HPS/Pl 125: Problem 3

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Consider the Gaussian wave packet given by

$$\psi(x) = (\frac{1}{\pi\omega^2})^{\frac{1}{4}} e^{\frac{-x^2}{2\omega^2}+ikx}$$

* *Note that all odd integrals dissapear and all integrals shown are taken over* $-\infty$ *to* ∞.

1 Part a

Compute the expectation values of $\hat{p}$ *and* $\hat{H}$ *on this state*

The expectation value of an operator $\hat{A}$ on a state ψ is given by

$$\langle \hat{A} \rangle = \int \psi^*(x)\hat{A}\psi(x)dx$$

over all space. Thus:

$$\langle \hat{p} \rangle = \int \psi^*(x)\hat{p}\psi(x)dx$$

$$= -i\hbar(\frac{1}{\pi\omega^2})^{\frac{1}{2}} \int e^{\frac{-x^2}{2\omega^2}-ikx} \frac{\partial}{\partial x}[e^{\frac{-x^2}{2\omega^2}+ikx}]dx$$

$$= -i\hbar(\frac{1}{\pi\omega^2})^{\frac{1}{2}} \int e^{\frac{-x^2}{2\omega^2}} (\frac{-x}{\omega^2} + ik)dx$$

$$= -i^2\hbar k(\frac{1}{\pi\omega^2})^{\frac{1}{2}} (\pi\omega^2)^{\frac{1}{2}}$$

$$\boxed{= \hbar k}$$

$$\langle \hat{H} \rangle = \int \psi^*(x)\hat{H}\psi(x)dx$$

$$= \frac{-\hbar^2}{2m}(\frac{1}{\pi\omega^2})^{\frac{1}{2}} \int e^{\frac{-x^2}{2\omega^2}-ikx} \frac{\partial^2}{\partial x^2}[e^{\frac{-x^2}{2\omega^2}+ikx}]dx$$

$$= \frac{-\hbar^2}{2m}(\frac{1}{\pi\omega^2})^{\frac{1}{2}}\int e^{\frac{-x^2}{\omega^2}}(\frac{x^2}{\omega^4} - \frac{1}{\omega^2} - \frac{2ikx}{\omega^2} - k^2)dx$$

$$= \frac{-\hbar}{2m}(\frac{1}{2\omega^2} - \frac{1}{\omega^2} - k^2)$$

$$\boxed{= \frac{\hbar^2}{2m}(\frac{1}{2\omega^2} + k^2)}$$

2 Part b

Collapse the wave function by multiplying it by a gaussian given as

$$(\frac{\omega^2}{\sigma^2} + 1)^{\frac{1}{4}} e^{\frac{-x^2}{2\sigma^2}}$$

Compute the expectation values of $\hat{p}$ and $\hat{H}$ on this state

$$\langle\hat{p}\rangle = \int (\frac{\omega^2}{\sigma^2} + 1)^{\frac{1}{4}} e^{\frac{-x^2}{2\sigma^2}} \psi^*(x)\hat{p}(\frac{\omega^2}{\sigma^2} + 1)^{\frac{1}{4}} e^{\frac{-x^2}{2\sigma^2}} \psi(x)dx$$

$$= -i\hbar(\frac{1}{\pi\sigma^2} + \frac{1}{\pi\omega^2})^{\frac{1}{2}}\int e^{-(\frac{1}{2\sigma^2}+\frac{1}{2\omega^2})x^2 - ikx}\frac{\partial}{\partial x}[e^{-(\frac{1}{2\sigma^2}+\frac{1}{2\omega^2})x^2 + ikx}]dx$$

$$= -i\hbar(\frac{1}{\pi\sigma^2} + \frac{1}{\pi\omega^2})^{\frac{1}{2}}\int e^{-(\frac{1}{\sigma^2}+\frac{1}{\omega^2})x^2}(\frac{x}{\sigma^2} - \frac{x}{\omega^2} + ik)dx$$

$$= -i\hbar(\frac{1}{\pi\sigma^2} + \frac{1}{\pi\omega^2})^{\frac{1}{2}}(ik)(\frac{\pi}{\frac{1}{\sigma^2} + \frac{1}{\omega^2}})^{\frac{1}{2}}$$

$$= \boxed{\hbar k}$$

$$\langle\hat{H}\rangle = \int (\frac{\omega^2}{\sigma^2} + 1)^{\frac{1}{4}} e^{\frac{-x^2}{2\sigma^2}} \psi^*(x)\hat{H}(\frac{\omega^2}{\sigma^2} + 1)^{\frac{1}{4}} e^{\frac{-x^2}{2\sigma^2}} \psi(x)dx$$

$$= \frac{-\hbar^2}{2m}(\frac{1}{\pi\sigma^2} + \frac{1}{\pi\omega^2})^{\frac{1}{2}}\int e^{-(\frac{1}{2\sigma^2}+\frac{1}{2\omega^2})x^2 - ikx}\frac{\partial^2}{\partial x^2}[e^{-(\frac{1}{2\sigma^2}+\frac{1}{2\omega^2})x^2 + ikx}]dx$$

$$= \frac{-\hbar^2}{2m}(\frac{1}{\pi\sigma^2} + \frac{1}{\pi\omega^2})^{\frac{1}{2}}\int e^{-(\frac{1}{\sigma^2}+\frac{1}{\omega^2})x^2}(k^2 + (\frac{1}{2\sigma^2} + \frac{1}{2\omega^2}) - 2ik(\frac{x}{2\sigma^2} + \frac{x}{2\omega^2}) + (\frac{x}{2\sigma^2} + \frac{x}{2\omega^2})^2)dx$$

$$= \frac{-\hbar^2}{2m}(\frac{1}{\pi\sigma^2} + \frac{1}{\pi\omega^2})^{\frac{1}{2}}(\frac{\pi}{\frac{1}{\sigma^2} + \frac{1}{\omega^2}})^{\frac{1}{2}}(-k^2 - \frac{1}{2\sigma^2} - \frac{1}{2\omega^2})$$

$$\boxed{= \frac{\hbar}{2m}(\frac{1}{2\sigma^2} + \frac{1}{2\omega^2} + k^2)}$$

3 Part c

Explain why the collapse law multiplies the ψ by a Gaussian instead of just replacing it with a gaussian.

We see by comparing the expectation of the momentum operator from parts a and b that multiplying by a gaussian preserves the expectation value of the momentum operator. If the expectation value of momentum were to change, the collapse law would be unphysical and create a discontinuity in both momentum and energy space. Simply replacing the wavefunction with a gaussian would not preseve the expectation value of momentum and would create a discontinuity in momentum space.

4 Part d

Explain why σ should not be made arbitrarily small.

Compare the expectation of the energy operator from parts a and b. We see that the system has increased in the expectation of energy by a gain which is given by $\frac{\hbar}{2m}(\frac{1}{2\sigma^2})$. If σ were to be made arbitrarily small, this gain would tend towards infinity, meaning that the system would have a sudden infinite spike in energy. This is unphysical and would violate observation. In other words, there is a bound in the precision of localization of a particle for this type of collapse law in order to avoid a system of infinite energy.