Outils d'optimisation pour les sciences des donnees et de la decision

Linearization Techniques of Binary Quadratic Programs

The Quadratic Formulation (QF) of a Binary Quadratic Program (BQP) with n variables and p constraints, reads as follows:

$$
\begin{aligned}
(\text{QF}) \quad \max\ & \sum_{i=1}^{n}\sum_{j=i}^{n} Q_{ij}x_i x_j + \sum_{i=1}^{n} L_i x_i \\
& x \in K \\
& x \in \{0,1\}^n,
\end{aligned}
$$

where $Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^n$, $K = \{x \in \mathbb{R}^n : Ax \geq b\}$, $A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$.

Glover-Woolsey Linear Formulation

$$
\begin{aligned}
\text{GW}: \quad \max\ & \sum_{i=1}^{n}\sum_{j=i+1}^{n} 2Q_{ij}y_{ij} + \sum_{i=1}^{n} L_i x_i \\
& \left.\begin{array}{l}
(1)\ y_{ij} \leq x_i \\
(2)\ y_{ij} \leq x_j \\
(3)\ y_{ij} \geq x_i + x_j - 1 \\
(4)\ y_{ij} \geq 0
\end{array}\right\} i,j = 1,\ldots,n \quad i<j \\
& x \in K \\
& x \in \{0,1\}^n
\end{aligned}
$$

The new variables y_{ij} take the place of the products between the original variables x_i and x_j. GW increases the size of the problem by adding: $n(n-1)/2$ variables and $4n(n-1)/2$ constraints.

Glover Linear Formulation

$$\text{G}: \quad \max \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} L_i x_i$$

$$\left.\begin{array}{l} (1)\ w_i \le Q_i^+ x_i \\ (2)\ w_i \le \sum_{j=1}^{n} Q_{ij} x_j - Q_i^-(1 - x_i) \end{array}\right\} \qquad i = 1, \ldots, n$$

$$x \in K$$

$$x \in \{0,1\}^n$$

$$Q_i^- = \sum_{j=1}^{n} \min\{0, Q_{ij}\}\ ,\ Q_i^+ = \sum_{j=1}^{n} \max\{0, Q_{ij}\} \tag{1}$$

The new variables w_i take the place of the products between the original variables x_i and x_j. G increases the size of the problem by adding: n variables ($\sum_{j=1}^{n} Q_{ij} x_j$ if $x_i = 1$, or 0 otherwise) and $4n$ constraints.

In case of minimization we have the following constraints:

$$\left.\begin{array}{l} (1)\ w_i \ge Q_i^- x_i \\ (2)\ w_i \ge \sum_{j=1}^{n} Q_{ij} x_j - Q_i^+(1 - x_i) \end{array}\right\} \qquad i = 1, \ldots, n$$

Case Study – Investigate the performance of the two different Linearization

- Consider the BQP data sets on the web page `http://qplib.zib.de/`). Analyse the instances describing their main features, with particular attention to the presence of binary variables and the different kind of linear constraints. Consider the instances with only binary variables and linear constraints. Solve the GW and the G linearization techniques using a Mixed Integer Linear Programming solver (e.g., CPLEX or Gurobi `https://www.gurobi.com/`). Analyse and compare the quality of the bound provided by the Linear Programming relaxation of the two Linearization Techniques.

- The dimension and the computational complexity increase with the number of binary variables. Analyse the variation of the computational time necessary to solve the instances in function of the number of binary variables. Determine the maximum dimension of the instances that can be solved to proven optimality within a computing time of 10 minutes.

- Discuss the advantages and the disadvantages of the two linearization teachniques. Discuss the results as you were in machine learning data center to convince the management to adopt one or the other linearization technique. [1]

[1] For further details on the linearization techniques see: `https://link.springer.com/article/10.1007/s10479-018-3118-2`