Analysis Lecture Notes

John Doe

Table of Contents

1. The Real and Complex Number Systems

1.1. Introduction

Lecture 1	Jan 1, 2025

We must predicate the main concepts of analysis on a well-defined concept of numbers. What do we mean when we say the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

tends to $\sqrt{2}$?

1.1.1. Example

We now show that the equation

$$p^2 = 2 \tag{1}$$

is not satisfied by any rational p. By contradiction, suppose there were such a p. Then we could write $p = \frac{m}{n}$ where $n, m \in \mathbb{Z}$ and n and m are coprime. Then (1) implies

$$m^2 = 2n^2, \tag{2}$$

which shows that m^2 is even. Note that if m were odd, m^2 would be odd, so m^2 being even implies m is even. But then we can write $m = 2k$ for some $k \in \mathbb{Z}$, giving us $2k^2 = n^2$, which by the same argument shows that n is even. But n and m were supposed to be coprime, a contradiction.

We now consider this more closely.

1.1.2. *Proposition*

Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. Then for every $p \in A \exists q \in \mathbb{Q} \cap A$ such that $p < q$, and $\forall p \in B \exists q \in \mathbb{Q} \cap B$ such that $q < p$.

Proof: Associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{3}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{4}$$

If $p \in A$, then $p^2 - 2 < 0$, so (3) shows that $q > p$, and (4) shows that $q^2 < 2$. Thus $q \in A$. If $p \in B$, then $p^2 - 2 > 0$ so (3) shows that $0 < q < p$ and (4) shows that $q^2 > 2$. Thus $q \in B$.

□

1.1.3. Remark

This shows that the rational number system has gaps, despite the density of $\mathbb{Q}$ in $\mathbb{Q}$. The real number system fills these gaps.

1.1.4. Definition: Order

Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y \qquad x = y \qquad y < x$$

is true.

ii) If $x, y, z \in S$, then if $x < y$ and $y < z$, then $x < z$.

The statement "$x < y$" may be read as "x is less than y" of "x is smaller than y".

1.1.5. Definition: Ordered Set

An **ordered set** is a set S in which an order is defined.

For example, $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

1.1.6. ***Theorem***

Suppose S is an ordered set with the least upper bound property, $B \subset S$, B is nonempty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

In particular, $\inf B$ exists in S.

Proof: Since B is bounded below, L is nonempty. SInce L consists of the $y \in S$ which satisfy the inequality $y \leq x$ for every $x \in B$, we see that *every* $x \in B$ is an *upper bound* of L. Thus, L is bounded above. Our hypothesis then implies that L has a supremum in S called α.

If $\gamma < \alpha$, then (see Definition 1.1.4) γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x \forall x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is a lower bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B, but β is not if $\beta > \alpha$.

$\square$