Definition: Multi-index

For $\alpha_1, ..., \alpha_n \geq 0$, we call $\alpha = (\alpha_1, ..., \alpha_n)$ a **multi-index**. Then define the following:

1. We let $\alpha! = \alpha_1! \cdots \alpha_n!$
2. For $x \in \mathbb{R}^n$, define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if $x_i \neq 0 \forall i$; otherwise $x^\alpha = 0$ if there exists j such that $x_j = 0$
3. Let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ be the *order* of α
4. Let $D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$

Theorem: Taylor's Formula over $\mathbb{R}^n$

Let $E \subseteq \mathbb{R}^n$ be open and let $\overline{B_r(a)} \subseteq E$. Suppose $f \in \mathcal{C}^k(E)$. Then

$$f(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=k} h_\alpha(x)(x-a)^\alpha$$

for any $x \in B_r(a)$, where α is a multi-index, and $h_\alpha : B_r(a) \to \mathbb{R}$ such that $h_\alpha(x) \to 0$ when $x \to a$ for all $|\alpha| = k$.

Proof:

Remark

Observe

$$|(x-a)^\alpha| = \left| \prod_{i=1}^n (x_i - a_i)_i^\alpha \right| = \prod_{i=1}^n |x_i - a_i|_i^\alpha \leq |x-a|^{|\alpha|}.$$

Therefore we have that

$$\left| \sum_{|\alpha|=k} h_\alpha(x)(x-a)^\alpha \right| \leq |x-a|^k \sum_{|\alpha|=k} |h_\alpha(x)|.$$

Let $x \in B_r(a)$ and define $[a, x] = \{\varphi(t) \mid t \in [0,1]\}$, where $\varphi : [-1,1] \to \mathbb{R}^n$ is defined by $\varphi(t) = a + t(x-a)$. Now let $g(t) = f(\varphi(t))$ so that $g(0) = f(a)$ and $g(1) = f(x)$. Thus $g \in \mathcal{C}^k([-1,1])$, meaning

$$g(1) = g(0) + \frac{1}{1!}g'(0) + \cdots + \frac{1}{(k-1)!}g^{(k-1)}(0) + \frac{1}{k!}g^{(k)}(c)$$

for some $c \in [0,1]$. As shown previously, we have

$$g^{(m)}(t) = \sum_{|\alpha|=m} D^\alpha f(\varphi(t)) \cdot (x-a)^\alpha$$

for $m = 1, ..., k$ by the Mixed Derivative Theorem. This implies

$$f(x) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=k} \frac{D^\alpha f(y)}{\alpha!}(x-a)^\alpha$$

for some $y \in [a, x]$. Now observe $D^\alpha f(y) = D^\alpha f(a) + (D^\alpha(y) - D^\alpha f(a))$, where $D^\alpha f(y) - D^\alpha f(a) \to 0$ as $y \to a$. Lastly define $h_\alpha(x) = \frac{D^\alpha f(y) - D^\alpha f(a)}{\alpha!} \to 0$ as $x \to a$, where y is a function of x.

□