Exercise sheet (4)
Functions of Complex Variables

[1] Show that

(a) $u = \dfrac{y}{x^2+y^2}$ is harmonic and find its conjugate "v"

and find then the analytic function $f(z) = u + iv$ interms of "z" $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

$\therefore u = \dfrac{r\sin\theta}{r^2} = \dfrac{\sin\theta}{r} \Rightarrow u_r = \dfrac{-\sin\theta}{r^2}$, $u_{rr} = \dfrac{2\sin\theta}{r^3}$

$u_\theta = \dfrac{\cos\theta}{r}$, $u_{\theta\theta} = \dfrac{-\sin\theta}{r}$

$\therefore u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = \dfrac{2\sin\theta}{r^3} - \dfrac{\sin\theta}{r^3} - \dfrac{\sin\theta}{r^3} = 0$

$\therefore f(z)$ is harmonic function

$\because f(z)$ is analytic $\therefore r u_r = v_\theta \Rightarrow \dfrac{-\sin\theta}{r} = v_\theta$

$r v_r = -u_\theta \Rightarrow v_r = -\dfrac{\cos\theta}{r^2} \rightarrow (1)$

$v = \dfrac{\cos\theta}{r} + h(r) \Rightarrow v_r = \dfrac{-\cos\theta}{r^2} + h'(r) \rightarrow (2)$

from (1) & (2) $h'(r) = 0$ $\therefore h(r) = K \Rightarrow v = \dfrac{\cos\theta}{r} + K$

$f(z) = \dfrac{\sin\theta}{r} + i\left(\dfrac{\cos\theta}{r} + K\right) = \dfrac{y}{x^2+y^2} + i\left(\dfrac{x}{x^2+y^2} + K\right)$

let $x = z$ & $y = 0$

$f(z) = i\left(\dfrac{z}{z^2} + K\right) = i\left(\dfrac{1}{z} + K\right)$

Subject : ....

(b) $v = \frac{x}{x^2+y^2}$, find v and $f(Z)$ interms of Z

$v = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$, $v_r = \frac{-\cos\theta}{r^2} \Rightarrow v_{rr} = \frac{2\cos\theta}{r^3}$

$v_\theta = \frac{-\sin\theta}{r} \Rightarrow v_{\theta\theta} = \frac{-\cos\theta}{r}$

$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = \frac{2\cos}{r^3} - \frac{\cos\theta}{r^3} - \frac{\cos\theta}{r^3} = 0$

$\therefore f(z)$ is harmonic function

$r u_r = v_\theta \Rightarrow u_r = \frac{v_\theta}{r} = \frac{-\sin\theta}{r^2} \rightarrow (1)$

$r v_r = -u_\theta \Rightarrow u_\theta = \frac{\cos\theta}{r} \rightarrow u = \frac{\sin\theta}{r} + h(r)$

$u_r = \frac{-\sin\theta}{r} + h'(r) \rightarrow (2)$ from (1) & (2) $h'(r) = 0$

$\therefore h(r) = K$

$\boxed{u = \frac{\sin\theta}{r} + K}$ $f(z) = \left(\frac{\sin\theta}{r} + K\right) + i\,\frac{\cos\theta}{r}$

$f(z) = \left(\frac{y}{x^2+y^2} + K\right) + i\,\frac{x}{x^2+y^2}$ let $x = z$, $y = 0$

$f(z) = K + i\,\frac{1}{z}$

Subject : ....

(c) $v = \ln\sqrt{x^2+y^2}$, find u and find f(Z) in terms of Z

$x^2+y^2 = r^2 \quad \therefore v = \ln r \quad , v_r = \frac{1}{r} \quad , v_{rr} = \frac{-1}{r^2}$

$v_\theta = 0 \quad , v_{\theta\theta} = 0$

$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = \frac{-1}{r^2} + \frac{1}{r^2} = 0 \quad \therefore f(z)$ is harmonic

$rv_r = -u_\theta \Rightarrow u_\theta = -1 \quad , ru_r = v_\theta \Rightarrow u_r = 0 \rightarrow (1)$

$u = -\theta + h(r) \Rightarrow u_r = h'(r) \rightarrow (2)$

from (1) & (2) $\quad h'(r) = 0 \quad \therefore h(r) = K \Rightarrow u = -\theta + K$

$f(Z) = -\theta + K + i\ln r = \left(\tan^{-1}\left(\frac{-y}{x}\right) + K\right) + i\ln\sqrt{x^2+y^2}$

let $x = Z$, $y = 0$

$\therefore f(Z) = (n\pi + K) + i\ln Z$

(d) $u = 2\tan^{-1}\left(\frac{y}{x}\right)$, find v and f(Z) in terms of Z

$u = 2\theta \quad , u_r = 0 \quad , u_{rr} = 0 \quad , u_\theta = 2 \quad , u_{\theta\theta} = 0$

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \therefore f(Z)$ is harmonic

$ru_r = v_\theta \Rightarrow v_\theta = 0$ & $rv_r = -u_\theta \Rightarrow v_r = \frac{-2}{r}$

$v = -2\ln r + h(\theta) \Rightarrow v_\theta = h'(\theta) = 0 \Rightarrow h(\theta) = K$

$v = -2\ln r + K \quad \therefore f(Z) = 2\theta + i(-\ln r^2 + K)$

$f(Z) = 2\tan^{-1}\left(\frac{y}{x}\right) + i(-\ln(x^2+y^2) + K)$

let $x = Z \quad y = 0$

$f(Z) = 2n\pi + i(-2\ln Z + K)$

(e) $f(z) = e^{(x^2-y^2)}(\cos 2xy + i\sin 2xy)$ is an entire function. Hence, find its derivatives in terms of z

$u = e^{x^2-y^2}\cos(2xy)$, $v = e^{x^2-y^2}\sin(2xy)$

$u_x = 2x e^{x^2-y^2}\cos(2xy) - 2y e^{x^2-y^2}\sin(2xy)$

$u_y = -2y e^{x^2-y^2}\cos(2xy) - 2x e^{x^2-y^2}\sin(2xy)$

$v_x = 2x e^{x^2-y^2}\sin(2xy) + 2y e^{x^2-y^2}\cos(2xy)$

$v_y = -2y e^{x^2-y^2}\sin(2xy) + 2x e^{x^2-y^2}\cos(2xy)$

$\because u_x = v_y$ and $u_y = -v_x$

$\therefore$ C.R. Equations satisfied on Z-Plane

$\because$ u, v continous on Z-Plane

$\therefore$ f(z) is differentiable every where

$\therefore$ f(z) is analytic every where $\therefore$ f(z) is entire

(f) $u = \dfrac{x}{x^2+y^2}$ is harmonic, find v and f(z) in terms of z

$u = \dfrac{\cos\theta}{r}$, $u_r = \dfrac{-\cos\theta}{r^2}$, $u_{rr} = \dfrac{2\cos\theta}{r^3}$

$u_\theta = \dfrac{-\sin\theta}{r}$, $u_{\theta\theta} = \dfrac{-\cos\theta}{r}$

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \dfrac{2\cos\theta}{r^3} - \dfrac{\cos\theta}{r^3} - \dfrac{\cos\theta}{r^3} = 0$

$\therefore$ f(z) is harmonic , $rv_r = -u_\theta \Rightarrow v_r = \dfrac{\sin\theta}{r^2}$ — (1)

$ru_r = v_\theta \Rightarrow v_\theta = \dfrac{-\cos\theta}{r} \Rightarrow v = \dfrac{-\sin\theta}{r} + h(r)$

$v_r = \dfrac{\sin\theta}{r^2} + h'(r)$ from (1) & (2) $h'(r) = 0$, $h(r) = K$

Subject : ....

$\therefore V = \frac{-\sin\theta}{r} + K \quad \therefore f(Z) = \frac{\cos\theta}{r} + i\left(\frac{-\sin\theta}{r} + K\right)$

$f(Z) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2} + K\right)$ let $x = Z$ $y = 0$

$f(Z) = \frac{1}{Z} + iK$

[2] Find & solve

i. $e^{2Z} = 1+\sqrt{3}i \Rightarrow 2Z = \ln(1+\sqrt{3}i) = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$

$Z = \frac{1}{2}\ln 2 + i\left(\frac{\pi}{6} + n\pi\right) = \ln\sqrt{2} + i\pi\left(\frac{1}{6} + n\right)$

ii. $\cos Z = \cosh 5 = \cos(x+iy)$

$\cos x \cosh y - i \sin x \sinh y = \cosh 5$

$\sin x \sinh y = 0$

$\sin x = 0$

$x = n\pi$

$n = 0, \pm 1, \pm 2, ---$

$\sinh y = 0$

$y = 0$

$\cos x = \cosh 5$

refused

$\cos x \cosh y = \cosh 5$

$\therefore x = n\pi$

$\cosh y = \frac{\cosh 5}{(-1)^n}$ n even $\cosh y > 0$

$\cosh y = \cosh 5$

$\therefore y = \pm 5$

$\therefore Z = n\pi \pm i5 \quad ; n = 0, \pm 2, \pm 4$

ضوئيا بـ CamScanner

Subject : .... / /

iii. $\cosh Z = -2 = \cosh x \cos y + i \sinh x \sin y$

$\sinh x \sin y = 0$, $\cosh x \cos y = -2$ — ①

$\sinh x = 0$

$x = 0$

Subs. in ①

$\cos y = -2$

refused

$\sin y = 0$

$y = n\pi$

subs. in ①

$\cosh x = \dfrac{-2}{\cos(n\pi)} = \dfrac{-2}{(-1)^n} = 2$

But $\cosh x$ can't be -ve ∴ n must be odd

$\therefore x = \pm \cosh^{-1}(2)$

$Z = \pm \cosh^{-1}(2) + i n\pi$, $n = \pm 1, \pm 3, \pm 5, ---$

iv. $\sin Z = 4 = \sin x \cosh y + i \cos x \sinh y$

$\cos x \sinh y = 0$, $\sin x \cosh y = 4$ → ①

$\cos x = 0$

$x = (n + \frac{1}{2})\pi$

Subs. in ①

$(-1)^n \cosh y = 4$

$\cosh y = \dfrac{4}{(-1)^n}$ n is even

$\cosh y = 4$

$y = \pm \cosh^{-1}(4)$

$\sinh y = 0$

$y = 0$

subs. in ①

$\sin x = 4$

refused

Subject:

v. $\left(\frac{e}{2}(1+\sqrt{3}i)\right)^{3\pi i} = \left(\frac{e}{\cancel{2}}(\cancel{2} e^{i(\frac{\pi}{3}+2\pi K)})\right)^{3\pi i}$

$= \left(e^{i\pi(\frac{1}{3}+2K)+1}\right)^{3\pi i} = e^{-\pi^2(1+6K)+3\pi i}$

$= e^{\pi^2(-6K-1)} e^{3\pi i} = -e^{\pi^2(-1-6K)}$, $K=0,\pm1,\pm2,--$

The Principal value @ $K=0$ $\therefore Z = -e^{-\pi^2} = \frac{-1}{e^{\pi^2}}$

$|Z| = \frac{1}{e^{\pi^2}}$, $\arg(Z) = \pi$

vi. $(\sqrt{3}+i)^{5\pi i} = e^{5\pi i \ln(\sqrt{3}+i)} = e^{5\pi i[\ln 2 + i(\frac{\pi}{6}+2\pi K)]}$

$= e^{i5\pi \ln 2 - \pi^2(5/6+10K)} = e^{\pi^2(-5/6-10K)} e^{i5\pi\ln 2}$

$|Z| = e^{\pi^2(-5/6)}$ $\Longrightarrow$ The Principle @ $K=0$

$\arg(Z) = \pi \ln 2^5 = \pi \ln 32$

vii. $(1+\sqrt{3}i)^{2+i} = e^{(2+i)\ln(1+\sqrt{3}i)} = e^{(2+i)(\ln 2 + i(\frac{\pi}{3}+2\pi k))}$

$= e^{2\ln 2 + i\ln 2 + i\frac{2\pi}{3} - \frac{\pi}{3} + i4\pi K - 2\pi K}$

$= e^{i(\ln 2 + \frac{2\pi}{3} + 4\pi K)} e^{2\ln 2 - \frac{\pi}{3} - 2\pi K}$

The Principle $|Z| = e^{\ln 4 - \frac{\pi}{3}}$

$\arg(Z) = \ln 2 + \frac{2\pi}{3}$