Some applications of affine Springer theory

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<u>Lecture 1</u> Geom. RT

RT $\longleftrightarrow$ Alg. Geom.

study representations / characters of 1) finite groups $G = \underline{G}(\mathbb{F}_q)$

$\underline{G}$ - reductive alg gp / $\mathbb{F}_q$

Eg. $G = GL(n,q) = GL_n(\mathbb{F}_q)$, $SO(n, \mathbb{F}_q)$, $Sp(2n, \mathbb{F}_q)$

2) locally cpt gp $\underline{G}(\mathbb{F}_q((t)))$ (or $G = \underline{G}(F)$, $F/\mathbb{Q}_p$ - finite) (local Langlands conjectures)

loop group

geometry in 1) Springer fibers — classical

2) affine Springer fibers (recent / in progress)

Character sheaves

3) Categorification of (affine) Springer theory

<u>Springer fibers</u>

$G/_k$ $\mathcal{B} := G/B$, B - Borel. flag space

B = upper $\triangle$ matrices

$G = SL_n / GL_n$ $GL_n = GL(V)$

$\mathcal{B} = \{(V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V)$ V_i vector subspace in $V = k^n\}$

For $g \in G$, $\mathcal{B}_g$ - fixed pts of g acting on $\mathcal{B}$

- projective variety

If $g = u \in \mathcal{U}$ - set of unipotent elts in G,

$\mathcal{B}_u$ - Springer fiber (or $\mathcal{B}_e$, $e \in \mathcal{N} \subset \mathfrak{g}$)

Ex. $e=0$, $u=1$, $\mathcal{B}_u = \mathcal{B}_e = \mathcal{B}$

u regular unipotent (1 Jordan block), $\mathcal{B}_u = pt$

$G = \mathbb{G}(\mathbb{F}_q)$. A rep'n of G can be constructed by parabolic induction.

$$B \to \underset{\substack{| \\ \text{diagonal matrices}}}{T} \quad , \theta : T \to \mathbb{C}^\times$$

$\mathrm{ind}_B^G(\theta) =: I_\theta$

Fact. 1) I_θ is irreducible for most θ

2) for $\theta = 1$, $\mathrm{End}(I_\theta) \simeq \mathbb{C}[W]$

$$\begin{array}{c} \| \\ \mathbb{C}[B \backslash G/B] \end{array}$$

$I_1 = \mathbb{C}[G/B]$.

$G = GL_n$, $W = \mathfrak{S}_n$, $\rho \in \mathrm{Irr}(\mathfrak{S}_n)$, get an irr. $I(\rho)$ of G,

$$I(\rho) = \mathrm{Hom}_W(\rho, I_1)$$

The character of I_θ can be computed by Frobenius formula, in particular,

for $u \in \mathcal{U}$ - unipotent element, $\chi_{I_\theta}(u) = \# \{ \gamma \in G/B : \underset{u \in \gamma B \gamma^{-1}}{\gamma^{-1} u \gamma \in B} \}$ $\quad \gamma^{-1}u\gamma \mapsto 1 \in T$

$= \# \mathcal{B}_u(\mathbb{F}_q)$

e.g. $\dim(I_\theta) = \# \mathcal{B}(\mathbb{F}_q) = \sum_{w\in W} q^{\ell(w)}$

$G = SL_2$, $1+q$

$G = SL_3$, $1+2q+2q^2+q^3$

Rmk $\# \mathcal{B}_u(\mathbb{F}_q)$ is given by Weil conjecture as $\mathrm{Tr}(Fr, H^*_{\acute{e}t}(\mathcal{B}_u))$.

e.g. for $u=1$, $G = GL_2$, $\mathcal{B} = \mathbb{P}^1$, $H^*(\mathbb{P}^1) = \underset{1}{\underset{\circlearrowleft}{H^0(\mathbb{P}^1)}} \oplus \underset{q}{\underset{\circlearrowleft}{H^2(\mathbb{P}^1)}}$

$H^*(\mathcal{B}) = \mathbb{C}[\mathfrak{t}] / (\mathbb{C}[\mathfrak{t}]^W_+)$

GL_n, $\mathbb{C}[t_1, \dots, t_n] / (\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}_+)$

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reg. rep'n of $\mathfrak{S}_n$, $\dim = n!$

Rmk $GL(2,q)$ also has complementary series

indexed by characters of $\mathbb{F}_{q^2}^\times$, have dim $q-1$.

Fact. The group W ($\mathfrak{S}_n$ for GL_n) acts on $H^*(\mathcal{B}_u)$ Springer action.

Be

Eg. $u=1$, $\mathcal{B}_u = \mathcal{B} = G/B \xleftarrow{U} G/T$

(over $\mathbb{C}$) $U \simeq \mathbb{C}^{n(n-1)/2}$

$B = T\cdot U$

$\mathbb{C}^N$ - contractible

$H^*(G/B) = H^*(G/T)$, $W = N_G(T)/T$ acts on G/T on the right.

over $\mathbb{C}$: can also identify $G/B(\mathbb{C}) \simeq K/C$, $G = GL_n$, $U_n / \prod_{(S^1)^n}$, $W = N(C)/C$

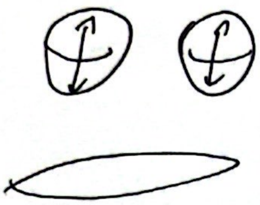
e.g. $G = SL_2$.

In general, $W = \langle s_i \rangle$, $G = GL_n$, $s_i = (i, i+1)$

$$\begin{array}{c} \mathcal{B} \\ \downarrow \mathbb{P}^1 \end{array}$$

fixing K, get a metric

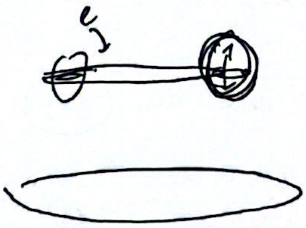
$$P_i = \{(V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq \underbrace{V_{i-1}}_{\dim i-1} \subsetneq \underbrace{V_{i+1}}_{\dim i+1} \subsetneq \cdots)\}$$



Generalization: topological construction of Springer action (KL)

$$\begin{tikzcd} \mathcal{B}_e \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{B} \arrow[d, "\pi_i"] \\ P_{e,i} & P_i \end{tikzcd}$$

$\pi_i^{-1}(P_{e,i})$



$\lim_{t \to \infty} b_t(e)(Z)$ - cycle in $\mathcal{B}_e$

$Z \subset \mathcal{B}_e$

$e \in \mathfrak{sl}_n$, e Jordan type $(1, n-1)$

$\mathcal{B}_e \supset L_i \simeq \mathbb{P}^1$.

fix $V_0, \cdots, V_{i-1}, V_{i+1}, \cdots, V_{n-1}$

$n-1$ lines

$s_i[L_i] = -[L_i]$, $s_i[L_j] = L_j$, $j \neq i \pm 1$

$s_i[L_{i\pm1}] = [L_{i\pm1}] + [L_i]$

The same action can be defined in G.

The input is always

$$\begin{array}{l} \tilde{G} \subset G \times B \xrightarrow{\pi} G \\ \shortparallel \\ \{(g,x) : g(x) = x\} \qquad , \quad \mathcal{B}_g = \pi^{-1}(g) \end{array}$$

$G \supset G^{reg}$ - matrices w/ unique Jordan block $\forall$ eigenvalue.

$$\begin{array}{l} W \curvearrowright \tilde{G}^{reg} \\ \quad \downarrow \quad - \quad \text{ramified } W:1 \text{ cover} \\ \quad G^{reg} \end{array}$$

Springer action is obtained by degenerating this action.

<u>Facts</u> $\mathcal{B}_u$ - connected equidimensional variety.

$u \in \mathcal{U}$, $\dim(\mathcal{B}_u) = \frac{1}{2} \operatorname{codim}_{\mathcal{U}} (G(u))$

orbit of u

$H^{i}(\mathcal{B}_u) = 0$ for i odd

Every irr. rep'n of W appears in $H^{top}(\mathcal{B}_u)$ for a unique u (up to conj'n)

For SL_n, this is in bijection w/ (σ_n) & unipotent conj. classes.

Deligne & Lusztig: defined a virtual rep'n $R_{T,\theta}$ for any max'l torus T,

character $\theta : T \to \mathbb{C}^\times$

e.g. $T = \mathbb{F}_{q^2}^\times \subset GL(2, \mathbb{F}_q)$, this is the complementary series

$\chi_{R_{T,\theta}}(u)$ is indep. of θ, for generic θ, $R_{T,\theta}$ is irreducible

Eg. $T = \mathbb{F}_{q^{n_1}}^\times \times \mathbb{F}_{q^{n_2}}^\times \times \dots \times \mathbb{F}_{q^{n_t}}^\times$, $\sum n_i = n$ (GL_n)

over $\overline{\mathbb{F}_q}$, T is conj. to the standard diag torus T_0

$$T = g T_0 g^{-1}$$

$$T = Fr(g) T_0 Fr(g^{-1})$$

$T \hookrightarrow$ conj. classes in W

$g^{-1} Fr(g) : T_0 \to T_0$ defines $w \in W$ $\quad T = T_w$

$\dim(R_{T,\theta}) = \pm Tr(w_0 Fr, H^*(G/B))$

Eg. for $G = GL_2$, $\mathbb{F}_{q^2}^\times \longleftrightarrow s \in \mathfrak{S}_2$

Rmk. $T = T_0$, $R_{T,\theta} = I_\theta$.

$$\begin{array}{cc} H^0 & H^2 \\ \circlearrowleft & \circlearrowleft \\ q & -1 \end{array}$$

$$1 - q = -\dim R_{T,\theta}$$

$$\chi(R_{T,\theta})(u) = Tr(Fr \cdot w, H^*(B_u))$$

all these $H^*(B_u)$ are stalks of an ℓ-adic complex.

$\underline{X}$ - var. $/\mathbb{F}_q$, $X = \underline{X}(\mathbb{F}_q)$, $\mathbb{C}[X]$

Interesting functions are often of the form $\varphi_{\mathcal{F}}(x) = Tr(Fr, \mathcal{F}_x)$, $x \in X(\mathbb{F}_q)$

$\mathcal{F}$: ℓ-adic cpx, $\mathcal{F}_x$ - stalk

Analogy. $X/\mathbb{C}$,

- local system on $X(\mathbb{C})$

$\sigma: X \to X$ autom.

L - equiv., $\sigma^*(L) \simeq L$.

$\forall x \in X^\sigma$, $L_x \xrightarrow[\sigma_x]{\sim} L_x$

$\phi(x) = Tr(\sigma_x)$

More generally, $\mathcal{F}$ can be a constructible sheaf or a complex of such.

then $\phi(x) = \sum (-1)^i \operatorname{Tr}(\phi_x, H^i(\mathcal{F}_x))$.

Now replace $X(\mathbb{C})$ by $X(\overline{\mathbb{F}_q})$

$\sigma \rightsquigarrow$ Frobenius. $X^{\phi} = \underline{X}(\mathbb{F}_q)$

$\mathcal{F} \in$ constructible complex of ℓ-adic sheaves on $X_{\mathbb{F}_q}$

for $f: \underline{X} \to \underline{Y}$, can pull back f^*

pushforward $f_!$

compatible w/ maps of functions Weil conjecture

Among all ℓ-adic complexes, there are perverse sheaves.

A local system on a smooth variety is perverse up to a shift.

$\pi: X \to Y$ proper map, $\mathcal{F}$ perverse irreducible on X, $\pi_!(\mathcal{F})$ is a sum of irred. perverse sheaves w/ shifts.

<u>Fact</u>. $G = GL_n$. the character of every irrep ρ of $G = GL(n,q)$

$\chi_\rho = \phi_{\mathcal{F}_\rho}$, $\mathcal{F}_\rho$ - irreducible perverse sheaf (character sheaf)

(Lusztig)

<u>$n=1$</u>. $GL(1) = \mathbb{F}_q^\times$ is abelian, irreps are 1-dim char.

$\chi(xy) = \chi(x)\chi(y)$. $m^*(\chi) = \chi \boxtimes \chi$

$\underline{G} \times \underline{G} \xrightarrow{m} \underline{G}$ $\chi = \phi_{\mathcal{L}_\chi}$, $m^*(\mathcal{L}_\chi) \simeq \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi$ satisfying a cocycle cond.

$\mathcal{L}_\chi$ local system

$\mathbb{G}_m \xrightarrow{K} \mathbb{G}_m$ $\qquad$ $K_*(\text{const}) \supseteq \mathbb{F}_q^\times$

$z \longmapsto z^{q-1}$ $\qquad$ $\theta : \mathbb{F}_q^\times \to \mathbb{C}^\times$

$\ker \simeq \mathbb{F}_q^\times$ $\qquad$ $K_*(\text{const})_\theta = \mathcal{L}_\theta$

Lecture 2. More on (irreducible) perverse sheaves

GM intersection cohomology

H^*, IH^* satisfies PD

$H^*(X, \mathbb{C}) = \pi_*(\underbrace{\underline{\mathbb{C}}}_{\text{const. sheaf}})$

$\pi : X \to pt$

$IH^*(X, \mathbb{C}) = \pi_*(\underbrace{IC}_{\text{intersection coh. sheaf}})$.

IC – unique irreducible perverse sheaf. constant on a dense open

Normalization : rmk. Shift degrees s.t. PD takes form $H^i \simeq (H^{-i})^*$.

$IC|_{\underset{\text{open dense}}{U}} = \underline{\mathbb{C}}[\dim X]$ $\quad(*)$

Now IC is characterized by $\quad \dim \operatorname{supp}(\mathcal{H}^i(IC)) < -i$ (except $i = -\dim X$)

$\dim \operatorname{supp} \mathcal{H}^i(D(IC)) < -i$ (except $i = -\dim X$)

D: Verdier duality (localized PD)

Replace $<$ by $\leq$, get def'n of perverse sheaf.

Back to $G = \underline{G}(\mathbb{F}_q)$, We defined rep'n I_θ of dim $\#(G/B) = [n!]_q$. $G = GL_n$

It's easy to write χ_{I_θ} as $\phi_{\mathcal{F}_\theta}$.

Frobenius induction formula says that

$$\begin{tikzcd} & \tilde{G} \arrow[dl, "\delta"'] \arrow[dr, "\pi"] & \\ T & & G \end{tikzcd}$$

$$\chi_{I_\theta} = \pi_* \delta^* (\theta)$$

(here $\pi_* = \pi_!$)

$$\theta = \underset{\substack{| \\ \text{trace of } Fr.}}{\phi} (\mathcal{L}_\theta)$$

$$\mathcal{F}_\theta := \pi_* \delta^* (\mathcal{L}_\theta)$$

<u>Fact</u>. (Cor. of dim. formula for $\mathcal{B}_u$) π is small

<u>Def</u>. A map $f: X \to Y$ is (semi) small if Y has a stratification $Y = \bigcup Y_i$,

s.t. for $y \in Y_i$, $\dim(f^{-1}(y)) < \frac{1}{2} \operatorname{codim}(Y_i)$ (except for open stratum Y_0) (small)

$\leq \frac{1}{2} \operatorname{codim}(Y_i)$ (semismall)

<u>Cor</u>. a) If X smooth irreducible, f onto semismall, proper, then $f_*(\underline{\mathbb{C}}[d])$ is perverse.

b) If f is small, birational, then $f_*(\underline{\mathbb{C}}[d]) = IC$

c) If f is small, $f_*(\underline{\mathbb{C}}[d]) = j_{!*}(\mathcal{L}[d])$

$\llcorner$ a local system on the open part.

replace $(*)$ by $\mathcal{F}|_U = \mathcal{L}[\dim X]$, $\mathcal{F} = j_{!*}(\mathcal{L}[d])$ $j_{!*}$: minimal (Gm) extn

<u>Cor</u>. $\mathcal{F}_\theta[d] = j_{!*}(\mathcal{L})$

$\llcorner$ a loc. system G^{rs} - reg. semisimple

$\tilde{G}^{rs} \to G^{rs}$ is a Galois W-cover.

This gives a way to <u>define</u> the action of W on $H^*(\mathcal{B}_u)$.

- uses that $j_{!*}$ is a functor.

$\pi^{rs} = \pi|_{\tilde{G}^{rs}} : \tilde{G}^{rs} \to G^{rs}$ $\quad$ W-cover, W acts by deck transformations

So it acts on $\pi^{rs}_*(\mathbb{C})$, hence on $j_{!*}(\pi^{rs}_*(\mathbb{C}[d])) = \mathcal{F}_1$

$H^*(\mathcal{B}_u) = \mathcal{F}_{\theta=1}|_u$

$W \curvearrowright \mathcal{F}_{\theta=1}$ decomposes it

$$\mathcal{F}_{\theta=1} = \bigoplus_{\rho \in \mathrm{Irr}(W)} j_{!*}(\mathcal{L}_\rho[d]^{\oplus \dim \rho})$$

$\mathcal{L}_\rho$: local system on G^{rs}.

$\mathcal{F}_\rho := j_{!*}(\mathcal{L}_\rho[d])$

Summary $I_{\theta=1} = \mathbb{C}[G/B]$

For $G = GL_n$, $\phi_{\mathcal{F}_\rho} = \pm \chi_{R\rho}$

$R\rho$: ps irr. rep. of $GL_n(\mathbb{F}_q)$

<u>NB</u>. This is not true for $G \neq GL_n$.

for $G = Sp(4)$, the dim. of the S-reps $\frac{q(q^2+1)}{2}$, $\frac{q(q^2-1)}{2}$, $\frac{q(q+1)^2}{2}$, 1, q^5

<u>Cor.</u> We have a canonical isom. $\mathcal{F}_\theta \simeq \mathcal{F}_{w(\theta)}$ $\quad w \in W$ $\quad (**)$

$$\mathcal{F}_\theta = j_{!*}(\pi^{rs}_*(s^*(I_\theta))), \qquad \mathcal{F}_{w(\theta)} = \pi^{rs}_*(s^*(\mathcal{L}_{w(\theta)}))$$

$\mathrm{Stab}_W(\theta)$ acts on $\mathcal{F}_\theta$

In particular, let $T \subset G$ be a nonsplit torus (e.g. $\mathrm{Res}_{\mathbb{F}_{q^n}|\mathbb{F}_q} \mathbb{G}_m \subset GL_n$)

$\theta : T \to \mathbb{C}^\times$

$\quad \overset{\shortparallel}{\underline{T}(\mathbb{F}_q)}$

$\mathcal{L}_\theta$ is still well-defined

$T \longleftrightarrow w_\circ \in W$

split torus

Over $\overline{\mathbb{F}}_q$, $\quad \underline{T}_{\overline{\mathbb{F}}_q} \simeq \underline{T}_{0,\overline{\mathbb{F}}_q}$

$\quad\quad\quad\quad\quad \circlearrowleft \quad\quad\quad \circlearrowleft$

$\quad\quad\quad\quad\quad \mathrm{Fr} \quad\quad\quad w_\circ \mathrm{Fr}_0$

$\mathrm{Fr}_0^*(\mathcal{L}_\theta) \simeq w^*(\mathcal{L}_\theta)$

over $\mathcal{U}$, standard $w_\circ \mathrm{Fr}_0$

Using $(**)$, we get $\mathcal{F}_\theta \simeq \mathrm{Fr}^*(\mathcal{F}_\theta)$

Using this Fr str., $\quad \phi_{\mathcal{F}_\theta} = \pm \chi_{R_{T,\theta}}$.

Let $G = SL_2(\mathbb{F}_q)$

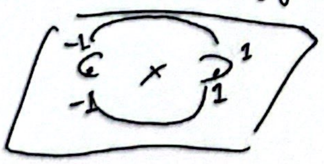
$\quad\quad \cup$

$\quad\quad T \simeq \mathbb{F}_q^\times$

$\quad\quad \theta^2 = 1,\ \theta \neq 1.$

This θ is also W-invariant. $\quad \mathcal{F}_\theta = \mathcal{F}_\theta^+ \oplus \mathcal{F}_\theta^-$. ($\mathbb{Z}/2 \curvearrowright$)

$\mathcal{F}_\theta^+$



$\mathbb{G}_m\ (/\mathbb{C})$

$\mathcal{L}_\theta$ is equiv. for $\overset{i:}{z \mapsto z^{-1}}$.

$\mathbb{G}_m \to \mathbb{G}_m$

normalize the isom.

$\mathcal{L}_\theta \simeq i^*(\mathcal{L}_\theta)$ to be 1 at 1

$\mathcal{F}_\theta^+$ is constant around $\underset{\text{unip.}}{\mathcal{U}}$, but looks like $j_{!*}(\mathrm{sgn})$ near $(-1)\mathcal{U}$.

$\mathcal{F}_\theta^-$ vice versa.

For an irr. rep ρ of $SL(2,q)$, (-1) acts by 1 or -1.

But $g \to -g$ swaps $\mathcal{F}_\theta^+$, $\mathcal{F}_\theta^-$.

On the other hand, $SL(2) \subset GL(2)$,

$$GL(2)/\mathbb{F}_q^\times \cdot SL(2) = \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \simeq \mathbb{Z}/2$$

acts on $SL(2)$ by outer automorphisms

$I_\theta = I_\theta^+ \oplus I_\theta^-$, swaps I_θ^+, I_θ^-.

$\theta^2 = 1$

E.g. $\underset{x}{\underset{\shortparallel}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}}$, $\underset{y}{\underset{\shortparallel}{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}}$ are $GL(2)$ but not $SL(2)$ conj. if $a \in \mathbb{F}_q$ - not a square

If $\mathcal{F}$ is a sheaf eqvt for $SL(2)$. Pick $g \in SL(2, \overline{\mathbb{F}}_q)$.

$g x g^{-1} = y$, $\quad Fr(g)\, x\, Fr(g)^{-1} \overset{?}{=} y$, $\quad g^{-1} Fr(g) \in Z_{SL(2)}(x)$

$$= \left\{ \begin{pmatrix} \pm 1 & t \\ & \pm 1 \end{pmatrix} \right\}$$

$g^{-1} Fr(g) = \begin{pmatrix} -1 & \underbrace{t}_{\text{doesn't matter}} \\ 0 & -1 \end{pmatrix}$

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow[\sim]{g} & \mathcal{F}_y \\ \circlearrowright & & \circlearrowleft \\ Fr_x & & Fr_y \end{array}$$

$Fr_y = \alpha_x \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} Fr_x$

$\alpha_x : Z(x) \to \mathrm{Aut}(\mathcal{F}_x)$

In particular, $\mathcal{F}_x$ is 1-dim'l , $\phi_{\mathcal{F}}(y) = \pm \phi_{\mathcal{F}}(x)$

$\phi_{\mathcal{F}}(x) = \phi_{\mathcal{F}}(y)$, $\mathcal{F} = \mathcal{F}_\theta^\pm$.

$\mathcal{U}_0 = SL(2)/Z(u)$, $\quad Z(u) = \left\{ \begin{pmatrix} \pm 1 & t \\ & \pm 1 \end{pmatrix} \right\}$ $\quad$ Carries an order 2

regular unipotent orbit

rk 1 loc. sys. $\mathcal{S}$,

$j : \mathcal{U}_0 \to \mathcal{U}$ $\quad j_!(\mathcal{S}) \xrightarrow{\sim} j_*(\mathcal{S})$

$(-1)^* \mathcal{S}$

deg

$$\chi_{I_\theta^+} = \frac{1}{2}\left(\phi_{F_\theta^+} + \phi_{F_\theta^-} + \phi_S + \phi_{(-1)^*S} \right) \qquad \frac{q+1}{2}$$

$$\chi_{I_\theta^-} = \frac{1}{2}\left(\phi_{F_\theta^+} + \phi_{F_\theta^-} - \phi_S - \phi_{(-1)^*S} \right) \qquad \frac{q+1}{2}$$

Rmk. $\frac{1}{2}\left(\phi_{F_\theta^+} - \phi_{F_\theta^-} + \phi_S - \phi_{(-1)^*S} \right) \qquad \frac{q-1}{2}$

$\frac{1}{2}\left(\phi_{F_\theta^+} - \phi_{F_\theta^-} - \phi_S + \phi_{(-1)^*S} \right) \qquad \frac{q-1}{2}$

are also irr. characters $\quad I'^{+}_{\theta'} \oplus I'^{-}_{\theta'}$

summands of a complementary series rep'n of $GL(2)$ restricted to $SL(2)$.

Next: RT of $G = \underline{G}(\underset{F}{\overset{\|}{\mathbb{F}_q((t))}})$.

local Langlands conj. L-packets & endoscopy

$$[\text{irrep. of } G] \longleftrightarrow [\, \mathrm{Gal}(F) \xrightarrow{\rho} \check{G} \,]$$

less roughly. + an irrep. of $\pi_0(\mathrm{Stab}_{\check{G}}(\rho))$

Lecture 3. Correction . $Sp(4,q)$ dim of PS irr. reps

$$1,\ q^4,\ \frac{q(q^2+1)}{2},\ \frac{q(q^2+1)}{2},\ \frac{q(1+q)^2}{2}$$

non PS $\longrightarrow \frac{q(q-1)^2}{2}$

(cuspidal irr. rep

the characters are $\frac{1}{2}(\alpha+\beta+\gamma+\delta)$, $\frac{1}{2}(\alpha+\beta-\gamma-\delta)$, $\frac{1}{2}(\alpha-\beta+\gamma-\delta)$, $\frac{1}{2}(\alpha-\beta-\gamma+\delta)$

$\alpha, \beta, \gamma, \delta = \phi_{F_i}$, F_i - irr. perv. sheaf

Recall for $SL(2,q)$. we described 4 irred. sheaves S, $\begin{pmatrix}-1 & \\ & -1\end{pmatrix}^* S$, $\mathcal{F}_\theta^+$, $\mathcal{F}_\theta^-$

θ: quadratic char.

Half signed sums are irred. char of dim $\frac{q+1}{2}, \frac{q+1}{2}, \frac{q-1}{2}, \frac{q-1}{2}$.

More generally, for $SL(n)$, can define n^2 sheaves including

1) S_n^i - order n rank 1 loc. sys on reg. unip. matrices

1') translates of S_n^i by the center

2) summands of $\mathcal{F}_\theta$, θ order n

χ - order n char. of $\mathbb{F}_q^\times$.

$\chi_n = (1, \chi, \chi^2, \ldots, \chi^{n-1})$ - char. of $(\mathbb{F}_q^\times)^n \supset T$

$\theta = \chi_n|_T$ $\quad$ T max'l torus of SL_n

their combinations w/ coefficients c_i, $c_i^n = 1$ are irred. chars.

<u>General story (Lusztig)</u>

All irred reps are partitioned into families, to each such, there corresponds Γ.

If G is classical, $\theta=1$, $\Gamma = (\mathbb{Z}/2)^n$

$\mathrm{Irr}\left(Sh^\Gamma(\Gamma)\right) \longleftrightarrow (\gamma, \psi)/\sim,\ \gamma\in\Gamma,\ \psi\in \mathrm{Irr}\,(Z_\Gamma(\gamma))$

If Γ is abelian, then $Sh^\Gamma(\Gamma) = \Gamma\times\Gamma^\vee$- rep'n $\quad (\mathrm{Irr}\,(Sh^\Gamma(\Gamma)) = \mathrm{Irr}\ \Gamma\times\Gamma^\vee\text{-rep})$

elts in the family $\longleftrightarrow$ $\mathrm{Irr}\,(Sh^\Gamma(\Gamma))$.

In the terms, one describes linear combinations of chars given by $\phi_{\mathcal{F}}$
$\quad$ $\mathcal{F}$ irred. p. sheaf

$K(Sh^{\Gamma}(\Gamma)) = \mathbb{Z}[\mathrm{Irr}(Sh^{\Gamma}(\Gamma))]$

$\cap$
i

$K_{\mathbb{C}}(Sh^{\Gamma}(\Gamma)) \simeq \mathbb{C}[\mathrm{Com}(\Gamma)]^{\Gamma}$, $\quad \mathrm{Com}(\Gamma) = \{(\gamma_1, \gamma_2) : \gamma_1\gamma_2 = \gamma_2\gamma_1\}$

$\mathcal{F} \mapsto f_{\mathcal{F}}(\gamma_1, \gamma_2) = \mathrm{tr}(\gamma_2, \mathcal{F}_{\gamma_1})$

$i : (\gamma_1, \gamma_2) \mapsto (\gamma_2, \gamma_1)$

(non-abelian Fourier transform)

The other basis is i (basis of irr.)

families (for $\theta = 1$) $\longleftrightarrow$ 2-sided cells in W $\longleftrightarrow$ a subset $\mathcal{U}/\sim$

$\quad$ special elts

$\Gamma \leftarrow Z(u)$, $u \in$ the corresponding orbit

Ex $G = GL_n$, every u-special

2-sided cell
RSK : $\mathfrak{S}_n \longleftrightarrow \{\tau_1, \tau_2\}$ standard tableaux of same shape

(τ_1, τ_2) $\mapsto$ shape λ, $\downarrow$ $\{\lambda\}$

Local systems on unip. orbits appear in Springer correspondence

$\mathrm{Irr}(W) \hookrightarrow (\mathcal{O}, \mathcal{L})$

$\mathcal{O}$ - G-orbit in $\mathcal{U}$. $\mathcal{L}$ irr. local system

$\mathcal{L} \leftrightarrow$ rep'n of $\pi_0(Z(u))$, $u \in \mathcal{O}$.

$\mathcal{L}$ appears $\Leftrightarrow [H_{top}(\mathcal{B}_u) : \rho] \neq 0$

Ex. subreg. case

$e = (1, n-1)$


$\mathcal{B}_e$

In general, for simple G, $\exists!$ orbit $\mathcal{O}$ of codim 2

A_n, D_n, E_n. lines in $\mathcal{B}_e$

$\updownarrow$

nodes, 2 lines intersect if nodes conn.

Ex. $G = G_2$

$\mathcal{B}_u$

$\pi_0 \longrightarrow \mathfrak{S}_3$ permuting 3 lines

2-rep'n of $\mathfrak{S}_3$ appear

$G = \underline{G}(\mathbb{F}_q((t)))$ - locally compact totally disc.

Consider smooth complex rep'ns

G has many open compact subgps, eg. $G(k[[t]]) = K_0 \supset K_n = \equiv 1 \mod t^n$.

$\forall \xi \in V$, $\xi \in V^{K_n}$ for some n.

<u>Fact</u> 1) An irr. rep. (ρ, V) is admissible, i.e. $\dim(V^{K_n}) < \infty$.

(fix Haar measure on G)

$H = C_c^\infty(G)$, algebra under convol'n. acts on V.

Fact $\Rightarrow$ $\rho(h)$ has finite rank, $\forall\, h \in H$.

Let $\chi_\rho \in (H^*)^G = \mathrm{Dist}^G(G)$

<u>Fact</u>. $\chi_\rho \big|_{G^{rs}}$ is a locally constant conj. invariant function.

<u>Q</u>. Do these come from perverse sheaves?

On LG - loop group? (Character sheaves on LG)

$LG(R) = G(R((t)))$

<u>Rmk</u> Say in geom. Satake etc., consider LG/L^+G, $L^+G(R) = G(R[[t]])$ reduce to finite dim geom.

Not here

Bouthier - Kazhdan - Varshavsky ; 2020, defined conj. equiv. perverse sheaves on LG.

<u>principal series</u>.

We said before for $G(\mathbb{F}_q)$ PS rep'ns (subrep. in $I_1 = \mathbb{C}[G/B]$)

$\longleftrightarrow$ irr. reps W.

From basic algebra, PS $\longleftrightarrow$ Irr Rep $(H_q = \mathbb{C}[B\backslash G/B])$

has a basis T_w, $w \in W$

$(T_s + 1)(T_s - q) = 0$, $T_u T_v = T_{uv}$, $\ell(uv) = \ell(u) + \ell(v)$

$\exists\ H_q \simeq \mathbb{C}[W]$

$G = \underline{G}(\mathbb{F}_q((t)))$.

Consider (ρ, V) appearing as a subquotient in $\mathbb{C}[G/I]$

$\Updownarrow$ $\quad$ I Iwahori

rep'ns of $\mathbb{C}[I \backslash G / I]$

$\quad\quad \|$

H_{aff} $\quad$ (extended) affine Hecke alg.

$$W_{aff} = \underbrace{W \ltimes \Lambda}_{\text{coweight lattice of } T} \qquad (\mathbb{Z}^n, GL_n)$$

$\{T_w\}$, $T_u T_v = T_{uv}$, $\ell(uv) = \ell(u) + \ell(v)$, $(T_s + 1)(T_s - q) = 0$.

$H_{aff,q} \rightsquigarrow \mathbb{C}[W_{aff}]$

nontrivial deformation.

$\{(\rho, V) \in \mathrm{Irr}(G) : V^I \neq 0\} \longleftrightarrow \mathrm{irr.}(H_{aff}^q)$

<u>Thm</u> (B, Kazhdan, Varshavsky)

$G = GL_n$ $\quad$ (ρ, V) irr, $V^I \neq 0$. $\quad$ $\chi_\rho(g) \doteq \Phi_{\mathcal{F}_\rho}(g)$, $g \in LG^{rs}$

conj. eq-wt perverse sheaf on LG.

To define $\mathcal{F}_\rho$, consider LG analogue of $\tilde{G} \to G$ $\quad$ where $\mathcal{F}_\rho = \bar{\rho} \overset{L}{\underset{W_{aff}}{\otimes}} \pi_*(\underline{\mathbb{C}})$

$\tilde{G} \subset G \times \mathcal{B}$

Replace $G \rightsquigarrow LG$

$B \rightsquigarrow I$

$\mathcal{B} \rightsquigarrow \mathcal{F}\ell$ - affine flag variety $\quad$ $L_n = t^{-1} L_0$

$G = GL_n$, $\quad$ $\mathcal{F}\ell = (\cdots \subset L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n)$, $\quad$ $\mathcal{F}\ell \xrightarrow{\mathcal{B}} G_n$

$$\begin{array}{ccc} \widetilde{LG} & & \mathcal{F}l_\gamma \text{ - affine Springer fiber.} \\ \pi \downarrow & & \downarrow \\ LG & & \ni \gamma \end{array}$$

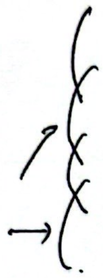
$\gamma \in G^{rs}$, $\mathcal{F}l_\gamma$ - finite equi-dim'l   many have ∞ many components.

$SL(n)$, $\gamma \in G^{rs} \subset G$. $\qquad \mathcal{F}l_\gamma = W_{aff}$

e.g. $\gamma = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ $\quad a \neq \pm 1$, $a \in \mathbb{C}$

e.g. $G = SL(2)$, $\quad \gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $\quad a = 1 + t$,

$\mathcal{F}l_\gamma$



$\mathbb{Z} = \left\{ \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} \right\}$

$W_{aff} \curvearrowright H^*(\mathcal{F}l_\gamma)$

$\curvearrowright \Pi_*(\mathbb{C})$

$\rho \in \mathrm{Rep}(H_{aff})$. $\qquad H_{aff,q} \rightsquigarrow \mathbb{C}[W_{aff}]$

$\rho \rightsquigarrow \bar{\rho}$

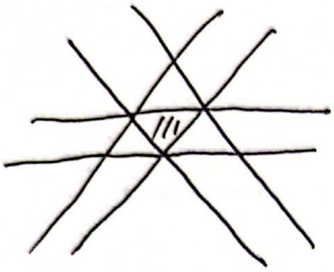
$$\chi_\rho(\gamma) = \mathrm{Tr}\left(\mathrm{Fr}, H_*(\mathcal{F}l_\gamma) \otimes_{W_{aff}} \bar{\rho}\right)$$

Ingredients in the proof:

1) projective resolution of $\mathbb{1}$ over W_{aff}

2) ——"—— for ρ over G
Schneider–Stühler

$(G = SL(3))$



complex for $H_*(\mathbb{R}^2)$

acted by W_{aff}

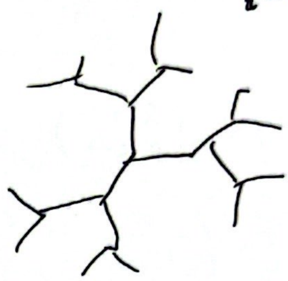
terms are projective.

2) in the case $\rho = \mathbb{1}$ trivial,

$\mathbb{R}^n \rightsquigarrow$ BT building

e.g.

$q=2$



<u>Lecture 4</u> Last time stated

<u>Thm</u> (B. Kazhdan, Varshavsky) $G = GL_n$, ρ, V - generated by V^I

$$\chi_\rho(g) = \phi_{\mathcal{F}_{\bar\rho}}(g)$$
$$\parallel$$
$$\mathrm{Tr}\left(Fr, H_c^*(Fl_g) \overset{L}{\underset{W_{aff}}{\otimes}} \bar\rho\right)$$

if $g \in G_c$

$\parallel$

$\{$ conj. to $GL_n(\mathcal{O})\}$

$g \in G_c \iff \overline{\{g^n : n \in \mathbb{Z}\}}$ - compact.

<u>Rmk.</u> char. values on G_c determine the rest

G semisimple, $g \notin G_c \rightsquigarrow P = P_g \ni g$

$\chi_\rho(g) \underset{\text{Deligne}}{=} \chi_{\rho'}(\bar g)$, $P_g \to L_g$

$g \mapsto \bar g$

Rmk. Perversity of $\mathcal{F}_{\bar{\rho}}$ $\Leftarrow$ Bouthier - Kazhdan - Varshavsky (2025)

Proof for g (geometrically) elliptic $\Leftrightarrow$ Fl_g is a scheme

- finitely many components

GL_n or SL_n: elliptic - irred. char. poly.

geom. ell. - irred. over $\overline{\mathbb{F}_q}((t))$.

g - elliptic $\Leftrightarrow$ $Z(g)/Z(G)$ is compact.

Rmk By a result of Schneider - Stühler (also D. thesis), for such g, can extend the def'n of $\chi_\rho(g)$

from admissible ρ to all finitely genrated smooth modules, s.t. 1) it is additive on SES.

2) If $K \subset G$ open compact, U finite dim'l rep'n, open kernel,

(e.g. $K = G(\mathcal{O}) \twoheadrightarrow G(\mathbb{F}_q)$, U pulledback from $G(\mathbb{F}_q)$)
$\mathcal{O} = \mathbb{F}_q[[t]]$

$V = \text{c-ind}_K^G(U)$ - smooth rep'n (eg. $\mathbb{C}_c[G/K]$)

$$\text{"}\chi_V\text{"}(g) = \int_{\gamma : \gamma g \gamma^{-1} \in K} \chi_U(\gamma g \gamma^{-1})\, d\gamma$$

$d\gamma$: vol $(K) = 1$

Frobenius induction formula

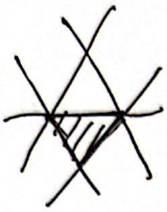
g - elliptic $\Leftrightarrow$ sum is finite. $\int$ - converges

Proof. RHS

(G - simply conn'd, simple), W_S parabolic subgp, $S \subsetneq$ affine Dynkin diagram

W_{aff} - Coxeter group

Resolution of $\mathbb{1}_{W_{aff}}$.



$$\cdots \to \bigoplus_{|S|=n-1} \mathrm{ind}_{W_S}^{W_{aff}}(1) \to \bigoplus_{|S|=2} \mathrm{ind}_{W_S}^{W_{aff}}(1) \to \mathbb{1}_{W_{aff}} \to 0$$

exact

computes $H_*(\mathbb{R}^n; pt)$

finite dim'l since g geom. elliptic.

$$RHS = \sum_S (-1)^{|S|-n} Tr\left(Fr, H_c^*(F\ell_g) \otimes_{W_S} \bar{P}\right)$$

$G = \underline{G}(F) \supset P$ - parahoric subgp

$\phantom{G = \underline{G}(F) \supset} \cup$

$\phantom{G = \underline{G}(F) \supset} I$ - Iwahori

$P_S \longleftrightarrow S$

fix I, standard parahorics $\longleftrightarrow S$

This matches expression for Tr coming from a resolution. [S.S.]

P_S^+ : pro-p-radical

$$\cdots \to \bigoplus_{S, |S|=2} \text{c-ind}_{P_S}^{G}\left(V^{P_S^+}\right) \to V \to 0$$

Ex (ρ, V) trivial - get complex for $H_*(\mathcal{B})$ - $\mathcal{B}$: BT building.

More precisely, fixing S, plug $V^{P_S^+} \curvearrowleft P_S$ into the Frobenius induction formula

$$\chi_V(g) = \int_{\gamma : \gamma g \gamma^{-1} \in P_S} \chi_U(\gamma g \gamma^{-1})\, d\gamma \qquad (*)$$

, get $Tr(Fr, H_c^*(F\ell_g) \otimes_{W_S} \bar{P})$ - by finite dim story from before.

$(*)$ is obtained by averaging over $\frac{G}{P_S}$ from a func. on P_S. $\phi(\mathcal{F}_{\bar{P}}|_{W_S}) = \chi_{V^{P_S^+}}$

P_S' char. sheaf on P_S/P_S^+.

Recall local Langlands conj.:

$\mathrm{Irr}(G) \to \{ \mathrm{Gal}(\bar{F}|F) \xrightarrow{u} \check{G} \}$

fibers L-packets $\leftrightarrow$ ψ: irr. rep. of π_0 of the image

$R_{u,\psi} \longleftarrow u, \psi$

Conj. (char's of L-packets)

involves stability, endoscopy

Let $f \in C^\infty(G^{rs})^G$

loc. const. func.

Def. f is stably invariant if $f(g) = f(g')$ when $g = \gamma g' \gamma^{-1}$, $\gamma \in G(\bar{F})$

Eg. $G = SL(n)$, f is stably invariant if $f \in C^\infty(G^{rs})^{GL_n(F)}$.

Conjecturally, $\sum_\psi \dim(\psi) \chi_{R_{u,\psi}}$ is stably invariant.

Also $\sum_\psi \mathrm{Tr}(\psi(\kappa)) \chi_{R_{u,\psi}}$ has endoscopic behaviours.

fix $\kappa \in \pi_0(\mathrm{Stab}_{\check{G}}(u))$

eg. for $SL(n)$, transforms under a character of GL_n.

In general, has similar behaviour on $G \cap \{\gamma g \gamma^{-1} : \gamma \in G(\bar{F})\}/G$.

- torsor over an abelian group A.

get a func. transforming by a character.

Idea: these combinations of characters should come as ϕ_F, F - perverse sheaf on LG.

(CS on LG)

Endoscopic property $\Leftarrow$ geom. properties of $\mathcal{F}$

Eg. stability $\Leftarrow$ $\pi_0(Z(g)) \curvearrowright \mathcal{F}_g$

Thm (Tô. - Varshavsky) This works for cuspidal depth zero L-packets.
(G - simple)

$\mathrm{Im}(u) = \langle s, F \rangle \in G^\vee$

s, F semisimple in $G^\vee$,

$s \in (G^\vee)^{r.s.}$, $\quad FsF^{-1} = s^q$.

$Z(u) = Z(s) \cap Z(F) = (T^\vee)^w$ – finite abelian gp.

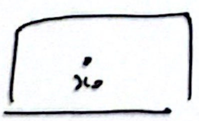
$Z(s) = T^\vee \subset G^\vee$
$\quad /\!/ \quad$ max. torus
$Z(s^q)$ – normalized by F.

$F \in wT^\vee$ for some $w \in W$

L-packet $\longleftrightarrow ((T^\vee)^w)^* \simeq (T)^w \longleftrightarrow \{\tilde{w}$ – lift of w to W_{aff} up to conj.$\}$

Construction: All elements in the L-packet are obtained by compact induction.

$u \rightsquigarrow \theta_u$ – character of a torus $\bar{T}_w \subset G(\mathbb{F}_q)$



$x_0 \mapsto x_0 + \lambda - w(\lambda)$

For every S, $W_S \longrightarrow W_{aff}$
$\downarrow$
$[w] \in W$

If S, u s.t. $(w) \cap \mathrm{Im}(W_S) \neq \emptyset$,

then $\bar{T}_w \subset P_S/P_S^+$, then get a rep'n of P_S/P_S^+

$\{\text{c-ind}_{P_S}^{G}(\underset{\widehat{P_S/P_S^+}}{R_{\bar{T}_w, \theta}})\}$ is the L-packet
$\simeq \rho_{S,u} \qquad \{S\} \longleftrightarrow T^w \ni s$

Work over $\overline{\mathbb{F}}_q$ $\qquad$ $\mathcal{L}_u = \mathcal{L}_\theta$ - loc. sys. on the torus

$$I \xrightarrow{\varpi} T \quad / \mathbb{F}_q$$

$$\mathrm{Av}_I^{LG} \, \varpi^*(\mathcal{L}_u) =: \mathcal{F}_u$$

Rmk. $\mathcal{F}|_g = H_c^*(\mathcal{F}\ell_g)$ (esp. if g - top. unipotent)

$$\mathcal{F}_u \circlearrowleft \Lambda$$
$$\cap$$
$$W_{aff}$$

For every $\sigma : \Lambda \to \overline{\mathbb{Q}}_\ell^\times$. $\sigma \in T^\vee$

$$\mathcal{F}_{u,\sigma} = \mathcal{F}_u \otimes_{\overline{\mathbb{Q}}_\ell[\Lambda]} \sigma$$ if $\sigma \in (T^\vee)^w$, then $\mathcal{F}_{u,\sigma}$ carries a Fr. str.

Thm a) $\langle \phi_{\mathcal{F}_{u,\sigma}} \rangle = \langle \chi_{\rho_{u,s}} \rangle$

$$\begin{array}{ccc} \langle \phi_{\mathcal{F}_{u,\sigma}} \rangle & = & \langle \chi_{\rho_{u,s}} \rangle \\ \updownarrow & & \updownarrow \\ (T^\vee)^w & \xleftrightarrow{FT} & T^w \end{array}$$

b) $\phi_{\mathcal{F}_{u,\sigma}}$ is an endoscopic function.

in particular for $\sigma = 1$, it's stably invariant

$$\Uparrow$$

Thm of Yun; compatibility b/w affine Springer action & $\pi_0(Z(g))$ action on $H_c^*(\mathcal{F}\ell_g)$

Lecture 5. Summary of last time

$LG(\mathbb{F}_q) = G(\mathbb{F}_q((t)))$

$u \in G^\vee$

$w \in W$, $\theta : \overline{T_w} \to \mathbb{C}^\times = \overline{\mathbb{Q}_\ell}^\times$

$u^q = FuF^{-1}$

torus / $\mathbb{F}_q$

$F \in G^\vee$

$\rightsquigarrow$ L-packet cuspidal, depth 0.

$\mathcal{L}_\theta$ local system on $T_{0,\mathbb{F}_q}$, $Fr^*(\theta) = w(\theta)$

$t^w = 0$

Recall that $R_{T,\theta}$ - DL rep'n of $G(\mathbb{F}_q)$

$g \in \mathcal{U}$. $\chi_{R_{T,\theta}}(g) = Tr(w \cdot Fr, H^*(\mathcal{B}_g))$

L-packet $\leftrightarrow$ $T_{\mathbb{C}}^w$.

$g \in LG$, top. unipotent ($g^n \to 1$, $n \to \infty$)

(1) $\chi_{\rho_s}(g) =$ "Tr"$(w_s \cdot Fr, H^*(Fl_g))$

(1) $\Leftrightarrow$ (2)

(2) $\phi_{\mathcal{F}_\sigma}(g) = Tr(Fr, H_c^*(Fl_g) \overset{L}{\underset{\mathbb{C}[\Lambda]}{\otimes}} \mathbb{C}_\sigma)$

$\sigma \in (T^\vee)^w$

$\chi_s \leftrightarrow \phi_{\mathcal{F}_\sigma}$ - finite FT

where "Tr"

$w \curvearrowright M \circlearrowleft \mathbb{C}[\Lambda]$

finite dim'l module $\leftrightarrow$ coh. sheaf on $T = Spec(\mathbb{C}[\Lambda])$

$w \curvearrowright \Lambda$

$\curvearrowright T$ finitely many fixed pts.

"Tr"$(w, M) := \sum_{t \in T^w} (\quad)$

$$\mathrm{Tr}\left(\mathrm{Fr},\ H_c^*(\mathcal{F}\ell_g) \overset{L}{\underset{W_{aff}}{\otimes}} \mathrm{sgn}\right)$$

$$\| \ [\mathrm{BKV}]$$

$$\Pi(g)$$

Π - projector to depth 0

$$\Pi|_p = \begin{cases} \mathrm{Id}, & p\text{- depth } 0 \\ 0, & \text{otherwise} \end{cases}$$

Yun's result $\Rightarrow$ endoscopic properties

$$\begin{array}{ccc} \mathbb{C}[W_{aff}] \curvearrowright H_c^*(\mathcal{F}\ell_g) & & \curvearrowleft \mathbb{C}[\pi_0(Z_{LG}(g))] \\ \cup & & \| \\ \mathbb{C}[\Lambda]^W & \longrightarrow & \mathbb{C}[\Lambda_\Gamma] \\ & & \circlearrowleft \\ & & \Gamma \to W \end{array}$$

g - reg. semisimple.

If g split, $\pi_0(Z_{LG}(g)) = \pi_0(LT) = \Lambda$

In general, $\Gamma = \mathrm{Gal}(\bar{F} \mid F_{un}) \curvearrowright \Lambda$

[Yun]
Then the two actions of $\mathbb{C}[\Lambda]^W$ coincide (on gr).

<u>General story.</u>

<u>Idea</u> to understand $\mathrm{Dist}^G(G) = H_G^*$

$H = C_c^\infty(G)$ $\qquad H_G = H/[H,H]$ - cocenter of H
$C(H)$

Should work $e(\mathrm{Sh}(LG_{\mathbb{F}_q})) \supset \mathrm{Fr}$
$\uparrow$
cat. cocenter

What's cat. co center?

$\mathcal{A}$ mon. cat., $\mathcal{A} \xrightarrow{F} \mathcal{T}$ F comm. if $F(XY) \xrightarrow{\sim} F(YX)$ + compats.

$$\mathcal{A} \to \mathcal{C}(\mathcal{A})$$

universal comm. functor

Ex. Γ finite group. $\mathcal{A} = \mathrm{Rep}(\Gamma)$, $\mathcal{C}(\mathcal{A}) = \mathrm{Sh}^{\Gamma}(\Gamma)$

More generally, $\mathcal{A} = \mathrm{Coh}(X)$, $\mathcal{C}(\mathcal{A}) = (\mathrm{Q})\mathrm{Coh}(\underbrace{I}(X))$,

alg. stack   inertia stack

$$I(X) = X \underset{X \times X}{\times} X$$

$$I(X) = \text{"}\{(x \in X, \kappa \in \mathrm{Aut}(x))\}\text{"}$$

eg. $X = pt/\Gamma$, $I(X) = \Gamma/\Gamma$.

$$\mathcal{C}(\mathrm{Sh}(LG)) = \mathrm{Coh}(I(\mathcal{L}))$$

Langlands parameters

$$I(\mathcal{L}) = \{(u, \kappa) : u : \mathrm{Gal} \to G^\vee, \ \kappa \text{ autom.}\}$$

endoscopic parameter

eg. stable dist. $\leftrightarrow$ $\kappa = 1$.

<u>unipotent inv. dist. story</u>

$H = H_{unip} \oplus H_{non\text{-}unip}$ $C = C_{unip} \oplus C_{nonunip}$.

$h \in H_{unip}$, $\rho(h) = 0$ if ρ is nonunip.

$h \notin$ $\rho(h) = 0$ ρ unip.

$\{\rho : \rho \text{ unip}\} \longleftrightarrow \{(F, m, \psi)/\sim\}$

Kazhdan - Lusztig

Ginzburg

Lusztig

$\cup$

$m \in G^\vee$ - unipotent

$F \in G^\vee$ - ss

$\{(\rho, V) : V^I \neq 0\}$

$FmF^{-1} = m^q$

$\mathcal{N} \subset \mathfrak{g}^\vee$

$\psi \in \mathrm{Irr}\,(\pi_0(Z(m, F)))\}$

$\mathcal{N}/G^\vee$

$\cap$

$\mathcal{L} = \mathcal{L}_{unip} \simeq \mathcal{U}/G^\vee$

unip. cone of $G^\vee$

$, \mathcal{I}(\mathcal{L}) = Z \subset G^\vee \times \mathfrak{g}^\vee / G^\vee$

$= \{(k, x) : Ad(k)(x) = x, \ x\text{-unip}\}$

$x = \log(m)$

$[q] = F : (k, x) \mapsto (k, qx)$

Thm (T, Ginzburg, K, B)

$$C_{unip} \simeq K_{[q]}(\mathrm{Coh}\,\mathcal{I}) = K(\mathrm{Coh}^{G_m \times G^\vee}(\{(k,x)\}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{C}$$

equt obj.

$\mathcal{C} \supset F$. $\quad K_F(\mathcal{C}) = K(\mathcal{C}^F) \otimes_{K[\mathbb{Z}\text{-mod}_{fd}]} \mathbb{C}$

Grothendieck gp

$[\lambda] \longmapsto \lambda$

$K[\mathbb{Z}\text{-mod}_{f.d.}] = \mathbb{Z}[\mathbb{C}^*] \longrightarrow \mathbb{C}$

$[\mathcal{F}] \longmapsto b_{\mathcal{F}}(x, y) = \mathrm{Tr}(y, \mathcal{F}_x)$

$$\simeq \bigoplus_{e \in \mathcal{N}/\sim} K^{Z_e}(Z_e) = \bigoplus_e \mathcal{O}_\ell(\mathrm{Com}(Z_e))$$

left

$Z_e = Z^{red}_{\zeta}(e)$

$K\,\mathrm{Coh}^{Z_e}(Z_e)$

$\mathcal{O}_\ell(\mathrm{Com}(H)) = \{ f \in \mathcal{O}(\{(x,y) : xy = yx, \ x,y \in H\})^H : \forall y = y_0,$
$x \mapsto f(x, y_0)$ is loc. const$\}$

$\mathcal{O}_r(\mathrm{Com}(H)) = \{ f \cdots : f(x, y)$ is loc. const$\}$, $\mathcal{O}_{r\ell} = \mathcal{O}_r \cap \mathcal{O}_\ell$

b) ($G = GL_n$. (w. Krylov)
Conj. in general).   Given F, x, ψ,

$\rightsquigarrow \rho_{F,x,\psi}$ - unip. rep'n

$$\chi_{\rho_{F,x,\psi}} \in C^*_{unip}$$

$$(f_e) \longmapsto \left\langle f_{e=x}\big|_{\{(g_1,g_2):\ g_2 = F g_1\}} , \psi \right\rangle$$

$$f_{e=x} \in \mathcal{O}_e(Z_x)$$

$$C = C_c \oplus C_{nc}$$

$$G = G_c \sqcup G_{nc}$$

$$C = C_c \oplus C_{nc}.$$

$$C^c_{uni} = C_c \cap C_{uni}$$

c) $C_c^{uni} \simeq \bigoplus \mathcal{O}_{e_n}(Z_e)$ Deligne : monodromy $\xleftrightarrow{\text{swap}}$ endoscopy

d) (conj.) $i : \mathcal{O}_{e_n} \circlearrowleft \quad (x,y) \mapsto (y,x)$

$i(\text{char.}) = \phi(\text{char. sheaf on } LG)$ & ϕ is endoscopic.

Conj. $C = K_F(I(\mathcal{L}))$, $\mathcal{L} = \mathrm{Hom}(\text{Gal curr}, G^\vee)/\hat{G}$

Weil-Deligne group

analogues of a–d in Thm.

Rmk.   Thm/Conj. above & cuspidal depth 0 agree.

Used B_e to compute $\mathrm{Tr}(Fr, H^*(\quad))$ to get char. values.

It also gives interesting categories.

KL realized rep'ns of H_{aff} via $K(\mathrm{Coh}(\mathcal{B}_e))$

1) $D^b(\mathrm{Coh}(\mathcal{B}_e)) \simeq D^b(A_e\text{-mod})$

derived Springer fiber

non-comm. Springer resol'n.

$A_e\text{-mod} \simeq \mathfrak{g}\text{-mod}\,|\,$cent. char. working over k of char p

quantum group @ root of 1 / $\mathbb{C}$

$\hat{\mathfrak{g}}_{crit}\text{-mod}^{I}_{\chi_e}$

fixed central char.

2) Constructible categorification (cat. H^{top})

M_e finite dim'l modules / finite W-algebras

IS conj.

microlocal sheaves on $\mathcal{B}_e$

$Z(e) \curvearrowright M_e$

$\searrow \Gamma_e \curvearrowright$

canonical qt

Claim. can be used to deduce classification of CS. (in D-mod., $\mathcal{F}$. Φ. O)

Expect: generalization to affine Springer fibers

Ex. $\gamma = ts$, $s \in \mathfrak{g}^{rs}$, $ts \in L\mathfrak{g}$,

Fl_γ

Thm Analogue of M_e for $e \rightsquigarrow \gamma$

$(\mu sh(Fl_\gamma)) \xleftrightarrow{KD} U_q\text{-mod}^{T^\vee}_{block} \simeq A_{e=0}\text{-mod}^{T^\vee}$

(for $\mathfrak{g}^\vee$)

Coherent cat'n. $D^b Coh(Fl_e)$ - cat'n of DAHA-modules

conj. relating $D^b Coh(Fl_e)$ w/ $D^b Coh(Fl_{e^\vee})$

$$\begin{array}{ccc} D\,Coh^{LG}(T^*Fl \times_{Lg} T^*Fl) & \sim & D\,Coh(Fl_e) \\ \text{conj.} \ \wr\wr & & \wr\wr \\ D\,Coh^{LG^\vee}(T^*Fl^\vee \times_{Lg^\vee} T^*Fl^\vee) & \sim & D\,Coh^{Z(e^\vee)}(Fl_{e^\vee}) \end{array}$$