This study treats the determination of eigenvalues and eigenvectors of large algebraic systems. The methods developed are applicable to finding the natural frequencies and modes of vibration of large structural systems.

Eigenvalues and Eigenvectors: Let ***A*** be an n-square matrix. A scalar λ is called an eigenvalue or characteristic root of ***A*** if there exists a non-zero column vector v such that,

$$Av = \lambda v$$

$$\text{or,} \left(A - \lambda I\right) v = 0$$

where, ***I*** is the n-square identity matrix. Every vector satisfying this relation is called an eigenvector or characteristic vector of ***A*** associated with the eigenvalue λ.

Spectrum: The set of all eigenvalues of a matrix ***A*** is called the spectrum of ***A***.

Eigenspace: If ***A*** be an n-square matrix and λ be an eigenvalue of ***A***, then the set of all vectors satisfying the relation $Av = \lambda v$ including the zero vector is called an eigenspace of ***A*** corresponding to λ.

Characteristic Matrix: If ***A*** be an n-square matrix and λ be an eigenvalue of ***A*** , then the matrix $A - \lambda I_n$ or $\lambda I_n - A$ is called a characteristic matrix of ***A***.

Characteristic Polynomial: If ***A*** be an n-square matrix and λ be an eigenvalue of ***A***, then the determinant of the characteristic matrix $A - \lambda I_n$ or $\lambda I_n - A$ is called a characteristic polynomial of ***A***.

$$i.e, \ \Delta = \left|A - \lambda I_n\right|$$

$$\text{or,} \Delta = \left|\lambda I_n - A\right|.$$

Characteristic Equation: If ***A*** be an n-square matrix and λ be an eigenvalue of ***A***, then the equation $\left|A - \lambda I_n\right| = 0$ or $\left|\lambda I_n - A\right| = 0$ is called a characteristic equation of ***A***.

NOTE: If A be an $n \times n$ matrix then,

1. An eigenvalue of A is a scalar such that $\left|A - \lambda I_n\right| = 0$
2. An eigenvectors of A corresponding to λ are the non-zero solutions of $\left(\left|A - \lambda I_n\right|\right) v = 0$.

Algebraic Multiplicity: The number of times an eigenvalue occurs is called its algebraic multiplicity. For example, if $\lambda = -2, -2, 0, 3, 3, 5$; then algebraic multiplicities of $-2, 0, 3, 5$ are $2, 1, 2, 1$ respectively.

Geometric Multiplicity: The geometric multiplicity of an eigenvalue is the dimension of the eigenspace associated with that eigenspace.

NOTE: The geometric multiplicity of an eigenvalue is either less than or, equal to the algebraic multiplicity of that eigenvalue.

NOTE: If ***A*** is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of ***A*** are the entries on the main diagonal of ***A***.

Example: If $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$, then its eigenvalues are, $a_{11}, a_{22}, \dots \dots, a_{nn}$.

Problem-01: Find the eigenvalues and associated eigenvectors of the matrix

$$A = \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix}.$$

Solution: The given matrix is,

$$A = \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix}$$

The characteristic matrix of ***A*** is,

$$A - \lambda I = \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} -3-\lambda & 2 & 2 \\ -6 & 5-\lambda & 2 \\ -7 & 4 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of $\boldsymbol{A}$ is,

$$\Delta = |A-\lambda I|$$

$$=\begin{vmatrix} -3-\lambda & 2 & 2 \\ -6 & 5-\lambda & 2 \\ -7 & 4 & 4-\lambda \end{vmatrix}$$

$$=(-3-\lambda)\{(5-\lambda)(4-\lambda)-8\}-2\{-6(4-\lambda)+14\}+2\{-24+7(5-\lambda)\}$$

$$=(-3-\lambda)(20-9\lambda+\lambda^2-8)-2(-24+6\lambda+14)+2(-24+35-7\lambda)$$

$$=(-3-\lambda)(\lambda^2-9\lambda+12)-2(6\lambda-10)+2(-7\lambda+11)$$

$$=-3\lambda^2+27\lambda-36-\lambda^3+9\lambda^2-12\lambda-12\lambda+20-14\lambda+22$$

$$=-\lambda^3+6\lambda^2-11\lambda+6$$

The characteristic equation of $\boldsymbol{A}$ is,

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$$

$$\Rightarrow \lambda^3-\lambda^2-5\lambda^2+5\lambda+6\lambda-6=0$$

$$\Rightarrow \lambda^2(\lambda-1)-5\lambda(\lambda-1)+6(\lambda-1)=0$$

$$\Rightarrow (\lambda-1)(\lambda^2-5\lambda+6)=0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\therefore\ \lambda=1,\ 2,\ 3$$

The eigenvalues of $\boldsymbol{A}$ are 1, 2,3.

2nd part:

Let $v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda = 1$.

$$\therefore \ (A - \lambda I) v_1 = 0$$

$$\Rightarrow \begin{pmatrix} -4 & 2 & 2 \\ -6 & 4 & 2 \\ -7 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$or, \left.\begin{array}{l} -4x + 2y + 2z = 0 \\ -6x + 4y + 2z = 0 \\ -7x + 4y + 3z = 0 \end{array}\right\}$$

$$or, \left.\begin{array}{l} -2x + y + z = 0 \\ -3x + 2y + z = 0 \\ -7x + 4y + 3z = 0 \end{array}\right\} \begin{array}{l} L_1' \to \frac{1}{2} L_1 \\ L_2' \to \frac{1}{2} L_2 \end{array}$$

$$or, \left.\begin{array}{r} -2x + y + z = 0 \\ y - z = 0 \\ y - z = 0 \end{array}\right\} \begin{array}{l} L_2' \to 2L_2 - 3L_1 \\ L_3' \to 2L_3 - 7L_1 \end{array}$$

$$or, \left.\begin{array}{r} -2x + y + z = 0 \\ y - z = 0 \\ 0 = 0 \end{array}\right\} \ L_3' \to L_3 - L_2$$

$$or, \left.\begin{array}{r} -2x + y + z = 0 \\ y - z = 0 \end{array}\right\} \ L_3' \to L_3 - L_2$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is z. Thus the system has only one independent solution.

Putting $z = 1$ then we get $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Thus the independent eigenvector is $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = 1$ and $\{(1,1,1)\}$ is a basis of the eigenspace corresponding to the eigenvalue $\lambda_1 = 1$.

Again, Let $v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda = 2$.

$$\therefore \ (A - \lambda I)v_2 = 0$$

$$\Rightarrow \begin{pmatrix} -5 & 2 & 2 \\ -6 & 3 & 2 \\ -7 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$or, \left.\begin{aligned} -5x + 2y + 2z &= 0 \\ -6x + 3y + 2z &= 0 \\ -7x + 4y + 2z &= 0 \end{aligned}\right\}$$

$$or, \left.\begin{aligned} -5x + 2y + 2z &= 0 \\ 3y - 2z &= 0 \\ 6y - 4z &= 0 \end{aligned}\right\} \begin{aligned} L_2^{'} &\to 5L_2 - 6L_1 \\ L_3^{'} &\to 5L_3 - 7L_1 \end{aligned}$$

$$or, \left.\begin{aligned} -5x + 2y + 2z &= 0 \\ 3y - 2z &= 0 \\ 3y - 2z &= 0 \end{aligned}\right\} L_3^{'} \to \frac{1}{2}L_3$$

$$or, \left.\begin{aligned} -5x + 2y + 2z &= 0 \\ 3y - 2z &= 0 \\ 0 &= 0 \end{aligned}\right\} L_3^{'} \to L_3 - L_2$$

$$or, \left.\begin{aligned} -5x + 2y + 2z &= 0 \\ 3y - 2z &= 0 \end{aligned}\right\}$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is *z*. Thus the system has only one independent solution.

Putting $z=3$ then we get $v_2=\begin{pmatrix}2\\2\\3\end{pmatrix}$.

Thus the independent eigenvector is $v_2=\begin{pmatrix}2\\2\\3\end{pmatrix}$ corresponding to the eigenvalue $\lambda_2=2$ and $\{(2,2,3)\}$ is a basis of the eigenspace corresponding to the eigenvalue $\lambda_2=2$.

Again, Let $v_3=\begin{pmatrix}x\\y\\z\end{pmatrix}$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda=3$.

$$\therefore\ (A-\lambda I)v_3=0$$

$$\Rightarrow\begin{pmatrix}-6&2&2\\-6&2&2\\-7&4&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$or,\ \left.\begin{aligned}-6x+2y+2z&=0\\-6x+2y+2z&=0\\-7x+4y+\ z&=0\end{aligned}\right\}$$

$$or,\ \left.\begin{aligned}-6x+2y+2z&=0\\0&=0\\10y-8z&=0\end{aligned}\right\}\quad\begin{aligned}L_2^{'}&\to L_2-L_1\\L_3^{'}&\to 6L_3-7L_1\end{aligned}$$

$$or,\ \left.\begin{aligned}-6x+2y+2z&=0\\10y-8z&=0\end{aligned}\right\}$$

$$or,\ \left.\begin{aligned}-6x+2y+2z&=0\\5y-4z&=0\end{aligned}\right\}L_2^{'}\to\frac{1}{2}L_2$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is *z*. Thus the system has only one independent solution.

Putting $z=5$ then we get $v_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

Thus the independent eigenvector is $v_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_3 = 3$ and $\{(3,4,5)\}$ is a basis of the eigenspace corresponding to the eigenvalue $\lambda_3 = 3$.

Problem-02: Find the eigenvalues and associated eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

Solution: The given matrix is,

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

The characteristic matrix of ***A*** is,

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of ***A*** is,

$$\Delta = |A - \lambda I|$$

$$=\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$=(1-\lambda)\{(-5-\lambda)(4-\lambda)+18\}+3\{3(4-\lambda)-18\}+3\{-18-6(-5-\lambda)\}$$

$$=(1-\lambda)(\lambda^2+\lambda-20+18)+3(12-3\lambda-18)+3(-18+30+6\lambda)$$

$$=(1-\lambda)(\lambda^2+\lambda-2)-9\lambda-18+36+18\lambda$$

$$=\lambda^2+\lambda-2-\lambda^3-\lambda^2+2\lambda-9\lambda-18+36+18\lambda$$

$$=-\lambda^3+12\lambda+16$$

The characteristic equation of $\boldsymbol{A}$ is,

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+12\lambda+16=0$$

$$\Rightarrow \lambda^3-12\lambda-16=0$$

$$\Rightarrow \lambda^3+2\lambda^2-2\lambda^2-4\lambda-8\lambda-16=0$$

$$\Rightarrow \lambda^2(\lambda+2)-2\lambda(\lambda+2)-8(\lambda+2)=0$$

$$\Rightarrow (\lambda+2)(\lambda^2-2\lambda-8)=0$$

$$\Rightarrow (\lambda+2)(\lambda+2)(\lambda-4)=0$$

$$\therefore\ \lambda=-2,\,-2,\,4$$

The eigenvalues of $\boldsymbol{A}$ are -2, -2,4.

2nd part:

Let $v_1=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda=-2$.

$$\therefore\ (A-\lambda I)v_1=0$$

$$\Rightarrow \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$or,\ \left.\begin{aligned} 3x-3y+3z=0 \\ 3x-3y+3z=0 \\ 6x-6y+6z=0 \end{aligned}\right\}$$

$$or,\ \left.\begin{aligned} 3x-3y+3z=0 \\ 0=0 \\ 0=0 \end{aligned}\right\} \begin{aligned} L_2' \to L_2 - L_1 \\ L_3' \to L_3 - 2L_1 \end{aligned}$$

$$or,\ x-y+z=0$$

There is 1 equation in 3 unknowns. So there are (3-1) = 2 free variables which are y andz. Thus the system has only two independent solutions.

Putting i). $y=1,\ z=0$ and ii). $y=0,\ z=1$ we get $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Thus two independent eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ corresponding to the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -2$ respectivelyand $\{(1,1,0),(1,0,-1)\}$ is a basis of the eigenspace corresponding to the double eigenvalue $\lambda = -2$.

Again, Let $v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda = 4$.

$$\therefore\ (A-\lambda I)v_2 = 0$$

$$\Rightarrow \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$or,\quad \left.\begin{array}{r} -3x-3y+3z=0 \\ 3x-9y+3z=0 \\ 6x-6y\qquad =0 \end{array}\right\}$$

$$or,\quad \left.\begin{array}{r} -3x-3y+3z=0 \\ -12y+6z=0 \\ -12y+6z=0 \end{array}\right\} \begin{array}{l} L_2^{'} \to L_2+L_1 \\ L_3^{'} \to L_3+2L_1 \end{array}$$

$$or,\quad \left.\begin{array}{r} -3x-3y+3z=0 \\ -12y+6z=0 \\ 0=0 \end{array}\right\} L_3^{'} \to L_3-L_2$$

$$or,\quad \left.\begin{array}{r} x+y-z=0 \\ 2y-z=0 \end{array}\right\} \begin{array}{l} L_1^{'} \to -\frac{1}{3}L_1 \\ L_3^{'} \to -\frac{1}{6}L_2 \end{array}$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is z. Thus the system has only one independent solution.

Putting $z=2$ then we get $v_3=\begin{pmatrix}1\\1\\2\end{pmatrix}$.

Thus the independent eigenvector is $v_3=\begin{pmatrix}1\\1\\2\end{pmatrix}$ corresponding to the eigenvalue $\lambda_3=4$ and $\{(1,1,2)\}$ is a basis of the eigenspace corresponding to the eigenvalue $\lambda_3=4$.

Problem-03: If $A=\begin{pmatrix}1&2&2\\1&2&-1\\-1&1&4\end{pmatrix}$, then find its

$$\left\{\begin{array}{c} i).\ eigenvalues \\ ii).\ a\lg ebraic\ multiplicities\ of\ eigenvalues \\ iii).\ geometric\ multiplicities. \end{array}\right.$$

Solution: The given matrix is,

$$A=\begin{pmatrix}1 & 2 & 2\\ 1 & 2 & -1\\ -1 & 1 & 4\end{pmatrix}$$

The characteristic matrix of ***A*** is,

$$A-\lambda I=\begin{pmatrix}1 & 2 & 2\\ 1 & 2 & -1\\ -1 & 1 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 2 & 2\\ 1 & 2 & -1\\ -1 & 1 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & 2 & 2\\ 3 & 2-\lambda & -1\\ -1 & 1 & 4-\lambda\end{pmatrix}$$

The characteristic polynomial of ***A*** is,

$$\Delta=\left|A-\lambda I\right|$$

$$=\begin{vmatrix}1-\lambda & 2 & 2\\ 3 & 2-\lambda & -1\\ -1 & 1 & 4-\lambda\end{vmatrix}$$

$$=(1-\lambda)\{(2-\lambda)(4-\lambda)+1\}-2\{(4-\lambda)-1\}+2\{1+(2-\lambda)\}$$

$$=(1-\lambda)(8-6\lambda+\lambda^2+1)-2(3-\lambda)+2(3-\lambda)$$

$$=(1-\lambda)(9-6\lambda+\lambda^2)-6+2\lambda+6-2\lambda$$

$$=(1-\lambda)(\lambda^2-6\lambda+9)$$

$$=(1-\lambda)(\lambda-3)(\lambda-3)$$

The characteristic equation of ***A*** is,

$$\left|A-\lambda I\right|=0$$

$$\Rightarrow(1-\lambda)(\lambda^2-6\lambda+9)=0$$

$$\Rightarrow (1-\lambda)(\lambda-3)(\lambda-3)=0$$

$$\therefore\ \lambda = 1,\ 3,\ 3$$

The eigenvalues of ***A*** are 1, 3,3.

The algebraic multiplicity of $\lambda = 1$ is 1. So its geometric multiplicity must be 1.

The algebraic multiplicity of $\lambda = 3$ is 2. So its geometric multiplicity may be 1 or, 2.

To become sure we use $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in the characteristic equation.

$$i.e,\ (A-\lambda I)v = 0$$

$$\Rightarrow \begin{pmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or, $\left.\begin{aligned} -2a+2b+2c &= 0 \\ a-b-c &= 0 \\ -a+b+c &= 0 \end{aligned}\right\}$

or, $\left.\begin{aligned} -2a+2b+2c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\} \begin{aligned} L_2^{'} &= 2L_2 + L_1 \\ L_3^{'} &= 2L_3 - L_1 \end{aligned}$

or, $-2a+2b+2c=0$

or, $a-b-c=0$

or, $a=b+c$

$$\therefore\ v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow v = \begin{pmatrix} b+c \\ b \\ c \end{pmatrix}$$

$$\Rightarrow v = b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\0\\1\end{pmatrix} \text{where, } b \neq 0 \;\; and \;\; c \neq 0$$

Here, the resulting eigenspace is 2-dimentional.

Hence, the geometric multiplicity of $\lambda = 3$ is2.

Problem-04: Find the eigenvalues of the matrix $A = \begin{pmatrix}-3 & 1 & -1\\-7 & 5 & -1\\-6 & 6 & -2\end{pmatrix}$.

Solution: The given matrix is,

$$A = \begin{pmatrix}-3 & 1 & -1\\-7 & 5 & -1\\-6 & 6 & -2\end{pmatrix}$$

The characteristic matrix of ***A*** is,

$$A - \lambda I = \begin{pmatrix}-3 & 1 & -1\\-7 & 5 & -1\\-6 & 6 & -2\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}-3 & 1 & -1\\-7 & 5 & -1\\-6 & 6 & -2\end{pmatrix} - \begin{pmatrix}\lambda & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{pmatrix}$$

$$= \begin{pmatrix}-3-\lambda & 1 & -1\\-7 & 5-\lambda & -1\\-6 & 6 & -2-\lambda\end{pmatrix}$$

The characteristic polynomial of ***A*** is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix}-3-\lambda & 1 & -1\\-7 & 5-\lambda & -1\\-6 & 6 & -2-\lambda\end{vmatrix}$$

$$= (-3-\lambda)\{(5-\lambda)(-2-\lambda)+6\} - \{-7(-2-\lambda)-6\} - \{-42+6(5-\lambda)\}$$

$$=(-3-\lambda)\left\{(\lambda^2-3\lambda-10)+6\right\}-(14+7\lambda-6)-(-42+30-6\lambda)$$
$$=(-3-\lambda)(\lambda^2-3\lambda-4)-(7\lambda+8)-(-12-6\lambda)$$
$$=-3\lambda^2+9\lambda+12-\lambda^3+3\lambda^2+4\lambda-7\lambda-8+12+6\lambda$$
$$=-\lambda^3+12\lambda+16$$

The characteristic equation of $\boldsymbol{A}$ is,

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+12\lambda+16=0$$

$$\Rightarrow \lambda^3-12\lambda-16=0$$

$$\Rightarrow \lambda^3-4\lambda^2+4\lambda^2-16\lambda+4\lambda-16=0$$

$$\Rightarrow \lambda^2(\lambda-4)+4\lambda(\lambda-4)+4(\lambda-4)=0$$

$$\Rightarrow (\lambda-4)(\lambda^2+4\lambda+4)=0$$

$$\Rightarrow (\lambda-4)(\lambda+2)(\lambda+2)=0$$

$$\therefore\ \lambda=-2,\ -2,\ 4$$

The eigenvalues of $\boldsymbol{A}$ are -2, -2,4.

Problem-05: Find the eigenvalues of the matrix $A=\begin{pmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{pmatrix}$.

Solution: The given matrix is,

$$A=\begin{pmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{pmatrix}$$

The characteristic matrix of $\boldsymbol{A}$ is,

$$A-\lambda I=\begin{pmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of ***A*** is,

$$\Delta = |A - \lambda I|$$

$$=\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix}$$

$$=(1-\lambda)\{-\lambda(4-\lambda)-0\}-0-2(0-2\lambda)$$

$$=(1-\lambda)(-4\lambda+\lambda^2)+4\lambda$$

$$=-4\lambda+\lambda^2+4\lambda^2-\lambda^3+4\lambda$$

$$=-\lambda^3+5\lambda^2$$

The characteristic equation of ***A*** is,

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+5\lambda^2=0$$

$$\Rightarrow \lambda^3-5\lambda^2=0$$

$$\Rightarrow \lambda^2(\lambda-5)=0$$

$$\therefore \; \lambda = 0,\ 0,\ 5$$

The eigenvalues of ***A*** are 0, 0,5.

Problem-06: Find the eigenvalues of the matrix $A=\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Solution: The given matrix is,

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The characteristic matrix of ***A*** is,

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of ***A*** is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & 1 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\{(2-\lambda)(4-\lambda)-0\} - \{3(4-\lambda)-0\} + 0$$

$$= (2-\lambda)^2(4-\lambda) - 3(4-\lambda)$$

$$= (4-\lambda)\{(2-\lambda)^2 - 3\}$$

$$= (4-\lambda)(4-4\lambda+\lambda^2-3)$$

$$= (4-\lambda)(\lambda^2-4\lambda+1)$$

The characteristic equation of ***A*** is,

$$|A - \lambda I| = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2-4\lambda+1) = 0$$

$$\therefore\ 4-\lambda=0 \quad or,\ \lambda^2-4\lambda+1=0$$

$$\Rightarrow\ \lambda=4 \quad or,\ \lambda=\frac{4\pm\sqrt{16-4}}{2}$$

$$or,\ \lambda=\frac{4\pm\sqrt{12}}{2}$$

$$or,\ \lambda=\frac{4\pm2\sqrt{3}}{2}$$

$$or,\ \lambda=2\pm\sqrt{3}$$

$$\therefore\ \lambda=4,\ 2\pm\sqrt{3}$$

The eigenvalues of $\boldsymbol{A}$ are $4,\ 2\pm\sqrt{3}$.

Problem-07: Find the eigenvalues of the matrix $A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & 1\end{pmatrix}$.

Solution: The given matrix is,

$$A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & 1\end{pmatrix}$$

The characteristic matrix of $\boldsymbol{A}$ is,

$$A-\lambda I=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & 1\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & 2 & 3\\ 0 & 1-\lambda & 2\\ 0 & -2 & 1-\lambda\end{pmatrix}$$

The characteristic polynomial of $\boldsymbol{A}$ is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left\{(1-\lambda)^2 + 4\right\}$$

$$= (1-\lambda)\left(\lambda^2 - 2\lambda + 1 + 4\right)$$

$$= (1-\lambda)\left(\lambda^2 - 2\lambda + 5\right)$$

The characteristic equation of $\boldsymbol{A}$ is,

$$|A - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2 - 2\lambda + 5\right) = 0$$

$$\therefore\ 1-\lambda = 0 \quad or,\ \lambda^2 - 2\lambda + 5 = 0$$

$$\Rightarrow\ \lambda = 1\ \ or,\ \lambda = \frac{2 \pm \sqrt{4-20}}{2}$$

$$or,\ \lambda = \frac{2 \pm \sqrt{-16}}{2}$$

$$or,\ \lambda = \frac{2 \pm \sqrt{16i^2}}{2}$$

$$or,\ \lambda = \frac{2 \pm 4i}{2}$$

$$or,\ \lambda = 1 \pm 2i$$

$$\therefore\ \lambda = 1,\ 1 \pm 2i$$

The eigenvalues of $\boldsymbol{A}$ are $1,\ 1 \pm 2i$.

Problem-08: Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

Solution: The given matrix is,

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}$$

The characteristic matrix of $\boldsymbol{A}$ is,

$$A-\lambda I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & 0 & 0\\ 0 & 1-\lambda & -1\\ 0 & 1 & 1-\lambda\end{pmatrix}$$

The characteristic polynomial of $\boldsymbol{A}$ is,

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}1-\lambda & 0 & 0\\ 0 & 1-\lambda & -1\\ 0 & 1 & 1-\lambda\end{vmatrix}$$

$$=(1-\lambda)\left\{(1-\lambda)^2+1\right\}$$

$$=(1-\lambda)\left(\lambda^2-2\lambda+2\right)$$

The characteristic equation of $\boldsymbol{A}$ is,

$$|A-\lambda I|=0$$

$$\Rightarrow(1-\lambda)\left(\lambda^2-2\lambda+2\right)=0$$

$$\therefore\ 1-\lambda=0\quad or,\ \lambda^2-2\lambda+2=0$$

$$\Rightarrow\ \lambda=1\ \ or,\ \lambda=\frac{2\pm\sqrt{4-8}}{2}$$

$$or,\ \lambda = \frac{2 \pm \sqrt{-4}}{2}$$

$$or,\ \lambda = \frac{2 \pm \sqrt{4i^2}}{2}$$

$$or,\ \lambda = \frac{2 \pm 2i}{2}$$

$$or,\ \lambda = 1 \pm i$$

$$\therefore\ \lambda = 1,\ 1 \pm i$$

The eigenvalues of ***A*** are $1, 1 \pm i$.

Problem-09: Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix}$

Solution: The given matrix is,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix}$$

The characteristic matrix of ***A*** is,

$$A - \lambda I = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} 2-\lambda & -1 & 0 & 0 \\ -2 & 3-\lambda & 0 & 0 \\ 2 & 0 & 4-\lambda & 2 \\ 1 & 3 & -2 & -1-\lambda \end{pmatrix}$$

The characteristic polynomial of $\boldsymbol{A}$ is,

$$\Delta = |A-\lambda I|$$

$$=\begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -2 & 3-\lambda & 0 & 0 \\ 2 & 0 & 4-\lambda & 2 \\ 1 & 3 & -2 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 3 & -2 & -1-\lambda \end{vmatrix}+1\begin{vmatrix} -2 & 0 & 0 \\ 2 & 4-\lambda & 2 \\ 1 & -2 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix}-2\begin{vmatrix} 4-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)\{(4-\lambda)(-1-\lambda)+4\}-2\{(4-\lambda)(-1-\lambda)+4\}$$

$$=(2-\lambda)(3-\lambda)\left(\lambda^2-3\lambda-4+4\right)-2\left(\lambda^2-3\lambda-4+4\right)$$

$$=(2-\lambda)(3-\lambda)\left(\lambda^2-3\lambda\right)-2\left(\lambda^2-3\lambda\right)$$

$$=\left(\lambda^2-3\lambda\right)\{(2-\lambda)(3-\lambda)-2\}$$

$$=\left(\lambda^2-3\lambda\right)\left(\lambda^2-5\lambda+6-2\right)$$

$$=\left(\lambda^2-3\lambda\right)\left(\lambda^2-5\lambda+4\right)n\times nA=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

$$=\lambda(\lambda-1)(\lambda-3)(\lambda-4)$$

The characteristic equation of $\boldsymbol{A}$ is,

$$|A-\lambda I|=0$$

$$\Rightarrow \lambda(\lambda-1)(\lambda-3)(\lambda-4)=0$$

$$\therefore\ \lambda=0\ ;\lambda-1=0\ ;\ \lambda-3=0\ ;\ \lambda-4=0$$

$$\Rightarrow\ \lambda=0\ ;\lambda=1\ ;\ \lambda=3\ ;\ \lambda=4$$

$$\therefore\ \lambda=0,1,3,4$$

The eigenvalues of $\boldsymbol{A}$ are $0,1,3,4$.

Exercise:

1. Find the eigenvalues of the matrix $A=\begin{pmatrix} 2 & -2 & 1 \\ 2 & -8 & -2 \\ 1 & 2 & 2 \end{pmatrix}$ $\quad$ $Ans:\ 0,\ 3,\ -7.$

2. Find the eigenvalues of the matrix $A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad$ $Ans:\ 1,\ 2,\ 2.$

3. Find the eigenvalues of the matrix $A=\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ $\quad$ $Ans:\ 5,\ -3,\ -3.$

4. Find the eigenvalues of the matrix $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$ $\quad$ $Ans:\ 4,\ 2\pm\sqrt{3}.$