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HOMEWORK 8

JESSE COBB - 3PM SECTION (MON,WED)

1. $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, E[X_i] = \mu, \mathrm{Var}(X_i) = \sigma^2$
   a. $X_i \overset{i.i.d}{\sim} \mathrm{Gamma}(r, \lambda)$
   $\bar{X}_n = \frac{nX_i}{n} = X_i$
   $\bar{X}_n \sim \mathrm{Gamma}(r, \lambda)$
   b. $E[\bar{X}_n] = E[\frac{1}{n}\sum_{i=1}^n X_i] = \frac{1}{n}\sum_{i=1}^n E[X_i] = \frac{1}{n}\sum_{i=1}^n \mu = \mu$
   $\mathrm{Var}(\bar{X}_n) = \mathrm{Var}(\frac{1}{n}\sum_{i=1}^n X_i) = \frac{1}{n^2}\mathrm{Var}(\sum_{i=1}^n X_i)$
   $= \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$
2. $M_X(t) = \begin{cases} (1-t)^{-\frac{3}{2}} & t < 1 \\ \infty & \text{otherwise} \end{cases}$
   $M_X'(t) = \frac{3}{2}(1-t)^{-\frac{5}{2}} \qquad E[X] = M_X(0) = \frac{3}{2}(1)^{-\frac{5}{2}} = \frac{3}{2}$
   $M_X''(t) = \frac{15}{4}(1-t)^{-\frac{7}{2}} \qquad \mathrm{Var}(X) = M_X''(0) - M_X'(0)^2 = \frac{15}{4} - (\frac{3}{2})^2 = \frac{3}{2}$
   $X \sim \mathrm{Gamma}(\frac{3}{2}, 1)$
3. $M_X(t) = (0.3 + 0.7e^{2t})^8$
   $Y = aX + b$
   $M_Y(t) = E[e^{(aX+b)t}] = e^{bt}M_X(at) = e^{bt}(0.3 + 0.7e^{2at})^8$
   $= (0.3 + 0.7e^t)^8$ when $a = \frac{1}{2}, b = 0$ which shows $Y \sim \mathrm{Bin}(8, .7)$
   $p_X(x) = P(X = x) = P(2Y = x) = P(Y = \frac{x}{2}) = \binom{8}{\frac{x}{2}}(.7)^{\frac{x}{2}}(.3)^{8-\frac{x}{2}}$
4. $f_X(x) = \frac{1}{2}e^{-|x|}$ if $-\infty < x < \infty$

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\begin{aligned}
M_X(t) = E[e^{Xt}] &= \int_{\mathbb{R}} e^{xt}\frac{1}{2}e^{-|x|}dx = \frac{1}{2}\int_{\mathbb{R}} e^{xt-|x|}dx \\
&= \frac{1}{2}\int_0^\infty e^{x(t-1)}dx + \frac{1}{2}\int_{-\infty}^0 e^{x(t+1)}dx \\
&= [\frac{e^{x(t-1)}}{2(t-1)}]_0^\infty + [\frac{e^{x(t+1)}}{2(t+1)}]_{-\infty}^0 \\
&= \lim_{r\to\infty} \frac{e^{r(t-1)}}{2(t-1)} - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} - \frac{e^{-r(t+1)}}{2(t+1)} \\
&= \lim_{r\to\infty} \frac{e^{r(t-1)}(t+1) - e^{-r(t+1)}(t-1) - 2}{2(t-1)(t+1)} \\
&= \frac{-1}{(t-1)(t+1)} \text{ when } -1 < t < 1
\end{aligned}
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$$M_X(t) = \begin{cases} (1-t^2)^{-1} & -1 < t < 1 \\ \infty & \text{otherwise} \end{cases}$$