SERIES SOLUTIONS AROUND SINGULAR POINTS

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Example 1:

Find a series solution in powers of x for the differential equation:

$$2x^2 y'' + 3xy' - (x^2+1)y = 0$$

Solution:

$x = 0$ is a regular singularity

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

Substitute in the differential equation

$$\sum_{n=0}^{\infty} 2(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s)a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+2} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

Step 1

Shifting the index of the third summation so that the powers of x are the same in all summations.

$$\sum_{n=0}^{\infty} 2(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s)a_n x^{n+s} - \sum_{n=2}^{\infty} a_{n-2} x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

Step 2

$$a_0\left(2s(s-1)+3s-1\right)x^s + a_1\left(2(s+1)s+3(s+1)-1\right)x^{s+1}$$

$$+\sum_{n=2}^{\infty}\left(a_n[2(n+s)(n+s-1)+3(n+s)-1]-a_{n-2}\right)x^{n+s} = 0$$

The coefficient of x to the least power equals zero **(Called the indicial equation)**, this equation is a quadratic equation in s and has two roots s_1 & s_2 .

Step 3

Classification:

Case 1: $If\quad s_1 - s_2\quad is\ a\ \ fraction$

$$y_{gs} = y_1(x,s_1) + y_2(x,s_2) \quad = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s_1} + \sum_{n=0}^{\infty} a_n(s_2)\, x^{n+s_2}$$

Case 2: $If\quad s_1 - s_2 = 0$

$$y_1(x) = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s_1} \qquad\qquad y_2(x) = y_1(x)\ \ln x + \sum_{n=1}^{\infty} a_n'(s_1)\ x^{n+s_1}$$

$$y_{gs} = C_1\ y_1(x) + C_2\ y_2(x)$$

Case 3: *If* $s_1 - s_2$ *is a positive integer*

Return back to our example,

$$\text{Coefficient of } x^{s} = 0 \Rightarrow a_0\left(2s(s-1)+3s-1\right) = 0$$

$$a_0\left(2s^2+s-1\right) = 0 \quad \Rightarrow (2s-1)(s+1)=0 \quad \Rightarrow s_1 = 1/2 \;\; \& \;\; s_2 = -1 \;\; \text{Case 1}$$

$$\text{Coefficient of } x^{s+1} = 0 \Rightarrow a_1\left(2s(s+1)+3(s+1)-1\right) = 0$$

$$\text{For } s_1 = 1/2 \Rightarrow \; 5a_1 = 0 \Rightarrow a_1 = 0$$

$$\text{For } s_2 = -1 \Rightarrow \; -a_1 = 0 \quad \Rightarrow a_1 = 0$$

$$\text{Coefficient of } x^{s+n} = 0$$

$$a_n\left(2(n+s)(n+s-1)+3(n+s)-1\right) - a_{n-2} = 0$$

Step 4

$$a_n = \frac{1}{2(n+s)^2+(n+s)-1}\, a_{n-2}$$

$$\boxed{a_n = \frac{1}{(2(n+s)-1)((n+s)+1)}\, a_{n-2} \quad , \quad n \ge 2}$$

Step 4

$$\text{For } s_1 = \frac{1}{2} \qquad \boxed{\Rightarrow \quad a_n = \frac{1}{n(2n+3)}\, a_{n-2} \quad , \quad n \ge 2}$$

$$a_2 = \frac{1}{(2)(7)}\, a_0$$

$$a_4 = \frac{1}{(4)(11)}\, a_2 = \frac{1}{(2)(4)(7)(11)}\, a_0$$

$$a_6 = \frac{1}{(6)(15)}\, a_4 = \frac{1}{(2)(4)(6)\times(7)(11)(15)}\, a_0$$

$$a_{2n} = \frac{1}{(2\times4\times6\times\ldots\times2n)(7\times11\times15\times\ldots\times(4n+3))}\, a_0 \quad , \quad n \ge 1$$

Step 5 - 1

$$\therefore\ y_1 = \sum_{n=0}^{\infty} a_n(s_1)\ x^{n+s_1} = \sqrt{x}\ \sum_{n=0}^{\infty} a_n(\tfrac{1}{2})\ x^n = \sqrt{x}\ \sum_{n=0}^{\infty} a_{2n}(\tfrac{1}{2})\ x^{2n}$$

$$y_1 = \sqrt{x}\left(a_0 + \sum_{n=1}^{\infty} a_{2n}(\tfrac{1}{2})\ x^{2n}\right) = a_0\sqrt{x}\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!\left(7\times 11\times 15\times \ldots \times (4n+3)\right)}\right)$$

Step 6 - 1

Repeat the same work for $s_2 = -1$

$$\boxed{\Rightarrow \quad a_n = \frac{1}{n(2n-3)}\, a_{n-2} \quad , \quad n \geq 2}$$

$$a_2 = \frac{1}{(2)(1)}\, a_0$$

$$a_4 = \frac{1}{(4)(5)}\, a_2 = \frac{1}{(2)(4)(1)(5)}\, a_0$$

$$a_6 = \frac{1}{(6)(9)}\, a_4 = \frac{1}{(2)(4)(6)\times(1)(5)(9)}\, a_0$$

$$a_{2n} = \frac{1}{(2\times 4\times 6\times \ldots \times 2n)(1\times 5\times 9\times \ldots \times (4n-3))}\, a_0 \quad , \quad n \geq 1$$

Step 5 - 2

$$\therefore\ y_2=\sum_{n=0}^{\infty}a_n(s_2)\ x^{n+s_2}\qquad =\frac{1}{x}\sum_{n=0}^{\infty}a_n(-1)\ x^{n}\qquad =\frac{1}{x}\sum_{n=0}^{\infty}a_{2n}(-1)\ x^{2n}$$

$$y_2=\frac{1}{x}\left(a_0+\sum_{n=1}^{\infty}a_{2n}(-1)\ x^{2n}\right)\qquad =a_0\frac{1}{x}\left(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n!\left(1\times5\times9\times\ldots\times(4n-3)\right)}\right)$$

Step 6 - 2

$$y_{gs}=C_1\ y_1(x)+C_2\ y_2(x)$$

$$\Rightarrow\quad y_{gs}=C_1\sqrt{x}\left(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n!\left(7\times11\times15\times\ldots\times(4n+3)\right)}\right)$$

$$+C_2\frac{1}{x}\left(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n!\left(1\times5\times9\times\ldots\times(4n-3)\right)}\right)$$

Note that we can replace a_0 by 1 **for calculations simplicity.**

Example 2:

Find a two linearly independent solutions in powers of x for the following differential equation:

$$x^2 y'' + x y' + x^2 y = 0$$

Solution:

$x = 0$ **is a regular singularity**

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+s} \;, \quad y' = \sum_{n=0}^{\infty} (n+s)\, a_n x^{n+s-1} \;, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n\, x^{n+s-2}$$

Substitute in the differential equation

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

Step 1

Shift the index of the third summation.

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0$$

Step 2

$$a_0\left(s(s-1)+s\right)x^s + a_1\left((s+1)s+(s+1)\right)x^{s+1}$$

$$+\sum_{n=2}^{\infty}\left(a_n[(n+s)(n+s-1)+(n+s)] + a_{n-2}\right)x^{n+s} = 0$$

Coefficient of $x^s = 0 \quad \Rightarrow a_0\left(s^2\right) = 0 \quad \Rightarrow s^2 = 0 \quad \Rightarrow s_1 = s_2 = 0 \quad Case\,2$

Step 3

Coefficient of $x^{s+1} = 0 \Rightarrow a_1\left((s+1)^2\right) = 0 \quad \Rightarrow a_1 = 0$

Coefficient of $x^{s+n} = 0$

$$a_n\left((n+s)(n+s-1)+(n+s)\right)+a_{n-2} = 0$$

$$\boxed{a_n = \frac{-1}{(n+s)^2}\, a_{n-2} \quad , \quad n \geq 2}$$

Step 4

$$y_1(x) = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s_1}$$

For $s_1 = 0$

$$\boxed{a_n = \frac{-1}{n^2}\, a_{n-2} \qquad , \qquad n \ge 2}$$

$$a_2 = \frac{-1}{2^2}\, a_0$$

$$a_4 = \frac{-1}{4^2}\, a_2 \;= \frac{(-1)^2}{2^2 \times 4^2}\, a_0$$

$$a_6 = \frac{-1}{6^2}\, a_4 \;= \frac{(-1)^3}{2^2 \times 4^2 \times 6^2}\, a_0$$

$$a_{2n} = \frac{(-1)^n}{(2\times 4\times 6\times \ldots \times 2n)^2}\, a_0 \quad , \quad n \ge 1 \qquad \Rightarrow a_{2n} = \frac{(-1)^n}{2^{2n}\times (n!)^2}\, a_0 \quad , \quad n \ge 0$$

Step 5

$$\boxed{\Rightarrow\; y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\times (n!)^2}\, x^{2n}}$$

Step 6 - 1

$$y_2(x) = y_1(x)\ \ln x + \sum_{n=1}^{\infty} a_n'(s_1)\ x^{n+s_1}$$

To find $a_n'(s_1)$ we have to solve the recurrence relation in terms of s and differentiate it.

$$\boxed{a_n = \frac{-1}{(n+s)^2}\ a_{n-2} \quad , \quad n \geq 2}$$

$$a_2(s) = \frac{-1}{(2+s)^2}\, a_0$$

$$a_4(s) = \frac{-1}{(4+s)^2}\, a_2 = \frac{(-1)^2}{(2+s)^2(4+s)^2}\, a_0$$

$$a_6(s) = \frac{-1}{(6+s)^2}\, a_4 = \frac{(-1)^3}{(2+s)^2(4+s)^2(6+s)^2}\, a_0$$

$$a_{2n}(s) = \frac{(-1)^n}{\left((2+s)(4+s)(6+s)\ldots(2n+s)\right)^2}\, a_0 \quad , \quad n \geq 1$$

$$a_{2n}(0) = \frac{(-1)^n}{\left((2)(4)(6)\ldots(2n)\right)^2}\, a_0 = \frac{(-1)^n}{2^{2n}(n!)^2}\, a_0$$

$$\ln\left(a_{2n}(s)\right) = \ln(-1)^n - 2\left(\ln(2+s) + \ln(4+s) + \ldots + \ln(2n+s)\right) + \ln a_0$$

Differentiate both sides w. r. t. s

$$\frac{a'_{2n}(s)}{a_{2n}(s)} = -2\left(\frac{1}{(2+s)} + \frac{1}{(4+s)} + \ldots + \frac{1}{(2n+s)}\right)$$

$$a'_{2n}(0) = -2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}\right)a_{2n}(0) \qquad = -\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\frac{(-1)^n}{2^{2n}(n!)^2}a_0$$

$$\Rightarrow a'_{2n}(0) = \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2}\, a_0$$

$$\boxed{y_2(x) = y_1(x)\ \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2}\, a_0\ x^{2n}}$$

Step 6 - 2

$$y_{gs} = C_1\ y_1(x) + C_2\ y_2(x)$$