Complex, Special Functions & Numerical Analysis

Special Functions

1) Gamma Function

$$\int_0^\infty t^{x-1} e^{-t} \, dt = \Gamma(x) \quad , \; x > 0$$

→ Properties :-

1. $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2. $\Gamma(x+1) = x\Gamma(x)$

3. $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$

4. $\Gamma(n+1) = n!$

5. $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$

→ Legendre's Duplication Formula

$$\sqrt{\pi}\, \Gamma(2x) = 2^{2x-1} \Gamma(x) \Gamma(x + \tfrac{1}{2})$$

2) Beta Function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad , \quad x,y > 0$$

$$= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta\,d\theta$$

→ Properties

$$\beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad , \quad \beta(x,y) = \beta(y,x)$$

Series Solutions

1. Check $p(x)$ & $q(x)$ are well defined or not
   if well defined → $x_0 = 0$ ordinary point
   if not well defined → $P(x)$ & $Q(x)$ well defined → regular Singularity
   else → irregular Singularity

2. In Case of Ordinary Substitute with:-

$$y = \sum_{n=0}^{\infty} a_n x^n \,,\; y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \,,\; y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

In case of Singular point Substitute with:

$$y = \sum_{n=0}^{\infty} a_n x^{n+s} \,,\; y' = \sum_{n=0}^{\infty} (n+s)a_n x^{n+s-1} \,,\; y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)\,a_n x^{n+s-2}$$

3. Make the power of X the same in all terms

→ Ordinary → let summation starts from 0

→ Singular → summation will start > 0

4. Get the Recurrence Relation in Ordinary

Make Coeff of $X^s = 0$ & get S_1 & S_2 in Singular before getting the recurrence relation

$S_1 - S_2$ = fraction → Case 1

$S_1 - S_2 = 0$ → Case 2

$S_1 - S_2$ = integer → Case 3

5. Substitute in Recurrence relation to have a general form for a_n

6. Getting the Solution

→ For Ordinary

$$y = a_0 + \sum_{K=1}^{\infty} a_{2K} x^{2K} + a_1 x + \sum_{K=1}^{\infty} a_{2K+1} x^{2K}$$

$$y_{gs} = C_1 y_1 + C_2 y_2$$

→ For Singular Case 1

$$y_1 = y(x, S_1) = \sum_{n=0}^{\infty} a_n(S_1)\, x^{n+S_1}$$

$$y_2 = y(x, S_2) = \sum_{n=0}^{\infty} a_n(S_2)\, x^{n+S_2}$$

→ For Singular Case 2

$$y_1 = y(x, S_1) = \sum_{n=0}^{\infty} a_n(S_1)\, x^{n+S_1}$$

$$y_2 = y_1 \ln|x| + \sum_{n=1}^{\infty} a_n'(S_1)\, x^{n+S_1}$$

* Note for Singular

$a_{n+2} = \ldots a_n \longrightarrow a_1 = 0 \qquad a_n = a_{2K}$

$a_{n+1} = \ldots a_n \longrightarrow a_n$

Bessel Function

$$x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$$

if ν is non-integer

$\longrightarrow y_{g.s.} = C_1 J_\nu(\lambda x) + C_2 J_{-\nu}(\lambda x)$

if ν is integer

$\longrightarrow y_{g.s} = C_1 J_\nu(\lambda x) + C_2 Y_{-\nu}(\lambda x)$

* Note for Substitution

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \ldots \dot{y}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \ldots \ddot{y}$$

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\,\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}$$

$$Y_{\nu}(x) = \frac{J_{\nu}(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

$$Y_N(x) = \lim_{\nu\to N} \frac{J_{\nu}(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

Bessel Proofs

① $J_{-N}(x) = (-1)^N J_N(x)$ & $J_n(x)$ is odd when n is odd & even when n is even.

$$J_{-N}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-N+1)} \left(\frac{x}{2}\right)^{2n-N}$$

to avoid undefined values $n-N+1 \geq 1$ $n \geq N$

let $n-N = K$

$$J_{-N}(x) = \sum_{n=N}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-N+1)} \left(\frac{x}{2}\right)^{2n-N}$$

$$J_{-N}(x) = \sum_{K=0}^{\infty} \frac{(-1)^{K+N}}{(K+N)!\,\Gamma(K+1)} \left(\frac{x}{2}\right)^{2K+N}$$

$\because (K+N)! = \Gamma(K+N+1)$ & $\Gamma(K+1) = K!$

$$J_{-N}(x) = \sum_{K=0}^{\infty} \frac{(-1)^{K+N}}{K!\,\Gamma(K+N+1)} \left(\frac{x}{2}\right)^{2K+N}$$

$$J_{-N}(x) = (-1)^N \sum_{K=0}^{\infty} \frac{(-1)^{K}}{K!\,\Gamma(K+N+1)} \left(\frac{x}{2}\right)^{2K+N}$$

$$\boxed{\therefore J_{-N}(x) = (-1)^N J_N(x)}$$

② $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+\frac{1}{2}}}{2^{2n+\frac{1}{2}} \, n! \, \Gamma(n+\frac{3}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{-\frac{1}{2}} . x^{2n+1}}{2^{-\frac{1}{2}} \, 2^{2n+1} \, n! \, \Gamma(n+\frac{3}{2})}$$

$$= \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{2^{2n+1} \, n! \, \Gamma(n+\frac{3}{2})}$$

$\rightarrow 2^{2n+1} \, n! \, \Gamma(n+\frac{3}{2}) = 2^{2n+1} \, \Gamma(n+1) \, \Gamma(n+\frac{3}{2}) \longrightarrow ①$

from Legendre's duplication formula

$\sqrt{\pi} \, \Gamma(2x) = 2^{2x-1} \, \Gamma(x) \, \Gamma(x+\frac{1}{2})$ let $x = n+1$

$\longrightarrow \sqrt{\pi} \, \Gamma(2n+2) = 2^{2n+1} \, \Gamma(n+1) \, \Gamma(n+\frac{3}{2}) \longrightarrow ②$

from 1 & 2

$\therefore 2^{2n+1} \, n! \, \Gamma(n+\frac{3}{2}) = \sqrt{\pi} \, \Gamma(2n+2)$

Substitute in $J_{1/2}(x)$

$$J_{1/2}(x) = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{\sqrt{\pi} \, \Gamma(2n+2)}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}$$

$$\therefore \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}$$

$$\boxed{\therefore J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$

③ $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n-\frac{1}{2}}}{2^{2n-\frac{1}{2}} \, n! \, \Gamma(n+\frac{1}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{-\frac{1}{2}} \quad x^{2n}}{2^{\frac{1}{2}} \, 2^{2n-1} \, n! \, \Gamma(n+\frac{1}{2})}$$

$\because n! = \Gamma(n+1) = n\Gamma(n)$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{1}{2x}} \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n}}{n \times 2^{2n-1} \, \Gamma(n) \Gamma(n+\frac{1}{2})}$$

from legendre's duplication formula

$\sqrt{\pi} \, \Gamma(2x) = 2^{2x-1} \, \Gamma(x) \, \Gamma(x+\frac{1}{2})$ let $x = n$

$\rightarrow \sqrt{\pi} \, \Gamma(2n) = 2^{2n-1} \, \Gamma(n) \, \Gamma(n+\frac{1}{2})$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{1}{2x}} \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n}}{\sqrt{\pi} \, n \, \Gamma(2n)} \times \frac{\sqrt{4}}{2}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{4}{2\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n}}{2n \Gamma(2n)}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n}}{\Gamma(2n+1)}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n}}{(2n)!}$$

$\because \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n}}{(2n)!}$

$$\boxed{\therefore J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

④ $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = \frac{d}{dx}\left(x^n \sum_{n=0}^{\infty} \frac{(-1)^K x^{2K+n}}{2^{2K+n} K!\, \Gamma(K+n+1)}\right)$$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^K x^{2K+2n}}{2^{2K+n} K!\, \Gamma(K+n+1)}\right)$$

$$= \sum_{n=0}^{\infty} \frac{2\cancel{(K+n)}\,(-1)^K x^{2K+2n-1}}{2^{2K+n} K!\, \cancel{(K+n)}\, \Gamma(x+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^K x^n \cdot x^{2K+(n-1)}}{2^{2K+(n-1)} K!\, \Gamma(x+(n-1)+1)}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n \sum_{n=0}^{\infty} \frac{(-1)^K x^{2K+(n-1)}}{2^{2K+(n-1)} K!\, \Gamma(x+(n-1)+1)}$$

$$\boxed{\therefore \frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)}$$

⑤ $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = \frac{d}{dx}\left(x^{-n} \sum_{K=0}^{\infty} \frac{(-1)^K x^{2K+n}}{2^{2K+n} K!\, \Gamma(K+n+1)}\right)$$

$$= \frac{d}{dx}\left(\sum_{K=0}^{\infty} \frac{(-1)^K x^{2K}}{2^{2K+n} K!\, \Gamma(K+n+1)}\right)$$

$$= \sum_{K=1}^{\infty} \frac{2\cancel{K}\,(-1)^K x^{2K-1}}{2^{2K+n} \cancel{K}(K-1)!\, \Gamma(K+n+1)}$$

let $m = K-1$
$\to K = m+1$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{2^{2m+(n+1)} m!\, \Gamma(K+m+2)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (-1)\, x^{-n} \cdot x^{2m+(n+1)}}{2^{2m+(n+1)} m!\, \Gamma(K+(m+1)+1)}$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+(n+1)}}{2^{2m+(n+1)} m!\, \Gamma(K+(m+1)+1)}$$

$$\boxed{\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)}$$

(6) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$

$\rightarrow J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \longrightarrow (1)$

$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$

$\rightarrow J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \longrightarrow (2)$

$\underline{(1) + (2)}$

$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \longrightarrow \#$

$\underline{(1) - (2)}$

$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \longrightarrow \#$

Properties of Bessel Function

1. $J_{-n}(x) = (-1)^n J_n(x)$ → J_n, J_{-n} are linearly independent.

2. $J_n(-x) = (-1)^n J_n(x)$ → n → even → even, odd → odd

3. $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ → Recurrence Rel.

4. $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ → $J_0' = -J_1$

   ↳ $\int J_{n+1}\,dx = \int J_{n-1}\,dx - 2J_n$ → $\int J_1\,dx = -J_0 + c$

5. $\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$

   power same as index

   or

6. $\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1}$

   power diff from index

7. $\int x^n J_{n-1}\,dx = x^n J_n + c$

   power - index = 1 → J_{n+1}

8. $\int x^{-n} J_{n+1}\,dx = -x^{-n} J_n + c$

   power + index = 1 → J_{n-1}

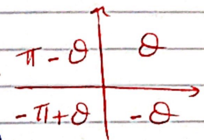
9. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Complex variables

1- $Z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$

2- $r = |z| = \sqrt{x^2+y^2}$

3- $\theta = \arg(z) = \tan^{-1}\frac{y}{x}$, $-\pi < \theta \leq \pi$



4- $\theta = \arg(z) \pm 2n\pi$

5- $\bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$

6- $z\bar{z} = x^2+y^2$

Linear Transformation $\omega = az+b$

① Shape of pre-image & image are the same
② Rotation with θ_a
③ Scaling with r_a
④ Translation with b

$$\omega = r_a e^{i\theta_a} Z + b$$

* $\arg(a) > 0 \rightarrow$ rotation anticlockwise
* $\arg(a) < 0 \rightarrow$ rotation clockwise

Reciprocal Transformation $\omega = \frac{1}{Z}$ (inversion)

$$\omega = \frac{1}{Z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

① $|\omega| = \frac{1}{|z|}$ ② $\arg(\omega) = -\arg(z)$

③ $x = \frac{u}{u^2+v^2}$ ④ $y = \frac{-v}{u^2+v^2}$

$$w = \frac{1}{z} \rightarrow z = \frac{1}{w} \rightarrow x^2+y^2 = \frac{1}{u^2+v^2}$$

$$A(x^2+y^2) + Bx + Cy + D = 0 \rightarrow D(u^2+v^2) + Bu - Cv + A = 0$$

$A \neq 0 \rightarrow$ circle $A = 0 \rightarrow$ line
$D \neq 0 \rightarrow$ not passing through origin
$D = 0 \rightarrow$ passing through origin.

Pre-image	Image
- Circle passes through origin	st line doesn't pass the origin
* Circle doesn't pass the origin	Circle doesn't pass the origin
- Line doesn't pass the origin	Circle passes the origin.
* Line passes the origin	Line passes the origin.

Differentiability

$f(z) = u(x,y) + i\,v(x,y)$ is differentiable only if it satisfies Cauchy-Riemann equations

$$U_x = V_y \quad \& \quad U_y = -V_x$$

$$\therefore f'(z) = U_x + iV_x = V_y - iU_y$$

Analytic & Harmonic Functions

① $f(z)$ is called analytic at a point z_0 if it is differentiable at z_0 & its neighborhood.

② $f(z)$ is called entire function if it is analytic everywhere (happens if it is differentiable everywhere)

③ $u(x,y)$ is called harmonic if it satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

* if $f(z) = u + iv$ is analytic

① $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$

② Replace $x \to z$, $y \to 0$

③ $f'(z)$ can be got as if it is real fn.

④ u is harmonic fn
v is harmonic fn
$\left.\begin{matrix} u_{xx}+u_{yy}=0 \\ v_{xx}+v_{yy}=0 \end{matrix}\right\}$

u & v are harmonic conjugates to each other

$u_x = v_y$ & $u_y = -v_x$

C-R Eqns

$u_x = v_y \longrightarrow r u_r = v_\theta$

$u_y = -v_x \longrightarrow r v_r = -u_\theta$

$\longrightarrow f'(z) = \frac{r}{z}(u_r + i v_r) = \frac{1}{z}(v_\theta - i u_\theta)$

Laplace Eqn

$u_{xx} + u_{yy} = 0 \longrightarrow r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$

De moivre Theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$z^{1/n} = (re^{i\theta})^{\frac{1}{n}} = r^{1/n} e^{i\left(\frac{\theta+2\pi K}{n}\right)} \quad , K=0,1,2,\dots,n-1$$

Trigonometric & Hyperbolic Functions

$e^{ix} = \cos x + i\sin x$ $\qquad$ $e^{-ix} = \cos x - i\sin x$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $\qquad$ $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$\sin iz = i\sinh z$ $\qquad$ $\sinh iz = i\sin z$

$\cos iz = \cosh z$ $\qquad$ $\cosh iz = \cos z$

$\tan iz = i\tanh z$ $\qquad$ $\tanh iz = i\tan z$

Logarithmic Functions

① $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$

to get all values :

$\ln z = \ln(re^{i(\theta+2\pi n)})$

$\ln z = \ln r + i(\theta + 2\pi n)$

② $z^a = e^{a\log z} = e^{a\log(re^{i\theta})}$

$= e^{a\log r + i(\theta+2\pi n)}$ → for all values.

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2\theta = 2\sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Hyperbolic Identities

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

Derivatives

$\sinh x \rightarrow \cosh x$ $\quad$ $\operatorname{csch} x \rightarrow -\operatorname{csch} x \coth x$

$\cosh x \rightarrow \sinh x$ $\quad$ $\operatorname{sech} x \rightarrow -\operatorname{sech} x \tanh x$

$\tanh x \rightarrow \operatorname{sech}^2 x$ $\quad$ $\coth x \rightarrow -\operatorname{csch}^2 x$

Numerical Analysis

1- ODE

① Euler - Cauchy Method

$$y_{n+1} = y_n + \Delta y \quad , \Delta y = hf(x,y)$$

① get: $x_0 , y_0 , x_n \longrightarrow y_n = ?$
$y' = f(x,y)$

② $h = \frac{x_n - x_0}{n}$ (step size) , $n = \frac{x_n - x_0}{h}$ (no of steps)

③

n	x_n	y_n	$\Delta y = hf(x,y)$	$y_{n+1} = y_n + \Delta y$
⋮	⋮ $+h$	⋮ y_{n+1}	⋮	⋮

④ error $= | y_{exact} - y_{E-C} | \quad \propto h$

② Runge - Kutta Method

$$y_{n+1} = y_n + \Delta y \quad , \Delta y = \frac{1}{6}(\omega_1 + 2\omega_2 + 2\omega_3 + \omega_4)$$

$\omega_1 = hf(x,y)$
$\omega_2 = hf(x + h/2 , y + \frac{\omega_1}{2})$
$\omega_3 = hf(x + h/2 , y + \omega_2/2)$
$\omega_4 = hf(x + h , y + \omega_3)$

n	x_n	y_n	$\omega_i = hf(x_n, y_n)$	Δy
0	$+h/2$ $+h/2$ $+h$	$+\omega_1/2$ $+\omega_2/2$ $+\omega_3$	$\omega_1 =$ $\omega_2 =$ $\omega_3 =$ $\omega_4 =$	ω_1 $2\omega_2$ $2\omega_3$ $2\omega_4$

③ System of ODE

get t_0, x_0, y_0 , t_n , $x_n = ?$, $y_n = ?$

$$y' = g(t, x, y) \quad , \quad x' = f(t, x, y)$$

* let $x' = y \rightarrow x'' = y'$

$$x_{n+1} = x_n + \Delta x_n \qquad \Delta x_n = \frac{1}{6}(w_1 + 2w_2 + 2w_3 + w_4)$$

$$y_{n+1} = y_n + \Delta y_n \qquad \Delta y_n = \frac{1}{6}(v_1 + 2v_2 + 2v_3 + v_4)$$

$$w_1 = hf(t_n, x_n, y_n) \qquad v_1 = hg(t_n, x_n, y_n)$$

$$w_2 = hf(t_n + \tfrac{h}{2}, x_n + \tfrac{w_1}{2}, y_n + \tfrac{v_1}{2}) \qquad v_2 = hg(t_n + \tfrac{h}{2}, x_n + \tfrac{w_1}{2}, v_n + \tfrac{v_1}{2})$$

$$w_3 = hf(t_n + \tfrac{h}{2}, x_n + \tfrac{w_2}{2}, y_n + \tfrac{v_2}{2}) \qquad v_3 = hg(t_n + \tfrac{h}{2}, x_n + \tfrac{w_2}{2}, v_n + \tfrac{v_2}{2})$$

$$w_4 = hf(t_n + h, x_n + w_3, y_n + v_3) \qquad v_4 = hg(t_n + h, x_n + w_3, v_n + v_3)$$

2- PDE

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} \rightarrow \text{forward}$$

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} \rightarrow \text{backward}$$

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 + h)}{2h} \rightarrow \text{central.}$$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0)$$

$$f''(x_0) = \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}$$

$$\therefore \quad U_{xx} = \frac{U(x+h,y) + U(x-h,y) - 2U(x,y)}{h^2}$$

$$U_{yy} = \frac{U(x,y+h) + U(x,y-h) - 2U(x,y)}{h^2}$$

Laplace Eqn. $\quad U_{xx} + U_{yy} = 0$

Poisson Eqn $\quad U_{xx} + U_{yy} = f(x,y)$

$$\therefore \quad \nabla^2 U = U_{xx} + U_{yy}$$

$$= \frac{U(x+h,y) + U(x-h,y) + U(x,y+h) + U(x,y-h) - 4U(x,y)}{h^2}$$

$$= \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} U$$

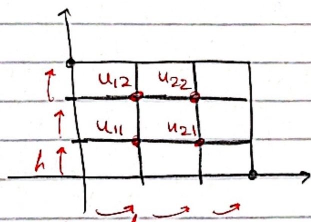
Get Eqns of $U_{11}, U_{12}, U_{21}, U_{22}$

$U_{11} =$ ———

$U_{12} =$ ———

$U_{21} =$ ———

$U_{22} =$ ———



→ Solve using Gauss - Seidal Method.

n	U_{11}	U_{12}	U_{21}	U_{22}
0				
1				
2				

نعوض في الـ Eqns

← وانا ماشية اللي لسه متغيرش

اعوض بالـ value القديمة

اللي اتغير اعوض بالـ value الجديدة

← نكمل الجدول لحد ما الـ values تثبت عند الـ decimal points اللي هو طالبها.