The second derivative

Example

Find the equation of the tangent line to the graph $y = \frac{1}{2}x^2$ at $x = 3$.

Solution

Letting $f(x) = \frac{1}{2}x^2$, we have $f'(x) = x$, so $f(3) = \frac{9}{2}$ and $f'(3) = 3$. Thus the tangent line has gradient 3 and passes through $(3, \frac{9}{2})$, and is given by

$$y - \frac{9}{2} = 3(x-3)$$

or, equivalently,

$$y = 3x - \frac{9}{2}.$$

Exercise 17

What is the equation of the tangent line to the graph of $y = \sqrt{9-x^2}$ at $x = \frac{3}{2}\sqrt{2}$?

Given a function $f(x)$, we can differentiate it to obtain $f'(x)$. It can be useful for many purposes to differentiate again and consider the **second derivative** of a function.

In functional notation, the second derivative is denoted by $f''(x)$. In Leibniz notation, letting $y = f(x)$, the second derivative is denoted by $\frac{d^2y}{dx^2}$.

The placement of the 2's in the notation $\frac{d^2y}{dx^2}$ may appear unusual. We consider that we have applied the differentiation operator $\frac{dy}{dx}$ twice to y:

$$\left(\frac{d}{dx}\right)^2 y = \frac{d^2}{dx^2}y = \frac{d^2y}{dx^2}$$

or that we have applied the differentiation operator $\frac{d}{dx}$ to $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

As we will see in the module *Applications of differentiation*, the second derivative can be very useful in curve-sketching. The second derivative determines the *convexity* of the graph $y = f(x)$ and can, for example, be used to distinguish maxima from minima.

The second derivative can also have a physical meaning. For example, if $x(t)$ gives position at time t, then $x'(t)$ is the velocity and the second derivative $x''(t)$ is the *acceleration* at time t. This is discussed in the module *Motion in a straight line*.

Example

Find the second derivative of $f(x) = x^2$.

Solution

We have $f'(x) = 2x$, and so $f''(x) = 2$.

This example implies that, if you ride your bike and your position is x^2 metres after x seconds, then your acceleration is a constant 2 m/s^2.

Given a graph $y = f(x)$, we have seen how to calculate the gradient of a tangent line to this graph. We can go further and find the *equation* of a tangent line.

Example

Let $y = x^7 + 3x^5 + x^{\frac{3}{2}}$. Find $\frac{d^2y}{dx^2}$.

Solution

The first derivative is

$$\frac{dy}{dx} = 7x^6 + 15x^4 + \frac{3}{2}x^{\frac{1}{2}},$$

so the second derivative is

$$\frac{d^2y}{dx^2} = 42x^5 + 60x^3 + \frac{3}{4}x^{-\frac{1}{2}}.$$

Exercise 18

Let $f(x) = (x^2 + 7)^{100}$, as in exercise 11. What is $f''(x)$?

Consider the tangent line to the graph $y = f(x)$ at $x = a$. This line has gradient $f'(a)$ and passes through the point $(a, f(a))$. Once we know a point on the line and its gradient, we can write down its equation:

$$y - f(a) = f'(a)(x - a).$$

(See the module *Coordinate geometry*.)