Cosmological Principle :

① Homogeneity : The universe looks the same at each point. This implies a uniform distribution of matter.

② Isotropy : There are no preferred directions in the universe.

① ≡ Does not matter from where you observe

② ≡ Does not matter what you look at your point of observation

Basics of Light

1 light year (ly) $\approx$ 10^6 m

speed of light $c \approx 3 \times 10^8$ m/s

Dynamics of the Universe

Galaxies are observed to be moving away from each other.

Cosmological redshift - occurs because the expansion of space stretches waves of light, increasing their wavelength.

$$1 + z = \frac{\lambda_{observed}}{\lambda_{emitted}}$$

If a galaxy emitted light at half its size its redshif $z = 1$.

Redshift follows Hubble's Law,

$$V = H_0 d$$

V = recession velocity

d = distance to obj

H_0 = Hubble's Const

As of now, $H_0 \approx$ 70 km/s per parsec

If $z < 0$ = blueshift

$z > 0$ = redshift

$z = 0$ = seems practically impossible in this universe.

<u>Parsec</u>

Parsec (pc) stands for parallax second.

$$1 \text{ parsec} = \frac{1 \text{ AU}}{\tan(1 \text{ arc second})} \approx 3.086 \times 10^{13} \text{ km}$$

1° = 60 arcminutes = 3600 arcseconds

Hubble - Lemaitre's Law

Strictly defined as $V = H_0 d$

But for small redshifts $z \ll 1$

(z is dimensionless)

and small velocities $V \ll 1$,

redshift can be approximated by the Doppler Effect.

$$z \approx V/c$$

and from Hubble's Law, $V = H_0 d$

$$\Rightarrow \quad z \approx \frac{H_0 d}{c}$$

This approximation does not hold for cosmological scales with large distances.

We will look at the relativistic model because I'm a fucking idiot.

$z \gg 1$,

The scale factor $a(t)$ describes how distances evolve over time.

$$\lambda_{obs} = \lambda_{emit} \frac{a(t_{obs})}{a(t_{emit})}$$

if the photon was emitted at t_{emit} and observed at t_{obs}

Ignoring the expansion of the universe, we have a special relativistic Doppler effect

$$1+z = \sqrt{\frac{1+v/c}{1-v/c}}$$

$v =$ radial velocity

This only works for local motion

From the Friedmann equations,

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

$$\Rightarrow \int \frac{da}{a} = \int H(t)\, dt$$

$$\ln\left(\frac{a_{then}}{a_{now}}\right) = \int_{t_0}^{t} H(t)\, dt$$

$$\frac{a_t}{a_0} = 1 + z = \exp\left(\int H(t)\, dt \right)$$

$$V_{effective} = V_{hubble} + V_{peculiar}$$

Composition of Universe

Non-relativistic particle

Relativistic Particle - Mass-energy does not dominate

$$E_{total}^2 = (mc^2)^2 + (pc)^2$$

$$= mc^2 + \left(1 + \frac{p^2}{m^2c^2}\right)$$

$$\approx mc^2 + \frac{p^2}{2m}$$

Metric of n-D plane

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \qquad (2D)$$

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \quad (3D)$$

$$\Delta s_n^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_2^2 + \dots + \Delta x_n^2$$

$$ds_n^2 = dx_1^2 + dx_2^2 + \dots + dx_n^2$$

$$L = \int_A^B ds$$

where L is the length of a curve in a nD plane

In 2D, we may use the polar transformation equations $x = r\cos\theta$, $y = r\sin\theta$

$$ds^2 = (d\, r\cos\theta)^2 + (d\, r\sin\theta)^2$$

$$d\, r\cos\theta = -r\sin\theta\, d\theta + \cos\theta\, dr$$

$$d\, r\sin\theta = r\cos\theta\, d\theta + \sin\theta\, dr$$

$$ds^2 = dr^2 + r^2 d\theta^2$$

Similarly, in 3D, we may use spherical coordinate transformations

$$x = r\sin\varphi\cos\theta ,$$

$$y = r\sin\varphi\sin\theta ,$$

$$z = r\cos\varphi .$$

Thus,

$$ds^2 = dr^2 + r^2\sin^2\varphi\, d\theta^2 + r^2 d\varphi^2$$

Metric of a spherical Surface

$$x^2 + y^2 + z^2 = R^2$$

In cylindrical polar coordinates

$$r^2 + z^2 = R^2 \quad \text{where} \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$\Rightarrow \quad r\,dr = -z\,dz$ on the surface of the sphere

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$ (derive it if you'd like)

In spherical polar coordinates

$$ds^2 = R^2 \sin^2\varphi \, d\theta^2 + R^2 d\varphi^2$$

Curvature

The Gaussian curvature of a surface is given by

$$K = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y \partial x}\right)^2}{\left(1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2\right)^2}$$

Pedantry: Deriving Gaussian Curvature

Consider a monge patch $\vec{X}(x,y) = \langle x, y, f(x,y) \rangle$

this is 3D space

We take partial derivatives

$$\vec{X}_x = \langle 1, 0, \frac{\partial f}{\partial x} \rangle \qquad \frac{\partial \vec{X}}{\partial x}$$

$$\vec{X}_y = \langle 0, 1, \frac{\partial f}{\partial y} \rangle \qquad \frac{\partial \vec{X}}{\partial y}$$

$\vec{X}_x$ and $\vec{X}_y$ span the tangent plane.

The surface unit normal is given by

$$\vec{N} = \frac{\vec{X}_x \otimes \vec{X}_y}{\| \vec{X}_x \otimes \vec{X}_y \|} = \frac{\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle}{\sqrt{1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2}}$$

Look at the First Fundamental Form (I)

I tells us how the surface stretches and angles deform locally.

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \vec{X}_x \cdot \vec{X}_x & \vec{X}_x \cdot \vec{X}_y \\ \vec{X}_y \cdot \vec{X}_x & \vec{X}_y \cdot \vec{X}_y \end{bmatrix}$$

$$ds^2 = E\,dx^2 + F\,dx\,dy + G\,dy^2$$

We may now look at the Second Fundamental Form (II)

II measures how much the surface bends relative to the space around it, or how fast the surface normal changes as we move along the surface.

$$\mathrm{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \vec{X}_{xx} \cdot \vec{N} & \vec{X}_{xy} \cdot \vec{N} \\ \vec{X}_{xy} \cdot \vec{N} & \vec{X}_{yy} \cdot \vec{N} \end{bmatrix}$$

Gaussian Curvature K is given by

$$K = \frac{\det(\mathrm{II})}{\det(\mathrm{I})}$$

$$K = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2}{\left(1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2\right)^2}$$

We may use various surfaces $\langle x, y, f(x,y) \rangle$ such as a sphere or hyperboloid

Hyperboloid : $\langle x, y, f = x^2 - y^2 \rangle$

$$K(\langle x, y, x^2 - y^2 \rangle) = \frac{-4}{(1 + 4x^2 + 4y^2)^2}$$

We see that the curvature is negative $\forall\, x, y \in \mathbb{R}$, although it is not constant

H_0, the Hubble's Constant has the units km/s/MPc

If two galaxies are 2 MPc apart, then they are moving away from each other at H_0 km/s.

A uniform distribution is used to estimate its value $H_0 \in [50, 100]$

Metric of an Isotropic and Homogenous Surface

An isotropic, homogenous 3D space is given by

$$ds^2 = \frac{dr^2}{1 - Kr^2} + r^2\left(\sin^2\varphi \, d\theta^2 + d\varphi^2\right)$$

where
$$K = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2}{\left(1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2\right)^2}$$

with suitable coordinate transformations, we may represent this metric in Cartesian coordinates.

$$ds^2 = \frac{1}{\left(1 + \frac{1}{4}K\right)}\left(dx^2 + dy^2 + dz^2\right)$$

However, if we assume the 3D space to be expanding with a scale factor $a(t)$

$$r(t) = a(t)\,\chi$$

χ = comoving distance

The comoving distance is measured in a coordinate system that expands w/ the universe.

$$ds^2 = a(t)\left[\frac{d\chi^2}{1 - K(t)\,a^2(t)\,\chi} + \chi^2\left(\sin^2\varphi\, d\theta^2 + d\varphi^2\right)\right]$$

let $K(t) = \dfrac{k}{a^2(t)}$

where k is time independent

how?

$k = K(t)\, a^2(t)$

$$ds^2 = a(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(\sin^2\varphi\, d\theta^2 + d\varphi^2\right)\right]$$

This metric exclude the relativistic effects on time and length.

(we will revisit this).

Relativity

Principle of Equivalence:

Gravitation is <u>locally</u> equivalent to acceleration.

From Einstein, we find out that

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

Robertson-Walker Metric

We have shown previously that a 3D homogenous and isotropic space is described by the equation

$$ds^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(\sin^2\varphi \, d\theta^2 + d\varphi^2\right)\right]$$

And from Special Relativity, we also know that

$$ds^2 = -c^2dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

$$\Rightarrow ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(\sin^2\varphi \, d\theta^2 + d\varphi^2\right)\right]$$

Pedantry: A result of the equivalence relation.

From SR, we know that

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

We can chose the SR metric since locally we may treat the space as the Minkowski space. We know this from the equivalence relation – GR reduces to SR locally.

At a particle's rest frame, $g_{\nu\nu} \approx \eta_{\nu\nu} = \text{diag}(-1, 1, 1, 1)$

Since we are treating the particle locally, we may introduce proper time, which is defined by the equation $ds^2 = -c^2 d\tau^2$

Thus, we get

$$-c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\Rightarrow d\tau^2 = dt^2 - \frac{dt^2}{c^2}\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right)$$

$$\Rightarrow d\tau^2 = dt^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}}$$

Reciprocating,

$$\Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \text{ (Lorentz factor)}$$

For the Robert- Walkerson metric, Einstein's equations produce the following two equations (which my dumbass will derive later)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \quad -(1)$$

and

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}p \quad -(2)$$

where $\dot{a} = \dfrac{da}{dt}$ (derivative w/rt cosmic time)

These are Alexander Friedmann's Equations

Pedantry: Deriving the Friedmann Equations from the Field Equations

Field Equations: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ — (1.1)

$$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} \quad \text{— (1.2)}$$

Equating (1.1) and (1.2),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$$

Apart from postulating the Field Equations, we will also postulate the stress energy tensor to be

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g_{\mu\nu}$$

The FLRW metric to be used is given by

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\varphi^2 + \sin^2\varphi \, d\theta^2\right)\right]$$

$$ds^2 = -c^2 dt + a^2(t)\, \tilde{g}_{ij}$$: time independent metric tensor

Deriving the Ricci Tensor for the FLRW metric

$R_{\mu\nu} = R_{00} + R_{0i} + R_{ij}$ where i and j are spatial indicies ($i, j = 1, 2, 3$)

$R^{\rho}_{\ \sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$: Riemann Curvature Tensor

$R_{\mu\nu} = R^{\eta}_{\ \mu\eta\nu} = \partial_\eta \Gamma^{\eta}_{\nu\mu} - \partial_\nu \Gamma^{\eta}_{\mu\eta} + \Gamma^{\eta}_{\eta\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\eta}_{\nu\lambda}\Gamma^{\lambda}_{\eta\mu}$: Ricci Tensor - Riemann Curvature Tensor contraction

$$R_{00} = \partial_\eta \Gamma^{\eta}_{00} - \partial_0 \Gamma^{\eta}_{0\eta} + \Gamma^{\eta}_{\eta\lambda}\Gamma^{\lambda}_{00} - \Gamma^{\eta}_{0\lambda}\Gamma^{\lambda}_{\eta 0}$$

let's assume we know nothing about the results of Christoffel symbols in the FLRW metric.

$$\Gamma^{\eta}_{00} = \frac{1}{2} g^{\eta\rho}\left[\partial_0 g_{\rho 0} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}\right]$$

$$= \frac{1}{2} g^{\eta 0}\left[\cancel{\partial_0 g_{00}}^{\,0} + \partial_0 g_{00} - \partial_0 g_{00}\right] + \Phi \quad (\Phi \text{ denote vanishing terms})$$

$$= 0$$

From this result we may also infer that $\Gamma^{\lambda}_{00} = 0$

$$\therefore \Gamma^{\eta}_{\eta\lambda}\Gamma^{\lambda}_{00} = 0$$

Now we may consider the term $-\partial_0 \Gamma^{\eta}_{0\eta}$

$$\Gamma^{\eta}_{0\eta} = \frac{1}{2} g^{\eta\sigma}\left[\partial_0 g_{\eta\sigma} + \cancelto{0}{\partial_\eta g_{0\sigma}} - \cancelto{0}{\partial_\sigma g_{\eta 0}}\right]$$

$$= \frac{1}{2} g^{kk} \frac{\partial}{\partial t} g_{kk}$$

$$= \frac{1}{2} g^{00} \frac{\partial}{\partial t} g_{00} + \sum_{k=1}^{3} \frac{1}{2} \frac{\$^{-1}}{a^2(t)} \frac{\partial}{\partial t} \$\, a^2(t)$$

Notation: \$ represents "something"

$$= 0 + \sum_{k=1}^{3} \frac{1}{2} \frac{\$^{-1}}{a^2(t)} \cdot 2a(t) \frac{\partial a(t)}{\partial t} \cdot \$$$

$$= 3\frac{\dot{a}}{a}$$

Looking at the entire term $-\partial_0 \, 3\frac{\dot{a}}{a}$

$$= -3\frac{\partial}{\partial t}\frac{\dot{a}}{a} = -3\left(\frac{\ddot{a}a - \dot{a}^2}{a^2}\right)$$

$$= -3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right)$$

The last term to consider is $-\Gamma^{\eta}_{0\lambda}\Gamma^{\lambda}_{\eta 0}$

$$\Gamma^{\lambda}_{\eta 0} = \frac{1}{2} g^{\lambda\rho}\left[\partial_0 g_{\rho\eta} + \partial_\eta g_{\rho 0} - \partial_\rho g_{\eta 0}\right]$$

$$= \frac{\dot{a}}{a}\delta^{\lambda}_{\eta}$$

and $\Gamma^{\eta}_{0\lambda} = \frac{\dot{a}}{a}\delta^{\eta}_{\lambda}$

$$-\Gamma^{\eta}_{0\lambda}\Gamma^{\lambda}_{0\eta} = -\sum_{i=1}^{3}\left(\frac{\dot{a}}{a}\right)^2 \delta^{i}_{i}\,\delta^{i}_{i}$$

$$= -3\left(\frac{\dot{a}}{a}\right)^2$$

$$\therefore R_{00} = -3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right) - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a} \quad - \quad 2.1$$

R_{i0} :

We know $R_{\mu\nu}$ is symmetric, $R_{\mu\nu} = R_{\nu\mu}$,

$\therefore R_{i0} = R_{0i}$ must be 0. — (2.2)

we also know space is isotropic, $R_{i0} \neq 0$ would suggest a coupling between time and spatial dimension i.

R_{ij} :

$$R_{ij} = R^{\lambda}_{i\lambda j} = \partial_\lambda \Gamma^{\lambda}_{ij} - \partial_j \Gamma^{\lambda}_{i\lambda} + \Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{ij} - \Gamma^{\lambda}_{j\sigma}\Gamma^{\sigma}_{\lambda i}$$

Going term by term, $\partial_\lambda \Gamma^{\lambda}_{ij} = \partial_0 \Gamma^{0}_{ij} + \partial_k \Gamma^{k}_{ij}$

$$\partial_0 \Gamma^{0}_{ij} = \partial_0 \frac{1}{2} g^{0\rho}\left[\partial_i g_{\rho j} + \partial_j g_{\rho i} - \partial_\rho g_{ij}\right]$$

$$= \partial_0 \frac{1}{2} g^{00}\left[\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}\right] = \partial_0 \frac{1}{2} g^{00}\left[0 + 0 - \partial_0 g_{ij}\right]$$

$$= \partial_0 \frac{1}{2c^2} \tilde{g}_{ij} \frac{\partial}{\partial t} a^2(t) = \partial_0 \frac{1}{\cancel{2}c^2} \tilde{g}_{ij} \cancel{2}\dot{a}a$$

$$= \frac{\ddot{a}a - \dot{a}^2}{c^2} \tilde{g}_{ij}$$

$$\left[g_{ij} = a^2(t)\,\tilde{g}_{ij}\right]$$

$$= \left(\frac{\ddot{a}a - \dot{a}^2}{a^2c^2}\right) g_{ij}$$

We'll jump a term and consider $\Gamma^{\lambda}_{j\sigma}\Gamma^{\sigma}_{\lambda i}$

We know $\Gamma \neq 0$ when Γ^{0}_{jk} or Γ^{j}_{0k}, aka we need at least one time index. Any Γ w/ two time indices vanishes.

here, either $\lambda = k$ and $\sigma = 0$ or $\lambda = 0$ and $\sigma = k$

Consider case ① : $\lambda = k, \sigma = 0$

$$\Gamma^{k}_{j0}\Gamma^{0}_{ki} \; ; \quad \Gamma^{k}_{j0} = \frac{1}{2} g^{k\eta}\left[\cancel{\partial_j g_{0\eta}}^{\,0} + \partial_0 g_{j\eta} - \cancel{\partial_\eta g_{0j}}^{\,0}\right]$$

$$= \frac{1}{2} g^{kk}\left[\partial_0 g_{jk}\right] = \frac{\dot{a}}{a}\delta^{k}_{j}$$

$$\Gamma^{0}_{kj} = \frac{1}{2} g^{0\eta}\left[\partial_k g_{j\eta} + \partial_j g_{k\eta} - \partial_\eta g_{jk}\right]$$

$$= \frac{1}{2} g^{00}\left[\cancel{\partial_k g_{j0}}^{0} + \partial_j \cancel{g_{k0}}^{0} - \partial_0 g_{jk}\right] + \Phi$$

$$= \frac{1}{2} \frac{+1}{c^2}\left[\partial_0 g_{jk}\right]$$

$$= \frac{1}{2c^2}\frac{\partial}{\partial t}\left(a^2 \tilde{g}_{jk}\right)$$

$$= \frac{1}{c^2}\dot{a}a\,\tilde{g}_{jk}$$

Thus $-\Gamma^{k}_{j0}\Gamma^{0}_{jk} = -\left(\frac{\dot{a}}{a}\delta^{j}_{k}\,\frac{1}{c^2}\dot{a}a\,\tilde{g}_{jk}\right) \times 2 \longleftarrow$ doubled to consider case 2

$$= -2\frac{\dot{a}^2}{c^2}\tilde{g}_{jk} = -2\left(\frac{\dot{a}^2}{c^2a^2}\right)g_{jk}$$

Now we look at the term $\Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{ij}$. Clearly $\Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{ij} = \Gamma^{\lambda}_{0\lambda}\Gamma^{0}_{ij} + \Phi$, (at least one temporal index)

$$\Gamma^{0}_{ij} = \frac{1}{2} g^{0\eta}\left[\partial_i g_{j\eta} + \partial_j g_{i\eta} - \partial_\eta g_{ij}\right]$$

$$= \frac{1}{2}g^{00}\left[\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij}\right]$$

$$= +\frac{1}{2}\frac{1}{c^2}\partial_0 g_{ij} = \frac{1}{2c^2}\frac{d}{dt}a^2\tilde{g}_{ij}$$

$$= \frac{\dot{a}a}{c^2}\tilde{g}_{ij} = \frac{\dot{a}}{ac^2}g_{ij}$$

The remaining terms $\partial_k \Gamma^{k}_{ij} - \partial_j \Gamma^{k}_{ik} = 2k\frac{g_{ij}}{a^2}$;

Thus $R_{ij} = \dfrac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{a^2c^2} g_{ij}$ — 2.3

Deriving the Ricci Tensor for the FLRW metric

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\Rightarrow R = g^{00} R_{00} + \cancel{\left(g^{0i} R_{0i}\right)} \xrightarrow{} 0 + g^{ij} R_{ij}$$

$$g^{00} R_{00} = \frac{-1}{c^2} \cdot \left(-3\frac{\ddot{a}}{a}\right) = \frac{3\ddot{a}}{ac^2}$$

$$g^{ij} R_{ij} = g^{ij}\left(\frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{a^2c^2}\right) g_{ij} \quad , \quad g^{ij} g_{ij} = \sum_{i=1}^{3} \delta^i_i$$

$$= \delta^i_j \left(\frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{a^2c^2}\right)$$

$$= 3\left(\frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{a^2c^2}\right)$$

$$\therefore R = \frac{3\ddot{a}}{ac^2} + 3\left(\frac{a\ddot{a} + 2\dot{a}^2 + kc^2}{a^2c^2}\right)$$

$$\Rightarrow R = \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2}\right) \quad - \quad 3$$

First Friedmann Equation

Consider $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

and $G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$$

Now consider G_{00}, the temporal index;

$$R_{00} - \frac{1}{2} R g_{00} = \frac{8\pi G}{c^2} T_{00}$$

$R_{00} = -3\frac{\ddot{a}}{a}$; [From Eq. 2.1]

$-\frac{1}{2} R g_{00} = \frac{-1}{2} \cdot \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2}\right)(-c^2)$; [From Eq. 3]

Now $T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$

and we know $T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu}$

$$T^{00} = \left(\rho + \frac{p}{c^2}\right) u^0 u^0 + p g^{00}$$

$\Rightarrow T^{00} = \left(\rho + \cancel{\frac{p}{c^2}}\right) - \cancel{\frac{p}{c^2}}$; $\because u^0 = 1$, in comoving coordinates $u^\nu = (1, 0, 0, 0)$

$= \rho$

$\therefore T_{00} = g_{00} g_{00} T^{00} = c^4 \rho$

Putting these together,

$$R_{00} - \frac{1}{2} R g_{00} = \frac{8\pi G}{c^2} T_{00}$$

$$\Rightarrow -3\left(\frac{\ddot{a}}{a}\right) + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2}\right) = \frac{8\pi G}{c^4} c^2 \rho$$

$$\Rightarrow 3\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) = \frac{8\pi G}{c^2} \rho \quad - (4.1)$$

The First Friedmann Equation

The Second Friedmann Equation

Like we did previously, we look at

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$$

But this time we consider spatial indices G_{ij};

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^2} T_{ij}$$

$$R_{ij} = \left(\frac{\ddot{a} a + 2\dot{a}^2 + 2kc^2}{a^2 c^2} \right) g_{ij} \; ; \left[\text{From Eq. } 2.3 \right]$$

$$-\frac{1}{2} R g_{ij} = -\frac{1}{2} \frac{6}{c^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2} \right) g_{ij}$$

and $T_{ij} = g_{i\eta} g_{j\nu} T^{\eta\nu}$; lets assume that η and ν are spatial indices

$$= g_{i\eta} g_{j\eta} \left(\left(\rho + \frac{p}{c^2} \right) u^{\eta} u^{\nu} + p g^{\eta\nu} \right)$$

$$= 0 + g_{i\eta} g_{j\eta} \, p g^{\eta\nu} \; ; \because u^{\nu} = (1, 0, 0, 0)$$

$$= p g_{i\eta} \delta^{\nu}_{j} = p g_{ij}$$

Putting these terms together, we get

$$\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2kc^2}{a^2} - 3\frac{\ddot{a}}{a} - 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{kc^2}{a^2}\right) \right] \cancel{g_{ij}} = \frac{8\pi G}{c^2} p(\cancel{g_{ij}})$$

$$\Rightarrow -\frac{2\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{kc^2}{a^2}\right) = \frac{8\pi G}{c^2} p$$

$$\Rightarrow \frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2} p \quad - (2.2)$$

The Second Friedmann Equation

The stress tensor $T^{\mu\nu}$ for perfect fluids is cosmologically useful.

$$T^{\mu\nu} = (\rho c^2 + p) u^\mu u^\nu + p g^{\mu\nu}$$

where ρ = mass-energy density;

p = hydrostatic pressure

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$$\tau = \int \sqrt{-g_{\mu\nu}\, dx^\mu dx^\nu}$$

The energy momentum tensor satisfies the continuity equation $T^{\mu\nu}{}_{;\mu} = 0$

Using Friedmann's Equations, the continuity Equation $T^{\mu\nu}{}_{;\mu} = 0$ simplifies to

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

#Pedantry: Deriving the continuity expression

(For now) postulate $T^{\mu\nu} = (\rho c^2 + p) u^\mu u^\nu + p g^{\mu\nu}$

Before we start working on differentiating $T^{\mu\nu}$, we need to look at u^μ OR u^ν.

At rest, the object only moves through time.

Thus $u^\nu = (1, 0, 0, 0)$

Differentiating $T^{\mu\nu}$:

$$\nabla_\nu T^{\mu\nu} = \nabla_\nu \left((\rho c^2 + p) u^\mu u^\nu \right) + \nabla_\nu (p g^{\mu\nu})$$

Lets do the easy component first, and apply the Leibnitz rule

$$\nabla_\nu (p g^{\mu\nu}) = \nabla_\nu (p) g^{\mu\nu} + \cancelto{0}{p \nabla_\nu g^{\mu\nu}} \quad \because \nabla_\nu g^{\mu\nu} = 0$$

The covariant derivative is defined on the Levi-Cevita connection, which includes the metric compatibility $\nabla_\lambda g^{\mu\nu} = 0$

$$\nabla_\nu (p(t)) g^{\mu\nu} = g^{0\nu} \nabla_0 p(t) \quad \because p(t) \text{ is only a function of time}$$

$$g^{0\nu} = 0 \;\forall\; \nu \neq 0$$

$$\Rightarrow \nabla_\nu (p(t)) g^{\mu\nu} = g^{00} \nabla_0 p(t) = -c^2 \dot{p}(t)$$

Thus $\nabla_\nu \left(p(t) g^{\mu\nu} \right) = -c^2 \dot{p}(t)$;

Now consider the term

$$\nabla_\nu \left((\rho c^2 + p) u^\mu u^\nu \right); \text{ applying the Leibnitz rule,}$$

$$\nabla_\nu \left((\rho c^2 + p) u^\mu u^\nu \right) = \nabla_\nu (\rho c^2 + p) u^\mu u^\nu + (\rho c^2 + p) \nabla_\nu (u^\mu) u^\nu + (\rho c^2 + p) u^\mu \nabla_\nu (u^\nu)$$

Going term by term,

$$\nabla_\nu (\rho c^2 + p) u^\nu u^\nu = \partial_\nu (\rho c^2 + p) u^\nu u^\nu \quad \because (\rho c^2+p) \text{ is a function}$$

$$= \partial_0 (\rho c^2 + p) u^0 u^\nu \quad \because (\rho c^2+p) \text{ is strictly a function of time } (x^0)$$

$$= (\dot\rho c^2 + \dot p)\, 1\, u^\nu$$

$$= (\dot\rho c^2 + \dot p)\ (1, 0, 0, 0)$$

Now, the next term,

$$(\rho c^2 + p) \nabla_\nu u^\nu . u^\nu$$

lets drop $(\rho c^2 + p)$ for now, its not relevant at the moment.

$$\nabla_\nu u^\nu u^\nu = \partial_\nu u^\nu u^\nu + \Gamma^\nu_{\nu\lambda} u^\lambda u^\nu$$

$$\partial_\nu u^\nu = \partial_0 u^0 \quad \because \text{The other } u^i \text{ components are } 0$$

$$u^0 = \text{const} \Rightarrow \partial_0 u^0 = 0$$

$$\Rightarrow \nabla_\nu u^\nu u^\nu = \Gamma^\nu_{\nu\lambda} u^\lambda u^\nu$$

$$\Gamma^\nu_{\nu\lambda} = \Gamma^0_{00} + \Gamma^i_{i0} \quad \because \text{Values of } u^{\lambda \neq 0} = 0$$

We know $\Gamma^\lambda_{\nu\nu} = \frac{1}{2} g^{\lambda\eta} \left[\frac{\partial g_{\eta\nu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\nu} - \frac{g_{\nu\nu}}{\partial x^\eta} \right]$ (by definition)

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \left[\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right] \quad \because g_{0\eta} = 0 \ \forall \eta \neq 0$$

$$g_{00} = -c^2 \quad (\text{const})$$

$$\Rightarrow \Gamma^0_{00} = 0$$

Now consider Γ^i_{i0} ; $i \neq 0$

Consider the Robertson-Walker Metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(\sin^2\varphi\, d\theta^2 + d\varphi^2\right)\right]$$

From this, we get $g_{00} = -c^2$; $g_{11} = a^2(t) \cdot \frac{1}{1-kr^2}$; $g_{22} = r^2 \sin^2\varphi$; $g_{33} = r^2$

Consider $\Gamma^1_{10} = \frac{1}{2} g^{11} \cdot \frac{\partial}{\partial x^0} g_{11}$ $\because \frac{\partial ?}{\partial x^j} = 0 \ \forall j \neq 0$ $\because$ g_{ii} terms are only time dependent

$g^{11} = \frac{1}{g_{11}}$ (Inverse relationship)

$g_{11} = \frac{a^2(t)}{1-kr^2}$

Thus $\Gamma^1_{10} = \frac{1}{2} \cdot \frac{\cancel{(1-kr^2)}}{a^2(t)} \cdot \frac{\partial}{\partial x^0} \frac{a^2(t)}{\cancel{(1-kr^2)}} = \frac{1}{\cancel{2}} \cdot \frac{1}{a^{\cancel{2}}(t)} \cdot \cancel{2a(t)}\, \dot{a}(t) = \frac{\dot{a}(t)}{a(t)}$

From the universe's isotropy, we can predict that $\Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30}$, but we're trying to be rigorous here dammit!

$$\Gamma^2_{20} = \frac{1}{2} g^{22}\, \partial_0 g_{22} = \frac{1}{2} \cdot \frac{\cancel{r^2 \sin^2\varphi}}{a^2(t)} \cdot \frac{\partial}{\partial t}\left(\frac{a^2(t)}{\cancel{r^2 \sin^2\varphi}}\right)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{a^{\cancel{2}}(t)} \cdot \cancel{2}\,\cancel{a(t)}\, \dot{a}(t) = \frac{\dot{a}(t)}{a(t)}$$

And finally,

$$\Gamma^3_{30} = \frac{1}{2} g^{33}\, \partial_0 g_{33} = \frac{1}{2} \cdot \frac{\cancel{r^2}}{a^2(t)} \frac{\partial}{\partial t} \frac{a^2(t)}{\cancel{r^2}} = \frac{\dot{a}(t)}{a(t)}$$

We have also just demonstrated the isotropic nature of the universe.

$$\therefore \Gamma^\mu_{\mu\nu} = 0 + 3\frac{\dot{a}}{a}$$

$$\Rightarrow \Gamma^\mu_{\mu\nu} u^\mu u^\nu = \Gamma^\mu_{\mu 0} u^\mu u^0$$

only u^0 terms are non-zero

$$= 3\frac{\dot{a}}{a} \cdot 1 \cdot 1$$

$$= 3\dot{a}/a$$

Now consider the term $(\rho c^2 + p)\, u^\mu \nabla_\mu u^\nu$

drop $(\rho c^2 + p)$ for now for convinience,

$u^\mu \nabla_\mu u^\nu$; u^0 is the only non-zero term

$= u^0 \nabla_0 u^\nu$

$= 1\left(\partial_0 u^\nu + \Gamma^\nu_{00} u^0 \right)$ $\because$ other lower indices of the Christoffel symbols go to zero since $g_{ij} = \delta_{ij}$. (some terms)

$\partial_0 u^0 = 0$ $\because$ u^0 is a constant

and $u^i = 0$, $i = 1, 2, 3$

We know $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\eta} \left[\frac{\partial g_{\eta\nu}}{\partial x^\mu} + \frac{\partial g_{\rho\mu}}{\partial x^\nu} - \frac{g_{\mu\nu}}{\partial x^\eta} \right]$ (by definition)

$$\Gamma^0_{00} u^0 = \frac{1}{2} g^{0\eta} \left[\frac{\partial g_{\eta 0}}{\partial x^0} + \frac{\partial g_{\rho 0}}{\partial x^0} - \frac{g_{00}}{\partial x^\eta} \right] \times 1$$

We have shown previously that $\Gamma^0_{00} = 0$

$\Rightarrow \Gamma^0_{00} u^0 = 0$

The remaining $\Gamma^i_{00} u^0$ terms go to zero $\because \Gamma^i_{00} = 0$

Thus $(\rho c^2 + p)\, u^\mu \nabla_\mu u^\nu = 0$

Thus, $\nabla_\mu T^{\mu\nu} = (\dot\rho c^2 + \dot p) + 3\frac{\dot a}{a}(\rho c^2 + p) + (-c^2 \dot p) = 0$

$= c^2(\dot\rho + \cancel{\dot p}) + c^2\, 3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) + \cancel{c^2(-\dot p)} = 0$

$= \cancel{c^2}\dot\rho + 3\cancel{c^2}\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) = 0$

$= \dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) = 0$

Equation of State

The equation of state of the fluid gives us the relationship

$$p = p(\rho)$$

$$p = w \rho c^2$$

where w is a dimless const, the EoS parameter of the fluid

Hereafter, $c^2 = 1$ (in the units we love)

$$\Rightarrow E = mc^2 \rightarrow E = m$$

and $p = w\rho$

EoS of Radiation

Radiation is relativistic and massless, and they have momentum instead. They thus exert pressure.

pressure is a third of energy density, this is experimental ?

$$p_r = \frac{1}{3} \rho_r$$

Thus, the EoS parameter of the radiation of the universe $w_r = 1/3$

EoS of Matter

Non-relativistic material is observed to be "cold" (negligible thermal motion) and pressureless.

$$p_m = 0$$

$$\Rightarrow \quad w_m = 0$$

The EoS parameter of the matter component of the universe is 0.

EoS of Dark Energy

Dark energy is believed to have negative pressure since it causes repulsive gravity and thus an accelerated expansion of the universe.

$$p_{de} = w_{de} \rho_{de}$$

where $w_{de} < 0$

w_{de} is unknown as of yet.
Its expected to be time dependent

If $w_{de} > -1$, quintessence

else if $w_{de} = -1$, vaccume energy

else if $w_{de} < -1$, phantom

Effective EoS

Since both pressure and energy densities are additive, we may define effective pressure and effective energy density.

$$p_{eff} = p_r + p_m + p_{de}$$

$$\rho_{eff} = \rho_r + \rho_m + \rho_{de}$$

and effective EoS parameter may be thus defined

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}}$$

The Friedmann continuity equation is defined for effective pressures and densities.

$$\dot{\rho}_{eff} + 3\frac{\dot{a}}{a}\left(\rho_{eff} + p_{eff}\right) = 0$$

$$p_{eff} = w_{eff}\,\rho_{eff}$$

$$\Rightarrow \dot{\rho}_{eff} + 3\frac{\dot{a}}{a}\left(1 + w_{eff}\right)\rho_{eff} = 0$$

ASSUME that the components interact strictly gravitationally.

$$\dot{\rho}_{r} + 3\frac{\dot{a}}{a}\left(1 + w_{r}\right)\rho_{r} = 0$$

lets integrate this equation

$$\frac{d\rho_r}{dt} + \frac{3\frac{da}{dt}}{a}\left(1 + \frac{1}{3}\right)\rho_r = 0$$

$$\Rightarrow \frac{d\rho_r}{dt} + \frac{4\frac{da}{dt}}{a}\rho_r = 0$$

$$\Rightarrow \frac{d\rho_r}{\cancel{dt}} = -4\frac{da}{\cancel{dt}} \cdot \frac{\rho_r}{a}$$

$$\Rightarrow \int \frac{1}{\rho_r}\, d\rho_r = \int -\frac{4}{a}\, da$$

$$\Rightarrow \ln \rho_r = -4 \ln a + C$$

$$\Rightarrow \rho_r = a^{-4}\rho_{r0} \quad \text{where } \rho_r(a=1) = \rho_{r0}$$

$$\rho_r = \rho_{r_0} a^{-4} \quad \text{where}$$

$$\rho_{r_0} = \rho_r(a=1)$$

similarly, we find that

$$\rho_m = \rho_{m_0} a^{-3}$$

$$\rho_{m_0} = \rho_m(a=1)$$

Matter density decreases with a^{-3}, but the volume of the universe scales with a^3. So this makes sense.

However, we see that energy density scales with a^{-4}. This extra $1/a$ term comes from the fact that the wavelength of photons is stretched, causing them to lose energy.

From $\dot{\rho}_{de} + 3\frac{\dot{a}}{a}(1 + w_{de})\rho_{de} = 0$,

we get

$$\rho_{de} = \rho_{de0} \exp\left(-3\int_1^a \frac{1+w_{de}}{a'} da'\right)$$

where ρ_{de0} is the value of ρ_{de} at the present universe

The Cosmological Constant

If we consider the universe to be static,

$a = a_0$ and $\dot{a} = 0$, $\ddot{a} = 0$,

Thus the Friedmann equations reduce to

$$① : \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho_0$$

$$\Rightarrow \frac{kc^2}{a_0^{\,2}} = \frac{8\pi G}{3}\rho_0$$

$$② : 2\left(\frac{\ddot{a}}{a_0}\right) + \left(\frac{\dot{a}}{a_0}\right)^2 + \frac{kc^2}{a_0^2} = \frac{-8\pi G}{c^2}p_0$$

$$\Rightarrow \frac{kc^2}{a_0^2} = \frac{-8\pi G}{c^2}p_0$$

$\rho_0 > 0$ and k curvature > 0

Thus p_0 must be negative, pressure of matter.

A constant was thus introduced to Einstein's Field Equations.

" Constant Lorentz Invariant " term $\Lambda g_{\mu\nu}$

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$, which

results in the Equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho \quad \&$$

$$2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{c^2} p$$

The Λ terms can be treated as a fluid contributing to the energy of the universe.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}(\rho + \rho_\Lambda)$$

$$2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}(p + p_\Lambda)$$

where $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$ and $p_\Lambda = -\frac{\Lambda c^4}{8\pi G}$

We may thus compute the EoS parameter of the Λ fluid:

$$w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = \frac{-\Lambda c^2}{8\pi G} \cdot \frac{8\pi G}{\Lambda c^4}$$

$$= -\frac{1}{c^2} < 0$$

Since $w_\Lambda < 0$, it represents a repulsive pressure, given $\Lambda > 0$.

$\lambda = \Lambda c^2$ is the cosmological constant.

Cosmological Parameters of the Universe

The Hubble Parameter

The Hubble parameter quantifies the change in distance scales or the expansion rate of the expanding universe.

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

H is the ratio between physical velocity and physical distance.

$$H = \frac{\frac{d(a\chi)}{dt}}{(a\chi)},$$

where χ = comoving distance

χ is a constant

Present value $H_0 = H(t_0) \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$

We have the relationship

$$v = H_0 d \quad \text{(Hubble's Law)}$$

Thus $t_H = H_0^{-1} = 13.96$ Gyr

This estimation of the age of the universe is called "Hubble Time" or "Hubble Age of the Universe".

Density Parameters and the Friedmann Equation

We may write the second Friedmann equation to be

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

Critical density ρ_c is the density of a flat universe (Gaussian curvature $k=0$).

$$\rho_c = \frac{3H^2}{8\pi G}$$

When averaged over scales of above 100Mpc,

$$\rho_c \approx \rho$$

We may define the density parameter:

$$\Omega = \frac{\rho}{\rho_c}$$

$$H^2 + \frac{Kc^2}{a^2} = \frac{3H^2}{3H^2} \cdot \frac{8\pi G}{3}\rho$$

$$\Rightarrow \frac{1}{H^2}\left(H^2 + \frac{Kc^2}{a^2}\right) = \frac{\rho}{\rho_c} = \Omega$$

$$\Rightarrow \quad \Omega = 1 + \frac{Kc^2}{a^2 H^2}$$

$$\Rightarrow \quad \Omega - 1 = \frac{Kc^2}{a^2 H^2}$$

is another form of the First Friedmann Equation

We also define

$$\Omega_k = -\frac{Kc^2}{a^2 H^2}$$

We may thus write the First Friedmann Equation to be

$$\Omega + \Omega_k = 1$$

" Density Parameter Form "

Since

$$\rho = \rho_r + \rho_m + \rho_{de}$$, we may write

$$\Omega = \Omega_r + \Omega_m + \Omega_{de},$$

Thus the Friedmann Equation comes out to be

$$(\Omega_r + \Omega_m + \Omega_{de}) + \Omega_k = 1$$

From the continuity equation we know

$$\rho_r = \rho_{r_0} a^{-4}, \quad \rho_m = \rho_{m_0} a^{-3}$$

$$\rho_{de} = \rho_{de_0} f(a), \text{ where } f(a) = e^{-3\int_1^a \frac{w_{de}}{a} da}$$

$$\rho = \rho_{r_0} a^{-4} + \rho_{m_0} a^{-3} + \rho_{de_0} f(a)$$

It follows that

$$\Omega = \Omega_{r0}\, a^{-4} + \Omega_{m0}\, a^{-3} + \Omega_{de0}\, f(a)$$

where $$\Omega_{i0} = \frac{\rho_{i0}}{\rho_{c0}}$$

Pertubed

Spacetime

Consider an unperturbed spatially flat FLRW metric

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \delta_{ij}\, dx^i dx^j\right)$$

where

$$d\tau \overset{\text{def}}{=} \frac{dt}{a(\tau)}$$

Small perturbations $h_{\mu\nu}$ are introduced

$$g_{\mu\nu} = a^2(\tau)\left[\eta_{\mu\nu} + h_{\mu\nu}\right]$$

$\eta_{\mu\nu} \equiv$ Minkowski metric

Considering the spatial components of the perturbation, we may decompose it:

$$h_{ij} = h_{ii} \quad \text{<trace part>} + h_{ij}(1-\delta_{ij}) \quad \text{<traceless>}$$

$$= \underset{\text{trace}}{\frac{1}{3} h \delta_{ij}} + \underset{\substack{\text{longitudanal} \\ \text{scalar}}}{h^{\parallel}_{ij}} + \underset{\substack{\text{transverse} \\ \text{vector}}}{h^{\perp}_{ij}} + \underset{\substack{\text{transverse} \\ \text{traceless} \\ \text{tensor}}}{h^{T}_{ij}}$$

① Trace part:

$$\frac{1}{3} h \delta_{ij} \; ; \; \text{where } h = \delta^{ij} h_{ij}$$

This component captures the volume dialation in space.

② Longitudinal (traceless) scalar part:

$$h^{\parallel}_{ij} \stackrel{\text{def}}{=} \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) \mu(\tau, \vec{x})$$

→ Symmetric and curl-free

$$\epsilon_{ijk}\,\partial_j\,\partial_l\,h^{\parallel}_{lk} = 0$$

→ Purely scalar.

③ Transverse (vector) part $h^{\perp}_{ij}$

→ defined via divergenceless vector field A_i

$$h^{\perp}_{ij} \stackrel{\text{def}}{=\!=} \partial_i A_j + \partial_j A_i$$

$$\left(\partial_i A_i = 0\right)$$

→ divergenceless in double derivatives

$$\partial_i\,\partial_j\,h^{\perp}_{ij} = 0$$

→ Contains the vector degrees of freedom that often decay in linear cosmology.

(4) Transverse traceless tensor h^T_{ij}

$$\partial^i h^T_{ij} = 0 ,$$

$$\delta^{ij} h^T_{ij} = 0 ;$$

Represents the gravitational wave modes.

* These are gauge invariant, physically observable degrees of freedom.

* Scalars, vectors and tensors transform independently under rotations and coordinate transforms.

Each type of perturbation may be studied independently.

This comes from Helmholtz (irreducable) decomposition for a symmetric tensor.

Gauges

Gauges are specific coordinate systems.

Not all perturbations are physical, some are just artifacts of the coordinate choice

Synchronous Gauge

Impose $h_{0\nu} = 0$

No time-time or time-space perturbations.

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \left(\delta_{ij} + h_{ij}(\tau, \vec{x})\right)dx^i\, dx^j\right]$$

No perturbations in g_{00} or g_{i0}

In this gauge, comoving observers measure the same cosmic time.

Newtonian Gauge

$$h_{i0} = 0,$$
$$h_{ij} \propto \delta_{ij}$$

$$ds^2 = a^2(\tau)\left[-(1+\Phi)d\tau^2 + (1-2\Psi)\,\delta_{ij}\,dx^i dx^j\right]$$

Φ : Newtonian Gravitational potential

Ψ : Spatial curvature perturbation

All scalar modes are captured by functions Φ and Ψ. These functions are termed Bardeen potentials.

Gauge Transformations

Consider the unperturbed FLRW metric

$$g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu} \quad ; \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

Introducing scalar perturbations,

$$g_{00} = -a^2(\tau)\left[1 + 2\psi(\vec{x}, \tau)\right]$$

$$g_{0i} = a^2(\tau)\, w_i(\vec{x}, \tau)$$

$$g_{ij} = a^2(\tau)\left[\{1 - 2\phi(\vec{x}, \tau)\}\delta_{ij} + \chi_{ij}(\vec{x}, \tau)\right]$$

χ_{ij} is considered traceless ; trace components are absorbed into ϕ.

$\psi, w_i, \phi, \chi_{ij} \ll 1$, they represent metric perturbations

Now we consider a general coordinate transform

$$x^\mu \to \tilde{x}^\mu = x^\mu + d^\mu(x^\nu)$$

$$\tilde{x}^0 = x^0 + \alpha(\vec{x}, \tau)$$

$$\vec{\tilde{x}} = \vec{x} + \vec{\nabla}\beta(\vec{x}, \tau) + \vec{\epsilon}(\vec{x}, \tau)$$

$$\vec{d}(\vec{x}) = \vec{\nabla}\beta(\vec{x}, \tau) + \vec{\epsilon}(\vec{x}, \tau)$$ # Helmholtz decomposition

$$\text{ST } \vec{\nabla}\cdot\vec{\epsilon} = 0 \quad (\text{divergence free})$$

$$\& \ \vec{\nabla}\times(\vec{\nabla}\beta) = 0 \ (\text{curl free})$$

We require that ds^2 is invariant under this transformation.

Deriving the Gauge Transformation Expressions

We consider the metric tensor in two different coordinate systems $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$.

* ds^2 must be invariant under gauge transformations.

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = \tilde{g}_{\mu\nu}\, d\tilde{x}^\mu d\tilde{x}^\nu \quad - (1)$$

We define a transformation as $x^\mu \to \tilde{x}^\mu$ where

$$\tilde{x}^\mu = x^\mu + \xi(x) \quad - (2)$$

$\xi(x)$ is small.

From (1), we write

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\nu} \quad - (3)$$

Equation (2) gives us the forward map for $\tilde{x}^\mu$

$$\tilde{x}^\mu(x) = x^\mu + \xi(x)$$

To evaluate Eq. (3), we require a reverse map.

let $F(x) = x^\nu + \xi(x)$

and $x^\nu = G(F(x)) \Rightarrow G^\nu(\tilde{x}) = \tilde{x}^\nu + \Delta^\nu(\tilde{x})$

Or $\tilde{x}^\nu = F(G(x))$

$\Rightarrow \tilde{x}^\nu = F(\tilde{x}^\nu + \Delta^\nu(\tilde{x})) = (\tilde{x}^\nu + \Delta^\nu(\tilde{x})) + \xi(\tilde{x}^\nu + \Delta^\nu(\tilde{x}))$

$\Rightarrow 0 = \Delta^\nu(\tilde{x}) + \xi(\tilde{x}^\nu + \Delta^\nu(\tilde{x}))$

We know that the Taylor expansion of a function w/ n degrees of freedom is

$$f(x+\delta) = f(x) + \frac{\partial f}{\partial x^i}(x)\,\delta^i + \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(x)\,\delta^i\delta^j + \cdots$$

Second order terms are ignored here since $\xi(x)$ and $\Delta(\tilde{x})$ are both small.

$$\Rightarrow 0 = \Delta^\nu(\tilde{x}) + \xi^\nu(\tilde{x}) + \Delta^\nu \frac{\partial}{\partial x^\nu}\xi^\nu(\tilde{x}) + O(\xi^2)$$

$$\Rightarrow 0 = \Delta^\nu(\tilde{x}) + \xi^\nu(\tilde{x}) + O(\xi^2)$$

$$\Rightarrow \Delta^\nu(\tilde{x}) = -\xi^\nu(\tilde{x}) + O(\xi^2)$$

Thus the reverse map

$$x^\nu = \tilde{x}^\nu - \xi^\nu(\tilde{x}) + O(\xi^2) \quad - \quad (4)$$

Now we consider $g_{\mu\nu}(x)$;

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x(\tilde{x}))$$ (Represent in new coordinates)

We just showed in Eq. 4, $x^\mu(\tilde{x}) = \tilde{x}^\mu - \xi^\mu(\tilde{x})$

$$g_{\mu\nu}(x) = g_{\mu\nu}(\tilde{x} - \xi(\tilde{x}))$$
$$= g_{\mu\nu}(\tilde{x}) + \frac{\partial}{\partial \tilde{x}^\rho} g_{\mu\nu}(\tilde{x})(-\xi^\rho(\tilde{x})) + O(\xi^2) \quad -5$$

Now we may revisit Eq. 3,

$$\tilde{g}_{\mu\nu} = g_{\alpha\beta}(x(\tilde{x})) \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu}$$

$\Rightarrow \tilde{g}_{\mu\nu} =$

$$g_{\alpha\beta}(\tilde{x}) - \xi^\rho(\tilde{x}) \partial_\rho g_{\alpha\beta}(\tilde{x}) \cdot \frac{\partial}{\partial \tilde{x}^\mu}(\tilde{x}^\alpha - \xi^\alpha(\tilde{x})) \cdot \frac{\partial}{\partial \tilde{x}^\nu}(\tilde{x}^\beta - \xi^\beta(\tilde{x})) + O(\xi^2)$$

$$= \left[g_{\alpha\beta}(\tilde{x}) - \xi^\rho(\tilde{x}) \partial_\rho g_{\alpha\beta}(\tilde{x})\right] \cdot \left(\delta^\alpha_\mu - \partial_\mu \xi^\alpha(\tilde{x})\right) \cdot \left(\delta^\beta_\nu - \partial_\nu \xi^\beta(\tilde{x})\right)$$

$$= \left[g_{\alpha\beta}(\tilde{x}) - \xi^\rho(\tilde{x}) \partial_\rho g_{\alpha\beta}(\tilde{x})\right] \cdot \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\nu \partial_\mu \xi^\alpha(\tilde{x}) - \delta^\alpha_\mu \partial_\nu \xi^\beta(\tilde{x}) + \partial_\mu \partial_\nu \xi^\alpha \xi^\beta\right)$$

$$= \left[g_{\alpha\beta}(\tilde{x}) - \xi^{\rho}(\tilde{x})\partial_{\rho} g_{\alpha\beta}(\tilde{x})\right] \cdot \left(\delta^{\alpha}_{\nu}\delta^{\beta}_{\gamma} - \delta^{\beta}_{\gamma}\partial_{\nu}\xi^{\alpha}(\tilde{x}) - \delta^{\alpha}_{\nu}\partial_{\gamma}\xi^{\beta}(\tilde{x}) + \partial_{\nu}\partial_{\gamma}\xi^{\alpha}\xi^{\beta}\right)$$

$O(\xi^2)$ terms are ignored, since ξ is small, and we require a linear solution

$$= \left[g_{\alpha\beta} - \xi^{\rho}\partial_{\rho} g_{\alpha\beta}\right]\left(\delta^{\alpha}_{\nu}\delta^{\beta}_{\gamma} - \delta^{\beta}_{\gamma}\partial_{\nu}\xi^{\alpha} - \delta^{\alpha}_{\nu}\partial_{\gamma}\xi^{\beta}\right)$$

$$= \delta^{\alpha}_{\nu}\delta^{\beta}_{\gamma} g_{\alpha\beta} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\gamma}\xi^{\rho}\partial_{\rho} g_{\alpha\beta} - \delta^{\beta}_{\gamma} g_{\alpha\beta}\partial_{\nu}\xi^{\alpha} - \delta^{\alpha}_{\nu} g_{\alpha\beta}\partial_{\gamma}\xi^{\beta}) + O(\xi^2)$$

$$= g_{\nu\gamma} - \xi^{\rho}\partial_{\rho} g_{\nu\gamma} - g_{\alpha\gamma}\partial_{\nu}\xi^{\alpha} - g_{\nu\beta}\partial_{\gamma}\xi^{\beta}$$

Thus, we have shown

$$\tilde{g}_{\nu\gamma}(\tilde{x}) = g_{\nu\gamma}(\tilde{x}) - \xi^{\rho}(\tilde{x})\partial_{\rho} g_{\nu\gamma}(\tilde{x}) + - g_{\alpha\gamma}(\tilde{x})\partial_{\nu}\xi^{\alpha}(\tilde{x}) - g_{\nu\beta}(\tilde{x})\partial_{\gamma}\xi^{\beta}(\tilde{x}) \quad \text{— (5)}$$

By definition, the perturbed metric is written as

$$g_{00} = -\, a^2(\tau)\left[1 + 2\psi(\vec{x},\tau)\right] \quad - \text{(6a)}$$

$$g_{0i} : a^2(\tau)\, w_i(\vec{x},\tau) \quad - \text{(6b)}$$

$$g_{ij} : a^2(\tau)\left[\{1 - 2\varphi(\vec{x},\tau)\}\delta_{ij} + \chi_{ij}(\vec{x},\tau)\right], \chi_{ii} = 0 \quad - \text{(6c)}$$

The resultant line element is

$$ds^2 = a^2(\tau)\Big[-\left(1 + 2\psi(\vec{x},\tau)\right)d\tau^2 + 2\, w_i(\vec{x},\tau)\, d\tau\, dx^i + \left\{(1 - 2\varphi(\vec{x},\tau))\delta_{ij} + \chi_{ij}(\vec{x},\tau)\right\} dx^i dx^j\Big]$$

Now we derive the transformations for each function individually using Equation (5).

$$\tilde{g}_{00} = g_{00} - \xi^\rho \partial_\rho g_{00} - g_{\alpha 0}\partial_0 \xi^\alpha - g_{0\beta}\partial_0 \xi^\beta \quad - \text{(5.1)}$$

Going term-by-term,

$$-\xi^\rho \partial_\rho g_{00} = -\xi^0 \partial_0 g_{00} - \xi^i \partial_i g_{00}$$

$$= -\alpha(\vec{x},\tau)\frac{\partial}{\partial\tau} - a^2(\tau)\left\{1 + 2\psi(\vec{x},\tau)\right\} +$$

$$-\left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)\partial_i \cdot - a^2(\tau)\left\{1 + 2\psi(\vec{x},\tau)\right\}$$

$$= \alpha(\vec{x},\tau)\left(2a(\tau)\dot{a}(\tau) + 2a^2(\tau)\dot{\psi}(\vec{x},\tau) + 4a(\tau)\dot{a}(\tau)\psi(\vec{x},\tau)\right.$$

$$\left. + \left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)\left(0 + a^2(\tau)\,\partial_i 2\psi(\vec{x},\tau)\right)\right.$$

$$= \alpha(\vec{x},\tau)\left(2a(\tau)\,\dot{a}(\tau) + 2a^2(\tau)\dot{\psi}(\vec{x},\tau) + 4\,a(\tau)\dot{a}(\tau)\psi(\vec{x},\tau)\right)$$
$$+ \left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)\left(0 + a^2(\tau)\,\partial_i 2\psi(\vec{x},\tau)\right)$$

$$= 2a\dot{a}\alpha + 2a^2\alpha\dot{\psi} + 4a\dot{a}\alpha\psi + (\vec{\nabla}\beta + \vec{\epsilon})\,a^2\,2\,\partial_i\psi$$

* All perturbations are small and thus $\sim O(\epsilon)$

Gauge transformations are also small and considered $\sim O(\epsilon)$

$$= 2a\dot{a}\alpha + O(\epsilon^2)$$

The next term

$$-g_{\alpha 0}\partial_0\xi^{\alpha} = -g_{00}\partial_0\xi^0 - g_{0i}\partial_0\xi^i$$

$$\Rightarrow -g_{\alpha 0}\partial_0\xi^{\alpha} = a^2(\tau)\left\{1 + 2\psi(\vec{x},\tau)\right\}\frac{\partial}{\partial\tau}\alpha(\vec{x},\tau) +$$
$$-w_i(\vec{x},\tau)\frac{\partial}{\partial\tau}\left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)$$

$$= a^2\dot{\alpha} + 2a^2\dot{\alpha}\psi - w_i\left(\vec{\nabla}\dot{\beta} + \dot{\vec{\epsilon}}\right)$$

$$= a^2\dot{\alpha} + O(\epsilon^2)$$

And the final term

$$-g_{0\beta}\partial_0\xi^{\beta} \text{ is similarly } = a^2\dot{\alpha} + O(\epsilon^2)$$

* Since we're dealing w/ linear perturbations, all $O(\epsilon^2)$ terms are dropped.

Putting everything together,

$$\tilde{g}_{00} = g_{00} + 2\dot{a}a\alpha + 2a^2\dot{\alpha}$$

$$\Rightarrow -a^2(1+2\tilde{\psi}) = -a^2(1+2\psi) + 2\dot{a}a\alpha + 2a^2\dot{\alpha}$$

$$\Rightarrow \quad \tilde{\psi} = \psi - \frac{\dot{a}}{a}\alpha - \dot{\alpha}$$

$$\Rightarrow \quad \tilde{\psi} = \psi - \mathcal{H}\alpha - \dot{\alpha}$$

$$\tilde{\psi}(\vec{x},\tau) = \psi(\vec{x},\tau) - \dot{\alpha}(\vec{x},\tau) - \mathcal{H}(\tau)\alpha(\vec{x},\tau) \qquad - (8)$$

From Equation (5), in spatio temporal indeces,

$$\tilde{g}_{i0} = g_{i0} - \xi^{\rho}\partial_{\rho} g_{i0} - g_{i\beta}\partial_0 \xi^{\beta} - g_{\alpha 0}\partial_i \xi^{\alpha} \qquad - (5.2)$$

Going term - by - term,

$$-\xi^{\rho}\partial_{\rho} g_{i0} = -\xi^0 \frac{\partial}{\partial\tau} g_{i0} - \xi^j \partial_j g_{i0}$$

Now $-\xi^0 \frac{\partial}{\partial\tau} g_{i0} =$

$$-\alpha(\vec{x},\tau)\frac{\partial}{\partial\tau} a^2(\tau)\, w_i(\vec{x},\tau)$$

α is of $O(\epsilon)$

and w is also of $O(\epsilon)$

$\therefore = O(\epsilon^2)$ and thus is subsequently dropped

And $-\xi^j \partial_j g_{i0} =$

$$-\left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)\partial_j\left(a^2(\tau)\, w_i(\vec{x},\tau)\right)$$

$$= O(\epsilon^2)$$

Please note the difference between the scalar ϵ and the vector field $\vec{\epsilon}(\vec{x},\tau)$ or $\epsilon_i(x_i,\tau)$

Thus $-\xi^\rho \partial_\rho g_{i0} = O(\epsilon^2)$

The next term

$$-g_{\alpha 0}\partial_i \xi^\alpha = -g_{00}\partial_i \xi^0 - g_{j0}\partial_i \xi^j$$

$$-g_{00}\partial_i \xi^0 = a^2(\tau)\{1+2\psi(\vec{x},\tau)\}\partial_i \alpha(\vec{x},\tau)$$

$$= a^2(\tau)\partial_i \alpha(\vec{x},\tau) + O(\epsilon^2) \quad \because \psi = O(\epsilon) \ \& \ \alpha = O(\epsilon)$$

$$-g_{j0}\partial_i \xi^j = -a^2(\tau)\, w_j(\vec{x},\tau)\,\partial_i\left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)$$

$$= O(\epsilon^2)$$

Thus, $-g_{\alpha 0}\partial_i \xi^\alpha = a^2(\tau)\partial_i \alpha(\vec{x},\tau) + O(\epsilon^2)$

Finally, we have the term

$$-g_{i\beta}\partial_0 \xi^\beta = -g_{i0}\partial_0 \xi^0 - g_{ij}\partial_0 \xi^j$$

$$-g_{i0}\partial_0 \xi^0 = -a^2(\tau)\, w_i(\vec{x},\tau)\,\frac{\partial}{\partial\tau}\,\alpha(\vec{x},\tau)$$

$$= O(\epsilon^2)$$

and

$$-g_{ij}\partial_0 \xi^j = -a^2(\tau)\Big[\{1+2\psi(\vec{x},\tau)\}\delta_{ij} + \chi_{ij}(\vec{x},\tau)\Big]\cdot \frac{\partial}{\partial\tau}\left(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\right)$$

$$= -a^2(\tau)\,\delta_{ij}\left(\nabla^j\mathring{\beta}(\vec{x},\tau) + \mathring{\epsilon}^j(\vec{x},\tau)\right)$$

$$= -a^2(\tau)\left(\nabla_i\mathring{\beta}(\vec{x},\tau) + \mathring{\epsilon}_i(\vec{x},\tau)\right)$$

Compiling all the terms,

$$a^2(\tau)\,\hat{w}_i(\vec{x},\tau) = a^2(\tau)\,w_i(\vec{x},\tau) - a^2(\tau)\,\partial_i\alpha(\vec{x},\tau) + - a^2(\tau)\left(\nabla_i\mathring{\beta}(\vec{x},\tau) + \mathring{\epsilon}_i(\vec{x},\tau)\right)$$

$$\hat{w}_i(\vec{x},\tau) = w_i(\vec{x},\tau) - \partial_i\alpha(\vec{x},\tau) - \left(\nabla_i\mathring{\beta}(\vec{x},\tau) + \mathring{\epsilon}_i(\vec{x},\tau)\right) \quad -9$$

Now we consider the strictly spatial indeces in Equation 5.

$$\tilde{g}_{ij} = g_{ij} - g_{i\alpha}\partial_j \xi^\alpha - g_{\beta j}\partial_i \xi^\beta - \xi^\rho \partial_\rho g_{ij} \quad (5.3)$$

In $g_{ij} = a^2(\tau)\left[\{1 + 2\psi(\vec{x},\tau)\}\delta_{ij} + \chi_{ij}(\vec{x},\tau)\right]$,

we have a traced component ψ and traceless component χ_{ij}

Tracing Equation (5.3),

$$\delta^{ij}\tilde{g}_{ij} = \delta^{ij}\left(g_{ij} - g_{i\alpha}\partial_j \xi^\alpha - g_{\beta j}\partial_i \xi^\beta - \xi^\rho \partial_\rho g_{ij}\right)$$

$$\delta^{ij}\hat{g}_{ij} = \hat{g}^i_i = \delta^{ij} a^2\left[\{1 - 2\tilde{\psi}\}\delta_{ij} + \tilde{\chi}_{ij}\right] = (1 - 2\tilde{\psi})$$

$$= \delta^i_i a^2(1 - 2\tilde{\psi}) + 0 = a^2(3 - 6\tilde{\psi})$$

The first two terms:

$$-\delta^{ij} g_{i\alpha}\partial_j \xi^\alpha = -g_{i\alpha}\partial^i \xi^\alpha = -g_{k\eta}\partial^k \xi^\eta$$

and $-\delta^{ij} g_{\beta j}\partial_i \xi^\beta = -g_{\beta j}\partial^j \xi^\beta = -g_{k\eta}\partial^k \xi^\eta$

Together, they are expressed as $-2g_{k\eta}\partial^k \xi^\eta$

$$-2g_{k\eta}\partial^k\xi^\eta = -2\left(g_{k0}\partial^k\xi^0 + g_{kl}\partial^k\xi^l\right)$$

$$g_{k0}\partial^k\xi^0 = a^2 w_k\,\delta^k_k\,\partial_k\alpha = O(\epsilon^2)$$

and

$$g_{kl}\,\partial^k\xi^l = a^2\Big[\{1+2\varphi\}\delta_{kl} + \chi_{kl}\Big]\delta^k_k\,\partial_k\left[\nabla^l\beta + \epsilon^l\right]$$

$$= a^2\,\mathrm{Tr}\left(\nabla^l\beta + \epsilon^l\right)$$

$$= \underline{\underline{a^2\nabla^2\beta}}$$

Thus,

$$-2g_{k\eta}\,\partial^k\xi^\eta = -2a^2\vec{\nabla}^2\beta$$

And finally,

$$\delta^{ij}\xi^\rho\partial_\rho g_{ij} = \xi^\rho\partial_\rho g^i_i$$

$$= \xi^0\partial_0 g^i_i + \xi^l\partial_l g^i_i$$

$$\xi^0\partial_0 g^i_i = \alpha\frac{\partial}{\partial\tau}a^2(3-6\varphi)$$

$$= 6a\dot{a}\alpha + O(\epsilon^2)$$

$$\xi^l\partial_l g^i_i = (\vec{\nabla}\beta + \vec{\epsilon})\,\partial_l\,a^2(3-6\varphi)$$

$$= (\vec{\nabla}\beta + \vec{\epsilon})(0 - 6a^2\partial_l\varphi)$$

$$= O(\epsilon^2)$$

Consolidating all these terms, we get

$$a^2(3-6\tilde{\psi}) = a^2(3-6\psi) - 2a^2\nabla^2\beta - 6a\dot{a}\alpha$$

$$\Rightarrow \tilde{\psi} = \psi - \frac{1}{3}a^2\nabla^2\beta - \frac{\dot{a}}{a}\alpha \; ; \quad \mathcal{H} = \frac{\dot{a}}{a}$$

$$\tilde{\psi}(\vec{x},\tau) = \psi(\vec{x},\tau) - \frac{1}{3}a^2(\tau)\nabla^2\beta(\vec{x},\tau) - \mathcal{H}(\tau)\alpha(\vec{x},\tau) \qquad - (10)$$

Finally, we must find the traceless component of Equation (5.3),

$$(5.3): \tilde{g}_{ij} = g_{ij} - g_{i\alpha}\partial_j\xi^\alpha - g_{\beta j}\partial_i\xi^\beta - \xi^\rho\partial_\rho g_{ij}$$

$$-g_{i\alpha}\partial_j\xi^\alpha = -g_{i0}\partial_j\xi^0 - g_{ik}\partial_j\xi^k$$

$$-g_{i0}\partial_j\xi^0 = -a^2 w_i\partial_j\alpha = O(\epsilon^2)$$

and
$$-g_{ik}\partial_j\xi^k = -a^2\left[\{1+2\psi\}\delta_{ik} + \chi_{ik}\right]\partial_j(\vec{\nabla}\beta + \vec{\epsilon})$$

$$= -a^2\partial_j\delta_{ik}(\nabla^k\beta + \epsilon^k) + O(\epsilon^2)$$

$$= -a^2\partial_j\xi_i$$

Thus $-g_{i\alpha}\partial_j\xi^\alpha = -a^2\partial_j\xi_i$

Similarly, $-g_{\beta j}\partial_i\xi^\beta = -a^2\partial_i\xi_j$

The last term,

$$-\xi^\rho \partial_\rho g_{ij} = -\xi^0 \partial_0 g_{ij} - \xi^k \partial_k g_{ij}$$

$$-\xi^0 \partial_0 g_{ij} = -\alpha \frac{\partial}{\partial \tau} a^2\left[\{1+2\psi\}\delta_{ij} + \chi_{ij}\right]$$

$$= -\alpha\,(2a\dot{a})\frac{\partial}{\partial\tau}\delta_{ij} + O(\epsilon^2)$$

δ_{ij} terms are trace only and thus dropped.

And $$-\xi^k \partial_k g_{ij} = -(\vec{\nabla}\beta + \vec{\epsilon})\,\partial_k\, a^2\left[\{1+2\psi\}\delta_{ij} + \chi_{ij}\right]$$

$$= \langle \delta_{ij} \text{ terms} \rangle + O(\epsilon^2)$$

Thus, our final expression for Eq. 5.3 is

$$\tilde{g}_{ij} = g_{ij} - a^2(\partial_i \xi_j + \partial_j \xi_i) + \text{trace terms}$$

To remove trace terms from $(\partial_i \xi_j + \partial_j \xi_i)$,

First we find $Tr(\partial_i \xi_j + \partial_j \xi_i)$

$$= \delta^{ij}(\partial_i \xi_j + \partial_j \xi_i)$$

$$= \partial^k \xi_k + \partial^k \xi_k = 2\partial^k \xi_k$$

We know that $$T_{\mu\nu}^{\text{Traceless}} = T_{\mu\nu} - \frac{1}{3}\delta_{\mu\nu}\left(\delta^{\mu\nu} T_{\mu\nu}\right)$$

Hence, we find the traceless expression of $(\partial_i \xi_j - \partial_j \xi_i)$ to be

$$\left(\partial_i \xi_j + \partial_j \xi_i - \frac{1}{3} \delta_{ij} 2 \partial^k \xi_k \right)$$

Ridding the Equation of all its traced components, we get

$$\tilde{\chi}_{ij} = \chi_{ij} - \left(\partial_i \xi_j + \partial_j \xi_i - \frac{2}{3} \delta_{ij} \partial^k \xi_k \right)$$

$$\partial_i \xi_j + \partial_i \xi_i$$
$$= \partial_i (\partial_j \beta - \epsilon_j) + \partial_j (\partial_i \beta - \epsilon_i)$$
$$= 2 \partial_i \partial_j \beta - (\partial_j \epsilon_i + \partial_i \epsilon_j)$$

and $\frac{2}{3} \delta_{ij} \partial^k \xi_k = \frac{2}{3} \delta_{ij} \partial^k \nabla_k \beta$ $\because$ ϵ^k is transverse

$= \frac{2}{3} \delta_{ij} \nabla^2 \beta$ $\because$ Trace term

$$\Rightarrow \tilde{\chi}_{ij} = \chi_{ij} - 2 \left\{ \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \beta - \frac{1}{2} \left(\partial_i \epsilon_j - \partial_j \epsilon_i \right) \right.$$

$$\tilde{\chi}_{ij}(\vec{x}, \tau) = \chi_{ij}(\vec{x}, \tau) - 2 \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \beta$$
$$- \left(\partial_i \epsilon_j - \partial_j \epsilon_i \right) \quad - (11)$$

Using Lie Derivatives

By definition, the perturbed metric is written as

$$g_{00} = -a^2(\tau)\left[1 + 2\psi(\vec{x},\tau)\right] \quad - (6a)$$

$$g_{0i} = a^2(\tau)\, w_i(\vec{x},\tau) \quad - (6b)$$

$$g_{ij} = a^2(\tau)\left[\{1 - 2\varphi(\vec{x},\tau)\}\delta_{ij} + \chi_{ij}(\vec{x},\tau)\right], \; \chi_{ii} = 0 \quad - (6c)$$

The resultant line element is

$$ds^2 = a^2(\tau)\Big[-(1 + 2\psi(\vec{x},\tau))\,d\tau^2 + 2\,w_i(\vec{x},\tau)\,d\tau\,dx^i + \{(1 - 2\varphi(\vec{x},\tau))\delta_{ij} + \chi_{ij}(\vec{x},\tau)\}\,dx^i dx^j\Big]$$

We define $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

where $\bar{g}_{\mu\nu}$ is the flat, unperturbed Minkowski space

and $\delta g_{\mu\nu}$ represents the pertubation

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$$

From the transformation in Eq. (2): $\tilde{x}^\mu = x^\mu + \xi^\mu(x)$

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_\xi\, g_{\mu\nu}(x) + O(\xi^2)$$

(By definition of the Lie derivative $\mathcal{L}_\xi$)

$$\tilde{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) + \mathcal{L}_\xi\left(\bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\right)$$

$$\tilde{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) + \mathcal{L}_\xi\left(\bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\right)$$

$\delta g_{\mu\nu}$ is small, $= O(\epsilon)$ and $\xi^\mu = O(\epsilon)$

thus $\mathcal{L}_\xi(\delta g_{\mu\nu}) = O(\epsilon^2)$, which is neglected.

Thus, $\tilde{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) + \mathcal{L}_\xi(\bar{g}_{\mu\nu}(x))$

$$\Rightarrow \tilde{\bar{g}}_{\mu\nu}(x) + \delta\tilde{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) + \mathcal{L}_\xi(\bar{g}_{\mu\nu}(x))$$

No matter the coordinates, the unperturbed FLRW background is the same

$$\tilde{\bar{g}}_{\mu\nu}(\tilde{x}) = \bar{g}_{\mu\nu}(x)$$

$$\Rightarrow \delta\tilde{g}_{\mu\nu}(x) = \delta g_{\mu\nu}(x) + \mathcal{L}_\xi \bar{g}_{\mu\nu}(x) \quad - \quad B.1$$

$$\mathcal{L}_\xi \bar{g}_{\mu\nu} = +\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad - \quad \text{Identity 1}$$

Covariant derivative

(Yet to prove identity)

If we consider only the temporal index in Eqation B.1, we may write

$$\delta\tilde{g}_{00} = \delta g_{00} + 2\nabla_0 \xi_0$$

$$\delta\tilde{g}_{00} = \tilde{g}_{00} - \bar{\tilde{g}}_{00} = -a^2(\tau)\{1 - 2\tilde{\psi}(\vec{x},\tau)\} - a^2(\tau)$$

$$\Rightarrow \delta\tilde{g}_{00} = 2a^2(\tau)\,\psi(\vec{x},\tau)$$

$$\text{and } \delta g_{00} = 2a^2(\tau)\,\psi(\vec{x},\tau)$$

$$2\nabla_0 \xi^0 = 2\Big[\partial_0\left(-a^2(\tau)\,\alpha(\vec{x},\tau)\right) - \Gamma^{\eta}_{00}\left(-a^2(\tau)\,\alpha(\vec{x},\tau)\right)\Big]$$

$$= 2\left[-2a\dot{a}\alpha - a^2\dot{\alpha} - \frac{\dot{a}}{a}(-a^2\alpha)\right]$$

Thus,

$$2a^2\hat{\psi} = 2a^2\psi + 2\left[-2a^2\frac{\dot{a}}{a}\alpha - a^2\dot{\alpha} + \frac{\dot{a}}{a}a^2\alpha\right]$$

$$\Rightarrow \tilde{\psi} = \psi - \dot{\alpha} - \alpha\mathcal{H}$$

$$\Rightarrow \tilde{\psi}(\vec{x},\tau) = \psi(\vec{x},\tau) - \dot{\alpha}(\vec{x},\tau) - \mathcal{H}(\tau)\,\alpha(\vec{x},\tau) \quad \text{— (B.2)}$$

This matches Eq. (8) derived conventionally.

The equations for the Gauge Transformations of the rest of the functions can be carried out similarly.

The Conformal - Newtonian Gauge

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)\,d\tau^2 + (1-2\phi)\,dx^i dx_i\right\}$$

in conformal time $dt = a(\tau)\,d\tau$

or $dt = a(\eta)\,d\eta$,

there are a lot of different notations here.

> Does not have gauge artifacts
> ψ and ϕ are gauge invariant
> In non-relativistic limits, ψ plays the role of the Newtonian gravitational potential.

A perturbed flat FLRW metric can be written as

$$ds^2 \overset{\text{def}}{=} g_{\mu\nu}\,dx^\mu dx^\nu$$
$$= g_{00}\,d^2x^0 + 2g_{0i}\,dx^0 dx^i + g_{ij}\,dx^i dx^j$$

$$ds^2 = a^2(\eta)\Big[-\{1+2\psi\} + 2w_i\,d\eta\,dx^i + \{(1-2\phi)\delta_{ij} + \chi_{ij}\}\,dx^i dx^j\Big]$$

$$ds^2 = a^2(\eta)\Big[-\{1+2\psi\} + 2\,w_i\,d\eta\,dx^i + \{(1-2\phi)\delta_{ij} + \chi_{ij}\}\,dx^i dx^j\Big]$$

There exist transformation laws

$$w_i = \partial_i E$$
$$\chi_{ij} = \partial_i \partial_j B$$

$$E \longmapsto E - \beta$$
$$B \longmapsto B + \alpha - \beta'$$
$$\phi \longmapsto \phi + \mathcal{H}\alpha + \alpha'$$
$$\psi \longmapsto \psi - \mathcal{H}\alpha$$

where $x^0 \longmapsto x^0 + \alpha$
$x^i \longmapsto x^i + \partial_j \beta$

We may gauge away B and E by $\beta = E$, $\alpha = \beta' - B$

And we are left with

$$ds^2 = -a^2(\eta)\Big[-(1+2\phi)\,d\eta^2 + (1-2\psi)\delta_{ij}\,dx^i dx^j\Big]$$

All non-index variables are functions of spacetime

$$(x^i, \eta)$$

Peculiar velocity

$$ds^2 = a^2(\eta)\left[-(1+2\Phi)\,d\eta^2 + (1-2\Psi)\,\delta_{ij}\,dx^i dx^j\right]$$

The fluid 4-velocity obeys

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$u^\mu = \frac{dx^\mu}{ds} \qquad (ds^2 = -d\tau^2\text{, proper time})$$

$$u^\mu = \frac{1}{a}(1-\Phi)\,;$$

$$u^i = \frac{1}{a}\delta u^i\,, \quad \delta u^i \ll 1$$

$$\Rightarrow \quad u_i = g_{ij} u^j = a^2(1-2\Psi)\frac{1}{a}\delta u^i \approx a\,\delta u^i$$

Utilising the Helmholtz decomposition,

$$\delta u_i = \underbrace{a\,\partial_i v(\vec{x},\eta)}_{\text{curl-free}} + \underbrace{a\,S_i}_{\text{div-free}}$$

In scalar perturbation theory, $S_i = 0$

$\because$ no anisotropic stress in Linear Perfect Fluids

- Primordial vorticity decays w/ $1/a^2$.
- Inflation produces 0 vector modes at the first order.

$\therefore S_i$ is dropped

Isotropy: A small vector δu^i can be written as the gradient of a scalar

$$\delta u^i = a\, \partial_i v(\vec{x}, \eta)$$

$\uparrow$ peculiar velocity

In a perfect fluid,

$$T^{\mu}_{\ \nu} = (\rho + p)\, u^{\mu} u_{\nu} + p\, \delta^{\mu}_{\ \nu}$$

$$\Rightarrow T^{i}_{\ 0} = (\rho + p)\, u^i u_0 + 0$$

$$= (\rho + p)\, \partial_i v$$

$T^{i}_{\ 0}$ gives us the momentum density.

Anisotropy (in off-diagonal terms) come entirely from perturbations (peculiar velocity)

Energy Density Perturbation

$$\rho_A = \bar{\rho}_A(\eta) + \delta\rho_A(\vec{x},\eta)$$

$$\Rightarrow \delta\rho_A(\vec{x},\eta) = \rho_A(\vec{x},\eta) - \bar{\rho}_A(\eta)$$

Defining a dimless contrast

$$\delta_A(\vec{x},\eta) = \frac{\delta\rho_A(\vec{x},\eta)}{\bar{\rho}_A(\eta)}$$

$$T^0{}_0 = (\rho + p)u^0 u_0 + p$$

$$u^0 u_0 \approx -1 \quad \because \quad u_0 = g_{00} u^0$$

$$\Rightarrow T_0{}^0 = -\rho$$

$$= -(\bar{\rho} + \delta\bar{\rho})$$

$$\Rightarrow \delta T^0_0 = -\delta\rho$$

Inflation Theory

Linear Perturbation Theory

In conformal time, the metric presents itself as

$$ds^2 = a^2(\eta)\Big[-(1+2\Phi)\,d\eta^2 + 2\,\partial_i B\, d\eta\, dx^i + \{(1-2\Psi)\,\delta_{ij} + 2\,\partial_i\partial_j E\}\,dx^i dx^j$$

– Bardeen

We will stick w/ the Newtonian Gauge

This representation is especially useful since in the linear order the Einstein equations for scalar, vector and tensor perturbations don't mix.

Postulate a few things, pedant these later –

Linearized Dynamics

Combining the Einstein Equations w/ energy momentum conservation, linearised around FLRW, and in the Fourier Space for a fluid EOS: $w = p/\rho$, and sound speed* $c_s^2 = \delta p/\delta\rho$, and anisotropic stress σ, Continuity:

$$\frac{\partial \delta}{\partial \eta} = -(1+w)\left(\theta - 3\frac{\partial \Phi}{\partial \eta}\right) - 3\mathcal{H}\left(c_s^2 - w\right)\delta$$

$$\left[\mathcal{H} = \frac{\mathring{a}(\eta)}{a(\eta)}\right]$$

Euler (momentum)

$$\frac{\partial \theta}{\partial \eta} = -\mathcal{H}(1-3w)\theta + k^2\left(\frac{c_s^2}{1+w}\delta + \Psi - \sigma\right)$$

The Einstein Constraints give a generalised Poisson relation

$$k^2\Phi = 4\pi G a^2 \sum_X \rho_X\left[\delta_X + 3\mathcal{H}(1+w_X)\frac{\theta_X}{k^2}\right],$$

$$\Phi - \Psi = 8\pi G a^2 \sum_X (\rho_X + P_X)\,\sigma_X$$

$\underline{\theta}$:

when a fluid's velocity is perturbed, we describe it w/ a velocity vector $\vec{v}(\vec{x}, \eta)$

However, in Fourier space, we work w/ its divergence

$v^i(\vec{x}, \eta)$:

$$\mathcal{F}[\vec{v}] = \int \frac{d^3k}{(2\pi)^3} \, v^i \, \mathrm{en}(i\vec{k} \cdot \vec{x})$$

and $\nabla \cdot \vec{v} = \partial_i v^i(\vec{x})$

$$\partial_i v^i(\vec{x}) \xmapsto{\mathcal{F}} i k_i v^i(\vec{x}) = \theta(k, \eta)$$

<u>δ:</u>

δ is known by many names

- energy desity contrast
- fractional energy density contrast

$$\delta_x = \frac{\rho_x - \bar{\rho}_x}{\bar{\rho}_x} = \frac{\delta \rho_x}{\bar{\rho}_x}$$

This dimless value makes it very easy to compare perturbations to the background.

$\delta \ll 1$: Linear Perturbations

Raw density perturbations ($\delta \rho$) have units which change based on the epoch.

We may build a Gauge-Invariant, comoving $\underbrace{\text{fractional energy density perturbation}}_{\text{same shi by the way,}}$

$$\Delta = \delta + 3\mathcal{H}(1+w)\frac{\theta}{k^2}$$

The R parameter is the baryon energy density : photon energy density ratio.

$$R = \frac{3\rho_b}{4\rho_\gamma} \quad \left[p_\gamma = \rho_\gamma / 3 \right]$$

Before recombination*, the sound speed of photon-baryon fluid is $C_s^2 = \dfrac{1}{3(1+R)}$

Taking an arbitrary density contrast $\delta(\vec{x})$, and Fourier Transform it.

$$\delta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \delta(k) \exp(i\vec{k}\cdot\vec{x})$$

Quantum fluctuations are of quantum origin and thus modelled statistically.

The power spectrum is the variance of modes:

$$\langle \delta(k) \delta^*(k') \rangle = (2\pi)^3 \delta^{(3)}(k-k') P(k)$$

Thus $P(k)$ tells us the average size of fluctuations at the wavenumber 'k'.

The statistical behavior of a bunch of particles is described by a distribution function.

How many photons are @ $\vec{x}$ w/ $\vec{p}$ @ time τ:

$$f(\vec{x}, \vec{p}, \tau)$$

The Boltzman equations model how f() changes w/ time τ.

$$\frac{\partial f_i}{\partial \tau} = \text{Collision term } C_i$$

The early universe is mostly smooth, and thus

$$f = \bar{f} + \delta f$$

Fourier Space

The resultant perturbation equations we get are coupled linear DEs, and solving them in the real space is non-trivial.

Carrying out a Fourier transform gives us a simple ODE for a given wavenumber 'k'.

$$u(\vec{x},t) \longmapsto \hat{u}(k,t)$$

$$\frac{d^n u(\vec{x},t)}{dx^n} = (ik)^n \hat{u}(k,t)$$

This only works if the equation is linear.

$$u(x,t)\,v(x,t) \neq \hat{u}(k,t)\,\tilde{v}(k,t)$$

$$= \int_{-\infty}^{\infty} \hat{u}(k-k',t)\,\hat{v}(k',t)\,dk'$$

For functions in more than one spatial component,

$$\nabla u \longmapsto (i\vec{k})\hat{u}$$

Perturbation of the Einstein Boltzman Equations

. C code in this notebook;

who would've ever guessed?

Monte Carlo Monte-Python next, I'm

sure.

CLASS - Cosmic Linear Anisotropy Solving System

General Layout:

- > source/ - Core .c files (background.c, perturbations.c, etc.)
- > build/ - stores .o output files w/ machine code (background.o, perturbations.o, etc.)
- > main/ - holds class.c, the main file that reads .ini files.
- > tests/ - developer tests: `a.ini` and expected output files
- > include/ - header files (background.h, trigonometric_integrals.h, etc.)
- > tools/ - holds .c files for mathematical tools used throughout CLASS.
  Eg: sparse.c (sparse matrices), quadrature.c (num integrals)

background.c

Defines the variables for the background universe.

perturbations.c

Mathematical Structure:

- CLASS uses spline interpolation from interpolation tables stored in structures.

Wrapper - Class - Call():

- Defined in /include/common.h.
- Wraps a function call, checking for errors
- Prevents error cascades
- Safely aborts errors.

Solvers:
(Found in /tools/)

- evolver_rkck.c : RK 4/5 solver
- evolver_ndf15.c : BDF solver
- quadrature.c : Solves singular integrals, whereas the evolvers solve for coupled ODEs. This has various rules to solve definite integrals.

Background.c

background.h - declares all variables and structs

The background struct that is defined is called pba and it holds all the interpolation tables →

Density Parameters Ω

$$\Omega_{x_0} = \frac{\rho_{x_0}}{\rho_{c_0}}$$

- photons
- baryonic
- Cold dark matter
- non-relativistic neutrinos
- interating dark matter
- interacting radiation

where $\rho_{c_0} = \frac{3H_0^2}{8\pi G}$ (critical)

There is only one place where the Friedmann Equations are defined.

$$H^2(a) = H_0^2\left[\frac{\Omega_{r_0}}{a^4} + \frac{\Omega_{m_0}}{a^3} + \frac{\Omega_{k_0}}{a^2} + \Omega_{DE}(a)\right]$$

where Ω_k is the Gaussian curvature

$$H^2(a) = \frac{8\pi G}{a^2}\rho(a) - \frac{k}{a^2}$$

and $\Omega_{DE}(a) = \Omega_{DE_0}\, e^{\left(-3\int \frac{1+w(a)}{a}\, da\right)}$

`Double` variables and interpolation tables are initialised

Interpolation tables: a, $H(a)$, $\rho_x(a)$, $p_x(a)$, Ω_x

Helper functions carry out the spline interpolation on the interpolation tables.

Workflow:

* Numerical integration on {B} and {C} variables.

{B} Scalar field used to model DE (ϕ, $\dot{\phi}$); Quintessence, NCDM (non-cold dark matter)

{C}
+ Comoving radial distance $\chi(a) = \int \frac{c}{a^2 H(a)}\, da$
+ Cosmic time $t(a) = \int \frac{da}{a H(a)}$
+ Conformal time $\tau(a) = \int \frac{da}{a^2 H(a)}$
+ Linear growth factor
+ Sound horizon $r_s(a) = \int \frac{c_s(a)}{a^2 H}\, da$

* {A} variables are defined using simple power scaling of `a` and {B} variables. {A} variables do not rely on {C} variables.

background.c functions

background_solve() • Called once during background_init(), numerically solves the Friedmann Equations[1], Conservation equations[2], and the Conformal time integral[3].

• Results are stored in the interpolation tables in

pba → background_table

pba → loga_table

background_functions() Returns background values at a given τ or a. Carries out spline interpolation multiple times in one call.

→ background_at_tau()

→ background_at_z()

```
double loga; // Our normalisation: loga ≡ log(a/a_0)
loga = -log(1+z);
array_interpolate_spline(); // Array interpolation helper function
```

May change based on interpolation mode (inter_normal, inter_closeby, ~_linear, etc.)

pba → bg_size; // Number of background quantities

Spline interpolation interpolates 'bg_size' number of points around the requested loga value.

The interpolated points are stored in 'pvecback', a vector.

1 $$H^2(a) = \frac{8\pi G}{3}\rho(a)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

2 $$\dot{\rho} + 3H(\rho + p) = 0 \iff \frac{d\rho}{da} = -\frac{3}{a}(\rho + p)$$

3 $$\tau(a) = \int_0^a \frac{da}{a^2 H(a)} \; ; \quad da = aH\,dt$$

line 1922

* loga_final = 0.; defines end of solver init w/ background_solve()

∘ Somewhat breaks the $\log(a/a_0) = \log(a)$ convention.

Generic solver uses loga_final to make the τ table which feeds start and end values to evolvers (RK45, BDF)

Perturbations.c

include perturbations.h // Variable definitions

include parallel.h // Cofigures logic for parallel computation OpenMP multithreading.

perturbations_init() → Runs once in the begining after background_init() and thermodynamics_init().

Params: struct precision* ppr; struct background* pba;
struct thermodynamics* pth;
struct perturbations* ppt;

} → Numerical settings for class.

* Integration tolerances:
  - background_tol
  - tol_thermo
  - tol_perturb_integration

* Sampling modes:
  - k_min, k_max (k modes)

* Interpolation density:
  - Number of background sampling points

/precisions.h → holds default values

/common.h → holds struct precision {

Uses macros to pull values

};

/macros_precision.h → Not to be edited. Holds macros that overwrite default values from *.ini parameters

perturbations_free() → At the end, when no more perturbation calls are to

① Integrate perturbation equations, store the terms contributing to the source functions in terms of τ.

② Perform "some manipulations" of these terms to get the actual source functions $S^X(k, \tau)$. They are stored in terms of τ in an interpolation table.

```
struct perturbations {
  int index_md_*;
```

Integers store the indices of various modes. Under these cols values stored make up the interpolation table.

```
  int index_ic_*;
```

Unused ics are given a value `-1`, telling CLASS that they are not to be generated.

```
  int ic_size;
};
```

pbt → Sources[index_md][index_ic * pbt → tp_size[index_md] + index_tp]

Index assignment is done by the function perturbation_indices().

~ 500 line function

A multidimensional model holds

Sources[modes][ICs][k][tp][τ]

5D worth of information.

We flatten this to a 1D array, keeping track of what indices hold what info.

A 2D array is made where the 1st array is the flattened 5D array and the 2nd holds the values that make up the interpolation table

Interpolation Table Data Structure Architecture

Sources[flat(mode, ic, k, tp), τ]

	τ
Scalar, ad, k_1, δ_m	⟨Appropriate table values⟩
Scalar, ad, k_2, δ_m	
Scalar, ad, k_1, ψ	
⋮	

Eg: ic = ad (adiabatic), isocurvature

modes = scalar, vector, tensor

tp = # type of perturbation (δ, θ, ψ, φ, ...)

k = Fourier modes

Solving

* perturbations_initial_conditions() → Evaluates the system (variables) at τ_{ini}, and populates the y vector.
~ 700 lines

Declares a bunch of local variables for computation.

- Has seperate sections of code for modes scalar, vector and tensor.
- Along w/ the structs (pba, pth, ppt), a 'k' and a 'tau' (τ_{ini}) make up the parameters.

Temperature Fluctuation

$$\Theta(\vec{k}, N, \tau) = \frac{\delta T}{T}(\vec{k}, \nu, \tau)$$

$$= \sum_{l=0}^{\infty} (-i)^l (2l+1)\, \Theta_l\, P_l(\nu)$$

→ # infinite DEs.

$$\dot{\Theta} = \frac{k}{2l+1}\left[l\,\Theta_{l-1} - (l+1)\,\Theta_{l+1}\right]$$

+ Source terms

* perturbations_approximations() → Optimisation step.

- Decides which approximations can be used when (@ a given time τ). Otherwise computing the Boltzman Hierarchy in its entirety would be very expensive.

require approximation

The Boltzman Equations

Multipoles are stored in terms of the perturbation vector `y` and its derivative `dy`. This is relavant for the Boltzman Hierarchy for photons.

The state vector y[] is served to the ODE solver. → Sampling times are saved into sources[] (for CMB or lensing) or transfer[] (for matter power spectrum)

A helper struct pv holds the indices

Thermodynamics

Distribution Function

$f(t, \vec{x}, \vec{p})$ – Each species of particles has its own distribution function.

$$\Delta N = \frac{g}{(2\pi)^3} f(t, \vec{x}, \vec{p}) (\Delta \vec{x})^3 (\Delta \vec{p})^3$$

g = # of internal degrees of freedom (spin/polarization)

(because quantum mechanics I don't understand)

① Number density n

$$n(t,\vec{x}) = \frac{g}{(2\pi\hbar)^3} \iiint f(t,\vec{x},\vec{p})\, d^3\vec{x}$$

Number of particles per unit real space.

② Energy density

$$\rho = \frac{g}{(2\pi\hbar)^3} \iiint E(\vec{p}) f(t,\vec{x},\vec{p})\, d^3\vec{p}$$

③ Pressure P

$$P = \frac{g}{(2\pi\hbar)^3} \iiint \frac{\vec{p}^{\,2}}{3E(\vec{p})} f(t,\vec{x},\vec{p})\, d^3\vec{p}$$

In general, the average of some quantity Q is

$$\langle Q \rangle = \frac{g}{(2\pi\hbar)^3} \iiint \frac{Q\, f(t, \vec{x}, \vec{p})\, d^3\vec{p}}{n}$$

$$\Rightarrow \quad \langle Q \rangle = \frac{\iiint Q\, f\, d^3\vec{p}}{\iiint f\, d^3\vec{p}}$$

From now on triple integrals will be implicit.

Monte Python

First the mathematics, code next.

We start with the basics of MCMC, and thus Bayesian analysis.

$$p(\theta \mid D) = \frac{p(D \mid \theta)\, p(\theta)}{p(D)}$$

here $p(D \mid \theta)$ = liklihood

$p(\theta)$ = prior

$p(D) = \int p(D \mid \theta)\, p(\theta)\, d\theta$ is the evidence

posterior $= p(D \mid \theta)\, p(\theta)$

We may treat the evidence as a sort of normalising constant for the posterior. But it is computationally expensive.

MCMC bypasses the computation of the evidence.

① Init Guess $\theta^{(0)}$,

② On the t^{th} iteration,

$$\theta^* = \theta^{(t)} + \mathcal{N}(0, \sigma^2)$$

or $\theta^* \sim \mathcal{N}(\theta^{(t)}, \sigma^2)$

(we add some Gaussian noise)

③ Compute log posterior difference

The evidence basically cancels out.

$$\Delta\ell = \ell(\theta^*) - \ell(\theta)$$

$$\pi_\Delta = \frac{\pi(\theta^*)}{\pi(\theta)} = \frac{P(\theta^* \mid D)\, P(\theta^*)}{P(\theta \mid D)\, P(\theta)}$$

④ We determine an acceptance ratio 'r'.

$$r = \exp(\Delta\ell)$$

⑤ If $r \geq 1$, posterior probability is clearly better. Else we accept θ^* w/ a probability r. ($0.01 r$ %).

An example of a toy Gaussian Liklihood is

$$p(D|\theta) = \mathcal{L}(\theta) = \prod \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)}$$

$$p(\theta=\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(\mu_0-\mu)^2}{2\tau^2}\right)}$$

where μ_0 = prior param

τ = prior Variance

MontePython likelihoods are vectors and involve the use of a covariance matrix instead of a variance – that encodes how each value is related to each other.

The liklihood module in MontePy

1. Reads data (observation + covariance)
2. Class computes theoretical data from a proposed θ (proposed by Monte Py).
3. Proposed parameters are compared to real data in the liklihood functions.

Cosmo parameters,

True cosmological content

$\Omega_b h^2$ baryonic density

$\Omega_c h^2$ CDM density

H_0, h Hubble param

n_s, A_s primordial power spectrum

priors are put on these to be sampled

Nuisance parameters,

From experiments and measurements not about the universe itself.

Thermal dust emission amplitude in Planck Liklihoods

Calibration factors

SZ Contamination amplitudes.

We don't care about their value, but they must be computed so they do not bias results.

Derived parameters

Functions of Cosmo parameters, sometime nuisance parameters.