# Group Theory Homework 1

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**Exercise: Dummit and Foote 1.1.9**

Let $G = \left\{a + b\sqrt{2} \in \mathbb{R} : a, b \in \mathbb{Q}\right\}$.

a) Prove that $G$ is a group under addition.
b) Prove that the nonzero elements of $G$ are a group under multiplication. [Rationalize the denominators to find multiplicative inverses.]

**Solution**

a) We begin by showing closure under the operation. Take $g_1, g_2 \in G$ so that $\exists a, b, c, d \in \mathbb{Q}$ such that $g_1 = a + b\sqrt{2}, g_2 = c + d\sqrt{2}$. Then $g_1 + g_2 = a + b\sqrt{2} + c + d\sqrt{2} = (a + c) + (b + d)\sqrt{2} \in G$, which follows from associativity, commutativity, and and the distribution law on the field $\mathbb{R}$, and the fact that $\mathbb{Q}$ is closed under addition.. In fact, the associativity of addition on $G$ trivially follows from the fact that $G \subset \mathbb{R}$ and addition is associative in $\mathbb{R}$. Further, $0 = 0 + 0\sqrt{2} \in G$ is an additive identity, which follows from the fact that 0 is the additive identity in $\mathbb{R}$ already.

Now to show every element has an inverse, notice given $g \in G$ with $g = a + b\sqrt{2}$ and $a, b \in \mathbb{Q}$, let $g' = (-a) + (-b)\sqrt{2} \in G$, so that

$$\begin{aligned} g + g' &= a + b\sqrt{2} + (-a) + (-b)\sqrt{2} \\ &= a + (-a) + b\sqrt{2} + (-b)\sqrt{2} \\ &= (a + (-a)) + (b + (-b))\sqrt{2} \\ &= 0 \\ &= (-a) + \left(-b\sqrt{2}\right) + a + b\sqrt{2} \\ &= g' + g \end{aligned}$$

which follows from addition being commutative and associative in $\mathbb{R}$.

Thus $G$ is a group under addition.

b) Again we show closure under the operation. Take $g_1, g_2 \in G$ so that $\exists a, b, c, d \in \mathbb{Q}$ such that $g_1 = a + b\sqrt{2}, g_2 = c + d\sqrt{2}$. Then $g_1 g_2 = \left(a + b\sqrt{2}\right)\left(c + d\sqrt{2}\right) = ac + ad\sqrt{2} + bc\sqrt{2} + 2bd = (ac + 2bd) + (ad + bc)\sqrt{2} \in G$ where our manipulations are valid since $\mathbb{R}$ is a field, and $ac + 2bd, ad + bc \in \mathbb{Q}$ follows from the closure of addition and multiplication on $\mathbb{Q}$. The associativity of multiplication follows from $G \subset \mathbb{R}$, and since $1 = 1 + 0\sqrt{2} \in G$ is the multiplicative identity in $\mathbb{R}$, it is the multiplicative identity here too.

Now let $g \in G$ with $g = a + b\sqrt{2}$ and $a, b \in \mathbb{Q}$, and let $g' = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$. Notice since $\mathbb{Q}$ is closed under multiplication and division by nonzero elements, we need only check that $a^2 - 2b^2 \neq 0$. But this is equivalent to $a = \pm b\sqrt{2}$. We need only check nonzero elements of $G$, so we can throw out the case that $a = b = 0$. But then our condition is equivalent to $\frac{a}{b} = \pm\sqrt{2}$, an impossibility since $\frac{a}{b} \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$. Thus we indeed have $g' \in G$, and

$$\begin{aligned}
gg' &= \left(a + b\sqrt{2}\right)\left(\frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}\right) \\
&= \left(\frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}\right)\left(a + b\sqrt{2}\right) \\
&= g'g \\
&= \frac{a^2}{a^2 - 2b^2} + \frac{ab\sqrt{2}}{a^2 - 2b^2} - \frac{ba\sqrt{2}}{a^2 - 2b^2} - \frac{2b^2}{a^2 - 2b^2} \\
&= \frac{a^2 - 2b^2}{a^2 - 2b^2} \\
&= 1.
\end{aligned}$$

Thus $G \setminus \{0\}$ is a group under multiplication.

**Exercise: Dummit and Foote 1.1.25**

Let $G$ be a group. Prove that if $x^2 = 1 \forall x \in G$, then $G$ is abelian.

**Solution**

Let $a, b \in G$. Then observe

$$\begin{aligned}
ab &= 1 \cdot ab \cdot 1 \\
&= b^2(ab)a^2 \\
&= b(ba)(ba)a \\
&= b(ba)^2 a \\
&= b \cdot 1 \cdot a \\
&= ba
\end{aligned}$$

so $G$ is abelian.

**Exercise: Dummit and Foote 1.1.32)**

If $x$ is an element of finite order $n$ in a group $G$, prove that the elements $1, x, x^2, ..., x^{n-1}$ are all distinct. Deduce that $|x| \leq |G|$.

**Solution**

Suppose by contradiction that $x^a = x^b$ for some $a, b \in \{0, 1, ..., n-1\}$ with $a \neq b$. Without loss of generality, suppose $a < b$. Then $x^{-a}x^a = 1 = x^{-a}x^b = x^{b-a}$. But since $b - a \leq (n-1) - 0 < n$, we must have that $x$ cannot be order $n$, a contradiction. Therefore $1, x, ..., x^{n-1}$ are all distinct. Further, each belongs to $G$, so $G$ includes at least these elements, and thus $n = |x| \leq |G|$.

**Exercise: Dummit and Foote 1.2.5)**

If $n$ is odd and $n \geq 3$, show that the identity is the only element of $D_{2n}$ which commutes with all the elements of $D_{2n}$.

**Solution**

Recall that $D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, sr = r^{-1}s \rangle$. Suppose $x \in D_{2n}$ commutes with all elements of $D_{2n}$.

Recall that $r$ and $s$ are generators of $D_{2n}$, and in particular $D_{2n} = \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\}$

First, consider the case that $x = r^k s$ for some $k \in \{0, ..., n-1\}$. Then since $x$ commutes universally,

$$\begin{aligned}
rx &= xr \\
\Longrightarrow r(r^k s) &= (r^k s)r \\
\Longrightarrow r^{k+1}s &= r^k(r^{-1}s) \\
&= r^{k-1}s \\
\Longrightarrow (r^{-k})r^{k+1} &= (r^{-k})r^{k-1} \\
\Longrightarrow r &= r^{-1} \\
\Longrightarrow r^2 &= 1
\end{aligned}$$

but by definition, $r$ has order $n \geq 3$, so this is a contradiction. Therefore, we must have $x = r^k$ for $k \in \{0, ..., n-1\}$:

$$\begin{aligned}
xs &= sx \\
\Longrightarrow r^k s &= sr^k \\
&= (sr)r^{k-1} \\
&= r^{-1}(sr)r^{k-2} \\
&\vdots \\
&= r^{-k}s \\
\Longrightarrow r^k &= r^{-k} \\
\Longrightarrow r^{2k} &= 1
\end{aligned}$$

Now since $r$ has order $n$, we must have that $n \mid 2k$. But since $k \leq n-1$, we must have either $n = 2k$ or $k = 0$. But the former situation is impossible since $n$ is odd, so it must be true that $k = 0$ and $x = r^0 = 1$. Thus any element in $D_{2n}$ that commutes with every other element must be the identity.