كلية الهندسة

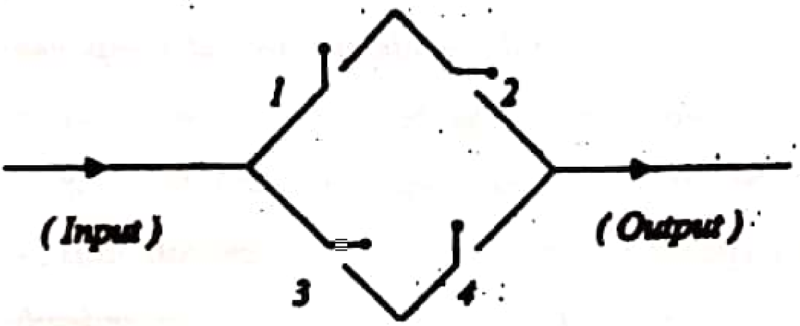
STUDENT MANUAL

CONTENTS

PROBABILITY : BASIC CONCEPTS

EXERCISES (1)

(1)

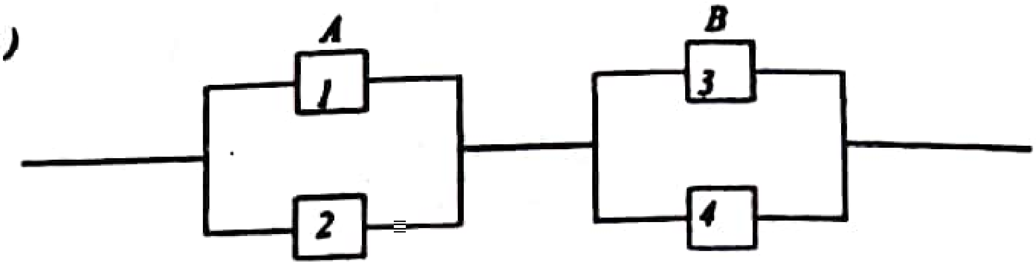


Electric Circuit

The Figure shows an electric circuit in which each of the switch i is independently closed with probability p_i and open with probability $1-p_i$, $i=1,2,3,4$. If a signal is fed to the input. What is the probability that it is transmitted to the output ?

(Answer : $p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4$)

(2)



A machine has two components A and B. The first component A is held stable by two bolts while the second component is also held stable by two bolts. The bolts are redundant safety features, in the sense that the component stays stable if at least one of its supporting bolts stays tight. Each bolt fails with probability p_i, $i=1,2,3,4$, and the bolts fail independently of one another. Calculate the probability that the machine suffers instability.

(Answer : $p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4$)

****

(3) *Two fair coins are tossed. Let,*

A = event that the tail will appear on the second throw,

B = event that the tail will appear on both throws,

Are the two events A and B independent ?

(Answer : A & B are not independent events.)

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(4) *A pair of dice I, II is thrown. Let,*

A = event that the outcome on die I is less than 5,

B = event that the outcome on die II is more than 5,

C = event that the sum of the outcome is more than 11.

Are the events A & B and B & C independent ?

(*Answer* : *A* & *B* are independent events.

B & *C* are not independent events.)

****

(5) A lot contains 20 items of which 6 are defective. Three items are drawn at random from the lot one after the other. Find the probability that all the three items are defective.

(*Answer* : $\frac{1}{57}$)

****

TREE - DIAGRAM

EXERCISES (2)

(1) Three machines A , B and C produce respectively 60 % , 30 % and 10 % of the total number of items of a factory. The percentage of defective output of these machines are 2 % , 3 % and 4 %. If an item is selected at random. Find the probability that the selected item is defective.

(Answer : 0.025)

(2) In a certain factory, there are three machines. Given the following table.

Machine A_i	No. of items produced / day	% Defective items
1	1000	1.0
2	1200	0.5
3	1800	0.5

An item is selected at random from the daily production of the factory. Find the probability that the selected item is defective. What is the probability that this defective item is of the production of machine A_3 ?

(Answer : 0.00625 – 0.36)

(3) The grade distribution of mathematics course of 3000 students for two faculties F_1 and F_2 of Ain Shams University is given by the following table :-

% Grade

		A	B	C	F
Ain Shams	F_1	20	30	40	10
University	F_2	15	25	50	10

where, F_1 has 1000 students and F_2 has 2000 students., If a student is selected at random from among the 3000 students, find the probability that the student received grade C and the Probability that he was from faculty F_1.

(Answer : $\frac{7}{15}$; $\frac{2}{7}$)

****

(4) A box contains 8 radio tubes of which 2 are defective. The tubes are tested one after the other until the two defective tubes are discovered. What is the probability that the process stopped on :

(I) second test (ii) third test.

If the process stopped on the third test, what is the probability that the first tube is non-defective ?

(i) P (process stopped on the third test) =

$$P(D_1 \cap D_2) = P(D_1) \,.\, P(D_2) = \frac{2}{8} \cdot \frac{1}{7} = \frac{1}{28}$$

(ii) • P (process stopped on the third test) =

$$P(ND_1 \cap D_2 \cap D_3) + P(D_1 \cap ND_2 \cap D_3)$$

$$= \frac{6}{8} \cdot \frac{2}{7} \cdot \frac{1}{6} + \frac{2}{8} \cdot \frac{6}{7} \cdot \frac{1}{6} = \frac{1}{14}$$

• P (first tube is non-defective | process stopped on the third test)

$$= \frac{1/28}{1/14} = \frac{1}{2}$$

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RANDOM VARIABLES AND DISTRIBUTIONS

EXERCISES (3)

(1) Three students have interviews scheduled for summer employment at an institute. In each case, the result of the interview will either be that a position is offered or not offered.

(i) List the experimental outcomes.

(ii) Define a r. v. that represents the number of offers made.

(iii) Show what value the r. v. will assume for each of the experimental outcomes.

---

(2) A continuous r. v. x has a density function given by :-

$$f(x) = \begin{cases} C & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) C

(ii) $P(2 < x < 2.5)$ (iii) $P(x < 1.6)$

****

(3) *A continuous r. v. x has a density function given by :-*

$$f(x) = \begin{cases} C\,(1+x) & 2<x<5 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) C

(ii) $P(x<4)$ (iii) $P(3<x<4)$

****

(4) A continuous r. v. x has a density function given by :-

$$f(x) = \begin{cases} k\sqrt{x} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k

(ii) $P(0.3 < x < 0.6)$

---

(5) Suppose that X & Y have the joint probability distribution as follows :-

Y \ X	1	2	3
1	0	1/6	1/12
2	c	1/9	0
3	2/15	1/4	1/18

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

****

(6) Suppose that X & Y have the joint probability distribution as follows :-

Y \ X	2	4
1	0.10	0.15
2	0.20	C
3	0.10	0.15

Calculate (i) c (ii) Marginal probability distributions (iii) Check independence of X & Y.

(7) *Two dice are thrown. The random variable X represents the minimum of the two faces and the random variable Y represents the number of sixes. Find*

(i) *The probability distribution of the two random variables X & Y.*

(ii) *The marginal probability distribution of X and Y.*

(iii) *The probability that X will exceed 5 given that $Y = 1$.*

(8) Two dice are thrown. The random variable X represents the square difference between the two faces and the random variable Y represents the number of sixes. Find

(i) The probability distribution of the two random variables X & Y.

(ii) The marginal probability distribution of X and Y.

(iii) The probability that X will exceed 9 given that $Y = 2$.

****

(9) The continuous r. v. x & y have a density function given by :-

$$f(x,y) = \begin{cases} C(6-x-y) & 0<x<2 \ \& \ 2<y<4 \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

(10) The continuous r. v. x & y have a density function given by :-

$$f(x,y) = \begin{cases} C & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

(11) The continuous r. v. x & y have a density function given by :-

$$f(x,y) = \begin{cases} C\,x & 0<x<1 \;\&\; 0<y<1-x \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) c (ii) Marginal probability distributions

(iii) Check independence of X & Y.

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STANDARD DEVIATION & CORRELATION COEFFICIENT

EXERCISES (4)

(1) The probability distribution of the continuous r. v. x is :-

$$f(x) = \begin{cases} c\,(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) $Var.(x)$ (iii) σ_x

****

(2) The probability distribution of the continuous r. v. x is :-

$$f(x) = \begin{cases} k\,/(1+x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k (ii) $Var.(x)$ (iii) σ_x

****

(3) Let x represent the outcome when a balanced die is tossed.
Find (i) $E(x)$ (ii) $E(2x^2-5)$ (iii) σ_x

****

(4) A continuous r. v. x has a density function given by :-

$$f(x) = \begin{cases} C\,(1+x) & 2<x<5 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) $E(2x)$ (iii) σ_{2x}

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(5) Show that $Cov.\,(a\,x\,,\,by) = a\,b\;Cov.\,(x,y)$

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(6) Let x & y be independent random variables with variances $\sigma_x^2 = 5$ and $\sigma_y^2 = 3$. Find the variance of the random variable $Z = -2x + 4y - 3$.

****

(7) Let x represent the number that occurs when a green die is tossed, and y represent the number occurs when a red die is tossed. Find the variance of the random variables :

$2x - y$ and $x + 3y - 5$.

****

(8) A continuous r. v. x has a density function given by :-

$$f(x,y) = \begin{cases} c\,xy & 0<x<1\ ,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) Var. (2x) (iii) $\rho_{x,y}$

****

(9) A continuous r. v. x has a density function given by :-

$$f(x) = \begin{cases} k\sqrt{x} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) c (ii) E(2x) (iii) σ_{2x}

****

(10) *Suppose that X & Y have the joint probability distribution as follows :-*

Y \ X	1	2	3
1	0	1/6	1/12
2	1/9	c	0
3	2/15	1/4	1/18

Calculate (i) c (ii) $\rho_{x,y}$

****

(11) Suppose that X & Y have the joint probability distribution as follows :-

Y \ X	2	4
1	c	0.15
2	0.20	0.10
3	0.10	0.15

Calculate (i) c (ii) $\rho_{x,y}$

****

(12) Two dice are thrown. The random variable X represents the maximum of the two faces and the random variable Y represents the number of sixes. Find

(i) The probability distribution of the two random variables X & Y

(ii) $\rho_{x,y}$

****

SPECIAL PROBABILITY DISTRIBUTIONS

EXERCISES (5)

(1) *In a single toss of 6 fair coins, find the probability of :*

(i) 2 Or more heads *(ii) fewer than 4 heads.*

Hint : Use Binomial distribution.

****

(2) *What is the probability of getting 9 exactly once in 3 throws of a pair of dice ?*

Hint : Use Binomial distribution.

****

(3) *Find the probability of guessing at least 6 of the 10 answers on a true false examination.*

Hint: Use Binomial distribution.

****

(4) *An insurance salesman sells policies to 5 men, all of identical age and in good health. According to the actuarial tables the probability that a man of this particular age will be alive 30 years hence is* $\frac{2}{3}$. *Find the probability that in 30 years*

(i) all 5 men *(ii) at least 1 man*

will be alive.

Hint: Use Binomial distribution.

****

(5) *If 3% of the electric bulbs manufactured by a company are defectives. Find the probability that in a sample of 100 bulbs :*

(i) more than 5 *(ii) between 1 and 3*

(iii) less than or equal to 2 *bulbs will be defective.*

Hint : Use Poisson distribution.

****

(6) If Z has standard normal distribution. Find

(i) $P(Z \geq -1.64)$ (ii) $P(-1.96 \leq Z \leq 1.96)$

(iii) $P(|Z| \geq 1)$.

****

(7) Find the values of Z such that :

(i) The area to the right of Z is 0.2266.

(ii) The area to the left of Z is 0.0314.

(iii) The area between -0.23 and Z is 0.5722.

(iv) The area between 1.15 and Z is $0.0.0730$.

(v) The area between $-Z$ and Z is 0.9000.

****

(8) *If the height of 300 students are normally distributed with mean 172 cm., and standard deviation 8 cm. How many students have heights :*

(i) Greater than 184 cm.

(ii) Less than or equal to 160 cm.

(iii) Between 164 and 180 cm.

****

(9) The marks of 300 students of Ain Shams University in mathematics examination are normally distributed with mean 172 marks, and standard deviation 8 marks. Determine how many students have marks :

(I) Between 164 and 180 marks.

(ii) Less than or equal to 164 marks.

(Given that $\Phi(-1) = 0.1587$)

Hint : $\Phi(1) + \Phi(-1) = 1$

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QUALITY CONTROL

EXERCISES (6)

(1) *The lengths of items produced on a machine are distributed normally with mean 4.05 cm and a standard deviation of 0.04 cm. What is the probability that the mean of a sample of 16 items taken at random differs from the over all mean by more than 0.01 cm ?*

(Answer : $P(-1 < z < 1) = 0.6826$)

****

(2) A process produces items whose lengths are distributed normally with mean 1.5 units and standard deviation of 0.04 units. Samples of 16 items are taken at random.
Between what limits do 90 % of the samples has a mean value $\mu = 1.5$ units and standard error = 0.01 units.

(Answer : 1.484 & 1.516)

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(3) A process produces items whose lengths are distributed normally with mean 0.25 units and standard deviation of 0.05 units.

(i) Calculate control limits for sample of size 25

(ii) Prepare a control chart.

(*Answer :* 0.2809 , 0.2696 , 0.25 , 0.2304 , 0.2191)

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(4) The lengths of items produced on a machine are distributed normally with standard deviation of 0.01 cm. What is the minimum size of a sample so that the standard error does not exceed 0.001 cm ?

(*Answer :* $n > 100$)

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MOMENT GENERATING FUNCTION

EXERCISES (7)

(1) Let Y be a r. v. for the distribution whose probability density function $f(Y)$ is given by :-

$$f(Y) = \frac{\left(\frac{1}{\beta}\right)^{Y} e^{-\frac{1}{\beta}}}{Y!} \quad , \quad \beta > 0 \ , \ Y = 0,1,2,\ldots$$

Find (i) $m_Y(t)$ (ii) $E(Y)$ (iii) $Var.(Y)$

Hint : Poisson distribution with parameter $\frac{1}{\beta}$.

****

(2) Let Y be a r. v. for the distribution whose probability density function $f(Y)$ is given by :-

$$f(Y) = \frac{\lambda^{\beta}}{\Gamma(\beta)} Y^{\beta-1} e^{-\lambda Y} \quad ; \quad Y, \lambda, \beta > 0$$

Find (i) $m_Y(t)$ (ii) $E(Y)$ (iii) $Var.(Y)$

Hint : Gamma distribution with parameters $\frac{1}{\lambda}$ and β.

****

(3) Let x be a r. v. of the distribution whose probability density function $f(x)$ is given by :-

$$f(x) = \frac{a}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2x^2} \quad ; \quad -\infty < x < \infty$$

Find (i) $m_x(t)$ (ii) $E(2x-1)$ (iii) $Var.(2x-3)$

Hint : Normal distribution $N(0, \frac{1}{a^2})$.

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(4) Let x be a r. v. of the distribution whose probability density function $f(x)$ is given by :-

$$f(x) = \frac{1}{2}\,\beta\, e^{-\beta|x|} \quad ; \quad -\infty < x < \infty$$

Find (i) $m_x(t)$ (ii) $E(x)$ (iii) $Var.(x)$

(5) Let x be a r. v. of the distribution whose probability density function $f(x)$ is given by :-

$$f(x) = \frac{1}{4}\, x\, e^{-\frac{x}{2}} \quad ; \quad x > 0$$

Find (i) $m_x(t)$ (ii) $E(x)$ (iii) $Var.(x)$

Hint : χ^2 - distribution with $n = 4$.

TRANSFORMATIONS

EXERCISES (8)

(1) Let $f(x_1, x_2) = \left(\frac{2}{3}\right)^{x_1+x_2}\left(\frac{1}{3}\right)^{2-x_1-x_2}, (x_1, x_2) = (0,0), (0,1)$

$, (1,0), (1,1).$

Find the joint p.d.f. of $x_1 - x_2$ and $x_1 + x_2$.

****

(2) Let X be a r. v. for the distribution whose probability density function $f(x)$ is given by :-

$$f(x) = \left(\frac{1}{2}\right)^x \; ; \quad x = 1, 2, 3, \ldots$$

Find the p. d. f. of $Y = X^3$.

(3) Let X_1 and X_2 have the p. d. f. given by :-

$$f(x_1, x_2) = \frac{x_1 x_2}{36} \quad , \quad x_1, x_2 = 1,2,3.$$

Find the joint p. d. f. of $x_1 x_2$ and x_2,

and the marginal p. d. f. of $x_1 x_2$.

$X_1 X_2$ \ X_2	1	2	3	$g_1(X_1 X_2)$
1	$\frac{1}{36}$	0	0	$\frac{1}{36}$
2	$\frac{2}{36}$	$\frac{2}{36}$	0	$\frac{4}{36}$
3	$\frac{3}{36}$	0	$\frac{3}{36}$	$\frac{6}{36}$
4	0	$\frac{4}{36}$	0	$\frac{4}{36}$
6	0	$\frac{6}{36}$	$\frac{6}{36}$	$\frac{12}{36}$
9	0	0	$\frac{9}{36}$	$\frac{9}{36}$

****

TRANSFORMATIONS

EXERCISES (9)

(1) Let $Y_1 = (X_1 - X_2)/2$, where X_1 and X_2 are independent r. v. each with probability density function $f(X)$ is given by :-

$$f(X) = \frac{1}{2} e^{-\frac{1}{2}X} , \quad 0 < X < \infty$$

Find the p. d. f. of Y_1.

(Answer: $g_1(y_1) = \frac{1}{2} e^{-|y_1|}$, $-\infty < y_1 < \infty$ double exponential distribution)

****

2) Let W be $N(0,1)$. Let V be chi-squared r. v. with r degrees of freedom. Let W and V be independent. Show that the p. d. f. of the r. v. $T = W/\sqrt{V/r}$ is given by :-

$$g(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\,\Gamma(r/2)} (1+t^2/r)^{-\frac{r+1}{2}} \; ; \; -\infty < t < \infty$$

where,

$$N(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; \sigma > 0, \; \infty < x, \mu < \infty$$

$$\chi^2(n) = [2^{n/2}\,\Gamma(\tfrac{n}{2})]^{-1}\, x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \; x > 0$$

****

(3) *Let U and V be two independent r. v. with r and s degrees of freedom respectively. Show that the p. d. f. of* $F = (U/r)/(V/s)$ *is given by :-*

$$g(f) = \frac{1}{\beta(\frac{r}{2},\frac{s}{2})} \left(\frac{r}{s}\right)^{\frac{r}{2}} f^{\frac{r}{2}-1} \left(1+\frac{r}{s}f\right)^{\frac{r+s}{2}} \; ; \; f > 0$$

where,

$$\chi^2(n) = [2^{n/2}\Gamma(\frac{n}{2})]^{-1} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \; , \; x > 0$$

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4) The joint p. d. f. of the two random variables x_1 and x_2 is given by :-

$$f(x_1, x_2) = \begin{cases} C\, e^{-x_1-x_2} & 0<x_1<\infty\,,\ 0<x_2<\infty \\ 0 & \text{otherwise} \end{cases}$$

Find

(i) C

(ii) The joint p. d. f. of $(x_1+x_2)/2$ and $x_1/(x_1+x_2)$.

(iii) The p. d. f. of $(x_1+x_2)/2$.

(iv) Probability that $x_1/(x_1+x_2)$ is greater than one given that $(x_1+x_2)/2$ is greater than zero.

(i) $\int_0^\infty \int_0^\infty C\, e^{-x_1}\, e^{-x_2}\, dx_1\, dx_2 = 1 \quad \Rightarrow \quad C = 1$

(ii) $Y_1 = (x_1+x_2)/2 \quad \Rightarrow X_1 = 2\,Y_1\,Y_2 \qquad \Rightarrow J(X_1, X_2 / Y_1, Y_2) = 4\,Y_1$

$Y_2 = x_1/(x_1+x_2) \quad \Rightarrow X_2 = 2\,Y_1\,(1 - Y_2)$

$$g(Y_1, Y_2) = \begin{cases} 4\,Y_1\, e^{-2Y_1} & 0<Y_1<\infty\,,\ 0<Y_2<1 \\ 0 & \text{otherwise} \end{cases}$$

$$A = \{(x_1, x_2) : 0 < x_1 < \infty\,,\ 0 < x_2 < \infty\}$$

$$B = \{(Y_1, Y_2) : 0 < Y_1 < \infty\,,\ 0 < Y_2 < 1\}$$

(iii) $g_1(Y_1) = \int_0^1 4\, Y_1\, e^{-2Y_1}\, dY_2 = \begin{cases} 4\,Y_1\, e^{-2Y_1} & 0<Y_1<\infty \\ 0 & \text{otherwise} \end{cases}$

(iv) $P(Y_2 > 1 \mid Y_1 > 0) = 0$

****

GOODNESS OF FIT

EXERCISES (10)

(1) A new container design has been adopted by a manufacturer. Color preferences in a sample of 150 individuals are as follows :

Category	Red	Blue	Green
Observed	40	64	46

Test using $\alpha = 0.01$ to see if the color preferences are the same.

Let H_o be the hypothesis that there is no difference between the actual and the expected results.

(Answer : $\chi^2_{cal.} = 6.24 < \chi^2_{2,0.01} = 9.21$ not reject H_o)

****

2) Grade distribution guidelines for a statistics course at a major university are as follows :

10 % A , 30 % B , 40 % C , 15 % D and 5 % F.

A sample 120 statistics grades at the end of a semester showed :

18 A , 30 B , 40 C , 22 D and 10 F.

Use $\alpha = 0.05$ and test to see if the actual grades differ significantly from the grade distribution guidelines.

Let H_0 be the hypothesis that there is no difference between the actual grades and the expected grades.

(Answer : $\chi^2_{cal} = 8.8889 < \chi^2_{4,0.05} = 9.488$ not reject H_0)

****

(3) Consumer panel preferences for three proposed store displays are

Category	Display A	Display B	Display C
Observed	43	53	39

Use $\alpha = 0.05$ and test to see if there is a preference among the three display designs.

Let H_o be the hypothesis that there is no difference between the actual and the expected results.

(Answer : $\chi^2_{cal.} = 2.3111 < \chi^2_{2,0.05} = 5.991$ not reject H_o)

****

(4) At Ontario University entering freshmen have historically selected the following colleges :

Business 15 % , Education 20 % , Engineering 30 % , Liberal Arts 25 % and Science 10 %.

Data obtained for the most recent class show that :

73 students selected business, 105 students selected education, 150 selected engineering, 124 chose liberal arts and 47 selected science. Use $\alpha = 0.10$ and test whether or not the historical percentages have changed.

Let H_o be the hypothesis that there is no difference between the actual and the expected results.

(Answer : $\chi^2_{cal} = 0.4913 < \chi^2_{4,0.10} = 7.779$ not reject H_o)

****

(5) The grade distribution in a sample of size $n = 100$ students for a course of mathematics in the faculty of engineering at Ain Shams University is given as follows :-

Grade	A	B	C	D	F
Actual grade result	5	20	50	10	5
Expected grade	np_1	np_2	Np_3	np_4	np_5

where, $p_1 = 10\%$, $p_2 = 20\%$, $p_3 = 40\%$, $p_4 = 20\%$ and $p_5 = 10\%$.

By using χ^2 - test, test to see if the actual grades differ from the expected grades or not.

($\chi^2_{5,0.05} = 11.070$, $\chi^2_{5,0.01} = 15.086$, $\chi^2_{4,0.05} = 9.488$; $\chi^2_{4,0.01} = 13.277$;

Let H_o be the hypothesis that there is no difference between the actual grades and the expected grades.

Grade	A	B	C	D	F
Actual grade O_j	5	20	50	10	5
Expected grade E_j	10	20	40	20	10

$$\chi^2_{cal.} = \sum_{j=1}^{5} \frac{(O_j - E_j)^2}{E_j}$$

$$= \frac{(5)^2}{10} + 0 + \frac{(10)^2}{40} + \frac{(10)^2}{20} + \frac{(5)^2}{10}$$

$$= 12.5 > \chi^2_{4,0.05} = 9.488 \quad (\text{ reject } H_o)$$

$$< \chi^2_{4,0.01} = 13.277 \quad (\text{not reject } H_o)$$

****

TEST OF INDEPENDENCE

EXERCISES (11)

(1) *The number of units of three different products sold by three salespersons over 3 – month period are shown as follows :-*

Salesperson \ Product	A	B	C
I	14	12	4
II	21	16	8
III	15	5	10

Use $\alpha = 0.05$ and test for the independence of salesperson and the type of product sold.

Let H_0 be the independent hypothesis of salesperson and type of product.

(Answer : $\chi^2_{cal.} = 6.3175 < \chi^2_{4,0.05} = 9.488$ not reject H_0)

****

(2) Starting positions for business and engineering graduates are classified by industry as follows :-

Degree Major \ Industry	Oil	Chemical	Electrical	Computer
Business	30	15	15	40
Engineering	30	30	20	20

Use $\alpha = 0.01$ and test for the independence of degree major and Industry type.

Let H_0 be the independent hypothesis of degree major and industry.

(Answer : $\chi^2_{cal.} = 12.381 < \chi^2_{3,0.01} = 11.341$ reject H_0)

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(3) A sport preference poll shows the following data for men and women :-

Sex \ Favorite	Baseball	Basketball	Football
Men	19	15	24
Women	16	18	16

Use $\alpha = 0.05$ and test for similar sport preferences by men and women.

Let H_0 be the similar hypothesis preferences of men and women.

(Answer : $\chi^2_{cal} = 1.5459 < \chi^2_{2,0.05} = 5.99147$ not reject H_0)

••••••••••••••••••••••••••

(4) Three suppliers provide the following data on defective parts :-

Supplier \ Part Quality	Good	Minor Defect	Major Defect
A	90	3	7
B	170	18	7
C	135	6	9

Use $\alpha = 0.05$ and test for the independence between the supplier And the part quality.

Let H_0 be the independent hypothesis between the supplier and the part quality.

(Answer : $\chi^2_{cal} = 7.7117 < \chi^2_{4,0.05} = 9.488$ not reject H_0)

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Series Solutions of Linear Differential Equations

Problems (2)

First: 1, 3, 5, 8, 10

Final Answers:

(1) $x = 0$ is an ordinary point.

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} 2^n \frac{3.5.7 \cdots (2n+1)}{(2n)!} x^{2n} \right]$$

$$+ a_1 \left[x + \sum_{n=1}^{\infty} \frac{2^{2n}.(n+1)!}{(2n+1)!} x^{2n+1} \right]$$

(3) $y = a_1 x + a_0 \left(1 - 2x^2 - x^4 - \frac{4}{3} x^6 + \dots\right)$

(5) $x = 0$ is a regular singular point.

$s_1 = \frac{1}{2}$, $s_2 = -\frac{1}{2}$

$y = \frac{1}{\sqrt{x}} \left(a_0 \sin x + a_1 \cos x\right)$

(8) $x = 0$ is a regular singular point.

$s_1 = s_2 = 0.$

$$y = A \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n + B \left[\ln x \sum_{n=0}^{\infty} \frac{x^n (-1)^n}{(n!)^2} \right.$$

$$\left. + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n x^n}{(n!)^2} \right]$$

(10) x is a regular singular point. $s_1 = 3$, $s_2 = -2$.

$$y = A\left[x^3 + \sum_{n=1}^{\infty} \frac{(2n+2)!\; x^{2n+3}}{2^{n+1}\, n!\,(7.9\ldots(2n+5))}\right] + B\left(\frac{1}{x^2} - \frac{1}{3}\right)$$

* * *

Second : 4, 7, 9, 14

Complete Answers:

(4) $x = 0$ is a regular singular point. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}.$$

Substitute into the eqn:

$$\sum_{n=0}^{\infty} 4a_n(n+s)(n+s-1)x^{n+s-1} + \sum_{n=0}^{\infty} 2a_n(n+s)x^{n+s-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

The indicial eqn is (coeff. of $x^{s-1} = 0$)

$$4a_0 s(s-1) + 2a_0 s = 0 \Rightarrow 2s^2 - s = 0$$

$$\Rightarrow s(2s-1) = 0 \Rightarrow \boxed{s_1 = \tfrac{1}{2}\,,\; s_2 = 0}$$

Equating the coefficient of x^{n+s-1} to zero, we get

$$4a_n(n+s)(n+s-1) + 2a_n(n+s) + a_{n-1} = 0$$

$$\Rightarrow 2a_n(n+s)(2n+2s-1) = -a_{n-1}$$

The reccurence relation is

$$\boxed{a_n = -\frac{a_{n-1}}{2(n+s)(2n+2s-1)} \quad ; \quad n \geq 1}$$

For $s = s_1 = \frac{1}{2}$:

$$a_n = -\frac{a_{n-1}}{2n(2n+1)}$$

$$n=1 \Rightarrow a_1 = -\frac{a_0}{2 * 1 * 3}$$

$$n=2 \Rightarrow a_2 = -\frac{a_1}{2 * 2 * 5} = \frac{a_0}{2^2 * 1.2 * 3.5}$$

$$\cdots\cdots$$

$$a_n = \frac{(-1)^n a_0}{2^n\ n!\ (3.5\ldots(2n+1))} \quad \text{for } n \geq 1$$

When $a_0 = 1$:

$$y_1 = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n\ n!\ [3.5\ldots(2n+1)]}\right]$$

For $s = s_2 = 0$:

$$a_n = -\frac{a_{n-1}}{2n(2n-1)} \quad ; \quad n \geq 1$$

$$n=1 \Rightarrow a_1 = -\frac{a_0}{2 * 1 * 1}$$

$$n=2 \Rightarrow a_2 = -\frac{a_1}{2 * 2 * 3} = \frac{a_0}{2^2 * 1.2 * 1.3}$$

$$\cdots\cdots\cdots$$

$$a_n = \frac{(-1)^n a_0}{2^n\ n!\ [1.3\ldots(2n-1)]}.$$

When $a_0 = 1$:

$$y_2 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n \, n! \, [1.3 \dots (2n-1)]}$$

The general solution is

$$y = A y_1 + B y_2 = A x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n n! \, [3.5 \dots (2n+1)]} \right]$$

$$+ B \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n n! \, [1.3 \dots (2n-1)]} \right]$$

—o-o-o—

(7) $x = 0$ is an ordinary point, so we let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substitute into the eqn :

$$(3 - x^2) \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 25 a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n$$

$$+ \sum_{n=0}^{\infty} 25 a_n x^n = 0$$

x^0 : $6 a_2 + 25 a_0 = 0 \quad \Rightarrow \quad a_2 = -\frac{25}{6} a_0$

x^1 : $18 a_3 - a_1 + 25 a_1 = 0 \quad \Rightarrow \quad a_3 = -\frac{4}{3} a_1$

Equate the coeff. of x^n $(n \geq 2)$ to zero:

$$3(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 25a_n = 0$$

The recourence relation (R.R.) is

$$a_{n+2} = \frac{(n-5)(n+5)}{3(n+1)(n+2)}\, a_n \;, \quad n \geq 2 .$$

Set $n = 2, 4, 6, \dots$:

$$a_4 = \frac{-3 \times 7}{3 \times 3 \times 4}\, a_2 = \frac{-3 \times 7}{3 \times 3 \times 4} \times \frac{-25}{6}\, a_0$$

$$= \frac{-3 \times 7}{3^2 \times 4!}\,(-25)\, a_0$$

$$a_6 = \frac{-1 \times 9}{3 \times 5 \times 6}\, a_4 = \frac{(-3)(-1) \times 7 \cdot 9}{3^3 \times 6!}\,(-25)\, a_0$$

$$\cdots\cdots\cdots\cdots$$

$$\Rightarrow \quad a_{2n} = \frac{(-3)(-1)\cdots(2n-7) \times 7 \cdot 9 \cdots (2n+3)}{3^n \times (2n)!}\,(-25)\, a_0$$

$$a_0 = 1 \Rightarrow y_1 = 1 - \frac{25}{6}x^2 - 25 \sum_{n=2}^{\infty} \frac{(-3)(-1)\cdots(2n-7) \times 7 \cdots (2n+3)}{3^n\,(2n)!}\, x^{2n}$$

Set $n = 3, 5, \dots$:

$$a_5 = \frac{(-2)(8)}{3(4)(5)}\, a_3 = -\frac{4}{15}\, a_3 = \frac{16}{45}\, a_1$$

$$a_7 = 0 \Rightarrow a_9 = a_{11} = \cdots = 0$$

$$a_1 = 1 \Rightarrow y_2 = x - \frac{4}{3}x^3 + \frac{16}{45}x^5 .$$

The general solution is

$$y = A y_1 + B y_2$$

(9) x is a regular singular point. so we let

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

Substitute into the differential eqn :

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s}$$

$$+ \sum_{n=0}^{\infty} (n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} 5(n+s)a_n x^{n+s} - \sum_{n=0}^{\infty} 4a_n x^{n+s} = 0$$

The indicial eqn (coeff. of $x^{s-1} = 0$) is

$$a_0 s(s-1) + a_0 s = 0$$

$$\Rightarrow s^2 = 0 \quad \Rightarrow \quad \boxed{s_1 = s_2 = 0}$$

So, we have the case of equal exponents.

Equate the coeff. of x^{n+s-1} to zero :

$$(n+s)(n+s-1)a_n - (n+s-1)(n+s-2)a_{n-1} + (n+s)a_n$$
$$+ 5(n+s-1)a_{n-1} - 4a_{n-1} = 0$$

The R.R. is

$$a_n = \frac{(n+s-1)(n+s+3)+4}{(n+s)^2} a_{n-1}$$

$$= \frac{(n+s)^2 + 2(n+s)+1}{(n+s)^2} a_{n-1}$$

$$\Rightarrow \quad a_n = \frac{(n+s+1)^2}{(n+s)^2} a_{n-1} \quad ; \quad n \geq 1$$

$n=1 \Rightarrow \quad a_1 = \frac{(s+2)^2}{(s+1)^2} a_0$

$n=2 \Rightarrow \quad a_2 = \frac{(s+3)^2}{(s+2)^2} a_1 = \frac{(s+3)^2}{(s+1)^2} a_0$

$\cdots\cdots\cdots\cdots\cdots$

$$a_n = \frac{(s+n+1)^2}{(s+1)^2} a_0$$

For $a_0 = 1$ we have

$$y(x,s) = x^s \left[1 + \sum_{n=1}^{\infty} \frac{(s+n+1)^2}{(s+1)^2} x^n \right]$$

$$y_1(x) = y(x,0) = 1 + \sum_{n=1}^{\infty} (n+1)^2 x^n .$$

$$= \sum_{n=0}^{\infty} (n+1)^2 x^n .$$

$$y_2(x) = \left.\frac{\partial y(x,s)}{\partial s}\right|_{s=0}$$

$$= x^s \ln x \left[1 + \sum_{n=1}^{\infty} \frac{(n+s+1)^2}{(s+1)^2} x^n \right] \Bigg|_{s=0}$$

$$+ x^s \sum_{n=1}^{\infty} x^n \frac{\partial}{\partial s} \left[\frac{(n+s+1)^2}{(s+1)^2} \right] \Bigg|_{s=0} \quad \ldots (1)$$

Let $\quad a_n(s) = \frac{(n+s+1)^2}{(s+1)^2}$

$$\Rightarrow \ln a_n(s) = 2 \left[\ln(n+s+1) - \ln(s+1) \right]$$

$$\Rightarrow \frac{a_n'(s)}{a_n(s)} = 2 \left(\frac{1}{n+s+1} - \frac{1}{s+1} \right)$$

$$\Rightarrow a_n'(0) = a_n(0)\cdot 2\left(\frac{1}{n+1} - 1\right) = 2(n+1)^2\left(\frac{-n}{n+1}\right)$$
$$= -2n(n+1).$$

From (1):

$$y_2 = y_1 \ln x - 2\sum_{n=1}^{\infty} n(n+1)x^n.$$

The general solution is

$$y = Ay_1 + By_2$$
$$= A\sum_{n=0}^{\infty}(n+1)^2x^n + B\left[\ln x \sum_{n=0}^{\infty}(n+1)^2x^n - 2\sum_{n=1}^{\infty} n(n+1)x^n\right]$$

---

(14) $(x - x^2)y'' - 3xy' - y = 0$

x is a regular singular point, so we let

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s-1} - \sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s}$$
$$- \sum_{n=0}^{\infty} 3a_n(n+s)x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

The indicial eqn is

$a_0 s(s-1) = 0 \Rightarrow s_1 = 1,\ s_2 = 0$

The R.R. is

$$a_n = \frac{n+s}{n+s-1}\, a_{n-1} \; ; \; n \geq 1$$

$n=1 \Rightarrow a_1 = \frac{s+1}{s} a_0$

$n=2 \Rightarrow a_2 = \frac{s+2}{s+1} a_1 = \frac{s+2}{s} a_0$

....

$$a_n(s) = \frac{s+n}{s} a_0 \quad \text{for } n \geq 1.$$

$$y(x,s) = x^s \left[1 + \sum_{n=1}^{\infty} \left(\frac{s+n}{s}\right) x^n \right]$$

$$y_1 = y(x,1) = x \left[1 + \sum_{n=1}^{\infty} (n+1) x^n \right] = \sum_{n=0}^{\infty} (n+1) x^{n+1}$$

Let $\bar{y}(x,s) = s\, y(x,s) = x^s \left[s + \sum_{n=1}^{\infty} (n+s) x^n \right]$

$$\Rightarrow y_2 = \left.\frac{\partial \bar{y}(x,s)}{\partial s}\right|_{s=0} = x^s \left[s + \sum_{n=1}^{\infty} (n+s) x^n \right] \ln x \Bigg|_{s=0}$$

$$+ \; x^s \left[1 + \sum_{n=1}^{\infty} x^n \right] \Bigg|_{s=0}$$

$$= \ln x \sum_{n=1}^{\infty} n x^n + \sum_{n=0}^{\infty} x^n .$$

The general solution is

$$y = A \sum_{n=0}^{\infty} (n+1) x^{n+1} + B \left[\ln x \sum_{n=1}^{\infty} n x^n + \sum_{n=0}^{\infty} x^n \right]$$

* * * *

<u>Third:</u>

Find series solutions in powers of x :

(1) $(x^2+1)y'' + xy' - y = 0$

(2) $3xy'' + y' - y = 0$

(3) $xy'' + y' - 4y = 0$

(4) $xy'' - y = 0$

Final Answers:

(1) $y = a_0\left[1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1.3.5\cdots(2n-3)}{2^n\, n!}x^{2n}\right]+a_1 x$

(2) $s_1=\frac{2}{3}$, $s_2=0$

$$y = A\,x^{2/3}\left[1+\sum_{n=1}^{\infty}\frac{x^n}{n! * 5.8.11\cdots(3n+2)}\right]$$

$$+B\left[1+\sum_{n=1}^{\infty}\frac{x^n}{n! * 1.4.7\cdots(3n-2)}\right]$$

(3) $s_1 = s_2 = 0$

$$y_1(x) = \sum_{n=0}^{\infty}\frac{4^n}{(n!)^2}x^n$$

$$y = A\,y_1(x) + B\left[y_1(x)\ln x + y_1(x)\left(-8x+20x^2 - \frac{1472}{27}x^3+\cdots\right)\right]$$

(4) $s_1 = 1$, $s_2 = 0$

$$y_1 = \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n!)^2(n+1)} .$$

$$y_2 = \ln x\sum_{n=1}^{\infty}\frac{x^n}{n!\,(n-1)!} + 1 - \sum_{n=1}^{\infty}\frac{H_n+H_{n-1}}{n!\,(n-1)!}x^n$$

* * * *

Special Functions

1,2. Gamma and Beta Functions

Problems pp. 13,14

First : 1,2, 3 , 6, 10 , 11

Final Answers :

1 - (i) $\dfrac{15\sqrt{\pi}}{8\, s^{7/2}}$ (ii) $\dfrac{\Gamma(0.67)}{s^{2/3}} = \dfrac{0.90333}{0.67\, s^{2/3}} = \dfrac{1.3483}{s^{2/3}}$

(iii) $\dfrac{\sqrt{\pi}}{2(s+3)^{3/2}}$ (2) 8.66

6) $\dfrac{\Gamma(0.8)}{5\,(2)^{0.8}} = 0.1337$ (10) $\dfrac{\pi}{\sqrt{2}}$ (11) $\dfrac{\pi}{\sqrt{2}}$

* * *

Second: 4 , 5 , 7 , 8 , 13

Complete Answers:

4) Put $b^{-x} = e^{-t} \Rightarrow -x \ln b = -t \Rightarrow x = \dfrac{t}{\ln b}$,

$dx = \dfrac{dt}{\ln b}$:

$$I = \int_0^\infty \left(\frac{t}{\ln b}\right)^a e^{-t}\,\frac{dt}{\ln b} = \frac{1}{(\ln b)^{a+1}} \int_0^\infty t^a e^{-t}\,dt$$

$$= \frac{\Gamma(a+1)}{(\ln b)^{a+1}} \quad ※$$

5) The curve $x^4 + y^4 = 1$ is symmetric w.r.t. x-axis and y-axis. So, the area inside the curve is formed from 4 symmetric parts:

$$A = 4\int_0^1 y\,dx = 4\int_0^1 (1-x^4)^{1/4}\,dx$$

Put $x^4 = t \Rightarrow x = t^{1/4} \Rightarrow dx = \frac{1}{4} t^{-\frac{3}{4}} dt$:

$$A = 4\int_0^1 (1-t)^{1/4} \cdot \frac{1}{4} t^{-\frac{3}{4}} dt = \int_0^1 t^{-\frac{3}{4}} (1-t)^{1/4} dt$$

Compare with $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$:

$$x - 1 = -\frac{3}{4}, \quad y - 1 = \frac{1}{4} \Rightarrow x = \frac{1}{4}, \; y = \frac{5}{4}$$

$$A = B\left(\frac{1}{4}, \frac{5}{4}\right) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{2})}$$

$$= \frac{\Gamma(\frac{1}{4}) \cdot \frac{1}{4}\Gamma(\frac{1}{4})}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{\pi}} \quad ※$$

---

7) Put $\ln x = -u \Rightarrow x = e^{-u}$, $dx = -e^{-u} du$:

$$\Rightarrow I = \int_{\infty}^{0} e^{-u/3} (-u)^5 (-e^{-u})\,du$$

$$= \int_{\infty}^{0} u^5 e^{-\frac{4u}{3}}\,du$$

Put $\frac{4u}{3} = t \Rightarrow u = \frac{3}{4} t$, $du = \frac{3}{4} dt$:

$$I = \int_{\infty}^{0} \left(\frac{3}{4} t\right)^5 e^{-t} \cdot \frac{3}{4} dt = -\left(\frac{3}{4}\right)^6 \int_0^{\infty} t^5 e^{-t} dt$$

Compare with $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$:

$$I = -\left(\frac{3}{4}\right)^6 \Gamma(6) = -\left(\frac{3}{4}\right)^6 \cdot 5! \cong -21.357$$

$\circ\!\!-\!\!\circ\!\!-\!\!\circ$

8) Put $3^{-x^2} = e^{-t} \Rightarrow -x^2 \ln 3 = -t \Rightarrow x = \frac{\sqrt{t}}{\sqrt{\ln 3}}$,

$dx = \frac{1}{2\sqrt{\ln 3}\sqrt{t}}\, dt$;

$$I = \int_0^\infty 3^{-x^2} dx = \int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{\ln 3}\sqrt{t}}\, dt$$

$$= \frac{1}{2\sqrt{\ln 3}} \int_0^\infty t^{-1/2} e^{-t}\, dt = \frac{1}{2\sqrt{\ln 3}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\ln 3}}$$

$\circ\!\!-\!\!\circ\!\!-\!\!\circ$

13) Put $x^6 = u \Rightarrow x = u^{1/6}$, $dx = \frac{1}{6} u^{-5/6} du$:

$$I = \int_0^\infty \frac{u^{1/6}}{1+u} \cdot \frac{1}{6} u^{-\frac{5}{6}}\, du = \frac{1}{6} \int_0^\infty \frac{u^{-2/3}}{1+u}\, du$$

Compare with $B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du$:

$x - 1 = -\frac{2}{3}$, $x + y = 1 \Rightarrow x = \frac{1}{3}$, $y = \frac{2}{3}$.

$$\Rightarrow I = \frac{1}{6} B\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{6} \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$$

$$= \frac{\pi}{6 \sin\frac{\pi}{3}} = \frac{\pi}{3\sqrt{3}} .$$

Third :

Evaluate each of the following integrals by using gamma and beta functions :

1) $\int_0^1 (\ln x)^{5/3} dx$

2) $\int_0^\infty x^2 e^{-x^2} dx$

3) $\int_0^{\pi/2} \sin^{3.04} x \, dx$

4) $\int_0^{2} x(8 - x^3)^{1/3} \, dx$

5) $\int_0^1 \sqrt[n]{1-x^n}\, dx$

6) $\int_0^\infty \frac{dx}{\sqrt{1+x^4}}$

7)* $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

8) Find $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \cdot \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

Final Answers:

(1) $-\Gamma\left(\frac{8}{3}\right)$

(2) $\frac{\sqrt{\pi}}{4}$

(3) $\frac{1}{3} B\left(2.02, \frac{1}{3}\right) \approx 0.663$

(4) $\frac{16\pi}{9\sqrt{3}}$

(5) $\frac{1}{2n} \frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}$

(6) $\frac{1}{4\sqrt{\pi}} \Gamma^2\left(\frac{1}{4}\right)$

(7) $\frac{\sqrt{2\pi}}{8} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$

(8) π

* * * *

Functions of a Complex Variable

Chapter I

Analytic Functions

EXERCISES I

pp. 12, 13

First: 1, 3-b, 4-b, 5-b

Final Answers:

3-b) The domain is the entire z-plane except $z=1$.

$$f(z) = \frac{1-x}{(1-x)^2+y^2} - \frac{y}{(1-x)^2+y^2}\, i$$

4-b) $|\operatorname{Arg} w| \leq \frac{3\pi}{4}$, $|w| \leq 8$

5-b) 2

* * *

Second: 2, 3-c, 4-a, 5-c

Complete Answers:

2) Since $\lim_{n \to \infty} \frac{1}{n} = 0$, the boundary of E is consisting of only one point : $z = 0$.

Note that every neighbourhood of the point $z=0$ contains points in E as well as points not in E.

3 - c) $f(z) = \dfrac{z+i}{(z+i)(z-i)}$

The function is not defined at $z = \pm i$.

When $z \neq \pm i$:

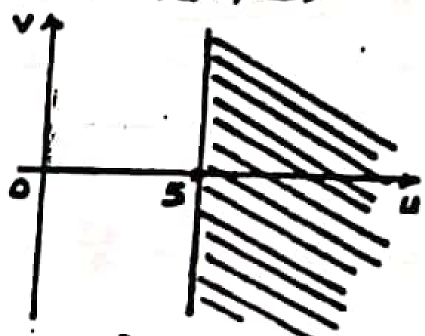
$$f(z) = \frac{1}{z-i} = \frac{1}{x+i(y-1)} \cdot \frac{x-i(y-1)}{x-i(y-1)}$$

$$= \frac{x}{x^2+(y-1)^2} - \frac{y-1}{x^2+(y-1)^2}\, i .$$

4 - a) Let $w = z + 5 \Rightarrow \operatorname{Re} w = \operatorname{Re}(z+5)$

$\Rightarrow \operatorname{Re} w = 5 + \operatorname{Re} z > 5$

The range is $\operatorname{Re} w > 5$.



5 - c) $$\lim = \lim_{\Delta z \to 0} \frac{(z_0^2 + 2z_0\Delta z + \Delta z^2) - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z_0\Delta z + \Delta z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$$

Third:

1 - Sketch each of the following sets and determine which of them is open, connected or closed. What is the boundary of each set :

(a) $\operatorname{Re} z > 1$ (b) $|z-2-i| \leq 2$

(c) $|z| < 1$ or $|z-4| < 1$ (d) $|z+i| > 2$

(e) $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\}$

2- Express each function in the form $u+iv$:

(a) $f(z) = \bar{z}^2 + (2-3i)\bar{z}$

(b) $f(z) = \bar{z}\,\mathrm{Re}\,z + z^2 + \mathrm{Im}\,z$

3) Describe the range of

(a) $f(z) = z^2$ for $|z| > 3$ (b) $f(z) = 3 - 2z$ for $\mathrm{Im}\,z > 4$

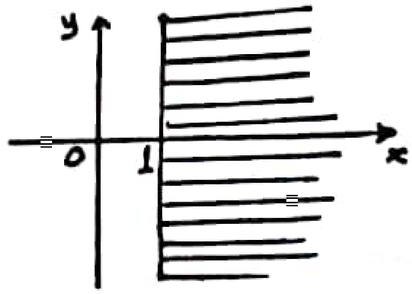
4) Find the following limits:

a) $\lim\limits_{z \to i} \dfrac{z^2+4z+2}{z+1}$ (b) $\lim\limits_{z \to i} \dfrac{z^4-1}{z-i}$

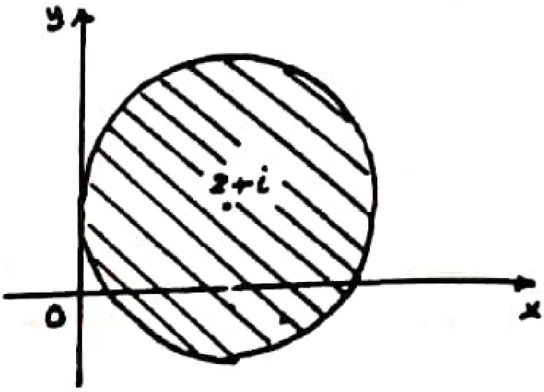
c) $\lim\limits_{z \to 1+i} \dfrac{z^2+z-1-3i}{z^2-2z+2}$ (Factorize!!)

Final Answers:

1-a) open and connected . The boundary is Re $z=1$



(a)

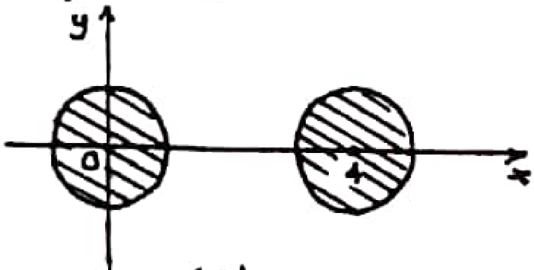


(b)

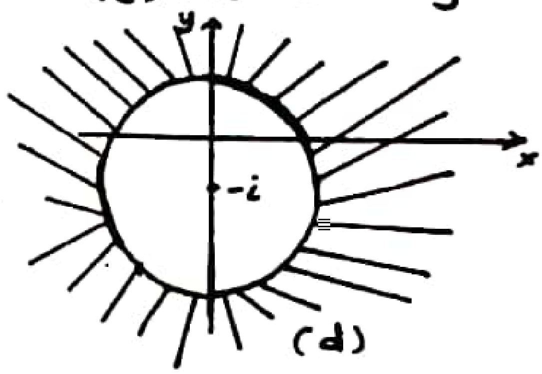
b) Closed and connected : $|z-(2+i)|=2$

c) open : $|z|=1$ or $|z-4|=1$

d) open and connected (e) The boundary : $z=1$



(c)



(d)

2-a) $(x^2+2x+3y-y^2)+i(-3x-2xy+2y)$

b) $(2x^2-y^2+y)+xyi$

3-a) $|w|>9$ (b) $\text{Im}\, w<-8$

4-a) $\frac{1}{2}(5+3i)$ (b) $-4i$ (c) $1-\frac{3}{2}i$

* * * *

EXERCISES II

PP. 30, 31

First: 1, 2, 3, 4-a, e, 5, 7-a, 8-a, c.

Final Answers:

4-a) Nowhere differentiable (e) Nowhere diff.

5) $f(z) = 3z^2 + 2z - 1$

8-a) $f(z) = 2x(1-y) + i(x^2 - y^2 + 2y + c)$

$= 2z + i(z^2 + c)$

c) $f(z) = \left(\dfrac{-x}{x^2+y^2} + c\right) + \dfrac{iy}{x^2+y^2} = -\dfrac{1}{z} + c$

* * *

Second: 4-d, f, 6, 7-d, 8-b, e, 9

Complete Answers:

4-d) $u = e^x \cos y$, $v = e^x \sin y$

$\Rightarrow u_x = e^x \cos y$, $u_y = -e^x \sin y$

$v_x = e^x \sin y$, $v_y = e^x \cos y$

$\Rightarrow u_x = v_y$, $u_y = -v_x$

Cauchy-Riemann equations are satisfied everywhere, and hence the given function is entire.

$$f'(z) = u_x + iv_x = e^x(\cos y + i\sin y)$$

f) $w = (x+iy)^2(x-iy) = (x^2+y^2)(x+iy)$
$= (x^3+xy^2) + i(x^2y+y^3)$

$$\Rightarrow u = x^3+xy^2 \; , \; v = x^2y+y^3$$

$$u_x = 3x^2+y^2 \; , \; u_y = 2xy$$
$$v_x = 2xy \; , \; v_y = x^2+3y^2$$

C-R eqns take the form:
$u_x = v_y \Rightarrow 3x^2+y^2 = x^2+3y^2 \Rightarrow x^2 = y^2$

This eqn is satisfied only when $y = \pm x$. (1)

$u_y = -v_x \Rightarrow 2xy = -2xy \Rightarrow xy = 0$
$\Rightarrow x = 0$ or $y = 0$ (2)

From (1), (2) we see that C-R eqns are satisfied only at $z = 0$.
The fn is differentiable at $z = 0$ and nowhere analytic:

$$f'(0) = u_x(0,0) + i\,v_x(0,0) = 0.$$

-o-o-o-

6) $u = e^{x^2-y^2}\cos 2xy \; , \; v = e^{x^2-y^2}\sin 2xy$

$\Rightarrow u_x = 2x\,e^{x^2-y^2}\cos 2xy - 2y\,e^{x^2-y^2}\sin 2xy$
$v_y = -2y\,e^{x^2-y^2}\sin 2xy + 2x\,e^{x^2-y^2}\cos 2xy$

Hence $u_x = v_y$ for all x, y (1)

$u_y = -2y e^{x^2-y^2} \cos 2xy - 2x e^{x^2-y^2} \sin 2xy$

$v_x = 2x e^{x^2-y^2} \sin 2xy + 2y e^{x^2-y^2} \cos 2xy$

$\Rightarrow \quad u_y = -v_x$ for all x, y (2)

From (1), (2) we that $f(z)$ is differentiable in the whole z-plane and consequently it is an entire fn.

$f'(z) = u_x + i v_x$

$$= 2e^{x^2-y^2}\left[(x \cos 2xy - y \sin 2xy) + i(x \sin 2xy + y \cos 2xy)\right]$$

7-d) Let $f(z) = u(x,y) + i\,v(x,y)$. Then

$$\overline{f(z)} = u - iv = u + i(-v)$$

Since $\overline{f(z)}$ is analytic, it must satisfy C-R eqns:

$u_x = (-v)_y \rightarrow u_x = -v_y$ (1)

$u_y = -(-v)_x \rightarrow u_y = v_x$ (2)

$f(z)$ is also analytic. Hence

$u_x = v_y$ (3)

$u_y = -v_x$ (4)

Adding (1), (3) we get $u_x = 0$
Adding (2), (4) we get $u_y = 0$

$$\Rightarrow u = \text{const} = c_1$$

Substituting in (1) and ( ), we find

$$V_y = 0, \ V_x = 0 \Longrightarrow V = \text{const} = c_2$$

Hence $f(z) = c_1 + ic_2 = \text{const}$ ⁂

-o-o-o-

8-b) $u_x = -\sin x \, \text{ch}\, y, \quad u_y = \cos x \, \text{sh}\, y.$

From C-R eqns:

$$V_y = -\sin x \, \text{ch}\, y \Rightarrow V = \int -\sin x \, \text{ch}\, y \, dy + h(x)$$

$$\Rightarrow V = -\sin x \, \text{sh}\, y + h(x)$$

$$V_x = -u_y \Rightarrow -\cos x \, \text{sh}\, y + h'(x) = -\cos x \, \text{sh}\, y$$

$$\Rightarrow h'(x) = 0 \Longrightarrow h(x) = c.$$

The corresponding analytic fn is

$$W = f(z) = \cos x \, \text{ch}\, y - i \sin x \, \text{sh}\, y + ic.$$

:) $u = \text{Arg}\, z = \theta \Rightarrow u_r = 0, \ u_\theta = 1, \ u_{rr} = 0, \ u_{\theta\theta} = 0$

$$\Rightarrow u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \Rightarrow u = \theta \text{ is harmonic.}$$

Using C-R eqns in polar form, we have

$v_\theta = r u_r \Rightarrow v_\theta = 0 \Rightarrow v = h(r)$

$v_r = -\frac{1}{r} u_\theta \Rightarrow h'(r) = -\frac{1}{r} \Rightarrow h(r) = -\ln r + c$
$= \ln\frac{1}{r} + C.$

$\Rightarrow f(z) = \theta + i\left(\ln\frac{1}{r} + c\right).$ $(r>0)$

---

1) Since u and v are conjugate harmonic fns, they satisfy C-R eqns:

$$u_x = v_y \quad , \quad u_y = -v_x \qquad (1)$$

Also, we have

$$u_{xx} + u_{yy} = 0 \quad , \quad v_{xx} + v_{yy} = 0 \qquad (2)$$

$(uv)_x = u v_x + v u_x \Rightarrow (uv)_{xx} = u v_{xx} + 2 u_x v_x + v u_{xx}$

$(uv)_y = u v_y + v u_y \Rightarrow (uv)_{yy} = u v_{yy} + 2 u_y v_y + v u_{yy}$

Use (2):

$$(uv)_{xx} + (uv)_{yy} = u(\underbrace{v_{xx} + v_{yy}}_{0}) + 2(u_x v_x + u_y v_y) + v(\underbrace{u_{xx} + u_{yy}}_{0})$$

Use (1):

$$(uv)_{xx} + (uv)_{yy} = 2\left(v_y v_x + (-v_x) v_y\right) = 0$$

Hence uv is a harmonic fn ※

* * *

Third:

1) Use C-R eqns to determine where each fn is differentiable, and where it is analytic. Find $f'(z)$ if it exists:

(a) $w = \operatorname{Im} z$

(b) $w = \frac{1}{z}$

(c) $w = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

d) $f(z) = r^{1/2}\cos\frac{\theta}{2} + i\,r^{1/2}\sin\frac{\theta}{2}$ $\quad (r > 0,\ -\pi < \theta < \pi)$

2) Show that each given fn is harmonic and find the corresponding analytic fn $w = u + iv$:

(a) $u = x^3 - 3xy^2 + y$

(b) $u = \ln^2 r - \theta^2$; $r > 0,\ -\pi < \theta < \pi$

(c) $v = e^y \sin x$

3) Let v be the harmonic conjugate of u. Prove that $h = u^2 - v^2$ is a harmonic function.

4) Let $f(z)$ be analytic and nonzero in a domain $\mathfrak{D}$.
Prove that $\ln|f(z)|$ is harmonic in $\mathfrak{D}$.

Final Answers:

1 - a) Nowhere (b) Differentiable and analytic everywhere except at $z = 0$. $f'(z) = -\frac{1}{z^2}$; $z \neq 0$

c) Diff. on the lines $x = 0$, $y = 0$. Nowhere analytic. Where it exists, $f'(z)$ is

$$f'(z) = (3x^2 + 3y^2 - 3) + 6xy\,i$$

d) Diff. and analytic in the domain $r > 0$, $-\pi < \theta < \pi$.

$$f'(z) = \frac{1}{2} r^{-\frac{1}{2}} \left(\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right).$$

2 - (a) $f(z) = (x^3 - 3xy^2 + y) + i(3x^2y - y^3 - x + c) = z^3 - iz + ic$

(b) $f(z) = \ln^2 r - \theta^2 + (2\theta \ln r + c)\,i$

(c) $f(z) = e^y(-\cos x + i \sin x)$

* * * *

Chapter II

Elementary Functions of a Complex Variable

EXERCISES III

PP. 42, 43, 44

First : 1, 2 – (i) , 3 – (i), (iv) , 4 – (ii) , 5 – (ii) , 7 – (i) , 8 – (i) , (vi) , (ix) , 9 – (i) , 10 – (i) , 11 – (i)

Final Answers :

2 – (i) $z = \ln 2 + i(2n+1)\pi$

4 – (ii) $\frac{4m-1}{2}\pi + 4i \; ; m = 0, \pm 1, \pm 2, \dots$

7 – (i) $\left(\pm \frac{\pi}{3} + 2n\pi\right) i \; ; \; n = 0, \pm 1, \pm 2, \dots$

8 – (i) $2n\pi i \; ; n = 0, \pm 1, \pm 2, \dots$

(vi) $\ln\sqrt{2} + i\left(-\frac{\pi}{4}\right)$

(ix) $e^{(4n+1)i} \; ; \; n = 0, \pm 1, \pm 2, \dots$

9 – (i) $z = i$

11 – (i) $z = -i \ln(2i + i\sqrt{3}) = \left(\frac{\pi}{2} + 2n\pi\right) - i \ln(2+\sqrt{3}) \; ;$

$n = 0, \pm 1, \pm 2, \dots$

* * *

Second : 2 – (iv) , 4 – (i) , 7 – (iii) , 8 – (vii) , (x) , 9 – (ii) , 10 – (ii) , 11 – (ii) , 12 – (i)

Complete Answers :

2 - iv) $e^{iz} = \overline{(e^{iz})} \Rightarrow e^{i(x-iy)} = \overline{(e^{i(x+iy)})}$

$$\Rightarrow e^{y+ix} = \overline{(e^{-y+ix})} \Rightarrow e^{y}e^{ix} = e^{-y}e^{-ix} \qquad \text{---}(1)$$

Note that
$$\overline{e^{ix}} = \overline{\cos x + i\sin x} = \cos x - i\sin x = e^{-ix}$$

From (1) :

$$y = -y \Rightarrow y = 0,$$

$$x = -x + 2n\pi \Rightarrow x = n\pi$$

The roots are

$$z = n\pi \; ; \; n = 0, \pm 1, \pm 2, \dots$$

4 - i) $\cos z = 2 \Rightarrow \cos(x + iy) = 2$

$$\Rightarrow \cos x \cos iy - \sin x \sin iy = 2$$

$$\Rightarrow \cos x \, ch\, y - i \sin x \, sh\, y = 2$$

$$\Rightarrow \cos x \, ch y = 2 \quad (1)$$
$$\sin x \, sh\, y = 0 \quad (2)$$

From (2) : $\sin x = 0$ or $sh\, y = 0$

$$\Rightarrow x = n\pi \quad \text{or} \quad y = 0$$

Put $x = n\pi$ in (1) : $(-1)^n ch\, y = 2$

Since $ch\, y \geq 1$ for all y . n must be even :

$$x = 2n\pi \quad , \quad y = ch^{-1} 2$$

Put $y = 0$ in (1) : $\cos x = 2$

It has no solution because

$|\cos x| \leq 1$ for all x

The roots are

$$z = 2n\pi + i\, ch^{-1} 2 \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$$

7-(ii) $sh\, z = i \Rightarrow sh(x+iy) = i$

$\Rightarrow sh\, x\, ch\, iy + ch\, x\, sh\, iy = i$

$\Rightarrow sh\, x \cos y + i\, ch\, x \sin y = i$

$\Rightarrow \quad sh\, x \cos y = 0 \quad (1)$

$\quad\quad ch\, x \sin y = 1 \quad (2)$

From (1) : $sh\, x = 0$ or $\cos y = 0$

$\Rightarrow x = 0$ or $y = \frac{2n+1}{2}\pi$; $n = 0, \pm 1, \pm 2, \ldots$

Put $x = 0$ in (2) : $\sin y = 1 \Rightarrow y = \frac{\pi}{2} + 2n\pi$

Put $y = \frac{2n+1}{2}\pi$ in (2) : $(-1)^n ch\, x = 1 \ldots \Rightarrow$

n must be even and $x = 0$.

$\Rightarrow x = 0$ and $y = \frac{4n+1}{2}\pi = \frac{\pi}{2} + 2n\pi$

The roots are

$$z = i\left(\frac{\pi}{2} + 2n\pi\right) \; ; \quad n = 0, \pm 1, \pm 2, \ldots$$

8-(vii) $|-3 + i\sqrt{27}| = \sqrt{9 + 27} = 6$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{\sqrt{27}}{-3}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$$\Rightarrow \operatorname{Ln}(-3 + i\sqrt{27}) = \ln 6 - \frac{\pi}{3} i$$

(x) $$\left[\frac{e}{2}(-1 - i\sqrt{3})\right]^{3\pi i} = e^{3\pi i \ln\left[\frac{e}{2}(-1 - i\sqrt{3})\right]} \quad \cdots (1)$$

$|-1 - i\sqrt{3}| = 2$, $\theta = \tan^{-1}\sqrt{3} = -\frac{2\pi}{3}$ (الربع الثالث)

$$\Rightarrow \ln\left[\frac{e}{2}(-1 - i\sqrt{3})\right] = \ln\left(\frac{e}{2}\right)(2) + i\left(-\frac{2\pi}{3} + 2n\pi\right)$$

$$= i\left(-\frac{2\pi}{3} + 2n\pi\right).$$

$$\Rightarrow \left[\frac{e}{2}(-1 - i\sqrt{3})\right]^{3\pi i} = e^{3\pi i \cdot i\left(-\frac{2\pi}{3} + 2n\pi\right)} = e^{2\pi^2 - 6n\pi^2}.$$

The principal value is $e^{2\pi^2}$.

9-(ii) $\ln r + i\theta = 1 + \pi i$ where $z = re^{i\theta}$

$$\Rightarrow \ln r = 1 \; , \; \theta = \pi + 2n\pi \quad \Rightarrow r = e.$$

$$\Rightarrow z = e\, e^{i(\pi + 2n\pi)}$$

$$= -e.$$

10 - ii) Let $w = \cos^{-1} z \Rightarrow \cos w = z \Rightarrow \dfrac{e^{iw} + e^{-iw}}{2} = z$

$\Rightarrow e^{iw} - 2z + e^{-iw} = 0 \Rightarrow e^{2iw} - 2z e^{iw} + 1 = 0$

Solving this quadratic eqn in e^{iw}, we get

$$e^{iw} = \frac{2z + \sqrt{4z^2 - 4}}{2} = z + \sqrt{z^2 - 1}.$$

Here $\sqrt{z^2 - 1}$ is a double-valued fn.

$\Rightarrow iw = \ln(z + \sqrt{z^2 - 1}) \Rightarrow w = -i \ln(z + \sqrt{z^2 - 1})$.

11 - ii) $\cos z = \quad \Rightarrow z = \cos^{-1}\sqrt{2}$

$\Rightarrow z = -i \ln(\sqrt{2} \pm 1)$

Since $\ln(\sqrt{2} - 1) = \ln \dfrac{1}{\sqrt{2} + 1} = -\ln(\sqrt{2} + 1)$,

hence

$$\begin{aligned} z &= \pm i \ln(1 + \sqrt{2}) \\ &= \pm i \left[\ln(1 + \sqrt{2}) + 2n\pi i\right] \\ &= 2n\pi \pm i \ln(1 + \sqrt{2}) \quad , \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

12 - i) Let $w = \tan^{-1} z \Rightarrow \tan w = z$

$\Rightarrow \dfrac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} = z$

$$\Rightarrow \frac{e^{2iw}-1}{e^{2iw}+1} = iz \Rightarrow e^{2iw} - 1 = ize^{2iw} + iz$$

$$\Rightarrow e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$$

$$\Rightarrow w = \tan^{-1} z = \frac{1}{2i} \ln\left(\frac{i-z}{i+z}\right) = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right).$$

$$\tan^{-1} 2i = \frac{i}{2} \ln \frac{3i}{-i} = \frac{i}{2} \ln(-3)$$

$$= \frac{i}{2}\left[\ln 3 + i(\pi + 2n\pi)\right]$$

$$= -\frac{2n+1}{2}\pi + \frac{i}{2}\ln 3 \; ; \quad n = 0, \pm 1, \dots$$

* * *

Third:

Find all roots of each of the following equations:

1) $e^z = 1 + i\sqrt{3}$

2) $\cos z = 0$

3) $\operatorname{ch} z = -2$

4) $\mathrm{Ln}(z-1) = \frac{i\pi}{2}$

Find all values of

5) (i) $\ln 3$ (ii) $\mathrm{Ln}(i\sqrt{2} - \sqrt{2})$

6) (i) $(-1)^{i}$ (ii) Principal value of $(1+i)^{i}$

7) (i) $\tan^{-1}(1+i)$ (ii) $\cosh^{-1} i$ (iii) $\operatorname{th}^{-1}(1+2i)$

Final Answers: Always $n = 0, \pm 1, \pm 2, \ldots$

1) $\ln 2 + i(1+2n)\pi$ (2) $\frac{\pi}{2} + 2n\pi$

3) $ch^{-1} 2 + i(2n-1)\pi$ or $\ln(2-\sqrt{3}) + i(2n+1)\pi$

4) $1+i$ (5)-(i) $\ln 3 + 2n\pi i$ (ii) $\ln 2 + \frac{3\pi}{4} i$

6)-(i) $e^{-(1+2n)\pi}$ (ii) $e^{-\frac{\pi}{4} + i \ln\sqrt{2}}$

7)-(i) $-\frac{1}{2}\left[\tan^{-1} 2 + (2n-1)\pi\right] + \frac{i}{4}\ln 5.$ (ii) $\ln(\sqrt{2} \pm 1) + i\left(\pm\frac{1}{2} + 2n\right)\pi$

(iii) $\frac{1}{4}\ln 2 + i\left(\frac{3}{8} + n\right)\pi$: $th^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right).$

* * * *

الزمن : 1 ساعة

الإجابة في ورقة الأسئلة

الفرقة الثانية كهرباء — رياضيات هندسية (٣) — < ١ > — الامتحان ثلاثة أسئلة في صفحة واحدة

الاسم (باللغة العربية):	الفصل:	رقم البحث:

1. Evaluate i) $\int_0^{\infty} 3^{5-x^2}\,dx$ ii) $\int_0^{\pi/2} (\tan\theta)^{3/2}\,d\theta$

2. Show that that the function: $u(x, y) = \cos 2x \cosh 2y$ is harmonic. and find its harmonic conjugate. If $f(Z) = u + iv$ Find $f'(Z)$.

3. Two dice are thrown. The random variable x represents the maximum of the two faces and the random variable y represents the number of sixes. Find
   (i) The probability distribution of the two random variables x & y.
   (ii) The marginal probability distribution of x and y, then check independence.

الزمن : ثلاث ساعات	رياضيات هندسية (٣)	يناير ٢٠١٠
الامتحان يحتوى على ستة أسئلة فى صفحتين		١ / ٢

1) (a) i) Evaluate $\int_0^1 x^4 \left(\ln \frac{1}{x}\right)^3 dx$ (3 marks)

ii) Show that $\beta(x, x+1) = \frac{1}{2} \frac{\Gamma^2(x)}{\Gamma(2x)}$ and hence show that

$$\int_0^{\pi/2} \left(\frac{1}{\sin^3 \theta} - \frac{1}{\sin^2 \theta} \right)^{\frac{1}{4}} \cos\theta \, d\theta = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{\pi}}$$ (5 marks)

(b) Show that the function $u = x^3 - 3xy^2 + y + 10$ is harmonic and find the corresponding analytic function $f(z) = u + iv$. (7 marks)

2) (a) Find all values of
i) $e^{2z-1} = 1 + i$ (2 marks) ii) $\sin z = -5$ (2 marks) iii) $(1-i)^i$ (2 marks)

(b) Under the function $w = \frac{1}{Z}$, find the image of:
i) The line $y = 2x$ ii) The circle $|z - 1| = 1$. Show the curves and their images graphically. Discuss the behavior of each curve near the origin under the transformation. (9 marks)

3) (a) Determine whether the set of the functions:
i) $\{e^{2x}, \cosh 2x, \sinh 2x\}$ ii) $\{3 + 2\sqrt{x}, 3x + 2\sqrt{x}, x - 1, x^2\}$
linearly independent or not and find the interval of validity in each case. Hence from (i) form a fundamental set of solutions for the D. E. $y'' - 4y = 0$, in at least two different forms. (8 marks)

(b) Solve the D. E. $x^3 y''' - 6y = 0$ (7 marks)

4) (a) Find the series solution about $x = 0$ of D.E. $y'' + xy = 0$, and write the interval of validity. (8 marks)

4) (b) For Bessel D.E. $x^2 y'' + x y' + (x^2 - \upsilon^2) y = 0$,

i) Find the indicial equation for its series solution near the origin.

ii) Write the form of the solution in the following cases:

(a) $\upsilon^2 = \frac{1}{4}$ (b) $\upsilon^2 = 0$ (c) $\upsilon^2 = 4$ (7 marks)

without finding it.

5) (a) Three machines A , B and C produce per day 3000 , 2000 and 1000 items, respectively of the total number of items of a factory. The number of defective items of these machines are 15 , 30 and 25 items, respectively. An item is selected at random from the daily production of the factory. Find the probability that the selected item is defective and the probability that this defective item is of the production of machine C. (6 marks)

(b) The joint probability function of two random variables X & Y is defined by:

$$f(x,y) = \begin{cases} \frac{C}{x} & 0 < y < x \ \& \ 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

i) C

ii) Probability that the random interval $(\frac{1}{x}, \frac{2}{x})$ includes the point $\frac{8}{3}$.

iii) Check independence of X & Y. (9 marks)

6) (a) Let X be a random variable representing the number divisible by 3 when a fair die is tossed 10 times. Determine:

i) $P(x \geq 1)$

ii) $E(3x + 1)$

iii) $\sigma_{2X - 1}$

iv) $m_x(t)$ (8 marks)

(b) The diameters of a sample of 400 washers produced by a machine are normally distributed with mean $\mu = 10$ mm.

i) Determine the standard deviation σ so that the standard error is 0.001.

ii) Prepare a Control chart. (Given that $\phi(-1.96) = 0.025$, $\phi(3.09) = 0.999$)

iii) Prove that $m_{\bar{x}}(t) = e^{10t(1 + 0.00002\, t)}$.

(7 marks)

جامعة عين شمس
كلية الهندسة
قسم الفيزيقا و الرياضيات الهندسية

الفرقة الثانية كهرباء
تخلفات
مارس ٢٠١١
٢ / ١

الزمن : ثلاث ساعات	رياضيات هندسية (٣)

الامتحان يحتوى على خمسة أسئلة فى صفحتين

1) (a) Evaluate: i) $\int_0^1 \sqrt[3]{x}\ \ln^5 x\ dx$ ii) $\int_0^{\pi/2} \left(\frac{1}{\sin^5\theta} - \frac{1}{\sin^4\theta} \right)^{\frac{1}{8}} \cos\theta\ d\theta$

(b) Show that the function: $V = \ln \frac{1}{r} - r\sin\theta$ is harmonic.

Find the analytic function $f(z) = u + iv$.

(20 marks)

2) (a) Find all values of :

(i) $e^{2Z-1} = 1 + i$ (ii) $\cos Z = -4i$ (iii) $(1 - i)^i$

(b) Under the mapping $w = \frac{1}{Z}$, find the image of:

(i) The line $y + 2x = 0$ (ii) The circle $|Z - 1| = 1$

Show the curves and their images graphically. Discuss the behavior of each curve near the origin under the transformation.

(20 marks)

3) (a) Find the general solution of: $x^2 y'' + x y' - m^2 y = 0$,

for any real number m^2.

Then use Frobenius method to solve: $x^2 y'' + x y' - 4y = 0$

and compare between the indicial and auxiliary equations.

(b) Classify the singular points of Legendre's equation:

$$(1 - x^2)\, y'' - 2x\, y' + m(m+1)\, y = 0.$$

Hence solve it in powers of x.

Discuss the solution in case of: i) $m = 1$ ii) $m = 2$.

(20 marks)

4) (a) Three faculties F_1, F_2 and F_3 have 1500 , 2000 and 3500 students, respectively of the total number of Ain Shams University. The number of students that have received grade A in Statistics course are 50 , 100 and 150 students, respectively. A student is selected at random from among the 7000 students. Find the probability that the student received grade A and the probability that he was from faculty F_1. (8 marks)

(b) The joint probability function of two random variables X & Y is defined by:

$$f(x,y) = \begin{cases} \dfrac{C}{y} & 0 < x < y \ \& \ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

i) C

ii) $P(\frac{1}{3} \le y \le \frac{1}{2})$.

iii) Check independence of X & Y. (12 marks)

5) (a) Let X be a random variable representing the number divisible by 2 when a fair die is tossed 10 times. Determine:

i) $P(x \ge 1)$ ii) $P(0 \le x \le 2)$

iii) $E(3x - 2)$ iv) σ_{2X+1} (10 marks)

(b) The lengths of a sample of 900 items produced by a machine are normally distributed with mean $\mu = 5.0$ mm and standard deviation of 0.03 mm.

i) Determine the percentage of items produced by a machine has lengths greater than 4.91 mm.

ii) Prepare a Control chart. (Given that $\phi(-2) = 0.0228$, $\phi(-3) = 0.0013$)

(10 marks)

1) (a) Evaluate: i) $\int_0^1 (\sqrt{x})^3 \left(\ln \frac{1}{x}\right)^4 dx$ ii) $\int_0^{\pi/2} \left(\frac{1}{\sin^5\theta} - \frac{1}{\sin^4\theta}\right) \cos\theta \, d\theta$

(b) Show that the function: $V = \ln \frac{1}{r} - r \sin\theta$ is harmonic.

Find the analytic function $f(z) = u + i\, v$.

(19 marks)

2) (a) Find all values of :

(i) $e^{3Z+1} = 1 - i$ (ii) $\cos Z = -4i$ (iii) $(1 + i)^{3i}$

(b) Under the mapping $w = \frac{1}{Z}$, find the image of:

(i) The line $y + 2x = 0$ (ii) The circle $|Z - i| = 1$

Show the curves and their images graphically. Discuss the behavior of each curve near the origin under the transformation.

(19 marks)

3) (a) Find the general solution of: $x^2 y'' + x y' - k^2 y = 0$,

for any real number k^2.

Then use Frobenius method to solve: $x^2 y'' + x y' - 4 y = 0$

and compare between the indicial and auxiliary equations.

(b) Classify the singular points of Legendre's equation:

$$(1 - x^2)\, y'' - 2\, x\, y' + k\,(k+1)\, y = 0.$$

Hence solve it in powers of x.

Discuss the solution in case of: i) $k = 1$ ii) $k = 2$.

(19 marks)

الفصل:

الاسم (باللغة العربية):

1\. a) Evaluate the following integrals: i) $\int_0^1 \sqrt[5]{x}\ \ln^3 x\ dx$ ii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

b) Find the values of m such that: $u = e^{mx} \cos 2y$ is harmonic and find v such that $f(z) = u + iv$ is an analytic function for one value of m.

2\. a) Three machines in a certain factory A_1, A_2 and A_3 produce respectively 3000, 2000 and 1000 items / day. The number of defective output items of these machines are 15, 20 and 10 items / day, respectively. If an item is selected at random from the daily production. Find the probability that the selected item is defective and the probability that it was from machine A_1.

b) The joint probability function of two random variables x & y is defined by:

$$f(x,y) = \begin{cases} c(x-2y) & x=2,3 \quad ; \quad y=-1/2, -3/2, -5/2 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(i) c (ii) $P(x \geq 3)$ (iii) Check independence of x & y.

الفصل:	الاسم (باللغة العربية):

1\. a) Evaluate the following integrals: i) $\int_0^1 \sqrt[3]{x}\ \ln^5 x\ dx$ ii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

b) Find the values of m such that: $v = e^{mx} \sin 3y$ is harmonic, and find u such that $f(z) = u + iv$ is an analytic function for one value of m.

2\. a) Three machines in a certain factory A_1, A_2 and A_3 produce respectively 1000, 2000 and 3000 items / day. The number of defective output items of these machines are 10, 20 and 15 items / day, respectively. If an item is selected at random from the daily production. Find the probability that the selected item is defective and the probability that it was from machine A_3.

b) The joint probability function of two random variables x & y is defined by :

$$f(x,y) = \begin{cases} c(y-2x) & x=-1/2, -3/2, -5/2 \ ; \ y=2,3 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(i) c (ii) $P(y \geq 3)$ (iii) Check independence of x & y.

جامعة عين شمس
كلية الهندسة
قسم الفيزيقا و الرياضيات الهندسية

الفرقة الثانية كهرباء

الزمن : ثلاث ساعات	رياضيات هندسية (٣)	يناير ٢٠١٢
الامتحان يحتوى على خمسة أسئلة في صفحتين		٢ / ١

1) (a) Evaluate: i) $\int_0^1 x^{1/3} (\ln x)^7 dx$ ii) $\int_0^3 x^3 \left(27 - x^3\right)^{2/3} dx$

(b) Prove that if V is the harmonic conjugate of u in a domain D, then

$W = u^2 - v^2$ is harmonic in D.

(20 marks)

2) (a) Show that the function $u = \ln \sqrt{x^2 + y^2}$ is harmonic in the domain

$r > 0$, $0 < \theta < 2\pi$. Find v such that the function

$f(z) = u + iv$ is analytic in that domain. ($Z = x + i\,y = r e^{i\theta}$).

(b) Find all values of Z , if:

(i) $e^{3Z+3} = 2 - 2i$ (ii) $\sin 2Z = 3\,i$

(c) Evaluate all values of (i) $\ln\left(3 - i\sqrt{27}\right)$ (ii) $(2 - 2i)^i$

(20 marks)

3) (a) Under the transformation $W = \dfrac{1}{Z}$, Find the image of:

(i) The circle $|Z + 2| = 2$ (ii) The line $y = x$.

Show the regions graphically and discuss the behavior of the two curves and their images near the origin.

(b) Find the series solution about $x = 0$ of the equation:

$y'' - x\,y = 0$, and write its interval of validity.

(20 marks)

4) (a) Three machines A_1, A_2 and A_3 produce 1000 , 1200 and 1800 items / day, respectively of the total number of items in a factory. The number of defective items are 50 , 48 and 36 items, respectively. Find the probability that the selected item is defective and the probability that this defective item is of the production of machine A_1.

(8 marks)

(b) The joint probability function of two random variables X & Y is defined by:

$$f(x, y) = \begin{cases} C & x < y < 2 - x \quad \& \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

i) C

ii) $P\left(\frac{1}{3} \le x \le \frac{1}{2}\right)$.

iii) Check independence of X & Y.

(12 marks)

5) (a) Let X be a random variable representing obtaining sum at most 5 when a fair die is tossed 10 times. Determine:

i) $P(x \ge 1)$

ii) $P(0 \le x \le 2)$

iii) $E(3x - 2)$

iv) σ_{2X+1}

(10 marks)

(b) The lengths of a sample of 10000 items produced by a machine are normally distributed with mean $\mu = 4.0$ mm and standard deviation of 0.03 mm.

i) Determine the percentage of items produced by a machine has lengths less than 3.91 mm.

ii) Prepare a Control chart. (Given that $\phi(-2) = 0.0228$, $\phi(-3) = 0.0013$)

(10 marks)

2)(a) To find V

$V_\theta = r u_r = 1 \Rightarrow V = h(r) + \theta$

$V_r = -\frac{1}{r} u_\theta = 0 \Rightarrow V_r = 0 + \frac{dh}{dr} \Rightarrow 0 = \frac{dh}{dr} \Rightarrow h(r) = c$

$\therefore V = \theta + C \Rightarrow f(z) = \ln r + i(\theta + C) = \ln\sqrt{x^2+y^2} + i(\tan^{-1}\frac{y}{x} + c)$

2(b)i) $e^{3z+3} = 2-2i \Rightarrow e^{3x+3iy+3} = \sqrt{8}\, e^{i(-\frac{\pi}{4}+2k\pi)} \Rightarrow$

$e^{3x+3} \cdot e^{3iy} = \sqrt{8}\, e^{i(-\frac{\pi}{4}+2k\pi)} \Rightarrow e^{3x+3} = \sqrt{8}$ and $3y = -\frac{\pi}{4} + 2k\pi$

$\therefore x = \frac{1}{3}(\ln 2\sqrt{2} - 3)$ and $y = -\frac{\pi}{12} + \frac{2}{3}k\pi$, $k = 0, \pm1, \pm2, \dots$

ii) $\sin 2z = 3i \Rightarrow \sin 2x \cosh 2y + i \cos 2x \sinh 2y = 3i \Rightarrow$

$\sin 2x \cosh 2y = 0$ and $\cos 2x \sinh 2y = 3$

$\sin 2x = 0$ or $\cosh 2y = 0$ refused $\Rightarrow \therefore 2x = n\pi \Rightarrow x = \frac{n}{2}\pi$ $n = 0, \pm1, \pm2, \dots$

$\therefore \cos n\pi \sinh 2y = 3 \Rightarrow (-1)^n \sinh 2y = 3 \Rightarrow y = \frac{1}{2}\sinh^{-1}(-1)^n 3$

$\therefore y = \frac{(-1)^n}{2}\sinh^{-1}(3)$ $n = 0, \pm1, \pm2, \dots$

(iii) $\ln(3 - i\sqrt{27}) \Rightarrow \ln(6 e^{i(-\frac{\pi}{3}+2k\pi)}) = \ln 6 + i(-\frac{\pi}{3} + 2k\pi)$, $k = 0, \pm1, \pm2, \dots$

c)ii) $(2-2i)^i = e^{i\ln(2-2i)} = e^{i\ln[2\sqrt{2}\, e^{i(-\frac{\pi}{4}+2k\pi)}]}$

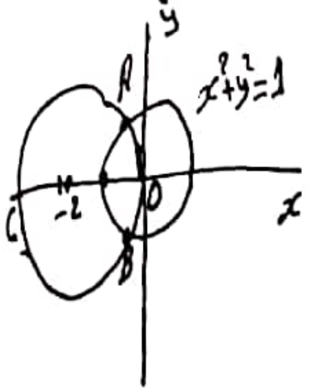
$= e^{i(\ln 2\sqrt{2})}\, e^{-(-\frac{\pi}{4}+2k\pi)} = e^{\frac{\pi}{4}-2k\pi}\, e^{i\ln(2\sqrt{2})}$, $k = 0, \pm1, \pm2, \dots$

$= e^{\frac{\pi}{4}-2k\pi}[\cos(\ln 2\sqrt{2}) + i\sin(\ln 2\sqrt{2})]$. #

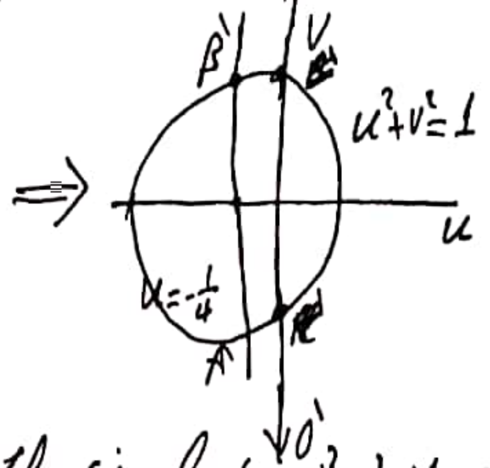
3) a) i) $w = \frac{1}{z} \Rightarrow u = \frac{x}{x^2+y^2}, \; v = \frac{-y}{x^2+y^2}$ or $x = \frac{u}{u^2+v^2}, \; y = \frac{-v}{u^2+v^2}$

$|z+2| = 2 \Rightarrow |x+2+iy| = 2 \Rightarrow (x+2)^2 + y^2 = 4 \Rightarrow x^2+y^2+4x = 0$

$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{4u}{u^2+v^2} = 0 \Rightarrow 1 + 4u = 0 \Rightarrow u = -\frac{1}{4}$



$\Rightarrow$

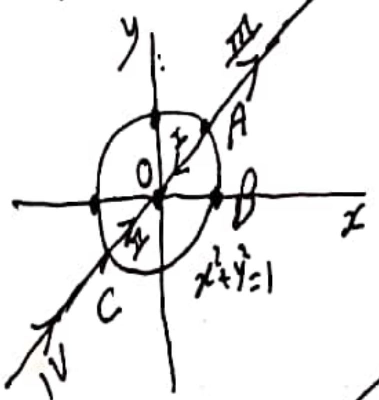


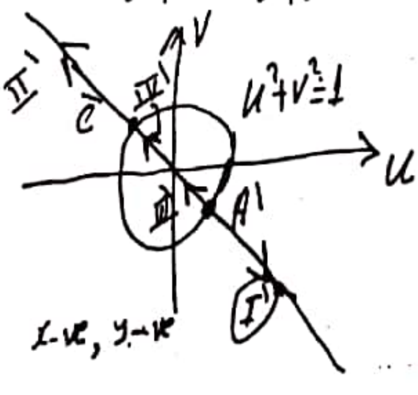
$(x,y) \to (0,0)$ on the circle $(x+2)^2+y^2=4 \Rightarrow y = \pm\sqrt{-4x-x^2}$

or $x^2+y^2 = -4x$

$u = \frac{x}{-4x} \to -\frac{1}{4}, \; v = \frac{+y}{+4x} = \frac{y}{4x}$ $\begin{cases} +\infty & y\text{-ve and } x\text{-ve} \\ -\infty & y\text{+ve and } x\text{-ve} \end{cases}$

3 a) ii) The line $y = x \Rightarrow \frac{u}{u^2+v^2} = \frac{-v}{u^2+v^2} \Rightarrow u = -v$





$0 \begin{cases} +\infty & x\text{-ve}, y\text{-ve} \\ -\infty & x\text{+ve}, y\text{+ve} \end{cases}$

3) b) $P(x) = 0, \; q(x) = -x$ analytic fns.

Let $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y = a_0\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right] + a_1\left[x + \frac{x^4}{12} + \frac{1}{504}x^7 + \cdots\right]$

interval of validity $(-\infty, \infty)$

1) a) i) let $\ln x = -t \Rightarrow x = e^{-t}$, $dx = -e^{-t}dt$ and $x=0 \Rightarrow t\to\infty$
$x = 1 \to t = 0$

$$\therefore I = \int_{\infty}^{0} e^{-\frac{t}{3}} (-t)^7 (-e^{-t})dt = -\int_0^{\infty} t^7 e^{-\frac{4}{3}t} dt \qquad (I)$$

let $t = \frac{3}{4}u \Rightarrow dt = \frac{3}{4}du \Rightarrow I = -\int_0^{\infty} (\frac{3}{4})^7 u^7 e^{-u} (\frac{3}{4})du$

$$= -(\tfrac{3}{4})^8 \int_0^{\infty} u^7 e^{-u} du = -(\tfrac{3}{4})^8 \Gamma(8) = -(\tfrac{3}{4})^8 (7!) \ \#$$

ii) $I = \int_0^3 x^3 (27 - x^3)^{\frac{2}{3}} dx$

let $x^3 = 27t \Rightarrow x = 3t^{\frac{1}{3}}$, $dx = t^{-\frac{2}{3}}dt$ and $x=0 \Rightarrow t=0$, $x=3 \Rightarrow t=1$

$$\therefore I = 243 \int_0^1 t^{\frac{1}{3}} (1-t)^{\frac{2}{3}} dt \Rightarrow I = 243\beta(x,y) = 243\beta(\tfrac{4}{3}, \tfrac{5}{3})$$

$$I = 243 \frac{\Gamma(\frac{4}{3})\Gamma(\frac{5}{3})}{\Gamma(3)} = 243 \frac{\frac{1}{3}\Gamma(\frac{1}{3})\,\frac{2}{3}\Gamma(\frac{2}{3})}{2!} = 27 \frac{\pi}{\sin\frac{\pi}{3}} = 18\sqrt{3} \ \#$$

1) b) $W = u^2 - v^2 \Rightarrow W_x = 2uu_x - 2vv_x \Rightarrow W_{xx} = 2uu_{xx} + 2u_x^2 - 2vv_{xx} - 2v_x^2$

Similarly $W_{yy} = 2uu_{yy} + 2u_y^2 - 2vv_{yy} - 2v_y^2$

v is Harmonic Conjugate for $u \Rightarrow u_x = v_y$ and $u_y = -v_x \Rightarrow$
and $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$

$$\therefore W_{xx} + W_{yy} = 2u(u_{xx} + u_{yy}) + 2u_x^2 + 2u_y^2 - 2v(v_{xx} + v_{yy}) - 2(-u_y)^2 - 2(u_x)^2 = 0$$

$\therefore W = u^2 - v^2$ is harmonic #

2(a) $u = \ln\sqrt{x^2+y^2} = \ln r \Rightarrow u_r = \frac{1}{r}$, $u_\theta = 0$, $u_{rr} = -\frac{1}{r^2}$

For u to be harmonic $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$??
R.H.S $= r^2(-\frac{1}{r^2}) + r(\frac{1}{r}) + 0 \Rightarrow 1 - 1 = 0 =$ L.H.S
u is harmonic. #

الاسم (باللغة العربية):	الفصل:

1. Evaluate the following integrals: i) $\int_0^1 \sqrt[5]{x}\ \ln^4 x\ dx$ ii) $\int_0^3 \frac{dx}{(27-x^3)^{1/3}}$ (6 marks)

2. Find the relation between A and B such that: $u = Ax^3 + Bxy^2 - 3x + 5y + 4$ is harmonic. Find in terms of Z the analytic function $f(Z) = u + iv$ when A=2. (6 marks)

3. A pair of dice I , II is thrown. Let, A = event that the outcome on die I is more than 5, B = event that the outcome on die II is more than 5, and C = event that the sum of the outcome is more than 9. Are the three events A^C, B^C and C independent ? (5 marks)

4. The joint probability function of two random variables x & y is defined by :

$$f(x,y) = \begin{cases} C\,(x^2+y) & x=-1,\ 2 \quad ; \quad y=1,4,5 \\ 0 & \text{otherwise} \end{cases}$$ Determine:

(i) C   (ii) $P(x+y>5)$ (iii) Check independence of x & y.   (5 marks)

$\boxed{3}$ $A=\{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

$B=\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}$

$C=\{(5,5),(6,4),(4,6),(5,6),(6,5),(6,6)\}$

$P(A)=\boxed{\frac{1}{6}}$ $P(B)=\boxed{\frac{1}{6}}$ $P(C)=\boxed{\frac{1}{6}}$

$A\cap B=\{(6,6)\} \Rightarrow P(A\cap B)=\boxed{\frac{1}{36}}=P(A).P(B)=\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$

$\Rightarrow$ A & B are Indep.

$A\cap C=\{(6,4),(6,5),(6,6)\} \Rightarrow P(A\cap C)=\frac{3}{36}$

$P(A\cap C)=\boxed{\frac{1}{12}} \neq P(A).P(C) \neq \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \Rightarrow$ A & C are not Indep

$B\cap C=\{(4,6),(5,6),(6,6)\} \Rightarrow P(B\cap C)=\frac{3}{36}$

$P(B\cap C)=\boxed{\frac{1}{12}} \neq P(B).P(C) \neq \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \Rightarrow$ B & C are not Indep.

$A\cap B\cap C=\{6,6\} \Rightarrow P(A\cap B\cap C)=\boxed{\frac{1}{36}}$

$P(A\cap B\cap C)=\boxed{\frac{1}{36}} \neq P(A)P(B)P(C) \neq \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$

$\Rightarrow$ A & B & C are not Indep.

$\Rightarrow$ A^c & B^c & C are not Indep.

$\boxed{4}$ (i) $\sum_y \sum_x c(x^2+y)=1$

$\therefore C\left[(\overset{x=-1}{2+5+6})+(\overset{x=2}{5+8+9})\right]=1 \Rightarrow \boxed{C=\frac{1}{35}}$ 1

(ii) $P(x+y>5)=f(2,4)+f(2,5)$

$=\frac{1}{35}[8+9] = \boxed{\frac{17}{35}}$ 1

(iii) $f_1(x)=\sum_y \frac{1}{35}(x^2+y)=\frac{1}{35}\left[(x^2+1)+(x^2+4)+(x^2+5)\right]$

$=\boxed{\frac{1}{35}[3x^2+10]}$, $x=-1,2$

$f_2(y)=\sum_x \frac{1}{35}(x^2+y)=\frac{1}{35}\left[(1+y)+(4+y)\right]$

$=\boxed{\frac{1}{35}[5+2y]}$, $y=1,4,5$

$f(x,y)=\frac{1}{35}(x^2+y) \neq f_1(x).f_2(y)$

$\neq \frac{1}{35}[3x^2+10]\cdot\frac{1}{35}[5+2y]$ 1

$\Rightarrow$ x & y are not independent

y \ x	-1	2	$f_2(y)$
1	$\frac{2}{35}$	$\frac{5}{35}$	$\frac{7}{35}$
4	$\frac{5}{35}$	$\frac{8}{35}$	$\frac{13}{35}$
5	$\frac{6}{35}$	$\frac{9}{35}$	$\frac{15}{35}$
$f_1(x)$	$\frac{13}{35}$	$\frac{22}{35}$	

2

$f(-1,1)=\frac{2}{35} \neq f_1(-1).f_2(1) \neq \left(\frac{13}{35}\right)\left(\frac{7}{35}\right) \Rightarrow$ x & y are not independent.

الفرقة الثانية كهرباء   رياضيات هندسية (٣)   < ٢ >   الامتحان أربعة أسئلة فى صفحة واحده

الاسم (باللغة العربية):	الفصل:

1. Evaluate the following integrals: i) $\int_0^1 \sqrt[7]{x}\ \ln^6 x\ dx$ ii) $\int_0^2 \frac{dx}{(32 - x^5)^{1/5}}$ (6 marks)

2. Find the relation between A and B such that: $V = B y^3 + A x^2 y + 3y - 5x + 2$ is harmonic. Find in terms of Z the analytic function $f(Z) = u + i V$ when B=3. (6 marks)

3. A pair of dice I , II is thrown. Let, A = event that the outcome on die I is less than 2, B = event that the outcome on die II is less than 2, and C = event that the sum of the outcome is less than 5. Are the three events A , B^C and C^C independent ? (5 marks)

4. The joint probability function of two random variables x & y is defined by :

$$f(x,y) = \begin{cases} K\ (x + y^2) & x = 1, 4, 5;\ y = -1, 2 \\ 0 & \text{otherwise} \end{cases}$$ Determine:

(i) K   (ii) P(x + y > 5)   (iii) Check independence of x & y.   (5 marks)

3 $A = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$
$B = \{(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$
$C = \{(1,1),(1,2),(2,1),(2,2),(1,3),(3,1)\}$

$P(A) = \boxed{\frac{1}{6}}$, $P(B) = \boxed{\frac{1}{6}}$, $P(C) = \boxed{\frac{1}{6}}$ 1

$A \cap B = \{(1,1)\} \Rightarrow P(A \cap B) = \boxed{\frac{1}{36}} = P(A).P(B) = (\frac{1}{6})(\frac{1}{6})$

$\Rightarrow$ A & B are Indep.

$A \cap C = \{(1,1),(1,2),(1,3)\} \Rightarrow P(A \cap C) = \frac{3}{36}$

$P(A \cap C) = \boxed{\frac{1}{12}} \neq P(A).P(C) \neq (\frac{1}{6})(\frac{1}{6}) \Rightarrow$ A & C are not Indep. 1

$B \cap C = \{(1,1),(2,1),(3,1)\} \Rightarrow P(B \cap C) = \frac{3}{36}$

$P(B \cap C) = \boxed{\frac{1}{12}} \neq P(B).P(C) \neq (\frac{1}{6})(\frac{1}{6}) \Rightarrow$ B & C are not Indep

$A \cap B \cap C = \{(1,1)\} \Rightarrow P(A \cap B \cap C) = \boxed{\frac{1}{36}}$

$P(A \cap B \cap C) = \boxed{\frac{1}{36}} \neq P(A)P(B)P(C) \neq (\frac{1}{6})(\frac{1}{6})(\frac{1}{6})$

$\Rightarrow$ A & B & C are not Indep.

$\Rightarrow$ A & B^c & C^c are not Indep.

1

4 (i) $\sum_y \sum_x K(x + y^2) = 1$

$\therefore K[\underbrace{(2+5+6)}_{y=-1} + \underbrace{(5+8+9)}_{y=2}] = 1 \Rightarrow \boxed{K = \frac{1}{35}}$ 1

(ii) $P(x+y>5) = f(4,2) + f(5,2)$
$= \frac{1}{35}[8+9] = \boxed{\frac{17}{35}}$ 1

(iii) $f_1(x) = \sum_y \frac{1}{35}(x+y^2) = \frac{1}{35}[(x+1)+(x+4)]$
$= \boxed{\frac{1}{35}[2x+5]}$, $x = 1,4,5$

$f_2(y) = \sum_x \frac{1}{35}(x+y^2) = \frac{1}{35}[(1+y^2)+(4+y^2)+(5+y^2)]$
$= \boxed{\frac{1}{35}[10+3y^2]}$, $y = -1, 5$

$f(x,y) = \frac{1}{35}(x+y^2) \neq f_1(x) \cdot f_2(y)$
$\neq \frac{1}{35}[2x+5] \cdot \frac{1}{35}[10+3y^2]$

$\Rightarrow$ x & y are not independent 1

x \ y	-1	2	$f_1(x)$
1	$\frac{2}{35}$	$\frac{5}{35}$	$\frac{7}{35}$
4	$\frac{5}{35}$	$\frac{8}{35}$	$\frac{13}{35}$
5	$\frac{6}{35}$	$\frac{9}{35}$	$\frac{15}{35}$
$f_2(y)$	$\frac{13}{35}$	$\frac{22}{35}$	

2

$f(1,-1) = \frac{2}{35} \neq f_1(1).f_2(-1)$
$\neq (\frac{7}{35}).(\frac{13}{35}) \Rightarrow$ x & y are not indep.

الاسم (باللغة العربية): Model Answer	الفصل:

1. Evaluate the following integrals: i) $\int_0^1 \sqrt[7]{x}\ \ln^6 x\ dx$ ii) $\int_0^2 \frac{dx}{(32 - x^5)^{1/5}}$ (6 marks)

2. Find the relation between A and B such that: $V = By^3 + Ax^2y + 3y - 5x + 2$ is harmonic. Find in terms of Z the analytic function $f(Z) = u + iV$ when B=3. (6 marks)

3. A pair of dice I , II is thrown. Let, A = event that the outcome on die I is less than 2, B = event that the outcome on die II is less than 2, and C = event that the sum of the outcome is less than 5. Are the three events A , B^C and C^C independent ? (5 marks)

4. The joint probability function of two random variables x & y is defined by :

$$f(x,y) = \begin{cases} K\,(x + y^2) & x = 1, 4, 5\ ;\ y = -1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(i) K (ii) $P(x+y>5)$ (iii) Check independence of x & y. (5 marks)

$\boxed{5}$

$A = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$

$B = \{(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$

$C = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$

1

$\Rightarrow P(A) = \frac{6}{36}$, $P(B) = \frac{6}{36}$, $P(C) = \frac{6}{36}$

1

$P(A\cap B) = \frac{1}{36} = \frac{6}{36}\cdot\frac{6}{36} = P(A)P(B)$

~~$\Rightarrow A, B$ are indep.~~

~~$\Rightarrow A^c, B^c$ are indep~~

1

$P(A\cap C) = \frac{3}{36} \neq \frac{6}{36}\cdot\frac{6}{36} = P(A)P(C)$

$\Rightarrow$ A, B, C are not indep.

$\Rightarrow$ A, B^c, C^c are not indep

1

$\boxed{4}$

y \ x	1	4	5	$f_2(y)$
-1	2C	5C	6C	13C
2	5C	8C	9C	22C
$f_1(x)$	7C	13C	15C	35C

2

$\because \sum_{\text{all } x}\sum_{\text{all } y} f(x,y) = 1 \Rightarrow \boxed{K = 1/35}$ — i)

1

$P(x+y>5) = \cancel{f(4,2)+f(5,2)}$

$= f(4,2) + f(5,2)$

$= \frac{8}{35} + \frac{9}{35} = \frac{17}{35}$

$\Rightarrow \boxed{P(x+y>5) = \frac{17}{35}}$ — ii)

1

iii) $f(1,-1) = 2/35$

1

$f_1(1) = \frac{7}{35}$, $f_2(-1) = \frac{13}{35}$

$\therefore \frac{7}{35}\cdot\frac{13}{35} \neq 2/35$

$\Rightarrow f(1,-1) \neq f_1(1)\cdot f_2(-1)$

$\Rightarrow$ x, y are not indep.

1

الاسم (باللغة العربية):	الفصل:

1. Evaluate the following integrals: i) $\int_0^1 \sqrt[7]{x}\,\ln^6 x\,dx$ ii) $\int_0^2 \frac{dx}{(32-x^5)^{1/5}}$ (6 marks)

2. Find the relation between A and B such that: $V = By^3 + Ax^2y + 3y - 5x + 2$ is harmonic. Find in terms of Z the analytic function $f(Z) = u + iV$ when B=3. (6 marks)

3. A pair of dice I, II is thrown. Let, A = event that the outcome on die I is less than 2, B = event that the outcome on die II is less than 2, and C = event that the sum of the outcome is less than 5. Are the three events A, B^C and C^C independent? (5 marks)

4. The joint probability function of two random variables x & y is defined by:

$$f(x,y) = \begin{cases} K\,(x+y^2) & x=1,4,5\,;\ y=-1,2 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(i) K (ii) $P(x+y>5)$ (iii) Check independence of x & y. (5 marks)

1) (a) i) Evaluate: $\int_0^1 x^3 (-\ln x)^{3/2}\, dx$ (4 marks)

ii) Prove that: $\beta(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$,

Hence show that: $\beta(x,x+1) = \dfrac{1}{2}\,\dfrac{\Gamma^2(x)}{\Gamma(2x)}$

and evaluate: $\int_0^{\pi/2} \left[\dfrac{1}{\sin^5\theta} - \dfrac{1}{\sin^4\theta}\right]^{1/6} \cos\theta \; d\theta$ (6 marks)

(b) Show that: $u = \ln(x^2 + y^2)$ is harmonic and find the analytic function $f(Z) = u + i v$ in terms of Z. (6 marks)

2) (a) Find all values of Z, where

(i) $e^{2Z+2i} = 4 + 4i$ (ii) $\sin 2Z = -5i$ (iii) $Z = (1+2i)^{3i}$ (6 marks)

(b) Find the image of:

(i) The line $4y = 2x + 1$ (ii) The circle $|Z - i| = 1$.

under the transformation $w = \dfrac{1}{Z}$. Show the curves and their images graphically.

Discuss the behavior of the line near $\pm\infty$ and how the image tends to zero.

Also, discuss the behavior of the circle near the origin. (10 marks)

3) (a) Determine whether the set of the functions:

(i) $\{\cos 2x,\ \sin 2x,\ e^{2ix}\}$ (ii) $\{5+3\sqrt{x},\ 5x+3\sqrt{x},\ 2x-2,\ 3x^2\}$.

Linear independent or not and find the interval of validity in each case.

Hence from (i) form a fundamental set of solutions for the Differential Equation:

$y'' + 4y = 0$, in at least two different forms. (8 marks)

(b) Find and classify the singular points of the Differential Equation:

$(1 - x^2)\, y'' - 4x\, y' + 10\, y = 0$

and solve it in powers of x. Find the interval of validity. (8 marks)

4) (a) Three items are drawn at random from a lot containing 12 items of which 4 are defective. Let A be an event of obtaining at least one item is defective. Find P (A). (6 marks)

(b) The diameters of a sample of 1600 washers produced by a machine are normally distributed with mean $\mu = 10$ mm.

i) Determine the standard deviation σ so that the standard error is 0.01.

ii) Prepare a Control chart. (Given that $\phi(-2) = 0.0228$, $\phi(-3) = 0.0013$) (10 marks)

5) (a) Four machines M_1, M_2, M_3 and M_4 produce 2000 , 1800 , 1200 and 1000 items /week, respectively of the total number of items in a factory. The number of defective items are 20 , 9 , 6 and 10 items, respectively. Find the probability that the selected item is defective and the probability that this defective item is of the production of machine M_4. (8 marks)

(b) The joint probability function of two random variables X & Y is defined by:

$$f(x,y) = \begin{cases} C\,xy & x = 0,1,3 \; ; \; y = 2,5,8 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(i) C (ii) $P(2x + y < 9)$ (iii) Check independence of x & y. (8 marks)

6) (a) Let X be a random variable representing obtaining sum at least 10 when a pair of dice is tossed 15 times. Determine:

i) $P(x \geq 1)$ ii) $E(3x - 2)$ iii) σ_{2X-5} (8 marks)

(b) The joint probability function of two random variables X & Y is defined by:

$$f(x,y) = \begin{cases} C & x \geq y \;\; \& \;\; x \leq 2 \;\; \& \;\; x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

i) C ii) $P(x + y > 1)$ iii) Check independence of X & Y. (8 marks)

EXAMINERS: Professor Reda El Barkouky and Professor Niveen Badra

[illegible]

نموذج اجابة [illegible]

[illegible] (3) [illegible]

الفرقة الاولى للعام الجامعي [illegible]

السؤال الاول

1(a) i) Evaluate $\int_0^1 x^3(-\ln x)^{\frac{3}{2}}\,dx$

solution

Put $\ln x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt \Rightarrow$

$x=0 \Rightarrow t \to \infty$ and $x = 1 \Rightarrow t = 0$

$I = \int_\infty^0 e^{-3t}(t)^{\frac{3}{2}}(-e^{-t})\,dt = \int_0^\infty e^{-4t}\, t^{\frac{3}{2}}\,dt$

$4t = u \Rightarrow t = \frac{u}{4} \Rightarrow dt = \frac{du}{4}$

and $I = \int_0^\infty e^{-u}\left(\frac{u}{4}\right)^{\frac{3}{2}}\frac{du}{4} = \frac{1}{32}\int_0^\infty u^{\frac{3}{2}}e^{-u}\,du = \frac{1}{32}\Gamma(x)$

where $x-1 = \frac{3}{2} \Rightarrow x = \frac{5}{2} \Rightarrow I = \frac{1}{32}\Gamma(\frac{5}{2}) \Rightarrow$

$I = \frac{3}{128}\Gamma(\frac{1}{2}) = \frac{3}{128}\sqrt{\pi}$ (4 marks).

الخطوات الرئيسية هي (1) التعويضات لتحويل التكامل الى صورة دالة جاما

(2) التحويل الى الصورة المناسبة $\Gamma(\frac{3}{2})$ ثم استخدام

الخاصية $\Gamma(\frac{1}{2})$.

1(a)(ii) Prove that $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \Rightarrow \beta(x,x+1) = \frac{1}{2}\frac{\Gamma^2(x)}{\Gamma(2x)}$

solution

$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-t}t^{x-1}\,dt\int_0^\infty e^{-u}u^{y-1}\,du = \int_0^\infty\int_0^\infty t^{x-1}u^{y-1}e^{-(t+u)}\,dt\,du$

Put $t = r\cos^2\theta,\ u = r\sin^2\theta \Rightarrow dt\,du = J\left(\frac{t,u}{r,\theta}\right)dr\,d\theta = 2r\sin\theta\cos\theta\,dr\,d\theta$

$\Gamma(x)\Gamma(y) = \int_0^{\frac{\pi}{2}}\int_0^\infty r^{x-1}r^{y-1}e^{-r}\cos^{2x-2}\theta\sin^{2y-2}\theta\,(2r\sin\theta\cos\theta)\,dr\,d\theta$

$= 2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\theta\sin^{2y-1}\theta\,d\theta\int_0^\infty r^{x+y-1}e^{-r}\,dr = \beta(x,y)\Gamma(x+y)$ #

$\beta(x,x+1) = \frac{\Gamma(x)\Gamma(x+1)}{\Gamma(2x+1)} = \frac{x\,\Gamma^2(x)}{2x\,\Gamma(2x)} = \frac{\Gamma^2(x)}{2\Gamma(2x)}$ #.

1) a) ii) Evaluate $\int_0^{\frac{\pi}{2}} \left[\frac{1}{\sin^3\theta} - \frac{1}{\sin^4\theta}\right]^{\frac{1}{6}} \cos\theta \, d\theta$

Solution

$t = \sin\theta \Rightarrow \cos\theta \, d\theta = dt \Rightarrow \theta = 0 \Rightarrow t = 0$ and

$\theta = \frac{\pi}{2} \Rightarrow t = 1.$

$$I = \int_0^1 \left[\frac{1}{t^5} - \frac{1}{t^4}\right]^{\frac{1}{6}} dt = \int_0^1 t^{-\frac{5}{6}} (1-t)^{\frac{1}{6}} dt = \beta(x,y)$$

$x - 1 = -\frac{5}{6}, \; y - 1 = \frac{1}{6} \Rightarrow x = \frac{1}{6}, \; y = \frac{7}{6}$

$$I = \beta\left(\overset{x}{\frac{1}{6}}, \overset{x+1}{\frac{7}{6}}\right) = \frac{1}{2} \frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}. \quad \text{(6 marks)} \; \#$$

b) $u = \ln(x^2+y^2) = \ln r^2 = 2\ln r$

Solution

$\frac{\partial u}{\partial r} = \frac{2}{r}, \; \frac{\partial u}{\partial \theta} = 0, \; \frac{\partial^2 u}{\partial r^2} = \frac{-2}{r^2}, \; \frac{\partial^2 u}{\partial \theta^2} = 0$

$r^2 u_{rr} + r u_r + u_{\theta\theta} = r^2\left(\frac{-2}{r^2}\right) + r\left(\frac{2}{r}\right) + 0 = 0$

u is harmonic

C.R. eqn's $u_r = \frac{v_\theta}{r}$ $\boxed{1}$ $\quad u_\theta = -r v_r$ $\boxed{2}$

$\boxed{1} \Rightarrow v_\theta = 2, \; u_\theta = 0 \Rightarrow v_r = 0$

$v = 2\theta + h(r) \Rightarrow v_r = \frac{dh}{dr} = 0 \Rightarrow h = c$

$f(z) = u + iv = \ln r^2 + (2\theta + c) i = \ln r^2 + (2\theta + c) i$

$= \ln(x^2+y^2) + i\left(2\tan^{-1}\frac{y}{x} + c\right)$

$f(x) = \ln x^2 + c \Rightarrow f(z) = \ln z^2 + c.$

\#

Q2) a) i) $e^{2Z+2i} = 4+4i \Rightarrow e^{2x+2i(y+1)} = \sqrt{32}\, e^{i\frac{\pi}{4}}$

$e^{2x} = \sqrt{32}$, $2(y+1) = \frac{\pi}{4} + 2K\pi$, $K = 0, \pm1, \pm2, \ldots$

$x = \frac{1}{4}\ln(32)$, $y = \frac{\pi}{8} + K\pi - 1$, $K = 0, \pm1, \pm2, \ldots$

(2 marks)

Another solution $2Z + 2i = \ln(4+4i) = \ln\sqrt{32} + i(\frac{\pi}{4} + 2K\pi)$

$\therefore 2x = \ln\sqrt{32}$, $2y + 2 = \frac{\pi}{4} + 2K\pi$. #

2) ii) $\sin 2Z = -5i \Rightarrow \sin(2x + 2yi) = -5i \Rightarrow$

$\sin 2x \cosh 2y + i \cos 2x \sinh 2y = -5i$

$\sin 2x \cosh 2y = 0$ and $\cos 2x \sinh 2y = -5$

$\sin 2x = 0$ $\cosh 2y = 0$ refused.

$2x = K\pi$, $K = 0, \pm1, \pm2, \ldots \Rightarrow \cos K\pi \sinh 2y = -5 \Rightarrow$

$\sinh 2y = -5/(-1)^K \Rightarrow y = (-1)^{K+1} \sinh^{-1}(5)$.

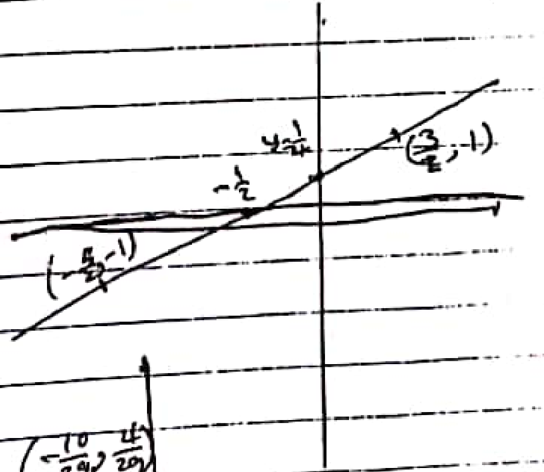
(2 marks)

) $Z = (1+2i)^{3i} \Rightarrow Z = e^{3i\ln(1+2i)} =$

$= e^{3i\ln\sqrt{5}}\, e^{-3(\alpha + 2K\pi)}$ where $\alpha = \tan^{-1} 2$.

(2 marks).

b) i) $4y = 2x + 1$



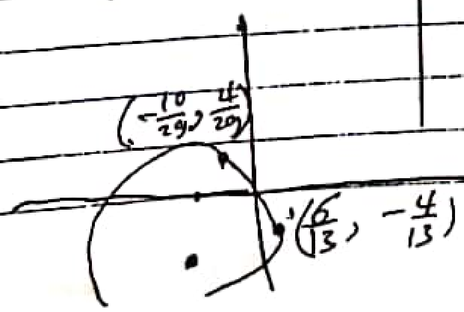
$= \frac{1}{Z} = \frac{x - iy}{x^2 + y^2}$

$\frac{1}{W} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$

e line $4y = 2x + 1 \Rightarrow$

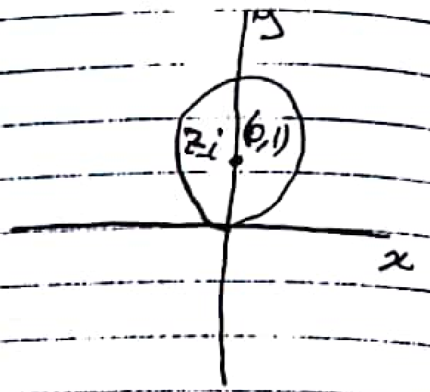
$u^2 + v^2 + 2u + 4v = 0$

$(u+1)^2 + (v+2)^2 = 5$



2(b)ii) $|z-i|=1 \Rightarrow |x+i(y-1)|=1 \Rightarrow \sqrt{x^2+(y-1)^2}=1$

$\Rightarrow x^2+y^2-2y=0$



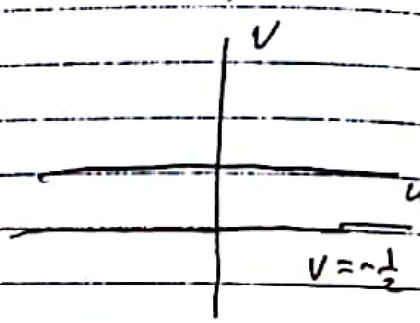
$x^2+y^2=\frac{1}{u^2+v^2}$, $y=\frac{-v}{u^2+v^2}$

$\frac{1}{u^2+v^2}+\frac{2v}{u^2+v^2}=0 \Rightarrow v=-\frac{1}{2}$

$(1,1) \Rightarrow (\frac{1}{2},-\frac{1}{2})$

$(-1,1) \Rightarrow (-\frac{1}{2},-\frac{1}{2})$

$(0,2) \Rightarrow (0,-\frac{1}{2})$.



a) $(\cos 2x, \sin 2x, e^{2ix})$

$$W=\begin{vmatrix} \cos 2x & \sin 2x & e^{2ix} \\ -2\sin 2x & 2\cos 2x & 2ie^{2ix} \\ -4\cos 2x & -4\sin 2x & -4e^{2ix} \end{vmatrix}=0$$ row (1) similar to row (3).

Linear dependent.

or $e^{2ix}=\cos 2x+i\sin 2x$

Thus e^{2ix} is a linear combination of $\cos 2x$, $\sin 2x$

$A(5+3\sqrt{x})+B(5x+3\sqrt{x})+C(2x-2)+d(3x^2)=0$

coefficient of $x^2 \Rightarrow 3d=0 \Rightarrow d=0$

coeff. of $x \Rightarrow 5B+2C=0 \Rightarrow C=-\frac{5}{2}B$

" of $\sqrt{x} \Rightarrow 3A+3B=0 \Rightarrow A=-B$

" of $1 \Rightarrow 5A-2C=0 \Rightarrow C=\frac{5}{2}A=-\frac{5}{2}B$

take for example $B=-2$, $A=2$, $C=5$, $d=0$

They are dependent

$y'' + 4y = 0$

Fundamental set of solutions $\{\sin 2x, \cos 2x\}$ or

$\{\sin 2x, e^{2ix}\}$ or $\{\cos 2x, e^{2ix}\}$.

5) $y'' - \frac{4x}{1-x^2} y' + \frac{10}{1-x^2} y = 0$

The singular pts are $x = \pm 1$.

$x = 0$ is an ordinary point

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 10 \sum_{n=0}^{\infty} a_n x^n = 0$$

Coeff. of $x^{(0)}$

$(2)(1)\, a_2 + 10 a_0 = 0 \Rightarrow a_2 = -5 a_0$

Coeff. of x

$(3)(2)\, a_3 - 4a_1 + 10 a_1 = 0 \Rightarrow a_3 = -a_1$

Coeff of x^2

$4(3)\, a_4 - 2(1)\, a_2 - 8 a_2 + 10 a_2 = 0 \Rightarrow a_4 = 0$

Coeff of x^3 : $5(4)\, a_5 - 3(2)\, a_3 - 12 a_3 + 10 a_3 = 0$

$a_5 = \frac{8}{20} a_3$, - - -

$y_{G.S} = a_0 [1 - 5x^2] + a_1 [x - x^3 + \frac{2}{5} x^5 - \cdots]$

a 12 items $\begin{cases} 4\ D \\ 8\ ND \end{cases}$ 3 items are drawn

$$P(\text{at least } 1D) = P(1D) + P(2D) + P(3D)$$

$$= \frac{{}^4C_1\ {}^8C_2}{{}^{12}C_3} + \frac{{}^4C_2\ {}^8C_1}{{}^{12}C_3} + \frac{{}^4C_3\ {}^8C_0}{{}^{12}C_3}$$

$$= \frac{(4)(28)}{220} + \frac{(6)(8)}{220} + \frac{(4)}{220} = \frac{164}{220} = \boxed{\frac{41}{55}} = \boxed{0.74545}$$

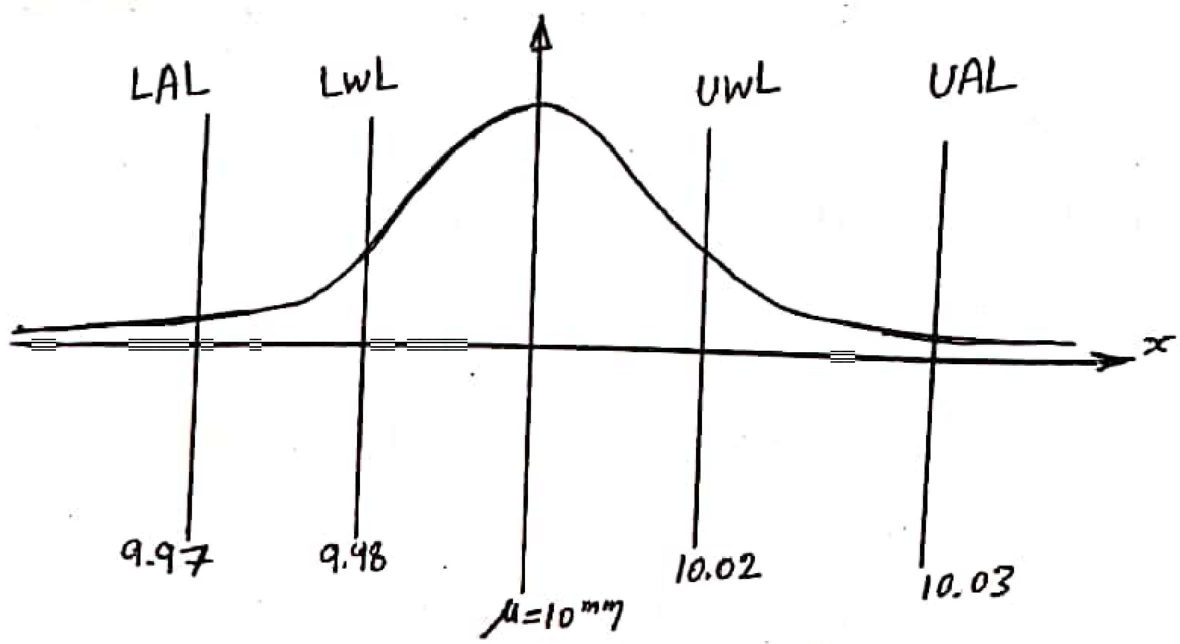
---

b $n = 1600$ $\mu = 10\ mm$

(i) Standard error $= \frac{\sigma}{\sqrt{n}} = \frac{\sigma}{\sqrt{1600}} = 0.01 \Rightarrow \boxed{\sigma = 0.4\ mm}$

(ii) Warning limits $= \mu \pm 2\frac{\sigma}{\sqrt{n}} = 10 \pm 2(0.01) = \boxed{9.98 \text{ \& } 10.02}$

Action limits $= \mu \pm 3\frac{\sigma}{\sqrt{n}} = 10 \pm 3(0.01) = \boxed{9.97 \text{ \& } 10.03}$



$\boxed{a}$

	item/week	defective items
M_1	2000/6000	20/2000
M_2	1800/6000	9/1800
M_3	1200/6000	6/1200
M_4	1000/6000	10/1000

$$P(D) = P(D|M_1)P(M_1) + P(D|M_2).P(M_2) + P(D|M_3)P(M_3) + P(D|M_4)P(M_4)$$

$$= \left(\frac{20}{2000}\right)\left(\frac{2000}{6000}\right) + \left(\frac{9}{1800}\right)\left(\frac{1800}{6000}\right) + \left(\frac{6}{1200}\right)\left(\frac{1200}{6000}\right) + \boxed{\left(\frac{10}{1000}\right)\left(\frac{1000}{6000}\right)}$$

$$= \boxed{\frac{3}{400}} = \boxed{0.0075}$$

$$P(M_4|D) = \frac{P(D|M_4)P(M_4)}{P(D)} = \frac{\left(\frac{10}{1000}\right)\left(\frac{1000}{6000}\right)}{\frac{3}{400}} = \frac{\frac{1}{600}}{\frac{3}{400}} = \boxed{\frac{2}{9}} = \boxed{0.22}$$

$\boxed{b}$

$$f(x,y) = \begin{cases} cxy & x = 0,1,3 \quad y = 2,5,8 \\ 0 & \text{otherwise} \end{cases}$$

(i) $$\sum_y \sum_x cxy = 1 \Rightarrow c\left[\overset{x=0}{(0+0+0)} + \overset{x=1}{(2+5+8)} + \overset{x=3}{(6+15+24)}\right] = 1$$

$$\Rightarrow \boxed{c = \frac{1}{60}}$$

(ii) $$P(2x+y<9) = \underbrace{f(0,2)+f(0,5)+f(0,8)}_{0} + f(1,2) + f(1,5) + f(3,2)$$

$$= \frac{1}{60}[2+5+6] = \boxed{\frac{13}{60}}$$

(iii) $$f_1(x) = \sum_y f(x,y) = \sum_{y=2,5,8} \frac{1}{60}xy = \frac{x}{60}[2+5+8] = \boxed{\frac{x}{4}}$$

$$f_2(y) = \sum_x f(x,y) = \sum_{x=0,1,3} \frac{1}{60}xy = \frac{y}{60}[0+1+3] = \boxed{\frac{y}{15}}$$

$$f(x,y) = \frac{1}{60}xy = f_1(x).f_2(y) = \left(\frac{x}{4}\right)\left(\frac{y}{15}\right) \Rightarrow x \,\&\, y \text{ are independent}$$

a $n=15$ $\quad p = P(\text{sum at least } 10) = P\{(5,5),(6,4),(4,6),(5,6),(6,5),(6,6)\} = |$

$$f(x) = {}^{15}C_x \left(\tfrac{1}{6}\right)^x \left(\tfrac{5}{6}\right)^{15-x}, \quad x = 0,1,\dots,15$$

i) $P(x \geq 1) = 1 - P(x<1) = 1 - f(0) = 1 - {}^{15}C_0 \left(\tfrac{1}{6}\right)^0 \left(\tfrac{5}{6}\right)^{15}$
$= 1 - 0.064905 = \boxed{0.935095}$

---

ii) $E(3X-2) = 3E(X) - 2 = 3[np] - 2 = 3\left((15)\left(\tfrac{1}{6}\right)\right) - 2$
$= 3\left(\tfrac{5}{2}\right) - 2 = \boxed{\tfrac{11}{2}} = \boxed{5.5}$

iii) $\sigma_{2x-5} = \sqrt{Var(2x-5)} = \sqrt{4\,Var(x)} = 2\sqrt{npq}$
$= 2\sqrt{(15)\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)} = \cancel{2}\left(\tfrac{5}{\cancel{6}_3}\right)\sqrt{3} = \boxed{\tfrac{5}{3}\sqrt{3}} = \boxed{2.88675}$

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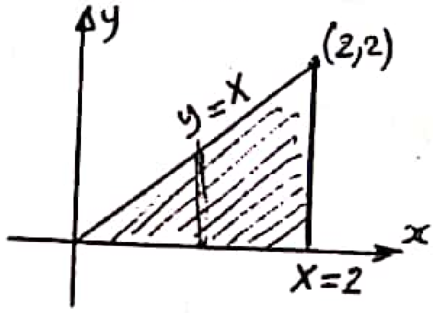
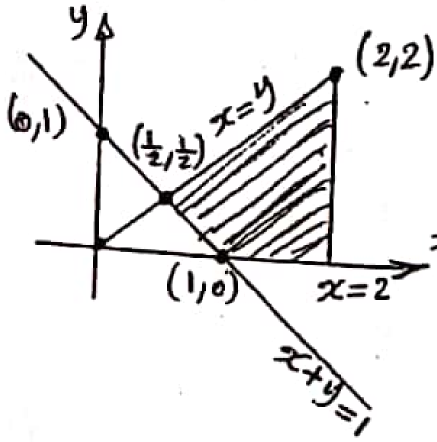
b $f(x,y) = \begin{cases} c & x \geq y \ \& \ x \leq 2 \ \& \ x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$



(i) $\iint c \, dx\,dy = 1 \Rightarrow c\left(\tfrac{1}{2}\right)(2)(2) = 1$
$\Rightarrow \boxed{c = \tfrac{1}{2}}$

(ii) $P(x+y>1) = 1 - \iint \tfrac{1}{2}\, dx\,dy$
$= 1 - \left(\tfrac{1}{2}\right)(1)\left(\tfrac{1}{2}\right) = \boxed{\tfrac{3}{4}}$



(iii) $f_1(x) = \int_0^x \tfrac{1}{2}\, dy = \boxed{\tfrac{1}{2}x}$

$f_2(y) = \int_y^2 \tfrac{1}{2}\, dx = \boxed{\tfrac{1}{2}(2-y)}$

$f(x,y) = \tfrac{1}{2} \neq f_1(x) \cdot f_2(y)$
$\neq \left(\tfrac{1}{2}x\right)\left(\tfrac{1}{2}(2-y)\right) \Rightarrow$ x & y are <u>not</u> indep.