & y' must be equal

Also $y'' \xrightarrow{*} x^2$ & $y' \xrightarrow{*} x$

② the independent Variable x in the Coefficient term $(\lambda^2x^2-\nu^2)$ of the dependent variable y → must be of a second degree

⇒ So, we have two Cases

* Steps of Solving Bessels Differential Equation :

① try to put the DE in the standard form

→ either by multiplying the DE by x^n or Constant

② Check For Cases 1 & 2 to get the Standard form

③ Determine the value of λ & ν

④ the Solution of bessel's differential solution :

$$y_{gs}=\begin{cases}C_1J_\nu(\lambda x)+C_2J_{-\nu}(\lambda x) & : \text{if } \nu \text{ is a fraction}\\ C_1J_\nu(\lambda x)+C_2Y_\nu(\lambda x) & : \text{if } \nu \text{ is an integer}\end{cases}$$

• Case ① : the Coefficients of y'' or y' is not in the proper form

we substitute the dependent variable with another one to reach to the proper form & find the general solution in terms of the new dependent variable, then we substitute back with the original one

Steps → ① let $y=x^\alpha u$

$\therefore y'=x^\alpha u'+\alpha x^{\alpha-1}u$

$\therefore y''=x^\alpha u''+2\alpha x^{\alpha-1}u'+\alpha(\alpha-1)x^{\alpha-2}u$

② Substitute in DE

③ Determine the value of α that brings the Coefficients of u'' & u' to its proper Form

④ Check for Case ②

⑤ Solve the DE & get $u_{gs}(x)$

⑥ Get $y_{gs}(x)$ as $y_{gs}=x^\alpha u_{gs}$

• Case ② : $(\lambda^2x^\beta-\nu^2)y$ — the independent variable isn't of second order

we substitute the independent variable with another one to reach to the proper form & find the general solution in terms of the new independent variable, then we substitute back with the original one

Steps → ① let $x^\beta=t^2$ $\quad\therefore t=x^{\beta/2}$

$\therefore y'=\frac{d}{dx}(y)=\frac{d}{dt}(y)\frac{dt}{dx}=\dot{y}\frac{dt}{dx}$

$\therefore y''=\frac{d}{dx}(y')=\frac{d}{dt}\left(\dot{y}\frac{dt}{dx}\right)\frac{dt}{dx}=\left(\ddot{y}\frac{dt}{dx}+\dot{y}\frac{d}{dt}\left(\frac{dt}{dx}\right)\right)\cdot\frac{dt}{dx}$

② Substitute in DE

③ Solve the DE & get $y_{gs}(t)$

④ Get $y_{gs}(x)$

* Properties of Bessel Function : ([Proofs])

① $J_{-n}(x)=(-1)^nJ_n(x)$: n is integer $\therefore J_{-n}$ & J_n are linearly dependent

② $J_n(-x)=(-1)^nJ_n(x)$: if n is even ⇒ $J_n(x)$ is even function; odd ⇒ $J_n(x)$ is odd function

③ $\frac{2n}{x}J_n(x)=J_{n-1}(x)+J_{n+1}(x)$: Reccurrence Relation

④ $2J_n'(x)=J_{n-1}(x)-J_{n+1}(x)$ ⟶ $J_0'(x)=-J_1(x)$

$\int J_{n+1}\,dx=\int J_{n-1}\,dx-2J_n$ ⟶ $\int J_1(x)\,dx=-J_0(x)+c$

⑤ $\frac{d}{dx}(x^nJ_n)=x^nJ_{n-1}$

⑥ $\frac{d}{dx}(x^{-n}J_n)=-x^{-n}J_{n+1}$

Same index → same sign ⇒ J_{n-1}; different sign ⇒ $-J_{n+1}$

different index : differentiate using product rule

* $\int x^mJ_n\,dx=$ ⑦ $m-n=1$ ⟶ $x^mJ_{n+1}+c$; ⑧ $m+n=1$ ⟶ $-x^mJ_{n-1}+c$

Otherwise : try to separate x^m into $x^\beta\cdot x^{m-\beta}$ that Satisfies the rule Condition & use integration by parts

$\int J_0(x)\,dx$ Can't be Solved except numerically

⑨ $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ & $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$: in terms of elementary Functions, Closed Form