Beta function

$$* \; B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt \qquad \text{"as } x,y>0\text{"}$$

$$= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} \, du$$

$$= 2 \int_0^{\pi/2} \sin^{2x-1}\theta \, \cos^{2y-1}\theta \, d\theta$$

$$* \; B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = B(y,x)$$

$$* \; \Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad \text{"Multiplication Rule"}$$

$$* \; \sqrt{\pi}\;\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2})$$

Ex :

$$1)\; B(\tfrac{1}{3}, \tfrac{2}{3}) = \frac{\Gamma(\frac{1}{3})\,\Gamma(\frac{2}{3})}{\Gamma(1)} = \frac{\frac{\pi}{\sin(\pi * \frac{1}{3})}}{1} = \frac{\pi}{\sqrt{3}/2}$$

"Note that $\sin(\pi * \frac{1}{3}) = \sin(\pi * \frac{2}{3})$"

$$2)\; B(\tfrac{4}{5}, \tfrac{6}{5}) = \frac{\Gamma(4/5)\,\Gamma(6/5)}{\Gamma(2)} = \Gamma(\tfrac{4}{5}) \cdot (\tfrac{1}{5}\Gamma(\tfrac{1}{5}))$$

$$= \frac{1}{5} \, \frac{\pi}{\sin(\pi/5)}$$

③ $\int_0^2 x^2 (16 - x^4)^{2/3} \, dx$

$= (16)^{2/3} \int_0^2 x^2 \left(1 - \frac{x^4}{16}\right)^{2/3} dx$

* بتأكد إن التعويض صح لما حدود التكامل تطلع زى الي الـ Formula

let $t = \frac{x^4}{16}$ $\qquad$ $x = 2 \Rightarrow t = 1$, $\quad x = 0 \Rightarrow t = 0$

$\Longrightarrow x^4 = 16t \Longrightarrow x = 2t^{1/4} \qquad \Longrightarrow dx = \frac{2}{4} t^{-3/4} \, dt$

$I = 16^{2/3} \int_0^1 (2t^{1/4})^2 (1-t)^{2/3} \cdot \frac{1}{2} t^{-3/4} \, dt$

$= \frac{16^{2/3} \cdot \cancel{2}^2}{\cancel{2}} \int_0^1 t^{-1/4} (1-t)^{2/3} \, dt = 2^{11/3} \, \beta\left(\frac{3}{4}, \frac{5}{3}\right)$

④ $\int_{-\infty}^{\infty} \frac{1}{1+x^4} \, dx$

* لما بلاقي التكامل من (-) قيمة للموجب بتاعها بشوف الأول الدالة even ولا odd

$= 2 \int_0^{\infty} \frac{dx}{1+x^4}$ $\qquad$ "even func."

let $u = x^4 \Longrightarrow x = u^{1/4}$ $\qquad$ $x = \infty \Rightarrow u = \infty$, $\quad x = 0 \Rightarrow u = 0$

$\Longrightarrow dx = \frac{1}{4} u^{-3/4} \, du$

$I = \frac{2}{4} \int_0^{\infty} \frac{u^{-3/4}}{(1+u)} \, du$ $\qquad$ $x - 1 = -3/4$, $\quad x + y = 1$

$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$

(5) $\displaystyle\int_0^{\pi/2} \tan^{3/4}\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^{3/4}\theta \, \cos^{-3/4}\theta \, d\theta$$

$2x-1 = 3/4 \qquad 2y-1 = -3/4$

$x = 7/8 \qquad y = \frac{1}{8}$

$$= \frac{1}{2} \beta(7/8, 1/8) = \frac{1}{2} \frac{\Gamma(7/8)\Gamma(1/8)}{\Gamma(1)} = \frac{1}{2} \frac{\pi}{\sin(\frac{7\pi}{8})}$$

(6) $\displaystyle\int_0^{\pi/2} \cos^2(2\theta) \, d\theta$

$$= \int_0^{\pi/2} [\cos^2\theta - \sin^2\theta]^2 \, d\theta$$

$$= \int_0^{\pi/2} \cos^4\theta \, d\theta + \int_0^{\pi/2} \sin^4\theta \, d\theta - 2\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= \frac{1}{2} \beta(\tfrac{1}{2}, \tfrac{5}{2}) + \frac{1}{2} \beta(\tfrac{5}{2}, \tfrac{1}{2}) - \beta(\tfrac{3}{2}, \tfrac{3}{2})$$

(7) $\displaystyle\int_0^{\pi/2} (1 - \sin^3\theta)^{1/4} \cos\theta \, d\theta$

let $t = \sin^3\theta$

$\theta = \frac{\pi}{2} \Rightarrow t = 1$

$\theta = 0 \Rightarrow t = 0$

$\Rightarrow \sin\theta = t^{1/3}$

$\Rightarrow \cos\theta \, d\theta = \frac{1}{3} t^{-2/3} \, dt$

* علشان كان صعب نبسط المسألة
لشكل الـ sin والـ cos بتاع الـ β
دورنا على تعويض مناسب علشان
تتحل بشكل مختلف للـ β

$$I = \frac{1}{3} \int_0^1 (1-t)^{1/4} \, t^{-2/3} \, dt$$

$$= \frac{1}{3} \beta(\tfrac{1}{3}, \tfrac{5}{4})$$

Assignment Question:

$$\int_0^{\pi/2} (\sin^{-3}\theta - \sin^{-2}\theta)^{1/4} \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{-3/4} (1-\sin\theta)^{1/4} \cos\theta \, d\theta$$

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(8) $$\int_0^1 \frac{1}{\sqrt{1+x^4}}$$

* خد بالك من الـ limits ، مختلفة عن الشكل التاني لـ β فمش هينفع تحل بيه

let $x^4 = \tan^2\theta$

$\Rightarrow x = \tan^{1/2}\theta \Rightarrow dx = \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$

$$I = \int_0^{\pi/4} \frac{1}{\sec\theta} \cdot \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin^{-1/2}\theta \cos^{-1/2}\theta \, d\theta$$

let $\alpha = 2\theta \Rightarrow \theta = \frac{\alpha}{2} \Rightarrow d\theta = \frac{1}{2}d\alpha$

$$I = \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\frac{\alpha}{2} \cos^{-1/2}\frac{\alpha}{2} \, d\alpha$$

$$= \frac{1}{4}\int_0^{\pi/2} \left[\frac{1}{2}\sin\alpha\right]^{-1/2} d\alpha$$

$$= \frac{1}{4}\sqrt{2}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\alpha \, d\alpha$$

(9) $\displaystyle\int_0^{\pi/2} (\tan^3\theta + \tan^5\theta)\, e^{-\tan^2\theta}\, d\theta$

let $t = \tan^2\theta$ $\quad$ $\begin{matrix}\theta = \pi/2 \\ \theta = 0\end{matrix} \implies \begin{matrix} t = \infty \\ t = 0\end{matrix}$

$\Rightarrow dt = 2\tan\theta \sec^2\theta\, d\theta$

$$I = \int_0^\infty (t^{3/2} + t^{5/2}) \cdot e^{-t} \frac{dt}{2t^{1/2}(1+t)}$$

$$= \int_0^\infty \frac{t^{3/2}\cancel{(1+t)}}{2t^{1/2}\cancel{(1+t)}} e^{-t}\, dt = \frac{1}{2}\int_0^\infty t e^{-t}\, dt = \frac{1}{2}\Gamma(2)$$

$$* \ \Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2}) \cdots\cdots \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma(\tfrac{1}{2})$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots\cdots 5\cdot 3\cdot 1}{2^n}\sqrt{\pi}$$

$$= \frac{(2n)!}{2^n\,[2n\cdot(2n-2)\cdot(2n-4)\cdots\cdots 6\cdot 4\cdot 2]}\sqrt{\pi}$$

$$= \frac{(2n)!}{2^n\cdot 2^n\,[n\cdot(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1]}\sqrt{\pi}$$

$$= \frac{(2n)!\,\sqrt{\pi}}{2^{2n}\cdot n!}$$
