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الاســـــراء

ملزمة (١١)

رياضـــة

Elementary Transformation

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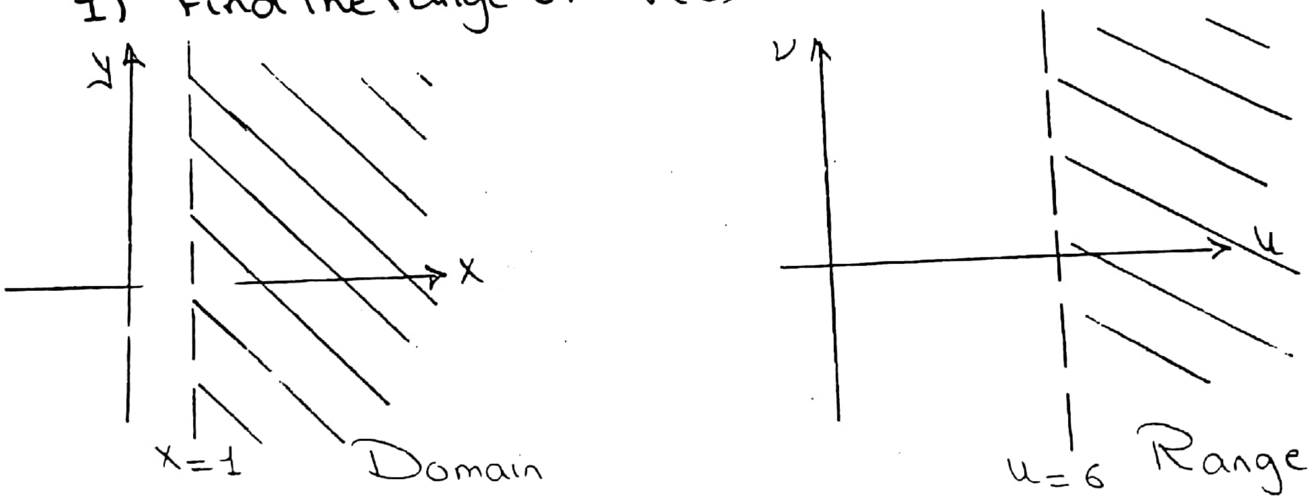
Elementary Transformation -

Mapping or transformation under a fn. $f(z)$ is to obtain the shape of the range (in the u-v plane) for a certain domain (in the x-y plane).

Examples :-

1) Find the range of $f(z) = z+5$ for $\operatorname{Re} z > 1$



or, Since the domain is $x > 1$ & we have

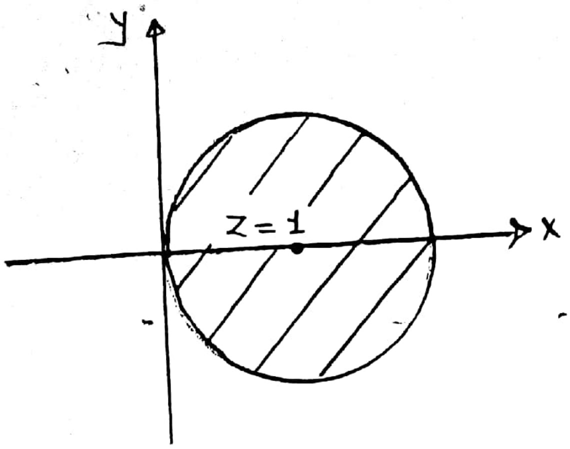
$f(z) = z+5 = x+5+iy \Rightarrow u = x+5$ & $v = y$

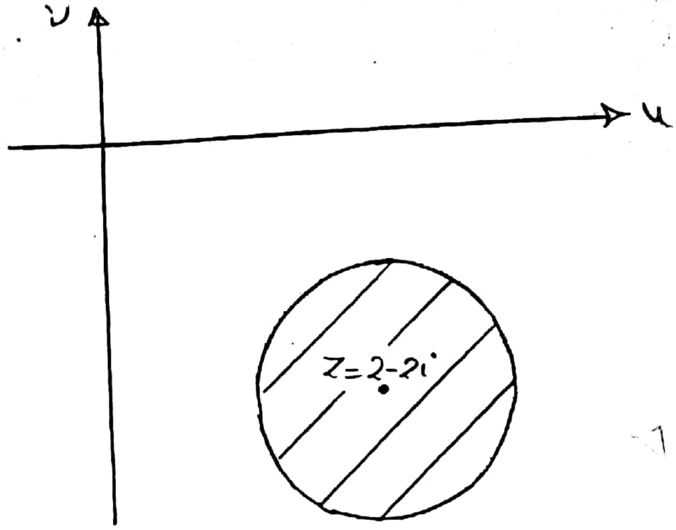
So for $x > 1 \Rightarrow x+5 > 6 \Rightarrow u > 6$ (Range)

2) Find the range of $f(z) = z+1-2i$ for the region $|z-1| \leqslant 1$

we have $f(z) = z+1-2i = (x+1) + i(y-2)$

$\Rightarrow u = x+1$ & $v = y-2$





- So adding a complex no. $b = \alpha + i\beta$ to z represents shifting to the right by α & up by β.

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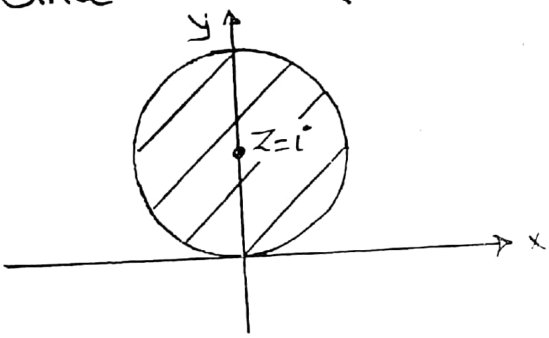
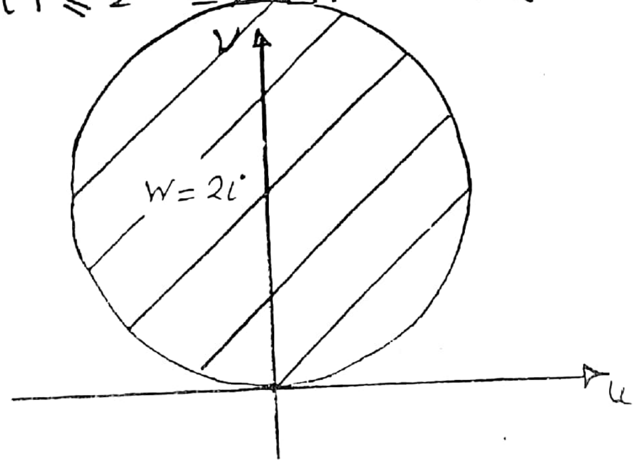
3) determine the region in the w plane into which $|z-i| \leq 1$ is mapped under $f(z) = 2z$.

Since $|z-i| \leq 1 \Rightarrow |2z-2i| \leq 2 \Rightarrow |w-2i| \leq 2$





- observe that a stretching by 2 occures.

or domain is $|z-i| \leq 1 \Rightarrow x^2 + (y-1)^2 \leq 1$

but $f(z) = 2(x+iy) \Rightarrow u = 2x$ & $v = 2y$

$\Rightarrow x^2 + (y-1)^2 \leq 1$ becomes.

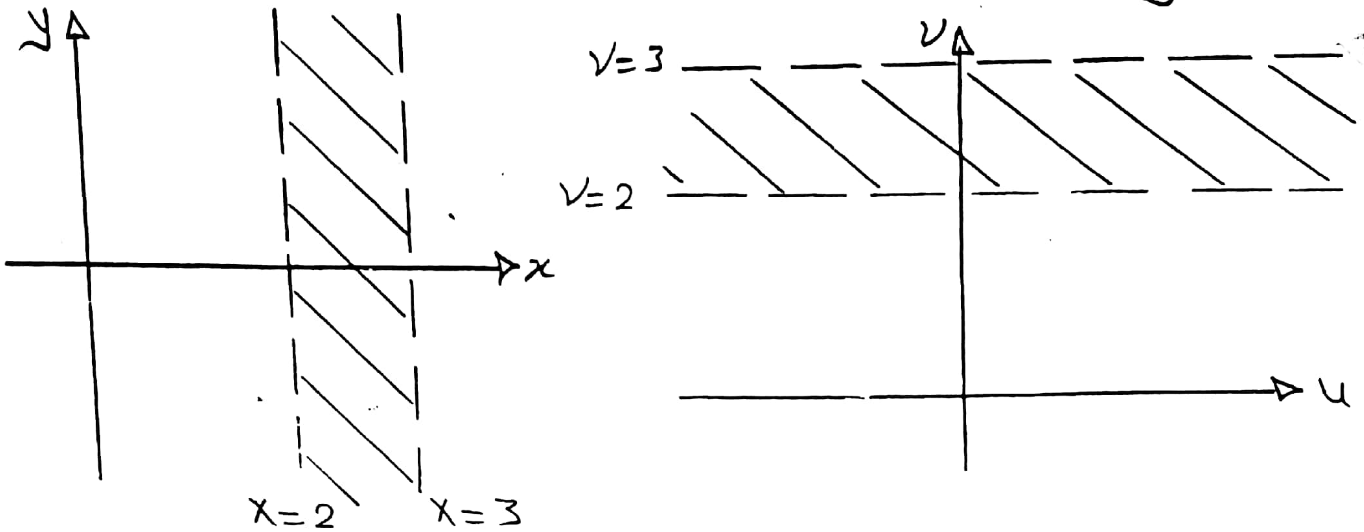
$\frac{u^2}{4} + (\frac{v}{2} - 1)^2 \leq 1 \Rightarrow u^2 + (v-2)^2 \leq 4$

$\Rightarrow |w-2i| \leq 2$

Find the image of $2 < Re\, z < 3$ under $f(z) = iz$

$$f(z) = iz = e^{i\pi/2} \cdot re^{i\theta} = re^{i(\theta + \pi/2)}$$

$\Rightarrow$ $f(z)$ is equivilant to adding an angle of $\pi/2$ to each point in the domain, i.e. rotation by $\pi/2$.



or, Since $f(z) = iz = i(x+iy) = -y + ix \Rightarrow \begin{matrix} u = -y \\ v = x \end{matrix}$

$\Rightarrow$ the image of $x=2$ is $v=2$ & the image of $x=3$ is $v=3$.

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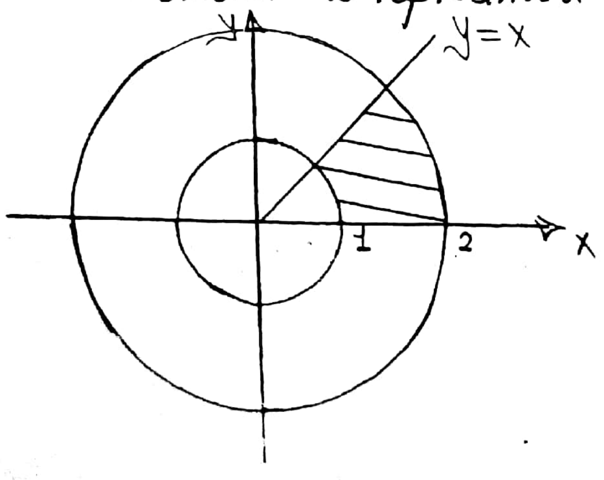
5) Find the image of $1 \leq |z| \leq 2$, $Re\, z \geq Im\, z \geq 0$ under $f(z) = \ln z$

This domain is represented as



$\Rightarrow$ the domain boundaries are $r=1$, $r=2$, $\theta = 0$, $\theta = \pi/4$

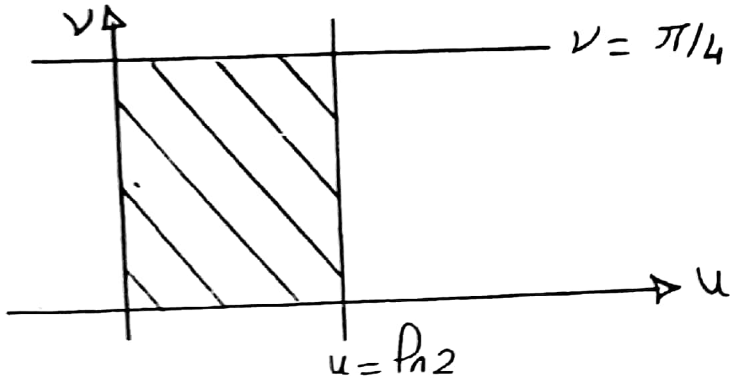
$f(z) = \ln z = \ln re^{i\theta} = \ln r + i\theta = u + iv$

The image of $r = 1 \Rightarrow u = 0$

" " " $r = 2 \Rightarrow u = \ln 2$

" " " $\theta = 0 \Rightarrow v = 0$

" " " $\theta = \pi/4 \Rightarrow v = \pi/4$



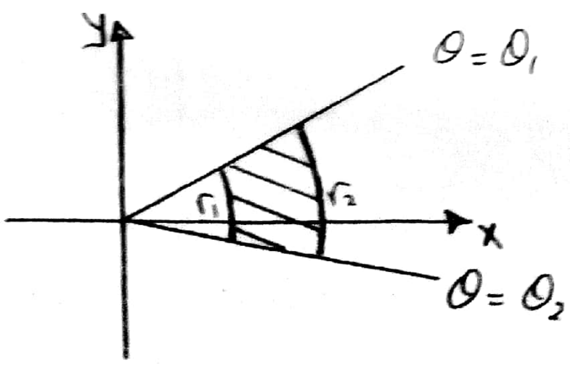
* Elementary Transformations:

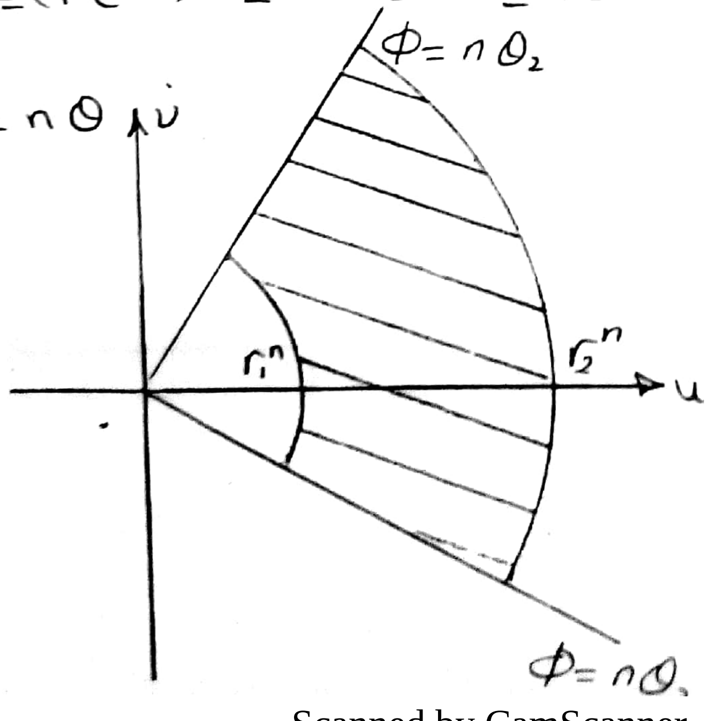
1) $f(z) = z^n$

Under this fn., if the domain is $r_1 < r < r_2$ & $\theta_1 < \theta < \theta_2$
the range is $r_1^n < R < r_2^n$ & $n\theta_1 < \phi < n\theta_2$.

Because we have $f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta} = Re^{i\phi}$

$\Rightarrow R = r^n$ & $\phi = n\theta$





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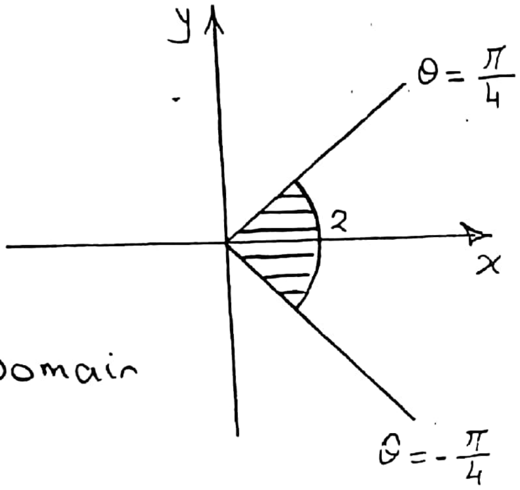
Example :- Find the range of $f(z) = z^3$ for $|\text{Arg}\, z| \leqslant \pi/4$,
$|z| \leqslant 2$.

Solution :- $f(z) = z^3 = (re^{i\theta})^3 = r^3 e^{i3\theta} \Rightarrow \begin{matrix} R = r^3 \\ \phi = 3\theta \end{matrix}$

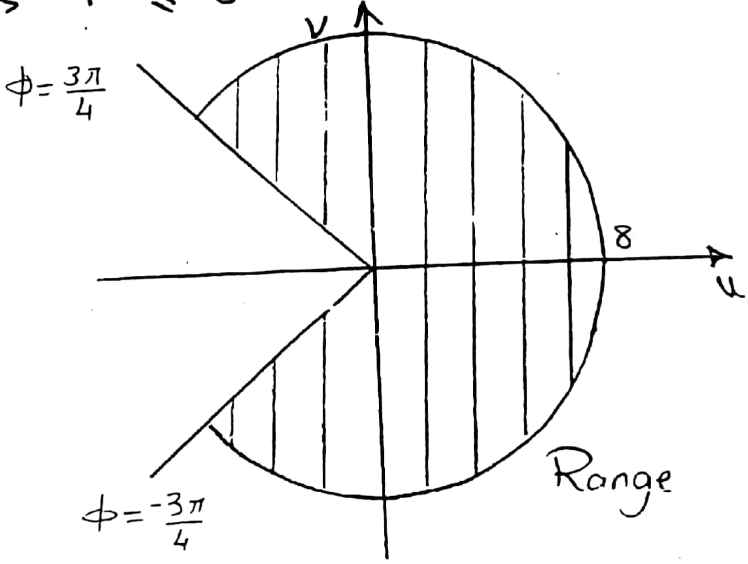
Since $|\text{Arg}(z)| \leqslant \pi/4 \Rightarrow |\theta| \leqslant \pi/4 \Rightarrow -\frac{\pi}{4} \leqslant \theta \leqslant \pi/4$

$\Rightarrow -\frac{3\pi}{4} \leqslant 3\theta \leqslant \frac{3\pi}{4} \Rightarrow -\frac{3\pi}{4} \leqslant \phi \leqslant \frac{3\pi}{4}$

& $|z| \leqslant 2 \Rightarrow r \leqslant 2 \Rightarrow r^3 \leqslant 8 \Rightarrow R \leqslant 8$

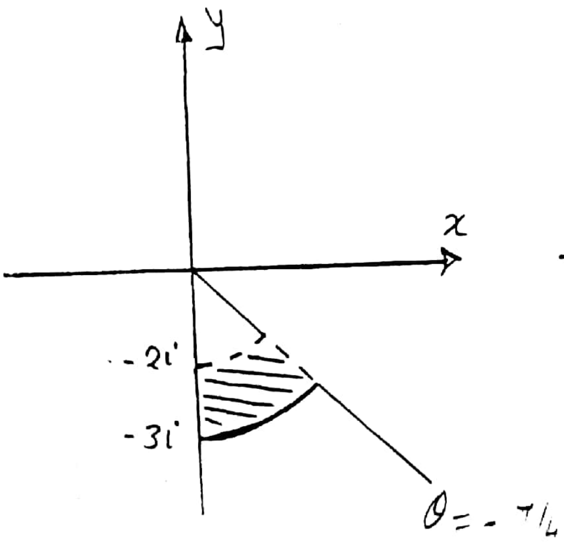


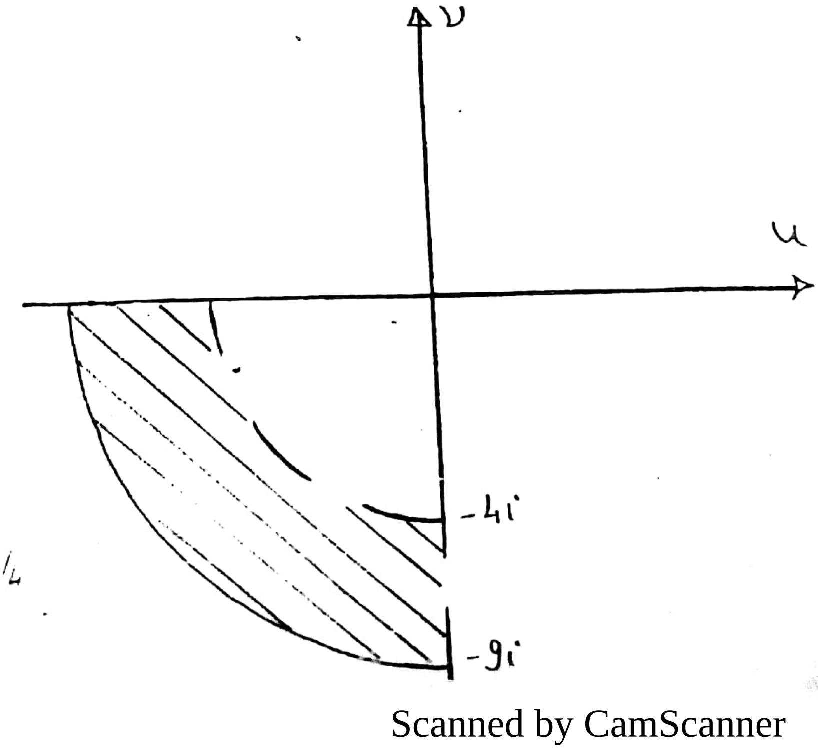
Domain



Range

Example :- Find the image of the region $2 < |z| \leqslant 3$,
$-\frac{\pi}{2} \leqslant \text{Arg}\, z < -\pi/4$ under $f(z) = z^2$





Linear Transformation

$$W = f(z) = az + b \quad ; \quad a \& b \text{ are complex.}$$

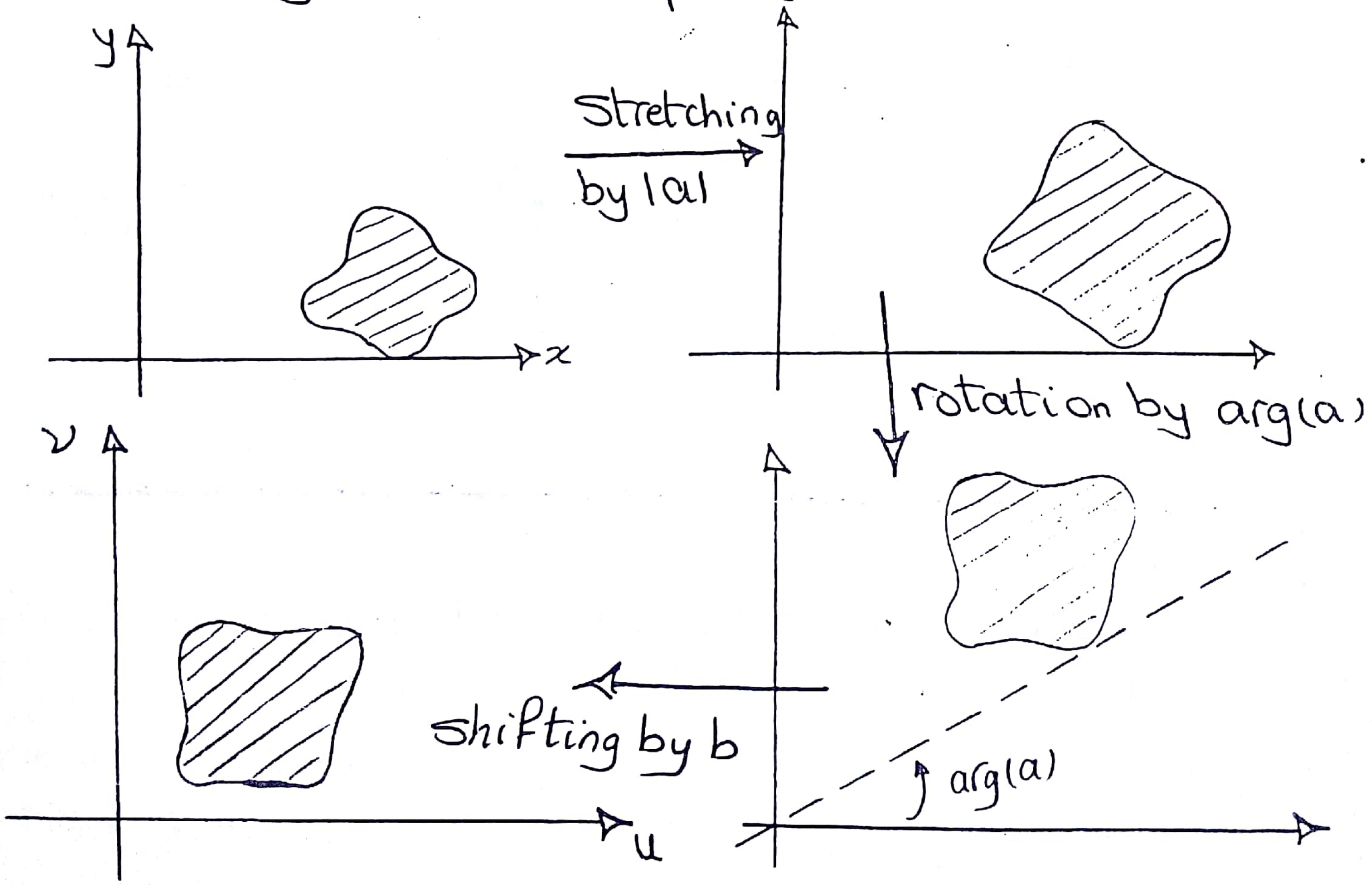
The effect of this mapping is the result of a 3 steps :-

$$f(z) = |a| \, e^{i \arg(a)} \cdot z + b$$

i) Stretching by $|a|$ " or Contraction if $|a| < 1$ "

ii) rotation the whole shape by $\arg(a)$.

iii) Shifting the whole shape by b.



Example :- Determine the region in the w-plane into which the region bounded by $x=0$, $y=0$, $x=2$ and $y=1$ is mapped by the fn.

$$w = (1+i)z + 1 + 2i$$

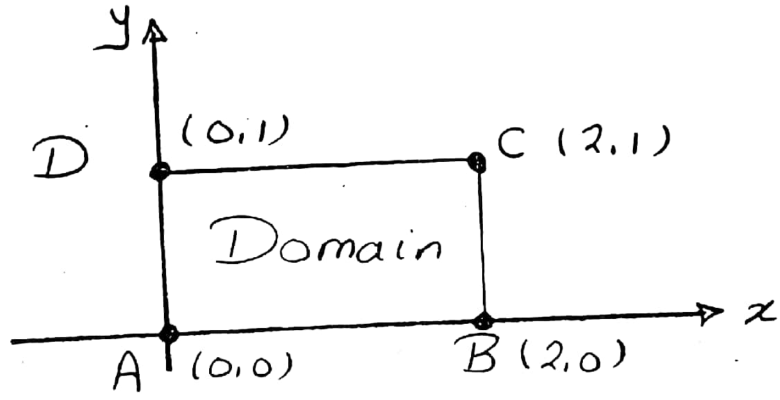
Solution :-

Set $z = x+iy$, $w = u+iv$

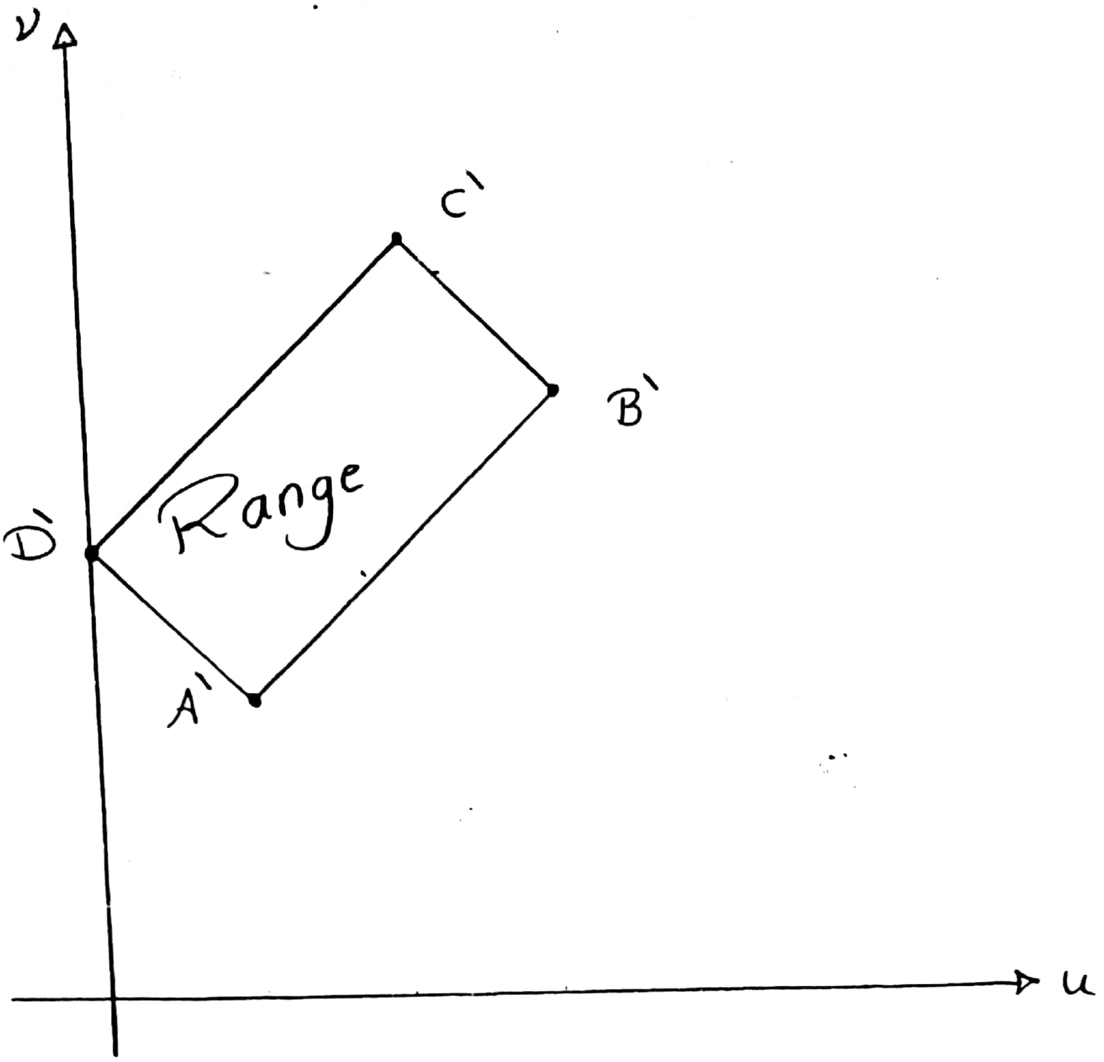
$$\Rightarrow \quad u+iv = (1+i)(x+iy) + 1 + 2i$$
$$= x + iy + ix - y + 1 + 2i$$
$$= (x-y+1) + i(x+y+2)$$

$$\Rightarrow \quad u = x-y+1 \quad \& \quad v = x+y+2$$

The Domain :-



Point A (0,0) is transformed to $u=1, v=2$
Point B (2,0) " " " $u=3, v=4$
Point C (2,1) " " " $u=2, v=5$
Point D (0,1) " " " $u=0, v=3$



Thus, the domain is enlarged (stretched) by

$|a| = |1+i| = \sqrt{2}$ & is rotated by $Arg(a) = \pi/4$

and then translated by $1+2i$.

So without translating the corner points one can deduce the image directly by rewritting $f(z)$ as

$$f(z) = \sqrt{2}\, e^{i\pi/4} \cdot z + 1+2i$$

Example : Find the region into which the half plane $y > 0$ is mapped by $w = (1+i)z$.

Solution

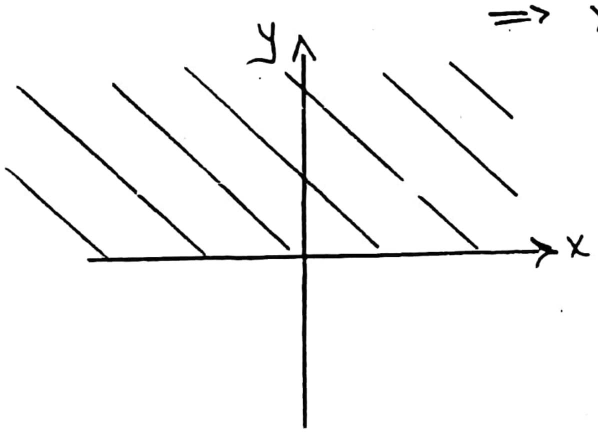
1) Algebrically we have

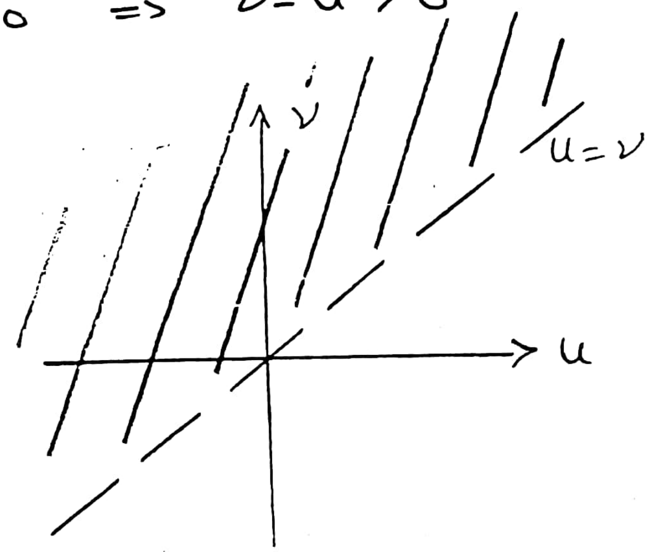
$$w = (1+i)z \Rightarrow z = \frac{w}{1+i} = \frac{u+iv}{1+i} \cdot \frac{1-i}{1-i}$$

$$z = \frac{u - iu + iv + v}{2} \Rightarrow x = \frac{u+v}{2} \ \& \ y = \frac{v-u}{2}$$

$$\Rightarrow y > 0 \text{ becomes } \frac{v-u}{2} > 0 \Rightarrow v - u > 0$$

$$\Rightarrow v > u$$





2) graphically : $w = (1+i)z = \sqrt{2}\, e^{i\pi/4} z$

$\Rightarrow$ enlarge (stretch) the region by $\sqrt{2}$ & then rotate it by $\pi/4$

Algebrically we can do this as follow

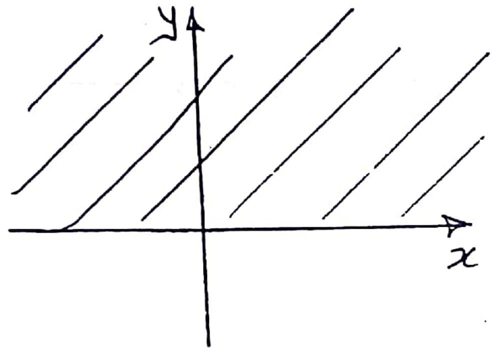
$|z-1-i|<1 \Rightarrow (x-1)^2+(y-1)^2<1 \Rightarrow$ put $x=v$, $y=1-u$

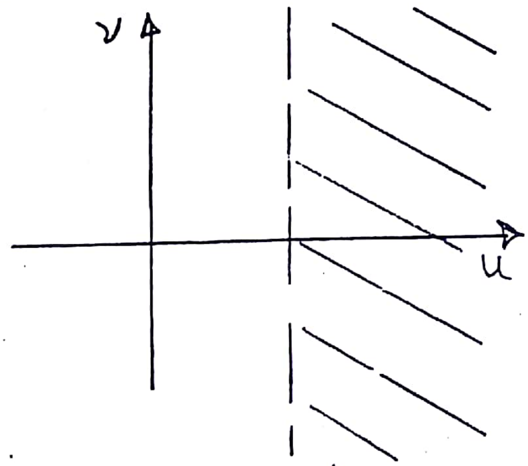
$\Rightarrow (v-1)^2+(-u)^2<1$

$\Rightarrow u^2+(v-1)^2<1 \Rightarrow |w-i|<1$

Example :- Find a transformation that maps the half plane $Im(z)>0$ into the region $R(w)>1$

Solution :-

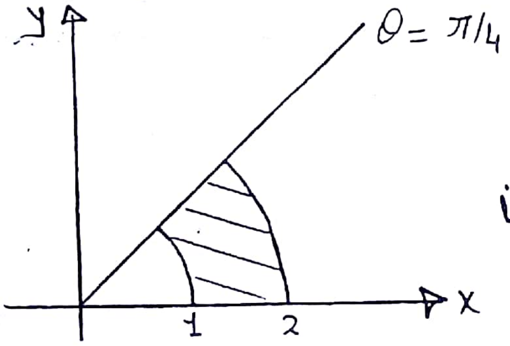




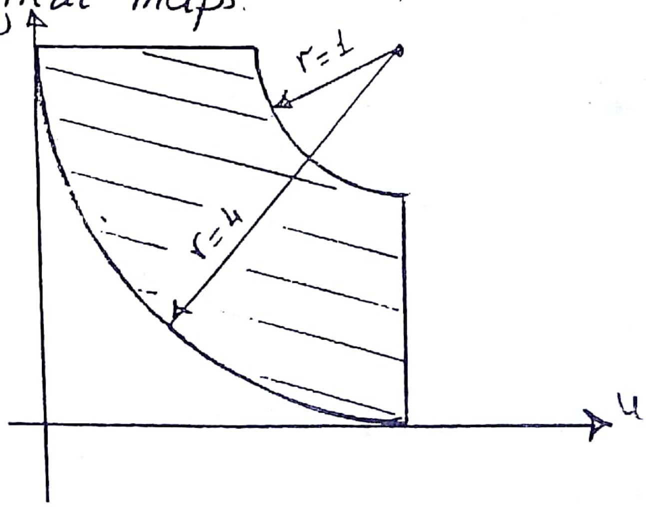
$\Rightarrow W = e^{-i\pi/2} z + 1$

$= -iz + 1$

Example : Find the transformation that maps.



into



Sol :-

$W = e^{i\pi} z^2 + (4+4i) = -z^2+4+4i$

$$w = f(z) = 1/z$$

Set $z = x + iy$ and $w = u + iv$.

$$\Rightarrow \quad u + iv = \frac{1}{x+iy} * \frac{x-iy}{x-iy}$$

$$\Rightarrow \quad u + iv = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

$$\Rightarrow \quad u = \frac{x}{x^2+y^2} \quad \& \quad v = \frac{-y}{x^2+y^2}$$

Example :- Show that the reciprocal mapping preserving lines and circles.

Solution :- Consider the domain in the x-y plane

$$a(x^2+y^2) + bx + cy + d = 0$$ which

represents circles or lines.

From $w = \frac{1}{z}$, we have $x = \frac{u}{u^2+v^2}$ & $y = \frac{-v}{u^2+v^2}$

Subsituting in the equation, we get

$$a\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + b\,\frac{u}{u^2+v^2} + c\,\frac{-v}{u^2+v^2} + d = 0$$

$$a\left(\frac{1}{u^2+v^2}\right)+\frac{bu}{u^2+v^2}-\frac{cv}{u^2+v^2}+d=0 \quad *(u^2+v^2)$$

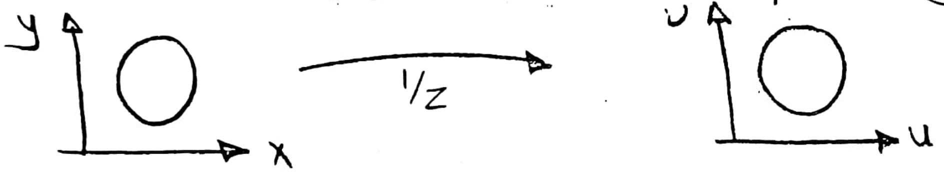
$$a+bu-cv+d(u^2+v^2)=0$$

$$\Rightarrow \quad d(u^2+v^2)+bu-cv+a=0$$

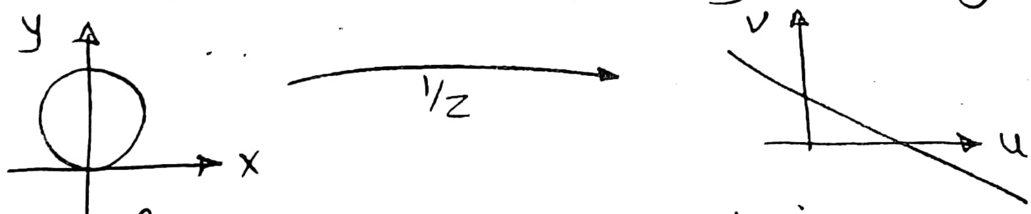
which is again an equation of circles or st. lines.

$\Rightarrow$ <u>The Result is :-</u>

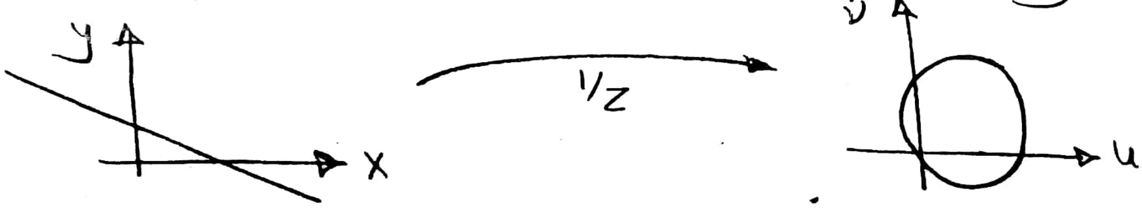
* A circle that doesn't pass by the origin is transformed to another circle that doesn't pass by the origin.



* A circle that passes by the origin is transformed to a st. line that doesn't pass by the origin.



* A st. line that doesn't pass by the origin is transformed to a circle that passes by the origin.



* A st. line that passes by the origin is transformed to st. line that passes by the origin.

Example :-

find the image of the vertical line $x = k$ & the horizontal line $y = k$ & the half plane $x > k$, under the reciprocal mapping $w = 1/z$.

Solution :-

① for the vertical line $x = k$, set $x = \frac{u}{u^2+v^2}$

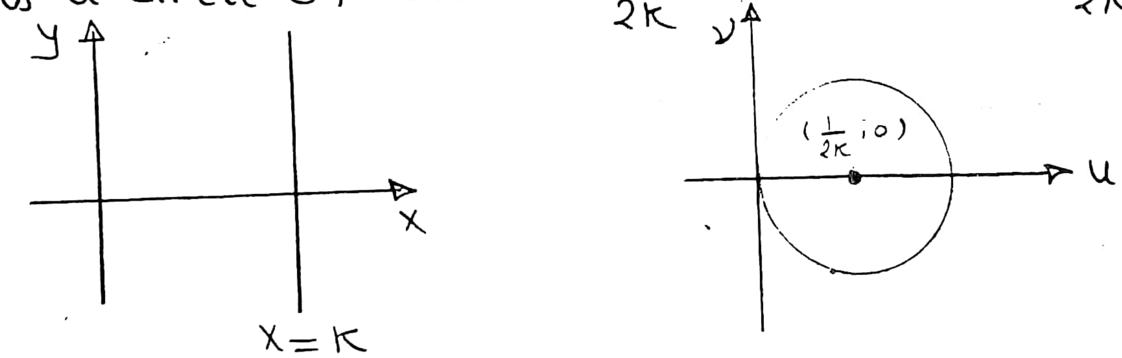
$$\Rightarrow \frac{u}{u^2+v^2} = k \Rightarrow u^2+v^2 = \frac{u}{k}$$

$$\Rightarrow u^2+v^2 - \frac{u}{k} = 0$$

$$\Rightarrow \left(u - \frac{1}{2k}\right)^2 - \frac{1}{4k^2} + v^2 = 0$$

$$\Rightarrow \left(u - \frac{1}{2k}\right)^2 + v^2 = \frac{1}{4k^2}$$

is a circle of center $\left(\frac{1}{2k}, 0\right)$ & radius $\frac{1}{2k}$

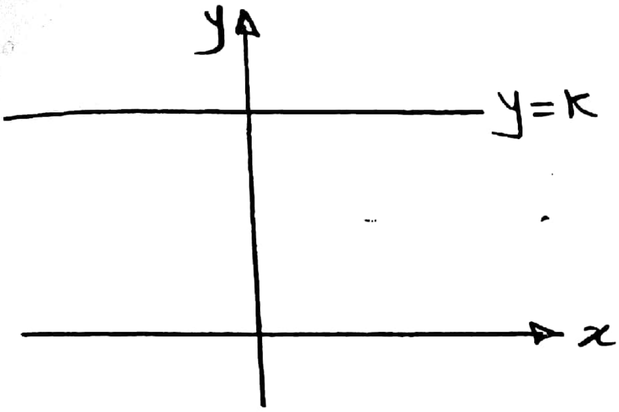


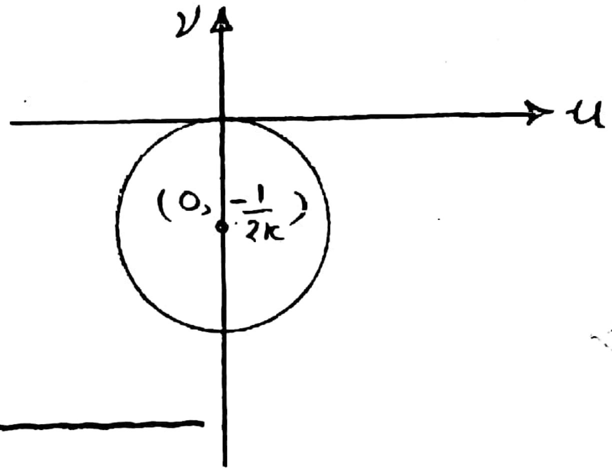
② for the horizontal line $y = k$, set $y = \frac{-v}{u^2+v^2}$

$$\Rightarrow \frac{-v}{u^2+v^2} = k \Rightarrow u^2+v^2+\frac{v}{k} = 0$$

$$\Rightarrow u^2 + \left(v + \frac{1}{2k}\right)^2 = \frac{1}{4k^2}$$

is a circle of center $(0, \frac{-1}{2k})$ and radius $\frac{1}{2k}$





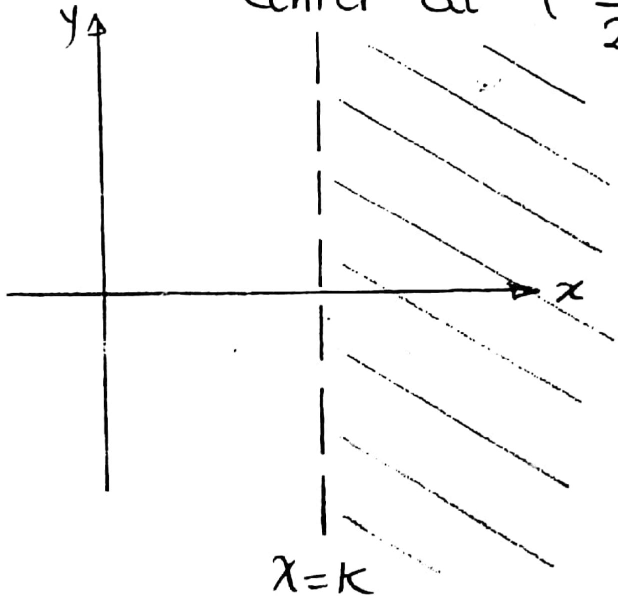
③ The half plane $x > k$

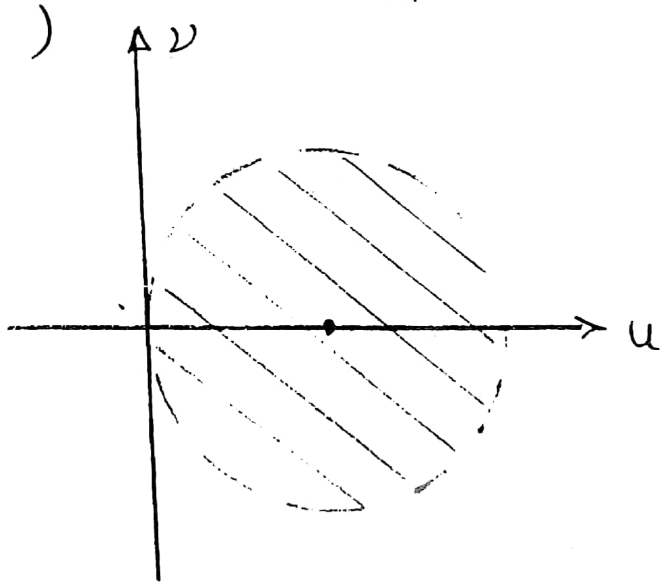
Set $x = \dfrac{u}{u^2+v^2} \Rightarrow \dfrac{u}{u^2+v^2} > k$

$$\Rightarrow \quad u^2 + v^2 < \frac{u}{k}$$

$$\Rightarrow \quad \left(u - \frac{1}{2k}\right)^2 + v^2 < \frac{1}{4k^2}$$

i.e. the interior of a circle of radius $\frac{1}{2k}$ & Center at $(\frac{1}{2k}, 0)$





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