In this chapter, we have a concentration to discuss about the linear combination, linear dependence and independence of vectors.

Linear Combination of Vectors: Let V be a vector space over the scalar field F and $v_1, v_2, v_3, \cdots\cdots, v_n \in V$, then any vector $v \in V$ is called a linear combination of the vectors $v_1, v_2, v_3, \cdots\cdots, v_n$ if there exists scalars $\alpha_1, \alpha_2, a_3, \cdots\cdots, a_n \in F$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots\cdots + \alpha_n v_n$$

Example: Observe that $(-13,-4,4) = 2(2,-1,2) + (-2)(4,1,0) + 3(-3,0,0)$

Thus $(-13,-4,4)$ is a linear combination of $(2,-1,2)$, $(4,1,0)$ and $(-3,0,0)$ in R^3.

The same vector may be a linear combination of more than one set of vectors, viz,

$(-13,-4,4) = (-13)(1,0,0) + (-4)(0,1,0) + 4(0,0,1)$

Problem-01: Write the vector $v = (5,9,5)$ as a linear combination of the vectors $v_1 = (2,1,4)$, $v_2 = (1,-1,3)$ and $v_3 = (3,2,5)$ in R^3.

Solution:

We know that, v will be a linear combination of the vectors v_1, v_2 and v_3 if there exists scalars x, y and z in R such that,

$$v = xv_1 + yv_2 + zv_3$$

$$\begin{aligned} i.e,\ (5,9,5) &= x(2,1,4) + y(1,-1,3) + z(3,2,5) \\ &= (2x, x, 4x) + (y, -y, 3y) + (3z, 2z, 5z) \\ &= (2x + y + 3z, x - y + 2z, 4x + 3y + 5z) \end{aligned}$$

Equating corresponding components and forming linear system, we get

$$\begin{aligned} 2x + y + 3z &= 5 \\ x - y + 2z &= 9 \\ 4x + 3y + 5z &= 5 \end{aligned}$$

The augmented matrix is,

$$\begin{pmatrix} 2 & 1 & 3 & \vdots & 5 \\ 1 & -1 & 2 & \vdots & 9 \\ 4 & 3 & 5 & \vdots & 5 \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 & 1 & 3 & \vdots & 5 \\ 0 & -3 & 1 & \vdots & 13 \\ 0 & 1 & -1 & \vdots & -5 \end{pmatrix} \begin{matrix} R_2' = 2R_2 - R_1 \\ R_3' = R_3 - 2R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 2 & 1 & 3 & \vdots & 5 \\ 0 & -3 & 1 & \vdots & 13 \\ 0 & 0 & -2 & \vdots & -2 \end{pmatrix} R_3' = 3R_3 + R_2$$

The reduced system is,

$$\begin{aligned} 2x+y+3z&=5 \\ -3y+z&=13 \\ -2z&=-2 \end{aligned}$$

By back substitution, From 3^{rd} equation we get,

$$z=1$$

From 2^{nd} equation we get,

$$-3y+1=13$$
$$\Rightarrow -3y=12$$
$$\Rightarrow y=-4$$

From 1^{st} equation we get,

$$2x-4+3=5$$
$$\Rightarrow 2x=6$$
$$\Rightarrow x=3$$

Therefore, the required linear combination is,

$$v=3v_1-4v_2+v_3 \qquad (Ans.)$$

Problem-02: Write the vector $(16,1,-11,-23)$ as a linear combination of the vectors $(2,0,-1,1)$, $(-1,1,2,0)$, $(1,1,0,-5)$ and $(1,0,0,-1)$ in R^4.

Solution:

We know that, $(16,1,-11,-23)$ will be a linear combination of the vectors $(2,0,-1,1)$, $(-1,1,2,0)$, $(1,1,0,-5)$ and $(1,0,0,-1)$ if there exists scalars x , y , z and t in R such that,

$$\begin{aligned} (16,1,-11,-23) &= x(2,0,-1,1)+y(-1,1,2,0)+z(1,1,0,-5)+t(1,0,0,-1) \\ &= (2x,0,-x,x)+(-y,y,2y,0)+(z,z,0,-5z)+(t,0,0,-t) \\ &= (2x-y+z+t,\, y+z,\, -x+2y,\, x-5z-t) \end{aligned}$$

Equating corresponding components and forming linear system, we get

$$\begin{aligned} 2x-y+z+t&=16 \\ y+z&=1 \\ -x+2y&=-11 \\ x-5z-t&=-23 \end{aligned}$$

The augmented matrix is,

$$\begin{pmatrix} 2 & -1 & 1 & 1 & \vdots & 16 \\ 0 & 1 & 1 & 0 & \vdots & 1 \\ -1 & 2 & 0 & 0 & \vdots & -11 \\ 1 & 0 & -5 & -1 & \vdots & -23 \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 1 & 1 & \vdots & 16 \\ 0 & 1 & 1 & 0 & \vdots & 1 \\ 0 & 3 & 1 & 1 & \vdots & -6 \\ 0 & 1 & -11 & -3 & \vdots & -62 \end{pmatrix} \begin{matrix} R_3^{'} = 2R_3 + R_1 \\ R_4^{'} = 2R_4 - R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 1 & 1 & \vdots & 16 \\ 0 & 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & -2 & 1 & \vdots & -9 \\ 0 & 0 & -12 & -3 & \vdots & -63 \end{pmatrix} \begin{matrix} R_3^{'} = R_3 - 3R_2 \\ R_4^{'} = R_4 - R_2 \end{matrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 1 & 1 & \vdots & 16 \\ 0 & 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & -2 & 1 & \vdots & -9 \\ 0 & 0 & 0 & -9 & \vdots & -9 \end{pmatrix} R_4^{'} = R_4 - 6R_3$$

The reduced system is,

$$\begin{aligned} 2x - y + z + t &= 16 \\ y + z \quad &= 1 \\ -2z + t &= -9 \\ -9t &= -9 \end{aligned}$$

By back substitution, from 4^{th} *equation we get,*

$$t = 1$$

From 3^{rd} *equation we get,*

$$-2z + 1 = -9$$
$$\Rightarrow -2z = -10$$
$$\Rightarrow z = 5$$

From 2^{nd} *equation we get,*

$$y + 5 = 1$$
$$\Rightarrow y = -4$$

From 1^{st} *equation we get,*

$$2x + 4 + 5 + 1 = 16$$
$$\Rightarrow 2x = 6$$
$$\Rightarrow x = 3$$

Therefore, the required linear combination is,

$$(16,1,-11,-23) = 3(2,0,-1,1) + (-4)(-1,1,2,0) + 5(1,1,0,-5) + (1,0,0,-1) \qquad (Ans.)$$

Problem-03: *Determine whether or not the vector* $(1,2,6)$ *as a linear combination of the vectors* $(2,1,0)$, $(1,-1,2)$ *and* $(0,3,-4)$ *in* R^3.

Solution:

We know that, $(1,2,6)$ *will be a linear combination of the vectors* $(2,1,0)$, $(1,-1,2)$ *and* $(0,3,-4)$ *if there exists scalars* x , y *and* z *in* R *such that,*

$$(1,2,6) = x(2,1,0) + y(1,-1,2) + z(0,3,-4)$$
$$= (2x, x, 0) + (y, -y, 2y) + (0, 3z, -4z)$$
$$= (2x + y, x - y + 3z, 2y - 4z)$$

Equating corresponding components and forming linear system, we get

$$\begin{aligned} 2x + y \quad &= 1 \\ x - y + 3z &= 2 \\ 2y - 4z &= 6 \end{aligned}$$

The augmented matrix is,

$$\begin{pmatrix} 2 & 1 & 0 & \vdots & 1 \\ 1 & -1 & 3 & \vdots & 2 \\ 0 & 2 & -4 & \vdots & 6 \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 & 1 & 0 & \vdots & 1 \\ 0 & -3 & 6 & \vdots & 3 \\ 0 & 2 & -4 & \vdots & 6 \end{pmatrix} R_2' = 2R_2 - R_1$$

$$\approx \begin{pmatrix} 2 & 1 & 0 & \vdots & 1 \\ 0 & -3 & 6 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 24 \end{pmatrix} R_3' = 3R_3 + 2R_2$$

The reduced system is,

$$\begin{aligned} 2x + y \quad &= 1 \\ -3y + \ 6 &= 3 \\ 0 &= 24 \end{aligned}$$

Since, $0 = 24$ arise, which is impossible, so the system of linear equations is inconsistent.

Thus, our assumption regarding the linear combination was wrong. Therefore, we conclude that, $(1,2,6)$ can't be a linear combination of the vectors $(2,1,0)$, $(1,-1,2)$ and $(0,3,-4)$ in R^3. (Ans).

Problem-04: *Express the matrix $B = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$ as a linear combination of the matrices $B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $B_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

Solution:

We know that, B will be a linear combination of the matrices B_1, B_2 and B_3 if there exists scalars x , y and z in R such that,

$$B = xB_1 + yB_2 + zB_3$$

$$i.e, \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} x & 0 \\ x & 0 \end{pmatrix}+\begin{pmatrix} y & -y \\ 0 & y \end{pmatrix}+\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$$

$$=\begin{pmatrix} x+y & -y+z \\ x-z & y \end{pmatrix}$$

Equating corresponding components and forming linear system, we get

$$\begin{aligned} x+y &= 5 \\ -y+z &= 1 \\ x-z &= -2 \\ y &= 3 \end{aligned}$$

Solving these equations we get, $x=2$, $y=3$ *and* $z=4$

Therefore, the required linear combination of the matrices is,

$$B=2B_1+3B_2+4B_3 \qquad \text{(Ans).}$$

Exercise:

Problem-01: *Express the vector* v, *if possible, as a linear combination (LC) of the vectors,* v_1, v_2, v_3 *where,*

a). $v=(2,3,4)$ *and* $v_1=(1,0,1)$, $v_2=(0,-1,1)$, $v_3=(-1,-1,-1)$

b). $v=(1,-2,5)$ *and* $v_1=(1,2,3)$, $v_2=(1,1,1)$, $v_3=(2,-1,1)$

c). $v=(2,-5,4)$ *and* $v_1=(1,-3,2)$, $v_2=(2,-1,1)$

d). $v=(15,17,0)$ *and* $v_1=(2,1,0)$, $v_2=(-1,-1,1)$, $v_3=(1,2,3)$

e). $v=(5,1,-7)$ *and* $v_1=(-1,-2,10)$, $v_2=(-1,0,2)$, $v_3=(1,-1,-2)$

f). $v=(3,9,-4,-2)$ *and* $v_1=(1,-2,0,3)$, $v_2=(2,3,-1,0)$, $v_3=(2,-1,2,1)$

ANS: a). $v=-3v_1+2v_2-5v_3$**; b).** $v=3v_1-6v_2+2v_3$**; c). not possible ;**

d). not possible ; e). not possible ; f). $v=\left(\frac{-7}{17}\right)v_1+\left(\frac{42}{17}\right)v_2-\left(\frac{13}{17}\right)v_3$.

Problem-02: *Express, if possible*

a). $P=\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ *as a LC of* $P_1=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $P_2=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ & $P_3=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

b). $M=\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$ *as a LC of* $M_1=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_2=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ & $M_3=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

c). $A=\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ *as a LC of* $A_1=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A_2=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ & $A_3=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

ANS: a). $P=2P_1-P_2+2P_3$**; b). not possible ; c). not possible.**

Linear Dependence (LD) of vectors: *Let* V *be a vector space over the scalar field* F. *The vectors* $v_1,v_2,v_3,\cdots\cdots,v_n\in V$ *are said to be linearly dependent over* F, *if there exist scalars* $\alpha_1,\alpha_2,a_3,\cdots\cdots,a_n\in F$, *not all of them zero, such that*

$$\alpha_1v_1+\alpha_2v_2+\alpha_3v_3+\cdots\cdots+\alpha_nv_n=0$$

Example: *Observe that* $3(1,-1,0)+2(1,3,-1)+(-1)(5,3,-2)=(0,0,0)$

Thus, the vectors $(1,-1,0)$, $(1,3,-1)$ *and* $(5,3,-2)$ *are linear dependent in* R^3.

Linear Independence (LI) of vectors: *Let* V *be a vector space over the scalar field* F *.The vectors* $v_1,v_2,v_3,\cdots\cdots,v_n\in V$ *are said to be linearly independent over* F, *if there exist scalars* $\alpha_1,\alpha_2,a_3,\cdots\cdots,a_n\in F$, *all of them zero i.e,* $\alpha_1=\alpha_2=a_3=\cdots\cdots=a_n=0$, *such that*

$$\alpha_1v_1+\alpha_2v_2+\alpha_3v_3+\cdots\cdots+\alpha_nv_n=0$$

Example: *The vectors* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$ *are linearly independent in* R^3.

Since, $x(1,0,0)+y\,(0,1,0)+z(0,0,1)=0$

$$\Rightarrow x=0; y=0\,\&\,z=0$$

Note: **1.** *Any set containing a zero vector is linearly dependent.*

2. *Any two vectors are linearly dependent iff one is scalar multiple of the other i.e, they are collinear.*

3. *The non- zero rows of a matrix in echelon form are linearly independent.*

4. *Every singleton having a nonzero vector is linearly independent.*

5. *Any empty set is linearly independent.*

6. *In plane, any two nonzero vectors are linearly dependent iff they are parallel.*

7. *In plane, any two nonzero vectors are linearly independent iff they are intersecting.*

Determination of Linear dependence & independence:

Let $\alpha_1v_1+\alpha_2v_2+\alpha_3v_3+\cdots\cdots+\alpha_nv_n=0$ *for some scalars* $\alpha_1,\alpha_2,a_3,\cdots\cdots,a_n$. *If we substitute the values of* $v_1,v_2,v_3,\cdots\cdots,v_n$ *we get an* $n\times n$ *system of linear equations which yields the following matrix equation:*

$$\begin{pmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$or, MX=0$$

At this stage there are two methods to test the linear dependence or independence as follows:

<u>Method-01: Use of Determinant:</u> *If* $|M|\neq 0$, *then* $X=0$ *i.e,* $\alpha_1=\alpha_2=\cdots\cdots=a_n=0$ *and the vectors are linearly independent.*

Again, if $|M|=0$, *then the vectors are linearly dependent.*

Method-02: Use of Echelon Matrix: *If we apply elementary row or column operations to obtain rk(M), then*

1). *if we get,* $rk(M)=n$, *then the vectors are linearly independent and*

2). *if we get,* $rk(M)<n$, *then the vectors are linearly dependent.*

Problem-01: *Check the linear dependency of S where,*

i). $S=\{(1,-1,3),(1,4,5),(2,-3,-7)\}\subset R^3$

ii). $S=\{(2,1,1),(3,-4,6),(4,-9,11)\}\subset R^3$

iii). $S=\{(2,1,3,-1),(2,3,1,2),(3,2,5,6),(-2,-7,3,-8)\}\subset R^4$

iv). $S=\{(1,1,2,-1),(1,2,5,0),(0,1,2,1),(2,1,2,5)\}\subset R^4$

Solution: *i). We have,* $S=\{(1,-1,3),(1,4,5),(2,-3,-7)\}\subset R^3$

Writing the vectors in S as column we form the following matrix,

$$M=\begin{pmatrix} 1 & 1 & 2 \\ -1 & 4 & -3 \\ 3 & 5 & -7 \end{pmatrix}$$

The determinant of this matrix is,

$$|M|=\begin{vmatrix} 1 & 1 & 2 \\ -1 & 4 & -3 \\ 3 & 5 & -7 \end{vmatrix}$$

$$=(-28+15)-(7+9)+2(-5-12)$$

$$=-13-16-120$$

$$=-149$$

Since $|M|\neq 0$, *so S is a linear independent subset of vectors from* R^3.

ii). *We have,* $S=\{(2,1,1),(3,-4,6),(4,-9,11)\}\subset R^3$

Writing the vectors in S as column we form the following matrix,

$$M=\begin{pmatrix} 2 & 3 & 4 \\ 1 & -4 & -9 \\ 1 & 6 & 11 \end{pmatrix}$$

The determinant of this matrix is,

$$|M|=\begin{vmatrix} 2 & 3 & 4 \\ 1 & -4 & -9 \\ 1 & 6 & 11 \end{vmatrix}$$

$$=2(-44+54)-3(11+9)+4(6+4)$$

$$=20-60+40$$

$$=0$$

Since $|M|=0$, *so S is a linear dependent subset of vectors from* R^3.

iii). We have, $S=\{(2,1,3,-1),(2,3,1,2),(3,2,5,6),(-2,-7,3,-8)\}\subset R^4$

Writing the vectors in S as row we form the following matrix,

$$M=\begin{pmatrix}2&1&3&-1\\2&3&1&2\\3&2&5&6\\-2&-7&3&-8\end{pmatrix}$$

Now we shall reduce this matrix into echelon form by elementary row operations,

$$\sim\begin{pmatrix}2&1&3&-1\\0&2&-2&3\\0&1&1&15\\0&-6&6&-9\end{pmatrix}\begin{matrix}R_2^{'}=R_2-R_1\\R_3^{'}=2R_3-3R_1\\R_4^{'}=R_4+R_1\end{matrix}$$

$$\sim\begin{pmatrix}2&1&3&-1\\0&2&-2&3\\0&0&4&27\\0&0&0&0\end{pmatrix}\begin{matrix}R_3^{'}=2R_3-R_2\\R_4^{'}=R_4+3R_2\end{matrix}$$

Since, the echelon form of the matrix contains a zero row so S is a linear dependent subset of vectors from R^4.

iv). We have, $S=\{(1,1,2,-1),(1,2,5,0),(0,1,2,1),(2,1,2,-5)\}\subset R^4$

Writing the vectors in S as row we form the following matrix,

$$M=\begin{pmatrix}1&1&2&-1\\1&2&5&0\\0&1&2&1\\2&1&2&5\end{pmatrix}$$

Now we shall reduce this matrix into echelon form by elementary row operations,

$$\sim\begin{pmatrix}1&1&2&-1\\0&1&3&1\\0&1&2&1\\0&-1&-2&7\end{pmatrix}\begin{matrix}R_2^{'}=R_2-R_1\\R_4^{'}=R_4-2R_1\end{matrix}$$

$$\sim\begin{pmatrix}1&1&2&-1\\0&1&3&1\\0&0&-1&0\\0&0&1&8\end{pmatrix}\begin{matrix}R_3^{'}=R_3-R_2\\R_4^{'}=R_4+R_2\end{matrix}$$

$$\sim\begin{pmatrix}1&1&2&-1\\0&1&3&1\\0&0&-1&0\\0&0&0&8\end{pmatrix}R_4^{'}=R_4+R_3$$

Since, the echelon form of the matrix does not contain any zero row so S is a linear independent subset of vectors from R^4.

Problem-03: Test the linear dependency of the functions $-2+t^2, 1+t-t^2, 3+2t, -1+2t^2 \ in R^3$

Solution: We have, $-2+t^2=-2+0.t+t^2$

$$1+t-t^2=1+t+(-1)t^2$$

$$3+2t=3+2t+0.t^2$$

$$-1+2t^2=-1+0.t+2t^2$$

The matrix of the coefficients is,

$$M=\begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \\ 3 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Now we shall reduce this matrix into echelon form by elementary row operations,

$$\sim\begin{pmatrix} -2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_2^{'}=2R_2+R_1 \\ R_3^{'}=2R_3+3R_1 \\ R_4^{'}=2R_4-R_1 \end{matrix}$$

$$\sim\begin{pmatrix} -2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} R_3^{'}=R_3-2R_2$$

Since, the echelon form of the matrix contains a zero row, so the given functions are linearly dependent in R^3.

Exercise:

Problem-01: Test the linear dependency of S where,

i). $S=\{(1,-2,1),(2,1,-1),(7,-4,1)\}\subset R^3$

ii). $S=\{(-1,1,1),(1,-4,2),(3,-5,1)\}\subset R^3$

iii). $S=\{(1,1,-1),(1,2,3),(3,5,-3)\}\subset R^3$

iv). $S=\{(1,4,0,-1),(2,0,0,3),(-1,4,0,-4)\}\subset R^4$

v). $S=\{(1,4,0,-1,2),(2,0,0,3,1),(-2,1,5,2,0)\}\subset R^5$

vi). $S=\{(1,-2,4,1),(2,1,0,-3),(3,-6,1,4)\}\subset R^4$

ANS: i). LD ; ii). LI ; iii). LD ; iv). LD ; v). LI ; vi). LI.

Problem-02: Test whether or not the following vectors are linearly dependent or not:

i). $(1,2,3),(2,1,-2),(3,3,1) \ in \ R^3$.

ii). $(2,1,3),(3,2,1),(1,1,-2)$ *in* R^3

iii). $t^2+2t+3, 3t^2-t+2, 5t^2+3t+8$ *in* R^3

iv). $t^2+t+2, 2t^2+t, 3t^2+2t+2$ *in* R^3

v). $(1,2,-1,-3),(3,1,-1,2),(4,-1,0,3)$ *in* R^4