[1] (a) $\int_0^{\pi/2} (\csc^3\theta - \csc^2\theta)^{1/5} \cos\theta\, d\theta$

$= \int_0^{\pi/2} (\sin^{-3}\theta\,(1-\sin\theta))^{1/5} \cos\theta\, d\theta = \int_0^{\pi/2} \sin^{-3/5}\theta\,(1-\sin\theta)^{1/5} \cos\theta\, d\theta$

let $t = \sin\theta \implies dt = \cos\theta\, d\theta$ $\quad \begin{matrix}\theta = \pi/2 \\ \theta = 0\end{matrix} \implies \begin{matrix} t = 1 \\ t = 0\end{matrix}$

$\therefore I = \int_0^1 t^{-3/5} (1-t)^{1/5}\, dt$ $\quad \therefore \begin{matrix} x-1 = -3/5 \\ y-1 = 1/5 \end{matrix} \implies \begin{matrix} x = 2/5 \\ y = 6/5 \end{matrix}$

$\implies I = \beta(2/5, 6/5) = \dfrac{\Gamma(2/5)\,\Gamma(6/5)}{\Gamma(8/5)}$

(b) $\int_0^{\pi/2} (\sec^3\theta - \sec^2\theta)^{1/4} \sin\theta\, d\theta$

$= \int_0^{\pi/2} (\cos^{-3}\theta\,(1-\cos\theta))^{1/4} \sin\theta\, d\theta = \int_0^{\pi/2} \cos^{-3/4}\theta\,(1-\cos\theta)^{1/4} \sin\theta\, d\theta$

let $t = \cos\theta \implies dt = -\sin\theta\, d\theta$ $\quad \begin{matrix}\theta = \pi/2 \\ \theta = 0\end{matrix} \implies \begin{matrix} t = 0 \\ t = 1\end{matrix}$

$\therefore I = -\int_1^0 t^{-3/4} (1-t)^{1/4}\, dt = \int_0^1 t^{-3/4} (1-t)^{1/4}\, dt$ $\quad \therefore \begin{matrix} x = 1/4 \\ y = 5/4 \end{matrix}$

$\implies I = \beta(1/4, 5/4) = \dfrac{\Gamma(1/4)\,\Gamma(5/4)}{\Gamma(1.5)} = \dfrac{0.25\,\Gamma^2(1/4)}{0.5\sqrt{\pi}} = \dfrac{\Gamma^2(1/4)}{2\sqrt{\pi}}$

(c) $\int_0^{\pi/2} (\tan^5 x + \tan^7 x)\, e^{-\tan^2 x}\, dx$

$= \int_0^{\pi/2} \tan^5 x\,(1+\tan^2 x)\, e^{-\tan^2 x}\, dx = \int_0^{\pi/2} \tan^5 x \cdot \sec^2 x \cdot e^{-\tan^2 x}\, dx$

let $t = \tan^2 x \implies dt = 2\tan x \sec^2 x\, dx$ $\quad \begin{matrix} x = \pi/2 \\ x = 0 \end{matrix} \implies \begin{matrix} t = \infty \\ t = 0 \end{matrix}$

$\therefore I = \int_0^{\pi/2} \tan^4 x \cdot e^{-\tan^2 x} \cdot \tan x \sec^2 x\, dx = \frac{1}{2}\int_0^{\infty} t^2 e^{-t}\, dt = \frac{1}{2}\Gamma(3)$

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(d) $\int_0^{\sqrt{2}} \frac{dx}{\sqrt{4+x^4}} = \int_0^{\sqrt{2}} \frac{dx}{2\sqrt{1+\frac{1}{4}x^4}}$ $\quad \begin{matrix} x=\sqrt{2} \\ x=0 \end{matrix} \Rightarrow \begin{matrix} \theta=\pi/4 \\ \theta=0 \end{matrix}$

let $\frac{1}{4}x^4 = \tan^2\theta \Rightarrow x = \sqrt{2}\tan^{1/2}\theta \Rightarrow dx = \frac{\sqrt{2}}{2}\tan^{-1/2}\theta \sec^2\theta d\theta$

$$I = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{1+\frac{1}{4}(\sqrt{2}\tan^{1/2}\theta)^4}} \cdot \frac{\sqrt{2}}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{\pi/4} \frac{\tan^{-1/2}\theta \sec^2\theta}{\sqrt{1+\tan^2\theta}} d\theta = \frac{\sqrt{2}}{4}\int_0^{\pi/4} \frac{\tan^{-1/2}\theta \sec^{\cancel{2}}\theta}{\cancel{\sec\theta}} d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{\pi/4} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta} \cdot \frac{1}{\cos\theta} d\theta = \frac{\sqrt{2}}{4}\int_0^{\pi/4} \sin^{-1/2}\theta \cos^{-1/2}\theta \, d\theta$$

let $\alpha = 2\theta \Rightarrow d\alpha = 2d\theta$ $\quad \begin{matrix} \theta=\pi/4 \\ \theta=0 \end{matrix} \Rightarrow \begin{matrix} \alpha=\pi/2 \\ \alpha=0 \end{matrix}$

$$I = \frac{\sqrt{2}}{8}\int_0^{\pi/2} \sin^{-1/2}\frac{\alpha}{2} \cos^{-1/2}\frac{\alpha}{2} \, d\alpha$$ " $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ "

$$= \frac{\sqrt{2}}{8}\int_0^{\pi/2} \left[\frac{1}{2}\sin\alpha\right]^{-1/2} d\alpha = \frac{\sqrt{2} * \sqrt{2}}{8_4}\int_0^{\pi/2} \sin^{-1/2}\alpha \, d\alpha$$

$\therefore \begin{matrix} 2x-1 = -1/2 \\ 2y-1 = 0 \end{matrix} \Rightarrow \begin{matrix} x = 1/4 \\ y = 1/2 \end{matrix}$ $\qquad \therefore I = \frac{1}{8}\beta(1/4, 1/2) = \frac{\Gamma(1/4)\Gamma(1/2)}{8\,\Gamma(3/4)}$

(e) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{9+x^4}} = \int_0^{\sqrt{3}} \frac{dx}{3\sqrt{1+\frac{1}{9}x^4}}$ $\quad \begin{matrix} x=\sqrt{3} \\ x=0 \end{matrix} \Rightarrow \begin{matrix} \theta=\pi/4 \\ \theta=0 \end{matrix}$

let $\frac{1}{9}x^4 = \tan^2\theta \Rightarrow x = \sqrt{3}\tan^{1/2}\theta \Rightarrow dx = \frac{\sqrt{3}}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$

$$I = \frac{1}{3}\int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{\sqrt{3}}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$$

$$= \frac{\sqrt{3}}{6}\int_0^{\pi/4} \frac{\tan^{-1/2}\theta \sec^{\cancel{2}}\theta}{\cancel{\sec\theta}} d\theta = \frac{\sqrt{3}}{6}\int_0^{\pi/4} \sin^{-1/2}\theta \cos^{-1/2}\theta \, d\theta$$

let $\alpha = 2\theta \Rightarrow d\alpha = 2d\theta$ $\quad \begin{matrix} \theta=\pi/4 \\ \theta=0 \end{matrix} \Rightarrow \begin{matrix} \alpha=\pi/2 \\ \alpha=0 \end{matrix}$

$$I = \frac{\sqrt{3}}{12}\int_0^{\pi/2} \sin^{-1/2}\frac{\alpha}{2} \cos^{-1/2}\frac{\alpha}{2} \, d\alpha = \frac{\sqrt{3}}{12}\int_0^{\pi/2} \left[\frac{1}{2}\sin\alpha\right]^{-1/2} d\alpha$$

$$= \frac{\sqrt{6}}{12}\int_0^{\pi/2} \sin^{-1/2}\alpha \, d\alpha = \frac{\sqrt{6}}{24}\beta(1/4, 1/2) = \frac{\sqrt{6}}{24}\,\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}$$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt[3]{1+x^6}} = 2\int_0^{\infty} \frac{dx}{\sqrt[3]{1+x^6}}$ "even function"

let $x^6 = t \implies x = t^{1/6} \implies dx = \frac{1}{6} t^{-5/6}\, dt$ $\qquad \begin{matrix} x=\infty \\ x=0 \end{matrix} \implies \begin{matrix} t=\infty \\ t=0 \end{matrix}$

$\therefore I = 2\displaystyle\int_0^{\infty} \frac{1}{\sqrt[3]{1+t}} \cdot \frac{1}{6} t^{-5/6}\, dt = \frac{1}{3}\int_0^{\infty} \frac{t^{-5/6}}{(1+t)^{1/3}}\, dt$

$\therefore \begin{matrix} x-1 = -5/6 \\ x+y = 1/3 \end{matrix} \implies \begin{matrix} x = 1/6 \\ y = 1/6 \end{matrix}$ $\qquad \therefore I = \frac{1}{3}\beta(1/6, 1/6) = \dfrac{\Gamma^2(1/6)}{3\Gamma(1/3)}$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{1+x^6}} = 2\int_0^{\infty} \frac{dx}{\sqrt{1+x^6}}$ "even function"

let $x^6 = t \implies x = t^{1/6} \implies dx = \frac{1}{6} t^{-5/6}\, dt$ $\qquad \begin{matrix} x=\infty \\ x=0 \end{matrix} \implies \begin{matrix} t=\infty \\ t=0 \end{matrix}$

$\therefore I = 2\displaystyle\int_0^{\infty} \frac{1}{\sqrt{1+t}} \cdot \frac{1}{6} t^{-5/6}\, dt = \frac{1}{3}\int_0^{\infty} \frac{t^{-5/6}}{(1+t)^{1/2}}\, dt$

$\therefore \begin{matrix} x-1 = -5/6 \\ x+y = 1/2 \end{matrix} \implies \begin{matrix} x = 1/6 \\ y = 1/3 \end{matrix}$ $\qquad \therefore I = \frac{1}{3}\beta(1/6, 1/3) = \dfrac{\Gamma(1/6)\Gamma(1/3)}{3\Gamma(1/2)}$

(h) $\displaystyle\int_0^1 \sqrt[n]{1-x^n}\, dx$ $\qquad \begin{matrix} x=1 \\ x=0 \end{matrix} \implies \begin{matrix} \theta=\pi/2 \\ \theta=0 \end{matrix}$

let $x^n = \sin^2(\theta) \implies x = (\sin\theta)^{2/n} \implies dx = \frac{2}{n}(\sin\theta)^{\frac{2}{n}-1} \cos\theta\, d\theta$

$I = \dfrac{2}{n}\displaystyle\int_0^{\pi/2} (1-\sin^2\theta)^{1/n} \cdot (\sin\theta)^{\frac{2}{n}-1} \cos\theta\, d\theta$

$= \dfrac{2}{n}\displaystyle\int_0^{\pi/2} (\cos^2\theta)^{1/n} \cdot (\sin\theta)^{\frac{2}{n}-1} \cos\theta\, d\theta = \frac{2}{n}\int_0^{\pi/2} (\sin\theta)^{\frac{2}{n}-1} (\cos\theta)^{\frac{2}{n}+1}\, d\theta$

$\therefore \begin{matrix} 2x-1 = \frac{2}{n}-1 \\ 2y-1 = \frac{2}{n}+1 \end{matrix} \implies \begin{matrix} x = \frac{1}{n} \\ y = \frac{1}{n}+1 \end{matrix}$

$\therefore I = \dfrac{1}{n}\beta\left(1/n, 1/n + 1\right) = \dfrac{\Gamma(1/n)\Gamma(1/n+1)}{n\,\Gamma(2/n+1)}$

(i) $\int_a^{\infty} e^{(2ax - x^2)} dx$

let $x = t + a \implies dx = dt$ $\quad$ $\begin{matrix} x = \infty \\ x = a \end{matrix} \implies \begin{matrix} t = \infty \\ t = 0 \end{matrix}$

$$I = \int_0^{\infty} e^{(2at + 2a^2 - (t+a)^2)} dt = \int_0^{\infty} e^{(2at + 2a^2 - t^2 - 2at - a^2)} dt$$

$$= \int_0^{\infty} e^{(a^2 - t^2)} dt = e^{a^2} \int_0^{\infty} e^{-t^2} dt$$

$\begin{matrix} t = \infty \\ t = 0 \end{matrix} \implies \begin{matrix} u = \infty \\ u = 0 \end{matrix}$

let $u = t^2 \implies t = \sqrt{u} \implies dt = \frac{1}{2} u^{-\frac{1}{2}} du$

$$\therefore I = e^{a^2} \int_0^{\infty} e^{-u} \cdot \frac{1}{2} u^{-\frac{1}{2}} du = \frac{e^{a^2}}{2} \Gamma\left(\frac{1}{2}\right)$$

(j) $\int_0^1 (\sqrt{x})^3 \left(\ln \frac{1}{x}\right)^4 dx$ $\quad$ $\begin{matrix} x = 1 \\ x = 0 \end{matrix} \implies \begin{matrix} t = 0 \\ t = \infty \end{matrix}$

let $\ln\left(\frac{1}{x}\right) = t \implies x = e^{-t} \implies dx = -e^{-t} dt$

$$\therefore I = \int_{\infty}^{0} (e^{-t})^{3/2} (t)^4 (-e^{-t}) dt = \int_0^{\infty} e^{-\frac{5}{2}t} \cdot t^4 dt$$

let $u = \frac{5}{2} t \implies dt = \frac{2}{5} du$ $\quad$ $\begin{matrix} t = \infty \\ t = 0 \end{matrix} \implies \begin{matrix} u = \infty \\ u = 0 \end{matrix}$

$$\therefore I = \int_0^{\infty} e^{-u} \cdot \left(\frac{2}{5} u\right)^4 \cdot \frac{2}{5} du = \left(\frac{2}{5}\right)^5 \int_0^{\infty} e^{-u} u^4 du$$

$$= \left(\frac{2}{5}\right)^5 \Gamma(5)$$

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(K) $\int_0^1 x^{m-1} \ln(\frac{1}{x})\,dx$ $\quad \begin{matrix} x=1 \\ x=0 \end{matrix} \Rightarrow \begin{matrix} t=0 \\ t=\infty \end{matrix}$

let $\ln(\frac{1}{x}) = t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$

$\therefore I = \int_\infty^0 (e^{-t})^{m-1} . t . (-e^{-t})dt = \int_0^\infty t . e^{-mt}\, dt$

let $mt = u \Rightarrow dt = \frac{1}{m} du$ $\quad \begin{matrix} t=\infty \\ t=0 \end{matrix} \Rightarrow \begin{matrix} u=\infty \\ u=0 \end{matrix}$

$\therefore I = \int_0^\infty (\frac{1}{m}u) . e^{-u} . \frac{1}{m} du = \frac{1}{m^2} \int_0^\infty u . e^{-u} = \frac{1}{m^2}\Gamma(2)$

(L) $\int_0^1 \sqrt[4]{1-x^4}\, dx$ $\quad \begin{matrix} x=1 \\ x=0 \end{matrix} \Rightarrow \begin{matrix} t=1 \\ t=0 \end{matrix}$

let $x^4 = t \Rightarrow x = t^{1/4} \Rightarrow dx = \frac{1}{4} t^{-3/4} dt$

$\therefore I = \int_0^1 (1-t)^{1/4} . (\frac{1}{4} t^{-3/4}) dt = \frac{1}{4} \int_0^1 t^{-3/4} (1-t)^{1/4} dt$

$\therefore \begin{matrix} x-1 = -3/4 \\ y-1 = 1/4 \end{matrix} \Rightarrow \begin{matrix} x = 1/4 \\ y = 5/4 \end{matrix}$ $\quad \therefore I = \frac{1}{4}\beta(1/4, 5/4) = \frac{\Gamma(1/4)\,\Gamma(5/4)}{4\,\Gamma(1.5)}$

[2] $I = \int_0^\infty x^a b^{-x} dx$

let $b^{-x} = e^{-t} \Rightarrow -x \ln(b) = -t$ $\quad \begin{matrix} x=\infty \\ x=0 \end{matrix} \Rightarrow \begin{matrix} t=\infty \\ t=0 \end{matrix}$

$\Rightarrow dx = \frac{1}{\ln b} dt \Rightarrow \ln(b) > 0 \Rightarrow \boxed{b>1}$ $\quad$ * علشان الـ limits تكون مظبوطة

$\therefore I = \int_0^\infty (\frac{t}{\ln b})^a e^{-t} . \frac{1}{\ln b} dt = \frac{1}{(\ln b)^{a+1}} \int_0^\infty t^a . e^{-t} dt$

$= (\frac{1}{\ln b})^{a+1} \Gamma(a+1)$ $\quad \therefore a+1 > 0 \quad \therefore \boxed{a > -1}$

[3] $I = \int_0^\infty x^c c^{-x} dx$

let $c^{-x} = e^{-t} \Rightarrow -x \ln c = -t$ $\qquad \begin{matrix} x=\infty \\ x=0 \end{matrix} \Rightarrow \begin{matrix} t=\infty \\ t=0 \end{matrix}$

$x = \frac{t}{\ln c} \Rightarrow dx = \frac{dt}{\ln c} \Rightarrow \ln c > 0 \Rightarrow \boxed{c > 1}$ ---- ①

$\therefore I = \int_0^\infty \left(\frac{t}{\ln c}\right)^c \cdot e^{-t} \cdot \frac{1}{\ln c} dt = \left(\frac{1}{\ln c}\right)^{c+1} \int_0^\infty t^c \cdot e^{-t} dt$

$= \left(\frac{1}{\ln c}\right)^{c+1} \Gamma(c+1)$ $\qquad \because c+1 > 0 \Rightarrow c > -1$ ---- ②

from ①, ② $\boxed{c > 1}$

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[4] i) $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \int_0^2 x^2 (2-x)^{-1/2} dx$

let $x = 2t \Rightarrow dx = 2dt$ $\qquad \begin{matrix} x=2 \\ x=0 \end{matrix} \Rightarrow \begin{matrix} t=1 \\ t=0 \end{matrix}$

$\therefore I = \int_0^1 (2t)^2 (2-2t)^{-1/2} \cdot 2dt = 2^2 * 2 * 2^{-1/2} \int_0^1 t^2 (1-t)^{-1/2} dt$

$= 2^{2.5} \beta(3, 1/2) = 2^{2.5} \left(\frac{\Gamma(3)\Gamma(1/2)}{\Gamma(3.5)} \right) = 2^{2.5} \left(\frac{2! \sqrt{\pi}}{(2.5)(1.5)(0.5)(\sqrt{\pi})} \right)$

$= 6.034$

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ii) $\int_0^1 (-\ln x)^{-1/2} dx$ $\qquad \begin{matrix} x=1 \\ x=0 \end{matrix} \Rightarrow \begin{matrix} t=0 \\ t=\infty \end{matrix}$

let $-\ln x = t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$\therefore I = \int_\infty^0 (t)^{-1/2} \cdot (-e^{-t}) dt = \int_0^\infty t^{-1/2} e^{-t} dt$

$\therefore I = \Gamma(\frac{1}{2}) = \sqrt{\pi}$

iii) $\int_0^\infty \frac{1}{1+x^4}\,dx$

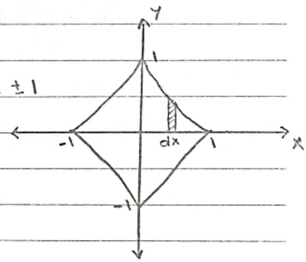
$x=\infty \Rightarrow t=\infty$
$x=0 \Rightarrow t=0$

let $x^4 = t \Rightarrow x = t^{1/4} \Rightarrow dx = \frac{1}{4} t^{-3/4}\,dt$

$\therefore I = \int_0^\infty \frac{\frac{1}{4} t^{-3/4}\,dt}{1+t} = \frac{1}{4}\beta(1/4, 3/4) = \frac{1}{4}\frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)}$

$= \frac{1}{4}\frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{4} = 1.1107$

[5] (a) $x^{2/3} + y^{2/3} = 1$



for $x=0 \Rightarrow y = \pm 1$, for $y=0 \Rightarrow x = \pm 1$

$y = (1 - x^{2/3})^{3/2}$

$A = 4\int_0^1 y\,dx = 4\int_0^1 (1-x^{2/3})^{3/2}\,dx$

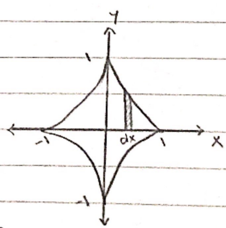
let $x^{2/3} = t \Rightarrow x = t^{3/2} \Rightarrow dx = \frac{3}{2}\sqrt{t}\,dt$

$\Rightarrow A = 4\int_0^1 (1-t)^{3/2} \cdot 3/2 \sqrt{t}\,dt = 6\int_0^1 t^{1/2}(1-t)^{3/2}\,dt$

$= 6\beta(\frac{3}{2}, \frac{5}{2}) = 6\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = 6\frac{0.5\sqrt{\pi} * (1.5)(0.5)(\sqrt{\pi})}{3!} = \frac{3}{8}\pi$

(b) $x^{2/5} + y^{2/5} = 1$ for $x=0 \Rightarrow y = \pm 1$, for $y=0 \Rightarrow x = \pm 1$

$y = (1 - x^{2/5})^{5/2}$



$A = 4\int_0^1 y\,dx = 4\int_0^1 (1-x^{2/5})^{5/2}\,dx$

let $x^{2/5} = t \Rightarrow x = t^{5/2} \Rightarrow dx = \frac{5}{2} t^{3/2}\,dt$

$\Rightarrow A = 4\int_0^1 (1-t)^{5/2} \cdot 5/2\, t^{3/2}\,dt = 10\beta(\frac{5}{2}, \frac{7}{2}) = \frac{15}{128}\pi$

[6] First method "by using legendre duplication formula":

let $x = n+1 \implies \sqrt{\pi}\,\Gamma(2n+2) = 2^{2n+1}\,\Gamma(n+1)\,\Gamma(n+\frac{3}{2})$

$$\therefore \Gamma(n+\tfrac{3}{2}) = \frac{\Gamma(2n+2)}{2^{2n+1}\,\Gamma(n+1)} \cdot \sqrt{\pi} \cdot \left(\frac{2n+2}{2n+2}\right) = \frac{(2n+2)!}{2^{2n+2}\,(n+1)!}$$

second method "by using the property $\Gamma(x+1) = x\,\Gamma(x)$":

$$\Gamma(n+\tfrac{3}{2}) = (n+\tfrac{1}{2}) * (n-\tfrac{1}{2}) * (n-\tfrac{3}{2}) \cdots\cdots \tfrac{3}{2} \cdot \tfrac{1}{2} \cdot \Gamma(\tfrac{1}{2})$$

$$= \frac{(2n+1)(2n-1)(2n-3)\cdots\cdots 3 * 1}{2^{n+1}} * \sqrt{\pi}$$

$$= \frac{(2n+2)!}{2^{n+1}\,[(2n+2)(2n)(2n-2)\cdots\cdots 4 \cdot 2]} \cdot \sqrt{\pi}$$

$$= \frac{(2n+2)!}{2^{n+1} * 2^{n+1}\,[(n+1)(n)(n-1)\cdots\cdots 2 \cdot 1]} \cdot \sqrt{\pi}$$

$$= \frac{(2n+2)!}{2^{2n+2}\,(n+1)!} \cdot \sqrt{\pi}$$