Find all the roots of :

① $z^4 + 16 = 0$

De Moivre theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \qquad k = 0, 1, 2, \ldots, n-1$$

Sol

$$\Rightarrow z^4 = (-16)$$

$$\Rightarrow z = (-16)^{1/4} = (16 e^{i\pi})^{1/4}$$

$$= (16)^{1/4} e^{i\left(\frac{\pi + 2\pi k}{4}\right)} \qquad k = 0, 1, 2, 3$$

at $k=0 \Rightarrow z_0 = 2e^{i(\pi/4)}$

$k=1 \Rightarrow z_1 = 2e^{i(3\pi/4)}$

$k=2 \Rightarrow z_2 = 2e^{i\left(\frac{5\pi}{4}\right)}$

$k=3 \Rightarrow z_3 = 2e^{i\left(\frac{7\pi}{4}\right)}$

Note :

$k = 0, 1, \ldots, n-1$ as $z_0 = z_n$, $z_1 = z_{n+1}$ ...

② $z^2 + 2z + (2+i) = 0$

$\Rightarrow z_{1,2} = \dfrac{-2 \pm \sqrt{4-4(2+i)}}{2}$

③ $\sin z = 2$

$\Rightarrow \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \, ch y + i \cos x \, sh y$

$\Rightarrow \sin x \, ch y + i \cos x \, sh y = 2 + i0$

Note:

$\cos iy = ch y$

$\sin iy = i\, sh y$

$ch\, iy = \cos y$

$sh\, iy = i \sin y$

$\Rightarrow \sin x \, ch y = 2$ ——— ①

$\Rightarrow \cos x \, sh y = 0$ ——— ②

$\therefore \cos x = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$

$n = 0, \pm 1, \pm 2, \ldots$

in ① $\Rightarrow \sin(2n+1)\frac{\pi}{2} \cdot ch y = 2$

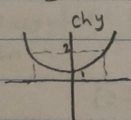
$\Rightarrow (-1)^n \cdot ch y = 2$

$sh y = 0$

$y = 0$

in ① $\Rightarrow \sin x = 2$ "refused"

"$-1 < \sin x < 1$"



$\therefore ch y = \dfrac{2}{(-1)^n}$

since $ch y \geq 1$ $\Rightarrow$ n can't be odd

$n = 2m \Rightarrow ch y = 2$ $\quad m = 0, \pm 1, \pm 2, \pm 3, \ldots$

$\Rightarrow y = \pm ch^{-1} 2$

$\therefore z = (4m+1)\frac{\pi}{2} + i(\pm ch^{-1} 2)$ $\quad m = 0, \pm 1, \pm 2, \ldots$

④ $e^{2z} = 1+i$

$\Rightarrow 2z = \ln(1+i)$ $\qquad \Rightarrow z = \frac{1}{2}\ln(1+i)$

$\because 1+i = \sqrt{2}\, e^{i(\frac{\pi}{4} + 2\pi k)}$, $k = 0, \pm1, \pm2, \ldots$

$\therefore z = \frac{1}{2}\ln(\sqrt{2}\, e^{i(\frac{\pi}{4}+2\pi k)}) = \frac{1}{2}\left[\ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi k)\right]$

$\therefore z = \frac{1}{4}\ln 2 + i\,\frac{1}{2}(\frac{\pi}{4} + 2\pi k)$

⑤ $(-2+i)^i = z$

$\Rightarrow z = e^{\ln z} = e^{\ln(-2+i)^i} = e^{i\ln(-2+i)}$

$\because -2+i = \sqrt{5}\, e^{i(\pi - \tan^{-1}\frac{1}{2} + 2\pi k)}$ $\qquad k = 0, \pm1, \pm2, \ldots$

$\Rightarrow \ln(-2+i) = \frac{1}{2}\ln(5) + i(\pi - \tan^{-1}\frac{1}{2} + 2\pi k)$

$\therefore i\ln(-2+i) = i(\frac{1}{2}\ln(5)) - (\pi - \tan^{-1}\frac{1}{2} + 2\pi k)$

$\therefore z = e^{i(\frac{1}{2}\ln 5) - (\pi - \tan^{-1}\frac{1}{2} + 2\pi k)}$

$\Rightarrow z = \underbrace{e^{-(\pi - \tan^{-1}\frac{1}{2} + 2\pi k)}}_{|z|} \cdot \underbrace{e^{i(\frac{1}{2}\ln 5)}}_{\arg(z)}$

⑥ Discuss differentiability of
$f(z) = z^2 \bar{z}$

$\Rightarrow f(z) = z^2 \bar{z} = (x+iy)^2 (x-iy) = (x^3 + xy^2) + i(x^2 y + y^3)$

$\therefore$ u, v cont on Z-plane

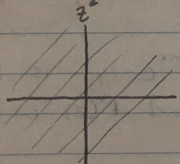
$u_x = 3x^2 + y^2$, $v_y = x^2 + 3y^2$ $\Rightarrow u_x = v_y$ at $2x^2 = 2y^2 \Rightarrow x^2 = y^2$
$\Rightarrow x = \pm y$ — ①

$u_y = 2xy$, $v_x = 2xy$ $\Rightarrow u_y = -v_x$ at $xy = 0$ — ②

① in ② $\Rightarrow \pm y^2 = 0 \Rightarrow y = 0 \Rightarrow x = 0$

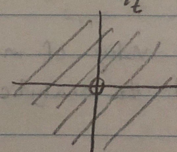
$\therefore$ C.R satisfied only at $(0,0)$
$\therefore$ $f(z)$ is differentiable at $(0,0)$

z^2

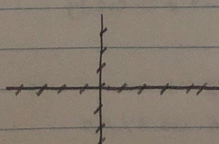


analytic everywhere
= entire fn

$1/z$

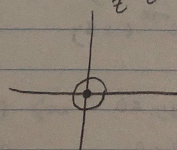


analytic everywhere
except $(0,0)$



not analytic

$z^2 \bar{z}$



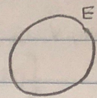
not analytic

* $f(z)$ is said to be analytic at a point z_0 if it is diff at z_0 and its neighbourhood

* $f(z)$ is said to be analytic at a point E if it is diff on E (open or closed)

diff $|z| \leq 1$

analytic $|z| < 1$



if $f(z)$ is analytic on a region then:

① you can replace $x \to z$, $y \to 0$ to write the fn and its derivatives in terms of z

② you can differentiate as a Real function

③ if $f(z) = u + iv$ is analytic

$\therefore$ u is harmonic fn $\quad u_{xx} + u_{yy} = 0$

v is harmonic fn $\quad v_{xx} + v_{yy} = 0$

u, v are harmonic conjigate $\quad u_x = v_y \ , \ u_y = -v_x$

Find the value of m so that $u = e^{mx} \cos 3y$ is harmonic, then find the analytic $f(z) = u + iv$ and $f'(z)$ in terms of z

Ans

$u_x = m e^{mx} \cos 3y \qquad u_y = -3 e^{mx} \sin 3y$

$u_{xx} = m^2 e^{mx} \cos 3y \qquad u_{yy} = -9 e^{mx} \cos 3y$

$u_{xx} + u_{yy} = 0 \implies (m^2 - 9) e^{mx} \cos 3y = 0$

$\therefore m^2 - 9 = 0$ OR $\cancel{e^{mx} = 0}$ OR $\cos 3y \neq 0$

$\therefore m = \pm 3$

at $m = 3$

$\Rightarrow u = e^{3x} \cos 3y \quad \Rightarrow u_x = 3e^{3x} \cos 3y$

$u_y = -3e^{3x} \sin 3y$

$\because f(z)$ is analytic

$\therefore u_x = v_y = 3e^{3x} \cos 3y \quad \text{--- } 1$

$-u_y = v_x = 3e^{3x} \sin 3y \quad \text{--- } 2$

2 $\int dx \Rightarrow v = e^{3x} \sin 3y + h(y)$

in 1 $3e^{3x} \cos 3y + h'(y) = v_y = 3e^{3x} \cos 3y$

$\therefore h'(y) = 0 \quad \Rightarrow h(y) = c$

$\therefore v = e^{3x} \sin 3y + c$

$\therefore f(z) = e^{3x} \cos(3y) + i(e^{3x} \sin 3y + c) \qquad x = z,\ y = 0$

$\Rightarrow f(z) = e^{3z} + c_1 \qquad$ "$c_1 = ic$"

$f'(z) = 3e^{3z}$