Differentiation of inverses

A clever use of the chain rule arises when we have a function f and its *inverse* function f^{-1}. Refer to the module *Functions II* for a discussion of inverse functions.

Letting $y = f(x)$, we can express x as the inverse function of y:

$$y = f(x), \qquad x = f^{-1}(y).$$

The composition of f and its inverse f^{-1}, by definition, is just x. That is, $f^{-1}(f(x)) = x$.

We can think of this diagrammatically as

$$x \to^{f} y \to^{f'} x$$

Using the chain rule, we can differentiate this composition of functions to obtain

$$\frac{dx}{dy} = \frac{dx}{dy} \cdot \frac{dy}{dx}$$

The derivative $\frac{dx}{dx}$ of x with respect to x is just 1, so we obtain the important formula

$$1 = \frac{dx}{dy}\frac{dy}{dx},$$

which can also be expressed as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad or \qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

This formula allows us to differentiate inverse functions, as in the following example.

Example

Let $y = \sqrt[3]{x}$. $\frac{dy}{dx}$.

Solution

(We assume that we only know the derivative of x^n when n is a positive integer.) The inverse function of the cube-root function is the cube function: if $y = \sqrt[3]{x}$, then $x = y^3$. We know the derivative of the cube function, $\frac{dx}{dy} = 3y^2$. We use this to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}.$$

Substituting $y = \sqrt[3]{x} = x^{\frac{1}{3}}$ gives

$$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3}x^{-\frac{2}{3}},$$

as expected.

The following exercises generalise the above example to find the derivative $\sqrt[n]{x} = x^{\frac{1}{n}}$ and then of any rational power of *x*.