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Faculty of engineering

Ain Shams University

2nd year

(5)

Complex Integral (Evaluation of Real integrals)

A.K

* Calculation of Real Integration using Complex integral

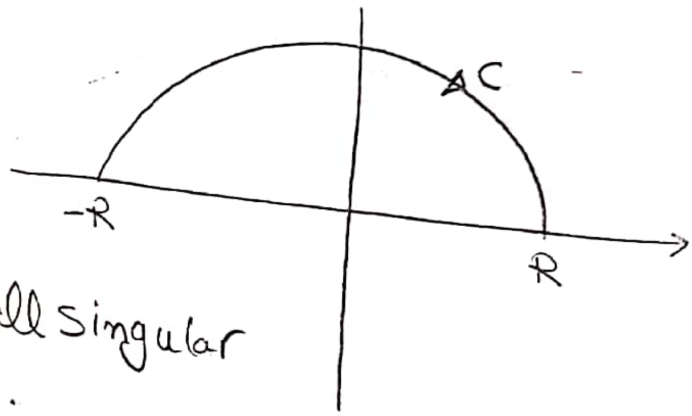
Case(1) $\displaystyle\int_{-\infty}^{\infty} \frac{P_m(x)}{Q_n(x)}\,dx$ if $n \geq m+2$

(i.e)

(درجة المقام $\geq$ درجة البسط $+2$)

$P(x)$ & $Q(x)$ are Polynomial.

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx \Rightarrow \oint_C \frac{P(z)}{Q(z)}\,dz.$$



Where C is closed Path contain all singular Points in upper half Plane.

Pb] $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+4)^2}\,dx$

درجة المقام $\leftarrow 4$ ✓

درجة البسط $\leftarrow 2$

$$\int_0^{\infty} \frac{x^2}{(x^2+4)^2}\,dx = \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx}_{I} \quad \text{(even)}$$

$$I = \oint_C \frac{z^2}{(z^2+4)^2}\,dz$$ (C: closed Path contain all s.P at upper half Plane)

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$$I = \oint_C \frac{Z^2}{(Z+2i)^2(Z-2i)^2}$$

$$= 2\pi i \operatorname{Res}\Big|_{Z=2i}$$

$$= 2\pi i \cdot \left[\frac{1}{1!} \lim_{Z \to 2i} \frac{d}{dZ} \frac{Z^2}{(Z+2i)^2} \right]$$

$$= 2\pi i \left[\lim_{Z \to 2i} \frac{2Z(2+2i)^2 - 2(Z+2i)Z^2}{(Z+2i)^4} \right] = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+4)^2} dx = \frac{1}{2} I = \boxed{\frac{\pi}{8}}$$

(b) $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

حل

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \oint_C \frac{dZ}{1+Z^6} \quad C: \text{upper half Plane}$$

Singular Points @ $Z^6+1=0 \Rightarrow Z^6=-1$, $Z=re^{i\theta}$

$$r^6 e^{i6\theta} = 1\, e^{i(\pi+2n\pi)} \qquad n=0,1,2,3 \dots$$

$r^6=1 \Rightarrow \boxed{r=1}$, $6\theta=\pi+2n\pi \Rightarrow \boxed{\theta = \pi/6 + \frac{n\pi}{3}}$

$n=0 \Rightarrow \theta=\pi/6 \Rightarrow Z_1 = 1e^{i\pi/6}$

$n=1 \Rightarrow \theta=\pi/6+\pi/3 \Rightarrow Z_2 = 1e^{i\pi/2}$ داخل الاطار

$n=2 \Rightarrow \theta=\pi/6+\frac{2\pi}{3} \Rightarrow Z_3 = 1e^{i5/6\pi}$

$n=3 \Rightarrow \theta=\pi/6+\pi \Rightarrow Z_4 = 1e^{i7/6\pi}$ ×

$n=4 \Rightarrow \theta=\pi/6+\frac{4}{3}\pi \Rightarrow Z_5 = 1e^{i9/6\pi}$ × خارج

$n=5 \Rightarrow \theta=\pi/6+\frac{5}{3}\pi \Rightarrow Z_6 = 1e^{i11/6\pi}$ ×

$$\oint_C = 2\pi i \sum Res = 2\pi i \left[Res_{Z_1} + Res_{Z_2} + Res_{Z_3} \right]$$

$$Res\Big|_{Z_1} = \frac{P(Z)}{Q'(Z)}\Bigg|_{Z_1} = \frac{1}{6Z^5}\Bigg|_{Z=1e^{i\pi/6}} = \frac{1}{6}\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$\text{Res}\Big|_{z_2} = \frac{P(z)}{Q'(z)}\Bigg|_{z=1e^{i\pi/2}} = \frac{1}{6z^5}\Bigg|_{z=1e^{i\pi/2}} = \frac{-1}{6}i$$

$$\text{Res}\Big|_{z_3} = \frac{P(z)}{Q'(z)}\Bigg|_{z=1e^{i5\pi/6}} = \frac{1}{6z^5}\Bigg|_{z=1e^{i5\pi/6}} = \frac{1}{6}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$\oint = 2\pi i \sum \text{Res} = \boxed{\frac{2\pi}{3}}$$

Pb $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)}$

$$\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2\ (x^2+4)} = \frac{1}{2} I$$

$$I = \oint_C \frac{dz}{(z^2+1)^2(z^2+4)} = \oint_C \frac{dz}{(z+i)^2(z-i)^2(z+2i)(z-2i)}$$

c: all upper half plane

$$I = 2\pi i\left[\text{Res}\Big|_{i} + \text{Res}\Big|_{2i}\right]$$

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$$\text{Res}\Big|_{i} = \frac{1}{1!}\lim_{Z\to i}\frac{d}{dZ}\frac{1}{(Z+i)^2(Z^2+4)}$$

$$= \frac{1}{1!}\lim_{Z\to i}\frac{-\left[2(Z+i)(Z^2+4)+2Z(Z+i)^2\right]}{(Z+i)^4(Z^2+4)^2}$$

$$= \frac{-\left[2(2i)(3)+2(i)(2i)^2\right]}{(2i)^4(3)^2} = -\frac{1}{36}i$$

$$\text{Res}\Big|_{2i} = \lim_{Z\to 2i}\frac{1}{(Z^2+1)^2(Z+2i)} = \frac{-1}{36}i$$

$$\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{1}{2}\cdot 2\pi i\sum \text{Res}$$

$$= \frac{1}{2}\cdot 2\pi i\left[\frac{-1}{36}i + \frac{-1}{36}i\right]$$

$$= \boxed{\frac{\pi}{18}}$$

* Case (2)

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta$$

← دالة بها فقط ارقام و sin و Cos

* Let $Z = e^{i\theta}$ $\Rightarrow dZ = ie^{i\theta} d\theta = iZ\, d\theta$

$$\boxed{d\theta = \frac{dZ}{iZ}}$$

* $\int_0^{2\pi} d\theta \Longrightarrow \oint_{|Z|=1} dZ$

* $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{Z - \frac{1}{Z}}{2i}$

* $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{Z + \frac{1}{Z}}{2}$

then Integration will be $\oint_{|Z|=1} f(Z)\, dZ = 2\pi i \sum Res.$

Note

$$\left.\begin{aligned} \sin^2\theta &= \tfrac{1}{2}(1 - \cos 2\theta) \\ \cos^2\theta &= \tfrac{1}{2}(1 + \cos 2\theta) \end{aligned}\right\} \quad \cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{Z^2 + \frac{1}{Z^2}}{2}$$

Pb $\int_0^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2}$

let $Z = e^{i\theta} \rightarrow dZ = i e^{i\theta} d\theta \rightarrow dZ = iZ d\theta \Rightarrow \boxed{d\theta = \frac{dZ}{iZ}}$

* $\int_0^{2\pi} d\theta \longrightarrow \oint_{|Z|=1} dZ$

* $\cos\theta = \frac{Z + \frac{1}{Z}}{2}$

$$I = \oint_{|Z|=1} \frac{dZ/iZ}{\left(5 + 4 \cdot \frac{Z + 1/Z}{2}\right)^2} = \oint_{|Z|=1} \frac{dZ}{iZ\left(5 + 2Z + \frac{2}{Z}\right)^2} * \left(\frac{Z}{Z}\right)$$

$$I = \oint \frac{-iZ\,dZ}{(5Z + 2Z^2 + 2)^2} = \oint \frac{-iZ\,dZ}{(2Z+1)^2 (Z+2)^2}$$

$$I = 2\pi i \operatorname{Res}\Big|_{Z=-1/2}$$

$$\operatorname{Res}\Big|_{Z=-1/2} = \frac{1}{1!} \lim_{Z \to -1/2} \frac{-iZ}{2^2 \cancel{(Z+\frac{1}{2})^2} (Z+2)^2} \cdot \cancel{(Z+\tfrac{1}{2})^2}$$

$$= \frac{-i}{4} \cdot \frac{(Z+2)^2 - 2(Z+2)Z}{(Z+2)^4}\Bigg|_{Z=-1/2} = \boxed{\frac{-5i}{27}}$$

$$I = 2\pi i \left(\frac{-5i}{27}\right) = \boxed{\frac{10\pi}{27}}$$

$$\int_0^{\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$$

الحل

Function is even ⇒ sym. about $\theta = 0, \pi, 2\pi \cdots$

$$\int_0^{\pi} = \frac{1}{2}\int_0^{2\pi}$$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{\frac{1}{2}[1-\cos 2\theta]}{5+4\cos\theta}\, d\theta$$

let $Z = e^{i\theta} \Rightarrow \boxed{d\theta = \frac{dZ}{iZ}}$, $\int_0^{2\pi} \Rightarrow \oint_{|Z|=1} dZ$

$$\cos\theta = \frac{Z + 1/Z}{2} \quad , \quad \cos 2\theta = \frac{Z^2 + 1/Z^2}{2}$$

$$I = \frac{1}{4}\oint_{|Z|=1} \frac{\left[1 - \frac{Z^2 + 1/Z^2}{2}\right]}{5 + \frac{4}{2}(Z + 1/Z)} \cdot \frac{dZ}{iZ} \quad * \frac{2Z^2}{2Z^2}$$

$$I = \frac{-i}{8}\oint \frac{[2Z^2 - Z^4 - 1]}{Z^2[5Z + 2Z^2 + 2]}$$

$$I = \frac{i}{8}\oint \frac{Z^4 - 2Z^2 + 1}{Z^2(2Z+1)(Z+2)}\, dZ$$

$$= \frac{i}{16}\oint \frac{(Z^2-1)^2}{Z^2(Z+1/2)(Z+2)}$$

$$I = \frac{i}{16} \cdot 2\pi i \left[\text{Res}\Big|_{-1/2} + \text{Res}\Big|_{0} \right]$$

$$\text{Res}\Big|_{-1/2} = \lim_{Z \to -1/2} \frac{(Z^2-1)^2}{Z^2(Z+2)} = 3/2$$

$$\text{Res}\Big|_{0} = \frac{1}{1!} \lim_{Z \to 0} \frac{d}{dZ} \frac{(Z^2-1)^2}{(Z+1/2)(Z+2)} = -5/2$$

↳ [Can solve by Calcultor]

$$I = \frac{i}{16} \cdot 2\pi i \, [-1] = \boxed{\pi/8}$$

Pb) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}$

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$$I = \int_0^{2\pi} \frac{\frac{1}{2}[1 + \cos 6\theta]}{5 - 4\cos 2\theta} \, d\theta$$

let $Z = e^{i\theta} \rightarrow d\theta = \frac{dZ}{iZ}$, $\displaystyle\int_0^{2\pi} d\theta \rightarrow \oint_{|Z|=1} dZ$

$$\cos 2\theta = \frac{Z^2 + 1/Z^2}{2}$$

$$\cos 6\theta = \frac{Z^6 + 1/Z^6}{2}$$

$$I = \oint_{|Z|=1} \frac{\frac{1}{2}\left[1 + \frac{Z^6 + 1/Z^6}{2}\right]}{5 - \frac{4}{2}\left(Z^2 + \frac{1}{Z^2}\right)} \cdot \frac{dZ}{iZ} \quad * \frac{2Z^6}{2Z^6}$$

$$= \frac{-i}{4} \oint \frac{2Z^6 + Z^{12} + 1}{Z^5 (5Z^2 - 2Z^4 - 2)}$$

$$I = \frac{i}{4} \oint_{|Z|=1} \frac{(Z^6+1)^2}{Z^5 (2Z^4 - 5Z^2 + 2)} \, dZ$$

$$I = i/4 \oint_{|Z|=1} \frac{(Z^6+1)^2}{Z^5 (2Z^2 - 1)(Z^2 - 2)}$$

$$I = i/8 \oint \frac{(Z^6+1)^2}{Z^5 (Z^2 - 1/2)(Z^2 - 2)}$$

$$= i/8 \oint \frac{(Z^6+1)^2}{Z^5 (Z - \frac{1}{\sqrt{2}})(Z + \frac{1}{\sqrt{2}})(Z - \sqrt{2})(Z + \sqrt{2})}$$

$$I = i/8 \; 2\pi i \left[\text{Res}\Big|_{\frac{1}{\sqrt{2}}} + \text{Res}\Big|_{-\frac{1}{\sqrt{2}}} + \text{Res}\Big|_{0} \right]$$

$$\text{Res}\Big|_{\frac{1}{\sqrt{2}}} = \lim_{Z \to \frac{1}{\sqrt{2}}} \frac{(Z^6+1)^2}{Z^5 (Z + \frac{1}{\sqrt{2}})(Z^2 - 2)} = -27/8$$

$$\text{Res}\Big|_{-\frac{1}{\sqrt{2}}} = \lim_{Z \to -1/2} \frac{(Z^6+1)^2}{Z^2 (Z - \frac{1}{\sqrt{2}})(Z^2 - 2)} = -27/8$$

$$\text{Res}\Big|_{Z=0} = \frac{1}{4!}\frac{d^4}{dZ^4}\frac{(Z^6+1)^2}{(Z^2-\frac{1}{2})(Z^2-2)} = 126$$

تفاضل ٤ مرات ليه هو احنا كفرة

, then $I = \frac{i}{8}(2\pi i)\sum \text{Res} = -\frac{477}{16}\pi$

Case "3"

$$\int_{-\infty}^{\infty} \frac{P_n(x)}{Q_m(x)} \cos kx \, dx$$

OR $$\int_{-\infty}^{\infty} \frac{P_n(x)}{Q_m(x)} \sin kx \, dx$$

rule

① $\int_{-\infty}^{\infty}$

② $m \geq n+2$ "Like case 1"

then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos kx = \text{Real} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ikx} dx$$

$$= \text{Real} \oint_{C} \frac{P(Z)}{Q(Z)} e^{ikZ} dZ = \text{Real}\{2\pi i \sum \text{Res}\}$$

upper half plane

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin kx \Rightarrow \text{Imag} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ikx}$$

$$= \text{Imag} \oint_{C} \frac{P(Z)}{Q(Z)} e^{ikZ} dZ = \text{Imag}\{2\pi i \sum \text{Res}\}$$

b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 4x}{x^2+4}\,dx$

$$\int_{-\infty}^{\infty} \frac{\cos 4x}{x^2+4}\,dx = \text{Real} \int_{-\infty}^{\infty} \frac{e^{i4x}}{x^2+4}\,dx$$

$$= \text{Real} \oint_{\substack{\text{upper half}\\ \text{Plane}}} \frac{e^{i4z}}{z^2+4}\,dz = \text{Real} \oint \frac{e^{i4z}}{(z+2i)(z-2i)}\,dz$$

$$= \text{Real}\left\{2\pi i \,\text{Res}\Big|_{2i}\right\}$$

$$\text{Res}\Big|_{2i} = \lim_{z\to 2i} \frac{e^{i4z}}{(z+2i)} = \frac{e^{-8}}{4i}$$

$$I = \text{Real}\left\{2\pi i \cdot \frac{e^{-8}}{4i}\right\} = \text{Real}\left\{\frac{\pi e^{-8}}{2}\right\} = \frac{\pi e^{-8}}{2}$$

Note if $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 4x}{x^2+4} = \text{Img}\left\{\frac{\pi e^{-8}}{2}\right\} = \text{Zero}$

$\Rightarrow$ we can note that function is odd $\rightarrow \displaystyle\int_{-\infty}^{\infty} \text{odd} = \underline{\underline{\text{Zero}}}$

$$\int_{-\infty}^{\infty} \frac{x \sin 3x}{x^4+4}\, dx$$

$$I = \text{Imag} \int_{-\infty}^{\infty} \frac{x e^{i3x}}{x^4+4}\, dx = \text{Imag} \oint_C \frac{z e^{i3z}}{z^4+4}\, dz$$

C ↳ upper half plane

S.P @ $z^4+4=0 \Rightarrow z^4=-4$ — let $z = re^{i\theta}$

$$r^4 e^{i4\theta} = 4 e^{i(\pi+2n\pi)} \qquad n = 0, \pm 1, \pm 2 \ldots$$

$$r^4 = 4 \Rightarrow \boxed{r = \sqrt{2}}$$

$$4\theta = \pi + 2n\pi \Rightarrow \boxed{\theta = \frac{\pi}{4} + \frac{n\pi}{2}}$$

$n=0 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow z_1 = \sqrt{2}\, e^{i\pi/4}$ (داخل المسار)

$n=1 \Rightarrow \theta = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \Rightarrow z_2 = \sqrt{2}\, e^{i3\pi/4}$ (داخل المسار)

$n=2 \Rightarrow \theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4} \Rightarrow z_3 = \sqrt{2}\, e^{\frac{i5\pi}{4}}$ (خارج المسار)

$n=3 \Rightarrow \theta = \frac{\pi}{4} + \frac{3\pi}{2} = \frac{7\pi}{4} \Rightarrow z_4 = \sqrt{2}\, e^{i7\pi/4}$ (خارج المسار)

$$\therefore I = \text{Imag} \left\{ 2\pi i \left[\text{Res}|_{z_1} + \text{Res}|_{z_2} \right] \right.$$

$$\text{Res}\Big|_{z_1} = \frac{P(z)}{Q'(z)}\Big|_{z_1} = \frac{z e^{i3z}}{4z^3}\Big|_{z=\sqrt{2}e^{i\pi/4}}$$

$$\boxed{\sqrt{2}e^{i\pi/4} = 1+i}$$

$$= \frac{\sqrt{2}e^{i\pi/4} \cdot e^{i3(1+i)}}{4(\sqrt{2}e^{i\pi/4})^3} = \frac{\sqrt{2}e^{i\pi/4}\, e^{-3+3i}}{4 \cdot 2^{3/2} \cdot e^{i3\pi/4}}$$

$$= \frac{1}{8} \cdot \underbrace{e^{i(-\pi/2)}}_{-i} \cdot e^{-3} \cdot e^{3i}$$

$$= \frac{-i}{8} e^{-3}\left[\cos 3 + i \sin 3\right]$$

$$\text{Res}\Big|_{z_2} = \frac{P(z)}{Q'(z)}\Big|_{z_2} = \frac{z e^{i3z}}{4z^3}\Big|_{z=\sqrt{2}e^{i3\pi/4}}$$

$$\boxed{\sqrt{2}e^{i3\pi/4} = -1+i}$$

$$= \frac{\sqrt{2}e^{i3\pi/4} \cdot e^{i3(-1+i)}}{4(\sqrt{2}e^{i\frac{3\pi}{4}})^3} = \frac{i}{8} e^{-3}\left(\cos 3 - i \sin 3\right]$$

$$I = \text{Img}\left\{2\pi i \sum \text{Res}\right\} = \text{Img}\left\{\frac{\pi i}{2} e^{-3} \sin 3\right\}$$

$$= \boxed{\frac{\pi}{2} e^{-3} \sin(3)}$$

Note $\int_{-\infty}^{\infty} \frac{x \cos 3x}{x^4+4} = \text{zero}$ "odd"

$$\int_{-\infty} \frac{x^2 \cos x}{(x^2+1)^2}\, dx$$

س1

$$I = \text{Real}\left\{ \int_{-\infty}^{\infty} \frac{x^2 e^{ix}}{(x^2+1)^2}\, dx \right\}$$

$$= \text{Real}\left\{ \oint_{\substack{c \\ \text{upper half} \\ \text{Plane}}} \frac{Z^2 e^{iZ}}{(Z^2+1)^2}\, dZ \right\}$$

$$I = \text{Real}\left\{ \oint_{c} \frac{Z^2 e^{iZ}}{(Z+i)^2 (Z-i)^2}\, dZ \right\}$$

$$= \text{Real}\left\{ 2\pi i \; \text{Res}\Big|_{Z=i} \right\}$$

$$\text{Res}\Big|_{Z=i} = \frac{1}{1!} \lim_{Z \to i} \frac{d}{dZ} \frac{Z^2 e^{iZ}}{(Z+i)^2}$$

$$= \frac{\left(2Ze^{iZ} + iZ^2 e^{iZ}\right)(Z+i)^2 - 2(Z+i)\left(Z^2 e^{iZ}\right)}{(Z+i)^4}\Bigg|_{Z=i}$$

$$= \frac{(2ie^{-1} - ie^{-1})(-4) - \left[2(2i)(-e^{-1})\right]}{16} = \text{Zero}$$

$$I = \text{Zero}$$