Lecture 5
Bessel function

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (\lambda^2 t^2 - \nu^2)y = 0$$

Bessel differential equation
- order ν (not to get confused with a ODE order)
- parameter λ

exp : $x^2 y'' + x y' + (25x^2 - \frac{1}{4})y = 0$

↳ order $\frac{1}{2}$ parameter $\lambda = 5$

⇒ We will peform a substitution for simplification making $\lambda = 1$

$\boxed{\text{we get}}$ $x^2 y'' + x y' + (x^2 - \nu^2)y = 0$

→ Bessel differential equation of order ν and parameter 1

→ this can be solved using series solution.

$$p(x) = \frac{1}{x} \quad , \quad q = \frac{x^2 - \nu^2}{x^2} \quad \text{at } x_0 = 0 \text{ singular point}$$

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$; $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$

$$y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

$$x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + (x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{n+s} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{\boxed{n+s+2}} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+s}$$

all series should now start from zero

remember $y = c_1 y_1 + c_2 y_2$

$$a_0\left(s(s-1)+s-\nu^2\right)x^s + a_1\left((s+1)s+(s+1)-\nu^2\right)x^{s+1} + \sum_{n=2}^{\infty}\left(a_n\left[(n+s)(n+s-1)+(n+s)-\nu^2\right]+a_{n-2}\right)x^{n+s} = 0$$

Coefficient of $x^s = 0 \Longrightarrow a_0\left(s(s-1)+s-\nu^2\right)=0 \quad a_0 \neq 0$

$\therefore s^2-\nu^2=0 \quad s^2=\nu^2$

$$* \boxed{s_1=\nu \ \& \ s_2=-\nu} *$$

Coff of $x^{s+n}=0$

$$a_n\left((n+s)(n+s-1)+(n+s)-\nu^2\right)+a_{n-2}=0$$

$$a_n\left((n+s)^2-(n+s)+(n+s)-\nu^2\right)=-a_{n-2}$$

to be able to find simple form for recurrence relation, it needs to be quadratic in $(n+s)$

$$a_n = \frac{-1}{((n+s)-\nu)((n+s)+\nu)}a_{n-2} \; ; \; n \geq 2$$

For $s_1=\nu$

$$a_n = \frac{-1}{n(2\nu+n)}a_{n-2} \; ; \; n\geq 2$$

$$a_2 = \frac{-1}{(2)(2\nu+2)}a_0$$

$$a_4 = \frac{(-1)^2}{(2)(4)(2\nu+2)(2\nu+4)}a_0$$

$$a_6 = \frac{-1}{(6)(2\nu+6)}a_4$$

$$a_{2n} = \frac{(-1)^n}{(2\times4\times6\times\cdots\times2n)((2\nu+2)(2\nu+4)\cdots(2\nu+2n))}a_0$$

$$a_{2n} = \frac{(-1)^n}{2^{2n}n!\times[(\nu+1)(\nu+2)\cdots(\nu+n)]}a_0$$

* to find compact form remember $\Gamma(\nu+n+1)=(\nu+n)\Gamma(\nu+n)$

$$a_{2n} = \frac{(-1)^n\,\Gamma(\nu+1)}{2^{2n}\,n!\,\Gamma(n+\nu+1)}a_0$$

$n \geq 0$

$$y_1 = \sum_{n=0}^{\infty}a_n(\nu)\,x^{n+\nu}$$

$$= \sum_{n=0}^{\infty}a_{2n}(\nu)\,x^{2n+\nu}$$

we took this arbitrary constant.

$\Rightarrow$ take : $2^{\nu}\Gamma(\nu+1)a_0 = 1$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu+1) x^{2n+\nu} 2^{\nu}}{2^{2n+\nu} \times n! \times \Gamma(n+\nu+1)} a_0$$

$$\boxed{y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \equiv J_\nu(x)}$$

Bessel function of the 1st Kind

we solve again for $S_2 = -\nu$

$$\boxed{y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu} \equiv J_{-\nu}(x)}$$

$y_{gs} = C_1 y_1 + C_2 y_2$ they should both (y_1 & y_2) be independent

$y_{gs}(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$

$y_{gs}(t) = C_1 J_\nu(\lambda t) + C_2 J_{-\nu}(\lambda t)$

If $\nu = 0$ we will have only one solution

If $\nu = N$ integer

* If ν integer they will be dependent and we will need a second linearly indep solution

Consider Bessel fn

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

$J_{-N}(x) = (-1)^N J_N(x)$

$\therefore C_1 J_N(x) + C_2 J_{-N}(x)$ doesn't represent the general solution

* Second linearly independent solution

$$Y_N(x) = \lim_{\nu \to N} \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

$(x^2 - 0)$

exp: $x^2 y'' + x y' + x^2 y = 0$ $\quad \nu = 0 \to$ integer

$$y = C_1 J_0(x) + C_2 Y_0(x)$$

- $x^2 y'' + x y' + 4(x^4 - 1) y = 0$

let $x^2 = t$ $\quad 2x\,dx = dt$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2x \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= 2\frac{dy}{dt} + 2x \frac{d}{dx}\frac{dy}{dt}$$

$$= 2\frac{dy}{dt} + 2x\,\dot{y}\,(2x)$$

$$4x^4 \frac{d^2y}{dt^2} + 2x^2 \frac{dy}{dt} + 2x^2 \frac{dy}{dt}$$

$$+ 4(x^4 - 1) y = 0$$

$$t^2 \ddot{y} + t\dot{y} + (t^2 - 1) y = 0$$

$$y = C_1 J_1(t) + C_2 Y_1(t)$$

$$y = C_1 J_1(x^2) + C_2 Y_1(x^2)$$

داخل في الامتحان

example 4.