## Definition of the derivative

The method we used in the previous section to find the gradient of a tangent to a graph at a point can actually be used to work out the gradient everywhere, simultaneously.

### Example

Let $f(x) = x^2$. What is the gradient of the tangent line to the graph $y = f(x)$ at a general point $(x, f(x))$ on this graph?

### Solution

We calculate the same limit as in previous examples, using the variable $x$ in place of a number:

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(2x + \Delta x\right) = 2x.
\end{aligned}$$

Thus, the gradient of the tangent line at the point $(x, f(x))$ is $2x$.

The **derivative** of a function $f(x)$ is the function $f'(x)$ which gives the gradient of the tangent to the graph y = f(x) at each value of x. It is often also denoted $\frac{dy}{dx}$. Thus

$$f^1(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The previous example shows that the derivative of $f(x) = x^2$ is $f\,0(x) = 2x$.

### Exercise 2

Show that the derivative of $f(x) = x^3$ is $f^0(x) = 3x^2$.

Functions which have derivatives are called **differentiable**. Not all functions are differentiable; in particular, to be differentiable, a function must be continuous. Almost all functions we meet in secondary school mathematics are differentiable. In particular, all polynomials, rational functions, exponentials, logarithms and trigonometric functions (such as sin, cos and tan) are differentiable.

Derivatives of exponential, logarithmic and trigonometric functions are discussed in the two modules *Exponential and logarithmic functions* and *The calculus of trigonometric functions*. We shall say a little more about which functions are differentiable and which are not in the *Appendix* to this module.

In the following example and exercises, we differentiate constant and linear functions.

### Exercise 3

Show that the derivative of a constant function $f(x) = c$, with $c$ a real constant, is given by $f'(x) = 0$.

### Example

What is the derivative of the linear function $f(x) = 3x + 7$?

### Solution

We calculate

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3(x + \Delta x) + 7 - 3x - 7}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3. \end{aligned}$$

Thus $f'(x) = 3$. (This answer makes sense, as the graph of $y = f(x)$ is the line $y = 3x + 7$, which has gradient 3.)

### Exercise 4

Show that the derivative of a linear function $f(x) = ax + b$, with $a, b$ real constants, is $f'(x) = a$.