MOTIVE BEHIND THE LESSON

The primarily motive for this website is to focus on differentiation calculus. In mathematics, Differential calculus is a subfield of calculus that studies the rates at which quantities change. The primary object of studying in differential calculus are the derivative of a function, related notions such as the differential, and their applications. The derivative of a function at a chosen input value describes the rate of change of the function near their input value. The process of finding a derivative is called DIFFERENTIATION. Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at the point, provided that the derivative exists and is defined at that point.

Differential calculus and and Integral calculus are connected by the fundamental theorem of calculus, which states that differentiation is the reverse process to integration.

Differentiation has applications in nearly all quantitative disciplines. In physics, the derivative of the displacement of a moving body with respect to time is the velocity of the body, and the derivative of the velocity with respect to time is acceleration. The derivative of the momentum of a body with respect to time equals the force applied to the body. Rearranging this derivative statement leads to the famous equation associated with Newtons second law of motion. $F = ma$

LET MOTIVATE OURSELVES WITH SOME REAL-LIFE SCENARIOS

> You ride your bicycle down a straight bike path. As you proceed, you keep track of your exact position — so that, at every instant, you know exactly where you are.
> *How fast were you going one second after you started?*

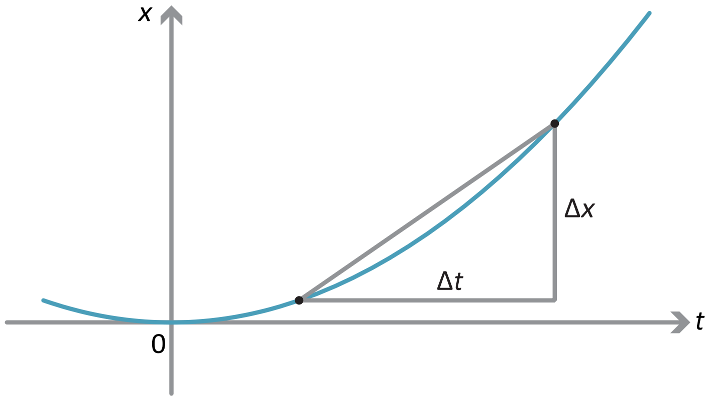
This is not a very realistic situation — it's hard to measure your position precisely at every instant! Nonetheless, let us suspend disbelief and imagine you have this information. (Perhaps you have an extremely accurate GPS or a high-speed camera.) By considering this question, we are led to some important mathematical ideas.

First, recall the formula for **average velocity**:

$$V = \frac{\Delta X}{\Delta t},$$

where Δx is the change in your position and Δt is the time taken. Thus, average velocity is the rate of change of position with respect to time.

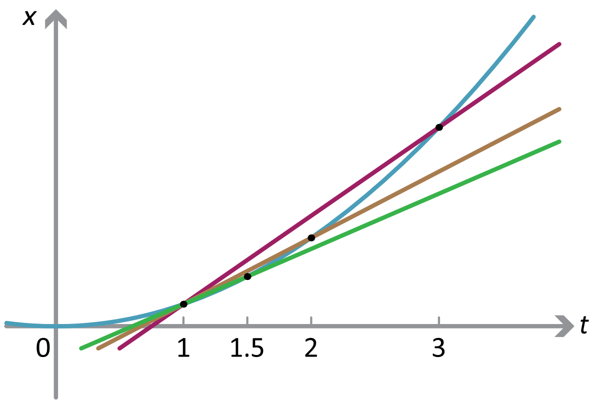
Draw a graph of your position $x(t)$ at time t seconds. Connect two points on the graph, representing your position at two different times. The *gradient* of this line is your average velocity over that time period.



$v = —$

$$Average\ Velocity\ v = \frac{\Delta x}{\Delta t}$$

Trying to discover your velocity at the one-second mark (t = 1), you calculate your *average velocity* over the period from t = 1 to a slightly later time $t = 1+\Delta t$. Trying to be more accurate, you look at shorter time intervals, with Δt smaller and smaller. If you really knew your position at every single instant of time, then you could work out your average velocity over any time interval, no matter how short. The three lines in the following diagram correspond to $\Delta t = 2$, $\Delta t = 1$ and $\Delta t = 0.5$.



Average velocity over shorter time intervals.

As Δt approaches 0, you obtain better and better estimates of your instantaneous velocity at the instant t = 1. These estimates correspond to the gradients of lines connecting closer and closer points on the graph.

In the limit, as $\Delta t \to 0$,

$$\lim_{\Delta t} \frac{\Delta x}{\Delta t}$$

gives the precise value for the instantaneous velocity at t = 1. This is also the gradient of the *tangent* to the graph at t = 1. **Instantaneous velocity** is the instantaneous rate of change of position with respect to time.

Although the scenario is unrealistic, these ideas show how you could answer the question of how fast you were going at t = 1. Given a function $x(t)$ describing your position at time t, you could calculate your exact velocity at time t = 1.

In fact, it is possible to calculate the instantaneous velocity at *any* value of t, obtaining a function which gives your instantaneous velocity at time t. This function is commonly denoted by $x0(t)$ or $\frac{dy}{dx}$ and is known as the *derivative* of $x(t)$ with respect to t.

In this module, we will discuss derivatives.

The more things change ...

Velocity is an important example of a derivative, but this is just one example. The world is full of quantities which change with respect to each other — and these rates of change can often be expressed as derivatives. It is often important to understand and predict how things will change, and so derivatives are important.

Here are some examples of derivatives, illustrating the range of topics where derivatives are found:

- Mechanics. We saw that the derivative of position with respect to time is velocity. Also, the derivative of velocity with respect to time is *acceleration*. And the derivative of momentum with respect to time is the (net) *force* acting on an object.
- Civil engineering, topography. Let $h(x)$ be the height of a road, or the altitude of a mountain, as you move along a horizontal distance x. The derivative $h0(x)$ with respect to distance is the *gradient* of the road or mountain.
- Population growth. Suppose a population has size $p(t)$ at time t. The derivative $p0(t)$ with respect to time is the *population growth rate*. The growth rates of human, animal and cell populations are important in demography, ecology and biology, respectively.
- Economics. In macroeconomics, the rate of change of the gross domestic product (GDP) of an economy with respect to time is known as the *economic growth rate*. It is often used by economists and politicians as a measure of progress.
- Mechanical engineering. Suppose that the total amount of energy produced by an engine is $E(t)$ at time t. The derivative $E0(t)$ of energy with respect to time is the *power* of the engine.

All of these examples arise from a more abstract question in mathematics:

- Mathematics. Consider the graph of a function $y = f(x)$, which is a curve in the plane. What is the *gradient* of a tangent to this graph at a point? Equivalently, what is the instantaneous rate of change of y with respect to x?

In this module, we discuss purely mathematical questions about derivatives. In the three modules *Applications of differentiation*, *Growth and decay* and *Motion in a straight line*, we discuss some real-world examples.

Therefore, although the *Motivation* scenario has focused on instantaneous velocity, which is an important motivating example, we now concentrate on calculating the gradient of a tangent to a curve.