Total Marks: 60 Marks 1/2

Important Rules:	تعليمات هامة
• Having a mobile, Smart Watch or earphones inside the examination hall is forbidden and is considered as a cheating behavior. • It is forbidden to have any references, notes, books, or any other materials even if it is not related to the exam content with you in the examination hall. • It is not allowable to use programable or graphical calculators.	• حيازة (المحمول- الساعات الذكية - سماعة الأذن) داخل لجنة الامتحان يعتبر حالة غش تستوجب العقاب. • لا يسمح بدخول أي كتب أو ملازم أو أوراق داخل اللجنة والمخالفة تعتبر حالة غش • ممنوع استخدام الآلات الحاسبة المبرمجة و التي تستطيع الرسم

Question (1): (12 Marks)

(A) Solve in terms of the Gamma function $\int_0^1 \sqrt[n]{1-x^m}\;dx$ where n and m are positive integers.

$$= \frac{1}{m}\cdot\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{n}+1\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{n}+1\right)}$$

Hence, use it to evaluate $\int_0^1 \sqrt{1-x^2}\;dx$.

$\frac{\pi}{4}$

[6 Marks]

(B) By two different methods (one of them is by using the Gamma function) evaluate

$$\int_0^{\pi/2} \sin^4 x\ \cos^5 x\ dx = \frac{8}{315}$$

$\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x\, dx$

$2x-1=4$

$x=5/2$

$2y-1=5 \Rightarrow y=3$

[6 Marks]

$\cos x = e^{-t}$

Question (2): (16 Marks)

(A) Find two linearly independent solutions in powers of " x " for the following differential equation:

$$2x^2 y'' + 3x\,y' - (1+x^2)\,y = 0$$

) **Use $e^x = z$ to solve in terms of Bessel functions the following differential equation:**

$$y'' + (3e^{2x} - 4)\, y = 0.$$

[5 Marks]

Subject : ....

(A) $\int_0^1 \sqrt[n]{1-x^m}\, dx$

let $x^m = t \Rightarrow x = t^{1/m} \Rightarrow dx = \frac{1}{m} t^{\frac{1}{m}-1}\, dt$

$$I = \int_0^1 \frac{1}{m}(1-t)^{\frac{1}{n}} t^{\frac{1}{m}-1}\, dt = \frac{1}{m}\beta\left(\frac{1}{m}, \frac{1}{n}+1\right)$$

$$= \frac{1}{m}\, \frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{n}+1\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{n}+1\right)}$$

$\int_0^1 \sqrt{1-x^2}\, dx \; : - m=2, n=2$

$$\therefore I = \frac{1}{2}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)} = \frac{1}{2}\, \frac{\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\pi}{4}$$

(B) $\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx$

First method: $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta$

$$\therefore I = \frac{1}{2}\beta\left(\frac{5}{2}, 3\right) = \frac{1}{2}\, \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{11}{2}\right)} = \frac{1}{2}\, \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)2\Gamma(2)}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{8}{315}$$

Second method: walli's Rule

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots 2.1}$$

GHARIB

[If m, n are both odd +ve integers or one odd +ve integer] c $m = 4, \ n = 5$

$$I = \frac{(4-1)(4-3)(5-1)(5-3)}{9(9-2)(9-4)(9-6)(9-8)} = \frac{\cancel{(3)}(1)(4)(2)}{9(7)(5)\cancel{(3)}(1)}$$

$$= \frac{8}{315}$$

Subject : .... / /

Q2

(A) $2x^2y'' + 3xy' - (1+x^2)y = 0$

$P(x) = \frac{3x}{2x^2} = \frac{3}{2}\frac{1}{x}$

$q(x) = \frac{-(1+x^2)}{2x^2}$

$\therefore x_0 = 0$ is regular singular Point

$y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$

$$\sum_{n=0}^{\infty} 2a_n (n+s)(n+s-1) x^{n+s} + \sum_{n=0}^{\infty} 3a_n (n+s) x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+s)^2 + (n+s) - 1\right] a_n x^{n+s} - \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0$$

$$(2s^2+s-1) a_0 x^s + (2(1+s)^2+s) a_1 x^{s+1} + \sum_{n=2}^{\infty} \left[\left[2(n+s)^2 + (n+s) - 1\right] a_n - a_{n-2}\right] x^{n+s}$$

Coeff. of $x^s = 0$ $\quad (2s^2+s-1) = 0 \Rightarrow s_1 = \frac{1}{2}, \; s_2 = -1$ case ①

Coeff. of $x^{s+1} = 0$ $\quad 5a_1(\frac{1}{2}) = -a_1(-1) = 0$

Coeff. of $x^{n+s} = 0$ $\quad \boxed{a_n = \frac{1}{(n+s-\frac{1}{2})(n+s+1)} a_{n-2} \quad n \geq 2}$

$$a_2 = \frac{1}{(s+\frac{3}{2})(s+3)} a_0$$

$$a_4 = \frac{1}{(s+\frac{7}{2})(s+5)} a_2 = \frac{1}{(s+\frac{3}{2})(s+\frac{7}{2})(s+3)(s+5)} a_0$$

$$a_6 = \frac{1}{(s+\frac{3}{2})(s+\frac{7}{2})(s+\frac{11}{2})(s+3)(s+5)(s+7)} a_0$$

$$a_{2K} = \frac{1}{\left[(s+\frac{3}{2})(s+\frac{7}{2}) \cdots (s+n-\frac{1}{2})\right]\left[(s+3)(s+5) \cdots (s+n+1)\right]} a_0$$

$K \geq 1$

For $S_1 = \frac{1}{2} \Rightarrow a_{2K}(\frac{1}{2}) = \frac{a_0}{[2 \cdot 4 \cdot 6 \cdots (2K)][\frac{7}{2} \cdot \frac{11}{2} \cdot \frac{15}{2} \cdots (2K+\frac{3}{2})]}$

For $S_2 = -1 \Rightarrow a_{2K}(-1) = \frac{a_0}{[\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} \cdots (2K-\frac{3}{2})][2 \cdot 4 \cdot 6 \cdots (2K)]}$

$$y_1(x, s_1) = \sum_{n=0}^{\infty} a_{2K}(\tfrac{1}{2}) x^{n+\frac{1}{2}}$$

$$y_2(x, s_2) = \sum_{n=0}^{\infty} a_{2K}(-1) x^{n-1}$$

$$y_{gs} = c_1 y_1 + c_2 y_2$$

$$= c_1 \sqrt{x} \sum_{n=0}^{\infty} a_{2K}(\tfrac{1}{2}) x^n + \frac{c_2}{x} \sum_{n=0}^{\infty} a_{2K}(-1) x^n$$

Subject : .... / /

use $e^x = z$ to solve interms of Bessel functions the following differential equations:

$$y'' + (3e^{2x} - 4)y = 0$$

let $e^x = z \Rightarrow \frac{dz}{dx} = e^x = z$

$y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = z \frac{dy}{dz}$

$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(z \frac{dy}{dz}\right) \frac{dz}{dx} = z\left[\frac{dy}{dz} + z \frac{d^2y}{dz^2}\right]$

$= z \frac{dy}{dz} + z^2 \frac{d^2y}{dz^2}$

$\therefore z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (3z^2 - 4)y = 0$

$\lambda = \sqrt{3}$, $v = \sqrt{4} = 2$ Positive integer number

$\therefore y = c_1 J_2(\sqrt{3}z) + c_2 Y_2(\sqrt{3}z)$

$= c_1 J_2(\sqrt{3}e^x) + c_2 Y_2(\sqrt{3}e^x)$