MAT3360 Oblig 1

Espen Lønes

February 18, 2021

Question 1)
a)

Prove that $\int_0^1 u(x)^2 dx \leq \frac{1}{2} \int_0^1 u'(x)^2 dx$
Where $u \in C_0^2((0,1))$ and $x \in (0,1)$. We start with.

$$u(x) = \int_0^x u'(y) dy$$

And use the Cauchy-Schwarts inequality.

$$|u(x)| = |\int_0^x u'(y) dy| \leq (\int_0^x 1^2 dy)^{\frac{1}{2}} (\int_0^x u'(y)^2 dy)^{\frac{1}{2}}$$

Square on both sides

$$u(x)^2 \leq x(\int_0^x u'(y)^2 dy)$$

Integrate from 0 to 1.

$$\int_0^1 u(x)^2 dx \leq \int_0^1 \int_0^x x u'(y)^2 dy \ dx$$

Change integration order.

$$\int_0^1 \int_y^1 x u'(y)^2 dx \ dy = \int_0^1 \frac{1}{2} u'(y)^2 - \frac{1}{2} y^2 u'(y)^2 \ dy$$

$-\frac{1}{2}y^2u'(y)^2 \leq 0$
Therefore.

$$\int_0^1 \frac{1}{2}u'(y)^2 - \frac{1}{2}y^2u'(y)^2 \; dy \leq \int_0^1 \frac{1}{2}u'(y)^2 dy = \frac{1}{2}\int_0^1 u'(x)^2 dx$$

and

$$\int_0^1 u(x)^2 dx \leq \frac{1}{2}\int_0^1 u'(x)^2 dx$$

b)
Prove

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}h\sum_{j=1}^{n+1}(D_-^h v_j)^2$$

Where $v \in D_{0,h}$. We start with.

$$v_j = h\sum_{k=1}^{j}(D_-^h v_k)$$

We then use the Cauchy-Schwarts inequality.

$$|h\sum_{k=1}^{j}(D_-^h v_k)| \leq (h\sum_{k=1}^{j} 1^2)^{\frac{1}{2}}(h\sum_{k=1}^{j}(D_-^h v_k)^2)^{\frac{1}{2}}$$

Square on both sides.

$$v_j^2 \leq (h\sum_{k=1}^{j} 1^2)(h\sum_{k=1}^{j}(D_-^h v_k)^2) = (hj)(h\sum_{k=1}^{j}(D_-^h v_k)^2)$$

Then sum from $j = 1$ to $j = n$ and multiply by h on both sides.

$$h\sum_{j=1}^{n} v_j^2 \leq h^3\sum_{j=1}^{n}\sum_{k=1}^{j} j(D_-^h v_k)^2$$

Change summation order.

$$h\sum_{j=1}^{n} v_j^2 \leq h^3\sum_{k=1}^{n+1}\sum_{j=k}^{n} j(D_-^h v_k)^2$$

$$\sum_{j=k}^{n+1} j \leq \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Inserting this gives us.

$$h\sum_{j=1}^{n} v_j^2 \leq h^3 \sum_{k=1}^{n+1} \frac{n(n+1)}{2} (D_-^h v_k)^2$$

Then using the fact that $h = \frac{1}{(n+1)}$

$$h\sum_{j=1}^{n} v_j^2 \leq h\frac{1}{(n+1)^2}\frac{n(n+1)}{2}\sum_{k=1}^{n+1}(D_-^h v_k)^2 = \frac{1}{2}h\frac{n}{(n+1)}\sum_{k=1}^{n}(D_-^h v_k)^2$$

$$lim_{n\to\infty}\frac{n}{(n+1)} = 1$$

So

$$\frac{n}{(n+1)} \leq 1$$

Giving us.

$$\frac{1}{2}h\frac{n}{(n+1)}\sum_{k=1}^{n+1}(D_-^h v_k)^2 \leq \frac{1}{2}h\sum_{k=1}^{n+1}(D_-^h v_k)^2 = \frac{1}{2}h\sum_{j=1}^{n+1}(D_-^h v_j)^2$$

Giving

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}h\sum_{j=1}^{n+1}(D_-^h v_j)^2$$

Question 2)

We have $-u''(x) + u(x) = f(x) \qquad (1)$
$u \in C_0^2((0,1))$ and $x \in (0,1)$ and f is given.
A solution u is on the form.

$$u(x) = u_h(x) + u_p(x)$$

Where $u_h(x)$ is the solution to the corresponding homogeneous equation: $-u''(x) + u(x) = 0$
And $u_p(x)$ is a particular solution.
We can find the form of $u_h(x)$ by observing that

$$u'(x) - u''(x) = 0 \iff u''(x) = u'(x) \iff u'(x) = C_1 e^x$$

Giving us.

$$u_h(x) = C_1 e^x + C_2$$

Since we are given $f(x) = 1$, $u_p(x)$ might be on the form $u_p(x) = Ax$.
Giving $u_p'(x) = A$ and $u_p''(x) = 0$. We must then find A s.t.

$$u'(x) - u''(x) = 1$$

$$A - 0 = 1 \iff A = 1$$

We then have.

$$u(x) = C_1 e^x + C_2 + x$$

Inserting for $u(0) = 0$ and $u(1) = 0$

$$u(0) = C_1 + C_2 = 0 \iff C_2 = -C_1$$

$$u(1) = C_1 e - C_1 = -1 \iff C_1 = \frac{1}{e-1} \iff C_2 = -\frac{1}{e-1}$$

Inserting C_1 and C_2 gives.

$$u(x) = \frac{1}{1-e} e^x - \frac{1}{1-e} + x = \frac{e^x - 1}{1-e} + x$$

Question 3)
a)

Show that if u solves (1) then

$$\int_0^1 u'(x)^2 dx = \int_0^1 f(x)u(x)dx$$

we start with (1) and multiply by u then integrate from 0 to 1. which gives.

$$\int_0^1 u(x)u'(x) - u(x)u''(x)dx = \int_0^1 f(x)u(x)dx$$

Left side:

$$\int_0^1 u(x)u'(x) - u(x)u''(x)dx = \int_0^1 u(x)u'(x)dx - \int_0^1 u(x)u''(x)dx$$

$$= \int_0^1 u(x)u'(x)dx - \left[u(x)u'(x)|_0^1 - \int_0^1 u'(x)u'(x)dx \right]$$

$$= \int_0^1 u(x)u'(x)dx + \int_0^1 u'(x)^2 dx$$

We now look at

$$I = \int_0^1 u(x)u'(x)dx = u(x)u(x)|_0^1 - \int u'(x)u(x)dx = -\int u'(x)u(x)dx = -I$$

$I = -I \iff I = 0$ So we have

$$\int_0^1 u'(x)^2 dx = \int_0^1 f(x)u(x)dx$$

b)

Show

$$\int_0^1 u(x)^2 dx \le \frac{1}{3}\int_0^1 f(x)^2 dx$$

We start by combining the results from 1a: $\int_0^1 u(x)^2 dx \le \frac{1}{2}\int_0^1 u'(x)^2 dx$ and 3a: $\int_0^1 u'(x)^2 dx = \int_0^1 f(x)u(x)dx$, to get.

$$\int_0^1 u(x)^2 dx \le \frac{1}{2}\int_0^1 f(x)u(x)dx$$

We use Cauchy-Schwartz on the rhs. to get.

$$\int_0^1 u(x)^2 dx \le \frac{1}{2}\left(\int_0^1 f(x)^2 dx\right)^2 \left(\int_0^1 u(x)^2 dx\right)^2$$

We then use the fact that $|ab| \le \frac{a^2}{2} + \frac{b^2}{2}$ to get.

$$\frac{1}{2}\left(\int_0^1 f(x)^2 dx\right)^2 \left(\int_0^1 u(x)^2 dx\right)^2 \le \frac{1}{2}\left(\frac{1}{2}\int_0^1 f(x)^2 dx + \frac{1}{2}u(x)^2 dx\right)$$

So

$$\int_0^1 u(x)^2 dx \le \frac{1}{4}\int_0^1 f(x)^2 dx + \frac{1}{4}u(x)^2 dx$$

$$\iff 4\int_0^1 u(x)^2 dx - \int_0^1 u(x)^2 dx \le \int_0^1 f(x)^2 dx$$

$$\iff \int_0^1 u(x)^2 dx \le \frac{1}{3}\int_0^1 f(x)^2 dx$$

Question 4)
a)

First we look at $|D_c^h u(x) - u'(x)|$.

$$|D_c^h u(x) - u'(x)| = \frac{u(x+h) - u(x-h)}{2h} - u'(x)$$

$$= \frac{1}{2h}[u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + c_1 h^4$$
$$- u(x) + hu'(x) - \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) - c_2 h^4] - u'(x)$$

$$= u'(x) + \frac{h^2}{6}u'''(x) + c_3 h^3 - u'(x) = \frac{h^2}{6}u'''(x) + c_3 h^3$$

Where $c_3 = \frac{c_1 - c_2}{2}$. We see that $h \leq 1$.
If we give $a = sup_{x\in[0,1]}|u'''(x)|$ We get.

$$|D_c^h u(x) - u'(x)| \leq |h^2\frac{a}{6} + c_3 h^3|$$

Use $h^2 \leq h^3$

$$|D_c^h u(x) - u'(x)| \leq h^2|\frac{a}{6} + c_3| = O(h^2)$$

Since $\frac{a}{6} + c_3$ is constant.

Then we look at $|D_+^h D_-^h u(x) - u''(x)|$

$$D_+^h D_-^h u(x) - u''(x) = D_+^h\left(\frac{u(x) - u(x-h)}{h}\right) - u''(x)$$

$$= \frac{1}{h}\left(D_+^h u(x) - D_+^h u(x-h)\right) - u''(x)$$

$$= \frac{1}{h}\left(\frac{u(x+h) - u(x)}{h} - \frac{u(x) - u(x-h)}{h}\right) - u''(x)$$

$$= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - u''(x)$$

$$= \frac{1}{h^2}[u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + c_1h^4$$
$$- u(x) + hu'(x) - \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) - c_2h^4 - 2u(x)] - u''(x)$$
$$= \frac{2}{h}u'(x) + \frac{h}{3}u'''(x) - h^2c_3 - u''(x)$$

Where the constant $c_3 = c_1 - c_2$
We define $a = sup_{x\in[0,1]}\frac{2}{h}u'(x)$, $b = sup_{x\in[0,1]}\frac{h}{3}u'''(x)$ and $c = sup_{x\in[0,1]}u''(x)$
And get.

$$|D_+^hD_-^hu(x) - u''(x)| \leq |a + b - h^2c_3 - c| = O(h^2)$$

b)

(1) is $f(x) = -u''(x) + u'(x)$
(2) is $-D_+^hD_-^hv_j + D_c^hv_j = f(x)$
And $u(x_j) = v_j$
We can then look at.

$$| - D_+^hD_-^hv_j + D_c^hv_j - f(x)| = | - D_+^hD_-^hu(x_j) + D_c^hu(x_j) + u''(x_j) - u'(x_j)|$$

We see that $| - D_+^hD_-^hu(x_j) + u''(x_j)| = |D_+^hD_-^hu(x_j) - u''(x_j)|$.
Using this and $|a + b| \leq |a| + |b|$. We then get.

$$|D_+^hD_-^hu(x_j) - u''(x_j) + D_c^hu(x_j) - u'(x_j)| \leq |D_+^hD_-^hu(x_j) - u''(x_j)| + |D_c^hu(x_j) - u'(x_j)|$$
$$= O(h^2)$$

So (2) approaches (1) proportional to h^2. Which is a reasonable approximation.

c)

Find an $n \times n$ matrix A and a vector $b \in \mathbb{R}^n$ s.t. (2) can be written $Av = b$

$$b = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ 0 \end{pmatrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_{n-1} \\ 0 \end{pmatrix}$$

We have $D_c^h v_j = \frac{v_{j+1} - v_{j-1}}{2h}$

and $D_+^h D_-^h v_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}$
Giving us that.

$$(2) \to -D_+^h D_-^h v_j + D_c^h v_j = -\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + \frac{v_{j+1} - v_{j-1}}{2h}$$

$$= \frac{-2v_{j+1} + 4v_j - 2v_{j-1} + hv_{j+1} - hv_{j-1}}{2h^2} = \frac{v_{j-1}(-2-h) + 4v_j + v_{j+1}(h-2)}{2h^2}$$

Giving us.

$$A = \frac{1}{2h^2} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & & \\ (-2-h) & 4 & (h-2) & 0 & & & \\ 0 & (-2-h) & 4 & (h-2) & & & \\ \vdots & & & \ddots & & & \\ & & & & & & \\ & & & 0 & (-2-h) & 4 & (h-2) \\ & & & & 0 & 0 & 1 \end{bmatrix}$$

d)

Since $h\sum_{j=1}^{n} v_j = \int_0^1 u(x_j)$ we can write. (same as in 3b)

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}h\sum_{j=1}^{n} f(x_j)v_j$$

Then use Cauchy-Schwartz.

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}\left(h\sum_{j=1}^{n} f(x_j)^2\right)^{\frac{1}{2}}\left(h\sum_{j=1}^{n} v_j^2\right)^{\frac{1}{2}}$$

Then use $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}\left(\frac{1}{2}h\sum_{j=1}^{n} f(x_j)^2\right)\left(\frac{1}{2}h\sum_{j=1}^{n} v_j^2\right)$$

$$4h\sum_{j=1}^{n} v_j^2 - h\sum_{j=1}^{n} v_j^2 \leq h\sum_{j=1}^{n} f(x_j)^2$$

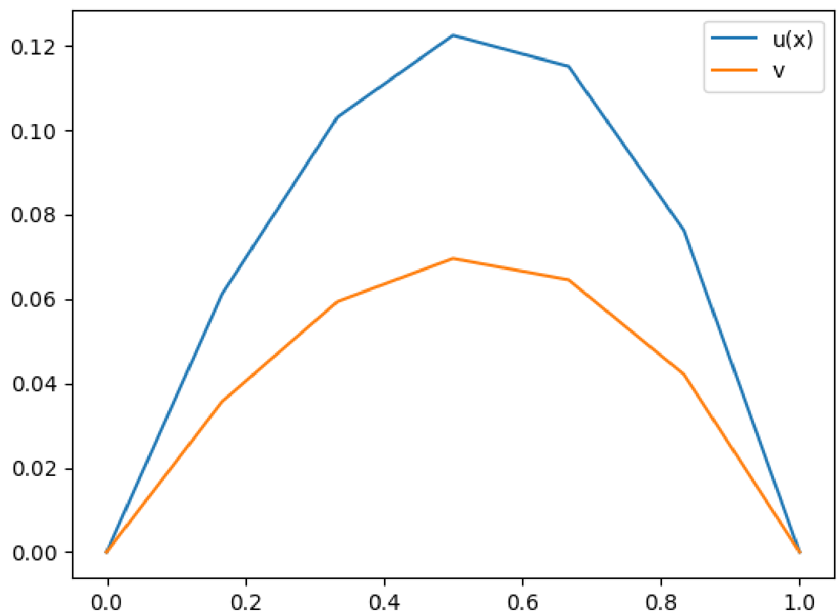
$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{3}h\sum_{j=1}^{n} f(x_j)^2$$

Question 5)
a)

Program was implemented in python 3:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import lstsq

def f(x):
    if x == 0 or x == 1:
        return 0
    else:
        return 1

def u(x):
    return ((np.exp(x) - 1) / (1 - np.exp(1))) + x


n = 7
h = 1.0 / (1 + n)
x = np.linspace(0, 1, n)
b = np.array([f(i)*2*(h**2) for i in x])

A = np.zeros((n,n))
A[0][0] = 1; A[-1][-1] = 1
for i in range(1, n-1):
    A[i][i-1] = - 2 - h
    A[i][i] = 4
    A[i][i+1] = h - 2

p, res, rnk, s = lstsq(A, b)

plt.plot(x, u(x), label="u(x)")
plt.plot(x, p, label="v")
plt.legend()
plt.show()
```

Plot given by program for $n = 7$



b)

Was not able to complete this task.