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HOMEWORK 2

JESSE COBB - 2PM SECTION

1.5.3 c. *Proof.* Solving by the contrapositive, let x be even, so that $x = 2k$ for some $k \in \mathbb{Z}$. We will prove that if this assumption is correct $4 \mid x^2$. Since $x = 2k$, $x^2 = (2k)^2 = 4k^2 = 2h$ where $h = k^2$ is an integer. Thus we've proved by the contrapositive that if $4 \nmid x^2$ then x is odd. □

d. *Proof.* Solving by the contrapositive, let both x and y be odd, so that $x = 2k + 1$ and $y = 2j + 1$ for some $k, j \in \mathbb{Z}$. We'll prove that if this assumption is correct xy is odd. Since $x = 2k + 1$ and $y = 2j + 1$, then $xy = (2k+1)(2j+1) = 4kj+2j+2k+1 = 2(2kj+j+k)+1 = 2h+1$ where $h = 2kj+j+k$ is an integer. Thus we've proved through the contrapositive that if $x + y$ is even then x or y have to be even. □

e. *Proof.* Solving by the contrapositive, let x and y have different parity then, without loss of generality, assume x is even and y is odd so that $x = 2k$ and $y = 2j+1$ for some $k, j \in \mathbb{Z}$. So that $x+y = 2k+2j+1 = 2(k+j)+1 = 2h+1$ where $h = k+j$ is an integer. Thus we've proved through the contrapositive that if $x + y$ is even then x and y must be the same parity. □

f. *Proof.* Solving by the contrapositive, let at least x or y be odd. Without loss of generality consider the following cases:
Case 1: If x is odd and y is even, such that $x = 2k + 1$ and $y = 2j$ for $k, j \in \mathbb{Z}$ then $xy = (2k + 1)2j = 4kj + 2j = 2(2kj + j) = 2h$ where $h = 2kj + j$ is an integer.
Case 2: If x and y are both even, such that $x = 2k$ and $y = 2j$ for $k, j \in \mathbb{Z}$ then $xy = (2k)2j = 4kj = 2(2kj) = 2h$ where $h = 2kj$ is an integer.
Thus we've proved by the contrapositive that if xy is odd then x and y must both be odd. □

g. *Proof.* Solving by the contrapositive, let x be odd, so that $x = 2k + 1$ for some $k \in \mathbb{Z}$. We'll show that $8 \mid x^2 - 1$ if the assumption is true. Since $x = 2k + 1$ then $x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k) = 4k(k + 1)$. Since $a(a + 1)$ is even for any integer $4k(k + 1) = 4 * 2(h) = 8h$ where $h \in \mathbb{Z}$ therefore $x^2 - 1 = 8h$ if x is odd. Thus we've proved through the contrapositive that if $8 \nmid x^2 - 1$ then x is even. □

h. *Proof.* Solving by the contrapositive, let $x \mid z$ so that $z = xk$ for some $k \in \mathbb{Z}$. Then $yz = y(xk) = x(yk) = xj$ where $j = yk$ is an integer. Thus we've proved by contrapositive that if $x \nmid yz$ then $x \nmid z$. □

1.5.4 d. *Proof.* Solving by the contrapositive, let the real number x be greater than or equal to 1, $x \geq 1$. Therefore $x - 1 \geq 0$ which implies $x^2 - x \geq 0$ since x is always positive. This is equivalent to $(x + 1)(x - 1) \geq 0$. Thus we've proved through the contrapositive that if $(x+1)(x-1) < 0$ then $x < 1$. □

e. *Proof.* Solving by the contrapositive, let the real number x then assume $3 \geq x \geq 1$. Since $3 \geq x$ then $0 \geq x-3$ and since $1 \leq x$ then $0 \leq x-1$. With these equivalence statement we can conclude the $0 \geq (x - 3)(x - 1) \implies$

$0 \geq x^2-4x+3 \implies -3 \geq x^2-4x \implies -3 \geq x(x-4)$. Thus we've proved through contrapositive that if $x(x-4) > 0$ then $x < 1$ or $x > 3$. $\square$

1.5.6 d. *Proof.* Let $a-b$ be odd such that $a-b = 2k+1$ for $k \in \mathbb{Z}$. Seeking a contradiction, assume $a+b$ is even such that $a+b = 2j$ for some integer j. Therefore $a-b = 2k+1 \implies a-b+2b = 2k+1 \implies 2j+2b = 2k+1 \implies 2(j+b) = 2k+1 \implies 2h = 2k+1$ where $h = j+b$ is an integer. This results in a contradiction showing that an even number ($2h$) is equal to an odd number ($2k+1$). Therefore the claim follows that if $a-b$ is odd then $a+b$ must be odd. $\square$

e. *Proof.* Let $a < b$ and $ab < 3$ for $a, b \in \mathbb{Z}$ where $a, b > 0$. Seeking a contradiction, assume $a \neq 1$. Since $ab < 3$ then $0 < b < 3$ which implies $0 < a < 2$. This means the only valid value for a is 1 which contradicts our assumption. Thus we've proved the claim if $a < b$ and $ab < 3$ then $a = 1$. $\square$

1.5.7 a. *Proof.* Assume a and b are positive integers. Therefore $a \mid b$ if and only if:

$$b = ak \text{ where } k \in \mathbb{Z} \iff$$
$$bc = ack \iff$$
$$ac \mid bc$$

Since the final statement is true all the previous statements are true including $a \mid b$. Thus we've proved the claim that $ac \mid bc$ if and only if $a \mid b$. $\square$

b. *Proof.* Let a and b be positive integers. We'll prove that $a+1 \mid b$ and $b \mid b+3$ if and only if $a = 2$ and $b = 3$.

($\implies$) Assume that $a+1 \mid b$ and $b \mid b+3$ so that $b = (a+1)k$ and $b+3 = bj$ for some $k, j \in \mathbb{Z}$. Then $(a+1)k+3 = (a+1)kj \implies ak+k+3 = akj+kj \implies 3 = ak(j-1)+k(j-1) \implies 3 = h(a+1) \implies a+1 \mid 3$. This requires $a+1$ to be 1 or 3, where 1 is impossible since it would require $a = 0$, therefore $a = 2$. Thus $b = 3k$ which means $3 \mid b$ and $3 = b(j-1) = bh$ where $b \mid 3$ which means that $b = 3$.

($\impliedby$) Now assume, for the reverse direction, that $a = 2$ and $b = 3$. This means $a+1 = 3 \implies a+1 \mid 3$ which means $a+1 \mid b$ and $b \mid b+3$ since $3 \mid 6$.

Thus we've proved that $a+1 \mid b$ and $b \mid b+3$ if and only if $a = 2$ and $b = 3$ by showing their implications in both directions are true. $\square$

1.5.9 *Proof.* Seeking a contradiction, for the natural number n assume $\frac{n}{n+1} \leq \frac{n}{n+2}$. Since $n \geq 1$ we can say that $\frac{1}{n+1} \leq \frac{1}{n+2} \implies n+2 \leq n+1 \implies 2 \leq 1$ which is a contradiction. Thus the claim stands that for any natural number n $\frac{n}{n+1} > \frac{n}{n+2}$. $\square$

1.5.10 *Proof.* Seeking a contradiction, assume that $\sqrt{5}$ is rational such that $\sqrt{5} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. Then $5 = \frac{a^2}{b^2} \implies a^2 = 5b^2$. By the fundamental theorem of arithmetic a should have k factors of the prime factor 5 where $k \in \mathbb{Z}$. Thus a^2 should have $2k$ prime factors of 5. Similarly b^2 should have $2j$ prime factors of 5 where $j \in \mathbb{Z}$. Therefore we have a contradiction as a^2 has $2k$ factors of 5 and $5b^2$ has $2j+1$ factors of 5. Thus the claim follows that $\sqrt{5}$ is irrational. $\square$

1.5.11 *Proof.* Assume for the real numbers x, y, and z between 0 and 1 that $0 < x < y < z < 1$. Seeking a contradiction we'll assume that the distance between each number, x, y, and z, is a space greater than or equal to $\frac{1}{2}$. So we say that $a = x - 0$ is the distance from 0 to x, $b = y - x$ is the distance from x to y, $c = z - y$ is the distance from y to z, where $a, b, c \in \mathbb{R}$ and $a + b + c < 1$. If we set $x \approx 0$ then $a \approx 0$ leaving us with $b + c < 1$. However we get a contradiction when stating that $b \geq \frac{1}{2}$ and $c \geq \frac{1}{2}$, as at their lowest values we still end with $\frac{1}{2} + \frac{1}{2} < 1$ which is false. Thus by contradiction we can conclude that at least two of the real numbers, x, y, and z, between 0 and 1 must be within $\frac{1}{2}$ units of one another. □

1.5.12 a. C, the proof is all over the place. This is neither a proof by contrapositive (if m is even m^2 is even), not a proof by contradiction (show contradiction of: if m^2 is even assume m is odd), and also not a direct proof as it assumes the opposite of what the proof asks for (m^2 is odd). It is partially correct as the claim is correct by the following proof:

Proof. Solving by the contrapositive, assume the integer m is even such that $m = 2k$ for some $k \in \mathbb{Z}$. Then $m^2 = (2k)^2 = 2 * 2k^2 = 2j$ where $j = 2k^2$ is an integer. Thus by proving the contrapositive we can say that if m^2 is odd m is odd. □

1.6.1 b. *Proof.* Since $15m + 12n = 3 \implies 5m + 4n = 1 \implies 5m = 1 - 4n$. Assume $m = 1$ and $n = -1$, then $15m + 12n + 3$ turns into $15 - 12 = 3$ which is true. Thus we've proved that there exists integers m and n that satisfy $15m + 12n = 3$. □

d. *Proof.* Since $15m + 12n = 1 \implies 3(5m + 4n) = 1 \implies 3k = 1$ where $k = 5m + 4n$ is an integer. This is a contradiction as $3 \nmid 1$. Thus we've proved that there exists no integers m, n that satisfy $15m + 12n = 1$. □

1.6.4 a. *Proof.* Suppose $x = 41$ then $x^2 + x + 41 = 41^2 + 41 + 41 = 43(41)$. This disproves the statement that for all integers x, $x^2 + x + 41$ is prime as $43(41)$ clearly has at least one other factor besides itself and 1. □

b. *Proof.* For all real numbers x there exists a real number y that satisfies the statement $x + y = 0$. Since $x + y = 0 \implies y = -x \in \mathbb{R}$. Thus we've proved that there always exists at least one real number y that satisfies $x + y = 0$. □

c. *Proof.* Suppose the real numbers $x = 2$ and $y = 1$. Then $y^x > x \implies 1 > 2$ which is clearly a contradiction. Thus we've disproved that for all real numbers $x > 1$ and $y > 0$, $y^x > x$. □

d. *Proof.* Suppose the integers $a = 4$, $b = 2$, and $c = 2$. Then this fulfills the assumption that $a \mid bc$, as $2 \cdot 4k$ for some $k \in \mathbb{Z}$ where $k = 1$. However this fails both $a \mid b$ and $a \mid c$ as $2 \neq 4j$ and $2 \neq 4l$ for some $j, l \in \mathbb{Z}$. Thus we have disproved the statement that if an integer a divides the product of two integers b and c then a must also divide one of the other integers b or c. □

1.6.6 a. *Proof.* The natural number n is greater than or equal to 1 by definition of a natural number. Since $n \geq 1 \implies 1 \geq \frac{1}{n}$. Thus we've proved for all natural numbers n, $\frac{1}{n} \leq 1$. □

b. *Proof.* Assume there exists a natural number n that $\frac{1}{n} < .13$. Based on assumptions $\frac{1}{.13} < n$. Therefore there exists a natural number M that $n > M$ as long as there exists a natural number $m \leq \frac{1}{.13}$ which is true since $7 \leq 1.13$. Thus we've proved there exists natural numbers M and n that satisfy $n > M$ and $\frac{1}{n} < .13$. □

e. *Proof.* Assume there exists a natural number n. If this is the case, there will also exist a natural number $k = n + 1$ which implies that $k > n$. Thus we've proved that there exists no largest natural number since there will always exist a natural number k that is 1 greater the natural number n. □

f. *Proof.* Assume there exists a positive real number x such that $x > 0$. If this is the case, there will also exist a positive real number $y = \frac{x}{2}$ which implies $y < x$ since both x and y are positive. Thus we've proved that there exists no largest natural number since there will always exist a real number y that is half the size of any real number x. □

i. *Proof.* We'll prove there exists a natural number K that for all greater real numbers r, so that $r > K$, then $\frac{1}{r^2} < 0.01$. $\frac{1}{r^2} < 0.01 \implies 100 < r^2 \implies 10 < r \vee -10 > r$. This proves that $K = 10$, therefore if $r > K = 10$ then $\frac{1}{r^2} < 0.01$. Thus we've proved that there exists a natural number K that for all greater real numbers r then $\frac{1}{r^2} < 0.01$. □

k. *Proof.* We'll prove there exists an odd integer M, so that $M = 2k + 1$ where $k \in \mathbb{Z}$, that for all greater real numbers r, so that $r > M$, then $\frac{1}{2r} < 0.01$. Since $\frac{1}{2r} < 0.01$ then $100 < 2r \implies 50 < r$ and $50 \leq 51$ then let $M = 51$ as $51 = 2(25) + 1$. Then all real numbers r greater than $M = 51$ is implicitly greater than 50 fulfilling $50 < r$ therefore implying $\frac{1}{2r} < 0.01$. Thus we've proved there exists an odd integer M that for all greater real numbers r then $\frac{1}{2r} < 0.01$. □

2.1.4
a. False
b. True
c. False
d. True
e. True
f. False
g. True
h. False
i. False
j. True

2.1.5
a. True
b. True
c. True
d. True
e. False
f. True
g. False
h. False
i. False
j. False
k. True
l. True

2.1.6
a. $A \subseteq B, B \not\subseteq C,$ and $A \subseteq C$
$A = \{1\}, B = \{1, 2\}, C = \{1, 3\}$

b. $A \subseteq B, B \subseteq C,$ and $C \subseteq A$
$A = \{1\}, B = \{1\}, C = \{1\}$
c. $A \not\subseteq B, B \not\subseteq C,$ and $A \subseteq C$
$A = \{1\}, B = \{2\}, C = \{1, 3\}$
d. $A \subseteq B, B \not\subseteq C,$ and $A \not\subseteq C$
$A = \{1\}, B = \{1, 3\}, C = \{2\}$

2.1.8 *Proof.* Assume the sets exist A, B, and C and let $A \subseteq B$ and $B \subseteq C$. Therefore all elements $x \in A$ and because $A \subseteq B$ all elements $x \in B$. Finally, since $B \subseteq C$ makes all all elements $x \in C$. Since A is entirely made up of all elements x and $x \in C$ then $A \subseteq C$. Thus we've shown $A \subseteq C$ if $A \subseteq B$ and $B \subseteq C$. □

2.1.11 a. *Proof.* Assume for all real numbers x that $\frac{3x}{4} - 2 > 10 \iff \frac{3x}{4} > 12 \iff x > \frac{48}{3} \iff x > 16 \iff x \in (16, \infty)$. □

b. *Proof.* We'll prove that $\{x \in \mathbb{R}, |x-4| = 2|x| - 2\} = \{-6, 2\}$ by showing $\{x \in \mathbb{R}, |x-4| = 2|x| - 2\} \subseteq \{-6, 2\}$ and $\{x \in \mathbb{R}, |x-4| = 2|x| - 2\} \supseteq \{-6, 2\}$. Assume that all real numbers x satisfy $|x-4| = 2|x| - 2 \implies (x-4)^2 = 4(|x|-1)^2 \implies x^2 - 8x + 16 = 4x^2 - 8|x| + 4 \implies 0 = 3x^2 - 8|x| + 8x - 12$.
Case 1: if $x \geq 0$ then $0 = 3x^2 - 12$ and therefore $0 = x^2 - 4 \implies x = 2$. x can only be 2 for this case
Case 2: if $x < 0$ then $0 = 3x^2 + 16x - 12$ and therefore $x = \frac{-16 \pm \sqrt{16^2 - 4(3)(-12)}}{6} = \frac{-16 \pm \sqrt{400}}{6} = -\frac{8}{3} \pm \frac{10}{3} = -6$ or $\frac{2}{3}$. Thus x can only be equal to 6 for this case. Thus we've shown that the only real numbers x that satisfy satisfy $|x-4| = 2|x| - 2$ are all of the elements of $\{-6, 2\}$. Therefore we've proved that $\{x \in \mathbb{R}, |x-4| = 2|x| - 2\} = \{-6, 2\}$. □

2.1.14 a. $\mathcal{P}(\{0, \triangle, \square\}) = \{\varnothing, \{0\}, \{\triangle\}, \{\square\}, \{0, \triangle\}, \{0, \square\}, \{\triangle, \square\}, \{0, \triangle, \square\}\}$
b. $\mathcal{P}(\{S, \{S\}\}) = \{\varnothing, \{S\}, \{\{S\}\}, \{S, \{S\}\}\}$

2.1.15 a. False
b. True
c. False
d. True