

# Lagrangian Mechanics

We want to now use the Euler-Lagrange equation to solve mechanics problems. To do this, we want to find some function  $L(t, x, \dot{x})$  such that evaluating the Euler-Lagrange equation gives Newton's Second Law for a particle subject to conservative forces,

$$\frac{d}{dt}(m\dot{x}) = -\frac{dU}{dx}.$$

Comparing with the form of Euler-Lagrange, we see we must have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}; \quad \frac{\partial L}{\partial x} = -\frac{dU}{dx}.$$

Solving the first equation by separation of variables gives

$$L = \frac{1}{2}m\dot{x}^2 + g(t, x).$$

Now since  $U$  is purely a function of  $x$ , our second PDE means we do not have to consider  $t$  dependence in our solution, meaning without loss of generality let  $g$  be a function of  $x$  alone. From here we deduce

$$\frac{\partial g}{\partial x} = -\frac{dU}{dx} \Rightarrow g(x) = -U(x),$$

implying that

$$L = T - U$$

works. This is called the **Lagrangian** of our system, and it gives us a powerful new formulation of mechanics. Importantly, because we did not consider our PDE solution in full generality, it is not unique in its implication of Newton's Second Law.

## Definition: Least Action Principle

Given a mechanical system described by  $N$  dynamical generalized coordinates  $q_k(t)$ , with  $k = 1, 2, \dots, N$ , define its **action** by

$$S[q_k(t)] = \int_{t_a}^{t_b} dt L(t, q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots).$$

(We assume the particle begins at some position  $(q_1, q_2, \dots)_a$  at time  $t_a$  and ends at position  $(q_1, q_2, \dots)_b$  at time  $t_b$ .) Also note that this is a generalized version of the function  $J[f]$  shown previously.

Now the **least action principle** states that, for trajectories  $q_k(t)$  where  $S$  is stationary, i.e.,

$$\delta S = \delta \int_{t_a}^{t_b} L(t, q_k, \dot{q}_k) dt = 0,$$

then the  $q_k(t)$ 's satisfy the equations of motions for the system with the given boundary conditions. Note this is just the proof we did earlier a—if to first order  $\delta S = 0$ , then  $S$  is an extrema, and thus the Euler-Lagrange equations apply.