Analysis Lecture Notes

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1. The Real and Complex Number Systems

1.1. Introduction

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We must predicate the main concepts of analysis on a well-defined concept of numbers. What do we mean when we say the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \dots$$

tends to $\sqrt{2}$?

1.1.1. Example

We now show that the equation

$$p^2 = 2 \tag{1}$$

is not satisfied by any rational p. By contradiction, suppose there were such a p. Then we could write $p = \frac{m}{n}$ where $n, m \in \mathbb{Z}$ and n and m are coprime. Then (1) implies

$$m^2 = 2n^2, (2)$$

which shows that m^2 is even. Note that if m were odd, m^2 would be odd, so m^2 being even implies m is even. But then we can write m=2k for some $k\in\mathbb{Z}$, giving us $2k^2=n^2$, which by the same argument shows that n is even. But n and m were supposed to be coprime, a contradiction.

We now consider this more closely.

1.1.2. Proposition

Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. Then for every $p \in A \exists q \in \mathbb{Q} \cap A$ such that p < q, and $\forall p \in B \exists q \in \mathbb{Q} \cap B$ such that q < p.

Proof: Associate with each rational p > 0 the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. (3)$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2}. (4)$$

If $p \in A$, then $p^2 - 2 < 0$, so (3) shows that q > p, and (4) shows that $q^2 < 2$. Thus $q \in A$. If $p \in B$, then $p^2 - 2 > 0$ so (3) shows that 0 < q < p and (4) shows that $q^2 > 2$. Thus $q \in B$.

1.1.3. Remark

This shows that the rational number system has gaps, despite the density of \mathbb{Q} in \mathbb{Q} . The real number system fills these gaps.

1.1.4. Definition: Order

Let S be a set. An **order** on S is a relation, denoted by <, with the following two properties:

i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y$$
 $x = y$ $y < x$

is true.

ii) If $x, y, z \in S$, then if x < y and y < z, then x < z.

The statement "x < y" may be read as "x is less than y" of "x is smaller than y".

1.1.5. Definition: Ordered Set

An **ordered set** is a set S in which an order is defined.

For example, $\mathbb Q$ is an ordered set if r < s is defined to mean that s - r is a positive rational number.

1.1.6. Theorem

Suppose S is an ordered set with the least upper bound property, $B \subset S$, B is nonempty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

In particular, $\inf B$ exists in S.

Proof: Since B is bounded below, L is nonempty. Since L consists of the $y \in S$ which satisfy the inequality $y \le x$ for every $x \in B$, we see that every $x \in B$ is an upper bound of L. Thus, L is bounded above. Our hypothesis then implies that L has a supremum in S called α .

If $\gamma < \alpha$, then (see <u>Definition 1.1.4</u>) γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x \forall x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is a lower bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B, but β is not if $\beta > \alpha$.