Evaluation of Real Integrals

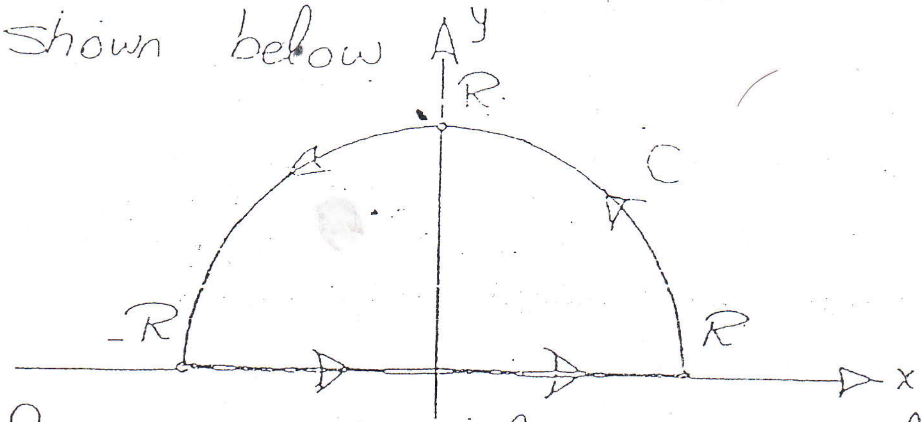
I) Integrals of Rational function $\frac{P(x)}{Q(x)}$

To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, where $f(x)$ is a rational fn $= \frac{P(x)}{Q(x)}$, such that:

(order of $P(x)$) $\leq$ 2 + (order of $Q(x)$),

we evaluate

$$\oint_C f(z)\,dz,$$ where C is the path shown below



R is large enough to include all singular Pts.

- By the Residue Theorm we have

$$\oint_C f(z)\,dz = 2\pi i \sum \text{Res. of S.P. inside}$$

$$= 2\pi i \sum \text{Res of S.P. in the upperhalf plane}$$

$$\frac{\quad}{Q(z)} \Big| \leq \frac{\quad}{R^2}$$

$$\Rightarrow \left| \int_{C_1} f(z)\,dz \right| \leq \frac{\pi M}{R}$$

$$\Rightarrow \text{as } R \to \infty \Rightarrow \left| \int_{C_1} f(z)\,dz \right| \to 0$$

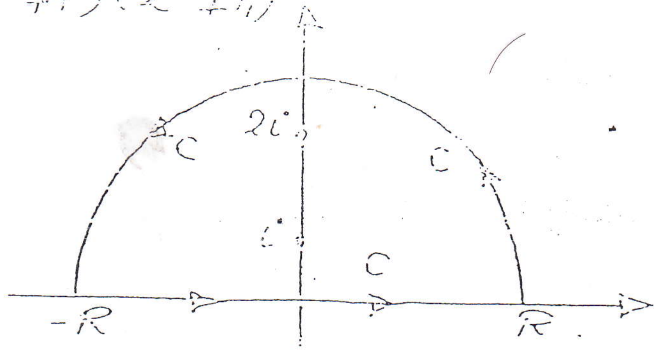
$$\Rightarrow \int_{C_1} f(z)\,dz \to 0$$

Example: Evaluate $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

Solution

$f(x) = \dfrac{x^2}{(x^2+1)(x^2+4)} \Rightarrow$ order of $P(x) \geq$ order of $Q(x) + 2 \Rightarrow$ we will evaluate

$$\oint_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz$$ using the Residue theorm $\Rightarrow$



$$\Rightarrow \oint \frac{z^2}{(z^2+1)(z^2+4)}\,dz = 2\pi i \left(\underset{z=i}{\text{Res}} + \underset{z=2i}{\text{Res}} \right)$$

- for $z = i$ (pole of order 1)

$$\underset{z=i}{\text{Res}} = \frac{1}{0!} \lim_{z \to i} \frac{z^2}{(z^2+1)(z^2+4)} (z-i)$$

$$= \frac{-1}{3} \lim_{z \to i} \frac{z-i}{z^2+1} \quad \text{using L'Hopital}$$

$$= -\frac{1}{3} \lim_{z \to i} \frac{1}{2z} = -\frac{1}{3}\left(\frac{1}{2i}\right) = \frac{i}{6}$$

- for $z = 2i$ (pole of order 1)

$$\underset{z=2i}{\text{Res}} = \frac{1}{0!} \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z^2+4)} (z-2i)$$

$$= \frac{-4}{-3} \lim_{z \to 2i} \frac{z-2i}{z^2+4} \quad \text{using L'Hopital}$$

$$= \frac{4}{3} \lim_{z \to 2i} \frac{1}{2z} = \frac{4}{3}\left(\frac{1}{4i}\right) = -\frac{i}{3}$$

$$\Rightarrow \oint_C \frac{z^2}{(z^2+1)(z^2+4)} \, dz = 2\pi i \left(\frac{i}{6} - \frac{i}{3}\right)$$

$$= 2\pi i\left(\frac{i}{6} - \frac{2i}{6}\right) = 2\pi i\left[\frac{-i}{6}\right] = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\Rightarrow \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx + \int_{C_1} \frac{z^2}{(z^2+1)(z^2+4)} \, dz = \frac{\pi}{3}$$

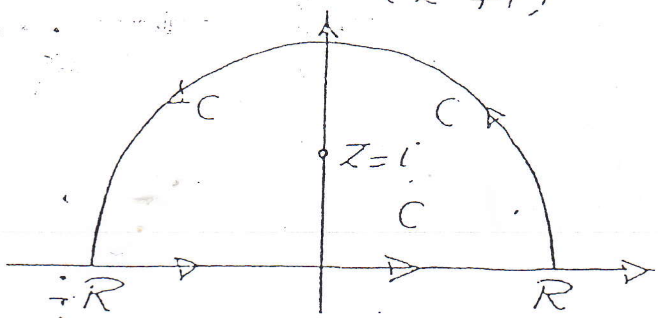
Take lim for $R \to \infty \Rightarrow$ the 2^{nd} integral $\to 0$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = \pi/3$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = \frac{1}{2}\left(\frac{\pi}{3}\right) = \pi/6$$

Example :- Evaluate $\int_0^{\infty} \frac{dx}{(x^2+1)^3}$

Solution we will find $\oint_C \frac{dz}{(z^2+1)^3}$ by the residue theorm



$$\oint_C \frac{dz}{(z^2+1)^3} = 2\pi i \operatorname*{Res}_{z=i}$$

for $z=i$ (pole of order $n=3$)

$$\operatorname*{Res}_{z=i} = \frac{1}{2!} \lim_{z \to i} \frac{d^2}{dz^2} \left(\frac{1}{(z^2+1)^3} (z-i)^3 \right)$$

$$= \frac{1}{2} \lim_{z \to i} \frac{d^2}{dz^2} \frac{1}{(z+i)^3} = \frac{1}{2} \lim_{z \to i} \frac{3(4)}{(z+i)^5}$$

$$= \frac{1}{2} \cdot \frac{12}{(2i)^5} = -\frac{3}{16} i$$

$$\Rightarrow \oint_C \frac{dz}{(z^2+1)^3} = 2\pi i \left(\frac{-3}{16} i \right) = \frac{3\pi}{8}$$

$$\Rightarrow \int_{-R}^{R} \frac{dx}{(x^2+1)^3} + \int_{C_1} \frac{dz}{(z^2+1)^3} = \frac{3\pi}{8}$$

Take lim for $R \to \infty$ the 2^{nd} integral $\to 0$ because order of $P(z) \leq$ order of $Q(z) + 2$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8} \Rightarrow \int_0^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{16}$$

...tional f^n of $\cos\theta$ & $\sin\theta$:-

$$= \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta ,$$

use the Subsitution of $z = e^{i\theta}$:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = i e^{i\theta} d\theta \implies d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} d\theta \implies \oint_{|z|=1} \frac{dz}{iz}$$

$$\int R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} f(z)\, dz$$

: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2}$

...m :- let $z = e^{i\theta} \implies d\theta = \frac{dz}{iz}$

$z = e^{i\theta}$

$\Rightarrow d\theta = \frac{1}{iz} dz$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$5 + 4\cos\theta = 5 + 2z + \frac{2}{z} = \frac{2z^2 + 5z + 2}{z}$$

$$\int_0^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2} = \oint_{|z|=1} \frac{1}{\left(\frac{2z^2+5z+2}{z}\right)^2} \cdot \frac{dz}{iz}$$

$$\frac{1}{i} \oint_{|z|=1} \frac{z}{(2z^2+5z+2)^2}\, dz = \frac{1}{i} \oint_{|z|=1} \frac{z}{(2z+1)^2 (z+2)^2}\, dz$$

$\therefore$ Singular points are $Z = -\frac{1}{2}$ & $Z = -2$, but only $Z = -\frac{1}{2}$ lies inside $|Z| = 1$

Using Cauchy integral formula:-

$$I = \frac{1}{i} \oint_{|Z|=1} \frac{Z/(Z+2)^2}{(2Z+1)^2} \, dZ = \frac{1}{4i} \oint_{|Z|=1} \frac{Z/(Z+2)^2}{(Z+\frac{1}{2})^2} \, dZ$$

$$= \frac{1}{4i} * \frac{2\pi i}{1!} * \frac{d}{dZ} \left(\frac{Z}{(Z+2)^2} \right) \Bigg|_{Z=-\frac{1}{2}}$$

$$= \frac{\pi}{2} \left(\frac{(Z+2)^{\cancel{2}} - 2Z\cancel{(Z+2)}}{(Z+2)^{\cancel{4}\,3}} \right) \Bigg|_{Z=-\frac{1}{2}} = \frac{10}{27}\pi$$

Example:- Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} \, d\theta$

Solution:-

Let $Z = e^{i\theta} \implies d\theta = \dfrac{dZ}{iZ}$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(Z + \frac{1}{Z}\right)$$

$$\cos 3\theta = \frac{1}{2}(e^{i3\theta} + e^{-i3\theta}) = \frac{1}{2}\left(Z^3 + \frac{1}{Z^3}\right)$$

$$I = \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} \, d\theta = \oint_{|Z|=1} \frac{\frac{1}{2}(Z^3 + 1/Z^3)}{5 - 2(Z + 1/Z)} \, \frac{dZ}{iZ}$$

$$= \frac{1}{2i} \oint_{|Z|=1} \frac{Z^3 + 1/Z^3}{-2Z^2 + 5Z - 2} \, dZ$$

$$= \frac{i}{4} \oint_{|Z|=1} \frac{Z^6 + 1}{Z^3 (Z^2 - \frac{5}{2}Z + 1)} \, dZ$$

$$= \frac{i}{4} \oint_{|Z|=1} \frac{Z^6 + 1}{Z^3 (Z - \frac{1}{2})(Z - 2)} \, dZ$$

Using the Residue theorm, only $z=0$ & $z=\frac{1}{2}$ lie inside $|z|=1$

* $z=\frac{1}{2}$ is a pole of order 1 $\Rightarrow$

$$\underset{z=\frac{1}{2}}{\operatorname{Res}} f(z) = \lim_{z\to\frac{1}{2}} \frac{z^6+i}{z^3(z-2)} = -\frac{65}{12}$$

* $z=0$ is a pole of order 3.

$$\underset{z=0}{\operatorname{Res}} f(z) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\frac{z^6+1}{(z-1/2)(z-2)}\right) = \frac{21}{4}$$

$$\Rightarrow \quad I = \frac{i}{4}\left(2\pi i \sum \operatorname{Res}\right) = \frac{\pi}{12}$$

Example:
$$\int_0^{2\pi}\frac{d\theta}{2-\cos\theta}$$

Solution: let $z=e^{i\theta} \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ & $d\theta = \frac{dz}{iz}$

$$I = \int_0^{2\pi}\frac{d\theta}{2-\cos\theta} = \oint_{|z|=1}\frac{1}{2-\frac{1}{2}(z+1/z)}\cdot\frac{dz}{iz}$$

$$= \frac{2}{i}\oint\frac{dz}{4z-z^2-1} = 2i\oint\frac{dz}{z^2-4z+1}$$

Roots of $z^2-4z+1=0$ are $z=2\pm\sqrt{3}$, only $z=2-\sqrt{3}$ lies inside $|z|=1$, if we use Cauchy formula

$$I = 2i\oint_{|z|=1}\frac{1/(z-2-\sqrt{3})}{(z-2+\sqrt{3})} = 2i * 2\pi i\left(\frac{1}{z-2-\sqrt{3}}\right)\Bigg|_{z=2-\sqrt{3}}$$

$$= -4\pi\left(\frac{1}{-2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

III) Fourier Integrals :-

They are integrals of the form

$$\int_{-\infty}^{\infty} f(x)\cos tx\,dx \quad \& \quad \int_{-\infty}^{\infty} f(x)\sin tx\,dx$$

* We evaluate $\oint_C f(z)e^{itz}\,dz$ exactly as Case (I)

$\Rightarrow \oint_C f(z)e^{itz}\,dz = 2\pi i \sum$ Res of S.P. in the upper half plane.

$\Rightarrow \int_{-R}^{R} f(x)e^{itx}\,dx + \int_{C_1} f(z)e^{itz}\,dz = \text{''}$

- Take $\lim R \to \infty$ the 2^{nd} integral $\to$ zero

- Finally we use the fact that

$$\int_{-\infty}^{\infty} f(x)\cos tx\,dx = \text{Real}\left(\int_{-\infty}^{\infty} f(x)e^{itx}\,dx\right)$$

$$\& \quad \int_{-\infty}^{\infty} f(x)\sin tx\,dx = \text{Imagenary}\left(\int_{-\infty}^{\infty} f(x)e^{itx}\,dx\right)$$

Example :- Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx$

Solution :- $\oint_C \frac{e^{iz}}{(z^2+1)^2}\,dz = 2\pi i\,(\text{Res}_{z=i})$

$\Rightarrow \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\,dx + \int_{C_1} \frac{e^{iz}}{(z^2+1)^2}\,dz = 2\pi i\,\text{Res}_{z=i}$

Take $\lim R \to \infty$, the 2^{nd} integral $\to$ zero

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\,dx = 2\pi i\,\text{Res}_{z=i}$$

$L^{-1}[F(s)] \qquad L^{-1}[\frac{d}{ds}F(s)] = -f(t)\,t$

$$\operatorname*{Res}_{\substack{z=i\\n=2}} \frac{e^{iz}}{(z^2+1)^2} = \frac{1}{1!} \lim_{z\to i} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)$$

$$= \lim_{z\to i} \frac{ie^{iz}(z+i)^2 - 2e^{iz}(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{i(z+i)e^{iz} - 2e^{iz}}{(z+i)^3} = \frac{-2e^{-1}-2e^{-1}}{(2i)^3} = -\frac{i}{2e}$$

$z=-i$ lies in the lower half plane.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\,dx = 2\pi i\cdot\left(\frac{-i}{2e}\right) = \frac{\pi}{e}$$

But $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx$ = Real of $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\,dx$

$$= \frac{\pi}{e}$$

Note that:- $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2}\,dx$ = zero because the fn is an odd fn.

Example:- Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin 3x}{x^4+4}\,dx$

Solution

Consider $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{i3x}}{x^4+4}\,dx$ which equals to $2\pi i \sum \operatorname{Res} \dfrac{ze^{i3z}}{z^4+4}$

The Poles are:-

$$z^4+4=0 \Rightarrow z^4=-4$$

$$(re^{i\theta})^4 = 4e^{i(\pi+2k\pi)}$$

$\Rightarrow r^4 = 4 \Rightarrow r = \sqrt{2}$

$4\theta = \pi + 2k\pi \Rightarrow \theta = \frac{\pi + 2k\pi}{4}$; $k = 0, 1, 2.$

Poles are $\sqrt{2}\, e^{i\left(\frac{\pi+2k\pi}{4}\right)}$; $k = 0, 1, 2, 3$

$\Rightarrow$ Poles are $\pm 1 \pm i$.

Poles lieing in the upper half plane are $\pm 1 + i$, both are of order 1.

$$\operatorname*{Res}_{z=1+i} = \lim_{z \to 1+i} \frac{z e^{i3z}}{z^4+4}(z-1-i) =$$

$$(1+i)\, e^{i3(1+i)} \lim_{z \to 1+i} \frac{z-1-i}{z^4+4}$$

Using L'Hopital $\Rightarrow = (1+i)\, e^{3i-3} \cdot \frac{1}{4(1+i)^3}$

$$\operatorname*{Res}_{z=1+i} = \frac{e^{-3} \cdot e^{3i}}{4(1+i)^2} = \frac{e^{-3} \cdot e^{3i}}{8i}.$$

Similarly, $\operatorname*{Res}_{z=-1+i} = \lim_{z=-1+i} \frac{z e^{i3z}}{z^4+4}(z+1-i)$

$$= \frac{e^{i3(-1+i)}}{4(-1+i)^2} = -\frac{e^{-3} \cdot e^{-i3}}{8i}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x e^{i3x}}{x^4+4}\, dx = 2\pi i \left(\frac{e^{-3} \cdot e^{3i}}{8i} - \frac{e^{-3} \cdot e^{-3i}}{8i} \right)$$

$$= \frac{\pi}{4} e^{-3} \left(\cos 3 + i \sin 3 - \cos 3 + i \sin 3 \right)$$

$$= i\, \frac{\pi}{2} e^{-3} \sin 3$$

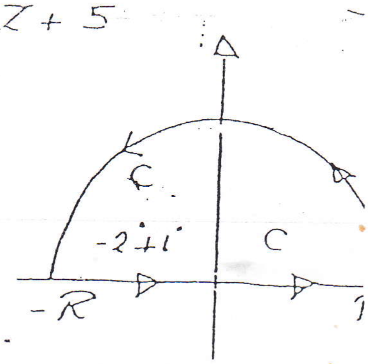
$$\therefore \int_{-\infty}^{\infty} \frac{x \sin 3x}{x^4+4}\, dx = \text{Imaginary Part} = \frac{\pi \sin 3}{2e^3}$$

Note that: $\int_{-\infty}^{\infty} \frac{x \cos 3x}{x^4+4}\, dx = 0$ because the fn. is odd.

Example Evaluate $\displaystyle\int_{-\infty} \frac{\sin 2x}{x^2+4x+5}\, dx$

Solution we will evaluate $\displaystyle\oint_C \frac{e^{i2z}}{z^2+4z+5}\, dz$ usi

the residue theorm



S.P. are $z^2+4z+5=0$

$$z=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$$

$$\Rightarrow \oint_C \frac{e^{i2z}}{z^2+4z+5}\, dz = 2\pi i \operatorname*{Res}_{z=-2+i}$$

$$\operatorname*{Res}_{z=-2+i} \text{ (pole of order } n=1) = \lim_{z\to -2+i} \frac{e^{i2z}(z+2-i}{z^2+4z+5}$$

$$= e^{i2(-2+i)} \lim_{z\to -2+i} \frac{z+2-i}{z^2+4z+5} \quad \text{using L'Hopita}$$

$$= e^{-4i-2} \lim_{z\to -2+i} \frac{1}{2z+4} = e^{-4i-2}\left(\frac{1}{-4+2i+4}\right)$$

$$= \frac{1}{2i} e^{-2} (\cos -4 + i \sin -4) = \frac{1}{2e^2}(-i\cos 4 - \sin 4$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{i2x}}{x^2+4x+5}\, dx + \int_{C_1} \frac{e^{i2z}\, dz}{z^2+4z+5} = \frac{1}{2e^2}(-i\cos 4 - \sin$$

Take lim $R\to\infty$, the 2^{nd} integral $\to$ zero

$$\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^2+4x+5}\, dx = \frac{1}{2e^2}(-i\cos 4 - \sin 4)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+4x+5}\, dx = \text{Imag. part} = -\frac{\cos 4}{2e^2}$$