Juba

$$\boxed{J_{-N}(x) = (-1)^N J_N(x)}$$

$$\rightarrow J_{-N}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, x^{2m-N}}{2^{2m-N} \, m! \, \Gamma(m-N+1)}$$

as m & N are two integer values

$\therefore$ $m - N + 1 \geq 1$ (to avoid undefined values)

then $m \geq N$

$$\therefore J_{-N}(x) = \sum_{m=N}^{\infty} \frac{(-1)^m \, x^{2m-N}}{2^{2m-N} \, m! \, \Gamma(m-N+1)}$$

$\rightarrow$ let $m - N = k$ $\therefore$ $m = k + N$

$$\therefore J_{-N}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+N} \, x^{k+m}}{2^{k+m} \, (k+N)! \, \Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^{k+N} \, x^{2k+N}}{2^{2k+N} \, (k+N)! \, \Gamma(k+1)}$$

$\therefore$ $(k+N)! = \Gamma(k+N+1)$ & $\Gamma(k+1) = k!$

$$\rightarrow J_{-N}(x) = \sum_{k=0}^{\infty} \frac{(-1)^N (-1)^k \, x^{2k+N}}{2^{2k+N} \, k! \, \Gamma(k+N+1)} = (-1)^N \sum_{k=0}^{\infty} \frac{(-1)^k \, x^{2k+N}}{2^{2k+N} \, k! \, \Gamma(k+N+1)}$$

$$= (-1)^N J_N(x)$$

Prove that : Jn(x) & J-n(x) are linearly dependent

$\boxed{J_N(-x) = (-1)^N J_N(x)}$

$\because$ $$J_N(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, (x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)}$$ $\because$ x is raised to power $2m+N$

So, when N is an even number, the series involves even powers of x only therefore $J_N(x)$ is an even function of x.

& when N is an odd number, the series involves odd powers of x only therefore $J_N(x)$ is an odd function of x.

Another Method :

"that could be wrong"

$$\rightarrow J_N(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, (-x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)} = \sum_{m=0}^{\infty} \frac{(-1)^m \, (-1)^{2m+N} \, (x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{3m+N} \, (x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)} = (-1)^N \sum_{m=0}^{\infty} \frac{(-1)^{3m} \, (x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)}$$

$$= (-1)^N \sum_{m=0}^{\infty} \frac{((-1)^3)^m \, (x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)} = (-1)^N \sum_{m=0}^{\infty} \frac{(-1)^m \, (x)^{2m+N}}{2^{2m+N} \, m! \, \Gamma(m+N+1)}$$

$$= (-1)^N J_n(x) \quad \#$$

Prove that : Jn(x) is an Odd Function when n is odd & it is an Even Function when n is even.

$$\boxed{J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x}$$

* by putting $x = m+1$ $\longrightarrow$ $\therefore\ \sqrt{\pi}\,\Gamma(2m+2) = 2^{2m+1}\,\Gamma(m+1)\,\Gamma(m+3/2)$

$$\rightarrow J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}\, m!\, \Gamma(m+3/2)} = \sum_{m=0}^{\infty} \frac{(-1)^m\, (x)^{-1/2}\, (x)^{2m+1}}{(2)^{-1/2}\, 2^{2m+1}\, \Gamma(m+1)\, \Gamma(m+3/2)}$$

$$= \sum_{m=0}^{\infty} \sqrt{\frac{2}{x}}\, \frac{(-1)^m\, x^{2m+1}}{\sqrt{\pi}\, \Gamma(2m+2)} = \sqrt{\frac{2}{x\pi}}\, \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m+1}}{(2m+1)!}}$$

$$= \sqrt{\frac{2}{\pi x}}\, \sin x$$

$$\boxed{\&\quad J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x}$$

$$\rightarrow J_{-1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m-\frac{1}{2}}}{2^{2m-\frac{1}{2}}\, m!\, \Gamma(m+1/2)} = \sum_{m=0}^{\infty} \frac{(-1)^m\, (x)^{-1/2}\, (x)^{2m}}{(2)^{1/2}\, 2^{2m-1}\, m\, \Gamma(m)\, \Gamma(m+1/2)}$$

$$= \sum_{m=0}^{\infty} \sqrt{\frac{1}{2x}}\, \frac{(-1)^m\, (x)^{2m}}{m\sqrt{\pi}\, \Gamma(2m)} = \sqrt{\frac{4}{2\pi x}}\, \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2m\, (2m-1)!}$$

$$= \sqrt{\frac{2}{x\pi}}\, \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{(2m)!} = \sqrt{\frac{2}{\pi x}}\, \cos x$$

$$\frac{d}{dx}\left(x^n\ J_n(x)\right) = x^n\ J_{n-1}(x)$$

$$\rightarrow \frac{d}{dx}\left(x^n J_n(x)\right) = \frac{d}{dx}\left(x^n \sum_{k=0}^{\infty} \frac{(-1)^k\ x^{2k+n}}{2^{2k+n}\ k!\ \Gamma(k+n+1)}\right)$$

$$= \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k\ x^{2k+2n}}{2^{2k+n}\ k!\ \Gamma(k+n+1)}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(2k+2n)\ (-1)^k\ x^{2k+2n-1}}{2^{2k+n}\ k!\ \Gamma(k+n+1)}$$

$$= \sum_{k=0}^{\infty} \frac{2\ \cancel{(k+n)}\ (-1)^k\ x^n\ x^{2k+(n-1)}}{2^{2k+n}\ k!\ \cancel{(k+n)}\ \Gamma(k+n)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k\ x^{2k+(n-1)}}{2^{2k+(n-1)}\ k!\ \Gamma(k+(n-1)+1)}$$

$$= x^n\ J_{n-1}(x)$$

$$\boxed{\frac{d}{dx}\left(x^{-n}\,J_n(x)\right) = -x^{-n}\,J_{n+1}(x) \qquad (II)}$$

$$\rightarrow \frac{d}{dx}\left(x^{-n} J_n(x)\right) = \frac{d}{dx}\left(x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k\, x^{2k+n}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}\right)$$

$$= \frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\, x^{2k}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}\right)$$

$$= \sum_{k=0}^{\infty}\frac{(2k)\,(-1)^k\, x^{2k-1}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}$$

$\ast$ we have to shift the index as k must be $k \geqslant 1$

$$= x^{-n}\sum_{k=1}^{\infty}\frac{\cancel{(k)}\,(-1)^k\, x^{2k+n-1}}{2^{2k+n-1}\,\cancel{(k)}\,(k-1)!\,\Gamma(k+n+1)}$$

$\rightarrow$ let $l = k-1$ $\therefore k = l+1$

$$= x^{-n}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}\, x^{2l+n+1}}{2^{2l+n+1}\, l!\,\Gamma(l+n+2)}$$

$$= -x^{-n}\sum_{l=0}^{\infty}\frac{(-1)^{l}\, x^{2l+(n+1)}}{2^{2l+(n+1)}\, l!\,\Gamma(l+(n+1)+1)}$$

$$= -x^{n} J_{n+1}(x)$$

$$\boxed{J_n(x) = \frac{x}{2n}\big(J_{n-1}(x) + J_{n+1}(x)\big) \qquad (III)}$$

$$\boxed{J'_n(x) = \frac{1}{2}\big(J_{n-1}(x) - J_{n+1}(x)\big) \qquad (IV)}$$

$$\rightarrow \frac{d}{dx}\left(x^n J_n(x)\right) = x^n J'_n(x) + (n)\,x^{n-1} J_n(x)$$

$$= x^n J_{n-1}(x) \qquad (* \, x^{-n})$$

$$\therefore \; J_{n-1}(x) = J'_n(x) + \frac{n}{x} J_n(x) \longrightarrow ①$$

$$\rightarrow \frac{d}{dx}\left(x^{-n} J_n(x)\right) = x^{-n} J'_n(x) - n\,x^{n-1} J_n(x)$$

$$= -x^{-n} J_{n+1}(x) \qquad (* \, x^{n})$$

$$\therefore \; -J_{n+1}(x) = J'_n(x) - \frac{n}{x} J_n(x) \longrightarrow ②$$

① + ②

$$\therefore \; J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x)$$

$$\therefore \; J'_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$$

① − ②

$$\therefore \; J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$\therefore \; J_n(x) = \frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right)$$