PHM212s: Special Functions, Complex Analysis & Numerical Analysis

Name:	**ID:**	**Deadline: Week 9**

Please, Solve each problem in its assigned place ONLY (the empty space below it)

Bessel Functions

1. Solve in terms of Bessel functions the following differential equations:

a) $x^2\ y'' + x\ y'\ + (x^2 - 9) y = 0$

b) $x^2\ y'' + x\ y'\ + (x^2 - 8) y = 0$

c) $x^2\ y'' + x\ y'\ + (3x^2 - 4) y = 0$

d) $x^2\ y'' + x\ y'\ + 4(x^4 - n^2) y = 0 \quad , \quad n \in I$

e) $x\ y'' + 3\ y'\ + x\ y = 0$

f) $4\,x\,y'' + 4\,y' + y = 0$

2. Find the solution of $x^2\,y'' + x\,y' + (4x^2 - 1)\,y = 0$ which is bounded at $x = 0$ and $y\,(2) = 5$

3. Show that:

a) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$

b) $Y_{1/2}(x) = -\sqrt{\dfrac{2}{\pi x}}\cos x$

c) $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\sin x}{x} - \cos x\right)$

d) $J_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\cos x}{x} + \sin x\right)$

4. Show that $y = x^n J_n(x)$ is a solution for the differential equation $x\,y'' + (1-2n)\,y' + x\,y = 0$ using two different methods.

1st Method:

2nd Method:

5. Show that

a) $J_n''(x) = \frac{1}{4}\left(J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x)\right)$

b) $\frac{d}{dx}\left(J_n^2(x)\right) = \frac{x}{2n}\left(J_{n-1}^2(x) - J_{n+1}^2(x)\right)$

c) $\frac{d}{dx}\left(x\ J_n(x)\, J_{n+1}(x)\right) = x\left(J_n^2(x) - J_{n+1}^2(x)\right)$

6. Solve the following integrals in terms of Bessel Functions:

a) $\int x^3\ J_2(x)\ dx$

b) $\int x^{-4}\ J_5(x)\ dx$

c) $\int x^4\ J_1(x)\ dx$

d) $\int \sqrt{x}\ J_{1/2}(x)\ dx$

e) $\int x^{-2}\ J_2(x)\ dx$

7. Solve in terms of Bessel functions the following differential equations:

$$y'' + x\,y = 0$$

8. Solve in terms of Bessel functions the following differential equations:

$$x\, y'' + y = 0$$

Extra Problems

9. Use the generating function of Bessel functions to show that:

a) $J_0(x) + 2J_2(x) + 2J_4(x) + \dots = 1$

b) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \dots$

c) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \dots$

10. Prove that $J_n(x) = \dfrac{1}{2\pi}\int_0^{2\pi} \cos\left(n\theta - x\sin\theta\right)\, d\theta$ for n is a non-negative integer.

11. Show that $\int x^m\ J_n(x)\ dx$ can be evaluated in a closed form if $m+n$ is odd and in terms of $\int J_0(x)\ dx$ (which can't be integrated in closed form) if $m+n$ is even where n and m are integers.

Best wishes,
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