HOMEWORK 6

JESSE COBB - 2PM SECTION

5.1.1 *Proof.* Let A be a set then $A \approx A$ since there exists a bijection $I_A : A \to A$ given by $I_A(x) = x$. Thus $\approx$ is a reflexive relation.
Let A and B be sets and $A \approx B$ so there exists a bijection $f : A \to B$. Then by definition there exists an inverse bijection $f^{-1} : B \to A$ so $B \approx A$. Thus $\approx$ is a symmetric relation.
Now Let A,B, and C be sets and let $A \approx B$ and $B \approx C$ so that there exists bijections $f : A \to B$ and $g : B \to C$. By definition there exists a bijection $g \circ f : A \to C$ that is a composite of two bijections. This implies $A \approx C$ so $\approx$ is a transitive relation.
Thus we've shown $\approx$ to be an equivalence relation as it is reflexive, symmetric, and transitive. $\square$

5.1.2 a. $A = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512\} \approx \mathbb{N}_{10}$
$f : \mathbb{N}_{10} \to A$ where $f(n) = 2^{n-1}$ so $\overline{\overline{A}} = 10$
c. $B = \{x \in \mathbb{Z} : x^2 < 11\} \approx \mathbb{N}_7$
$g : \mathbb{N}_7 \to B$ where $g(n) = n - 4$ so $\overline{\overline{B}} = 7$
d. $C = \{(x, y) \in \mathbb{N} : x + y < 6\} \approx \mathbb{N}_{10}$
$h : \mathbb{N}_{10} \to C$ where $h(1), h(2), h(3), h(4), h(5), h(6), h(7), h(8), h(9), h(10)$
$= (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)$ so $\overline{\overline{C}} = 10$

5.1.13 *Proof.* Let $x \in \mathbb{N}_r$ where r is a positive integer. By definition there exists an identity bijection $f : \mathbb{N}_r \to \mathbb{N}_r$ where $f(n) = n$ for all $n \in \mathbb{N}_r$. Due to this definition of the identity bijection there exists $g : \mathbb{N}_r - \{x\} \to \mathbb{N}_{r-1}$ where $g(m) = f(m) = m \in \mathbb{N}_r - \{0\}$ where there is 1 less one-to-one mapping so g is a one-to-one correspondence on $\mathbb{N}_{r-1}$. Thus we've proved that $\mathbb{N} - \{0\} \approx \mathbb{N}_{r-1}$. $\square$

5.1.14 *Proof.* Let $\overline{\overline{A}} = n$ and $\overline{\overline{B}} = r$ where $r < n$. This implies there exists bijections $f : \mathbb{N}_n \xrightarrow{1-1} A$ and $g : B \xrightarrow{1-1} \mathbb{N}_r$. Now, seeking a contradiction, assume there exists an injection $h : A \xrightarrow{1-1} B$. This implies there exists an injection $g \circ (f \circ g) : \mathbb{N}_n \xrightarrow{1-1} \mathbb{N}_r$ which is a contradiction by the Pigeonhole Principle as there is no injection from $\mathbb{N}_n \to \mathbb{N}_r$ where $r < n$. Then h can't possibly be an injection. Thus we've proved there exists no injection from A to B if $\overline{\overline{A}} = n$ and $\overline{\overline{B}} = r$ where $r < n$. $\square$

5.2.2 a. *Proof.* Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}$ and $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots\}$ where $B \subsetneq A$. There exists a function $f : A \to B$ where $f(x) = \frac{1}{1+\frac{1}{x}}$. Let $x = \frac{1}{\frac{1}{y}-1} \in A$, note that:

$$\begin{aligned} f(x) &= f(\frac{1}{\frac{1}{y}-1}) \\ &= \frac{1}{1+\frac{1}{\frac{1}{\frac{1}{y}-1}}} \\ &= \frac{1}{1+\frac{1}{y}-1} \\ &= \frac{1}{\frac{1}{y}} \\ &= y \end{aligned}$$

So we've proved f to be a surjection as there always exists an $x \in A$ for all values of $y \in B$ such that $f(x) = y$. Next let $x_1, x_2 \in A$ and note:

$$\begin{aligned} f(x_1) = f(x_2) &\implies \frac{1}{1+\frac{1}{x_1}} = \frac{1}{1+\frac{1}{x_2}} \\ &\implies 1 + \frac{1}{x_2} = 1 + \frac{1}{x_1} \\ &\implies \frac{1}{x_2} = \frac{1}{x_1} \\ &\implies x_1 = x_2 \end{aligned}$$

So we've proved f to be an injection as any two equal elements in B are mapped to by the same element in A. Thus f is a bijection showing that $A \approx B$, but since $B \subsetneq A$ then A must be an infinite set. Thus we've proved A to be an infinite set. $\square$

b. *Proof.* Let $\mathbb{N} - \mathbb{N}_{15} = \{16, 17, 18 \ldots\}$ and $\mathbb{N} - \mathbb{N}_{16} = \{17, 18, 19 \ldots\}$ exist as well as a function $f : \mathbb{N} - \mathbb{N}_{15} \to \mathbb{N} - \mathbb{N}_{16}$ where $f(x) = x + 1$. Let $x = y - 1 \in \mathbb{N} - \mathbb{N}_{15}$ so that $f(x) = f(y-1) = y - 1 + 1 = y \in \mathbb{N} - \mathbb{N}_{16}$. Thus we've shown f to be a surjection as there always exists an x for all y such that $f(x) = y$. Next note that if, for $x_1, x_2 \in \mathbb{N} - \mathbb{N}_{15}$:

$$\begin{aligned} &f(x_1) = f(x_2) \\ &\qquad \implies x_1 + 1 = x_2 + 1 \\ &\qquad \implies x_1 = x_2 \end{aligned}$$

So we've shown f to be an injection as any two equal elements in $\mathbb{N} - \mathbb{N}_{16}$ are mapped to by two equal elements in $\mathbb{N} - \mathbb{N}_{15}$. Thus f is a bijection so $\mathbb{N} - \mathbb{N}_{15} \approx \mathbb{N} - \mathbb{N}_{16}$. Since $\mathbb{N} - \mathbb{N}_{16} \subsetneq \mathbb{N} - \mathbb{N}_{15}$, this shows that $\mathbb{N} - \mathbb{N}_{15}$ is an infinite set. $\square$

5.2.3 c. *Proof.* Let $3\mathbb{Z} = \{\ldots -6, -3, 0, 3, 6 \ldots\}$ and there exist a function $f : \mathbb{Z} \to 3\mathbb{Z}$ where $f(x) = 3x$. Let $x = \frac{y}{3} \in \mathbb{Z}$. Note that: $f(x) = f(\frac{y}{3}) = 3(\frac{y}{3}) = y$. Thus we've shown f to be a surjection as there always exists an $x \in \mathbb{Z}$ for all $y \in 3\mathbb{Z}$ such that $f(x) = y$. Next note when $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$ then $3x_1 = 3x_2 \implies x_1 = x_2$. Thus we've shown f to be an injection as all same images in $3\mathbb{Z}$ have the same preimages in $\mathbb{Z}$. Since f is a bijection $\mathbb{Z} \approx 3\mathbb{Z}$. Since $\mathbb{N} \approx \mathbb{Z}$ then $\mathbb{N} \approx 3\mathbb{Z}$ by transitivity. Thus $3\mathbb{Z}$ is denumerable. $\square$

e. *Proof.* Let $A = \{x : x \in \mathbb{Z} \text{ and } x < -12\}$. Then there exists a function $f : \mathbb{N} \to A$ given by $f(x) = -x - 11$. Let $x = -y - 11$ then $f(x) = f(-y-11) = -(-y-11) - 11 = y$. Thus we've shown f to be a surjection as there always exists and an $x \in \mathbb{N}$ for all $y \in A$ such that $f(x) = y$. Next if $x_1, x_2 \in \mathbb{N}$ then $f(x_1) = f(x_2) \implies -x_1 - 11 = -x_2 - 11 \implies x_1 = x_2$. Thus we've shown f to be an injection as each value in $\mathbb{N}$ is mapped to a unique value in A. Since there exists a bijection, f, between $\mathbb{N}$ and A then $\mathbb{N} \approx A$ and therefore A is denumerable. $\square$

f. *Proof.* Let $A = \mathbb{N} - \{5, 6\}$. Then there exists a function $f : \mathbb{N} \to A$ given by $f(x) = \begin{cases} x & x < 5 \\ x+2 & x \geq 5 \end{cases}$. For the case of $x < 5$ let $x = y$ so that $f(x) = f(y) = y$ and for the case that $x \geq 5$ let $x = y - 2$ so $f(x) = f(y-2) = y - 2 + 2 = y$ so we've shown f to be a surjection as there always exists an $\in \mathbb{N}$ for either case for any $y \in A$. Then if $x_1 = x_2$ where $x_1, x_2 < 5$ then $x_1 = x_2$. If $x_1, x_2 \geq 5$ then $x_1 + 2 = x_2 + 2 \implies x_1 = x_2$. Then finally, without loss of generality, if $x_1 < 5$ and $x_2 \geq 5$ then $x_1 = x_2 + 2$ which is impossible as $x_2 > x_1$. Thus we've shown f to be an injection for all $x_1, x_2 \in \mathbb{N}$. Since f is a bijection between $\mathbb{N}$ and A we can say that A is denumerable. $\square$

5.2.4 a. *Proof.* Let $f : (0,1) \to (1, \infty)$ be a function given by $f(x) = \frac{1}{x}$. Let $x = \frac{1}{y} \in (0,1)$ then note $f(x) = f(\frac{1}{y}) = \frac{1}{\frac{1}{y}} = y$, thus f is a surjection as there always exists an $x \in (,1)$ for all $y \in (1, \infty)$ so that $f(x) = y$. Next assume for $x_1, x_2 \in (0,1)$ that $f(x_1) = f(x_2) \implies \frac{1}{x_1} = \frac{1}{x_2} \implies x_2 = x_1$, thus f is an injection for all $x_1, x_2 \in (0,1)$. Since there exists a bijection, f, between $(0,1)$ and $(1, \infty)$ then $(0,1) \approx (1, \infty)$. Since $\overline{\overline{(0,1)}} = \mathfrak{c}$ then $\overline{\overline{(1,\infty)}} = \mathfrak{c}$. $\square$

b. *Proof.* Let $f : (0,1) \to (a, \infty)$ be a function given by $f(x) = \frac{1}{x} + a - 1$. Let $x = \frac{1}{y-a+1} \in (0,1)$ then note $f(x) = f(\frac{1}{y-a+1}) = a + \frac{1}{\frac{1}{y-a+1}} = a - 1 + y - a + 1 = y$, thus f is a surjection as there always exists an $x \in (0,1)$ for any $y \in (a, \infty)$ such that $f(x) = y$. Additionally if $x_1, x_2 \in (0,1)$ and we assume $f(x_1) = f(x_2)$ then $\frac{1}{x_1} + a - 1 = \frac{1}{x_2} + a - 1 \implies \frac{1}{x_1} = \frac{1}{x_2} \implies x_2 = x_1$ so f is an injection. Since a bijection, f, exists between $(0,1)$ and (a, ∞) then $(0,1) \approx (a, \infty)$. Thus $\overline{\overline{(a,\infty)}} = \mathfrak{c}$. $\square$

5.3.1 $f(28) = 9$

5.3.4 *Proof.* Let $f : \mathbb{N} \xrightarrow{bijection} A$ and $g : \mathbb{N} \xrightarrow{biject} B$. Define $h : \mathbb{N} \to A \cup B$:

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & 2 \nmid n \\ g(\frac{n}{2}) & 2 \mid n \end{cases}$$

Let $n = 2y - 1 \in \mathbb{N}$ where $y\mathbb{N}$ and therefore $2 \nmid n$ then note:

$$\begin{aligned} h(n) &= h(2y-1) \\ &= f(\frac{2y-1+1}{2}) \\ &= f(y) \in A \\ &= f(y) \in A \cup B \end{aligned}$$

Now let $n = 2y \in \mathbb{N}$ where $y \in \mathbb{N}$ so $2 \mid n$ then note:

$$\begin{aligned} h(n) &= h(2y) \\ &= g(\frac{2y}{2}) \\ &= g(y) \in B \\ &= g(y) \in A \cup B \end{aligned}$$

Thus we've shown h to be a surjection as there exists an $n \in \mathbb{N}$ for any $f(y), g(y) \in A \cup B$. Next assume $n_1, n_2 \in \mathbb{N}$, first let $2 \nmid n_1, n_2$ so:

$$\begin{aligned} h(n_1) = h(n_2) &\implies f(\frac{n_1+1}{2}) = f(\frac{n_2+1}{2}) \\ &\implies (f^{-1} \circ f)(\frac{n_1+1}{2}) = (f^{-1} \circ f)(\frac{n_2+1}{2}) \\ &\implies \frac{n_1+1}{2} = \frac{n_2+1}{2} \\ &\implies n_1 = n_2 \end{aligned}$$

Similarly if $2 \mid n_1, n_2$ note:

$$\begin{aligned} h(n_1) = h(n_2) &\implies g(\frac{n_1}{2}) = f(\frac{n_2}{2}) \\ &\implies (g^{-1} \circ g)(\frac{n_1}{2}) = (g^{-1} \circ g)(\frac{n_2}{2}) \\ &\implies \frac{n_1}{2} = \frac{n_2}{2} \\ &\implies n_1 = n_2 \end{aligned}$$

Without loss of generality $2 \mid n_1$ and $2 \nmid n_2$ is not possible as g and f map to disjoint sets. Thus we've shown h to be an injection. Thus h is a bijection from $\mathbb{N}$ to $A \cup B$ so $\mathbb{N} \approx A \cup B$. Thus we've proven $A \cup B$ to be denumerable. □

5.3.13 a. *Proof.* Let $A = \mathbb{R} - \mathbb{Q}$ be the set of all irrationals. Since $\mathbb{R} = \mathbb{Q} \cup A$ by definiton, then A must be uncountable as we know $\mathbb{R}$ is uncountable so it can't be the union of two countable sets and $\mathbb{Q}$ is countable. Thus A, the set of all irrationals, must be uncountable. □

5.4.1 a. $\mathbb{N} \subsetneq \mathbb{N} - \{0\}$
$\overline{\overline{\mathbb{N}}} = \overline{\overline{\mathbb{N} - \{0\}}} = \aleph_0$

b. $\mathbb{N} \subsetneq \mathbb{Z}$
$\overline{\overline{\mathbb{N}}} = \overline{\overline{\mathbb{Z}}} = \aleph_0$

c. $\mathbb{R} - \mathbb{Q} \subsetneq \mathbb{R}$
$\overline{\overline{\mathbb{R} - \mathbb{Q}}} = \overline{\overline{\mathbb{R}}} = \mathfrak{c}$

d. $\mathbb{N} - \{0\} \times \mathbb{N} \subsetneq \mathbb{N} \times \mathbb{N}$
$\overline{\overline{\mathbb{N} - \{0\} \times \mathbb{N}}} = \overline{\overline{\mathbb{N} \times \mathbb{N}}} = \aleph_0^2$

5.4.5 *Proof.* Let A be a set. We'll prove that there exists no largest cardinal number by showing that $\overline{\overline{A}} < \overline{\overline{\mathscr{P}(A)}}$. So for any set there is a set with a larger cardinal number. To prove this we will show that the function $f : A \to \mathscr{P}(A)$ given by $f(x) = \{x\}$ is an injection. If $x, y \in A$ and $f(x) = f(y)$ then $\{x\} = \{y\} \implies x = y$ which shows f's bijectivity. This shows that $\overline{\overline{A}} \leq \overline{\overline{\mathscr{P}(A)}}$. Now to show $\overline{\overline{A}} \neq \overline{\overline{\mathscr{P}(A)}}$. Seeking a contradiction suppose $A \approx \mathscr{P}(A)$ so that there exists a bijection $g : A \xrightarrow[onto]{1-1} \mathscr{P}(A)$. Now define set $B = \{y \in A : y \notin g(y)\} \in \mathscr{P}(A)$. Since g is a surjection there exists a $z \in A$ so that $g(z) = B$, now consider the following cases.
Case 1: If $z \in B$ then, by definition of B, $z \notin g(z)$ which is a contradiciton as $g(z) = B$.
Case 2: If $z \notin B$ then, by definition of B, $z \in g(z)$ which is a contradiciton as $g(z) = B$.
Thus we've shown that it is not possible for g to be a surjection so $\overline{\overline{A}} \neq \overline{\overline{\mathscr{P}(A)}}$. Thus we've shown that $\overline{\overline{A}} < \overline{\overline{\mathscr{P}(A)}}$ for any set A. We can use this fact to show that there exists no largest cardinal number as any cardinal number is attached to the size of a set but there is no largest size for a set as $\overline{\overline{A}} < \overline{\overline{\mathscr{P}(A)}}$. Thus we've proved there exists no largest cardinal number. □