Geometric Satake for p-adic group

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Talk 1

	F	$\mathcal{O}_F$	$\mathfrak{p}$	$\mathcal{O}_F/\mathfrak{p}$	$\lvert\cdot\rvert_v$
no. thy	$\mathbb{Q}$	$\mathbb{Z}$	$2, 3, 5, \cdots$	$\mathbb{F}_2, \mathbb{F}_3, \cdots$	$v \in \{p\} \cup \{\infty\}$ $\lvert$
alg curves	$\mathbb{F}_p(X)$	$\mathbb{F}_p[X]$	$x, X-1, X+2, \cdots, p(x)$	$\mathbb{F}_{p^r}$, $r = \deg p(x)$	

F_v

$\mathbb{Q}_p \supset \mathbb{Z}_p = \{ \sum a_i p^i : a_i \in \{0, 1, \cdots, p-1\} \}$

$\mathbb{F}_{p^r}((t)) \supset \mathbb{F}_{p^r}[\![t]\!] = \{ \sum a_i t^i : a_i \in \mathbb{F}_{p^r} \}$

Some questions related to $\mathbb{Q}$ $\Leftarrow$ question for $\mathbb{Q}_p$

$\Leftarrow$ question for $\mathbb{F}_{p^r}((t))$

$\Leftarrow$ geom. of alg curves

$$\mathrm{Gal}_F = \mathrm{Aut}(\bar{F} | F) = \varprojlim_{\substack{E|F \\ \text{finite sep.}}} \mathrm{Aut}(E|F)$$

$\uparrow$

$\bar{F} \hookrightarrow \bar{F}_v$

Gal_{F_v}

$$1 \to I_v \to \mathrm{Gal}_{F_v} \to \langle \mathrm{Frob}_v \rangle \to 1$$

Understand Gal_F via rep'n.

$$\mathbb{A}_F = \prod' F_v$$
$$\check{\mathcal{O}}_F = \prod \mathcal{O}_v$$

$$\{ \text{1-dim'l rep's of } \mathrm{Gal}_F \} \xrightarrow{\text{CFT}} \{ \text{chars of } F^\times \backslash \mathbb{A}_F^\times \}$$

Langlands program

$$\left\{ \rho : \mathrm{Gal}_F \to GL_2 \right\} \longrightarrow \left\{ f_\rho : GL_2(F) \backslash GL_2(\mathbb{A}_F) / GL_2(\mathcal{O}) \to \mathbb{C} \right\}$$

$\rho(I_v) = 1$ at (almost) all v

Hecke eigenfunction

$$\circlearrowright \bigotimes_v \mathcal{H}_v$$

$\mathbb{C}[C_{a,b}]_{a,b\in\mathbb{Z},\, a>b}$

$$\mathcal{H}_v = \mathbb{C}\left(GL_2(\mathcal{O}_v) \backslash GL_2(F_v) / GL_2(\mathcal{O}_v) \right)$$

$Nv = p$

$$h_1 * h_2(x) = \int_{GL_2(F_v)} h_1(xy^{-1})\, h_2(y)\, dy$$

$$\mathrm{vol}\left(GL_2(\mathcal{O}_v)\right) = 1$$

$C_{a,b}$ = char. function of

$$GL_2(\mathcal{O}_v) \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} GL_2(\mathcal{O}_v)$$

$$\boxed{T_v = C_{1,0}} \qquad \boxed{S_v = C_{1,1}}$$

$$T_v(f_\rho) = (Nv)^{-1/2}\, \mathrm{tr}\left(\rho(\mathrm{Frob}_v)\right) f_\rho$$

<u>Satake isom</u>. $\mathcal{H}_v = \mathbb{C}[T_v, S_v^{\pm 1}] \xrightarrow[\sim]{S}$ conj. inv't functions on GL_2

$\mathbb{C}[\mathrm{tr}, \det{}^{\pm 1}]$

$$S(T_v) = (Nv)^{-1/2}\, \mathrm{tr}$$
$$S(S_v) = \det$$

$$T_v * T_v = C_{2,0} + (p+1) S_v$$

$$\mathrm{tr} \cdot \mathrm{tr} = \chi_{\mathrm{Sym}^2 V} + \det$$

$$S^{-1}(\chi_{\mathrm{Sym}^2 V}) = p(C_{2,0} + S_v)$$

$$F = F_v \supset \mathcal{O} = \mathcal{O}_v$$

$$\begin{array}{ccc} gK & \longmapsto & g\,\mathcal{O}^2 \\ GL_2(F_v)/GL_2(\mathcal{O}_v) & = & \{\text{lattices in } F^2\} \\ \| & & \\ K & & \text{Sub } \mathcal{O}\text{-mod } \Lambda \text{ of } F^2 \\ & & \text{s.t. } \Lambda \otimes_{\mathcal{O}} F = F^2 \end{array}$$

"<u>Thm</u>" This set has some algebro-geom. str.

$$\left(\begin{array}{lll} F = \mathbb{F}_q((\varpi)), & \text{Beauville-Laszlo} & \\ F = \mathbb{Q}_p, & \text{Z.}, & \text{Bhatt-Scholze} \end{array} \right)$$

<u>Ex</u>. $K\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}K/K = \{\Lambda \subset \mathcal{O}^2 : \ell(\mathcal{O}^2/\Lambda) = 1\}$

$$\begin{array}{c} \text{S}\!\uparrow \\ \{\text{1-dim'l quotients of } \mathbb{F}_p^2 = \mathcal{O}^2/\varpi\mathcal{O}^2\} \simeq \mathbb{P}^1(\mathbb{F}_p) \end{array}$$

<u>Ex</u>. $K\begin{pmatrix} \varpi^2 & \\ & 1 \end{pmatrix}K \cup K\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix}K/K \subset G(2,4)(\mathbb{F}_p)$

$\wr\!\wr$

$\{\Lambda \subset \mathcal{O}^2 : \ell(\mathcal{O}^2/\Lambda) = 2\}$

$F = \mathbb{F}_p((t))$, $\left(\simeq \mathcal{O}^2/\varpi^2\mathcal{O}^2 \simeq \mathbb{F}_p^4 \twoheadrightarrow L \quad \dim_{\mathbb{F}_p} L = 2\right)$

$F = \mathbb{Q}_p$, $\mathcal{O}^2/\varpi^2\mathcal{O}^2 \twoheadrightarrow \mathcal{O}^2/\Lambda$

$\not\cong$ v.s $/\mathbb{F}_p$

$\widetilde{Gr}_2 = \{ \Lambda \subset \Lambda' \subset \mathcal{O}^2 : \ \ell(\mathcal{O}^2/\Lambda') = \ell(\Lambda'/\Lambda) = 1 \}$

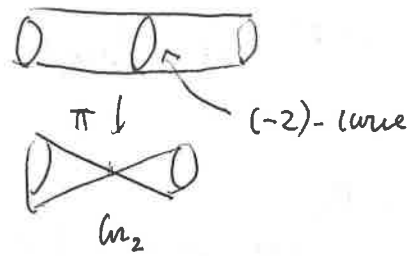
$\downarrow$

$\{ \Lambda \subset \mathcal{O}^2 : \ \ell(\mathcal{O}^2/\Lambda) = 2 \}$

$\cong$

$Gr_2(\mathbb{F}_p)$

$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))(\mathbb{F}_p)$



$\widetilde{Gr}_2 \leadsto V \otimes V$

$\cong$

$\mathrm{Sym}^2 V \oplus \det V$

$\pi_* \mathbb{Q}_\ell$

$= \mathbb{Q}_{\ell, Gr_2} \oplus \delta[-2]$

In general,

$\Lambda \subset F^2$

$Gr_{\vec{\mu}} = \{ \Lambda_2 \overset{1}{\subset} \Lambda_{2-1} \overset{1}{\supset} \Lambda_{2-2} \subset \cdots \subset \mathcal{O}^2 \}$

$\downarrow$

$GL_2(F)/GL_2(\mathcal{O}) = \{ \Lambda \subset \mathcal{O}^2 \}$

Def A monoidal additive cat.

Sat^0 Obj.: $Gr_{\vec{\mu}}$ $\vec{\mu} = (1, -1, \cdots)$

$\uparrow$

(Fontaine-Kamnitzer-Kuperberg)

Mor.: $\mathrm{Hom}(Gr_{\vec{\mu}}, Gr_{\vec{\nu}}) = \mathbb{Q}_\ell\text{-span}$ irred. comp. of $\subset Gr_{\vec{\mu}} \times Gr_{\vec{\nu}}$

$\|$

$\left\{ \begin{matrix} \Lambda_i' & \supset & \cdots & \supset \mathcal{O}^2 \\ \| & & & \| \\ \Lambda_i'' & \subset & \cdots & \supset \mathcal{O}^2 \end{matrix} \right\}$

Fact: these are half dim. subvar.

$$\mathrm{Hom}(\mathrm{Gr}_{\vec{\mu}}, \mathrm{Gr}_{\vec{\nu}}) \times \mathrm{Hom}(\mathrm{Gr}_{\vec{\nu}}, \mathrm{Gr}_{\vec{\lambda}}) \longrightarrow \mathrm{Hom}(\mathrm{Gr}_{\vec{\mu}}, \mathrm{Gr}_{\vec{\lambda}})$$

is given by intersection product of alg cycles

$$\mathrm{Gr}_{\vec{\mu}} \otimes \mathrm{Gr}_{\vec{\nu}} - \mathrm{Gr}_{\vec{\mu}\vec{\nu}}$$

<u>Thm</u> (Geometric Satake)

$$\mathrm{Sat}^{M} (= \text{Idem. completion of } \mathrm{Sat}^{\circ}_{G}) \simeq \mathrm{Rep}(GL_2)$$

$$\mathrm{Gr}_1 \longleftrightarrow \mathrm{Std}$$

$$\mathrm{Gr}_{-1} \longleftrightarrow \mathrm{Std}^*$$

$F = \mathbb{F}_q((\varpi))$, Lusztig - Drinfeld - Ginzburg - Mirković - Vilonen

$F = \mathbb{Q}_p$, Z.

$$\left(F = \mathbb{F}_p((\varpi)) \text{ case} \right) \xrightarrow[\text{Lusztig - Yun}]{\text{Lusztig}} \begin{matrix}\text{numerical result of} \\ \text{affine Hecke alg.}\end{matrix} \xrightarrow{Z.} F = \mathbb{Q}_p \text{ case}$$

<u>Talk 2</u>

$$\mathrm{Gr}_{1,1} = \{\Lambda_2 \subset \Lambda_1 \subset \Lambda_0 = \mathcal{O}^2\} \longleftrightarrow c : V \otimes V \simeq V \otimes V$$

$$v \otimes w \longmapsto w \otimes v$$

$$\mathrm{Hom}(\mathrm{Gr}_{1,1}, \mathrm{Gr}_{1,1})$$

$= \mathbb{Q}$- span of irred. comp. of

$$[\Delta] - [Z] \longleftrightarrow c$$

$$\left\{ \begin{matrix} \Lambda_2 & \subset \Lambda_1 & \subset \\ \| & & \Lambda_0 \\ \Lambda_2' & \subset \Lambda_1' & \subset \end{matrix} \right\} \subset \mathrm{Gr}_{1,1} \times \mathrm{Gr}_{1,1}$$

$$\Delta \cup'' Z = \{\Lambda_2 = \varpi \Lambda_0\}$$

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Gr}^{\circ}_{1,-1} = \{\Lambda_2 \supset \Lambda_1 \subset \Lambda_0\} \arrow[dl] \arrow[dr] & & & \mathrm{Gr}^{\circ}_{-1,1} \arrow[dl] \arrow[dr] & \\
\mathrm{Gr}_0 = \{\Lambda_0\} = \bullet & & \mathrm{Gr}_{1,-1} = \{\Lambda_2 \supset \Lambda_1 \subset \Lambda_0\} \arrow[r, "\sim"] & \mathrm{Gr}_{-1,1} = \{\Lambda_2 \subset \Lambda_1 \supset \Lambda_0\} & & \bullet
\end{tikzcd}$$

$\mathrm{Rep}(GL_2)$ $\qquad 1 \to \mathrm{Std} \otimes \mathrm{Std}^* \xrightarrow{c} \mathrm{Std}^* \otimes \mathrm{Std} \to 1 \qquad \dim \mathrm{Std} = 2$

Sat^m : self-intersection # of this (-2)-curve

$F \supset \mathcal{O} \ni \varpi$, $\qquad \mathbb{F}_p = \mathcal{O}/\varpi$

R perfect $\mathbb{F}_p$-alg. $\qquad W_{\mathcal{O}}(R) = W(R) \otimes_{W(k)} \mathcal{O} = \begin{cases} W(R), & F = \mathbb{Q}_p \\ R[[t]], & F = \mathbb{F}_p((t)) \end{cases}$

$\circlearrowright \sigma \qquad \circlearrowright \sigma$

$D_R = \mathrm{Spec}\, W_{\mathcal{O}}(R)$ "family of discs parametrized by Spec R"

$\mathcal{E}$ v.b. on D_R, $\quad {}^{\sigma}\mathcal{E} := \sigma^* \mathcal{E}$.

Def $\vec{\mu} = (1, -1, -1, 1, \ldots)$ A rk two $\mathcal{O}$-shtuka on Spec R is a chain

$$\mathcal{E}_r \overset{\subset 1}{\underset{\supset -1}{\dashrightarrow}} \mathcal{E}_{r-1} \cdots \to \mathcal{E}_0 \simeq {}^{\sigma}\mathcal{E}_r$$

$\mathrm{Sht}_{\vec{\mu}}(R) = \{$the cat. of all such shtukas on Spec R$\}$

Rmk i) Shtukas were invented by Drinfeld, in global function field setting, as generalization of elliptic modules.

Is there any analogue in no. field setting ?

ii) What we defined are shtukas w/ singularities at the closed pt $s \in D$. One can define those w/ singularities at $\eta \in D$, or even moving along D.

in mixed char., $\longrightarrow$ Breuil-Kisin module.

$\mathrm{Sht}_\mu(R)$ $\qquad\qquad\qquad\qquad \mu = 1$

$= \{ \mathcal{E} \hookrightarrow {}^\sigma\mathcal{E} \}$

$\overset{\text{Gabber}}{\simeq}$ $\{$ 1-dim'l p-divisible group of height $2 / \mathrm{Spec}\, R \}$ $\quad (F = \mathbb{Q}_p)$

$\vec{\mu} = (\mu_r, \dots, \mu_1)$ $\qquad\qquad \sigma(\vec{\mu}) := \{ \mu_{r-1}, \dots, \mu_1, \mu_r \}$

$\mathrm{Sht}_{\vec{\mu}}(R) = \{ \mathcal{E}_r \dashrightarrow \mathcal{E}_{r-1} \dashrightarrow \mathcal{E}_0 = {}^\sigma\mathcal{E}_r \}$

$\downarrow$ partial Frob. $\qquad\qquad \downarrow$

$\mathrm{Sht}_{\sigma(\vec{\mu})}(R)$ $\qquad \mathcal{E}_{r-1} \dashrightarrow \cdots \to \mathcal{E}_0 = {}^\sigma\mathcal{E}_r \to {}^\sigma\mathcal{E}_{r-1}$

$$\widetilde{\mathrm{Sht}}_\mu = \{ \mathcal{E}_r \to \cdots \to \mathcal{E}_0 \simeq {}^\sigma\mathcal{E}_r + \text{trivialization of } \mathcal{E}_0 \}$$

$$\begin{array}{ccc} & \widetilde{\mathrm{Sht}}_\mu & \\ \swarrow & & \searrow \\ \mathrm{Sht}_\mu & & \mathrm{Gr}_{\vec{\mu}} \end{array}$$

One can define

$$\mathrm{Hom}_{Sat^\circ}(\mathrm{Gr}_{\vec{\mu}}, \mathrm{Gr}_{\vec{\nu}}) \longrightarrow \mathrm{Corr}(\mathrm{Sht}_{\vec{\mu}}, \mathrm{Sht}_{\vec{\nu}})$$

$$\begin{tikzcd}
 & \mathrm{Sht}_{1,-1} \arrow[dl] \arrow[dr] & & & \mathrm{Sht}^\circ_{-1,1} \arrow[dl] \arrow[dr] & \\
\mathrm{Sht}_0 & & \mathrm{Sht}^\circ_{1,-1} \arrow[r] \arrow[d, equal] & \mathrm{Sht}_{-1,1} \arrow[d, equal] & & \mathrm{Sht}_0 \\
 & & \left\{ \begin{smallmatrix} & \Lambda_1 & \\ \cap & & \cap \\ \Lambda_2 & & \sigma\Lambda_2 \end{smallmatrix} \right\} \arrow[r, "\text{p. Frob.}"] & \left\{ \begin{smallmatrix} & \Lambda_1 & \\ \subset & & \cup \\ \Lambda_2 & & \sigma\Lambda_2 \end{smallmatrix} \right\} & &
\end{tikzcd}$$

<u>Meta theorem</u>. (V. Lafforgue)

$$1 \longrightarrow \mathrm{Std} \otimes \mathrm{Std}^* \xrightarrow{\gamma \times 1} \mathrm{Std} \otimes \mathrm{Std}^* \longrightarrow 1$$

γ is an elt in GL_2

<u>Notation</u>

$W = V_1 \otimes \cdots \otimes V_2$ rep'n of GL_2 $(\hat{G})$

$\rightsquigarrow \mathrm{Gr}_W \rightsquigarrow \mathrm{Sht}_W$

[In general, G_{grp} is non-split (unramified), $\sigma \curvearrowright \hat{G}$, $\sigma(V)$ is the rep twist of V by σ ($G = GL_2$, σ acts trivially)]

<u>Construction</u>.

$a \in \mathrm{Hom}(V^* \otimes \sigma(V), W)$

$$\begin{tikzcd}
 & \mathrm{Sht}^\circ_{V\otimes V^*} \arrow[dl] \arrow[dr] & & & \\
\mathrm{Sht}_0 & & \mathrm{Sht}_{V\otimes V^*} \arrow[r, "\text{p. Frob}"] & \mathrm{Sht}_{V^*\otimes\sigma(V)} \arrow[r, bend left, "\mathrm{Corr}"] & \mathrm{Sht}_W
\end{tikzcd}$$

$$\operatorname{Hom}\left(V^* \otimes \sigma(V), W\right) \overset{\Xi}{\rightsquigarrow} \Xi(a) : \hat{G} \to W, \quad \hat{G}\text{- equiv.}$$

$\hat{G} \circlearrowleft$ $\hat{G}$ σ-conjugacy

$$\overset{\ell}{\rightsquigarrow} \operatorname{Cor}\left(\mathrm{Sht}_0, \mathrm{Sht}_W\right)$$

$$\Xi(a)(g) = \left[1 \longrightarrow V \otimes V^* \xrightarrow{(g \times \sigma) \otimes 1} V^* \otimes \sigma(V) \xrightarrow{a} W \right]$$

<u>Meta theorem</u>.

$$\ell(a) = \Xi(a) \in \left(\mathcal{O}_{\hat{G}} \otimes W\right)^{\hat{G}}$$

<u>Ex</u>. $G = GU(3)/\mathbb{Z}_p$, $W = \Lambda^2 \mathrm{Std}$, G unitary similitude group of signature $(1,2)$ assoc. to $\underbrace{E|\mathbb{Q}}$ quadratic imaginary

p unramified prime

$$\begin{tikzcd}[column sep=small]
 & \left\{ \begin{smallmatrix} A \to A' \\ A \in \mathrm{Sh}_K + \cdots \\ A' \in \mathrm{Sh}_{K'} \end{smallmatrix} \right\} \arrow[dl] \arrow[dr] \arrow[d] & \\
\mathrm{Sh}'_K \arrow[dd] & \ell(a) \arrow[dl] \arrow[dr] & \mathrm{Sh}_K \arrow[dd] \\
 & & \\
\mathrm{Sht}_0 & & \mathrm{Sht}_W
\end{tikzcd}$$

$K \subset G(\mathbb{A}_f)$, $\mathrm{Sh}_K = \left\{ \begin{array}{l} (A, \iota : \mathcal{O}_E \otimes \mathbb{Z}_{(p)} \to \operatorname{End} A \\ \lambda : A \to A^\vee \\ \eta \text{ level str.} \end{array} \right.$ satisfying Kottwitz signature condition $\Big\}$ mod p fiber

$$\begin{tikzcd}
\mathrm{Sh}_K \arrow[d, "\text{fake } p\text{-div. gp}"'] \\
\mathrm{Sht}_W
\end{tikzcd}$$

Thm (Liang Xiao, Z.)

$H^* e(a)$:

$$C\left({}_{G'(\mathbb{Q})}\backslash G'(\mathbb{A}_f)/K\right) \to H_c^2(Sh_K, \mathcal{O}_\ell(1))$$

this is Hecke equivariant, given by $\Xi(a)$.

More precisely:

$$H^* e(a): C\left(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K\right) \underset{\mathcal{O}_{\hat{G}}^{\mathrm{Int}_\sigma \hat{G}}}{\otimes} \underset{\substack{\downarrow \\ \Xi(a)}}{\left(\mathcal{O}_{\hat{G}} \otimes W\right)^{\mathrm{Int}_\sigma \hat{G}}} \longrightarrow H_c^2(Sh_K)$$

s.t.

$$H^{\cdot} e(a)^{[\pi_f]} \times H^{\cdot} e(a)^{[\pi_f]} : \tilde{W}^{[\pi_f]} \otimes \tilde{W}^{[\pi_f]} \longrightarrow H_c^2(Sh_K) \times H_c^2(Sh_K)$$

$$\big\downarrow \text{intersection}$$

$$\Xi(a)\left(\mathrm{rec}(\pi_{f,p})\right) \times \Xi(a)\left(\mathrm{rec}(\pi_{f,p})\right)$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \\ W & \otimes & W & \xrightarrow[\hat{G}\text{-equiv.}]{} & \mathcal{O}_\ell \end{array}$$