Section 8

Complex Mapping

exp: Find the image $x+y=1$ under the transformation $w=\dfrac{1}{z}$; Show the region graphically

sol: $w=\dfrac{1}{z} \Rightarrow z=\dfrac{1}{w}$

$$x+iy=\frac{1}{u+iv} * \frac{u-iv}{u-iv}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$$\boxed{x=\frac{u}{u^2+v^2} \qquad y=\frac{-v}{u^2+v^2}}$$

لو كان ده اللي هو عايزه

thus, $u=\dfrac{x}{x^2+y^2}$, $v=\dfrac{-y}{x^2+y^2}$

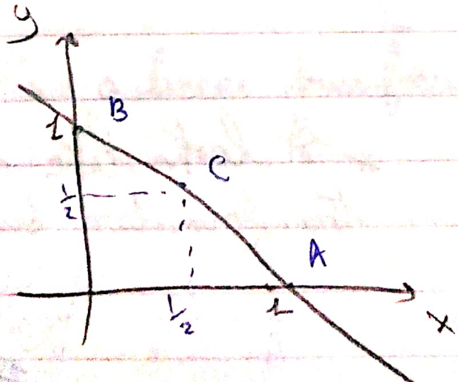
$$\Rightarrow x+y=1 \Rightarrow \left[\frac{u}{u^2+v^2}+\frac{-v}{u^2+v^2}=1\right]*(u^2+v^2)$$

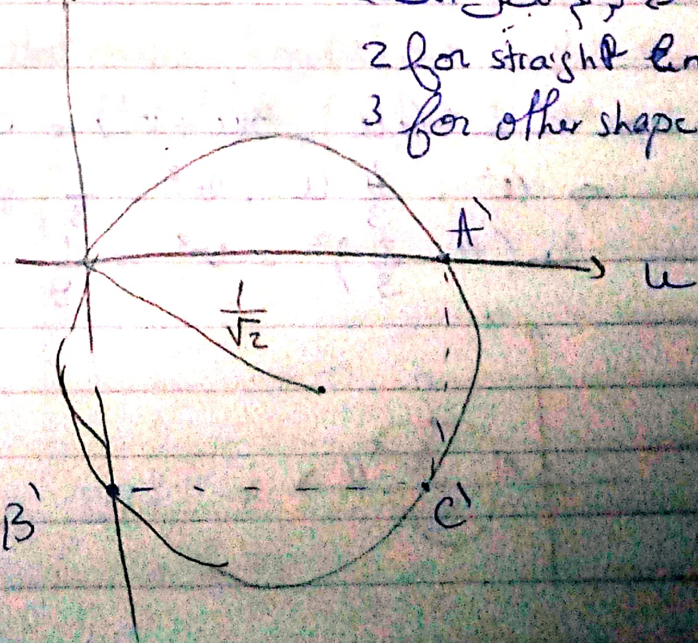
$\hookrightarrow u-v=u^2+v^2$

$u^2+v^2-u+v=0$ (circle/ellipse)

$\hookrightarrow (u-\frac{1}{2})^2-(\frac{1}{2})^2+(v+\frac{1}{2})^2-(\frac{1}{2})^2=0$ (complete square)

$$\Rightarrow (u-\tfrac{1}{2})^2+(v+\tfrac{1}{2})^2=\tfrac{1}{2}$$





لازم نبين النقط
2 for straight line
3 for other shape

A: $z=1$: A': $w=1$

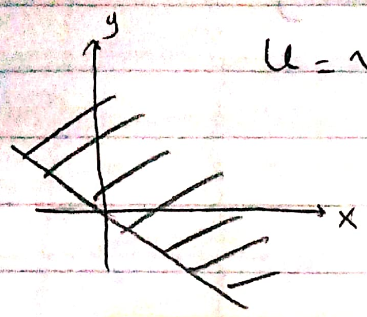
B: $z=i$ B': $w=-i$

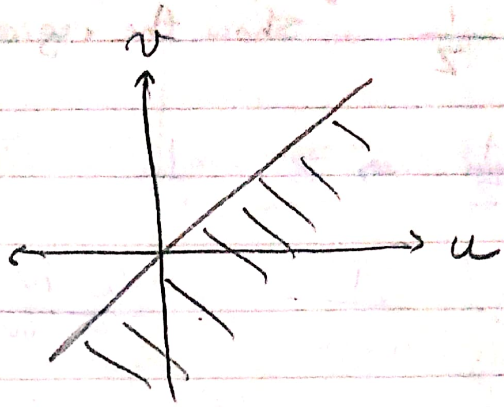
C: $z=\frac{1}{2}+\frac{1}{2}i$ C': $w=1-i$

Exp 2 $x+y \geqslant 0$

$$\frac{u}{u^2+v^2} + \frac{-v}{u^2+v^2} \geqslant 0 \Rightarrow u - v \geqslant 0$$

$u = v$





Exp 3: $|z-2| \leq 1$

$|z - z_0| = r \Rightarrow$ Circle r, z_0

radius = 1 center = 2

Sol: $|(x+iy) - 2| = 1$ $\quad z = \sqrt{x^2+y^2}$

$|(x-2) + iy| = 1$

$(x-2)^2 + y^2 = 1^2$

$\Rightarrow x^2 + y^2 - 4x + 4 = 1$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{4u}{u^2+v^2} + 3 = 0$$

$$\Rightarrow \left[\frac{1}{u^2+v^2} - \frac{4u}{u^2+v^2} + 3 = 0\right] * (u^2+v^2)$$

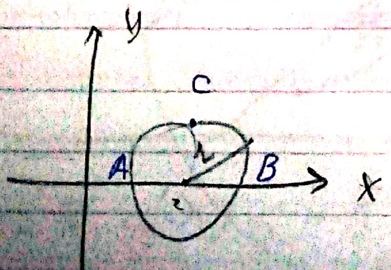
$\Rightarrow 1 - 4u + 3(u^2+v^2) = 0$

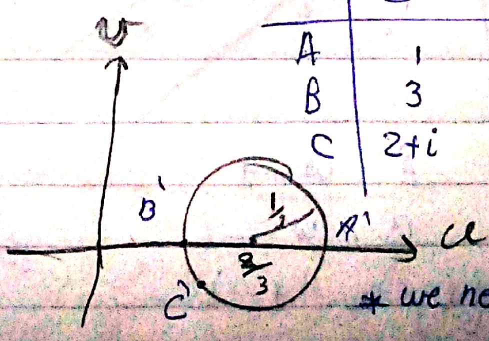
$\Rightarrow u^2 - \frac{4}{3}u + v^2 + \frac{1}{3} = 0$

$\hookrightarrow (u - \frac{2}{3})^2 + v^2 = \frac{1}{9}$

* It looks like linear tranformation but it's not

	z	w
A	1	1
B	3	$\frac{1}{3}$
C	$2+i$	$\frac{2}{5} - \frac{1}{5}i$





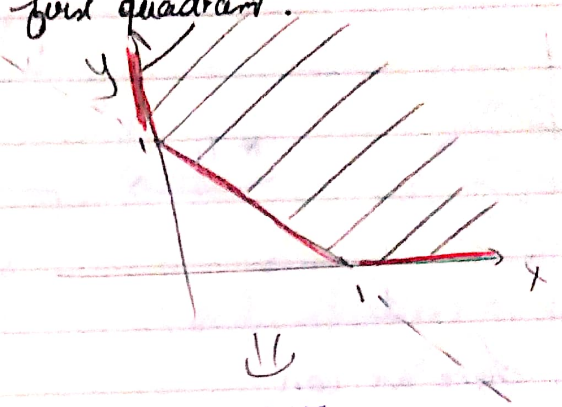
* we need 3 point

NB: 1- Circle → Circle } If the pre image didn't pass by origin
straight line → straight line

2- If the preimage circle passed by origine, we can't take the origin point because $w=\frac{1}{z}$ will lead to infinity and we don't know it's + or − $\therefore$ Better take other points

exp: $x+y \geq 1$ in the first quadrant.

$\omega=\frac{1}{z}$



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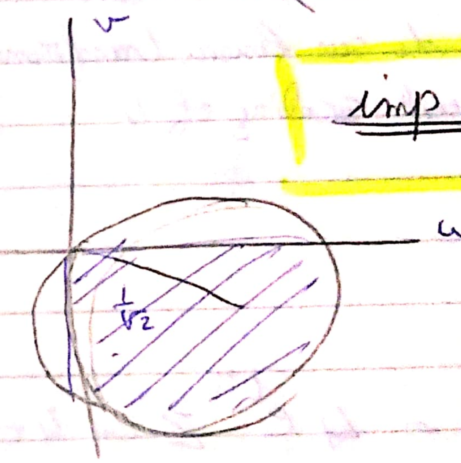


imp

$z=1+i$

$\omega=\frac{1}{2}-\frac{1}{2}i$

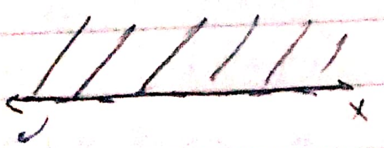
$\therefore$ inside the circle

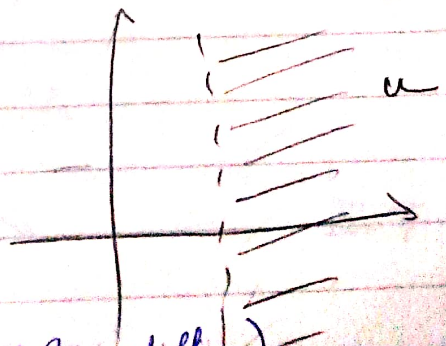
adding that $y \geq 0 \Rightarrow u \geq 0$

$x \geq 0 \Rightarrow v \leq 0$

exp: Find a linear transformation that maps. $Img\{z\} > 0$ $y>0$

$Re\{\omega\} > 1$ $u > 1$





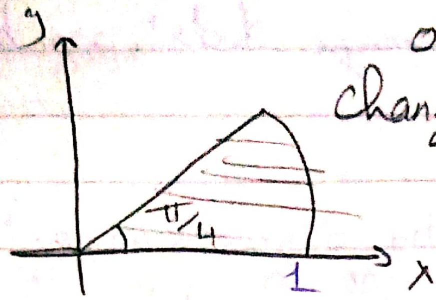
dashed because:
y: It doesn't equal zero

→ let $r=1$ (assume no scaling & no shifting)

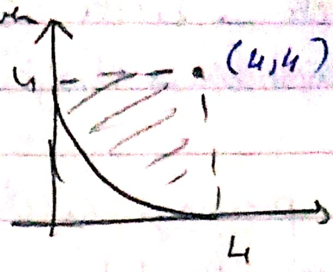
$w = r_a e^{i\theta_a} z + b = 1e^{i(-\frac{\pi}{2})} z + 1 + i\cdot 0$

$w = -i\cdot z + 1$

Exp 6:



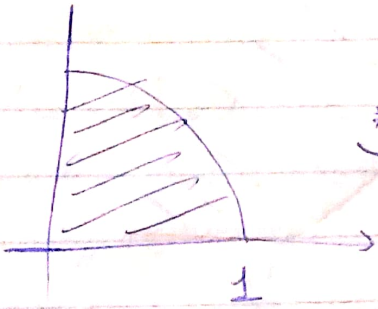
only the power changes $\underline{\underline{\theta}}$ $\Longrightarrow$



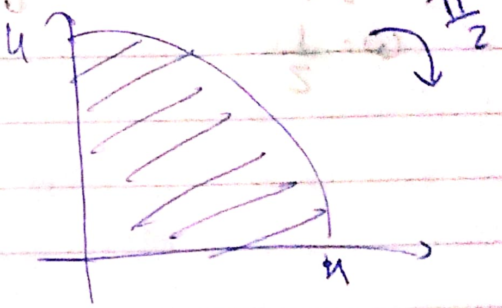
linear tranformation won't change the θ enclosed

$\Downarrow$

If z^2

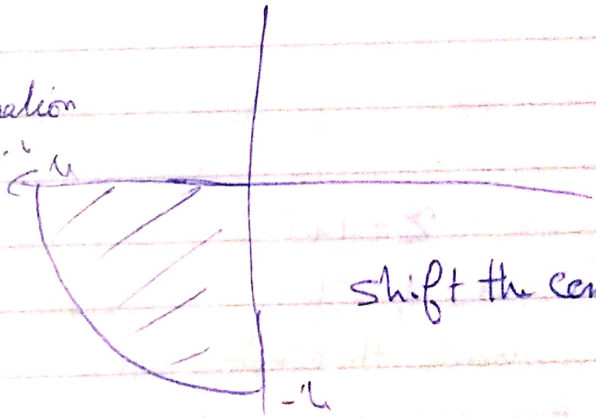


$*4$



from here we can do our linear tranformation

we need scaling of 4



shift the center

$\therefore w = 4e^{i\pi} z^2 + 4 + 4i$

* Differentiability:

Cond 1

u, v are Continuous on z plane

$\boxed{1}$ $\omega = z^2 = (x+iy)^2$
$= (x^2 - y^2) + i(2xy)$

Cond 2

→ Cauch. R eq :

$u_x = 2x$ $\qquad v_y = 2x$

$u_y = -2y$ $\qquad v_x = 2y$

$u_x = v_y$? yes ✓
$u_y = -v_x$? yes ✓

∴ Cauchy-Rieman's eq are satisfied $\forall x, y$

∴ $F(z)$ is differentiable on z plane

$\Rightarrow f'(z) = u_x + i v_x = 2x + i(2y)$ #

$\frac{dw}{dz}$ not dy dx

$\boxed{2}$ $\omega = \frac{1}{z} = \frac{1}{x+iy} * \frac{x-iy}{x-iy}$

$\omega = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$

$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2} \; ----= v_y$

$u_y = \frac{-2xy}{(x^2+y^2)^2} \; ----= -v_x$

Cauch R eq are satisfied → $z \neq 0$

$\boxed{\text{But}}$ on z plane u, v are not Continuous because they are not continuous on (0,0)
Not differentiable or differentiable on plane z except (0,0)

exp 3 : $w = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

u, v are Cont on z plane

$u_x = 3x^2 + 3y^2 - 3$ $\quad V_y = 3y^2 + 3x^2 - 3$

$u_y = 6xy$ $\quad V_x = 6xy$

$u_x = V_y$ ✓

for $u_y = -V_x \Rightarrow 6xy = -6xy$

at $x = 0$ or $y = 0$ only : on axis.

C.R eq are satisfied on x axis & y axis only.

$\Rightarrow F(z)$ is differentiable only x & y axis. #