# *SERIES SOLUTIONS FOR LINEAR DIFFERENTIAL EQUATIONS*

*DR. Makram Roshdy Eskaros*

makram_eskaros@eng.asu.edu.eg

# BASIC PRINCIPLES

It is required to solve the following differential equation

$$\boxed{y'' + p(x)\, y' + q(x)\, y = 0}$$

And it is required to find a solution in the form of infinite series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

Note that the solution is convergent on the interval $|x - x_0| < R$ where $R = \lim_{x \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$.

Note also that this solution is called solution around the point $x_0$

It is also called a series solution in powers of $(x - x_0)$

***Classification:***

1) If both $p(x_0)$ *and* $q(x_0)$ are well defined, $x_0$ is called an ***ordinary point***, otherwise, $x_0$ is called a ***singular point*** (***singularity of the differential equation***).

2) If $x_0$ is a singularity but both $P(x)=(x-x_0)p(x_0)$ *and* $Q(x)=(x-x_0)^2 q(x_0)$ are well defined, $x_0$ is called a regular singular point (***Regular singularity***), otherwise it is called ***Irregular singularity***.

## ***THEOREM:***

1) If $x_0$ is an ordinary point, $\Rightarrow$ $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a solution.

2) If $x_0$ is a regular singularity, $\Rightarrow$ $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+s}$ is a solution where s is a constant called ***the singularity exponent***.

3) If $x_0$ is an irregular singularity, No series solution can be obtained.

# SOLUTION AROUND AN ORDINARY POINT

***Illustrative example:***

Find a series solution in powers of x for the differential equation $y'' + y = 0$ .

***Solution:***

$x = 0$ is an ordinary point $\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n \;,\; y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \;,\; y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$

Substitute in the differential equation

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 1

Shifting the index of the first summation so that the powers of x are the same in the two summations.

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 2

$$\Rightarrow \sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2}+a_n\Big)x^n = 0$$

All the coefficients of $x^n$ must vanish.

$$\Rightarrow (n+2)(n+1)a_{n+2}+a_n=0$$

$$\boxed{\Rightarrow\ a_{n+2}=\frac{-1}{(n+2)(n+1)}a_n \quad ,\ n\geq 0}$$

Step 3

Solve the Recurrence relation twice

$$a_2=\frac{-1}{(2)(1)}a_0$$

$$a_4=\frac{-1}{(4)(3)}a_2 =\frac{(-1)^2}{(4)(3)(2)(1)}a_0=\frac{(-1)^2}{4!}a_0$$

$$a_6=\frac{-1}{(6)(5)}a_4 =\frac{(-1)^3}{6!}a_0$$

$$\ldots\ a_{2n}=\frac{(-1)^n}{(2n)!}a_0 \quad ,\ n\geq 0$$

$$a_3=\frac{-1}{(3)(2)}a_1 =\frac{-1}{3!}a_1$$

$$a_5=\frac{-1}{(5)(4)}a_3 =\frac{(-1)^2}{5!}a_1$$

$$a_7=\frac{-1}{(7)(6)}a_5 =\frac{(-1)^3}{7!}a_1$$

$$\ldots\ a_{2n+1}=\frac{(-1)^n}{(2n+1)!}a_1 \quad ,\ n\geq 0$$

Step 4

$$y=\sum_{n=0}^{\infty} a_n\ x^n = \sum_{n=0}^{\infty} a_{2n}\ x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}\ x^{2n+1}$$

$$\Rightarrow\ y\ =\ a_0 \sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n+1}}{(2n+1)!}$$

$$\boxed{\Rightarrow\ y\ =\ a_0\ \cos x\ +\ a_1\ \sin x}$$

**Step 5**

**Remember the technique of homogeneous differential equations with constant coefficients**

$$y'' + y = 0\ \Rightarrow (D^2+1)\,y = 0\ \Rightarrow m^2 + 1 = 0\ \Rightarrow m = \pm i$$

$$\boxed{\Rightarrow\ y\ =\ C_1 \cos x\ +\ C_2\ \sin x}$$

**Which is the same result obtained by the technique of series solutions.**

## Example:

Find two linearly independent solutions in powers of x for $(1-x^2)y''-4xy'+4y = 0$.

## Solution:

$x=0$ is an ordinary point $\Rightarrow y=\sum_{n=0}^{\infty} a_n x^n \ , \ y'=\sum_{n=1}^{\infty} n\, a_n x^{n-1} \ , \ y''=\sum_{n=2}^{\infty} n(n-1)\, a_n \ x^{n-2}$

Substitute in the differential equation

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 4\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

Step 1

Shifting the index of the first summation so that the powers of x are the same in all summations.

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - 4\sum_{n=0}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

Step 2

$$\sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2}-\big(n(n-1)+4n-4\big)a_n\Big)x^n = 0$$

Coefficient of $x^n$ equals zero for all n

$$\Rightarrow (n+2)(n+1)a_{n+2}-(n^2+3n-4)a_n=0 \quad , \quad n\geq 0$$

$$\boxed{\Rightarrow \quad a_{n+2}=\frac{(n-1)(n+4)}{(n+2)(n+1)}a_n \qquad , \qquad n\geq 0}$$

Step 3

Solve the Recurrence relation twice

$$a_2=\frac{(-1)(4)}{(2)(1)}a_0$$

$$a_4=\frac{1(6)}{(4)(3)}a_2 = \frac{(-1\times 1)(4\times 6)}{4!}a_0$$

$$a_6=\frac{(3)(8)}{(6)(5)}a_4 = \frac{(-1\times 1\times 3)(4\times 6\times 8)}{6!}a_0$$

$$a_3 = 0$$

$$a_5=a_7 = \dots a_{2n+1}=0$$

$$a_{2n}=\frac{(-1\times 1\times \dots\times 2n-3)(4\times 6\times \dots\times 2n+2)}{(2n)!}a_0 \quad , \quad n\geq 1$$

Step 4

Step 5

$$y=\sum_{n=0}^{\infty}a_n\ x^n \ = \ \sum_{n=0}^{\infty}a_{2n}\ x^{2n} \ + \ \sum_{n=0}^{\infty}a_{2n+1}\ x^{2n+1}$$

$$y \ = \ a_0 \left(1+\frac{(-1\times 1\times \ldots\times 2n-3)(4\times 6\times \ldots\times 2n+2)}{(2n)!}x^{2n}\right)+\ a_1\ \ x$$

***Extra Exercise:***

**Find the general solution in powers of x for the following differential equation:**

$$y'' + x\,y = 0$$

***(Hint: This equation is called Ayri's differential equation.)***