Eigenvalue & Cayley Hamilton Theorem

Eigenvalue:

A scalar $\boldsymbol{\lambda}$ is called an *eigenvalue* of the $\boldsymbol{n} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$ is there is a nontrivial solution $\boldsymbol{x}$ of $\boldsymbol{Ax} = \boldsymbol{\lambda x}$. Such an $\boldsymbol{x}$ is called eigenvector corresponding to the eigenvalue $\boldsymbol{\lambda}$**.**

Characteristic Matrix:

To find the eigenvalue of $n \times n$ matrix A. we write $Av = \lambda v$ as $Av = \lambda I v$

$$\Rightarrow (\lambda I - A) v = 0$$

The matrix $\lambda I - A$, where I is an $n \times n$ matrix. is called the characteristic matrix of A.

Characteristic Polynomial:

The determinant of the characteristic matrix $\lambda I - A$ is a polynomial in λ and is called the characteristic polynomial of A and is denoted by

$$\det(\lambda I - A) \text{ i.e } |\lambda I - A|$$

Characteristic Equation:

The equation $|\lambda I - A| = 0$ is called the characteristic equation of A. The roots of this equation are called the characteristic roots or eigenvalues of A.

Problem Solving:

Q: Find the eigenvalues of the matrix $A = \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix}$.

Soln: The given matrix is,

$$A = \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix}$$

The characteristic matrix of A is,

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda+3 & -2 & -2 \\ 6 & \lambda-5 & -2 \\ 7 & -4 & \lambda-4 \end{pmatrix}$$

The characteristic polynomial of A is,

$$|\lambda I - A| = \begin{vmatrix} \lambda+3 & -2 & -2 \\ 6 & \lambda-5 & -2 \\ 7 & -4 & \lambda-4 \end{vmatrix}$$

The characteristic equation of A is,

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda+3 & -2 & -2 \\ 6 & \lambda-5 & -2 \\ 7 & -4 & \lambda-4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+3)\{(\lambda-5)(\lambda-4)-8\} - (-2)(6\lambda-24+14) + (-2)(-24-7\lambda+35) = 0$$

$$\Rightarrow (\lambda+3)(\lambda^2-9\lambda+12) + 12\lambda - 20 - 22 + 14\lambda = 0$$

$$\Rightarrow \lambda^3 - 9\lambda^2 + 12\lambda + 3\lambda^2 - 27\lambda + 36 + 26\lambda - 42 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 5\lambda(\lambda-1) + 6(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-5\lambda+6) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1, 2, 3$$

Thus, the eigenvalues of A are 1, 2, 3.

Q: Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

Soln: The given matrix is,

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

The characteristic matrix of A is,

$$A - \lambda I = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix}$$

The characteristic equation of A is,

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-4\lambda+\lambda^2) - 0\cdot(0-0) - 2(0-2\lambda) = 0$$

$$\Rightarrow -4\lambda + \lambda^2 + 4\lambda^2 - \lambda^3 + 4\lambda = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 = 0$$

$$\Rightarrow \lambda^2(\lambda - 5) = 0$$

$$\therefore \lambda = 0, 0, 5$$

Thus the eigenvalues of A are, 0, 0, 5.

Q: Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Soln: The given matrix is,

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The characteristic matrix of A is,

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{pmatrix}$$

The characteristic polynomial of A is,

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix}$$

The characteristic equation of A is,

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)^2(\lambda-4) + (-3)(\lambda-4) = 0$$

$$\Rightarrow (\lambda-4)(\lambda^2-4\lambda+4-3) = 0$$

$$\Rightarrow (\lambda-4)(\lambda^2-4\lambda+1) = 0$$

$\therefore\ \lambda = 4$ and $\lambda^2 - 4\lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}.$$

Thus, the eigenvalues of A are, $4, 2+\sqrt{3}, 2-\sqrt{3}$. Ans.

Q: Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$.

Soln: The given matrix is,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

The characteristic matrix of A is,

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda-1 & -2 & -3 \\ 0 & \lambda-1 & -2 \\ 0 & 2 & \lambda-1 \end{pmatrix}$$

The characteristic polynomial is,

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -3 \\ 0 & \lambda-1 & -2 \\ 0 & 2 & \lambda-1 \end{vmatrix}$$

The characteristic equation of A is,

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -2 & -3 \\ 0 & \lambda-1 & -2 \\ 0 & 2 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)\left\{(\lambda-1)^2+4\right\} - 2\cdot 0 - 3\cdot 0 = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-2\lambda+1+4) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-2\lambda+5) = 0$$

$\therefore \lambda = 1$ and $\lambda^2 - 2\lambda + 5 = 0$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-4\cdot 5}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{(4i)^2}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

Thus, the eigenvalues of A are, $1, 1+2i, 1-2i$.

Exercise: ① Find the eigenvalues of the matrix $A=\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$

Ans: $\lambda = 2, 2, 6$.

② Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -8 & -2 \\ 1 & 2 & 2 \end{pmatrix}$.

Ans: $\lambda = 0, 3, -7$.

③ Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

Ans: $\lambda = -2, \lambda = 4$.

Cayley Hamilton Theorem:

Every square matrix satisfies its own characteristic equation i.e. if the characteristic equation of the nth order matrix is

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

then Cayley-Hamilton theorem states that

$$f(A) = A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI = 0$$

where I is the nth order unit matrix and O is the nth order zero matrix.

Example-1:

Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and verify Cayley-Hamilton theorem for it.

Solution:-

The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda+1 & -1 \\ -3 & -1 & \lambda-1 \end{bmatrix}$$

The determinant of the matrix $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda+1 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2-1-1) + 2(-2\lambda+2-3) - 3(2+3\lambda+3)$$

$$= (\lambda-1)(\lambda^2-2) - 4\lambda - 2 - 9\lambda - 15$$

$$= \lambda^3 - 2\lambda - \lambda^2 + 2 - \cancel{4\lambda}\, 13\lambda - 17$$

$$= \lambda^3 - \lambda^2 - 15\lambda - 15$$

Therefore, the characteristic equation of A is

$$\lambda^3 - \lambda^2 - 15\lambda - 15 = 0$$

Now in order to verify Cayley-Hamilton theorem we have to show that

$$A^3 - A^2 - 15A - 15I = 0$$

Now, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix}$$

$$\therefore A^3 - A^2 - 15A - 15I = \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - 15\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44-14-15-15 & 33-3-30-0 & 53-8-45-0 \\ 33-3-30-0 & 6-6+15-15 & 21-6-15-0 \\ 53-8-45-0 & 21-6-15-0 & 41-11-15-15 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O$$

i.e. $A^3 - A^2 - 15A - 15I = O$

Hence the Cayley-Hamilton theorem is verified.

Ex-2: Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & -2 & -2 \\ -3 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{bmatrix}$$

Now the determinant of the matrix $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -2 \\ -3 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left\{(\lambda-1)^2 - 0\right\} + 2(-3\lambda+3) - 2(3+\lambda-1)$$

$$= (\lambda-1)^3 - 6\lambda + 6 - 4 - 2\lambda$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 - 8\lambda + 2$$

$$= \lambda^3 - 3\lambda^2 - 5\lambda + 1$$

Therefore, the characteris equation of A is

$$\lambda^3 - 3\lambda^2 - 5\lambda + 1 = 0$$

Now using Cayley-Hamilton theorem we get

$$A^3 - 3A^2 - 5A + I = 0$$

Multiplying the above Cayley-Hamilton equation on both sides by A^{-1} we have

$$A^2 - 3A - 5I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = 3A + 5I - A^2$$

Now $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$A^{-1} = 3A + 5I - A^2$$

$$= 3\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+5-9 & 6+0-6 & 6+0-4 \\ 9+0-6 & 3+5-7 & 0+0-6 \\ 3+0-5 & 3+0-4 & 3+5-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$