BESSEL FUNCTIONS 2

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Example 1:

Solve in terms of Bessel functions the following differential equation:

$$x^2 y'' + x y' + (9x^2 - 1/4) y = 0$$

Solution:

We have $\upsilon = 1/2 \quad , \quad \lambda = 3$

$$\Rightarrow \quad y = C_1 \, J_{1/2}(3x) + C_2 \, J_{-1/2}(3x)$$

Example 2:

Solve in terms of Bessel functions the differential equation $x y'' + y' + x y = 0$

Solution:

We can consider the equation $x^2 y'' + x y' + x^2 y = 0 \quad \Rightarrow x^2 y'' + x y' + (x^2 - 0) y = 0$

$$\Rightarrow \quad y = C_1 \, J_0(x) + C_2 \, Y_0(x)$$

Example 3:

Solve in terms of Bessel functions the following differential equation:

$$x^2 y'' + x\,y' + 4(x^4 - 1)\,y = 0$$

Solution:

$$\text{Let} \quad x^2 = t \qquad \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \Rightarrow \frac{dy}{dx} = 2x\,\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\,\frac{dy}{dt}\right) = 2\,\frac{dy}{dt} + 2x\,\frac{d}{dx}\left(\frac{dy}{dt}\right) = 2\,\frac{dy}{dt} + 2x\,\frac{d}{dt}\left(\frac{dy}{dt}\right)(2x)$$

Substitute in the differential equation

$$4\,x^4\,\frac{d^2y}{dt^2} + 2\,x^2\,\frac{dy}{dt} + 2\,x^2\,\frac{dy}{dt} + 4(x^4 - 1)\,y = 0 \qquad \Rightarrow t^2\,\frac{d^2y}{dt^2} + t\,\frac{dy}{dt} + (t^2 - 1)\,y = 0$$

$$\Rightarrow \quad y = C_1\,J_1(t) + C_2\,Y_1(t) \qquad\qquad \Rightarrow \quad y = C_1\,J_1(x^2) + C_2\,Y_1(x^2)$$

Example 4:

Solve in terms of Bessel functions the following differential equation:

$$x\,y'' + 3\,y' + x\,y = 0$$

Solution:

$$\boxed{\text{Let}\quad y = x^{\alpha}\,u} \qquad \boxed{\Rightarrow\; y' = x^{\alpha}\,u' + \alpha\,x^{\alpha-1}\,u}$$

$$\boxed{\Rightarrow\; y'' = x^{\alpha}\,u'' + 2\,\alpha\,x^{\alpha-1}\,u' + \alpha(\alpha-1)\,x^{\alpha-2}\,u}$$

Substitute in the differential equation

$$x^{\alpha+1}\,u'' + 2\,\alpha\,x^{\alpha}\,u' + \alpha(\alpha-1)\,x^{\alpha-1}\,u$$

$$+3\,x^{\alpha}\,u' + 3\,\alpha\,x^{\alpha-1}\,u + x^{\alpha+1}\,u = 0$$

$$x^{\alpha+1}\,u'' + (2\,\alpha+3)\,x^{\alpha}\,u' + \left([\alpha(\alpha-1)+3\alpha]\,x^{\alpha-1} + x^{\alpha+1}\right)u = 0$$

$$2\alpha+3=1 \;\Rightarrow\; \alpha=-1 \qquad \Rightarrow u'' + \frac{1}{x}\,u' + \left(\frac{-1}{x^{2}}+1\right)u = 0$$

$$\Rightarrow x^2 u'' + x u' + \left(x^2 - 1\right) u = 0$$

$$\therefore \quad u_{gs} = C_1 J_1(x) + C_1 Y_1(x)$$

but $y = x^{\alpha} u$

$$\boxed{\Rightarrow y_{gs} = \frac{1}{x}\left(C_1 J_1(x) + C_1 Y_1(x)\right)}$$

$\boxed{\text{Prove that } J_{-N}(x) = (-1)^N J_N(x) \text{ for any positive integer } N}$

Proof:

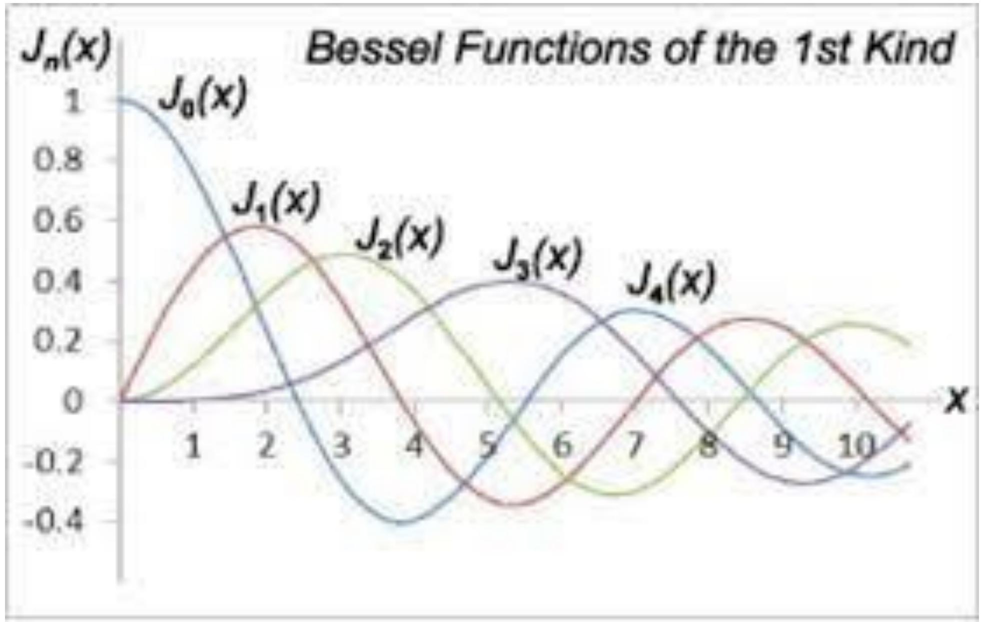
$$J_{-N}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{n!\,\Gamma(n-N+1)} \qquad = \sum_{n=N}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-N}}{n!\,\Gamma(n-N+1)}$$

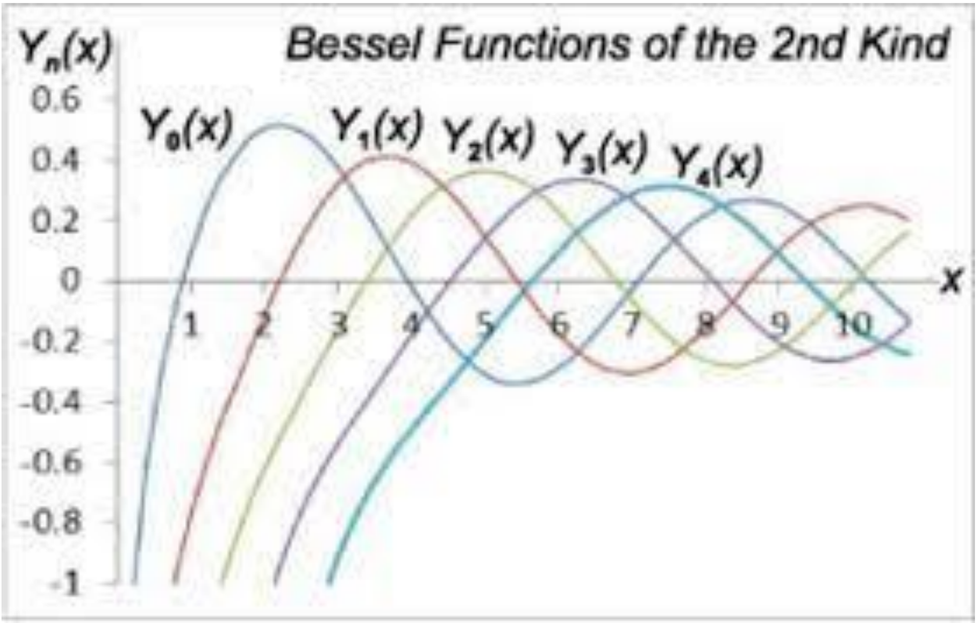
Let $n - N = k$

$$\Rightarrow J_{-N}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+N} \left(\frac{x}{2}\right)^{2k+2N-N}}{(k+N)!\,\Gamma(k+1)}$$

$$\Rightarrow J_{-N}(x) = (-1)^N \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+N}}{k!\,\Gamma(k+N+1)} \qquad = (-1)^N J_N(x)$$

Graph of Bessel functions:





From the graphs we can conclude the following facts:

$$J_0(0)=1 \quad \& \quad J_n(0)=0 \quad \vee n \qquad and \; J_\infty(x)=0 \qquad\qquad y=\frac{A}{\sqrt{x}}\sin(x+\varphi)$$

$J_n(x)$ is a bounded function but $Y_n(x)$ is unbounded at $x=0$

Bessel functions canbe represented as damped sine function for large x

Show that $\boxed{J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\ \sin x}$ $\boxed{\&\quad J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\ \cos x}$

Proof:

$$J_{1/2}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n+\frac{1}{2}}}{n!\ \Gamma(n+\frac{1}{2}+1)}=\sqrt{\frac{2}{\pi\ x}}\ \sum_{n=0}^{\infty}\frac{(-1)^n\ \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}}{n!\ \Gamma(n+\frac{1}{2}+1)}\ \sqrt{\frac{\pi\ x}{2}}$$

$$=\sqrt{\frac{2}{\pi\ x}}\ \sum_{n=0}^{\infty}\frac{(-1)^n\ \ x^{2n+1}\ \ \sqrt{\pi}}{2^{2n+1}n!\ \ \Gamma(n+\frac{3}{2})}$$

Using Legendre's duplication formula $\sqrt{\pi}\,\Gamma(2x)=\ 2^{2x-1}\ \Gamma(x)\,\Gamma(x+\frac{1}{2})$

$$\Rightarrow\quad 2^{2n+1}\ n!\ \ \Gamma(x+\tfrac{3}{2})\ =\sqrt{\pi}\ \ (2n+1)!$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi\ x}}\ \sum_{n=0}^{\infty}\frac{(-1)^n\ \ x^{2n+1}}{(2n+1)!}\quad=\sqrt{\frac{2}{\pi\ x}}\ \ \sin x$$

Similarly, we can prove that $J_{-1/2}(x)=\sqrt{\dfrac{2}{\pi x}}\cos x$

Show that $\boxed{\dfrac{d}{dx}\left(x^n\ J_n(x)\right) = x^n\ J_{n-1}(x) \qquad (I)}$

Proof:

$$\frac{d}{dx}\left(x^n\ J_n(x)\right) = \frac{d}{dx}\left(x^n \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+n}}{k!\,\Gamma(k+n+1)}\right)$$

$$= \frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+2n}}{2^{2k+n}k!\,\Gamma(k+n+1)}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k\,2(k+n)\ x^{2k+2n-1}}{2^{2k+n}k!\,\Gamma(k+n+1)}$$

$$= x^n\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+n-1}}{2^{2k+n-1}k!\,\Gamma(k+n)} = x^n\sum_{k=0}^{\infty}\frac{(-1)^k\ \left(\frac{x}{2}\right)^{2k+n-1}}{k!\,\Gamma(k+(n-1)+1)} = x^n\ J_{n-1}(x)$$

Similarly, we can prove that

$$\frac{d}{dx}\left(x^{-n}\ J_n(x)\right) = -x^{-n}\ J_{n+1}(x) \qquad (II)$$

Using I and II we can show that

$$J_n(x) = \frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right) \qquad (III)$$

$$J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) \qquad (IV)$$

Using I and II we can show that

We can use III to express $J_n(x)$ in terms of $J_0(x)$ and $J_1(x)$ only.

We can also use III to express $J_{\frac{2n+1}{2}}(x)$ sinx and cosx.

We can use IV to evaluate $\int J_n(x)$.

We can Use I and II as follows

$$\boxed{\int x^{n}\, J_{n-1}(x)\, dx = x^{n}\, J_{n}(x) + C}$$

$$\boxed{\int x^{-n}\, J_{n+1}(x)\, dx = -\,x^{-n}\, J_{n}(x) + C}$$

Example:

$$\textit{Evaluate}\quad \int x^{4}\, J_{1}(x)\, dx \quad \textit{in terms of}\ \ J_{0}(x)\ \textit{and}\ \ J_{1}(x)\ \textit{only}$$