Numerical Analysis (1)

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Agenda

- Introduction
- First Order ODE Numerical Methods
- Higher Order ODE Numerical Methods

Numerical Methods

Methods of solution for
Mathematical problems by formulating them
into a number of arithmetic operations

Numerical Analysis : Introduction

Why ?

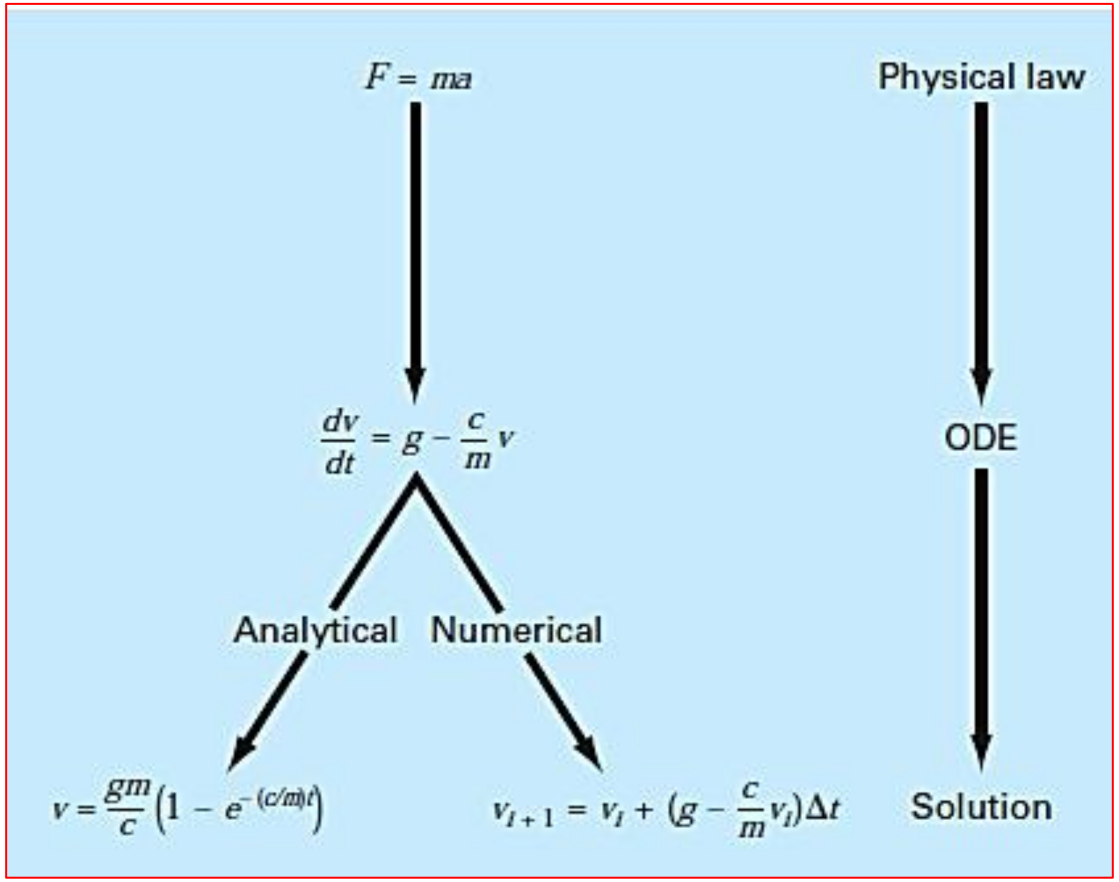
Exact Methods are limited

Solution of Exact Methods may be implicit functions

Geometrical Methods are not accurate

Numerical Analysis : Introduction

How ?

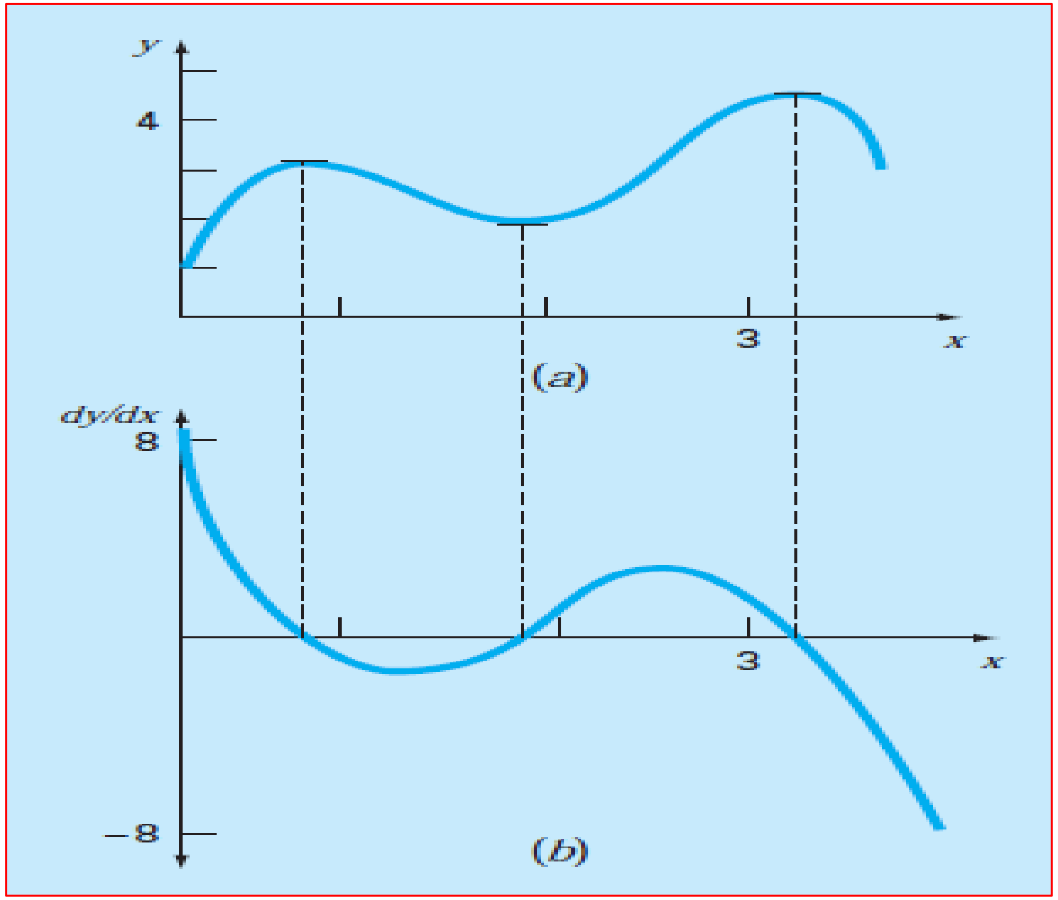


Numerical Analysis : Introduction

Idea ?

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$



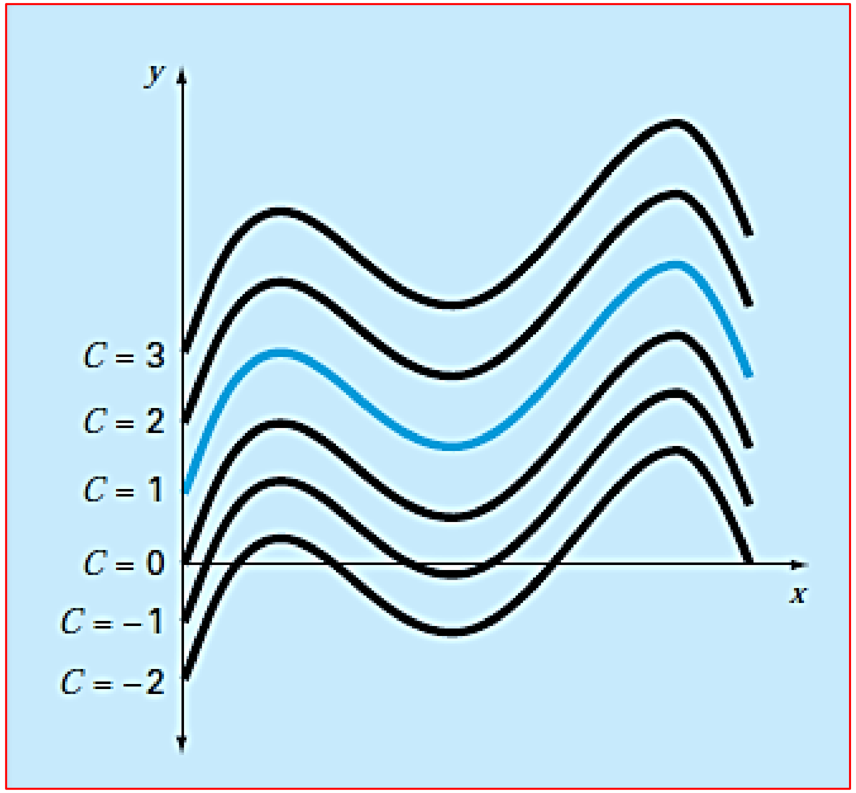
Idea ?

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

at $x = 0,\ y = 1.$

IVP

system of 1 or more Differential Equations, together with 1 or more initial conditions.



Idea ?

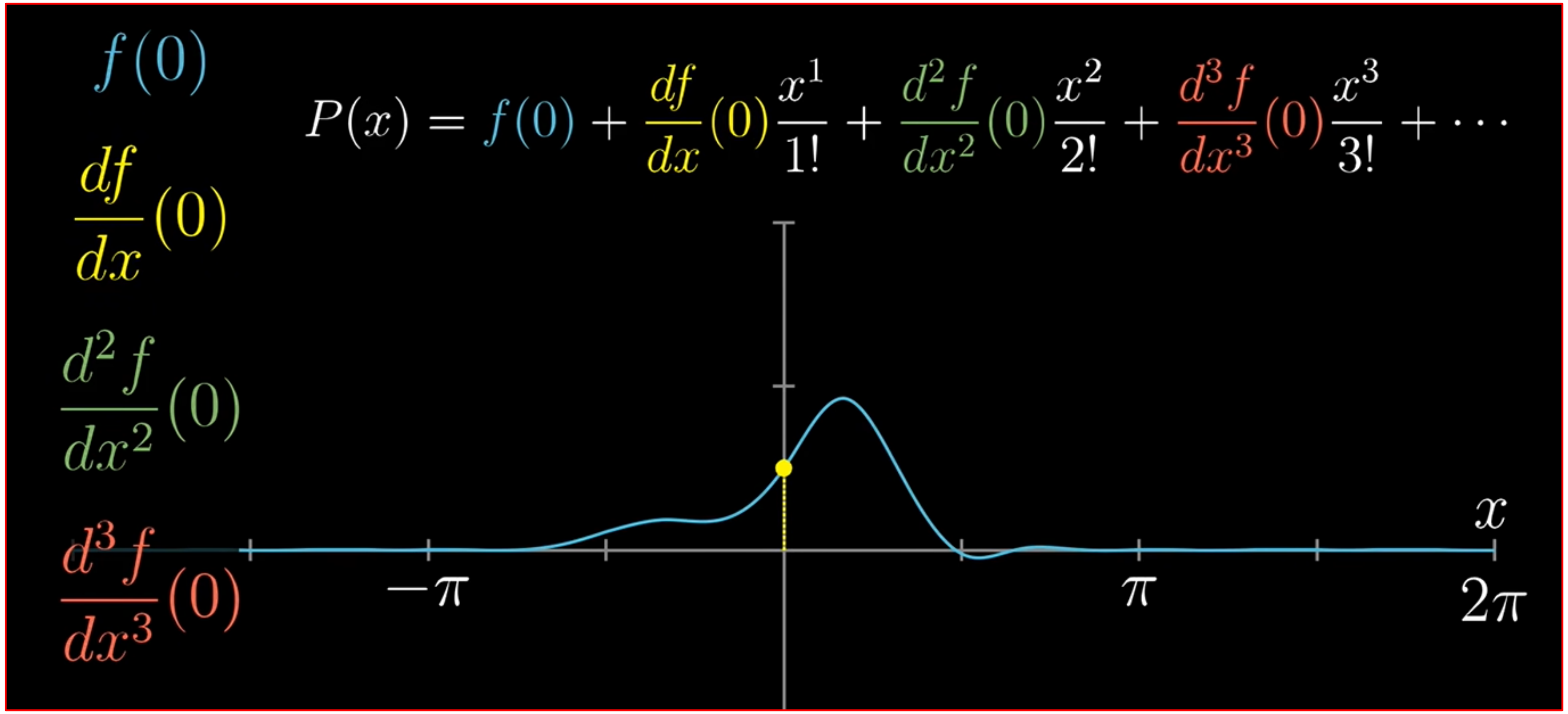
Solution of IVP

Numerical methods gets the solution in a tabular form

n	x_n	y_n
0	0.0	0.000
1	0.2	0.000
2	0.4	0.04
3	0.6	0.128
4	0.8	0.274
5	1.0	0.488

Numerical Analysis : Introduction

Series Approximation



Numerical Analysis : Introduction

Maclaurin Series

$$f(x) = e^x \text{ can be approximated near } x = 0$$

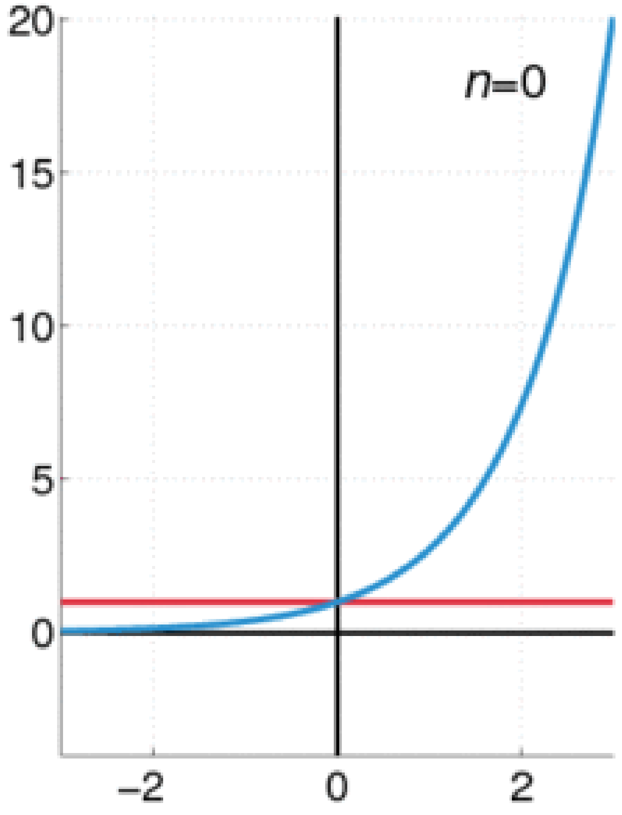
$$f(x) \cong e^0 = a = 1$$

$$f(x) \cong a + bx = 1 + f'(x=0)\,x = 1 + x$$

$$f(x) \cong a + bx + cx^2 = 1 + x + \frac{f''(x=0)}{2!}x^2$$

Near $x = 0$

$$\mathrm{f}(x) \cong \mathrm{P}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x=0)}{k!}\, x^k$$

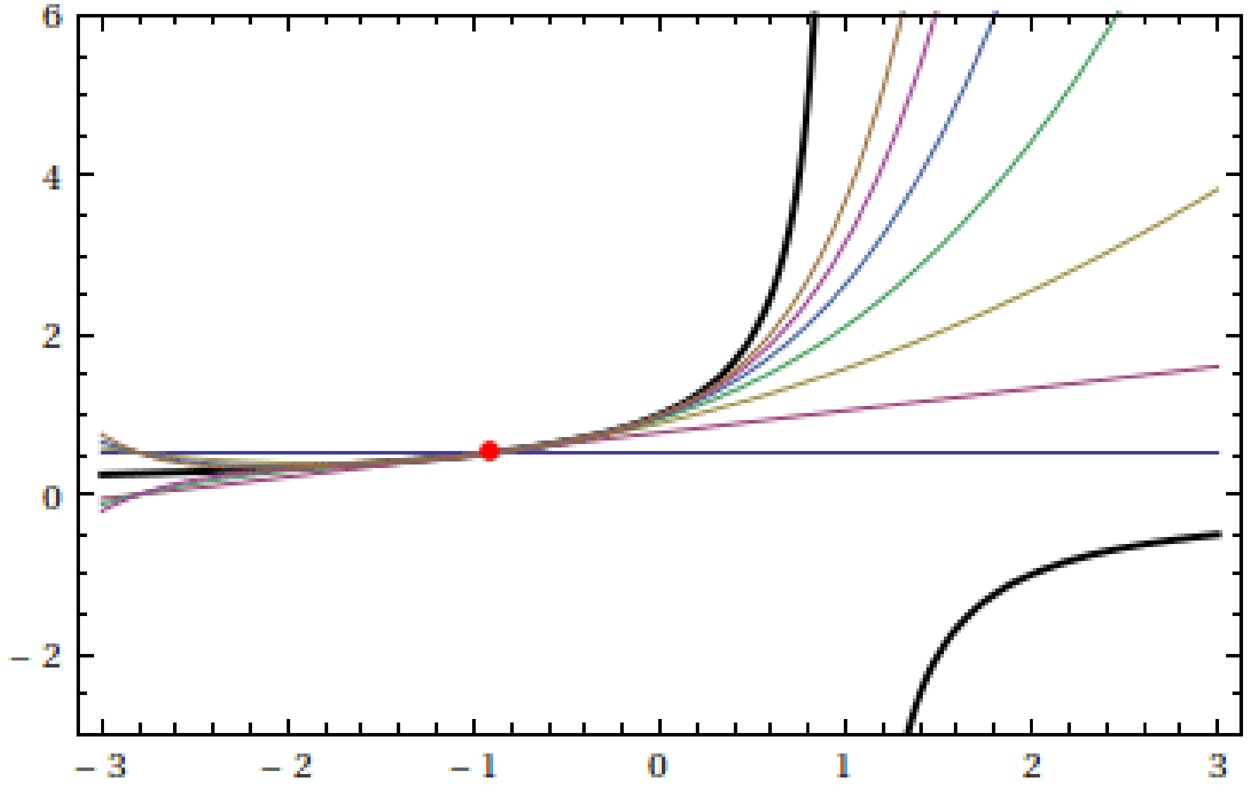


Numerical Analysis : Introduction

Taylor Series

Near $x = a$

$$f(x) \cong \sum_{k=0}^{\infty} \frac{f^{(k)}(x=a)}{k!}(x-a)^k$$



$$f(x) \cong f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Numerical Analysis : Introduction

Taylor Series

The nth order Taylor polynomial of f centered at $x = a$ is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This degree n polynomial approximates $f(x)$ near $x = a$ and has the property that $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0 \ldots n$.

Numerical Analysis : Introduction

Taylor Series

Near $x = a$

$$f(x) \cong T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k = \sum_{k=0}^{n} c_k \, (x-a)^k$$

This approximation is valid as long as

$$\lim_{k\to\infty} \left| \frac{c_{k+1}(x-a)^{k+1}}{c_k (x-a)^k} \right| = L$$

Exists & **Smaller than 1**

Taylor Series

Error $E_{n,a}(x) = f(x) - \mathrm{T_n}(\mathrm{x}) = 0 + 0 + \cdots + \frac{f^{(k+1)}(a)}{k+1!}(x-a)^{k+1} + \frac{f^{(k+2)}(a)}{k+2!}(x-a)^{k+2} \ldots$

$E_{n,a}(x)$ **is of order** $(x-a)^{k+1}$

ODE

$$f\left(x, y, y', y'', \dots, y^{(m)}\right) = 0$$

Order => Highest derivative

Degree => Power of the highest derivative

Solution => all the functions y(x) that satisfy the ODE

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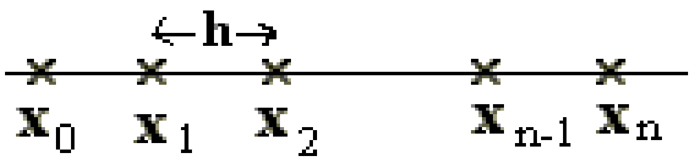
- Higher Order ODE Numerical Methods

ODE

To solve an ODE numerically over the interval [a,b]

Step 1 Let $x_0 = a, x_n = b$ and subdivide the interval into n equal parts such that $x_i = x_0 + ih \quad , i = 1, 2, 3, \ldots n$



ODE

To solve an ODE numerically over the interval [a,b]

Step 2 **Define y_i for each x_i according to the ODE Method**

Step 3 **Solve the defined equation and get all y_i**

$$\text{New value} = \text{old value} + \text{slope} \times \text{step size}$$

Numerical Analysis

ODE Methods

Single Step

compute new value y_{i+1} using only a single step (only previous y_i)

Multi- Step

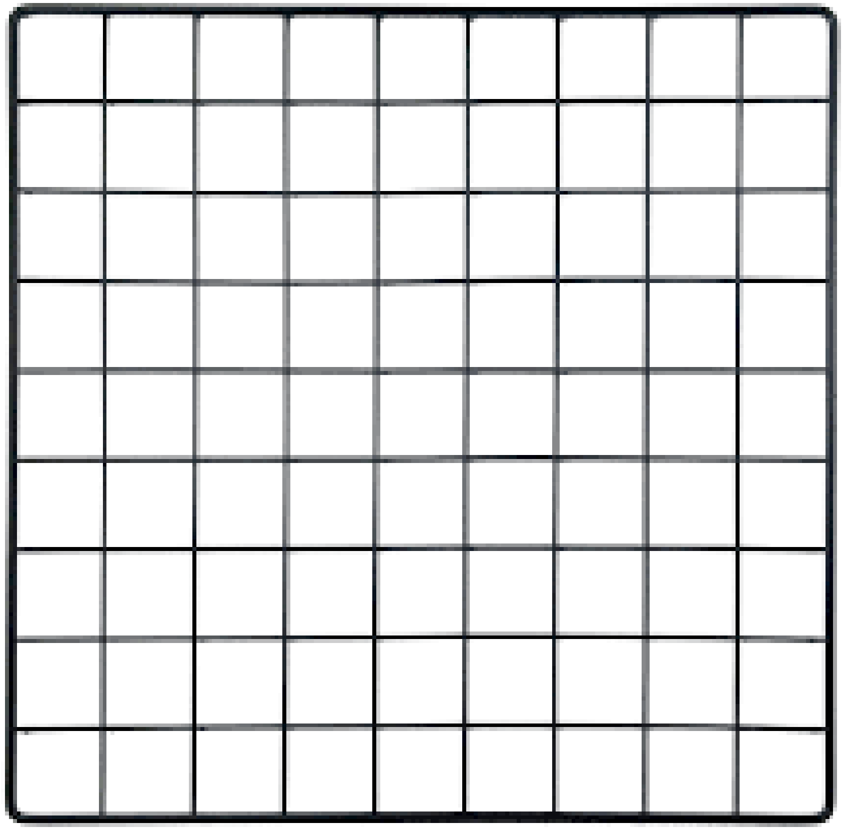
uses, in each step, values from two or more previous steps.

ODE : Euler-Cauchy Method

1- Method

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0,$$



Then!

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1),$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$$\text{New value} = \text{old value} + \text{slope} \times \text{step size}$$

2- Example 1

Apply Euler-Cauchy method with h = 0.1 to find the solution of the initial-value problem

$$y' = 1 - x + 4y,\ y(0) = 1$$

on the interval [0, 0.5] Given the exact solution below compute the error $E_n = y(x_n) - y_n$ in each step. Use 5 decimal points in your calculations.

$$y(x) = \frac{1}{4}x - \frac{3}{16} + \frac{19}{16}e^{4x}$$

ODE : Euler-Cauchy Method

2- Example 1

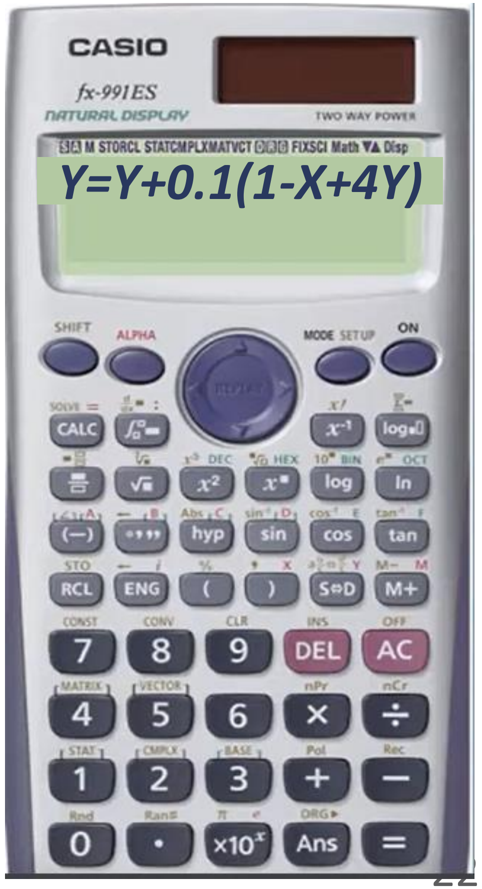
$$y' = 1 - x + 4y,\ y(0) = 1 \qquad y(x) = \frac{1}{4}x - \frac{3}{16} + \frac{19}{16}e^{4x}$$

From Euler formula $y_{n+1} = y_n + h\, f(x_n, y_n)$, we have

$$y_{n+1} = y_n + 0.1\,(1 - x_n + 4\,y_n),\ n = 0, 1, 2, \dots$$

The numerical solution is given in the following table:

n	x_n	y_n	f_n	$y_{n+1} = y_{approx}$	y_{exact}	Error = $y_{exact} - y_{approx}$
0	0					
1	0.1					
2	0.2					
3	0.3					
4	0.4					



ODE : Euler-Cauchy Method

2- Example 1

$$y' = 1 - x + 4y,\ y(0) = 1 \qquad y(x) = \frac{1}{4}x - \frac{3}{16} + \frac{19}{16}e^{4x}$$

From Euler formula $y_{n+1} = y_n + h\, f(x_n, y_n)$, we have

$$y_{n+1} = y_n + 0.1\,(1 - x_n + 4\,y_n),\ n = 0, 1, 2, \ldots$$

The numerical solution is given in the following table:

n	x_n	y_n	f_n	$y_{n+1} = y_{approx}$	y_{exact}	Error = $y_{exact} - y_{approx}$
0	0	1.0	5.0	1.5	1.60904	0.10904
1	0.1	1.5	6.9	2.19	2.50533	0.31533
2	0.2	2.19	9.56	3.146	3.83014	0.68414
3	0.3	3.146	13.284	4.4744	5.79423	1.31923
4	0.4	4.4744	18.4976	6.32416	8.71200	2.38784

2- Example 2

Apply Euler-Cauchy method witl $h = 0.2$ to find the solution of the initial-value problem

$$y' = x + y \quad , y(0) = 0$$

on the interval [0, 1] Given the exact solution below compute the error $E_n = y(x_n) - y_n$ in each step. Use 3 decimal points in your calculations.

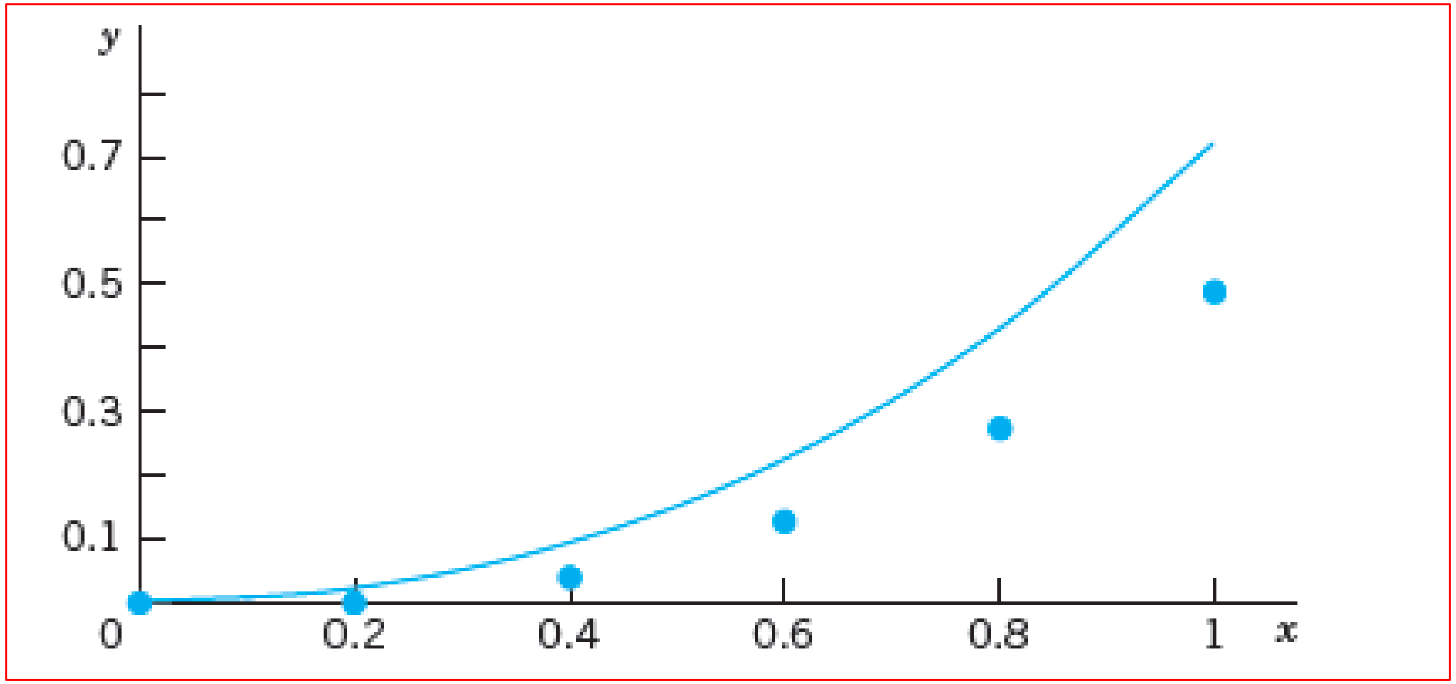
$$y = e^x - x - 1$$

2- Example 2

$$y' = x + y\ , \qquad y(0) = 0 \qquad y = e^{x} - x - 1$$

n	x_n	y_n	$y(x_n)$	Error
0	0.0	0.000	0.000	0.000
1	0.2	0.000	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230

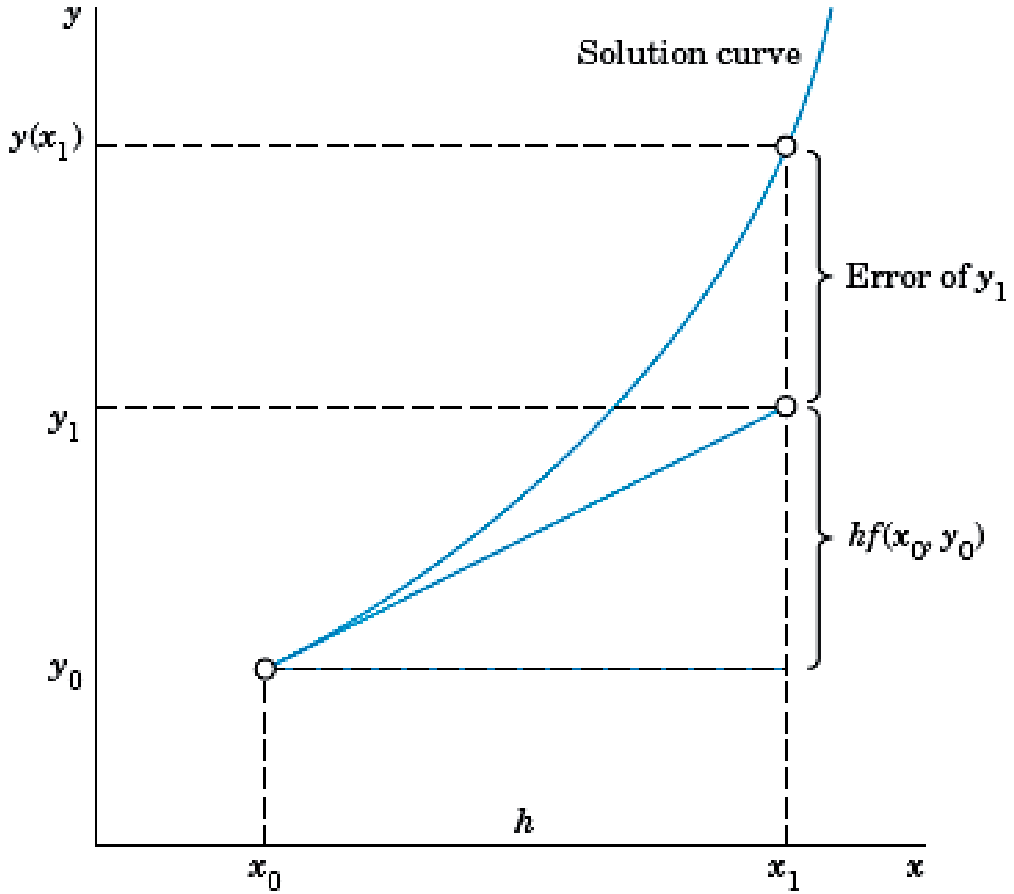
2- Example 2



ODE : Euler-Cauchy Method

3- Error

$$y(x+h) = y(x) + \frac{h}{1!}y'(x) + \frac{h^2}{2!}y''(x) + \ldots + \frac{h^n}{n!}y^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi)$$



1- Round-off Error

2- Truncation Error

Local

$$\frac{f'(x_i, y_i)}{2!}h^2 + \cdots + O(h^{n+1})$$

Global

Remark !

Euler - Cauchy is a first order method

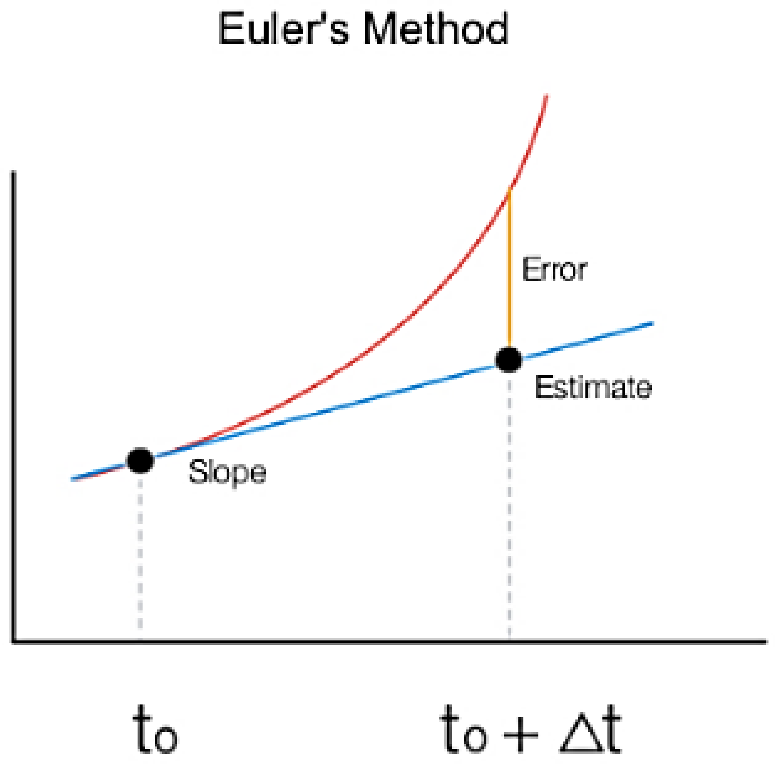
$O_E(h)$ ↳ 1st order

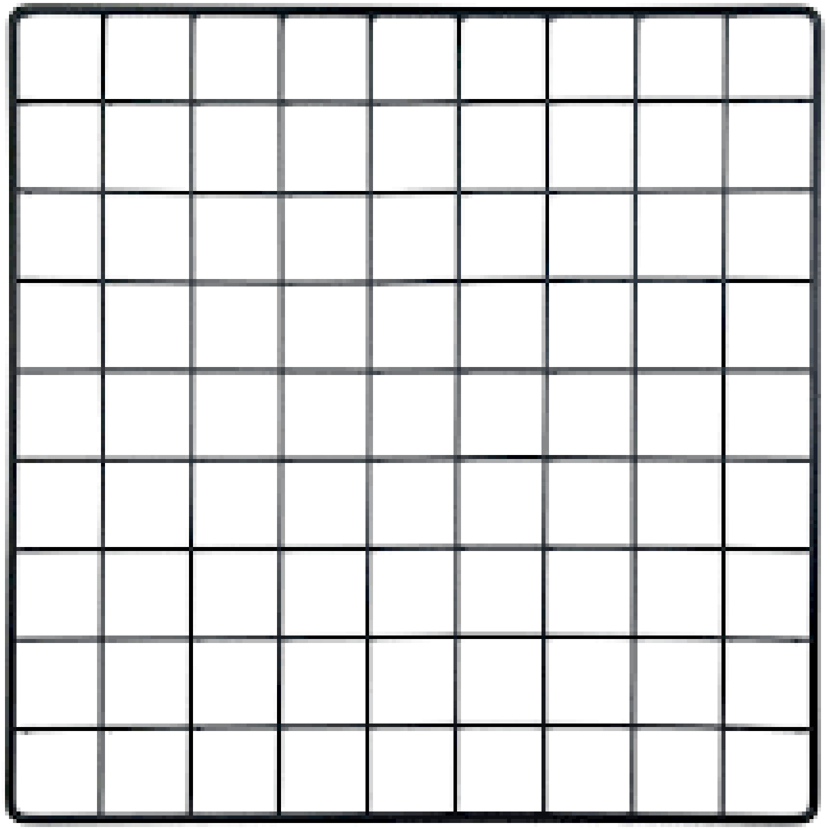
The error between the y_{exact} and y_{euler} is considerably large
And decreasing the step size h   n ↑↑
to reach an acceptable approximate y,
will make h very small leading to many operations

Improvements in Euler Method

Improvements in Euler Method

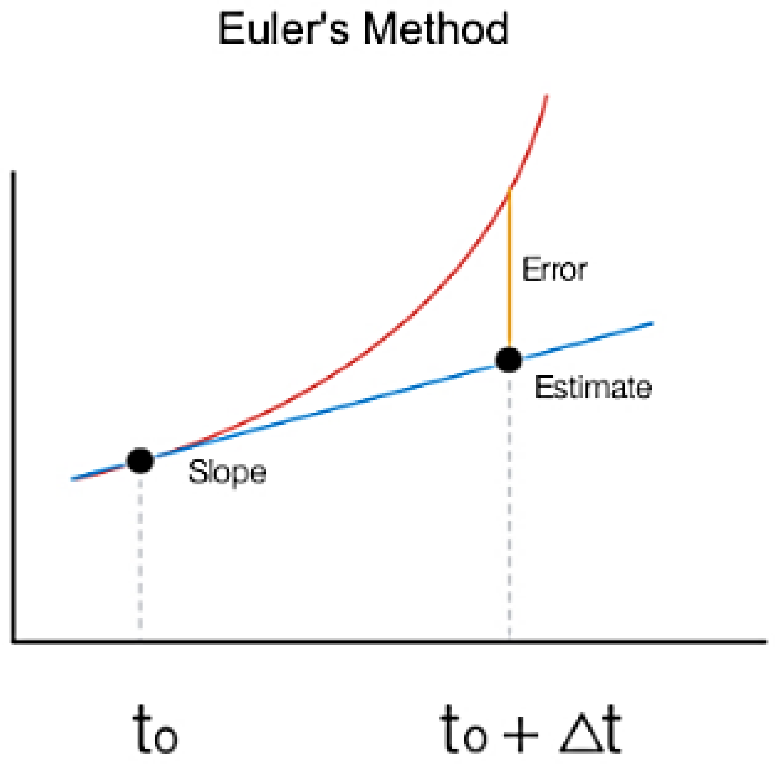
Improved Euler

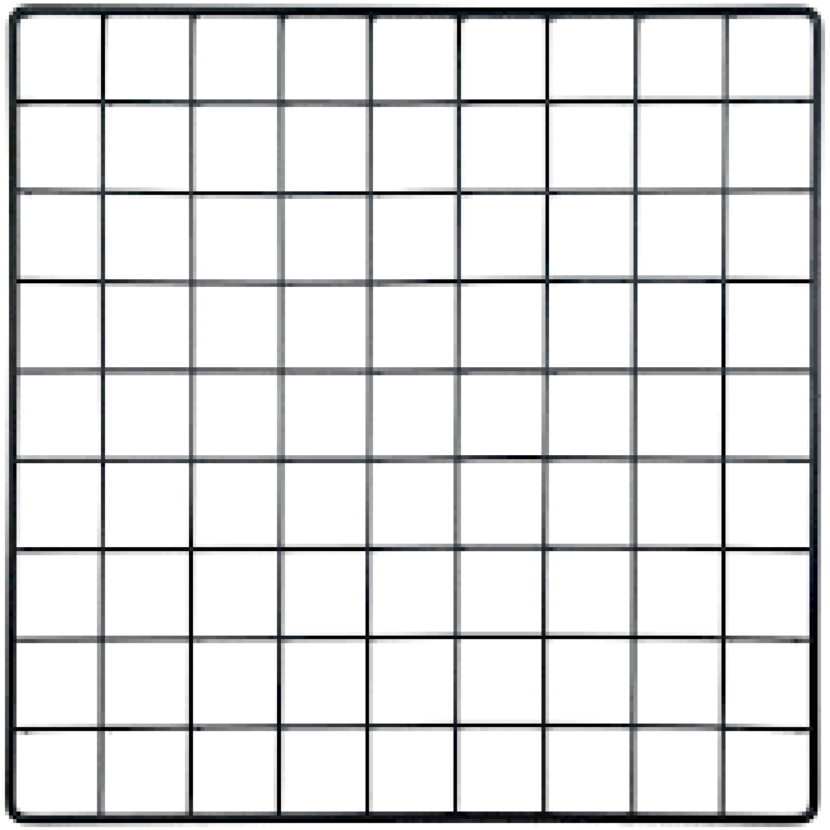




Improvements in Euler Method

Runge-Kutta 4





Improvements in Euler Method

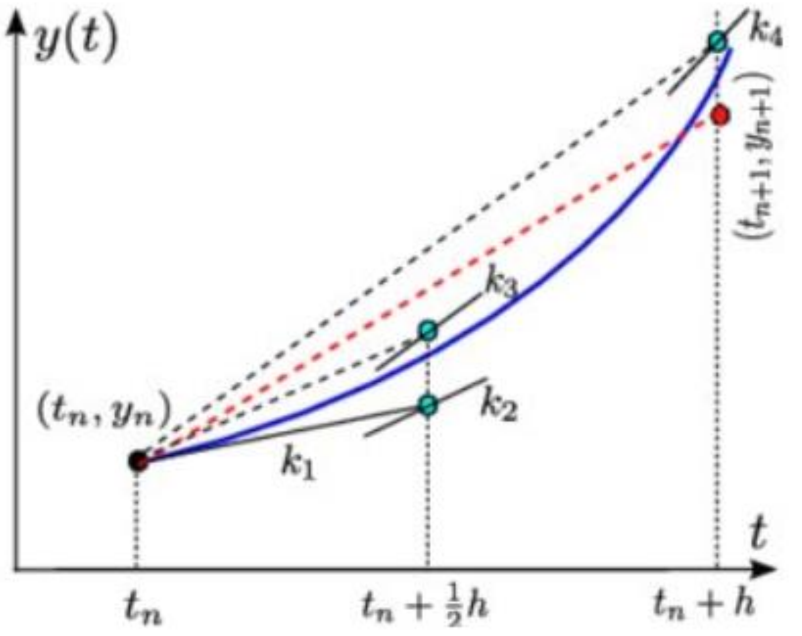
Improved Euler

Runge-Kutta 4

1- Method

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0,$$



Calculate

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

Remark

$$y_{n+1} = y_n + \Phi$$

2- Example 1

Apply Runge-Kutta 4 method with h = 0.1 to find the solution of the initial-value problem

$$y' = 1 - x + 4y,\ y(0) = 1$$

Find y(0.2).

ODE : Runge-Kutta Method

2- Example 1

Take $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

Calculate $k_1 = h\,f\,(x_0, y_0)$

Calculate $k_2 = h\,f\,(x_0 + ½\,h,\ y_0 + ½\,w_1)$

Calculate $k_3 = h\,f\,(x_0 + ½\,h,\ y_0 + ½\,w_2)$

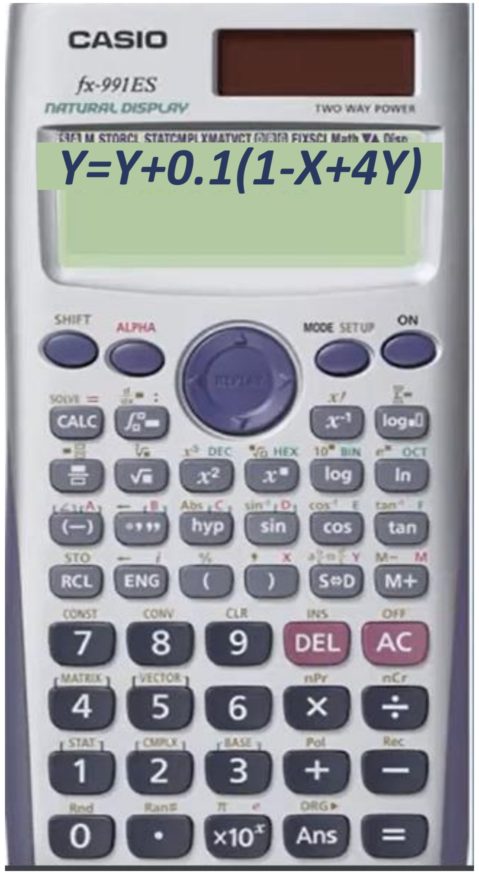
Calculate $k_4 = h\,f\,(x_0 + h,\ y_0 + w_3)$

Calculate $\Delta y = k_1 + 2\,k_2 + 2\,k_3 + k_4$

Calculate $y_1 = y\,(x_0 + h) = y\,(0 + 0.1) = y\,(0.1) = y(0) + 1/6\,\Delta y$

n	x	y	$k = h\,(1 - x + 4\,y)$	Δ y
0				
			sum	

$y\,(0.1) = 1.00000 + (1/6)\,(3.65360) = 1.60893$



ODE : Runge-Kutta Method

2- Example 1

Take $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

Calculate $k_1 = h\, f\,(x_0, y_0)$

Calculate $k_2 = h\, f\,(x_0 + ½\, h,\ y_0 + ½\, w_1)$

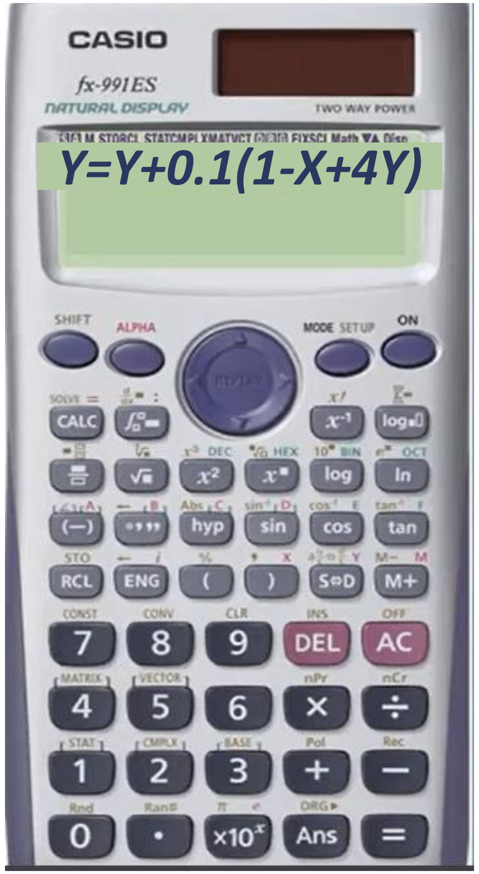
Calculate $k_3 = h\, f\,(x_0 + ½\, h,\ y_0 + ½\, w_2)$

Calculate $k_4 = h\, f\,(x_0 + h,\ y_0 + w_3)$

Calculate $\Delta\, y = k_1 + 2\, k_2 + 2\, k_3 + k_4$

Calculate $y_1 = y\,(x_0 + h) = y\,(0 + 0.1) = y\,(0.1) = y(0) + 1/6\,\Delta\, y$

n	x	y	W = h (1 – x + 4 y)	Δ y
0	0.00	1.00000	0.500000	0.50000
	0.05	1.25000	0.595000	1.19000
	0.05	1.29750	0.614000	1.22800
	0.10	1.61400	0.735600	0.73560
			sum	3.65360



y (0.1) = 1.00000 + (1/6) (3.65360) = 1.60893

ODE : Runge-Kutta Method

2- Example 1

Take $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

Calculate $k_1 = h\, f\,(x_0, y_0)$

Calculate $k_2 = h\, f\,(x_0 + ½\, h,\; y_0 + ½\, w_1)$

Calculate $k_3 = h\, f\,(x_0 + ½\, h,\; y_0 + ½\, w_2)$

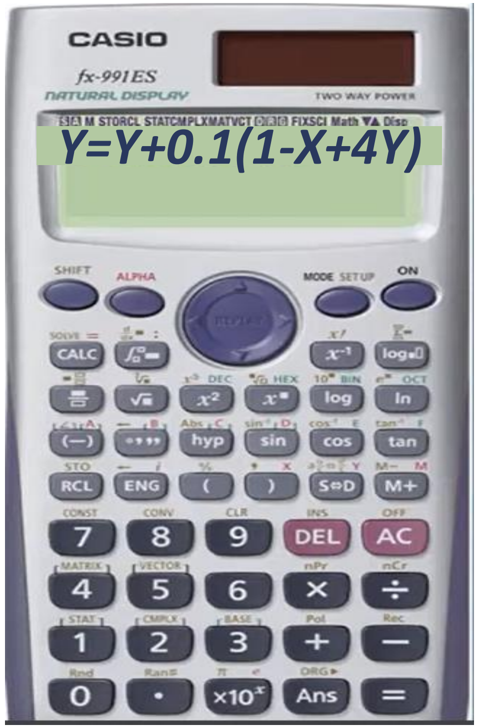
Calculate $k_4 = h\, f\,(x_0 + h,\; y_0 + w_3)$

Calculate $\Delta\, y = k_1 + 2\, k_2 + 2\, k_3 + k_4$

Calculate $y_1 = y\,(x_0 + h) = y\,(0 + 0.1) = y\,(0.1) = y(0) + 1/6\, \Delta\, y$

n	x	y	W = h (1 – x + 4 y)	Δ y
1	0.10	1.60893	0.733572	0.733572
	0.15	1.97572	0.875288	1.75058
	0.15	2.04657	0.903628	1.80726
	0.20	2.51250	1.08502	1.08502
			sum	5.37643

$y\,(0.2) = 1.60893 + (1/6)\,(5.37643) = 2.50500$



ODE : Runge-Kutta Method

3- Error

Method	Function Evaluation per Step	Global Error	Local Error
Euler	1	$O(h)$	$O(h^2)$
Improved Euler	2	$O(h^2)$	$O(h^3)$
RK (fourth order)	4	$O(h^4)$	$O(h^5)$

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- Higher Order ODE Numerical Methods

Next time ☺

The second order ordinary differential equation

IVP

$$x'' = f(t, x, x') \qquad x(t_0) = a, x'(t_0) = b$$

Let $x' = y$ **Then.**

$$x' = y \qquad , x(t_0) = a$$

$$y' = f(t, x, y) \quad , \; y(t_0) = b$$

2 First Order Differential Eq.

2 First Order Differential Eq. solved using RK4

$$x' = y \qquad , x(t_0) = a \qquad\qquad y' = f(t, x, y) \quad , \; y(t_0) = b$$

$$v_1 = h\, f(t_n, x_n, y_n),$$

$$v_2 = h\, f(t_n + \frac{h}{2}, x_n + \frac{v_1}{2}, y_n + \frac{w_1}{2}),$$

$$v_3 = h\, f(t_n + \frac{h}{2}, x_n + \frac{v_2}{2}, y_n + \frac{w_2}{2})$$

$$v_4 = h\, f(t_n + h, x_n + v_3, y_n + w_3)$$

$$w_1 = h\, g(t_n, x_n, y_n)$$

$$w_2 = h\, g(t_n + \frac{h}{2}, x_n + \frac{v_1}{2}, y_n + \frac{w_1}{2})$$

$$w_3 = h\, g(t_n + \frac{h}{2}, x_n + \frac{v_2}{2}, y_n + \frac{w_2}{2})$$

$$w_4 = h\, g(t_n + h, x_n + v_3, y_n + w_3)$$

$$x_{n+1} = x_n + \tfrac{1}{6}\Delta x = x_n + \tfrac{1}{6}(v_1 + 2v_2 + 2v_3 + v_4)$$

$$y_{n+1} = y_n + \tfrac{1}{6}\Delta y = y_n + \tfrac{1}{6}(w_1 + 2w_2 + 2w_3 + w_4)$$

2 First Order Differential Eq. solved using RK4

n	t_n	x_n	y_n	$v = h\,f(x_n, y_n)$	$w = h\,g(x_n, y_n)$	Δx	Δy
0	t_0	x_0	y_0	v_1	w_1	v_1	w_1
1	$t_0 + h/2$	$x_0 + v_1/2$	$y_0 + w_1/2$	v_2	w_2	$2v_2$	$2w_2$
2	$t_0 + h/2$	$x_0 + v_2/2$	$y_0 + w_2/2$	v_3	w_3	$2v_3$	$2w_3$
3	$t_0 + h$	$x_0 + v_3$	$y_0 + w_3$	v_4	w_4	v_4	w_4
					sum	Sum (1)	Sum (2)

Numerical Analysis: ODE

Example

Use Runge – Kutta method with step size **h = 0.25** to obtain **x (0.5)** with 5 decimal places for the IVP

$$x'' - t\,x' - x = 0,\; x(0) = 0,\; x'(0) = 1$$

Solution

Assuming $x' = y$. Therefore $x'' = y'$. The equivalent system is

$x' = y = f(t, x, y)$, $y' = x + t\,y = g(t, x, y)$ such that $x(0) = 0$, $y(0) = 1$.

Numerical Analysis: ODE

$x' = y = f(t, x, y), \ y' = x + t\,y = g(t, x, y)$ such that $x(0) = 0, y(0) = 1.$

n	t	x	y	v	w	Δx	Δy
0	0.000						
	0.125						
	0.125						
	0.25						

$x(0.25) = x(0) + (1/6)(\Delta x) = 0 + (1/6) * (1.53154) = 0.25526$

$y(0.25) = y(0) + (1/6)(\Delta y) = 1 + (1/6) * (0.38289) = 1.0638$

Numerical Analysis: ODE

$x' = y = f(t, x, y), \; y' = x + t\,y = g(t, x, y)$ such that $x(0) = 0, y(0) = 1.$

n	t	x	y	v	w	Δx	Δy
0	0.250						
	0.375						
	0.375						
	0.5						

$X(0.5) = x(0.250) + (1/6)(\Delta x) = 0.25526 + (1/6) * (1.73118) = 0.54379$

$y(0.5) = y(0.250) + (1/6)(\Delta y) = 1.0638 + (1/6) * (1.24878) = 1.2719$

Thank you ☺