Definition: Group

A **group** is an ordered pair $(G, \star)$, where G is a set and $\star$ is a binary operation on G satisfying

1. The operation is associative: $(a \star b) \star c = a \star (b \star c) \forall a, b, c \in G$
2. G has an identity: $\exists e \in G$ such that $e \star a = a \star e = a \forall a \in G$
3. Every element $a \in G$ is invertible: $\forall a \in G \exists a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$

Theorem: **Orbit-Stabilizer Theorem**

Let G be a group acting on a set X, with $x \in X$. Then the map

$$\begin{gathered} G/G_x \longrightarrow G \cdot x \\ aG_x \mapsto a \cdot x \end{gathered}$$

is well-defined and bijective, and therefore $|G \cdot x| = [G : G_x]$.

Proof: Let $a, b \in G$. Then

$$\begin{aligned} aG_x = bG_x &\iff b^{-1}a \in G_x \\ &\iff b^{-1}a \cdot x = x \\ &\iff a \cdot x = b \cdot x. \end{aligned}$$

Observe the map is well-defined by $(\Longrightarrow)$ and injective by $(\Longleftarrow)$.

For surjectivity, note for any $a \in G$, $a \cdot x$ is the image of aG_x.

$\square$