A Lie-Theoretic Derivation of Linear Gauge Transformations

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May 2025

1 Conventions

Spacetime coordinates in this paper are denoted by x^μ. Greek letters such as μ, ν, λ index x over both temporal and spatial coordinates, i.e. $\mu \in \{0, 1, 2, 3\}$. Roman letters such as i, j, k index x strictly over spatial coordinates. That is, considering x^i, $i \in \{1, 2, 3\}$. We will use the comoving coordinate system $x^\mu = (\tau, \vec{t})$. Here τ represents comoving time, defined as $\mathrm{d}t = a(\tau)\mathrm{d}\tau$.

The model presented by Bertschinger et. al. for a perturbed flat ($\Omega = 1$) FLRW metric, as given below, will be considered in this paper.

$$\begin{aligned} g_{00} &= -a^2(\tau)[1 + 2\psi(\vec{x}, \tau)], \\ g_{0i} &= a^2(\tau) w_i(\vec{x}, \tau), \\ g_{ij} &= a^2(\tau)[\{1 - 2\phi(\vec{x}, \tau)\}\delta_{ij} + \chi_{ij}(\vec{x}, \tau)], \qquad \chi_{ii} = 0 \end{aligned} \tag{1}$$

where the functions ψ, w_i, ϕ and χ_{ij} represent metric perturbations about the FLRW metric. They are all assumed to be small compared to unity: $\psi, w_i, \phi, \chi_{ij} = \mathcal{O}(\mathcal{E})$.

2 Gauge Transformations

We consider a general infinitesimal coordinate transformation from a coordinate system x^μ to another $\hat{x}^\mu$.

$$x^\mu \mapsto \hat{x}^\mu + \xi^\mu(x^\nu). \tag{2}$$

The temporal and spatial components of the are written separately as

$$\begin{aligned} \xi^0 &= \alpha(\vec{x}, \tau), \\ \vec{\xi} &= \vec{\nabla}\beta(\vec{x}, \tau) + \vec{\epsilon}(\vec{x}, \tau), \quad \vec{\nabla} \cdot \vec{\epsilon} = 0 \end{aligned} \tag{3}$$

where ξ^i is written as the sum of a solenoidal function $\nabla^i \beta$ and transverse function ϵ^i.

2.1 Deriving the Transformation Equation for the Metric Tensor using Line Element Invariance

The metric tensor in the transformed coordinate system is denoted as $\hat{g}_{\mu\nu}(x)$. We require that the line element is invariant under this coordinate system.

$$\mathrm{d}s^2 = g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \hat{g}_{\mu\nu}\,\mathrm{d}\hat{x}^\mu\,\mathrm{d}\hat{x}^\nu \tag{4}$$

To begin simplifying Eq.(4), we require a reverse map to Eq.(2).

Consider a function F^μ defined as

$$F^\mu(x) = \hat{x}^\mu = x^\mu + \xi^\mu(x) \tag{5}$$

and a function G^μ such that

$$\begin{aligned} x^\mu &= G^\mu(F(x)), \\ \text{and let } G^\mu(\hat{x}) &= \hat{x}^\mu + \Delta^\mu(\hat{x}) \end{aligned} \tag{6}$$

From Eq.(5) and Eq.(6), we may write

$$\begin{aligned} & \hat{x}^\mu = F^\mu(G(\hat{x})) \\ \Rightarrow \quad & \hat{x}^\mu = F^\mu(\hat{x} + \Delta(\hat{x})) \\ \Rightarrow \quad & \hat{x}^\mu = \hat{x}^\mu + \Delta^\mu(\hat{x}) + \xi^\mu(\hat{x}^\mu + \Delta^\mu(\hat{x})) \\ \Rightarrow \quad & 0 = \Delta^\mu(\hat{x}) + \xi^\mu(\hat{x}^\mu + \Delta^\mu(\hat{x})) \end{aligned}$$

We may expand $\xi^\mu(\hat{x}^\mu + \Delta^\mu(\hat{x}))$ using the Taylor Series. Since ξ and Δ are both small, we may truncate the expansion to exclude the non-linear terms. The Taylor Series Expansion of a function with n degrees of freedom is given by the following expression.

$$f(x+\delta) = f(x) + \frac{\partial^\mu f(x)}{\partial x^i}\delta^i + \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^i \partial x^j}\delta^i\delta^j + \dots$$

Thus, we write

$$\begin{aligned} & 0 = \Delta^\mu(\hat{x}) + \xi^\mu(\hat{x}) + \Delta^\mu \frac{\partial \xi^\mu(\hat{x})}{\partial x^\mu} + \mathcal{O}(\xi^2) \\ \Rightarrow \quad & \Delta^\mu(\hat{x}) = -\xi^\mu(\hat{x}) + \mathcal{O}(\mathcal{E}^2) \end{aligned}$$

since $\xi, \Delta = \mathcal{O}(\mathcal{E})$. We have thus derived a reverse map for Eq.(5).

$$x^\mu = \hat{x}^\mu - \xi^\mu(\hat{x})$$

Revisiting Eq.(4), we may now express $\hat{g}_{\mu\nu}$ as below.

$$\hat{g}_{\mu\nu}(\hat{x}) \quad = \quad g_{\alpha\beta}(x(\hat{x}))\frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu} \tag{7}$$

Substituting for $\hat{x}$ from Eq.(2.1),

$$\begin{aligned} & \hat{g}_{\mu\nu}(\hat{x}) = \big(g_{\alpha\beta}(\hat{x}) - \xi^\rho(\hat{x})\hat{\partial}_\rho\, g_{\alpha\beta}(\hat{x})\big)\cdot\frac{\partial}{\partial \hat{x}^\mu}\big(\hat{x}^\alpha - \xi^\alpha(\hat{x})\big)\cdot\frac{\partial}{\partial \hat{x}^\beta}\big(\hat{x}^\beta - \xi^\beta(\hat{x})\big) + \mathcal{O}(\xi^2) \\ \Rightarrow \quad & \hat{g}_{\mu\nu}(\hat{x}) = \big(g_{\alpha\beta}(\hat{x}) - \xi^\rho(\hat{x})\hat{\partial}_\rho\, g_{\alpha\beta}(\hat{x})\big)\cdot\big(\delta^\alpha_\mu - \hat{\partial}_\mu\,\xi^\alpha(\hat{x})\big)\cdot\big(\delta^\beta_\nu - \hat{\partial}\,\xi^\beta(\hat{x})\big) \end{aligned}$$

Expanding the expression on the left, we get

$$\begin{aligned} \Rightarrow \quad & \hat{g}_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu\big(g_{\alpha\beta}(\hat{x}) - \xi^\rho(\hat{x})\hat{\partial}_\rho\, g_{\alpha\beta}(\hat{x})\big) - \delta^\beta_\nu\, g_{\alpha\beta}(\hat{x})\hat{\partial}_\mu\,\xi^\alpha(\hat{x}) + \delta^\alpha_\mu\, g_{\alpha\beta}(\hat{x})\hat{\partial}_\nu\,\xi^\beta(\hat{x}) + \mathcal{O}(\xi^2) \\ \Rightarrow \quad & \hat{g}_{\mu\nu}(\hat{x}) = g_{\mu\nu}(\hat{x}) - \xi^\rho(\hat{x})\hat{\partial}_\rho\, g_{\mu\nu}(\hat{x}) - g_{\alpha\nu}(\hat{x})\hat{\partial}_\mu\,\xi^\alpha(\hat{x}) - g_{\mu\beta}(\hat{x})\hat{\partial}\,\xi^\beta(\hat{x}) \end{aligned} \tag{8}$$

We have thus arrived at the gauge transformation equation. Before further simplifying this equation, we will first derive the transformation equations for each of the functions ψ, ϕ, w_i and χ_{ij} individually.

Deriving the Gauge Transformation Equation for ψ : First, we just consider the temporal indices

$$\hat{g}_{00} = g_{00} - \xi^{\rho}\hat{\partial}_{\rho}\, g_{00} - g_{\alpha 0}\hat{\partial}_0\, \xi^{\alpha} - g_{0\beta}\hat{\partial}_0\, \xi^{\beta}$$

Expanding the right-hand-side term by term,

$$\begin{aligned}
-\xi^{\rho}\hat{\partial}_{\rho}\, g_{00} &= -\xi^{0}\hat{\partial}_{0}\, g_{00} \, - \xi^{i}\hat{\partial}_{i}\, g_{00} \\
\text{where, } -\xi^{0}\hat{\partial}_{0}\, g_{00} &= -\alpha(\vec{x},\tau)\frac{\partial}{\partial\tau}\Big(-a^2(\tau)\big(1+2\psi(\vec{x},\tau)\big)\Big) \\
&= \alpha(\vec{x},\tau)\big(2a\dot{a}(\tau) + 2a^2(\tau)\dot{\psi}(\vec{x},\tau) + 4a\dot{a}(\tau)\psi(\vec{x},\tau)\big) \\
\text{and } -\xi^{i}\hat{\partial}_{i} g_{00} &= -\big(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\frac{\partial}{\partial x^i}\Big(-a^2(\tau)\big(1+2\psi(\vec{x},\tau)\big)\Big)
\end{aligned}$$

Keep in mind that since both $\xi, \alpha, \psi = \mathcal{O}(\mathcal{E})$, and $\mathcal{E}$ is small. As a result, all $\mathcal{O}(\mathcal{E}^2)$ terms will be dropped in the final equation.

$$\text{Thus, } -\xi^{\rho}\partial_{\rho}\, g_{00} \quad = \quad 2a\dot{a}(\tau) + \mathcal{O}(\mathcal{E}^2)$$

Moving onto the next two identical terms $-g_{\alpha 0}\hat{\partial}_0\, \xi^{\alpha} = g_{0\beta}\hat{\partial}_0\, \xi^{\beta} = g_{0\eta}\hat{\partial}_0\xi^{\eta}$, we may write

$$\begin{aligned}
-g_{0\eta}\hat{\partial}_0\, \xi^{\eta} &= -g_{00}\hat{\partial}_0\, \xi^{0} - g_{0i}\hat{\partial}_0\, \xi^{i} \\
&= \Big(a^2(\tau)\big(1+2\psi(\vec{x},\tau)\frac{\partial}{\partial\tau}\alpha(\vec{x},\tau)\Big) - \Big(w_i(\vec{x},\tau)\frac{\partial}{\partial\tau}\big(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\big)\Big) \\
&= a^2(\tau)\dot{\alpha} + 2a^2\dot{\alpha}\psi - w_i\big(\vec{\nabla}\dot{\beta} + \dot{\vec{\epsilon}}\big) \\
&= a^2(\tau)\dot{\alpha}(\vec{x},\tau) + \mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

Hence $-g_{\alpha 0}\hat{\partial}_0\, \xi^{\alpha} - g_{0\beta}\hat{\partial}_0\, \xi^{\beta} = -2g_{0\eta}\hat{\partial}_0\xi^{\eta} = 2a^2(\tau)\dot{\alpha}(\vec{x},\tau)$. Putting all the terms together,

$$\begin{aligned}
& \hat{g}_{00} = g_{00} + 2a\dot{a}\alpha + 2a^2\dot{\alpha} \\
\Rightarrow \quad & -a^2\big(1+2\hat{\psi}\big) = -a^2\big(1+2\psi\big) + 2a\dot{a}\alpha + 2a^2\dot{\alpha} \\
\Rightarrow \quad & \hat{\psi} = \psi - \frac{\dot{a}}{a}\alpha - \dot{\alpha}
\end{aligned}$$

We have thus derived the gauge transformation equation for the function ψ

$$\hat{\psi}(\vec{x},\tau) = \psi(\vec{x},\tau) - \mathcal{H}(\tau)\alpha(\vec{x},\tau) - \dot{\alpha}(\vec{x},\tau) \tag{9}$$

Deriving the Gauge Transformation Equation for w_i : Now, we consider the spatio-temporal indices g_{i0}.

$$\hat{g}_{i0} = g_{i0} - \xi^{\rho}\hat{\partial}_{\rho}\, g_{i0} - g_{i\beta}\hat{\partial}_0\, \xi^{\beta} - g_{\alpha 0}\hat{\partial}_i\, \xi^{\alpha}$$

Expanding the right-hand side term-by-term,

$$\begin{aligned}
\xi^{\rho}\hat{\partial}_{\rho}\, g_{i0} &= -\xi^{0}\frac{\partial}{\partial\tau}g_{i0} \, - \xi^{j}\hat{\partial}_j\, g_{i0} \\
\text{where, } -\xi^{0}\frac{\partial}{\partial\tau}g_{i0} &= -\alpha(\vec{x},\tau)a^2(\tau)w_i(\vec{x},\tau) = \mathcal{O}(\mathcal{E}^2) \\
\text{and } -\xi^{j}\hat{\partial}_j\, g_{i0} &= -\big(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\big)\hat{\partial}_j\, a^2(\tau)w_i(\vec{x},\tau) = \mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

Thus $\xi^\rho \hat{\partial}_\rho\, g_{i0} = \mathcal{O}(\mathcal{E}^2)$. This term is thus dropped. Considering the next term, $-g_{i\beta}\hat{\partial}_0\xi^\beta$,

$$\begin{aligned}
-g_{i\beta}\hat{\partial}_0\xi^\beta &= -g_{i0}\hat{\partial}_0\xi^0 - g_{ij}\hat{\partial}_0\xi^j \\
\text{where, } -g_{i0}\hat{\partial}_0\xi^0 &= -a^2(\tau)w_i(\vec{x},\tau)\frac{\partial}{\partial\tau 0}\alpha(\vec{x},\tau) = \mathcal{O}(\mathcal{E}^2) \\
\text{and } -g_{ij}\hat{\partial}_0\xi^j &= -a^2(\tau)\bigg(\big(1-2\phi(\vec{x},\tau)\big)\delta_{ij} + \chi_{ij}(\vec{x},\tau)\bigg)\frac{\partial}{\partial\tau}\big(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\big) \\
&= -a^2(\tau)\delta_{ij}\big(\nabla^j\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}^j(\vec{x},\tau)\big) + \mathcal{O}(\mathcal{E}^2) \\
&= -a^2(\tau)\big(\nabla_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big) + \mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

Thus, we may drop the $\mathcal{O}(\mathcal{E}^2)$ terms and express $-g_{i\beta}\hat{\partial}_0\xi^\beta$ as $-g_{i\beta}\hat{\partial}_0\xi^\beta = \bigg(-a^2(\tau)\big(\nabla_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big)\bigg)$. Now we consider the final term $-g_{\alpha 0}\hat{\partial}_i\xi^\alpha$,

$$\begin{aligned}
-g_{\alpha 0}\hat{\partial}_i\xi^\alpha &= -g_{00}\hat{\partial}_i\xi^0 - g_{j0}\hat{\partial}_i\xi^j \\
\text{where, } -g_{00}\hat{\partial}_i\xi^0 &= a^2(\tau)\big(1+2\psi(\vec{x},\tau)\big)\hat{\partial}_i\alpha(\vec{x},\tau) \\
&= a^2(\tau)\hat{\partial}_i\alpha(\vec{x},\tau) + \mathcal{O}(\mathcal{E}^2) \\
\text{and } -g_{j0}\hat{\partial}_i\xi^j &= -a^2(\tau)w_i(\vec{x},\tau)\hat{\partial}_i\big(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\big) \\
&= \mathcal{O}(\mathcal{E}^2) \\
\text{Thus, } -g_{\alpha 0}\hat{\partial}_i\xi^\alpha &= a^2(\tau)\hat{\partial}_i\alpha(\vec{x},\tau) + \mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

The resultant final equation comes out to be

$$\begin{aligned}
&\hat{g}_{i0} = g_{i0} + a^2(\tau)\hat{\partial}_i\alpha(\vec{x},\tau) - a^2(\tau)\big(\nabla_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big) \\
\Rightarrow\ &a^2(\tau)\hat{w}_i(\vec{x},\tau) = a^2(\tau)w_i(\vec{x},\tau) + a^2(\tau)\hat{\partial}_i\alpha(\vec{x},\tau) - a^2(\tau)\big(\nabla_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big) \\
\Rightarrow\ &\hat{w}_i(\vec{x},\tau) = w_i(\vec{x},\tau) + \hat{\partial}_i\alpha(\vec{x},\tau) - \big(\nabla_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big) \qquad (10)
\end{aligned}$$

Deriving the Gauge Transformation Equation for ϕ and χ_{ij} : Now, having derived the gauge transformation equation for the function w_i, we constrain ourselves to consider only the spatial components of the general gauge transformation equation.

$$\hat{g}_{ij} = g_{ij} - \xi^\rho\hat{\partial}_\rho\, g_{ij} - g_{\alpha j}\hat{\partial}_i\xi^\alpha - g_{i\beta}\hat{\partial}_j\xi^\beta \qquad (11)$$

The term g_{ij} has been defined as to have a traced component ϕ and a traceless component χ_{ij}. Tracing Eq.(11),

$$\delta^{ij}\hat{g}_{ij} = \delta^{ij}\bigg(g_{ij} - \xi^\rho\hat{\partial}_\rho\, g_{ij} - g_{\alpha j}\hat{\partial}_i\xi^\alpha - g_{i\beta}\hat{\partial}_j\xi^\beta\bigg)$$

We see that

$$\begin{aligned}
\delta^{ij}\hat{g}_{ij} = \hat{g}^i_i &= \delta^{ij}a^2\bigg(\big(1-2\hat{\phi}\big)\delta_{ij} + \chi_{ij}\bigg) \\
&= \delta^i_i\, a^2\big(1-2\hat{\phi}\big) \\
&= a^2\big(3-6\hat{\phi}\big)
\end{aligned}$$

The two terms $\big(-\delta^{ij}g_{\alpha j}\hat{\partial}_i\xi^{\alpha}\big)$ and $\big(-\delta^{ij}g_{i\beta}\hat{\partial}_j\xi^{\beta}\big)$ are identical and are summed and equated to $\big(-2g_{k\eta}\hat{\partial}^k\xi^{\eta}\big)$. This term may then as expanded,

$$\begin{aligned}
-2g_{k\eta}\hat{\partial}^k\xi^{\eta} &= -2\big(g_{ko}\hat{\partial}^k\xi^0 + g_{kl}\hat{\partial}^k\xi^l\big) \\
\text{where, } g_{ko}\hat{\partial}^k\xi^0 &= a^2(\tau)w_k(\vec{x},\tau)\delta^k_k\hat{\partial}_k\alpha(\vec{x},\tau) = \mathcal{O}(\mathcal{E}^2) \\
\text{and } g_{kl}\hat{\partial}^k\xi^l &= a^2(\tau)\bigg(\big(1-2\phi(\vec{x},\tau)\big)\delta_{kl}+\chi_{kl}\bigg)\delta^k_k\hat{\partial}_k\big(\nabla^l\beta(\vec{x},\tau)+\epsilon^l(\vec{x},\tau)\big)+\mathcal{O}(\mathcal{E}^2) \\
&= a^2(\tau)\cdot\mathrm{Tr}\big(\nabla^l\beta(\vec{x},\tau)+\epsilon^l(\vec{x},\tau)\big)+\mathcal{O}(\mathcal{E}^2) \\
&= a^2(\tau)\nabla^2\beta(\vec{x},\tau)+\mathcal{O}(\mathcal{E}^2) \qquad \text{Where } \mathrm{Tr}(\cdot) \text{ is the trace function} \\
\text{Thus, } -2g_{k\eta}\hat{\partial}^k\xi^{\eta} &= -2a^2(\tau)\nabla^2\beta(\vec{x},\tau)+\mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

The final term $\delta^{ij}\xi^{\rho}\hat{\partial}_{\rho}\,g_{ij}$ is expressed as

$$\begin{aligned}
\delta^{ij}\xi^{\rho}\hat{\partial}_{\rho}\,g_{ij} = \xi^{\rho}\hat{\partial}_{\rho}\,g^i_i &= \xi^0\hat{\partial}_0\,g^i_i + \xi^l\hat{\partial}_l\,g^i_i \\
\text{where } \xi^{\rho}\hat{\partial}_{\rho}\,g^i_i &= \alpha(\vec{x},\tau)\frac{\partial}{\partial\tau}a^2(\tau)\big(3-6\phi(\vec{x},\tau)\big) \\
&= 6a\dot{a}(\tau)\alpha(\vec{x},\tau)+\mathcal{O}(\mathcal{E}^2) \\
\text{and } \xi^l\hat{\partial}_l\,g^i_i &= \big(\vec{\nabla}\beta(\vec{x},\tau)+\vec{\epsilon}(\vec{x},\tau)\big)\hat{\partial}_l a^2(\tau)\big(3-6\phi(\vec{x},\tau)\big) \\
&= \big(\vec{\nabla}\beta(\vec{x},\tau)+\vec{\epsilon}(\vec{x},\tau)\big)\cdot\big(0-6a^2(\tau)\hat{\partial}_l\phi(\vec{x},\tau)\big) \\
&= \mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

Consolidating all these terms, we get

$$\begin{aligned}
a^2\big(3-6\hat{\phi}\big) &= a^2\big(3-6\phi\big)-2a^2\nabla^2\beta-6a\dot{a}\alpha \\
\Rightarrow\quad \hat{\phi} &= \phi+\frac{1}{3}\nabla^2\beta+\frac{\dot{a}}{a}\alpha
\end{aligned}$$

We have thus derived the gauge transformation for the function ϕ,

$$\hat{\phi}(\vec{x},\tau) = \phi(\vec{x},\tau)+\frac{1}{3}\nabla^2\beta(\vec{x},\tau)+\mathcal{H}(\tau)\alpha(\vec{x},\tau) \tag{12}$$

For the final function χ_{ij}, we must isolate the traceless component of Eq.(11). We may express the terms of this equation as a sum of strictly spatial and spatio-temporal indices.

$$\begin{aligned}
\hat{g}_{ij} &= g_{ij}-\xi^{\rho}\hat{\partial}_{\rho}\,g_{ij}-g_{\alpha j}\hat{\partial}_i\xi^{\alpha}-g_{i\beta}\hat{\partial}_j\xi^{\beta} \\
\text{Firstly, } -\xi^{\rho}\hat{\partial}_{\rho}\,g_{ij} &= -\xi^0\hat{\partial}_0\,g_{ij}-\xi^k\hat{\partial}_k\,g_{ij} \\
&= \bigg[-\alpha\frac{\partial}{\partial\tau}a^2\bigg(\big(1-2\phi\big)\delta_{ij}+\chi_{ij}\bigg)\bigg]+\bigg[-\big(\vec{\nabla}\beta+\vec{\epsilon}\big)\hat{\partial}_k\,a^2\bigg(\big(1-2\phi\big)\delta_{ij}+\chi_{ij}\bigg)\bigg]
\end{aligned}$$

We know that δ_{ij} terms are trace-only and can thus be dropped dropped. Trace terms are grouped and henceforth denoted by $\mathcal{T}$.

$$\begin{aligned}
\text{Thus, } -\xi^{\rho}\hat{\partial}_{\rho}\,g_{ij} &= \bigg[-2\alpha a\dot{a}\frac{\partial}{\partial\tau}\delta_{ij}+\mathcal{O}(\mathcal{E}^2)\bigg]+\bigg[\big(\vec{\nabla}\beta+\vec{\epsilon}\big)\hat{\partial}_k\,a^2\delta_{ij}+\mathcal{O}(\mathcal{E}^2)\bigg] \\
&= \bigg[\mathcal{T}+\mathcal{O}(\mathcal{E}^2)\bigg]+\bigg[\mathcal{T}+\mathcal{O}(\mathcal{E}^2)\bigg] \\
&= \mathcal{T}+\mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

Now we consider the term $-g_{i\beta}\hat{\partial}_j\xi^\beta$,

$$\begin{aligned}
-g_{i\beta}\hat{\partial}_j\xi^\beta &= -g_{i0}\hat{\partial}_j\xi^0 - g_{ik}\hat{\partial}_j\xi^k \\
\text{where,} \ -g_{i0}\hat{\partial}_j\xi^0 &= -a^2(\tau)w_i(\vec{x},\tau)\hat{\partial}_j\,\alpha(\vec{x},\tau) = \mathcal{O}(\mathcal{E}^2) \\
\text{and} \ -g_{ik}\hat{\partial}_j\xi^k &= a^2(\tau)\bigg(\big(1-2\phi(\vec{x},\tau)\big)\delta_{ik} + \chi_{ik}(\vec{x},\tau)\bigg)\hat{\partial}_j\big(\vec{\nabla}\beta(\vec{x},\tau) + \vec{\epsilon}(\vec{x},\tau)\big) \\
&= -a^2(\tau)\hat{\partial}_j\delta_{ik}\big(\nabla^k\beta(\vec{x},\tau) + \epsilon^k(\vec{x},\tau)\big) + \mathcal{O}(\mathcal{E}^2) \\
&= -a^2(\tau)\hat{\partial}_j\,\xi_i + \mathcal{O}(\mathcal{E}^2)
\end{aligned}$$

Thus, $-g_{i\beta}\hat{\partial}_j\xi^\beta = -a^2(\tau)\hat{\partial}_j\,\xi_i$. Similarly, $-g_{\alpha j}\hat{\partial}_i\xi^\alpha = -a^2(\tau)\hat{\partial}_i\,\xi_j$. With these terms, we my write the equation

$$\hat{g}_{ij} \quad = \quad g_{ij} - a^2(\tau)\bigg(\hat{\partial}_i\xi_j + \hat{\partial}_j\xi_i\bigg) + \mathcal{T} + \mathcal{O}(\mathcal{E}^2)$$

Now, we must remove residual trace terms from the expression $\hat{\partial}_i\xi_j - \hat{\partial}_j\xi_i$. To remove the traced component of a tensor $T_{\mu\nu}$, we must subtract its trace from itself. We define the identity

$$T_{\mu\nu}^{\text{Traceless}} = T_{\mu\nu} - \frac{1}{n}\delta_{\mu\nu}\big(\delta^{\mu\nu}T_{\mu\nu}\big) \tag{13}$$

where n is degrees of freedom the indices μ and ν are constraint to. Now let $T_{ij} = \hat{\partial}_i\xi_j - \hat{\partial}_j\xi_i$. Applying the properties of Eq.(13) to T_{ij},

$$\begin{aligned}
T_{ij}^{\text{Traceless}} &= T_{ij} - \frac{1}{3}\delta_{ij}\big(\delta^{ij}\,T_{ij})\big) \\
&= \hat{\partial}_i\xi_j + \hat{\partial}_j\xi_i - \frac{1}{3}\delta_{ij}\cdot 2\hat{\partial}^k\xi_k
\end{aligned}$$

We may finally revise the equation to group all the trace terms under $\mathcal{T}$, thus written as

$$\hat{\chi}_{ij} = \chi_{ij} - a^2(\tau)\bigg(\hat{\partial}_i\xi_j + \hat{\partial}_j\xi_i - \frac{2}{3}\delta_{ij}\hat{\partial}^k\big(\nabla_k\beta\big)\bigg) + \mathcal{T} + \mathcal{O}(\mathcal{E}^2)$$

The $\epsilon(\vec{x},\tau)$ term in ξ^k is dropped due to its transverse nature. Henceforth the $\mathcal{T}$ and $\mathcal{O}(\mathcal{E}^2)$ terms will be dropped from the equation altogether.

$$\begin{aligned}
\hat{\chi}_{ij} &= \chi_{ij} - a^2(\tau)\bigg(\hat{\partial}_i\xi_j + \hat{\partial}_j\xi_i - \frac{2}{3}\delta_{ij}\hat{\partial}^k\xi_k\bigg) \\
&= \chi_{ij} - a^2(\tau)\bigg(\hat{\partial}_i\big(\hat{\partial}_j\beta + \epsilon_j\big) + \hat{\partial}_j\big(\hat{\partial}_i\beta + \epsilon_i\big) - \frac{2}{3}\delta_{ij}\hat{\partial}^k\big(\nabla_k\beta + \epsilon_k\big)\bigg) \\
&= \chi_{ij} - \bigg(2\hat{\partial}_i\hat{\partial}_j\beta + \big(\hat{\partial}_j\epsilon_i + \hat{\partial}_i\epsilon_j\big) - \frac{2}{3}\delta_{ij}\nabla^2\beta\bigg)
\end{aligned}$$

$\delta_{ij}\partial^k$ acts like the divergence function ∇^k, and ϵ_k is transverse. We algebraically simplify the expression on the right hand side, resulting in the gauge transformation equation for the function χ_{ij}.

$$\hat{\chi}_{ij}(\vec{x},\tau) = \chi_{ij}(\vec{x},\tau) - 2\bigg(\hat{\partial}_i\hat{\partial}_j - \frac{1}{3}\delta_{ij}\beta(\vec{x},\tau)\bigg) - \big(\hat{\partial}_i\epsilon_j(\vec{x},\tau) + \hat{\partial}_j\epsilon_i(\vec{x},\tau)\big) \tag{14}$$

2.2 Deriving the Gauge Transformation Equation using the Lie Derivative

In the previous subsection, we derived the gauge transformation Eq.(8) equation by utilising the invariance of the line element ds^2 under gauge transformations. In this section, we will derive the same gauge transformation equation using Lie Derivatives. The Lie Derivative evaluates the change of a tensor field T along the flow defined by another vector X. Since this change is coordinate invariant, the Lie Derivative is defined on any differentiable manifold. It does not require any connection, metric or inner product. The Lie Derivative of T along X is denoted as $\mathcal{L}_X T$.

We will now derive the Lie Derivative for the perturbed spacetime considered in this paper. Sean M. Caroll defines the Lie Derivative of a field as,

$$g_{\mu\nu}(x) \mapsto \hat{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - \mathcal{L}_\xi g_{\mu\nu}(x) \tag{15}$$

Defining the Lie Derivative of the Metric Tensor: Consider smooth, differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$. Let ξ^μ as defined in Eq.(2) be a vector field on $\mathcal{M}$ that generates an infinitesimal coordinate transformation ${}_\sigma\phi$.

$$ {}_\sigma\phi : \mathcal{M} \to \mathcal{N}$$

$$ {}_\sigma\phi : x^\mu \mapsto x^\mu + \sigma\xi^\mu(x) + \mathcal{O}(\sigma^2) \tag{16}$$

where the scalar σ is the "flow parameter" – denotes how far we've moved along the vector flow ξ^μ. If we consider a point p on $\mathcal{M}$, then utilising Eq.(16) we may write

$$p \mapsto {}_\sigma\phi(p)$$

where ${}_\sigma\phi(p) \in \mathcal{N}$. In local coordinates $\{x^i\}$ on $\mathcal{M}$ and $\{y^j\}$ on $\mathcal{N}$, we may represent the transformation as

$$ {}_\sigma\phi(p) = \big(y^1 = \varphi^1\big(x^1(p), \ldots, x^m(p)\big), \ldots, y^n = \varphi^n\big(x^1(p), \ldots, x^m(p)\big)\big) \tag{17}$$

We may now define the *pullback* denoted ${}_\sigma\phi^*$. The pullback drags covariant objects backward along the map ${}_\sigma\phi$. To exemplify this, we consider a scalar field $f(\cdot)$ and a point p. Here, the pullback just evaluates $f(\cdot)$ at the moved point ${}_\sigma\phi(p)$.

$$({}_\sigma\phi^* \circ f)(p) = f({}_\sigma\phi(p)) \tag{18}$$

Now if we are to consider a covariant vector $X_\nu(p)$ instead of a scalar field, since a vector carries directional information, we will require a Jacobian factor.

$$\big({}_\sigma\phi^* \circ X\big)_\nu(p) = X_\eta({}_\sigma\phi(p)) \cdot J^\eta_\nu = X_\eta({}_\sigma\phi(p)) \cdot \frac{\partial_\sigma\phi^\eta(p)}{\partial x^\nu} \tag{19}$$

Applying these principles we can now derive the pullback of the perturbed metric tensor described in Eq.(1).

$$\big({}_\sigma\phi^* \circ g\big)_{\mu\nu}(x) = \frac{\partial({}_\sigma\phi^\lambda)}{\partial x^\nu}\frac{\partial({}_\sigma\phi^\kappa)}{\partial x^\mu} g_{\kappa\lambda}({}_\sigma\phi(x)) \tag{20}$$

Notice how Eq.(20) is similar to Eq.(7), although they are not equivalent because of the σ term defined in Eq.(16). Expanding the terms of Eq.(20) to the first order in σ,

$$
\begin{aligned}
\left({}_\sigma\phi * g\right)_{\mu\nu}(x) &= \left(\delta_\mu^\lambda + \sigma\,\partial_\mu\xi^\lambda + \mathcal{O}(\sigma^2)\right)\left(\delta_\nu^\kappa + \sigma\,\partial_\nu\xi^\kappa + \mathcal{O}(\sigma^2)\right)\left(g_{\lambda\kappa}(x) + \sigma\xi^\rho\partial_\rho g_{\lambda\kappa}(x) + \mathcal{O}(\sigma^2)\right) \\
&= g_{\mu\nu}(x) + \sigma\left(\xi^\rho\,\partial_\rho g_{\mu\nu}(x) \;+\; (\partial_\mu\xi^\lambda)\,g_{\lambda\nu}(x) \;+\; (\partial_\nu\xi^\kappa)\,g_{\mu\kappa}(x)\right) + \mathcal{O}(t^2)
\end{aligned} \tag{21}
$$

The derivative of the pullback of the perturbed metric with respect to σ at $\sigma = 0$ gives us the Lie Derivative of the perturbed metric.

$$
\mathcal{L}_\xi\, g_{\mu\nu}(x) = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left({}_\sigma\phi^* g_{\mu\nu}(x)\right)\Bigg|_{\sigma=0} = \xi^\rho\,\partial_\rho g_{\mu\nu} + (\partial_\mu\xi^\lambda)\,g_{\lambda\nu} + (\partial_\nu\xi^\kappa)\,g_{\mu\kappa} \tag{22}
$$

We thus derive a gauge transformation equation

$$
\begin{aligned}
\hat{g}_{\mu\nu}(x) &= g_{\mu\nu}(x) - \mathcal{L}_\xi g_{\mu\nu}(x) \\
&= g_{\mu\nu}(x) - \xi^\rho(x)\partial_\rho\, g_{\mu\nu}(x) - g_{\lambda\nu}(x)\partial_\mu\xi^\lambda(x) - g_{\mu\kappa}(x)\partial_\nu\,\xi^\kappa(x)
\end{aligned} \tag{23}
$$

From Eq.(22), it's clear that we may rewrite Eq.(8) as

$$
\hat{g}_{\mu\nu}(\hat{x}) = g_{\mu\nu}(\hat{x}) - \mathcal{L}_\xi g_{\mu\nu}(\hat{x}) \tag{24}
$$

We thus see that both Eq.(23) and Eq.(24) are identical.

The difference between the line element invariance derivation and the pullback derivation : The derivation carried out by asserting the invariance of the line element under a general coordinate transformation is henceforth referred to as the "passive derivation", since the field $g_{\mu\nu}$ is kept fixed, and the coordinates are relabelled by the general coordinate transformation (2) : $\hat{x}^\mu = x^\mu + \xi^\mu$.

On the other hand, the derivation that utilises the pullback of the field $g_{\mu\nu}$ is termed the "active derivation" since $g_{\mu\nu}$ is dragged along the flow of ξ. This gives us the the new metric $\hat{g}_{\mu\nu}$ in the same coordinate system x. This change is highlighted in the use of the variable $\hat{x}$ in Eq.(24) and the use of x in Eq.(23), although both x and $\hat{x}$ in this context are dummy variables, ensuring that the two equations are equivalent.

2.3 Simplifying the Gauge Transformation Equation given by Bertschinger

In the previous sections we have derived and confirmed the gauge transformation equation given by Bertschinger in Eq.(23) and Eq.(24). For the ease of exposition, we relabel $\hat{x}$ in Eq.(24) and Eq.(8) to x. We will henceforth be working with the equation

$$
\hat{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - \xi^\rho(x)\partial_\rho\, g_{\mu\nu}(x) - g_{\alpha\nu}(x)\partial_\mu\xi^\alpha(x) - g_{\mu\beta}(x)\partial_\nu\,\xi^\beta(x) \tag{25}
$$

which is, in essence Eq.(23).

We define

$$
g_{\mu\nu}(x) = \overline{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \tag{26}
$$

where $\overline{g}_{\mu\nu}$ is the flat, unperturbed Minkowski space, and $\delta g_{\mu\nu}$ represents the scalar perturbation. Expanding the terms in Eq.(23),

$$
\begin{aligned}
&\hat{\overline{g}}_{\mu\nu}(x) + \delta\hat{g}_{\mu\nu}(x) = \overline{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) - \mathcal{L}_\xi g_{\mu\nu}(x) \\
\Rightarrow\quad &\delta\hat{g}_{\mu\nu}(x) = \delta g_{\mu\nu}(x) - \mathcal{L}_\xi g_{\mu\nu}(x) \quad \because \hat{\overline{g}}_{\mu\nu}(x) = \overline{g}_{\mu\nu}(x)
\end{aligned} \tag{27}
$$

$\hat{\overline{g}}_{\mu\nu}(x) = \overline{g}_{\mu\nu}(x)$ since the background – the Minkowski space is gauge invariant.

Simplifying the Lie Derivative Expression : We consider the covariant derivatives of ξ in both indices μ and ν.

$$\begin{aligned}
\nabla_\mu \xi_\nu &= \partial_\mu \xi_\nu - \Gamma^\lambda_{\mu\nu}\xi_\lambda \\
&= \partial_\mu\big(g_{\eta\nu}\xi^\eta\big) - \Gamma^\lambda_{\mu\nu}\big(g_{\eta\lambda}\xi^\eta\big) \\
&= \big(\partial_\mu g_{\eta\nu}\big)\xi^\eta + g_{\eta\nu}\big(\partial_\mu \xi^\eta\big) - \Gamma^\lambda_{\mu\nu}\big(g_{\eta\lambda}\xi^\eta\big)
\end{aligned} \tag{28}$$

From the Levi-Cevita connection, we know

$$\begin{aligned}
&\nabla_\lambda g_{\mu\nu} = 0 \\
\Rightarrow\quad &\partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\mu\lambda}g_{\lambda\nu} - \Gamma^\rho_{\lambda\nu}g_{\mu\lambda} = 0 \\
\Rightarrow\quad &\partial_\lambda g_{\mu\nu} = \Gamma^\rho_{\mu\lambda}g_{\lambda\nu} + \Gamma^\rho_{\lambda\nu}g_{\mu\lambda}
\end{aligned} \tag{29}$$

Applying results from Eq.(29) to $\nabla_\mu\xi_\nu$,

$$\begin{aligned}
\nabla_\mu \xi_\nu &= \Big(\cancel{\Gamma^\rho_{\mu\nu}g_{\eta\rho}\xi^\eta} + \Gamma^\rho_{\mu\eta}\, g_{\rho\nu}\xi^\eta\Big) + g_{\eta\nu}\partial_\mu\xi^\eta - \cancel{\Gamma^\lambda_{\mu\nu}g_{\eta\lambda}\xi^\eta} \\
&= \Gamma^\rho_{\mu\eta}\, g_{\rho\nu}\xi^\eta + g_{\eta\nu}\partial_\mu\xi^\eta
\end{aligned} \tag{30}$$

Similarly,

$$\nabla_\nu\xi_\mu = \Gamma^\rho_{\nu\eta}\, g_{\mu\rho}\xi^\eta + g_{\mu\eta}\partial_\nu\xi^\eta \tag{31}$$

Adding Eq.(30) and Eq.(31), we get

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = g_{\eta\nu}\partial_\nu\xi^\eta + g_{\mu\eta}\partial_\nu\xi^\eta + \Gamma^\rho_{\mu\eta}\, g_{\rho\nu}\xi^\eta + \Gamma^\rho_{\nu\eta}\, g_{\mu\rho}\xi^\eta$$

From the Levi-Cevita Connection, we can show that $\partial_\eta\, \xi^\eta g_{\mu\nu} = \Gamma^\rho_{\mu\eta}\, g_{\rho\nu}\xi^\eta + \Gamma^\rho_{\nu\eta}\, g_{\mu\rho}\xi^\eta$. Thus,

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = g_{\eta\nu}\partial_\nu\xi^\eta + g_{\mu\eta}\partial_\nu\xi^\eta + \partial_\eta\, \xi^\eta g_{\mu\nu} \tag{32}$$

We see that this directly matches the expression of the Lie Derivative we have expressed in Eq.(22). Thus, we conclude that

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \tag{33}$$

Using the result from Eq.(33) in Eq.(27), we get a simplified gauge transformation equation

$$\delta\hat{g}_{\mu\nu} = \delta g_{\mu\nu} - \Big(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu\Big). \tag{34}$$

We may demonstrate the validity of Eq.(34) by deriving the gauge transformation equations for the functions ψ, ϕ, w_i and χ_{ij}. **It is important to keep in mind that the covariant derivatives in Eq.**(33) **only take into account the unperturbed background $\bar{g}_{\mu\nu}$. The perturbed piece is dropped since both $\delta g_{\mu\nu}$ and ξ^μ are of $\mathcal{O}(\mathcal{E})$. All non-linear terms are ignored.**

Re-deriving the gauge transformation equation for ψ : Consider the temporal indices of Eq.(34).

$$\begin{aligned}
&\delta\hat{g}_{00} = \delta g_{00} - 2\Big(\nabla_0\xi_0\Big) \\
\Rightarrow\quad &-a^2(\tau)\cdot 2\hat{\psi}(\vec{x},\tau) = -a^2(\tau)\cdot 2\psi(\vec{x},\tau) - 2\Big(\nabla_0\xi_0\Big)
\end{aligned}$$

Consider the piece of the Lie Derivative

$$
\begin{aligned}
\nabla_0\xi_0 &= \partial_0\xi_0 - \Gamma^\lambda_{00}\xi_\lambda \\
&= \partial_0\big(g_{00}\xi^0\big) - \Big(\Gamma^0_{00}\xi_0 + \cancelto{0}{\Gamma^l_{00}\xi_l}\Big) \\
&= \frac{\partial}{\partial\tau}\big(-a^2(\tau)\alpha(\vec{x},\tau)\big) - \left(\frac{\dot{a}(\tau)}{a(\tau)}\cdot\Big(-a^2(\tau)\alpha(\vec{x},\tau)\Big)\right) \\
&= -2a\dot{a}(\tau)\alpha(\vec{x},\tau) - a^2(\tau)\dot{\alpha}(\vec{x},\tau) + a\dot{a}(\tau)\alpha(\vec{x},\tau) \\
&= -a^2(\tau)\dot{\alpha}(\vec{x},\tau) - a\dot{a}(\tau)\alpha(\vec{x},\tau)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& -a^2(\tau)\cdot 2\hat{\psi}(\vec{x},\tau) = -a^2(\tau)\cdot 2\psi(\vec{x},\tau) + a^2(\tau)\dot{\alpha}(\vec{x},\tau) + a\dot{a}(\tau)\alpha(\vec{x},\tau) \\
\Rightarrow\quad & \hat{\psi}(\vec{x},\tau) = \psi(\vec{x},\tau) - \dot{\alpha}(\vec{x},\tau) - \mathcal{H}(\tau)\alpha(\vec{x},\tau) \qquad (35)
\end{aligned}
$$

Re-deriving the gauge transformation equation for w_i : Consider the spatio-temporal indices of Eq.(34).

$$
\begin{aligned}
& \delta\hat{g}_{i0} = \delta g_{i0} - \Big(\nabla_0\xi_i + \nabla_i\xi_0\Big) \\
\Rightarrow\quad & a^2(\tau)\hat{w}_i = a^2(\tau)w_i - \Big(\nabla_i\xi_0 + \nabla_0\xi_i\Big)
\end{aligned}
$$

Evaluating the covariant derivatives individually,

$$
\begin{aligned}
\nabla_i\xi_0 &= \partial_i\big(\overline{g}_{00}\xi^0\big) - \Gamma^\lambda_{i0}\xi_\lambda \\
&= -a^2(\tau)\partial_i\alpha(\vec{x},\tau) - \Big(\cancelto{0}{\Gamma^0_{i0}\xi_0} + \Gamma^j_{io}\big(\overline{g}_{jk}\,\xi^k\big)\Big)
\end{aligned}
$$

Consider the piece

$$
\begin{aligned}
\Gamma^j_{io}\big(\overline{g}_{jk}\,\xi^k\big) &= \frac{1}{2}\overline{g}^{j\eta}\Big(\partial_i\overline{g}_{\eta 0} + \partial_0\overline{g}_{i\eta} - \cancelto{0}{\partial_\eta\overline{g}_{i0}}\Big)\big(\overline{g}_{jk}\,\xi^k\big) \\
&= \frac{1}{2}\overline{g}^{jl}\Big(\cancelto{0}{\partial_i\overline{g}_{l0}} + \partial_0\overline{g}_{il}\Big)\big(\overline{g}_{jk}\,\xi^k\big) \\
&= \frac{1}{2}\cdot 2a\dot{a}(\tau)\Big(a^2(\tau)\delta_{jk}\big(\nabla_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big)\Big) \\
&= -a\dot{a}(\tau)\big(\partial_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big)
\end{aligned}
$$

$$
\Rightarrow \nabla_i\xi_0 = -a^2(\tau)\partial_i\alpha(\vec{x},\tau) - a\dot{a}(\tau)\big(\partial_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big)
$$

Now consider the second covariant derivative

$$\begin{aligned}
\nabla_0\xi_i &= \partial_0\xi_i - \Gamma^\lambda_{i0}\xi_\lambda \\
&= \partial_0\big(g_{ij}\xi^j\big) + a\dot{a}(\tau)\big(\partial_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big) \qquad \because -\Gamma^\lambda_{i0}\xi_\lambda = -a\dot{a}(\tau)\big(\partial_k\beta + \epsilon_k\big) \\
&= \partial_0\left(a^2(\tau)\delta_{ij}\xi^j\right) - a\dot{a}(\tau)\big(\partial_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big) \\
&= \left(2a\dot{a}(\tau)\big(\partial_i\beta(\vec{x},\tau) + \epsilon_i(\vec{x},\tau)\big) + a^2(\tau)\big(\partial_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big)\right) - a\dot{a}(\tau)\big(\partial_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big) \\
&= a\dot{a}(\tau)\big(\partial_k\beta(\vec{x},\tau) + \epsilon_k(\vec{x},\tau)\big) + a^2(\tau)\big(\partial_i\dot{\beta}(\vec{x},\tau) + \dot{\epsilon}_i(\vec{x},\tau)\big)
\end{aligned}$$

Alltogether, we get the equation

$$\hat{w}_i = w_i + \partial_i\alpha(\vec{x},\tau) - \big(\partial_i\beta(\vec{x},\tau) - \epsilon_i(\vec{x},\tau)\big) \tag{36}$$

Re-deriving the gauge transformation equation for ϕ : Consider the trace of the spatial indices of Eq.(34).

$$\begin{aligned}
& \delta^{ij}\hat{g}_{ij} = \delta^{ij}g_{ij} - \delta^{ij}\big(\nabla_i\xi_j + \nabla_j\xi_i\big) \\
\Rightarrow\quad & -2a^2(\tau)\hat{\phi}(\vec{x},\tau)\delta^{ij}\delta_{ij} = -2a^2(\tau)\phi(\vec{x},\tau)\delta^{ij}\delta_{ij} - \delta^{ij}\big(\nabla_i\xi_j + \nabla_j\xi_i\big) \\
\Rightarrow\quad & 6a^2(\tau)\hat{\phi}(\vec{x},\tau) = 6a^2(\tau)\phi(\vec{x},\tau) - \delta^{ij}\big(\nabla_i\xi_j + \nabla_j\xi_i\big)
\end{aligned}$$

Consider the term

$$\begin{aligned}
\nabla_i\xi_j &= \partial_i\xi_j - \Gamma^\lambda_{ij}\xi_\lambda \\
\text{Where, } \partial_i\xi_j &= \partial_i a^2(\tau)\delta_{jk}\xi^k \\
&= a^2(\tau)\nabla^2\beta(\vec{x},\tau) \qquad \because \epsilon_j \text{ is transverse, } \partial_i\epsilon_j = 0 \\
\text{And } \Gamma^\lambda_{ij}\xi_\lambda &= \Gamma^0_{ij}\xi_0 + \cancelto{0}{\Gamma^l_{ij}\xi_l} \\
&= \frac{1}{2}\bar{g}^{0\rho}\Big(\partial_i\bar{g}_{\rho j} + \partial_j\bar{g}_{i\rho} - \partial_\rho\bar{g}_{ij}\Big)\big(\bar{g}_{00}\xi^0\big) \\
&= \frac{1}{2}\bar{g}^{00}\Big(\cancelto{0}{\partial_i\bar{g}_{0j}} + \cancelto{0}{\partial_j\bar{g}_{i0}} - \partial_0\bar{g}_{ij}\Big)\alpha(\vec{x},\tau) \\
&= a\dot{a}(\tau)\delta_{ij}\alpha(\vec{x},\tau) \\
\\
\text{Thus, } \nabla_i\xi_j &= a^2(\tau)\nabla^2\beta(\vec{x},\tau) + a\dot{a}(\tau)\delta_{ij}\alpha(\vec{x},\tau)
\end{aligned}$$

It's clear that $\nabla_i\xi_j = \nabla_j\xi_i$. Substituing the expanded expression for $\nabla_i\xi_j + \nabla_j\xi_i = 2\nabla_i\xi_j$ into the equation,

$$\begin{aligned}
& -6a^2(\tau)\hat{\phi}(\vec{x},\tau) = -6a^2(\tau)\phi(\vec{x},\tau) - 2a^2(\tau)\nabla^2\beta(\vec{x},\tau) - 2a\dot{a}(\tau)\alpha(\vec{x},\tau)\delta^i_i \\
\Rightarrow\quad & 6\hat{\phi}(\vec{x},\tau) = 6\phi(\vec{x},\tau) + 2\nabla^2\beta(\vec{x},\tau) + 6\mathcal{H}(\tau)\alpha(\vec{x},\tau) \\
\Rightarrow\quad & \hat{\phi}(\vec{x},\tau) = \phi(\vec{x},\tau) + \frac{1}{3}\nabla^2\beta(\vec{x},\tau) + \mathcal{H}(\tau)\alpha(\vec{x},\tau)
\end{aligned} \tag{37}$$

We see that the gauge transformation equations for the functions ψ, w_i and ϕ match the published equations.