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Journal:	*IEEE Antennas and Wireless Propagation Letters*
Manuscript ID:	Draft
Manuscript Type:	Original Manuscript
Date Submitted by the Author:	n/a
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Keywords:	Integral equations, Method of Moments, Computational Electromagnetics, Algorithms

Generalized Equivalence Integral Equations

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***Abstract*—A generalized equivalence integral equation (GEIE) approach to formulating surface scattering problems is proposed. GEIE approach invokes the generalized surface field equivalence to partially fill the volume originally occupied by the scatterer with judiciously selected materials, as opposed to the conventional replacement of the scatterer by the free-space. The type and shape of the material inclusions can be selected to allow for a numerically efficient construction of the modified Green's function. Introduction of impenetrable and lossy materials confines the field interactions along the scatterer surface and reduces the coupling between the distant parts of the scatterer, which essentially makes the impedance matrix banded. The presence of lossy materials also resolves the non-uniqueness problem of the electric and magnetic field integral equations by eliminating the internal resonances. The formulation provides a pathway for developing fast iterative and direct electromagnetic integral equation solvers.**

***Index Terms*— Algorithms, computational electromagnetics, integral equations, moment methods.**

I. Introduction

INTEGRAL equations-based techniques, such as the Method of Moments (MoM) [1] are often the choice for the analysis of various problems of computational electromagnetics (CEM). However, there is a number of difficulties in using integral equation formulations for complex problems. In particular, conventional formulations lead to a high computational complexity stemming from the fact that the resulting matrix equations involve dense matrices. These matrices can also have a large condition number leading to slow convergence. Furthermore, the obtained solutions can be unstable because of spurious resonances. A combination of these deficiencies may substantially limit the applicability of integral equation methods for solving complex electromagnetic problems.

Several methods exist to overcome these difficulties. Improving the matrix sparsity can be achieved by a proper selection of expansion and testing functions in the MoM, e.g. [2,3], but the existing methods are geometry dependent. Fast algorithms for matrix-vector multiplication, such as the Fast Multipole Method and its modifications [4-6], MultiLevel Matrix Decomposition Algorithm (MLMDA) [7], non-uniform grids interpolation method (NGIM) [8], and other related methods, allow computing the field in O($N\log N$) operations for a problem discretized into N unknowns. These methods, however, typically do not automatically account for the physical behavior of the solution. Methods also exist for improving the equation stability, e.g. the combined field integral equation for overcoming the problem of internal resonances [1]. However, such a formulation has to involve equations of several types and cannot be easily used for objects comprising open parts. In addition to iterative solvers fast direct solvers are also being developed [9,10]. Direct solvers are generally preferred for resonant problems, for problems where high accuracy is required, and for multiple excitation problems. However, no provably fast direct solvers have been proposed for high-frequency scattering from general geometries [11].

Many of the difficulties in solving integral equations can be traced to the fact that the conventional formulations are based on the free-space (Love's) equivalence principle, which leads to a strong coupling between distant parts of the geometry. It appears, therefore, advantageous to seek alternative forms of the integral equations that are based on more general equivalent configurations.

In this paper, we propose a generalized equivalence integral equation (GEIE) formulation of computational electromagnetics to facilitate the construction of fast iterative and direct solvers. This formulation stems from the observation that, based on the high-frequency asymptotic considerations for large essentially convex scatterers, one can expect the coupling between distant parts of the scatterer to be approximately localized. Towards reducing the unwanted interactions, the GEIE is based on constructing an equivalent situation while introducing an impenetrable "physical" obstacle in the region originally occupied by the scatterer. Such approach results in a suppression of the distant coupling for essentially convex scatterers and effectively reduces the dimensionality of the problems under study. The goal of this work is to present the main ideas of the formulation and analyze the resulting equations and solutions. To that end, we study an example problem of a two-dimensional (2D) scattering from an impenetrable "essentially circular" cylinder. The GEIE approach reduces this problem to a quasi-1D cyclic case involving coupling only along the scatterer surface, which results in a number of properties important for the construction of fast and stable electromagnetic solvers.

Manuscript received August 21, 2012. This work was supported in part by the United States - Israel Binational Science Foundation Grant No. 2008077.

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II. Integral Equation Formulation

Consider scattering from an arbitrary shaped body described by a closed perfectly electrically conducting (PEC) surface S residing in free space. Integral equation formulations of the scattering problem are conventionally derived using the Love's equivalence principle [1], in which the scatterer is replaced by equivalent currents radiating in free space as shown in Fig. 1(a). The currents produce the scattered field outside the body, while cancelling the incident field inside. This equivalent situation can be generalized by introducing arbitrary lossy materials or auxiliary boundary conditions in the zero-field zone, i.e., inside the scatterer's volume (Fig. 1(b)). GEIE is introduced by requiring that the equivalent currents radiate in the presence of those material inclusions.

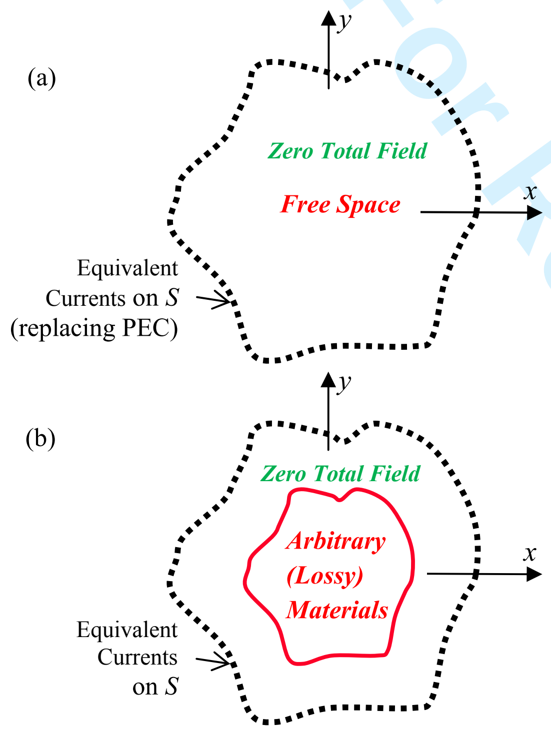

Fig. 1. Scattering from an arbitrary shaped PEC body (a) Love's (free space) equivalence principle, (b) Generalized equivalence principle.

To demonstrate the main features of the proposed GEIE approach, we consider the problem of two-dimensional scattering from a z-invariant PEC cylinder with its cross-section described by a closed contour S (see Fig. 2). It is assumed that S can be circumscribed by a circle of radius R_o and can inscribe a concentric circle of radius R_i, where $h = R_o - R_i$ is small or comparable to the wavelength. The cylinder is illuminated by a z-polarized electric field $\mathbf{E}^{\text{inc}}(\mathbf{r}) = \hat{\mathbf{z}}E^{\text{inc}}$, where $\hat{\mathbf{z}}$ is the unit vector along the z-direction and $\mathbf{r} = (\rho, \varphi)$ denotes an observation point with its polar coordinates. A harmonic time dependence $e^{j\omega t}$ is assumed and suppressed.

The incident field induces a z-directed surface current $\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}}J(\mathbf{r})$, $\mathbf{r} \in S$, which satisfies the electric field integral equation (EFIE) [1]

$$-E^{\text{inc}} = E^{s}[J](\mathbf{r}) = \int_{S} G_{\gamma}(\mathbf{r},\mathbf{r}')J(\mathbf{r}')d\mathbf{r}', \quad \mathbf{r} \in S \qquad (1)$$

where $G_{\gamma}(\mathbf{r},\mathbf{r}')$ denotes the Green's function with subscript γ. In the conventional formulations, the Green's function is given for the free space by $G_{\gamma}(\mathbf{r},\mathbf{r}') = -(k\eta/4)H_0^{(2)}(k\,|\,\mathbf{r}-\mathbf{r}'\,|)$, where $H_0^{(2)}$ denotes the zero-order Hankel function of the second kind, while k and η are the intrinsic wavenumber and impedance, respectively. In the GEIE, an arbitrary material is placed inside the area originally occupied by the scatterer. Such generalization is allowed, since the total field comprising the incident field and the scattered field produced by $\mathbf{J}(\mathbf{r})$ should be zero within S. For the configuration in Fig. 2, we introduce inside S an impenetrable circular cylinder of radius R_i with a surface impedance boundary condition $E_z = Z_s H_{\varphi}$, where Z_s is the surface impedance. The impenetrable cylinder effectively eliminates the line-of-sight coupling between the opposite faces of the scatterer. The surface impedance is chosen to be lossy ($\mathrm{Re}\{Z_s\} > 0$) and can be designed to suppress creeping and surface wave phenomena at large distances from the source, thus, further reducing the coupling between the distant parts of the scatterer. The presence of the loss within S also eliminates the internal resonance phenomena, thus enabling the use of the EFIE for electrically large closed scatterers. Discretization of the GEIE formulation via the MoM leads to essentially banded matrices with rank deficient off-diagonal blocks as shown in the Sec. III.

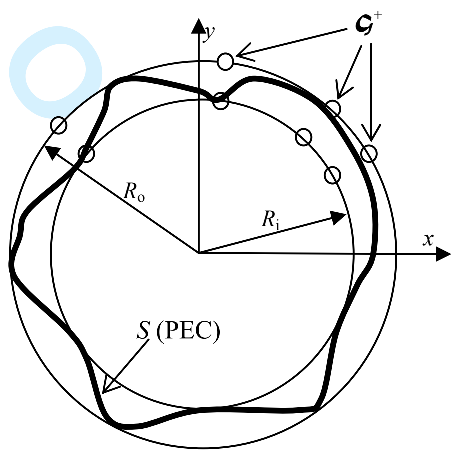

Fig. 2. Geometry of the scatterer and the non-uniform polar grid for counter-clockwise propagating field corresponding to the sources located at $\varphi = 0$.

III. Analysis and Discussion

A. Test problem outline

To illustrate the main features of the GEIE formulation presented in Section II, we consider scattering from a corrugated PEC cylinder defined by $\rho = a + b\cos(m\varphi)$. The cylinder's average radius, corrugation amplitude, and rate are $a = 8.5\lambda$, $b = \lambda/4$, and $m = 12$, respectively, where λ denotes the wavelength. Generalized equivalent configuration is constructed by introducing a circular cylinder of radius $R_i = 8\lambda$ with an impedance boundary condition.

B. *Modified Green's function*

The GEIE approach is based on a modified Green's function used in Eq. (1). In particular, the Green's function here describes the electric field radiated by a filamentary current near an impedance cylinder. The computation of $G_g(\mathbf{r},\mathbf{r}')$ can be performed by different means, e.g. via the Mie series [12] or via asymptotic approximations [13]. An important property of the modified Green's function for electrically large and smooth bodies is that its asymptotic behavior is known. Specifically, in the proximity of the impedance cylinder surface, the wave propagation can be approximated by that in a lossy/leaky azimuthal transmission line, i.e.

$$\tilde{G}_{\mathrm{g}}(\mathbf{r},\mathbf{r}') = \tilde{G}_{\mathrm{g}}(\mathbf{r},\mathbf{r}')e^{-jkR_i|\varphi-\varphi'|} \qquad (2)$$

where $\tilde{G}_{\mathrm{g}}(\mathbf{r},\mathbf{r}')$ is a slowly varying function of its arguments. This form allows precomputing the compensated Green's function $\tilde{G}_{\mathrm{g}}(\mathbf{r},\mathbf{r}')$ for a small fixed ($O(1)$) number of points on a non-uniform grid $\mathcal{G} = \{\tilde{\mathbf{r}}_n\}$ and interpolate it to all required source-observer locations.

In Fig. 3, we compare the free space Green's function to the generalized one constructed with various values of the surface impedance. One can see that while the free space Green's function is not localized, the generalized one with resistive and inductive surface impedances is approximately exponentially decaying away from the source. As expected, the real positive value of the surface impedance $Z_s = 50\,\Omega$ provides a high degree of localization, purely imaginary impedance $Z_s = 500j\,\Omega$ results in a lower localization, while $Z_s = -500j\,\Omega$ supports a surface wave, thus eliminating the localization. Since localization is desired to obtain banded matrices with a minimal bandwidth, we should employ surface impedance values with a positive real part.

C. *Matrix properties*

As we have seen the use of the impenetrable lossy auxiliary object produces effectively banded matrices. Next, towards the construction of fast solvers, we study the number of degrees of freedom (DoF) needed to represent interactions between distant parts of the scatterer. Specifically, we consider the interaction between a given subdomain and the rest of the scatterer with the goal to analyze its rank. We employ the MoM representation of the scattering problem using the conventional EFIE and the proposed GEIE formulations. To that end, the problem is discretized employing the conventional pulse basis functions and point matching with 10 basis function per linear wavelength. The GEIE formulation is constructed by introducing a circular cylinder of radius $R_i = 8\lambda$ with surface impedance $Z_s = 50\,\Omega$.

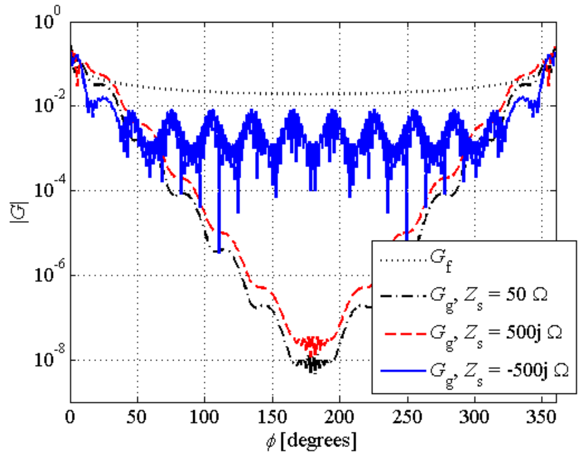

Fig. 3. Free space and generalized Green's functions with $R_i = 8\lambda$ for various values of the surface impedance versus the azimuthal angle of the observation point for source point located at $\varphi' = 0$. Both source and observation points located on the surface *S* of the corrugated cylinder.

The coupling of a subdomain comprising M basis functions with the rest of the scatterer is described by an $(N-M)\times M$ off-diagonal block of the MoM matrix. A singular value decomposition (SVD) of this block allows us to quantify the number of DoF involved in the interaction between this subdomain and the rest of the scatterer. More specifically the number of DoF is determined as the number of most significant singular values satisfying $\sigma_n/\sigma_1 > \tau$, where τ is a prescribed threshold. Singular value spectra obtained for the conventional EFIE and the GEIE formulations for subdomains of roughly 10λ and 20λ in size is shown in Fig. 4. For the conventional EFIE and these subdomain sizes, the SVD reveals roughly 20 and 36 strong singular values followed by an exponentially decaying tail. This behavior is consistent with the well-known properties of the free-space radiation that

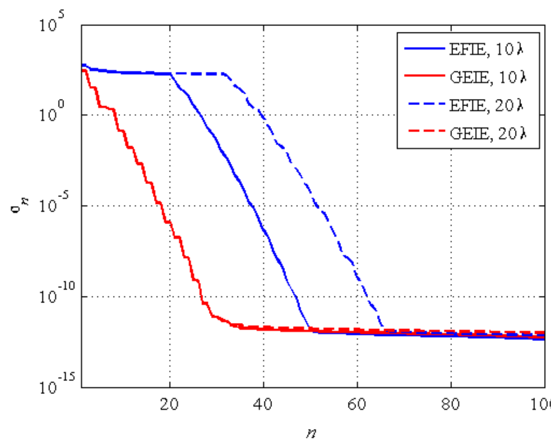

Fig. 4. Distribution of singular values of the blocks describing the radiation from the subdomains of 10λ and 20λ in size ($M = 100$ and 200 unknowns) towards the rest of the scatterer discretized by the MoM using the conventional EFIE and GEIE with $R_i = 8\lambda$ and $Z_s = 50\,\Omega$.

requires roughly two DoF per linear wavelength of the radiating aperture. On the other hand, for the GEIE formulation an exponential decay is obtained starting from the second singular value independent of the domain size, i.e. the number of DoF is of $O(1)$ for any problem size. This behavior indicates that GEIE formulation facilitates a strong compression of the interactions between different subdomains. In fact, the GEIE effectively reduces the 2D problem to a periodic quasi-1D case, which can be used to construct fast direct solvers.

D. Solution properties

Finally, we examine the stability of the MoM-based numerical solution for the incident field produced by a unit filamentary current $I^{\rm inc} = 1$ located at $(x, y) = (10\lambda, 0)$. The resulting surface current density on the scatterer surface S is presented in Fig. 5. One can observe the instability of the conventional EFIE producing inaccurate currents in the shadow zone. This instability is due to the presence of internal resonances in the effective PEC cavity formed by the interior of the original surface of the scatterer. The numerical problem discretization effectively couples these resonances to the exterior problem, thus leading to the unstable behavior linked to the use of the free space Green's function.

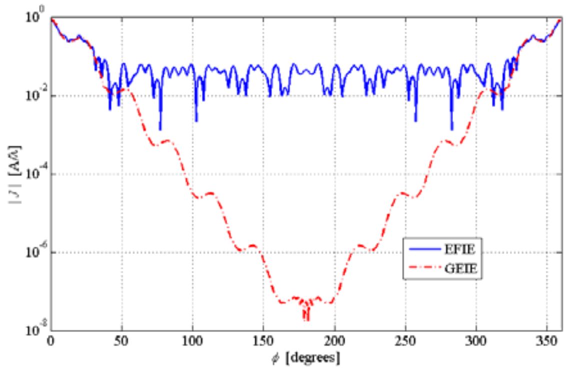

Fig. 5. Surface current density vs. azimuthal angle for the corrugated cylinder illuminated by a filamentary current computed by the MoM with $N = 535$ unknowns using the EFIE and GEIE with $R_i = 8\lambda$ and $Z_s = 50\,\Omega$.

On the other hand, the GEIE-based formulation provides stable results even in the deep shadow zone (close to $\varphi = 180^o$) and even for very large problem sizes. Such behavior supports our assertion that the use of the generalized Green's function effectively creates a reduced dimensionality problem, in which there are no direct interactions through the scatterer interior region. The effective interior problem is the one comprising a waveguide formed between the auxiliary cylinder and the original scatterer surface. Since the auxiliary cylinder is chosen to have a positive resistive impedance such a waveguide can only support a lossy propagation, which eliminates any effects of internal resonances and the associated instabilities typical for the conventional EFIE. Importantly, this is achieved without involving the combined field integral equation approach and without possible inaccuracies appearing due to the use of the magnetic field integral equation.

IV. Conclusion

We presented a GEIE formulation for solving problems of the electromagnetic scattering from electrically large objects. The GEIE is based on a generalized physical equivalence principle in which an auxiliary object is inserted in the region originally occupied by the scatterer, the boundary conditions on the scatter are satisfied, and a modified Green's function is used instead of the free-space Green's function. The object is chosen to be characterized by a resistive impedance boundary condition and be impenetrable. The GEIE approach leads to several important properties of the resulting impedance matrix and solution. The auxiliary impenetrable object effectively reduces the problem dimensionality and confines the interactions only to a reduced domain making the impedance matrix essentially banded. Moreover, the off-diagonal blocks of the impedance matrix have a substantially reduced rank. In particular for the considered example the rank of the impedance sub-matrix characterizing the interaction of a subdomain to the rest of the scatterer is of $O(1)$, independent of the subdomain size. This behavior represents a significant improvement as compared to conventional approaches, in which the corresponding rank increases with the subdomain size. The presence of a lossy auxiliary object also eliminates internal resonances characteristic of the conventional EFIE. Thus, the resulting solutions are stable everywhere even for electrically large problems.

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