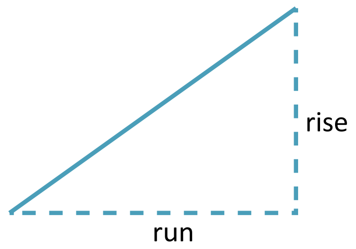
The gradient of secants and tangents to a graph

Consider a function $f : \mathbf{R} \to \mathbf{R}$ and its graph $y = f(x)$, which is a curve in the plane. We wish to find the *gradient* of this curve at a point. But first we need to define properly what we mean by the gradient of a curve at a point!

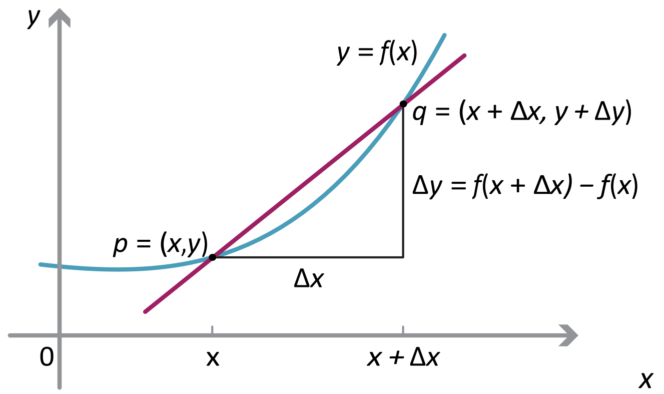
The module *Coordinate geometry* defines the gradient of a line in the plane: Given a non-vertical line and two points on it, the **gradient** is defined as $\frac{rise}{run}$



Gradient of a line.

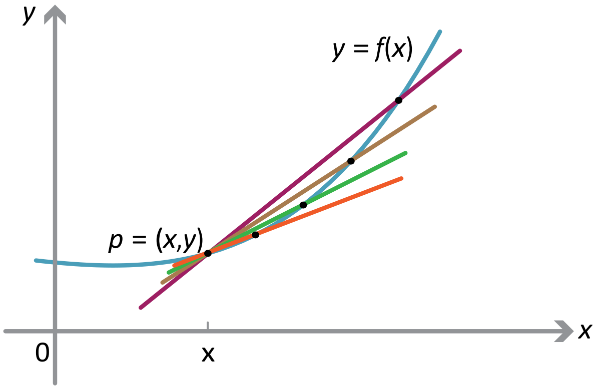
Now, given a *curve* defined by $y = f(x)$, and a point p on the curve, consider another point q on the curve near p, and draw the line pq connecting p and q. This line is called a **secant line**.

We write the coordinates of p as (x,y), and the coordinates of q as $(x+\Delta x, y+\Delta y)$. Here Δx represents a small change in x, and Δy represents the corresponding small change in y.



Secant connecting points on the graph $y = f(x)$ at x and $x + \Delta x$.

As Δx becomes smaller and smaller, the point q approaches p, and the secant line pq approaches a line called the **tangent** to the curve at p. We define the **gradient of the curve** at p to be the gradient of this tangent line.



Secants on $y = f(x)$ approaching the tangent line at x.

Note that, in this definition, the approximation of a tangent line by secant lines is just like the approximation of instantaneous velocity by average velocities, as discussed in the *Motivation* section.

With this definition, we now consider how to compute the gradient of the curve $y = f(x)$ at the point $p = (x,y)$.

Taking $q = (x + \Delta x, y + \Delta y)$ as above, the secant line pq has gradient

$$\frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Note that the symbol Δ on its own has no meaning: Δx and Δy refer to change in x and y, respectively. You cannot cancel the Δ's!

As $\Delta x \to 0$, the gradient of the tangent line is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We also denote this limit by $\frac{\Delta y}{\Delta x}$;

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

The notation $\frac{}{dx}$ indicates the instantaneous rate of change of y with respect to x, and is not a fraction. For our purposes, the expressions dx and dy have no meaning on their own, and the d's do not cancel!

The gradient of a secant is analogous to average velocity, and the gradient of a tangent is analogous to instantaneous velocity. Velocity is the instantaneous rate of change of position with respect to time, and the gradient of a tangent to the graph $y = f(x)$ is the instantaneous rate of change of y with respect to x