الإســـــــراء

ملزمة (٣)

رياضـــة

The Beta Function B (X,Y)

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(2) The Beta function $\beta(x,y)$;

Definition :-

For $x>0$ & $y>0$ we define

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad ; \; x>0 \;\&\; y>0$$

But, for $x \leq 0$ or $y \leq 0$ this integral diverges.

Example :- Evaluate 1) $\int_0^1 \sqrt{\frac{x}{1-x}}\,dx$

2) $\int_0^4 x^2 \sqrt[3]{16-x^2}\,dx$

Solution :-

1) $\int_0^1 \sqrt{\frac{x}{1-x}}\,dx = \int_0^1 x^{1/2}(1-x)^{-1/2}\,dx$

$= \beta\left(\frac{3}{2}, \frac{1}{2}\right)$

we will know later that $\beta\left(\frac{3}{2}\cdot\frac{1}{2}\right) = \pi/2$.

2) $\int_0^4 x^2 \sqrt[3]{16-x^2}\,dx$

let $x^2 = 16t \Rightarrow x = 4\sqrt{t} \Rightarrow dx = \frac{2}{\sqrt{t}}\,dt$

$I = \int_0^1 16t \sqrt[3]{16-16t} \cdot \frac{2}{\sqrt{t}}\,dt$

$= 16\sqrt[3]{16} \int_0^1 t^{1/2}(1-t)^{1/3}\,dt$

$= 16\sqrt[3]{16}\,\beta\left(\frac{3}{2}\cdot\frac{4}{3}\right).$

Beta function $\beta(x,y)$;

Definition:- Only when $x>0$ & $y>0$, we define

$$\boxed{\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,}$$

else, the integral will diverge.

- If we use the subsitution $t = \sin^2\theta$, we get another form for the beta fn.

$$\boxed{\beta(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta \quad ; \quad x>0 \ \& \ y>0}$$

- If we use the subsitution $t = \frac{u}{1+u}$, we get another form for the beta fn.

$$\boxed{\beta(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\,du \quad ; \quad x>0 \ \& \ y>0}$$

- Note that : $\beta(x,y) = \beta(y,x)$

- The Relation between the beta & gamma functions:-

$$\boxed{\beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \ ;}$$

this relation is valid for all values of x & y.

How to Calculate $\beta(x,y)$?

The relation between the gamma & Beta fns :-

$$\boxed{\beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad ; \text{ for all } x \,\&\, y.}$$

Proof:-

$$\Gamma(x)\,\Gamma(y) = \int_0^\infty t^{x-1} e^{-t}\,dt \cdot \int_0^\infty u^{y-1} e^{-u}\,du$$

$$= \int_0^\infty\!\!\int_0^\infty t^{x-1}\, u^{y-1}\, e^{-(u+t)}\,dt\,du$$

Let $t = r\cos^2\theta$ & $u = r\sin^2\theta$

So, when $t, u : 0 \to \infty$ then $r : 0 \to \infty$ & $\theta : 0 \to \pi/2$.

$$dt\,du = J\left(\frac{t,u}{r,\theta}\right) dr\,d\theta = \begin{vmatrix} \frac{\partial t}{\partial r} & \frac{\partial t}{\partial \theta} \\ \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$= \begin{vmatrix} \cos^2\theta & -2r\cos\theta\sin\theta \\ \sin^2\theta & 2r\sin\theta\cos\theta \end{vmatrix} dr\,d\theta$$

$$= 2r\sin\theta\cos^3\theta + 2r\cos\theta\sin^3\theta \quad dr\,d\theta$$

$$= 2r\sin\theta\cos\theta\cdot(\cos^2\theta + \sin^2\theta)\,dr\,d\theta$$

$$= 2r\sin\theta\cos\theta\,dr\,d\theta$$

$$\Rightarrow \Gamma(x)\,\Gamma(y) = \int_0^\infty\!\!\int_0^\infty t^{x-1}\, u^{y-1}\, e^{-(u+t)}\,dt\,du$$

$$= \int_0^{\frac{\pi}{2}}\!\!\int_0^\infty (r\cos^2\theta)^{x-1}\,(r\sin^2\theta)^{y-1}\, e^{-r}\, 2r\sin\theta\cos\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} 2 r^{x+y-1} e^{-r} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} \, dr \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} \left(\int_0^{\infty} r^{x+y-1} e^{-r} \, dr \right) d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} \, \Gamma(x+y) \, d\theta$$

$$= \Gamma(x+y) \cdot 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} \, d\theta$$

$$= \Gamma(x+y) \, \beta(x,y)$$

$$\Rightarrow \Gamma(x)\,\Gamma(y) = \Gamma(x+y)\,\beta(x,y)$$

$$\Rightarrow \beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Example :-

Evaluate

i) $\beta\left(\frac{3}{2}, \frac{1}{2}\right)$ ii) $\beta(4,7)$

iii) $\beta\left(\frac{1}{4}, \frac{11}{4}\right)$ iv) $\beta(3,-2)$

v) $\beta\left(\frac{3}{4}, -\frac{5}{4}\right)$. vi) Show that $\beta(x,n) = \dfrac{(n-1)!}{(x+n-1)\cdots(x+1)x}$

Solution :-

i) $$\beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1!}$$

$$= \frac{1}{2}\sqrt{\pi}\,\sqrt{\pi} = \pi/2.$$

ii) $\beta(4,7) = \dfrac{\Gamma(4)\Gamma(7)}{\Gamma(11)} = \dfrac{3!\ 6!}{10!} = \dfrac{1}{840}.$

iii) $\beta\left(\frac{1}{4}, \frac{11}{4}\right) = \dfrac{\Gamma(1/4)\ \Gamma(11/4)}{\Gamma(3)}$

$= \dfrac{\Gamma(1/4)\ .\ 7/4\ \Gamma(7/4)}{2!} = \dfrac{\Gamma(1/4)\ 7/4 \cdot 3/4\ \Gamma(3/4)}{2}$

$= \dfrac{21}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \dfrac{21}{32} \cdot \dfrac{\pi}{\sin \pi/4} = \dfrac{21\pi\sqrt{2}}{32}.$

iv) $\beta(3,-2) = \dfrac{\Gamma(3)\ \Gamma(-2)}{\Gamma(1)} = 2!\,(\pm\infty) = \pm\infty$

v) $\beta\left(\frac{3}{4}, -\frac{5}{4}\right) = \dfrac{\Gamma(3/4).\ \Gamma(-5/4)}{\Gamma(-1/2)}$

$= \dfrac{\dfrac{\Gamma(7/4)}{3/4} \cdot \dfrac{\Gamma(-1/4)}{-5/4}}{\dfrac{\Gamma(1/2)}{-1/2}} = \dfrac{-\frac{4}{3}\Gamma(7/4) \cdot \frac{4}{5}\ \dfrac{\Gamma(3/4)}{-1/4}}{-2\sqrt{\pi}}$

$= \dfrac{\frac{64}{15}\Gamma(7/4) \cdot \dfrac{\Gamma(7/4)}{3/4}}{-2\sqrt{\pi}} = -\dfrac{128}{45\sqrt{\pi}}\left(\Gamma(7/4)\right)^2$

vi) $\beta(x,n) = \dfrac{\Gamma(x)\ \Gamma(n)}{\Gamma(x+n)} = \dfrac{\Gamma(x)\ (n-1)!}{\Gamma(x+n)}$

$= \dfrac{\Gamma(x)\ (n-1)!}{(x+n-1)\ \Gamma(x+n-1)} = \dfrac{\Gamma(x)\ (n-1)!}{(x+n-1)(x+n-2)\,\Gamma(x+n-2)}$

$= \dfrac{\cancel{\Gamma(x)}\ (n-1)!}{(x+n-1)(x+n-2)\ \cdots\ (x+1)\,x\,\cancel{\Gamma(x)}}$

$= \dfrac{(n-1)!}{(x+n-1)(x+n-2)\cdots(x+1)\,x}.$

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xamples:

Evaluate these integrals :

1) $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^0\theta \cos^6\theta \, d\theta = \frac{1}{2} \beta(x, y)$

where, $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$2y - 1 = 6 \Rightarrow y = 7/2$

$\Rightarrow I = \frac{1}{2} \beta(\frac{1}{2}, \frac{7}{2}) = \frac{1}{2} \frac{\Gamma(1/2)\Gamma(7/2)}{\Gamma(4)}$

$= \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{2}) \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)}{3!} = \frac{5}{32}\pi$

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2) $\int_0^{\pi/2} \sin^{3.04}\theta \, d\theta$

$\Rightarrow 2x - 1 = 3.04 \Rightarrow x = 2.02$

$2y - 1 = 0 \Rightarrow y = 1/2$

$I = \frac{1}{2}\beta(x, y) = \frac{1}{2}\beta(2.02, \frac{1}{2}) = \frac{1}{2} \frac{\Gamma(2.02)\Gamma(1/2)}{\Gamma(2.52)}$

$= \frac{\sqrt{\pi}}{2} \cdot \frac{1.02\,\Gamma(1.02)}{1.52\,\Gamma(1.52)} = \frac{51}{152}\sqrt{\pi} \cdot \frac{\Gamma(1.02)}{\Gamma(1.52)}$

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3) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$

$I = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}} \, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta \, d\theta$

$\Rightarrow 2x - 1 = \frac{1}{2} \Rightarrow x = 3/4$ & $2y - 1 = -\frac{1}{2} \Rightarrow y = 1/4$

$I = \frac{1}{2}\beta(3/4, 1/4) = \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)}$

$= \frac{1}{2}\Gamma(\frac{3}{4})\Gamma(\frac{1}{4}) = \frac{1}{2} \cdot \frac{\pi}{\sin \pi/4} \quad \frac{\pi}{\sqrt{2}}$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

Since $f(x) = \frac{1}{1+x^4}$ is an even fn , then

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\int_0^{\infty} \frac{dx}{1+x^4} ;$$

let $u = x^4 \Rightarrow x = u^{1/4} \Rightarrow dx = \frac{1}{4} u^{-3/4} du$

$$I = 2\int_0^{\infty} \frac{1}{1+u} \cdot \frac{1}{4} u^{-3/4} du .$$

$$= \frac{1}{2}\int_0^{\infty} \frac{u^{-3/4}}{1+u} du = \frac{1}{2} \beta\left(\tfrac{1}{4}, \tfrac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$$

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5) $\int_0^2 x(8-x^3)^{1/3} dx$

let $x^3 = 8t \Rightarrow x = 2t^{1/3} \Rightarrow dx = \frac{2}{3} t^{-2/3} dt$

$$I = \int_0^1 2t^{1/3}(8-8t)^{1/3} \cdot \frac{2}{3} t^{-2/3} dt .$$

$$= \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3} dt = \frac{8}{3} \beta\left(\tfrac{2}{3}, \tfrac{4}{3}\right)$$

$$= \frac{8}{3} \frac{\Gamma(2/3)\,\Gamma(4/3)}{\cancel{1}\ \cancel{\Gamma(2)}} = \frac{8}{3} \Gamma(2/3) \cdot \tfrac{1}{3}\Gamma(1/3)$$

$$= \frac{8}{9}\Gamma(2/3)\,\Gamma(1/3) = \frac{8}{9} \cdot \frac{\pi}{\sin \pi/3} = \frac{16\pi}{9\sqrt{3}}$$

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6) $\int_0^3 \sqrt{x}\,(81-x^4)^{5/8} dx$

let $x^4 = 81t \Rightarrow x = 3t^{1/4} \Rightarrow dx = \frac{3}{4} t^{-3/4} dt$

$$I = \int_0^1 \sqrt{3}\, t^{1/8} (81-81t)^{5/8} \cdot \frac{3}{4} t^{-3/4} dt$$

$$I = \frac{3\sqrt{3}}{4} (81)^{5/8} \int_0^1 t^{-5/8} (1-t)^{5/8} \, dt$$

$$= \frac{3\sqrt{3}}{4} \, 9\sqrt{3} \; \beta\left(\frac{3}{8}, \frac{13}{8}\right)$$

$$= \frac{81}{4} \frac{\Gamma(3/8)\,\Gamma(13/8)}{\Gamma(2)} = \frac{81}{4} \Gamma\left(\frac{3}{8}\right) \cdot \frac{5}{8} \Gamma(5/8)$$

$$= \frac{81}{4} \cdot \frac{5}{8} \cdot \frac{\pi}{\sin \frac{3\pi}{8}} = \frac{405\,\pi}{32 \sin \frac{3\pi}{8}}$$

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7) $$I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

Let $x^4 = \tan^2\theta \Rightarrow x = \sqrt{\tan\theta} \Rightarrow dx = \frac{\sec^2\theta}{2\sqrt{\tan\theta}} \, d\theta$

$$I = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{\sec^2\theta}{2\sqrt{\tan\theta}} \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}\,\sqrt{\tan\theta}} \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sqrt{\tan\theta}} \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{d\theta}{\cos\theta \cdot \sqrt{\frac{\sin\theta}{\cos\theta}}} = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin\theta \, \cos\theta}}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{1}{2}\sin 2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Let $\phi = 2\theta \Rightarrow d\theta = \frac{d\phi}{2}$

$$I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{2\sqrt{\sin\phi}} = \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} (\sin\phi)^{-1/2} \, d\phi$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \, \frac{\Gamma(1/4)}{\Gamma(3/4)}.$$

Example :- Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\Gamma(\frac{1}{n})\sqrt{\pi}}{\Gamma(\frac{1}{n}+\frac{1}{2}) \cdot n}$; $n > 0$

Solution :-

$$L.H.S = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

Let $t = x^n \Rightarrow x = t^{1/n} \Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

at $x = 0 \Rightarrow t = 0$ " Since n is $+ve$ "

at $x = 1 \Rightarrow t = 1$

$$\Rightarrow \int = \int_0^1 \frac{1}{n} \frac{t^{\frac{1}{n}-1} \, dt}{\sqrt{1-t}}$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-1/2} \, dt$$

$$= \frac{1}{n} \beta\left(\frac{1}{n}, \frac{1}{2}\right) = \frac{1}{n} \frac{\Gamma(\frac{1}{n}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n}+\frac{1}{2})}$$

$$= \frac{\Gamma(\frac{1}{n}) \sqrt{\pi}}{n \Gamma(\frac{1}{n}+\frac{1}{2})}$$

* Evaluate these integrals :-

1) $\int_0^2 \sqrt{x}\,(16-x^4)^{5/8}\, dx$

2) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

3) $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

4) $\int_1^e \frac{1}{x} \sqrt[n]{\frac{1}{\ln x} - 1}\, dx$; n is apostive integer. & $n \geq 2$

Solution :-

1) let $x^4 = 16t \Rightarrow x = 2t^{1/4} \Rightarrow dx = \frac{2}{4} t^{-3/4}\, dt$

$$I = \int_0^{16} \sqrt{2}\, t^{1/8} (16)^{5/8} (1-t)^{5/8} \cdot \frac{1}{2} t^{-3/4}\, dt$$

$$= 4 \int_0^{16} t^{-5/8} (1-t)^{5/8}\, dt$$

$$= 4\, \beta\left(\frac{3}{8}, \frac{13}{8}\right) = 4 \cdot \frac{\Gamma(3/8)\,\Gamma(\frac{13}{8})}{\Gamma(2)}$$

$$= 4\, \Gamma\left(\frac{3}{8}\right) \cdot \frac{5}{8} \Gamma(5/8) = \frac{5}{2} \frac{\pi}{\sin \frac{3\pi}{8}}$$

2) $$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta)^{1/2} (\cos\theta)^{-1/2}\, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \frac{\pi}{\sin \pi/4} = \frac{\pi}{\sqrt{2}}$$

3) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = 2\int_{0}^{\infty} \frac{dx}{1+x^6}$$, since the fn is even

let $x^6 = u \longrightarrow x = u^{1/6} \longrightarrow dx = \frac{1}{6} u^{-5/6} du$

$$I = 2\int_0^{\infty} \frac{1}{1+u} \cdot \frac{1}{6} u^{-5/6} du$$

$$= \frac{1}{3}\int_0^{\infty} \frac{u^{-5/6}}{1+u} du = \frac{1}{3}\beta\left(\frac{1}{6}, \frac{5}{6}\right)$$

$$= \frac{1}{3} \frac{\Gamma(1/6)\,\Gamma(5/6)}{\Gamma(1)} = \frac{1}{3} \cdot \frac{\pi}{\sin \pi/6} = \frac{2\pi}{3}$$

4) let $\ln x = u \Rightarrow du = \frac{1}{x} dx$

$$I = \int_1^e \frac{1}{x}\sqrt[n]{\frac{1}{\ln x} - 1}\, dx$$

$$= \int_0^1 \sqrt[n]{\frac{1}{u} - 1}\, du = \int_0^1 \sqrt[n]{\frac{1-u}{u}}\, du$$

$$= \int_0^1 \frac{(1-u)^{1/n}}{u^{1/n}} du = \int_0^1 u^{-1/n}(1-u)^{1/n} du$$

$$= \beta\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$$

$$= \frac{\Gamma(1-\frac{1}{n})\,\Gamma(1+\frac{1}{n})}{\Gamma(2)} = \Gamma(1-\tfrac{1}{n})\,\Gamma(1+\tfrac{1}{n})$$

$$= \Gamma(1-\tfrac{1}{n}) \cdot \frac{1}{n}\Gamma(\tfrac{1}{n})$$

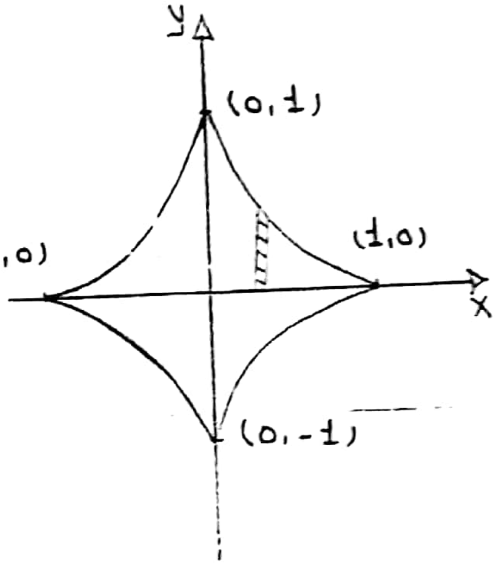
$$= \frac{1}{n} \frac{\pi}{\sin(\pi/n)}$$

xample :-

Show that the area enclosed by the Curve

$x^{2/3} + y^{2/3} = 1$ equals $\frac{3\pi}{8}$.

Solution



$$\text{Area} = 4 A_1$$

$$= 4 \int_0^1 y \, dx$$

$$= 4 \int_0^1 (1 - x^{2/3})^{3/2} \, dx$$

let $t = x^{2/3} \rightarrow x = t^{3/2}$

$$\rightarrow dx = \frac{3}{2} t^{1/2} \, dt$$

$$\text{Area} = 4 \int_0^1 (1-t)^{3/2} \cdot \frac{3}{2} t^{1/2} \, dt$$

$$= 6 \int_0^1 t^{1/2} (1-t)^{3/2} \, dt$$

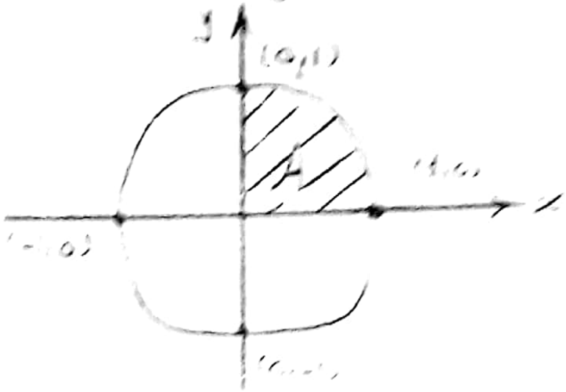
$$= 6 \, \beta\left(\frac{3}{2}, 5/2\right)$$

$$= 6 \cdot \frac{\Gamma(3/2) \, \Gamma(5/2)}{\Gamma(4)} = \frac{6}{3!} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{8} \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \frac{3}{8} \pi$$

Example :- Show that the area enclosed by the curve $x^4 + y^4 = 1$ is $\Gamma^2(\frac{1}{4})/2\sqrt{\pi}$

Solution :- The curve of $x^4 + y^4 = 1$ is as shown :-



Area enclosed $= 4A = 4 \int_0^1 y\, dx$

Since $x^4 + y^4 = 1 \Rightarrow y^4 = 1 - x^4 \Rightarrow y = (1 - x^4)^{1/4}$

Area $= 4 \int_0^1 (1 - x^4)^{1/4}\, dx$

Let $t = x^4 \Rightarrow x = t^{1/4} \Rightarrow dx = \frac{1}{4} t^{-3/4}\, dt$

$\Rightarrow$ Area $= 4 \int_0^1 (1-t)^{1/4} \cdot \frac{1}{4} t^{-3/4}\, dt$

$= \int_0^1 t^{-3/4} (1-t)^{1/4}\, dt$

$= \beta(\frac{1}{4}, 5/4) = \dfrac{\Gamma(1/4)\,\Gamma(5/4)}{\Gamma(3/2)}$

$= \dfrac{\Gamma(1/4) \cdot 1/4\, \Gamma(1/4)}{1/2\, \Gamma(1/2)} = \dfrac{\Gamma^2(1/4)}{2\sqrt{\pi}}$

Example :- Evaluate these integrals

1) $\int_0^\infty \frac{dx}{\sqrt{1+x^4}}$ 2) $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Solution :-

1) let $x^4 = u \rightarrow x = u^{1/4} \rightarrow dx = \frac{1}{4} u^{-3/4} du$

$$I = \int_0^\infty \frac{1}{4} \frac{u^{-3/4}\, du}{\sqrt{1+u}} = \frac{1}{4} \int_0^\infty \frac{u^{-3/4}}{(1+u)^{1/2}} du$$

$$= \frac{1}{4} \beta\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{4} \frac{\Gamma^2(1/4)}{\sqrt{\pi}}$$

2) let $x^4 = \tan^2\theta \Rightarrow x = \sqrt{\tan\theta} \Rightarrow dx = \frac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta$

$$I = \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}\; 2\sqrt{\tan\theta}} d\theta$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta \cdot 2\sqrt{\tan\theta}} d\theta = \frac{1}{2} \int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos\theta \cdot \sqrt{\frac{\sin\theta}{\cos\theta}}} = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin\theta \cos\theta}}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{1}{2} \sin 2\theta}} \Rightarrow \text{let } \phi = 2\theta \Rightarrow d\theta = \frac{d\phi}{2}$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{d\phi}{2 \cdot \frac{1}{\sqrt{2}} \sqrt{\sin\phi}} = \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} (\sin\phi)^{-1/2} d\phi$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \cdot \beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

Example :- Evaluate $\int_a^b (x-a)^p (b-x)^q \, dx$

with Conditions on p & q.

Solution Let $u = x - a \Rightarrow du = dx$

$$\int = \int_0^{b-a} u^p (b-a-u)^q \, du$$

$$= \int_0^{b-a} u^p (b-a)^q \left(1 - \frac{u}{b-a}\right)^q du$$

Let $t = \frac{u}{b-a} \Rightarrow du = (b-a)\, dt$

$$\int = (b-a)^q \int_0^1 (b-a)^p t^p (1-t)^q (b-a)\, dt$$

$$= (b-a)^{q+p+1} \int_0^1 t^p (1-t)^q \, dt$$

$$= (b-a)^{q+p+1} \beta(p+1, q+1) \quad ; \quad p > -1, \; q > -1$$

Example :- Under what Conditions does the integral $\int_0^1 x^m (\ln x)^n \, dx$ exist

hence, find (i) $\int_0^1 \sqrt{\frac{\ln x}{x}} \, dx$ (ii) $\int_0^1 \sqrt{\frac{\ln x}{x^3}} \, dx$

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Solution:- Let $\ln x = -t \Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t}\,dt$, $x = 0 \to t = \infty$

$x = 1 \to t = 0$

$$\int = \int_{\infty}^{0} (e^{-t})^m (-t)^n (-e^{-t})\,dt$$

$$= \int_0^\infty e^{-(m+1)t} (-1)^n \cdot t^n\,dt$$

Let $u = (m+1)t$ $\quad du = (m+1)\,dt$

$$= (-1)^n \int_0^\infty e^{-u} \frac{u^n}{(m+1)^n} \cdot \frac{du}{m+1} \qquad m+1 > 0,\ m > -1$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty u^n e^{-u}\,du = \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

$\therefore n+1 > 0$

$n > -1$

i) $\int_0^1 \sqrt{\frac{\ln x}{x}}\,dx = \int_0^1 x^{-\frac{1}{2}} (\ln x)^{1/2}\,dx$

$m = -\frac{1}{2}$ & $n = \frac{1}{2}$

$$\Rightarrow \int = \frac{\sqrt{-1}}{(\frac{1}{2})^{3/2}} \Gamma(\tfrac{3}{2})$$

$$= i \sqrt{8} \cdot \frac{1}{2} \Gamma(\tfrac{1}{2}) = \frac{i\sqrt{8\pi}}{2}$$

$$= i\sqrt{2\pi}$$

ii) $\int_0^1 \sqrt{\frac{\ln x}{x^3}}\,dx \Rightarrow m = -\frac{3}{2} < -1$

$\Rightarrow$ Integration diverges

Example:- Evaluate $\int_0^3 (27-x^3)^m dx$ for all values of m.

Solution:- $\int_0^3 (27-x^3)^m dx$

$$= (27)^m \int_0^3 \left(1-\frac{x^3}{27}\right)^m dx$$

let $t = \frac{x^3}{27} \Rightarrow x = 3t^{1/3} \Rightarrow dx = t^{-2/3} dt$

$$I = (27)^m \int_0^1 (1-t)^m \, t^{-2/3} dt$$

$= (27)^m \beta(m+1, 1/3)$; for $m+1 > 0$ $\Rightarrow m > -1$

Else, $(m \leq -1)$ the integration diverges.