AIN SHAMS UNIVERSITY
FACULTY OF ENGINEERING
SPECIALIZED ENGINEERING PROGRAMS
JUNIOR COMMUNICATION ENGINEERING PROGRAM

SPRING 2022	Assignment #3	Total: 5 marks

PHM212s: Special Functions, Complex Analysis & Numerical Analysis

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Name: **ID:** **Deadline: Week 9**

Please, Solve each problem in its assigned place ONLY (the empty space below it)

Bessel Functions

1. Solve in terms of Bessel functions the following differential equations:

a) $x^2\,y'' + x\,y' + (x^2 - 9)y = 0 \rightarrow \lambda = 1 \ \& \ \nu = 3$

$\therefore y = C_1 J_3(x) + C_2 Y_3(x)$

b) $x^2\,y'' + x\,y' + (x^2 - 8)y = 0 \rightarrow \lambda = 1 \ \& \ \nu = \sqrt{8}$

$\therefore y = C_1 J_{\sqrt{8}}(x) + C_2 J_{-\sqrt{8}}(x)$

c) $x^2\,y'' + x\,y' + (3x^2 - 4)y = 0 \rightarrow \lambda = \sqrt{3} \ \& \ \nu = 2$

$\therefore y = C_1 J_2(\sqrt{3}x) + C_2 Y_2(\sqrt{3}x)$

d) $x^2\,y'' + x\,y' + 4(x^4 - n^2)y = 0 \quad , \quad n \in I$

$\rightarrow$ let $t^2 = x^4$

$\therefore t = x^2 \ \& \ x = t^{1/2} \Rightarrow \frac{dt}{dx} = 2x = 2t^{1/2}$

• $y' = \frac{d}{dx}(y) = \frac{d}{dt}(y) * \frac{dt}{dx} = 2\dot{y}\,t^{1/2}$

• $y'' = \frac{d}{dx}(y') = \frac{d}{dt}(2\dot{y}t^{1/2}) * \frac{dt}{dx}$

$= (\dot{y}t^{-1/2} + 2\ddot{y}t^{1/2})(2t^{1/2}) = 2\dot{y} + 4t\ddot{y}$

$\rightarrow t(4t\ddot{y} + 2\dot{y}) + t^{1/2}(2\dot{y}t^{1/2}) + (4t^2 - 4n^2)y = 0$

$4t^2\ddot{y} + 2t\dot{y} + 2t\dot{y} + (4t^2 - 4n^2)y = 0 \qquad (\div 4)$

$\therefore t^2\ddot{y} + t\dot{y} + (t^2 - n^2)y = 0 \rightarrow \lambda = 1 \ \& \ \nu = n$

$\therefore y = C_1 J_n(x) + C_2 Y_n(x)$

e) $x\,y'' + 3\,y' + x\,y = 0$

$\rightarrow$ let $y = x^\alpha u$ $\quad \therefore y' = \alpha x^{\alpha-1}u + x^\alpha u' \quad \therefore y'' = x^\alpha u'' + 2\alpha x^{\alpha-1}u' + \alpha(\alpha-1)x^{\alpha-2}u$

by subs. in DE:

$x^{\alpha+1}u'' + 2\alpha x^\alpha u' + \alpha(\alpha-1)x^{\alpha-1}u + 3\alpha x^{\alpha-1}u + 3x^\alpha u' + x^{\alpha+1}u = 0$

$\rightarrow x^{\alpha+1}u'' + (2\alpha+3)x^\alpha u' + [\alpha(\alpha-1)x^{\alpha-1} + 3\alpha x^{\alpha-1} + x^{\alpha+1}]u = 0 \quad (\div x^{\alpha-1})$

$x^2u'' + (2\alpha+3)xu' + [\alpha(\alpha-1) + 3\alpha + x^2]u = 0 \quad \therefore 2\alpha + 3 = 1 \rightarrow \alpha = -1$

$x^2u'' + xu' + [x^2-1]u = 0 \rightarrow \lambda = 1 \ \& \ \nu = 1$

$\therefore u_{gs} = c_1 J_1(x) + c_2 Y_1(x)$

$\therefore y_{gs} = \frac{1}{x} c_1 J_1(x) + \frac{1}{x} c_2 Y_1(x)$

f) $4\,x\,y'' + 4\,y' + y = 0$ $\left(* \frac{x}{4}\right)$

$\rightarrow x^2y'' + xy' + \left(\frac{x}{4}\right)y = 0$

$\rightarrow$ let $t^2 = x$

$\therefore t = x^{1/2}$ & $x = t^2 \Rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2}t^{-1}$

$\cdot\ y' = \frac{d}{dx}(y) = \frac{d}{dt}(y) * \frac{dt}{dx} = \frac{1}{2}t^{-1}\dot{y}$

$\cdot\ y'' = \frac{d}{dx}(y') = \frac{d}{dt}\left(\frac{1}{2}t^{-1}\dot{y}\right) * \frac{dt}{dx}$

$= \left(\frac{1}{2}t^{-1}\ddot{y} - \frac{1}{2}t^{-2}\dot{y}\right) * \frac{1}{2}t^{-1}$

$= \frac{1}{4}t^{-2}\ddot{y} - \frac{1}{4}t^{-3}\dot{y}$

$\rightarrow 4t^2\left(\frac{1}{4}t^{-2}\ddot{y} - \frac{1}{4}t^{-3}\dot{y}\right) + 4\left(\frac{1}{2}t^{-1}\dot{y}\right) + y = 0$

$\ddot{y} - t^{-1}\dot{y} + 2t^{-1}\dot{y} + y = 0$ $\left(* t^2\right)$

$t^2\ddot{y} + t\dot{y} + t^2y = 0 \rightarrow \lambda = 1 \ \& \ \nu = 0$

$\therefore y_{gs} = c_1 J_0(t) + c_2 Y_0(t)$

$\therefore y_{gs} = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x})$

2. Find the solution of $x^2\,y'' + x\,y' + (4x^2 - 1)\,y = 0$ which is bounded at $x = 0$ and $y(2) = 5$ $\rightarrow \lambda = 2 \ \& \ \nu = 1$

means $C_2 = 0$

$\therefore y_{gs} = C_1 J_1(2x) + C_2 Y_1(2x)$ & at $x = 2$, $y = 5$

$\therefore C_1 = \frac{5}{J_1(4)} \rightarrow y = \frac{5}{J_1(4)} J_1(2x)$

3. Show that:

a) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$$\rightarrow J_{-1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \ x^{2m-\frac{1}{2}}}{2^{2m-\frac{1}{2}}\ m!\ \Gamma(m+1/2)} = \sum_{m=0}^{\infty} \frac{(-1)^m\ (x)^{-1/2}\ (x)^{2m}}{(2)^{1/2}\ 2^{2m-1}\ m\ \Gamma(m)\ \Gamma(m+1/2)}$$

$$= \sum_{m=0}^{\infty} \sqrt{\frac{1}{2x}} \frac{(-1)^m\ (x)^{2m}}{m\sqrt{\pi}\ \Gamma(2m)} = \sqrt{\frac{4}{2\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m}}{2m\ (2m-1)!}$$

$$= \sqrt{\frac{2}{x\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m}}{(2m)!} = \sqrt{\frac{2}{\pi x}}\ \cos x$$

b) $Y_{1/2}(x) = -\sqrt{\dfrac{2}{\pi x}}\cos x$ $\rightarrow$ $Y_{1/2}(x) = \dfrac{J_{1/2}(x)\cos(\pi/2) - J_{-1/2}(x)}{\sin(\pi/2)} = -J_{-1/2}(x)$

$\therefore Y_{1/2}(x) = -\sqrt{\dfrac{2}{\pi x}}\cos x$

c) $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\sin x}{x} - \cos x\right)$ $\longrightarrow$ $n+1 = 3/2$ $\therefore n = \frac{1}{2}$

$\rightarrow J_{1/2}(x) = x\left[J_{-1/2}(x) + J_{3/2}(x)\right]$ $\therefore J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$

$\therefore J_{3/2}(x) = \frac{1}{x}\left[\sqrt{\frac{2}{\pi x}}\sin x\right] - \sqrt{\frac{2}{\pi x}}\cos x$ $\therefore J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

d) $J_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\cos x}{x} + \sin x\right)$ $\longrightarrow$ $n-1 = -3/2$ $\therefore n = -1/2$

$\rightarrow J_{-1/2}(x) = -x\left[J_{-3/2}(x) + J_{1/2}(x)\right]$ $\therefore J_{-3/2}(x) = \frac{-1}{x}J_{-1/2}(x) - J_{1/2}(x)$

$\therefore J_{-3/2}(x) = \frac{-1}{x}\left[\sqrt{\frac{2}{\pi x}}\cos x\right] - \sqrt{\frac{2}{\pi x}}\sin x$ $\therefore J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$

4. Show that $y = x^n J_n(x)$ is a solution for the differential equation $x\,y'' + (1-2n)\,y' + x\,y = 0$ using two different methods.

1st Method:

$\rightarrow x^2 y'' + (1-2n)x y' + x^2 y = 0$

$\rightarrow$ let $y = x^\alpha u$ $\therefore y' = \alpha x^{\alpha-1}u + x^\alpha u'$

$\therefore y'' = x^\alpha u'' + 2\alpha x^{\alpha-1}u' + \alpha(\alpha-1)x^{\alpha-2}u$

by Subs in DE :

$x^{\alpha+2}u'' + 2\alpha x^{\alpha+1}u' + \alpha(\alpha-1)x^\alpha u$

$+ (1-2n)x^{\alpha+1}u' + (1-2n)\alpha x^\alpha u + x^{\alpha+2}u = 0$ $(\times x^{-\alpha})$

$x^2 u'' + \left[2\alpha + (1-2n)\right]x u' + \left[\alpha(\alpha-1) + \alpha(1-2n) + x^2\right]u = 0$

$\Rightarrow 2\alpha + \cancel{1} - 2n = \cancel{1}$ $\therefore \alpha = n$

$\therefore x^2 u'' + x u' + (x^2 - n^2)u = 0 \longrightarrow \lambda = 1$ & $\nu = n$

$\therefore u_{gs} = C_1 J_n(x) + C_2 Y_n(x)$

$\therefore y_{gs} = C_1 x^n J_n(x) + C_2 Y_n(x)$

@ $C_1 = 1$ & $C_2 = 0$

then, $y = x^n J_n(x)$ is a solution

<u>2nd Method:</u> → Let $y = x^n J_n(x)$ is a solution

$$\therefore y' = \frac{d}{dx}(y) = \frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$$\therefore y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(x^n J_{n-1}(x)\right) = \frac{d}{dx}\left[x \cdot x^{n-1} J_{n-1}(x)\right] = x^n J_{n-2}(x) + x^{n-1} J_{n-1}(x)$$

$$\Rightarrow L.H.S = x^{n+1} J_{n-2}(x) + x^n J_{n-1}(x) + (1-2n) x^n J_{n-1} x + x^{n+1} J_n(x) = \qquad (\times x^{-n})$$

$$x J_{n-2}(x) + (2-2n) J_{n-1}(x) + x J_n(x) =$$

$$\cancel{x}\left(\frac{2(n-1)}{\cancel{x}}\right) J_{n-1}(x) + 2(1-n) J_{n-1}(x) =$$

$$J_{n-1}(x)\left[\cancel{2n} - \cancel{2} + \cancel{2} - \cancel{2n}\right] = 0 = R.H.S \ \#$$

5. Show that

a) $J_n''(x) = \frac{1}{4}\left(J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x)\right)$

$$\rightarrow J_n'(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x)$$

$$\rightarrow J_n''(x) = \frac{1}{2} J_{n-1}'(x) - \frac{1}{2} J_{n+1}'(x)$$

$$= \frac{1}{2}\left[\frac{1}{2} J_{n-2}(x) - \frac{1}{2} J_n(x)\right] - \frac{1}{2}\left[\frac{1}{2} J_n(x) - \frac{1}{2} J_{n+2}(x)\right]$$

$$= \frac{1}{4} J_{n-2}(x) - \frac{1}{4} J_n(x) - \frac{1}{4} J_n(x) + \frac{1}{4} J_{n+2}(x)$$

$$= \frac{1}{4}\left(J_{n-2}(x) - 2 J_n(x) + J_{n+2}(x)\right)$$

b) $\frac{d}{dx}\left(J_n^2(x)\right) = \frac{x}{2n}\left(J_{n-1}^2(x) - J_{n+1}^2(x)\right)$

$$\rightarrow \frac{d}{dx}\left(J_n^2(x)\right) = 2 J_n(x) \cdot J_n'(x)$$

$$= \cancel{2}\left[\frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right)\right]\left[\frac{1}{\cancel{2}}\left(J_{n-1}(x) - J_{n+1}(x)\right)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - \cancel{J_{n-1}(x) J_{n+1}(x)} + \cancel{J_{n-1}(x) J_{n+1}(x)} - J_{n+1}^2(x)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

c) $\frac{d}{dx}\left(x\ J_n(x)\, J_{n+1}(x)\right) = x\left(J_n^2(x) - J_{n+1}^2(x)\right)$

$\rightarrow \frac{d}{dx}\left(x J_n(x) J_{n+1}(x)\right) = \frac{d}{dx}\left(x \cdot x^{n+1} \cdot x^{-(n+1)}\ J_n(x)\ J_{n+1}(x)\right)$

$= \frac{d}{dx}\left(x^{-n} J_n(x) \cdot x^{n+1} J_{n+1}(x)\right)$

$= \left(x^{-n} J_n(x)\right)\left(x^{n+1} J_n(x)\right) + \left(-x^{-n} J_{n+1}(x)\right)\left(x^{n+1} J_{n+1}(x)\right)$

$= x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

6. Solve the following integrals in terms of Bessel Functions:

a) $\int x^3\ J_2(x)\ dx = x^3 J_3 + C$

b) $\int x^{-4}\ J_5(x)\ dx = -x^{-4} J_4 + C$

c) $\int x^4\ J_1(x)\ dx = \int (x^2)(x^2 J_1(x))\, dx$

$U = x^2 \quad dV = x^2 J_1(x)\, dx$
$dU = 2x\, dx \quad V = x^2 J_2(x)$

$\therefore I = x^4 J_2(x) - 2\int x^3 J_2(x)\, dx$

$= x^4 J_2(x) - 2x^3 J_3 + C$

d) $\int \sqrt{x}\ J_{1/2}(x)\ dx = \int \cancel{\sqrt{x}} \cdot \frac{\sqrt{2}}{\sqrt{\pi}\,\cancel{\sqrt{x}}} \sin x\, dx = \sqrt{\frac{2}{\pi}} \int \sin x\, dx = -\sqrt{\frac{2}{\pi}} \cos x + C$

$= -\sqrt{x}\, J_{-1/2}(x) + C$

e) $\int x^{-2}\ J_2(x)\ dx = \int (x^{-1})(x^{-1} J_2(x))\, dx$

$U = x^{-1} \quad dV = x^{-1} J_2(x)\, dx$
$dU = -x^{-2} dx \quad V = -x^{-1} J_1(x)$

$U = -x^{-3} \quad dV = x^{0} J_1(x)\, dx$
$dU = 3x^{-4} dx \quad V = -x^{0} J_0(x)\, dx$

$\therefore I = -x^{-2} J_1(x) - \int x^{-3} J_1(x)\, dx = -x^{-2} J_1(x) + \int -x^{-3} \cdot (x^0) J_1(x)\, dx$

$= -x^{-2} J_1(x) + x^{-3} J_0(x) + \int 3x^{-4} J_0(x)\, dx$

7. Solve in terms of Bessel functions the following differential equations:

$$y'' + x\,y = 0$$

$\rightarrow x^2 y'' + x^3 y = 0$

$\rightarrow$ let $y = x^\alpha u$ $\quad \therefore y' = \alpha x^{\alpha-1} u + x^\alpha u'$ $\quad \therefore y'' = x^\alpha u'' + 2\alpha x^{\alpha-1} u' + \alpha(\alpha-1) x^{\alpha-2} u$

* by Subs in DE :

$\therefore x^{\alpha+2} u'' + 2\alpha x^{\alpha+1} u' + \alpha(\alpha-1) x^\alpha u + x^{\alpha+3} u = 0$

$x^2 u'' + 2\alpha x u' + [x^3 + \alpha(\alpha-1)] u = 0$ $\quad \rightarrow 2\alpha = 1 \quad \therefore \alpha = 1/2$

$\therefore x^2 u'' + x u' + [x^3 - \frac{1}{4}] u = 0$

$\rightarrow$ let $t^2 = x^3$ $\therefore t = x^{3/2}$ & $x = t^{2/3}$ $\Rightarrow \frac{dt}{dx} = \frac{3}{2} x^{1/2} = \frac{3}{2} t^{1/3}$

- $u' = \frac{d}{dx}(u) = \frac{d}{dt}(u) * \frac{dt}{dx} = \frac{3}{2} t^{1/3} \dot{u}$

- $u'' = \frac{d}{dx}(u') = \frac{d}{dt}(\frac{3}{2} t^{1/3} \dot{u}) * \frac{dt}{dx} = (\frac{3}{2} t^{1/3} \ddot{u} + \frac{1}{2} t^{-2/3} \dot{u}) * \frac{3}{2} t^{1/3} = \frac{9}{4} t^{2/3} \ddot{u} + \frac{3}{4} t^{-1/3} \dot{u}$

* by Subs in DE :

$\therefore t^{4/3} (\frac{9}{4} t^{2/3} \ddot{u} + \frac{3}{4} t^{-1/3} \dot{u}) + t^{2/3} (\frac{3}{2} t^{1/3} \dot{u}) + [t^2 - \frac{1}{4}] u = 0$

$\frac{9}{4} t^2 \ddot{u} + \frac{3}{4} t \dot{u} + \frac{3}{2} t \dot{u} + [t^2 - 1/4] u = 0$

$t^2 \ddot{u} + t \dot{u} + [4/9\, t^2 - 1/9] u = 0 \longrightarrow \lambda = 2/3$ & $\nu = 1/3$

$\therefore u_{gs} = C_1 J_{1/3}(2/3\, t) + C_2 J_{-1/3}(2/3\, t)$

$\therefore u_{gs} = C_1 J_{1/3}(2/3\, x^{3/2}) + C_2 J_{-1/3}(2/3\, x^{3/2})$

$\therefore y_{gs} = C_1 \sqrt{x}\, J_{1/3}(2/3\, x^{3/2}) + C_2 \sqrt{x}\, J_{-1/3}(2/3\, x^{3/2})$

8. Solve in terms of Bessel functions the following differential equations:

$$x\, y'' + y = 0$$

$\rightarrow x^2 y'' + x y = 0$

$\rightarrow$ let $y = x^\alpha u$ $\therefore y' = \alpha x^{\alpha-1} u + x^\alpha u'$ $\therefore y'' = x^\alpha u'' + 2\alpha x^{\alpha-1} u' + \alpha(\alpha-1) x^{\alpha-2} u$

* by Subs in DE :

$\therefore x^{\alpha+2} u'' + 2\alpha x^{\alpha+1} u' + \alpha(\alpha-1) x^\alpha u + x^{\alpha+1} u = 0$

$x^2 u'' + 2\alpha\, x u' + [x + \alpha(\alpha-1)] u = 0$ $\rightarrow 2\alpha = 1$ $\therefore \alpha = 1/2$

$\therefore x^2 u'' + x u' + [x - \frac{1}{4}] u = 0$

$\rightarrow$ let $t^2 = x$ $\therefore t = x^{1/2}$ & $x = t^2 \Rightarrow \frac{dt}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2} t^{-1}$

- $u' = \frac{d}{dx}(u) = \frac{d}{dt}(u) * \frac{dt}{dx} = \frac{1}{2} t^{-1} \dot{u}$

- $u'' = \frac{d}{dx}(u') = \frac{d}{dt}(\frac{1}{2} t^{-1} \dot{u}) * \frac{dt}{dx} = (\frac{1}{2} t^{-1} \ddot{u} - \frac{1}{2} t^{-2} \dot{u}) * \frac{1}{2} t^{-1} = \frac{1}{4} t^{-2} \ddot{u} - \frac{1}{4} t^{-3} \dot{u}$

* by Subs in DE :

$\therefore t^4 (\frac{1}{4} t^{-2} \ddot{u} - \frac{1}{4} t^{-3} \dot{u}) + t^2 (\frac{1}{2} t^{-1} \dot{u}) + [t^2 - \frac{1}{4}] u = 0$

$\frac{1}{4} t^2 \ddot{u} - \frac{1}{4} t \dot{u} + \frac{1}{2} t \dot{u} + [t^2 - 1/4] u = 0$

$t^2 \ddot{u} + t \dot{u} + [4t^2 - 1] u = 0 \rightarrow \lambda = 2$ & $\nu = 1$

$\therefore u_{gs} = C_1 J_1(2t) + C_2 Y_1(2t)$

$\therefore u_{gs} = C_1 J_1(2\sqrt{x}) + C_2 Y_1(2\sqrt{x})$

$\therefore y_{gs} = C_1 \sqrt{x}\, J_1(2\sqrt{x}) + C_2 \sqrt{x}\, Y_1(2\sqrt{x})$