Theorem: Galois Subextensions

Let K/F be a finite Galois extension. Then:

1. If $H \leq G$, K^H/F is Galois if and only if $H \trianglelefteq G$. In this case, $\mathrm{Gal}(K^H/F) \cong \frac{G}{H}$;
2. If $F \subseteq E \subseteq K$, E/F is Galois if and only if $\mathrm{Gal}(K/E) \trianglelefteq G$. In this case $\mathrm{Gal}(E/F) \cong \frac{\mathrm{Gal}(K/F)}{\mathrm{Gal}(K/E)}$.

Example: Galois Correspondence

Let $\zeta_3 = e^{\frac{2\pi i}{3}}$. Then the splitting field of $f(X) = X^3 - 2 \in \mathbb{Q}[X]$ is $K = \mathbb{Q}\left(\sqrt[3]{2}, \zeta_3\right)$, where $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. Now consider generators of $\mathrm{Gal}(K/\mathbb{Q})$, $\sigma_1 : K \to K$ and $\sigma_2 : K \to K$, defined by

$$\sigma_1(\zeta_3) = \zeta_3, \quad \sigma_1\left(\sqrt[3]{2}\right) = \sqrt[3]{2}\zeta_3, \quad \sigma_2(\zeta_3) = \zeta_3^2, \quad \sigma_2\left(\sqrt[3]{2}\right) = \sqrt[3]{2}.$$

Thus, the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ are given by $\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_1\sigma_2\rangle, \langle\sigma_1^2\sigma_2\rangle$, implying the following correspondence (which comes from comparing the generators of $\mathrm{Gal}(K/\mathbb{Q})$ with basis elements):

$$\begin{tikzcd}
 & K & & \\
\mathbb{Q}(\zeta_3) \arrow[ur, "\langle\sigma_1\rangle"] & \mathbb{Q}\left(\sqrt[3]{2}\right) \arrow[u, "\langle\sigma_2\rangle"] & \mathbb{Q}\left(\sqrt[3]{2}\zeta_3\right) \arrow[ul, "\langle\sigma_1^2\sigma_2\rangle"'] & \mathbb{Q}\left(\sqrt[3]{2}\zeta_3^2\right) \arrow[ull, "\langle\sigma_1\sigma_2\rangle"'] \\
 & \mathbb{Q} \arrow[ul, "\frac{S_3}{\langle\sigma_1\rangle}"] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$