Quiz (1) A. Model Answer

Q1: $\int_0^{\pi/2} (\cot^5\theta + \cot^7\theta)\, e^{-\cot^2\theta}\, d\theta$ $\frac{1}{2}$, let $t = \cot^2\theta \Rightarrow dt = -2\cot\theta\csc^2\theta\, d\theta$

$$\Rightarrow I = \int_\infty^0 \left(t^{5/2} + t^{7/2}\right) e^{-t} \frac{dt}{-2t^{1/2}(1+t)} \quad \tfrac{1}{2} \quad = \frac{1}{2}\int_0^\infty \frac{t^2(1+t)}{(1+t)} e^{-t}\, dt \qquad 2$$

$$= \frac{1}{2}\int_0^\infty t^2 e^{-t}\, dt \quad \tfrac{1}{2} \quad = \frac{1}{2}\Gamma(3) = 1. \quad \tfrac{1}{2}$$

Q2: $(x^2+1)y'' + xy' - y = 0$, $x_0 = 0$ —— (*) 3

1st $\Rightarrow$ $P(x) = \frac{x}{x^2+1}$, $q(x) = \frac{-1}{x^2+1}$ $\Rightarrow$ P, q are defined at $x_0 = 0$ $\Rightarrow$ Ordinary Point. $\frac{1}{2}$

2nd $\Rightarrow$ let $y = \sum_{n=0}^\infty a_n x^n$ be a solution $\Rightarrow y' = \sum_{n=1}^\infty n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^\infty n(n-1)a_n x^{n-2}$

3rd $\Rightarrow$ Subs. in (*) $\Rightarrow (x^2+1)\sum_{n=2} n(n-1)a_n x^{n-2} + x\sum_{n=1} n a_n x^{n-1} - \sum_{n=0} a_n x^n = 0$

$$\Rightarrow \sum_{n=2} n(n-1)a_n x^n + \sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=1} n a_n x^n - \sum_{n=0} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0} n(n-1)a_n x^n + \sum_{n=0} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0} n a_n x^n - \sum_{n=0} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^\infty \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n + n a_n - a_n \right] x^n = 0$$

4th $\Rightarrow$ Coef. $(x^n) = 0 \Rightarrow (n+2)(n+1)a_{n+2} = -\left[n(n-1) + n - 1\right] a_n$ 1 Rec. Reln.

$$\Rightarrow \boxed{a_{n+2} = \frac{-(n-1)(n+1)}{(n+2)(n+1)} a_n = \frac{-(n-1)}{(n+2)} a_n, \quad n \geq 0}$$

5th $\Rightarrow$ $n=0 \Rightarrow a_2 = \frac{(-1)(-1)}{2} a_0$, $n=2 \Rightarrow a_4 = \frac{(-1)(1)}{4} a_2 = \frac{(-1)^2(-1\cdot 1)}{2\cdot 4} a_0$

$n=4 \Rightarrow a_6 = \frac{(-1)(3)}{6} a_4 = \frac{(-1)^3(-1\cdot 1\cdot 3)}{2\cdot 4\cdot 6} a_0 \Rightarrow \boxed{a_{2k} = \frac{(-1)^k(-1\cdot 1\cdot 3 \cdots (2k-3))}{(2\cdot 4\cdot 6 \cdots 2k)} a_0, \quad k \geq 1}$ $\frac{1}{2}$

$n=1 \Rightarrow a_3 = 0 = a_5 = a_7 = \cdots a_{2k+1}$ $\frac{1}{2}$

6th $\Rightarrow$ $y = a_0 + \sum_{k=1}^\infty a_{2k} x^{2k} + a_1 x$ $\frac{1}{2}$

$$= a_0\left[1 + \sum_{k=1}^\infty \frac{(-1)^k(-1\cdot 1\cdot 3 \cdots (2k-3))}{2^k \cdot k!} x^{2k}\right] + a_1\left[x\right]$$

Quiz (1) B Model Answer

Q1. $\int_{-\infty}^{\infty} \frac{1}{(4+x^4)^3} dx = 2\int_0^{\infty} \frac{1}{(4+x^4)^3} dx$ (even fn.) 1/2

let $4t = x^4 \Rightarrow x = \sqrt{2}\, t^{1/4} \Rightarrow dx = \frac{\sqrt{2}}{4} t^{-3/4} dt$. 1/2

$\Rightarrow I = \frac{2\cdot\sqrt{2}}{4}\int_0^{\infty} \frac{t^{-3/4}}{(4t+4)^3} dt = \frac{2\sqrt{2}}{4^4}\int_0^{\infty} \frac{t^{-3/4}}{(1+t)^3} dt$ 1/2 $= \frac{2\sqrt{2}}{4^4}\beta(\frac{1}{4}, \frac{11}{4})$

0.0… $= \frac{\sqrt{2}}{128}\frac{2\sqrt{2}}{4^4}\cdot\frac{\Gamma(\frac{1}{4})\Gamma(\frac{11}{4})}{\Gamma(3)}$ 1/2 $= \frac{\cancel{2}\sqrt{2}}{4^4\cdot(\cancel{2})}\frac{7}{4}\cdot\frac{3}{4}\cdot\frac{\pi}{\sin\pi/4} = \frac{42}{4^6}\pi$

Q2: $(2x^2+1)y'' + 6xy' + 2y = 0$, $x_0 = 0$ —— (*)

1, $p(x) = \frac{6x}{2x^2+1}$, $q(x) = \frac{2}{2x^2+1} \Rightarrow p, q$ are defined at $x_0 \Rightarrow$ Ordinary Point 1/2

2, let $y = \sum_{n=0}^{\infty} a_n x^n$ be a soln. $\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2} n(n-1)a_n x^{n-2}$

3, Subs. in (*) $\Rightarrow (2x^2+1)\sum_{n=2} n(n-1)a_n x^{n-2} + 6x\sum_{n=1} n a_n x^{n-1} + 2\sum_{n=0} a_n x^n = 0$

$\Rightarrow \sum_{n=2} 2n(n-1)a_n x^n + \sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=1} 6n a_n x^n + \sum_{n=0} 2a_n x^n = 0$

$\Rightarrow \sum_{n=0} 2n(n-1)a_n x^n + \sum_{n=0} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0} 6n a_n x^n + \sum_{n=0} 2a_n x^n = 0.$

$$\Rightarrow \sum_{n=0}\left[(n+2)(n+1)a_{n+2} + [2n(n-1)+6n+2]a_n\right] x^n = 0$$

4, Coef. $(x^n) = 0 \Rightarrow (n+2)(n+1)a_{n+2} = -(2n^2+4n+2)a_n$ 1

$$\Rightarrow \boxed{a_{n+2} = \frac{(-2)(n+1)^{\cancel{2}}}{(n+2)\cancel{(n+1)}}a_n = \frac{(-2)(n+1)}{(n+2)}a_n, \quad n \geq 0}$$ Rec. Rel.

5, $n=0 \Rightarrow a_2 = \frac{-2(1)}{2}a_0$ | $n=1 \Rightarrow a_3 = \frac{(-2)(2)}{3}a_1$

$n=2 \Rightarrow a_4 = \frac{(-2)(3)}{4}a_2 = \frac{(-2)^2(1.3)}{2.4}a_0$ | $n=3 \Rightarrow a_5 = \frac{(-2)(4)}{5}a_3 = \frac{(-2)^2(2.4)}{3.5}a_1$

⋮

$\Rightarrow a_{2k} = \frac{(-2)^k(1.3.5\ldots(2k-1))}{2.4.6\ldots 2k}a_0$ 1/2 | $a_{2k+1} = \frac{(-2)^k(2.4.6\ldots 2k)}{3.5.7\ldots(2k+1)}a_1$. 1/2

$\Rightarrow y = a_0 + \sum_{k=1} a_{2k}x^{2k} + a_1 x + \sum_{k=1} a_{2k+1}x^{2k+1}$

$$\boxed{y = a_0\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k(1.3.5\ldots(2k-1))}{k!}x^{2k}\right] + a_1\left[x + \sum_{k=1}^{\infty}\frac{(-4)^k k!}{3.5.7\ldots(2k+1)}x^{2k+1}\right]}$$ 1/2