MAT3360 Oblig 2

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Question 1)
a)

We have $u_k(x,t) = T_k(t)X_k(x)$, with the ansatz $X_k(x) = e^{ik\pi x}$
By doing separation of variables we get, $\frac{T'}{T} = \frac{X'''}{X} = -\lambda$.
Giving us the two equations.

$$-X''' = \lambda X$$

$$T' = -\lambda T$$

The first gives us:

$$-(ik\pi)^3 e^{ik\pi x} = \lambda e^{ik\pi x} \iff \lambda_k = i(k\pi)^3$$

Further the second equation has the solution:

$$T_k(t) = e^{-lambda_k t} = e^{-i(k\pi)^3 t}$$

We then get:

$$u_k(x,t) = e^{-i(k\pi)^3 t} * e^{ik\pi x}$$

Using $e^{i\alpha} = cos(\alpha) + isin(\alpha)$, gives:

$$u_k = [cos(k^3\pi^3 t) - isin(k^3\pi^3 t)] * [cos(k\pi x) + isin(k\pi x)]$$

$$= cos(k^3\pi^3 t)cos(k\pi x) + sin(k^3\pi^3 t)sin(k\pi x) + i[cos(k^3\pi^3 t)sin(k\pi x) - sin(k^3\pi^3 t)cos(k\pi x)]$$

Gives the family of real solutions:

$$u_k(x,t) = cos(k^3\pi^3 t)cos(k\pi x) + sin(k^3\pi^3 t)sin(k\pi x)$$

b)

We can form new solutions by taking linear combinations of solutions.

$$u(x,t) = \sum_{k=1}^{N} c_k u_k(x,t)$$

First we have $f(x) = cos(\pi x)$. Using the initial condition $u(x,0) = f(x)$. We get:

$$u(x,0) = \sum_{k=1}^{N} c_k u_k(x,0) = \sum_{k=1}^{N} c_k cos(k\pi x) = f(x) = cos(\pi x)$$

Which is solved by having $c_1 = 1$ and all other $c_k = 0$. Giving:

$$u(x,t) = \sum_{k=1}^{N} c_k u_k(x,t) = u_1 = cos(\pi^3 t)cos(\pi x) + sin(\pi^3 t)sin(\pi x)$$

Could not find solution for $f(x) = sign(x)$

Did not do c)

Question 2)
a)

$$< u, D_{-}v >_{\Delta x} = \Delta x \sum_{j=0}^{n} u_j D_{-} v_j = \Delta x \sum_{j=0}^{n} u_j \frac{v_j - v_{j-1}}{\Delta x} = \sum_{j=0}^{n} u_j (v_j - v_{j-1})$$

$$= \sum_{j=0}^{n} u_j\ v_j - \sum_{j=0}^{n} u_j\ v_{j-1} = \sum_{j=0}^{n} u_j\ v_j - [\frac{1}{2} u_0 v_{-1} + \sum_{j=1}^{n} u_j\ v_{j-1} + \frac{1}{2} u_{n+1} v_n]$$

We have $u_0 = u_{n+1}$ and $v_{-1} = v_n$.

$$= \sum_{j=0}^{n} u_j\ v_j - [\sum_{j=1}^{n} u_j\ v_{j-1} + u_{n+1} v_n] = \sum_{j=0}^{n} u_j\ v_j - \sum_{j=1}^{n+1} u_j\ v_{j-1}$$

Showing that $\sum_{j=0}^{n} u_j\ v_{j-1} = \sum_{j=1}^{n+1} u_j\ v_{j-1}$ Which further shows:

$$\sum_{j=0}^{n} u_j\ v_{j-1} = \sum_{j=0}^{n} u_{j+1}\ v_j$$

Now continuing the calculation we have:

$$< u, D_{-}v >_{\Delta x} = \sum_{j=0}^{n} u_j\ v_j - \sum_{j=0}^{n} u_{j+1}\ v_j = \sum_{j=0}^{n} u_j\ v_j - u_{j+1}\ v_j = \sum_{j=0}^{n} (u_j - u_{j+1}) v_j$$

$$= -\Delta x \sum_{j=0}^{n} \frac{(u_{j+1} - u_j)}{\Delta x} v_j = - < D_{+}u, v >_{\Delta x}$$

b)

$$D_+v_j - D_-v_j = \frac{v_{j+1} - v_j}{\Delta x} - \frac{v_j - v_{j-1}}{\Delta x} = D_+v_j - D_+v_{j-1}$$
$$= \Delta x D_+(\frac{v_j - v_{j-1}}{\Delta x}) = \Delta x D_+ D_- v_j$$

c)

Start with the RHS of the equation:

$$-\frac{\Delta x}{2}||D_+D_-v||^2_{\Delta x} = -\frac{\Delta x^2}{2}\sum_{j=0}^{n}(D_+D_-v_j)^2 = -\frac{\Delta x^2}{2}\sum_{j=0}^{n}(v_{j+1} - 2v_j + v_{j-1})^2$$

$$= -\frac{1}{2\Delta x^2}\sum_{j=0}^{n} v_{j+1}v_{j+1} - 2v_jv_{j+1} + v_{j+1}v_{j-1} - 2v_{j+1}v_j + 4v_jv_j - 2v_{j+1}v_{j-1} - 2v_jv_{j-1} + v_{j-1}v_{j-1}$$

$$= -\frac{1}{2\Delta x^2}\sum_{j=0}^{n} v_jv_j - 2v_jv_{j+1} + v_{j+2}v_j - 2v_{j+1}v_j + 4v_jv_j - 2v_{j+1}v_j + v_{j+2}v_j - 2v_jv_{j-1} + v_jv_j$$

$$= -\frac{1}{2\Delta x^2}\sum_{j=0}^{n} v_j\ (2v_{j+2} - 6v_{j+1} + 6v_j - 2v_{j-1})$$

$$= -\frac{1}{\Delta x^2}\sum_{j=0}^{n} v_j\ (v_{j+2} - 3v_{j+1} + 3v_j - v_{j-1})$$

Then starting from the LHS:

$$- < v, D_+D_-D_+v >_{\Delta x} = -\Delta x \sum_{j=0}^{n} v_j * D_+D_-D_+v_j$$

$$= -\sum_{j=0}^{n} v_j * D_+D_-(v_{j+1} - v_j) = -\frac{1}{\Delta x}\sum_{j=0}^{n} v_j * D_+(v_{j+1} - 2v_j + v_{j-1})$$

$$= -\frac{1}{\Delta x^2}\sum_{j=0}^{n} v_j\ (v_{j+2} - 2v_{j+1} + v_j - (v_{j+1} - 2v_j + v_{j-1}))$$

$$= -\frac{1}{\Delta x^2}\sum_{j=0}^{n} v_j\ (v_{j+2} - 3v_{j+1} + 3v_j - v_{j-1})$$

We see that RHS=LHS.

d)

We need to find D_p(for D_+) and D_m(for D_-) to satisfy $D_p\ v = b_p$ and $D_m\ v = b_m$.
Where $v = (v_0, ..., v_n)^T$.

$b_p = (\frac{v_1 - v_0}{\Delta x}, ..., \frac{v_{n+1} - v_n}{\Delta x} = \frac{v_0 - v_n}{\Delta x})^T$
We then get:

$$D_p = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & 0 & ... & & & \\ 0 & -1 & 1 & 0 & ... & & \\ 0 & 0 & -1 & 1 & 0 & & \\ \vdots & & & \ddots & & & \\ & & & & & & \\ 0 & ... & & & & -1 & 1 \\ 1 & 0 & & ... & & 0 & -1 \end{bmatrix}$$

Using similar logic we get:

$$D_m = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & ... & & & 0 & -1 \\ -1 & 1 & 0 & ... & & & \\ 0 & -1 & 1 & 0 & ... & & \\ 0 & 0 & -1 & 1 & 0 & ... & \\ \vdots & & & \ddots & & & \\ & & & & & & \\ 0 & ... & & & & -1 & 1 \end{bmatrix}$$

e)

We have $D_-D_+D_-u_j = \frac{-u_{j-2}+3u_{j-1}-3u_j+u_{j+1}}{\Delta x^3}$
And using the Fourier series:

$$u(x_{j+1}) = u(x_j+\Delta x) = u(x_j)+\Delta x u'(x_j)+\frac{\Delta x^2}{2}u''(x_j)+\frac{\Delta x^3}{6}u'''(x_j)+c_1\Delta x^4$$

$$u(x_{j-1}) = u(x_j-\Delta x) = u(x_j)-\Delta x u'(x_j)+\frac{\Delta x^2}{2}u''(x_j)-\frac{\Delta x^3}{6}u'''(x_j)+c_2\Delta x^4$$

$$u(x_{j-2}) = u(x_{j-1}-\Delta x) = u(x_{j-1})-\Delta x u'(x_{j-1})+\frac{\Delta x^2}{2}u''(x_{j-1})-\frac{\Delta x^3}{6}u'''(x_{j-1})+c_3\Delta x^4$$

Using this we get:

$$D_-D_+D_-u(x_j) - u'''(x_j) = \frac{-u_{j-2}+3u_{j-1}-3u_j+u_{j+1}}{\Delta x^3} - u'''(x_j)$$

$$\begin{gathered}
= \frac{1}{\Delta x^3}\ [\ -u(x_{j-1})+\Delta x u'(x_{j-1})-\frac{\Delta x^2}{2}u''(x_{j-1})+\frac{\Delta x^3}{6}u'''(x_{j-1})-c_3\Delta x^4 \\
+3u(x_j)-3\Delta x u'(x_j)+\frac{3\Delta x^2}{2}u''(x_j)-\frac{3\Delta x^3}{6}u'''(x_j)+3c_2\Delta x^4 \\
-3u(x_j) \\
+u(x_j)+\Delta x u'(x_j)+\frac{\Delta x^2}{2}u''(x_j)+\frac{\Delta x^3}{6}u'''(x_j)+c_1\Delta x^4\ ] - u'''(x_j)
\end{gathered}$$

$$\begin{gathered}
= \frac{1}{\Delta x^3}\ [\ -u(x_{j-1})+\Delta x u'(x_{j-1})-\frac{\Delta x^2}{2}u''(x_{j-1})+\frac{\Delta x^3}{6}u'''(x_{j-1})-c_3\Delta x^4 \\
-2\Delta x u'(x_j)+2\Delta x^2 u''(x_j)-\frac{1}{3}\Delta x^3 u'''(x_j)+3c_2\Delta x^4 \\
+u(x_j)+c_1\Delta x^4\ ]\ - u'''(x_j)
\end{gathered}$$

$$= \frac{1}{\Delta x^3} \; [\; -u(x_j) + \Delta x u'(x_j) - \frac{\Delta x^2}{2} u''(x_j) + \frac{\Delta x^3}{6} u'''(x_j) - c_2 \Delta x^4$$

$$+ \Delta x u'(x_{j-1}) - \frac{\Delta x^2}{2} u''(x_{j-1}) + \frac{\Delta x^3}{6} u'''(x_{j-1}) - c_3 \Delta x^4$$

$$-2\Delta x u'(x_j) + 2\Delta x^2 u''(x_j) - \frac{1}{3} \Delta x^3 u'''(x_j) + 3c_2 \Delta x^4$$

$$+ u(x_j) + c_1 \Delta x^4 \;] \; - u'''(x_j)$$

$$= \frac{1}{\Delta x^3} \; [\; \Delta x u'(x_{j-1}) - \frac{\Delta x^2}{2} u''(x_{j-1}) + \frac{\Delta x^3}{6} u'''(x_{j-1}) - c_3 \Delta x^4$$

$$-\Delta x u'(x_j) + \frac{3}{2} \Delta x^2 u''(x_j) - \frac{1}{6} \Delta x^3 u'''(x_j) + 2c_2 \Delta x^4$$

$$+ c_1 \Delta x^4 \;] \; - u'''(x_j)$$

Which gives:

$$u'''(x_j) = D_- D_+ D_- u_j + \Delta x \; (u'(x_{j-1}) - u'(x_j)) + O(\Delta x^2)$$

$$\Rightarrow$$

$$D_- D_+ D_- u_j - u'''(x_j) = O(\Delta x)$$

Question 3)
a)

$$u_t = u_{xxx} \Rightarrow 0 = u_{xxx}(x,t) - u_t(x,t)$$

$$= \frac{1}{2} D_- D_+ D_- \, (u(x,t) - u(x,t+\Delta t)) + O(\Delta x) - \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t)$$

Evaluation at x_j^m (where $u(x_j, t_m) = u(x_j^m)$) gives:

$$\frac{1}{2} D_- D_+ D_- \, (u(x_j^m) - v_j^m - (u(x_j^{m+1}) - v_j^{m+1})) - \frac{u(x_j^{m+1} - v_j^{m+1}) - (u(x_j^m) - v_j^m)}{\Delta t} = O(\Delta x + \Delta t)$$

So it is reasonable to say that $v_j^m \approx u(x_j^m)$ for small Δx and small Δt. (Making (2) a reasonable scheme for (1)).

b)

Start with:

$$< \frac{v^{m+1} - v^m}{t}, v^{m+1} + v^m >_{\Delta x} = < \frac{1}{2} D_- D_+ D_- \, (v^{m+1} - v^m), v^{m+1} + v^m >_{\Delta x}$$

$$\iff$$

$$\Delta x \sum_{j=0}^{n} (\frac{v^{m+1} - v^m}{t})(v^{m+1} + v^m) = \Delta x \sum_{j=0}^{n} (\frac{1}{2} D_- D_+ D_- \, (v^{m+1} - v^m))(v^{m+1} + v^m)$$

$$\iff$$

$$\frac{\Delta x}{\Delta t} \sum_{j=0}^{n} (v_j^{m+1})^2 - (v_j^m)^2$$

$$= \frac{\Delta x}{2\Delta x^3} \sum_{j=0}^{n} [v_{j+1}^{m+1} - 3v_j^{m+1} + 3v_{j-1}^{m+1} - v_{j-2}^{m+1} + v_{j+1}^m - 3v_j^m + 3v_{j-1}^m - v_{j-2}^m](v_j^{m+1} + v_j^m)$$

$$\iff$$

$$E^{m+1} - E^m = \frac{\Delta t}{2\Delta x^2} \sum_{j=0}^{n} [v_{j+1}^{m+1} - 3v_j^{m+1} + 3v_{j-1}^{m+1} - v_{j-2}^{m+1} + v_{j+1}^m - 3v_j^m + 3v_{j-1}^m - v_{j-2}^m](v_j^{m+1} + v_j^m)$$

Have to show:

$$\frac{\Delta t}{2\Delta x^2}\sum_{j=0}^{n}[v_{j+1}^{m+1}-3v_j^{m+1}+3v_{j-1}^{m+1}-v_{j-2}^{m+1}+v_{j+1}^{m}-3v_j^{m}+3v_{j-1}^{m}-v_{j-2}^{m}](v_j^{m+1}+v_j^m)\leq 0$$

I did not get any further.

c)

The matrix form of $D_-D_+D_-$ is:

$$\frac{1}{\Delta x^3}\begin{bmatrix} -3 & 1 & 0 & ... & & & 0 & -1 & 3 \\ 3 & -3 & 1 & 0 & ... & & & 0 & -1 \\ -1 & 3 & -3 & 1 & 0 & ... & & & 0 \\ 0 & -1 & 3 & -3 & 1 & 0 & ... & & 0 \\ \vdots & & & \ddots & & & & & \\ & & & & & & & & \\ 0 & ... & & & & -1 & 3 & -3 & 1 \\ 1 & 0 & ... & & & 0 & -1 & 3 & -3 \end{bmatrix}$$

We then get:

$$A = I - \frac{\Delta t}{2\Delta x^3}\begin{bmatrix} -3 & 1 & 0 & ... & & & 0 & -1 & 3 \\ 3 & -3 & 1 & 0 & ... & & & 0 & -1 \\ -1 & 3 & -3 & 1 & 0 & ... & & & 0 \\ 0 & -1 & 3 & -3 & 1 & 0 & ... & & 0 \\ \vdots & & & \ddots & & & & & \\ & & & & & & & & \\ 0 & ... & & & & -1 & 3 & -3 & 1 \\ 1 & 0 & ... & & & 0 & -1 & 3 & -3 \end{bmatrix}$$

$$= \frac{\Delta t}{2\Delta x^3}\begin{bmatrix} 4 & -1 & 0 & ... & & & 0 & 1 & -3 \\ -3 & 4 & -1 & 0 & ... & & & 0 & 1 \\ 1 & -3 & 4 & -1 & 0 & ... & & & 0 \\ 0 & 1 & -3 & 4 & -1 & 0 & ... & & 0 \\ \vdots & & & \ddots & & & & & \\ & & & & & & & & \\ 0 & ... & & & & 1 & -3 & 4 & -1 \\ -1 & 0 & ... & & & 0 & 1 & -3 & 4 \end{bmatrix}$$

We can call this $\frac{\Delta t}{2\Delta x^3}B$, where if B is invertible then so is A.
We see that we can do Gaussian elimination on B:

$$B \sim \begin{bmatrix} 4 & -1 & 0 & ... & & & 0 & 1 & -3 \\ -3 & 4 & -1 & 0 & ... & & & 0 & 1 \\ 1 & -3 & 4 & -1 & 0 & ... & & & 0 \\ 0 & 1 & -3 & 4 & -1 & 0 & ... & & 0 \\ \vdots & & & \ddots & & & & & \\ & & & & & & & & \\ 0 & ... & & & & 1 & -3 & 4 & -1 \\ -1 & 0 & ... & & & 0 & 1 & -3 & 4 \\ 4 & -1 & 0 & ... & & & 0 & 1 & -3 \\ -3 & 4 & -1 & 0 & ... & & & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & ... & & & 0 \\ 0 & 1 & -3 & 4 & -1 & 0 & ... & & 0 \\ \vdots & & & \ddots & & & & & \\ & & & & & & & & \\ 0 & ... & & & 0 & 1 & -3 & 4 & -1 \\ 0 & ... & & & & 0 & 1 & c_1 & c_2 \\ 0 & ... & & & & & 0 & 1 & c_3 \\ 0 & ... & & & & & & 0 & 1 \end{bmatrix}$$

We now see that B has full rank meaning it is invertible $\iff$ A is invertible.

d)

Could not get it to work.