SPECIAL FUNCTIONS

1. The Gamma Function

The gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad z > 0. \tag{1.1}$$

This integral diverges for $z \leq 0$.

Properties of the gamma function

1. $\Gamma(1) = 1$

Proof

Set $z = 1$ in (1.1) :

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = - e^{-t} \Big|_0^\infty = 1 .$$

2. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof

By (1.1) we have

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt .$$

Set $t = y^2$, so that $dt = 2y dy$:

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty y^{-1} e^{-y^2} . 2y dy = 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}$$

Here we used the well-known result

$$\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$$

3. The reccurence formula for the gamma function :

$$\Gamma(z + 1) = z\, \Gamma(z).$$

Proof

By (1.1) we have

$$\Gamma(z + 1) = \int_0^\infty t^z e^{-t}\, dt .$$

Integration by parts yields

$$\Gamma(z + 1) = -t^z e^{-t} \Big|_0^\infty + z \int_0^\infty t^{z-1} e^{-t}\, dt .$$

Since the integral on the right is $\Gamma(z)$ and $\lim_{t \to \infty} t^z/e^t = 0$ for $z > 0$, we obtain that

$$\Gamma(z + 1) = z \ \ (z) .$$

4. $\Gamma(n + 1) = n!$ for any positive integer n

Proof

Applying the reccurence formula repeatedly, we have

$$\begin{aligned} \Gamma(n + 1) &= n\,\Gamma(n) \\ &= n(n - 1)\,\Gamma(n - 1) = \ldots \\ &= n(n - 1)(n - 2) \ldots 3.2.1\ \Gamma(1) \\ &= n!\ \Gamma(1) \end{aligned}$$

But $\Gamma(1) = 1$, and so $\Gamma(n + 1) = n!$.

Not

Because of property 4, the gamma function is a generalization of the factorial. $\Gamma(z)$ is defined for noninteger z , while $z!$ is defined only for integer z.

5. $L[t^{-\frac{1}{2}}] = \sqrt{\frac{\pi}{s}}$

<u>Proof</u>

By the definition of Laplace transform :

$$L[t^{-\frac{1}{2}}] = \int_0^\infty t^{-\frac{1}{2}} e^{-st}\, dt , \quad s > 0.$$

Set $u = st$, so that $dt = \frac{1}{s}\, du$:

$$L[t^{-\frac{1}{2}}] = \int_0^\infty \left(\frac{u}{s}\right)^{-\frac{1}{2}} e^{-u} . \frac{1}{s}\, du$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty u^{-\frac{1}{2}} e^{-u} = \frac{1}{\sqrt{s}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \sqrt{\frac{\pi}{s}} . \qquad \text{(by property 2)}$$

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In what follows we give a table of value of the gamma function for $x \in [1,2]$ in steps of 0.02. For values of x not cited in the table, we can use linear interpolation to find $\Gamma(x)$:

TABLE

of the Gamma Function

(In Steps of 0.02)

x	(x)	x	(x)	x	(x)
1.00	1.000000	1.34	0.892216	1.68	0.905001
1.02	0.988844	1.36	0.890185	1.70	0 908639
1.04	0.978438	1.38	0.888537	1.72	0.912581
1.06	0.968744	1.40	0.887264	1.74	0.916826
1.08	0.959725	1.42	0.866356	1.76	0.921275
1.10	0.951351	1.44	0.885805	1.78	0.926227
1.12	0.943590	1.46	0.885604	1.80	0.931384
1.14	0.936416	1.48	0.885747	1.82	0.936845
1.16	0.929803	1.50	0.886227	1.84	0.942612
1.18	0.923728	1.52	0.887039	1.86	0.948667
1.20	0.918169	1.54	0.888178	1.88	0.955017
1.22	0.913106	1.56	0.889390	1.90	0.961766
1.24	0.908521	1.58	0.891420	1.92	0.968774
1.26	0.904397	1.60	0.893515	1.94	0.976099
1.28	0.900718	1.62	0.895924	1.69	0.983743
1.30	0.897471	1.64	0.898642	1.98	0.991708
1.32	0.894640	1.66	0.901668	2.00	1.000000

By using this table and the reccurence formula we can evaluate $\Gamma(x)$ for any $x > 0$. The reccurence relation is used repeatedly until we arrive at a tabular value.

Example

a. $\Gamma(4) = 3! = 6$

b. $\Gamma(5.42) = (4.42)(3.42)(2.42)(1.42)\,\Gamma(1.42)$

From the above table we find $\Gamma(1.42) = 0.866356$.

Hence

$$\Gamma(5.42) = 45.003726$$

c. $$\Gamma\left(\frac{5}{3}\right) = \frac{3}{2} \cdot \frac{1}{2}\ \Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}\ .$$

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Definition of $\Gamma(x)$ for negative x

The definition (1.1) cannot be used for defining $\Gamma(x)$ when $x < 0$, since it converges only for $x > 0$. We agree to extend the domain of definition of $\Gamma(x)$ to $x < 0$ by use of the reccurence formula $\Gamma(x+1) = x\,\Gamma(x)$ as follows:

a. Let $-1 < x < 0$. We define

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \qquad x + 1 > 0.$$

For example,

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}$$

b. Let $-2 < x < -1$. We apply the reccurence formula twice:

$$\Gamma(x) = \frac{\Gamma(x+2)}{x(x+1)}, \qquad x + 2 > 0.$$

For example,

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{3}\sqrt{\pi}$$

c. In general, if $-m < x < -(m-1)$, we set

$$\Gamma(x) = \frac{\Gamma(x+m)}{x(x+1)\ldots(x+m-1)} \qquad x+m > 0 \qquad (1.2)$$

d. Rewrite the reccurence formula in the form

$$\frac{1}{\Gamma(x)} = \frac{x}{\Gamma(x+1)}$$

Putting $z = 0$, we obtain

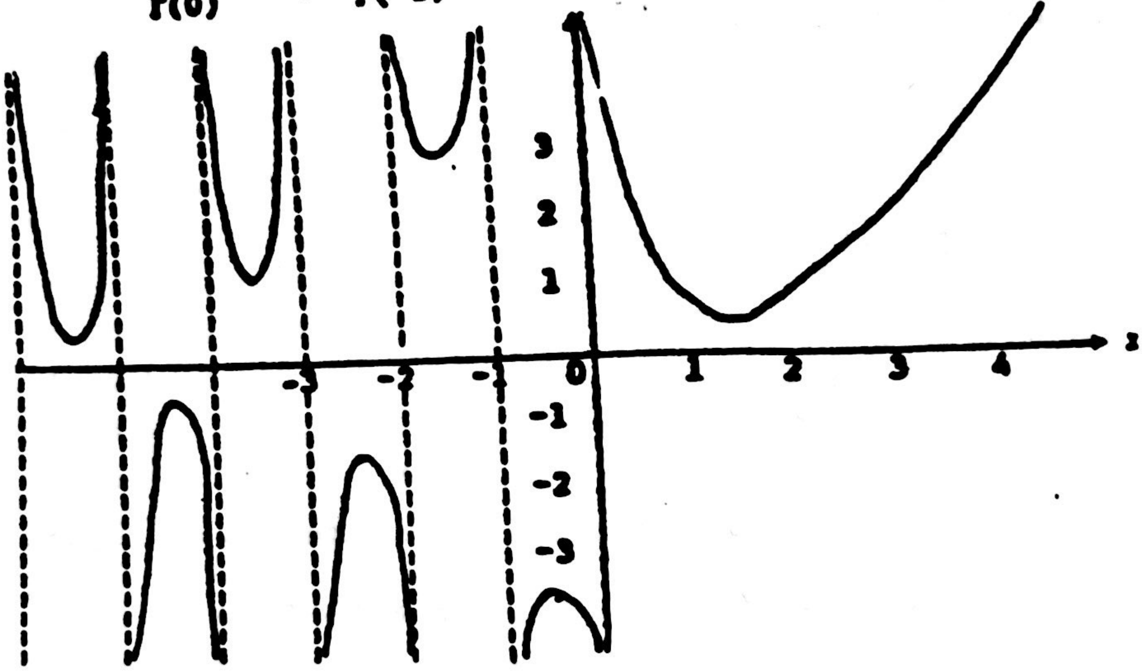
$$\frac{1}{\Gamma(0)} = \frac{0}{\Gamma(1)} = \frac{0}{1} = 0 .$$

Putting $z = -1$, we obtain

$$\frac{1}{\Gamma(-1)} = \frac{-1}{\Gamma(0)} = 0.$$

The process can be continued and we find

$$\frac{1}{\Gamma(0)} = 0, \quad \frac{1}{\Gamma(-1)} = 0, \quad \frac{1}{\Gamma(-2)} = 0, \quad \frac{1}{\Gamma(-3)} = 0, \ldots \quad (1.3)$$



Example Evaluate $\Gamma(-4.3)$.

Solution

Use (1.2) such that $1 < z + n < 2$, and then use the table of values of $\Gamma(z)$ for $1 < z < 2$:

$$\Gamma(-4.3) = \frac{\Gamma(1.7)}{(-4.3)(-3.3)(-2.3)(-1.3)(-0.3)(0.7)}$$

$$= -\frac{0.908639}{8.909901} = -0.101981.$$

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The multiplication formula

If x is any real number, which is not an integer or zero, then

$$\Gamma(x)\,\Gamma(1-x) = \pi/\sin \pi x. \qquad (1.4)$$

The proof of this formula lies outside the scope of this book. It is called the multiplication formula.

Example

$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) = \frac{\pi}{\sin\frac{\pi}{4}} = \pi\sqrt{2} \qquad (x = \tfrac{1}{4})$$

$$\Gamma(-\tfrac{1}{6})\,\Gamma(\tfrac{7}{6}) = \frac{\pi}{\sin\frac{7\pi}{6}} = 2\pi \qquad (x = \tfrac{7}{6})$$

$$\Gamma(-1.42)\,\Gamma(2.42) = \frac{\pi}{\sin 2.42\,\pi} \qquad (x = 2.42)$$

$$= \frac{\pi}{\sin 0.42\,\pi} = \frac{\pi}{0.968\;583\;16}$$

$$= 3.243\;493.$$

Example

Evaluate $\int_0^\infty x^{\frac{3}{2}}\, 5^{-x}\, dx$.

Solution

Set $5^x = e^t \longrightarrow x \ln 5 = t \longrightarrow dx = \frac{dt}{\ln 5}$;

$$I = \int_0^\infty \left(\frac{t}{\ln 5}\right)^{3/2} . \; e^{-t} . \frac{dt}{\ln 5}$$

$$= \frac{1}{(\ln 5)^{5/2}} \int_0^\infty t^{3/2}\, e^{-t}\, dt .$$

Comparing with (1.1), we find

$$I = \frac{1}{(\ln 5)^{3/2}}\,\Gamma\left(\tfrac{5}{2}\right) = \frac{1}{(\ln 5)^{3/2}} \cdot \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{3\sqrt{\pi}}{4(\ln 5)^{3/2}}\,.$$

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II – The Beta Function

The beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt\ ; \quad x>0,\ y>0. \tag{2.1}$$

It is clear by setting $u = 1 - t$, that $B(y,x) = B(x,y)$.

If we set $t = \sin^2\theta$, so that $dt = 2\sin\theta\cos\theta\,d\theta$, we obtain another form for beta function:

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,d\theta. \tag{2.2}$$

Other equivalent form for beta function is obtained by setting $t = u/(1+u)$ in (2.1):

$$B(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\,du \tag{2.3}$$

Relation between beta and gamma functions:

By (1.1) we have

$$\Gamma(x)\,\Gamma(y) = \int_0^{\infty} t^{x-1}\,e^{-t}\,dt.\int_0^{\infty} u^{y-1}\,e^{-u}\,du$$

$$= \int_0^{\infty}\int_0^{\infty} t^{x-1}\,u^{y-1}\,e^{-(t+u)}\,dt\,du.$$

This double integral is taken over the first quadrant of the tu-plane. If we set

$$t = r\cos^2\theta \quad , \quad u = r\sin^2\theta$$

then r will vary from 0 to ∞, and θ from 0 to $\frac{\pi}{2}$.

$$J\left(\frac{t,u}{r,\theta}\right) = \begin{vmatrix} \frac{\partial t}{\partial r} & \frac{\partial t}{\partial \theta} \\ \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos^2\theta & -2r\cos\theta\sin\theta \\ \sin^2\theta & 2r\sin\theta\cos\theta \end{vmatrix}$$

$$= 2r\sin\theta\cos\theta.$$

Hence

$$dt\,du = 2r\sin\theta\cos\theta\,dr d\theta.$$

The integral becomes

$$\Gamma(x)\Gamma(y) = \int_0^{\pi/2}\int_0^{\infty}(r\cos^2\theta)^{x-1}(r\sin^2\theta)^{y-1}e^{-r}.\ 2r\sin\theta\cos\theta\,dr d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{\infty} r^{x+y-1}\,e^{-r}\cos^{2x-1}\theta\,\sin^{2y-1}\theta\,dr d\theta$$

$$= 2\int_0^{\pi/2}\cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta.\int_0^{\infty} r^{x+y-1}\,e^{-r}\,dr$$

Taking (1.1) and (2.2) into consideration, we find

$$\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y)\ .$$

From here we obtain the important formula :

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}\ . \qquad (2.4).$$

<u>Example</u>

a. $B(2,4) = \frac{\Gamma(2)\,\Gamma(4)}{\Gamma(6)} = \frac{1!\ 3!}{5!} = \frac{1}{20}$

b. $B(\frac{1}{4},\frac{7}{4}) = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{7}{4})}{\Gamma(2)} = \frac{\Gamma(\frac{1}{4}).\ \frac{3}{4}\,\Gamma(\frac{3}{4})}{1!}$

$$= \frac{3}{4}\,\Gamma(\tfrac{3}{4})\,\Gamma(\tfrac{1}{4}) = \frac{3}{4}\cdot\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{3\sqrt{2}}{4}\pi .$$

c. $$B(1.42,\ 3) = \frac{\Gamma(1.42)\,\Gamma(3)}{\Gamma(4.42)} = \frac{2!\ \ \Gamma(1.42)}{(3.42)(2.42)(1.42)\,\Gamma(1.42)}$$

$$= 0.170\ 177$$

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Now we prove that

$$W_x = \int_0^{\pi/2} \sin^x\theta\, d\theta = \int_0^{\pi/2} \cos^x\theta\, d\theta = \frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+1)}\cdot\frac{\sqrt{\pi}}{2} \qquad (2.5)$$

for any real number $x > -1$.

In fact, comparison with (2.2) gives

$$W_x = \frac{1}{2}\,B(\tfrac{x}{2}+\tfrac{1}{2}\ ,\ \tfrac{1}{2}) = \frac{1}{2}\,\frac{\Gamma(\frac{x}{2}+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{x}{2}+1)}$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, formula (2.5) follows.

<u>Example</u>

a. $$\int_0^{\pi/2} \sin^{\frac{3}{2}}\theta\, d\theta = \frac{\Gamma(\frac{3}{4}+\frac{1}{2})}{\Gamma(\frac{3}{4}+1)}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{\Gamma(1.25)}{\Gamma(1.75)}\cdot\frac{\sqrt{\pi}}{2}$$

The value of $\Gamma(1.25)$ and $\Gamma(1.75)$ can be found from the table of values of $\Gamma(x)$.

b. $$\int_0^{\pi/2} \cos^6\theta\, d\theta = \frac{\Gamma(\frac{7}{2})}{\Gamma(4)}\cdot\frac{\sqrt{\pi}}{2} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\ \Gamma(\frac{1}{2})}{3!}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{5}{32}\pi$$

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Legendre duplication formula:

$$\sqrt{\pi}\ \Gamma(2x) = 2^{2x-1}\ \Gamma(x)\ \Gamma(x+\tfrac{1}{2})\ . \tag{2.6}$$

Proof!

By (2.2) when $y = x$, we have

$$B(x,x) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2x-1}\theta d\theta$$

$$= \frac{2}{2^{2x-1}} \int_0^{\pi/2} \sin^{2x-1} 2\theta d\theta. \tag{i}$$

By (2.4) :

$$B(x,x) = \frac{\Gamma^2(x)}{\Gamma(2x)}\ . \tag{ii}$$

In the integral on the right of (i) set $\phi = 2\theta$ use the symmetry of sin about $\phi = \frac{\pi}{2}$ and formula (2.5):

$$\int_0^{\pi/2} \sin^{2x-1} 2\theta d\theta = \frac{1}{2}\int_0^{\pi} \sin^{2x-1}\phi d\phi$$

$$= \int_0^{\pi/2} \sin^{2x-1}\phi d\phi$$

$$= \frac{\Gamma(\frac{2x-1}{2}+\frac{1}{2})}{\Gamma(\frac{2x-1}{2}+1)} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})} \cdot \frac{\sqrt{\pi}}{2} \tag{iii}$$

Substitute from (ii) and (iii) into (i) to obtain

$$\frac{\Gamma^2(x)}{\Gamma(2x)} = \frac{1}{2^{2x-1}} \cdot \frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})} \cdot \sqrt{\pi}$$

Multiplying both sides by $2^{2x-1}\ \Gamma(2x)\,\Gamma(x+\frac{1}{2})/\Gamma(x)$, we obtain formula (2.5).

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Example Evaluate $\int_0^2 \sqrt{x}\,(16 - x^4)^{5/8}\,dx.$

Solution

Set $x^4 = 16t \longrightarrow x = 2t^{1/4} \longrightarrow dx = \frac{1}{2}t^{-3/4}\,dt$:

$$I = \int_0^1 \sqrt{2}\; t^{1/8}(16 - 16t)^{5/8} \cdot \frac{1}{2}\,t^{-\frac{3}{4}}\,dt$$

$$= 4\int_0^1 t^{-5/8}(1 - t)^{5/8}\,dt$$

Compare with (2.1):

$$x - 1 = -\frac{5}{8} \longrightarrow x = \frac{3}{8}\,, \quad y - 1 = \frac{5}{8} \longrightarrow y = \frac{13}{8}\,.$$

Hence,

$$I = 4B\left(\frac{3}{8}, \frac{13}{8}\right) = 4\,\frac{\Gamma(\frac{3}{8})\,\Gamma(\frac{13}{8})}{\Gamma(2)}$$

$$= 4\cdot\frac{5}{8}\;\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{3}{8}\right) = \frac{5}{2}\cdot\frac{\pi}{\sin(3\pi/8)}$$

$$\approx 8.501\,088.$$

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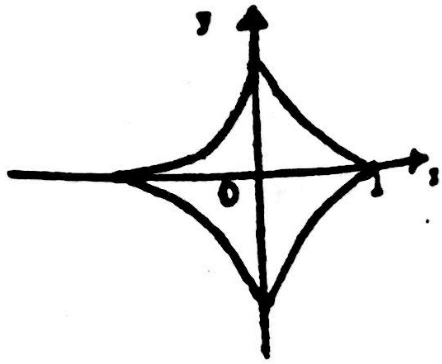
Example Find the area enclosed by the astroid $x^{2/3} + y^{2/3} = 1$



Solution

Because of the symmetry,

we have

$$A = 4\int_0^1 y\,dx = 4\int_0^1 (1 - x^{2/3})^{3/2}\,dx.$$

Set $x^{2/3} = t \longrightarrow x = t^{3/2}$, $dx = \frac{3}{2} t^{1/2}\, dt$;

$$A = 4\int_0^1 (1 - t)^{3/2} \cdot \frac{3}{2} t^{1/2}\, dt = 6\int_0^1 t^{1/2}(1 - t)^{3/2}\, dt$$

$$= 6B(\tfrac{3}{2} , \tfrac{5}{2}) = \tfrac{3}{8}\pi .$$

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PROBLEMS

1. Prove that $L[\, t^n \,] = \dfrac{\Gamma(n + 1)}{s^{n+1}}$ for any real number $n > -1$. Hence, find Laplace transforms of the following function :

(i) $t^{5/2}$ (ii) $\dfrac{1}{\sqrt[3]{t}}$ (iii) $\sqrt{t}\; e^{-3t}$

2. Given that n is a positive integer, show that

$$B(x,n) = \frac{(n-1)\,!}{(x + 1)\; .\; .\; .\; (x+n-1)} .$$

Hence evaluate $B(0.1 \;,\; 3)$.

3. Use Legendre duplication formula to show that

$$\Gamma(n + \tfrac{1}{2}) = \frac{(2n)!}{2^{2n} \,.\, n!}\sqrt{\pi}$$

for any nonnegative integer n .

4. Show that

$$\int_0^\infty x^a\, b^{-x}\, dx = \frac{\Gamma(a + 1)}{(\ln b)^{a+1}}$$

where $a > -1$ and $b > 1$.

5. Show that the area enclosed by the curve

$x^4 + y^4 = 1$ is $\Gamma^2(\frac{1}{4})/(2\sqrt{\pi})$

Using gamma and beta function, evaluate the following integrals:

6. $\int_0^\infty x^3 e^{-2x^3}\, dx$

7. $\int_0^1 \sqrt[3]{x}\, \ln^5 x\, dx$

8. $\int_0^\infty 3^{-x^2} dx$

9. $\int_0^{\pi/2} \sin^{3.04} x\, dx$

10. $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

11. $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

12. $\int_0^2 x(8 - x^3)^{1/3}\, dx$

13. $\int_0^\infty \frac{x\,dx}{1+x^6}$

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III - BESSEL FUNCTIONS

The differential equation

$$x^2y'' + xy' + (x^2 - n^2)\, y = 0 \qquad (3.1)$$

where n is some positive real number or 0 , is called Bessel equation of order n

Note The word "order" here refers to the parameter n, and not to the usual concept of order, which is 2 for equation (3.1).

Equation (3.1) has only one regular singular point at $x = 0$, and hence there exists a series solution conver-