بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

PHM113s:

Differential and Partial Differential Equations

Revision on DE

﴿يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖ﴾

Ahmed Juba

* order of differential equation
  it's the order of the highest order derivative involved in the differential equation

* degree of differential equation
  it's the exponent of the highest order derivative involved in the differential equation

First order differential equations

$$\frac{dy}{dx} = y' = f(x, y)$$

1. Separable equations $[y' = f(y)\,g(x)]$

1. Separate the functions → $\frac{1}{f(y)}dy = g(x)\,dx$
2. integrate both sides $\int \frac{1}{f(y)}dy = \int g(x)\,dx + C$

* if $[y' = f(ax + by + c)]$
  1. Substitute with $(z = ax + by + c)$
  2. differentiate both sides to subs. y' with $f(z')$

* homogenous equations $[y' = f(\frac{y}{x})]$
  1. Substitute with ⇒ $v = \frac{y}{x}$ & $y' = v + v'x$
  2. then: $v + v'x = f(v)$

2. Exact differential Equations $[M\,dx + N\,dy = 0]$

1. if $M_y = N_x$ ⟶ ∴ Exact
2. $\int M\,dx + \int N\,dy = C$
3. Omit One of the repeated term

Caution

* Integrating Factor
  1. if $M_y \neq N_x$ ⟶ ∴ not exact
  2. Calculate $M_y - N_x$
  3. Find μ
     - if $\frac{M_y - N_x}{N} = f(x) \rightarrow \mu_x = e^{\int f(x)dx}$
     - if $\frac{M_y - N_x}{M} = g(y) \rightarrow \mu_y = e^{-\int g(y)dy}$
  4. Multiply μ to the DE
     $\mu M\,dx + \mu N\,dy = 0$
  5. Solve as Exact

3. Linear Equations $[y' + p(x)\,y = q(x)]$

1. Find $\mu = e^{\int p(x)dx}$
2. then $\mu y = \int \mu\, q(x)\,dx + c$

* Bernoulli's Equations $[y' + p(x)\,y = q(x)\,y^n]$
  1. Find $\mu = e^{\int (1-n)p(x)dx}$
  2. then $\mu y^{1-n} = \int (1-n)\mu\, q(x)\,dx + c$

Second order differential equations

$y'' = f(x, y, y')$

① Variable Coefficients

Case ①: y is absent — $y'' = f(x, y')$

1. let $y' = z$ & $y'' = z'$
2. Substitute & Solve as first order DE of z (but twice)

Case ②: x is absent — $y'' = f(y, y')$

1. let $y' = z$ & $y'' = z\dfrac{dz}{dy}$
2. Substitute & Solve as first order DE of z (but twice)

② Constant Coefficients

$[\,a_0 y'' + a_1 y' + a_2 y = \square\,]$

- $= 0$ → Ⓐ homogenous equation
- $= f(x)$ → Ⓑ non homogenous equation

A] homogenous equation

1. Calculate y_{CF}
   1) Replace each $y'' \to m^2$, $y' \to m$, $y \to 1$
   2) Solve the quadratic equation $(a_0 m^2 + a_1 m + a_2 = 0)$ to get m
2. if the roots are
   - Real & distinct: $y_{CF} = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
   - Real & repeated: $y_{CF} = C_1 e^{mx} + C_2 x e^{mx}$
   - Complex distinct: $y_{CF} = e^{ax}[C_1 \cos bx + C_2 \sin bx]$
3. $y_{G.S} = y_{CF}$

B] non homogenous equation

① Variation of parameters method

1. Find y_{CF} as a homogenous equation
   $\Rightarrow y_{CF} = C_1 y_1 + C_2 y_2$
2. Find $\omega = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}$
3. Find $y_{PI} = \left[-\displaystyle\int \frac{y_2 f(x)}{\omega}\,dx\right] y_1 + \left[\displaystyle\int \frac{y_1 f(x)}{\omega}\,dx\right] y_2$
4. $y_{G.S} = y_{CF} + y_{PI}$

② Operator method [D-method]

Valid for 5 cases only, depends on $f(x)$

$D = \dfrac{d}{dx}$ → $Dy = y'$, $D^2 y = y''$ • $\dfrac{1}{D} = \int dx$

1. Find y_{CF} as a homogenous equation
   $\Rightarrow y_{CF} = C_1 y_1 + C_2 y_2$
2. Replace each $d^2/dx^2 \to D^2$, $d/dx \to D$
   $(a_0 D^2 + a_1 D + a_2) y = f(x)$ → $L(D)$
3. $y_{PI} = \dfrac{1}{L(D)} f(x)$
4. $y_{G.S} = y_{CF} + y_{PI}$

* **Case(1): $f(x) = e^{ax}$**
- let $D = a$ (only if $L(a) \neq 0$) ⟶ $y_{PI} = \dfrac{1}{L(a)} e^{ax}$

* **Case(2): $f(x) = \sin(ax)$ or $\cos(ax)$**
- let $D^2 = -a^2$ ⟶ $D = \dfrac{d}{dx}$
- multiply by the conjugate [to omit D from denominator]

* **Case(3): $f(x) = \sinh(ax)$ or $\cosh(ax)$**
- let $D^2 = a^2$ ⟶ $D = \dfrac{d}{dx}$
- multiply by the conjugate [to omit D from denominator]

* **Case(4): $f(x) = x^n$** (polynomial function)
- $y_{PI} = \dfrac{1}{L(D)} x^n$ — expand:
  - $\dfrac{1}{1-(\;)} \longrightarrow 1 + (\;) + (\;)^2 + \dots$
  - $\dfrac{1}{1+(\;)} \longrightarrow 1 - (\;) + (\;)^2 + \dots$

* **Case(5): $f(x) = e^{ax} g(x)$** & if $L(D) = 0$
- $y_{PI} = e^{ax} \dfrac{1}{L(D+a)} g(x)$
- ⊖ $\sinh x$, $\cosh x$ ⊕ & $L(D) = 0$ ⟹ $\dfrac{e^x \mp e^{-x}}{2}$
- Img $\sin x$, real $\cos x$ & $L(D) = 0$ ⟹ e^{imx} {real part, img part}
  - → $\cos mx = \mathrm{Re}\, e^{imx}$
  - → $\sin mx = \mathrm{Im}\, e^{imx}$

Laplace transform & it's inverse

*Laplace transform properties:

- *linearity $\Rightarrow A\,f_1(t) + B\,f_2(t) \longleftrightarrow A\,F_1(s) + B\,F_2(s)$
- *integration $\Rightarrow \int_0^t f(u)\,du \longleftrightarrow \frac{1}{s}F(s)$
- *time reversal $\Rightarrow f(-t) \longleftrightarrow F(-s)$
- *time scaling $\Rightarrow f(\frac{t}{a}) \longleftrightarrow a\,F(as)$

*Laplace transform theorems:

(1) $\Rightarrow L[e^{-at} f(t)] = F(s+a)$

:first shift theorem

(2) $\Rightarrow L[t\,f(t)] = -\frac{d}{ds}F(s)$

Collary $\rightarrow L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

(3) $\Rightarrow L[\frac{f(t)}{t}] = \int_s^\infty F(s)\,ds$

→only if: ($\lim_{t\to 0} \frac{f(t)}{t}$ exists)

[Convolution theorem]

(4) $\Rightarrow L^{-1}[F(s)G(s)] = f * g = \int_0^t f(t-u)\,g(u)\,du = \int_0^t f(v)\,g(t-v)\,dv$

two elementary functions

$L^{-1}[G(s)]$

$L^{-1}[F(s)]$

→Replace s with u in function & s with t-u in the other function

*to Find Laplace of piecewise function: $f(t) = \begin{cases} g(t) & 0<t<k \\ h(t) & k<t<\infty \end{cases}$

↳use the main definition: $L[f(t)] = \int_0^k e^{-st} g(t)\,dt + \int_k^\infty e^{-st} h(t)\,dt$

*if F(s) isn't an elementary function & can't be simplified

↳use theorem ② to get an elementary function then divide by t

$$\Rightarrow f(t) = \frac{L^{-1}[-\frac{d}{ds}F(s)]}{t}$$

*Laplace transform of elementary functions:

- $a \longrightarrow \frac{a}{s}$
- $e^{kt} \longrightarrow \frac{1}{s-k}$
- $\sinh(kt) \longrightarrow \frac{k}{s^2-k^2}$
- $\cosh(kt) \longrightarrow \frac{s}{s^2-k^2}$
- $\sin(kt) \longrightarrow \frac{k}{s^2+k^2}$
- $\cos(kt) \longrightarrow \frac{s}{s^2+k^2}$
- $t^n \longrightarrow \frac{n!}{s^{n+1}}$

*inverse Laplace transform of elementary functions:

- $\frac{a}{s} \longrightarrow a$
- $\frac{1}{s-k} \longrightarrow e^{kt}$
- $\frac{k}{s^2-k^2} \longrightarrow \sinh(kt)$
- $\frac{s}{s^2-k^2} \longrightarrow \cosh(kt)$
- $\frac{k}{s^2+k^2} \longrightarrow \sin(kt)$
- $\frac{s}{s^2+k^2} \longrightarrow \cos(kt)$
- $\frac{n!}{s^{n+1}} \longrightarrow t^n$

- $\frac{(s+a)}{(s+a)^2+k^2} \longrightarrow e^{-at}\cos kt$
- $\frac{k}{(s+a)^2+k^2} \longrightarrow e^{-at}\sin kt$
- $\frac{n!}{(s+a)^{n+1}} \longrightarrow e^{-at}t^n$

Solving differential equations using Laplace transform

Steps:
1. take the Laplace transform of the differential equation using the derivative property
2. Put intial Conditions into the resulting equation
3. Solve for the output variables
4. Get result from Laplace transform table
   (if result is in a form that isn't in the tables you'll need to use the inverse La place transform)

the differention property of the Laplace trans form:

- $L[y] = Y(S)$
- $L[y'] = SY(S) - Y(0)$
- $L[y''] = S^2 Y(S) - SY(0) - Y'(0)$
- generally: $L[y^n] = S^n \underbrace{Y(S)}_{L[y]} - S^{n-1} y(0) - S^{n-2} y'(0) \dots S\, y^{(n-2)}(0) - y^{(n-1)}(0)$

integration $\rightarrow$ $L\left[\int_0^t f(u)\,du\right] = \frac{1}{S} F(S)$

Convolution $\rightarrow$ $L\left[\int_0^t f(t-u)\,g(u)\,du\right] = L\left[f(t) * g(t)\right] = F(S)\cdot G(S)$

Simultaneous differential equation

using laplace transform

you wish to find $X(t)$ & $Y(t)$

$$a_1\dot{x}+a_2x+a_3y=f(t) \rightarrow (1)$$

$$b_1\dot{y}+b_2y+b_3x=g(t) \rightarrow (2)$$

- Steps

1. take the Laplace transform of the two differential equations to get $X(s)$ & $Y(s)$ in each equation
2. omit one of the two functions either $X(s)$ or $Y(s)$ by subtracting the two equation
3. Solve the differential equation
4. Substitue in the other equation & Solve it

using operator method

Ex

$$\dot{x}+y=f(t)$$

$$\dot{y}+x=g(t)$$

Steps:

(1) rewrite the equation on the operator form

$$Dx+y=f(t) \quad (1)$$

$$x+Dy=g(t) \quad (2)$$

(2) omit on of them

$$Dx+y=f(t)$$

$$(-)\ Dx+D^2y=Dg(t)$$

$$\therefore (1-D^2)y=f(t)-Dg(t)$$

differential equation of y

(3) Solve the DE

$$y_{G.S}=y_{C.F}+y_{P.I}$$

(4) subs. $y_{G.S}$ in (2)

Heaviside Unit Step Function & Periodic Function

* **Second Shift theorem:**

$L[H(t-k)\,f(t-k)] = e^{-ks}F(s)$

(OR)

$L[H(t-k)\,f(t)] = e^{-ks}\,L[f(t+k)]$

* **inverse laplace for heaviside:**

$L^{-1}[e^{-ks}F(s)] = H(t-k)\,f(t-k)$

* **Steps:**
1] Find $L^{-1}[F(s)] = f(t)$
2] find $e^{-ks} \longrightarrow H(t-k) \rightarrow ①$
3] Shift $f(t)$ to $f(t-k) \rightarrow ②$
4] multiply ①x② $\longrightarrow L^{-1}[e^{-ks}F(s)] = H(t-k)\,f(t-k)$

* **Laplace transform of periodic function:** $F(s) = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\,dt$

* **Steps:**
1] Find $f(t)$ in form of piecewise function
2] Find T (the period of the function)
3] Evalute the integral

So if $\rightarrow f(t) = A$ $a < t < b$

Function of t $g(t)$ lower boundary   upper boundary

then
- in order to calculate $L[f(t)] \rightarrow$ it must be in this form

$\rightsquigarrow f(t) = A\,[H(t-a) - H(t-b)]$

* you have to write this function to be $g(t-k)$ in order to be able to apply transform (k: a, b)

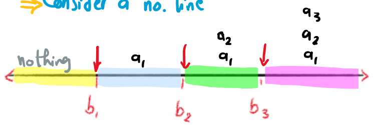
* **to write a heaviside function in form of piecewise function?**

$f(t) = a_1H(t-b_1) + a_2H(t-b_2) + a_3H(t-b_3) + \cdots$

starts at b_1 till ∞ starts at b_2 till ∞ starts at b_3 till ∞

Functions of t

⇒ Consider a no. line



① then $\rightarrow f(t) = \begin{cases} a_1 & b_1 < t < b_2 \\ a_1 + a_2 & b_2 < t < b_3 \\ a_1 + a_2 + a_3 & t > b_3 \end{cases}$

② then graph $f(t)$
Substitute with boundaries in a_n & sketch

* **to find $L[f(t)]$**

* **Steps:**
1] Find $f(t)$ in form of heaviside unit step function
2] Replace A in form of $g(t-k)$
3] Factorize $f(t)$ into simplest form
4] Apply laplace transform for each term
$L[H(t-k)\,g(t-k)] = e^{-kt}G(s)$
5] Summation of all terms $= F[s]$

Fourier Series

$$F(X) = \underbrace{\frac{a_0}{2}}_{\text{Constant term}} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right)$$

Amplitudes

Harmonics of $f(x)$

- $a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\,dx$ (average)
- $a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{n\pi x}{T}\right)dx$
- $b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\left(\frac{n\pi x}{T}\right)dx$

* Fourier series for the even function is the Cosine Series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{T}x$$

- $a_0 = \frac{2}{T}\int_{0}^{T} f(x)\,dx$
- $a_n = \frac{2}{T}\int_{0}^{T} f(x)\cos\left(\frac{n\pi x}{T}\right)dx$
- $b_n = 0$

* Fourier series for the odd function is the Sine Series :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{T}x$$

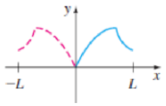
- $a_0 = 0$
- $a_n = 0$
- $b_n = \frac{2}{T}\int_{0}^{T} f(x)\sin\left(\frac{n\pi x}{T}\right)dx$

Half-Range Expansion

* if $f(x)$ is defined on the interval $(0,L)$

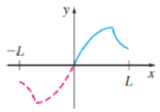
1] Cosine Fourier Series : → reflect the graph of f about y-axis onto $(-L,0)$

→ the function is now even on $(-L,L)$ & it's represented by a Cosine Fourier series



2] Series Fourier Series : → reflect the graph of f about the origin onto $(-L,0)$

→ the function is now odd on $(-L,L)$ & it's represented by a Sine Fourier series



Partial differential Equations

1. **the order:** it's the order of the highest derivative occuring in the equation
2. **Number of Variables:** it's the no. of independent variables of the unknown function in the PDE.
3. **Linearity:** it means all terms involving u & any of its derivatives can be expressed as a linear combination in which constants of u-terms are independent of u.

   it's an expression constructed from a set of terms by multiplying each term by a constant & adding the results.
4. **homogeneity:** if all terms of a PDE contain the function u or its partial derivatives then it's called homogeneous

types of Second order linear PDEs:

- A is Coefficient of u_{xx}
- B is Coefficient of u_{xy}
- C is Coefficient of u_{yy}

$$B^2(x,y) - 4A(x,y)C(x,y) \begin{cases} = 0 & \textit{Parabolic} \\ > 0 & \textit{Hyperbolic} \\ < 0 & \textit{Elliptic} \end{cases}$$

* General Solution:

the equation contains only one of u derivatives (u_x, u_y, u_{xy}, u_{xx}, u_{yy})

⇓

1 direct integration

- take the derivative of u alone in one side
- integrate both side W.R.T the parameter of u derivative
- the general Solution of u = function of this parameter + function of other Variables (arbitary Function)

the equation Contains u & one of its derivatives (u_x, u_y, u_{xx}, u_{yy}) but with respect to certain one independent variable

⇓

2 Converts it to ODE

- Compare the PDE to ODE ($y' \to u_x, u_y$; $y'' \to u_{xx}, u_{yy}$)
- Solve as ODE
- replace the arbitrary constants (c_1, c_2) in ODE with arbitrary functions ($h(x)$, $g(y)$) & the $Y_{G.s} \to U_{G.s}$

the equation contains derivatives of u of the Same order of differentiation

⇓

3 Assuming

- Assume that $u(x,y) = f(ax+y)$ ($u_x = af'$, $u_y = f'$, $u_{xx} = a^2f''$, $u_{yy} = f''$, $u_{xy} = af''$)
- Replace u & it's derivatives with f & it's derivatives
- Solve the DE & Find the value of a

* Finite String - Wave equation: $u_{tt}(x,t) = c^2 u_{xx}$ ($0<x<L$, $t>0$)

use general solution to solve if the Conditions met.

- **Boundary Condition** ⇒ $u(0,t) = 0$, $u(L,t) = 0$
- **initial Condition** ⇒ $u(x,0) = f(x)$, $u_t(x,0) = g(x)$
- **General Solution** ⇒

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

→ if $f(x)$, $g(x)$ are sine functions we can use method of Compare with the series instead of integration

* Method of Separation (with an example) (wave equation $u_{tt} = c^2u_{xx}$)

Steps →

1) Assume the solution of the form: $u(x,t) = X_{(x)}T_{(t)}$ ($u_{xx} = TX''$, $u_{tt} = T''X$)
2) Substitute with solution into the equation: $T''X = c^2TX''$
3) Separate each variable with it's derivative: $\frac{X''}{X} = \frac{T''}{c^2T}$
4) Equate both sides with constant k: $\frac{X''}{X} = \frac{T''}{c^2T} = k$
5) Simplify both sides: $X'' - kX = 0$, $T'' - kc^2T = 0$
6) Solve the ODEs
7) Study the Cases of the constant k to shun from the trivial Solutions ($k = m^2$, $k = 0$, $k = -m^2$)
8) Combine the Solutions to the ODEs to get $u(x,y)$

*Remember:

1] Completing the Square:

$$ax^2+bx \xrightarrow{\div a} x^2+\frac{b}{a}x \xrightarrow[\frac{1}{2}\text{Co. of } x \;\; \& \; -(\text{Co. of } x)^2]{\sqrt{(x)^2}\;\&} \left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2$$

2] Partial Fraction: $= \frac{P(x)}{q(x)}$

Step Zero: if
- degree of $P(x)$ > degree of $q(x)$ → long divison first
- degree of $P(x)$ < degree of $q(x)$ → partial fraction

step 1]: factorize the denominator ($q(x)$) to simplest form

step 2]: write out the partial fraction of each term

ex:

$$\frac{1}{(x+2)^3}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+3)^3}$$

$$\frac{1}{(x)\underbrace{(x^2+2x+2)}_{\text{quad.}}}=\frac{A}{x}+\frac{Bx+C}{x^2+2x+2}$$

step 3]: multiply by denominator for both sides

step 4]: Calculate the Constants

* **Cover up rule:** Only if the fraction denominator is composed of linear functions

Ex:

$$\frac{f(x)}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

For each constant you remove the partial fraction denominator term from orginial fraction & subs. with constant which makes the term $=0$

$$\therefore A=\frac{f(a)}{(a-b)(a-c)} \;\;\&\;\; B=\frac{f(b)}{(b-a)(b-c)} \;\;\&\;\; C=\frac{f(c)}{(c-a)(c-b)}$$

* Useful Relations:

Product to Sum:

$$\sin(\;)\cos(\;)=\frac{1}{2}\left[\sin(+)+\sin(-)\right]$$

$$\cos(\;)\sin(\;)=\frac{1}{2}\left[\sin(+)-\sin(-)\right]$$

$$\cos(\;)\cos(\;)=\frac{1}{2}\left[\cos(+)+\cos(-)\right]$$

$$\sin(\;)\sin(\;)=\frac{1}{2}\left[\cos(-)-\cos(+)\right]$$

sincos $=\frac{1}{2}$sin $++-$

cossin $=\frac{1}{2}$sin $+--$

coscos $=\frac{1}{2}$cos $++-$

sinsin $=\frac{1}{2}$cos $--+$

Addition or Subt.:

$$\sin(a\pm b)=\sin(a)\cos(b)\pm\cos(a)\sin(b)$$

$$\cos(a\pm b)=\cos(a)\cos(b)\mp\sin(a)\sin(b)$$

double angles:

$$\sin(2x)=2\sin x\cos x$$

$$\cos(2x)=\cos^2x-\sin^2x=1-2\sin^2x=2\cos^2x-1$$

$$\tan(2x)=\frac{2\tan x}{1-\tan^2x}$$

* Quick integration by parts technique:

$$\int \underbrace{u}_{\text{must be Polynomial Function}}\,dv = uv - u'v^{-1} + u''v^{-2} - u'''v^{-3} + \cdots$$

d/dx

v^{-1}: $\int v\,dx$

v^{-2}: $\int v^{-1}\,dx$

v^{-3}: $\int v^{-2}\,dx$

*Remember:

For any integer (n) (0, 1, 2, ...)

$$\left\{\begin{aligned}&\sin(n\pi)=0\\&\cos(n\pi)=(-1)^n\end{aligned}\right\}$$

n

(we can separate it to odd series & even series)

- even: $2n$
- odd:
  - $2n-1$ (Counter starts from 1)
  - $2n+1$ (Counter starts from 0)