A Report on The Aharonov-Bohm Effect

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July 2025

1 Introduction

The Aharonov-Bohm Effect was first introduced by Yakir Aharonov and David Bohm in 1959[1] and experimentally corroborated a year later by R. G. Chambers[2]. Aharonov and Bohm proposed a result that showed that a charged particle can be influenced by the potentials of an electromagnetic field in a completely field-free region.

Maxwell's equations allow for the description of the electric field $\vec{E}$ and magnetic field $\vec{B}$ in terms of a scalar potential or electric potential ϕ and a vector potential $\vec{A}$ as given below.

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \end{aligned} \tag{1}$$

It is observed that such a description leaves room for gauge freedom. This can be demonstrated by considering an arbitrary well-behaved scalar function χ, and defining new potentials

$$\begin{aligned} \vec{A}' &= \vec{A} + \nabla\chi \\ \phi' &= \phi - \frac{\partial \chi}{\partial t} \end{aligned} \tag{2}$$

and subsequently applying them to the potential description of the electrical and magnetic fields in Eq.(1), we see that

$$\begin{aligned} \vec{B}' &= \nabla \times \vec{A}' = \nabla \times \left(\vec{A} + \nabla\chi\right) = \left(\nabla \times \vec{A}\right) + \cancelto{0}{\nabla \times \left(\nabla\chi\right)} = \vec{B} \\ \vec{E}' &= -\nabla\phi' - \frac{\partial \vec{A}'}{\partial t} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \cancelto{0}{\left(\partial_t \nabla\chi - \nabla\partial_t\chi\right)} = \vec{E}. \end{aligned} \tag{3}$$

Additionally, classically, only the fields $\vec{E}$ and $\vec{B}$ are observable. It is for these reasons that the potentials have been regarded purely as mathematical auxiliaries and physical relevance is reserved exclusively for the fields. The Aharonov-Bohm Effect argues that quantum-mechanically, the potentials do carry physical significance by inducing measurable phase shifts in charged particles.

Numerous authors, including Aharonov and Bohm themselves have suggested that the AB-effect constitutes a breach of locality as formulated within Einstein's relativistic framework, since the charged particle seems to be affected by a field despite residing in a field-free region. However, more recently this notion has been challenged with both theoretical explanations and experimental work. This paper aims to explore the theoretical underpinnings of one such explanation posited by Lev Vaidman in 2012[3].

2 The Standard Interpretation of the Aharonov-Bohm Effect

This section covers very simply the fundamentals of the Aharonov-Bohm effect, as outlined by the first paper[1] on the idea from 1959.

Consider a system in a multiply-connected region with electrical fields $\vec{E}_{\text{in}} \neq 0$ and $\vec{E}_{\text{out}} = 0$ and magnetic fields $\vec{B}_{\text{in}} \neq 0$ and $\vec{B}_{\text{out}} = 0$. In order to describe the behaviour of a particle in an electromagnetic field, we also consider the Lagrangian of a particle with charge q and mass m, expressed as

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A} - q\phi \tag{4}$$

The time evolution of such a system over a time t is given by a propagator $\mathcal{K}$, expressed here in Feynman's path integral formalism below.

$$\mathcal{K}(x_b, t_b; x_a, t_a) = \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar} S[x(t)]} \tag{5}$$

where $S = \int_{t_a}^{t_b} \mathrm{d}t \; \mathcal{L}(x, \dot{x}, t)$ is the action integral of the system being described. For brevity of exposition, the free propagator $\mathcal{K}_0$ of the particle's motion, given by the term $\left(\frac{1}{2}m\dot{\vec{x}}^2\right)$ in the Lagrangian can be isolated from the interactive component, allowing the propagator to be denoted as

$$\mathcal{K}(t) = \mathcal{K}_0 \int_0^t \mathcal{D}[x(t)] \exp\left(\frac{i}{\hbar} S_{\text{interactive}}\right) \tag{6}$$

2.1 The Magnetic Aharonov-Bohm Effect

The interactive phase of the propagator from Eq. (6), expressed in terms of the Lagrangian outlined in Eq. (4) is considered, and expressed below.

$$\exp\left(\frac{iq}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\,\dot{x}^\mu A_\mu\right)=\exp\left(\frac{iq}{\hbar}\int_{x_a}^{x_b}x^\mu A_\mu\right) \tag{7}$$

where $A_\mu = \big(\phi, -\vec{A}\big)$ and $x^\mu = \big(t, x, y, z\big)$. We assume that each particle indexed by $j \in \mathbb{N}$ follows a path C_j. The final propagator $\mathcal{K}_j$ of such a particle is thus given as

$$\mathcal{K}_j = \mathcal{K}_0\int_0^t \mathcal{D}[x(t)]\,\exp\left(\frac{iq}{\hbar}\int_{C_j}\mathrm{d}x^\mu A_\mu\right). \tag{8}$$

Considering two particles, the resultant phase difference ${}^{\vec{B}}\Delta\varphi_{\mathrm{AB}}$ is expressed as

$$\begin{aligned}\Delta\varphi &= \frac{q}{\hbar}\left(\int_{C_1}\mathrm{d}x^\mu A_\mu - \int_{C_2}\mathrm{d}x^\mu A_\mu\right)\\ &= \frac{q}{\hbar}\oint \mathrm{d}x^\mu A_\mu \qquad \because C = C_1 \cup \big(-C_2\big).\end{aligned} \tag{9}$$

Applying Stokes' Theorem to the integral above, we are left with the expression

$$\begin{aligned}{}^{\vec{B}}\Delta\varphi_{\mathrm{AB}} &= \frac{q}{\hbar}\iint_\Sigma \big(\nabla \times A_\mu\big)\,\mathrm{d}\Sigma\\ &= \frac{q}{\hbar}\Phi_B\end{aligned} \tag{10}$$

where Φ_B is the magnetic flux. It is clearly demonstrated that the phase of charged particles is affected by a magnetic flux without field interactions.

2.2 The Electrical Aharonov-Bohm Effect

There are many variants of the Aharonov-Bohm effect, including a Gravitational Aharonov-Bohm effect. The closest electromagnetic analogue to this is the Electrical Aharonov-Bohm effect, which is elaborated on in this section. Consider a new system in the same multiply-connected region, however this time in a gauge where $\vec{A} = 0$ in the region where $\vec{E}_{\mathrm{out}} = 0$ and $\vec{B}_{\mathrm{out}} = 0$. This gauge maintains $\vec{B}_{\mathrm{out}} = 0$. The Lagrangian in this system thus simplifies to

$$\mathcal{L}' = \frac{1}{2}m\dot{\vec{x}}^2 - q\phi. \tag{11}$$

The propagator of this system is given by the equation

$$\mathcal{K}_{\mathrm{electric}} = \mathcal{K}_0\int \mathcal{D}[x(t)]\exp\left(\frac{iq}{\hbar}\int \mathrm{d}t\;\phi(t)\right) \tag{12}$$

which, in the context of two interfering particles results in a phase difference ${}^{\vec{E}}\Delta\varphi_{\mathrm{AB}}$

$${}^{\vec{E}}\Delta\varphi_{\mathrm{AB}} = \frac{q}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\;\phi(t) \tag{13}$$

3 Nonlocality and the Aharonov-Bohm Effect

As previously mentioned, the Aharonov-Bohm effect has sparked numerous debates with regard to its supposed violation of Einstein's notion of local action. Many authors have provided explanations to confine the nature of the effect within the boundaries of locality. A convincing semi-classical argument was offered by L. Vaidman in 2012[3], the essence of which has been outlined in the following section.

3.1 Vaidman's Local Results

Vaidman offered explanations of both the magnetic and electric AB effects, by quantising the source of the fields and entangling a charged particle (here to be considered an electron with a charge e) with the quantised sources.

Localising the magnetic AB effect Two counter-rotating cylinders of radius r and length L are considered. Charges $+Q$ and $-Q$ homogenously populate the surfaces of each cylinder respectively. The cylinders are said to counter-rotate at a relative surface velocity of v, resulting in a surface current $\mathcal{I}_\sigma = Q\sigma$ where $\sigma = \frac{Q}{2\pi r L}$ is the surface charge density. The system can be described quantum-mechanically as

$$|\Psi_B(t=0)\rangle = \frac{1}{\sqrt{2}}\big(\,|R\rangle_e + |L\rangle_e\,\big) \otimes |\Psi_0\rangle_c \tag{14}$$

where $|R\rangle_e$ and $|L\rangle_e$ are the electron states in the left and right paths, and $|\Psi_0\rangle_c$ is the initial state of the solenoidal cylinders. The resulting propagator of the state $|\Psi_B(t)\rangle$ is

$$\mathcal{K}_B(t) = |R\rangle\,\langle R|_e \otimes \mathcal{K}_R(t) + |L\rangle\,\langle L|_e \otimes \mathcal{K}_L(t) \tag{15}$$

where $\mathcal{K}_{R,L}$ are the propagators of the state $|\Psi_0\rangle_c$. The operators $|R\rangle\,\langle R|_e$ and $|L\rangle\,\langle L|_e$ project the state onto the Hilbert spaces of state $|R\rangle_e$ and $|L\rangle_e$ respectively. The chronology of the electron's motion is enumerated below.

1. The electron wave packet approaches the solenoid radially. The flux of the electron at this time is zero since the magnetic field generated by its motion does not intersect the surface of the cylinder.

2. The electron wave packet enters splits into superposition of two wave packets that encircle the solenoid from both sides at a radius R.

3. The two wave packets leave the circular path in nearly the same direction and interfere towards two detectors A and B.

Consider the electron in the second stage – as it splits into two wave packets in superposition. It is assumed that the electron travels at a constant velocity u as it completes its circular motion, as a result of which the electron generates

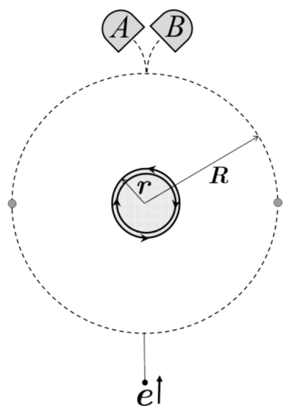

Figure 1: Vaidman's Solenoidal System

a current $\mathcal{I}_{e^-} = \frac{e}{\mathcal{T}}$ where $\mathcal{T} = \frac{2\pi R}{u}$ is the time period of its motion around the circumference of its motion. The well-known result of the axial magnetic field generated by a current carrying coil, better known as the Biot-Savart law is applied in this context to determine the magnetic flux through the solenoid generated by the moving electron.

$$\vec{B}_z(z) = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \mathrm{d}\theta \, \frac{R\big(\hat{\theta} \times (\vec{z} - \vec{R})\big)}{\big(R^2 + z^2\big)^{3/2}} = \frac{\mu_0 I R^2}{2\big(R^2 + z^2\big)^{3/2}} \tag{16}$$

$$\Phi_z(z) = \int \vec{B} \cdot \mathrm{d}\vec{A} = \frac{\mu_0 e u R}{4\pi\big(R^2 + z^2\big)^{3/2}} \cdot \pi r^2 = \frac{\mu_0 e u R r^2}{4\big(R^2 + z^2\big)^{3/2}} \tag{17}$$

Vaidman asserts that the electron exerts an electromotive force on the solenoidal cylinders, bringing about a change in their angular velocity. A relationship between the magnetic flux generated by the electron and the electromotive force it generates as a result can be established using Faraday's Law

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \Rightarrow E = \frac{-1}{2\pi r} \dot{\Phi}_B \tag{18}$$

and Coulomb's Law

$$\vec{F} = (\mathrm{d}Q)\vec{E}. \tag{19}$$

The change in momentum – or the impulse – that the solenoid experiences as a result of the electron's magnetic flux is thus given by the expression

$$\Delta p = \int F \, \mathrm{d}t = \mathrm{d}Q \int \mathrm{d}t \, \cdot \frac{-1}{2\pi r} \frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{20}$$

The change in velocity as a result of the impulse can subsequently be determined. the expression below models a small perturbation, and the negative sign can be dropped inconsequentially.

$$\begin{aligned} \mathrm{d}(\delta v) &= \frac{\Delta p}{M} = \frac{1}{M}\left(\frac{\Phi(z)\,\mathrm{d}Q}{2\pi r}\right) \\ \Rightarrow \quad \delta v &= \frac{1}{M}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}Q\,\frac{\Phi(z)}{2\pi r} \quad \text{, where } \mathrm{d}Q = Q\frac{\mathrm{d}z}{L} \end{aligned} \tag{21}$$

The electron completes a motion of half the circumference of the circle of radius R with tangential velocity u. Thus the time it spends in uniform circular motion is $\frac{\pi R}{u}$. The displacement of the solenoidal cylinders δx can be determined by the expression

$$\begin{aligned} \delta x &= \delta v\left(\frac{\pi R}{u}\right) \\ &= \frac{\mu_0}{4\cancel{\pi}}\frac{\cancel{u}Qer}{M\cancel{R}L}\frac{\cancel{\pi}\cancel{R}}{\cancel{u}} = \frac{\mu_0 Qer}{4ML}. \end{aligned} \tag{22}$$

Using the de Broglie Hypothesis one may relate the wavelength of matter λ to its momentum Mv with the help of Planck's constant h.

$$\lambda = \frac{h}{Mv} \tag{23}$$

Applying the de Broglie Hypothesis to the solenoidal cylinders and computing the wavenumber $k = \frac{2\pi}{\lambda}$, lets one determine the phase difference of the cylinders using the formula $\Delta\varphi = k(\delta x)$ that is derived from the wavefunction ansatz $\psi(x) = Ae^{ikx}$. As shown previously in Eq.(15), the state of the electron is entangled with the state of the solenoidal cylinders.

$$|\Psi_B(T)\rangle = \mathcal{K}_B(T)\,|\Psi_B(0)\rangle = \frac{1}{\sqrt{2}}\big(\,|R\rangle_e \otimes \mathcal{K}_R\,|\Psi_0\rangle_c + |L\rangle_e \otimes \mathcal{K}_L\,|\Psi_0\rangle_c\big) \tag{24}$$

By applying the eigenvalue-eigenvector relation, the propagators $\mathcal{K}_{R,L}$ can be expressed as phase factors $e^{(i\varphi_{R,L})}$. Additionally, the tensor product of the state $|\Psi_o\rangle_c$ being common to all terms can be isolated, following the distributive property of addition and multiplication.

$$|\Psi_B(T)\rangle = \frac{1}{\sqrt{2}}\big(e^{(i\varphi_R)}\,|R\rangle_e + e^{(i\varphi_L)}\,|L\rangle_e\big) \otimes |\Psi_0\rangle_c \tag{25}$$

Demonstrably, the phase picked up by the solenoidal cylinders is expressed in the final states of the electron, while the cylinders return to their original state $|\Psi_o\rangle_c$. Therefore, the resultant phase difference of the interfering electron wave packets is

$$ {}^{B}\Delta\varphi_{\text{local}} = 4\frac{Mv}{\hbar}\frac{\mu_0 Q e r}{4ML} = \frac{e}{\hbar}\left(\frac{\mu_0 Q v r}{L}\right). \tag{26} $$

An additional coefficient 4 is introduced to account for the two solenoidal cylinders in each branch. Recall the phase difference initially hypothesised by Aharonov and Bohm (10). In Viadman's system, the flux from an individual cylinder Φ_B is

$$ \Phi_B = \int \vec{B}\cdot \mathrm{d}\vec{A} = \big(\mu_0 \mathcal{I}_e\big)\big(\pi r^2\big), \tag{27} $$

where the surface current $\mathcal{I} = \frac{Qv}{2\pi r L}$. The hypothesised Aharonov-Bohm phase difference is determined to be

$$ {}^{B}\Delta\varphi_{\text{AB}} = \frac{e}{\hbar}\frac{\mu_0 Q v r}{L}. \tag{28} $$

It is evident that ${}^{B}\Delta\varphi_{\text{local}} = {}^{B}\Delta\varphi_{\text{AB}}$.

Localising the electric AB effect Consider a one-dimensional interferometer as given in figure 2. Initially, two massive, charged particles are placed symmetrically on a perpendicular axis at equal, large distances from mirror A, each with a constant initial velocity towards A. At a distance r, the charges spend a time T near mirror A and then bounce back symmetrically. The chronology of the motion of an electron in this system is enumerated below.

1. A electron wave packet splits into two wave packets, where each wave packet approaches mirror A and B.
2. Each electron wave packet spends a time $\tau > T$ at its respective mirror before being reflected. The charged particles reside near the electron during this time, and the electron resides in a region where electric fields from both particles cancel out, thus availing it a field-free region.
3. The wave packets, once reflected interfere coherently.

The initial state of the system $|\Psi_E(0)\rangle$ is given by the expression

$$ |\Psi_E(0)\rangle = |\Psi_e\rangle \otimes |\Psi_0\rangle_Q \tag{29} $$

where $|\Psi_e\rangle = \frac{1}{\sqrt{2}}\big(|A\rangle_e + |B\rangle_e\big)$ is the state of the electron in superposition of the two branches $|A\rangle$ and $|B\rangle$, and $|\Psi_0\rangle_Q$ is the initial state of the two charged particles. The Coulombic potential between each particle and the electron is

$$ V(x) = -\frac{k_c e Q}{x} \tag{30} $$

where k_c is Coulomb's constant. It is evident from this expression that the potential $V(x)$ may be neglected when the charged particles and the electron

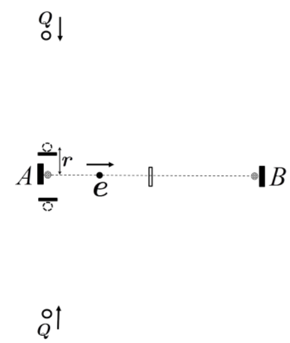

Figure 2: Vaidman's 1D Interferometer

are far apart. With that conclusion, one can state that $|\Psi_0\rangle_Q$ does not undergo any change when the electron is in state $|B\rangle_e$. However, as demonstrated in the rest of this section, $|\Psi_0\rangle_Q$ is affected by the wave packet $|A\rangle_e$.

$$\mathcal{K}_E(t) = |A\rangle \langle A|_e \otimes \mathcal{K}_Q(t) + |B\rangle \langle B|_e \otimes \mathbb{I}_Q \tag{31}$$

Eq.(31) defines the propagator for the state $|\Psi_E(0)\rangle$. This ensures that the propagator $\mathcal{K}_Q$ of the state of the two charges $|\Psi_o\rangle_Q$ only acts in the subspace selected by the projection operator $|A\rangle \langle A|_e$. $\mathcal{K}_E$ has no effect when it is in $|B\rangle_e$.

During the time T that the charged particles near the mirror, they develop a small shift of position δx. Assume the electron to be near the mirror if its distance from the mirror $x \in [0, d]$, where $d \ll r$. It is for this reason, the change in velocity of the charged particles δv is asserted to be small and constant. The change in the kinetic energy of the charged particles is equated to the Coulombic potential between the particles and the electron as given in Eq.(30).

$$\mathcal{T}_e = V(x = r) = -\frac{k_c e Q}{r} \tag{32}$$

The gain in the kinetic energy $\mathcal{T}_e$ of an individual charge can be expressed as

$$\mathcal{T}_e = \delta\left(\frac{1}{2}Mv^2\right) = \frac{1}{2}M\big((v + \delta v)^2 - v^2\big).$$

Complying with the assumption that δv is small and thus $\delta v \ll v$, the change in kinetic energy is simply written as

$$\mathcal{T}_e = Mv\delta v. \tag{33}$$

Since the charges remain close to the electron for time T, the centroids of their wave-packets are displaced by

$$\delta x = T\delta v. \tag{34}$$

Equating the expressions for $\mathcal{T}_e$ from Eq.(33) and Eq.(32), and substituting for δv with $\frac{\delta x}{T}$

$$
\begin{aligned}
-\frac{k_c eQ}{r} &= Mv\delta v \\
\Rightarrow \quad \delta x &= -\frac{k_c eQ}{rMv}T.
\end{aligned} \tag{35}
$$

To observe interference, this change must be much smaller than the intrinsic position uncertainty of the charges. Although this statement is fairly intuitive, is can be formally as is done below.

Consider the state of the two charges $|\Psi_0\rangle_Q$ mentioned in Eq.(29). Applying the state propagator in Eq.(31), this state may evolve to $\mathcal{K}_Q(T)\,|\Psi_0\rangle = |\Psi_T\rangle$ or remain in $|\Psi_0\rangle_Q$. Let the intrinsic position uncertainty of these two states be Δx. One may use a Gaussian curve with a standard deviation Δx to represent the position of such states.

$$
\langle x|\Psi_0\rangle = \psi_0(x) \quad = \quad \frac{1}{\sqrt{2\pi(\Delta x)^2}} \exp\left(\frac{-x^2}{4(\Delta x)^2}\right) \tag{36}
$$

$$
\langle x|\Psi_T\rangle = \psi_T(x) \quad = \quad \frac{1}{\sqrt{2\pi(\Delta x)^2}} \exp\left(\frac{-\big(x-\delta x\big)^2}{4(\Delta x)^2}\right) \tag{37}
$$

Simply put, $\psi_T(x) = \psi_0(x - \delta x)$. The condition necessary for coherence is $\langle\Phi_T|\Phi_0\rangle \approx 1$.

$$
\begin{aligned}
\langle\Phi_T|\Phi_0\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x \ \psi_T^*(x)\psi_0(x) = \exp\left(\frac{-(\delta x)^2}{8(\Delta x)^2}\right) \\
\Rightarrow \quad & \exp\left(\frac{-(\delta x)^2}{8(\Delta x)^2}\right) \approx 1 \\
\Rightarrow \quad & \delta x \ll \Delta x
\end{aligned} \tag{38}
$$

Revisiting de Broglie's hypothesis and the formula $\Delta\varphi = k(\delta x)$, the phase difference picked up by the charged particles, which is subsequently expressed as the relative phase in the entangled electron's superposition – as demonstrated in Eq.(25) – is computed to be

$$
{}^{E}\Delta\varphi_{\mathrm{local}} = -2\frac{k_c eQ}{\hbar r}T. \tag{39}
$$

Recall the result in Eq.(13). The gauge in which this result was obtained asserts $\vec{E} = \nabla\phi(t)$, allowing one to treat ϕ as the Coulombic potential $U(x) = \frac{-k_c\mathcal{Q}}{x}$. Assuming the interfering quantum to be an electron ($q = e$) and a constant Coulombic potential over time T, the Aharonov-Bohm phase difference is

$$ {}^{E}\Delta\varphi_{\mathrm{AB}} = \frac{e}{\hbar} \cdot -2\frac{k_c Q}{r} \int_0^T dt = -2\frac{k_c e Q}{\hbar r} T. \tag{40} $$

It has thus been demonstrated that ${}^{E}\Delta\varphi_{\mathrm{local}} = {}^{E}\Delta\varphi_{\mathrm{AB}}$.

3.2 Shortcomings of Vaidman's Localisation

Four years following Vaidman's paper, Aharonov et. al. published a paper[4] refuting several of Vaidman's claims. It was pointed out that Vaidman's local interpretation suggest all fluxes locally affect his electron, including remote fluxes. This disagrees with experiment, which shows that the phase of the electron is only affected by fluxes that are enclosed within the paths of the electron[5][6][7]. Aharonov et. al. also questions the basis on which Vaidman assumes electron velocities to be constant in his paper.

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