System of Linear Equations

Linear Equation: An equation in which the power of each unknown is one is called a linear equation. The general form of a linear equation is defined as,

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots\dots + a_nx_n = b \;\dots\;\dots\;\dots \quad (1)$$

where, $a_1, a_2, a_3, \dots\;\dots\;\dots, a_n$ and b real numbers and $x_1,\; x_2,\; x_3,\; \dots\;\dots\;\dots\; x_n$ are unknowns(or variables) which is to be determined.

If $b = 0$ then the equation (1) is called a homogeneous linear equation and if $b \neq 0$ then it is called a non-homogeneous linear equation.

Example: 1. $ax + by = c$ (non-homogeneous) represents a straight line.

2. $ax + by + cz + d = 0$ (homogeneous) represents a plane.

Solutions of Linear Equation: A solution of the linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots\dots + a_nx_n = b$$

in n variables is a sequence of n numbers, $\alpha_1, \alpha_2,\; \alpha_3,\; \dots\;\dots\;\dots\;, \alpha_n$ such that the equation is satisfied when we substitute $x_1 = \alpha_1,\; x_2 = \alpha_2,\; \dots\;\dots\;\dots\;, x_n = \alpha_n$.

The set of all such solutions of this linear equation is called a solution set.

System of Linear Equations: A finite set of linear equations is known as a system of linear equations. So,

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m \end{array}\right\} \; where,\; a_{ij}, b_i \in R,\; m \geq 2$$

is a system of m linear equations in n variables $x_1,\; x_2,\; x_3,\; \dots\;\dots\;\dots\; x_n$.

If $b_i = 0$, then the above system is called a homogeneous system of linear equations and if $b_i \neq 0$, then it is called a non-homogeneous system of linear equations.

Classification of System of Linear Equations: Regarding the nature of solutions, systems of linear equations are classified as follows:

1. **Inconsistent:** A system of linear equations is called an inconsistent if it has no solution.
2. **Consistent:** A system of linear equations is called consistent if it has one or more solution. It is also classified as,

   a). Unique: A system of linear equations is called unique if it has only one solution.

   b).Redundant:A system of linear equations is called redundant if it has more than one solution.

Solution of System of Linear Equations: A system of m linear equations in n unknowns is,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots\dots\dots (1)\ where,\ a_{ij}, b_i \in R,\ m \geq 2$$

A sequence of numbers $\alpha_1,\ \alpha_2,\ \alpha_3, \dots\ \dots\ \dots\ \alpha_n$ is called solution of the system of linear equations (1) if $x_1 = \alpha_1,\ x_2 = \alpha_2,\ \dots\ \dots\ \dots\ , x_n = \alpha_n$ is a solution of every equations in the system.

Free Variables: If a system of m linear equations in n unknowns is,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots\dots\dots (1)\ where,\ a_{ij}, b_i \in R,\ m \geq 2$$

and its echelon form is,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{13}x_3 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m3}x_3 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots\dots\dots (2)\ where,\ a_{ij}, b_i \in R,\ m \geq 2$$

The variables which do not appear at the beginning of any equation of (2) are called free variables.

Trivial and non-trivial Solution: A homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots&\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= 0 \end{aligned}\right\} \dots\dots\dots (1)$$

The above system of linear equations (1) has a solution, namely zero n-tuple $O = (0,0,0,\dots\dots,0,)$ called **zero** or **trivial solution** and any other solution, if it exists, is called **non-zero** or **non-trivial solution**.

Augmented matrix: Consider a non-homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots&\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots\dots\dots (1) \; where,\; a_{ij}, b_i \in R,\; m \geq 2$$

we can write it as $AX = B$.

Where, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$

Here, $\boldsymbol{A}$ is called the coefficient matrix, $\boldsymbol{X}$ is called the column matrix of the variables and $\boldsymbol{B}$ is called the column matrix of the constants.

The matrix $[A \;\vdots\; B] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{pmatrix}$ is called augmented matrix of the system of linear equations (1). The augmented matrix is also denoted by A^* or $[A, B]$ or $[A\ B]$.

NOTE:

1. If the coefficient matrix and augmented matrix have the same rank, then the system of linear equations is said to be consistent.
2. If the coefficient matrix and augmented matrix have the different rank, then the system of linear equations is said to be inconsistent and it has no solution.

3. If the coefficient matrix and augmented matrix have the same rank and the rank is equal to the number of variables then the system of linear equations has a unique solution.
4. If the coefficient matrix and augmented matrix have the same rank and the rank is less than the number of variables then the system of linear equations has infinite number of solutions.

Gauss Elimination Method: Consider a non-homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m \end{array}\right\} \dots\dots\dots (1) \; where, \; a_{ij}, b_i \in R, \; m \geq 2$$

we can write it as $AX = B$.

Where, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$

The augmented matrix is,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{pmatrix}$$

reduce this matrix into the following form,

$$\begin{pmatrix} a_{11}' & a'_{12} & \dots & a'_{1n} & \vdots & l_1 \\ 0 & a_{22}' & \dots & a'_{2n} & \vdots & l_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & a_{mn}' & \vdots & l_m \end{pmatrix}$$

then the reduced system is,

$$\begin{array}{l} 1.x_1 + a'_{1n}.x_2 + \dots\dots + a'_{1n}.x_n = l_1 \\ 0.x_1 + 1.x_2 + \dots\dots + a'_{2n}.x_n = l_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0.x_1 + 0.x_2 + \dots\dots + 1.x_n = l_m \end{array}$$

Now, by back substitution, we solve for, $x_n, \; x_{n-1}, \; \dots\dots, x_1$.

This process which eliminates unknowns from succeeding equations is known as Gauss elimination.

NOTE:

1. If an equation $0x_1 + 0x_2 + \dots\dots + 0x_n = b,\ b \neq 0$ occurs, then the system is inconsistent and has no solution.
2. If an equation $0x_1 + 0x_2 + \dots\dots + 0x_n = 0$ occurs, then the equation can be deleted without affecting the solution.
3. If the number of equations is equal to the number of variables, then the system has a unique solution.
4. If the number of equations is less than the number of variables, then the system has a infinitely many solutions.

Gauss Jordan Elimination Method: Consider a non-homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots&\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots\dots\dots (1)\ \textit{where},\ a_{ij}, b_i \in R,\ m \geq 2$$

we can write it as $AX = B$.

Where, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$

The augmented matrix is,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{pmatrix}$$

reduce this matrix into the following form,

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \vdots & l_1 \\ 0 & 1 & \dots & 0 & \vdots & l_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & 1 & \vdots & l_m \end{pmatrix}$$

then the reduced system is,

$$\begin{aligned}
1.x_1 + 0.x_2 + \dots \dots + 0.x_n &= l_1 \\
0.x_1 + 1.x_2 + \dots \dots + 0.x_n &= l_2 \\
\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\
0.x_1 + 0.x_2 + \dots \dots + 1.x_n &= l_m
\end{aligned}$$

Thus, we get $x_1 = l_1, x_2 = l_2, \dots \dots x_n = l_m$.

Problem-01: Show that the following system of linear equations is not consistent

$$\begin{aligned}
x + y + z &= -3 \\
3x + y - 2z &= -2 \\
2x + 4y + 7z &= 7
\end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned}
x + y + z &= -3 \\
3x + y - 2z &= -2 \\
2x + 4y + 7z &= 7
\end{aligned}\right\} \dots \dots \dots (1)$$

the system (1) can be written as,

$$AX = B \dots \dots \dots (2)$$

where, $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix}$.

The augmented matrix is,

$$[A, B] = \begin{pmatrix} 1 & 1 & 1 & \vdots & -3 \\ 3 & 1 & -2 & \vdots & -2 \\ 2 & 4 & 7 & \vdots & 7 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 1 & 1 & \vdots & -3 \\ 0 & -2 & -5 & \vdots & 7 \\ 0 & 2 & 5 & \vdots & 13 \end{pmatrix} \begin{matrix} R_2' \to R_2 - 3R_1 \\ R_3' \to R_3 - 2R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 1 & 1 & \vdots & -3 \\ 0 & -2 & -5 & \vdots & 7 \\ 0 & 0 & 0 & \vdots & 20 \end{pmatrix} \; R_3' \to R_3 + R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} x+y+z&=-3 \\ -2y-5z&=7 \\ 0&=20 \end{aligned}\right\}$$

Since, $0=20$ occurs, which is not possible so the given system of linear equations is inconsistent. **(Showed)**

Problem-02: Show that the following system of linear equations is consistent

$$\begin{aligned} 2x-y+3z&=8 \\ -x+2y+z&=4 \\ 3x+y-4z&=0 \end{aligned}$$

and find the solution.

Solution: The given system of linear equations is,

$$\left.\begin{aligned} 2x-y+3z&=8 \\ -x+2y+z&=4 \\ 3x+y-4z&=0 \end{aligned}\right\} \dots\ \dots\ \dots\ (1)$$

the system (1) can be written as,

$$AX=B \dots\ \dots\ \dots\ (2)$$

where, $A=\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B=\begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$.

The augmented matrix is,

$$[A,\ B]=\begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ -1 & 2 & 1 & \vdots & 4 \\ 3 & 1 & -4 & \vdots & 0 \end{pmatrix}$$

$$\approx\begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 5 & -17 & \vdots & -24 \end{pmatrix} \begin{matrix} R_2' \to 2R_2+R_1 \\ R_3' \to 2R_3-3R_1 \end{matrix}$$

$$\approx\begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & -76 & \vdots & -152 \end{pmatrix} R_3' \to 3R_3-5R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} 2x - y + 3z &= 8 \\ 3y + 5z &= 16 \\ -76z &= -152 \end{aligned}\right\}$$

By back substitution we get, $z = 2$, $y = 2$, $x = 2$.

Hence the given system is consistent and the solution is,

$$x = 2,\ y = 2,\ z = 2.$$

Problem-03: Show that the following system of linear equations is not consistent

$$\begin{aligned} x + y + 2z + w &= 5 \\ 2x + 3y - z - 2w &= 2 \\ 4x + 5y + 3z \quad &= 7 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} x + y + 2z + w &= 5 \\ 2x + 3y - z - 2w &= 2 \\ 4x + 5y + 3z \quad &= 7 \end{aligned}\right\} \ldots\ \ldots\ \ldots\ (1)$$

the system (1) can be written as,

$$AX = B \ldots\ \ldots\ \ldots\ (2)$$

where, $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$.

The augmented matrix is,

$$[A, B] = \begin{pmatrix} 1 & 1 & 2 & 1 & \vdots & 5 \\ 2 & 3 & -1 & -2 & \vdots & 2 \\ 4 & 5 & 3 & 0 & \vdots & 7 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 1 & 2 & 1 & \vdots & 5 \\ 0 & 1 & -5 & -4 & \vdots & -8 \\ 0 & 1 & -5 & -4 & \vdots & -13 \end{pmatrix} \begin{matrix} R_2' \to R_2 - 2R_1 \\ R_3' \to R_3 - 4R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 1 & 2 & 1 & \vdots & 5 \\ 0 & 1 & -5 & -4 & \vdots & -8 \\ 0 & 0 & 0 & 0 & \vdots & -5 \end{pmatrix} R_3' \to R_3 - R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} x+y+2z+w&=5\\ y-5z-4w&=-8\\ 0&=-5 \end{aligned}\right\}$$

Since, $0=-5$ occurs, which is not possible so the given system of linear equations is inconsistent. **(Showed)**

Problem-04: Find for what values of λ, μ the system of linear equations

$$\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu \end{aligned}$$

has i). a unique solution, ii). no solution, iii). infinite solutions.

Solution: The given system of linear equations is,

$$\left.\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu \end{aligned}\right\}\ldots\ \ldots\ \ldots\ (1)$$

the system (1) can be written as,

$$AX=B \ldots\ \ldots\ \ldots\ (2)$$

where, $A=\begin{pmatrix} 1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda \end{pmatrix}$, $X=\begin{pmatrix} x\\ y\\ z \end{pmatrix}$ and $B=\begin{pmatrix} 6\\ 10\\ \mu \end{pmatrix}$.

The augmented matrix is,

$$[A,\ B]=\begin{pmatrix} 1 & 1 & 1 & \vdots & 6\\ 1 & 2 & 3 & \vdots & 10\\ 1 & 2 & \lambda & \vdots & \mu \end{pmatrix}$$

$$\approx\begin{pmatrix} 1 & 1 & 1 & \vdots & 6\\ 0 & 1 & 2 & \vdots & 4\\ 0 & 1 & \lambda-1 & \vdots & \mu-6 \end{pmatrix}\begin{matrix} R_2' \to R_2-R_1\\ R_3' \to R_3-R_1 \end{matrix}$$

$$\approx\begin{pmatrix} 1 & 1 & 1 & \vdots & 6\\ 0 & 1 & 2 & \vdots & 4\\ 0 & 0 & \lambda-3 & \vdots & \mu-10 \end{pmatrix} R_3' \to R_3-R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} x + y + z &= 6 \\ y + 2z &= 4 \\ (\lambda - 3)z &= \mu - 10 \end{aligned}\right\}$$

If $\lambda \neq 3$, then the given system of equations possesses a unique solution for any value of μ.

If $\lambda = 3$ and $\mu \neq 10$, then the given system of equations possesses no solution.

If $\lambda = 3$ and $\mu = 10$, then the given system of equations possesses infinite solutions.

Problem-05: Solve the following system of linear equations

$$\begin{aligned} 5x - 6y + 4z &= 15 \\ 7x + 4y - 3z &= 19 \\ 2x + y + 6z &= 46 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} 5x - 6y + 4z &= 15 \\ 7x + 4y - 3z &= 19 \\ 2x + y + 6z &= 46 \end{aligned}\right\} \ldots\ \ldots\ \ldots\ (1)$$

the system (1) can be written as,

$$AX = B \ldots\ \ldots\ \ldots\ (2)$$

where, $A = \begin{pmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 15 \\ 19 \\ 46 \end{pmatrix}$.

The augmented matrix is,

$$[A, B] = \begin{pmatrix} 5 & -6 & 4 & \vdots & 15 \\ 7 & 4 & -3 & \vdots & 19 \\ 2 & 1 & 6 & \vdots & 46 \end{pmatrix}$$

$$\approx \begin{pmatrix} 5 & -6 & 4 & \vdots & 15 \\ 0 & 62 & -43 & \vdots & -10 \\ 0 & 17 & 22 & \vdots & 200 \end{pmatrix} \begin{matrix} R_2' \to 5R_2 - 7R_1 \\ R_3' \to 5R_3 - 2R_1 \end{matrix}$$

$$\approx\begin{pmatrix} 5 & -6 & 4 & \vdots & 15 \\ 0 & 62 & -43 & \vdots & -10 \\ 0 & 0 & 2095 & \vdots & 12570 \end{pmatrix} R_3' \to 62R_3 - 17R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} 5x-6y+4z&=15 \\ 62y-43z&=-10 \\ 2095z&=12570 \end{aligned}\right\}$$

By back substitution, we get $z=6$, $y=4$, $x=3$.

Hence the required result is, $x=3$, $y=4$, $z=6$.

Problem-06: Solve the following system of linear equations

$$\begin{aligned} x+y-2z+s+3t&=1 \\ 2x-y+2z+2s+6t&=2 \\ 3x+2y-4z-3s-9t&=3 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} x+y-2z+s+3t&=1 \\ 2x-y+2z+2s+6t&=2 \\ 3x+2y-4z-3s-9t&=3 \end{aligned}\right\} \ldots \ldots \ldots (1)$$

the system (1) can be written as,

$$AX=B \ldots \ldots \ldots (2)$$

where, $A=\begin{pmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix}$ and $B=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

The augmented matrix is,

$$[A, B]=\begin{pmatrix} 1 & 1 & -2 & 1 & 3 & \vdots & 1 \\ 2 & -1 & 2 & 2 & 6 & \vdots & 2 \\ 3 & 2 & -4 & -3 & -9 & \vdots & 3 \end{pmatrix}$$

$$\approx\begin{pmatrix}1 & 1 & -2 & 1 & 3 & \vdots & 1\\ 0 & -3 & 6 & 0 & 0 & \vdots & 0\\ 0 & -1 & 2 & -6 & -18 & \vdots & 0\end{pmatrix}\begin{matrix}R_2^{'}\to R_2-2R_1\\ R_3^{'}\to R_3-3R_1\end{matrix}$$

$$\approx\begin{pmatrix}1 & 1 & -2 & 1 & 3 & \vdots & 1\\ 0 & -3 & 6 & 0 & 0 & \vdots & 0\\ 0 & 0 & 0 & -18 & -54 & \vdots & 0\end{pmatrix}R_3^{'}\to 3R_3-R_2$$

$$\approx\begin{pmatrix}1 & 1 & -2 & 1 & 3 & \vdots & 1\\ 0 & 1 & -2 & 0 & 0 & \vdots & 0\\ 0 & 0 & 0 & 1 & 3 & \vdots & 0\end{pmatrix}\begin{matrix}R_2^{'}\to -\frac{1}{3}R_2\\ R_3^{'}\to -\frac{1}{18}R_3\end{matrix}$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned}x+y-2z+s+3t&=1\\ y-2z&=0\\ s+3t&=0\end{aligned}\right\}$$

There are 3 equations in 5 unknowns, so there are (5-3)=2free variables which are *z* and *t*. Thus the system is consistent with an infinite number of solutions.

We put $z=a$ and $t=b$. So by back substitution we have, $s=-3b$, $y=2a$, $x=1$.

Hence the required result is, $x=1$, $y=2a$, $z=a$, $s=-3b$, $t=b$.

Problem-07: Solve the following system of linear equations

$$\begin{aligned}2x+3y+5z+t&=3\\ 3x+4y+2z+3t&=-2\\ x+2y+8z-t&=8\\ 7x+9y+z+8t&=0\end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned}2x+3y+5z+t&=3\\ 3x+4y+2z+3t&=-2\\ x+2y+8z-t&=8\\ 7x+9y+z+8t&=0\end{aligned}\right\}\ldots\ \ldots\ \ldots\ (1)$$

the system (1) can be written as,

$$AX=B\ldots\ \ldots\ \ldots\ (2)$$

where, $A=\begin{pmatrix} 2 & 3 & 5 & 1 \\ 3 & 4 & 2 & 3 \\ 1 & 2 & 8 & -1 \\ 7 & 9 & 1 & 8 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ and $B=\begin{pmatrix} 3 \\ -2 \\ 8 \\ 0 \end{pmatrix}$.

The augmented matrix is,

$$[A,\ B]=\begin{pmatrix} 2 & 3 & 5 & 1 & \vdots & 3 \\ 3 & 4 & 2 & 3 & \vdots & -2 \\ 1 & 2 & 8 & -1 & \vdots & 8 \\ 7 & 9 & 1 & 8 & \vdots & 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 8 & -1 & \vdots & 8 \\ 3 & 4 & 2 & 3 & \vdots & -2 \\ 2 & 3 & 5 & 1 & \vdots & 3 \\ 7 & 9 & 1 & 8 & \vdots & 0 \end{pmatrix} R_1 \leftrightarrow R_3$$

$$\approx \begin{pmatrix} 1 & 2 & 8 & -1 & \vdots & 8 \\ 0 & -2 & -22 & 6 & \vdots & -26 \\ 0 & -1 & -11 & 3 & \vdots & -13 \\ 0 & -5 & -55 & 15 & \vdots & -56 \end{pmatrix} \begin{matrix} R_2^{'} \to R_2 - 3R_1 \\ R_3^{'} \to R_3 - 2R_1 \\ R_4^{'} \to R_4 - 7R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 8 & -1 & \vdots & 8 \\ 0 & -2 & -22 & 6 & \vdots & -26 \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 18 \end{pmatrix} \begin{matrix} R_3^{'} \to 2R_3 - R_2 \\ R_4^{'} \to 2R_4 - 5R_2 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 8 & -1 & \vdots & 8 \\ 0 & 1 & 11 & -3 & \vdots & 13 \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 18 \end{pmatrix} R_2^{'} \to -\frac{1}{2}R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} x+2y+8z-t &= 8 \\ y+11z-3t &= 13 \\ 0 &= 0 \\ 0 &= 18 \end{aligned}\right\}$$

$$\text{or,}\quad \left.\begin{aligned} x+2y+8z-t &= 8\\ y+11z-3t &= 13\\ 0 &= 18 \end{aligned}\right\}$$

Since, $0=18$ occurs, which is not possible so the given system of linear equations is inconsistent and it has no solution.

Problem-08: Solve the following system of linear equations

$$\begin{aligned} x+2y-2z-t &= 0\\ 2x+5y-3z-t &= 1\\ 3x+8y-4z-t &= 2\\ x+5y+z+2t &= 3 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} x+2y-2z-t &= 0\\ 2x+5y-3z-t &= 1\\ 3x+8y-4z-t &= 2\\ x+5y+z+2t &= 3 \end{aligned}\right\} \dots\dots\dots (1)$$

the system (1) can be written as,

$$AX=B \dots\dots\dots (2)$$

where, $A=\begin{pmatrix} 1 & 2 & -2 & -1\\ 2 & 5 & -3 & -1\\ 3 & 8 & -4 & -1\\ 1 & 5 & 1 & 2 \end{pmatrix}$, $X=\begin{pmatrix} x\\ y\\ z\\ t \end{pmatrix}$ and $B=\begin{pmatrix} 0\\ 1\\ 2\\ 3 \end{pmatrix}$.

The augmented matrix is,

$$[A,\ B]=\left(\begin{array}{cccc:c} 1 & 2 & -2 & -1 & 0\\ 2 & 5 & -3 & -1 & 1\\ 3 & 8 & -4 & -1 & 2\\ 1 & 5 & 1 & 2 & 3 \end{array}\right)$$

$$\approx \left(\begin{array}{cccc:c} 1 & 2 & -2 & -1 & 0\\ 0 & 1 & 1 & 1 & 1\\ 0 & 2 & 2 & 2 & 2\\ 0 & 3 & 3 & 3 & 3 \end{array}\right) \begin{array}{l} R_2' \to R_2-2R_1\\ R_3' \to R_3-3R_1\\ R_4' \to R_4-R_1 \end{array}$$

$$\approx\begin{pmatrix}1 & 2 & -2 & -1 & \vdots & 0\\ 0 & 1 & 1 & 1 & \vdots & 1\\ 0 & 0 & 0 & 0 & \vdots & 0\\ 0 & 0 & 0 & 0 & \vdots & 0\end{pmatrix}\begin{matrix} R_3^{'}\to R_3-2R_1\\ R_4^{'}\to R_4-3R_2\end{matrix}$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} x+2y-2z-t&=0\\ y+\ \ z+t&=1\\ 0&=0\\ 0&=0\end{aligned}\right\}$$

or, $$\left.\begin{aligned} x+2y-2z-t&=0\\ y+\ \ z+t&=1\end{aligned}\right\}$$

There are 2 equations in 4 unknowns, so there are (4-2) = 2 free variables which are z and t. Thus the system is consistent with an infinite number of solutions.

We put $z=a$ and $t=b$. So by back substitution we have, $y=1-a-b$, $x=4a+3b-2$.

Hence the required result is, $x=4a+3b-2$, $y=1-a-b$, $z=a$, $t=b$.

Exercise:

1. Solve the following system of linear equations
$$\begin{aligned} 3x+5y-7z&=13\\ 4x+y-12z&=6\\ 2x+9y-3z&=20\end{aligned}$$
2. Solve the following system of linear equations
$$\begin{aligned} x+y+z&=3\\ x+2y+3z&=4\\ x+4y+9z&=6\end{aligned}$$
3. Solve the following system of linear equations
$$\begin{aligned} 2x-3y+4z&=1\\ 3x+4y-5z&=1\\ 5x-7y+2z&=3\end{aligned}$$
4. Determine for what value of a the following system of linear equations
$$\begin{aligned} x+y-z&=1\\ 2x+3y+az&=3\\ x+ay+3z&=2\end{aligned}$$

hasi). no solution, ii). more than one solution, iii). a unique solution.

5. Determine for what values of λ and μ the following system of linear equations

$$2x+3y+z=5$$
$$3x-y+\lambda z=2$$
$$x+7y-6z=\mu$$

hasi). no solution, ii). more than one solution, iii). a unique solution.

6. Solve the following system of linear equations

$$x+y+z+t=4$$
$$x+2y+2z-t=4$$
$$x+4y+9z+2t=16$$