

## Definition: Group

A **group** is an ordered pair  $(G, \star)$ , where  $G$  is a set and  $\star$  is a binary operation on  $G$  satisfying

1. *Associativity*:  $(a \star b) \star c = a \star (b \star c) \forall a, b, c \in G$
2. *Identity*:  $\exists e \in G$  such that  $e \star a = a \star e = a \forall a \in G$
3. *Invertibility*:  $\forall a \in G \exists a^{-1} \in G$  such that  $a \star a^{-1} = a^{-1} \star a = e$

## Theorem: Orbit-Stabilizer Theorem

Let  $G$  be a group acting on a set  $X$ , with  $x \in X$ . Then the map

$$\begin{aligned} G/G_x &\longrightarrow G \cdot x \\ aG_x &\longmapsto a \cdot x \end{aligned}$$

is well-defined and bijective, and therefore  $|G \cdot x| = [G : G_x]$ .

**Proof:** Let  $a, b \in G$ . Then

$$\begin{aligned} aG_x = bG_x &\iff b^{-1}a \in G_x \\ &\iff b^{-1}a \cdot x = x \\ &\iff a \cdot x = b \cdot x. \end{aligned}$$

Observe the map is well-defined by  $(\implies)$  and injective by  $(\impliedby)$ .

For surjectivity, note for any  $a \in G$ ,  $a \cdot x$  is the image of  $aG_x$ .  $\square$