Secants of the parabola $f(x) = x^2$

Secant between points	Δx	$\Delta y = f(x+\Delta x) - f(x)$	Gradient of secant $\frac{\Delta y}{\Delta x}$
$x = 1$, $x = 3$	2	8	4
$x = 1$, $x = 2$	1	3	3
$x = 1$, $x = 1.5$	0.5	1.25	2.5
$x = 1$, $x = 1.1$	0.1	0.21	2.1
$x = 1$ $x = 1.001$	0.001	0.002001	2.001

As Δx approaches 0, the gradients of the secants approach 2. It turns out that indeed the gradient of the tangent at $x = 1$ is 2. To see why, consider the interval of length Δx, from $x = 1$ to $x = 1+\Delta x$. We have

$$\Delta y = f(1+\Delta x) - f(1)$$

$$= (1+\Delta x)^2 - 1^2$$

$$= 2\Delta x + (\Delta x)^2,$$

so that

$$\frac{\Delta y}{\Delta x} = \frac{2\Delta x + (\Delta x)^2}{\Delta x}$$

$$= 2 + \Delta x$$

In the limit, as $\Delta x \to 0$, we obtain the instantaneous rate of change

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2 + \Delta x) = 2$$

(So, if you were riding your bike and your position was $f(x) = x^2$ metres after x seconds, then your instantaneous velocity after 1 second would be 2 metres per second.)

There's nothing special about the point $x = 1$ or the function $f(x) = x^2$, as the following example illustrates.

Example

Let $f(x) = x^3$. What is the gradient of the tangent line to the graph $y = f(x)$ at the point $(2,8)$?

Solution

The gradient at $x = 2$ is given by the limit

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{(2+\Delta x)^3 - 2^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{8 + 12(\Delta x) + 6(\Delta x)^2 + (\Delta x)^3 - 8}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(12 + 6(\Delta x) + (\Delta x)^2\right) = 12.\end{aligned}$$

Thus the gradient of the tangent line to $y = f(x)$ at $(2,8)$ is 12.

Exercise 1

Using a similar method, find the gradient of the tangent line to $y = x^4$ at $(-1,1)$, and find the equation of this line.

In the examples so far, we have been given a curve, and we have found the gradient of the curve at one particular point on the curve. But we can also find the gradient at all points simultaneously, as the next section illustrates.