Analytic Functions

$\nabla f = f_x \hat{i} + f_y \hat{j}$

$f'(x) = \frac{dy}{dx} = \frac{\text{change in } y}{\text{change in } x}$

Definition:

A function $f(z)$ is called analytic at a point z_0 if it is differentiable at z_0 and on a neighborhood of z_0.

$f_x = \frac{\partial f}{\partial x}$ in u ?? $\rightarrow \varepsilon$ $\quad f_y = \frac{\partial f}{\partial y}$ in v $\rightarrow i\varepsilon$

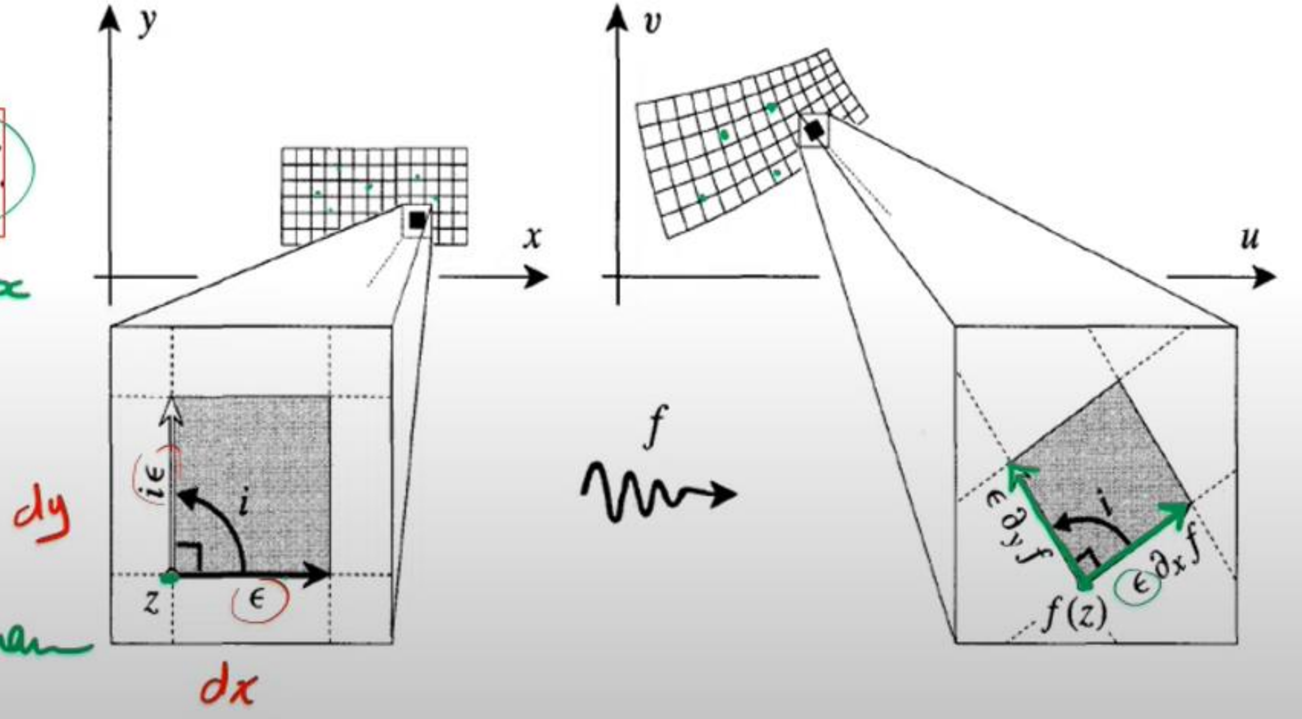
$$\boxed{(\text{change in } x)\cdot(\text{rate of change with } x \text{ of the image } f) = \epsilon\, \partial_x f.}$$

$\epsilon . f_x$

$$\boxed{i\partial_x(u + iv) = \partial_y(u + iv)}$$

$i u_x - v_x = u_y + i v_y$

$\left.\begin{array}{l} u_y = -v_x \\ v_y = u_x \end{array}\right\}$ Cauchy-Riemann



Harmonic Functions

Definition:

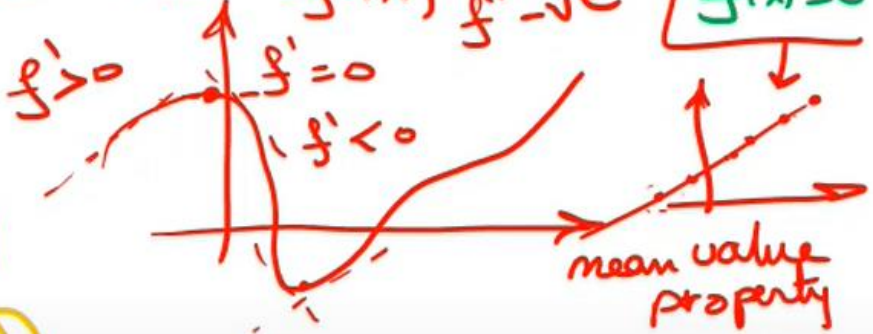
A function $u(x,y)$ is called harmonic on a certain domain $''D''$ if it satisfies Laplace's equation $\boxed{u_{xx} + u_{yy} = 0}$ on D

$\Delta \; \boxed{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0}$

linear function

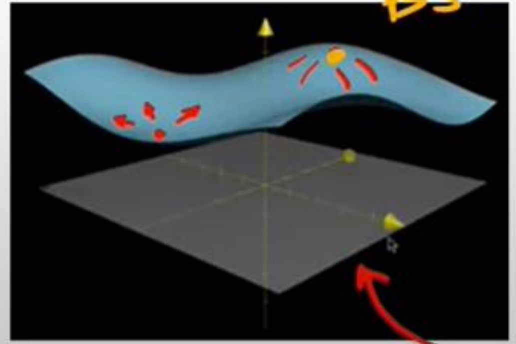
$f''(x) = 0 \xrightarrow{\int dx} f'(x) = c \xrightarrow{\int dx} f(x) = cx + k$

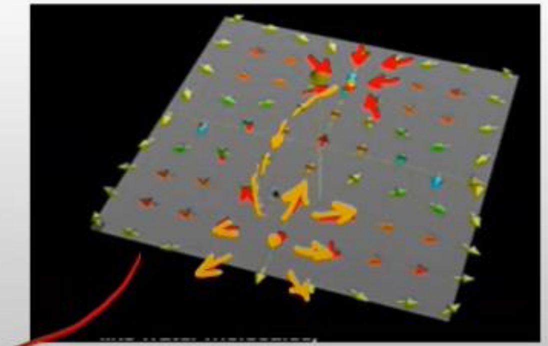
$f(x)$ $\quad f''$ -ve $\quad \boxed{f''(x) = 0}$



$f(x,y)$ $\qquad \nabla f = f_x \hat{i} + f_y \hat{j}$ $\qquad \Delta \to \nabla \cdot (\nabla f)$

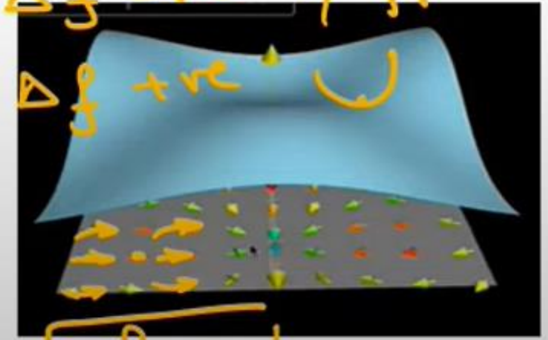
Δf -ve



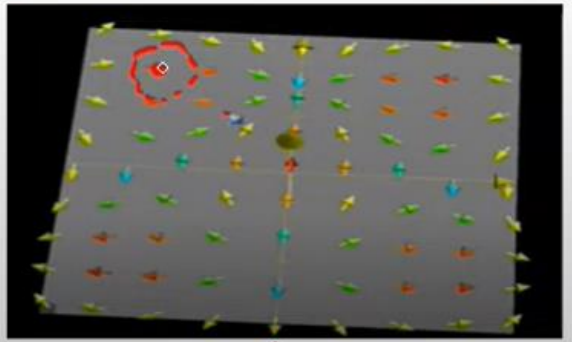


Δf -ve

Δf +ve



$\boxed{\Delta f = 0}$



mean value property

Harmonic $\longrightarrow$ mean value property

Differentiation in polar coordinates

Next time ☺

In polar coordinates $f(z) = u(r,\theta) + i\,v(r,\theta)$

Cauchy – Riemann equations are $r\,u_r = v_\theta \quad \& \quad r\,v_r = -u_\theta$

→ you can prove it!..

$U(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

Proof:

$\boxed{U_x = V_y}$ ① $\boxed{U_y = -V_x}$ ②

$U_r = U_x \cos\theta + U_y \sin\theta$

$U_y \to -V_x$

$V_\theta = V_x \frac{\partial x}{\partial \theta} + V_y \frac{\partial y}{\partial \theta} = V_x(-r\sin\theta) + V_y(r\cos\theta)$

$V_x \to -U_y$ ②, $V_y \to U_x$ ①

$= U_x\, r\cos\theta + U_y\, r\sin\theta$

$V_\theta = r\,(U_x\cos\theta + U_y\sin\theta)$

U_r

$\boxed{V_\theta = r\,U_r}$

U, V: x, y → r, θ

Differentiation in polar coordinates

$U_x = V_y$, $V_x = -U_y$ $\quad U_{xx} + U_{yy} = 0$

Laplace's equation is $\boxed{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0}$

$\boxed{r . u_r = v_\theta \quad \& \quad r v_r = - u_\theta}$

① ②

$\frac{\partial ①}{\partial r} + \frac{\partial ②}{\partial \theta}$

$U_r + r U_{rr} = V_{\theta r}$

$r V_{r\theta} = -U_{\theta\theta}$ $\quad \frac{-U_{\theta\theta}}{r}$

$\equiv \; r U_{rr} + U_r + \frac{1}{r} U_{\theta\theta} = 0$

$\boxed{f'(z) = \frac{r}{z}\left(u_r + i\, v_r\right) = \frac{1}{z}\left(v_\theta - i\, u_\theta\right)}$

$f'(z) = U_x + i V_x$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$f'(z) = U_x + i V_x$

$= (\cos\theta - i\sin\theta)(U_r + i V_r)$

$\cos\theta - i\sin\theta = e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{r}{z}$

$U_x = U_r \frac{\partial r}{\partial x} + U_\theta \frac{\partial \theta}{\partial x}$

$= U_r \cos\theta - U_\theta \frac{\sin\theta}{r}$

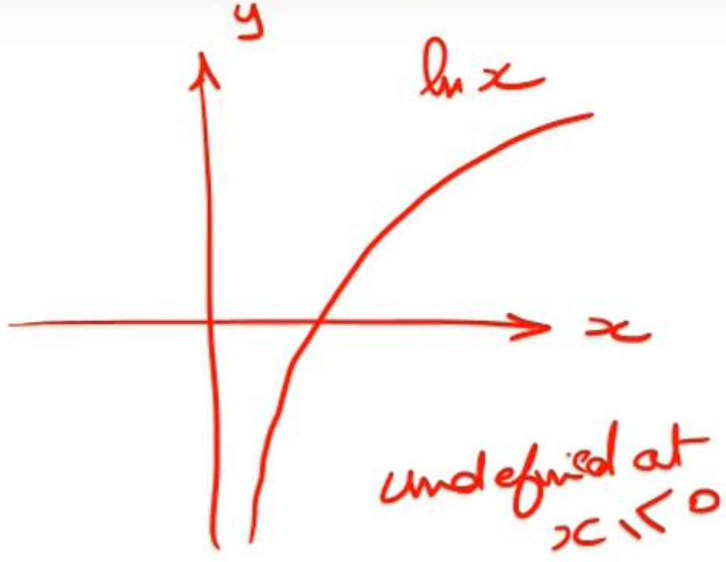
$r V_r$

$U \left\langle \begin{matrix} r \left\langle \begin{matrix} x \\ y \end{matrix} \right. \\ \theta \left\langle \begin{matrix} x \\ y \end{matrix} \right. \end{matrix} \right.$

$V_x = U_r(-\sin\theta) + r V_r \left(\frac{\cos\theta}{r}\right)$

Example 7:

Show that $\dfrac{d}{dz}(\ln z) = \dfrac{1}{z}$



Solution:

$$\ln z = \ln(r\,e^{i\theta}) = \overset{u}{\ln r} + i\,\overset{v}{\theta}$$

$$u_r = \frac{1}{r} \quad , \quad v_\theta = 1 \quad \Rightarrow \quad r\,u_r = v_\theta$$ ✓

$$v_r = 0 \quad , \quad u_\theta = 0 \quad \Rightarrow \quad r\,v_r = -\,u_\theta$$ ✓

$\Rightarrow f(z) = \ln z$ is differentiable everywhere except at $z = 0$ or the negative real axis.

$$f'(z) = \frac{r}{z}\left(u_r + i\,v_r\right) = \frac{r}{z}\left(\frac{1}{r}\right) = \frac{1}{z}$$

Functions of Complex Variables

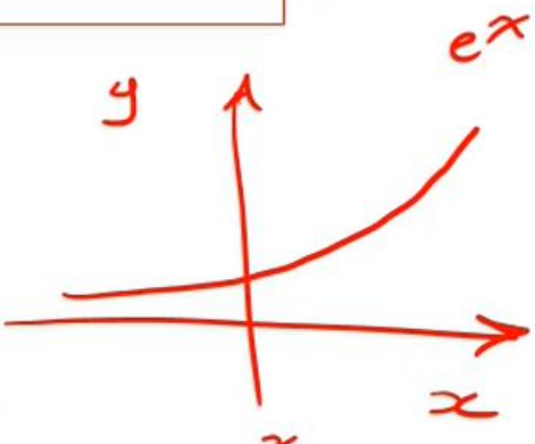
Exponential Function

$|w| = e^{x}$

$\arg(w) = y$

$$w = e^{x+iy} = e^{x}\,(\cos y + i \sin y)$$

$$u(x, y) = e^{x}\cos y \qquad v(x, y) = e^{x}\sin y$$



$e^x \neq 0$

$\therefore\ |w| \neq 0$

Caushy - Reeman Equns

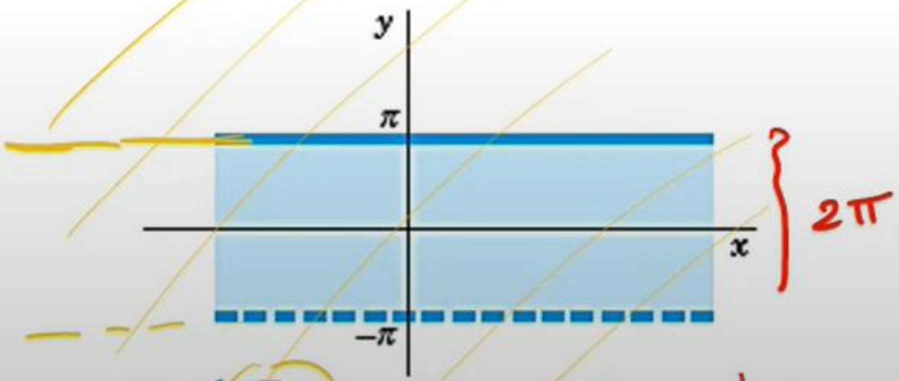
Entire Function

analytic, everywhere Differential

Remark!

Periodic Function

Principal region



$-\pi < \arg(z) < \pi$

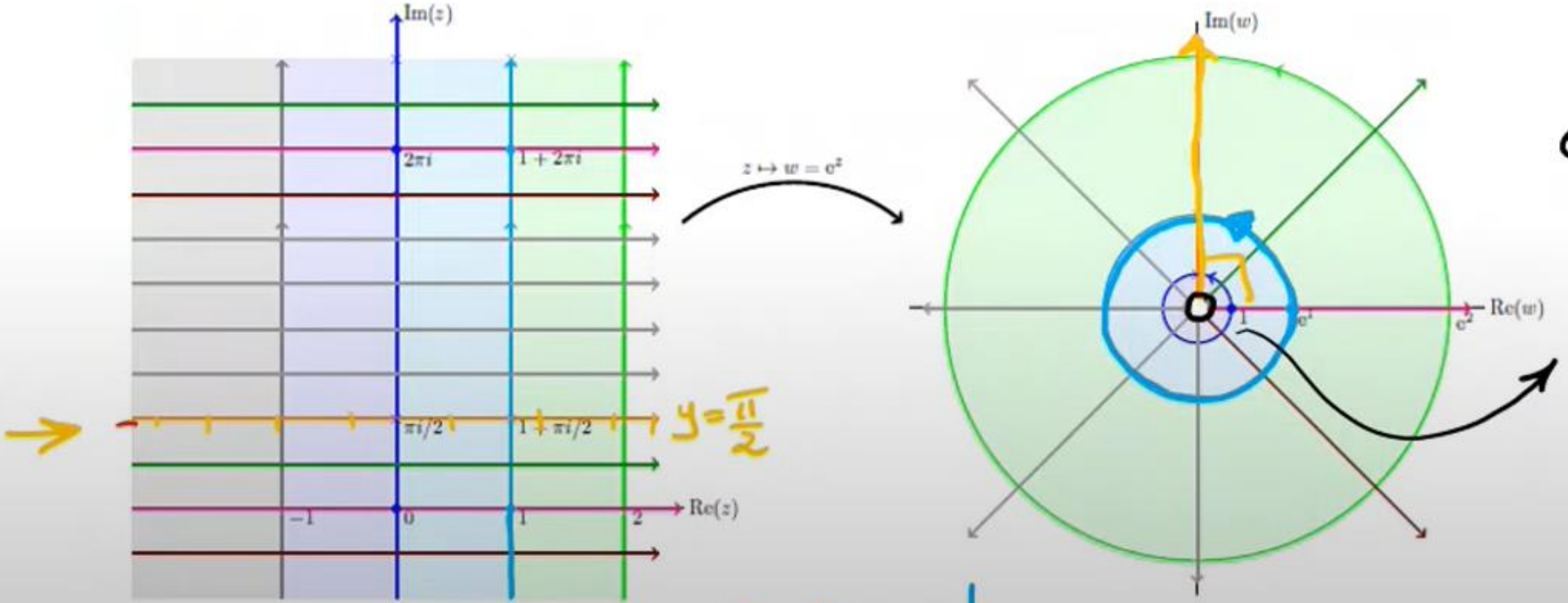
single value function inverse

Functions of Complex Variables

Exponential Function

$|w| = e^x$ $\quad$ $\arg(w) = y$

$$w = e^{x+iy} = e^{x}\ (\cos y + i \sin y)$$



$e^x \neq 0$

$|w| \neq 0$

$x = 1$ $\quad$ $|w| = e^1$

$-\pi < \arg(w) \le \pi$

Example 1:

Find all the values of z such that $e^{4z} = 1+i$ and specify principle value.

Solution:

Take ln to both sides

$$4z = ln(1+i) = ln(\sqrt{2}\ e^{i(\frac{\pi}{4}+2\pi k)}) = ln\sqrt{2} + i(\tfrac{\pi}{4}+2\pi k)$$

$$\Rightarrow z = \frac{1}{4}\left(ln\sqrt{2} + i\left(\tfrac{\pi}{4}+2\pi k\right)\right)$$

Functions of Complex Variables

Trigonometric and Hyperbolic Function

$$e^{ix} = \cos x + i \sin x, \quad (1)$$ Euler

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i \sin x = e^{-ix} \quad (2)$$

$\frac{(1)+(2)}{2}$

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}),$$

$\frac{(1)-(2)}{2i}$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

$$ch x = \frac{e^{x} + e^{-x}}{2}$$

$$sh x = \frac{e^{x} - e^{-x}}{2}$$

$$\sin iz = i \sinh z, \qquad \cos iz = \cosh z, \qquad \tan iz = i \tanh z$$
$$\sinh iz = i \sin z, \qquad \cosh iz = \cos z, \qquad \tanh iz = i \tan z$$

$$ch\, ix = \cos x$$

$$sh\, ix = i \sin x$$

Relation Between Trig. with Hyperbolic with exp.

Example 2:

$\sin x \sin y \rightleftharpoons -\operatorname{sh} x \operatorname{sh} y$

Trig. Hyperbolic

Show that :

a- $\cos z = (\cos x \cosh y - i \sin x \sinh y)$

b- $|\cos z|^2 = (\cos^2 x + \sinh^2 y)$

Solution : sol.1

a- L.H.S. $(\cos z) = \cos(x+iy) = \dfrac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$

$L.H.S = \frac{1}{2}\left(e^{ix} e^{-y} + e^{-ix} e^{y}\right) = \frac{1}{2}\left[(\cos x + i\sin x) e^{-y} + (\cos x - i \sin x) e^{y}\right]$

$= \underbrace{\dfrac{e^{y} + e^{-y}}{2}}_{ch y} \cos x - i \underbrace{\left(\dfrac{e^{y} - e^{-y}}{2}\right)}_{sh y} \sin x = R.H.S.$

sol 2

$\cos z = \cos(x+iy)$

$\cos x \underbrace{\cos iy}_{ch y} - \sin x \underbrace{\sin iy}_{i sh y}$

b- $|\cos z|^2 = (\cos x \, ch y)^2 + (\sin x \, sh y)^2$

$= \cos^2 x \, ch^2 y + \sin^2 x \, sh^2 y$

$(1 + sh^2 y) = \cos^2 x + \cos^2 x \, sh^2 y + \sin^2 x \, sh^2 y$

$\cos^2 x + \sin^2 x = 1$

$ch^2 x - sh^2 x = 1$

$= \cos^2 x + sh^2 y$

Example 3:

Show that $f(z) = \sin z$ is differentiable everywhere and $\frac{d}{dz}(\sin z) = \cos z.$

Solution:

$$w = \sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$u_x = \cos x \cosh y \;,\quad v_y = \cos x \cosh y \qquad \Rightarrow u_x = v_y \;\; everywhere$$

$$u_y = \sin x \sinh y \;,\quad v_x = -\sin x \sinh y \qquad \Rightarrow u_y = -v_x \;\; everywhere$$

$$\Rightarrow f(z) = \sin z \;\; \text{is differentiable everywhere.}$$

$$\Rightarrow f'(z) = u_x + i v_x = \cos x \cosh y - i \sin x \sinh y$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos(x + iy)$$

$$\boxed{\therefore \; \frac{d}{dz}(\sin z) = \cos z.}$$

Example 4:

Find all the values of z such that $\sin z = \cosh 2$

Solution:

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$ $-1 < \sin x < 1$

$$\Rightarrow \sin x \cosh y + i \cos x \sinh y = \cosh 2$$

Real Im

$$\cos x \sinh y = 0 \quad (1) \qquad \text{and} \qquad \sin x \cosh y = \cosh 2 \quad (2)$$

(1)

odd number

$$\cos x \sinh y = 0 \quad \Rightarrow \boxed{y=0} \ \text{or} \ \boxed{x=(2n+1)\tfrac{\pi}{2}}$$

in (2) $\quad y \neq 0 \Rightarrow \sin x = \cosh 2 \quad$ (Refused) $\quad -1 < |\sin x| < 1$, $\quad ch\,2 > 1$

$$x=(2n+1)\tfrac{\pi}{2} \Rightarrow \sin(2n+1)\tfrac{\pi}{2}\cosh y = \cosh 2$$

$$(-1)^n \cosh y = \cosh 2$$ $\quad 1, -1, 1, -1 \rightarrow (-1)^n = (-1, 1, -1, \ldots)\ n = 1, 2, 3 \ldots$

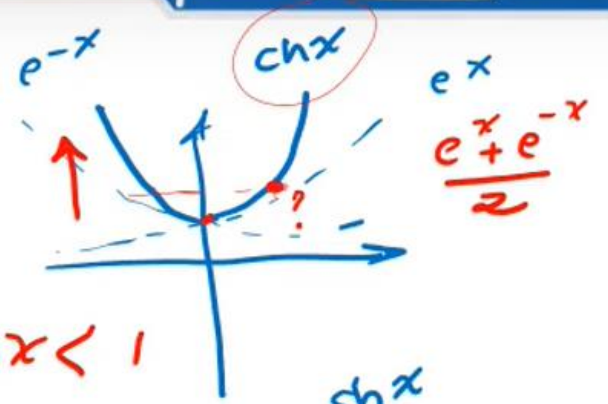
If n is odd, $\cosh y = -\cosh 2$ (Refused) $\quad \cosh y > 0$

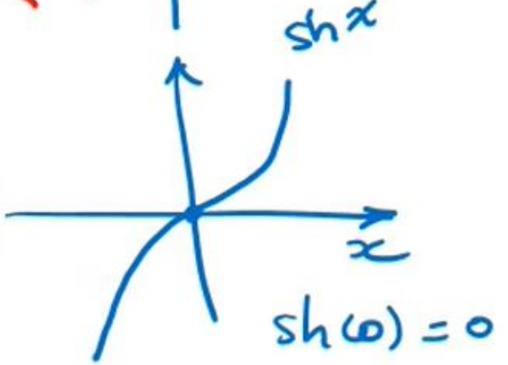
If n is even, $\cosh y = \cosh 2 \Rightarrow y = \pm 2$

$$\boxed{\Rightarrow z = (4k+1)\tfrac{\pi}{2} \pm 2i}$$

$x = (2(2k)+1)\frac{\pi}{2} = \frac{(4k+1)\pi}{2}$

$y = \pm 2$





Functions of Complex Variables

Logarithmic Function

$re^{i\theta}$

$$\ln z = \ln r + i\theta$$

Principle value

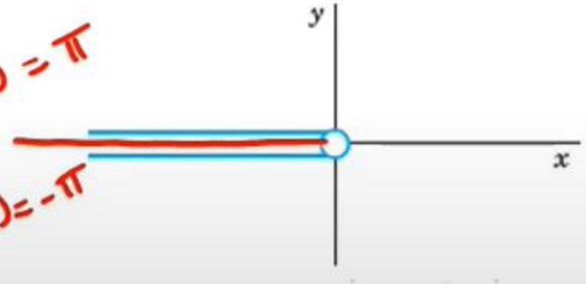
$(-\pi, \pi]$

$$\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z$$

multivalued function

$$\ln z = \operatorname{Ln} z \pm 2n\pi i$$

$\arg(z) = \pi$



$\arg(z) = -\pi$

$z \neq 0$

-ve real axis

Remark

$$e^{\ln z} = z \qquad \ln(e^z) = z \pm 2n\pi i,$$

Functions of Complex Variables

Logarithmic Function

Remarks.

ln

1. Since $\arg(z)$ has infinitely many possible values, so does $\log_e(z)$.

$z>0$ $r \neq 0$

2. $\log_e(0)$ is not defined. (Both because $\arg(0)$ is not defined and $\log_e(|0|)$ is not defined.)

3. Choosing a branch for $\arg(z)$ makes $\log(z)$ single valued. The usual terminology is to say we have chosen a branch of the log function.

4. The principal branch of log comes from the principal branch of arg. That is,

$$\log(z) = \log(|z|) + i\arg(z), \quad \text{where } \boxed{-\pi < \arg(z) \leq \pi} \text{ (principal branch).}$$

Logarithmic Function

$1 =$

Example 6:

Find $\log_e(1)$ $= \ln(e^{i0}) = i0 \ln e = 1 * 0 = \boxed{0}$ principl branch

$\ln(e^{i2\pi}) = i(2\pi \ln e = 2\pi i$ $\quad 2n\pi i$

Compute all the values of $\log_e(i)$.

$\ln i = \ln(e^{i(\frac{\pi}{2}+2n\pi)}) = i\left(\frac{\pi}{2} + 2n\pi\right) \underbrace{\ln e}_{1} = \left(\frac{\pi}{2} \pm 2n\pi\right) i$

$n = 0, 1, 2, \ldots$

Functions of Complex Variables

Logarithmic Function

$$z^a = e^{a\log_e(z)}.$$

Example 5:

Compute all the values of $\sqrt{2i}$.

$$(2i)^{1/2} = e^{\ln(2i)^{1/2}} = e^{\frac{1}{2}\ln(2i)} = e^{\frac{1}{2}[\ln(2 e^{i(\frac{\pi}{2}+2n\pi)})]}$$

$$\frac{1}{2}\ln 2$$

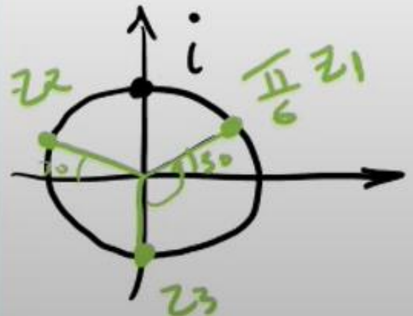
$$= e^{(\frac{1}{2}(\frac{\pi}{2}+2n\pi))i} = e^{(\frac{\pi}{4}+\frac{2n\pi}{2})i}$$

$$= \sqrt{2}\, e$$

$|z| = \sqrt{2}$

$\arg(z) = \frac{\pi}{4} + n\pi$

Compute all the cube roots of i.



$$i^{1/3} = e^{\ln i^{1/3}} = e^{\frac{1}{3}\ln i} = e^{\frac{1}{3}(\frac{\pi}{2}+2n\pi)i}$$

$$i^{1/3} = e^{\frac{1}{3}\ln(e^{i\frac{\pi}{2}+2n\pi})} = e$$

$|z| = 1$

$\arg(z) = \frac{\pi}{6} + \frac{2n\pi}{3}$

$n = 0, 1, 2$