$=$

HOMEWORK 6

JESSE COBB - 3PM SECTION (MON,WED)

1. m deductible, X denotes true cost of damages, and Y denote amount of money actually paid by you. $X \sim \text{Exp}(\lambda)$ where $\lambda > 0$.
   a. $Y = g(X) = \begin{cases} X & 0 \le Y \le m \\ m & m < Y \end{cases}$
   b. $E[Y] = E[g(X)] = \int_0^m \lambda x e^{-\lambda x} dx + \int_m^\infty \lambda e^{-\lambda x} dx$
   $= [-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}]_0^m + [-e^{-\lambda x}]_m^\infty$
   $= -me^{-\lambda m} - \frac{e^{-\lambda m}}{\lambda} + \frac{1}{\lambda} + e^{-\lambda m}$
   $= e^{-\lambda m}(-m - \frac{1}{\lambda} + 1) + \frac{1}{\lambda}$
   c. $F_Y(y) = \begin{cases} 1 - e^{-\lambda y} & 0 \le y \le m \\ 1 - e^{-\lambda m} & m < y \\ 0 & \text{otherwise} \end{cases}$
   d. Y is continuous in the sense that it is measured via density functions and that there are an uncountably infinite number of valid random variables Y. Though it does exhibit some discrete tendencies such as grouping of values. Therefore Y is neither.
2. Triangle: $(0,0), (1,0), (0,1)$. (X, Y) are uniformly distributed.
   a. $f_{X,Y}(x,y) = \frac{1}{\frac{1}{2}} = 2$
   $f_X(x) = \int_0^{1-x} 2dy = 2y]_0^{1-x} = 2 - 2x$
   $f_Y(y) = \int_0^{1-y} 2dx = 2x]_0^{1-y} = 2 - 2y$
   b. $E[X] = \int_0^1 x(2-2x)dx = \int_0^1 2x - 2x^2 dx = x^2 - \frac{2x^3}{3}]_0^1 = \frac{1}{3}$
   $E[Y] = \int_0^1 y(2-2y)dy = \int_0^1 2y - 2y^2 dy = y^2 - \frac{2y^3}{3}]_0^1 = \frac{1}{3}$
   c. $\int_0^1 \int_0^{1-x} 2xydydx = \int_0^1 xy^2]_0^1 dx = \int_0^1 xdx = 1$
   d. $P(X > Y) = \frac{1}{2}$, since $y = x$ line cuts region in half.
3. $f_{X,Y}(x,y) = \begin{cases} c \cdot (\frac{y}{x})^4 & (x,y) \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$
   $\mathcal{R}$ is the region in first quadrant under $y = \min\{x, 1\}$ and $c > 0$.
   a. $\int_0^1 \int_0^x c \cdot (\frac{y}{x})^4 dydx + \int_1^\infty \int_0^1 c \cdot (\frac{y}{x})^4 dydx$
   $= \frac{c}{5} \int_0^1 \frac{y^5}{x^4}]_0^x dx + \frac{c}{5} \int_1^\infty \frac{y^5}{x^4}]_0^1 dx$
   $= \frac{c}{5} \int_0^1 xdx + \frac{c}{5} \int_1^\infty \frac{1}{x^4} dx$
   $= \frac{c}{10} - \frac{c}{15}[x^{-3}]_1^\infty$
   $= \frac{c}{10} + \frac{c}{15} = \frac{5c}{30} = \frac{c}{6} = 1 \implies c = 6$
   b. $P(X + Y \ge 2) = 6 \int_1^2 \int_{2-x}^1 (\frac{y}{x})^4 dydx + 6 \int_2^\infty \int_0^1 (\frac{y}{x})^4 dydx$
   c. $f_X(x) = 6 \int_0^x (\frac{y}{x})^4 dy + 6 \int_0^1 (\frac{y}{x})^4 dy = \frac{6x}{5} + \frac{6}{5x^4}$
   d. $f_Y(y) = 6 \int_y^1 (\frac{y}{x})^4 dx + 6 \int_1^\infty (\frac{y}{x})^4 dx = -2\frac{y^4}{x^3}]_y^1 - 2\frac{y^4}{x^3}]_1^\infty$
   $= -2[y^4 - y] - 2[-y^4] = -2y^4 + 2y + 2y^4 = 2y$

e. $E[X] = 6\int_0^1 \int_0^x \frac{y^4}{x^3} dydx + 6\int_1^\infty \int_0^1 \frac{y^4}{x^3} dydx = \frac{6}{5}\int_0^1 \frac{y^5}{x^3}]_0^x dx + \frac{6}{5}\int_1^\infty \frac{y^5}{x^3}]_0^1 dx$
$= \frac{6}{5}\int_0^1 x^2 dx + \frac{6}{5}\int_1^\infty \frac{1}{x^3} dx = \frac{2}{5}[x^3]_0^1 - \frac{3}{5}[\frac{1}{x^2}]_1^\infty$
$= \frac{2}{5} + 1 = \frac{7}{5}$

f. $E[Y] = \int_0^1 2y^2 dy = \frac{2y^3}{3}]_0^1 = \frac{2}{3}$

4. Let $Y - X^\beta$ where $X \sim \text{Exp}(1)$ and $\beta = 3$.

a. $P(Y > s+t | Y > s) = \frac{P(Y>s+t)}{P(Y>s)}$
$P(Y > s+t) = P(X > \sqrt[3]{s+t}) = 1 - F_X(\sqrt[3]{s+t}) = e^{-\sqrt[3]{s+t}}$
$P(Y > s) = P(X > \sqrt[3]{s}) = 1 - F_X(\sqrt[3]{s}) = e^{-\sqrt[3]{s}}$
$P(Y > t) = P(X > \sqrt[3]{t}) = 1 - F_X(\sqrt[3]{t}) = e^{-\sqrt[3]{t}}$
$P(Y > s+t | Y > s) = e^{-\sqrt[3]{s+t} + \sqrt[3]{s}} \neq P(Y > t)$
Not memoryless

b. $E[Y] = E[X^3] = \int_0^\infty x^3 e^{-x} dx = 6$
$E[X^6] = \int_0^\infty x^6 e^{-x} dx = 720$
$\text{Var}(Y) = \text{Var}(X^3) = E[X^6] - E[X^3]^2 = 720 - 36 = 684$

5. $p_{X,Y}(x,y) = \begin{cases} (y-1)(\frac{1}{2})^{x+y} & x \in \{1, 2\ldots\}, y \in \{2, 3\ldots\} \\ 0 & \text{otherwise} \end{cases}$

a. $\displaystyle\sum_{y=2}^{\infty}\sum_{x=1}^{\infty}(y-1)(\frac{1}{2})^{x+y} = 1$ Valid PMF

b. $p_X(x) \displaystyle\sum_{y=2}^{\infty}(y-1)(\frac{1}{2})^{x+y} = 2^{-x}$

$p_Y(y) \displaystyle\sum_{x=1}^{\infty}(y-1)(\frac{1}{2})^{x+y} = 2^{-y}(y-1)$

c. $E[X] = \displaystyle\sum_{y=2}^{\infty}\sum_{x=1}^{\infty}x(y-1)(\frac{1}{2})^{x+y} = 2$

$E[Y] = \displaystyle\sum_{y=2}^{\infty}\sum_{x=1}^{\infty}y(y-1)(\frac{1}{2})^{x+y} = 4$

d. $E[XY] = \displaystyle\sum_{y=2}^{\infty}\sum_{x=1}^{\infty}xy(y-1)(\frac{1}{2})^{x+y} = 8$
$E[XY] = E[X]E[Y] = 8$

6. $f(x,y) = \begin{cases} c(1-y) & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$

a. $c\int_0^1 \int_0^y (1-y) dxdy = c\int_0^1 y - y^2 dy = c[\frac{1}{2} - \frac{1}{3}] = \frac{c}{6} = 1 \implies c = 6$

b. $P(X < \frac{1}{2} | Y > \frac{2}{3}) = \frac{P(X<\frac{1}{2} \cap Y > \frac{2}{3})}{P(Y>\frac{2}{3})}$
$P(X < \frac{1}{2} \cap Y > \frac{2}{3}) = 6\int_{\frac{2}{3}}^1 \int_0^y (1-y) dxdy - 6\int_{\frac{2}{3}}^1 \int_{\frac{1}{2}}^y (1-y) dxdy$
$= 6\int_{\frac{2}{3}}^1 (y - y^2) dy - 6\int_{\frac{2}{3}}^1 \frac{3y}{2} - y^2 - \frac{1}{2} dy$
$= 6[\frac{y^2}{2} - \frac{y^3}{3}]_{\frac{2}{3}}^1 - 6[\frac{3y^2}{4} - \frac{y^3}{3} - \frac{y}{2}]_{\frac{2}{3}}^1$
$= \frac{7}{27} - \frac{1}{27} = \frac{6}{27} = \frac{2}{9}$

$P(Y > \frac{2}{3}) = 6\int_{\frac{2}{3}}^{1}\int_0^y (1-y)dxdy = \frac{7}{27}$

$P(X < \frac{1}{2}|Y > \frac{2}{3}) = \frac{6}{7}$

c. $E[X] = 6\int_0^1\int_0^y x(1-y)dxdy = \frac{1}{4}$

d. $E[Y] = 6\int_0^1\int_0^y y(1-y)dxdy = \frac{1}{2}$

7. $f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2) & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

a. $P(X < 2Y) = \int_0^1\int_{\frac{x}{2}}^1 \frac{12}{7}(xy+y^2)dydx = \frac{13}{14}$

b. $f(x) = \int_0^1 \frac{12}{7}(xy+y^2)dy = \frac{12}{7}(\frac{1}{3}+\frac{x}{2})$

$f(y) = \int_0^1 \frac{12}{7}(xy+y^2)dx = \frac{12}{7}(y^2+\frac{y}{2})$

c. $E[X^2Y] = \int_0^1\int_0^1 \frac{12}{7}x^2y(xy+y^2)dydx = \frac{2}{7}$