HPS/Pl 125: Problem 2

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In Maudlin's figure 14, the wave function shown is of the form

$$\psi(x) = Ae^{i\alpha x}$$

where A and α are real constants, and the wave function is normalized.

1 Part a

Show that $\psi(x)$ is an eigenstate of the momentum operator

$\psi(x)$ is an eigenstate of the momentum operator iff $\hat{p}\psi(x) = k\psi(x)$ for some constant k, where k is the eigenvalue of the momentum operator, $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}[Ae^{i\alpha x}] = -i\hbar(i\alpha Ae^{i\alpha x}) = \hbar\alpha Ae^{i\alpha x} = \hbar\alpha\psi(x)$$

Therefore, ψx is an eigenstate of the momentum operator with eigenvalue $k = \hbar\alpha$.

2 Part b

Use the time-dependent schrödinger equation to determine the way this state evolves over time, $\psi(x, t)$

The time-dependent Schrödinger equation (with $V = 0$) is given by

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x,t)]$$

Using seperation of variables, write $\psi(x, t)$ as the product of a spatial component and a temporal component, $\psi(x, t) = \psi(x)\phi(t)$ — note that $\psi(x)$ is the wave function given in the problem statement. Substituting:

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)]$$

$$i\hbar\psi(x)\frac{\partial}{\partial t}\phi(t) = -\frac{\hbar^2}{2m}\phi(t)\frac{\partial^2}{\partial x^2}\psi(x)$$

Seperate all time and spatial components to opposite sides of the equation:

$$\frac{i\hbar}{\phi(t)}\frac{\partial}{\partial t}\phi(t) = -\frac{\hbar^2}{2m\psi(x)}\frac{\partial^2}{\partial x^2}\psi(x)$$

We know the derivative of $\psi(x)$ is $i\alpha\psi(x)$, so:

$$\frac{i\hbar}{\phi(t)}\frac{\partial}{\partial t}\phi(t) = -\frac{\hbar^2}{2m\psi(x)}(-\alpha^2\psi(x)) = \frac{\hbar^2\alpha^2}{2m}$$

$$\frac{1}{\phi(t)}\frac{\partial}{\partial t}\phi(t) = \frac{-i\hbar\alpha^2}{2m}$$

$$\ln(\phi(t)) = \frac{-i\hbar\alpha^2}{2m}t + C$$

$$\phi(t) = e^C e^{\frac{-i\hbar\alpha^2}{2m}t}$$

Recombine the spatial and temporal components and absorb all constants into one normalizing constant to get the full wave function:

$$\psi(x,t) = Ce^{i\alpha x}e^{\frac{-i\hbar\alpha^2}{2m}t}$$

3 Part c

Describe what figure 14 would look like if you animated it to show the time evolution of this wave function

As I see it, there are two options for this animation depending on how you conceive of the lines. The first option would be as follows: the lines are placed on the regions of the wavefunction with equal phase, and animated to follow these lines of equal phase, such that the lines are always associated with the same phase throughout the duration of the animation. If this is the case, then the lines should “move along with the wavefunction”, meaning that they will slide to the right at a constant speed off the page. The second option is that the lines are drawn on regions of space with equal phase, but not affixed to the wavefunction, such that they are stationary in space. In this case, we should see periodic behavior of the labeled phase of each line. The phase at each line will oscillate between 0 and 2π as the wavefunction evolves over time. At each line, we should see oscillation at the same rate, and each time one of the labels returns to its original phase, so too should all the others.