# Deformation theory and a theorem of Mori

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Fix an algebraically closed field $k$.

<u>Def</u>. A <u>Fano manifold</u> $X/k$ is a smooth, conn'd proper variety s.t.

$\det(T_X) = \omega_X^{-1}$ is ample.

<u>Examples</u> (1) curves $\to \mathbb{P}^1$

(2) surfaces $\to$ Del Pezzo surfaces — $\mathbb{P}^1 \times \mathbb{P}^1$ / $\mathbb{P}^2$ / blow up of $\mathbb{P}^2$ at $n \leq 8$ general pts

(3) $X_d \subset \mathbb{P}^n$ a smooth hypersurface of degree $d$ is Fano if $d \leq n$.

<u>Theorem</u> (Mori) Every Fano mfd of dim $\geq 1$ is "uniruled": every closed pt $p$ of $X$ is contained in the image of finite morphism $f: \mathbb{P}^1 \to X$.

<u>Obstruction theory</u>.

Let $R$ be a local, complete, noetherian ring. w/ alg. closed residue field $k$.

Let $\mathcal{C}_R$ = cat. of local, artinian $R$-algs, residue field $k$.

<u>Infinitesimal ext'n</u>. $A' \xrightarrow{q} A$ s.t. $\ker(q) = N$ with $m_{A'} \cdot N = 0$.

— a f-dim'l $k$-vec. sp.

$$\begin{tikzcd}
\Sigma: \arrow[d, "u"] & 0 \arrow[r] & N \arrow[r] \arrow[d, "u_N"] & A' \arrow[r] \arrow[d, "u_{A'}"] & A \arrow[r] \arrow[d, "u_A"] & 0 \\
\tilde{\Sigma}: & 0 \arrow[r] & \tilde{N} \arrow[r] & \tilde{A}' \arrow[r] & \tilde{A} \arrow[r] & 0
\end{tikzcd}$$

Let $F : \mathcal{C}_R \to$ Sets be a functor w/ $F(k) = \{\cdot\}$

Def. situation $(\Sigma, x \in F(A))$

Obstruction theory $(\mathcal{O}, \omega)$

$\mathcal{O}$ a f.dim'l $k$-vec. sp.

$(N \mapsto N \otimes_k \mathcal{O})$

$\omega$ is a rule $(\Sigma, x) \mapsto \omega_{\Sigma, x} \in N \otimes_k \mathcal{O}$.

(i) suitably natural, i.e.

$\forall\ u : (\Sigma, x) \to (\tilde{\Sigma}, \tilde{x})$,

$u : (\Sigma, x) \to (\tilde{\Sigma}, \tilde{x})$, $u : \Sigma \to \tilde{\Sigma}$ s.t.
$u_*(x) = \tilde{x}$

the image of $\omega_{\Sigma, x}$ under $N \otimes_k \mathcal{O} \to \tilde{N} \otimes_k \mathcal{O}$ is $\omega_{\tilde{\Sigma}, \tilde{x}}$.

(ii) $\omega_{\Sigma, x}$ equals $0$ iff $x$ is the image of an elt $x' \in F(A')$.

Exer. (i) $\Rightarrow$ "if" in (ii).

Examples. (1) Let $F = h_S$, $S = R[[x]]/I = \widehat{R[x_1, \dots, x_2]} / \langle f_1, \dots, f_s \rangle$.

$$I/I^2 \to \hat{\Omega}_{R[[x]]/R} \otimes_{R[[x]]} S$$
$$\shortparallel$$
$$R[[x]]\{\text{"}dx_1\text{"}, \dots \text{"}dx_2\text{"}\}$$

$$\operatorname{Hom}_{R[[x]]}(\hat{\Omega}_{R[[x]]/R}, k) \to \operatorname{Hom}_S(I/I^2, k)$$

$$(\phi : dx_i \mapsto c_i) \mapsto \left(f_j \mapsto \sum_{i=1}^{2} \overline{\frac{\partial f_j}{\partial x_i}}\, c_i\right)$$

$\mathcal{O} :=$ Coker ( ———— )

$$\begin{tikzcd}
 & & R[[x]] \arrow[d] \arrow[dd, dashed, bend right, "v'"'] \arrow[ddr, bend left, "v"] & & \\
 & & S \arrow[dr, "w"] & & \\
0 \arrow[r] & N \arrow[r] & A' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$\exists$ $R$-alg. homo.

$$v' : R[[x]] \longrightarrow A'$$

Every lift is of the form $v' + \partial$,

$$\partial : \text{"}dx_i\text{"} \longmapsto \text{elt in } N$$

$$\phi : \hat{\Omega}_{R[[x]]/R} \longrightarrow N$$

$v' + \partial$ factors through $S$ iff $f_1, \dots, f_s \longmapsto 0$.

$\rightsquigarrow$ $S$-module homomorphism $I/I^2 \longrightarrow N$

$$f_j \mapsto (v' + \partial)(f_j)$$

<u>Upshot</u>: elt $w \in \operatorname{Hom}_S(I/I^2, N) \big/ \operatorname{Hom}_{R[[x]]}(\hat{\Omega}, N)$ is independent of the choice of $v'$.

$$\left( \operatorname{Hom}(I/I^2, k) \big/ \operatorname{Hom}(\hat{\Omega}, k) \right) \otimes_k N.$$

This obstruction vanishes iff $w$ extends to an $R$-alg hom. $S \longrightarrow A'$.

<u>Example</u> Let $C$ be a smooth, proj. conn'd curve $/k$, Let $Z \subset C$ be an effective Cartier divisor. Let $X$ be a smooth $k$-scheme. Let $f_0 : C \to X$ be a $k$-morphism.

Denote $g := f_0|_Z : Z \to X$.

$R = k$. $\quad F : A \longmapsto \left\{ f_A : C \otimes_k A \longrightarrow X \otimes_k A \ \text{ s.t. (i) } f \equiv f_0 \bmod m_A \text{, (ii) } f|_{Z \times \operatorname{Spec} A} = g \times \operatorname{Id}_{\operatorname{Spec} A} \right\}$

(with $f_A$ over $\operatorname{Spec} A$)

<u>Obstruction theory</u>. $\mathcal{O} = H^1(C, f^* T_X \otimes I_Z)$

Given a deformation situation

$$0 \to N \to A' \to A \to 0. \qquad f_A : C_A \to X_A$$

Let $U \subset C$ be an affine open. $f = f_A = f_{A'}$ as maps of top. spaces

$$\begin{tikzcd} Z_A \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & Z_{A'} \arrow[d, phantom, "\cap"] \\ U_A \arrow[r, hook] \arrow[dr, "f_A|_{U_A}"'] & U_{A'} \arrow[d, dashed, "\tilde{f}_{A,u}"] \\ & X \end{tikzcd}$$

$\leftarrow$ NOT unique

$$\mathcal{O}_X \to f_* \mathcal{O}_{U_A}$$

Other choices differ by a derivation $\Omega_X \to N \otimes f_* \mathcal{O}_{U_A} \otimes I_Z$

$\{U_\beta\}$ open affine cov. of $C$.

For every $\beta$, choose $\tilde{f}_{A, U_\beta}$

On $U_\beta \cap U_\gamma$, $\tilde{f}_{A,U_\beta}|_{U_\beta \cap U_\gamma} \to \tilde{f}_{A,U_\gamma}|_{U_\beta \cap U_\gamma}$

is an elt in $\Omega_X \to N \otimes f_* \mathcal{O}_{U_\alpha \cap U_\beta} \otimes I_Z$

$$\rightsquigarrow \omega_{\Sigma, f} \in \check{H}^1(C_A, \mathcal{H}om_{\mathcal{O}_{C_A}}(f_A^* \Omega_X, N \otimes_k I_{Z_A}))$$

$$= H^1(C, \mathcal{H}om_{\mathcal{O}_C}(f^* \Omega_X, N \otimes_k I_Z))$$

$$= N \otimes_k H^1(C, f^* T_X \otimes I_Z)$$

<u>Fact</u>. Let $F = h_S$ be a pro-representable functor on $\mathcal{C}_R$.

Let $\mathcal{O}_{can} = \mathrm{Hom}(I/I^2, k) / \mathrm{Hom}(\hat{\Omega}_{R[[X]]/R}, k)$

Let $\mathcal{O}$ be any other obstruction theory, then $\exists!\ \psi : \mathcal{O}_{can} \to \mathcal{O}$ s.t. every

$w_{\Sigma, x} =$ image $w_{\Sigma, x, can}$ under $\psi$.

$$\begin{tikzcd}
0 \arrow[r] & I/mI \arrow[r] \arrow[d] & R[[X]]/m_{R[[X]]} I + m^c \arrow[r] \arrow[d, dashed] & R[[X]]/I + m^c_{R[[X]]} = S/m_S^c \arrow[d, "x"] & \\
0 \arrow[r] & N \arrow[r] & A' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

(hypothesis, $I \subset m^2_{R[[X]]}$)

$m^c_{A'} = 0$

And $\psi$ is injective.

$$\begin{tikzcd}
w \in I/m_{R[[X]]} I \otimes_k \mathcal{O} \arrow[d, mapsto] \arrow[d] \\
0 \in k \otimes_k \mathcal{O}
\end{tikzcd}$$

It does not extend after taking pushout.

$I/m_{R[[X]]} I \leftarrow$ free $k$-vec. space w/ basis the images of a minimal set of gen's for $I$.

$S = R[[x_1, \ldots, x_r]] / \langle f_1, \ldots, f_s \rangle$

$t_F = \text{"}dx_1\text{"}^\vee, \ldots, \text{"}dx_r\text{"}^\vee$

$\dim_k t_F = r =$ min # of gen's,

$\dim_k \mathcal{O} \geq s =$ min # of relations

$\dim S \geq \dim R + \dim_k t_F - \dim_k \mathcal{O}$

Example For

$$\begin{tikzcd} C \arrow[r, "f_0"] & X \\ Z \arrow[u, phantom, "\cup"] \arrow[ur, "g"'] & \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, phantom, "\subset"] & C_A \arrow[r, "f_A"] & X \\ Z \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] & Z_A \arrow[u, phantom, "\cup"] \arrow[ur, "g_A"'] & \end{tikzcd}$$

$\mathcal{O} = H^1(C, f^* T_X \otimes I_Z)$

$t_F = H^0(C, f^* T_X \otimes I_Z)$

$\dim S \geq \dim R + r - s \geq \dim R + \dim t_F - \dim \mathcal{O}$

$\dim S/m_R S \geq \dim t_F - \dim \mathcal{O}$ and equality $\Rightarrow$ $S$ is $R$-flat (miracle flatness)

$\dim S \geq (h^0 - h^1)(C, f^* T_X \otimes I_Z)$

$$\overset{\text{R.R.}}{=} \deg(f^* T_X) + \dim(X)\left(1 - g(C) - \deg(Z)\right)$$

$$\begin{array}{c} \deg(f^* T_X) \\ \shortparallel \\ \deg(f^* \det(T_X)) \end{array}$$

$\deg f^* \mathcal{O}_X(1)$

If this dim. is positive, then $\dim_{[f_0]} \mathrm{Hom}^{\text{degree}}(C, X; g: Z \to X)$ is positive.

$\uparrow$

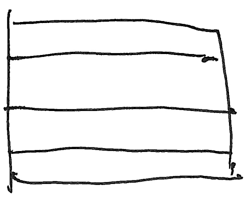
q-proj.

So there is an affine curve $[f_0] \in B \subset \mathrm{Hom}$

$$\begin{tikzcd} \bar{B} \times C \arrow[rr, dashed, bend left, "\tilde{f}"] & B \times C \arrow[l, phantom, "\supset"] \arrow[r, "f"] & X \\ & B \times Z \arrow[u, phantom, "\cup"] \arrow[r, "pr_Z"'] & Z \arrow[u, "g"'] \end{tikzcd}$$

$\bar{B} \times C$ ?

Rigidity Lemma




If $\tilde{f}$ is regular on $B \times \{p_i\}$, which is contracted, then $\tilde{f}$ factors through $\bar{B} \times C \xrightarrow{pr_C} C$.

In pos. char., use Frob. to make dim positive!