AIN SHAMS UNIVERSITY
FACULTY OF ENGINEERING
ENG. PHYSICS & MATH. DEPARTMENT
Electronics and Communications Engineering Program
Junior Students.

Mid Term Examination

Spring 2022	*Exam Time: 60 minutes.*

PHM212s: Complex, Special Functions and Numerical Analysis

The Exam Consists of <u>THREE Questions</u> in <u>THREE Pages</u>. Answer All Questions	*Total Marks: 20 Marks*

MODEL ANSWER		

General Instructions:

- ***Please read the examination paper carefully.***
- ***Be sure to solve each question in its paper (you can use the back).***
- ***Programmable & Graphical Calculators are NOT Allowed.***

Question no. 1 (6 marks)

a) Use the Gamma function to evaluate the following integral:

$$\int_0^1 x^{4/3}\ \ln^3 x\ \ dx$$

[3 Marks]

$$\rightarrow let \quad -t=\ln x \Rightarrow x=e^{-t} \Rightarrow dx=-e^{-t}dt \cdots \boxed{1}$$

$$\Rightarrow I=\int_\infty^0 \left(e^{-t}\right)^{4/3}\left(-t\right)^3\left(-e^{-t}dt\right)$$

$$=-\int_0^\infty t^3 e^{-\frac{7}{3}t}dt \qquad \cdots \boxed{0.5}$$

$$\rightarrow let \quad \frac{7}{3}t=u \Rightarrow t=\frac{3}{7}u \Rightarrow dt=\frac{3}{7}du$$

$$\Rightarrow I=-\left(\frac{3}{7}\right)^4\int_0^\infty u^3 e^{-u}du \qquad \cdots \boxed{0.5}$$

$$=-\left(\frac{3}{7}\right)^4\Gamma(4)=-\left(\frac{3}{7}\right)^4(3!) \quad \cdots \boxed{1}$$

b) Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{x^4}{1+x^6}\,dx$ using the Gamma function.

[3 Marks]

$$\rightarrow I=2\int_0^\infty \frac{x^4}{1+x^6}\ dx \quad (even\ function) \quad \cdots \boxed{0.5}$$

$$\rightarrow let \quad x^6=u \Rightarrow x=u^{1/6} \Rightarrow dx=\frac{1}{6}u^{-5/6}du \quad \cdots \boxed{1}$$

$$\Rightarrow I=2\int_0^\infty \frac{\left(u^{1/6}\right)^4}{1+u}\ \frac{1}{6}u^{-5/6}du$$

$$=\frac{2}{6}\int_0^\infty \frac{u^{-1/6}}{1+u}\ du \quad \cdots \boxed{0.5}$$

$$=\frac{1}{3}\beta\left(\frac{5}{6},\frac{1}{6}\right)=\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{6}\right)}{3\Gamma(1)} \quad \cdots \boxed{1}$$

$$=\frac{2\pi}{3}$$

Question no. 2 (7 marks)

a) Find and classify the singularities of the following differential equation:

$$(x-x^2)^2 y''+3\ x\,y'+(1-x^2)y=0$$

[3 Marks]

$$\to p(x)=\frac{3x}{(x(1-x))^2}=\frac{3}{x(1-x)^2},$$

$$\to q(x)=\frac{1-x^2}{(x(1-x))^2}=\frac{1+x}{x^2(1-x)}$$

$$\Rightarrow x_0\in\mathbb{R}-\{0,1\}\quad \text{are ordinary points}$$

$$\Rightarrow x_0=0\ \&\ x_0=1\quad \text{are singular points}\quad \cdots\boxed{1}$$

$$\to \text{at } x_0=0$$

$$\Rightarrow P(x)=x\,p(x)=\frac{3}{(1-x)^2},$$

$$Q(x)=x^2\,q(x)=\frac{1+x}{(1-x)}$$

$$\Rightarrow P(x)\ \&\ Q(x)\ \text{are defined}$$

$$\Rightarrow x_0=0 \text{ is Regular singular point}\quad \cdots\boxed{1}$$

$$\to \text{at } x_0=1$$

$$\Rightarrow P(x)=(x-1)\,p(x)=\frac{-3}{x(1-x)},$$

$$Q(x)=(x-1)^2\,q(x)=\frac{1-x^2}{x^2}$$

$$\Rightarrow P(x) \text{ is not defined}$$

$$\Rightarrow x_0=1 \text{ is IRRegular singular point}\quad \cdots\boxed{1}$$

B) Solve in terms of Bessel functions the following differential equation:

$$x^2 y''+\ x\,y'+(x^3-4)y=0$$

[4 Marks]

$$\to let\quad t^2=x^3\Rightarrow t=x^{3/2}\Rightarrow x=t^{2/3}$$

$$\Rightarrow \frac{dt}{dx}=\frac{3}{2}x^{1/2}=\frac{3}{2}t^{1/3}\quad \cdots\boxed{1}$$

$$\to y'=\frac{dy}{dx}=\frac{dy}{dt}*\frac{dt}{dx}=\dot{y}*\left(\frac{3}{2}t^{1/3}\right)$$

$$\Rightarrow y''=\frac{dy'}{dx}=\frac{dy'}{dt}*\frac{dt}{dx}=\frac{3}{2}\left(\frac{1}{3}t^{-2/3}\dot{y}+t^{1/3}\ddot{y}\right)*\left(\frac{3}{2}t^{1/3}\right)$$

$$\Rightarrow y''=\frac{3}{4}t^{-1/3}\dot{y}+\frac{9}{4}t^{2/3}\ddot{y}\quad \cdots\boxed{1}$$

$$\Rightarrow \left(t^{2/3}\right)^2\left(\frac{3}{4}t^{-1/3}\dot{y}+\frac{9}{4}t^{2/3}\ddot{y}\right)+\left(t^{2/3}\right)\left(\frac{3}{2}t^{1/3}\dot{y}\right)+\left(t^2-4\right)y=0$$

$$\Rightarrow \frac{9}{4}t^2\ddot{y}+\frac{9}{4}t\,\dot{y}+\left(t^2-4\right)y=0$$

$$\Rightarrow t^2\ddot{y}+t\,\dot{y}+\left(\frac{4}{9}t^2-\frac{16}{9}\right)y=0\quad \cdots\boxed{1}$$

$$\Rightarrow y_{g.s}=c_1J_{4/3}\left(\frac{2}{3}t\right)+c_2J_{-4/3}\left(\frac{2}{3}t\right)$$

$$\Rightarrow y_{g.s}=c_1J_{4/3}\left(\frac{2}{3}x^{3/2}\right)+c_2J_{-4/3}\left(\frac{2}{3}x^{3/2}\right)\quad \cdots\boxed{1}$$

<u>Question no. 3 (7 marks)</u>

Find two linearly independent solutions in powers of “ x ” for the following differential equations:

$$(3-x^2)\,y'' - x\,y' + 9\,y = 0$$

$$\to p(x)=\frac{-x}{3-x^2},\ q(x)=\frac{9}{3-x^2}$$

$$\Rightarrow p(x)\,\&\,q(x) \text{ are both defined at } x_0=0 \Rightarrow x_0=0 \text{ is an ordinary point} \quad \cdots\boxed{1}$$

$$\to let\ \text{y}=\sum_{n=0}^{\infty} a_n x^n \Rightarrow y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \quad \cdots\boxed{1}$$

$$\Rightarrow (3-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 9\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 9a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 9a_n x^n = 0$$

$$\to coef\,(x^n)=0$$

$$\Rightarrow 3(n+2)(n+1)a_{n+2} - n(n-1)a_n - n a_n + 9a_n = 0$$

$$\Rightarrow 3(n+2)(n+1)a_{n+2} = \left[n(n-1)+n-9\right]a_n$$

$$\Rightarrow a_{n+2} = \frac{(n-3)(n+3)}{3(n+2)(n+1)}a_n,\ n\ge 0 \quad \cdots\boxed{2}$$

$$\to n=0 \Rightarrow a_2=\frac{-3*3}{3*2*1}a_0,\quad \to n=2 \Rightarrow a_4=\frac{-1.5}{3*4*3}a_2=\frac{(-3*-1)(3*5)}{3^2(4*3*2*1)}a_0,$$

$$\to n=4 \Rightarrow a_6=\frac{1.7}{3*6*5}a_4=\frac{(-3*-1*1)(3*5*7)}{3^3(6*5*4*3*2*1)}a_0,$$

$$\Rightarrow a_{2m}=\frac{\left[-3*-1*1*\ldots*(2m-5)\right]\left[3*5*7*\ldots*(2m+1)\right]}{3^m(2m)!}a_0,\ m\ge 1 \quad \cdots\boxed{1}$$

$$\to n=1 \Rightarrow a_3=\frac{-2*4}{3.3.2}a_1=\frac{-4}{9}a_1,\ n=3 \Rightarrow a_5=0=a_7=a_9=\cdots=a_{2m+1} \quad \cdots\boxed{1}$$

$$\Rightarrow \boxed{y=a_0\left(1+\sum_{m=1}^{\infty}\frac{\left[-3*-1*1*\ldots*(2m-5)\right]\left[3*5*7*\ldots*(2m+1)\right]}{3^m(2m)!}x^{2m}\right)+a_1\left(x-\frac{4}{9}x^3\right)} \quad \cdots\boxed{1}$$

Best Wishes,

Dr. Makram Roshdy, Dr. Betty Nagy.