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HOMEWORK 3

JESSE COBB - 2PM SECTION

2.2.1 $A = \{1, 3, 5, 7, 9\}, B = \{0, 2, 4, 6, 8\}$,
$C = \{1, 2, 4, 5, 7, 8\}, D = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$

d. $A - (B - C) = A - \{0, 6\} = \{1, 3, 5, 7, 9\} = A$

f. $A \cup (C \cap D) = A \cup \{1, 2, 5, 7, 8\} = \{1, 2, 3, 5, 7, 8, 9\}$

h. $A \cap (B \cup C) = A \cap \{0, 1, 2, 4, 5, 6, 7, 8\} = \{1, 5, 7\}$

2.2.2 $A = [3, 8), B = [2, 6], C = (1, 4), D = (5, \infty)$

b. $A \cup B = [2, 8)$

d. $A \cap B = [3, 6]$

f. $A - B = (6, 8)$

2.2.5 $A = [3, 8), B = [2, 6], C = (1, 4), D = (5, \infty)$
$C \cap D = \varnothing$
C and D is the only disjoint pair

2.2.6 d. $A \nsubseteq B \cup C, B \nsubseteq A \cup C$, and $C \subseteq A \cup B$
$A = \{1, 4\}, B = \{2, 3\}, C = \{1, 2\}$

2.2.7 q. Prove if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Proof. Assume all elements $x \in A$, and since $A \subseteq B$ then $x \in B$. By the definition of union $x \in A \cup C$ for any set C with the addition of all elements $y \in C - A$. Since $x \in B$ then all $x \in B \cup C$ and all $y \in C$ so all $y \in B \cup C$. Since the elements x and y make up all of $A \cup C$ and $x, y \in B \cup C$ we've proved that $A \cup C \subseteq B \cup C$ if $A \subseteq B$. □

2.2.8 h. Prove $(A \cap B)^c = A^c \cup B^c$.

Proof.

$$\begin{aligned} & x \in (A \cap B)^c && \iff \\ & x \notin A \cap B && \iff \\ & x \notin A \text{ or } x \notin B && \iff \\ & x \in A^c \text{ or } x \in B^c && \iff \\ & x \in A^c \cup B^c \end{aligned}$$

Since the last statement is true, so are all previous statements. Thus $(A \cap B)^c$ if and only if $A^c \cup B^c$. □

2.2.9 b. *Proof.* Assume for all $x \in A$ and $A \subseteq B \cup C$ and $A \cap B = \varnothing$, which implies $x \in B \cup C$ and $x \notin B$. Since $x \notin B$ and x is in the union of $B \cup C$ this implies $x \in C$. Thus we've proved that $A \subseteq C$ if $A \subseteq B \cup C$ and $A \cap B = \varnothing$. □

2.2.10 a. *Proof.* Assume $x \in C$, $y \in D$, $C \subseteq A$, and $D \subseteq B$. This implies $x \in A$ and $y \in B$. By definition of intersection $z \in C \cap D$ where $z = x = y$. $z \in A \cap B$ since $x \in A$ and $y \in B$. Thus we've proved that $C \cap D \subseteq A \cap B$. □

d. *Proof.* Assume $x \in C$, $y \in D$, $C \subseteq A$, and $D \subseteq B$. This implies $x \in A$ and $y \in B$. $D - A$ includes all elements y that are not elements of A which includes elements x. $B - C$ includes all elements y that are not in C which is only elements x. In summary $B - C$ includes y, not x, and any elements $z \in B - D$. While $D - A$ includes only y, not x, and not $w \in A - C$. Thus we've proved $D - A \subseteq B - C$ if $C \subseteq A$ and $D \subseteq B$. □

2.2.11 b. Statement: if $A \cap C \subseteq B \cap C$ then $A \subseteq C$
Counterexample: $A = \{4\}, B = \varnothing, C = \varnothing$

f. Statement: $A - (B - C) = (A - B) - C$
Counterexample: $A = \{4, 13\}, B = \varnothing, C = \{4\}$

2.2.13 a. $A = \{1, 3, 4\}, B = \{a, e, k, n, r\}$
$A \times B = \{(1, a), (1, e), (1, k), (1, n), (1, r)$
$(3, a), (3, e), (3, k), (3, n), (3, r)$
$(5, a), (5, e), (5, k), (5, n), (5, r)\}$
$B \times A = \{(a, 1), (a, 3), (a, 5), (e, 1), (e, 3), (e, 5)$
$(k, 1), (k, 3), (k, 5), (n, 1), (n, 3), (n, 5)$
$(r, 1), (r, 3), (r, 5)\}$

2.2.15 b. *Proof.* Seeking a contradiction, assume $A \times \varnothing \neq \varnothing$. Assuming this there must be at least one element $x \in A \times \varnothing$. Which implies the existence of the ordered pair (a, b) where $a \in A$ and $b \in \varnothing$, though this is a contradiction as there can be no $b \in \varnothing$. Thus we've proved by contradiction that $A \times \varnothing = \varnothing$. □

2.2.16 a. $(A \times B) \cup (C \times D) \neq (A \cup B) \times (C \cup D)$
$A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$

2.3.1 b. $\mathscr{A} = \{\{1, 3, 5\}, \{2, 4, 6\}, \{7, 9, 11, 13\}, \{8, 10, 12\}\}$
$\bigcup\limits_{A \in \mathscr{A}} A = \{n \in \mathbb{N} : n \leq 13\}$ $\qquad \bigcap\limits_{A \in \mathscr{A}} A = \varnothing$

h. $\mathscr{A} = \{A_r = [-\pi, r) : r \in (0, \infty)\}$
$\bigcup\limits_{A \in \mathscr{A}} A = [-\pi, \infty)$ $\qquad \bigcap\limits_{A \in \mathscr{A}} A = [-\pi, 0]$

l. $\mathscr{C} = \{C_n = [n, n + 1) : n \in \mathbb{Z}\}$
$\bigcup\limits_{C \in \mathscr{C}} C = (-\infty, \infty)$ $\qquad \bigcap\limits_{C \in \mathscr{C}} C = \varnothing$

m. $\mathscr{A} = \{A_n = (n, n + 1) : n \in \mathbb{Z}\}$
$\bigcup\limits_{A \in \mathscr{A}} A = \{x \in \mathbb{R} : x \notin \mathbb{Z}\}$ $\qquad \bigcap\limits_{A \in \mathscr{A}} A = \varnothing$

n. $\mathscr{D} = \{D_n = (-n, \frac{1}{n}) : n \in \mathbb{N}\}$
$\bigcup\limits_{D \in \mathscr{D}} D = (-\infty, 1)$ $\qquad \bigcap\limits_{D \in \mathscr{D}} D = (-1, 0]$

2.3.2 b. Pairwise Disjoint
h. Not Pairwise Disjoint
l. Pairwise Disjoint
m. Pairwise Disjoint
n. Not Pairwise Disjoint

2.3.3 a. For every set B in the family $\mathscr{A}$, $B \subseteq \bigcup_{A \in \mathscr{A}} A$.

Proof. Let $B \in \mathscr{A}$ and $x \in B$. Since $B \in \mathscr{A}$, by the definition of union $x \in \bigcup_{A \in \mathscr{A}} A$. Thus we've proved $B \subseteq \bigcup_{A \in \mathscr{A}} A$ if $B \in \mathscr{A}$. □

b. If $A \subseteq B$ for all $A \in \mathscr{A}$, then $\bigcup_{A\in\mathscr{A}} A \subseteq B$.

Proof. Let $A \subseteq B$ for any set $A \in \mathscr{A}$. If $x \in \bigcup_{A\in\mathscr{A}} A$ then $x \in B$. Thus we've proved that $\bigcup_{A\in\mathscr{A}} A \subseteq B$ if $A \subseteq B$ for all $A \in \mathscr{A}$. $\square$

2.3.12 $X = \{1, 2, 3, 4, \ldots 20\}$

a. $\displaystyle\bigcup_{A\in\mathscr{A}} A = X \qquad \bigcap_{A\in\mathscr{A}} A = \{1\}$

$\mathscr{A} = \{\{1\}, X\}$

b. $\displaystyle\bigcup_{B\in\mathscr{B}} B = X$ 4 disjoint subsets of X

$\mathscr{B} = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}, \{11, 12, 13, 14, 15\}, \{16, 17, 18, 19, 20\}\}$

c. $\displaystyle\bigcup_{C\in\mathscr{C}} C = X$ 20 disjoint subsets of X

$\mathscr{C} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\},$
$\{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}\}$

2.3.16 $\mathscr{A} = \{A_i : i \in \mathbb{N}\} \quad k, m \in \mathbb{N} \quad k \le m$

d. $\displaystyle\bigcap_{i=1}^{\infty} A_i \subseteq \bigcap_{i=k}^{m} A_i$

Proof. Let $x \in \bigcap_{i=1}^{\infty} A_i$ which implies $x \in A_i$ for any $A_i \in \mathscr{A}$. This implies that $x \in \bigcap_{i=k}^{m} A_i$ since $k \ge 1$ and $m < \infty$. Thus we've proved that $\bigcap_{i=1}^{\infty} A_i \subseteq \bigcap_{i=k}^{m} A_i$ if $\mathscr{A} = \{A_i : i \in \mathbb{N}\}$, and $k, m \in \mathbb{N}$, and $k \le m$. $\square$

2.3.18 Nested family $\mathscr{A} = \{A_i : i \in \mathbb{N}\}$

b. $\bigcap_{i=1}^{\infty} A_i = (-\infty, 1] \implies A_i = (-\infty, 1 + \frac{1}{i})$

d. $\bigcap_{i=1}^{\infty} A_i = \varnothing \implies A_i = (0, \frac{1}{n})$

2.4.4 b. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $3 = 4(1)^2 - 1$ is true, which proves the base case holds. Next assume $3 + 11 + 19 + \ldots (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$. Then $3 + 11 + 19 + \ldots (8n - 5) + (8(n + 1) - 5) = 4n^2 - n + (8(n + 1) - 5)$ by the inductive hypothesis. This implies $4n^2 - n + (8(n + 1) - 5) = 4n^2 + 7n + 3 = 4(n + 1)^2 - (n + 1)$ so the statement works for the $n + 1$ case. Thus we've proved by the Principle of Mathematical Induction, that $3 + 11 + 19 + \ldots (8n - 5) = 4n^2 - n$ for all natural numbers n. $\square$

c. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2 \implies 2^1 = 2^2 - 2$ which proves the base case holds. Now assume $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ for all natural numbers n. Then $\sum_{i=1}^{n} 2^i + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1}$ by the inductive hypothesis. Thus $2^{n+1} - 2 + 2^{n+1} = 2 \cdot 2^{n+1} - 2 = 2^{(n+1)+1} - 2$ which proves that the $n + 1$ case is true. Thus we've proved, by the Principle of Mathematical Induction, that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ for all $n \in \mathbb{N}$. $\square$

d. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $1\cdot1! + 2\cdot2! + 3\cdot3! + \ldots n \cdots n! = (n+1)! - 1 \implies 1\cdot1! = 2! - 1$ so the base case holds. Now assume $1\cdot1! + 2\cdot2! + 3\cdot3! + \ldots n\cdot n! = (n+1)! - 1$ for all $n \in \mathbb{N}$. Then $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots n \cdot n! + (n + 1) \cdot (n + 1)! = (n + 1)! - 1 + (n + 1) \cdot (n + 1)!$ by the induction hypothesis. Then $(n + 1)! - 1 + (n + 1) \cdot (n + 1)! = (n + 2)(n + 1)! - 1 = (n + 2)! - 1$ which proves the $n+1$ case. Thus we've proved, by the Principle of Mathematical

Induction, that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots n \cdots n! = (n+1)! - 1$ for all natural numbers n. $\square$

e. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $1^3 + 2^3 + 3^3 + \ldots n^3 = [\frac{n(n+1)}{2}]^2 \implies 1^3 = [\frac{2}{2}]^2$ which proves the base case is true. Now assume $1^3 + 2^3 + 3^3 + \ldots n^3 = [\frac{n(n+1)}{2}]^2$ for all $n \in \mathbb{N}$. Then $1^3 + 2^3 + 3^3 + \ldots n^3 + (n+1)^3 = [\frac{n(n+1)}{2}]^2 + (n+1)^3$ by the induction hypothesis. Then:

$$\begin{aligned}
[\frac{n(n+1)}{2}]^2 + (n+1)^3 &= \frac{(n(n+1))^2 + 4(n+1)^3}{4} \\
&= \frac{n^2(n+1)^2 + 4(n+1)(n+1)^2}{4} \\
&= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= [\frac{(n+1)(n+2)}{2}]^2
\end{aligned}$$

which proves the $n+1$ case. Thus we've proved, by the Principle of Mathematical Induction, that $1^3 + 2^3 + 3^3 + \ldots n^3 = [\frac{n(n+1)}{2}]^2$ for all natural numbers n. $\square$

2.4.5 a. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $3 \mid n^3 + 5n + 6 \implies 3 \mid 1^3 + 5(1) + 6$ which is true since $1^3 + 5(1) + 6 = 12 = 3(4)$ where $4 \in \mathbb{Z}$ which proves the base case. Now assume $3 \mid n^3 + 5n + 6$ which implies $n^3 + 5n + 6 = 3k$ for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then $(n+1)^3 + 5(n+1) + 6 = n^3 + 3n^2 + 8n + 12 = (n^3 + 5n + 6) + 3n^2 + 3n + 6 = 3k + 3n^2 + 3n + 6$ by the induction hypothesis. Then $3k + 3n^2 + 3n + 6 = 3(k + n^2 + n + 2) = 3j$ where $j = k + n^2 + n + 2$ is an integer. This proves that the $n+1$ case is true. Thus we've proved, by the Principle of Mathematical Induction, that $3 \mid n^3 + 5n + 6$ for all natural numbers n. $\square$

j. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $3^n \geq 1 + 2^n \implies 3^1 \geq 1 + 2^n$ which proves the base case. Now assume $3^n \geq 1 + 2^n$ for all natural numbers n. Then $3^{n+1} = 3 \cdot 3^n \geq 3(1 + 2^n)$ by the induction hypothesis. Then $3(1 + 2^n) = (1 + 2^n) + (1 + 2^n) + (1 + 2^n) = 3 + 2^n + 2 \cdot 2^n = 3 + 2^n + 2^{n+1} > 1 + 2^{n+1}$ which implies that $3^{n+1} \geq 1 + 2^{n+1}$ which proves the $n+1$ case. Thus we've proved, by the Principle of Mathematical Induction, that $3^n \geq 1 + 2^n$ for all natural numbers n. $\square$

q. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $|A| = 1$ so that $A = \{a\}$, then the power set $\mathscr{P}(A) = \{\varnothing, \{a\}\}$ which implies $|\mathscr{P}(A)| = 2$ which proves the base case true as $|\mathscr{P}(A)| = 2^{|A|} = 2$. Now assume that if $|A| = n$ then $|\mathscr{P}(A)| = 2^n$ for all natural numbers n. Now let the set B have $n+1$ elements where there exists an element $x \in B$ but $x \notin A$. Then $\mathscr{P}(B)$ has 2^n elements excluding x and 2^n elements including x by the inductive hypothesis, so $|\mathscr{P}(B)| = 2 \cdot 2^n = 2^{n+1}$, so we've proved the $n+1$ case. Thus we've proved, by the Principle of Mathematical Induction, that $\mathscr{P}(A)$ has 2^n elements if A has n elements for all $n \in \mathbb{N}$. $\square$

2.4.6 c. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 5$ that $(n+1)! > 2^{n+3} \implies 6! > 2^8$ which is true since $720 > 256$ thus proving the base case. Now assume $(n+1)! > 2^{n+3}$ for all natural numbers $n \geq 5$. Then:

$$\begin{aligned}(n+2)! &= (n+2)(n+1)! \\ &> (n+2)2^{n+3} \\ &> 2 \cdot 2^{n+3} \\ &= 2^{n+4}\end{aligned}$$

which proves $(n+2)! > 2^{n+4}$ by the induction hypothesis, proving the $n+1$ case true. Thus we've proved, by the Principle of Mathematical Induction, that $(n+1)! > 2^{n+3}$ for all natural numbers $n \geq 5$. $\square$

e. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 4$ that $n! > 3n \implies 4! > 3(4)$ which is true since $24 > 12$ which proves the base case. Now assume $n! > 3n$ for all natural numbers $n \geq 4$. Then:

$$\begin{aligned}(n+1)! &= (n+1)n! \\ &> (n+1)3n \\ &> 3(n+1)\end{aligned}$$

which proves $(n+1)! > 3(n+1)$ by the induction hypothesis, proving the $n+1$ case true. Thus we've proved, by the Principle of Mathematical Induction, that $n! > 3n$ for all natural numbers $n \geq 4$. $\square$

2.4.7 a. *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $(\bigcap_{i=1}^n A_i)^c = \bigcup_{i=1}^n A_i^c \implies (A_i)^c = A_i^c$ which proves the base case. Now assume $(\bigcap_{i=1}^n A_i)^c = \bigcup_{i=1}^n A_i^c$ for all $n \in \mathbb{N}$ where the indexed family $\{A_i : i \in \mathbb{N}\}$ exists. Then:

$$\begin{aligned}\left(\bigcap_{i=1}^{n+1} A_i\right)^c &= \left(\bigcap_{i=1}^{n} A_i \cap A_{n+1}\right)^c \\ &= \left(\bigcap_{i=1}^{n} A_i\right)^c \cup A_{n+1}^c \\ &= \bigcup_{i=1}^{n} A_i^c \cup A_{n+1}^c \\ &= \bigcup_{i=1}^{n+1} A_i^c\end{aligned}$$

which proves $(\bigcap_{i=1}^{n+1} A_i)^c = \bigcup_{i=1}^{n+1} A_i^c$ by the induction hypothesis, proving the $n+1$ case true. Thus we've proved, by the Principle of Mathematical Induction, that $(\bigcap_{i=1}^n A_i)^c = \bigcup_{i=1}^n A_i^c$ for all natural numbers n. $\square$

2.4.9 *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that 1 point requires $\frac{1^2-1}{2} = 0$ points, which proves the base case. Now assume that for n points, with no three collinear points, the amount of line segments to join all of the points is $\frac{n^2-n}{2}$ for all natural numbers n. Then $n+1$ points requires an additional n points since the new point will have to connect to each n previously existing points. Then $n+1$ points requires $\frac{n^2-n}{2}+n = \frac{n^2-n+2n}{2} = \frac{(n+1)^2-(n+1)}{2}$ points which proves the $n+1$ case by the induction hypothesis. Thus we've proved, by the Principle of Mathematical Induction, that for any natural number n points, where no three points are collinear, they require $\frac{n^2-n}{2}$ line segments to connect all points. $\square$

2.4.10 *Proof.* We'll prove the claim using mathematical induction. First note that when $n = 1$ that $2^{(1)} - 1 = 1$ implies that with 1 disc it takes a minimum of 1 move to the disc to another peg without any larger discs ever being on top of a smaller disc. This is true as any possible move will achieve the desired result with a single disc confirming our base case. Now assume that any natural number n of discs can achieve the same result of being stacked in descending size on another peg then its starting position in $f(n) = 2^n - 1$ moves, this implies the recursive formula of $f(n) = 2f(n-1)+1$ as each additional disc requires 1 additional move to move the new disc and another $f(n-1)$ moves to then move the stack back on top of the new largest disc. Then $f(n+1) = 2f(n)+1 = 2(2^n-1)+1$ by the induction hypothesis. Then $2(2^n-1)+1 = 2\cdot 2^n - 2 + 1 = 2^{n+1}-1$ which proves the $n+1$ case. Thus we've proved, by the Principle of Mathematical Induction, that for any natural number n of disc the discs can all be moved to a new peg (out of three), without ever having a larger disc above a smaller disc, in $2^n - 1$ moves. $\square$

2.5.1 a. *Proof.* We'll prove the claim using complete induction. Suppose $n \geq 11$, then note that $11 = 2(3) + 5(1)$ and $12 = 2(1) + 5(2)$ verifying our base cases. Now we'll assume that $n \geq 13$, and that all natural numbers k where $n-1 \geq k \geq 11$ can be written as $k = 2s+5t$ for some $s, t \in \mathbb{N}$. We'll demonstrate that the same is true for n. Since $n \geq 13$ then $n-2 \geq 11$, so $n-2 = 2s+5t$ by assumption. Then $n = 2s+5t+2 = 2(s+1)+5t$ where $s+1, t \in \mathbb{N}$ and the statement is true for n. Thus we've proved, by the Principle of Complete Induction, that any natural number $n \geq 11$ can be written as $n = 2s+5t$ for some $s, t \in \mathbb{N}$. $\square$

2.5.10 *Proof.* Let the set $\mathbb{Z}^- = \{-1, -2, -3\ldots\}$ exist and also allow the set $A \subseteq \mathbb{Z}^-$ to exist. Assume any set A is nonempty, so that $A \neq \varnothing$, so that there exists an element $a \in A$. Now suppose a set B exists where $B = \{-a : a \in A\}$. Since the elements of A are all negative integers the elements of B must be only positive integers meaning $B \subseteq \mathbb{N}$. Since $B \subseteq \mathbb{N}$ we may apply the Well Ordering Principle, since it applies to all subsets of $\mathbb{N}$, and say that there exists an element b that is the smallest element $b \in B$ where $b \leq -a$. Inversely this implies the existence of a $-b \in A$ where $-b \geq a$. Thus we've proved that any subset A of $\mathbb{Z}^-$ must have a largest element as its inverse $B \subseteq \mathbb{N}$ must have a smallest element, by the Well Ordering Principle. $\square$