Lagrangian Mechanics

We want to now use the Euler-Lagrange equation to solve mechanics problems. To do this, we want to find some function $L(t, x, \dot{x})$ such that evaluating the Euler-Lagrange equation gives Newton's Second Law for a particle subject to conservative forces,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = -\frac{\mathrm{d}U}{\mathrm{d}x}.$$

Comparing with the form of Euler-Lagrange, we see we must have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}; \quad \frac{\partial L}{\partial x} = -\frac{\mathrm{d}U}{\mathrm{d}x}.$$

Solving the first equation by separation of variables gives

$$L = \frac{1}{2}m\dot{x}^2 + g(t, x).$$

Now since U is purely a function of x, our second PDE means we do not have to consider t dependence in our solution, meaning without loss of generality let g be a function of x alone. From here we deduce

$$\frac{\partial g}{\partial x} = -\frac{\mathrm{d}U}{\mathrm{d}x} \Rightarrow g(x) = -U(x),$$

implying that

$$L = T - U$$

works. This is called the **Lagrangian** of our system, and it gives us a powerful new formulation of mechanics. Importantly, because we did not consider our PDE solution in full generality, it is not unique in its implication of Newton's Second Law.

Definition: Least Action Principle

Given a mechanical system described by N dynamical generalized coordinates $q_k(t)$, with $k = 1, 2, \ldots, N$, define its **action** by

$$\boxed{S[q_k(t)] = \int_{t_a}^{t_b} \mathrm{d}t\, L(t, q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots).}$$

(We assume the particle begins at some position $(q_1, q_2, \ldots)_a$ at time t_a and ends at position $(q_1, q_2, \ldots)_b$ at time t_b.) Also note that this is a generalized version of the function $J[f]$ shown previously.

Now the **least action principle** states that, for trajectories $q_k(t)$ where S is stationary, i.e.,

$$\delta S = \delta \int_{t_a}^{t_b} L(t, q_k \dot{q}_k)\, \mathrm{d}t = 0,$$

then the $q_k(t)$'s satisfy the equations of motions for the system with the given boundary conditions. Note this is just the proof we did earlier a—if to first order $\delta S = 0$, then S is an extrema, and thus the Euler-Lagrange equations apply.