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Faculty of engineering

Ain Shams University

2nd year

(8)

Complex Variable

1. Cauchy Riemann Equation
2. Harmonic Equation
3. Important Proofs

A.K

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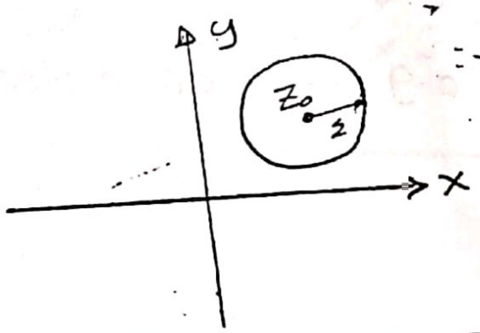
rtant Definitions

(1) Neighborhood of a Point Z_0

$\Rightarrow$ set of Points Z such that $|Z - Z_0| < \epsilon$, $\epsilon > $ Zero

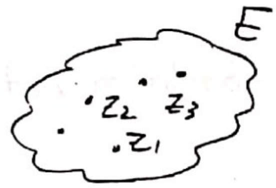
مجموعة النقط المجاورة للنقطة Z_0 حيث هي داخل دائرة مركزها Z_0

ونصف قطرها ابسلون صغير :



(2) interior, exterior and boundary Points

$\rightarrow$ First define $E \Rightarrow$ set of Points in Z-Plane



(a) interior Point Z_0

Z_0 is Called interior Point for E if we Can find neighborhood of Z_0 which Lies Completly in E

اذا امكن ايجاد نقط مجاورة لل Z_0 تقع بالكامل داخل E

$\boxed{\text{Ex}}$ if $E = [1, 10]$

Point $2, 3, 1.1, 9.9, \dots$ are interior Point

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b) Exterior Point Z_0

Z_0 is exterior Point for E if we can find a neighborhood of Z_0 we Lies outside E

يمكن ايجاد نقط مجاورة لـ Z_0 تقع بالكامل خارج E

$\boxed{Ex}$ $E = [1, 10]$

$Z = 11, 13, 100, 0.5, 0.9$ ---- are exterior Point

c) boundary Point Z_0

Z_0 is boundary Point if every neighborhood of Z_0 must Contain both Point inside E & outside E

النقط المجاورة لـ Z_0 يجب ان يكون بعضها داخل الـ E والبعض الاخر خارج E

$\boxed{Ex}$ $E = [1, 10]$

$Z = 1 \& 10$ are boundary Points

Type of sets E

① Closed set $E = [a, b]$

set E Contains all its boundary Points

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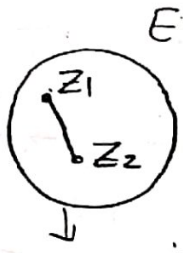
open set $E =]a, b[$

set E doesn't contain any of its boundary points

3) Semi. Closed set $E =]1, 2]$

4) Semi - open set $E = [1, 2[$

Connected set



Connected

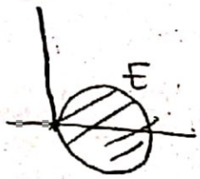
Not Connected

Connected set E if any Line between two Points in E Lies inside E

⊛ bounded set

Points of E lies in closed Regoin

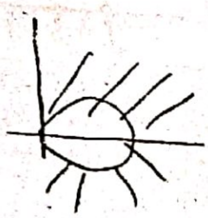
$E = |z-1| < 1$



bounded

منطقة مغلقة

$E = |z-1| > 1$



Not bounded

منطقة مفتوحة

B

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Pb ← important

sketch each of following sets and determine which of them is open, Connected or Closed. what is the boundary of each set?. and if bounded or not bounded set?

$\boxed{1}$ $Re(Z) > 1$

$E \Rightarrow]1, \infty[$ → open set

→ not bounded

→ Connected

→ boundary at $x=1$

$\boxed{2}$ $|Z-2-i| \leq 2$

Center $(2,1)$

radius (2)

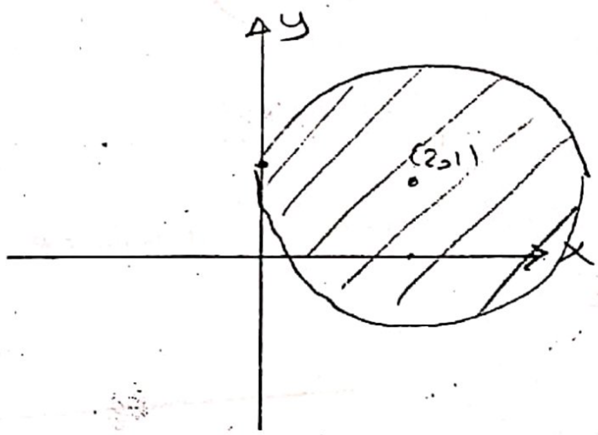
→ Closed set

→ bounded

→ Conneted

→ bounded at

$(x-2)^2+(y-1)^2=4$ or at $|Z-2-i|=2$



$\boxed{3}$ $|Z|<1$ or $|Z-4|<1$

2 circles

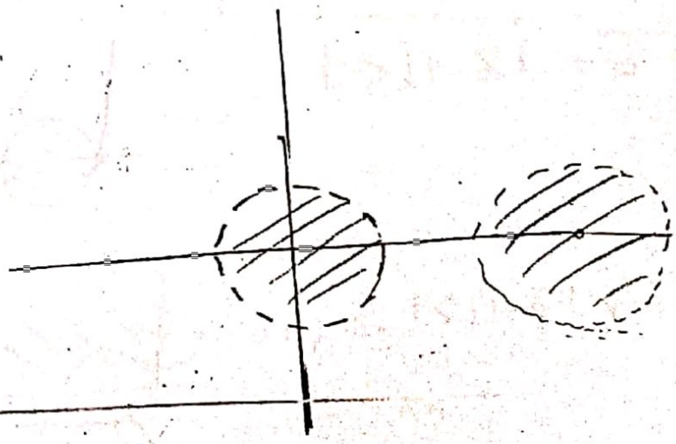
→ open set

→ bounded

→ Not Connected

→ boundary at

$|Z|=1$ & $|Z-4|=1$



$\boxed{4}$

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$|Z+i| > 2$

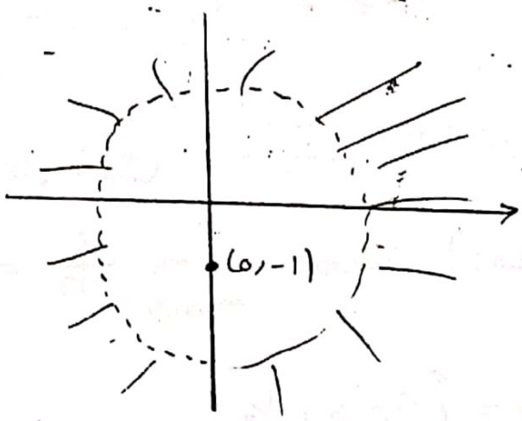
→ open set

→ Not bounded

→ Not Connected

→ boundary at

$|Z+i| = 2$



(b) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$ find boundary Point

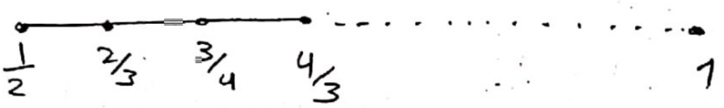
- هنا مجموعة من الارقام ال discrite وهي على شكل $\frac{n}{n+1}$

→ to get Last number $\lim_{n \to \infty} \frac{n}{n+1} = 1$

∴ set $E = \{ 1/2, 2/3, 3/4, \dots\dots 1 \}$

each Point is boundary, why ??

لا يمكن رسم من عندها دائرة تحتوي على نقط داخل ال E وخارجها



∴ all Point on E are boundary Points including 1

boundary $E \cup \{1\}$

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P6)

$E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ find boundary Points

اكو

مجموعة من الارقام discrite على شكل $\frac{1}{n}$

to get Last number $\lim_{n\to\infty} \frac{1}{n} = \text{Zero}$

$E = \{1, \frac{1}{2}, \frac{1}{3}, \dots\dots \text{Zero}\}$

all Points on E are boundary Point including Zero

boundary $E \cup \{0\}$

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...chy - Reimann equation & harmonic Function

$f(Z) = U + iV$ is <u>differentiable</u> ($f'(Z)$) iff it satisfies Cauchy Reiman (C-R) Conditions

⇒ <u>Conditions</u>

$$U_x = V_y \quad \& \quad U_y = -V_x$$

<u>or for Polar</u>

$$U_r = \frac{1}{r} V_r \quad \& \quad U_\theta = -r V_r$$

then $f'(Z) = U_x + i V_x$

$$f'(Z) = \frac{r}{Z}(U_r + i V_r) \quad \text{"Polar"}$$

⇒ $f(Z)$ is Analytic in Regoin if it's differtiable in this Regoin

تحقق C-R في منطقة

⇒ $f(Z)$ is Entire function: if Analytic everywhere ( )

تحقق C-R عند كل النقاط

⇒ if $f(Z)$ is Analytic function

① Can diff. direct from Z (i.e) $f(Z) = Z^2 \rightarrow f'(Z) = 2Z$

② if $f(Z)$ in terms of (x, y) or (r, θ)

to Make it in terms of Z

set $x = Z$, $y = 0$ (rectangular form)

$r = Z$, $\theta = 0$ (Polar form)

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Pb) Discuss differentiability for

① $f(z) = z^2$

② $f(z) = |z|^2$

③ $f(z) = \bar{z}$

① $f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{(2xy)}_{v}$

$$\left.\begin{array}{ll} u_x = 2x & v_x = 2y \\ u_y = -2y & v_y = 2x \end{array}\right\} \quad \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$$

$f(z)$ is Analytic every where → entire

$f'(z) = u_x + i v_x = 2x + i(2y) = 2(x+iy) = 2z$

$f(z) = z^2 \Rightarrow f'(z) = 2z$

* other soln

$f(z) = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2(\cos 2\theta + i \sin 2\theta)$

$u = r^2 \cos 2\theta, \quad v = r^2 \sin 2\theta$

$u_r = 2r\cos 2\theta \qquad v_r = 2r\sin 2\theta$

$u_\theta = -2r^2 \sin 2\theta \qquad v_\theta = 2r^2 \cos 2\theta$

$u_r = \frac{1}{r} v_\theta, \quad u_\theta = -r v_r$

Analytic every where

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$z) = \frac{r}{z}(u_r + i v_r) = \frac{r}{z}(2r\cos 2\theta + i2r\sin 2\theta)$

$$= \frac{2r^2}{z}(\cos 2\theta + i\sin \theta)$$

$$= \frac{2r^{\cancel{2}}}{\cancel{r}e^{i\theta}} \cdot e^{i2\theta} = 2re^{i\theta} = \boxed{2z}$$

---

2) $f(z) = |z|^2 = x^2 + y^2 + i(0)$

$u = x^2 + y^2$, $v = 0$

$u_x = 2x$ $\quad v_x = 0$

$u_y = 2y$ $\quad v_y = 0$

$u_x = v_y$ @ $x = 0$, $u_y = -v_x$ @ $y = 0$

function is diff only at $x = 0, y = 0$ $(0,0)$

$f'(z) = u_x + iv_x = 2x + i(0)\Big|_{x=0} = \underline{\text{Zero}}$

---

3) $f(z) = \bar{z} = x - iy$

$u = x$, $v = -y$

$u_x = 1$ $\quad v_x = 0$

$v_y = 0$ $\quad v_y = -1$ $\quad \Rightarrow u_x \neq v_y$

function Not diff.

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Pb) determine where the following functions are differentiable & where they are analytic & find $f'(z)$ if diff

1) $w = 2y - ix$

2) $e^x(\cos y + i \sin y)$

3) $z^2 \bar{z}$

4) $x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

5) $\frac{1}{z}$

---

1) $w = 2y - ix$

$u = 2y$, $v = -x$

$$\left.\begin{array}{ll} u_x = 0 & v_x = -1 \\ u_y = 2 & v_y = 0 \end{array}\right\} \quad u_y \neq -v_x$$

Not diff

---

2) $w = e^x(\cos y + i \sin y)$

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

$u_x = v_y$ & $u_y = -v_x$

entire function

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$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$

$= e^x(\cos y + i \sin y) = e^x e^{iy} = e^{i(x+y)} = e^z$

Note since $f(z)$ is entire set $y=0$, $x=z$

$f(z) = e^z(\cos(0) + i\sin(0)) = e^z$

$f'(z) = e^z$ #

3] $w = z^2 \bar{z} = (re^{i\theta})^2 (re^{-i\theta}) = r^3 e^{i\theta}$

$w = r^3(\cos\theta + i\sin\theta)$

$u = r^3\cos\theta$, $v = r^3 \sin\theta$

$u_r = 3r^2\cos\theta$ $\quad$ $v_r = 3r^2 \sin\theta$

$u_\theta = -r^3 \sin\theta$ $\quad$ $v_\theta = r^3\cos\theta$

$\rightarrow u_r = \frac{1}{r} v_\theta \Rightarrow 3r^2\cos\theta = r^2\cos\theta \Rightarrow 2r^2\cos\theta = 0$

$\left.\begin{array}{l} r=0 \\ \cos\theta = 0 \end{array}\right\}$ — 1

$\rightarrow u_\theta = -r v_r \Rightarrow \cancel{+}r^3\sin\theta = \cancel{+}3r^3\sin\theta$

$2r^3\sin\theta = 0 \rightarrow r=0$, $\rightarrow \sin\theta = 0$ — 2

1 & 2 sastifies only at $\boxed{r=0} \Rightarrow (x,y) = (0,0)$

$f'(z) = \frac{r}{z}(v_r + i v_\theta) = \underline{\text{zero}}$

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$\boxed{4}$ $u = x^3 + 3xy^2 - 3x$, $v = y^3 + 3x^2y - 3y$

$u_x = 3x^2 + 3y^2 - 3$ $\quad v_x = 6xy$

$u_y = 6xy$ $\quad v_y = 3y^2 + 3x^2 - 3$

$u_x = v_y$ ✓

$u_y = -v_x \Rightarrow 6xy = -6xy \Rightarrow 12xy = 0$

$x = 0$ OR $y = 0$

$f(z)$ diff only @ line $x=0$, $y=0$

$f'(z) = u_x + i v_x = 3x^2 + 3y^2 - 3 + i(6\underbrace{xy}_{\text{zero}})$

$\boxed{5}$ $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$u = \frac{1}{r}\cos\theta$, $v = -\frac{1}{r}\sin\theta$

$u_r = \frac{-1}{r^2}\cos\theta$ $\quad v_r = \frac{1}{r^2}\sin\theta$

$u_\theta = \frac{-1}{r}\sin\theta$ $\quad v_\theta = \frac{-1}{r}\cos\theta$

$u_r = \frac{1}{r}v_\theta \Rightarrow$ ✓

$u_\theta = -r v_r \Rightarrow$ ✓ Function Analaytic Except at $\boxed{r=0}$

$f'(z) = \frac{r}{z}(u_r + i v_r) = \frac{r}{z}\left(\frac{-1}{r^2}\cos\theta + i\frac{1}{r^2}\sin\theta\right)$

$= \frac{-r/r^2}{z}(\cos\theta + i\sin\theta) = \frac{e^{-i\theta}}{r^2e^{i\theta}} = \frac{-1}{r^2}e^{-i(2\theta)}$

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$f(z) = \frac{-1}{(r^2 e^{i\theta})^2} = \frac{-1}{z^2}$

2b) Prove that function $f(z) = e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$ is entire & find it's derivative

$u = e^{x^2-y^2}\cos 2xy \qquad v = e^{x^2-y^2}\sin 2xy$

$u_x = e^{x^2-y^2}[2x\cos 2xy - 2y\sin 2xy]$

$u_y = e^{x^2-y^2}[-2y\cos 2xy - 2x\sin 2xy]$

$v_x = e^{x^2-y^2}[2x\sin 2xy + 2y\cos 2xy]$

$v_y = e^{x^2-y^2}[-2y\sin 2xy + 2x\cos 2xy]$

$u_x = v_y$

$u_y = -v_x$

entire function

get $f(z)$ set $x = z$, $y = 0$

$f(z) = e^{z^2-0}[\cos(0) + i\sin(0)] = e^{z^2}$

$f'(z) = 2ze^{z^2}$ #

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Pb) Prove that an analytic function $f(z)$ must be Constant

if

① $\overline{f(z)}$ is analytic

② $Re(f(z)) = const$

③ $|f(z)| = const$

④ $|f(z)| = Analytic$

∴ $f(z)$ Analytic $\quad u_x = v_y$ & $u_y = -v_x$ — Ⓐ

Ⓘ $\overline{f(z)}$ is analytic $\Rightarrow \overline{f(z)} = \underbrace{u}_{U} - i\underbrace{v}_{V}$

$$U_x = u_x \quad , V_x = -v_x$$

$$U_y = u_y \quad , V_y = -v_y$$

$U_x = V_y \Rightarrow u_x = -v_y$ — ①

$U_y = -V_x \Rightarrow u_y = v_x$ — ②

Ⓐ & ① $\Rightarrow v_y = -v_y \Rightarrow v_y = zero$ $\quad$ ∴ $v_x = v_y = zero$

Ⓐ & ② $\Rightarrow v_x = -v_x \Rightarrow v_x = zero$ $\quad$ ∴ $\boxed{V = Const}$

also

Ⓐ & ① $u_x = -u_x$

Ⓐ & ② $u_y = -u_y$ $\quad u_x = u_y = 0 \Rightarrow \boxed{u = const}$

∴ $f(z) = const$

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Pb) Re(f(z)) ... From

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+ $Re(f(z)) = Const \Rightarrow \boxed{u = Const} \Rightarrow u_x = u_y = 0$

From (A)

$$\left.\begin{array}{l} u_x = v_y \Rightarrow v_y = 0 \\ u_y = -v_x \Rightarrow v_x = 0 \end{array}\right\} \boxed{v = Const}$$

$$f(z) = Const$$

---

(3) $|f(z)| = Const \Rightarrow |u + iv| = c \Rightarrow u^2 + v^2 = c$

Diff w.r.t $x \Rightarrow 2u u_x + 2v v_x = 0$ — (1)

Diff w.r.t $y \Rightarrow 2u u_y + 2v v_y = 0$ — (2)

From (A) $\rightarrow u_x = v_y , v_x = -u_y$ in (1) & (2)

$2u v_y + 2v v_x = 0$ — (3) $* (u)$

$2u(-v_x) + 2v v_y = 0$ — (4) $* (v)$

---

add) $2(u^2 + v^2) v_y = 0 \Rightarrow v_y = 0$

subtract) $2(u^2 + v^2) v_x = 0 \Rightarrow v_x = 0$

$\boxed{v = Const}$

$$\left.\begin{array}{l} u_x = v_y = 0 \\ u_y = -v_x = 0 \end{array}\right\} \Rightarrow \boxed{u = Const}$$

OR

$u^2 + v^2 = 0$

$\boxed{u = v = 0 = Const}$

---

(15)

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(4) $|f(z)|$ = Analytic $\Rightarrow |f(z)| = \underbrace{\sqrt{u^2+v^2}}_{U} + \underbrace{i(0)}_{V}$

$$U_x = \frac{2uu_x + 2vv_x}{2\sqrt{u^2+v^2}} \qquad V_x = 0$$

$$U_y = \frac{2uu_y + 2v\cancel{v_x}}{2\sqrt{u^2+v^2}} = \qquad V_y = 0$$

$$U_x = V_y \Rightarrow \frac{2uu_x + 2vv_x}{2\sqrt{u^2+v^2}} = 0 \Rightarrow \boxed{uu_x + vv_x = 0} \quad (1)$$

$$U_y = -V_x \Rightarrow \frac{2uu_y + 2vv_y}{2\sqrt{u^2+v^2}} = 0 \Rightarrow \boxed{uu_y + v\cancel{v_x} = 0} \quad (2)$$

$v_x = -u_y$ in (1) $\Rightarrow uu_x - vu_y = 0$ — (3)

$u_x = v_y$ in (2) $\Rightarrow uu_y + vu_x = 0$ — (4)

$$\left.\begin{aligned} (3)*u + (4)*v &\Rightarrow (u^2+v^2)u_x = 0 \Rightarrow u_x = 0 \\ (4)*u - (3)*v &\Rightarrow (u^2+v^2)u_y = 0 \Rightarrow u_y = 0 \end{aligned}\right\} \boxed{u = \text{Const}}$$

$$\left.\begin{aligned} u_x = v_y &\Rightarrow v_y = 0 \\ u_y = -v_x &\Rightarrow v_x = 0 \end{aligned}\right\} v\ \text{const}$$

$\boxed{\text{OR: } u^2+v^2 = \text{zero} \quad u=0,\ v=0 = \text{Const}}$

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...onic Function :

...nction f is harmonic if (rectangular form $f(x,y)$)

$$\boxed{f_{xx} + f_{yy} = 0}$$

in Polar form ($f(r,\theta)$)

$$\boxed{f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0}$$

$\Rightarrow$ if $f(z) = u + iv \Rightarrow$ Analaytic and u, v are harmonic, they are called harmonic conjected

(i.e) if he say u, v are harmonic conject

1. $f(z) = u + iv \Rightarrow$ Analaytic $\Rightarrow u_x = v_y$, $v_x = -u_y$
2. u, v harmonic $\Rightarrow u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

Pb Show that each of following functions is harmonic & find corresponding analytic function $f(z) = u + iv$.

1. $u = Arg(z)$
2. $u = \cos x \cosh y$
3. $u = \dfrac{y}{x^2 + y^2}$

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① $u = Arg(z) = \theta$

$u_r = 0 \ , u_{rr} = 0 \ , u_\theta = 1 , u_{\theta\theta} = 0$

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \Rightarrow$ harmonic

$\therefore u_r = \frac{1}{r} V_\theta \Rightarrow 0 = \frac{1}{r} V_\theta \Rightarrow V_\theta = 0 \Rightarrow V_r = h(r)$

$u_\theta = -r V_r \Rightarrow 1 = -r V_r \ , \ V_r = h'(r)$

$\therefore -\frac{1}{r} = h'(r) \Rightarrow h(r) = -\ln r + c = \boxed{\ln \frac{1}{r} + c}$

$\therefore V = \ln \frac{1}{r}$

$$\boxed{f(z) = \theta + i(\ln \frac{1}{r} + c}$$

$\boxed{2}$ $u = \cos x \, ch y$

$u_x = -\sin y \, ch y$ $\qquad$ $u_y = \cos x \, sh y$

$u_{xx} = -\cos x \, ch y$ $\qquad$ $u_{yy} = \cos x \, ch y$

$u_{xx} + u_{yy} = 0 \Rightarrow$ harmonic

$u_x = V_y \Rightarrow V_y = -\sin x \, ch y \Rightarrow \boxed{V = -\sin x \, sh y + h(x)}$ — ①

$u_y = -V_x \Rightarrow V_x = -\cos x \, sh y$ from ① get V_x

$-\cos x \, sh(y) + h'(x) = -\cos x \, sh(y) \Rightarrow h'(x) = 0$

$h(x) = const$

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$v = -\sin x \, sh y + C$

$f(z) = \cos x \, ch y - i \sin x \, sh y + C$

set $x = z, \ y = 0$

$f(z) = \cos z \, ch(0) - i \sin z \, sh(0) + C$

$= \cos z + C$

if required $f'(z) = -\sin z$

---

3 $u = \frac{y}{x^2+y^2} = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$ "Polar"

$u_r = \frac{-\sin\theta}{r^2}$, $u_{rr} = \frac{+2\sin\theta}{r^3}$

$u_\theta = \frac{\cos\theta}{r}$, $u_{\theta\theta} = -\frac{\sin\theta}{r}$

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{2\sin\theta}{r^3} - \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} = 0$

$\boxed{\text{harmonic}}$

$u_r = \frac{1}{r} v_\theta \Rightarrow \frac{-\sin\theta}{r^2} = \frac{1}{r} v_\theta$

$v_\theta = \frac{-\sin\theta}{r} \Rightarrow \boxed{v = \frac{\cos\theta}{r} + h(r)}$ — (1)

$u_\theta = -r v_r \Rightarrow \frac{\cos\theta}{r} = -r v_r \Rightarrow \frac{-\cos\theta}{r^2} = v_r \rightarrow$ from (1)

$\frac{-\cos\theta}{r^2} = \frac{-\cos\theta}{r^2} + h'(r) \Rightarrow h'(r) = 0 \Rightarrow h(r) = \text{const}$

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$$V = \frac{\cos\theta}{r} + c$$

$$f(z) = \frac{\sin\theta}{r} + i\left(\frac{\cos\theta}{r}\right) + c$$

if required $f'(z)$

get $f(z)$ in terms of $z \Rightarrow$ set $r = z$, $\theta = 0$

$$f(z) = \frac{\sin(0)}{z} + i\left(\frac{\cos 0}{z}\right) + c$$

$$f(z) = \frac{i}{z} + c$$

$$f'(z) = \frac{-i}{z^2}$$

P.b) Show that if v is harmonic conject of u in Domain D.

then uv is harmonic

sol

$\because u, v$ harmonic conj

$\boxed{1}$ $u_x = v_y$, $u_y = -v_x$ — (1)

$\boxed{2}$ $u_{xx} + u_{yy} = 0$ — (2)

$\boxed{3}$ $v_{xx} + v_{yy} = 0$ — (3)

$$\frac{\partial}{\partial x}(uv) = u_x v + u v_x$$

$$\frac{\partial^2}{\partial x^2}(uv) = u_{xx} v + u_x v_x + u v_{xx} + u_x v_x$$

$$= u_{xx} v + 2u_x v_x + u v_{xx} \quad \text{— (A)}$$

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so on

$$\frac{d^2}{dy^2}(uv) = u_{yy}v + 2u_y v_y + u v_{yy} \quad — (B)$$

From (2) $u_{xx} = -u_{yy}$

From (3) $v_{xx} = -u_{yy}$

From (1) $u_x = v_y$ & $v_x = -u_y$

} in (A)

$$\frac{d^2}{\partial x^2}(uv) = -u_{yy}v - 2u_y v_y - u v_{yy} \quad — (C)$$

$$\frac{d^2}{\partial x^2}(uv) + \frac{d^2}{dy^2}(uv) \rightarrow B+C = \text{Zero}$$

so uv harmonic

Pb) let $f(z)$ Analytic at domain D, u, v harmonic Conject

1. Prove that $\ln|f(z)|$ is harmonic?

* دليل
مهم ☹

أكيد

$$f = \ln|f(z)| = \ln(\sqrt{u^2+v^2}) = \frac{1}{2}\ln(u^2+v^2)$$

$$f_x = \frac{1}{2}\,\frac{2uu_x + 2vv_x}{u^2+v^2} = \frac{uu_x + vv_x}{u^2+v^2}$$

Bd, 1

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$$f_{xx} = \frac{(u_x^2 + uu_{xx} + v_x^2 + vv_{xx})(u^2+v^2) - (2uu_x + 2vv_x)(uu_x + \dots}{(u^2+v^2)^2}$$

$$= \frac{(u_x^2 + uu_{xx} + v_x^2 + vv_{xx})(u^2+v^2) - 2(uu_x + vv_x)^2}{(u^2+v^2)^2} \quad — (1)$$

and so on

$$F_{yy} = \frac{(u_y^2 + uu_{yy} + v_y^2 + vv_{yy})(u^2+v^2) - 2(uu_y + vv_y)^2}{(u^2+v^2)^2} \quad — (2)$$

$\because f(z)$ Analytic $\therefore u_x = v_y,\ u_y = -v_x$

harmonic Conj $\Rightarrow u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

$$f_{xx} + f_{yy} = \frac{(u^2+v^2)\left(u_x^2 + u_y^2 + u(\underbrace{\cancel{u_{xx} + u_{yy}}}_{zero}) + v_x^2 + v_y^2 + v(\underbrace{\cancel{v_{xx} + v_{yy}}}_{zero})\right)}{(u^2+v^2)^2}$$

$$- \frac{2(uu_x + vv_x)^2 - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$$

use $u_y = -v_x$

$v_x = v_y$

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$$f_{xx}+f_{yy}=\frac{(u^2+v^2)\left(u_x^2+v_x^2+u_x^2+v_x^2\right)-2\left[(uu_x+vv_x)^2+(uv_x-vu_x)^2\right]}{(u^2+v^2)^2}$$

$$=\frac{2(u^2+v^2)(u_x^2+v_x^2)-2\left[u^2u_x^2+v^2v_x^2+\cancel{2uvu_xv_x}+u^2v_x^2-\cancel{2uvu_xv_x}+v^2u_x^2\right]}{(u^2+v^2)^2}$$

$$=\frac{2(u^2+v^2)(u_x^2+v_x^2)-2\left[u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x)^2\right]}{(u^2+v^2)^2}$$

$$=\frac{2u^2\cancel{(u_x^2+v_x^2)}+2v^2\cancel{(u_x^2+v_x)^2}-2u^2\cancel{(u_x^2+v_x^2)}-\cancel{2v^2(u_x^2+v_x^2)}}{(u^2+v^2)^2}$$

$$=\text{Zero}$$

harmonic

23)

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ركز بيانات هامة

2017

Pb] Prove that the Polar form for Laplace's equation is given by

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

الحل

$$u = f(x,y), \quad x = r\cos\theta, \quad y = r\sin\theta \qquad u\begin{cases} x\begin{cases} r \\ \theta \end{cases} \\ y\begin{cases} r \\ \theta \end{cases} \end{cases}$$

$$u_r = u_x \cdot x_r + u_y \cdot y_r$$

$$= u_x\cos\theta + u_y\sin\theta \quad -(1) \qquad u_r\begin{cases} x\begin{cases} r \\ \theta \end{cases} \\ y\begin{cases} r \\ \theta \end{cases} \\ \theta \end{cases}$$

$$u_{rr} = u_{rx} \cdot x_r + u_{ry} \cdot y_r$$

$$= (u_{xx}\cos\theta + u_{yx}\sin\theta)\cos\theta$$

$$+ (u_{xy}\cos\theta + u_{yy}\sin\theta)\sin\theta$$

$$u_{rr} = u_{xx}\cos^2\theta + u_{yx}\sin\theta\cos\theta + u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta \rightarrow (2)$$

$$u_\theta = u_x \cdot x_\theta + u_y \cdot y_\theta \qquad u_\theta\begin{cases} x\begin{cases} r \\ \theta \end{cases} \\ y\begin{cases} r \\ \theta \end{cases} \\ r \\ \theta \end{cases}$$

$$= u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$u_{\theta\theta} = u_{\theta x} \cdot x_\theta + u_{\theta y} \cdot y_\theta + \frac{\partial u_\theta}{\partial\theta}$$

$$= (u_{xx}(-r\sin\theta) + u_{yx} r\cos\theta)(-r\sin\theta)$$

$$(u_{xy}(-r\sin\theta) + u_{yy} r\cos\theta)(r\cos\theta)$$

$$+ r u_x\cos\theta + r u_y\sin\theta$$

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$$u_{\theta\theta} = r^2\sin^2\theta\, u_{xx} - r^2\sin\theta\cos\theta\, u_{yx} - r^2\sin\theta\cos\theta\, u_{xy}$$

$$+ u_{yy}\, r^2\cos^2\theta - r u_x\cos\theta - r u_y\sin\theta \quad -(3)$$

use (1) & (2) & (3)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = u_{xx}(\sin^2\theta + \cos^2\theta) + u_{yy}(\sin^2\theta + \cos^2\theta)$$

$$= u_{xx} + u_{yy}$$

∴ Laplace equation in rectangular form $u_{xx} + u_{yy} = 0$

$$\text{or } u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \#$$

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-2016

Pb) starting from $f'(z) = u_x + i v_x$ Prove that $f'(z) = \frac{r}{z}(u_r +$ [illegible]
and then use it to find derivative of $\ln(z)$ for $z \neq 0$

ج1

$f'(z) = u_x + i v_x$

$r = \sqrt{x^2+y^2} \Rightarrow r_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \Rightarrow \boxed{r_x = \frac{x}{r}}$ — ①

$\theta = \tan^{-1}\left(\frac{y}{x}\right) \rightarrow \theta_x = \frac{-y/x^2}{1+y^2/x^2} = \frac{-y}{x^2+y^2} \Rightarrow \boxed{\theta_x = \frac{-y}{r^2}}$ — ②

∵ $f(z)$ differentiable ⇒ satisfy C-R condition

$u_r = \frac{1}{r} v_\theta \Rightarrow \boxed{r u_r = v_\theta}$ — ③

$u_\theta = -r v_r \Rightarrow \boxed{r v_r = -u_\theta}$ — ④

∵ $u = f(r, \theta)$ & $r = g(x, y)$ & $\theta = h(x, y)$ and same for v

u and v < r < x, y ; θ < x, y

$u_x = u_r \cdot r_x + u_\theta \cdot \theta_x$ from ① & ② & ④

$= u_r \frac{x}{r} + (-r v_r)\left(\frac{-y}{r^2}\right)$

$\boxed{u_x = \frac{1}{r}(x u_r + y v_r)}$ — Ⓐ

$v_x = v_r \cdot r_x + v_\theta \cdot \theta_x$ ① & ② & ③

$= v_r \cdot \frac{x}{r} + (r u_r)\cdot\left(\frac{-y}{r^2}\right)$

$\boxed{v_x = \frac{1}{r}[x v_r - y u_r]}$ — Ⓑ

$\therefore f'(z) = u_x + i v_x$ use Ⓐ & Ⓑ

$f'(z) = \frac{1}{r}(x u_r + y v_r + i x v_r - i u_r y]$

$= \frac{1}{r}\left(u_r(x-iy) + i v_r(x-iy)\right]$

$f'(z) = \frac{x-iy}{r}(u_r + i v_r) = \frac{\bar{z}}{r}(u_r + i v_r) * \frac{z}{z}$

$f'(z) = \frac{z\bar{z}}{rz}(u_r + i v_r) = \frac{r^2}{rz}(u_r + i v_r)$ | $z\bar{z} = x^2 + y^2 = r^2$

$\boxed{f'(z) = \frac{r}{z}(u_r + i v_r)}$

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Pb] Prove Polar form for Cauchy Remain Condition

$$u_r = \frac{1}{r} v_\theta , \quad u_\theta = -r v_r$$

حل

$u = f(x,y)$, $x = r\cos\theta$

$v = g(x,y)$, $y = r\sin\theta$

$$u \begin{cases} x \begin{cases} r \\ \theta \end{cases} \\ y \begin{cases} r \\ \theta \end{cases} \end{cases}$$

$u_r = u_x \cdot x_r + u_y \cdot y_r$

$u_r = u_x(\cos\theta) + u_y(\sin\theta)$ — ①

$u_\theta = u_x \cdot x_\theta + u_y \cdot y_\theta$

$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$ — ②

and so on

$v_r = v_x(\cos\theta) + v_y(\sin\theta)$ — ③

$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$ — ④

use Cauchy Remain Condition

$$u_x = v_y , \quad u_y = -v_x \quad \text{— Ⓐ}$$

Ⓐ in ①

$\Rightarrow u_r = v_y(\cos\theta) - v_x(\sin\theta)$

from ④ $\frac{1}{r} v_\theta = -v_x\sin\theta + v_y\cos\theta$

$\Rightarrow \boxed{u_r = \frac{1}{r} v_\theta}$

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use (A) in (2)

$$u_\theta = V_y(-r\sin\theta) + (-V_x)(r\cos\theta)$$

from (3)

$$-rV_r = -V_x(r\cos\theta) - V_y(r\sin\theta)$$

$$\boxed{u_\theta = -rV_r}$$

(#)

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