

## Theorem: Galois Subextensions

Let  $K/F$  be a finite Galois extension, then:

1. if  $H \leq G$ ,  $K^H/F$  is Galois if and only if  $H \trianglelefteq G$ . In this case  $\text{Gal}(K^H/F) \cong \frac{G}{H}$ ;
2. if  $F \subseteq E \subseteq K$ ,  $E/F$  is Galois if and only if  $\text{Gal}(K/E) \trianglelefteq G$ . In this case  $\text{Gal}(E/F) \cong \frac{\text{Gal}(K/F)}{\text{Gal}(K/E)}$ .

## Example: Galois Correspondence

Let  $\zeta_3 = e^{\frac{2\pi i}{3}}$  then the splitting field of  $f(X) = X^3 - 2 \in \mathbb{Q}[X]$  is  $K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$  where  $\text{Gal}(K/\mathbb{Q}) \cong S_3$ . Then we can observe the following generators of  $\text{Gal}(K/\mathbb{Q})$ ,  $\sigma_1 : K \rightarrow K$  and  $\sigma_2 : K \rightarrow K$ . They are defined by

$$\sigma_1(\zeta_3) = \zeta_3, \quad \sigma_1(\sqrt[3]{2}) = \sqrt[3]{2}\zeta_3, \quad \sigma_2(\zeta_3) = \zeta_3^2, \quad \sigma_2(\sqrt[3]{2}) = \sqrt[3]{2}.$$

Then the subgroups of  $\text{Gal}(K/\mathbb{Q})$  are given by  $\langle \sigma_1 \rangle$ ,  $\langle \sigma_2 \rangle$ ,  $\langle \sigma_1\sigma_2 \rangle$ ,  $\langle \sigma_1^2\sigma_2 \rangle$  such that we have the following correspondence by checking which generators of  $\text{Gal}(K/\mathbb{Q})$  fixes which basis elements:

