Numerical Analysis (2)

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Agenda

- Introduction
- Finite Difference Method
- Solution of PDE using Finite Difference Method
    - Steps of Solution
    - Laplace Equation Example
    - Poisson Equation Example

PDE

$U(x,y) = ?$

Partial differential equations are used to characterize engineering systems where the behavior of a physical quantity is couched in terms of its rate of change with respect to two or more independent variables.

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x$$

$$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = x$$

$U(x,y) = \checkmark$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

$\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $U(x,y)$

PDE: Introduction

Why ?

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

Many Applications in Engineering

steady state ??

$\boxed{T(x,y)}$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

Laplace ✓

$T(x,t)$

$$\frac{\partial T}{\partial t}=k'\frac{\partial^2 T}{\partial x^2}$$

Heat

$y(x,t)$

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Wave

PDE: Introduction

Why ?

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

$$\left(B^2 - 4AC\right)$$

method of solution

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad < 0 \longrightarrow \textbf{Elliptic}$$

$$\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2} \qquad = 0 \longrightarrow \textbf{Parabolic}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \qquad > 0 \longrightarrow \textbf{Hyperbolic}$$

Idea ?

$$U_{xx} + U_{yy} = 0$$

$$U(x,y) \checkmark$$

$\boxed{\text{BVP}}$

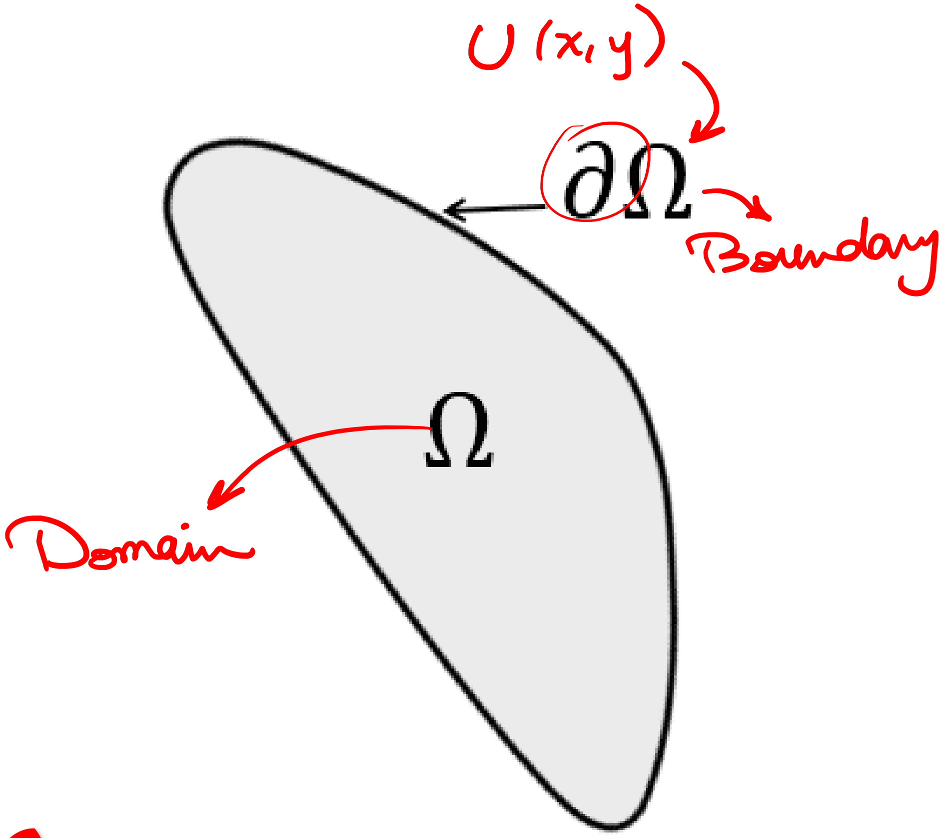
Partial Differential Equation, together with boundary conditions.

$$U_{xx} + U_{yy} = 0$$

$U(x,y) = \checkmark$

at values of x, y on Boundary

Dirichlet Conditions



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PDE: Finite Difference Method

Idea ?

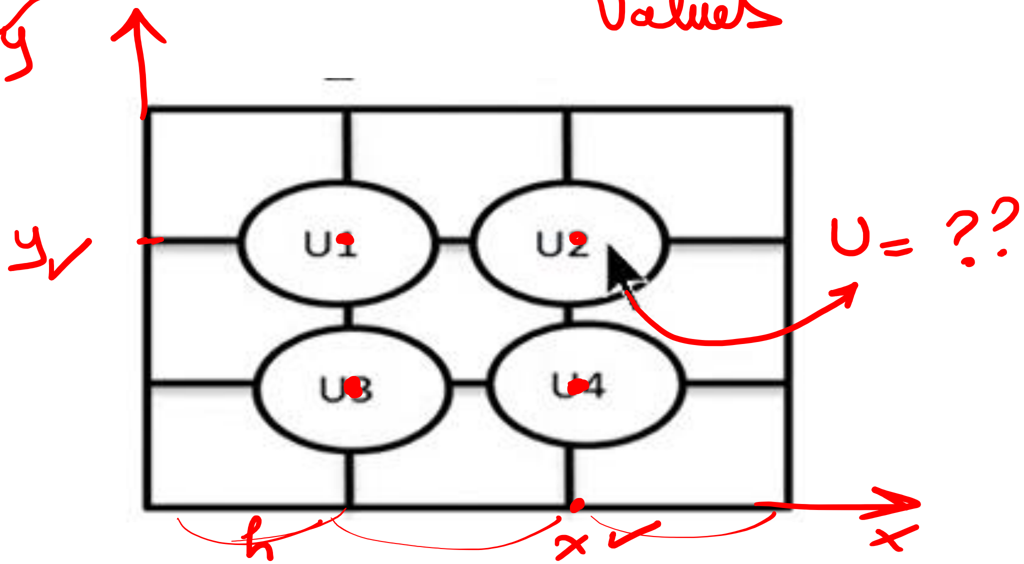
Find an approximate solution for PDE

$$u_{xx} + u_{yy} = 0$$

$$u_{xx} + u_{yy} = f(x, y)$$

U (x, y)

in the form of values



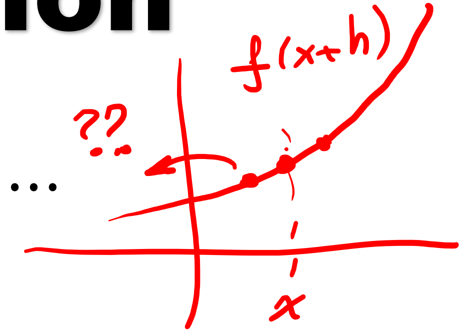
PDE: Finite Difference Method

Idea ?

Taylor Series Approximation

at x near a

$$\boxed{f(x)} = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$



at (x+h) right

= different symbols

$a \rightarrow x$

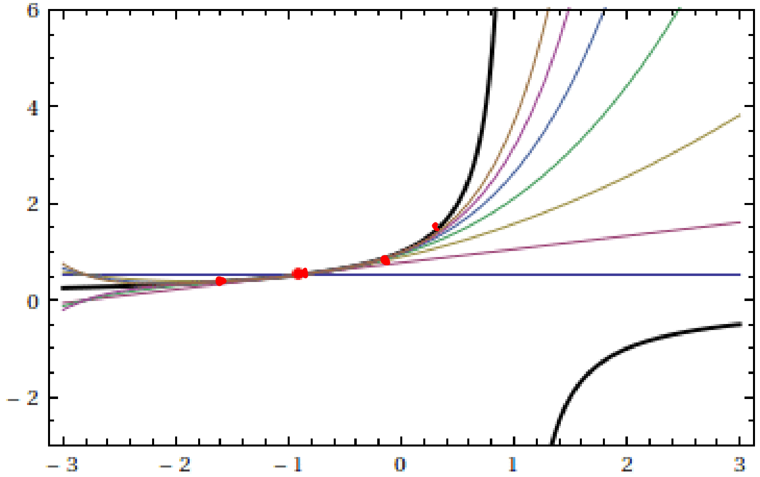
near a $\rightarrow$ x+h

x $\rightarrow$ (x+h)

$$f(x+h) = f(x) + \frac{(h)}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) \ \ldots$$

left

$$f(x-h) = f(x) - \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) \ldots$$



PDE: Finite Difference Method

Idea ?

Taylor

P.D.E.

$U_{xx} + U_{yy} = 0$

Alg. Equation

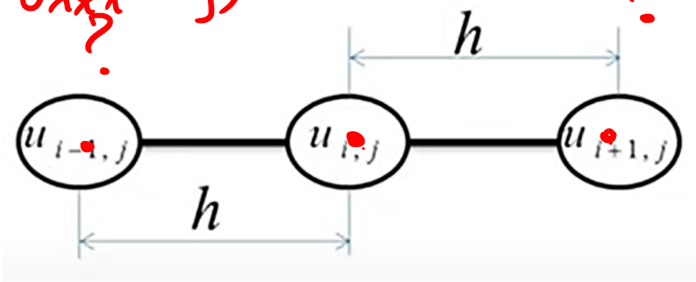
neglect

I right

$$u(x+h,y) = u(x,y) + h\,u_x(x,y) + \frac{h^2}{2}\,u_{xx}(x,y) + \frac{h^3}{6}\,u_{xxx}(x,y)$$

① $\boxed{U_x(x,y)} = \dfrac{U(x+h,y) - U(x,y)}{h} - \dfrac{h}{2} U_{xx}(x,y) - \dfrac{h^2}{6} U_{xxx}(x,y)$

Foward



II left

$$u(x-h,y) = u(x,y) - h\,u_x(x,y) + \frac{h^2}{2}\,u_{xx}(x,y) - \frac{h^3}{6}\,u_{xxx}(x,y)$$

② $\boxed{U_x(x,y)} = \dfrac{U(x,y) - U(x-h,y)}{h} + \dfrac{h}{2} U_{xx}(x,y)$ ----

Backward

central

③ $(I - II) \div 2h$ $\quad \boxed{U_x(x,y)} = \dfrac{U(x+h,y) - U(x-h,y)}{2h} + \dfrac{h^2}{6} U_{xxx}(x,y)$ ....

PDE: Finite Difference Method

Idea ?

Approximation for $u_x(x, y)$	Formula
Forward – difference approximation	$u_x(x,y) \approx \dfrac{u(x+h,y)-u(x,y)}{h}$
Backward – difference approximation	$u_x(x,y) \approx \dfrac{u(x,y)-u(x-h,y)}{h}$
Central – difference approximation	$\boxed{u_x(x,y)} \approx \dfrac{u(x+h,y)-u(x-h,y)}{2h}$

more accurate

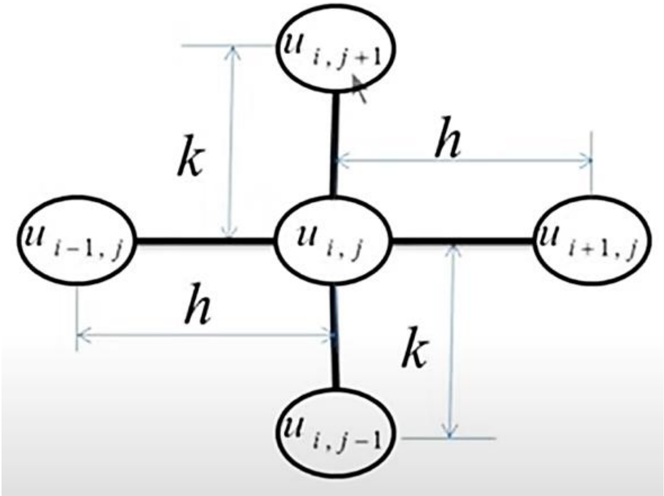
right left

$(I + II) \div h^2 \rightarrow$ $$\boxed{u_{xx}(x,y)} \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$$

PDE: Finite Difference Method

Idea ?

$$u(x, y + k) = u(x, y) + k\, u_y(x, y) + \frac{k^2}{2} u_{yy}(x, y) + \cdots$$



$$u(x, y - k) = u(x, y) - k\, u_y(x, y) + \frac{k^2}{2} u_{yy}(x, y) + \cdots$$

PDE: Finite Difference Method

Idea ?

Approximation for $u_y(x, y)$	Formulas
Forward – difference approximation	$u_y(x,y) \approx \dfrac{u(x,y+k)-u(x,y)}{k}$
Backward – difference approximation	$u_y(x,y) \approx \dfrac{u(x,y)-u(x,y-k)}{k}$
Central – difference approximation	$\boxed{u_y(x,y)} \approx \dfrac{u(x,y+k)-u(x,y-k)}{2k}$

$$\boxed{u_{yy}(x,y)} \approx \frac{\overset{\text{up}}{u(x,y+k)} - 2u(x,y) + \overset{\text{down}}{u(x,y-k)}}{k^2}$$

Idea ?

Find an approximate solution for PDE

$$u(x+h,y) - 2u(x,y) + u(x-h,y)/h^2$$

$$u_{xx} + u_{yy} = 0$$

$$u(x,y+h) - 2u(x,y) + u(x,y-h)/h^2$$

right up left down

h^2

$$[u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)] = 0.$$

$$u_{xx} + u_{yy} = f(x,y)$$

$$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = h^2 f(x,y).$$

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PDE: Steps of Solution

Idea ?

Step 1

Temperature

Laplace

$$U_{xx} + U_{yy} = 0$$

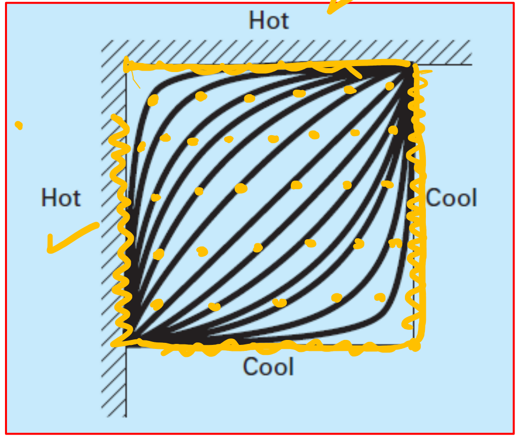
Poisson

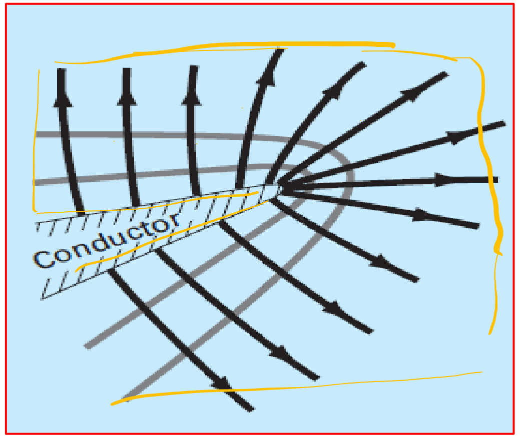
$$U_{xx} + U_{yy} = f(x, y)$$

$$(x, y)|_{\partial\Omega} = u_0(x, y)$$

$$(x, y) \in \Omega = (a, b)x(c, d)$$

Define the BVP





$T(x+h, y+h) = \checkmark$

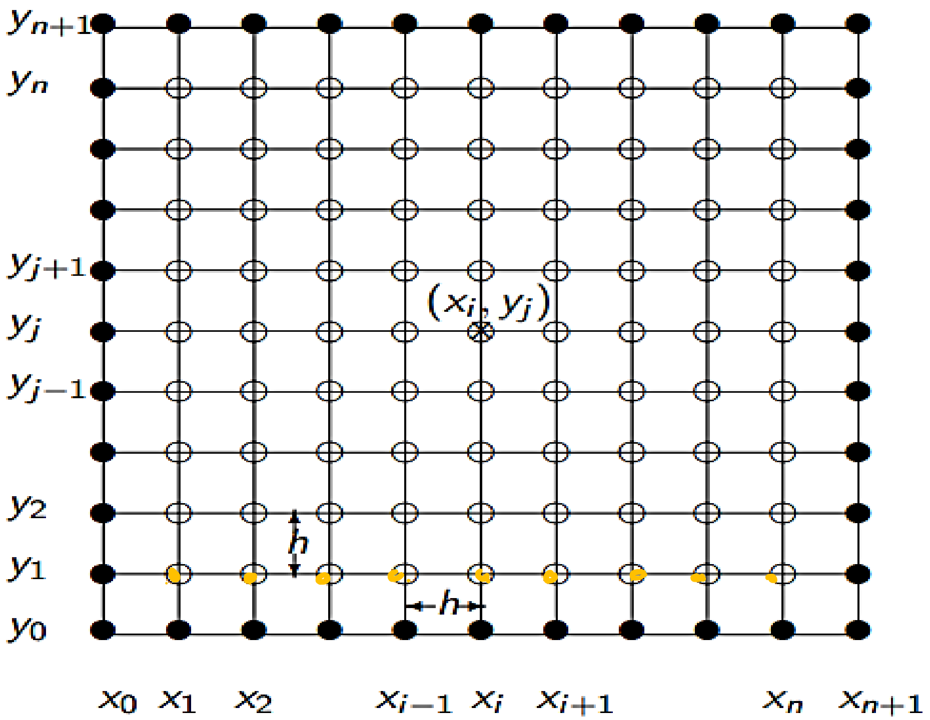
PDE: Steps of Solution

Idea ?

Step 2

Generate the grid
according to the mesh size h

equal mesh size in x, y



$u(x,y)$

PDE: Steps of Solution

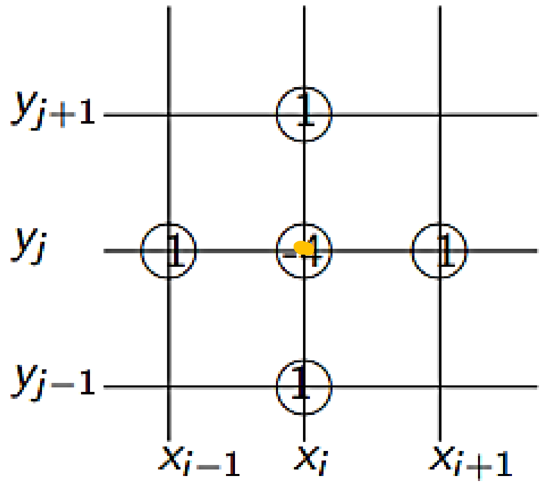
Idea ?

Step 3

Use Finite Difference to convert PDE to Linear System of Algebraic Equations

stencil

$$\nabla^2 u(x,y) \approx \frac{1}{h^2}\begin{bmatrix} & 1 \; \text{up} & \\ 1 \; \text{left} & -4 & \text{right} \; 1 \\ & 1 \; \text{down} & \end{bmatrix} u(x,y) = 0$$



Laplace Equation $\longrightarrow U_{xx}(x_i, y_j) + U_{yy}(x_i, y_j) = 0$

$\hookrightarrow U(x_i, y_{j+1}) + U(x_{i+1}, y_j) + U(x_{i-1}, j) + U(x_i, y_{j-1}) - 4U(x_i, y_j) = 0$

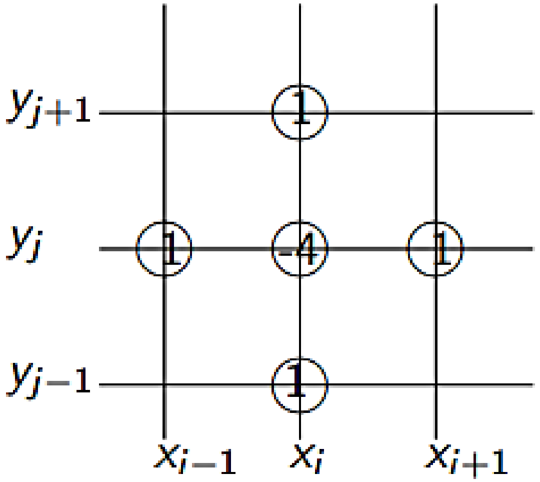
PDE: Steps of Solution

Idea ?

Step 3

Use Finite Difference to convert PDE to Linear System of Algebraic Equations

$$\nabla^2 u(x,y) \approx \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u(x,y) = f(x,y)$$



$$\left[U(x_{i+1}, j) + U(x_{i-1}, j) + U(x_i, y_{j+1}) + U(x_i, y_{j-1}) - 4U(x_i, y_j)\right] = h^2 f(x,y)$$

PDE: Steps of Solution

Idea ?

Step 4

Solve Linear System of Algebraic Equations

Gauss-Seidel Method

$$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{gathered}$$

$|a_{11}| > |a_{12} + \cdots a_{1n}|$

$|a_{22}| >$

$|a_{33}| >$

PDE: Steps of Solution

Idea ?

Step 4

Solve Linear System of Algebraic Equations

Gauss-Seidel

Converges faster!! than Jacobi

initial value $x_1 = x_2 = \dots x_n = \text{zero}$

$$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)}).$$

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$

$$x_3 = \dots \quad \vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

$$(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}).$$

PDE: Steps of Solution

Idea ?

Step 4

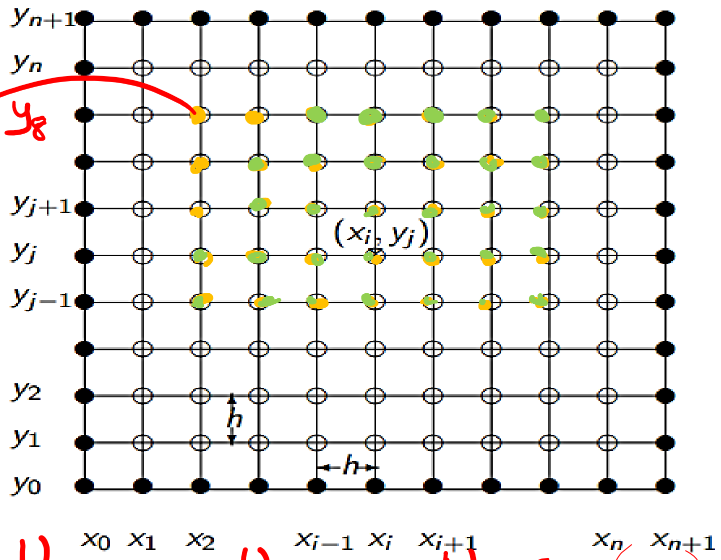
Solve Linear System of Algebraic Equations

Why?

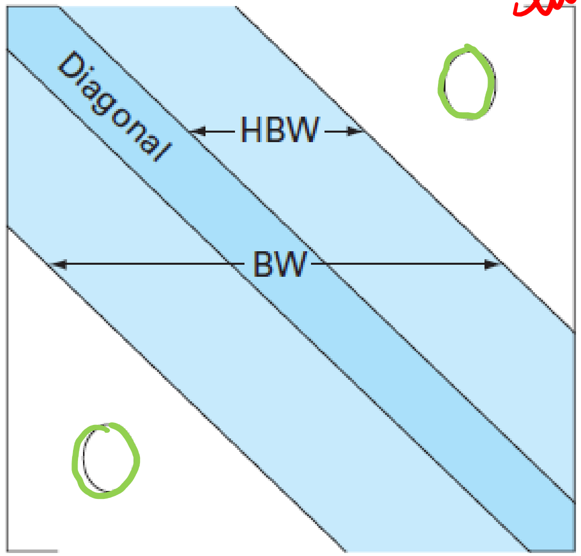
Gauss-Seidel

more efficient diagonally dominant linear system

at $U_{2,8}$ =



$$\left[U_{2,9} + U_{2,7} + U_{3,8} + U_{1,8} - 4\, U_{2,8} = 0 \right]$$



PDE: Steps of Solution

Idea ?

Step 1 Define the PDE

Step 2 Decompose the domain

Step 3 Use Finite Difference to convert PDE to Linear System of Algebraic Equations

Step 4 Solve Linear System of Algebraic Equations

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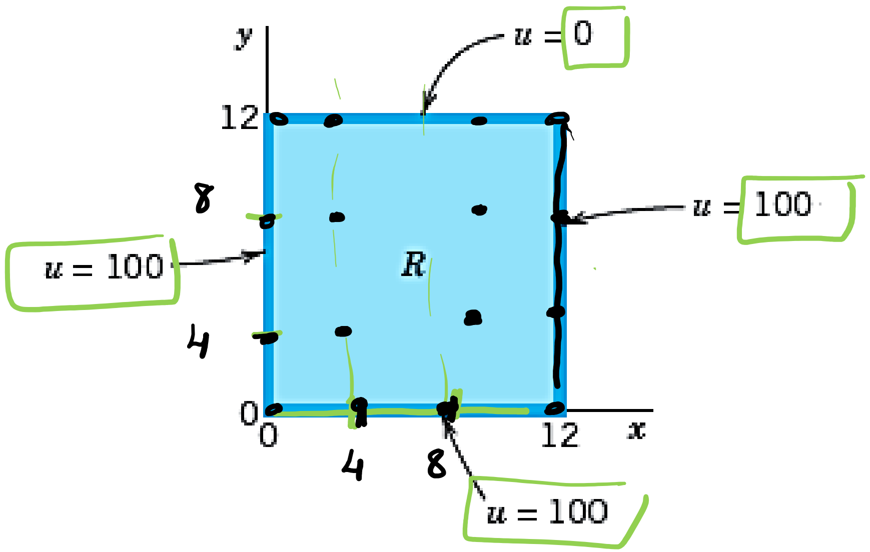
PDE: Example 1 Laplace Equation

Example 1:

The four sides of a square plate of side 12 cm, made of homogeneous material, are kept at constant temperature as shown in Figure. Using a (very wide) grid of mesh 4 cm and applying Gauss–Seidel (Liebmann) iteration method, find the steady-state temperature at the mesh points (which satisfies Laplace equation).



$U(x,y) = ?$ Dirishlet Problem

$$\left.\begin{aligned} &U_{xx} + U_{yy} = 0 \\ &U(x,0) = 100 \\ &U(0,y) = 100 \\ &U(12,y) = 100 \\ &U(x,12) = 0 \end{aligned}\right\}$$

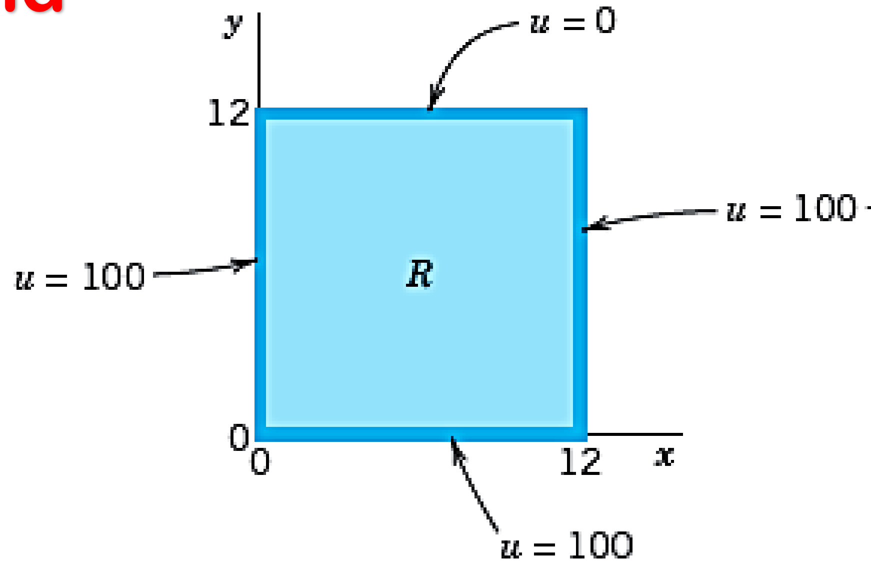
PDE: Example 1 Laplace Equation

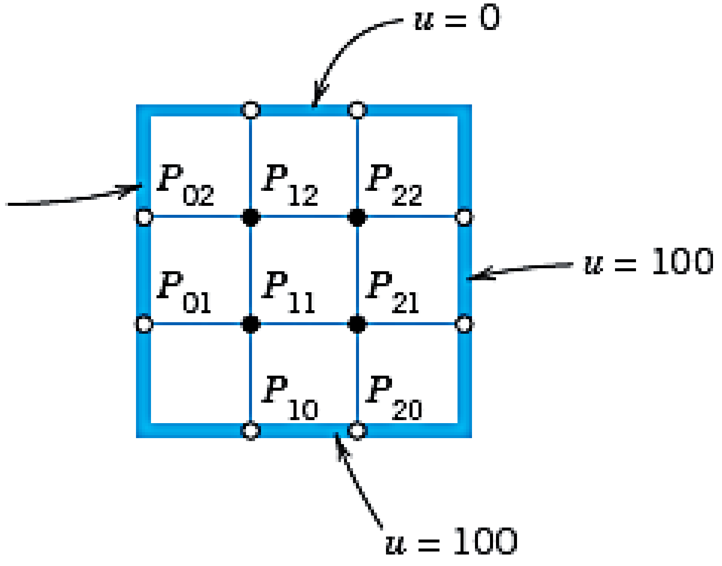
Solution

PDE

$$u_{xx} + u_{yy} = 0\,, \quad u(0,y) = u(x,0) = u(12,y) = 100 \text{ and } u(x,12) = 0,$$

Mesh Grid





PDE: Example 1 Laplace Equation

Gauss - Seidel Method

$$
\begin{aligned}
-4u_{11} + u_{21} + u_{12} \phantom{+ u_{22}} &= -200 \\
-u_{11} - 4u_{21} \phantom{+ u_{12}} + u_{22} &= -200 \\
u_{11} \phantom{+ u_{21}} - 4u_{12} + u_{22} &= -100 \\
u_{21} + u_{12} - 4u_{22} &= -100.
\end{aligned}
$$

$$
\begin{aligned}
u_{11} &= \phantom{0.25u_{11} +} 0.25\overset{100}{u_{21}} + 0.25\overset{50}{u_{12}} \phantom{+ 0.25u_{22}} + 50 \\
u_{21} &= 0.25u_{11} \phantom{+ 0.25u_{21} + 0.25u_{12}} + 0.25\overset{50}{u_{22}} + 50 \\
u_{12} &= 0.25u_{11} \phantom{+ 0.25u_{21} + 0.25u_{12}} + 0.25u_{22} + 25 \\
u_{22} &= \phantom{0.25u_{11} +} 0.25u_{21} + 0.25u_{12} \phantom{+ 0.25u_{22}} + 25.
\end{aligned}
$$

at start

$U_{11}^{(0)} = 100$ $\qquad U_{12}^{(0)} = 50$

$U_{21}^{(0)} = 100$ $\qquad U_{22}^{(0)} = 50$

PDE: Example 1 Laplace Equation

- For n = 0:
- For n = 1:
- For n = 2:

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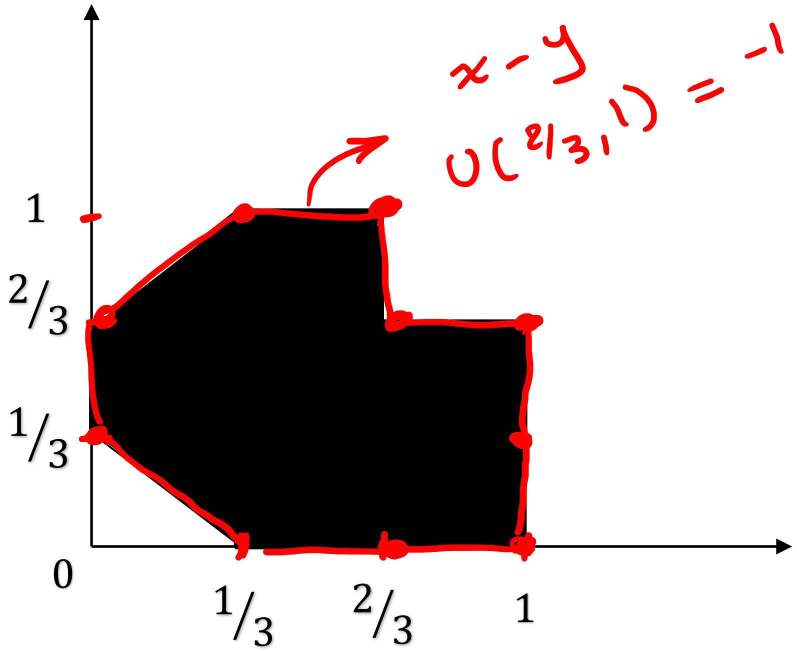
PDE: Example 2 Poisson Equation

Example 2:

Consider the Dirichlet problem $\nabla^2 u(x,y) = 9\,(x^2+y^2)$ **in** R , $u(x,y) = x - y$ **in** ∂R **where** ∂R **is the boundary of R and R is the region defined by the following figure with a mesh grid size h=1/3.**



$x - y$

$u(2/3, 1) = -1/3$

PDE: Example 2 Poisson Equation

Solution

$U_{xx}+U_{yy}=h^2 f(x,y)$

$h=\frac{1}{3}$

at U_{11}:

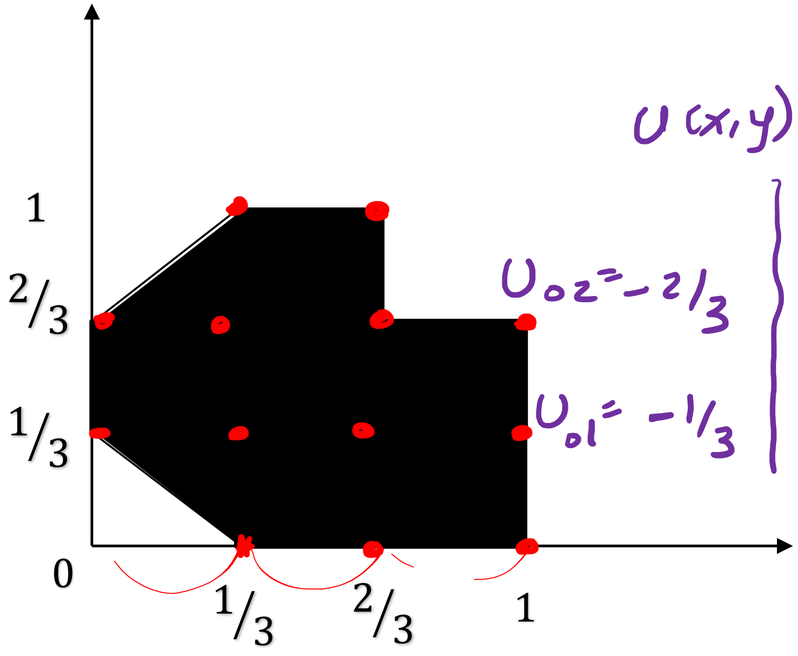
$$\left(\tfrac{1}{3}+U_{21}-\tfrac{1}{3}+U_{12}-4U_{11}\right)=\left(\tfrac{1}{3}\right)^2 \cancel{9}\underbrace{\left(\left(\tfrac{1}{3}\right)^2+\left(\tfrac{1}{3}\right)^2\right)}_{2/9}$$

PDE

Poisson

$$u_{xx}+u_{yy}=9\left(x^2+y^2\right)\text{in R}\,,\,u(x,y)=x-y\text{ in }\partial\text{R}$$

Mesh Grid

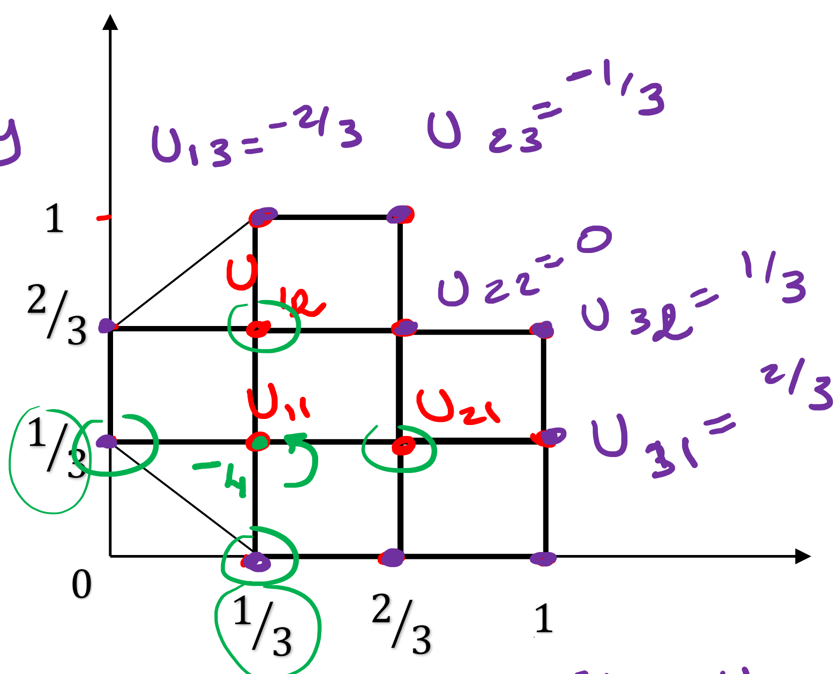


$U(x,y)$ y

$U_{02}=-2/3$ 2

$U_{01}=-1/3$ 1

3



$U_{13}=-2/3$ $U_{23}=-1/3$

$U_{22}=0$

$U_{32}=1/3$

$U_{31}=2/3$

$U_{1,0}=1/3$ $U_{2,0}=2/3$ $U_{3,0}=1$

PDE: Example 2 Poisson Equation

$$\left(-4u_{11} + u_{21} + u_{12} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} = \frac{1}{\cancel{9}} * \cancel{9}\left(\frac{1}{9} + \frac{1}{9}\right)\right)$$

$$\left(u_{11} - 4u_{21} + \frac{2}{3} + 0 + \frac{2}{3} = \frac{1}{\cancel{9}} * \cancel{9}\left(\frac{4}{9} + \frac{1}{9}\right)\right)$$

$$\left(u_{11} - 4u_{12} + 0 - \frac{2}{3} - \frac{2}{3} = \frac{1}{9} * 9\left(\frac{1}{9} + \frac{4}{9}\right)\right)$$

Gauss Seidel

$$U_{11}^{(0)} = U_{21}^{(0)} = U_{12}^{(0)} = 0$$

$$U_{11} = \left(\frac{2}{9} - U_{21} - U_{12}\right)\left(-\frac{1}{4}\right)$$

$$U_{21} = \left(\frac{7}{9} - U_{11}\right)\left(-\frac{1}{4}\right)$$

$$U_{12} = \left(\frac{14}{9} - U_{11}\right)\left(-\frac{1}{4}\right)$$

PDE: Example 2 Poisson Equation

$U_{11} = \checkmark$

then $U_{21} = \checkmark$

$U_{12} = \checkmark$

$$\left\{\begin{array}{l} -4u_{11} + u_{21} + u_{12} = \dfrac{2}{9} = 0.222 \\ u_{11} - 4u_{21} = -\dfrac{7}{9} = -0.778 \\ u_{11} - 4u_{12} = \dfrac{17}{9} = 1.889 \end{array}\right.$$

to get exact solution:

$$\boxed{u_{11}^{(3)} = -0.142,\ u_{21}^{(3)} = 0.159,\ u_{12}^{(3)} = -0.508}$$

Thank you ☺