***Theorem:* Galois Subextensions**

Let K/F be a finite Galois extension, then:
1. if $H \leq G$, K^H/F is Galois if and only if $H \trianglelefteq G$. In this case $\mathrm{Gal}(K^H/F) \cong \frac{G}{H}$;
2. if $F \subseteq E \subseteq K$, E/F is Galois if and only if $\mathrm{Gal}(K/E) \trianglelefteq G$. In this case $\mathrm{Gal}(E/F) \cong \frac{\mathrm{Gal}(K/F)}{\mathrm{Gal}(K/E)}$.

Example: Galois Correspondence

Let $\zeta_3 = e^{\frac{2\pi i}{3}}$ then the splitting field of $f(X) = X^3 - 2 \in \mathbb{Q}[X]$ is $K = \mathbb{Q}\left(\sqrt[3]{2}, \zeta_3\right)$ where $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. Then we can observe the following generators of $\mathrm{Gal}(K/\mathbb{Q})$, $\sigma_1 : K \to K$ and $\sigma_2 : K \to K$. They are defined by

$$\sigma_1(\zeta_3) = \zeta_3, \quad \sigma_1\left(\sqrt[3]{2}\right) = \sqrt[3]{2}\zeta_3, \quad \sigma_2(\zeta_3) = \zeta_3^2, \quad \sigma_2\left(\sqrt[3]{2}\right) = \sqrt[3]{2}.$$

Then the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ are given by $\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_1\sigma_2\rangle, \langle\sigma_1^2\sigma_2\rangle$ such that we have the following correspondence by checking which generators of $\mathrm{Gal}(K/\mathbb{Q})$ fixes which basis elements:

$$\begin{tikzcd}
 & K & & & \\
\mathbb{Q}(\zeta_3) \arrow[ur, "\langle\sigma_1\rangle"] & & \mathbb{Q}\left(\sqrt[3]{2}\right) \arrow[ul, "\langle\sigma_2\rangle"'] & \mathbb{Q}\left(\sqrt[3]{2}\zeta_3\right) \arrow[ull, "\langle\sigma_1^2\sigma_2\rangle"'] & \mathbb{Q}\left(\sqrt[3]{2}\zeta_3^2\right) \arrow[ulll, "\langle\sigma_1\sigma_2\rangle"'] \\
 & \mathbb{Q} \arrow[ul, "\frac{S_3}{\langle\sigma_1\rangle}"] \arrow[ur] \arrow[urr] \arrow[urrr] & & &
\end{tikzcd}$$