ملزمة

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Faculty of engineering

Ain Shams University

2nd year

(3)

Power Series solution:

A.K

* Series Solution of Linear diff equation

$$y'' + P(x)\, y' + q(x)\, y = 0$$

General solution for 2nd order diff equation

$$y = C_1\, y_1(x) + C_2\, y_2(x)$$

⇒ in series solution we solve about Point (x_0) "in Power of $(x-x_0)$".

The solution Method defend on $P(x), q(x)$ value at Point x_0.

Case (1): if $P(x_0), q(x_0)$ are Analaytic " لهم قيمة معرفة

then x_0 is ordinary Point, use Power Series Method

Case (2): if $P(x_0), q(x_0)$ are Not Analaytic, then x_0 is singular Point, use frobenuis Method.

① Power Series Method x_0 is ordinary Point

let $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$y' = \sum_{n=1}^{\infty} a_n (n) (x-x_0)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) (x-x_0)^{n-2}$$

* the interval of validity of solution is $|x - x_0| < R$

, $R \to$ distance between x_0 and nearst singular Point .

$\boxed{Pb}$ $y'' + y = 0$ in Power of x ---

اولا

① $P(x) = 0$, $q(x) = 1$, $x_0 = $ zero

x_0 is ordinary Point $\to$ use Power Series Method

② set $y = \sum_{0}^{\infty} a_n x^n$, $y' = \sum_{1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{2}^{\infty} n(n-1) a_n x^{n-2}$

③ substitue in equation

$$\sum_{2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{0}^{\infty} a_n x^n = 0$$

④ make all x have same Power $\Rightarrow x^n$

$$\sum_{0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{0}^{\infty} a_n x^n = 0$$

⑤ Make all $\sum$ start from same Point

← هنا, تحققت لوحدها

⑥ all Coeff. of Power of $x = 0$

$$\sum_{0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

Note

$\sum_{n=2}^{\infty} n x^{n-2}$

set $m = n - 2$

$n = m + 2$

$\sum_{m+2=2}^{\infty} (m+2) x^m$

$\sum_{m=0}^{\infty} (m+2) x^m$

$\sum_{n=0}^{\infty} (n+2) x^n$

الى ازودة جوة اطلعه

على $\sum_{n}^{\infty}$ والعكس صحيح

Coeff of $x^n = 0$

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$\boxed{a_{n+2} = \frac{-a_n}{(n+1)(n+2)}} \longrightarrow \text{reccurence Relation}$$

7) get all a_n interms of a_0, a_1

set $\boxed{n=0}$

$$a_2 = \frac{-a_0}{(1)(2)}$$

$\boxed{n=2}$

$$a_4 = \frac{-a_2}{(3)(4)} = \frac{(-1)^2 a_0}{(4)(3)(2)(1)}$$

$\boxed{n=4}$

$$a_6 = \frac{-a_4}{(5)(6)} = \frac{(-1)^3 a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$\boxed{a_{2n} = \frac{(-1)^n}{(2n)!}}$$

$\boxed{n=1}$

$$a_3 = \frac{-a_1}{(2)(3)}$$

$\boxed{n=3}$

$$a_5 = \frac{-a_3}{(4)(5)} = \frac{(-1)^2 a_1}{(5)(4)(3)(2)}$$

$\boxed{n=5}$

$$a_7 = \frac{-a_5}{6\cdot 7} = \frac{(-1)^3 a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$

8) $$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x + \sum_{n=1}^{\infty} (a_{2n} x^{2n} + a_{2n+1} x^{2n+1})$$

3)

$$y = a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n} + a_1 x + \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{2n} + a_1 x + \sum_{1}^{\infty} \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1}$$

$$= a_0 \underbrace{\left[1 + \sum_{1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right]}_{y_1(x)} + a_1 \underbrace{\left[x + \sum_{1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right]}_{y_2(x)}$$

Note: $1 + \sum_{1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$

$x + \sum_{1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$

$$\boxed{y = c_1 \cos x + c_2 \sin x}$$ #

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$\underline{Pb}$ $y'' - 2xy' - 6y = 0$; around $x_0 = $ zero

حل

1) $P(x) = -2x$, $q(x) = -6$ → Analaytic @ $x_0 = 0$

x_0 is ordinary Point , use Power Series Method.

2) $y = \sum_0 a_n x^n$, $y' = \sum_1 n a_n x^{n-1}$, $y'' = \sum_2 n(n-1) a_n x^{n-2}$

3) sub. in equation

$$\sum_2 n(n-1) a_n x^{n-2} - 2x \sum_1 n a_n x^{n-1} - 6 \sum_0 a_n x^n = 0$$

$$\sum_2 n(n-1) a_n x^{n-2} - 2 \sum_1 n a_n x^n - 6 \sum_0 a_n x^n = 0$$

4) make all x have same Power ⇒ (n-2)

$$\sum_2 n(n-1) a_n x^{n-2} - 2 \sum_3 (n-2) a_{n-2} x^{n-2} - 6 \sum_2 a_{n-2} x^{n-2} = 0$$

5) make all $\sum$ start from same Point (3)

$$2(1)a_2 + \sum_3 n(n-1) a_n x^{n-2} - 2 \sum_3 (n-2) a_{n-2} x^{n-2} - 6a_0$$
$$- 6 \sum_3 a_{n-2} x^{n-2} = 0$$

$$2a_2 - 6a_0 + \sum_{n=3}^{\infty} \left[n(n-1) a_n - 2(n-2) a_{n-2} - 6 a_{n-2} \right] x^{n-2} = 0$$

$\boxed{6}$ all Coeff. of Power of x = zero

- Coeff of x^0 = zero $\Rightarrow 2a_2 - 6a_0 = 0 \Rightarrow \boxed{a_2 = 3a_0}$

Coeff of x^{n-2} = zero $\Rightarrow$

$$n(n-1)a_n - a_{n-2}\left[2(n-2)+6\right] = 0$$

$$n(n-1)a_n - a_{n-2}\left[2(n+1)\right]$$

Reccurance Relation $\boxed{a_n = \frac{2(n+1)}{n(n-1)} a_{n-2}}$ $n \geq 3$

$\boxed{7}$ get all a_n in terms of a_0, a_1

$\boxed{n=3}$

$$a_3 = \frac{2(4)}{(3)(2)} a_1$$

$\boxed{n=5}$

$$a_5 = \frac{2(6)}{5(4)} a_3 = \frac{2^2(4)(6)}{(5)(4)(3)(2)}$$

$\boxed{n=7}$

$$a_7 = \frac{2(8)\,a_5}{7 \cdot 6} = \frac{2^3(4)(6)(8)}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

$$a_2 = \frac{(2) \cdot 3a_0}{(2)}$$

$\boxed{n=4}$

$$a_4 = \frac{2(5)\,a_2}{4 \cdot 3} = \frac{2^2(3)(5)}{4 \cdot 3 \cdot 2}$$

$\boxed{n=6}$

$$a_6 = \frac{2(7)\,a_4}{6 \cdot 5} = \frac{2^3(3 \cdot 5 \cdot 7)\,a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

$\boxed{7}$

$$a_{2n+1} = \frac{2^n\left[4.6.8\ldots(2n+2)\right]}{(2n+1)!} \quad n \geq 1$$

$$a_{2n} = \frac{2^n\,(3.5.7\ldots(2n+1))}{(2n)!} \quad n \geq 1$$

$\boxed{8}$ $y = a_0 + a_1 x + a_2 x^2 \ldots$

$$= a_0 + a_1 x + a_0 \sum_{n=1}^{\infty} a_{2n} x^{2n} + a_1 \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0\left[1 + \sum_{n=1}^{\infty} a_{2n} x^{2n}\right] + a_1\left[x + \sum_{1}^{\infty} a_{2n+1} x^{2n+1}\right]$$

Pb). $(1-x^2)y'' - 2xy' + 6y = 0$ about $x_0 = \text{zero}$

أكيد

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{6}{(1-x^2)} y = 0$$

$\boxed{1}$ $P(x) = \frac{-2x}{(1-x^2)}$, $q(x) = \frac{6}{1-x^2}$, $P(x), q(x)$ Analytic

at $x_0 = \text{zero}$ $\Rightarrow$ x_0 is ordinary Point $\Rightarrow$ "Power series Method.

$\boxed{2}$ $y = \sum_0 a_n x^n$, $y' = \sum_1 a_n (n) x^{n-1}$, $y'' = \sum_2 a_n (n)(n-1) x^{n-2}$

[3] Sub. in equation:

$$(1-x^2)\sum_{2} n(n-1)a_n x^{n-2} - 2x\sum_{1} n a_n x^{n-1} + 6\sum_{0} a_n x^n = 0$$

$$\sum_{2} n(n-1) a_n x^{n-2} - \sum_{2} n(n-1) a_n x^n - 2\sum_{1} n a_n x^n + 6\sum_{0} a_n x^n = 0$$

[4] Make all x Same Power $\Rightarrow x^n$ (shift $x^{n-2} \rightarrow x$)

$$\sum_{0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{2} n(n-1) a_n x^n - 2\sum_{1} n a_n x^n + 6\sum_{0} a_n x^n = 0$$

[5] all Σ Start from Same Point (2)

$$(2)(1)a_2 + (3)(2)a_3 x + \sum_{2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{2} n(n-1)a_n x^n$$
$$- 2a_1 x^1 - 2\sum_{2} n a_n x^n + 6a_0 + 6a_1 x + 6\sum_{2} a_n x^n = 0$$

$$2a_2 + 6a_0 + \left[(2)(3)a_3 + 4a_1\right] x$$

$$+\sum_{2}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 6a_n\right] x^n = 0$$

$$(2a_2+6a_0) + (2\cdot 3\, a_3 + 4a_1)\, x$$

$$+\sum_{2}\left[(n+1)(n+2)a_{n+2} - \left(n(n-1)+2n-6\right)a_n\right] x^n = 0$$

[9]

$\boxed{6}$ all Coeff of Power of x^n = Zero

Coeff of $x^0 \Rightarrow 2a_2 + 6a_0 = 0 \rightarrow \boxed{a_2 = -3a_0}$

Coeff of $x^1 \Rightarrow 2\cdot 3a_3 + 4a_1 = 0 \rightarrow \boxed{a_3 = \frac{-4}{3\cdot 2} a_1}$

Coeff of $x^n \Rightarrow (n+1)(n+2)a_{n+2} - (n^2+n-6)a_n = 0$

Reccurance Relation $\boxed{a_{n+2} = \frac{(n-2)(n+3)}{(n+1)(n+2)} a_n}$ $n \geq 2$

$\boxed{7}$ a_n in terms of a_0, a_1

$a_2 = -3a_0$

$a_3 = \frac{-4}{3\cdot 2} a_0 = \frac{(-1)(4)}{3\cdot 2} a_0$

$\boxed{n=2}$

$a_4 = \text{Zero}$

so $a_6 = a_8 = a_{10} = \underline{\text{Zero}}$

$\boxed{n=3}$

$a_5 = \frac{1\cdot 6}{5\cdot 4} a_3 = \frac{-1\cdot 4\cdot 6}{5\cdot 4\cdot 3\cdot 2} a_0$

$= \frac{(-1\cdot 1)(4\cdot 6)}{5\cdot 4\cdot 3\cdot 2} a_0$

$\boxed{n=5}$

$a_7 = \frac{(3)(8)}{6\cdot 7} = \frac{(-1\cdot 1\cdot 3)(4\cdot 6\cdot 8)\, a_0}{6\cdot 7\cdot 5\cdot 4\cdot 3\cdot 2}$

$$a_{2n+1} = \frac{(-1\cdot 1\cdot 3 \cdots (2n-3))\,(4\cdot 6\cdot 8 \cdots (2n+2))}{(2n+1)!}$$

$\boxed{10}$

$\boxed{8}$ $y = \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$$= a_0 + a_1 x + (-3a_0) x^2 + a_1 \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 \left[1 - 3x^2\right] + a_1 \left[x + \sum_{1}^{\infty} a_{2n+1} x^{2n+1}\right]$$

Pb] $(1-x^2)y'' - 2xy' + 2y = 0$ in Power of x

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{2}{(1-x^2)} y = 0$$

$\boxed{1}$ $P(x) = \frac{-2x}{(1-x^2)}$, $Q(x) = \frac{2}{1-x^2}$ $\Rightarrow$ Analaytic at $x=0$

x_0 is ordinary Point, use Power series Method.

$\boxed{2}$ let $y = \sum_0 a_n x^n$, $y' = \sum_1 n a_n x^{n-1}$, $y'' = \sum_2 n(n-1) a_n x^{n-2}$

$$(1-x^2) \sum_2 n(n-1) a_n x^{n-2} - 2x \sum_1 n a_n x^{n-1} + 2 \sum_0 a_n x^n = 0$$

$$\sum_2 n(n-1) a_n x^{n-2} - \sum_2 n(n-1) a_n x^n - 2 \sum_1 n a_n x^n + 2 \sum_0 a_n x^n = 0$$

$\boxed{3}$ make all in Power of x^n (shift $x^{n-2} \to x^n$)

$$\sum_0 (n+1)(n+2) a_{n+2} x^n - \sum_2 n(n-1) a_n x^n - 2 \sum_1 n a_n x^n + 2 \sum_0 a_n x^n = 0$$

$\boxed{11}$

$\boxed{4}$ start all Σ from same Point (2)

$$(2)a_2 + (2)(3)a_3 x + \sum_2 (n+1)(n+2)a_{n+2}x^n - \sum_2 n(n-1)a_n x^n$$

$$-2a_1 x - 2\sum_2 n a_n x^n + 2a_0 + 2a_1 x + 2\sum_2 a_n x^n = 0$$

$$(2a_2 + 2a_0) + (6a_3)x + \sum_2^{\infty}\begin{bmatrix}(n+1)(n+2)a_{n+2} - n(n-1)a_n \\ -2n a_n + 2a_n\end{bmatrix} x^n = 0$$

$\boxed{5}$ all Coeff of x^n = Zero

Coeff of $x^0 = 0 \Rightarrow 2a_2 + 2a_0 = 0 \Rightarrow \boxed{a_2 = -a_0}$

Coeff of $x^1 = 0 \Rightarrow \boxed{a_3 = 0}$

Coeff of $x^n = 0$

$$(n+1)(n+2)a_{n+2} - [n(n-1) + 2n - 2]a_n = 0$$

$$a_{n+2} = \frac{n^2 + n - 2}{(n+1)(n+2)} a_n = \frac{\cancel{(n+2)}(n-1)}{\cancel{(n+2)}(n+1)} a_n$$

$$\boxed{a_{n+2} = \frac{n-1}{n+1} a_n} \qquad n \geq 2$$

Reccurance Relation.

6) get a_n in terms of a_0, a_1

$a_3 = 0 \Rightarrow a_5 = a_7 = a_9 = \cdots = \text{zero}$

$a_2 = -a_0$

$\boxed{n=2}$ $a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0$

$\boxed{n=4}$ $a_6 = \frac{3}{5} a_4 = -\frac{\cancel{3}}{\cancel{3} \cdot 5} a_0 = \frac{-1}{5} a_0$

$\boxed{n=6}$ $a_8 = \frac{5}{7} a_6 = \frac{-\cancel{5}}{\cancel{5} \cdot 7} a_0 = \frac{-1}{7} a_0$

$$\boxed{\therefore \; a_{2n} = \frac{-1}{(2n-1)}}$$

7) $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$= a_0 + a_1 x + a_0 \sum_{1}^{\infty} a_{2n} x^{2n}$

$= a_0 \left[1 + \sum_{1}^{\infty} a_{2n} x^{2n} \right] + a_1 x$ #

Pb] $y'' + xy = 0$; in Power of x

$x_0 = 1$.

[1] $P(x) =$ zero, $q(x) = 1$ ⇒ Analaytic

x_0 is ordinary Point ⇒ use Power series Method

[2] $y = \sum_0 a_n x^n$, $y' = \sum_1 n a_n x^{n-1}$, $y'' = \sum_2 n(n-1) a_n x^{n-2}$

[3] Sub. in equation

$$\sum_2 n(n-1) a_n x^{n-2} + \sum_0 a_n x^{n+1} = 0$$

[4] Make all same Power x^{n-2} $[\, x^{n+1} \to x^{n-2}$

$$\sum_2 n(n-1) a_n x^{n-2} + \sum_3 a_{n-3} x^{n-2} = 0$$

[5] Start all from (3)

$$2(1) a_2 + \sum_3 \left[n(n-1) a_n + a_{n-3} \right] x^{n-2} = 0$$

[6] Coeff of x^n = zero

Coeff of $x^0 = 0$ ⇒ $\boxed{a_2 = 0.}$

Coeff of x^{n-2} = zero

$$n(n-1) a_n + a_{n-3} = 0$$

Reccurance Relation $\boxed{a_n = \frac{-a_{n-3}}{n(n-1)}}$ $n \geq 3$

$\boxed{7}$ get a_n in terms of a_0, a_1

$a_2 = 0 \Rightarrow a_5, a_8, a_{11} = \text{zero}$

$\boxed{n=3}$

$$a_3 = \frac{-a_0}{3(2)}$$

$\boxed{n=4}$

$$a_4 = \frac{-a_1}{(4)(3)}$$

$\boxed{n=6}$

$$a_6 = \frac{-a_3}{6 \cdot 5} = \frac{(-1)^2 a_0}{(3 \cdot 6)(2 \cdot 5)}$$

$\boxed{n=7}$

$$a_7 = \frac{-a_4}{7 \cdot 6} = \frac{(-1)^2 a_1}{(3 \cdot 6)(4 \cdot 7)}$$

$\boxed{n=9}$

$$a_9 = \frac{-a_6}{9 \cdot 8} = \frac{(-1)^3 a_0}{(3 \cdot 6 \cdot 9)(2 \cdot 5 \cdot 8)}$$

$\boxed{n=10}$

$$a_{10} = \frac{-a_7}{10 \cdot 9} = \frac{(-1)^3 a_1}{(10 \cdot 7 \cdot 4)(9 \cdot 6 \cdot 3}$$

$$\boxed{a_{3n} = \frac{(-1)^n}{(3 \cdot 6 \cdots (3n))(2 \cdot 5 \cdots (3n-1))}}$$

$$a_{3n+1} = \frac{(-1)^n}{(3 \cdot 6 \cdots (3n))(4 \cdot 7 \cdots (3n+1))}$$

$$y = \sum a_n x^n$$

$$= a_0\left[1 + \sum_{n=1}^{\infty} a_{3n} x^{3n}\right] + a_1\left[x + \sum_{n=1}^{\infty} a_{3n+1} x^{3n+1}\right]$$