HOMEWORK 5

JESSE COBB - 2PM SECTION

3.2.9 d. $\mathbb{Z}_7 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$

3.2.10 c. Congruent to 2(mod 4) and congruent to 8(mod 6).
$14 = 2(\text{mod } 4)$ and $14 = 8(\text{mod } 6)$
$-10 = 2(\text{mod } 4)$ and $-19 = 8(\text{mod } 6)$

3.4.1 a. $6 + 6$ in $\mathbb{Z}_7$
$\bar{6} + \bar{6} = 1\bar{2} = \bar{5}$

i. 2^{25} in $\mathbb{Z}_7$
$2^{\bar{2}5} = 2^{\bar{3}^8}\bar{2} = \bar{1}^8\bar{2} = \bar{2}$

j. 5^{23} in $\mathbb{Z}_7$
$5^{\bar{2}3} = 2\bar{5}^{11}\bar{5} = 4^{\bar{1}1}\bar{5} = 2^{\bar{2}2}\bar{5} = 2^{\bar{3}^7}\bar{1}0 = \bar{1}^7\bar{3} = \bar{3}$

k. 4^{44} in $\mathbb{Z}_7$
$4^{\bar{4}4} = 2^{\bar{8}8} = 2^{\bar{3}^{29}}\bar{2} = \bar{1}^{29}\bar{2} = \bar{2}$

l. 2^{26} in $\mathbb{Z}_7$
$2^{\bar{2}5} = 2^{\bar{3}^8}\bar{2}^2 = \bar{1}^8\bar{2}^2 = \bar{4}$

3.4.7 a. $238 + 496 - 44$ in $\mathbb{Z}_9$
$2\bar{3}8 + 4\bar{9}6 - \bar{4}4 = 6\bar{9}0 = \bar{9}0 + 4\bar{5}0 + \bar{9}0 + \bar{4}5 + \bar{1}5 = \bar{1}5 = \bar{6}$

4.1.1 c. $R = \{(1, 2), (2, 1)\}$
R is a function from A to B
$A = \{1, 2\}, B = \mathbb{N}$ or $\mathbb{Z}$

e. $R = \{(x, y) \in \mathbb{N}^2 : x \leq y\}$
R is not a function on $\mathbb{N}$
Since $1 \leq 2$ and $1 \leq 1$ so $f(1) = 1, 2$

f. $R = \{(x, y) \in \mathbb{Z}^2 : y^2 = x\}$
R is not a function on $\mathbb{Z}$
Since $1 = (-1)^2 = (1)^2$ so $f(1) = -1, 1$
and $x \geq 0$ since $y^2 \geq 0$ so $\text{Dom}(R) \neq \mathbb{Z}$

i. $R = \{(a, 3), (b, 2), (c, 1)\}$
R is not a function from A to B
Since $A = \{a, b, c, d\}$ so $\text{Dom}(R) \neq A$

4.1.2 $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \pm\sqrt{x}$ is not a function as $\mathbb{R} \neq \text{Dom}(R)$ since $\sqrt{x}$ is only defined when $x \geq 0$. Furthermore the function almost always has 2 outputs for every input, an example being $f(1) = -1, 1$ which deviates from the rules of a function.

4.1.7 *Proof.* Let $f : A \to B$ and $g : C \to D$ be functions. Now suppose $\text{Dom}(f) = \text{Dom}(g)$ and for all $x \in \text{Dom}(f), f(x) = g(x)$.
($\subseteq$): Let $(x, y) \in f$ this means that $f(x) = y$. Since $f(x) = g(x) \implies g(x) = y$ this means $(x, y) \in g$. Thus $f \subseteq g$.
($\subseteq$): Let $(x, y) \in g$ this means that $g(x) = y$. Since $f(x) = g(x) \implies f(x) = y$ this means $(x, y) \in f$. Thus $g \subseteq f$.

Thus we've shown that $f = g$ by double containment, if $\mathrm{Dom}(f) = \mathrm{Dom}(g)$ and for all $x \in \mathrm{Dom}(f), f(x) = g(x)$. $\square$

4.1.13 $f : \mathbb{Z} \to \mathbb{Z}_6$

a. $f(3) = \{\ldots - 3, 3, 9, 15 \ldots\} = \{6k + 3 : k \in \mathbb{Z}\}$
b. Image of $6 = f(6) = f(0) = \bar{0} = \{6k : k \in \mathbb{Z}\}$
c. A pre-image of $\bar{3} = 3$
d. All pre-images of $\bar{1} = 6k + 1$ where $k \in \mathbb{Z}$

4.1.17 $\overline{\overline{A}} = m, \overline{\overline{B}} = n$

a. A function f from A to B has n^m possible forms.
b. A function f with only one element in the domain from A to B has n possible forms.

4.1.18 a. *Proof.* Let $f : A \to B$ where xTy iff $f(x) = f(y)$ where T is a relation on A. Let $x \in A$. Since f is a function each value of x maps to to a single value so $f(x) = f(x)$. Thus xTx so T is reflexive. Let $x, y \in A$. Assume xTy so that $f(x) = f(y)$. This implies $f(y) = f(x)$ so yTx. Then T is symmetric. Now let $x, y, z \in A$ and xTy and yTz so that $f(x) = f(y)$ and $f(y) = f(z)$. This implies $f(x) = f(z)$ so xTz. Then T is transitive. Thus we've proved T to be an equivalence relation on A. $\square$

b. $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$.
$\bar{0} = \{0\} \qquad \bar{2} = \{4\} \qquad \bar{4} = \{16\}$

4.2.1 g. f^{-1} exists only when f is bijective
$f = \frac{1}{1-x}$
if $x = 1 - \frac{1}{y}$ then $f(x) = \frac{1}{1-(1-\frac{1}{y})} = y$ so f is onto
$f : \mathbb{R} - \{1\} \to \mathbb{R} - \{0\}$
Assume $x_1, x_2 \in \mathbb{R} - \{0\}$ and $f(x_1) - f(x_2)$ then:
$\frac{1}{1-x_1} = \frac{1}{1-x_2} \implies 1 - x_1 = 1 - x_2 \implies x_1 = x_2$
f is bijective so f^{-1} exists: $f^{-1} : \mathbb{R} - \{0\} \to \mathbb{R} - \{1\}$
$f^{-1}(x) = 1 - \frac{1}{x}$

4.2.2 b. $f(x) = x^2 + 2x \qquad g(x) = 2x + 1$
$\mathrm{Dom}(f) = \mathbb{R} \quad \mathrm{Rng}(f) = [-1, \infty)$
$\mathrm{Dom}(g) = \mathbb{R} \quad \mathrm{Rng}(g) = \mathbb{R}$
$(f \circ g)(x) = (2x + 1)^2 + 2(2x + 1) = 4x^2 + 8x + 3$
$(g \circ f)(x) = 2(x^2 + 2x) + 1 = 2x^2 + 4x + 1$

4.2.3 b. $f \circ g : \mathbb{R} \to [-1, \infty)$
$g \circ f : \mathbb{R} \to [-1, \infty)$

4.2.6 *Proof.* Let $f : A \to B$ and $I_B : B \to B$ where $I_B(z) = z$. Now let $x \in A$ so that $f(x) = y$ where $y \in B$, so $I_B(y) = y$. This implies that $I_B(f(x)) = f(x)$. Thus $I_B \circ f = f$ if $f : A \to B$ and $I_B : B \to B$. $\square$

4.2.7 *Proof.* We can say that $\mathrm{Dom}(f \circ f^{-1}) = \mathrm{Dom}(f^{-1}) = \mathrm{Rng}(f)$. This shows that $\mathrm{Dom}(f \circ f^{-1}) = C = \mathrm{Dom}(I_C)$ since $I_C : C \to C$ is bijective by definition. Let $x \in C$ so thatt $I_C(x) = x$ and $f^{-1}(x) \in A$. So $f(f^{-1}(x)) \in C$. So $f(f^{-1}(x)) = x = I_C(x)$. Thus we've shown that $f \circ f^{-1} = I_C$ if f and f^{-1} are functions and $\mathrm{Rng}(f) = C$. $\square$

4.3.1 b. *Proof.* Let $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = -x + 1000$. Suppose $x, y \in \mathbb{Z}$ and let $x = (1000 - y) \in \mathbb{Z}$ and note that:

$$\begin{aligned} f(x) &= f(1000 - y) \\ &= -(1000 - y) + 1000 \\ &= y - 1000 + 1000 \\ &= y \end{aligned}$$

We've shown that for all $y \in \mathbb{Z}$ there exists an $x \in \mathbb{Z}$ such that $f(x) = y$. Thus f is a surjection. □

d. *Proof.* Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. Now suppose $x, y \in \mathbb{R}$ and let $x = \sqrt[3]{y} \in \mathbb{R}$ and note that:

$$\begin{aligned} f(x) &= f(\sqrt[3]{y}) \\ &= (\sqrt[3]{y})^3 \\ &= y \end{aligned}$$

We've shown that for any $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that $f(x) = y$. Thus we've proved that f is a surjection. □

e. *Proof.* We'll show that $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = \sqrt{x^2 + 5}$, is not a surjection. We'll show this by contradiction and state that $f(x) = -1$. Note that:

$$\begin{aligned} -1 = \sqrt{x^2 + 5} &\implies (-1)^2 = x^2 + 5 \\ &\implies 1 - 5 = x^2 \\ &\implies \sqrt{-4} = x \end{aligned}$$

We've shown that $f(x) = -1$ is a contradiction as $x = \sqrt{-4} \notin \mathbb{R}$ and $-1 \in \mathbb{R}$. Thus we've shown that f is not a surjection by showing a value that is not mapped onto that is in the codomain. □

h. *Proof.* Let $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x - y$, is a surjection. Let $x \in \mathbb{R}$ and $y = 0 \in \mathbb{R}$ so $(x, y) = (x, 0)$. Then $f(x, y) = x - 0 = x \in \mathbb{R}$. Thus we've proved that f is a surjection. □

4.3.2 b. *Proof.* Let $x_1, x_2 \in \mathbb{Z}$ and assume for $f : \mathbb{Z} \to \mathbb{Z}$, given by $f(x) = -x + 1000$, that $f(x_1) = f(x_2)$. Note that:

$$\begin{aligned} f(x_1) = f(x_2) &\implies -x_1 = 1000 = -x_2 + 1000 \\ &\implies -x_1 = -x_2 \\ &\implies x_1 = x_2 \end{aligned}$$

We've shown that if $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Z}$ then $x_1 = x_2$. Thus we've shown that f is an injection. □

d. *Proof.* Let $x_1, x_2 \in \mathbb{Z}$ and assume for $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^3$, that $f(x_1) = f(x_2)$. Note that:

$$\begin{aligned} f(x_1) = f(x_2) &\implies x_1^3 = x_2^3 \\ &\implies x_1 = x_2 \end{aligned}$$

We've shown that if $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R}$ then $x_1 = x_2$. Thus we've shown that f is an injection. □

e. *Proof.* To show that $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = \sqrt{x^2 + 5}$, is not an injection we simply show the case of $f(x) = \sqrt{6}$. Since $f(-1) = f(1) = \sqrt{6}$ where $-1, 1 \in \mathbb{R}$, we've proved that f is not an injection. $\square$

h. *Proof.* To show that $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x-y$ is not an injection we simply show the case $f(x, y) = 0$. Since $f(1, -1) = f(2, -2) = 0$ where $(1, -1), (2, -2) \in \mathbb{R}^2$ but $(1, -1) \neq (2, -2)$ we have proved that f is not an injection. $\square$

4.3.5 *Proof.* Let $f : A \xrightarrow{\text{onto}} B$ and $g : B \xrightarrow{\text{onto}} C$. Then $\text{Rng}(f) = B$ and $\text{Dom}(g) = B$. Let $b \in B$ so that $f(a) = b$ for some $a \in A$ and $g(b) = c \in C$. Then $g(f(a)) = f(b) = c$ so we've proved that $g \circ f$ is a surjection. $\square$

4.3.6 *Proof.* Let $f : A \to B$, $g : B \to C$, and $g \circ f : A \xrightarrow{1-1} C$ be functions. Now let $x_1, x_2 \in A$ and let $f(x_1) = f(x_2)$. Note that:

$$\begin{aligned} f(x_1) = f(x_2) &\implies g(f(x_1)) = g(f(x_2)) \\ &\implies (g \circ f)(x_1) = (g \circ f)(x_2) \\ &\implies x_1 = x_2 \end{aligned}$$

We've shown that if $f(x_1) = f(x_2)$ then $x_1 = x_2$. Thus we've proved that if $g \circ f$ is an injection then f must also be an injection. $\square$

4.3.7 *Proof.* Let $f : A \xrightarrow{1-1} B$ be a function. Assume there exists a restriction of f, $f|_D$, and $x_1, x_2 \in D$. Since $D \subseteq A$, by the definition of a restriction, then $x_1, x_2 \in A$. Then $f(x_1) = f(x_2) \implies x_1 = x_2$. Since this is true for any x_1 and x_2 in D then $f|_D$ is an injection. Thus we've shown any restriction of an injection is, itself, an injection. $\square$

4.3.8 *Proof.* Let $h : A \xrightarrow{\text{onto}} C$, $g : B \xrightarrow{\text{onto}} D$, $A \cap B = \varnothing$ and $C \cap D = \varnothing$. Then $\text{Dom}(h \cup g) = A \cup B$ and $\text{Rng}(h \cup g) = C \cup D$.
Let $y \in C$ then $(h \cup g)(x) = h(x)$ and $h(x)$ is a surjection so for all y there exists an $x \in A$ so that $(h \cup g)(x) = y$.
Let $y \in D$ then $(h \cup g)(x) = g(x)$ and $g(x)$ is a surjection so for all y there exists an $x \in B$ so that $(h \cup g)(x) = y$.
Thus we've shown that in every possible case of $y \in C \cup D$ there exists an $x \in A \cup B$ so that $(h \cup g)(x) = y$, thus we've proved that $h \cup g$ is an injection. $\square$