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HOMEWORK 1

JESSE COBB - 2PM SECTION

1.1.1 e. This is a proposition.

$$\begin{aligned} P &\equiv \text{"}\pi \text{ is rational"} && \equiv \text{False} \\ Q &\equiv \text{"17 is a prime"} && \equiv \text{True} \\ R &\equiv \text{"}7 < 13\text{"} && \equiv \text{True} \\ S &\equiv \text{"81 is a perfect square} && \equiv \text{True} \end{aligned}$$

$$(P \wedge Q) \vee (R \wedge S) \text{ is True}$$

j. This is not a proposition (paradox)

$$\begin{aligned} P &\equiv \text{"There are more than three false statements in this book"} \equiv \text{True} \\ Q &\equiv \text{"This statement is one of them"} \\ Q &\equiv \neg(P \wedge Q) \\ Q &\equiv \neg P \vee \neg Q \\ \neg P &\equiv \text{ False} \\ Q &\equiv \neg Q : \text{paradox} \end{aligned}$$

1.1.2 c. Solve $P \wedge Q$ and $P \vee Q$

$$\begin{aligned} P &\equiv \text{"}5^2 + 12^2 = 13^2\text{"} && \equiv \text{True} \\ Q &\equiv \text{"}\sqrt{2} + \sqrt{3}\sqrt{2+3}\text{"} && \equiv \text{False} \\ P \wedge Q &\equiv \text{False} \\ P \vee Q &\equiv \text{True} \end{aligned}$$

1.1.3 c. $P \wedge \neg Q$

P	Q	$\neg Q$	$P \wedge \neg Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

l. $(P \wedge Q) \vee (R \wedge \neg S)$

P	Q	R	S	$\neg S$	$P \wedge Q$	$R \wedge \neg S$	$(P \wedge Q) \vee (R \wedge \neg S)$
T	T	T	T	F	T	F	T
T	T	T	F	T	T	T	T
T	T	F	T	F	T	F	T
T	T	F	F	T	T	F	T
T	F	T	T	F	F	F	F
T	F	T	F	T	F	T	T
T	F	F	T	F	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	T	F	T	F	T	T
F	T	F	T	F	F	F	F
F	T	F	F	T	F	F	F
F	F	T	T	F	F	F	F
F	F	T	F	T	F	T	T
F	F	F	T	F	F	F	F
F	F	F	F	T	F	F	F

1.1.7 c. Julius Caesar was born in 1492 or 1493 and died in 1776

$$\begin{aligned} P &\equiv \text{"Julius Caesar was born in 1492"} && \equiv \text{False} \\ Q &\equiv \text{"Julius Caesar was born in 1493"} && \equiv \text{False} \\ R &\equiv \text{"Julius Caesar died in 1776"} && \equiv \text{False} \\ (P \vee Q) \wedge R & && \equiv \text{False} \end{aligned}$$

g. It is not the case that both -5 and 13 are elements of $\mathbb{N}$, but 4 is in the set of all rational numbers

$$\begin{aligned} P &\equiv \text{"}-5 \in \mathbb{N}\text{"} && \equiv \text{False} \\ Q &\equiv \text{"}13 \in \mathbb{N}\text{"} && \equiv \text{True} \\ R &\equiv \text{"}4 \in \mathbb{Q}\text{"} && \equiv \text{True} \\ \neg(P \wedge Q) \wedge R & && \equiv \text{True} \end{aligned}$$

1.1.10 c. $(P \wedge Q) \vee (\neg P \vee \neg Q) \equiv (P \wedge Q) \vee \neg(P \wedge Q)$: Tautology

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \wedge Q) \vee \neg(P \wedge Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

1.1.11 e. Roses are read and violets are blue $(P \wedge Q)$

$$\begin{aligned} P &\equiv \text{"Roses are red"} \\ Q &\equiv \text{"Violets are blue"} \\ \neg(P \wedge Q) &\equiv \neg P \vee \neg Q \end{aligned}$$

Denial: Roses are not red or violets aren't blue.

i. The function g has a relative max at $x = 2$ or $x = 4$ and a relative min at $x = 3$ $((P \vee Q) \wedge R)$

$$P \equiv \text{"}g \text{ has a relative max at } x = 2\text{"}$$
$$Q \equiv \text{"}g \text{ has a relative max at } x = 4\text{"}$$
$$R \equiv \text{"}g \text{ has a relative min at } x = 3\text{"}$$
$$\neg((P \vee Q) \wedge R) \equiv \neg(P \vee Q) \vee \neg R \equiv (\neg P \wedge \neg Q) \vee \neg R$$

Denial: g doesn't have a relative min at $x = 3$ or g doesn't have a relative max at both $x = 2$ and $x = 4$

1.1.12 a. $\neg\neg P \vee \neg Q \wedge \neg S \equiv (\neg(\neg P)) \vee ((\neg Q) \wedge (\neg S))$

1.1.13 a. Truth Table for $A \veebar B$

A	B	$A \veebar B$
T	T	F
T	F	T
F	T	T
F	F	F

b. $(A \vee B) \wedge \neg(A \wedge B)$

A	B	$A \vee B$	$\neg(A \wedge B)$	$(A \vee B) \wedge \neg(A \wedge B)$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

Proof. $A \veebar B \equiv (A \vee B) \wedge \neg(A \wedge B)$ is demonstrated by the equivalent outcomes of their truth tables. □

1.2.2 b. "If the moon is made of cheese, then 8 is an irrational number"
Converse - "If 8 is an irrational number, then the moon is made of cheese."
Contrapositive - "If 8 is not an irrational number, then the moon isn't made of cheese."

d. "The differentiability of f is sufficient for f to be continuous."
Converse - "If f is continuous the function f is also differentiable."
Contrapositive - "If f isn't continuous then it isn't differentiable."

1.2.5 c. "If $7 + 6 = 14$, then $5 + 5 = 10$"
$(7 + 6 = 14) \implies (5 + 5 = 10) \equiv$ True

f. "If Euclid's birthday was April 2, then rectangles have four sides."
"Euclid's birthday was April 2" $\implies$ "rectangles have four sides"$\equiv$ True

g. "5 is prime if $\sqrt{2}$ is not irrational"
"$\sqrt{2}$ is not irrational" $\implies$ "5 is prime"$\equiv$ True

h. "$1 + 1 = 2$ is sufficient for $3 > 6$"
$(1 + 1 = 2) \implies (3 > 6) \equiv$ False

1.2.6 b. "$7 + 5 = 12$ if and only if $1 + 1 = 2$"
$(7 + 5 = 12) \iff (1 + 1 = 2) \equiv$ True

c. "$5 + 6 = 6 + 5$ iff $7 + 1 = 10$
$(5 + 6 = 6 + 5) \iff (7 + 1 = 10) \equiv$ False

g. "$x^2 \geq 0$ if and only if $x \geq 0$"
$(x^2 \geq 0) \iff (x \geq 0) \equiv$ False

1.2.7 b. $(\neg P \implies Q) \vee (Q \iff P)$

P	Q	$\neg P$	$\neg P \implies Q$	$Q \iff P$	$(\neg P \implies Q) \vee (Q \iff P)$
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	T	T

e. $(P \wedge Q) \vee (Q \wedge R) \implies (P \vee R) \equiv Q \wedge (P \vee R) \implies (P \vee R)$

P	Q	R	$P \vee R$	$Q \wedge (P \vee R)$	$Q \wedge (P \vee R) \implies (P \vee R)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	F	F	T

1.2.10 b. "If n is prime, then $n = 2$ or n is odd."
$(n \text{ is prime}) \implies ((n = 2) \vee (n \mod 2 = 1))$

1.2.12 b. Prove: $(P \wedge Q) \implies R \equiv (P \wedge \neg R) \implies \neg Q$

$$(P \wedge Q) \implies R$$
$$\neg(P \wedge Q) \vee R$$
$$\neg P \vee \neg Q \vee R$$
$$(\neg P \vee R) \vee \neg Q$$
$$\neg(P \wedge \neg R) \vee \neg Q$$
$$(P \wedge \neg R) \implies \neg Q$$

1.2.13 a. The converse is true: "A function f is integrable iff it is continous"
b. The converse is false: "A function f is differentiable if it is continuous."
c. The contrapositive is false: Impossible
d. The contrapositive is true: "A function f is differentiable if it is continuous."

1.3.1 f. $((\forall \text{Person} \in \text{All People})(\text{Person is not honest})) \vee$
$((\forall \text{Person} \in \text{All People})(\text{Person is honest}))$
g. $(\exists \text{Person} \in \text{All People})(\text{Person is not honest}) \wedge$
$(\exists \text{Person} \in \text{All People})(\text{Person is honest})$
h. $(\forall x \in \mathbb{R})(x \neq 0 \implies (x > 0 \vee x < 0))$
i. $(\forall x \in \mathbb{Z})(x > -4 \vee x < 6)$
j. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x > y)$
k. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x \leq y)$
l. $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(y > x \implies (\exists z \in \mathbb{R})(x < z \wedge z < y))$
m. $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x > 0 \implies x < y)$
n. $\neg((\exists \text{Person 1} \in \text{All People})(\forall \text{Person 2} \in \text{All People}),$
$(\text{Person 1 loves Person 2}))$
o. $(\forall \text{Person 1} \in \text{All People})(\exists \text{Person 2} \in \text{All People}),$
$(\text{Person 1 loves Person 2})$

p. $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(x > 0 \implies 2^y = x)$

1.3.2 f. $((\exists \text{Person} \in \text{All People})(\text{Person is honest})) \wedge$
$((\exists \text{Person} \in \text{All People})(\text{Person is not honest}))$
There exists a person that is honest and is not honest.

g. $(\forall \text{Person} \in \text{All People}, \text{Person is honest}) \vee$
$(\forall \text{Person} \in \text{All People}, \text{Person is not honest})$
Everyone is honest or not honest.

h. $(\exists x \in \mathbb{R})(x \neq 0 \wedge (x \leq 0 \wedge x \geq 0))$
There exists a non-zero real number x that equal to zero.

i. $(\exists x \in \mathbb{Z})(x \leq -4 \wedge x \geq 6)$
There exists a real number x that is both less than or equal to -4 and greater than or equal to 6.

j. $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x \leq y)$
There exists an integer x that is less than or equal to every integer.

k. $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x > y)$
There exists an integer x that is greater than every integer.

l. $(\exists x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x \wedge (\forall z \in \mathbb{R})(x \geq z \vee z \geq y))$
There exists an integer x and a greater integer y that all real numbers are less than or equal to x or greater than or equal to y.

m. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x > 0 \wedge x \geq y)$
For all real numbers there exists a real number less than it.

n. $(\exists \text{Person 1} \in \text{All People})(\forall \text{Person 2} \in \text{All People})$,
(Person 1 loves Person 2)
Someone loves everyone.

o. $(\exists \text{Person 1} \in \text{All People})(\forall \text{Person 2} \in \text{All People})$,
(Person 1 hates Person 2)
Someone hates everyone.

p. $(\exists x \in \mathbb{R})(((\forall y \in \mathbb{R})(x > 0 \wedge 2^y \neq x)) \vee ((\exists y \in \mathbb{R})(\exists z \in \mathbb{R})((x > 0 \wedge y \neq z) \implies (2^y = x \wedge 2^z = x))))$
There exists a positive real number x that doesn't satisfy the equation $2^y = x$ for any real number y or there exists at least two unique real numbers (y and z) that satisfy $2^y = x$ and $2^z = x$

1.3.6 a. $(\exists x \in \{17\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$
$(\exists x \in \{6\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$
$(\exists x \in \{24\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$
$(\exists x \in \{2, 3, 7, 26\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$

b. $(\exists x \in \{17\})(x \text{ is odd} \wedge x > 8) \equiv \text{True}$
$(\exists x \in \{6\})(x \text{ is odd} \wedge x > 8) \equiv \text{False}$
$(\exists x \in \{24\})(x \text{ is odd} \wedge x > 8) \equiv \text{False}$
$(\exists x \in \{2, 3, 7, 26\})(x \text{ is odd} \wedge x > 8) \equiv \text{False}$

c. $(\forall x \in \{17\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$
$(\forall x \in \{6\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$
$(\forall x \in \{24\})(x \text{ is odd} \implies x > 8) \equiv \text{True}$
$(\forall x \in \{2, 3, 7, 26\})(x \text{ is odd} \implies x > 8) \equiv \text{False}$

d. $(\forall x \in \{17\})(x \text{ is odd} \wedge x > 8) \equiv \text{True}$
$(\forall x \in \{6\})(x \text{ is odd} \wedge x > 8) \equiv \text{False}$
$(\forall x \in \{24\})(x \text{ is odd} \wedge x > 8) \equiv \text{False}$
$(\forall x \in \{2, 3, 7, 26\})(x \text{ is odd} \wedge x > 8) \equiv \text{False}$

1.3.8
a. False, $x > 0, x = -1 \in \mathbb{R}$
b. True, $x > 0, x \in \mathbb{N}$
c. False, $x = 3x + 2, x \in \mathbb{N}$
d. False, $\ln(3)/2 = \ln(x)/x$
e. False, $\ln(3) = \ln(x)/x$
f. True, $x = \frac{7}{5}$
g. False, $(x + 5)(x + 1) \geq 0$
h. True, $x(x + 4) + 5 \geq 0$
i. True, $x = 1 \in \mathbb{N}, 41$ is prime
j. False, No infinitely predictable prime sequence
k. False, $x = -10^{100} \in \mathbb{R}$
l. True, Real numbers can always be refined

1.3.9
a. All natural numbers x are greater than or equal to 1.
b. There exists only a single real number x that is equal to 0.
c. If a natural number x is prime and not 2 then x is odd.
d. There exists only a single real number that satisfies $\ln x = 1$.
e. There doesn't exist a real number x that satisfies $x^2 < 0$.
f. There exists only a single real number x that satisfies $x^2 = 0$.
g. If a natural number x is odd then x^2 must be odd.

1.3.10
a. True, $y = -x$
b. False, All real numbers only have 1 opposite.
c. False, $x^2 \geq 0$
d. False, Positive multiplied by negative will always be negative.
e. True, $x = 0$
f. False, No smallest number exists.
g. True, $x = y$
h. False, $y = -1, -2$
i. False, $y = \pm\sqrt{x}$
j. True, Function passes horizontal line test.
k. False, $(x, y) = (0, 1), (0, 2)$

1.3.13 Denials of $(\exists! x)P(x)$?
a. False, no case for any 2 existences proving $P(x)$
b. True
c. True
d. False, no case for all false

1.4.5
c. *Proof.* Assume that x and y are even, so that $x = 2k$ and $y = 2j$ where $k, j \in \mathbb{Z}$. Then $xy = 2k(2j) = 4(kj) = 4l$ where $l = kj$ is an integer. Thus we've proved that $4|xy$ when x and y are even. ☐

d. *Proof.* Assume that x and y are even, so that $x = 2k$ and $y = 2j$ where $k, j \in \mathbb{Z}$. Then $3x - 5y = 3(2k) - 5(2j) = 2(3k - 5j) = 2l$ where $l = 3k - 5j$ is an integer. Thus we've proved that $3x - 5y$ is even if x and y are even. ☐

e. *Proof.* Assume that x and y are odd, so that $x = 2k + 1$ and $y = 2j + 1$ where $k, j \in \mathbb{Z}$. Then $x + y = 2k + 1 + 2j + 1 = 2k + 2j + 2 = 2(k + j + 1) = 2l$ where $l = k + j + 1$ is an integer. Thus we've proved that $x + y$ is even if x and y are odd. ☐

f. *Proof.* Assume that x and y are odd, so that $x = 2k + 1$ and $y = 2j + 1$ where $k, j \in \mathbb{Z}$. Then $3x - 5y = 3(2k + 1) - 5(2j + 1) = 6k - 10j - 2 =$

$2(3k-5j-1) = 2l$ where $l = 3k-5j-1$ is an integer. Thus we've proved that $3x-5y$ is even if x and y are odd. ☐

g. *Proof.* Assume that x and y are odd, so that $x = 2k+1$ and $y = 2j+1$ where $k, j \in \mathbb{Z}$. Then $xy = (2k+1)(2j+1) = 4kj+2k+2j+1 = 2(kj+k+j)+1 = 2l+1$ where $l = kj+k+j$ is an integer. Thus we've proved that xy is odd if x and y are odd. ☐

h. *Proof.* Assume that x is even and y is odd, so that $x = 2k$ and $y = 2j+1$ where $k, j \in \mathbb{Z}$. Then $x+y = 2k+2j+1 = 2(k+j)+1 = 2l+1$ where $l = k+j$ is an integer. Thus we've proved that $x+y$ is odd if x is even and y is odd. ☐

i. *Proof.* Without loss of generality, assume x is even and y and z are odd such that $x = 2k$, $y = 2j+1$, and $z = 2l+1$ where $x, y, z \in \mathbb{Z}$. Then, the sum of x, y, and z, $x+y+z = 2k+2j+1+2l+1 = 2k+2j+2l+2 = 2(k+j+l+1) = 2h$ where $h = k+j+l+1$ is an integer. Thus we've proved that the sum of x, y, and z is even if exactly one is even. ☐

1.4.6 a. *Proof.* Let a and b be real numbers. Consider the following cases:
Case 1: If $a \geq 0$ and $b \geq 0$ then $|a| = a$and $|b| = b$. Then $|ab| = ab = |a||b|$.
Case 2: If $a \geq 0$ and $b < 0$ then $|a| = a$and $|b| = -b$. Then $|ab| = -(ab) = a(-b) = |a||b|$.
Case 3: If $a < 0$ and $b \geq 0$ then $|a| = -a$and $|b| = b$. Then $|ab| = -(ab) = (-a)b = |a||b|$.
Case 4: If $a < 0$ and $b < 0$ then $|a| = -a$and $|b| = -b$. Then $|ab| = ab = (-a)(-b) = |a||b|$.
Thus we've proved that $|ab| = |a||b|$ by proving its truth for every case for all real numbers a and b. ☐

b. *Proof.* Let a and b be real numbers. Consider the following cases:
Case 1: If $a \geq b$ then $a-b \geq 0$ and $b-a \leq 0$. This means $|a-b| = a-b = -(b-a) = |b-a|$.
Case 2: If $a < b$ then $a-b < 0$ and $b-a > 0$. This means $|a-b| = -(a-b) = b-a = |b-a|$. Thus we've proved that $|a-b| = |b-a|$ by proving its truth for every case for all real numbers a and b. ☐

d. *Proof.* Let a and b be real numbers so $a \leq |a|$, $-a \leq |a|$, $b \leq |b|$, and $-b \leq |b|$,. Consider the following cases:
Case 1: If $a+b \geq 0$ then $|a+b| = a+b \leq |a|+|b|$.
Case 2: If $a+b < 0$ then $|a+b| = -(a+b) = -a-b \leq |a|+|b|$. Thus we've proved that $|a+b| \leq |a|+|b|$ by proving its truth for every case for all real numbers a and b. ☐

e. *Proof.* Let a and b be real numbers and $|a| \leq b$ so that $a \leq |a|$, $b \geq 0$ and $-b \leq 0$. Consider the following cases:
Case 1: If $a \geq 0$ then $|a| = a$ so that $a \leq b$. By assumptions $-b \leq |a| \leq b$ is true and therefore $-b \leq a \leq b$.
Case 2: If $a < 0$ then $|a| = -a$ so that $-a \leq b$ and implicitly $a \leq b$ as $-a \leq |a|$. Because $-a \leq b$ then we can say that $a \geq -b$ therefore $-b \leq a \leq b$.
Thus we've proved that $-b \leq a \leq b$ by proving its truth for every case for all real numbers a and b when $|a| \leq b$. ☐

f. *Proof.* Let a and b be real numbers and $-b \leq a \leq b$. Consider the following cases:

Case 1: If $a \geq 0$ then $|a| = a$ so that $|a| \leq b$.
Case 2: If $a < 0$ then $|a| = -a$. The assumed equivalence $-b \leq a$ can be morphed into $b \geq -a$ which is $b \geq |a|$. Thus we've proved that $|a| \leq b$ by proving its truth for every case for all real numbers a and b when $-b \leq a \leq b$. □

1.4.7 c. *Proof.* Assume that a is odd, so that $a = 2k + 1$ where $k \in \mathbb{Z}$. Then $a+2 = 2k+1+2 = 2k+2+1 = 2(k+1)+1 = 2j+1$ where $j = k+1$ is an integer. Thus we've proved that $a+2$ is odd if a is odd. □

d. *Proof.* Let a be a real number. Consider the following cases:
Case 1: If we assume a is odd, so that $a = 2k + 1$ where $k \in \mathbb{Z}$, then $a(a+1) = (2k+1)(2k+1+1) = (2k+1)(2k+2) = 2((2k+1)(k+1)) = 2j$ where $j = (2k+1)(k+1)$ is an integer.
Case 2: If we assume b is even, so that $a = 2k$ where $k \in \mathbb{Z}$, then $a(a+1) = 2k(2k+1) = 2(k(2k+1)) = 2j$ where $j = k(2k+1)$ is an integer.
Thus we have proved that $a(a+1)$ for all integers a by proving its truth for all cases. □

e. *Proof.* Assume a is an integer. We'll prove that $1|a$ for all integers. If $1|a$ then $a = 1k$ for some $k \in \mathbb{Z}$. Which is equivalent to $a = k$. Thus we've proved $1|a$ by showing there always exists an integer k that satisfies $a = 1k$. □

f. *Proof.* Assume a is an integer. We'll prove that $a|a$ for all integers. If $a|a$ then $a = ak$ for some $k \in \mathbb{Z}$. Which is equivalent to $1 = k$. Thus we've proved $a|a$ by showing there always exists an integer k that satisfies $a = ak$. □

g. *Proof.* Assume a and b are positive integer and $a|b$ so that $b = ak$ for some $k \in \mathbb{Z}$. We'll prove that if the assumptions are true then $a \leq b$. Since both a and b are positive the equivalence statement $b = ak$ must use a $k \geq 1$, because if $k < 1$ a and b would have opposite parities or would require $b = 0$ making it no longer positive, therefore $k \in \mathbb{N}$. If $b = ak$ then $\frac{b}{k} = a$ where $k \geq 1$ and b and a are positive. Thus we've proved $a \leq b$ if both a and b are positive integers and $a|b$. □

h. *Proof.* Assume $a|b$ so that $b = ak$ for some $k \in \mathbb{Z}$ and there exists an integer c, then $a|bc$ so that $bc = aj$ for some $j \in \mathbb{Z}$. This equivalence implies $bc = c(ak)$ and $bc = a(ck)$ where $ck = j$. Thus we've proved that if $a|b$ then $a|bc$ for any integers a, b, and c. □

i. *Proof.* Assume a and b are positive integers and $ab = 1$, $a = \frac{1}{b}$. Since a is a positive integer $\frac{1}{b}$ must be a positive integer which is only true if $b = 1$ as any greater denominator will make $\frac{1}{b} \notin \mathbb{Z}$. Any lesser denominator would make b negative or $b = 0$ and $\frac{1}{0}$ can't exist. If $b = 1$ then by $a = \frac{1}{b} = 1$. Thus we've proved that $a = b = 1$ if a and b are positive integers and $ab = 1$. □

j. *Proof.* Assume a and b are positive integers, $a|b$ so that $b = ak$, and $b|a$ so that $a = bj$ for some $k, j \in \mathbb{Z}$. We'll prove $a = b$ if the assumptions are true. Since $a = bj$ then $a = akj$ which simplifies to $1 = kj$ and since $k, j \in \mathbb{Z}$ then $k, j = 1$ since they must each be positive. Therefore $b = ak$ can be simplified to $b = a$. Thus we've proved that $a = b$ if, for any positive integers a and b, $a|b$ and $b|a$. □

k. *Proof.* Assume for integers a, b, c, and d that $a|b$ so that $b = ak$ and $c|d$ so that $d = cj$ where $k, j \in \mathbb{Z}$. We'll prove that $ac|bd$ so that $bd = ach$ for some $h \in \mathbb{Z}$ if the assumptions are true. Based on the assumption $bd = ach$ is equivalent to $(ak)(cj) = ach$ therefore $(ac)(kj) = (ac)h$ which is true if $h = kj$. Thus we've proved that $ac|bd$ if, for some integers a, b, c, and d, that satisfy $a|b$ and $c|d$. □

1.4.8 a. *Proof.* Assume n is a natural number. We'll prove $n^2 + n + 3$ is odd. Consider the following cases:
Case 1: If n is odd, so that $n = 2k+1$ for some $k \in \mathbb{Z}$ greater than or equal to 0, then $n^2 + n + 3 = (2k + 1)^2 + 2k + 1 + 3 = 4k^2 + 4k + 1 + 2k + 4 = 4k^2 + 6k + 4 + 1 = 2(2k^2 + 3k + 2) + 1 = 2j + 1$ where $j = 2k^2 + 3k + 2$ is an integer greater than or equal to 2.
Case 2: If n is even, so that $n = 2k$ for some $k \in \mathbb{Z}$ greater than or equal to 1, then $n^2+n+3 = (2k)^2+2k+3 = 4k^2+2k+2+1 = 2(2k^2+k+1)+1 = 2j + 1$ where $j = 2k^2 + k + 1$ is an integer greater than or equal to 3.
Thus we've proved that $n^2 + n + 3$ is odd if $n \in \mathbb{N}$ by showing all cases are true. □

b. *Proof.* Assume n is a natural number. We'll prove n^2+n+3 is odd. Since $n^2 + n + 3 = n(n + 1) + 3$ and $a(a + 1)$ is even for any $a \in \mathbb{Z}$ then $n(n + 1)$ is even. Since $x + y$ is odd when x is even and y is odd, and $n(n + 1)$ is even and 3 is odd (since $3 = 2(1) + 1$), then $n^2 + n + 3$ is odd. Thus we've proved $n^2 + n + 3$ is odd for any integer n and natural numbers are within the domain of integers so it is true for all natural numbers n as well. □

1.4.9 a. *Proof.* Assume x and y are both nonnegative real numbers. The statement $\frac{x+y}{2} \geq \sqrt{xy}$ is true iff:

$$\begin{aligned}
\frac{(x+y)^2}{4} &\geq xy && \iff \\
(x+y)^2 &\geq 4xy && \iff \\
x^2 + y^2 + 2xy &\geq 4xy && \iff \\
x^2 + y^2 - 2xy &\geq 0 && \iff \\
(x-y)^2 &\geq 0 &&
\end{aligned}$$

Because the final statement is true for all nonnegative real numbers all previous statements are true including $\frac{x+y}{2} \geq \sqrt{xy}$. Thus we've proved for $\frac{x+y}{2} \geq \sqrt{xy}$ for all nonnegative real numbers. □

b. *Proof.* Assume a, b, and c are integers that satisfy $a|b$ and $a|b + c$ so that $b = ak$ and $b + c = aj$ for some $k, j \in \mathbb{Z}$. The statement $a|3c$ is true iff

$$\begin{aligned}
3c &= ah && \iff \\
3(aj - b) &= ah && \iff \\
3aj - 3b &= ah && \iff \\
3aj - 3ak &= ah && \iff \\
3(j - k) &= h && \text{for some } h \in \mathbb{Z}
\end{aligned}$$

Since the final statement is true all previous statements are true including $a|3c$ for all integers a, b, and c that satisfy $a|b$ and $a|b + c$. □

c. *Proof.* Assume a, b, and c are integers that satisfy $ab > 0$ and $bc < 0$. The statement "$ax^2 + bx + c = 0$ has two real solutions" is true iff:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{has two real solutions, iff}$$
$$0 < b^2 - 4ac \quad \iff$$
$$4ac < b^2 \quad \text{since } ab(bc) < 0 \text{ and therefore } 4ac < 0$$

Since the final statement is true all previous statements are true including "$ax^2 + bx + c = 0$ has two real solutions." Thus we have proved that $ax^2 + bx + c = 0$ has two real solutions if integers a, b, and c satisfy $ab > 0$ and $bc < 0$. □

1.4.11 b. C, this claim is correct but incorrect proof as it claims $b = aq$ for some integer q but also claims $c = aq$ for the same integer q which is not a correct assumption. It should be $c = ap$ for some new integer p, which would make the claim incorrect. But $b + c = al$ can be turned into $aq + ap = al$ which simplifies to $q + p = l$ which is true so the claim is correct.

c. F, all one must do to debunk this claim is plug in any large magnitude negative number such as -100 for x to prove this wrong. The multiplication by x is the step that incorrect logic is used as for any negative x the equality should be switched.

d. A, good proof by working backward.