<u>Inverse of a Matrix:</u> A square matrix *A* is said to be invertible if there exists a unique matrix *B* such that *AB*=*BA*=*I*, where *I* is the identity matrix. Then *B* is called the inverse of A and it is denoted by *B*= A^{-1}.

Mathematically,

$$A^{-1} = \frac{adj.A}{|A|} \qquad ;where, |A| \neq 0$$

Note:1.A matrix ***A***has inverse iff it is non-singular. $i.e.\ |A| \neq 0$.

2. The inverse of a matrix, if it exists, is unique.

<u>Cofactors of a square matrix:</u>If***A*** be any $n \times n$ square matrix

$$i.e,\ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The determinant of ***A*** is,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Then the cofactor of its any entry a_{ij} isdefined as,

$$A_{ij} = (-1)^{i+j} \begin{pmatrix} \det er\min ent\ of\ submatrix\ made\ by \\ delating\ ith\ row\ and\ jth\ column\ of\ A \end{pmatrix}.$$

<u>Adjoint matrix:</u>Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be any $n \times n$ square matrix and A_{ij} be the cofactors of entries a_{ij}, then the matrix of cofactors from ***A*** is,

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

The transpose of this matrix is called the adjoint of $\boldsymbol{A}$ and is denoted by adj($\boldsymbol{A}$).

$$i.e,\ adj(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^t = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Block Matrix: A matrix $\boldsymbol{A}$ may be partitioned into a system of similar matrices, called blocks, by a set of horizontal and vertical lines. The matrix $\boldsymbol{A}$ is then called a block matrix.

Problem-01: Find the adjoint matrix of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$.

Solution: The given matrix is,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

The cofactors of each elements of $|A|$ are,

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 6-1 = 5 \ ; \quad A_{12} = -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -(4-3) = -1 \ ; \quad A_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2-9 = -7 \ ;$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -(4-3) = -1 \ ; \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = 2-9 = -7 \ ; \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) = 5$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2-9 = -7 \ ; \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -(1-6) = 5 \ ; \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1 .$$

The matrix of cofactors is,

$$\begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix}$$

$$\therefore\ adj.A = \begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix}^t$$

$$= \begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix}$$

This is required adjoint matrix.

Problem-02: Find the adjoint matrix of the matrix $A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$.

Solution: The given matrix is,

$$A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$$

The cofactors of each elements of $|A|$ are,

$A_{11} = -4$; $A_{12} = 1$; $A_{13} = 4$; $A_{21} = -3$; $A_{22} = 0$; $A_{23} = 4$; $A_{31} = -3$; $A_{32} = 1$; $A_{33} = 3$.

The matrix of cofactors is,

$$\begin{pmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{pmatrix}$$

$$\therefore\ adj.A = \begin{pmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{pmatrix}^t$$

$$=\begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$$

This is required adjoint matrix.

Problem-03: Find the inverse of the matrix $A=\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

Solution: The given matrix is,

$$A=\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

The determinant of ***A*** is,

$$|A|=\begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$=2(12-2)-3(16-1)+4(8-3)$$

$$=20-45+20$$

$$=-5$$

Since, $|A|\neq 0$. So the given matrix is a non-singular matrix and it has an inverse matrix.

The cofactors of each elements of $|A|$ are,

$A_{11}=10$; $A_{12}=-15$; $A_{13}=5$; $A_{21}=-4$; $A_{22}=4$; $A_{23}=-1$; $A_{31}=-9$; $A_{32}=14$; $A_{33}=-6$.

The matrix of cofactors is,

$$\begin{pmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{pmatrix}$$

$$\therefore\ adj.A=\begin{pmatrix}10 & -15 & 5\\ -4 & 4 & -1\\ -9 & 14 & -6\end{pmatrix}^t$$

$$=\begin{pmatrix}10 & -4 & -9\\ -15 & 4 & 14\\ 5 & -1 & -6\end{pmatrix}$$

Therefore,

$$A^{-1}=\frac{adj.A}{|A|}$$

$$=-\frac{1}{5}\begin{pmatrix}10 & -4 & -9\\ -15 & 4 & 14\\ 5 & -1 & -6\end{pmatrix}$$

This is required inverse matrix.

Problem-04: If $A=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 5\\ 1 & 5 & 12\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}$ then find AB^{-1}.

Solution: The given matrices are,

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 5\\ 1 & 5 & 12\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}$$

The determinant of ***B*** is,

$$|B|=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{vmatrix}$$

$$=1(18-12)-1(9-3)+1(4-2)$$

$$=6-6+2$$

$$=2$$

Since, $|B|\neq 0$. So it is a non-singular matrix and it has an inverse matrix.

The cofactors of each elements of $|B|$ are,

$B_{11}=6$; $B_{12}=-6$; $B_{13}=2$; $B_{21}=-5$; $B_{22}=8$; $B_{23}=-3$; $B_{31}=1$; $B_{32}=-2$; $B_{33}=1$.

The matrix of cofactors is,

$$\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\therefore\ adj.A=\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}^t$$

$$=\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\therefore\ B^{-1}=\frac{adj.B}{|B|}$$

$$=\frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Therefore,

$$AB^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ -2 & 4 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & -1/2 & 3/2 \end{pmatrix}$$

This is required answer.

Problem-05: Find the inverse of the matrix $A=\begin{pmatrix}1&-1&0&2\\0&1&1&-1\\2&1&2&1\\3&-2&1&6\end{pmatrix}$ by using block matrix.

Solution: The given matrix is,

$$A=\begin{pmatrix}1&-1&0&2\\0&1&1&-1\\2&1&2&1\\3&-2&1&6\end{pmatrix}$$

Now we form the block matrix $M=(A\vdots I)$ and reduce ***M*** to echelon form.

$$M=\left(\begin{array}{cccc|cccc}1&-1&0&2&1&0&0&0\\0&1&1&-1&0&1&0&0\\2&1&2&1&0&0&1&0\\3&-2&1&6&0&0&0&1\end{array}\right)$$

$$\approx\left(\begin{array}{cccc|cccc}1&-1&0&2&1&0&0&0\\0&1&1&-1&0&1&0&0\\0&3&2&-3&-2&0&1&0\\0&1&1&0&-3&0&0&1\end{array}\right)\begin{array}{l}R_3'=R_3-2R_1\\R_4'=R_4-3R_1\end{array}$$

$$\approx\left(\begin{array}{cccc|cccc}1&-1&0&2&1&0&0&0\\0&1&1&-1&0&1&0&0\\0&0&-1&0&-2&-3&1&0\\0&0&0&1&-3&-1&0&1\end{array}\right)\begin{array}{l}R_3'=R_3-3R_2\\R_4'=R_4-R_2\end{array}$$

$$\approx\left(\begin{array}{cccc|cccc}1&-1&0&2&1&0&0&0\\0&1&1&-1&0&1&0&0\\0&0&1&0&2&3&-1&0\\0&0&0&1&-3&-1&0&1\end{array}\right)\begin{array}{l}R_3'=-R_3\\R_4'=R_4-R_2\end{array}$$

In echelon form, the left half of ***M*** is in triangular form; hence ***A*** is invertible. Further row reduce ***M*** to row canonical form.

$$\approx \left(\begin{array}{cccc:cccc} 1 & 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1' = R_1 + 2R_2 \\ R_2' = R_2 - R_3 \end{array}$$

$$\approx \left(\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & -3 & -4 & 2 & 0 \\ 0 & 1 & 0 & 0 & -5 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1' = R_1 - 2R_3 \\ R_2' = R_2 + R_4 \end{array}$$

$$\approx \left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -5 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right) R_1' = R_1 - R_2$$

The final block matrix is in the form $\left(I \vdots A^{-1}\right)$

$$\therefore\ A^{-1} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{pmatrix}$$

This is required inverse matrix.

NOTE: Any matrix ***A*** of order more than 3×3 has an inverse matrix if it does not contain zero row or zero column in its echelon form.

Also, Any matrix ***A*** of order more than 3×3 has an inverse matrix if the echelon form of ***A*** reduces into triangular form.

Problem-06: Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ by using block matrix.

Solution: The given matrix is,

$$A=\begin{pmatrix}1 & -1 & 2 & 1\\ 3 & 0 & 2 & 2\\ 2 & 1 & -1 & 1\\ 1 & 0 & 1 & 1\end{pmatrix}$$

Now we form the block matrix $M=(A\vdots I)$ and reduce ***M*** to echelon form.

$$M=\left(\begin{array}{cccc|cccc}1 & -1 & 2 & 1 & 1 & 0 & 0 & 0\\ 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0\\ 2 & 1 & -1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1\end{array}\right)$$

$$\approx\left(\begin{array}{cccc|cccc}1 & -1 & 2 & 1 & 1 & 0 & 0 & 0\\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0\\ 0 & 3 & -5 & -1 & -2 & 0 & 1 & 0\\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1\end{array}\right) \begin{array}{c} R_2' = R_2-3R_1\\ R_3' = R_3-2R_1\\ R_4' = R_4-R_1\end{array}$$

$$\approx\left(\begin{array}{cccc|cccc}1 & -1 & 2 & 1 & 1 & 0 & 0 & 0\\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 3\end{array}\right) \begin{array}{c} R_3' = R_3-R_2\\ R_4' = 3R_4-R_2\end{array}$$

$$\approx\left(\begin{array}{cccc|cccc}1 & -1 & 2 & 1 & 1 & 0 & 0 & 0\\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3\end{array}\right) R_4' = R_4+R_3$$

In echelon form, the left half of ***M*** is in triangular form; hence ***A*** is invertible. Further row reduce ***M*** to row canonical form.

$$\approx\left(\begin{array}{cccc|cccc}1 & -1 & 2 & 0 & 0 & 2 & -1 & -3\\ 0 & 3 & -4 & 0 & -2 & -1 & 1 & 3\\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3\end{array}\right) \begin{array}{c} R_1' = R_1-R_4\\ R_2' = R_2+R_4\\ R_3' = -R_3\end{array}$$

$$\approx\left(\begin{array}{cccc:cccc} 1 & -1 & 0 & 0 & 2 & 0 & 1 & -3 \\ 0 & 3 & 0 & 0 & -6 & 3 & -3 & 3 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right) \begin{array}{l} R_1^{'} = R_1 - 2R_3 \\ R_2^{'} = R_2 + 4R_3 \end{array}$$

$$\approx\left(\begin{array}{cccc:cccc} 1 & -1 & 0 & 0 & 2 & 0 & 1 & -3 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right) R_2^{'} = \frac{1}{3}R_2$$

$$\approx\left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right) R_1^{'} = R_1 + R_2$$

The final block matrix is in the form $\left(I \vdots A^{-1}\right)$

$$\therefore\ A^{-1} = \begin{pmatrix} 0 & 1 & 0 & -2 \\ -2 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & -2 & 1 & 3 \end{pmatrix}$$

This is required inverse matrix.

Exercise:

1. Find the adjoint matrix of the matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{pmatrix}$.
2. Find the adjoint matrix of the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.
3. Find the inverse of the matrix $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$.

4. Find the inverse of the matrix $A=\begin{pmatrix}-1 & 2 & -3\\ 2 & 1 & 0\\ 4 & -2 & 5\end{pmatrix}$.

5. If $A=\begin{pmatrix}-1 & 2 & -3\\ 2 & 1 & 0\\ 4 & -2 & 5\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1 & -1\\ 0 & 2 & 1\\ 5 & 2 & -3\end{pmatrix}$ then find $A^{-1}B$.

6. If $A=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 1 & 5 & 7\end{pmatrix}$ then show that $AA^{-1}=I$.

7. Find the inverse of the matrix $A=\begin{pmatrix}-1 & -3 & 3 & -1\\ 1 & 1 & -1 & 0\\ 2 & -5 & 2 & -3\\ -1 & 1 & 1 & 0\end{pmatrix}$ by using block matrix.

8. Find the inverse of the matrix $A=\begin{pmatrix}0 & -1 & -2 & -3\\ 1 & 1 & 4 & 4\\ 1 & 3 & 7 & 9\\ 1 & -2 & -4 & -6\end{pmatrix}$ by using block matrix.