Echelon Form of a Matrix: A matrix $\boldsymbol{A}$ is said to in echelon form if

1. all the non-zero rows, if any precede the zero rows.
2. the number of zero preceding the first non-zero element in a row is less than the number of such zero in the succeeding row.

Example: $\begin{pmatrix} 0 & -6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$; $\begin{pmatrix} 5 & 4 & 0 & 1 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Reduced Echelon Form of a Matrix: A matrix $\boldsymbol{A}$ is said to be in reduced echelon form if

1. all the non-zero rows, if any precede the zero rows.
2. the number of zero preceeding the first non-zero element in a row is less than the number of such zero in the succeeding row and
3. the first non-zero element in a row is unity.

Example: $\begin{pmatrix} 1 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Row Reduced Echelon Form of a Matrix: A matrix $\boldsymbol{A}$ is said to be in row reduced echelon form if

1. $\boldsymbol{A}$ is an echelon form,
2. each pivot element is 1 and
3. each pivot element is the only non-zero entry in its columns.

Example: $\begin{pmatrix} 1 & 0 & 0 & 5 & 6 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Pivot Element: The first non-zero entry of a non-zero row is called the pivot element of that row.

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$; here, 1, 5 and 7 are the pivot elements of the 1st, 2nd and 3rd rows respectively.

Normal Form of a Matrix: Every non-zero matrix ***A*** of order $m \times n$ can be reduced by the application of elementary row and column operations into equivalent matrix of one of the following forms:

1. $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, **2.** $\begin{pmatrix} I_r \\ O \end{pmatrix}$, **3.** $\begin{pmatrix} I_r & O \end{pmatrix}$ and **4.** $\left(I_r\right)$

where I_r is $r \times r$ identity matrix and O null matrix of any order.

These four forms are called normal or canonical form of ***A.***

Elementary Operations or Transformation on a Matrix: Let ***A*** be any matrix. Consider the following operations on the matrix ***A***.

1. Interchange of any two rows or two columns of ***A***.
2. Multiply each of the element of a row or a column of ***A*** with a non-zero scalar.
3. Add scalar multiple of one row of ***A*** to another row or one column of ***A*** two another column.

Each of the above operations are called elementary transformation or operations of ***A***.

Rank of a Matrix: Let ***M*** be a matrix. The number of pivot elements in an echelon form of ***M*** is called the rank of ***M*** and it is denoted by $\rho(M)$.

Alternatively, the number of non-zero rows in a row echelon form of a matrix is called the rank of the matrix.

Problem-01: Find the rank, echelon form and row reduced echelon form of the following matrix.

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$$

Solution: Given that,

$$A=\begin{pmatrix}1 & 2 & -1 & 2 & 1\\ 2 & 4 & 1 & -2 & 3\\ 3 & 6 & 2 & -6 & 5\end{pmatrix}$$

Now we shall reduce the matrix to echelon form by the elementary row operations.

$$\therefore \quad A=\begin{pmatrix}1 & 2 & -1 & 2 & 1\\ 2 & 4 & 1 & -2 & 3\\ 3 & 6 & 2 & -6 & 5\end{pmatrix}$$

$$\approx\begin{pmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & -6 & 1\\ 0 & 0 & 5 & -12 & 2\end{pmatrix} \quad \begin{matrix}R_2^{'}=R_2-2R_1\\ R_3^{'}=R_3-3R_1\end{matrix}$$

$$\approx\begin{pmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & -6 & 1\\ 0 & 0 & 0 & -6 & 1\end{pmatrix} \quad R_3^{'}=3R_3-5R_2$$

This matrix is in echelon form.

Since it contains 3 non-zero rows so the rank of the matrix ***A*** is, $\rho(A)=3$.

Again,

$$\approx\begin{pmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 1 & -2 & \frac{1}{3}\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{pmatrix} \quad \begin{matrix}R_2^{'}=\frac{1}{3}R_2\\ R_3^{'}=-\frac{1}{6}R_3\end{matrix}$$

$$\approx\begin{pmatrix}1 & 2 & 0 & 0 & \frac{4}{3}\\ 0 & 0 & 1 & -2 & \frac{1}{3}\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{pmatrix} \quad R_1^{'}=R_1+R_2$$

$$\approx \begin{pmatrix} 1 & 2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \quad R_2^{'} = R_2 + 2R_3$$

This matrix is in row reduced echelon form.

Problem-02: Find the rank, echelon form, row reduced echelon form and normal form of the following matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Solution: Given that,

$$A = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Now we shall reduce the matrix to echelon form by the elementary row operations.

$$\therefore \quad A = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & 1 \end{pmatrix} \quad R_3^{'} = R_1 - 3R_3$$

$$\approx \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 5 & -9 \\ 0 & 0 & -2 & -1 \end{pmatrix} \quad \begin{matrix} R_3^{'} = R_3 - 2R_2 \\ R_4^{'} = R_2 - 2R_4 \end{matrix}$$

$$\approx \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & -23 \end{pmatrix} \quad R_4' = 5R_4 + 2R_3$$

This matrix is in echelon form.

Since it contains 4 non-zero rows so the rank of the matrix ***A*** is, $\rho(A) = 4.$

Again,

$$\approx \begin{pmatrix} 1 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} R_1' = \frac{R_1}{3} \\ R_2' = \frac{R_2}{2} \\ R_3' = \frac{R_3}{5} \\ R_4' = \frac{-R_4}{23} \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1' = R_1 + \frac{2R_2}{3}$$

$$\approx \begin{pmatrix} 1 & 0 & 0 & \frac{6}{5} \\ 0 & 1 & 0 & \frac{23}{10} \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} R_1' = R_1 - \frac{2R_3}{3} \\ R_2' = R_2 - R_3 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} R_1' = R_1 - \frac{6R_4}{5} \\ R_2' = R_2 - \frac{23R_4}{10} \\ R_3' = R_3 + \frac{9R_4}{5} \end{matrix}$$

This is row reduced echelon form of the matrix ***A***.

This is also normal form of the matrix ***A.***

$i.e,(I_4)$

Problem-03: Find the rank, echelon form, row reduced echelon form and normal form of the following matrix.

$$A=\begin{pmatrix}1 & 3 & -2 & -1\\ 2 & 6 & -4 & -2\\ 1 & 3 & -2 & 1\\ 2 & 6 & 1 & -1\end{pmatrix}$$

Solution: Given that,

$$A=\begin{pmatrix}1 & 3 & -2 & -1\\ 2 & 6 & -4 & -2\\ 1 & 3 & -2 & 1\\ 2 & 6 & 1 & -1\end{pmatrix}$$

Now we shall reduce the matrix to echelon form by the elementary row operations.

$$\therefore\ A=\begin{pmatrix}1 & 3 & -2 & -1\\ 2 & 6 & -4 & -2\\ 1 & 3 & -2 & 1\\ 2 & 6 & 1 & -1\end{pmatrix}$$

$$\approx\begin{pmatrix}1 & 3 & -2 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & 5 & 1\end{pmatrix}\quad \begin{matrix}R_2^{'}=R_2-2R_1\\ R_3^{'}=R_3-R_1\\ R_4^{'}=R_4-2R_1\end{matrix}$$

$$\approx\begin{pmatrix}1 & 3 & -2 & -1\\ 0 & 0 & 5 & 1\\ 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}\quad R_2\leftrightarrow R_4$$

This matrix is in echelon form.

Since it contains 3 non-zero rows so the rank of the matrix ***A*** is, $\rho(A)=3.$

$$\approx \begin{pmatrix} 1 & 3 & -2 & -1 \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_2^{'} = \frac{1}{5}R_2 \\ R_3^{'} = \frac{1}{2}R_3 \end{array}$$

$$\approx \begin{pmatrix} 1 & 3 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1^{'} = R_1 + 2R_2$$

$$\approx \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1^{'} = R_1 + \frac{3}{5}R_3 \\ R_2^{'} = R_2 - \frac{1}{5}R_3 \end{array}$$

This is row reduced echelon form of the matrix ***A***.

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C_2^{'} = C_2 - 3C_1$$

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C_2 \leftrightarrow C_3$$

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C_3 \leftrightarrow C_4$$

$$\approx \begin{pmatrix} I_3 & O \\ O & O \end{pmatrix}$$

This is the normal form of the matrix ***A***.

Problem-04: Find the rank, echelon form, row reduced echelon form and normal form of the following matrix.

$$A=\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

Solution: Given that,

$$A=\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

Now we shall reduce the matrix to echelon form by the elementary row operations.

$$\therefore\ A=\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

$$\approx\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{pmatrix} \qquad \begin{matrix} R_2^{'}=R_2-R_1 \\ R_3^{'}=R_3-2R_1 \\ R_4^{'}=R_4-3R_1 \end{matrix}$$

$$\approx\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} R_3^{'}=R_3+3R_2 \\ R_4^{'}=R_4+R_2 \end{matrix}$$

This matrix is in echelon form.

Since it contains 2 non-zero rows so the rank of the matrix ***A*** is, $\rho(A)=2.$

$$\approx \begin{pmatrix} 1 & 0 & -5 & -5 & 0 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_1' = R_1 - 3R_2$$

This is row reduced echelon form of the matrix ***A***.

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} C_3' = C_3 + 5C_1 \\ C_4' = C_4 + 5C_1 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} C_3' = C_3 - 2C_2 \\ C_4' = C_4 - C_2 \\ C_5' = C_5 + C_2 \end{matrix}$$

$$\approx \begin{pmatrix} I_2 & O \\ O & O \end{pmatrix}$$

This is the normal form of the matrix ***A***.

Problem-05: Find the rank, echelon form, row reduced echelon form and normal form of the following matrix.

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \\ 5 & 3 & 5 & 3 \end{pmatrix}$$

Solution: Given that,

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \\ 5 & 3 & 5 & 3 \end{pmatrix}$$

Now we shall reduce the matrix to echelon form by the elementary row operations

$$\therefore\quad A=\begin{pmatrix}1&2&1&2\\2&1&2&1\\3&2&3&2\\3&3&3&3\\5&3&5&3\end{pmatrix}$$

$$\approx\begin{pmatrix}1&2&1&2\\0&-3&0&-3\\0&-4&0&-4\\0&-3&0&-3\\0&-7&0&-7\end{pmatrix}\qquad \begin{matrix}R_2^{'}=R_2-2R_1\\R_3^{'}=R_3-3R_1\\R_4^{'}=R_4-3R_1\\R_5^{'}=R_5-5R_1\end{matrix}$$

$$\approx\begin{pmatrix}1&2&1&2\\0&-3&0&-3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\qquad \begin{matrix}R_3^{'}=3R_3-4R_2\\R_4^{'}=R_4-R_2\\R_5^{'}=3R_5-7R_2\end{matrix}$$

This matrix is in echelon form.

Since it contains 2 non-zero rows so the rank of the matrix ***A*** is, $\rho(A)=2.$

$$\approx\begin{pmatrix}1&2&1&2\\0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\qquad R_2^{'}=-\frac{1}{3}R_2$$

$$\approx\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\qquad R_1^{'}=R_1-2R_2$$

$$\approx\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad R_1^{'}=R_1-2R_2$$

This is row reduced echelon form of the matrix ***A***.

$$\approx\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad \begin{matrix}C_3^{'}=C_3-C_1\\C_4^{'}=C_4-C_2\end{matrix}$$

$$\approx\begin{pmatrix}I_2&O\\O&O\end{pmatrix}$$

This is the normal form of the matrix ***A***.

Exercise: Find the rank, echelon form, row reduced echelon form and normal form of the following matrices:

1. $\begin{pmatrix}1&2&3&2&1\\3&1&-5&-2&1\\7&8&-1&2&5\end{pmatrix}$,**2.**$\begin{pmatrix}1&3&-1&2\\0&11&-5&3\\2&-5&3&1\\4&1&1&5\end{pmatrix}$, **3.**$\begin{pmatrix}1&2&-3&-2&-3\\1&3&-2&0&-4\\3&8&-7&-2&-11\\2&1&-9&-10&-3\end{pmatrix}$.