**AIN SHAMS UNIVERSITY**
**FACULTY OF ENGINEERING**
*SPECIALIZED ENGINEERING PROGRAMS*
*JUNIOR COMMUNICATION ENGINEERING PROGRAM*

| SPRING 2022 | Assignment #2 | Total: 5 marks |
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**PHM212s: Special Functions, Complex Analysis & Numerical Analysis**

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| Name: | ID: | Deadline: Week 6 |
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***Please, Solve each problem in its assigned place ONLY (the empty space below it)***

# Series Solutions for Linear Differential Equations

Find series solutions in powers of 'x 'for the following differential equations: $x_0 = 0$

**1.** $y'' - 2xy' - 6y = 0$

• $p(x) = -2x \longrightarrow p(x_0) = 0$  • $q(x) = -6 \longrightarrow q(x_0) = -6$

at $x_0 \Rightarrow p(x)$ & $q(x)$ are defined $\therefore x = 0$ is an ordinary point

let $y = \sum_{n=0}^{\infty} a_n x^n$ is a Solution, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$

$$\therefore \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} - (2n+6) a_n \right] x^n = 0$$ , Coefficients of $x^n = 0$

thus $(n+2)(n+1) a_{n+2} - (2n+6) a_n = 0$  $\therefore \boxed{a_{n+2} = \frac{2(n+3)}{(n+2)(n+1)} a_n}$ (for $n \geq 0$)

for $n=0$: $a_2 = \frac{(2)(3)}{(2)(1)} a_0$ ($k=1$)

for $n=1$: $a_3 = \frac{2(4)}{(3)(2)} a_1$ ($k=1$)

for $n=2$: $a_4 = \frac{(2)(5)}{(4)(3)} a_2 = \frac{(2)^2 (3 \cdot 5)}{(4 \cdot 3 \cdot 2 \cdot 1)} a_0$ ($k=2$)

for $n=3$: $a_5 = \frac{2(6)}{(5)(4)} a_3 = \frac{2^2 (4 \cdot 6)}{(5 \cdot 4 \cdot 3 \cdot 2)} a_1$ ($k=2$)

for $n=4$: $a_6 = \frac{(2)(7)}{(6)(5)} a_4 = \frac{(2)^3 (3 \cdot 5 \cdot 7)}{(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} a_0$ ($k=3$)

for $n=5$: $a_7 = \frac{2(8)}{(7)(6)} a_5 = \frac{2^3 (4 \cdot 6 \cdot 8)}{(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} a_1$ ($k=3$)

For even values of $n$: $a_{2k} = \frac{(2)^k [3 \cdot 5 \cdot 7 \ldots (2k+1)]}{(2k)!} a_0$ ($k \geq 1$)

For odd values of $n$: $a_{2k+1} = \frac{2^k [4 \cdot 6 \cdot 8 \cdots (2k+2)]}{(2k+1)!} a_1$ ($k \geq 1$)

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 + \sum_{k=1}^{\infty} \frac{(2)^k [3 \cdot 5 \cdot 7 \ldots (2k+1)]}{(2k)!} a_0 x^{2k} + a_1 x + \sum_{k=1}^{\infty} \frac{2^k [4 \cdot 6 \cdot 8 \cdots (2k+2)]}{(2k+1)!} a_1 x^{2k+1}$$

$$= a_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{(2)^k [3 \cdot 5 \cdot 7 \ldots (2k+1)]}{(2k)!} x^{2k} \right] + a_1 \left[ x + \sum_{k=1}^{\infty} \frac{2^k [4 \cdot 6 \cdot 8 \cdots (2k+2)]}{(2k+1)!} x^{2k+1} \right]$$

**2.** $x^2\, y'' + x\, y' + (x^2 - \frac{1}{4})y = 0$

• $p(x) = \frac{1}{x} \longrightarrow p(x_0) = \infty$ , $q(x) = \frac{x^2 - 1/4}{x^2} = 1 - \frac{1}{4x^2} \longrightarrow q(x_0) = -\infty$ at $x_0 \Rightarrow p(x)$ & $q(x)$ are undefined $\therefore x = 0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$

$$\rightarrow x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+s} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} + \sum_{n=0}^{\infty} (-1/4)\, a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} + \sum_{n=0}^{\infty} (-1/4) a_n x^{n+s} = 0$$

$$\left[\begin{array}{l} \left[(s)(s-1) + (s) - 1/4\right] a_0 x^{s} \\ + \left[(s+1)(s) + (s+1) - 1/4\right] a_1 x^{s+1} \end{array}\right] + \sum_{n=2}^{\infty} \left[\begin{array}{l} (n+s)(n+s-1) a_n + (n+s) a_n \\ + a_{n-2} - 1/4\, a_n \end{array}\right] x^{n+s} = 0$$

$*$ Coefficients of $x^{s} = 0 \longrightarrow (s^2 - s + s - 1/4)\, a_0 = 0 \quad \therefore s_1 = \frac{1}{2}$ , $s_2 = -1/2 \longrightarrow s_1 - s_2 = \frac{1}{2} - (-1/2) = 1$ (Case 3)

**3.** $x\, y'' - x\, y' - y = 0$

• $p(x) = 1 \longrightarrow p(x_0) = 1$ , $q(x) = \frac{1}{x} \longrightarrow q(x_0) = \infty$

at $x_0 \Rightarrow q(x)$ is undefined $\therefore x_0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$

$$\rightarrow x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} - x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} - \sum_{n=1}^{\infty} (n+s-1) a_{n-1} x^{n+s-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+s-1} = 0$$

$$(s)(s-1) a_0 x^{s-1} + \sum_{n=1}^{\infty} \left[\begin{array}{l} (n+s)(n+s-1) a_n \\ -(n+s-1) a_{n-1} \\ - a_{n-1} \end{array}\right] x^{n+s-1} = 0$$

$*$ Coefficients of $x^{s-1} = 0 \longrightarrow (s)(s-1)\, a_0 = 0$

$\therefore s_1 = 0$ & $s_2 = 1$

$\therefore s_1 - s_2 = -1$ (Case 3)

**4.** $(1-x^2)\,y'' - 4\,x\,y' + 4y = 0$

• $p(x) = \dfrac{-4x}{1-x^2} \longrightarrow p(x_0) = 0$ , $q(x) = \dfrac{4}{1-x^2} \longrightarrow q(x_0) = 4$

at $x_0 \Longrightarrow p(x)$ & $q(x)$ are defined ∴ $x_0$ is an ordinary point

let $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution , $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ , $y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$

$$\rightarrow (1-x^2) \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - (4x) \sum_{n=1}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} (n)(n-1) a_n x^n - \sum_{n=1}^{\infty} 4 n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n)(n-1) a_n x^n - \sum_{n=0}^{\infty} 4 n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ a_{n+2}(n+2)(n+1) - a_n \left[ (n)(n-1) + 4n - 4 \right] \right] x^n = 0$$ , Coefficients of $x^n = 0$

Thus $a_{n+2}(n+2)(n+1) - a_n \left\{ \underbrace{n^2 - n + 4n + 4}_{(n-1)(n+4)} \right\} = 0$ ∴ $\boxed{a_{n+2} = \dfrac{(n-1)(n+4)}{(n+2)(n+1)} a_n}$ (For $n \geq 0$)

for $n=0$ : $a_2 = \dfrac{(-1)(4)}{(2)(1)} a_0$ ($k=1$)

for $n=1$ : $a_3 = \dfrac{(0)(5)}{(3)(2)} a_1 = 0$

for $n=2$ : $a_4 = \dfrac{(1)(6)}{(4)(3)} a_2$ ($k=2$)

$= \dfrac{(-1 \cdot 1)(4 \cdot 6)}{(1 \cdot 2 \cdot 3 \cdot 4)} a_0$

For $n = 3, 5, 7 \dots$ : $a_5, a_7, a_9 = 0$

For $n=4$ : $a_6 = \dfrac{(3)(8)}{(6)(5)} a_4$ ($k=3$)

$= \dfrac{(-1 \cdot 1 \cdot 3)(4 \cdot 6 \cdot 8)}{(1 \cdot 2 \cdot 3 \cdot 5 \cdot 6)} a_0$

For even values of $n$ : $a_{2k} = \dfrac{[-1 \cdot 1 \cdot 3 \dots (2k-3)][4 \cdot 6 \cdot 8 \dots (2k+2)]}{(2k)!} a_0$ (For $k \geq 1$)

For odd values of $n$ : $a_{2k+1} = 0$ (For $k \geq 1$)

$$\therefore\ y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{[-1 \cdot 1 \cdot 3 \dots (2k-3)][4 \cdot 6 \cdot 8 \dots (2k+2)]}{(2k)!} x^{2k} \right] + a_1 x$$

**5.** $4\,x\,y'' + 2\,y' + y = 0$

• $p(x) = \frac{1}{2x} \longrightarrow p(x_0) = \infty$ , $q(x) = \frac{1}{4x} \longrightarrow q(x_0) = \infty$ at $x_0 \Rightarrow p(x)$ & $q(x)$ are undefined ∴ $x_0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n)\, x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n)\, x^{n+s-2}$

$$\rightarrow 4x \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s-2} + 2\sum_{n=0}^{\infty} (n+s)(a_n)\, x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} 4(n+s)(n+s-1)\, a_n x^{n+s-1} + \sum_{n=0}^{\infty} 2(n+s)(a_n)\, x^{n+s-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+s-1} = 0$$

$$\Big(4(s)(s-1) + 2(s)\Big) a_0 x^{s-1} + \sum_{n=1}^{\infty} \left[ \Big[4(n+s)(n+s-1) + 2(n+s)\Big] a_n + a_{n-1} \right] x^{n+s-1} = 0$$

* Coefficients of $x^{s-1} = 0 \longrightarrow 2s\,\big(2(s-1)+1\big) = 0$

$s_1 = 0$ , $s_2 = 1/2$

$s_1 - s_2 = -1/2 \longrightarrow$ Case (1)

* Coefficients of $x^{n+s-1} = 0$

$$\boxed{a_n = \frac{-\frac{1}{2}}{(n+s)\left[2(n+s)-1\right]}\, a_{n-1}}$$

$n \geq 1$

* For $n=1$ : $a_1 = \frac{-1/2}{(s+1)(2s+1)}\, a_0$

* For $n=2$ : $a_2 = \frac{-1/2}{(s+2)(2s+3)}\, a_1 = \frac{(-1/2)^2}{(s+1)(s+2)(2s+1)(2s+3)}\, a_0$

* For $n=3$ : $a_3 = \frac{-1/2}{(s+3)(2s+5)}\, a_2 = \frac{(-1/2)^3}{(s+1)(s+2)(s+3)(2s+1)(2s+3)(2s+5)}\, a_0$

* For any Value of $n \geq 1$

$$a_n = \frac{(-1/2)^n}{(s+1)(s+2)(s+3)\cdots(s+n)\cdot(2s+1)(2s+3)(2s+5)\cdots(2s+2n-1)}$$

For $s_1 = 0$

$$∴\ a_n(s_1) = \frac{(-1)^n}{2^n\left[(1)(2)(3)\cdots(n)\right] * \left[(1)(3)(5)\cdots(2n-1)\right]}\, a_0$$

$$= \frac{(-1)^n}{\left[(2)(4)(6)\cdots(2n)\right]\left[(1)(3)(5)\cdots(2n-1)\right]}\, a_0$$

$$= \frac{(-1)^n}{(2n)!}\, a_0 \longrightarrow @\, n=0 : a_n(s_1) = \frac{(-1)^0}{(0)!} a_0 = a_0$$

$$∴\ y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^n$$

For $s_2 = 1/2$

$$∴\ a_n(s_2) = \frac{(-1)^n}{(2)^n\left[(3/2)(5/2)(7/2)\cdots(n+1/2)\right]\left[(2)(4)(6)\cdots(2n)\right]}\, a_0$$

$$= \frac{(-1)^n}{\left[(1)(3)(5)(7)\cdots(2n+1)\right]\left[(2)(4)(6)\cdots(2n)\right]}\, a_0$$

$$= \frac{(-1)^n}{(2n+1)!}\, a_0 \longrightarrow @\, n=0 : a_n(s_2) = \frac{(-1)^0}{(0)!} a_0 = a_0$$

$$∴\ y_2 = \sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^n$$

$$\Longrightarrow y_{gs} = \sum_{n=0}^{\infty} x^n \left[ C_1 \frac{(-1)^n}{(2n)!} + C_2 \sqrt{x}\, \frac{(-1)^n}{(2n+1)!} \right] \quad \#$$

**6.** $(3 - x^2)\, y'' - x\, y' + 25\, y = 0$

$\cdot\, p(x) = \dfrac{-x}{3-x^2} \longrightarrow p(x_0) = 0$ , $\cdot\, q(x) = \dfrac{25}{3-x^2} \longrightarrow q(x_0) = \dfrac{25}{3}$ at $x_0 \longrightarrow p(x)$ & $q(x)$ are defined $\therefore x_0$ is an ordinary point

let $y = \sum_{n=0}^{\infty} a_n x^n$ , $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ , $y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$

$$\longrightarrow (3-x^2) \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 25 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} (n)(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 25 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n)(n-1) a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 25 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ a_{n+2} \left[3(n+2)(n+1)\right] - a_n \left[\underbrace{(n)(n-1) + n - 25}_{(n-5)(n+5)}\right] \right] x^n = 0$$ , Coefficients of $x^n = 0$

$$\therefore a_{n+2}\left[3(n+2)(n+1)\right] - a_n\left[(n-5)(n+5)\right] = 0 \qquad \therefore \boxed{a_{n+2} = \frac{(n-5)(n+5)}{3(n+2)(n+1)}\, a_n} \quad (\text{for } n \geq 0)$$

for $n=0$ : $a_2 = \dfrac{(-5)(5)}{(3)(2)(1)} a_0$ $\quad k=1$

for $n=1$ : $a_3 = \dfrac{(-4)(6)}{3(3)(2)} a_1 = -\dfrac{4}{3} a_1$

for $n=2$ : $a_4 = \dfrac{(-3)(7)}{(3)(4)(3)} a_2$ $\quad k=2$

$$= \frac{(-5\cdot -3)(5\cdot 7)}{(3)^2 (4\cdot 3\cdot 2\cdot 1)} a_0$$

For $n=3$ : $a_5 = \dfrac{(-2)(8)}{3(5)(4)} a_3$

$$= \frac{(-2\cdot -4)(6\cdot 8)}{3^2(5\cdot 4\cdot 3\cdot 2)} a_1$$

$$= \frac{16}{45} a_1$$

For $n=4$ : $a_6 = \dfrac{(-1)(9)}{3(6)(5)} a_4$ $\quad k=3$

$$= \frac{(-5\cdot -3\cdot -1)(5\cdot 9\cdot 7)}{(3)^3 (6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)} a_0$$

For $n=5,7,9$ : $a_7 = a_9 = a_{11} = $ Zero

For even values of $n$ : $a_{2k} = \dfrac{\left[-5\cdot -3\cdot -1 \ldots (2k-7)\right]\left[5\cdot 7\cdot 9 \cdot \ldots \cdot (2k+3)\right]}{3^k\,(2k)!} a_0$ $\quad (k \geq 1)$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{\left[-5\cdot -3\cdot -1 \ldots (2k-7)\right]\left[5\cdot 7\cdot 9\cdot \ldots \cdot (2k+3)\right]}{3^k\,(2k)!} x^{2k} \right]$$

$$+ a_1 \left[ x - \frac{4}{3} x^3 + \frac{16}{45} x^5 \right]$$

**7.** $(x^4 - x^2)\, y'' + 2\, x^3\, y' + 6\, y = 0$

- $p(x) = \frac{2x^3}{x^2(x^2-1)} = \frac{2x}{x^2-1} \longrightarrow p(x_0) = 0$, $q(x) = \frac{6}{x^4-x^2} \longrightarrow q(x_0) = \infty$ at $x_0 \longrightarrow q(x)$ is undefined $\therefore x_0$ is a regular Singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a Solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n) x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n)\, x^{n+s-2}$

$$\longrightarrow (x^4-x^2)\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + 2x^3\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + 6\sum_{n=0}^{\infty}a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s+2} - \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}2(n+s)a_n x^{n+s+2} + \sum_{n=0}^{\infty}6a_n x^{n+s} = 0$$

$$\sum_{n=2}^{\infty}(n+s-2)(n+s-3)a_{n-2}x^{n+s} - \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=2}^{\infty}2(n+s-2)a_{n-2}x^{n+s} + \sum_{n=0}^{\infty}6a_n x^{n+s} = 0$$

$$\begin{bmatrix}[-(s)(s-1)+6]\,a_0\, x^{s} \\ + \\ [-(s+1)(s)+6]\,a_1\, x^{s+1}\end{bmatrix} + \sum_{n=2}^{\infty}\begin{bmatrix}[(n+s-2)(n+s-3)+2(n+s-2)]\,a_{n-2} \\ + \\ [-(n+s)(n+s-1)+6]\,a_n\end{bmatrix} x^{n+s} = 0$$

* Coefficients of $x^s = 0$ : $[-(s)(s-1)+6]\,a_0 = 0$ $\therefore s_1 = 3$ & $s_2 = -2 \longrightarrow s_1 - s_2 = 3-(-2) = 5$ (Case (3))

**8.** $x\, y'' + (\,1 + x\,)\, y' + 2\, y = 0$

- $p(x) = \frac{1+x}{x} = \frac{1}{x} + 1 \longrightarrow p(x_0) = \infty$, $q(x) = \frac{2}{x} \longrightarrow q(x_0) = \infty$

at $x_0 \Rightarrow p(x)$ & $q(x)$ are undefined $\therefore x_0$ is a regular Singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a Solution, $y' = \sum_{n=0}^{\infty}(n+s)(a_n)x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty}(n+s)(n+s-1)(a_n)\,x^{n+s-2}$

$$\to x\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + x\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + 2\sum_{n=0}^{\infty}a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s} + \sum_{n=0}^{\infty}2a_n x^{n+s} = 0$$

$$\to \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$

$$+ \sum_{n=1}^{\infty}(n+s-1)a_{n-1}x^{n+s-1} + \sum_{n=1}^{\infty}2a_{n-1}x^{n+s-1} = 0$$

$$[(s)(s-1)+(s)]\,a_0 x^{s-1} + \sum_{n=1}^{\infty}\begin{bmatrix}[(n+s)(n+s-1)+(n+s)]\,a_n \\ + \\ [(n+s-1)+2]\,a_{n-1}\end{bmatrix} x^{n+s-1} = 0$$

* Coefficients of $x^{s-1} = 0$ : $[(s)(s-1)+(s)]\,a_0 = 0$

$s^2 = 0 \longrightarrow s_1 = s_2 = 0$ (Case 2")

* Coefficients of $x^{n+s-1} = 0$ :

$[(n+s)(n+s-1)+(n+s)]\,a_n + [(n+s-1)+2]\,a_{n-1} = 0$

$[(n+s)^2]\,a_n + [n+s+1]\,a_{n-1} = 0 \longrightarrow \boxed{a_n = \frac{(-1)(n+s+1)}{(n+s)^2}\,a_{n-1}}$

$(n \geq 1)$

* For $n=1$ : $a_1 = \frac{(-1)(s+2)}{(s+1)^2}\,a_0$

* For $n=2$ : $a_2 = \frac{(-1)(s+3)}{(s+2)^2}\,a_1 = \frac{(-1)^2(s+2)(s+3)}{(s+1)^2(s+2)^2}\,a_0$

* For $n=3$ : $a_3 = \frac{(-1)(s+4)}{(s+3)^2}\,a_2 = \frac{(-1)^3(s+2)(s+3)(s+4)}{(s+1)^2(s+2)^2(s+3)^2}\,a_0$

$\Rightarrow$ For any Values of $n \geq 1$ : $a_n = \frac{(-1)^n(s+2)(s+3)(s+4)\cdots(s+n+1)}{(s+1)^2(s+2)^2(s+3)^2\cdots(s+n)^2}\,a_0$

For $s = 0$

$\therefore a_n(s_1) = \frac{(-1)^n\,(n+1)!}{(n^2)!} \longrightarrow y_1 = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)!}{(n^2)!}\,x^n$

$y_2 = y_1 \ln x + \sum_{n=1}^{\infty} a_n'(s_1)\, x^{n+s_1}$

$\therefore \frac{a_n'(s)}{a_n(s)} = \left(\frac{1}{s+2} + \frac{1}{s+3} + \frac{1}{s+4} + \cdots + \frac{1}{s+n+1}\right)$

$- 2\left(\frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3} + \cdots + \frac{1}{s+n}\right)$

$a_n'(0) = \left[\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right) - 2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right] a_n(0)$

$\Longrightarrow y_{gs} = C_1 y_1 + C_2 y_2$ #

**9.** $x\,y'' + y' + y = 0$

$\cdot\, p(x) = \frac{1}{x} \longrightarrow p(x_o) = \infty$ , $q(x) = \frac{1}{x} \longrightarrow q(x_o) = \infty$   at $x_o \Rightarrow p(x)$ & $q(x)$ are undefined   $\therefore x_o$ is a regular singularity

Let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n)\, x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n)\, x^{n+s-2}$

$$x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+s-1}$$

$$\left[(s)(s-1) + s\right] a_o x^{s-1} + \sum_{n=1}^{\infty} \left[ \begin{array}{l} \left[(n+s)(n+s-1) + (n+s)\right] a_n \\ + \; a_{n-1} \end{array} \right] x^{n+s-1} = 0$$

$*$ Coefficients of $x^{s-1} \longrightarrow \left[(s)(s-1) + s\right] = 0$

$s^2 = 0 \longrightarrow s_1 = s_2 = 0$   (Case (2))

$*$ Coefficients of $x^{n+s-1} \longrightarrow \left[(n+s)(n+s-1) + (n+s)\right] a_n + a_{n-1} = 0$

$$\boxed{a_n = \frac{(-1)}{(n+s)^2}\, a_{n-1}} \quad (n \geq 1)$$

$*$ For $n=1$ : $a_1 = \frac{-1}{(s+1)^2} a_o$

$*$ For $n=2$ : $a_2 = \frac{-1}{(s+2)^2} a_1 = \frac{(-1)^2}{(s+1)^2 (s+2)^2} a_o$

$*$ For $n=3$ : $a_3 = \frac{-1}{(s+3)^2} a_2 = \frac{(-1)^3}{(s+1)^2 (s+2)^2 (s+3)^2} a_o$

$*$ for any value of $n \geq 1$ :

$$a_n = \frac{(-1)^n}{(s+1)^2 (s+2)^2 (s+3)^2 \cdots (s+n)^2} a_o$$

For $s = 0$

$\therefore a_n = \frac{(-1)^n}{(n!)^2} \longrightarrow$ @ $n=0$ : $a_n(s_1) = \frac{(-1)^0}{(0)!} a_o = a_o$   $\therefore y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n$

$\longrightarrow y_2 = y_1 \ln x + \sum_{n=1}^{\infty} a_n'(s_1)\, x^{n+s_1}$

$\therefore \frac{a_n'(s_1)}{a_n(s_1)} = -2\left[\frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3} + \cdots + \frac{1}{s+n}\right]$

$\therefore a_n'(0) = -2\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right] a_n(0) = -2 H_n \frac{(-1)^n}{(n!)^2}$

$$\Longrightarrow y_{gs} = C_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n + C_2 \left[ \ln x \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n - \sum_{n=1}^{\infty} H_n \frac{(-1)^n}{(n!)^2} x^n \right] \quad \#$$

**10.** $(1-x^2)\,y'' - 2\,x\,y' + 2y = 0$

$\cdot\ p(x) = \frac{-2x}{1-x^2} \longrightarrow p(x_0) = 0$ , $\cdot\ q(x) = \frac{2}{1-x^2} \longrightarrow q(x_0) = 2$ at $x_0 \longrightarrow p(x)$ & $q(x)$ are defined $\therefore x_0$ is an ordinary point

let $y = \sum_{n=0}^{\infty} a_n x^n$ , $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ , $y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$

$$\longrightarrow (1-x^2) \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} (n)(n-1) a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n)(n-1) a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ a_{n+2} \left[ (n+2)(n+1) \right] - a_n \left[ \underbrace{(n)(n-1) + 2n - 2}_{(n-1)(n+2)} \right] \right] = 0$$ , Coefficients of $x^n = 0$

$\therefore a_{n+2}(n+2)(n+1) - a_n(n-1)(n+2) = 0$

$\therefore$ $\boxed{a_{n+2} = \frac{(n-1)(n+2)}{(n+2)(n+1)} a_n}$ (For $n \geq 0$)

for $n=0$ : $a_2 = \frac{(-1)(2)}{(2)(1)} a_0$ ($k=1$)

for $n=2$ : $a_4 = \frac{(1)(4)}{(4)(3)} a_2$ ($k=2$)

$= \frac{(-1\cdot 1)(2\cdot 4)}{(4\cdot 3\cdot 2\cdot 1)} a_0$

For $n=4$ : $a_6 = \frac{(3)(6)}{(6)(5)} a_4$ ($k=3$)

$= \frac{(-1\cdot 1\cdot 3)(2\cdot 4\cdot 6)}{(6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)} a_0$

for $n=1$ : $a_3 = \frac{(0)(3)}{(3)(2)} a_1 = 0$

For $n = 3, 5, 7 \cdots$ : $a_5, a_7, a_9 = 0$

For even values of $n$ : $a_{2k} = \frac{\left[-1\cdot 1\cdot 3\cdot \ldots \cdot (2k-3)\right]\left[2\cdot 4\cdot 6\cdot \ldots \cdot (2k)\right]}{(2k)!} a_0$ (For $k \geq 1$)

For odd values of $n$ : $a_{2k+1} = 0$ (For $k \geq 1$)

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{\left[-1\cdot 1\cdot 3\cdot \ldots \cdot (2k-3)\right]\left[2\cdot 4\cdot 6\cdot \ldots \cdot (2k)\right]}{(2k)!} \right] + a_1 x$$

**11.** $(x - x^2)\, y'' + (1 - 5x)\, y' - 4y = 0$

• $p(x) = \frac{1-5x}{x(1-x)} \longrightarrow p(x_0) = \infty$ , $q(x) = \frac{-4}{x(1-x)} \longrightarrow q(x_0) = \infty$   at $x_0 \Rightarrow p(x)$ & $q(x)$ are undefined   $\therefore x_0$ is a regular singularity

Let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n) x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n)\, x^{n+s-2}$

$$\rightarrow x\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} - x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - 5x\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - 4\sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} - \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} 5(n+s)a_n x^{n+s} - \sum_{n=0}^{\infty} 4a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=1}^{\infty}(-1)(n+s-1)(n+s-2)a_{n-1} x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + \sum_{n=1}^{\infty}(-5)(n+s-1)a_{n-1}x^{n+s-1} + \sum_{n=1}^{\infty}(-4)a_{n-1}x^{n+s-1} = 0$$

$$[(s)(s-1)+s]\,a_0 x^{s-1} + \sum_{n=1}^{\infty}\left[\begin{matrix}[(n+s)(n+s-1)+(n+s)]\,a_n \\ + \\ [(-1)(n+s-1)(n+s-2)+(-5)(n+s-1)-4]\,a_{n-1}\end{matrix}\right] x^{n+s-1} = 0$$

*Coefficients of $x^{s-1} = 0 \longrightarrow s[(s-1)+1] = 0 \quad \therefore s_1 = s_2 = 0$ (Case 2)

*Coefficients of $x^{n+s-1} = 0 \longrightarrow$

$$[(n+s)(n+s-1)+(n+s)]\,a_n - [(n+s-1)(n+s-2)+(5)(n+s-1)+4]\,a_{n-1} = 0$$
$$[(n+s)^2]\,a_n = [(n+s-1)[(n+s-1)-1] + 5(n+s-1) + 4]\,a_{n-1}$$
$$= [(n+s-1)^2 + 4(n+s-1) + 4]\,a_{n-1}$$
$$= [(n+s+1)^2]\,a_{n-1}$$

$$\boxed{a_n = \frac{(n+s+1)^2}{(n+s)^2}\,a_{n-1}} \quad (n \geq 1)$$

* For $n=1$: $a_1 = \frac{(s+2)^2}{(s+1)^2}\,a_0$

* For $n=2$: $a_2 = \frac{(s+3)^2}{(s+2)^2}\,a_1 = \frac{(s+2)^2(s+3)^2}{(s+1)^2(s+2)^2}\,a_0$

* For $n=3$: $a_3 = \frac{(s+4)^2}{(s+3)^2}\,a_2 = \frac{(s+2)^2(s+3)^2(s+4)^2}{(s+1)^2(s+2)^2(s+3)^2}\,a_0$

* For any value of $n \geq 1$:

$$a_n[s] = \frac{[(s+2)(s+3)(s+4)\cdots(s+n+1)]^2}{[(s+1)(s+2)(s+3)\cdots(s+n)]^2}\,a_0$$

$$\therefore a_n[s_1] = \frac{[(n+1)!]^2}{(n!)^2} = \left(\frac{(n+1)\cancel{n!}}{\cancel{n!}}\right)^2 = (n+1)^2 a_0 \longrightarrow @\, n=0 : a_n(s) = a_0$$

$$\rightarrow y_1 = \sum_{n=0}^{\infty} a_n(s_1)\, x^{n+s} \qquad \therefore y_1 = \sum_{n=0}^{\infty}(n+1)^2 x^n$$

$$\rightarrow y_2 = y_1 \ln x + \sum_{n=1}^{\infty} a_n'(s)\, x^{n+s}$$

$$\therefore \frac{a_n'(s)}{a_n(s)} = 2\left[\cancel{\frac{1}{s+2}} + \cancel{\frac{1}{s+3}} + \cancel{\frac{1}{s+4}} + \cdots + \frac{1}{s+n+1}\right] - 2\left[\frac{1}{s+1} + \cancel{\frac{1}{s+2}} + \cancel{\frac{1}{s+3}} + \cdots + \cancel{\frac{1}{s+n}}\right]$$

$$\therefore a_n'(0) = 2\left[\frac{1}{n+1} - 1\right] a_n(0)$$
$$= 2\left[\frac{-n}{\cancel{n+1}}\right](n+1)^{\cancel{2}} = -2n(n+1)$$

$$\therefore y_2 = \sum_{n=0}^{\infty}(1+n)^2 x^n \ln x - \sum_{n=1}^{\infty} 2n(n+1)\, x^n$$

$$\Rightarrow y_{gs} = C_1 \sum_{n=0}^{\infty}(1+n)^2 x^n + C_2\left[\sum_{n=0}^{\infty}(1+n)^2 x^n \ln x - \sum_{n=1}^{\infty} 2n(n+1)\, x^n\right] \quad \#$$

**12.** $(x - x^3)\, y'' + 3\,(1 - x^2)\, y' + 8\,x\,y = 0$

• $p(x) = \frac{3\cancel{(1-x^2)}}{x\cancel{(1-x^2)}} = \frac{3}{x} \longrightarrow p(x_0) = \infty$, $q(x) = \frac{8x}{x(1-x^2)} = \frac{8}{1-x^2} \longrightarrow q(x_0) = 8$ at $x_0 \Rightarrow p(x)$ is undefined $\therefore x_0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n)\, x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n)\, x^{n+s-2}$

$$x\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} - x^3\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + 3\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - x^2\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + 8x\sum_{n=0}^{\infty}a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty}-(n+s)(n+s-1)a_n x^{n+s+1} + \sum_{n=0}^{\infty}3(n+s)a_n x^{n+s-1} + \sum_{n=0}^{\infty}-(n+s)a_n x^{n+s+1} + \sum_{n=0}^{\infty}8a_n x^{n+s+1} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=2}^{\infty}-(n+s-2)(n+s-3)a_{n-2} x^{n+s-1} + \sum_{n=0}^{\infty}3(n+s)a_n x^{n+s-1} + \sum_{n=2}^{\infty}-(n+s-2)a_{n-2} x^{n+s-1} + \sum_{n=2}^{\infty}8a_{n-2}x^{n+s-1} = 0$$

$$x^{s-1}\left[(s)(s-1)+3s\right]a_0 + x^{s}\left[(s+1)(s)+3(s+1)\right]a_1 + \sum_{n=2}^{\infty}\left[\left[(n+s)(n+s-1)+3(n+s)\right]a_n - \left[(n+s-2)(n+s-3)+(n+s-2)+8\right]a_{n-2}\right]x^{n+s-1} = 0$$

* Coefficient of $x^{s-1} = 0 \longrightarrow \left[(s)(s-1)+3s\right]a_0 = 0$

$$a_0\left[(s)(s+2)\right] = 0$$

$\therefore\ s_1 = 0\ \&\ s_2 = -2$

$\longrightarrow s_1 - s_2 = 2$ (Case 3)

**13.** $x^2\, y'' + x\, y' + (x^2 - 1)\,y = 0$

• $p(x) = \frac{x}{x^2} = \frac{1}{x} \longrightarrow p(x_0) = \infty$, $q(x) = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2} \longrightarrow q(x_0) = -\infty$ at $x_0 \Rightarrow p(x)\ \&\ q(x)$ are undefined $\therefore x_0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n)\, x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n)\, x^{n+s-2}$

$$x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + x\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + x^2\sum_{n=0}^{\infty}a_n x^{n+s} - \sum_{n=0}^{\infty}a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s} + \sum_{n=0}^{\infty}a_n x^{n+s+2} + \sum_{n=0}^{\infty}-a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s} + \sum_{n=2}^{\infty}a_{n-2} x^{n+s} + \sum_{n=0}^{\infty}-a_n x^{n+s} = 0$$

$$x^{s}\left[(s)(s-1)+s-1\right]a_0 + x^{s+1}\left[(s+1)(s)+(s+1)-1\right]a_1 + \sum_{n=2}^{\infty}\left[\left[(n+s)(n+s-1)+(n+s)-1\right]a_n + a_{n-2}\right]x^{n+s} = 0$$

* Coefficient of $x^{s} = 0 \longrightarrow \left[(s)(s-1)+(s-1)\right]a_0 = 0$

$$a_0\left[(s-1)(s+1)\right] = 0$$

$\therefore\ s_1 = 1\ \&\ s_2 = -1 \longrightarrow s_1 - s_2 = 2$ (Case 3)

**14.** $x(1-x)y'' - 3xy' - y = 0$

• $p(x) = \frac{-3\cancel{x}}{\cancel{x}(1-x)} = \frac{-3}{1-x} \longrightarrow p(x_0) = -3$ , $q(x) = \frac{-1}{x(1-x)} \longrightarrow q(x_0) = \infty$ at $x_0 \Rightarrow q(x)$ is undefined ∴ $x_0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n) x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n) x^{n+s-2}$

$$\rightarrow x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} - x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} - 3x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} -(n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (-3)(n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} (-1) a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=1}^{\infty} (-1)(n+s-1)(n+s-2) a_{n-1} x^{n+s-1} + \sum_{n=1}^{\infty} (-3)(n+s-1) a_{n-1} x^{n+s-1} + \sum_{n=1}^{\infty} (-1) a_{n-1} x^{n+s-1} = 0$$

$$x^{s-1} \Big[ (s)(s-1) \Big] a_0 + \sum_{n=1}^{\infty} \begin{bmatrix} [(n+s)(n+s-1)] a_n \\ - \\ [(n+s-1)(n+s-2) + 3(n+s-1) + 1] a_{n-1} \end{bmatrix} x^{n+s-1} = 0$$

* Coefficients of $x^{s-1} = 0$ : $(s)(s-1)a_0 = 0$ ∴ $s_1 = 0$ & $s_2 = 1$ $s_1 - s_2 = -1$ (Case 3,)

**15.** $xy'' - y = 0$

• $p(x) = 0 \longrightarrow p(x_0) = 0$ , $q(x) = \frac{-1}{x} \longrightarrow q(x_0) = -\infty$ at $x_0 \Rightarrow q(x)$ is undefined ∴ $x_0$ is a regular singularity

let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution, $y' = \sum_{n=0}^{\infty} (n+s)(a_n) x^{n+s-1}$ , $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)(a_n) x^{n+s-2}$

$$\rightarrow x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} - \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} (-1) a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=1}^{\infty} (-1) a_{n-1} x^{n+s-1} = 0$$

$$x^{s-1} \Big[ (s)(s-1) \Big] a_0 + \sum_{n=1}^{\infty} (-1) a_{n-1} x^{n+s-1} = 0$$

* Coefficients of $x^{s-1} \longrightarrow (s)(s-1)a_0 = 0$ ∴ $s_1 = 0$ & $s_1 = 1$ ∴ $s_1 - s_2 = -1$ (Case 3)