An useful expression for collision term for a two-to-two process $(1+2\to 3+4)$ is that [1]:

$$C_{1+2\to 3+4} \approx \frac{T}{32\pi^4}\int_{4m^2}^{\infty} s^{1/2}\left(s-4m^2\right)\sigma_{1+2\to 3+4} K_1\left(\frac{s^{1/2}}{T}\right) ds\,, \tag{1}$$

where $C_{1+2\to 3+4}$ is defined by

$$C_{1+2\to 3+4} = \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 f_1^{\rm eq} f_2^{\rm eq} \left|\mathcal{M}\right|^2 (2\pi)^4 \delta^{(4)}\left(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu\right)\,, \tag{2}$$

$$f_i^{\rm eq} = g_i e^{-E_i/T}\,. \tag{3}$$

To derive it, we can start from the basic conceptions. The Boltzmann equation for $m_1 = m_2 \equiv m$ and $n_1 = n_2$ is

$$a^{-3}\frac{d\left(n\cdot a^3\right)}{dt} = -\left\langle \sigma v\right\rangle \left(n^2 - n_{\rm eq}^2\right)\,. \tag{4}$$

Generally, the dimensionless variable Y is defined by

$$Y \equiv \frac{n}{s}\,, \tag{5}$$

where the entropy density s is given by

$$s = \frac{2\pi^2}{45} g_{s*} T^3\,, \tag{6}$$

The total entropy per comoving volume

$$S = sa^3 \tag{7}$$

is constant in absence of entropy production. Therefore, applying this equation, we obtain

$$\dot{Y} = -s\left\langle\sigma v\right\rangle\left(Y^2 - Y_{\rm eq}^2\right)\,. \tag{8}$$

Now, we have to compute the thermally averaged cross section $\langle\sigma v\rangle$.

$$\begin{aligned}\left\langle\sigma v\right\rangle =& \frac{1}{\int f_1^{\rm eq} d\Lambda_1 f_2^{\rm eq} d\Lambda_2}\int f_1^{\rm eq} d\Lambda_1 f_2^{\rm eq} d\Lambda_2\, \sigma v\,,\\ =& \frac{1}{\int f_1^{\rm eq} d\Lambda_1 f_2^{\rm eq} d\Lambda_2} C_{1+2\to 3+4}\end{aligned} \tag{9}$$

where

$$d\Lambda_i = d^3\mathbf{p}_i/\left(2\pi\right)^3\,, \tag{10}$$

We already know that the denominator is exactly the production of number density in equilibrium

$$n_1^{(0)} n_2^{(0)} = \frac{g_1 g_2}{2\pi^2} m^4 T^2\left[K_2\left(\frac{m}{T}\right)\right]^2\,. \tag{11}$$

And cross section and the "relative velocity" σv is defined by

$$\sigma = \frac{1}{4E_1 E_2 v}\int d\Pi_3 d\Pi_4 \left(2\pi\right)^4 \delta^{(4)}\left(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu\right)\left|\mathcal{M}\right|^2 \tag{12}$$

$$
\begin{aligned}
v &= \left[\left|v_1 - v_2\right|^2 - \left|v_1 \times v_2\right|^2\right]^{1/2} \\
&= \frac{1}{E_1 E_2}\sqrt{\left(p_1^\mu p_{2\mu}\right)^2 - m^4}
\end{aligned} \tag{13}
$$

Now we should compute the numerator. It's necessary to choose a proper variables set to simplify the multiple integrals (we have use p^μ to denote 4-vector and p to denote the magnitude of 3-vector of $\mathbf{p}$):

$$
d\Lambda_1 d\Lambda_2 = \frac{1}{(2\pi)^6} p_1^2 dp_1 d\Omega_1 p_2^2 dp_2 d\Omega_2 \,. \tag{14}
$$

First, we can perform a rotation so that the polar angle θ of the particle 2 can be the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$.Therefore,

$$
d\Lambda_1 d\Lambda_2 = \frac{1}{(2\pi)^6} 4\pi p_1 E_1 dE_1 \, 4\pi p_2 E_2 dE_2 \, \frac{1}{2} d\cos\theta \,.
$$

Defining that

$$
\begin{aligned}
E_+ = E_1 + E_2 \,, \qquad E_- = E_1 - E_2 \,, \\
s_{\mathrm{M}} = 2m^2 + 2E_1 E_2 - 2p_1 p_2 \cos\theta \,,
\end{aligned} \tag{15}
$$

we know that

$$
dE_+ dE_- = 2 dE_1 dE_2 \,.
$$

Because E_+, E_- is not explicit function of θ, we just need to be concerned about $\partial s_{\mathrm{M}}/\partial\theta$. Therefore,

$$
d\Lambda_1 d\Lambda_2 = \frac{1}{(2\pi)^6} 2\pi^2 E_1 E_2 dE_+ dE_- ds_{\mathrm{M}} \,. \tag{16}
$$

The integration limits can be solved by the MMA Reduce function, which are

$$
\begin{gathered}
s_{\mathrm{M}} \geq 4m^2 \\
E_+ \geq \sqrt{s_{\mathrm{M}}} \\
|E_-| \leq \sqrt{1 - 4m^2/s_{\mathrm{M}}}\sqrt{E_+^2 - s_{\mathrm{M}}}
\end{gathered} \tag{17}
$$

Therefore

$$
\begin{aligned}
C_{1+2\to 3+4} =& \frac{1}{(2\pi)^6} \int d^3p_1 d^3p_2 \, e^{-E_+/T} \sigma v \\
=& \frac{1}{(2\pi)^6} \int 2\pi^2 E_1 E_2 v \, dE_+ dE_- ds_{\mathrm{M}} \, e^{-E_+/T} \sigma \\
=& \frac{2\pi^2}{(2\pi)^6} \int dE_+ dE_- ds_{\mathrm{M}} \sqrt{\left(p_1^\mu p_{2\mu}\right)^2 - m^4} e^{-E_+/T} \sigma \\
=& \frac{2\pi^2}{(2\pi)^6} \int dE_+ dE_- ds_{\mathrm{M}} \sqrt{\left(\frac{s_{\mathrm{M}}}{2} - m^2\right)^2 - m^4} e^{-E_+/T} \sigma \\
=& \frac{2\pi^2}{(2\pi)^6} \int ds_{\mathrm{M}} dE_+ \left(s_{\mathrm{M}} - 4m^2\right) \sqrt{E_+^2 - s_{\mathrm{M}}} e^{-E_+/T} \sigma \\
=& \frac{T}{32\pi^4} \int_{4m^2}^\infty ds_{\mathrm{M}} \, \left(s_{\mathrm{M}} - 4m^2\right) \sqrt{s_{\mathrm{M}}} \sigma_{1+2\to 3+4} K_1\left(\frac{\sqrt{s_{\mathrm{M}}}}{T}\right) \,.
\end{aligned} \tag{18}
$$

Applying this result, we are able to compute the collision term for the process $\nu_R+\overline{\nu_R}\to\chi+\overline{\chi}$, where ν_R denotes the right-handed neutrino and χ denotes the dark matter fermion. It's necessary for us to calculate the cross-section firstly. ν_R couples to χ through a dark matter scalar ϕ. The interaction terms can be written as

$$\mathcal{L}_{\text{int}} = -y\,\chi\nu_R\phi\,. \tag{19}$$

Therefore, the process $\nu_R+\overline{\nu_R}\to\chi+\overline{\chi}$ has a contribution of t-channel.

The invariant matrix element $\mathcal{M}$ is given by

$$\mathcal{M} = \left(-iy\bar{u}\left(k\right)u\left(p\right)\right)\frac{i}{q^2-m_\phi^2}\left(-iy\bar{v}\left(p'\right)v\left(k'\right)\right)\,. \tag{20}$$

The Mandelstam variables are defined by equations below:

$$s = \left(p+p'\right)^2\,, \tag{21}$$

$$t = \left(p-k\right)^2\,, \tag{22}$$

$$u = \left(p-k'\right)^2\,. \tag{23}$$

Since q can be expressed by

$$q = p-k\,, \tag{24}$$

we can rewrite the equation for $\mathcal{M}$:

$$\mathcal{M} = \left(-iy\bar{u}\left(k\right)u\left(p\right)\right)\frac{i}{t-m_\phi^2}\left(-iy\bar{v}\left(p'\right)v\left(k'\right)\right)\,. \tag{25}$$

The averaged squared amplitude is

$$\begin{aligned}&\frac{1}{4}\sum_{\text{spins}}\left|\mathcal{M}\right|^2\\ =&\frac{y^4}{4\left(t-m_\phi^2\right)^2}\text{tr}\ \left[\left(\not{k}+m_\chi\right)\left(\not{p}+m_{\nu_R}\right)\right]\text{tr}\ \left[\left(\not{k}'-m_\chi\right)\left(\not{p}'-m_{\nu_R}\right)\right]\\ =&y^4\left(\frac{t-m_\chi^2}{t-m_\phi^2}\right)^2\text{(we have asssumed that } m_{\nu_R}=0\text{).}\end{aligned} \tag{26}$$

The cross-section is defined by

$$\frac{d}{d\Omega}\sigma_{1+2\to3+4} = \frac{1}{4E_1E_2v_{\text{Møller}}}\frac{\left|\mathbf{p}_3\right|}{\left(2\pi\right)^2 4E_{\text{cm}}}\left|\mathcal{M}\right|^2\,, \tag{27}$$

$$\frac{d}{d\Omega}\sigma_{2\nu_R\to2\chi} = \frac{y^4}{64\pi^2 s}\sqrt{1-\frac{4m_\chi^2}{s}}\left(\frac{t-m_\chi^2}{t-m_\phi^2}\right)^2\,. \tag{28}$$

$$t = \left(p-k\right)^2 = p^2+k^2-2p\cdot k = m_\chi^2-\frac{s}{2}+\frac{s}{2}\sqrt{1-\frac{4m_\chi^2}{s}}\cos\theta \tag{29}$$

$$
\begin{aligned}
\frac{d}{d\Omega}\sigma_{2\nu_R\to 2\chi} =&\frac{y^4}{64\pi^2 s}\sqrt{1-\frac{4m_\chi^2}{s}}\frac{1}{\left(1+\frac{2(m_\phi^2-m_\chi^2)/s}{1-\sqrt{1-\frac{4m_\chi^2}{s}}\cos\theta}\right)^2} \\
=&\frac{y^4}{64\pi^2 s}\sqrt{1-\frac{4m_\chi^2}{s}}+\mathcal{O}\left(\frac{m_\phi^2-m_\chi^2}{s}\right).
\end{aligned} \tag{30}
$$

$$
\sigma_{2\nu_R\to 2\chi} = \frac{y^4}{16\pi s}\sqrt{1-\frac{4m_\chi^2}{s}}+\mathcal{O}\left(\frac{m_\phi^2-m_\chi^2}{s}\right) \tag{31}
$$

Now, we just need to replace the σ the collision term with this express. For the 0-order approximation, we have

$$
\begin{aligned}
C_{2\nu_R\to 2\chi} =&\frac{y^4 T}{512\pi^5}\int_{4m_\chi^2}^{\infty} ds\,\sqrt{s-4m_\chi^2}K_1\left(\frac{\sqrt{s}}{T}\right) \\
=&\frac{y^4 T^4}{512\pi^5}\int_{\frac{4m_\chi^2}{T^2}}^{\infty} dx\,\sqrt{x-\frac{4m_\chi^2}{T^2}}K_1\left(\sqrt{x}\right) \\
=&\frac{y^4}{128\pi^5}\left[m_\chi T K_1\left(\frac{m_\chi}{T}\right)\right]^2
\end{aligned} \tag{32}
$$

For high- and low-temperature limits:

$$
\lim_{T\to\infty} C_{2\nu_R\to 2\chi} \approx \frac{y^4}{128\pi^5}T^4\,, \tag{33}
$$

$$
\lim_{T\to 0} C_{2\nu_R\to 2\chi} \approx \frac{y^4}{256\pi^4}T^3 m_\chi e^{-2m_\chi/T}\,. \tag{34}
$$

Monte Carlo Method

To obtain the collision term, a general way and almost useful in every situation is calculating it via numerical method, especially the *Monte Carlo Method.* Two related references [4, 2] give the methods. The phase space

$$
d\Phi_n \equiv (2\pi)^4\,\delta^{(4)}\left(p-\sum_{i=1}^{n}p_i\right)\prod_{i=1}^{n}\frac{d^3 p_i}{(2\pi)^3\,2E_i}\,. \tag{35}
$$

Using the technology of induction, one can obtain that

$$
\int d\Phi_n = \frac{1}{(2\pi)^{3n-4}}\int_{\mu_{n-1}}^{M_n-m_n} dM_{n-1}\cdots\int_{\mu_2}^{M_3-m_3} dM_2\left(\int\prod_{i=1}^{n-1}d\Omega_i\right)\frac{1}{2^n M_n}\prod_{i=2}^{n}P_i\,, \tag{36}
$$

where

$$
P_i \equiv \frac{\sqrt{\lambda\left(M_i^2, M_{i-1}^2, m_i^2\right)}}{2M_i}\,, \tag{37}
$$

$$
\lambda\left(x,y,z\right) \equiv \left(x-y-z\right)^2-4yz\,, \tag{38}
$$

$$\mu_i \equiv \sum_{j=1}^{i} m_j \, , \tag{39}$$

and $d\Omega_i$ is defined as the differential solid angle in the rest frame of $k_{i+1} \equiv \sum_{j=1}^{i+1} p_j$. More generally, the phase space for $m \to n$ process is

$$\prod_{i=1}^{m} \frac{d^3 p_i}{(2\pi)^3 \, 2E_i} \prod_{j=1}^{n} \frac{d^3 p'_j}{(2\pi)^3 \, 2E'_j} (2\pi)^4 \, \delta^4 \left(p_{in} - p_{out}\right) \tag{40}$$

By introducing the parameter

$$\kappa = \sum_{i}^{m} p_i \tag{41}$$

The phase space can be written as

$$\int d^4\kappa \prod_{i=1}^{m} \frac{d^3 p_i}{(2\pi)^3 \, 2E_i} \delta^4 \left(\kappa - \sum_{i}^{m} p_i\right) \theta\left(\kappa^2 - {\mu'}_n^2\right) d\Phi'_n(\kappa) \ . \tag{42}$$

It is natural to rewrite the integration as

$$\frac{1}{(2\pi)^4} \int d^4\kappa \, d\Phi_m(\kappa) \, \theta\left(\kappa^2 - {\mu'}_n^2\right) \theta\left(\kappa^2 - \mu_m^2\right) d\Phi'_n(\kappa) \ . \tag{43}$$

Now, applying this result for a $2 \to 2$ process, the collision term under Boltzmann approximation is

$$\begin{aligned} C^n(T) =& \frac{1}{(2\pi)^4} \int d^4\kappa \, d\Phi_2(\kappa) \, \theta\left(\kappa^2 - \mu_2^2\right) \theta\left(\kappa^2 - {\mu'}_2^2\right) d\Phi'_2(\kappa) \exp\left[-\left(E_1 + E_2\right)/T\right] |\mathcal{M}|^2 \\ =& \frac{1}{(2\pi)^4} \int d^4\kappa \int \frac{d\Omega_1}{(2\pi)^2} \frac{\sqrt{\lambda\left(\kappa^2, m_1^2, m_2^2\right)}}{8\kappa^2} \int \frac{d\Omega'_1}{(2\pi)^2} \frac{\sqrt{\lambda\left(\kappa^2, {m'}_1^2, {m'}_2^2\right)}}{8\kappa^2} \\ & \times \theta\left(\kappa^2 - \mu_2^2\right) \theta\left(\kappa^2 - {\mu'}_2^2\right) \exp\left[-\kappa^0/T\right] |\mathcal{M}|^2 \end{aligned} \tag{44}$$

Example 1. $2\nu \to 2e$

A typical squared amplitude of neutrino annihilation is[3]:

$$|\mathcal{M}|^2 = G\left(p_1 \cdot p_2\right)\left(p'_1 \cdot p'_2\right) \, , \tag{45}$$

which is equivalently

$$|\mathcal{M}|^2 = \frac{G}{4}\left(\kappa^2 - m_1^2 - m_2^2\right)\left(\kappa^2 - {m'}_1^2 - {m'}_2^2\right) \ . \tag{46}$$

Learning from the results, we understand that our Monte Carlo integration code may be correct, except for 10% of the coefficient. I have thought of a possible reason. Assuming that the samples are evenly distributed in the phase space, there are only about 30 kinds of assignments for each

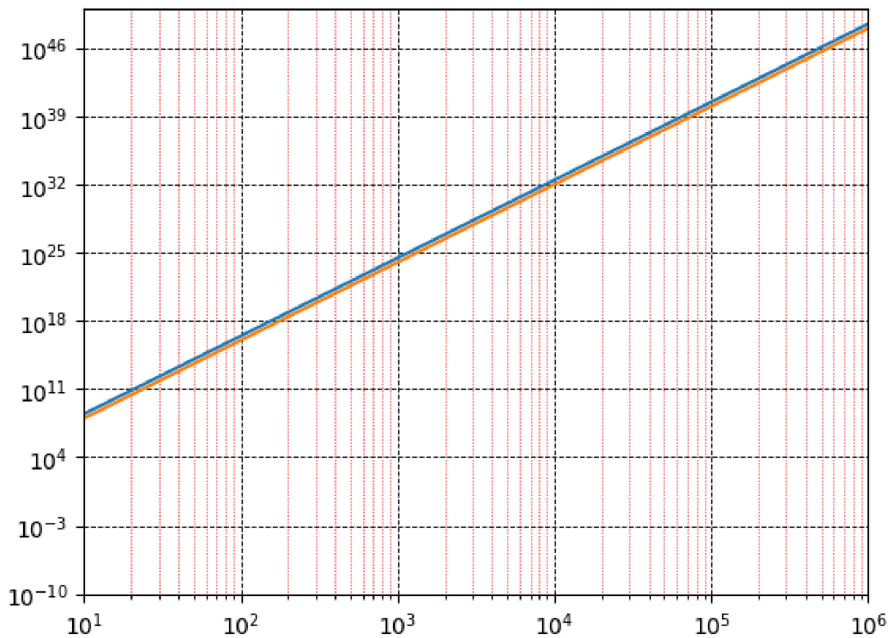

Figure 1: Calculated from 4D Monte Carlo. Apparently, the numeric result shows the power law correctly. However, the coefficient is about 10% of $3/\left(8\pi^5\right)$, which is obtained with 10^6 samples generated.

coordinate. Considering that the integrand does not explicitly contain the angles, the collision term is then

$$\begin{aligned} C^n\left(T\right) =& \frac{\Omega_{\vec{\kappa}}}{\left(2\pi\right)^4}\int dE k^2 dk\, \frac{\sqrt{\lambda\left(\kappa^2, m_1^2, m_2^2\right)}}{8\pi\kappa^2}\frac{\sqrt{\lambda\left(\kappa^2, {m'}_1^2, {m'}_2^2\right)}}{8\pi\kappa^2} \\ &\times\theta\left(\kappa^2-\mu_2^2\right)\theta\left(\kappa^2-{\mu'}_2^2\right)\exp\left[-E/T\right]\left|\mathcal{M}\right|^2 \end{aligned} \tag{47}$$

Later, we find that the 4D spherical Monte Carlo integration fits the analytic results well. And it is equivalent to 2D Monte Carlo integration. And we find that there are only $\frac{4\pi}{3}\times\frac{1}{8}\times\frac{1}{4}\simeq 13\%$ samples that are valid and should be taken into account in the Cartesian 4D Monte Carlo integration, which is restricted by $\kappa^2>0$. These condition may make the integration converge slow.

Example 2. $2\nu_R\to 2\chi$

With our derived formula, we can test whether our Monte Carlo method agree with the eq.(32) .

$$C^n_{2\nu_R\to2\chi}\left(T\right)=\frac{1}{\left(2\pi\right)^4}\int d^4\kappa\, d\Phi_2\left(\kappa\right)\theta\left(\kappa^2-\mu_2^2\right)\theta\left(\kappa^2-{\mu'}_2^2\right)d\Phi'_2\left(\kappa\right)\exp\left[-\left(E_1+E_2\right)/T\right]\left|\mathcal{M}\right|^2\ , \tag{48}$$

where

$$\left|\mathcal{M}\right|^2=y^4\,,\qquad \text{for } m_\phi=m_\chi\ . \tag{49}$$

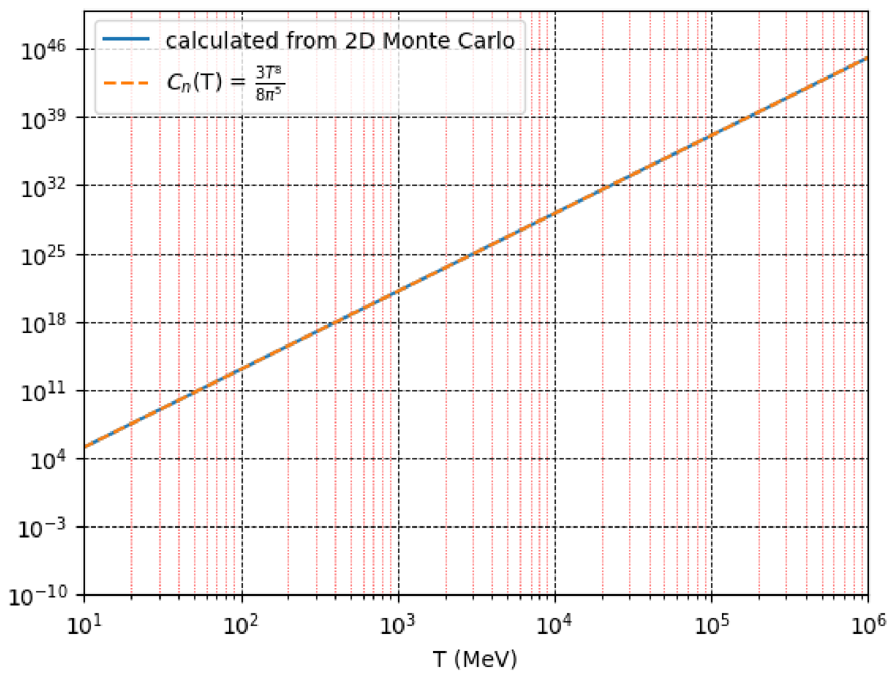

Figure 2: Obtained from 2D Monte Carlo integration

$$
\begin{aligned}
C^{n}_{2\nu_R\to 2\chi}(T) =& \frac{|\mathcal{M}|^2}{(2\pi)^4}\int d^4\kappa \int \frac{d\Omega_1}{(2\pi)^2}\frac{\sqrt{\lambda\left(\kappa^2,0,0\right)}}{8\kappa^2}\int \frac{d\Omega_1'}{(2\pi)^2}\frac{\sqrt{\lambda\left(\kappa^2,m_\phi^2,m_\phi^2\right)}}{8\kappa^2} \\
&\times\theta\left(\kappa^2-(2m_\phi)^2\right)\exp\left[-\kappa^0/T\right] \\
=& \frac{|\mathcal{M}|^2}{(2\pi)^4}\int d^4\kappa\,\frac{\sqrt{\lambda\left(\kappa^2,0,0\right)}}{8\pi\kappa^2}\frac{\sqrt{\lambda\left(\kappa^2,m_\phi^2,m_\phi^2\right)}}{8\pi\kappa^2}\theta\left(\kappa^2-(2m_\phi)^2\right)\exp\left[-\kappa^0/T\right]
\end{aligned}
\tag{50}
$$

Example 3. semi-Compton scattering

The amplitude of $e^- + \gamma \to e^- + Z'$ is relatively complex to compute the collision term analytically. It can be a good example to apply our Monte Carlo method.

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{16\pi\alpha g_{Z'}^2\left(2+r_{m,e}\right)}{3}\left(\frac{m_e^2}{2p_1\cdot p_2}+\frac{m_e^2}{m_{Z'}^2-2p_1\cdot k_2}\right)\left(\frac{m_e^2}{2p_1\cdot p_2}+\frac{m_e^2}{m_{Z'}^2-2p_1\cdot k_2}+1\right) \\
&-\frac{16\pi\alpha g_{Z'}^2\left(2+r_{m,e}\right)}{3}\frac{r_{m,e}m_e^4}{\left(2p_1\cdot p_2\right)\left(m_{Z'}^2-2p_1\cdot k_2\right)}-\frac{8\pi\alpha g_{Z'}^2}{3}\left(\frac{2p_1\cdot p_2}{m_{Z'}^2-2p_1\cdot k_2}+\frac{m_{Z'}^2-2p_1\cdot k_2}{2p_1\cdot p_2}\right)
\end{aligned}
\tag{51}
$$

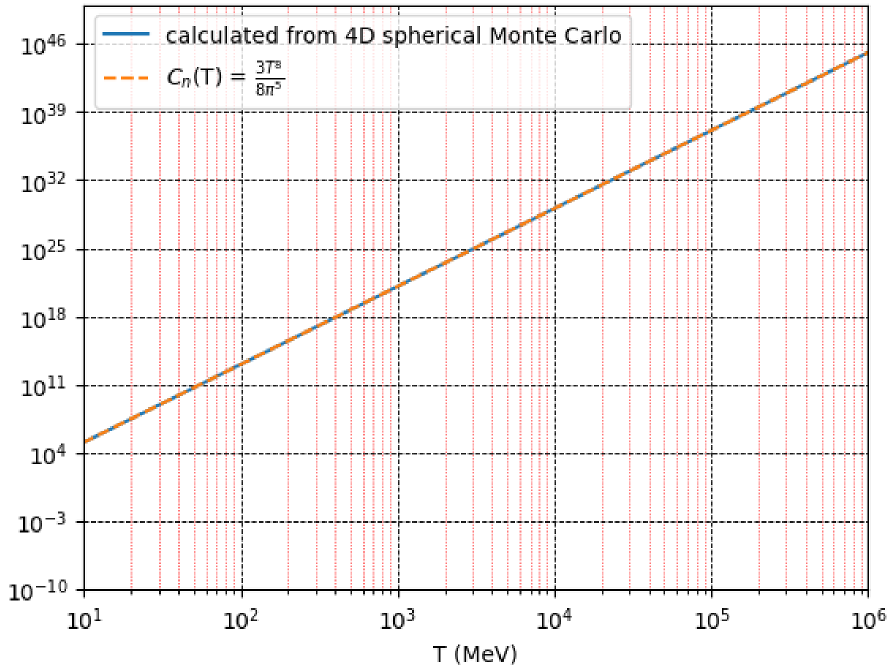

Figure 3: Obtained from 4D spherical Monte Carlo integration

$$
\begin{aligned}
C_s^n(T) =& \frac{1}{(2\pi)^4}\int d^4\kappa\, d\Phi_2(\kappa)\,\theta\left(\kappa^2-\mu_2^2\right)\theta\left(\kappa^2-{\mu'}_2^2\right) d\Phi_2'(\kappa)\exp\left[-\left(E_1+E_2\right)/T\right]|\mathcal{M}|^2 \\
=& \frac{\Omega_{\tilde{\kappa}}}{(2\pi)^4}\int d\kappa^0\tilde{\kappa}^2 d\tilde{\kappa}\,\frac{1}{\pi}\frac{\sqrt{\lambda\left(\kappa^2,m_1^2,m_2^2\right)}}{8\kappa^2}\int\frac{d\cos\theta}{2\pi}\frac{\sqrt{\lambda\left(\kappa^2,{m'}_1^2,{m'}_2^2\right)}}{8\kappa^2} \\
&\times\theta\left(\kappa^2-{\mu_2'}^2\right)\exp\left[-\kappa^0/T\right]|\mathcal{M}|^2 \\
=& \frac{\Omega_{\tilde{\kappa}}}{(2\pi)^6}\int_{\mu_2'}^{\infty} dE\int_{{\mu_2'}^2}^{E^2} ds\,\left(E^2-s\right)^{1/2}\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{8s}\int_{-1}^{1} d\cos\theta\,\frac{\sqrt{\lambda\left(s,{m'}_1^2,{m'}_2^2\right)}}{8s} \\
&\times\exp\left[-E/T\right]|\mathcal{M}|^2
\end{aligned}
\tag{52}
$$

The kinematic parameters is given by

$$2p_1\cdot p_2 = s - m_e^2 = \kappa^2 - m_e^2 \tag{53}$$

$$p_1\cdot k_2 = E_1E_2' - PK\cos\theta \tag{54}$$

$$E_1 = \sqrt{P^2+m_e^2}\,, E_2' = \sqrt{K^2+m_{Z'}^2} \tag{55}$$

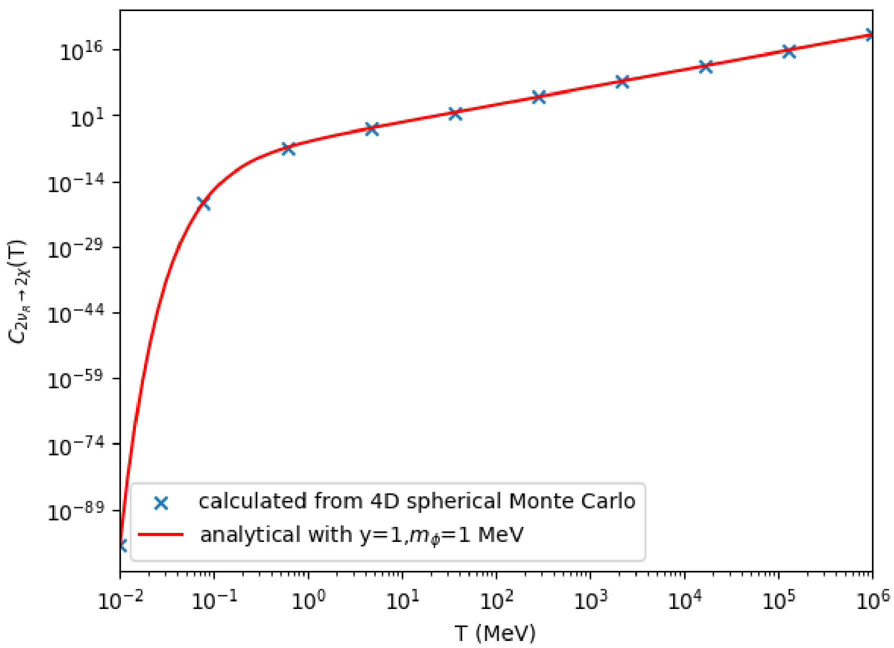

Figure 4: Numerical calculation of eq.(32) via 4D spherical Monte Carlo integration

$$P = \frac{\sqrt{\lambda\left(s, m_e^2, 0\right)}}{2\sqrt{s}}\,, \qquad K = \frac{\sqrt{\lambda\left(s, m_e^2, m_{Z'}^2\right)}}{2\sqrt{s}} \tag{56}$$

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