بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

PHM212s:

Complex, Special Functions and Numerical Analysis

Series Solutions for Linear DEs

﴿يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٍ﴾

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* Series Solutions :

→ It's a method used to solve second order linear differential equation in form of :

$$y'' + p(x)\,y' + q(x)\,y = 0$$

→ the solution is in the form of infinite series : $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

this solution is called
- solution around the point x_0
- a series solution in powers of $(x-x_0)$

→ we will solve for $x_0 = 0$

Otherwise, we have to do some substitutions

let $t = x - x_0 \Rightarrow x = t + x_0$

$\therefore \frac{dt}{dx} = 1$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \dot{y}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(\dot{y}) = \frac{d}{dt}(\dot{y})\cdot\frac{dt}{dx} = \ddot{y}$$

→ then substitute in DE & the solution will be $\sum_{n=0}^{\infty} a_n t^n$

* Classification :

if both $p(x_0)$ & $q(x_0)$ are well defined

↓

x_0 is an ordinary point

↓

$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a solution

at least one is not defined

↓

x_0 is a singular point (singularity)

- at least one is not defined → x_0 is irregular singularity → No series solution can be obtained
- if both $P(x) = (x-x_0)\,p(x)$ & $Q(x) = (x-x_0)^2\,q(x)$ are well defined → x_0 is a regular singularity → $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+s}$ is a solution

the two roots of the indicial equation (s_1, s_2)

Case (1): $s_1 - s_2$ is a fraction

$\therefore y_1 = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s_1}$

$\therefore y_2 = \sum_{n=0}^{\infty} a_n(s_2)\, x^{n+s_2}$

→ $y_{gs} = c_1 y_1 + c_2 y_2$

Case (2): $s_1 - s_2 = 0$

$\therefore y_1 = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s_1}$

$\therefore y_2 = y_1 \ln x + \sum_{n=1}^{\infty} a'_n(s_1)\, x^{n+s_1}$

→ $y_{gs} = c_1 y_1 + c_2 y_2$

Case (3): $s_1 - s_2$ is a positive integer

(will not be studied)

* Steps of Solving around an ordinary point :

1) Compare with $y'' + p(x)y' + q(x)y = 0$

2) Let $y = \sum_{n=0} a_n x^n$ is a solution
- $y' = \sum_{n=1}^{\infty} (n) a_n x^{n-1}$
- $y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$

3) Substitute in the DE
- x^n حاول تخلى كله : Replace each n with n+2
- $\sum_{n=0}^{\infty}$ حاول تخلى كله

4) Coefficients of $x^n = 0$ → Find a relation between a_{n+2} & a_n (Recurrence Relation)

5) Try to find a general form for any a_n in terms of a_0 & a_1 (arbitrary constants of DE)
- Subs. with 0,2,4 to get a general form for even values of n
- Subs. with 1,2,3 to get a general form for odd values of n

Careful : recurrence may be defined for $n \geq 0$ & the general form's defined for $k \geq 1$

6) the Solution is $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$

$$y = a_0 \left[1 + \sum_{k=1}^{\infty} a_{2k} x^{2k}\right] + a_1 \left[x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1}\right]$$

* Steps of Solving around a Regular Singularity :

1) Compare with $y'' + p(x)y' + q(x)y = 0$

2) Let $y = \sum_{n=0} a_n x^{n+s}$ is a solution
- $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$
- $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$

3) Substitute in the DE
- x^{s} حاول تخلى كله : Replace each n with n−2 ($x^{n+s} \to x^{n+s-2}$)

  اقل رقم موجود : n+s−2 مثلا
- $\sum_{n=2}^{\infty}$ (or n=1) حاول تخلى كله اعلى index موجود

وده عن طريق أني هخرج terms برة الـ $\sum$ عند قيم الـ n اللي اقل من اعلى index

4) Coefficients of x^s & x^{s+1} & .... & $x^{n+s} = 0$ (From least power to highest)
- Find a relation between a_n & a_{n-2} (or a_{n-1})

  (factorization Same as indicial equation, but replace each s with n+s)
- the Coefficient of x to the least power equals zero is Called the indicial equation & this is quadratic equation in s & has two roots s_1 & s_2

5) Solve the indicial equation to specify the Case ① ②

6) Try to find a general form for a_n in n & s in terms of a_0

7) the Solution is $y_{gs} = C_1 y_1 + C_2 y_2$

* Case 1 :

→ if $s_1 - s_2$ is a fraction

$$\therefore y_1 = \sum_{n=0}^{\infty} a_n(s_1) x^{n+s_1} \quad \& \quad y_2 = \sum_{n=0}^{\infty} a_n(s_2) x^{n+s_2}$$

* Case 2 :

→ if $s_1 - s_2 = 0$

$$\therefore y_1 = \sum_{n=0}^{\infty} a_n(s_1) x^{n+s_1} \quad \& \quad y_2 = y_1 \ln x + \sum_{n=1}^{\infty} a_n'(s_1) x^{n+s_1}$$

$$a_n'(s_1) = \frac{d}{ds}\left[a_n(s)\right]\Big|_{s=s_1}$$

→ $\ln[a_n(s)] = \_\_\_$ (differentiate both sides w.r.t s)

$$\therefore a_n'(s) = [\ \dots\ ] \cdot a_n(s)$$