MAT3360 Oblig 1

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Question 1)
a)

Prove that $\int_0^1 u(x)^2 dx \leq \frac{1}{2}\int_0^1 u'(x)^2 dx$
Where $u \in C_0^2((0,1))$ and $x \in (0,1)$. We start with.

$$u(x) = \int_0^x u'(y)dy$$

And use the Cauchy-Schwarts inequality.

$$|u(x)| = |\int_0^x u'(y)dy| \leq (\int_0^x 1^2 dy)^{\frac{1}{2}}(\int_0^x u'(y)^2 dy)^{\frac{1}{2}}$$

Square on both sides

$$u(x)^2 \leq x(\int_0^x u'(y)^2 dy)$$

Integrate from 0 to 1.

$$\int_0^1 u(x)^2 dx \leq \int_0^1 \int_0^x xu'(y)^2 dy\ dx$$

Change integration order.

$$\int_0^1 \int_y^1 xu'(y)^2 dx\ dy = \int_0^1 \frac{1}{2}u'(y)^2 - \frac{1}{2}y^2 u'(y)^2\ dy$$

$-\frac{1}{2}y^2 u'(y)^2 \leq 0$
Therefore.

$$\int_0^1 \frac{1}{2}u'(y)^2 - \frac{1}{2}y^2 u'(y)^2\ dy \leq \int_0^1 \frac{1}{2}u'(y)^2$$

and

$$\int_0^1 u(x)^2 dx \leq \frac{1}{2}\int_0^1 u'(x)^2 dx$$

b)
Prove

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}h\sum_{j=1}^{n+1}(D_-^h v_j)^2$$

Where $v \in D_{0,h}$. We start with.

$$v_j = h\sum_{k=1}^{j}(D_-^h v_k)$$

We then use the Cauchy-Schwarts inequality.

$$|h\sum_{k=1}^{j}(D_-^h v_k)| \leq (h\sum_{k=1}^{j}1^2)^{\frac{1}{2}}(h\sum_{k=1}^{j}(D_-^h v_k)^2)^{\frac{1}{2}}$$

Square on both sides.

$$v_j^2 \leq (h\sum_{k=1}^{j}1^2)(h\sum_{k=1}^{j}(D_-^h v_k)^2) = (hj)(h\sum_{k=1}^{j}(D_-^h v_k)^2)$$

Then sum from $j = 1$ to $j = n$ and multiply by h on both sides.

$$h\sum_{j=1}^{n} v_j^2 \leq h^3\sum_{j=1}^{n}\sum_{k=1}^{j} j(D_-^h v_k)^2$$

Change summation order.

$$h\sum_{j=1}^{n} v_j^2 \leq h^3\sum_{k=1}^{n+1}\sum_{j=k}^{n} j(D_-^h v_k)^2$$

$$\sum_{j=k}^{n+1} j \leq \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Inserting this gives us.

$$h\sum_{j=1}^{n} v_j^2 \leq h^3\sum_{k=1}^{n+1}\frac{n(n+1)}{2}(D_-^h v_k)^2$$

Then using the fact that $h = \frac{1}{(n+1)}$

$$h\sum_{j=1}^{n} v_j^2 \leq h\frac{1}{(n+1)^2}\frac{n(n+1)}{2}\sum_{k=1}^{n+1}(D_-^h v_k)^2 = \frac{1}{2}h\frac{n}{(n+1)}\sum_{k=1}^{n}(D_-^h v_k)^2$$

$$lim_{n\to\infty}\frac{n}{(n+1)} = 1$$

So

$$\frac{n}{(n+1)} \leq 1$$

Giving us.

$$\frac{1}{2}h\frac{n}{(n+1)}\sum_{k=1}^{n+1}(D_-^h v_k)^2 \leq \frac{1}{2}h\sum_{k=1}^{n+1}(D_-^h v_k)^2 = \frac{1}{2}h\sum_{j=1}^{n+1}(D_-^h v_j)^2$$

Giving

$$h\sum_{j=1}^{n} v_j^2 \leq \frac{1}{2}h\sum_{j=1}^{n+1}(D_-^h v_j)^2$$

Question 2)

We have $-u''(x) + u(x) = f(x) \qquad (1)$
$u \in C_0^2((0,1))$ and $x \in (0,1)$ and f is given.
A solution u is on the form.

$$u(x) = u_h(x) + u_p(x)$$

Where $u_h(x)$ is the solution to the corresponding homogeneous equation: $-u''(x) + u(x) = 0$
And $u_p(x)$ is a particular solution.
We can find the form of $u_h(x)$ by observing that

$$u'(x) - u''(x) = 0 \iff u''(x) = u'(x) \iff u'(x) = C_1 e^x$$

Giving us.

$$u_h(x) = C_1 e^x + C_2$$

Since we are given $f(x) = 1$, $u_p(x)$ might be on the form $u_p(x) = Ax$.
Giving $u_p'(x) = A$ and $u_p''(x) = 0$. We must then find A s.t.

$$u'(x) - u''(x) = 1$$

$$A - 0 = 1 \iff A = 1$$

We then have.

$$u(x) = C_1 e^x + C_2 + x$$

Inserting for $u(0) = 0$ and $u(1) = 0$

$$u(0) = C_1 + C_2 = 0 \iff C_2 = -C_1$$

$$u(1) = C_1 e - C_1 = -1 \iff C_1 = \frac{1}{e-1} \iff C_2 = -\frac{1}{e-1}$$

Inserting C_1 and C_2 gives.

$$u(x) = \frac{1}{1-e}e^x - \frac{1}{1-e} + x = \frac{e^x - 1}{1-e} + x$$

Question 3)
a)

Show that if u solves (1) then

$$\int_0^1 u'(x)^2 dx = \int_0^1 f(x)u(x)dx$$

we start with (1) and multiply by u then integrate from 0 to 1. which gives.

$$\int_0^1 u(x)u'(x) - u(x)u''(x)dx = \int_0^1 f(x)u(x)dx$$

Left side:

$$\int_0^1 u(x)u'(x) - u(x)u''(x)dx = \int_0^1 u(x)u'(x)dx - \int_0^1 u(x)u''(x)dx$$

$$= \int_0^1 u(x)u'(x)dx - \left[u(x)u'(x)|_0^1 - \int_0^1 u'(x)u'(x)dx \right]$$

$$= \int_0^1 u(x)u'(x)dx + \int_0^1 u'(x)^2 dx$$

We now look at

$$I = \int_0^1 u(x)u'(x)dx = u(x)u(x)|_0^1 - \int u'(x)u(x)dx = -\int u'(x)u(x)dx = -I$$

$I = -I \iff I = 0$ So we have

$$\int_0^1 u'(x)^2 dx = \int_0^1 f(x)u(x)dx$$

b)