## Properties of the derivative

We now consider various properties of differentiation. As we proceed, we will be able to differentiate wider and wider classes of functions.

Throughout this section and the next, we will be manipulating limits as we compute derivatives. We therefore recall some basic rules for limits; see the module *Limits and continuity* for details. The following hold provided the limits $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist*.

- The limit of a sum (or difference) is the sum (or difference) of the limits:

$$\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

- The limit of a product is the product of the limits:

$$\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$$

  This includes the case of multiplication by a constant *c*:

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$$

### The derivative of a constant multiple

Suppose we want to differentiate $4x^7$. Rather than returning to the definition of a derivative, we can use the following theorem.

*Theorem*

*Let f be a differentiable function and let c be a constant. Then the derivative of c f* (*x*) *is c f O*(*x*). *That is,*

$$\frac{d}{dx}[cf(x)] = cfo(x)$$

This fact can also be written in Leibniz notation as

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

*Proof*

The derivative of $c\,f(x)$ is given by $\lim_{\Delta x \to 0} \frac{cf(x+\Delta x)-cf(x)}{\Delta x} = c \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = cfo(x)$

$\Delta x \to 0$ $\Delta x$ $x \to$

We may factor out the $c$, since it is just a constant. □

## Example

What is the derivative of $4x^7$?

## Solution

The theorem tells us that the derivative of $4x^7$ is 4 times the derivative of $x^7$. Hence,

$$\frac{d}{dx}¡4x^7¿ = 4\frac{d}{dx}¡x^7¿$$
$$= 4 \cdot 7x^6$$
$$= 28x^6.$$

## The derivative of a sum

A function like $f(x) = x^3 + 5x^2$ can be differentiated from first principles; alternatively, we can use the following theorem.

*Theorem*

*Suppose both f and g are differentiable functions. Then the derivative of f(x)+g(x) is fo(x)+go(x) and, similarly, the derivative of f (x)−g(x) is fo(x)-go(x). That is,*

$$\frac{d}{dx}[f(x) + g(x)] = fo(x) + go(x),$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

In Leibniz notation, these statements can be written as

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}, \qquad \frac{df}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$$

*Proof*

We prove the first statement. The derivative of ***f*** **(*x*)+*g*(*x*)** is given by

$$\lim_{\Delta x \to 0} \frac{\big(f(x+\Delta x)+g(x+\Delta x)\big)-\big(f(x)+g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-g(x)}{\Delta x}$$

$$= f'(x)+g'(x)$$

Note that, as both the limits in the second line exist, we are retrospectively justified in splitting the first limit into two pieces.

**Exercise**

Find the derivative of f(x)=$x^3+5x^2$.

**Solution**

$$\boldsymbol{f'(x) = \frac{d}{dx}(x^3+5x^2)}$$

$$\boldsymbol{= \frac{d}{dx}(x^3) + \frac{d}{dx}(5x^2) \quad (derivative\ of\ sum)}$$

$$\boldsymbol{= \frac{d}{dx}(x^3) + 5\frac{d}{dx}(x^2) \quad (derivative\ of\ a\ constant\ multiple)}$$

$$\boldsymbol{= 3x^2 + 10x \qquad (derivative\ of\ x^n)}$$

*Note.* The solution to the previous example shows every individual step explicitly and states which theorems are used. Once proficient with these properties of the derivative, however, there is no need to justify each step in this way. It is common, for instance, to go straight from

$$f(x) = x^3 + 5x^2 \qquad \text{to} \qquad f'(x) = 3x^2 + 10x.$$

### Exercise 7

Let $f(x) = \dfrac{x^5 + \sqrt{x}}{x^2}$. Find $f'(x)$.

## Linearity of the derivative

The two previous theorems (for the derivative of a sum and the derivative of a constant multiple) can be summarised as follows. If $f, g$ are differentiable functions and $a, b$ are constants, then

$$\frac{d}{dx}[mf(x) + ng(x)] = mf'(x) + ng'(x).$$

The same fact can be written in Leibniz notation as

$$\frac{d}{dx}(mf + ng) = m\frac{df}{dx} + n\frac{dg}{dx}.$$

This property is sometimes expressed by saying that ‘differentiation is linear’.