HPS/Pl 125: Problem 7

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HPS/Pl 125, WI '25

February, 2025

Complete exercise 2.71 and 2.75 from Nielsen and Chuang.

1 2.71

Show that $\mathrm{Tr}(\rho^2) \leq 1$ *with equality iff* ρ *is a pure state.*

Begin with the definition of ρ: $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. Then:

$$\rho^2 = (\sum_i p_i |\psi_i\rangle \langle\psi_i|)(\sum_j p_j |\psi_j\rangle \langle\psi_j|) = \sum_{i,j} p_i p_j |\psi_i\rangle \langle\psi_i|\psi_j\rangle \langle\psi_j|$$

Note from equation 2.157 of the text that in the spectral decomposition of the density matrix, the state-space vectors form an orthogonal basis. Therefore, unless $i = j$, $\langle\psi_i|\psi_j\rangle = 0$. Thus:

$$\rho^2 = \sum_i p_i^2 |\psi_i\rangle \langle\psi_i|$$

Applying the trace and using equation 2.153:

$$\mathrm{Tr}(\rho^2) = \sum_i p_i^2$$

In a pure state, the only non-zero p_i is 1. Therefore, $\mathrm{Tr}(\rho^2) = 1$ in a pure state. In a mixed state, we have multiple non-zero p_i which are subject to the condition $\sum_i p_i = 1$. Thus each $p_i < 1$. If $p_i < 1$, then $p_i^2 < p_i$. Therefore, $\sum_i p_i^2 < \sum_i p_i = 1$. Thus, in the case of a mixed state, $\mathrm{Tr}(\rho^2) < 1$. Therefore $\mathrm{Tr}(\rho^2) = 1$ iff ρ is a pure state.

2 2.75

Find the reduced density operators for each particle in each of the four Bell states:

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

First, find the density operator for each Bell state:

$$\rho_{\beta_{00}} = |\beta_{00}\rangle\langle\beta_{00}| = \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

$$\rho_{\beta_{01}} = |\beta_{01}\rangle\langle\beta_{01}| = \frac{|01\rangle\langle 01| + |10\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 10|}{2}$$

$$\rho_{\beta_{10}} = |\beta_{10}\rangle\langle\beta_{10}| = \frac{|00\rangle\langle 00| - |11\rangle\langle 00| - |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

$$\rho_{\beta_{11}} = |\beta_{11}\rangle\langle\beta_{11}| = \frac{|01\rangle\langle 01| - |10\rangle\langle 01| - |01\rangle\langle 10| + |10\rangle\langle 10|}{2}$$

Apply the reduced density operator (partial trace) to each Bell state for each particle.

$$\rho^1 = \mathrm{Tr}_2(\rho), \quad \rho^2 = \mathrm{Tr}_1(\rho)$$

Note that the following general pattern will emerge:

$$\mathrm{Tr}_2(|AB\rangle\langle CD|) = \mathrm{Tr}_2(|A\rangle\langle C| \otimes |B\rangle\langle D|) = |A\rangle\langle C| \,\mathrm{Tr}(|B\rangle\langle D|) = |A\rangle\langle C| \langle D|B\rangle$$

Since the qubit basis is orthogonal, $\langle D|B\rangle = 0$ unless $D = B$. If $D = B$, then $\langle D|B\rangle = 1$. Therefore:

$$\rho^1_{B_{00}} = \mathrm{Tr}_2\left(\frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2}\right) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \boxed{\frac{I}{2}}$$

Examine the symmetry here. We can see that if we had traced out particle 1 instead of 2, the positions of the outer products in the numerator would've been swapped, i.e $\frac{|1\rangle\langle 1| + |0\rangle\langle 0|}{2}$. Therefore the reduced density operator for particle 2 is the same as for particle 1, $\boxed{\frac{I}{2}}$

$$\rho^1_{B_{01}} = \mathrm{Tr}_2\left(\frac{|01\rangle\langle 01| + |10\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 10|}{2}\right) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \boxed{\frac{I}{2}}$$

An equal symmetry argument applies here as well, so the reduced density operator for particle 2 is the same as for particle 1, $\boxed{\frac{I}{2}}$

$$\rho^1_{B_{10}} = \mathrm{Tr}_2\left(\frac{|00\rangle\langle 00| - |11\rangle\langle 00| - |00\rangle\langle 11| + |11\rangle\langle 11|}{2}\right) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \boxed{\frac{I}{2}}$$

Symmetry applies here as well, so the reduced density operator for particle 2 is the same as for particle 1, $\boxed{\frac{I}{2}}$

$$\rho^1_{B_{11}} = \mathrm{Tr}_2\left(\frac{|01\rangle\langle 01| - |10\rangle\langle 01| - |01\rangle\langle 10| + |10\rangle\langle 10|}{2}\right) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \boxed{\frac{I}{2}}$$

Symmetry applies here as well, so the reduced density operator for particle 2 is the same as for particle 1, $\boxed{\frac{I}{2}}$