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[1] $y'' + \frac{6x}{x^2+4} y' + \frac{1}{x^2+4} y = 0$

$\therefore$ $x=0$ is an ordinary point $\Longrightarrow y = \sum_{x=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

substituting in the differential equation

$\Longrightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$\Longrightarrow \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$\therefore \left[n(n-1) + 6n + 1\right] a_n + 4(n+2)(n+1) a_{n+2} = 0$

$\therefore a_{n+2} = -\frac{n^2+5n+1}{4(n+1)(n+2)} a_n = \frac{-(n+0.21)(n+4.8)}{4(n+1)(n+2)} a_n$

$a_2 = \frac{(-1)(0.21)(4.8)}{4(1)(2)} a_0$

$a_4 = \frac{(-1)^2(2.21 * 0.21)(6.8 * 4.8)}{4^2 (3*1)(4*2)} a_0$

$a_6 = \frac{(-1)^3(4.21 * 2.21 * 0.21)(8.8 * 6.8 * 4.8)}{4^3 (5*3*1)(6*4*2)} a_0$

$a_{2n} = \frac{(-1)^n(0.21 * 2.21 \ldots (2n-1.79))(4.8 * 6.8 \ldots (2n+2.8))}{4^n (2n)!} a_0$

$a_3 = \frac{(-1)(1.21)(5.8)}{4(2)(3)} a_1$

$a_5 = \frac{(-1)^2(3.21 * 1.21)(7.8 * 5.8)}{4^2 (4*2)(5*3)} a_1$

$a_7 = \frac{(-1)^3(5.21 * 3.21 * 1.21)(9.8 * 7.8 * 5.8)}{4^3 (6*4*2)(7*5*3)} a_1$

$a_{2n+1} = \frac{(-1)^n(1.21 * 3.21 \ldots (2n-0.79))(5.8 * 7.8 \ldots (2n+3.8))}{4^n (2n+1)!} a_1$

$\therefore y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=1}^{\infty} a_{2n} x^{2n} + \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}$

$= a_0 + a_1 x + \sum_{n=1}^{\infty} \frac{(0.21 * 2.21 \ldots (2n-1.79))(4.8 * 6.8 \ldots (2n+2.8))}{(-1)^n 4^n (2n)!} a_0 x^{2n}$

$+ \sum_{n=1}^{\infty} \frac{(1.21 * 3.21 \ldots (2n-0.79))(5.8 * 7.8 \ldots (2n+3.8))}{(-1)^n 4^n * (2n+1)!} a_1 x^{2n+1}$

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[2] $y'' + \frac{1-5x}{x(1-x)} y' - \frac{4}{x(1-x)} y = 0$

$\therefore x = 0$ is a regular singularity

$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$

$\therefore \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$

$-5 \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} - 4 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} - \sum_{n=1}^{\infty} (n+s-1)(n+s-2) a_{n-1} x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$

$-5 \sum_{n=1}^{\infty} (n+s-1) a_{n-1} x^{n+s-1} - 4 \sum_{n=1}^{\infty} a_{n-1} x^{n+s-1} = 0$

$\therefore a_0 [s(s-1) + s] x^{s-1} + \sum_{n=1}^{\infty} \left(\left[(n+s)(n+s-1) + (n+s) \right] a_n - \left[(n+s-1)(n+s-2) + 5(n+s-1) + 4 \right] a_{n-1} \right) x^{n+s-1} = 0$

$\therefore s(s-1) + s = 0 \Rightarrow s^2 = 0 \Rightarrow s_1 = s_2 = 0$ "case (2)"

$\therefore y_1(x) = \sum_{n=0}^{\infty} a_n(s_1) x^{n+s_1}$ $\qquad y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} a_n'(s_1) x^{n+s_1}$

$\Rightarrow \left[(n+s)(n+s-1) + (n+s) \right] a_n - \left[(n+s-1)(n+s-2) + 5(n+s-1) + 4 \right] a_{n-1} = 0$

$\therefore a_n = \frac{\left[(n+s-1)(n+s-1-1) + 5(n+s-1) + 4 \right]}{\left[(n+s)(n+s-1+1) \right]} a_{n-1} = \frac{\left[(n+s-1)^2 + 4(n+s-1) + 4 \right]}{(n+s)^2} a_{n-1}$

$\therefore a_n = \frac{(n+s-1+2)^2}{(n+s)^2} a_{n-1} = \frac{(n+s+1)^2}{(n+s)^2} a_{n-1}$

for $s = 0 \Rightarrow a_n(0) = \frac{(n+1)^2}{n^2} a_{n-1}(0)$

$a_1(0) = \frac{(2)^2}{(1)^2} a_0(0)$ $\qquad a_3(0) = \frac{(2.3.4)^2}{(1.2.3)^2} a_0$

$a_2(0) = \frac{(2.3)^2}{(1.2)^2} a_0(0)$ $\qquad a_n(0) = \frac{((n+1)!)^2}{(n!)^2} a_0 \quad n \geq 0$

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$$\therefore y_1(x) = \sum_{n=0}^{\infty} \frac{((n+1)!)^2}{(n!)^2} a_0 x^n$$

$$a_1(s) = \frac{(s+2)^2}{(s+1)^2} a_0(s) \qquad a_3(s) = \frac{((s+2)(s+3)(s+4))^2}{((s+1)(s+2)(s+3))^2} a_0(s)$$

$$a_2(s) = \frac{((s+2)(s+3))^2}{((s+1)(s+2))^2} a_0(s) \qquad a_n(s) = \frac{((s+2)(s+3)\cdots(s+n+1))^2}{((s+1)(s+2)\cdots(s+n))^2} a_0(s)$$

$$\ln a_n(s) = 2[\ln(s+2) + \ln(s+3) + \cdots + \ln(s+n+1)] + \ln a_0 - 2[\ln(s+1) + \ln(s+2) + \cdots + \ln(n+s)]$$

Differentiate both sides with respect to s

$$\Rightarrow \frac{a_n'(s)}{a_n(s)} = 2\left[\frac{1}{s+2} + \frac{1}{s+3} + \cdots + \frac{1}{s+n+1}\right] - 2\left[\frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{s+n}\right]$$

$$\Rightarrow a_n'(0) = \left[2\left[\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right] - 2\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]\right] a_n(0)$$

$$\therefore y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty}\left(2\left[\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right] - 2\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]\right) a_n(0)\, x^n$$

$$y_{GS} = c_1 y_1(x) + c_2 y_2(x)$$

[3] (i) $y'' - \frac{2x}{(1-x)(1+x)} y' + \frac{6}{(1-x)(1+x)} y = 0$

$\therefore x=0$ is an ordinary Point $\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

substitute in the differential equation

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$

$\therefore (n+2)(n+1) a_{n+2} - [n(n-1) + 2n - 6] a_n = 0$

$\therefore a_{n+2} = \frac{n^2+n-6}{(n+2)(n+1)} a_n = \frac{(n-2)(n+3)}{(n+2)(n+1)} a_n$

$a_2 = \frac{(-2)(3)}{(2)(1)} a_0 = -3a_0$

$a_4 = 0$ $\therefore a_4 = a_6 = a_{2n} = 0$
For $n \geq 2$

$a_3 = \frac{(-1)(4)}{(3)(2)} a_1$

$a_5 = \frac{(-1.1)(4.6)}{(3.5)(2.4)} a_1$

$a_7 = \frac{(-1.1.3)(4.6.8)}{(3.5.7)(2.4.6)} a_1$

$\therefore a_{2n+1} = \frac{(-1.1.3 \ldots (2n-3))(4.6.8 \ldots (2n+2))}{(2n+1)!} a_1$
$n \geq 1$

$\therefore y = a_0[1-3x^2] + a_1 x + \sum_{n=1}^{\infty} \frac{(-1.1.3 \ldots (2n-3))(4.6.8 \ldots (2n+2))}{(2n+1)!} a_1 x^{2n+1}$

$x=1$, $x=-1$ are regular singularity points

(ii) $y'' - \frac{2x}{1-x^2} y' + \frac{12}{1-x^2} y = 0$

$\therefore x = 0$ is an ordinary point $\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

substitute in the differential equation

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2 \sum_{n=1}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{x=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - 2 \sum_{n=0}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore (n+2)(n+1) a_{n+2} - [n(n-1) + 2n - 12] a_n = 0$$

$$\therefore a_{n+2} = \frac{n^2+n-12}{(n+2)(n+1)} a_n = \frac{(n+4)(n-3)}{(n+2)(n+1)} a_n$$

$a_2 = \frac{(4)(-3)}{(2)(1)} a_0$ | $a_3 = \frac{(5)(-2)}{(3)(2)} a_1 = -\frac{5}{3} a_1$

$a_4 = \frac{(6.4)(-1.-3)}{(4.2)(3.1)} a_0$ | $a_5 = 0$

$a_6 = \frac{(8.6.4)(-3.-1.1)}{(6.4.2)(5.3.1)} a_0$ | $\therefore a_5 = a_7 = a_{2n+1} = 0$ for $n \geq 2$

$$\therefore a_{2n} = \frac{(4.6.8 \ldots (2n+2))(-3.-1.1 \ldots (2n-5))}{(2n)!} a_0 \quad \text{for } n \geq 1$$

$$\therefore y = a_0 + a_1\left[x - \frac{5}{3}x^3\right] + \sum_{n=1}^{\infty} \frac{(4.6.8 \ldots (2n+2))(-3.-1.1 \ldots (2n-5))}{(2n)!} a_0 \, x^{2n}$$

$x = 1$, $x = -1$ are regular singularity points

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[4] (i) $(1-x^2)y'' - 2xy' + 2y = 0$

$\therefore x=0$ is an ordinary point $\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

substitute in the differential equation

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore (n+2)(n+1) a_{n+2} - [n(n-1) + 2n - 2] a_n = 0$$

$$\therefore a_{n+2} = \frac{n^2+n-2}{(n+2)(n+1)} a_n = \frac{\cancel{(n+2)}(n-1)}{\cancel{(n+2)}(n+1)} a_n = \frac{n-1}{n+1} a_n$$

$a_2 = \frac{-1}{1} a_0$

$a_4 = \frac{(-1.1)}{(1.3)} a_0$

$a_6 = \frac{(-1.1.3)}{(1.3.5)} a_0$

$\therefore a_{2n} = \frac{(-1.1.3 \ldots (2n-3))}{(1.3.5 \ldots (2n-1))} a_0 \quad (n \geq 1)$

$a_3 = 0$

$\therefore a_3 = a_5 = a_7 = a_{2n+1} = 0$ for $n \geq 1$

$$\therefore y = a_0 + a_1 x + \sum_{n=1}^{\infty} \frac{(-1.1.3 \ldots (2n-3))}{(1.3.5 \ldots (2n-1))} a_0 x^{2n}$$

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(ii) $(3-x^2)y'' - xy' + 16y = 0$

$p(x) = \frac{-x}{3-x^2}$, $q(x) = \frac{16}{3-x^2}$ $\Rightarrow$ at $x=0$ $p(x) = 0$, $q(x) = \frac{16}{3}$

$\therefore x=0$ is an ordinary point $\Rightarrow$ let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = n(n-1) a_n x^{n-2}$

Substitute in the differential equation

$$\Rightarrow 3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 3\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

coeff $x^n = 0$ $\Rightarrow$ $3(n+2)(n+1)a_{n+2} - \left[n(n-1) + n - 16\right] a_n = 0$

$$\therefore a_{n+2} = \frac{n^2-16}{3(n+2)(n+1)} a_n = \frac{(n+4)(n-4)}{3(n+2)(n+1)} a_n$$

$a_2 = \frac{(4)(-4)}{3(2)(1)} a_0$ | $a_3 = \frac{(5)(-3)}{3(3)(2)} a_1$

$a_4 = \frac{(4.6)(-4.-2)}{3^2(2.4)(1.3)} a_0$ | $a_5 = \frac{(5.7)(-3.-1)}{3^2(3.5)(2.4)} a_1$

$a_6 = 0 \Rightarrow \therefore a_{2n} = 0 \quad n \geq 3$ | $a_7 = \frac{(5.7.9)(-3.-1.1)}{3^3(3.5.7)(2.4.6)} a_1$ $\quad n \geq 1$

$$\therefore a_{2n+1} = \frac{(5.7.9 \dots (2n+3))(-3.-1.1 \dots (2n-5))}{3^n (2n+1)!} a_1$$

$$\therefore y = a_0 + a_2x^2 + a_4x^4 + a_1\left[x + \sum_{n=1}^{\infty} \frac{(5.7 \dots (2n+3))(-3.-1 \dots (2n-5))}{3^n (2n+1)!} x^{2n+1}\right]$$

(iii) $(2-x^2)y'' - xy' + 25y = 0$

$P(x) = \frac{-x}{2-x^2}$, $q(x) = \frac{25}{2-x^2}$ $\Rightarrow$ at $x=0$ $P(x)=0$, $q(x) = 12.5$

$\therefore$ $x=0$ is an ordinary point $\Rightarrow$ let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

substitute in the differential equation

$$\Rightarrow 2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 25 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + 25 \sum_{n=0}^{\infty} a_n x^n$$

Coeff $x^n = 0$ $\Rightarrow$ $2(n+2)(n+1) a_{n+2} - [n(n-1) + n - 25] a_n = 0$

$$\therefore a_{n+2} = \frac{n^2 - 25}{2(n+2)(n+1)} a_n = \frac{(n+5)(n-5)}{2(n+2)(n+1)} a_n$$

$a_2 = \frac{(5)\ (-5)}{2\ (2)\ (1)} a_0$

$a_4 = \frac{(5.7)\ (-5.-3)}{2^2 (2.4)\ (1.3)} a_0$

$a_6 = \frac{(5.7.9)(-5.-3.-1)}{2^3\ (2.4.6)(1.3.5)} a_0$

$n \geq 1$

$a_{2n} = \frac{(5.7.9 \ldots (2n+3))(-5.-3 \ldots (2n-7))}{2^n\ (2n)!} a_0$

$a_3 = \frac{(6)\ (-4)}{2\,(3)\ (2)} a_1$

$a_5 = \frac{(6.8)(-4.-2)}{2^n\ (3.5)(2.4)} a_1$

$a_7 = 0 \Rightarrow a_{2n+1} = 0$ $\quad n \geq 3$

$$\therefore y = a_0 + a_1 x + a_3 x^3 + a_5 x^5 + \sum_{n=1}^{\infty} \frac{(5.7.9 \ldots (2n+3))(-5.-3 \ldots (2n-7))}{2^n\ (2n)!} a_0 x^{2n}$$

(iv) $xy'' + y' + xy = 0$

$p(x) = \frac{1}{x}, \quad q(x) = 1 \Rightarrow$ at $x=0$ $\quad p(x) = \infty$

$P(x) = 1, \quad Q(x) = x^2 \Rightarrow$ at $x=0$ $\quad P(x) = 1, \quad Q(x) = 0$

$\therefore x = 0$ is a regular singular point $\Rightarrow$ let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$

$$y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

Substitute in the differential equation

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s-1} = 0$$

$$[s(s-1)+s] a_0 x^{s-1} + [(s+1)s + s+1] a_1 x^s + \sum_{n=2}^{\infty} \begin{bmatrix} (n+s)(n+s-1)a_n \\ +(n+s)a_n + a_{n-2} \end{bmatrix} x^{n+s-1} = 0$$

Coeff $x^{s-1} = 0 \Rightarrow s^2 a_0 = 0 \Rightarrow s_1 = s_2 = 0 \qquad a_0 \neq 0 \qquad$ "case (2)"

Coeff $x^s = 0 \Rightarrow [(s+1)^2] a_1 = 0 \Rightarrow \therefore a_1 = 0$

Coeff $x^{n+s-1} = 0 \Rightarrow [(n+s)(n+s-1) + (n+s)] a_n + a_{n-2} = 0$

$$\therefore a_n = \frac{-1}{(n+s)^2} a_{n-2}$$

$$a_n(0) = \frac{-1}{n^2} a_{n-2}(0)$$

$$a_2(0) = \frac{-1}{(2)^2} a_0$$

$$a_4(0) = \frac{(-1)^2}{(2.4)^2} a_0 \Longrightarrow a_{2n}(0) = \frac{(-1)^n}{2^{2n}(n!)^2} a_0(0), \quad n \geq 0$$

$$a_6(0) = \frac{(-1)^3}{(2.4.6)^2} a_0$$

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$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} a_0 x^{2n} \quad , \quad y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} a_n'(s_1)\, x^{n+s_1}$$

$$a_n(s) = \frac{-1}{(n+s)^2} a_{n-2}(s)$$

$$a_2(s) = \frac{-1}{(s+2)^2} a_0 \qquad a_4(s) = \frac{(-1)^2}{((s+2)(s+4))^2} a_0$$

$$a_6(s) = \frac{(-1)^3}{((s+2)(s+4)(s+6))^2} a_0 \qquad a_{2n}(s) = \frac{(-1)^n}{((s+2)(s+4)(s+6)\cdots(s+2n))^2} a_0$$

$$\ln a_{2n}(s) = \ln(-1)^n + \ln a_0 - 2[\ln(s+2) + \ln(s+4) + \cdots + \ln(s+2n)]$$

Differentiate both sides with respect to s

$$\frac{a_{2n}'(s)}{a_{2n}(s)} = -2\left[\frac{1}{s+2} + \frac{1}{s+4} + \cdots + \frac{1}{s+2n}\right]$$

$$a_{2n}'(0) = -2\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right] a_{2n}(0) = -\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right] a_{2n}(0)$$

$$\therefore a_{2n}'(0) = -H_n * \frac{(-1)^n}{2^{2n}(n!)^2} a_0 = \frac{(-1)^{n+1}}{2^{2n}(n!)^2} H_n a_0$$

$$\therefore y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2} a_0 x^{2n}$$

$$y_{GS} = C_1 y_1(x) + C_2 y_2(x)$$

(v) $y'' + 4y = 0$

$p(x) = 0$, $q(x) = 4$ $\Rightarrow$ $x = 0$ is an ordinary point

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substitute in the differential equation

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

coeff $x^n = 0$ $\Rightarrow$ $(n+1)(n+2) a_{n+2} + 4 a_n = 0$

$$\therefore a_{n+2} = \frac{-4}{(n+1)\ (n+2)} a_n$$

$a_2 = \frac{-4}{(1)\ (2)} a_0$ | $a_3 = \frac{-4}{(2)\ (3)} a_1$

$a_4 = \frac{(-4)^2}{(1.3)\ (2.4)} a_0$ | $a_5 = \frac{(-4)^2}{(2.4)\ (3.5)} a_1$

$a_6 = \frac{(-4)^3}{(1.3.5)(2.4.6)} a_0$ | $a_7 = \frac{(-4)^3}{(2.4.6)(3.5.7)} a_1$

$a_{2n} = \frac{(-4)^n}{(2n)!} a_0$ $n \geq 0$ | $a_{2n+1} = \frac{(-4)^n}{(2n+1)!} a_1$ $n \geq 0$

$$\therefore y = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} a_0 x^{2n} + \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} a_1 x^{2n+1}$$