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HOMEWORK 4

JESSE COBB - 2PM SECTION

3.1.2 $T = \{(3,1),(2,3),(3,5),(2,2),(1,6),(2,6),(1,2)\}$

a. $\text{Dom}(T) = \{1,2,3\}$

b. $\text{Rng}(T) = \{1,2,3,5,6\}$

c. $T^{-1} = \{(1,3),(3,2),(5,3),(2,2),(6,1),(6,2),(2,1)\}$

d. $(T^{-1})^{-1} = \{(3,1),(2,3),(3,5),(2,2),(1,6),(2,6),(1,2)\}$

3.1.3 W on $\mathbb{R}$, $(x,y) \in W$

b. $y = x^2 + 3$
$\text{Dom}(W) = \mathbb{R}$
$\text{Rng}(W) = \{y \in \mathbb{R} : y \geq 3\}$

d. $y = \frac{1}{x^2}$
$\text{Dom}(W) = \mathbb{R} - \{0\}$
$\text{Rng}(W) = (0, \infty)$

3.1.5 a. $R = \{(x,y) \in \mathbb{R}^2 : y = 6x\}, \text{Dom}(R) = \mathbb{R}, \text{Rng}(R) = \mathbb{R}$

Proof. Let $x \in \text{Dom}(R)$. Then there exists $y \in \mathbb{R}$ so that $(x,y) \in R$ so that $y = 6x$ where $6x$ is defined for all real numbers. Thus $x \in \mathbb{R}$ so $\text{Dom}(R) \subseteq \mathbb{R}$. Now let $x \in \mathbb{R}$ and if we let $y = 6x$ there is always a defined y so $x \in \text{Dom}(R)$, therefore $\mathbb{R} \subseteq \text{Dom}(R)$ and thus $\mathbb{R} = \text{Dom}(R)$.
Now let $y \in \text{Rng}(R)$. Then there exists $x \in \mathbb{R}$ so that $(x,y) \in R$ so that $y = 6x$. Then since $y = 6x$, $y \in \mathbb{R}$ since $x \in \mathbb{R}$ and therefore $\text{Rng}(R) \subseteq \mathbb{R}$. Now let $y \in \mathbb{R}$ and let $x = \frac{y}{6}$ now to find a relation xRy we say $y = 6(\frac{y}{6}) = y$ which shows that $y \in \text{Rng}(R)$. Therefore $\mathbb{R} \subseteq \text{Rng}(R)$ and hence $\mathbb{R} = \text{Rng}(R)$. $\square$

b. $R = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}, \text{Dom}(R) = \mathbb{R}, \text{Rng}(R) = [0, \infty)$

Proof. Let $x \in \text{Dom}(R)$. Then there exists $y \in \mathbb{R}$ so that $(x,y) \in R$ so that $y \geq x^2$. Since in x^2, x is defined for all real number so $x \in \mathbb{R}$ and $\text{Dom}(R) \subseteq \mathbb{R}$. Now let $x \in \mathbb{R}$. Since x^2 is defined for all real numbers for $y \geq x^2$ then $x \in \text{Dom}(R)$ and thus $\mathbb{R} = \text{Dom}(R)$.
Now let $y \in \text{Rng}(R)$. Then there exists $x \in \mathbb{R}$ so that $(x,y) \in R$ so that $y \geq x^2$. Then since $x^2 \geq 0$ then $y \geq 0$ so $y \in [0,\infty)$ so $\text{Rng}(R) \subseteq [0,\infty)$. Now let $y \in [0,\infty)$. To find an x that is related to this y let $x = \sqrt{y}$ then $y \geq x^2 = \sqrt{y}^2 = y$. This shows that a relationship is defined for all $y \in [0,\infty)$ so that $[0,\infty) = \text{Rng}(R)$. $\square$

3.1.6 b. $R_2 = \{(x,y) \in \mathbb{R}^2 : y = -5x + 2\}$
$R_2^{-1} = \{(x,y) \in \mathbb{R}^2 : \frac{2-x}{5} = y\}$

3.1.7 $R = \{(1,5),(2,2),(3,4),(5,2)\}, S = \{(2,4),(3,4),(3,1),(5,5)\}$,
$T = \{(1,4),(3,5),(4,1)\}$

a. $R \circ S = \{(3,5),(5,2)\}$

e. $S \circ R = \{(1,5),(2,4),(5,4)\}$

3.1.8 e. $\{(x,y) \in \mathbb{R}^2 : y = -5x+2\} \circ \{(x,y) \in \mathbb{R}^2 : y = x^2+2\}$
$= \{(x,y) \in \mathbb{R}^2 : y = -5x^2 - 8\}$

f. $\{(x,y) \in \mathbb{R}^2 : y = x^2+2\} \circ \{(x,y) \in \mathbb{R}^2 : y = -5x+2\}$
$= \{(x,y) \in \mathbb{R}^2 : y = 25x^2 - 20x + 6)\}$

3.1.11 a. *Proof.* ($\subseteq$) Assume $x \in \text{Rng}(R^{-1})$ then there exists a y so that $(y,x) \in R^{-1}$. By definition of inverse set $(x,y) \in R$ therefore $x \in \text{Dom}(R)$. This shows that $\text{Rng}(R^{-1}) \subseteq \text{Dom}(R)$.
($\supseteq$) Now assume $x \in \text{Dom}(R)$ then there exists a y so that $(x,y) \in R$. By definition of inverse set $(y,x) \in R^{-1}$ therefore $x \in \text{Rng}(R^{-1})$. So that $\text{Rng}(R^{-1}) \supseteq \text{Dom}(R)$. Thus we've proved that $\text{Rng}(R^{-1}) = \text{Dom}(R)$ by showing they are subsets of each other. $\square$

3.2.1 d. $\{(x,y) \in \mathbb{N}^2 : x < y\}$
Not Reflexive.
Not Symmetric.
Transitive.

f. $\{(x,y) \in \mathbb{N}^2 : x \neq y\}$
Not Reflexive.
Symmetric.
Not Transitive.

g. $\{(x,y) \in \mathbb{N}^2 : x \mid y\}$
Reflexive.
Not Symmetric.
Transitive.

3.2.6 b. *Proof.* In order to prove that R is an equivalence statement, consider the following:
Reflexive: Let $x \in \mathbb{N}$ so that its 10's digit can be written as the natural number $0 \leq a \leq 9$. Then x also has the same a for its 10's digit. Therefore $(x,x) \in R$ so the relation is reflexive.
Symmetric: Let $x, y \in \mathbb{N}$ so that x, y's 10's digit can be written as the natural numbers $0 \leq a \leq 9$ and $0 \leq b \leq 9$ respectively. Assume $(x,y) \in R$ so that $a = b$, then $b = a$. Thus $(y,x) \in R$ so the relation is symmetric.
Transitive: Now let $x, y, z \in \mathbb{N}$ so that x, y, z's 10's digit can be written as the natural numbers $0 \leq a \leq 9$, $0 \leq b \leq 9$, and $0 \leq c \leq 9$ respectively. Now assume $(x,y) \in R$ and $(y,z) \in R$ so that $a = b$ and $b = c$, thus $a = c$. Thus $(x,z) \in R$ so the relation is transitive.
Thus the R is an equivalence relation as it is reflexive, symmetric, and transitive. $\square$
$1 \in 1\bar{0}6 \cap 1 < 50$
$200 \in 1\bar{0}6 \cap 150 < 200 < 300$
$1001 \in 1\bar{0}6 \cap 1000 < 1001$
$30 \in 6\bar{3}5 \cap 30 < 50$
$230 \in 6\bar{3}5 \cap 150 < 230 < 300$
$1031 \in 6\bar{3}5 \cap 1000 < 1031$

c. *Proof.* In order to prove that V is an equivalence statement, consider the following:
Reflexive: Let $x \in \mathbb{R}$. Since $x = x$ this implies that $(x,x) \in V$ so that V is reflexive.

Symmetric: Let $x, y \in \mathbb{R}$. Now assume that $(x, y) \in V$ so that $x = y$ or $xy = 1$. Since $y = x$ and $yx = 1$ is true by assumptions then $(y, x) \in V$ so that V is symmetric.
Transitive: Let $x, y, z \in \mathbb{R}$. Now assume that $(x, y) \in V$ and $(y, z) \in V$ so that $x = y$ or $xy = 1$ and $y = z$ or $yz = 1$. In the case that $y = z$ simply by substitution $x = z$ or $xz = 1$. In the case that $yz = 1$ if $x = y$ then $xz = 1$ and if $xy = 1$ then $x = z$. Thus $(x, z) \in V$ so that V is transitive.
Thus we've shown that V is an equivalence relation as it is reflexive, symmetric, and transitive. □
$\bar{3} = \{3, \frac{1}{3}\}$
$-\bar{\frac{2}{3}} = \{-\frac{2}{3}, -\frac{3}{2}\}$
$\bar{0} = \{0\}$

d. *Proof.* In order to prove that R is an equivalence statement, consider the following:
Reflexive: Let $a \in \mathbb{N} - \{1\}$. Then $a = m2^n$ where $m \in \mathbb{N}$,$n \in \mathbb{N} \cup \{0\}$ and $2 \nmid m$. By this definition a and a will have the same value for n (numbers of prime factors of 2) so $(a, a) \in R$. Thus the relationship is reflexive.
Symmetric: Let $a, b \in \mathbb{N} - \{1\}$. Then $a = m2^n$ and $b = x2^y$ where $m, x \in \mathbb{N}$,$n, y \in \mathbb{N} \cup \{0\}$ and $2 \nmid m, x$. Now assume $(a, b) \in R$ so that $n = y$ (their number of prime factors of 2 are the same) then $y = n$ and therefore $(b, a) \in R$. Thus the relation is symmetric.
Transitive: Let $a, b, c \in \mathbb{N} - \{1\}$. Then $a = m2^n$, $b = x2^y$, and $c = k2^j$ where $m, x, k \in \mathbb{N}$,$n, y, j \in \mathbb{N} \cup \{0\}$ and $2 \nmid m, x, k$. Now assume $(a, b) \in R$ and $(b, c) \in R$ so that $n = y$ and $y = j$. Then $n = j$ therefore a and c have the same number of prime factors of 2 so $(a, c) \in R$. This shows that the relation is transitive.
Thus we've shown that R is an equivalence relation as it is reflexive, symmetric, and transitive. □
$2, 3, 5 \in \bar{7}$
$2, 3, 5 \in \bar{10}$
$8, 24, 40 \in \bar{72}$

i. *Proof.* In order to prove that T is an equivalence statement, consider the following:
Reflexive: Let $x \in \mathbb{R}$. Since $\sin(x) = \sin(x)$ then $(x, x) \in T$. Thus we've shown that T is reflexive.
Symmetric: Let $x, y \in \mathbb{R}$. Now assume $(x, y) \in T$ so that $\sin(x) = \sin(y)$ and thus $\sin(y) = \sin(x)$ which shows that $(y, x) \in T$. Thus we've shown that T is symmetric.
Transitive: Now let $x, y, z \in \mathbb{R}$. Now assume $(x, y) \in T$ and $(y, z) \in T$ so that $\sin(x) = \sin(y)$ and $\sin(y) = \sin(z)$. Then through substitution $\sin(x) = \sin(z)$ so $(x, z) \in T$ which shows that T is transitive.
Thus we've shown that T is an equivalence relation as it is reflexive, symmetric, and transitive. □
$\bar{0} = \{y : y = 2n\pi, n \in \mathbb{Z}\}$
$\bar{\frac{\pi}{2}} = \{y : y = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}\}$
$\bar{\frac{\pi}{4}} = \{y : y = 2n\pi + \frac{\pi}{4} \vee y = 2n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}\}$

3.2.7 *Proof.* In order to prove that R is an equivalence statement, consider the following:

Reflexive: Let $x \in \mathbb{Q}$. Then $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Since $pq = qp$ then $(x, x) \in R$. Thus we've shown that R is reflexive.
Symmetric: Let $x, y \in \mathbb{Q}$. Then $x = \frac{p}{q}$ and $y = \frac{s}{t}$ for $p, q, s, t \in \mathbb{Z}$ and $q, t \neq 0$. Now assume $(x, y) \in R$ so that $pt = qs$. Since $qs = pt$ then $(y, x) \in R$. Thus we've shown that R is reflexive.
Transitive: Now let $x, y, z \in \mathbb{Q}$. Then $x = \frac{p}{q}$, $y = \frac{s}{t}$, and $z = \frac{j}{k}$ for $p, q, s, t, j, k \in \mathbb{Z}$ and $q, t, k \neq 0$. Now assume $(x, y), (y, z) \in R$ so that $pt = qs$ and $sk = tj$. Then $pt = qs \implies p(\frac{sk}{j}) = qs \implies pk = qj$ so that $(x, z) \in R$. Thus we've shown that R is transitive.
Thus we've shown that R is an equivalence relation as it is reflexive, symmetric, and transitive. □

$\overline{\frac{2}{3}} = \{y : y = \frac{a}{b}, a = 2n, b = 3n, n \in \mathbb{Z}, n \neq 0\}$

3.3.2 a. $A = \{1, 2, 3, 4\}$, $\mathscr{P} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$
$\varnothing \notin \mathscr{P}$
$\{1, 2\} \cap \{2, 3\} \neq \varnothing$
$\mathscr{P}$ is not a partition.

c. $A = \{1, 2, 3, 4, 5, 6, 7\}, \mathscr{P} = \{\{1, 3\}, \{5, 6\}, \{2, 4\}, \{7\}\}$
$\varnothing \notin \mathscr{P}$
$\bigcup_{B \in \mathscr{P}} B = A$
Pairwise disjoint
$\mathscr{P}$ is a partition.

e. $A = \mathbb{R}, \mathscr{P} = (-\infty, -1) \cup [-1, 1] \cup (1, \infty)$
$\varnothing \notin \mathscr{P}$
$\bigcup_{B \in \mathscr{P}} B = \mathbb{R}$
Pairwise disjoint
$\mathscr{P}$ is a partition.

3.3.3 a. *Proof.* Assume $\mathscr{P} = \{\{-x, x\} : x \in \mathbb{N} \cup \{0\}\}$. Since $\varnothing \neq \{-x, x\}$ where $x \in \mathbb{N} \cup \{0\}$ then $\varnothing \notin \mathscr{P}$. Next if we assume $x, y \in \mathbb{N} \cup \{0\}$ are not equal $(x \neq y)$ then $\{-x, x\} \cap \{-y, y\} = \varnothing$ since $x \notin \{-y, y\}$ since it is not equal to y or $-y$ (Since $x \geq 0$). Finally $\bigcup_{A \in \mathscr{P}} A \subseteq \mathbb{Z}$ since for an element $x \in \bigcup_{A \in \mathscr{P}} A$ $x \in \mathbb{Z}$ and $\mathbb{Z} \subseteq \bigcup_{A \in \mathscr{P}} A$ since any integer $n \in \{\pm n, \mp n\}$ so $n \in \bigcup_{A \in \mathscr{P}} A$. Then $\bigcup_{A \in \mathscr{P}} A = \mathbb{Z}$. Thus we've proved $\mathscr{P}$ is a partition of $\mathbb{Z}$ □

3.3.7 a. R on $\mathbb{N}$
$\mathscr{P} = \{\{1, 2, \ldots 9\}, \{10, 11, \ldots 99\}, \{100, 101, \ldots 999\}, \ldots\}$
R has a relation between all numbers of n digits (ex. all 2 digit numbers are related to each other and to themselves).

c. R on $\mathbb{R}$
$\mathscr{P} = \{(-\infty, 0), \{0\}, (0, \infty)\}$
R has a relation between all negative real numbers, a relation between all positive real numbers, and a relation between each real number and itself.

3.3.9 d. $A = \{1, 2, 3, 4, 5\}$
$\mathscr{P} = \{\{1, 2\}, \{3, 4, 5\}\}$
$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5),$
$(1, 2), (2, 1), (3, 4), (4, 3), (3, 5), (5, 3), (4, 5), (5, 4)\}$