The N_{eff} bound on the dark photon model

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Dark photon is a dark neutral gauge boson arising from $U(1)$ extensions of the Standard Model(SM). In a generic framework, the SM with extension is invariant under $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_X$, where X denote a new gauge boson[2]. The interaction between dark photon and SM particles is basically through its kinetic mixing with photon. The corresponding Lagrangian reads

$$\mathcal{L} \supset -\frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}\hat{B}_{\mu\nu}\hat{B}^{\mu\nu} - \frac{1}{4}\hat{X}_{\mu\nu}\hat{X}^{\mu\nu} - \frac{\epsilon}{2}\hat{B}_{\mu\nu}\hat{X}^{\mu\nu} + \sum_{\psi}\overline{\psi}\gamma^{\mu}iD_{\mu}\psi\,, \tag{1}$$

where W, B denote the $SU(2)_L$ and $U(1)_Y$ gauge field strength tensors.The covariant derivative is defined as

$$D_{\mu} = \partial_{\mu} - ig\sum_{a=1}^{3}\frac{\sigma_a}{2}W_{a,\mu} - ig'Q_Y\hat{B}_Y - ig_{Z'}Q_X\hat{X}_{\mu}\,, \tag{2}$$

where Q_Y and Q_X denote the charge of ψ under $U(1)_Y$ and $U(1)_X$.

Because our Lagrangian is still not diagonal, the X_{μ} is not physical vector. The vector

$$\hat{V} = \left(W_1, W_2, W_3, \hat{B}, \hat{X}\right)^T \tag{3}$$

is transformed to make Lagrangian diagonal by $\mathbb{L}$, and mass eigenstates by $\mathbb{O}$, which can be regarded as a generalized Weinberg rotation in the $SU(2)_L \times U(1)_Y \times U(1)_X$ model with two angles:

$$\mathbb{L} = \begin{pmatrix} \mathbb{I}_3 & & \\ & 1 & \epsilon \\ & 0 & \sqrt{1-\epsilon^2} \end{pmatrix}\,, \quad \mathbb{O} = \begin{pmatrix} \mathbb{I}_2 & & & \\ & s_W & c_W & 0 \\ & c_W c_\theta & -s_W c_\theta & -s_\theta \\ & c_W s_\theta & -s_W s_\theta & c_\theta \end{pmatrix}\,. \tag{4}$$

After transformation the physical vector is

$$V_{\text{ph}} = \left(W_1, W_2, A, Z, Z'\right)^T\,. \tag{5}$$

Now we are concerned with the interactions of A, Z, and $Z\prime$.The related Lagrangian is

$$\mathcal{L} \supset eQ_{\text{em}}^{(\psi)}\bar{\psi}\not{A}\psi + g_Z^{(\psi)}\bar{\psi}\not{Z}\psi + g_{Z'}^{(\psi)}\bar{\psi}\not{Z}'\psi\,. \tag{6}$$

The effective couplings $g_Z^{(\psi)}$ and $g_{Z'}^{(\psi)}$ are given by

$$g_Z^{(\psi)} = c_\theta g\left[c_W Q_{\text{em}}^{(\psi)} - \frac{Q_Y^{(\psi)}}{c_W}\right] - s_\theta\left[g_{Z'}Q_X^{(\psi)} - gQ_Y^{(\psi)}\frac{s_W}{c_W}\epsilon\right] + O\left(\epsilon^2\right) \tag{7}$$

$$g_{Z'}^{(\psi)} = s_\theta g \left[c_W Q_{\text{em}}^{(\psi)} - \frac{Q_Y^{(\psi)}}{c_W} \right] + c_\theta \left[g_{Z'} Q_X^{(\psi)} - g Q_Y^{(\psi)} \frac{s_W}{c_W} \epsilon \right] + O\left(\epsilon^2\right) \tag{8}$$

Regarding that[2]

$$\theta \simeq \tan\theta = \frac{\epsilon m_Z^2 s_W}{m_{Z'}^2 - m_Z^2} + \mathcal{O}\left(\epsilon^2\right), \tag{9}$$

we have

$$g_Z \simeq \frac{g}{c_W} \left(Q_{\text{em}}^{(\psi)} - Q_Y^{(\psi)} - s_W^2 Q_{\text{em}}^{(\psi)} \right) \tag{10}$$

and

$$g_{Z'} \simeq g_X^{(\psi)} - \epsilon g s_W \frac{c_W^2 Q_{\text{em}}^{(\psi)} - Q_Y r_m}{c_W \left(1 - r_m\right)}, \tag{11}$$

where we defined that Q_Y as half of the hypercharge and $r_m \equiv m_{Z'}^2/m_Z^2$, for example $Q_Y^{e_L} = -\frac{1}{2}$ and $Q_Y^{e_R} = -1$.

To produce dark photon below 1 MeV(it means the lightest SM charged fermion can not produce a single Z' by annihilation), the main possible channel is

$$\text{A (nnihiltion)} : e^- + e^+ \to \gamma + Z' \tag{12}$$

and

$$\text{B (remsstrahlung)} : e^- + \gamma \to Z' + e^-. \tag{13}$$

The squared amplitude for situation A($p_1 + p_2 \to k_1 + k_2$) is given by

$$\begin{aligned} \frac{1}{12} \sum_{\text{spins}} |\mathcal{M}|^2 &= -\frac{4e^2 g_{Z'}^2 \left(2m_e^2 + m_{Z'}^2\right)}{3} \left[\frac{m_e^2}{\Delta t^2} + \frac{1}{\Delta t} + \frac{m_e^2}{\Delta u^2} + \frac{1}{\Delta u} + \frac{2m_e^2 - m_{Z'}^2}{\Delta t \Delta u} \right] + \frac{2e^2 g_{Z'}^2}{3} \left(\frac{\Delta t}{\Delta u} + \frac{\Delta u}{\Delta t} \right) \\ &= -\frac{4e^2 g_{Z'}^2 \left(2 + r_{m,e}\right)}{3} \left[\left(\frac{1}{t_e} + \frac{1}{u_e} \right) \left(\frac{1}{t_e} + \frac{1}{u_e} + 1 \right) - \frac{r_{m,e}}{t_e u_e} \right] + \frac{2e^2 g_{Z'}^2}{3} \left(\frac{t_e}{u_e} + \frac{u_e}{t_e} \right). \end{aligned} \tag{14}$$

Here, we have defined that

$$r_{m,e} \equiv m_{Z'}^2/m_e^2, \tag{15}$$

$$s_e \equiv s/m_e^2 \tag{16}$$

$$t_e \equiv \Delta t/m_e^2 = \left(p_1 - k_1\right)^2/m_e^2 - 1 = -2p_1 \cdot k_1 = -\frac{s_e - r_{m,e}}{2} \left(1 - \sqrt{1 - \frac{4}{s_e}} \cos\theta \right) \tag{17}$$

$$u_e \equiv \Delta u/m_e^2 = \left(p_1 - k_2\right)^2/m_e^2 - 1 = -\frac{s_e - r_{m,e}}{2} \left(1 + \sqrt{1 - \frac{4}{s_e}} \cos\theta \right) \tag{18}$$

The collision term is

$$C_{\text{A}}^n = \int d\Pi_1 d\Pi_2 d\Pi_1' d\Pi_2' f_1^{\text{eq}} f_2^{\text{eq}} \left|\mathcal{M}\right|^2 (2\pi)^4 \delta^{(4)} \left(p_1^\mu + p_2^\mu - k_1^\mu - k_2^\mu\right), \tag{19}$$

where in our concerning situation

$$f_i^{\text{eq}} = N_e e^{-E_i/T}, \tag{20}$$

and we defined that

$$d\Pi_i = \frac{d^3 p_i}{(2\pi)^3 2E_i}\,, d\Pi_i' = \frac{d^3 k_i}{(2\pi)^3 2E_i'}\,. \tag{21}$$

First, we can integrate with respect to the final states to obtain the invariant cross section:

$$\begin{aligned}
\sigma =& \frac{1}{4E_1E_2v}\int d\Pi_1' d\Pi_2' (2\pi)^4 \delta^{(4)}\left(p_1^\mu + p_2^\mu - k_1^\mu - k_2^\mu\right)|\mathcal{M}|^2 \\
=& \frac{1}{4E_1E_2v}\int d\Pi_k \frac{\pi}{\sqrt{m_{Z'}^2 + \boldsymbol{k}^2}}\delta\left(E_1 + E_2 - E_1' - E_2'\right)|\mathcal{M}|^2 \left(\boldsymbol{k}_2 = -\boldsymbol{k}_1 \equiv \boldsymbol{k}\right) \\
=& \frac{1}{16\pi^2 sv}\int \frac{kdkd\Omega}{\sqrt{m_{Z'}^2 + k^2}}\delta\left(E_1 + E_2 - E_1' - E_2'\right)|\mathcal{M}|^2 \\
=& \frac{s-r}{32\pi^2 s^2 v}\int d\Omega\,|\mathcal{M}|^2 \\
=& \frac{\pi\alpha}{3s_e^2\left(s_e - r_{m,e}\right)v\,m_e^2}\Big[-r_{m,e}^2 + \left(-8 - 4r_{m,e} + r_{m,e}^2\right)s_e^{1/2} + 2r_{m,e}s_e + \left(2 - r_{m,e}\right)s_e^{3/2} \\
&- s_e^2 + s_e^{5/2}\Big]
\end{aligned} \tag{22}$$

The collision term is then

$$C_{\mathrm{A}}^n = \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3} f_1^{\mathrm{eq}} f_2^{\mathrm{eq}} \sigma v \equiv \int d\Lambda_1 d\Lambda_2 f_1^{\mathrm{eq}} f_2^{\mathrm{eq}}\,\sigma v\,. \tag{23}$$

According to Gelmini's method[1]:

$$d\Lambda_1 d\Lambda_2 = \frac{1}{(2\pi)^6} 4\pi p_1 E_1 dE_1\, 4\pi p_2 E_2 dE_2\, \frac{1}{2} d\cos\theta\,.$$

Defining that

$$\begin{aligned}
E_+ = E_1 + E_2\,, \qquad E_- = E_1 - E_2\,, \\
s = 2m^2 + 2E_1E_2 - 2p_1p_2\cos\theta\,,
\end{aligned} \tag{24}$$

we know that

$$dE_+ dE_- = 2dE_1 dE_2\,.$$

Because E_+, E_- is not explicit function of θ, we just need to be concerned about $\partial s/\partial\theta$. Therefore,

$$d\Lambda_1 d\Lambda_2 = \frac{1}{(2\pi)^6} 2\pi^2 E_1 E_2 dE_+ dE_- ds\,. \tag{25}$$

The integration limits can be solved by the MMA Reduce function, which are

$$\begin{aligned}
s \geq 4m_e^2 \\
E_+ \geq \sqrt{s} \\
|E_-| \leq \sqrt{1 - 4m_e^2/s}\sqrt{E_+^2 - s}
\end{aligned} \tag{26}$$

Therefore,

$$
\begin{aligned}
C_{\mathrm{A}}^{n} =&\frac{1}{(2\pi)^6}\int 2\pi^2 E_1E_2\, dE_+dE_-ds\, e^{-E_+/T}\sigma v\\
=&\frac{1}{64\pi^4}\int s\sqrt{1-4m_e^2/s}\sqrt{E_+^2-s}\, dE_+e^{-E_+/T}ds\,\sigma v\\
=&\frac{T}{64\pi^4}\int s^{3/2}\sqrt{1-4m_e^2/s}K_1\left(\frac{\sqrt{s}}{T}\right)\sigma v\, ds\\
\simeq&\frac{\alpha g_{Z'}^2 m_e^4 T_e^3}{768\pi^2}e^{-2/T_e}\left(8-4r_{m,e}+\left(30+3r_{m,e}\right)T_e-\left(24+2r_{m,e}\right)T_e^2\right)\text{(low temperature)}\ ,\\
\simeq&\frac{\alpha g_{Z'}^2 m_e^4 T_e^5}{64\pi^2}\qquad\text{(high temperature)}\ ,
\end{aligned}
\tag{27}
$$

where

$$T_e \equiv T/m_e\,. \tag{28}$$

The inverse of annihilation situation is a bit different since produced photon have different temperature with dark photon. But, if we postulate that dark photon share the temperature with photon at initial time, we will get to similar result:

$$C_{\mathrm{A}}^{n}\left(T\right)=C_{\mathrm{inv(A)}}^{n}\left(T\right) \tag{29}$$

As for B situation, the squared amplitude can be obtained by crossing symmetry

$$\left|\mathcal{M}\left(e^-+\gamma\to Z'+e^-\right)\right|^2\xrightarrow[p_2\to -k_1]{k_1\to -p_2}-\left|\mathcal{M}\left(e^-+e^+\to Z'+\gamma\right)\right|^2\ , \tag{30}$$

where the minus is arising from the external line of fermion.

$$
\begin{aligned}
&\left|\mathcal{M}\left(e^-+\gamma\to Z'+e^+\right)\right|^2\\
=&-\frac{4e^2g_{Z'}^2\left(2+r_{m,e}\right)}{3}\left[\left(\frac{1}{s_e-1}+\frac{1}{u_e}\right)\left(\frac{1}{s_e-1}+\frac{1}{u_e}+1\right)-\frac{r_{m,e}}{\left(s_e-1\right)u_e}\right]+\frac{2e^2g_{Z'}^2}{3}\left(\frac{s_e-1}{u_e}+\frac{u_e}{s_e-1}\right)\,.
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
u_e =&\left(p_1-k_2\right)^2/m_e^2-1=r_{m,e}-2p_1\cdot k_2\\
=&r_{m,e}+\frac{1}{2s_e}\left[-\left(1+s_e\right)\left(r_{m,e}+s_e-1\right)+\left(1-s_e\right)\cos\theta\sqrt{1+\left(s_e-r_{m,e}\right)^2-2\left(r_{m,e}+s_e\right)}\right]
\end{aligned}
\tag{32}
$$

Now, we try to produce the corresponding collision term

$$
\begin{aligned}
\sigma =&\frac{1}{4E_1E_2v}\int d\Pi_1' d\Pi_2'\left(2\pi\right)^4\delta^{(4)}\left(p_1^\mu+p_2^\mu-k_1^\mu-k_2^\mu\right)\left|\mathcal{M}\right|^2\\
=&\frac{1}{4E_1E_2v}\int d\Pi_k\frac{\pi}{\sqrt{m_{Z'}^2+\boldsymbol{k}^2}}\delta\left(E_1+E_2-E_1'-E_2'\right)\left|\mathcal{M}\right|^2\left(\boldsymbol{k}_2=-\boldsymbol{k}_1\equiv\boldsymbol{k}\right)\\
=&
\end{aligned}
\tag{33}
$$

$$C_{\mathrm{B}}^{n}=\int d\Pi_1 d\Pi_2 d\Pi_1' d\Pi_2' f_1^{\mathrm{eq}} f_2^{\mathrm{eq}} \left|\mathcal{M}\right|^2 (2\pi)^4 \delta^{(4)}\left(p_1^\mu+p_2^\mu-k_1^\mu-k_2^\mu\right)\,, \tag{34}$$

The Boltzmann equation of A situation is given by

$$\frac{dn_{Z'}}{dt}+3Hn_{Z'}=N_e N_\gamma C_{\mathrm{A}}^{n}(T)\left(e^{2\mu_e/T}-e^{\mu_{Z'}/T}\right)\,, \tag{35}$$

or equivalently,

$$\frac{dn_{Z'}a^3}{da}=\frac{a^2}{H}N_e N_\gamma C_{\mathrm{A}}^{n}(T)\left(e^{2\mu_e/T}-e^{\mu_{Z'}/T}\right)\,, \tag{36}$$

for simplicity, we have taken that

$$a=\frac{1}{T} \tag{37}$$

The number density of DP (dark photon) in equilibrium is given by

$$n_{Z'}=\delta_{\mathrm{BE}}^{n} e^{\mu_{Z'}/T} n_B^{(0)}\left(m_{Z'},T\right)\,, \tag{38}$$

where $n_B^{(0)}$ denote the number density of Boltzmann massive particle with zero chemical potential

$$n_B^{(0)}\left(m_{Z'},T\right)=\frac{m_{Z'}^2 T K_2\left(\frac{m_{Z'}}{T}\right)}{2\pi^2} \tag{39}$$

and the constant δ_{BE}^{n}(actually varying with temperature) is used to denote the difference between BE distribution and MB distribution:

$$\delta_{\mathrm{BE}}^{n}=\zeta(3)\simeq 1.2\,. \tag{40}$$

According to reference[2], the coupling between neutrinos and DP is much lower than electron and DP. So we are only concerned about the energy delivered to EM plasma which may cause the deviation of N_{eff}. The Boltzmann equations of our concerning particles are

$$\frac{d\rho_{Z'}}{dt}+4H\rho_{Z'}=\left(\rho_{Z'}-3P_{Z'}\right)H+N_e N_\gamma\left(C_{\mathrm{A}}^{\rho}-C_{\mathrm{inv(A)}}^{\rho}\right) \tag{41}$$

$$\frac{d\rho_e}{dt}+4H\rho_e=\left(\rho_e-3P_e\right)H-N_{Z'}N_\gamma\left(C_{\mathrm{A}}^{\rho}-C_{\mathrm{inv(A)}}^{\rho}\right)+N_\nu\left(C_{\nu\nu\to ee}^{\rho}-C_{\nu\nu\leftarrow ee}^{\rho}\right) \tag{42}$$

$$\frac{d\rho_\nu}{dt}+4H\rho_\nu=-N_e\left(C_{\nu\nu\to ee}^{\rho}-C_{\nu\nu\leftarrow ee}^{\rho}\right) \tag{43}$$

Defining that

$$\rho_{\mathrm{inv}}=N_{Z'}\rho_{Z'}+N_e\rho_e+N_\nu\rho_\nu, \tag{44}$$

where

$$N_{Z'}=3\,,\qquad N_e=2\,,\qquad N_\nu=3, \tag{45}$$

we can obtain that

$$\frac{d\rho_{\mathrm{inv}}}{dt}+4H\rho_{\mathrm{inv}}=N_{Z'}\left(\rho_{Z'}-3P_{Z'}\right)H+N_e\left(\rho_e-3P_e\right)H\,, \tag{46}$$

which is equivalently

$$\frac{d\rho_{\text{inv}}a^4}{da} = a^3\left[N_{Z'}\left(\rho_{Z'} - 3P_{Z'}\right) + N_e\left(\rho_e - 3P_e\right)\right] . \tag{47}$$

As massive particles nearly loss all of its energy when universe expands and gets colder and colder, we can regard ρ_{inv} as the energy density of neutrinos. Therefore,

$$\begin{aligned} N_{\text{eff}} =& \frac{\rho_{\text{inv}}a^4}{\rho_\nu^{\text{st}}a^4} = 3 + \frac{1}{\rho_\nu^{\text{st}}a^4}\left[\int_0^{a_{\text{dec},\nu}} a^3 da\, N_{Z'}\left(\rho_{Z'} - 3P_{Z'}\right) + \int_{a_{\text{dec},\nu}}^{\infty} a^3 da\, N_{Z'}\left(\rho_{Z'} - 3P_{Z'}\right)\right] \\ &+ \text{contribution of electron}\,, \end{aligned} \tag{48}$$

where

$$\rho_\nu^{\text{st}} = \frac{7\pi^2 T_\nu^4}{240} \tag{49}$$

The contribution from DP is

$$\Delta N_{\text{eff}} = \frac{1}{\rho_\nu^{\text{st}}a^4}\left[\int_0^{a_{\text{dec},\nu}} a^3 da\, N_{Z'}\left(\rho_{Z'} - 3P_{Z'}\right) + \int_{a_{\text{dec},\nu}}^{\infty} a^3 da\, N_{Z'}\left(\rho_{Z'} - 3P_{Z'}\right)\right] \tag{50}$$

Because the energy transferred to neutrino from DP is through EM plasma as a medium, we can expect that ΔN_{eff} will barely change after neutrino's decoupling, which means

$$\Delta N_{\text{eff}} = \frac{\int_0^{a_{\text{dec},\nu}} a^3 da\, N_{Z'}\left(\rho_{Z'} - 3p_{Z'}\right)}{\rho_\nu^{\text{st}}a^4} . \tag{51}$$

The energy and pressure density are given by

$$\rho_{Z'} = \delta_{\text{BE}}^{\rho} e^{\mu_{Z'}/T} \frac{m_{Z'}^3 T K_1\left(\frac{m_{Z'}}{T}\right) + 3m_{Z'}^2 T^2 K_2\left(\frac{m_{Z'}}{T}\right)}{2\pi^2} , \tag{52}$$

$$p_{Z'} = \delta_{\text{BE}}^{p} e^{\mu_{Z'}/T} \frac{m_{Z'}^2 T^2 K_2\left(\frac{m_{Z'}}{T}\right)}{2\pi^2} , \tag{53}$$

where

$$\delta_{\text{BE}}^{\rho} = \delta_{\text{BE}}^{p} \simeq 1.08 . \tag{54}$$

$$\Delta N_{\text{eff}} = \frac{\delta_{\text{BE}}^{\rho} N_{Z'} m_{Z'} \int_0^{a_{\text{dec},\nu}} da\, Y_{Z'} K_1\left(\frac{m_{Z'}}{T}\right)/K_2\left(\frac{m_{Z'}}{T}\right)}{\delta_{\text{BE}}^{n}\rho_\nu^{\text{st}}a^4} , \tag{55}$$

where

$$Y_{Z'} \equiv n_{Z'}a^3 . \tag{56}$$

The result produced from the above dilution-resistant effect is kind of out of our expectation. Moreover, there might be other contributions in addition to the dilution-resistant effect.

First of all, we try to check the number density to understand the behavior of DP under the kinetic mixing.

Noticing that eq.(36) can be written as

$$\frac{dY_{Z'}}{da} = \text{collision term} \propto a^4 C_{\text{A}}^{n}\left(T\right)\left(1 - \frac{Y_{Z'}}{Y_{Z'}^{\text{eq}}}\right) \tag{57}$$

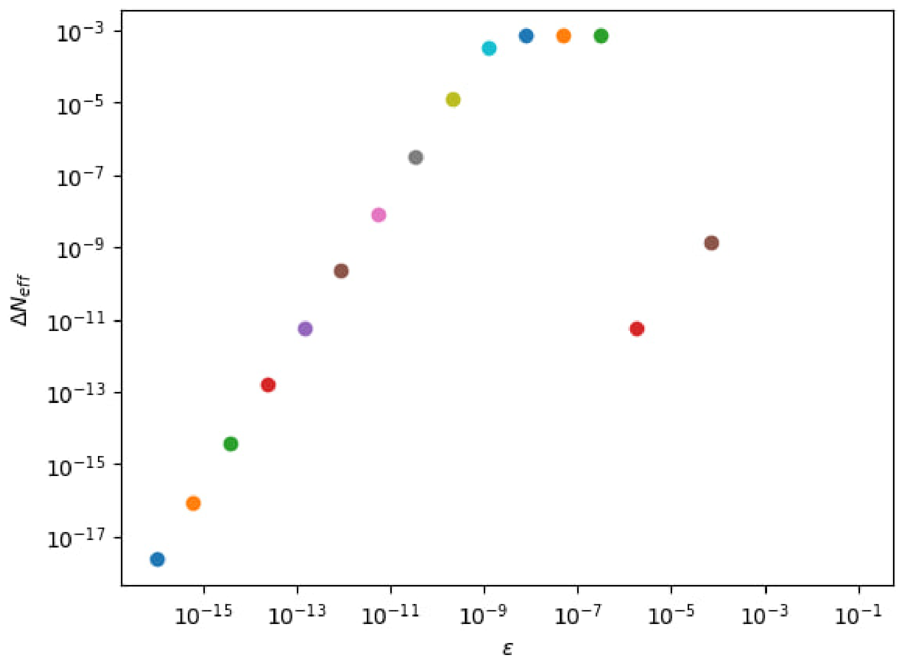

Figure 1: $N_{\rm eff}$ produced from the eq.(55)

For high temperature, the approximation of the collision term is

$$\text{collision term} \propto g_{Z'}^2 \left(m_e a\right)^{-1} \left(1 - \frac{Y_{Z'}}{Y_{Z'}^{\rm eq}}\right) , \tag{58}$$

here,

$$Y_{Z'}^{\rm eq} \simeq \frac{\delta_{\rm BE}^n}{\pi^2} . \tag{59}$$

At the starting moment,

$$Y_{Z'}\left(a\right) \simeq \lambda_1 g_{Z'}^2 \ln\left(\frac{a}{a_0}\right) , \tag{60}$$

where

$$\lambda_1 = \frac{3\sqrt{5} N_e N_\gamma \alpha}{128\pi^{7/2}\sqrt{g G_N} m_e} \tag{61}$$

For low temperature, the approximation of the collision term is

$$\text{collision term} \propto g_{Z'}^2 m_e^5 a e^{-2m_e a} \left(1 - \frac{Y_{Z'}}{Y_{Z'}^{\rm eq}}\right) , \tag{62}$$

here,

$$Y_{Z'}^{\rm eq} \simeq \frac{\delta_{\rm BE}^n \left(m_e a\right)^{3/2}}{2\sqrt{2}\pi^{3/2}} e^{-m_e a} . \tag{63}$$

Therefore,

$$\text{collision term} \simeq -\alpha g_{Z'}^2 N_e N_\gamma \delta_{\rm BE}^n \sqrt{\frac{5}{256 g G_N \pi^4 m_e a}} e^{-m_e a} Y_{Z'} \,. \tag{64}$$

The asymptotic solution is

$$Y_{Z'} \simeq Y_{Z'}\,(a = \infty) \tag{65}$$

References

[1] Paolo Gondolo and Graciela Gelmini. Cosmic abundances of stable particles: Improved analysis. *Nucl. Phys. B*, 360:145–179, 1991.

[2] Shao-Ping Li and Xun-Jie Xu. Production rates of dark photons and z'in the sun and stellar cooling bounds. *Journal of Cosmology and Astroparticle Physics*, 2023(09):009, 2023.