Lecture 1
Special functions
"Gamma and Beta"

"attendance is taken"

10% → Quizzes
20% → midterm
→ attendance
10 → assign

Definition of Gamma function:

$$\boxed{\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \quad , x > 0}$$

* It's a function of x only because it's a definite integration
  t is a dummy variable.

* It's special because you need to calculate when you want to know the value at a certain value (not just substitution)

* You need to see the limit

* The integration need to be convergent (not divergent (∞))
  i.e domain $x > 0$
* when the integration diverges, it's improper integrals, when $x \not> 0$

* You need all kinds of functions we ever took

$$\int_1^{\infty} \frac{1}{x}\, dx = \ln x \Big|_1^{\infty}$$

$\ln \infty - \ln 1$

$\infty - 0$

$= \infty$

$\therefore$ Div

$$\int_1^{\infty} \frac{1}{x^2}\, dx = -\frac{1}{x} \Big|_1^{\infty}$$

$= -\frac{1}{\infty} + 1$

$= 0 + 1 = 1$

$\therefore$ Conv.

Properties of the gamma function:

1) $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty y^{-1} e^{-y^2} \cdot 2y\,dy$

$= 2\int_0^\infty e^{-y^2}\,dy = \boxed{\sqrt{\pi} = \Gamma(\frac{1}{2})}$

$$\boxed{\text{Thus: } \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$$

remember:

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$

2) $\boxed{\Gamma(1) = 1}$ proof: let $x = 1$ $\therefore \Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$

$= -\frac{1}{e^t} = -\frac{1}{\infty} + \frac{1}{1}$

3) $\boxed{\Gamma(x+1) = x\,\Gamma(x)}$

proof: $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$ (solved using by parts)

$= \cancel{-t^x e^{-t}\Big|_0^\infty} + x\int_0^\infty t^{x-1} e^{-t}\,dt$

$\infty \cdot 0 \Rightarrow$ limits

$= x\,\Gamma(x)$

$u = t^x \quad dv = e^{-t}\,dt$

$du = x t^{x-1}\,dt \quad v = -e^{-t}$

$\lim_{t\to\infty} -\frac{t^x}{e^t} = \frac{\infty}{\infty}$ l'Hôp

$= \frac{x t^{x-1}}{e^t} = 0 \quad \boxed{\forall x}$

Note that when you combine property 2 and 3 we get

$\Gamma(2) = 1\,\Gamma(1) = 1$

$\Gamma(3) = 2 \cdot 1$

$\Gamma(4) = 3.2.1 = 3!$

we get

$\boxed{4-\ \Gamma(n+1)=n! \quad (\text{for } n \text{ positive integer})}$

thus $0! = 1$ because $0! = \Gamma(1) = 1$

i.e: Gamma function is a generalization of the factorial function.

Factorial → integers

Gamma funct → any value

Pro: Gamma F allowed us to solve numerically higher orders without reccuring do many by parts integratio

exp: show that $L\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}$

sol: $L\{t^{-1/2}\} = \int_0^\infty t^{-1/2} e^{-st} dt$, $s>0$

let $u = st$, $dt = \frac{1}{s} du$

Remember: $L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt = F(s)$

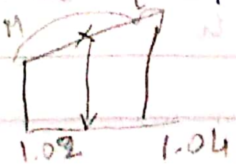
$L\{t^{-1/2}\} = \int_0^\infty \left(\frac{u}{s}\right)^{-\frac{1}{2}} e^{-u} \frac{1}{s} du$

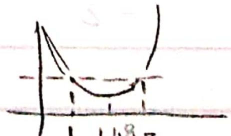
$= \int_0^\infty \frac{u^{-1/2} e^{-u}}{\sqrt{s}} du$

$\therefore \frac{\Gamma(\frac{1}{2})}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$

exp2: the table (solving by linear interpolation)

exp: the table has 1.02 & 1.03, we want 1.03
So we get their both value and average them
$(0.988844 + 0.978438) : 2 = 0.983641$





↳ even if the value isn't within the range we can get it

exp: $\Gamma(4.3) = \Gamma(3.3+1) = 3.3\,\Gamma(3.3)$

$= 3.3 \times 2.3\,\Gamma(2.3) = 3.3 \times 2.3 \times 1.3\,\Gamma(1.3)$

↳ $= 0.897471$ from table

$\therefore \Gamma(4.3) = 3.3 \times 2.3 \times 1.3 \times 0.897471 = 8.855346$

↳ If we want to find gamma for $x\,[0,1]$ ie for the interval

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \qquad \text{exp: } \Gamma\left(\frac{1}{4}\right) = \frac{\Gamma(1.25)}{1/4}$$

$\Gamma(1/4) = 4\,\Gamma(1.25)$

↳ from table $= \dfrac{0.908521 + 0.904397}{2}$

$= 4 * 0.906459 = 3.625836$

* Definition of the gamma function for negative values of x

Notice at $x=0$ $\quad \Gamma(0) = \int_0^\infty \frac{1}{t} e^{-t}\,dt$ (due) always undefined

but: If we use $\Gamma(x+1) = x\,\Gamma(x)$ (we didn't specify a domain for x) no restriction on x

$\rightarrow \Gamma(0) = \dfrac{\Gamma(1)}{0} = \dfrac{1}{0} = \infty$

↳ defined as infinity

From which we find: (Valid for all x

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad x+1 > 0$$

exp. $\Gamma(-1) = \dfrac{\Gamma(0)}{-1} = -\infty \qquad \Gamma(-3) = \dfrac{\Gamma(-2)}{-3} = -\infty$

$\Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty \qquad \Gamma(-4) = \dfrac{\Gamma(-3)}{-4} = \infty$

* Thus for negative integers, the gama is either $\pm\infty$

→ However if x is a negative fraction, it'll have a value

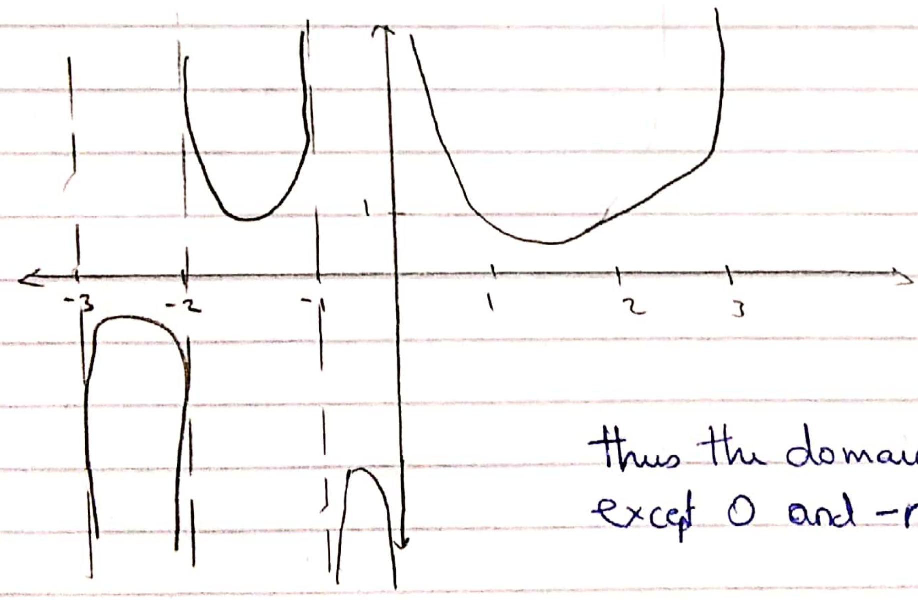
exp: $\Gamma'(-\frac{1}{2}) = \dfrac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$

exp: $\Gamma'(-3.4) = \dfrac{\Gamma'(x+1)}{x} = \dfrac{\Gamma'(-2.4)}{-3.4} = \dfrac{\Gamma'(-1.4)}{-3.4 \times -2.4}$

$= \dfrac{\Gamma'(-0.4)}{-3.4 \times -2.4 \times -1.4} = \dfrac{\Gamma'(0.6)}{-3.4 \times -2.4 \times 1.4 \times -0.4}$

$= \dfrac{\Gamma'(1.6)}{-3.4 \times -2.4 \times 1.4 \times -0.4 \times 0.6}$ → From table =

* Sketching Gamma function (+ve & -ve values)



thus the domain of x is $]-\infty, \infty[$ except 0 and $-n$