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ملزمة (7)

رياضـــــــة

Complex variables functions

ثانية كهرباء

Complex Variables & Functions -

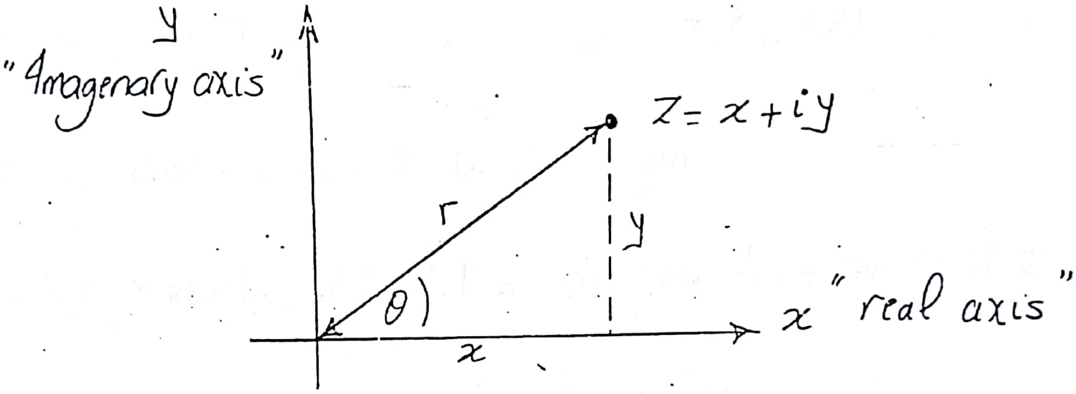
- We define $i = \sqrt{-1} \Rightarrow i^2 = -1,\ i^3 = -i$ and $i^4 = 1$.

- Any Complex number z may be written in 2 forms :-

1) Rectangular form :- $z = x + iy$ where

$$x = Re(z) = r\cos\theta = \frac{1}{2}(z + \bar{z})$$

$$y = Im(z) = r\sin\theta = \frac{1}{2i}(z - \bar{z})$$



Complex - plane

Since $z = x + iy = r\cos\theta + i\, r\sin\theta = r(\cos\theta + i\sin\theta)$

$= r\,cis\,\theta = re^{i\theta} \Rightarrow$ Polar form

2) Polar form :-

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$$

where ,

$r =$ Modulus $z = |z| = \sqrt{x^2+y^2}$ "must be ≥ 0"

$\theta = \tan^{-1}\frac{y}{x} = \arg(z) = \operatorname{Arg}(z) + 2n\pi$, where

$\operatorname{Arg}(z)$ is the principle value of θ which must satisfy

$$-\pi < \operatorname{Arg}(z) \leq \pi$$

Also remember that :- If $z = x+iy = re^{i\theta}$

1) Complex Conjugate of $z = \bar{z} = x - iy = re^{-i\theta}$

2) $|z| = \sqrt{x^2+y^2} = r \Rightarrow z\bar{z} = |z|^2$

$|z| =$ distance from z to the origin.

3) More generally, $|z - z_0| =$ distance from the point z to the point z_0 in the complex plane.

4) $\overline{(z_1 z_2)} = \bar{z}_1 . \bar{z}_2$

$|z_1 z_2| = |z_1| . |z_2|$

5) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

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1)

$$Z = 3e^{i\frac{5\pi}{4}}$$

$$\Rightarrow Z = 3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{3}{\sqrt{2}} - i\frac{3}{\sqrt{2}}$$

2) $Z = \text{Cis}\,\pi$

$$\Rightarrow Z = \cos\pi + i\sin\pi = -1$$

3) $Z = 7\,\text{Cis}\left(-\frac{7\pi}{6}\right)$

$$\Rightarrow Z = 7\left(\cos -\frac{7\pi}{6} + i\sin -\frac{7\pi}{6}\right) = -\frac{7\sqrt{3}}{2} + i\frac{7}{2}$$

4) $Z = \dfrac{3\,\text{Cis}(\pi/4)}{5\,\text{Cis}(\pi/12)} + 2i$

$$\Rightarrow Z = \frac{3e^{i\pi/4}}{5e^{i\pi/12}} + 2i = \frac{3}{5}e^{i(\pi/6)} + 2i$$

$$= \frac{3}{5}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) + 2i = \frac{3\sqrt{3}}{10} + i\frac{3}{10} + 2i$$

$$= \frac{3\sqrt{3}}{10} + i\frac{23}{10}$$

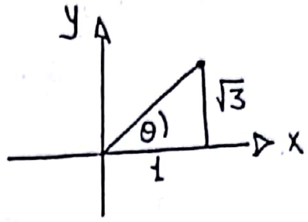
5) $Z = (1+i)^7$

$$\Rightarrow Z = (1+i)^7 = \left(\sqrt{2}e^{i\pi/4}\right)^7 = 8\sqrt{2}\,e^{i\,7\pi/4}$$

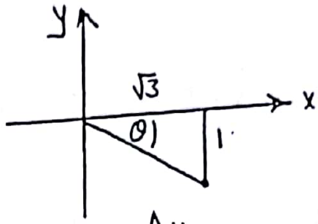
$$= 8\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 8 - i8$$

Example:- Find the Polar form of these Complex no. :-

1) $Z = 1 + i\sqrt{3}$

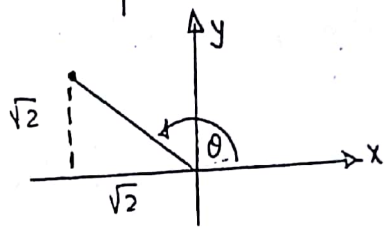


$Z = 2e^{i(\pi/3 + 2k\pi)}$



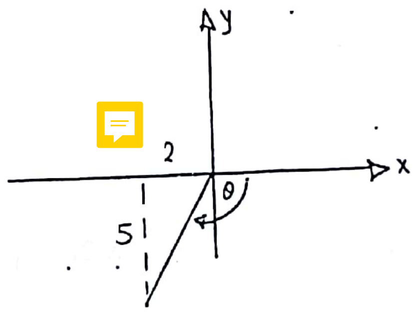
2) $Z = \sqrt{3} - i$

$Z = 2e^{i(-\pi/6 + 2k\pi)}$



3) $Z = -\sqrt{2} + i\sqrt{2}$

$Z = 2e^{i(\pi - \pi/4)} = 2e^{i(3\pi/4 + 2k\pi)}$



4) $Z = -2 - i5$

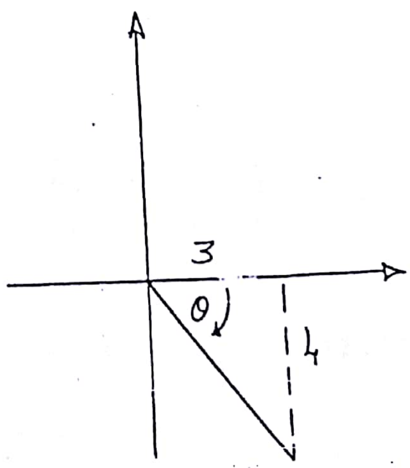
$Z = \sqrt{29}\, e^{-i(\pi - \tan^{-1} 5/2 + 2k\pi)}$

5) $Z = 5 \Rightarrow Z = 5e^{i(0 + 2k\pi)} = 5e^{i2k\pi}$

6) $Z = -3i \Rightarrow Z = 3e^{i(-\pi/2 + 2k\pi)}$

7) $Z = \sqrt{2}i \Rightarrow Z = \sqrt{2}e^{i(\pi/2 + 2k\pi)}$

8) $Z = -\frac{1}{2} \Rightarrow Z = \frac{1}{2}e^{i(\pi + 2k\pi)}$



9) $Z = 3 - 4i \Rightarrow$

$Z = 5e^{i(-\tan^{-1}\frac{4}{3} + 2k\pi)}$

; where $k = 0, \pm 1, \pm 2, \ldots$

De Moivre Theorm :-

1) If $z = x+iy = re^{i\theta} = r\,cis\,\theta$

$$\Rightarrow z^n = (x+iy)^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n cis(n\theta) = r^n(\cos n\theta + i\sin n\theta)$$

2) The n roots of the equation $z^n = z_0$, where n is postive integer, is found as follow

Let the roots be $z = re^{i\theta}$

$$\Rightarrow z^n = z_0$$

$$r^n e^{in\theta} = r_0 e^{i(Arg(z_0)+2k\pi)}$$

$$\Rightarrow r^n = r_0 \longrightarrow r = r_0^{1/n}$$

$$n\theta = Arg(z_0) + 2k\pi$$

$$\Rightarrow \theta = \frac{Arg(z_0)+2k\pi}{n}$$

The required n roots are $r_0^{\frac{1}{n}} cis\left(\frac{Arg(z_0)+2k\pi}{n}\right)$

; $k = 0, 1, 2, \ldots, (n-1)$.

Example:-

Solve the equation $z^4 + 16 = 0$

Solution:- $z^4 + 16 = 0 \Rightarrow z^4 = -16$

$$\Rightarrow r^4 e^{i4\theta} = 16\, e^{i(\pi + 2k\pi)}$$

$$\Rightarrow r^4 = 16 \Rightarrow r = 2 \quad \text{and} \quad 4\theta = \pi + 2k\pi$$

$$\Rightarrow \theta = \frac{\pi + 2k\pi}{4} \; ; k = 0, 1, 2, 3$$

The 4 roots are $z = 2\, e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)}$; $k = 0, 1, 2, 3.$

$$= \pm\sqrt{2} \pm i\sqrt{2}$$

Example:-

Solve $z^2 - 1 + \sqrt{3}i = 0$

Solution: let the required 2 roots be $z = re^{i\theta} \Rightarrow$

$$r^2 e^{i2\theta} = 1 - \sqrt{3}i = 2\, e^{i(-\pi/3 + 2k\pi)}$$

$$\Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$$

$$2\theta = -\frac{\pi}{3} + 2k\pi \Rightarrow \theta = -\frac{\pi}{6} + k\pi$$

The roots are $\sqrt{2}\, e^{i\left(-\frac{\pi}{6} + k\pi\right)}$; $k = 0, 1.$

Example :- Evaluate $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40}$

Solution :- It is much more easier to use the polar form :: $1+i\sqrt{3} = 2e^{i\pi/3}$ &

$1-i = \sqrt{2}\,e^{-i\pi/4}$

$\Rightarrow \frac{1+i\sqrt{3}}{1-i} = \frac{2e^{i\pi/3}}{\sqrt{2}\,e^{-i\pi/4}} = \sqrt{2}\,e^{i\frac{7\pi}{12}}$

$\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40} = \left(\sqrt{2}\,e^{i\,7\pi/12}\right)^{40} = 2^{20}\,e^{i\left(\frac{70\pi}{3}\right)}$

$= 2^{20}\left(\cos\frac{70\pi}{3} + i\sin\frac{70\pi}{3}\right) = 2^{20}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= -2^{19} - i\,2^{19}\sqrt{3}$

Example : Find the domain of $f(z) = \frac{z^3+1}{\sqrt{2}z^5 - \sqrt{3} - i}$

Solution :- The dominator must not equal to zero

But $\sqrt{2}z^5 - \sqrt{3} - i = 0$ when $z^5 = \frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\Rightarrow r^5 e^{i5\theta} = \sqrt{2}\,e^{i\left(\frac{\pi}{6}+2k\pi\right)}$

$\Rightarrow r^5 = \sqrt{2}$ & $5\theta = \frac{\pi}{6} + 2k\pi$

$\Downarrow$ $\Downarrow$

$r = 2^{1/10}$ $\theta = \frac{\pi}{30} + \frac{2\pi}{5}k$

Domain = Complex plane $- \left\{ 2^{\frac{1}{10}} e^{i\left(\frac{\pi}{30} + \frac{2\pi}{5}k\right)} \right\}$

$; k = 0, 1, 2, 3, 4.$

Function of Complex Variable :-

If $f(z)$ is a Complex function in the Complex Variable z,

then

$$f(z) = f(x+iy) = u + iv = Re^{i\phi}$$

where, $u = Re(f(z)) = u(x,y)$

$v = Im(f(z)) = v(x,y)$

$$R = \sqrt{u^2 + v^2} = |f(z)|$$

$$\phi = arg(f(z))$$

for example ; ① Consider $f(z) = z^2$

$$\Rightarrow f(z) = f(x+iy) = (x+iy)^2 = x^2 - y^2 + i\,2xy$$

$$\Rightarrow u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

OR $f(z) = f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$$\Rightarrow R = r^2 \quad \text{and} \quad \phi = 2\theta$$

② Consider $f(z) = 1/z$

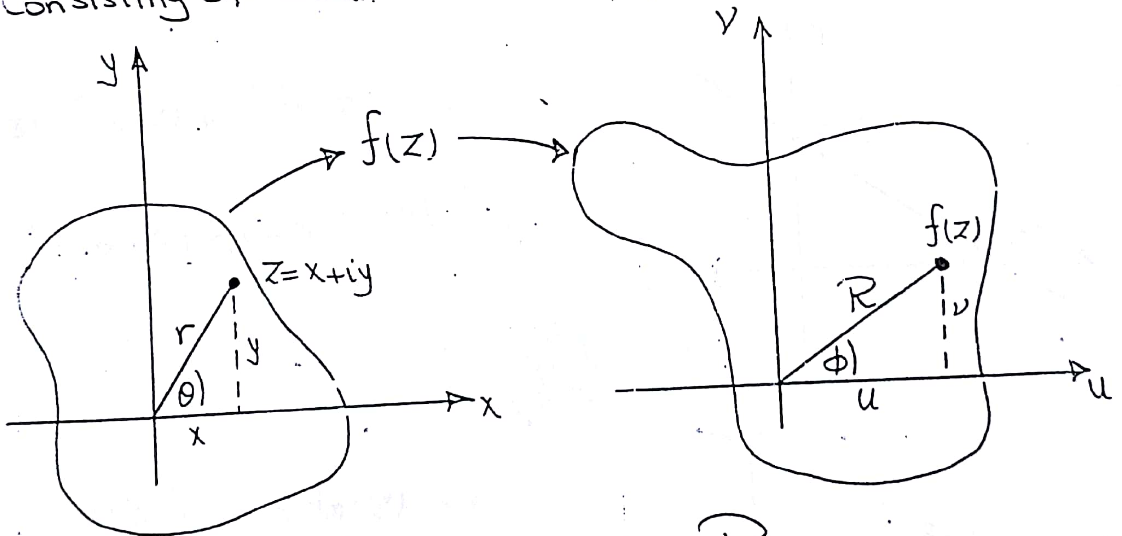
$$\Rightarrow f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2} \ \& \ v = \frac{-y}{x^2+y^2}$$

$$f(re^{i\theta}) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \Rightarrow R = \frac{1}{r} \ \& \ \phi = -\theta.$$

Domain of the Complex fn $f(z)$:- is the values of $z = x + iy$ under which $f(z)$ is defined.

- The domain of $f(z)$ is represented in a Complex plane Consisting of x axis and y axis.

- **Range of the Complex fn $f(z)$:-** is the Corresponding values of $f(z)$, it is represented in a Complex plane Consisting of u axis and v axis.

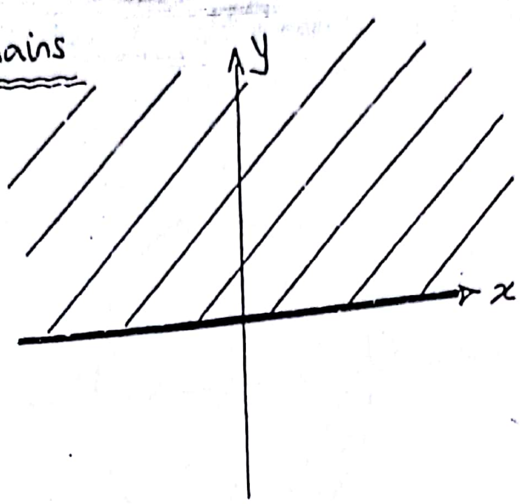


Domain Range

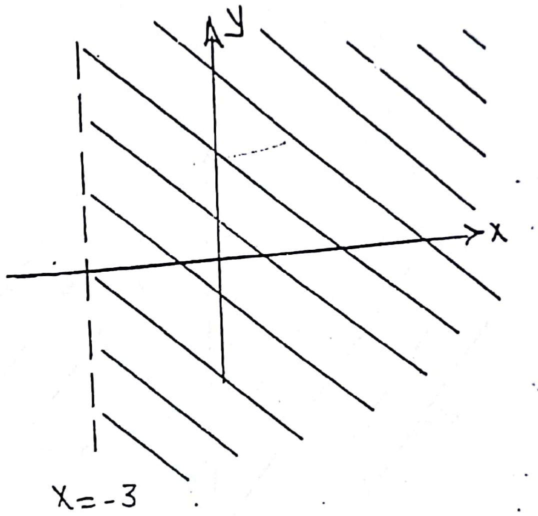
The Transformation from a domain to a range under the fn $f(z)$ is Called mapping or transformation.

Examples:: Sketch these Domains

1) $\operatorname{Im}(z) \geq 0$

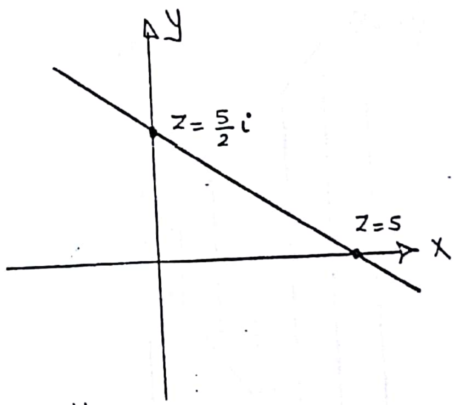


2) $\operatorname{Re}(z) > -3$



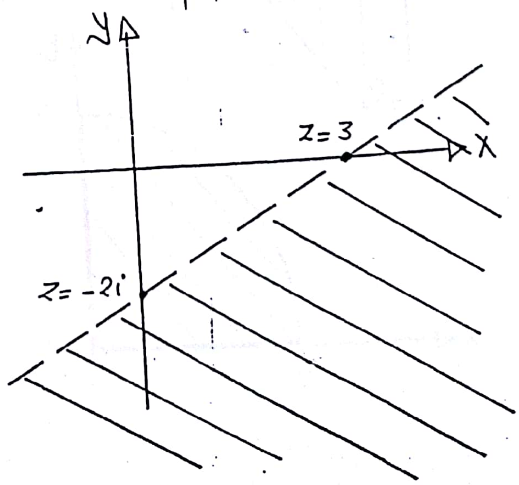
3) $\operatorname{Re}(z) + 2\operatorname{Im}(z) = 5$

$\Rightarrow x + 2y = 5$ " st. line "



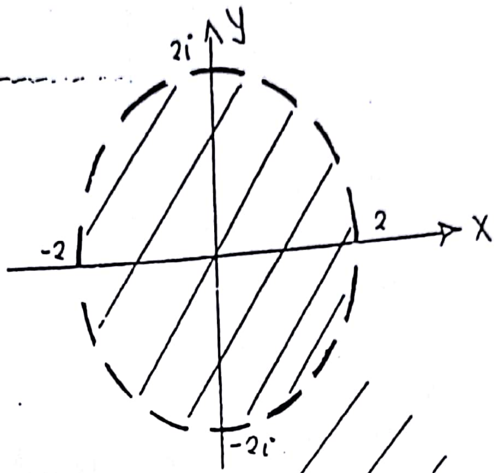
4) $2\operatorname{Re}(z) - 3\operatorname{Im}(z) > 6$

$\Rightarrow 2x - 3y > 6$

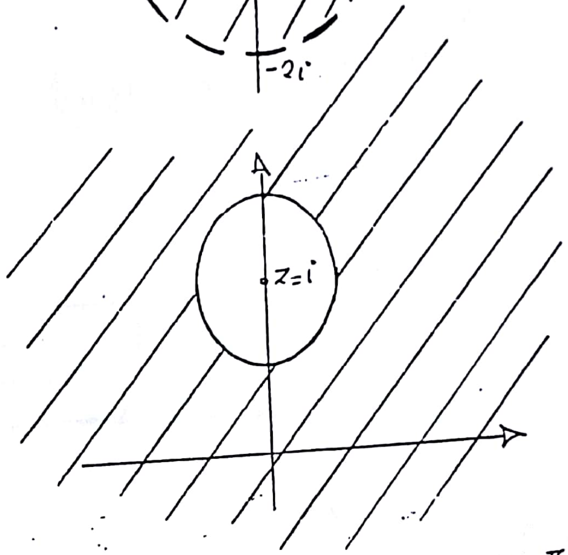


Note :- $|z-z_0| = a$ is an equation of circle with center at z_0 and radius a.

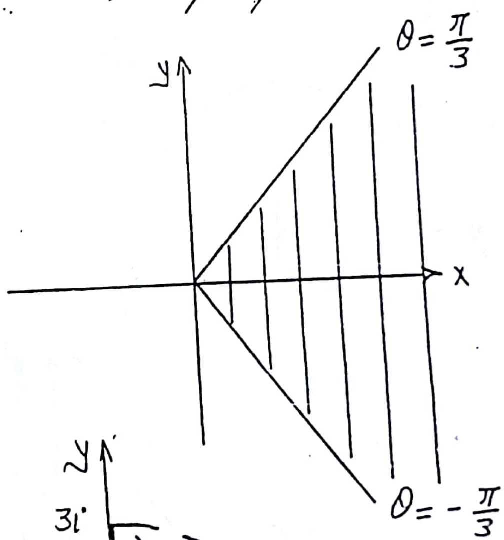
) $|z| < 2$



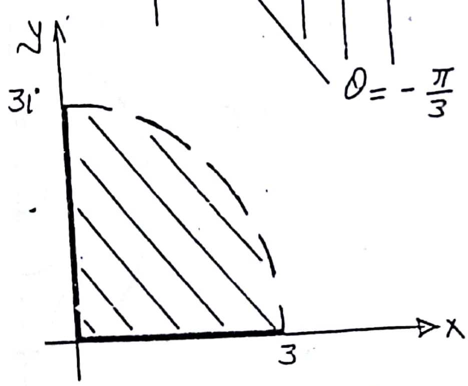
5) $|z-i| \geq \frac{1}{2}$



7) $-\frac{\pi}{3} \leq arg(z) \leq \frac{\pi}{3}$



8) $|z| < 3$, $0 \leq arg z \leq \pi/2$



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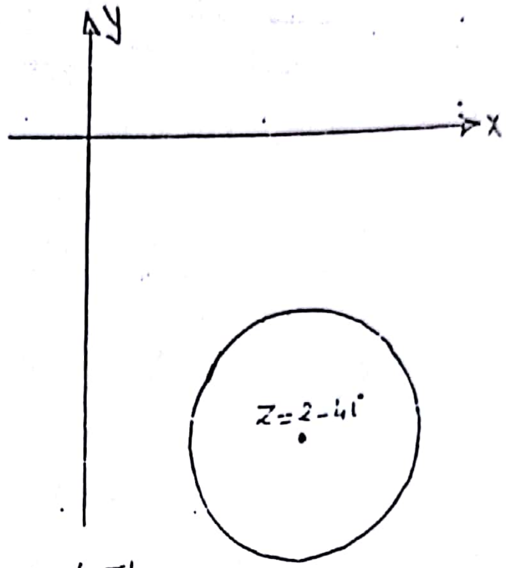
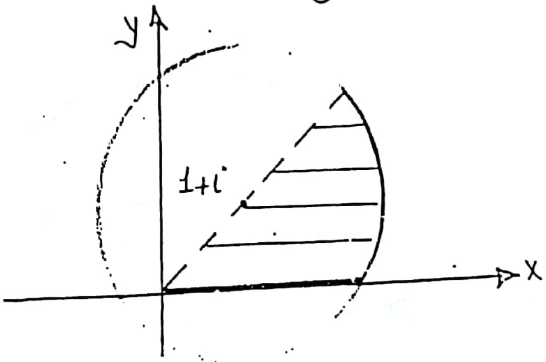
9) $|z-2+4i| = 1$

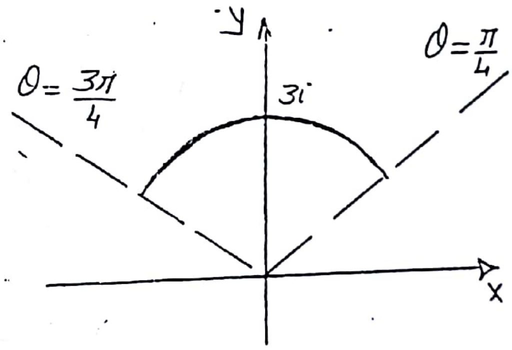
is a circle of radius 1 & center at $z = 2-4i$



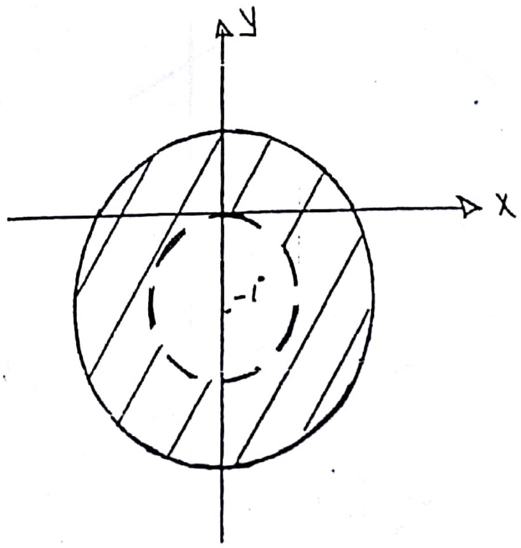
10) $|z-1-i| \leqslant 2$ and $0 \leqslant Arg(z) < \pi/4$



11) $|z| = 3$ & $\frac{\pi}{4} < Arg(z) < \frac{3\pi}{4}$



12) $1 < |z+i| \leqslant 2$



Example:- Find the image of the region

$0 \leqslant \arg(z) \leqslant \pi/3$ & $0 < |z| \leqslant 3$

under the transformation i) $f(z) = z^2$

ii) $f(z) = \frac{1}{z}$

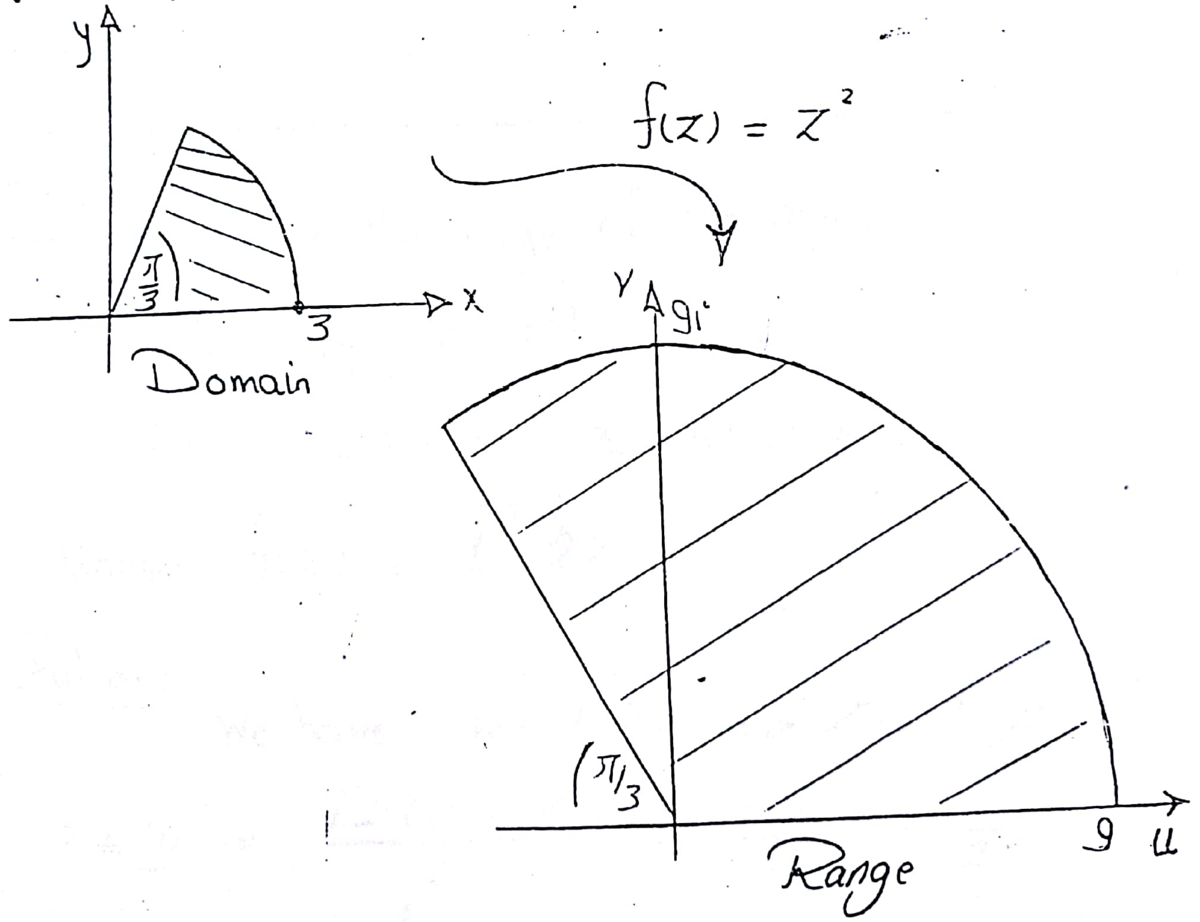
Solution:- i) We have $w = f(z) = z^2$

$\Rightarrow w = (re^{i\theta})^2 = r^2 e^{i2\theta} = Re^{i\phi}$

$\Rightarrow R = r^2$ & $\phi = 2\theta$

$\because 0 \leqslant \arg(z) \leqslant \frac{\pi}{3} \Rightarrow 0 \leqslant \phi \leqslant \frac{2\pi}{3}$

$\because 0 < |z| \leqslant 3 \Rightarrow 0 < R \leqslant 9$



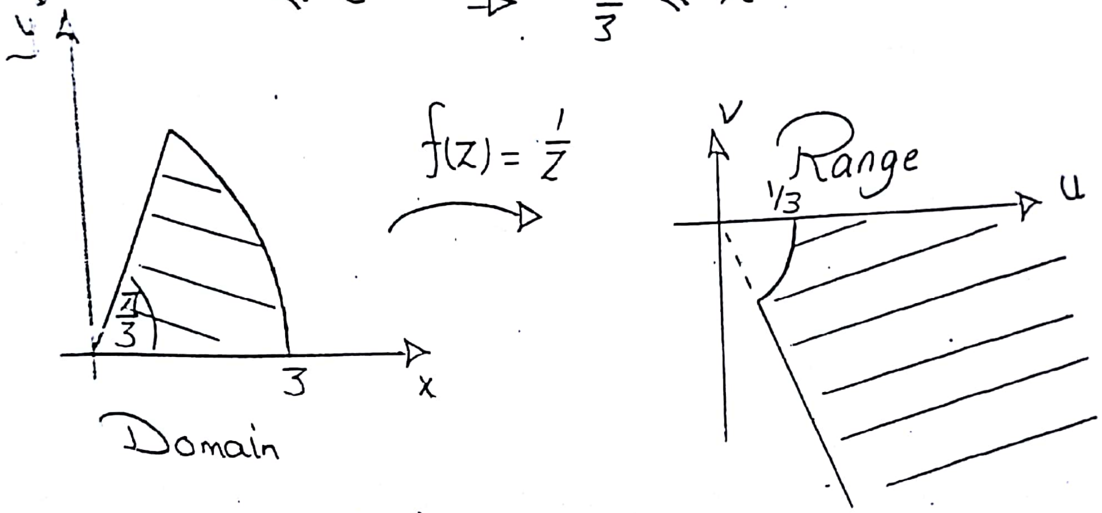
If $f(z) = \frac{1}{z} \Rightarrow$

$$W = f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = Re^{i\phi}$$

$\Rightarrow R = \frac{1}{r}$ & $\phi = -\theta$

$\because \; 0 \leqslant \theta \leqslant \pi/3 \Rightarrow 0 \leqslant \phi \leqslant -\pi/3$

$\because \; 0 < r \leqslant 3 \Rightarrow \frac{1}{3} \leqslant R < \infty$



Example:- Find the image of the region

i) $0 \leqslant Re(iz) < 1$

ii) $2Re(z) - 3Im(z) \leqslant 6$

Under $f(z) = 1 - 2z$

Solution:- We have $W = 1 - 2z \Rightarrow z = \frac{1-W}{2}$

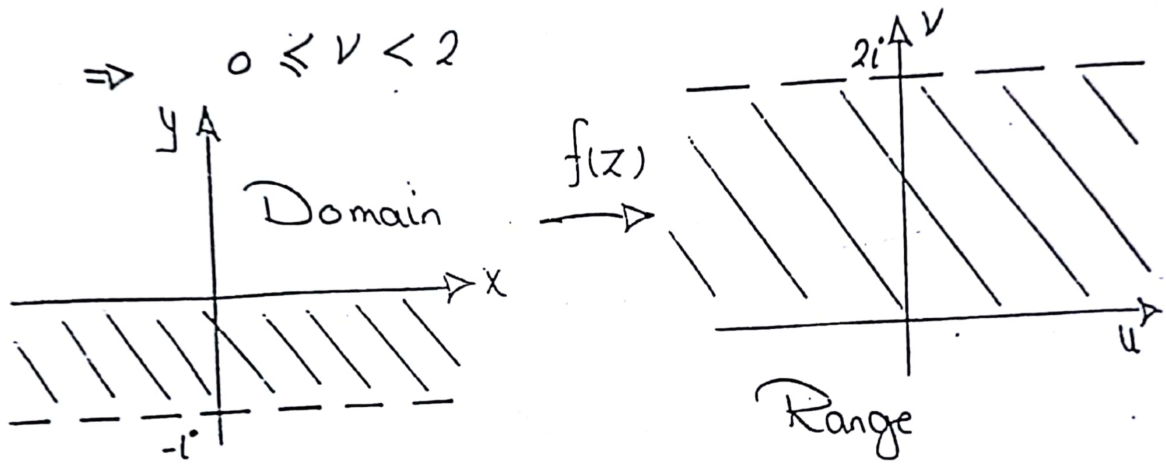
$$x + iy = \frac{1 - (u + iv)}{2} = \frac{1-u}{2} - i\,\frac{v}{2}$$

$\Rightarrow \; x = \frac{1-u}{2} \;\; \& \;\; y = -\frac{v}{2}$

i) we have $\mathrm{Re}(iz) = \mathrm{Re}(ix - y) = -y$

$\Rightarrow$ when $0 \leqslant -y < 1 \Rightarrow 0 \leqslant \frac{v}{2} < 1$

$\Rightarrow \quad 0 \leqslant v < 2$



ii) $2\mathrm{Re}(z) - 3\,\mathrm{Im}(z) \leqslant 6 \Rightarrow 2x - 3y \leqslant 6$

$\Rightarrow \quad (1-u) + \frac{3v}{2} \leqslant 6 \Rightarrow 2 - 2u + 3v \leqslant 12$

$\Rightarrow \quad 3v - 2u \leqslant 10$