Eigenvalues and Eigenvectors: *Let **A** be an n-square matrix. A scalar λ is called an eigenvalue or characteristic root of **A** if there exists a non-zero column vector v such that,*

$$Av=\lambda v \quad or \quad (A-\lambda I)v=0$$

*Where, **I** is the n-square identity matrix. Every vector satisfying this relation is called an eigenvector or characteristic vector of **A** associated with the eigenvalue λ.*

Spectrum: *The set of all eigenvalues of a matrix **A** is called the spectrum of **A**.*

Eigenspace: *If **A** be an n-square matrix and λ be an eigenvalue of **A**, then the set of all vectors satisfying the relation $Av=\lambda v$ including the zero vector is called an eigenspace of **A** corresponding to λ.*

Characteristic Matrix: *If **A** be an n-square matrix and λ be an eigenvalue of **A**, then the matrix $A-\lambda I_n$ or $\lambda I_n - A$ is called a characteristic matrix of **A**.*

Characteristic Polynomial: *If **A** be an n-square matrix and λ be an eigenvalue of **A**, then the determinant of the characteristic matrix $A-\lambda I_n$ or $\lambda I_n - A$ is called a characteristic polynomial of **A**.*

$$i.e,\ \Delta=|A-\lambda I_n|$$

$$or,\quad \Delta=|\lambda I_n - A|.$$

Characteristic Equation: *If **A** be an n-square matrix and λ be an eigenvalue of **A**, then the equation $|A-\lambda I_n|=0$ or $|\lambda I_n - A|=0$ is called a characteristic equation of **A**.*

NOTE: *If A be an $n\times n$ matrix then,*

1. *An eigenvalue of A is a scalar such that $|A-\lambda I_n|=0$*
2. *An eigenvectors of A corresponding to λ are the non-zero solutions of $(A-\lambda I_n)v=0$.*

Algebraic Multiplicity: *The number of times an eigenvalue occurs is called its algebraic multiplicity. For example, if $\lambda=-2,-2,0,3,3,5$; then algebraic multiplicities of $-2,0,3,5$ are $2,1,2,1$ respectively.*

Geometric Multiplicity: *The geometric multiplicity of an eigenvalue is the dimension of the eigenspace associated with that eigenvalue.*

NOTE: *The geometric multiplicity of an eigenvalue is either less than or, equal to the algebraic multiplicity of that eigenvalue.*

NOTE: *If **A** is an $n\times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of **A** are the entries on the main diagonal of **A**.*

Example: *If* $A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$*, then its eigenvalues are,* a_{11}, a_{22}, … … , a_{nn}.

NOTE:

1). Every polynomial equation of odd degree with real coefficients does posses a real solution.

2). $\lambda = 0$ *is an eigenvalue of a matrix* A *iff* A *is singular.*

3). All eigenvalues of a matrix A *are non-zero iff* A *is non-singular.*

4). If a matrix A *has an eigenvalue* λ *and* $k \in N$, *then* λ^k *will become an eigenvalue of* A^k.

5). Same eigenvalues belong to similar matrices.

6). Characteristic values of a symmetric matrix are real numbers.

7). Characteristic values of a skew-symmetric matrix are either zeroes or imaginary numbers.

8). Characteristic values of a Hermitian matrix are real numbers.

9). Characteristic values of an orthogonal matrix are 1 or -1.

10). The eigenvalues of a matrix A *are same as the eigenvalues of* A^t.

11). For, a non-singular matrix, the inverse of an eigenvalue will be the eigenvalue of the inverse matrix.

Cayley-Hamilton Theorem: *Every square matrix satisfies its characteristics equation.*

Problem-01: *Verify Cayley-Hamilton Theorem for* $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$.

Solution: *The given matrix is,*

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

The characteristic matrix of A *is,*

$$A - \lambda I = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{pmatrix}$$

The characteristic polynomial of A *is,*

$$\Delta = |A - \lambda I|$$

$$=\begin{vmatrix}1-\lambda & 3 & 7\\ 4 & 2-\lambda & 3\\ 1 & 2 & 1-\lambda\end{vmatrix}$$

$$=(1-\lambda)\{(2-\lambda)(1-\lambda)-6\}-3\{4(1-\lambda)-3\}+7\{8-(2-\lambda)\}$$

$$=(1-\lambda)(\lambda^2-3\lambda-4)-3(1-4\lambda)+7(6+\lambda)$$

$$=-\lambda^3+4\lambda^2+\lambda-4-3+12\lambda+42+7\lambda$$

$$=-\lambda^3+4\lambda^2+20\lambda+35$$

The characteristic equation of A is,

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+4\lambda^2+20\lambda+35=0$$

$$\Rightarrow \lambda^3-4\lambda^2-20\lambda-35=0$$

To verify Cayley-Hamilton Theorem we need to show:

$$A^3-4A^2-20A-35I=0$$

Now we have,

$$A^2=\begin{pmatrix}1&3&7\\4&2&3\\1&2&1\end{pmatrix}\begin{pmatrix}1&3&7\\4&2&3\\1&2&1\end{pmatrix}$$

$$=\begin{pmatrix}1+12+7 & 3+6+14 & 7+9+7\\ 4+8+3 & 12+4+6 & 28+6+3\\ 1+8+1 & 3+4+2 & 7+6+1\end{pmatrix}$$

$$=\begin{pmatrix}20&23&23\\15&22&37\\10&9&14\end{pmatrix}$$

$$A^3=AA^2$$

$$=\begin{pmatrix}1&3&7\\4&2&3\\1&2&1\end{pmatrix}\begin{pmatrix}20&23&23\\15&22&37\\10&9&14\end{pmatrix}$$

$$=\begin{pmatrix}135&152&232\\140&163&208\\60&76&111\end{pmatrix}$$

$$\therefore A^3-4A^2-20A-35I=\begin{pmatrix}135&152&232\\140&163&208\\60&76&111\end{pmatrix}-4\begin{pmatrix}20&23&23\\15&22&37\\10&9&14\end{pmatrix}-20\begin{pmatrix}1&3&7\\4&2&3\\1&2&1\end{pmatrix}-35\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}135&152&232\\140&163&208\\60&76&111\end{pmatrix}-\begin{pmatrix}80&92&92\\60&88&148\\40&36&56\end{pmatrix}-\begin{pmatrix}20&60&140\\80&40&60\\20&40&20\end{pmatrix}-\begin{pmatrix}35&0&0\\0&35&0\\0&0&35\end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the Cayley-Hamilton Theorem is verified for the given matrix.

Problem-02: *Find the inverse of the matrix* $A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ *by using Cayley-Hamilton Theorem.*

Solution: *The given matrix is,*

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

The characteristic matrix of A *is,*

$$A-\lambda I=\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}-\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}$$

The characteristic polynomial of A *is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)\{(-1-\lambda)(1-\lambda)-1\}-2\{2(1-\lambda)-3\}+3\{2-(-1-\lambda)\}$$

$$=(1-\lambda)(\lambda^2-2)-2(-1-2\lambda)+3(3+\lambda)$$

$$=-\lambda^3+\lambda^2+2\lambda+2+4\lambda+9+3\lambda$$

$$=-\lambda^3+\lambda^2+9\lambda+11$$

The characteristic equation of A *is,*

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+\lambda^2+9\lambda+11=0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 9\lambda - 11 = 0$$

By Cayley-Hamilton Theorem we can write:

$$A^3 - A^2 - 9A - 11I = 0$$

Multiplying both sides by A^{-1} *we have,*

$$A^2 - A - 9I - 11A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{11}\left(A^2 - A - 9I\right) \cdots\cdots\cdots (1)$$

Now

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+9 & 2-2+3 & 3+2+3 \\ 2-2+3 & 4+1+1 & 6-1+1 \\ 3+2+3 & 6-1+1 & 9+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{pmatrix}$$

From (1) we have,

$$A^{-1} = \frac{1}{11}\left(\begin{pmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} - 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{11}\left(\begin{pmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right)$$

$$= \frac{1}{11}\begin{pmatrix} 4 & 1 & 5 \\ 1 & -2 & 5 \\ 5 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4/11 & 1/11 & 5/11 \\ 1/11 & -2/11 & 5/11 \\ 5/11 & 5/11 & 1/11 \end{pmatrix}$$

This is required inverse matrix of the given matrix. Similarly, we can find A^{-2} *and* A^{-3}. *(Ans.)*

Exercise: *Verify the Cayley-Hamilton Theorem for the following matrices and also find inverse matrices* A^{-1}, A^{-2} & A^{-3}:

$$\textbf{a).}\ A=\begin{pmatrix}2 & -2 & 1\\ 2 & -8 & -2\\ 1 & 2 & 2\end{pmatrix}\ ;\ \textbf{b).}\ A=\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}\ ;\ c).\ A=\begin{pmatrix}0 & 1 & 2\\ 2 & 3 & 0\\ 1 & 1 & -2\end{pmatrix}.$$

Problem-03: *Find the eigenvalues and associated eigenvectors of the matrix*

$$A=\begin{pmatrix}-3 & 2 & 2\\ -6 & 5 & 2\\ -7 & 4 & 4\end{pmatrix}.$$

Solution: *The given matrix is,*

$$A=\begin{pmatrix}-3 & 2 & 2\\ -6 & 5 & 2\\ -7 & 4 & 4\end{pmatrix}$$

*The characteristic matrix of **A** is,*

$$A-\lambda I=\begin{pmatrix}-3 & 2 & 2\\ -6 & 5 & 2\\ -7 & 4 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 2 & 2\\ -6 & 5 & 2\\ -7 & 4 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}-3-\lambda & 2 & 2\\ -6 & 5-\lambda & 2\\ -7 & 4 & 4-\lambda\end{pmatrix}$$

*The characteristic polynomial of **A** is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}-3-\lambda & 2 & 2\\ -6 & 5-\lambda & 2\\ -7 & 4 & 4-\lambda\end{vmatrix}$$

$$=(-3-\lambda)\{(5-\lambda)(4-\lambda)-8\}-2\{-6(4-\lambda)+14\}+2\{-24+7(5-\lambda)\}$$

$$=(-3-\lambda)(20-9\lambda+\lambda^2-8)-2(-24+6\lambda+14)+2(-24+35-7\lambda)$$

$$=(-3-\lambda)(\lambda^2-9\lambda+12)-2(6\lambda-10)+2(-7\lambda+11)$$

$$=-3\lambda^2+27\lambda-36-\lambda^3+9\lambda^2-12\lambda-12\lambda+20-14\lambda+22$$

$$=-\lambda^3+6\lambda^2-11\lambda+6$$

*The characteristic equation of **A** is,*

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$
$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$$
$$\Rightarrow \lambda^2(\lambda-1) - 5\lambda(\lambda-1) + 6(\lambda-1) = 0$$
$$\Rightarrow (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$
$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$
$$\therefore\ \lambda = 1,\ 2,\ 3$$

The eigenvalues of A are 1, 2, 3.

2nd part:

Let $v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *be a non-zero eigenvector corresponding to the eigenvalue* $\lambda = 1$.

$$\therefore\ (A - \lambda I)v_1 = 0$$

$$\Rightarrow \begin{pmatrix} -4 & 2 & 2 \\ -6 & 4 & 2 \\ -7 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$or,\ \left.\begin{aligned} -4x + 2y + 2z &= 0 \\ -6x + 4y + 2z &= 0 \\ -7x + 4y + 3z &= 0 \end{aligned}\right\}$$

$$or,\ \left.\begin{aligned} -2x + y + z &= 0 \\ -3x + 2y + z &= 0 \\ -7x + 4y + 3z &= 0 \end{aligned}\right\} \begin{aligned} L_1' &\to \frac{1}{2}L_1 \\ L_2' &\to \frac{1}{2}L_2 \end{aligned}$$

$$or,\ \left.\begin{aligned} -2x + y + z &= 0 \\ y - z &= 0 \\ y - z &= 0 \end{aligned}\right\} \begin{aligned} L_2' &\to 2L_2 - 3L_1 \\ L_3' &\to 2L_3 - 7L_1 \end{aligned}$$

$$or,\ \left.\begin{aligned} -2x + y + z &= 0 \\ y - z &= 0 \\ 0 &= 0 \end{aligned}\right\} L_3' \to L_3 - L_2$$

$$or,\ \left.\begin{aligned} -2x + y + z &= 0 \\ y - z &= 0 \end{aligned}\right\} L_3' \to L_3 - L_2$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is z. Thus the system has only one independent solution.

Putting $z = 1$ *then we get* $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Thus the independent eigenvector is $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = 1$ and $\{(1,1,1)\}$ is a basis of the eigenspace corresponding to the eigenvalue $\lambda_1 = 1$.

Again, Let $v_2 = \begin{pmatrix}x\\y\\z\end{pmatrix}$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda = 2$.

$$\therefore\ (A - \lambda I)v_2 = 0$$

$$\Rightarrow \begin{pmatrix}-5 & 2 & 2\\-6 & 3 & 2\\-7 & 4 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$or,\ \left.\begin{aligned}-5x+2y+2z&=0\\-6x+3y+2z&=0\\-7x+4y+2z&=0\end{aligned}\right\}$$

$$or,\ \left.\begin{aligned}-5x+2y+2z&=0\\3y-2z&=0\\6y-4z&=0\end{aligned}\right\}\ \begin{aligned}L_2' &\to 5L_2 - 6L_1\\L_3' &\to 5L_3 - 7L_1\end{aligned}$$

$$or,\ \left.\begin{aligned}-5x+2y+2z&=0\\3y-2z&=0\\3y-2z&=0\end{aligned}\right\}\ L_3' \to \frac{1}{2}L_3$$

$$or,\ \left.\begin{aligned}-5x+2y+2z&=0\\3y-2z&=0\\0&=0\end{aligned}\right\}\ L_3' \to L_3 - L_2$$

$$or,\ \left.\begin{aligned}-5x+2y+2z&=0\\3y-2z&=0\end{aligned}\right\}$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is z. Thus the system has only one independent solution.

Putting $z = 3$ then we get $v_2 = \begin{pmatrix}2\\2\\3\end{pmatrix}$.

Thus the independent eigenvector is $v_2 = \begin{pmatrix}2\\2\\3\end{pmatrix}$ corresponding to the eigenvalue $\lambda_2 = 2$ and $\{(2,2,3)\}$ is a basis of the Eigen space corresponding to the eigenvalue $\lambda_2 = 2$.

Again, let $v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *be a non-zero eigenvector corresponding to the eigenvalue* $\lambda = 3$.

$$\therefore \ (A - \lambda I) v_3 = 0$$

$$\Rightarrow \begin{pmatrix} -6 & 2 & 2 \\ -6 & 2 & 2 \\ -7 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$or, \left.\begin{aligned} -6x + 2y + 2z &= 0 \\ -6x + 2y + 2z &= 0 \\ -7x + 4y + \ z &= 0 \end{aligned}\right\}$$

$$or, \left.\begin{aligned} -6x + 2y + 2z &= 0 \\ 0 &= 0 \\ 10y - 8z &= 0 \end{aligned}\right\} \begin{aligned} L_2' &\to L_2 - L_1 \\ L_3' &\to 6L_3 - 7L_1 \end{aligned}$$

$$or, \left.\begin{aligned} -6x + 2y + 2z &= 0 \\ 10y - 8z &= 0 \end{aligned}\right\}$$

$$or, \left.\begin{aligned} -6x + 2y + 2z &= 0 \\ 5y - 4z &= 0 \end{aligned}\right\} L_2' \to \frac{1}{2} L_2$$

There are 2 equations in 3 unknowns. So there is (3-2) = 1 free variable which is z. Thus the system has only one independent solution.

Putting $z = 5$ *then we get* $v_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

Thus the independent eigenvector is $v_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ *corresponding to the eigenvalue* $\lambda_3 = 3$

and $\{(3,4,5)\}$ *is a basis of the Eigen space corresponding to the eigenvalue* $\lambda_3 = 3$.

Problem-04: *If a matrix is,* $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

Then, (a). Find an spectrum of A.

(b). Find eigenvalues of A^T, A^3, A^{-1}, A^{-3}.

(c). Find A^{-1} *by using Cayley-Hamilton Theorem.*

Solution: *The given matrix is,*

$$A=\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}$$

*The characteristic matrix of **A** is,*

$$A-\lambda I=\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & -3 & 3\\ 3 & -5-\lambda & 3\\ 6 & -6 & 4-\lambda\end{pmatrix}$$

*The characteristic polynomial of **A** is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}1-\lambda & -3 & 3\\ 3 & -5-\lambda & 3\\ 6 & -6 & 4-\lambda\end{vmatrix}$$

$$=(1-\lambda)\{(-5-\lambda)(4-\lambda)+18\}+3\{3(4-\lambda)-18\}+3\{-18-6(-5-\lambda)\}$$

$$=(1-\lambda)(\lambda^2+\lambda-20+18)+3(12-3\lambda-18)+3(-18+30+6\lambda)$$

$$=(1-\lambda)(\lambda^2+\lambda-2)-9\lambda-18+36+18\lambda$$

$$=\lambda^2+\lambda-2-\lambda^3-\lambda^2+2\lambda-9\lambda-18+36+18\lambda$$

$$=-\lambda^3+12\lambda+16$$

*The characteristic equation of **A** is,*

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+12\lambda+16=0$$

$$\Rightarrow \lambda^3-12\lambda-16=0 \qquad \cdots\ \cdots\cdots (1)$$

$$\Rightarrow \lambda^3+2\lambda^2-2\lambda^2-4\lambda-8\lambda-16=0$$

$$\Rightarrow \lambda^2(\lambda+2)-2\lambda(\lambda+2)-8(\lambda+2)=0$$

$$\Rightarrow (\lambda+2)(\lambda^2-2\lambda-8)=0$$

$$\Rightarrow (\lambda+2)(\lambda+2)(\lambda-4)=0$$

$$\therefore\ \lambda=-2,\ -2,\ 4$$

*The eigenvalues of **A** are -2, -2, 4.*

(a). The spectrum is, $\{-2,4\}$.

(b). The eigenvalues of the matrix A^T *are, -2, -2, 4.*

The eigenvalues of the matrix A^3 *are,* $(-2)^3,(-2)^3,4^3$ *or* $-8,-8,64$.

The eigenvalues of the matrix A^{-1} *are,* $(-2)^{-1},(-2)^{-1},4^{-1}$ *or* $-\frac{1}{2},-\frac{1}{2},\frac{1}{4}$.

The eigenvalues of the matrix A^{-3} *are,* $(-2)^{-3},(-2)^{-3},4^{-3}$ *or* $-\frac{1}{8},-\frac{1}{8},\frac{1}{64}$.

(c). From Eq. (1) we have,

$$\lambda^3-12\lambda-16=0$$

By Cayley-Hamilton Theorem we have,

$$A^3-12A-16I=0$$
$$\Rightarrow A^2-12I-16A^{-1}=0$$
$$\Rightarrow -16A^{-1}=-A^2+12I$$
$$\Rightarrow A^{-1}=\frac{1}{16}A^2-\frac{3}{4}I \qquad \cdots\ \cdots\cdots (2)$$

Now, $$A^2=\begin{pmatrix}1&-3&3\\3&-5&3\\6&-6&4\end{pmatrix}\begin{pmatrix}1&-3&3\\3&-5&3\\6&-6&4\end{pmatrix}$$

$$=\begin{pmatrix}1-9+18&-3+15-16&3-9+12\\3-15+18&-9+25-18&9-15+12\\6-18+24&-18+30-24&18-18+16\end{pmatrix}$$

$$=\begin{pmatrix}10&-4&6\\6&-2&6\\12&-12&16\end{pmatrix}$$

From Eq. (2) we have,

$$A^{-1}=\frac{1}{16}\begin{pmatrix}10&-4&6\\6&-2&6\\12&-12&16\end{pmatrix}-\frac{3}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{5}{8}&-\frac{1}{4}&\frac{3}{8}\\\frac{3}{8}&-\frac{1}{8}&\frac{3}{8}\\\frac{3}{4}&-\frac{3}{4}&1\end{pmatrix}-\begin{pmatrix}\frac{3}{4}&0&0\\0&\frac{3}{4}&0\\0&0&\frac{3}{4}\end{pmatrix}$$

$$=\begin{pmatrix}-1/8&-1/4&3/8\\3/8&-7/8&3/8\\3/4&-3/4&1/4\end{pmatrix}$$

Problem-05: *If a matrix is,* $A=\begin{pmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{pmatrix}$

Then, (a). Find an spectrum of A.

(b). Find eigenvalues of A^T, A^3, A^{-1}, A^{-3}.

(c). Find A^{-1} *by using Cayley-Hamilton Theorem.*

Solution: *The given matrix is,*

$$A=\begin{pmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{pmatrix}$$

*The characteristic matrix of **A** is,*

$$A-\lambda I=\begin{pmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{pmatrix}-\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} -\lambda & 4 & 2 \\ -3 & 8-\lambda & 3 \\ 4 & -8 & -2-\lambda \end{pmatrix}$$

*The characteristic polynomial of **A** is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix} -\lambda & 4 & 2 \\ -3 & 8-\lambda & 3 \\ 4 & -8 & -2-\lambda \end{vmatrix}$$

$$=-\lambda\{(8-\lambda)(-2-\lambda)+24\}-4\{-3(-2-\lambda)-12\}+2\{24-4(8-\lambda)\}$$

$$=-\lambda\left(\lambda^2-6\lambda-16+24\right)-4(6+3\lambda-12)+2(24-32+4\lambda)$$

$$=-\lambda\left(\lambda^2-6\lambda+8\right)-4(3\lambda-6)+2(4\lambda-8)$$

$$=-\lambda^3+6\lambda^2-8\lambda-12\lambda+24+8\lambda-16$$

$$=-\lambda^3+6\lambda^2-12\lambda+8$$

*The characteristic equation of **A** is,*

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-12\lambda+8=0$$

$$\Rightarrow \lambda^3-6\lambda^2+12\lambda-8=0 \qquad \cdots\ \cdots\cdots (1)$$

$$\Rightarrow \lambda^3-2\lambda^2-4\lambda^2+8\lambda+4\lambda-8=0$$

$$\Rightarrow \lambda^2(\lambda-2)-4\lambda(\lambda-2)+4(\lambda-2)=0$$
$$\Rightarrow (\lambda-2)(\lambda^2-4\lambda+4)=0$$
$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-2)=0$$
$$\therefore\ \lambda = 2,\ 2,\ 2$$

*The eigenvalues of **A** are 2, 2, 2.*

(a). *The spectrum is,* $\{2\}$.

(b). *The eigenvalues of the matrix* A^T *are, 2, 2, 2.*

The eigenvalues of the matrix A^3 *are,* $2^3, 2^3, 2^3$ *or* $8, 8, 8$.

The eigenvalues of the matrix A^{-1} *are,* $2^{-1}, 2^{-1}, 2^{-1}$ *or* $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$.

The eigenvalues of the matrix A^{-3} *are,* $2^{-3}, 2^{-3}, 2^{-3}$ *or* $\frac{1}{8}, \frac{1}{8}, \frac{1}{8}$.

(c). *From Eq. (1) we have,*

$$\lambda^3-6\lambda^2+12\lambda-8=0$$

By Cayley-Hamilton Theorem we have,

$$A^3-6A^2+12A-8I=0$$
$$\Rightarrow A^2-6A+12I-8A^{-1}=0 \qquad (\text{Multipying by } A^{-1})$$
$$\Rightarrow -8A^{-1}=-A^2+6A-12I$$
$$\Rightarrow A^{-1}=\frac{1}{8}A^2-\frac{3}{4}A+\frac{3}{2}I \qquad \cdots\ \cdots\cdots\ (2)$$

Now,
$$A^2=\begin{pmatrix}0 & 4 & 2\\ -3 & 8 & 3\\ 4 & -8 & -2\end{pmatrix}\begin{pmatrix}0 & 4 & 2\\ -3 & 8 & 3\\ 4 & -8 & -2\end{pmatrix}$$
$$=\begin{pmatrix}0-12+8 & 0+32-16 & 0+12-4\\ 0-24+12 & -12+64-24 & -6+24-6\\ 0+24-8 & 16-64+16 & 8-24+4\end{pmatrix}$$
$$=\begin{pmatrix}-4 & 16 & 8\\ -12 & 28 & 12\\ 16 & -32 & -12\end{pmatrix}$$

From Eq. (2) we have,

$$A^{-1}=\frac{1}{8}\begin{pmatrix}-4 & 16 & 8\\ -12 & 28 & 12\\ 16 & -32 & -12\end{pmatrix}-\frac{3}{4}\begin{pmatrix}0 & 4 & 2\\ -3 & 8 & 3\\ 4 & -8 & -2\end{pmatrix}+\frac{3}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{2} & 2 & 1 \\ -\frac{3}{2} & \frac{7}{2} & \frac{3}{2} \\ 2 & -4 & -\frac{3}{2} \end{pmatrix}-\begin{pmatrix} 0 & 3 & \frac{3}{2} \\ -\frac{9}{4} & 6 & \frac{9}{4} \\ 3 & -6 & -\frac{3}{2} \end{pmatrix}+\begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -1 & -\frac{1}{2} \\ \frac{3}{4} & 2 & -\frac{3}{4} \\ -1 & 2 & \frac{3}{2} \end{pmatrix}$$

Problem-06: *If* $A=\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$, *then find its* $\begin{cases} i).\ eigenvalues \\ ii).\ algebraic\ multiplicities\ of\ eigenvalues \\ iii).\ geometric\ multiplicities. \end{cases}$

Solution: *The given matrix is,*

$$A=\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

The characteristic matrix of A *is,*

$$A-\lambda I=\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}-\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} 1-\lambda & 2 & 2 \\ 3 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of A *is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix} 1-\lambda & 2 & 2 \\ 3 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix}$$

$$=(1-\lambda)\{(2-\lambda)(4-\lambda)+1\}-2\{(4-\lambda)-1\}+2\{1+(2-\lambda)\}$$

$$=(1-\lambda)(8-6\lambda+\lambda^2+1)-2(3-\lambda)+2(3-\lambda)=(1-\lambda)(9-6\lambda+\lambda^2)-6+2\lambda+6-2\lambda$$

$$=(1-\lambda)(\lambda^2-6\lambda+9)$$

$$=(1-\lambda)(\lambda-3)(\lambda-3)$$

The characteristic equation of A *is,*

$$|A-\lambda I|=0$$
$$\Rightarrow (1-\lambda)\left(\lambda^2-6\lambda+9\right)=0$$
$$\Rightarrow (1-\lambda)(\lambda-3)(\lambda-3)=0$$
$$\therefore\ \lambda=1,\ 3,\ 3$$

The eigenvalues of $\mathcal{A}$ are 1, 3, 3.

The algebraic multiplicity of $\lambda=1$ is 1. So its geometric multiplicity must be 1.

The algebraic multiplicity of $\lambda=3$ is 2. So its geometric multiplicity may be 1 or, 2.

To become sure we use $v=\begin{pmatrix}a\\b\\c\end{pmatrix}$ *in the characteristic equation.*

$$i.e,\ (A-\lambda I)v=0$$

$$\Rightarrow \begin{pmatrix}-2 & 2 & 2\\ 1 & -1 & -1\\ -1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=0$$

or, $$\left.\begin{aligned}-2a+2b+2c&=0\\ a-b-c&=0\\ -a+b+c&=0\end{aligned}\right\}$$

or, $$\left.\begin{aligned}-2a+2b+2c&=0\\ 0&=0\\ 0&=0\end{aligned}\right\}\begin{aligned}L_2'&=2L_2+L_1\\ L_3'&=2L_3-L_1\end{aligned}$$

or, $-2a+2b+2c=0$

or, $a-b-c=0$

or, $a=b+c$

$$\therefore\ v=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\Rightarrow v=\begin{pmatrix}b+c\\b\\c\end{pmatrix}$$

$$\Rightarrow v=b\begin{pmatrix}1\\1\\0\end{pmatrix}+c\begin{pmatrix}1\\0\\1\end{pmatrix}\text{where, } b\neq 0\ \text{ and }\ c\neq 0$$

Here, the resulting eigenspace is 2-dimentional.

Hence, the geometric multiplicity of $\lambda=3$ is 2.

Problem-07: *Find the eigenvalues of the matrix* $A=\begin{pmatrix}-3 & 1 & -1\\ -7 & 5 & -1\\ -6 & 6 & -2\end{pmatrix}$.

Solution: *The given matrix is,*

$$A=\begin{pmatrix}-3 & 1 & -1\\ -7 & 5 & -1\\ -6 & 6 & -2\end{pmatrix}$$

The characteristic matrix of ***A*** *is,*

$$A-\lambda I=\begin{pmatrix}-3 & 1 & -1\\ -7 & 5 & -1\\ -6 & 6 & -2\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 1 & -1\\ -7 & 5 & -1\\ -6 & 6 & -2\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}-3-\lambda & 1 & -1\\ -7 & 5-\lambda & -1\\ -6 & 6 & -2-\lambda\end{pmatrix}$$

The characteristic polynomial of ***A*** *is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}-3-\lambda & 1 & -1\\ -7 & 5-\lambda & -1\\ -6 & 6 & -2-\lambda\end{vmatrix}$$

$$=(-3-\lambda)\{(5-\lambda)(-2-\lambda)+6\}-\{-7(-2-\lambda)-6\}-\{-42+6(5-\lambda)\}$$

$$=(-3-\lambda)\{(\lambda^2-3\lambda-10)+6\}-(14+7\lambda-6)-(-42+30-6\lambda)$$

$$=(-3-\lambda)(\lambda^2-3\lambda-4)-(7\lambda+8)-(-12-6\lambda)$$

$$=-3\lambda^2+9\lambda+12-\lambda^3+3\lambda^2+4\lambda-7\lambda-8+12+6\lambda$$

$$=-\lambda^3+12\lambda+16$$

The characteristic equation of ***A*** *is,*

$$|A-\lambda I|=0$$

$$\Rightarrow -\lambda^3+12\lambda+16=0$$

$$\Rightarrow \lambda^3-12\lambda-16=0$$

$$\Rightarrow \lambda^3-4\lambda^2+4\lambda^2-16\lambda+4\lambda-16=0$$

$$\Rightarrow \lambda^2(\lambda-4)+4\lambda(\lambda-4)+4(\lambda-4)=0$$

$$\Rightarrow (\lambda-4)(\lambda^2+4\lambda+4)=0$$

$$\Rightarrow (\lambda-4)(\lambda+2)(\lambda+2)=0$$

$\therefore\ \lambda = -2, -2, 4$

The eigenvalues of A are -2, -2, 4.

Problem-08: *If a matrix is,* $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$.

Then, (a). Find an spectrum of A.

(b). Find eigenvalues of A^T, A^3, A^{-1}, A^{-3}.

(c). Find A^{-1} *by using Cayley-Hamilton Theorem.*

Solution: *The given matrix is,*

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

The characteristic matrix of A is,

$$A - \lambda I = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{-\lambda(4-\lambda) - 0\} - 0 - 2(0 - 2\lambda)$$

$$= (1-\lambda)(-4\lambda + \lambda^2) + 4\lambda$$

$$= -4\lambda + \lambda^2 + 4\lambda^2 - \lambda^3 + 4\lambda$$

$$= -\lambda^3 + 5\lambda^2$$

The characteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 = 0$$

$$\Rightarrow \lambda^2(\lambda - 5) = 0$$

$\therefore\ \lambda = 0,\ 0,\ 5$

*The eigenvalues of **A** are 0, 0,5.*

***(a).** The spectrum is,* $\{0,5\}$.

***(b).** The eigenvalues of the matrix* A^T *are, 0, 0, 5.*

The eigenvalues of the matrix A^3 *are,* $0^3, 0^3, 5^3$ *or* $0, 0, 125$.

Since, 0 is the eigenvalue of the matrix A so it is a singular matrix and we know that singular matrix does not possess inverse matrix. So, the eigenvalues of matrices $A^{-1}\ \&\ A^{-3}$ *are not possible.*

***(c).** Since, the matrix is singular so its inverse does not exist.*

Problem-09: *Find the eigenvalues of the matrix* $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Then, (a). Find an spectrum of A.

(b). Find eigenvalues of A^T, A^3, A^{-1}, A^{-3}.

(c). Find A^{-1} *by using Cayley-Hamilton Theorem.*

Solution: *The given matrix is,*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

*The characteristic matrix of **A** is,*

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

*The characteristic polynomial of **A** is,*

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & 1 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\{(2-\lambda)(4-\lambda) - 0\} - \{3(4-\lambda) - 0\} + 0$$

$$= (2-\lambda)^2(4-\lambda) - 3(4-\lambda)$$

$$=(4-\lambda)\left\{(2-\lambda)^2-3\right\}$$
$$=(4-\lambda)(4-4\lambda+\lambda^2-3)$$
$$=(4-\lambda)(\lambda^2-4\lambda+1)$$

The characteristic equation of A is,

$|A-\lambda I|=0$

$$\Rightarrow (4-\lambda)(\lambda^2-4\lambda+1)=0$$
$$\therefore\ 4-\lambda=0 \quad or,\ \lambda^2-4\lambda+1=0$$
$$\Rightarrow\ \lambda=4 \quad or,\ \lambda=\frac{4\pm\sqrt{16-4}}{2}$$
$$or,\ \lambda=\frac{4\pm\sqrt{12}}{2}$$
$$or,\ \lambda=\frac{4\pm2\sqrt{3}}{2}$$
$$or,\ \lambda=2\pm\sqrt{3}$$
$$\therefore\ \lambda=4,\ 2+\sqrt{3}, 2-\sqrt{3}$$

The eigenvalues of A are $4,\ 2+\sqrt{3}, 2-\sqrt{3}.$.

(a). *The spectrum is,* $\left\{4,\ 2+\sqrt{3}, 2-\sqrt{3}\right\}$.

(b). *The eigenvalues of the matrix* A^T *are,* $4,\ 2+\sqrt{3}, 2-\sqrt{3}$.

The eigenvalues of the matrix A^3 *are,* $4^3,\left(2+\sqrt{3}\right)^3,\left(2-\sqrt{3}\right)^3$ *or* $64,\left(2+\sqrt{3}\right)^3,\left(2-\sqrt{3}\right)^3$.

The eigenvalues of the matrix A^{-1} *are,* $4^{-1},\left(2+\sqrt{3}\right)^{-1},\left(2-\sqrt{3}\right)^{-1}$.

The eigenvalues of the matrix A^{-3} *are,* $4^{-3},\left(2+\sqrt{3}\right)^{-3},\left(2-\sqrt{3}\right)^{-3}$.

(c). *Try yourself.*

Problem-10: *Find the eigenvalues of the matrix* $A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & 1\end{pmatrix}$.

Solution: *The given matrix is,*

$$A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & 1\end{pmatrix}$$

The characteristic matrix of A is,

$$A-\lambda I=\begin{pmatrix}1&2&3\\0&1&2\\0&-2&1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3\\0&1&2\\0&-2&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda&2&3\\0&1-\lambda&2\\0&-2&1-\lambda\end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}1-\lambda&2&3\\0&1-\lambda&2\\0&-2&1-\lambda\end{vmatrix}$$

$$=(1-\lambda)\left\{(1-\lambda)^2+4\right\}$$

$$=(1-\lambda)\left(\lambda^2-2\lambda+1+4\right)$$

$$=(1-\lambda)\left(\lambda^2-2\lambda+5\right)$$

The characteristic equation of A is,

$$|A-\lambda I|=0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2-2\lambda+5\right)=0$$

$$\therefore\ 1-\lambda=0\quad or,\ \lambda^2-2\lambda+5=0$$

$$\Rightarrow\ \lambda=1\ \ or,\ \lambda=\frac{2\pm\sqrt{4-20}}{2}$$

$$or,\ \lambda=\frac{2\pm\sqrt{-16}}{2}$$

$$or,\ \lambda=\frac{2\pm\sqrt{16i^2}}{2}$$

$$or,\ \lambda=\frac{2\pm 4i}{2}$$

$$or,\ \lambda=1\pm 2i$$

$$\therefore\ \lambda=1,\ 1\pm 2i$$

The eigenvalues of A are $1, 1\pm 2i$.

Problem-11: *Find the eigenvalues of the matrix* $A=\begin{pmatrix}1&0&0\\0&1&-1\\0&1&1\end{pmatrix}$.

Solution: *The given matrix is,*

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}$$

*The characteristic matrix of **A** is,*

$$A-\lambda I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & 0 & 0\\ 0 & 1-\lambda & -1\\ 0 & 1 & 1-\lambda\end{pmatrix}$$

*The characteristic polynomial of **A** is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}1-\lambda & 0 & 0\\ 0 & 1-\lambda & -1\\ 0 & 1 & 1-\lambda\end{vmatrix}$$

$$=(1-\lambda)\left\{(1-\lambda)^2+1\right\}$$

$$=(1-\lambda)\left(\lambda^2-2\lambda+2\right)$$

*The characteristic equation of **A** is,*

$$|A-\lambda I|=0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2-2\lambda+2\right)=0$$

$$\therefore\ 1-\lambda=0 \quad or,\ \lambda^2-2\lambda+2=0$$

$$\Rightarrow\ \lambda=1 \ \ or,\ \lambda=\frac{2\pm\sqrt{4-8}}{2}$$

$$or,\ \lambda=\frac{2\pm\sqrt{-4}}{2}$$

$$or,\ \lambda=\frac{2\pm\sqrt{4i^2}}{2}$$

$$or,\ \lambda=\frac{2\pm 2i}{2}$$

$$or,\ \lambda=1\pm i$$

$$\therefore\ \lambda=1,\ 1\pm i$$

*The eigenvalues of **A** are* $1, 1\pm i$.

Problem-12: *Find the eigenvalues of the matrix* $A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix}$

Solution: *The given matrix is,*

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix}$$

The characteristic matrix of ***A*** *is,*

$$A-\lambda I=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{pmatrix}-\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} 2-\lambda & -1 & 0 & 0 \\ -2 & 3-\lambda & 0 & 0 \\ 2 & 0 & 4-\lambda & 2 \\ 1 & 3 & -2 & -1-\lambda \end{pmatrix}$$

The characteristic polynomial of ***A*** *is,*

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -2 & 3-\lambda & 0 & 0 \\ 2 & 0 & 4-\lambda & 2 \\ 1 & 3 & -2 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 3 & -2 & -1-\lambda \end{vmatrix}+1\begin{vmatrix} -2 & 0 & 0 \\ 2 & 4-\lambda & 2 \\ 1 & -2 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix}-2\begin{vmatrix} 4-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)\{(4-\lambda)(-1-\lambda)+4\}-2\{(4-\lambda)(-1-\lambda)+4\}$$

$$=(2-\lambda)(3-\lambda)(\lambda^2-3\lambda-4+4)-2(\lambda^2-3\lambda-4+4)$$

$$=(2-\lambda)(3-\lambda)(\lambda^2-3\lambda)-2(\lambda^2-3\lambda)$$

$$=\left(\lambda^2-3\lambda\right)\left\{(2-\lambda)(3-\lambda)-2\right\}$$
$$=\left(\lambda^2-3\lambda\right)\left(\lambda^2-5\lambda+6-2\right)$$
$$=\left(\lambda^2-3\lambda\right)\left(\lambda^2-5\lambda+4\right)$$
$$=\lambda(\lambda-1)(\lambda-3)(\lambda-4)$$

The characteristic equation of A *is,*

$$|A-\lambda I|=0$$
$$\Rightarrow \lambda(\lambda-1)(\lambda-3)(\lambda-4)=0$$
$$\therefore\ \lambda=0\,;\lambda-1=0\,;\ \lambda-3=0\ ;\ \lambda-4=0$$
$$\Rightarrow\ \lambda=0\,;\lambda=1\,;\ \lambda=3\,;\ \lambda=4$$
$$\therefore\ \lambda=0,1,3,4$$

The eigenvalues of A *are* $0,1,3,4$. *(Ans)*

Exercise: *Find an spectrum &eigenvalues of* A^T, A^3, A^{-1}, A^{-3}. *Also find* A^{-1} *by using Cayley-Hamilton Theorem. Where,*

a). $A=\begin{pmatrix}2&-2&1\\2&-8&-2\\1&2&2\end{pmatrix}$ *b).* $A=\begin{pmatrix}1&2&3\\0&2&3\\0&0&2\end{pmatrix}$ *c).* $A=\begin{pmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{pmatrix}$ *d).* $A=\begin{pmatrix}0&1&0\\0&0&1\\4&-17&8\end{pmatrix}$

e). $A=\begin{pmatrix}3&1&1\\2&4&2\\1&1&3\end{pmatrix}$ *f).* $A=\begin{pmatrix}4&6&6\\1&3&2\\-1&-4&-3\end{pmatrix}$ *g).* $A=\begin{pmatrix}1&2&-1\\0&-2&0\\0&-5&2\end{pmatrix}$ *h).* $A=\begin{pmatrix}5&0&0\\-1&3&0\\0&2&-1\end{pmatrix}$

i). $A=\begin{pmatrix}1&2&3\\0&-4&2\\0&0&7\end{pmatrix}$ *j).* $A=\begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}$ *k).* $A=\begin{pmatrix}1&2&2\\1&2&-1\\-1&1&4\end{pmatrix}$ *l).* $A=\begin{pmatrix}3&2&4\\2&0&2\\4&2&3\end{pmatrix}$