Automorphic Lifting

Richard Taylor

Lecture 1.

Fermat's LT

Galois rep'ns

automorphic lifting

Thm (Wiles) $a, b, c \in \mathbb{Z} \neq 0$, $n \in \mathbb{Z} > 2, \Rightarrow a^n + b^n \neq c^n$.

Sketch of proof.

$n=4$. Fermat C17]

$n=3$ Euler C18] descent

Sufficient to treat the case $n = \ell$ a prime > 3.

WLOG a, b, c are pairwise coprime.

WLOG b even., and a, c is odd (as ℓ odd)

WLOG $a \equiv -1$ (4). (replace a, b, c by their negatives otherwise)

Frey: E $y^2 = x(x - a^\ell)(x + b^\ell)$

bad reduction at p, $\Leftrightarrow$ $p \mid a^\ell b^\ell (a^\ell + b^\ell)$

$\therefore p \mid abc$

$$j = \frac{2^8 (a^{2\ell} + b^{2\ell} + a^\ell b^\ell)}{(abc)^{2\ell}}$$

$p \mid a$, $y^2 = x^2(x + b^\ell)$

$p \neq 2$, $p \mid b$, $y^2 = x^2(x - a^\ell)$ 

$p \mid c$ $y^2 = x(x-a^\ell)^2$ 

} semistable reduction

$p=2$, $x = 4X$, $y = 8Y + 4X$

$$Y^2 + XY = X^3 + \frac{1}{4}\left(b^\ell - (1+a^\ell)\right)X^2 - \frac{a^\ell b^\ell}{16} X$$

rat'l tangent directions at (0,0)

mod 2: $Y^2 + XY = X^3$ if $a \equiv -1 \ (8)$

$Y^2 + XY = X^3 + X^2$ if $a \equiv 3 \ (8)$

→ single double pt at $(0,0)$

irrat'l tangent directions at $(0,0)$

everywhere semistable reduction.

$p \mid abc$, Tate: $E(\overline{\mathbb{Q}_p}) = \left(\overline{\mathbb{Q}_p}^\times / q^{\mathbb{Z}}\right) \otimes \delta$

$\delta : G_{\mathbb{Q}_p} \to \{\pm 1\}$, $\delta^2 = 1$

$$j = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n$$

$c_n \in \mathbb{Z}$ usual q-expansion

$\delta = 1$ if tangent directions at double pt in char. p are in $\mathbb{F}_p$

$q \in p\mathbb{Z}_p$

$\delta = -1$ otherwise

$v_p(q^{-1}) = v_p(j)$

$v_p(q) \equiv 0 \ (2\ell)$ if $p \neq 2$

$\equiv -8 \ (2\ell)$ if $p = 2$

$E[\ell](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/\ell\mathbb{Z})^2$

$\bar{\rho}_{E,\ell} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}/\ell\mathbb{Z})$

unramified at p, $p \nmid abc$

E supersingular @ ℓ

$\ell \nmid abc$, $\bar{\rho}_{E,\ell}|_{G_{\mathbb{Q}_\ell}} = \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \bar{\sigma}$, $\bar{\sigma} : G^{ab}_{\mathbb{Q}_{\ell^2}} \longrightarrow \mathbb{F}_{\ell^2}^\times$ $\qquad \mathbb{Q}_{\ell^2} | \mathbb{Q}_\ell$ unramified quad

$$\begin{tikzcd} G^{ab}_{\mathbb{Q}_{\ell^2}} \arrow[r] & \mathbb{F}_{\ell^2}^\times \\ \mathbb{Q}_{\ell^2}^\times \arrow[u, "\mathrm{Art}_{\mathbb{Q}_{\ell^2}}"] \arrow[r, two heads] & \mathbb{F}_{\ell^2}^\times \times \ell^{\mathbb{Z}} \end{tikzcd}$$

$\uparrow$ mod out $1 + \ell\mathbb{Z}_{\ell^2}$

$$\begin{pmatrix} \bar{\chi}^{-1}\bar{\epsilon}_\ell & * \\ 0 & \bar{\chi} \end{pmatrix}$$

$\uparrow$ ordinary redn @ ℓ

$\epsilon_\ell : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_\ell^\times$ cyclotomic char.

Ψ

σ

$\sigma \zeta_{\ell^n} = \zeta_{\ell^n}^{\epsilon_\ell(\sigma)}$, $\bar{\epsilon}_\ell = \epsilon_\ell \bmod \ell$

$\bar{\chi}$ unramified character

ordinary case: $\bar{\rho}_{E,\ell}|_{G_{\mathbb{Q}_\ell^{nr}}} = \begin{pmatrix} \bar{\epsilon}_\ell, & b \\ 0 & 1 \end{pmatrix}$ $\quad \mathbb{Q}_\ell^{nr} | \mathbb{Q}_\ell$ max'l unramified ext.

$\overset{\shortparallel}{\overline{\mathbb{Q}}_\ell^{I_{\mathbb{Q}_\ell}}}$

$b \in Z^1(I_{\mathbb{Q}_\ell}, \mathbb{F}_\ell(\bar{\epsilon}_\ell))$

$\mathcal{O}_{\mathbb{Q}_\ell^{nr}}^\times / (\mathcal{O}^\times_{\mathbb{Q}_\ell^{nr}})^\ell \subset \mathbb{Q}_\ell^{nr\times} / (\mathbb{Q}_\ell^{nr\times})^\ell \ni$ - - - - $[b] \in H^1(I_{\mathbb{Q}_\ell}, \mathbb{F}_\ell(\bar{\epsilon}_\ell)) / \mathbb{F}_\ell^\times$ does not depend on choice of basis splitting field for $\bar{\rho}_{E,\ell}\big|_{G_{\mathbb{Q}_\ell^{nr}}} = \mathbb{Q}_{n}^{r}(\zeta_\ell, \sqrt[\ell]{\alpha})$

$\alpha \in \mathcal{O}^\times_{\mathbb{Q}_\ell^{nr}}$

$q \in \mathbb{Q}_p^\times$

$p \mid abc,$ $\quad E[\ell](\overline{\mathbb{Q}_p}) = \underbrace{\left(\overline{\mathbb{Q}_p}^\times / q^{\mathbb{Z}}\right)[\ell]}_{} \otimes \delta$, $\delta^2 = 1$

δ unramified

basis over $\mathbb{F}_\ell$:

$\zeta_\ell, q^{1/\ell}$

$$\bar{\rho}_{E,\ell}\big|_{G_{\mathbb{Q}_p}} = \begin{pmatrix} \bar{\epsilon}_\ell & b \\ 0 & 1 \end{pmatrix} \otimes \delta$$

$[b] \in H^1(G_{\mathbb{Q}_p}, \mathbb{F}_\ell(\bar{\epsilon}_\ell)) / \mathbb{F}_\ell^\times$

$\updownarrow$ $\quad$ is Kummer

$q \in \left(\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^\ell\right) / \mathbb{F}_\ell^\times$

$p \neq 2, \ell$: $\quad \ell \mid v_p(q)$, $\quad \therefore q \equiv$ unit mod ℓ^{th} powers

$\therefore \overline{\mathbb{Q}_p}^{\ker \bar{\rho}_{E,\ell}|_{\mathbb{Q}_p}}$ unramified

$p = 2$: ramified, ramification index ℓ, $I_2 \twoheadrightarrow \begin{pmatrix} 1 & \mathbb{F}_\ell \\ 0 & 1 \end{pmatrix}$

$p = \ell$: $\quad [b] \longleftrightarrow$ elt of $\mathbb{Z}_\ell^\times$

unramified away from 2ℓ

$\bar{\rho}_{E,\ell} : I_2 \to \begin{pmatrix} 1 & \mathbb{F}_\ell \\ 0 & 1 \end{pmatrix}$ (peu ramified)

$$\bar{\rho}_{E,\ell}\big|_{G_{\mathbb{Q}_\ell}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \bar{\theta} \quad \text{or} \quad \begin{pmatrix} \bar{\chi}^{-1}\bar{\epsilon}_\ell & \overset{\text{elt of}}{b} \\ 0 & \bar{\chi} \end{pmatrix} \quad \mathcal{O}^\times_{\mathbb{Q}_\ell^m} / \left(\mathcal{O}^\times_{\mathbb{Q}_\ell^m}\right)^\times$$

$\bar{\chi}$ unramified.

Thm. Any $\overset{\ell \geq 3}{\bar{r}} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_\ell)$ as above is reducible.

Thm (Mazur) $\bar{r}$ is irreducible $\Leftarrow \begin{cases} \ell > 3 \\ \# E[2](\mathbb{Q}) = 4 \\ E/\mathbb{Q} \end{cases}$

Lecture 2

$$\left.\begin{array}{l} a, b, c \in \mathbb{Z}_{\neq 0} \\ n \in \mathbb{Z}_{>2} \end{array}\right. \Rightarrow a^n + b^n \neq c^n$$

$n = \ell$ a prime > 3, a, b, c coprime, b even, $a \equiv -1\ (4)$

$E \quad y^2 = x(x - a^\ell)(x - b^\ell)$, $\bar{r}_{E,\ell} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_\ell)$

$\bar{r}_{E,\ell}$ unramified away from 2 and ℓ

$\bar{r}_{E,\ell}\big|_{G_{\mathbb{Q}_\ell}} \cong \begin{pmatrix} \bar{\varepsilon}_\ell & * \\ 0 & 1 \end{pmatrix} \otimes \delta$, δ unramified, $\delta^2 = 1$, ramified

$$\bar{\rho}_{E,\ell}\big|_{G_{\mathbb{Q}_\ell}} \cong \begin{cases} \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \bar{\theta} \quad , & \bar{\theta} : I_\ell \twoheadrightarrow \mathbb{F}_{\ell^2}^{\times} \\ \text{or} \quad \begin{pmatrix} \bar{\chi}^{-1}\bar{\varepsilon}_\ell & * \\ 0 & \bar{\chi} \end{pmatrix} & \bar{\chi} \text{ unramified, peu-ramifié} \end{cases}$$

$$\begin{tikzcd} H^1(G_{\mathbb{Q}_\ell^{\mathrm{nr}}}, \mathbb{F}_\ell(\bar{\varepsilon}_\ell)) & \\ \mathbb{Q}_\ell^{\mathrm{nr}\times}/(\mathbb{Q}_\ell^{\mathrm{nr}\times})^\ell \arrow[u, phantom, sloped, "\cong"] & {*} \\ \mathbb{Z}_\ell^{\mathrm{nr}\times}/(\mathbb{Z}_\ell^{\mathrm{nr}\times})^\ell \arrow[u, phantom, sloped, "\subset"] \arrow[ur, bend right] & \end{tikzcd}$$

Thm. $\bar{\rho}_{E,\ell}$ absolutely irreducible.

Pf. If not $\bar{\rho}_{E,\ell} \sim \begin{pmatrix} \bar{\chi}_1 & * \\ 0 & \bar{\chi}_2 \end{pmatrix}$ over $\mathbb{F}_\ell$

$\det \bar{\rho}_{E,\ell}(c) = -1$

$\bar{\chi}_1, \bar{\chi}_2$ unramified away from ℓ.

$\bar{\chi}_1$ eigenspace

$\bar{\chi}_i\big|_{I_{\mathbb{Q}_\ell}} = 1$ or $\bar{\varepsilon}_\ell$ $\qquad$ $\ker(\bar{\rho}_{E,\ell}(c) \pm 1)$ defined $/\mathbb{F}_\ell$.

$\therefore$ $\bar{\chi}_i$ or $\bar{\chi}_i \bar{\varepsilon}_\ell^{-1} : G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ unramified everywhere.

$\therefore \bar{\chi}_i = 1$ or $\bar{\varepsilon}_\ell$

$\det \bar{\rho}_{E,\ell} = \bar{\varepsilon}_\ell$ $\qquad$ $\therefore$ either $\bar{\rho}_{E,\ell} \sim \begin{pmatrix} \bar{\varepsilon}_\ell & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell \end{pmatrix}$ – look at $I_{\mathbb{Q}_2}$

$\bar{\rho}_{E,\ell}\big|_{I_{\mathbb{Q}_2}} = 1 \oplus \bar{\varepsilon}_\ell$, impossible. #

$\mu_\ell \hookrightarrow E[\ell]$

Choose m max'l s.t. $\mu_{\ell^m} \hookrightarrow E$ (else $\mu_{\ell^\infty} \hookrightarrow E$

$\Rightarrow$ Frob$_p$ has eigenvalue p on T_pE)

$E' = E/\mu_{\ell^m}$

Riemann hyp.

$\Rightarrow E'$ has a $\mathbb{Q}$-pt of order ℓ

$\bar{\nu}_{E',\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell \end{pmatrix}$ non-split

$\# E'[2](\mathbb{Q}) = 4$

Thm (Mazur) If $E'/\mathbb{Q}$ is an elliptic curve w/ $\# E'[2](\mathbb{Q}) = 4$ and if $\ell > 3$ is a prime, then E' does not have a $\mathbb{Q}$-pt of exact order ℓ.

Cor. $\bar{\nu}_{E,\ell}$ is irreducible.

explicit arguments for $\ell = 5, 7, 13$.

Idea of proof. Suppose $\bar{\nu}_{E',\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell \end{pmatrix}$

1) E has everywhere semistable reduction.

2) $\bar{\nu}_{E',\ell}$ not split

$P \in E'[\ell](\mathbb{Q}) \setminus \{0\}$

E' bad reduction at p, $\bar{\nu}_{E',\ell}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \bar{\varepsilon}_\ell & * \\ 0 & 1 \end{pmatrix} \otimes \delta$, $\delta^2 = 1$

δ unramified.

$\therefore$ either $\bar{\nu}_{E',\ell}|_{G_{\mathbb{Q}_p}} \sim \bar{\varepsilon}_\ell \oplus 1$ Case I

or $\ell \mid p-1$ and $p \mapsto$ a root of unity in $(\mathbb{Q}_p^\times / q^{\mathbb{Z}})(\delta)$ Case II.

$p = \ell$, good reduction

$$\Rightarrow \bar{\nu}_{E',\ell}\Big|_{G_{\mathbb{Q}_\ell}} \sim \begin{pmatrix} \bar{\varepsilon}_\ell & * \\ 0 & 1 \end{pmatrix}$$

in fact $\sim 1 \oplus \bar{\varepsilon}_\ell$

If in case I $\forall p$, $L = \underbrace{\bar{\mathbb{Q}}^{\ker \bar{\nu}_{E,\ell}} \supset \mathbb{Q}(\zeta_\ell)}_{\substack{\text{unramified everywhere} \\ \text{degree } \ell}} \supset \mathbb{Q}$

$\xRightarrow{\text{class field theory}}$ $Cl(\mathbb{Z}[\zeta_\ell]) \ni \mathcal{C}$ of exact order ℓ.

and $^{\sigma}\mathcal{C} = \mathcal{C}^{\bar{\varepsilon}_\ell(\sigma)^{-1}}$ $\forall \sigma \in Gal(\mathbb{Q}(\zeta_\ell)|\mathbb{Q})$

$\#$ Herbrand's thm

$a \in \mathcal{C}$

find a generator for a using Gauß sums.

2 cusps $0, \infty$ $\downarrow$

$X_0(\ell)(\mathbb{Q}) \ni x = (E', \langle P \rangle)$

$Jac(X_0(1)) = J_0(\ell)$

$[x] - [0] \in J_0(\ell)(\mathbb{Q})$

$$x \bmod p = \begin{cases} \infty & \text{in case II} \qquad [x]-[0] \text{ reduces to } [\infty]-[0] \\ 0 & \text{in case I} \qquad [x]-[0] \text{ reduces to } 0 \end{cases}$$

$[\infty]-[0] \in J_0(\ell)(\mathbb{Q})$ is torsion of exact order numerator of $\frac{\ell-1}{12} > 1$

if $\ell = 11$ or $\ell > 13$

Mazur found $J_0(\ell) \twoheadrightarrow J'$ s.t.

1) $[\infty]-[0]$ has same order.

2) $\# J'(\mathbb{Q}) < \infty$

the hard work

$$\Downarrow$$

$p > 2$

$p \neq \ell$

$$\begin{tikzcd} J'(\mathbb{Q}) \arrow[r, hook] & J'(\mathbb{F}_p) \end{tikzcd}$$

$$\Downarrow$$

$[x]-[0] \longmapsto [\infty]-[0]$ @ p of type II

0 @ p of type I

$\Rightarrow$ either type I everywhere or type II everywhere.

3 has to be a prime of bad reduction

$\uparrow$

Case I

because $\ell \nmid 3-1=2$

if not, $\mathbb{Z}/\ell \hookrightarrow E'(\mathbb{Q})$

$$\wr$$

$E'(\mathbb{F}_3)$

$\uparrow$

order $\leq 1+3+2\sqrt{3} < 8$

STP

Thm A. If $\ell \geq 3$ and $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_\ell)$ is a cts repn

w/ 1) $\det \bar{\rho} = \bar{\varepsilon}_\ell$

2) $\bar{\rho}$ unramified away from $2, \ell$

3) $\bar{\rho}|_{G_{\mathbb{Q}_2}} = \begin{pmatrix} \bar{\varepsilon}_\ell & * \\ 0 & 1 \end{pmatrix} \otimes \delta$, $\delta^2=1$, δ unramified

4) $\bar{r}|_{G_{\mathbb{Q}_\ell}} = \begin{cases} \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \bar{\sigma} & , \bar{\sigma} \text{ as before} \\ \text{or } \begin{pmatrix} \bar{\chi}^{-1}\bar{\varepsilon}_\ell & * \\ 0 & \bar{\chi} \end{pmatrix} & , \bar{\chi} \text{ unramified} \\ & * \text{ peu-ramifié.} \end{cases}$

then $\bar{r}$ is reducible

<u>Thm B.</u> Suppose $\bar{r} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_3)$ is a cts rep'n satisfying

1) - 4) or thm A, then $\bar{r} \sim \begin{pmatrix} \bar{\varepsilon}_3 & * \\ 0 & 1 \end{pmatrix}$

$L = \overline{\mathbb{Q}}^{\ker \bar{r}}$

$$\begin{tikzcd} L \arrow[d, dash, "\text{finite Galois}"] \\ \mathbb{Q} \end{tikzcd}$$

$$|D_{L|\mathbb{Q}}|^{\frac{1}{[L:\mathbb{Q}]}} < 8.25$$

<u>Poitou</u> $\Rightarrow$ $L|\mathbb{Q}$ small degree.

$\downarrow$

Minkowski's thm

If $[L:\mathbb{Q}] = n$ and L is totally complex,

$$|D_{L|\mathbb{Q}}|^{\frac{1}{n}} \geq \frac{\pi}{4}\frac{n^2}{(n!)^{2/n}}$$

<u>Lecture 3</u> Thm B Suppose $\bar{r} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_3)$ satisfying

1) $\det \bar{r} = \bar{\varepsilon}_3$, 2) $\bar{r}$ unramified away from 6

3) $\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} \bar{\varepsilon}_3 & * \\ 0 & 1 \end{pmatrix} \otimes \delta$, $\delta^2 = 1$, δ unramified,

case A

4) $\bar{r}|_{G_{\mathbb{Q}_3}} = \mathrm{Ind}_{G_{\mathbb{Q}_9}}^{G_{\mathbb{Q}_3}} \bar{\theta}$, $\bar{\theta} : G_{\mathbb{Q}_9} \twoheadrightarrow \mathbb{F}_9^\times$ totally ramified

or $\begin{pmatrix} \bar{\chi}^{-1}\bar{\varepsilon}_3 & * \\ 0 & \bar{\chi} \end{pmatrix}$ ← Case B $*$ unramified, $\bar{\chi}$ unramified. Case C $*$ ramified peuramifié

Then $\bar{r} \sim \begin{pmatrix} \bar{\varepsilon}_3 & * \\ 0 & 1 \end{pmatrix}$ is reducible.

Pf. $L = \bar{\mathbb{Q}}^{\ker \bar{r}} | \mathbb{Q}$ lack of ramification $\Rightarrow |D_{L|\mathbb{Q}}|$ is small

$\Rightarrow [L:\mathbb{Q}]$ small

$K|\mathbb{Q}_p$ finite, $\mathcal{O}_K \rhd \mathfrak{p}_K$, $\mathcal{O}_K/\mathfrak{p}_K = k$.

$\mathcal{O}_{\bar{K}} \rhd \mathfrak{p}_{\bar{K}}$, $\mathcal{O}_{\bar{K}}/\mathfrak{p}_{\bar{K}} = \bar{k}$

$$\begin{tikzcd} & \text{inertia} \arrow[d] & \mathrm{Gal}(\bar{K}|K) \arrow[d, phantom, "\shortparallel"] & & \\ 1 \arrow[r] & I_K \arrow[r] & G_K \arrow[r] & G_k \arrow[r] \arrow[d, phantom, "\shortparallel"] & 1 \\ & & & \overline{\langle \mathrm{Frob}_K \rangle} & \end{tikzcd}$$

$(\mathrm{Frob}_K(\alpha))^{\# k} = \alpha$

$P_K \triangleleft I_K$

$\overset{?}{\text{Sylow pro-}p\text{ subgp}}$ $\quad I_K/P_K \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$

$\sigma \in I_K/P_K$

$\mathrm{Frob}_K \sigma = (\# k)^{-1} \sigma$

$L|K$ finite Galois, $\sigma \in \mathrm{Gal}(L|K)$

$$\mathcal{O}_L \rhd \underset{\overset{\parallel}{\mathfrak{p}_L^{v_L(\mathfrak{A}_\sigma)}}}{\mathfrak{A}_\sigma} = \langle \sigma\alpha - \alpha : \alpha \in \mathcal{O}_L \rangle$$

(Fontaine's notation)

$$i_{L|K}(\sigma) = v_L(\mathfrak{A}_\sigma)$$

$$i_{L|K}(\tau\sigma\tau^{-1}) = i_{L|K}(\sigma)$$

$$i_{L|K}(\sigma\tau) \geq \min(i_{L|K}(\sigma), i_{L|K}(\tau))$$

$$\sigma\tau\alpha - \alpha = [\sigma(\tau\alpha) - \tau\alpha] + [\tau\alpha - \alpha]$$

$$\mathrm{Gal}(L|K)_i = \{\sigma \in \mathrm{Gal}(L|K) : i_{L|K}(\sigma) \geq i\}$$

$$\mathrm{Gal}(L|K)_0 = \mathrm{Gal}(L|K)$$

$$\mathrm{Gal}(L|K)_1 = I_{L|K} \twoheadleftarrow I_K$$

$$\mathrm{Gal}(L|K)_2 = P_{L|K} \twoheadleftarrow P_K$$

$$\mathrm{Gal}(L|K)_i = \{id\}, \ i \gg 0$$

$$\mathrm{Gal}(L|K)_i / \mathrm{Gal}(L|K)_{i+1} \cong \mathrm{Gal}(k_L|k_K) \qquad \text{if } i=0$$

$$\hookrightarrow k_L^\times \qquad \text{if } i=1$$

$$\sigma \longmapsto \frac{\sigma(\pi_L)}{\pi_L} \bmod (\pi_L) = \mathfrak{p}_L$$

$$\hookrightarrow k_L \qquad \text{if } i>1$$

$$\sigma \longmapsto \frac{\sigma(\pi_L) - \pi_L}{\pi_L^i} \bmod \mathfrak{p}_L$$

$e = e_{L|K} = \# I_{L|K}$, $\mathfrak{p}_K \mathcal{O}_L = \mathfrak{p}_L^e$

$$u_{L|K}(\sigma) = \frac{1}{e} \sum_{\tau \in \mathrm{Gal}(L|K)_1} \min\left(i_{L|K}(\tau), i_{L|K}(\sigma)\right) \in \frac{1}{e}\mathbb{Z}$$

$u_{L|K}(\sigma\tau\sigma^{-1}) = u_{L|K}(\tau)$, $u_{L|K}(\sigma\tau) \geq \min\left(u_{L|K}(\sigma), u_{L|K}(\tau)\right)$

$\mathrm{Gal}(L|K)^u = \{\sigma \in \mathrm{Gal}(L|K) : u_{L|K}(\sigma) \geq u\} \triangleleft \mathrm{Gal}(L|K)$

for any $u \in \mathbb{R}_{\geq 0}$

$\mathrm{Gal}(L|K)^0 = \mathrm{Gal}(L|K)$

$\mathrm{Gal}(L|K)^u = I_{L|K}$ if $u \in (0, 1]$.

$\mathrm{Gal}(L|K)^u \subset P_{L|K}$ if $u > 1$.

Prop. If $M \supset L \supset K$ are finite Galois,

then $\mathrm{Gal}(M|K)^u \twoheadrightarrow \mathrm{Gal}(L|K)^u$

$$G_K^u := \varprojlim_{\substack{L|K \\ \text{finite} \\ \text{Galois}}} \mathrm{Gal}(L|K)^u .$$

$\mathcal{D}_{L|K}^{-1} = \{\alpha \in L : \mathrm{tr}(\alpha \mathcal{O}_L) \subset \mathcal{O}_K\} \supset \mathcal{O}_L$

fractional ideal

$\mathcal{D}_{L|K} = \{\alpha \in L : \alpha \mathcal{D}_{L|K}^{-1} \subset \mathcal{O}_L\} \triangleleft \mathcal{O}_L$

different.

$$D_{L|K} = N_{L|K}\, \mathcal{D}_{L|K}.$$

Prop. $$v_L(\mathcal{D}_{L|K}) = e \max_{1\neq\sigma\in \mathrm{Gal}(L|K)} u_{L|K}(\sigma) - \max_{1\neq\sigma\in \mathrm{Gal}(L|K)} i_{L|K}(\sigma)$$

Cor $$\frac{1}{[L:K]} v_K(D_{L|K}) = \max_{1\neq\sigma\in \mathrm{Gal}(L|K)} u_{L|K}(\sigma) - \frac{1}{e} \max_{1\neq\sigma\in \mathrm{Gal}(L|K)} i_{L|K}(\sigma).$$

Global.

$K|\mathbb{Q}$ finite

$L|K$ finite Galois - same defn of $\mathcal{D}_{L|K}$, $D_{L|K}$.

$w|v$ place of K

decomp. gp $\nearrow$
$$\begin{tikzcd} \mathrm{Gal}(L|K)_w \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Gal}(L_w|K_v) \\ \mathrm{Stab}_{\mathrm{Gal}(L|K)}(w) & \end{tikzcd}$$

$\mathrm{Gal}(L|K)$ acts transitively on places of L above v.

$$\mathcal{D}_{L|K,w} \xrightarrow{\sim} \mathcal{D}_{L_w|K_v}$$

$$D_{L|K,v} = \prod_{w|v} D_{L_w|K_v}$$

$$\frac{1}{\underset{\underset{(\#\text{ places above } v)[L_w:K_v]}{\parallel}}{[L:K]}} v(D_{L|K}) = \max_{1\neq\sigma\in \mathrm{Gal}(L_w|K_v)} u_{L_w|K_v}(\sigma) - \frac{1}{e}\max_{1\neq\sigma\in \mathrm{Gal}(L_w|K_v)} i_{L_w|K_v}(\sigma)$$

eg. if v tamely ramified in L, get $1-\frac{1}{e}$

$\bar{\rho}$, $L = \bar{\mathbb{Q}}^{\ker \bar{\rho}}$, $D_{L|\mathbb{Q}}$, $n=[L:\mathbb{Q}]$

totally complex. $\bar{\rho}(c) \sim \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Case A locally at 3, Ind $\bar{\theta}$, $\bar{\theta}$. has order 8, $\mathbb{F}_9^\times$

tamely ramified

locally at 2, I_2 acts $\begin{pmatrix}1&*\\0&1\end{pmatrix}$ order 3 or 1.

tamely ramified

$$|D_{L|\mathbb{Q}}|^{\frac{1}{n}} \leq 2^{1-1/3}\, 3^{1-1/8} < 4.16$$

Minkowski: $|D_{L|\mathbb{Q}}|^{\frac{1}{n}} \geq \frac{\pi}{4} \frac{n^2}{(n!)^{2/n}} \Rightarrow n \leq 14$

↯ $16 = [L_w|\mathbb{Q}_3] \,\big|\, [L:\mathbb{Q}] = n$

Case B. $\bar{\rho}|_{G_{\mathbb{Q}_3}} = \begin{pmatrix}\bar{\chi}^{-1}\bar{\varepsilon}_3 & *\\ 0 & \bar{\chi}\end{pmatrix}$ tamely ramified

$\therefore e_3 = 2$

$$|D_{L|\mathbb{Q}}|^{\frac{1}{n}} \leq 2^{2/3}\, 3^{1-1/2} < 2.75$$

$\bar{\chi}_i$ unramified away from 3

Minkowski $\Rightarrow n < 6$, $2|n \Rightarrow n = 2$ or 4. $\bar{\chi}_i = \bar{\varepsilon}_3$ or 1

Gal$(L|\mathbb{Q})$ abelian, and $\bar{\rho}$ is semisimple $\therefore \bar{\rho} = \bar{\chi}_1 \oplus \bar{\chi}_2$

$\bar\chi_1 \bar\varepsilon_3^{-1}$ or $\bar\chi_1$ unramified everywhere $\therefore$ trivial

$\Rightarrow$ $\bar\rho = 1 \oplus \bar\varepsilon_3$.

Case C. $\bar\rho|_{G_{\mathbb{Q}_3}} \sim \begin{pmatrix} \bar\chi^{-1}\bar\varepsilon_3 & * \\ 0 & \bar\chi \end{pmatrix}$ $\bar\chi$ nr, $*$ ramified but peu ramifié

$w|3$

$G_K^{3/2}$ fixes L_w for $w|3$.

$$\frac{1}{[L:\mathbb{Q}]} v_3(D_{L|\mathbb{Q}}) \le 3/2$$

$$|D_{L|\mathbb{Q}}|^{\frac{1}{n}} \le 2^{2/3}\, 3^{3/2} < 8.25$$

Poitou: $n \le 14$, $6|n$ $\Rightarrow$ $n = 6$ or 12

Lecture 4. Thm If $\bar\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\bar{\mathbb{F}}_3)$ cts s.t.

1) $\det \bar\rho = \bar\varepsilon_3$

2) unramified outside 6

3) $\bar\rho|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} \bar\varepsilon_3 & * \\ 0 & 1 \end{pmatrix} \otimes \delta$, $\delta^2 = 1$, δ unramified

4) $\bar\rho|_{G_{\mathbb{Q}_3}} \sim$
   1) —
   2) —
   3) $\begin{pmatrix} \bar\chi^{-1}\bar\varepsilon_3 & * \\ 0 & \bar\chi \end{pmatrix}$ $\bar\chi$ unramified wildly ramified but peu ramifié

then $\bar\rho \sim \begin{pmatrix} \bar\varepsilon_3 & * \\ 0 & 1 \end{pmatrix}$

Pf. $L = \overline{\mathbb{Q}}^{\ker \bar{\rho}}$, $|D_{L|\mathbb{Q}}|^{\frac{1}{[L:\mathbb{Q}]}} \leq 2^{2/3} 3^{u}$

$w | 3$ $\nearrow$

$\mathrm{Gal}(L_w | \mathbb{Q}_3)^{u}$

u max'l s.t. this $\neq \{1\}$

Claim. $u \leq \frac{3}{2}$

$$\mathbb{Q}_3^m(\zeta_3, \{\sqrt[3]{\alpha_i}\}), \quad \begin{array}{ccc} \alpha_i \in & \mathcal{O}_{\mathbb{Q}_3^m}^{\times} & \subset (\mathbb{Q}_3^m)^{\times} \\ \cap & \cup & \\ & 1 + 3\mathcal{O}_{\mathbb{Q}_3^m} & \end{array}$$

Need to show $\mathrm{Gal}(\overline{\mathbb{Q}_3} | \mathbb{Q}_3)^{\frac{3}{2}+\varepsilon}$ acts trivially on $\underbrace{\mathbb{Q}_3^m(\zeta_3, \sqrt[3]{\alpha_i})}_{L_i}$, $\forall i$

$\mathrm{Gal}(L_i | \mathbb{Q}_3^m)$

$\alpha_i \equiv 1 \ (3)$

$\alpha_i \not\equiv 1 \ (9)$

$\alpha_i \equiv 1 \pmod 9$

$\Rightarrow \alpha_i$ a cube in $\mathbb{Q}_3^m$

$$\pi = \frac{\zeta_3 - 1}{\sqrt[3]{\alpha_i} - 1}$$

$$\alpha_i^{1/3} = 1 + \frac{1}{3}(\alpha_i - 1) + \frac{\frac{1}{3} \cdot \frac{-2}{3}}{2!}(\alpha_i - 1)^2 + \cdots$$

$\Rightarrow L_i = \mathbb{Q}_3^m(\zeta_3)$ tamely-ramified

min. poly.

$$\frac{(\alpha_i - 1)^2}{9} x^6 + (\alpha_i - 1) x^5 + (4 - \alpha_i) x^4 + (\alpha_i - 10) x^3 + 12x^2 - 9x + 3$$

Eisenstein

$i_{L_i | \mathbb{Q}_3^m}(\sigma) = 1$

$\downarrow$

$\sigma, \sigma\tau, \sigma\tau^2$ have order 2.

$\notin \mathrm{Gal}(L_i | \mathbb{Q}_3^m)^{1+\varepsilon}, \forall \varepsilon > 0$

$[L_i : \mathbb{Q}_3^m] = 6$, $\mathcal{O}_{L_i} = \mathcal{O}_{\mathbb{Q}_3^m}[\pi]$

$\sigma\zeta_3 = \zeta_3^{-1}$, $\sigma\sqrt[3]{\alpha_i} = \sqrt[3]{\alpha_i}$

$\mathrm{Gal}(L_i | \mathbb{Q}_3^m) = \langle \sigma, \tau : \sigma^2 = 1, \tau^3 = 1, \sigma\tau\sigma^{-1} = \tau^2 \rangle$ $\tau\zeta_3 = \zeta_3$, $\tau\sqrt[3]{\alpha_i} = \zeta_3\sqrt[3]{\alpha_i}$

$$i_{L_i|\mathbb{Q}_3^m}(\tau) = v_{L_i}(\tau\pi - \pi) = v_{L_i}\left(\frac{\zeta_3 - 1}{\zeta_3\sqrt[3]{\alpha_i} - 1} - \frac{\zeta_3 - 1}{\sqrt[3]{\alpha_i} - 1}\right)$$

$$= v_{L_i}\left(\pi\left(\frac{\sqrt[3]{\alpha_i} - 1}{\zeta_3\sqrt[3]{\alpha_i} - 1} - 1\right)\right)$$

$$= v_{L_i}\left(\pi\left(\frac{(\zeta_3 - 1)\sqrt[3]{\alpha_i}}{\zeta_3\sqrt[3]{\alpha_i} - 1}\right)\right)$$

$$= v_{L_i}\left(\pi\,\tau_\pi\,\sqrt[3]{\alpha_i}\right) = 2$$

$$u_{L_i|\mathbb{Q}_3^m}(\tau) = \frac{1}{6}(1+1+1+2+2+2) = \frac{3}{2}$$

same for τ^2

$$\max_{1\neq\rho\in Gal(L_i|\mathbb{Q}_3^m)} u_{L|K}(\rho) = \frac{3}{2}$$

$$\therefore\ Gal(L_i|\mathbb{Q}_3^m)^{\frac{3}{2}+\varepsilon} = \{1\},\ \forall\varepsilon>0.$$

$$|D_{L|\mathbb{Q}}|^{\frac{1}{[L:\mathbb{Q}]}} \leq 2^{2/3}\,3^{3/2} < 8.25$$

Poitou $\Rightarrow [L:\mathbb{Q}] \leq 14$, $6 \mid [L:\mathbb{Q}] \Rightarrow [L:\mathbb{Q}] = 6$ or 12

$L > \mathbb{Q}(\zeta_3)$, $[L:\mathbb{Q}(\zeta_3)] = 3$ or 6

$Gal(L|\mathbb{Q}(\zeta_3))$ has a unique Sylow 3-subgp, say $\langle\sigma\rangle$

and $\langle\sigma\rangle \trianglelefteq Gal(L|\mathbb{Q})$, $\ker(\bar{\tau}(\sigma)-1) \overset{\shortparallel}{} \ker(\bar{\tau}(\sigma^2)-1)$ line in $\bar{\mathbb{F}}_3^2$

$$\begin{array}{c}\ker(\bar{\tau}(\sigma)-1)\\ \shortparallel \\ \ker(\bar{\tau}(\sigma^2)-1)\end{array}$$

∴ $\ker(\bar{r}(\sigma)-1)$ is invariant by $\underset{\tau}{\underset{\cup}{\mathrm{Gal}(L|\mathbb{Q})}}$

$$\tau \ker(\bar{r}(\sigma)-1) = \ker(\bar{r}(\tau\sigma\tau^{-1})-1) = \ker(\bar{r}(\sigma)-1)$$

∴ $\bar{r} \sim \begin{pmatrix} \bar{\chi}_1 & * \\ 0 & \bar{\chi}_2 \end{pmatrix}$

$\bar{\chi}_i$ unramified away from 3 and $\bar{\chi}_i$ or $\bar{\chi}_i \bar{\varepsilon}_3^{-1}$ is unramified @ 3

$\bar{\chi}_i = 1, \bar{\varepsilon}_3$
both occur

∴ $\bar{r} \sim \begin{pmatrix} \bar{\varepsilon}_3 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_3 \end{pmatrix}$ ← restrict to $I_{\mathbb{Q}_3}$
has inv't line
has a line where σ acts as $\bar{\varepsilon}_3(\sigma)$,
$\forall \sigma \in I_{\mathbb{Q}_3}$
∴ $\bar{r}|_{I_{\mathbb{Q}_3}} = 1 \oplus \bar{\varepsilon}_3$, # $\bar{r}$ wildly ramified @ 3.

$$\bar{r}|_{G_{\mathbb{Q}_\ell}} = \begin{cases} \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \bar{\theta} \\ \\ \begin{pmatrix} \bar{\chi}^{-1}\bar{\varepsilon}_\ell & * \\ 0 & \bar{\chi} \end{pmatrix} \end{cases}$$ ← true for $\bar{r}_{E,\ell}$ if $E|\mathbb{Q}_\ell$
has reduction at ℓ.

Fontaine - Laffaille $F|\mathbb{Q}_\ell$ unramified finite, $\overset{\mathcal{O}_F}{\diagup}$ G_F

$L|\mathbb{Q}_\ell$ finite, integers $\mathcal{O}$, max'l ideal λ, $\mathbb{F} = \mathcal{O}/\lambda$, $G_F \to GL_n(\mathcal{O}/\lambda^m)$

$\mathcal{MF}$ = Category of f.g. $\mathcal{O}_F \underset{\mathbb{Z}_\ell}{\otimes} \mathcal{O}$-modules M w/ decreasing filtration

$\mathrm{Fil}^i M$ w/ $\mathrm{Fil}^0 M = M$

$\mathrm{Fil}^{\ell-1} M = (0)$

together w/ $\mathrm{Frob}_\ell^{-1} \otimes 1$ - linear maps $\Phi^i : \mathrm{Fil}^i M \to M$

s.t. $\Phi^i \big|_{\mathrm{Fil}^{i+1} M} = \ell \, \Phi^{i+1}$

and $\langle \mathrm{Im}\, \Phi^i ;\ i = 0, \dots, \ell-1 \rangle_{\mathcal{O}_F \otimes \mathcal{O}} = M$

- $\mathcal{MF}$ is an abelian cat.

- $\exists\ G :\ \mathcal{MF} \longrightarrow$ f.g. $\mathcal{O}$-modules w/ a cts linear action of G_F

  exact covariant compatible w/ $\otimes$ products when defined.

s.t. 1) G.. is fully faithful, commutes w/ filtered inverse limits

2) The essential image is closed under $\oplus$, subobjects + quotients

3) $\lg_{\mathcal{O}}(M) = [F : \mathbb{Q}_\ell] \lg_{\mathcal{O}} G(M)$

4) $\mathrm{Fil}^i(M) = (0) \Rightarrow$ $G(M)$ is unramified

5) If $0 \leq n < \ell - 1$, then $\mathcal{O}(-n) = G\big((\mathcal{O}_F \otimes \mathcal{O})[-n]\big)$

$$\uparrow$$

$$\mathrm{gr}^i (\mathcal{O}_F \otimes \mathcal{O})[-n] = \begin{cases} 0, & i \neq n \\ \mathcal{O}_F \otimes \mathcal{O}, & i = n \end{cases}$$

$$\Phi^i = \ell^{n-i} \left(\mathrm{Frob}_\ell^{-1} \otimes 1 \right)$$

6) If $\ell>2$ and $\mathcal{O}=\mathbb{Z}_\ell$ $E|F$ is an elliptic curve w/ good reduction,

$$(T_\ell E)^\vee \cong G(M_E)$$

M_E is free rk 2 over $\mathcal{O}_F\otimes\mathcal{O}$,

$gr^i M_E$ is rank 1 if $i=0,1$

0, else

$\Lambda^2 M_E = (\mathcal{O}_F\otimes\mathcal{O})(-1)$

7) If $F=\mathbb{Q}_\ell$ and M a $\mathcal{O}_F\otimes\underset{\mathcal{O}/\lambda}{\overset{\shortparallel}{\mathbb{F}}}$ -module in $\mathcal{MF}$

w/ $gr^i M \cong \begin{cases} \mathcal{O}_F\otimes\mathbb{F}, & \text{if } i=0,1 \\ 0, & i>1 \end{cases}$ eg. $M_E/\ell M_E$

then $G(M)\cong \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}}\bar\theta$, where

$$\begin{tikzcd} \bar\theta|_{I_{\mathbb{Q}_{\ell^2}}}: I_{\mathbb{Q}_{\ell^2}} \arrow[r, two heads] & I_{\mathbb{Q}_{\ell^2}}^{ur}/\mathbb{Q}_{\ell^2} \arrow[r, phantom, "\cong"] & \mathbb{Z}_{\ell^2}^\times \arrow[l, bend right, "\text{Art}"'] \arrow[d, "\text{reduction}"] \\ & & \mathbb{F}_{\ell^2}^\times \end{tikzcd}$$

or $\begin{pmatrix}\bar\chi_1\bar\varepsilon_\ell & * \\ 0 & \bar\chi_2\end{pmatrix}$ w/ $\bar\chi_i$ unramified and $*$ peu-ramifié.

$\bar r|_{G_{\mathbb{Q}_\ell}}$ in image of G...

<u>Lecture 5</u> <u>Thm B</u>. $\mathbb{F}|\mathbb{F}_3$ finite ext, $\bar r: G_{\mathbb{Q}} \to GL_2(\mathbb{F})$

- $\det\bar r = \bar\varepsilon_3^{-1}$
- $\bar r$ unramified outside 6
- $\bar r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix}1 & * \\ 0 & \bar\varepsilon_3^{-1}\end{pmatrix}\otimes\delta$, $\delta^2=1$, δ unramified
- $\bar r|_{G_{\mathbb{Q}_3}} = G(M)$, where $M\in\mathcal{MF}_{\mathbb{Q}_3}$, $\mathrm{rk}_{\mathbb{F}}\, gr^i M = \begin{cases}1, & i=0,1 \\ 0, & \text{else}\end{cases}$

then $\bar r \sim \begin{pmatrix}1 & * \\ 0 & \bar\varepsilon_3^{-1}\end{pmatrix}$

Thm B'. Suppose $L | \mathbb{Q}_3$ finite ext, integers $\mathcal{O}$, and

$r : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O})$ cts rep. s.t.

1) $\det r = \varepsilon_3^{-1}$

2) r unramified outside 6

3) $r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_3^{-1} \end{pmatrix} \otimes \delta$, $\delta^2 = 1$, δ unramified

4) $r|_{G_{\mathbb{Q}_3}} = \mathbb{G}(M)$, $M \in \mathcal{MF}_{\mathbb{Q}_3, \mathcal{O}}$, $\mathrm{rk}_{\mathcal{O}}\, \mathrm{gr}^i M = \begin{cases} 1, & \text{if } i = 0, 1 \\ 0, & \text{else} \end{cases}$

then $(r \otimes L)^{ss} = 1 \oplus \varepsilon_3^{-1}$.

Pf. Consider $G_{\mathbb{Q}}$-stable $\mathcal{O}$-modules $\Lambda \subset \mathcal{O}^{\oplus 2}$ s.t. $\mathrm{Im}\left(\Lambda \longrightarrow \left((\mathcal{O}/\lambda)^{\oplus 2}\right)\right)$

$\cup$

$\left((\mathcal{O}/\lambda)^{\oplus 2}\right)^{G_{\mathbb{Q}}}$

$\wr\wr$

$\mathcal{O}/\lambda \ni \bar{e} \neq 0$

$r \bmod \lambda \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_3^{-1} \end{pmatrix}$

$\mathcal{X} = \{ \text{such } \Lambda \}$ ordered by inclusion

Chain $\{\Lambda_i\}$, $\Lambda = \cap \Lambda_i \in \mathcal{X}$

$e_i \in \Lambda_i \subset \mathcal{O}^{\oplus 2}$, $\longmapsto \bar{e}$ convergent subsequence

$e_j \to e \in \Lambda$

$\therefore e \mapsto \bar{e}$

$\exists\, e \in L$ s.t. $e \mapsto \bar{e}$

Zorn $\Rightarrow \exists\, \Lambda \in \mathcal{X}$ minimal.

2 choices : Λ rank 2, $\Lambda \longrightarrow \overbrace{\Lambda \otimes_{\mathcal{O}} \mathbb{F}}$ action on here has a line L on which $G_{\mathbb{Q}}$ acts by 1

$$\underset{\underset{e}{\downarrow}}{\Lambda'} = \ker\left(\Lambda \longrightarrow (\Lambda \otimes \mathbb{F})/L\right) \subsetneq \Lambda$$

$\Rightarrow \Lambda' \in \mathfrak{X}$ ✡

$\therefore \operatorname{rk} \Lambda = 1$ $\qquad$ $\underset{\underset{\mathcal{O}}{\wr\wr}}{\Lambda} \subset \mathcal{O}^2$ is invt by $G_{\mathbb{Q}}$

$\therefore \rho \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ $\qquad$ χ_i unramified at 2.

$\chi_i = G(M_i)$, $\operatorname{rk}_{\mathcal{O}} M_i = 1$, $\operatorname{gr}^j M_i = \begin{cases} 0 & \text{for all but one } j \\ 1 & , \ j = 0, 1 \end{cases}$

$j = 0$, χ_i unramified @ 3

$j = 1$ $\chi_i \otimes \varepsilon_3$ unramified @ 3

as $\mathbb{Q}$ has no everywhere unramified ext, $\chi_i = 1$ or ε_3^{-1}

$\therefore \rho \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_3^{-1} \end{pmatrix}$ or $\begin{pmatrix} \varepsilon_3^{-1} & * \\ 0 & 1 \end{pmatrix}$

<u>Thm A</u> $\quad \ell \geq 3$ prime, $\bar\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_\ell)$ cts rep.

s.t. $\det \bar\rho = \bar\varepsilon_\ell^{-1}$, $\bar\rho$ unramified outside 2ℓ,

$\bar\rho|_{G_{\mathbb{Q}_2}} = \begin{pmatrix} 1 & * \\ 0 & \bar\varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta$, $\delta^2 = 1$, δ ur.

$\bar\rho|_{G_{\mathbb{Q}_\ell}} = G(M)$, $M \in \mathcal{MF}_{\mathbb{Q}_\ell}$, $\operatorname{rk}_{\mathbb{F}_\ell} \operatorname{gr}^i M = \begin{cases} 1 ; & i = 0, 1 \\ 0 ; & \text{else.} \end{cases}$

$\Rightarrow \bar\rho$ reducible

Step 1. Show $\exists\, L \,|\, \mathbb{Q}_\ell$ finite, $\mathcal{O} = \mathcal{O}_L{}^{\triangleright(\lambda)}$, $r : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$

s.t. 1) $\det r = \varepsilon_\ell^{-1}$

2) r unramified outside 2ℓ

3) $r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta$, $\delta^2 = 1$, δ nr

4) $r|_{G_{\mathbb{Q}_\ell}} = G(M)$, $\operatorname{rk}_{\mathcal{O}} \operatorname{gr}^i M = \begin{cases} 1, & i = 0, 1 \\ 0, & \text{else} \end{cases}$

5) $r \bmod \lambda \cong \bar{r}$.

Step 2. $\exists$ M a number field, and for each prime μ of M,

a cts rep. $r_\mu : G_{\mathbb{Q}} \to GL_2(M_\mu)$ s.t. each r_μ satisfies

1)–4) (ℓ replaced by residue char. of μ)

- $r \cong r_\lambda$ for some $\lambda \,|\, \ell$.
- If $p \nmid 2 N(\mu\mu')$, then $\operatorname{tr} r_\mu(\mathrm{Frob}_p) = \operatorname{tr} r_{\mu'}(\mathrm{Frob}_p)$, $\forall \mu, \mu'$.

then choose $\mu | 3$, $r_\mu^{ss} \sim 1 \oplus \varepsilon_3^{-1}$

$p \nmid 6\ell$, $\operatorname{tr} r_\mu(\mathrm{Frob}_p) = 1 + p$

$\Rightarrow \operatorname{tr}(r = r_\lambda)(\mathrm{Frob}_p) = 1 + p = \operatorname{tr}(1 \oplus \varepsilon_\ell^{-1})(\mathrm{Frob}_p)$

Cebotarev Thm $\Rightarrow$ $\operatorname{tr} r = \operatorname{tr}(1 \oplus \varepsilon_\ell^{-1}) \Rightarrow r^{ss} \cong 1 \oplus \varepsilon_\ell^{-1} \Rightarrow \bar{r}$ reducible

$\mathcal{C}_{\mathcal{O}}$ = cat. of noetherian complete local $\mathcal{O}$-algs R

s.t. $\mathbb{F} = \mathcal{O}/\lambda \xrightarrow{\sim} R/m$.

$D : \mathcal{C}_{\mathcal{O}} \longrightarrow \text{Sets}$

$$R \longmapsto \left\{ \begin{array}{l} r : G_{\mathbb{Q}} \to GL_2(R) \text{ cts rep.} \\ r \bmod m_R = \bar{r} \\ \det r = \varepsilon_\ell^{-1} \\ r \text{ unramified outside } 2, \ell \\ r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta, \ \delta^2 = 1, \ \delta \text{ m} \\ \forall J \lhd R \text{ open}, \ \underset{\substack{\text{w/ action} \\ \text{of } G_{\mathbb{Q}_\ell}}}{(R/J)^2} = G(M_J) \end{array} \right\} \Big/ \sim$$

$\sim$ = conjugation by elements of $\ker(GL_2(R) \to GL_2(\mathbb{F}))$.

Prop. If $\bar{r}$ is absolutely irreducible, then D is representable

$(R^{univ}, [r^{univ}])$

1) $\underline{\dim R^{univ} \geq H^1(G_{\mathbb{Q}} \cdots) - H^2(G_{\mathbb{Q}}, \cdots) = 1}$

2) R^{univ} a f.g. $\mathcal{O}$-module.

$\Big] \ \exists\, R^{univ} \longrightarrow \mathcal{O}_{L'}$
L'/L finite.

$\phi \circ r^{univ}$ will do.

Lecture 6

$\mathcal{C}_{\mathcal{O}}$ = complete local noeth. $\mathcal{O}$-algs s.t. $\mathcal{O}/\lambda \xrightarrow{\sim} R/m$.

$L/\mathbb{Q}_\ell$ finite
$\mathcal{O} = \mathcal{O}_L$
$\mathcal{O}/\lambda = \mathbb{F}$

$S = \{2, \ell\}$

$$\begin{tikzcd} \mathbb{Q}_S \arrow[d, dash, "\text{max'l ext. unramified outside } S"] \\ \mathbb{Q} \end{tikzcd}$$

$D : \mathcal{C}_{\mathcal{O}} \longrightarrow \underline{Sets}$, $\quad G_{\mathbb{Q},S} := Gal(\mathbb{Q}_S|\mathbb{Q})$

$$R \longmapsto \left\{ \begin{array}{l} \text{cts liftings } r : G_{\mathbb{Q},S} \to GL_2(R) \text{ of } \bar{r} \\ \text{s.t. A) } \det r = \varepsilon_\ell^{-1} \\ \quad\;\; \text{B) } r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \\ \quad\;\; \text{C). } \forall J \lhd R \text{ open, } r \bmod J \in \text{Im of } G \end{array} \right\} \Big/ \sim$$

$r \sim ArA^{-1}$

for $A \in \ker(GL_2(R) \to GL_2(R/m_R))$.

Prop. D is represented by $(R^{univ}, [D^{univ}])$.

Ideas of proof. forgets A, B, C. – call it D_0

D_0 is pro-representable if $\bar{r}$ is absolutely irred. $(R_0^{univ}, [r_0^{univ}])$

it is representable if $\forall\, H \underset{open}{\subset} G_{\mathbb{Q},S}$

$$H / \overline{\langle h_1 h_2 h_1^{-1} h_2^{-1},\ h_1^{\ell} : h_1, h_2 \in H \rangle} \text{ is finite.}$$

$$\begin{tikzcd} \mathbb{Q}_S \arrow[d, dash, "H"'] \\ K \arrow[d, dash, "\text{finite}"'] & \cong \text{ some class gp} \\ \mathbb{Q} \end{tikzcd}$$

to impose A, B, C, we need to check:

i) $\bar{\rho}$ has the property and the property preserved under conjugation.

ii) ρ is a lifting of $\bar{\rho}$ to R w/ the property, and if $f: R \to S$,

then $f(\rho)$ has the property.

iii) If ρ is a lifting to R and if $I_i \lhd R$ are nested ideals w/

$\bigcap_i I_i = (0)$ and if $\rho \bmod I_i$ has the property, so does ρ.

iv) If R_1, R_2 and ρ_1, ρ_2 are two liftings w/ the property, and if

$I_i \lhd R_i$ and $R_1/I_1 \cong R_2/I_2$ and $\rho_1 \bmod I_1 \cong \rho_2 \bmod I_2$, then

we get $\rho = \rho_1 \times \rho_2 : G_{\mathbb{Q}} \longrightarrow GL_2\left(R_1 \times_{R_1/I_1} R_2\right)$, then ρ has the property.

v). If $R \overset{f}{\hookrightarrow} S$ and ρ is a lifting of $\bar{\rho}$ to R, and if $f \circ \rho$ has the property, so does ρ.

$\mathcal{X} = \left\{ I \lhd R_0^{univ} : \rho_0^{univ} \bmod I \text{ has properties } A, B, C \right\}$

ordered by inclusion.

iii) + Zorn's lemma $\Rightarrow$ $\mathcal{X}$ has a minimal elt I^{min}.

iv) $\Rightarrow$ $\forall I, J \in \mathcal{X}$, $I \cap J \in \mathcal{X}$, $\therefore I^{min} \subset I$, $\forall I \in \mathcal{X}$.

$R^{univ} := R_0^{univ} / I^{min}$.

$r_0^{univ} \rightsquigarrow r$ (R, r)

$$\begin{tikzcd} R_0^{univ} \arrow[r, "\phi"] & R \end{tikzcd}$$

r_0^{univ} mod $\ker\phi$ $\quad r$

$$\begin{tikzcd} R_0^{univ} \arrow[r] \arrow[dr, two heads, "\exists"'] & R_0^{univ}/\ker\phi \arrow[r, hook] & R \\ & R^{min} \arrow[u] & \end{tikzcd}$$

v) $\Rightarrow$ r_0^{un} mod $\ker\phi$ has A, B, C $\therefore$ $\ker\phi \in \mathcal{X}$ $\therefore$ $\ker\phi \supset I^{min}$

Need to check A, B, C satisfy i) – v)

eg. $\sigma \in G_{\mathbb{Q}}$, $\bar{r}(\sigma) = I_2$

D $r(\sigma) \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ does not satisfy iv).

$R_1 = \mathbb{F}[\varepsilon]$, $\varepsilon^2 = 0$, $r_1(\sigma) = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$

$R_2 = \mathbb{F}[\delta]$, $\delta^2 = 0$, $r_2(\sigma) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$] satisfy B.

$R = R_2 \times_{\mathbb{F}} R_2 = \mathbb{F}[\varepsilon, \delta]$, $\varepsilon^2 = 0$
$\delta^2 = 0$
$\varepsilon\delta = 0$

$r(\sigma) = \begin{pmatrix} 1 & \varepsilon \\ \delta & 1 \end{pmatrix}$ $\quad r(\sigma)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $\varepsilon y = 0$
$\delta x = 0$

$\rightarrow x, y \in \mathfrak{m}$ $\therefore \begin{pmatrix} x \\ y \end{pmatrix}$ not part of basis

B. r lifts $\bar{r}$ and if $\phi \in G_{\mathbb{Q}_2}$ lifts Frob_2,

then $\exists$ basis e_1, e_2 of R^2 s.t. $r(\phi) e_1 = \alpha e_1$ $\quad \alpha \equiv 1 \ (\mathfrak{m})$
$r(\phi) e_2 = \beta e_2$ $\quad \beta \equiv 2 \ (\mathfrak{m})$

$r(\phi)$ satisfies its characteristic polynomial $f(X)$

$f(X)$ mod $m \equiv (X-1)(X-2)$.

$f(X) = (X-\alpha)(X-\beta)$, $\alpha \equiv 1\ (m)$, $\beta \equiv 2\ (m)$ (Hensel)

$$e_\beta = \frac{r(\phi)-\alpha}{\beta-\alpha}, \qquad e_\beta^2 = e_\beta \qquad r(\phi)\, e_\beta = \beta e_\beta$$

$$e_\alpha = \frac{r(\phi)-\beta}{\alpha-\beta}, \qquad e_\alpha^2 = e_\alpha \qquad r(\phi)\, e_\alpha = \alpha e_\alpha$$

$$e_\alpha + e_\beta = 1, \quad e_\alpha e_\beta = 0$$

$$R^2 = \underset{\langle e_1 \rangle}{\underset{\parallel}{\operatorname{Im} e_\alpha}} \oplus \underset{\langle e_2 \rangle}{\underset{\parallel}{\operatorname{Im} e_\beta}}$$

- If $r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$ and suppose e_1, e_2 are as above,

then $r|_{G_{\mathbb{Q}_2}}$ w.r.t. e_1, e_2 has the form $\begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$: $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$

s.t. r has the form $\begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$ w.r.t. $\begin{matrix} ae_1 + ce_2 \\ be_1 + de_2 \end{matrix}$

$$\rightsquigarrow \quad \underset{a\alpha e_1 + c\beta e_2}{\underset{\parallel}{r(\phi)(ae_1 + ce_2)}} = ae_1 + ce_2$$

$\Rightarrow c(\beta - 1) = 0 \Rightarrow c = 0$.

$\therefore$ w.r.t. e_1, e_2, $r|_{G_{\mathbb{Q}_2}}$ looks like

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & -\frac{b}{ad} \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$$

B. iii) choose basis e_1, e_2 of R^2, $\rho(\phi) e_1 = \alpha e_1$ $\quad \alpha - 1 \in m$

$\rho(d) e_2 = \beta e_2 \quad \beta - 2 \in m$

ρ mod I_i satisfies B. w.r.t. e_1 mod I_i, e_2 mod I_i,

$\rho|_{G_{Q_2}} = \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$ $\quad \therefore \rho|_{G_{Q_2}}$ w.r.t. e_1, e_2 is $\begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$.

iv) e_1, e_2 ~~of~~ R_1^2

f_1, f_2 of R_2^2 as before. rescaling f_1, f_2

WLOG e_1 mod $I_1 = f_1$ mod I_2

e_2 mod $I_1 = f_2$ mod I_2

$(e_1, f_1), (e_2, f_2)$

$\sigma \in G_{Q_2}$, $\rho(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w.r.t. this basis

image of a in R_1 is 1

a $\quad R_2$ is 1 $\quad \Rightarrow a = 1$

Similarly $c = 0$, $d = \varepsilon_\ell(\sigma)^{-1}$.

v) $e_1, e_2 \cdots$ $f(e_1), f(e_2)$ basis of S^2

$\hat{\imath}$ for $|_{G_{Q_2}} = \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$ w.r.t. this basis

$\sigma \in G_{Q_2}$, $\rho(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $f(a) = 1$ $\quad f(d) = \varepsilon_\ell^{-1}(\sigma)$

$f(c) = 0$. $\quad \Rightarrow a = 1, c = 0, d = \varepsilon_\ell^{-1}(\sigma)$.

$\subseteq$ everything follows as Im G preserved under $\oplus$, sub, qt.

Lecture 7 $\ell > 3$, $S = \{2, \ell\}$

$\bar{r}: G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$, $\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$

$\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta$, $\delta^2 = 1$, δ nr

$\bar{r}|_{G_{\mathbb{Q}_\ell}} = G(\bar{M})$,

$\bar{M} \in \mathcal{MF}$, $\dim_{\mathbb{F}_\ell} gr^i \bar{M} = \begin{cases} 1, & i = 0,1 \\ 0, & i > 1 \end{cases}$

fix such

look for $L | \mathbb{Q}_\ell$, $\mathcal{O}$, $\mathcal{O}/\lambda = \mathbb{F}$,

$r: G_{\mathbb{Q},S} \longrightarrow GL_2(\mathcal{O})$ w/ same properties

$\mathcal{C}_\mathcal{O}$ = complete noeth. local $\mathcal{O}$-algs R s.t. $\mathcal{O}/\lambda \xrightarrow{\sim} R/m$

$R \in Ob(\mathcal{C}_\mathcal{O})$, consider $r: G_{\mathbb{Q},S} \longrightarrow GL_2(R)$ s.t. $r \bmod m = \bar{r}$

s.t. A) $\det r = \varepsilon_\ell^{-1}$

B) $r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta$. $\delta^2 = 1$, δ nr

c) $\forall J \triangleleft R$ open ideal, $r \bmod J \in Im\, G$

write $r \sim A r A^{-1}$ if $A \in \ker(GL_2(R) \longrightarrow GL_2(\mathbb{F}))$

$r^{univ}: G_{\mathbb{Q},S} \longrightarrow GL_2(R^{univ})$ (we want $R^{univ} \longrightarrow \mathcal{O}_{\bar{L}}$)

$m \triangleleft R^{univ}$ max ideal, $m/\langle \lambda, m^2 \rangle$ $\overset{\mathbb{F}}{\text{has}}$ a basis $\bar{y}_1, \dots, \bar{y}_d$ for some d.

$y_i \in m \longmapsto \bar{y}_i$

claim:

$\mathcal{O}[[x_1, \dots, x_d]] \twoheadrightarrow R^{univ}$, kernel I

$x_i \longmapsto y_i$

If $\overset{\text{Lemma:}}{A}, B \in \mathcal{C}_{\mathcal{O}}$ and $\phi : A \to B$ and $\phi : m_A/(\lambda, m_A^2) \twoheadrightarrow m_B/(\lambda, m_B^2)$

then $\phi : A \twoheadrightarrow B$

Pf. $\phi : m_A^i \to m_B^i$, $\phi : m_A^i/m_A^{i+1} \to m_B^i/m_B^{i+1}$ surj. $i=0$

$i=1$

$$\begin{tikzcd}
m_A^i/m_A^{i+1} \arrow[r] & m_B^i/m_B^{i+1} \\
m_A/m_A^2 \times m_A^{i-1}/m_A^i \arrow[u, twoheadrightarrow] \arrow[r] & m_B/m_B^2 \times m_B^{i-1}/m_B^i \arrow[u, twoheadrightarrow]
\end{tikzcd}$$

$b \in B$, find $a_i \in A$ s.t. $\phi(a_i) \equiv b \ (m_B^i)$, $a_i \equiv a_{i-1} \ (m_A^{i-1})$

$a_i \to a \in A$

$$\mathrm{Hom}_{\mathbb{F}}(\overset{m_{R^{univ}}}{m}/\langle \lambda, m^2 \rangle, \mathbb{F}) \underset{\sim}{\rightleftarrows} H^1_{\mathcal{L}}(G_{\mathbb{Q},S}, \mathrm{ad}^0 \bar{r})$$

$$\mathrm{Hom}_{\mathbb{F}}(I/\underset{m_{\mathcal{O}[[x]]}}{m}I, \mathbb{F}) \hookrightarrow H^2_{\mathcal{L}}(G_{\mathbb{Q},S}, \mathrm{ad}^0 \bar{r})$$

$$\text{Krull dim } R^{univ} \geq 1 + \dim H^1_{\mathcal{L}}(G_{\mathbb{Q},S}, \mathrm{ad}^0 \bar{r}) - \dim H^2_{\mathcal{L}}(\overset{\text{in fact} = 1}{G_{\mathbb{Q},S}}, \mathrm{ad}^0 \bar{r})$$

kernel I can be gen. by $\dim_{\mathbb{F}} I/mI$ elts (Nakayama's Lemma)

We will show R^{univ} is finite $/\mathcal{O}$

$\Rightarrow \exists \ R^{univ} \to \bar{L}$

$\mathrm{Hom}_{\mathbb{F}}\left(m/\langle \lambda, m^2\rangle, \mathbb{F}\right)$

$\cong$

$\mathrm{Hom}_{\mathcal{O}\text{-alg}}\left(R^{univ}, \mathbb{F}[\varepsilon]\right)$ $\quad \varepsilon^2 = 0$

$\frac{1}{\varepsilon}\phi\big|_m$

$\uparrow$

ϕ

b

$\updownarrow$

$\mathcal{O} + m$

$a + b \mapsto \underset{\underset{\mathbb{F}}{\uparrow}}{\bar{a}} + f(b)\varepsilon$

$\cong$

$$\left\{ \begin{array}{c} \rho : G_{\mathbb{Q},S} \to GL_2(\mathbb{F}[\varepsilon]) \\ \rho \bmod \varepsilon = \bar\rho \\ \text{Satisfying } A, B, C. \end{array} \right\} \Big/ \text{Conj. by } 1_2 + \varepsilon M_{2\times 2}(\mathbb{F})$$

$\rho = (1 + \phi\varepsilon)\bar\rho$

$\cong$

$$\left\{ \begin{array}{c} \phi : G_{\mathbb{Q},S} \to M_{2\times 2}(\mathbb{F}), \\ \phi \in Z^1(G_{\mathbb{Q},S}, \mathrm{ad}^0\bar\rho), \\ \mathrm{res}_2[\phi] \in H^1(G_{\mathbb{Q}_2}, \mathrm{ad}^0\bar\rho) \quad (B) \\ \in L_2 \subset \\ \mathrm{res}_\ell[\phi] \in L_\ell \subset H^1(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0\bar\sigma) \quad (C) \end{array} \right\}$$

$\overline{\text{Coboundaries}}$

$(1 + \phi(\sigma)\varepsilon)\,\bar\rho(\sigma)\,(1 + \phi(\tau)\varepsilon)\,\bar\rho(\tau)$

$= (1 + \phi(\sigma\tau)\varepsilon)\,\bar\rho(\sigma\tau)$

$\Leftrightarrow \phi(\sigma) + \mathrm{ad}\,\bar\rho(\sigma)(\phi(\tau)) = \phi(\sigma\tau)$

$\det(1 + \phi(\sigma)\varepsilon)\,\bar\rho(\sigma) = \bar\varepsilon_\ell(\sigma)^{-1}$

$(1 + \mathrm{tr}\,\phi(\sigma)\varepsilon)\,\bar\varepsilon_\ell(\sigma)^{-1} = \bar\varepsilon_\ell(\sigma)^{-1}$

$A \Leftrightarrow \mathrm{tr}\,\phi(\sigma) = 0, \ \forall \sigma$

$\mathrm{ad}^0\bar\rho = \{A \in \mathrm{ad}\,\bar\rho : \mathrm{tr}\,A = 0\}$

$(1 + \varepsilon A)(1 + \phi(\sigma)\varepsilon)\,\bar\rho(\sigma)(1 - \varepsilon A)$

$= \left(1 + \phi(\sigma)\varepsilon + A\varepsilon - \bar\rho(\sigma)A\bar\rho(\sigma)^{-1}\varepsilon\right)\bar\rho(\sigma)$

$\ell > 2$, $A = \left(A - \frac{\mathrm{tr}A}{2}\right) + \frac{\mathrm{tr}A}{2}$, $\phi + A - \mathrm{ad}\,\bar\rho(A)$

WLOG $A \in \mathrm{ad}^0\bar\rho$

S finite $\quad L_v \subset H^1(G_{\mathbb{Q}_v}, M)$

$$H^1_{\{L_v\}}(G_{\mathbb{Q},S}, M) = \{x \in H^1(G_{\mathbb{Q},S}, M) : res_v x \in L_v, \forall v \in S\}$$

Selmer groups.

$$Hom_{\mathbb{F}}(m/\langle \lambda, m^2\rangle, \mathbb{F}) \cong H^1_{\{L_2, L_\ell\}}(G_{\mathbb{Q},S}, ad^0\bar{\rho})$$

@ 2, $\phi \in Z^1(G_{\mathbb{Q}_2}, ad^0\bar{\rho})$, when is $(1+\phi\varepsilon)\bar{\rho}$ a lift of $\bar{\rho}|_{G_{\mathbb{Q}_2}}$ of type B.

Choose basis $\bar{e}_1, \bar{e}_2$ of $\bar{\rho}$, s.t. $\bar{\rho}(\varphi)e_1 = e_1$, $\bar{\rho}(\varphi)e_2 = 2e_2$

$\varphi \mapsto Frob_2$

$$\bar{\rho}|_{G_{\mathbb{Q}_2}} = \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell^{-1} \end{pmatrix}$$

$\uparrow$

w.r.t. $\{e_1, e_2\}$

We need to find a basis $\begin{pmatrix} 1+a\varepsilon \\ b\varepsilon \end{pmatrix}$, $\begin{pmatrix} c\varepsilon \\ 1+d\varepsilon \end{pmatrix}$ of $\mathbb{F}[\varepsilon]^2$ w.r.t. which

$G_{\mathbb{Q}_2}$ acts $\begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$

$$(1+\phi(\sigma)\varepsilon)\bar{\rho}(\sigma)\begin{pmatrix} 1+a\varepsilon \\ b\varepsilon \end{pmatrix} = \begin{pmatrix} 1+a\varepsilon \\ b\varepsilon \end{pmatrix}$$

$$(1+\phi(\sigma)\varepsilon)\bar{\rho}(\sigma)\begin{pmatrix} c\varepsilon \\ 1+d\varepsilon \end{pmatrix} = \bar{\varepsilon}_\ell(\sigma)^{-1}\begin{pmatrix} c\varepsilon \\ 1+d\varepsilon \end{pmatrix} + *\begin{pmatrix} 1+a\varepsilon \\ b\varepsilon \end{pmatrix}.$$

$$(1+\phi(\sigma)\varepsilon)\bar{\rho}(\sigma)\left(1+\varepsilon\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) = \left(1+\varepsilon\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right)\begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell(\sigma)^{-1} \end{pmatrix}$$

$$\left(1-\varepsilon\begin{pmatrix}a&c\\b&d\end{pmatrix}\right)(1+\phi(\sigma)\varepsilon)\left(1+\varepsilon\,\mathrm{ad}\,\bar{r}(\sigma)\begin{pmatrix}a&c\\b&d\end{pmatrix}\right)=\underline{\begin{pmatrix}1&*\\0&\bar{\varepsilon}_\ell(\sigma)^{-1}\end{pmatrix}\bar{r}(\sigma)^{-1}}$$

$$=\begin{pmatrix}1&*\\0&1\end{pmatrix}$$

$$\phi(\sigma)+\mathrm{ad}\,\bar{r}(\sigma)\begin{pmatrix}a&c\\b&d\end{pmatrix}-\begin{pmatrix}a&c\\b&d\end{pmatrix}=\begin{pmatrix}0&*\\0&0\end{pmatrix}$$

WLOG $a+d=0$.

$\mathrm{ad}^0\bar{r}$ as a rep of $G_{\mathbb{Q}_2} \supset \begin{pmatrix}0&\mathbb{F}\\0&0\end{pmatrix}=\mathrm{Hom}(\bar{\varepsilon}_\ell^{-1},1)\cong\mathbb{F}(\bar{\varepsilon}_\ell)$

$\mathrm{ad}^0\bar{r}/\mathrm{Hom}(\bar{\varepsilon}_\ell^{-1},1)\supset\left\{\begin{pmatrix}a&0\\0&-a\end{pmatrix}\right\}\cong\mathbb{F}.$

$\mathrm{ad}^0\bar{r}/(\mathrm{Hom}(\bar{\varepsilon}_\ell^{-1},1),\mathbb{F})\cong\mathbb{F}(\bar{\varepsilon}_\ell^{-1})$

$L_2=\mathrm{Im}\left(H^1(G_{\mathbb{Q}_2},\mathrm{Hom}(\bar{\varepsilon}_\ell^{-1},1))\to H^1(G_{\mathbb{Q}_2},\mathrm{ad}^0\bar{r})\right)$ will do.

$\dim_{\mathbb{F}}L_2=?$

$$\begin{tikzcd}
Q \arrow[d] & 0 \arrow[r] & \mathrm{Hom}(\bar{\varepsilon}_\ell^{-1},1) \arrow[r] & \mathrm{ad}^0\bar{r} \arrow[r] & Q \arrow[r] & 0. \\
H^0(G_{\mathbb{Q}_2},\mathrm{Hom}(\bar{\varepsilon}_\ell^{-1},1))=0 \arrow[d] \\
H^0(G_{\mathbb{Q}_2},\mathrm{ad}^0\bar{r}) \arrow[r] & H^0(G_{\mathbb{Q}_2},Q) \arrow[r] \arrow[d, phantom, "\cong"] & H^1(G_{\mathbb{Q}_2},\mathbb{F}(\bar{\varepsilon}_\ell)) \arrow[r] \arrow[d, phantom, "\cong"] & L_2 \arrow[r] & 0 \\
 & \mathbb{F} & \mathbb{Q}_2^\times\otimes_{\mathbb{Z}}\mathbb{F}\underset{\mathrm{val}_2}{\cong}\mathbb{F}.
\end{tikzcd}$$

$$\Rightarrow\ \dim_{\mathbb{F}}L_2=\dim_{\mathbb{F}}H^0(G_{\mathbb{Q}_2},\mathrm{ad}^0\bar{r})$$

Lecture 8

$S = \{2, \ell\}$, $\bar{r}: G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$

A $\det \bar{r} = \varepsilon_\ell^{-1}$

B ramification @ 2

C ramification @ ℓ

$L / \mathbb{Q}_\ell$ finite

$\mathcal{O} = \mathcal{O}_L$

$\mathcal{O}/\lambda = \mathbb{F}$

$R \in \mathcal{C}_\mathcal{O}$, $r: G_{\mathbb{Q},S} \longrightarrow GL_2(R)$

A $\det \bar{r} = \varepsilon_\ell^{-1}$

B ram @ 2

C ram @ ℓ

$$r^{univ}: G_{\mathbb{Q},S} \longrightarrow GL_2(R^{univ})$$

$$\mathrm{Hom}_{\mathbb{F}}\left(m/(\lambda, m^2), \mathbb{F}\right)$$

$r: G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}[\varepsilon])$

$r \bmod \varepsilon = \bar{r}$

$r = (1 + \phi\varepsilon)\bar{r}$

$\phi \in Z^1(G_{\mathbb{Q},S}; ad^0 \bar{r})$

$$\cong \frac{\left\{ \phi \in Z^1(G_{\mathbb{Q},S}; ad^0\bar{r}) : \ \phi|_{G_{\mathbb{Q}_2}} \in \tilde{L}_2 \subset Z^1(G_{\mathbb{Q}_2}, ad^0 \bar{r}) \supset \text{cobdries}, \ \ \phi|_{G_{\mathbb{Q}_\ell}} \in \tilde{L}_\ell \subset Z^1(G_{\mathbb{Q},S}, ad^0 \bar{r}) \right\}}{\text{coboundaries}}$$

$L_2 \subset H^1(G_{\mathbb{Q}_2}; ad^0 \bar{r})$

$\tilde{L}_2$ = preimage in Z^1

$$=: H^1_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r})$$

$r|_{G_{\mathbb{Q}_\ell}} \in \mathrm{Im}(G)$

$\bar{r}|_{G_{\mathbb{Q}_\ell}} = G(\bar{M})$, $\bar{M}$ 2-dim'l $\mathbb{F}$-vec. sp., $\mathrm{Fil}^0 \bar{M} = \bar{M}$, $\mathrm{Fil}^1 \bar{M} =$ 1-dim

$\mathrm{Fil}^2 \bar{M} = 0$

$\overline{\Phi}^1 : \mathrm{Fil}^1 \bar{M} \to \bar{M}$ linear

$\overline{\Phi}^0 : \bar{M}/\mathrm{Fil}^1 \bar{M} \to \bar{M}$

$\overline{\Phi}^0 + \overline{\Phi}^1 : \mathrm{gr}^{\cdot} \bar{M} \xrightarrow{\sim} \bar{M}$.

$r|_{G_{\mathbb{Q}_\ell}} = G(\overset{\mathbb{F}[\varepsilon]}{\overset{\curvearrowright}{M}})$

$$0 \to \bar{M} \to M \to \bar{M} \to 0$$

$$0 \to \mathrm{Fil}^1 \bar{M} \to \mathrm{Fil}^1 M \to \mathrm{Fil}^1 \bar{M} \to 0$$

$\Rightarrow$ M is free $| \mathbb{F}[\varepsilon]$ rk 2

$\mathrm{Fil}^1 M$ is free $| \mathbb{F}[\varepsilon]$ rk 1

$\Phi : \mathrm{gr}^{\cdot} M \xrightarrow{\sim} M$.

$$\begin{tikzcd}
\mathrm{Ext}^1_{MF, \mathbb{F}}(\bar{M}, \bar{M}) \arrow[d, leftrightarrow] \arrow[dr, "\theta"] & \\
\mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_\ell}]}(\bar{r}, \bar{r}) \arrow[r, phantom, "\cong"] & H^1(G_{\mathbb{Q}_\ell}, \mathrm{ad}\, \bar{r})
\end{tikzcd}$$

$L_\ell = \mathrm{Im}\,\theta \cap H^1(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0 \bar{r})$

$\cap$

$H^1(G_{\mathbb{Q}_\ell}, \mathrm{ad}\, \bar{r})$

$$\boxed{\dim L_2 = \dim H^0(G_{\mathbb{Q}_2}, \mathrm{ad}^0 \bar{r})}$$

$M = \mathbb{F}[\varepsilon]^2$. $\mathrm{Fil}^1 M = \mathbb{F}[\varepsilon]$.

$e_1, e_2, \varepsilon e_1, \varepsilon e_2,$ $\Phi = \begin{pmatrix} \overline{\Phi} & \alpha \\ 0 & \overline{\Phi} \end{pmatrix}$ $\alpha \in M_{2\times 2}(\mathbb{F})$.

$= \mathrm{Hom}(\mathrm{gr}\, \bar{M}, \bar{M})$

If $\beta \in \mathrm{End}_{\mathbb{F}}(\bar{M})$, $e_i \mapsto e_i + \varepsilon \beta e_i$

$\beta \, \mathrm{Fil}^1 \bar{M} \subset \mathrm{Fil}^1 \bar{M}$

$$\underline{\Phi} \rightsquigarrow \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\Phi} & \alpha \\ 0 & \bar{\Phi} \end{pmatrix} \begin{pmatrix} 1 & \mathrm{gr}\,\beta \\ 0 & 1 \end{pmatrix}$$

$$\alpha \sim \alpha - \beta \bar{\Phi} + \bar{\Phi}\, \mathrm{gr}\,\beta .$$

$$\begin{tikzcd}
0 & \mathrm{Ext}^1_{MF,\mathbb{F}}(\bar{M},\bar{M}) \arrow[l] & \mathrm{Hom}_{\mathbb{F}}(\mathrm{gr}\,\bar{M},\bar{M}) \arrow[l] & \mathrm{Fil}^0\,\mathrm{Hom}_{\mathbb{F}}(\bar{M},\bar{M}) \arrow[l] \\
& & \mathrm{Hom}_{\mathbb{F}[G_{\mathbb{Q}_\ell}]}(\bar{r},\bar{r}) \arrow[r, "\cong"] \arrow[d, equal] & \mathrm{Hom}_{MF}(\bar{M},\bar{M}) \arrow[u] \\
& & H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}\,\bar{r}) & 0 \arrow[u]
\end{tikzcd}$$

$\alpha \in \mathrm{Hom}_{\mathbb{F}}(\mathrm{gr}\,\bar{M},\bar{M})$, $\beta \in \mathrm{Fil}^0\,\mathrm{Hom}_{\mathbb{F}}(\bar{M},\bar{M})$

$\mathrm{Hom}_{\mathbb{F}}(\bar{M},\bar{M}) \in MF$

$$\dim \mathrm{Ext}^1_{MF,\mathbb{F}}(\bar{M},\bar{M}) = (\dim \bar{M})^2 - \dim \mathrm{Fil}^0\,\mathrm{Hom}_{\mathbb{F}}(\bar{M},\bar{M}) + \dim H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}\,\bar{r})$$

$\Lambda^2_{\mathbb{F}[\varepsilon]} M \cong \mathbb{F}[\varepsilon]$ in deg 1

$\underline{\Phi}$ acts by $a \in \mathbb{F}[\varepsilon]^\times$

$G(\Lambda^2_{\mathbb{F}[\varepsilon]} M) = \varepsilon_\ell^{-1} \chi_a$, χ_a unramified

$\mathrm{Frob}_\ell \mapsto a$

$\therefore a \in 1 + \varepsilon \mathbb{F}$.

$$\dim L_\ell = \dim \mathrm{Ext}^1_{MF,\mathbb{F}}(\bar M, \bar M) - 1$$

$$= (\dim \bar M)^2 - \dim \mathrm{Fil}^0 \mathrm{Hom}_{\mathbb F}(\bar M, \bar M) + \dim H^0(G_{\mathbb{Q}_\ell}, ad^0 \bar r)$$

$$= 4 - 3 + \dim H^0(G_{\mathbb{Q}_\ell}, ad^0 \bar r)$$

$$= 1 + \dim H^0(G_{\mathbb{Q}_\ell}, ad^0 \bar r)$$

$$d = \dim H^1_{\mathcal L}(G_{\mathbb{Q},S}, ad^0 \bar r)$$

$\mathcal{O}[[x_1, \dots, x_d]] \twoheadrightarrow R^{univ}$, kernel I

I is gen. by $\dim_{\mathbb F} I/mI$ elts.

will show $\mathrm{Hom}_{\mathbb F}(\underbrace{I/mI}_{\lambda \neq 0}, \mathbb{F}) \hookrightarrow H^2_{\mathcal L}(G_{\mathbb{Q},S}, ad^0 \bar r)$

$J/mI = \ker \lambda$, $\quad \begin{matrix} J \triangleleft \mathcal{O}[[\underline{x}]] \\ \cap \\ I \end{matrix}$

Is there a lifting of r^{univ} to $\mathcal{O}[[\underline{x}]]/J \twoheadrightarrow R^{univ}$?

$$\begin{tikzcd} \mathcal{O}[[\underline{x}]]/J \arrow[r, two heads] & R^{univ} \arrow[l, bend left, "s"] \end{tikzcd}$$

satisfying A – C.

No.

$$\mathcal{O}[[\underline{x}]]/J = I/J \oplus \mathrm{Im}\, s \cong R^{univ}$$

$$m/J = I/J \oplus s(m/I) \qquad (m^2, \lambda)/J = (0) \oplus (m^2, \lambda)/I$$

$m/(m^2, \lambda) \cong I/J \oplus m/(m^2, \lambda)$, 矛 to $J \subsetneq I$.

wlog $\det \tilde{r}(\sigma) = \varepsilon_\ell(\sigma)^{-1}$

$\exists\ \tilde{r} : G_{\mathbb{Q},S} \to GL_2(\mathcal{O}[[x]]/J)$ a cts set theoretic lifting of r^{univ}.

$$G^2_{\mathbb{Q},S} \longrightarrow \underset{\times\ \det=1}{\left(1 + M_{2\times 2}(I/J)\right)} \overset{A \mapsto A-1}{\cong} \underset{+}{\left(M_{2\times 2}(I/J)\right)^{tr=0}}$$

G profinite gp,

H a closed sub gp.

$\exists$ cts set theoretic section

$G/H \longrightarrow G$

$$\phi(\sigma,\tau) = \tilde{r}(\sigma)\,\tilde{r}(\tau)\,\tilde{r}(\sigma\tau)^{-1}$$

$$\mathrm{ad}\,\bar{r}(\sigma)\left(\phi(\tau,\rho)\right)\phi(\sigma,\tau\rho)$$

$$= \phi(\sigma,\tau)\,\phi(\sigma\tau,\rho)$$

$$\tilde{r}(\sigma)\tilde{r}(\tau)\tilde{r}(\rho)\tilde{r}(\tau\rho)^{-1}\tilde{r}(\sigma)^{-1}\tilde{r}(\sigma)\tilde{r}(\tau\rho)\tilde{r}(\sigma\tau\rho)^{-1}$$

✓

$$= \tilde{r}(\sigma)\tilde{r}(\tau)\tilde{r}(\sigma\tau)^{-1}\tilde{r}(\sigma\tau)\tilde{r}(\rho)\tilde{r}(\sigma\tau\rho)^{-1}$$

$$\phi \in Z^2(G_{\mathbb{Q},S}, \mathrm{ad}^\circ\bar{r}) \longrightarrow H^2(G_{\mathbb{Q},S}, \mathrm{ad}^\circ\bar{r})$$

$\tilde{r} \rightsquigarrow (1+\beta(\sigma))\tilde{r}(\sigma)$, $\quad \beta(\sigma) \in M_{2\times 2}(I/J)$

$$\phi \rightsquigarrow \phi + \partial\beta$$

$$(\sigma,\tau) \mapsto \phi(\sigma,\tau) + \beta(\sigma) + \mathrm{ad}\,\bar{r}(\sigma)(\beta(\tau)) - \beta(\sigma\tau)$$

r^{univ} lifts $\Leftrightarrow$ $[\phi] \in H^2(G_{\mathbb{Q},S}, \mathrm{ad}^\circ\bar{r})$ is trivial.

Lecture 9

$S = \{2, \ell\}$, $\quad \bar{\rho} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F})$

R^{univ}

A $\quad \det \rho = \varepsilon_\ell^{-1}$

B $\quad \rho|_{G_{\mathbb{Q}_2}}$

C $\quad \rho|_{G_{\mathbb{Q}_\ell}}$

$$\begin{tikzcd} 0 \arrow[dr] & & \\ & I \arrow[dr] & \\ & & \mathcal{O}[[x_1, \dots, x_d]] \arrow[r, two heads] & R^{univ} \end{tikzcd}$$

$d = \dim H^1_{\mathcal{L}}(G_{\mathbb{Q},S} ; ad^\circ \bar{\rho})$

$\mathcal{L} = \{L_v\}_{v \in S} \qquad L_v < H^1(G_{\mathbb{Q}_v}, ad^\circ \bar{\rho})$

$Hom_{\mathbb{F}}(I/mI, \mathbb{F}) \ni f \neq 0$, $\ker f = J_f/mI$, $I/J_f \xrightarrow[\sim]{f} \mathbb{F}$

$\exists$ a lift of ρ^{univ} to $\mathcal{O}[[\underline{x}]]/J_f$ satisfying A, B, C.

$\exists\ \tilde{\rho} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathcal{O}[[\underline{x}]]/J_f)$ cts set theoretic lift of ρ^{univ}

w/ $\det \tilde{\rho}(\sigma) = \varepsilon_\ell(\sigma)^{-1}$.

$\phi(\sigma,\tau)$

$\shortparallel$

$f(\tilde{\rho}(\sigma)\tilde{\rho}(\tau)\tilde{\rho}(\sigma\tau)^{-1} - 1) \in ad^\circ \bar{\rho}$, $\quad \phi \in Z^2(G_{\mathbb{Q},S}, ad^\circ \bar{\rho})$

$\exists\ \tilde{\rho}$ cts hom. lifting $\rho^{univ} \iff [\phi] = 0$ in $H^2(G_{\mathbb{Q},S}, ad^\circ \bar{\rho})$

Claim. For $v = 2$ or ℓ, $\exists\ \hat{\rho}_v : G_{\mathbb{Q}_v} \longrightarrow GL_2(\mathcal{O}[[\underline{x}]]/J_f)$

lifting $\rho^{univ}|_{G_{\mathbb{Q}_v}}$ satisfying B resp. C.

$v=2$ $\quad r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & \overset{\psi\varepsilon_\ell^{-1}}{\overset{\shortparallel}{*}} \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta, \ \delta^2 = 1.$ δ non-ramified.

$$\psi : G_{\mathbb{Q}_2} \longrightarrow R \quad , \psi \in Z^1_{cts}(G_{\mathbb{Q}_2}, R(\varepsilon_\ell))$$

Need $\quad Z^1_{cts}(G_{\mathbb{Q}_2}, \mathcal{O}[[\underline{x}]]/J_f(\varepsilon_\ell)) \twoheadrightarrow Z^1_{cts}(G_{\mathbb{Q}_2}, R^{univ}(\varepsilon_\ell))$

STP

$$\begin{tikzcd}
H^1_{cts}(G_{\mathbb{Q}_2}, \mathcal{O}[[\underline{x}]]/J_f(\varepsilon_\ell)) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & H^1_{cts}(G_{\mathbb{Q}_2}, R^{univ}(\varepsilon_\ell)) \arrow[d, phantom, "\cong"] \\
\widehat{\mathbb{Q}_2^\times} \otimes_{\widehat{\mathbb{Z}}} \mathcal{O}[[\underline{x}]]/J_f \arrow[rd, "val_2", "\wr"'] & \widehat{\mathbb{Q}_2^\times} \otimes_{\widehat{\mathbb{Z}}} R^{univ} \arrow[rd, "val_2", "\wr"'] & \\
& \mathcal{O}[[\underline{x}]]/J_f \arrow[r, two heads] & R^{univ}
\end{tikzcd}$$

Kummer

$\mathbb{Z}_2^\times$ pro 2-gp

$v=\ell$ $\quad \hat{r}_\ell : G_{\mathbb{Q}_\ell} \to GL_2(\mathcal{O}[[\underline{x}]]/J_f)$ $\quad \det \hat{r}_\ell = \varepsilon_\ell^{-1}$

look for $\hat{r}_{\ell,n} : G_{\mathbb{Q}_\ell} \to GL_2(\mathcal{O}[[\underline{x}]]/J_f + m^n)$

s.t. $\hat{r}_{\ell,n} \bmod m^{n-1} = \hat{r}_{\ell,n-1}$

$\hat{r}_\ell = \varprojlim \hat{r}_{\ell,n}$

$\hat{r}_{\ell,n} \bmod I + m^n = r^{univ}|_{G_{\mathbb{Q}_\ell}} \bmod I+m^n$

$\det \hat{r}_{\ell,n} = \varepsilon_\ell^{-1}$

$\hat{r}_{\ell,n}$ in image of $\mathfrak{G}$.

$\hat{r}_\ell$ lifts $r^{univ}|_{G_{\mathbb{Q}_\ell}}$

$\forall\, K \underset{open}{\triangleleft} \mathcal{O}[[\underline{x}]]$
$\quad \cup$
$\quad J_f$

$\hat{r}_\ell \bmod K$ in image $\mathfrak{G}$

$$\begin{tikzcd}
GL_2\Big(\overset{\hat{r}_{\ell,n-1}}{\mathcal{O}[[x]]/J_f + m^{n-1}} \times \overset{r^{univ}|_{G_{\mathbb{Q}_\ell}} \bmod I+m^n}{\mathcal{O}[[\underline{x}]]/I + m^n}\Big) & & G_{\mathbb{Q}_\ell} \arrow[ll, "\hat{r}_{\ell,n-1} \times r^{univ}|_{G_{\mathbb{Q}_\ell}} \bmod I+m^n"'] \\
\mathcal{O}[[\underline{x}]]/I+m^{n-1} & \mathcal{O}[[\underline{x}]]/(J_f+m^{n-1}) \cap (I+m^n) \arrow[u, phantom, "\cong"] & \mathcal{O}[[\underline{x}]]/J_f + m^n \arrow[l, two heads]
\end{tikzcd}$$

$R \in \mathcal{C}_\mathcal{O}$ artinian.

$K \triangleleft R$

$\nu : G_{\mathbb{Q}_\ell} \to GL_2(R/K)$, $\det = \varepsilon_\ell^{-1}$, in image of $\mathcal{G}$.

$\Rightarrow \exists\ \tilde{\nu} : G_{\mathbb{Q}_\ell} \to GL_2(R)$ $\det\ \varepsilon_\ell^{-1}$ in image of $\mathcal{G}$

$\mathcal{G}(M)$, M is an R/K-module.

$$\mathrm{Fil}^i M \Big/ m\,\mathrm{Fil}^i M \cong \begin{cases} \mathbb{F}^2, & i \le 0 \\ \mathbb{F}, & i = 1 \\ 0, & i > 1 \end{cases}$$

$\therefore\ M = \mathrm{Fil}^0 M \cong (R/k)^2$

$\mathrm{Fil}^1 M \cong R/K$

$\mathrm{Fil}^2 M = 0$

Choose e_1 basis of $\mathrm{Fil}^1 M$ + extend to basis e_1, e_2 of M

$\Phi^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Phi^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \ell a \\ \ell b \end{pmatrix}$ s.t. $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$ span $(R/k)^2$

i.e. $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(R/K)$

$\Phi^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\Lambda^2_{R/K} M \cong R/K = \mathrm{Fil}^1 \Lambda^2_{R/K} M$

$\Phi_{\Lambda^2} = ad - bc$ $\qquad \mathrm{Fil}^2 \Lambda^2_{R/K} M = 0$

$\therefore \det \rho = \varepsilon_\ell^{-1}$ (unramified character Frob $\mapsto ad-bc$)

$\therefore ad-bc=1$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2(R/k)$$

$SL_2(R) \twoheadrightarrow SL_2(R/k)$ lift $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ to $SL_2(R)$ + use it to construct

$\widetilde{M}/R$ lifting M.

ρ^{univ}, $\tilde{\rho} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathcal{O}[[x]]/J_f)$

cts, $\det \tilde{\rho} = \bar{\varepsilon}_\ell^{-1}$, set theoretic lift.

$$\phi(\sigma,\tau) = f\left(\tilde{\rho}(\sigma)\tilde{\rho}(\tau)\tilde{\rho}(\sigma\tau)^{-1} - 1\right) \qquad \partial\phi = 0$$

$$\psi_2(\sigma) = f\left(\tilde{\rho}(\sigma)\hat{\rho}_2(\sigma)^{-1} - 1\right), \quad \sigma \in G_{\mathbb{Q}_2}$$

$$\psi_\ell(\sigma) = f\left(\tilde{\rho}(\sigma)\hat{\rho}_\ell(\sigma)^{-1} - 1\right), \quad \sigma \in G_{\mathbb{Q}_\ell}$$

$$\phi\big|_{G_{\mathbb{Q}_v}} = \partial\psi_v$$

$v = 2, \ell$

$$\tilde{\rho} \longmapsto \left(1 - f^{-1}\beta(\sigma)\right)\tilde{\rho}(\sigma) \qquad \beta \in C^1(G_{\mathbb{Q},S}; ad^0\bar{\rho})$$

$$\phi \longmapsto \phi + \partial\beta$$

$$\psi_v \longmapsto \psi_v + \beta\big|_{G_{\mathbb{Q}_v}}$$

r^{univ} has a lift satisfying B + C,

if we can choose β s.t. $\phi + \partial\beta = 0$.

$$\psi_v + \beta|_{G_{\mathbb{Q}_v}} \in \tilde{L}_v \subset Z^1(G_{\mathbb{Q}_v}, ad^0\bar{r})$$

$\uparrow$ preimage of

$$L_v \subset H^1(G_{\mathbb{Q}_v}, ad^0\bar{r})$$

$$\begin{tikzcd}
\beta \arrow[d, mapsto] & C^1(G_{\mathbb{Q},S}, ad^0\bar{r}) \arrow[d] \\
(\partial\beta, (\beta|_{G_{\mathbb{Q}_v}})) \quad (\phi, (\psi_v)) \arrow[d, mapsto] & C^2(G_{\mathbb{Q},S}, ad^0\bar{r}) \oplus \bigoplus_{v\in S} C^1(G_{\mathbb{Q}_v}, ad^0\bar{r})/\tilde{L}_v \arrow[d] \\
(\partial\phi, (\phi|_{G_{\mathbb{Q}_v}} - \partial\psi_v)) & C^3(G_{\mathbb{Q},S}; ad^0\bar{r}) \oplus \bigoplus_{v\in S} C^2(G_{\mathbb{Q}_v}, ad^0\bar{r})
\end{tikzcd}$$

$H^2_L(G_{\mathbb{Q},S}; ad^0\bar{r})$ = cohomology in middle of this diagram

$\therefore$ $\mathrm{Hom}_{\mathbb{F}}(I/mI, \mathbb{F}) \hookrightarrow H^2_L(G_{\mathbb{Q},S}; ad^0\bar{r})$

Krull dim $R^{univ} \leq 1 + \dim H^1_R(G_{\mathbb{Q},S}; ad^0\bar{r}) - \dim H^2_L(G_{\mathbb{Q},S}; ad^0\bar{r})$

Lecture 10 $\quad S = \{2, \ell\}$, $\bar{r} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F})$

$r^{univ} : G_{\mathbb{Q},S} \longrightarrow GL_2(R^{univ})$

$\det r^{univ} = \varepsilon_\ell^{-1}$

$r^{univ}|_{G_{\mathbb{Q}_2}}$ ---

$r^{univ}|_{G_{\mathbb{Q}_\ell}}$ ...

$0 \to I \to \mathcal{O}[[x_1, \dots, x_d]] \twoheadrightarrow R^{univ}$, d minimal, $= \dim_{\mathbb{F}} H^1_{\mathcal{L}}(G_{\mathbb{Q},S} ; \mathrm{ad}^0 \bar{\rho})$

$$I/mI \hookrightarrow H^2_{\mathcal{L}}(G_{\mathbb{Q},S} ; \mathrm{ad}^0 \bar{\rho})$$

$\Rightarrow$ Krull dim of $R^{univ} \geq 1 + \dim H^1_{\mathcal{L}}(G_{\mathbb{Q},S} ; \mathrm{ad}^0 \bar{\rho})$

$- \dim H^2_{\mathcal{L}}(G_{\mathbb{Q},S} , \mathrm{ad}^0 \bar{\rho})$

$\mathcal{L} = \{L_2, L_\ell\}$

$$\begin{tikzcd} L_v \subset H^1(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho}) \\ \tilde{L}_v \subset Z^1(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho}) \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd}
 & C^0(G_{\mathbb{Q},S} ; \mathrm{ad}^0\bar{\rho}) \arrow[d, "\partial"] & & \\
 & C^1(G_{\mathbb{Q},S} ; \mathrm{ad}^0\bar{\rho}) \arrow[d] & & \\
(\phi, (\phi_v)_{v\in S}) \arrow[d, mapsto] & C^2(G_{\mathbb{Q},S} ; \mathrm{ad}^0\bar{\rho}) \oplus \bigoplus_{v\in S} C^1(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho})/\tilde{L}_v \arrow[d] & \bigoplus_{v\in S} C^1(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho})/\tilde{L}_v \arrow[d] \arrow[l] & \\
(\partial\phi, (\phi|_{G_{\mathbb{Q}_v}} - \partial\phi_v)_{v\in S}) & C^3(G_{\mathbb{Q},S} ; \mathrm{ad}^0\bar{\rho}) \oplus \bigoplus_{v\in S} C^2(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho}) \arrow[d] & \bigoplus_{v\in S} C^2(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho}) \arrow[d] & 0 \arrow[l] \\
0 \leftarrow C^\circ \leftarrow & \vdots \arrow[d] & \vdots & \\
 & C^i(G_{\mathbb{Q},S} ; \mathrm{ad}^0\bar{\rho}) \oplus \bigoplus_{v\in S} C^{i-1}(G_{\mathbb{Q}_v}, \mathrm{ad}^0\bar{\rho}) & &
\end{tikzcd}$$

$H^i_{\mathcal{L}}(G_{\mathbb{Q},S} ; \mathrm{ad}^0\bar{\rho})$ = cohomology of this complex.

$\underline{\underline{\ell \neq 2}}$

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & H^0(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & 0 \arrow[dll, rounded corners] \\
 & H^1_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & H^1(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & \bigoplus_{v \in S} H^1(G_{\mathbb{Q}_v}, ad^0 \bar{r})/L_v \arrow[dll, rounded corners] \\
 & H^2_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & H^2(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & \bigoplus_{v \in S} H^2(G_{\mathbb{Q}_v}; ad^0 \bar{r}) \arrow[dll, rounded corners] \\
 & H^3_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) \arrow[r] & (0) & \\
 & & (\ell \neq 2) \arrow[u] &
\end{tikzcd}$$

$H^i_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) = (0)$, for $i > 3$.

$$H^0_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) = H^0(G_{\mathbb{Q},S}; ad^0 \bar{r})$$

$$H^3_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r}) = H^0(G_{\mathbb{Q},S}; \underset{\underset{ad^0 \bar{r}}{\wr\wr}}{(ad^0 \bar{r})^\vee}(1))^\vee \leftarrow \text{Poitou - Tate duality}$$

$$\| $$

$$H^0_{\mathcal{L}^\perp}(G_{\mathbb{Q},S}; (ad^0 \bar{r})^\vee(1))^\vee$$

$tr: ad^0 \times ad^0 \longrightarrow \mathbb{F}$

$(A, B) \longmapsto tr(AB)$

non-degenerate.

$$\dim_{\mathbb{F}} H^2_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r})$$

$$= \dim_{\mathbb{F}} H^1_{\mathcal{L}^\perp}(G_{\mathbb{Q},S}; (ad^0 \bar{r})^\vee(1))^\vee$$

$$h^i := \dim_{\mathbb{F}} H^i(G_{\mathbb{Q},S}; ad^0 \bar{r})$$

$$h^i_{\mathcal{L}} := \dim_{\mathbb{F}} H^i_{\mathcal{L}}(G_{\mathbb{Q},S}; ad^0 \bar{r})$$

$$h^i_v := \dim_{\mathbb{F}} H^i(G_{\mathbb{Q}_v}; ad^0 \bar{r}), \quad \ell_v = \dim L_v$$

$$H^1(G_{\mathbb{Q}_v}, \overset{\overset{L_v}{\cap}}{ad^0 \bar{r}}) \times H^1(G_{\mathbb{Q}_v}, \overset{\overset{L_v^\perp}{\cap}}{(ad^0 \bar{r})^\vee}(1))$$

$$\longrightarrow H^2(G_{\mathbb{Q}_v}, \mathbb{F}(1)) \cong \mathbb{F}$$

$L_v^\perp$ annihilator of L_v

$\mathcal{L}^\perp = \{L_v^\perp\}$

$$h^1_{\mathcal{L}} - h^1 + \sum_{v\in S}\left(h^1_v - \ell_v\right) - h^2_{\mathcal{L}} + h^2 - \sum_{v\in S} h^2_v + h^3_{\mathcal{L}} = 0$$

$$1 + h^1_{\mathcal{L}} - h^2_{\mathcal{L}} = 1 + h^0 - h^3_{\mathcal{L}} + \underbrace{(h^1 - h^2 - h^0)}_{\dim ad^0\bar{r} - \dim (ad^0\bar{r})^{c=1}} - \sum_{v\in S} \overbrace{\left(h^1_v - h^2_v - h^0_v\right)}^{\begin{cases}\dim ad^0\bar{r} & \text{if } v=\ell\\ 0 & \text{if } v\neq\ell\end{cases}}$$

$$+ \sum_{v\in S}\left(\ell_v - h^0_v\right)$$

$$= \underbrace{1 + h^0}_{0} - \overset{=0\ =\ \dim H^0(G_{\mathbb{Q},S},\, ad^0\bar{r}(1))}{h^3_{\mathcal{L}}} + \underbrace{\sum_{\substack{v\in S\\ v\neq\ell}}\left(\ell_v - h^0_v\right)}_{0} + \underbrace{\left(\ell_\ell - h^0_\ell\right)}_{\dim\left(ad^0\bar{m}/\mathrm{Fil}^0 ad^0\bar{m}\right)\ =\ 1} - \underbrace{\dim\left(ad^0\bar{r}\right)^{c=1}}_{1}$$

as $\bar{r} \not\cong \bar{r}(1)$ (looking at det) $\ell > 3$

as $\bar{r}$ abs. irred.

$\bar{r}|_{G_{\mathbb{Q}_\ell}} = G(\bar{m})$

Krull dim $R^{univ} \geq 1$.

Suppose. $\mathcal{O} \to R^{univ}$ is finite

$\uparrow$ kernel must be 0 (as else R^{univ} is finite over $\mathcal{O}/\lambda^n$ some n, so R^{univ} has dim 0)

Going up: $\mathfrak{p} \triangleleft R^{univ}$ prime

$\mathfrak{p} \cap \mathcal{O} = (0)$.

$\mathcal{O} \hookrightarrow R^{univ}/\mathfrak{p}$ field of fractions: $L \underset{\text{finite}}{\hookrightarrow} L'$.

τ^{univ} pushed forward to L' is a lift of $\bar{\tau}$ satisfying A, B, C.

Suppose $F | \mathbb{Q}$ is a finite Galois ext'n s.t.
- $\bar{\tau} | \mathbb{Q}$ is unramified @ ℓ
- $F \cap \bar{\mathbb{Q}}^{\ker \bar{\tau}} = \mathbb{Q}$

$$\Downarrow$$

$$\bar{\tau}(G_{\mathbb{Q}}) = \bar{\tau}(G_F)$$

$\bar{\tau}|_{G_F}$

R_F^{univ} - the universal deformation ring parametrizing lifts of $\bar{\tau}|_{G_{F,S}}$

s.t. A $\det \tau = \varepsilon_\ell^{-1}$

C FL cond'n at all $v | \ell$.

$R_F^{univ} \longrightarrow R^{univ}$

<u>Claim</u>: $R_F^{univ} \longrightarrow R^{univ}$ is finite.

$\tau^{univ}|_{G_F}$

∴ Sufficient to prove R_F^{univ} finite $/ \mathcal{O}$.

<u>Lecture 11</u>. $S = \{2, \ell\}$

$\bar{\tau} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$

$$\left.\begin{array}{l} (R^{univ}, \tau^{univ}) \quad \text{Krull dim } R^{univ} \geq 1. \\ R^{univ} \text{ finite over } \mathcal{O} \end{array}\right] \Rightarrow \begin{array}{l} \bar{\tau} \text{ has an } \ell\text{-adic lift} \\ \text{similarly unramified.} \end{array}$$

$F | \mathbb{Q}$ finite Galois, unramified @ ℓ, R_F^{univ} for $\bar{\tau}|_{G_F}$

$F \cap \bar{\mathbb{Q}}^{\ker \bar{\tau}} = \mathbb{Q}$

$$\begin{tikzcd} R_F^{univ} \arrow[r, "\uparrow"'] & R^{univ} \end{tikzcd}$$

Claim, finite.

STP R^{univ}/m_F is finite over $\mathbb{F}$

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[rr] \arrow[dr] & & GL_2(R^{univ}/m_F) \\ & \mathrm{Gal}(\overline{\mathbb{Q}}^{\ker \bar{r}} F \mid \mathbb{Q}) \arrow[ur] & \end{tikzcd}$$

finite

Lemma Suppose $R \supset S$ in $\mathcal{C}_{\mathcal{O}}$, $r : \Gamma \longrightarrow GL_n(R)$

$I \lhd R$

Γ a group

s.t. $\bar{r} = r \bmod m_R$ is abs. irred.

$\mathrm{tr}(r)$ is valued in S.

If $r \bmod I$ is valued in $GL_n(S/I \cap S)$, then $\exists\, A \in M_{n\times n}(I)$ w/ $(1+A)\, r\, (1+A)^{-1}$ valued in $GL_n(S)$.

Cor R_r^{univ} is topologically gen. over $\mathcal{O}$ by $\mathrm{tr}\, r^{univ}(\sigma)$ for $\sigma \in G_{\mathbb{Q},S}$

Pf. $S \subset R^{univ}$ be the subalg. gen. by $\mathrm{tr}\, r^{univ}(\sigma)$ for $\sigma \in G_{\mathbb{Q},S}$

$\exists \ A \in M_{n\times n}(m_{R^{univ}}), \quad (1+A)\, r^{univ} (1+A)^{-1} : G_{\mathbb{Q},S} \to GL_2(S)$

$$\begin{tikzcd} R^{univ} \arrow[r] \arrow[rr, bend right, "id"'] & S \arrow[r, phantom, "\subset"] & R^{univ} \end{tikzcd}$$

$\Rightarrow S = R^{univ}$

<u>Pf of L</u>. $\underset{\substack{\cup \\ (I,0)}}{\mathcal{X}} = \left\{ (J,A) : \begin{array}{l} J \triangleleft R \\ J \subset I \\ A \in M_{n\times n}(I/J) \end{array} \ \text{s.t.} \ \begin{array}{l} (1+A)(r \bmod J)(1+A)^{-1} \\ \text{valued in } GL_n(S/J\cap S) \end{array} \right\}$

Use Zorn's Lemma. $(J,A) \leq (J',A')$ if $J \subset J'$ and $A \bmod J' = A'$.

Chain's have lower bounds. $\mathcal{C} \subset \mathcal{X}$ a chain.

$J = \bigcap_{(J_i,A_i)\in\mathcal{C}} J_i, \quad \exists!\ A \in M_{n\times n}(I/J)$

lifting $A_i \in M_{n\times n}(I/J_i)$,

$\forall\ (J_i, A_i) \in \mathcal{C}$

$\therefore \exists \ (J,A)$ in $\mathcal{X}$ minimal.

Want $J = (0)$. If not, $\exists\ J' \subset J$ w/ $\dim_{\mathbb{F}} J/J' = 1$.

Replace r by $r^{new} = \tilde{A}\, r\, \tilde{A}^{-1}$, $\tilde{A}$ any lift of A

S by $S^{new} = S/S\cap J'$

R by $R/J' \times_{R/J} S/S\cap J$

If Lemma true for R^{new}, S^{new}, τ^{new}, then $\exists\, B \in 1 + M_{n\times n}(J/J')$

w/ $B\tilde{A}\,\tau\,\tilde{A}^{-1}B^{-1}$ valued in $GL_n(S/J'\cap S)$.

∴ reduced to case: $R/I \cong S/S\cap I$.

$\dim_{\mathbb{F}} I = 1$.

If $I \subset S \Rightarrow R = S$ ✓

∴ $R = \underset{\underset{m_S \oplus 0}{\triangledown}}{S \oplus I}$, $(s,a)(s',a') = (ss', sa' + as')$

Sufficient to treat $S/m_S \oplus \bar{I} \supset S/m_S$ $\rightsquigarrow$ $A \in 1 + M_{n\times n}(I)$

this will do for $S \oplus I$ too.

i.e. $R = \mathbb{F}[\varepsilon]$, $\varepsilon^2 = 0$

$S = \mathbb{E}$.

$$\begin{tikzcd}
\mathbb{F}[\Gamma] \arrow[r, "\tau"] \arrow[dr, two heads, "\bar{\tau}"'] & M_{n\times n}(\mathbb{F}[\varepsilon]) \arrow[d] \\
& M_{n\times n}(\mathbb{F})
\end{tikzcd}$$

$\mathrm{tr}\,\tau \in \mathbb{E}$.

as $\bar{\tau}$ abs. irred.

$x \in \ker\bar{\tau}$, $y \in \mathbb{F}[\Gamma]$

$$\underset{\underset{\mathrm{tr}\left(\frac{\tau(x)}{\varepsilon}\bar{\tau}(y)\right)\varepsilon}{\shortparallel}}{\mathrm{tr}\,\tau(xy)} = \mathrm{tr}\,\bar{\tau}(xy) = \mathrm{tr}(0\,\bar{\tau}(y)) = 0$$

∴ $\mathrm{tr}\left(\frac{\tau(x)}{\varepsilon}A\right) = 0$, $\forall A \in M_{n\times n}(\mathbb{F})$ ∴ $\frac{\tau(x)}{\varepsilon} = 0$. ∴ $\tau(x) = 0$.

$$\begin{tikzcd} M_{n\times n}(\mathbb{F}) \arrow[r, "\tau"] \arrow[dr, "\sim"'] & M_{n\times n}(\mathbb{F}[\varepsilon]) \arrow[d] \\ & M_{n\times n}(\mathbb{F}) \end{tikzcd}$$

$\operatorname{tr} \tau \subset \mathbb{F}$

$\tau(x) = x + \phi(x)\varepsilon$, ϕ $\mathbb{F}$-linear $(\operatorname{tr}\phi = 0)$

$$\phi(xy) = x\phi(y) + \phi(x)y$$

WTS $\exists$ $A \in M_{n\times n}(\mathbb{F})$ w/

$\tau(x) = (1 + A\varepsilon)\, x\, (1 - A\varepsilon)$, i.e. $\phi(x) = Ax - xA$.

e_{ij} = elementary mat. in (i,j) entry.

$$A = \sum_{j=1}^{n} \phi(e_{j1})\, e_{1j}$$

$$\begin{aligned} A e_{ik} - e_{ik} A &= \sum_{j=1}^{n} \phi(e_{j1}) e_{1j} e_{ik} - \sum_{j=1}^{n} e_{ik}\, \phi(e_{j1})\, e_{1j} \\ &= \phi(e_{i1}) e_{1k} - \sum_{j=1}^{n} \phi(e_{ik} e_{j1}) e_{1j} + \sum_{j=1}^{n} \phi(e_{ik}) e_{j1} e_{1j} \\ &= \phi(e_{i1}) e_{1k} - \phi(e_{i1}) e_{1k} + \phi(e_{ik}) \\ &= \phi(e_{ik}) \end{aligned}$$

$R^{univ} / m_F R^{univ}$ (f.g) gen. by $\operatorname{tr} \tau^{univ}(\sigma)$, $\sigma \in \operatorname{Gal}(E|\mathbb{Q})$, $E = F_{\bar{\mathbb{Q}}}^{\ker \bar{\tau}}$

$m = [E : \mathbb{Q}]$, $\tau^{univ}(\sigma)^m = 1$

$\Rightarrow$ $\mathrm{Tr}\, \rho^{univ}(\sigma)$ is integral over $\mathbb{F}$

$\therefore$ $R^{univ}/m_F R^{univ}$ is finite over $\mathbb{F}$

<u>Lecture 12</u>. $F \,|\, \mathbb{Q}$ finite Galois ext'n, $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$

$F \cap \bar{\mathbb{Q}}^{\ker \bar{\rho}} = \mathbb{Q}$. ↖ ℓ unramified in F.

$$\begin{tikzcd}
R_F^{univ} \arrow[r] & R^{univ} \\
\text{def. ring for } \bar{\rho}|_{G_F} \arrow[u] & \text{finite}
\end{tikzcd}$$

R^{univ} $\subset$ def. rings for $\bar{\rho}$

lift ρ of $\bar{\rho}$

"as unramified as $\bar{\rho}$ is"

dim $R^{univ} \geq 1$.

Show R^{univ} finite $/\mathcal{O}$.

STP $R^{univ}/m_F R^{univ}$ finite over $\mathbb{F}$.

$$\begin{tikzcd}
\rho^{univ}: G_{\mathbb{Q}} \arrow[rr] \arrow[dr] & & GL_2(R^{univ}/m_F R^{univ}) \\
& \mathrm{Gal}(E|\mathbb{Q}) \arrow[ur] \arrow[d, dash] & \\
& F\bar{\mathbb{Q}}^{\ker\bar{\rho}} &
\end{tikzcd}$$

$\mathrm{Gal}(E|\mathbb{Q})$ ← finite

$R^{univ}/m_F R^{univ}$ $\subset$ gen. by $\mathrm{Tr}\,\rho^{univ}(\sigma)$ for $\sigma \in \mathrm{Gal}(E|\mathbb{Q})$

STP $\forall\, \sigma \in \mathrm{Gal}(E|F)$ that $\mathbb{F}[\mathrm{Tr}\,\rho^{univ}(\sigma)] \subset R^{univ}/m_F R^{univ}$.

$\uparrow$ finite $/\mathbb{F}$

$A = \mathbb{F}[x_{ij}]\,/$ equations saying $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^m = 1$.

$\cup$

$\mathbb{F}[T = x_{11} + x_{22}]$

finite $/\mathbb{F}$

$$f(T) = \prod_{\substack{\zeta_1, \zeta_2 \in \overline{\mathbb{F}} \\ \zeta_1^m = \zeta_2^m = 1}} \left(T - (\zeta_1 + \zeta_2)\right)$$

$\forall\ \mathfrak{p}$ a prime of A, $f(T) \bmod \mathfrak{p} = 0$.

$\therefore$ $f(T)$ is nilpotent in A

$\therefore$ $f(T)^n = 0$ in A.

$S = \{2, \ell\}$

$\bar{r} : G_{\mathbb{Q}, S} \longrightarrow GL_2(\mathbb{F}_\ell)$, $\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$

$\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell^{-1} \end{pmatrix} \otimes \delta$, δ unramified

$\bar{r}|_{G_{\mathbb{Q}_\ell}}$ FL graded pieces in deg 0, 1

- for some $F|\mathbb{Q}$ finite Galois, unramified @ ℓ.

- $F \cap \overline{\mathbb{Q}}^{\ker \bar{r}} = \mathbb{Q}$, R_F^{univ} finite $/\mathcal{O}$ $\Rightarrow \exists\ r : G_{\mathbb{Q}, S} \to GL_2(\overline{\mathbb{Q}}_\ell)$

  lifting $\bar{r}$ similarly unramified.

- r fits into a compatible system. $\to r_3$ ✗

g

$\circlearrowright$

$GL_n(\mathbb{A}_F^{\infty}) \times U_\infty$

F # field, $\underset{\substack{\uparrow \\ \text{cusp.}}}{A_0} \left(GL_n(F) \underset{\text{discrete}}{\backslash} GL_n(\mathbb{A}_F)\right) = \{\phi : GL_n(F) \backslash GL_n(\mathbb{A}_F) \to \mathbb{C} :$

smooth: $\forall\ x \in GL_n(\mathbb{A}_F)$,

$\exists\ x^\infty \in W \subset GL_n(\mathbb{A}_F^{\infty})$, $\phi|_{W \times GL_n(F_\infty)}$

factors as $W \times GL_n(F_\infty) \twoheadrightarrow GL_n(F_\infty) \xrightarrow{\text{smooth}} \mathbb{C}$

- ϕ is finite under the action of

$GL_n(\hat{\mathcal{O}}_F) \times U_\infty$,

$U_\infty = \prod_{v|\infty} U_v$, $U_v = \begin{cases} O(n), & v \text{ real} \\ U(n), & v \text{ cpx} \end{cases}$

$\langle \phi(-u) : u \in GL_n(\hat{\mathcal{O}}_F) \times U_\infty \rangle$ is finite dim'l.

$\mathfrak{g}_0 = \mathrm{Lie}\, GL_n(F_\infty) = \prod_{v|\infty} \mathfrak{g}_{0,v}$, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$

$x \in \mathfrak{g}_0$, $(x\phi)(g) = \frac{d}{dt} \phi(g \exp(tx))\Big|_{t=0}$

extend $\mathbb{C}$-linearly to $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \prod_{v|\infty} \mathfrak{g}_v = \prod_{\tau : F \hookrightarrow \mathbb{C}} \mathfrak{g}_\tau^{\cong M_{n\times n}(\mathbb{C})}$$

$U(\mathfrak{g})$ acts

$\cup$

Z centre

$\wr\wr$

$\bigotimes_{\tau : F \hookrightarrow \mathbb{C}} Z_\tau$

$Z_\tau \overset{HC}{\cong} \mathbb{C}[X_1, \dots, X_n]^{S_n}$

$\uparrow$ acts on the irred. alg. rep. of GL_n w/ h.w. $a_1 \geq \dots \geq a_n$, $a_i \in \mathbb{Z}$

by $\{X_1, \dots, X_n\}$ acts as $\left\{a_1 + \frac{n-1}{2}, a_2 + \frac{n-3}{2}, \dots, a_n + \frac{1-n}{2}\right\}$

- ϕ is finite under Z
- ϕ slowly increasing:

  $\exists\, c, r$ s.t. $|\phi(g)| \leq c \|g\|^r$. $\|g\| = \prod_v \max\{|g_{ij}|_v, |(g^{-1})_{ij}|_v\}$
- $\int_{N_m(F)\backslash N_m(\mathbb{A}_F)} \phi(ng)\,dn = 0$, $\forall g \in GL_n(\mathbb{A}_F)$, $N_m = \left\{\begin{pmatrix} 1_m & * \\ 0 & 1_{n-m}\end{pmatrix}\right\}$, $\forall m = 1, \dots, n-1$

$x \in \mathrm{Lie}\, U_\infty \Rightarrow \frac{d}{dt}\phi(g \; \exp(tX))\Big|_{t=0} = (X\phi)(g)$

- $X \in \mathfrak{g}$

$g \in GL_n(\mathbb{A}_F^{\infty}) \times U_\infty$, $g(X\phi) = \mathrm{ad}(g_\infty)(X)(g.\phi)$

$$\mathcal{A}_0\left(GL_n(F)\backslash GL_n(\mathbb{A}_F)\right) = \bigoplus \pi$$

irred.

$(\mathfrak{g}, \; GL_n(\mathbb{A}_F^{\infty}) \times U_\infty)$ -module

cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$

$\pi \not\cong \pi'$ unless $\pi = \pi'$.

$\pi \cong \bigotimes_v{}' \pi_v$, $v \nmid \infty$, π_v is a smooth rep. of $GL_n(F_v)$

for all but finitely many v. $v | \infty$. $(\mathfrak{g}_v, U_\infty)$-module

π_v is unramified, $\pi_v^{GL_n(O_{F_v})} \neq (0)$

$\therefore$ 1-dim'l

$0 \neq e_v \in \pi_v^{GL_n(O_{F,v})}$

$\pi \cong \bigotimes_v{}' \pi_v = \varinjlim_{S \text{ finite}} \bigotimes_{v \in S} \pi_v$ $\quad S' \supset S \quad \bigotimes_{v\in S} \longrightarrow \bigotimes_{v \in S'} \pi_v$

indep. of choice of e_v.

$\bigotimes_{v \in S} x_v \mapsto \left(\bigotimes_{v\in S} x_v\right) \otimes \left(\bigotimes_{v \in S' \setminus S} e_v\right)$

Z acts on π by a hom. $Z \longrightarrow \mathbb{C}$

$$\begin{tikzcd} Z \arrow[d, leftrightarrow, bend right, "\wr"'] \\ \bigotimes_{\tau: F \hookrightarrow \mathbb{C}} \mathbb{C}[X_1, \dots, X_n]^{\mathfrak{S}_n} \end{tikzcd}$$

$HC_\tau(\pi_\infty)$ = mult. set of n cpx numbers

$\forall \tau : F \hookrightarrow \mathbb{C}$

Def π is called <u>algebraic</u> if $HC_\tau(\pi_\infty) \subset \mathbb{Z}$, $\forall \tau$

π is called <u>regular</u> if $HC_\tau(\pi_\infty)$ has n-distinct elts, $\forall \tau$

Thm. Suppose F is CM. and π is regular algebraic. Let $i : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$.

$\uparrow$ $\exists\, c \in \operatorname{Aut}(F)$ $\forall \tau : F \hookrightarrow \mathbb{C}$, $\tau c = c \tau$

Then $\exists\ r_{\ell,i}(\pi) : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ cts rep. s.t. $\forall v \nmid \ell$,

$$r_{\ell,i}(\pi)\Big|^{ss}_{W_{F_v}} = \underset{\overset{\uparrow}{LLC}}{rec}(\pi_v)^{ss}$$

and if π_v is unramified, then $r_{\ell,i}(\pi)$ is unramified.

$U \subset GL_n(\mathbb{A}_F^\infty)$ open cpt.

$$GL_n(F) \backslash GL_n(\mathbb{A}_F) / U \times U_\infty (F_\infty^\times)^\circ$$

X_U $\sim$ real mfd. $H^*(X_U, \mathbb{C})$ computed in terms of aut. reps (e.g. alg.)

Lecture 13

$\pi = \otimes'_v \pi_v$ cusp. auto. rep. of $GL_n(\mathbb{A}_F)$

F # field

Strong Multiplicity One Theorem (SMOT): π, π', $\pi_v \cong \pi'_v$ for all but finitely many v

$\Rightarrow \pi = \pi' \subset \mathcal{A}_0(GL_n(F) \backslash GL_n(\mathbb{A}_F))$

π regular algebraic $\Rightarrow$ ...

$Z \curvearrowright \pi_\infty$

v finite place of F,

$$\begin{tikzcd} 0 \arrow[dr] & & & \\ & I_{F_v} \arrow[r] & G_{F_v} \arrow[r, two heads] & G_{k(v)} \arrow[d, equal] \\ & & \mathrm{Frob}_v \arrow[d, maps to] & \langle \mathrm{Frob}_v \rangle \arrow[d, "v"', "\wr"] \\ & & 1 & \hat{\mathbb{Z}} \end{tikzcd}$$

$\leftarrow$ residue field at v

$, \quad (\mathrm{Frob}_v \alpha)^{\# k(v)} = \alpha \quad , \alpha \in \overline{k(v)}$

Weil gp

$W_{F_v} = \{ \sigma \in G_{F_v} : v(\sigma) \in \mathbb{Z} \} \xrightarrow{v} \mathbb{Z}$

$\triangledown$ open

$P_{F_v} \triangleleft I_{F_v} \leftarrow$ usual top.

pro-p-gp

$t_\ell : I_{F_v} \twoheadrightarrow \mathbb{Z}_\ell$ unique up to $\mathbb{Z}_\ell^\times$-multiples

$I_{F_v} / P_{F_v} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$

A <u>WD rep</u> of W_{F_v} over a field L of char. 0 is (ρ, N)

V/L a fin. dim'l vec. sp. $\rho : W_{F_v} \longrightarrow \mathrm{Aut}(V/L)$ w/ open kernel

$N \in \mathrm{End}(V/L)$ s.t. $\rho(\sigma) N \rho(\sigma)^{-1} = \# k(v)^{-v(\sigma)} N$, $\forall \sigma \in W_{F_v}$

($\Rightarrow$ N nilpotent)

We call (ρ, N) <u>F-semisimple</u> if ρ is semisimple

<u>Semisimple</u> if ρ is semisimple and $N=0$.

<u>Lemma</u>

$\exists !$ $u \in \mathrm{Aut}(V/L)$ unipotent which commutes w/ $\mathrm{Im}\,\rho$ and N s.t.

$\sigma \longmapsto \rho(\sigma) u^{-v(\sigma)}$ is a semisimple rep'n

$$(\rho N)^{F\text{-}ss} = (\rho u^{-v(-)}, N) \ , \ (\rho, N)^{ss} = (\rho u^{-v(-)}, 0)$$

<u>$v \nmid \ell$</u> Choose $t_\ell : I_{F_v} \twoheadrightarrow \mathbb{Z}_\ell$

$\phi \in W_{F_v}$, $v(\phi) = 1$

then $\exists$ fully faithful functor

$$WD : \left\{ \begin{array}{l} \text{fin. dim'l } \overline{\mathbb{Q}}_\ell \text{ vec. sp} \\ \text{w/ linear action of } G_{F_v} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{WD reps of } W_{F_v} \\ \text{over } \overline{\mathbb{Q}}_\ell \end{array} \right\}$$

If you change t_ℓ or ϕ, then WD changes by a natural isom.

$v \mid \ell$. To a de Rham rep'n r of G_{F_v} on a fin. dim'l $\overline{\mathbb{Q}}_\ell$ - vec. sp.,

we can associate a WD rep. $WD(r)$ of W_{F_v} over $\overline{\mathbb{Q}}_\ell$ (<u>NOT</u> fully faithful)

and if $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_\ell / \mathbb{Q}_\ell$ a multiset $HT_\tau(r) \subset \mathbb{Z}$ (Hodge - Tate #'s)

$v \mid \ell$. If $\bar{r} : G_{F_v} \longrightarrow GL_n(\bar{\mathbb{F}}_\ell)$ Fontaine - Laffailles

$G(M) = \bar{r}$, $M \,/\, k(v) \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell$

$\tau : k(v) \hookrightarrow \bar{\mathbb{F}}_\ell$

$M_\tau = M \otimes_{k(v)\otimes \bar{\mathbb{F}}_\ell} \bar{\mathbb{F}}_\ell \;\; \nearrow_{\tau\otimes id}$, $M = \bigoplus M_\tau$

$HT_\tau(\bar{r})$ multiset $\subset \mathbb{Z}$

$\rotatebox[origin=c]{90}{$\in$}$

i multiplicity $\dim_{\bar{\mathbb{F}}_\ell} gr^i M_\tau$

$r : G_{F_v} \longrightarrow GL_n(\mathcal{O}_{\bar{\mathbb{Q}}_\ell})$ is FL, then

$$\begin{tikzcd}
\bar{r} = (r \bmod \lambda_{\bar{\mathbb{Q}}_\ell}) \arrow[r] & HT_\tau(\bar{r}) \arrow[d, phantom, "\parallel"] \\
r \otimes \bar{\mathbb{Q}}_\ell \text{ is de Rham} \arrow[r] & HT_\tau(r \otimes \bar{\mathbb{Q}}_\ell)
\end{tikzcd}$$

(FL $\Rightarrow$ de Rham)

$\exists$ a natural bijection

$$\begin{tikzcd}
rec : \left\{ \begin{array}{l} \text{isom. classes of} \\ \text{irred. sm. reps} \\ \text{of } GL_n(F_v) \text{ over } \mathbb{C} \end{array} \right\} \arrow[d, "S"'] \\
\left\{ \begin{array}{l} \text{iso. classes of} \\ n\text{-dim'l } F\text{-ss} \\ WD \text{ reps of } W_{F_v} \text{ over } \mathbb{C} \end{array} \right\}
\end{tikzcd}$$

$\pi^{GL_n(\mathcal{O}_{F_v})} \neq 0 \iff rec(\pi)$ is unramified.

Thm. Suppose F is a CM field and π $\overset{= \otimes'_v \pi_v}{}$ is a regular algebraic cuspidal auto rep. of $GL_n(\mathbb{A}_F)$ $\quad (GL_n(\mathbb{A}_F^\infty) \times U_\infty, \mathfrak{g})$

Suppose $i : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$ as fields, then $\exists\; r_{\ell,i}(\pi) : G_F \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$ cts semisimple unramified a.e.

s.t. $\forall\, v \nmid \ell$, $WD(r_{\ell,i}(\pi))^{ss} \cong i\, rec(\pi_v)^{ss}$

$\uparrow$ over determines $r_{\ell,i}(\pi)$

Often know more.

eg. $n=2$, F totally real:

$$WD(r_{\ell,i}(\pi))^{F\text{-}ss} \cong i\,\mathrm{rec}(\pi_v), \ \forall v$$

$$\left(v|\ell \Rightarrow r_{\ell,i}(\pi)\big|_{G_{F_v}} \text{ is de Rham} \right)$$

$$\begin{tikzcd} F \arrow[r, "\tau"] \arrow[d] & \overline{\mathbb{Q}}_\ell \\ F_{v(\tau)} \arrow[ur] & \end{tikzcd}$$

$\forall \tau: F \hookrightarrow \overline{\mathbb{Q}}_\ell$

$$HT_\tau(r_{\ell,i}(\pi)) = HT_\tau\left(r_{\ell,i}(\pi)\big|_{G_{F_{v(\tau)}}}\right)$$

$$= -HC_{i^{-1}\circ\tau}(\pi_\infty)$$

(regular algebraic)

We will call $r: G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ <u>automorphic</u> if it arises in this way for some π, i.

<u>weight</u> 0 $\longleftrightarrow$ $HC_\tau(\pi) = \{0, -1, \cdots, 1-n\}$, $\forall \tau$.

<u>level prime to ℓ</u> $\longleftrightarrow$ π_v is unramified for all $v|\ell$.

We call $\bar{r}: G_F \longrightarrow GL_n(\overline{\mathbb{F}}_\ell)$ automorphic if $\exists$ $r: G_F \longrightarrow GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})$ w/ $r \bmod \lambda_{\overline{\mathbb{Q}}_\ell} \cong \bar{r}$ and $r \otimes \overline{\mathbb{Q}}_\ell$ automorphic.

$\left.\begin{array}{l}\underline{\text{weight } 0} \\ \underline{\text{level prime to } \ell}\end{array}\right\}$ if can choose r s.t. these apply.

Conj. (Fontaine-Mazur) If $r: G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ is cts and

1) r unramified a.e.

2) $r|_{G_{F_v}}$ is de Rham, $\forall v|\ell$ and $HT_\tau(r)$ has n distinct elts, $\forall \tau$

3) r irreducible

then r is automorphic.

(consequence of cyclic base change)

Thm If $E|F$ is a soluble Galois ext'n of CM fields, and if $r: G_F \to GL_n(\overline{\mathbb{Q}}_\ell)$ is cts w/ $r|_{G_E}$ irreducible, then r is automorphic $\iff$ $r|_{G_E}$ is automorphic.

"Pf": reduce $E|F$ cyclic of prime order

Lecture 14

$v: G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

$r: G_F \longrightarrow GL_n(\overline{\mathbb{F}}_\ell)$ automorphic

Thm. (ALT) F totally real no. field, $\ell > 3$ a prime unramified in F $(\Rightarrow [F(\zeta_\ell):F] > 2)$

prime ord. $\begin{matrix} E \\ | \\ F \end{matrix}$ $\langle\sigma\rangle$ r

$\mathcal{A}_0(E) \ni \psi\sigma$

$\mathcal{A}_0(F) \ni \varphi$

$r \circ \mathrm{conj}_\sigma \cong r$

$\Rightarrow$ r descends to G_F.

$S = S_1 \sqcup S_\ell$ a finite set of finite places, $S_\ell = \{v|\ell\}$

$r: G_{F,S} \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell)$ cts rep. s.t.

↳ conj. to $GL_2(\mathcal{O}_{\overline{\mathbb{Q}}_\ell}) \twoheadrightarrow GL_2(\overline{\mathbb{F}}_\ell)$ $\quad G_{F,S} \longrightarrow GL_2(\overline{\mathbb{F}}_\ell)$ semisimplification $\overline{r}$

1) $\bar{r}|_{G_{F(\zeta_\ell)}}$ is absolutely irred. and $\bar{r}$ is automorphic of wt 0

$HC_\tau(\pi) = \{0,-1\}$, $\forall \tau$

level $U_0(S_1)$ $\leftarrow$ [$\bar{r} = \bar{r}_{\ell,i}(\pi)$ some $i : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$, π regular alg. cusp. auto. rep.

π_v unramified if $v \notin S_1$, if $v \in S_1$, then $\pi_v^{Iw_v} \neq 0$

$Iw_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,v}) : v | c \right\}$]

2) If $v | \ell$, then $r|_{G_{F,v}}$ is Fontaine-Laffaille w/ $HT_\tau(\bar{r}) = \{0,1\}$, $\forall \tau$.

3) If $v \in S_1$, then $r|_{I_{F_v}}$ unipotent. (or trivial)

Then r is automorphic of wt 0 and level $U_0(S_1)$.

Thm. Suppose $\ell > 3$ and $\bar{r} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$, w/ $S = \{2, \ell\}$, s.t.

- $\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$
- $\bar{r}|_{G_{\mathbb{Q}_\ell}}$ is FL, HT no's $\{0,1\}$
- $\bar{r}|_{G_{\mathbb{Q}_2}} = \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell \end{pmatrix} \otimes \delta$, δ unramified

Then there exists $F|\mathbb{Q}$ a finite totally real Galois ext'n in which ℓ is ramified, w/ $F \cap \bar{\mathbb{Q}}^{\ker \bar{r}}(\zeta_\ell) = \mathbb{Q}$ s.t. $\bar{r}|_{G_F}$ is automorphic of wt 0 and level $U_0(S_1)$ for some finite set of places S_1 of F containing primes above 2.

Pf. Choose $M|\mathbb{Q}$ an imaginary quad. field in which 2 and l are unramified, but ramified at some prime >3.

Choose a prime p which splits in M, $p>3$, $p \nmid l$,

Look for a finite ext'n $N|M$ and a cts char. $\theta: \mathbb{A}_M^\times \longrightarrow N^\times$ s.t.

1) $\theta|_{M^\times} = \mathrm{Id}$

2) $\theta|_{\mathbb{A}_{\mathbb{Q}}^\times}(x) = \|x\|^{-1} x_\infty \, \delta_{M|\mathbb{Q}}(x)$ $\quad \delta_{M|\mathbb{Q}} : \mathbb{A}^\times / \mathbb{Q}^\times (N_{M|\mathbb{Q}} \mathbb{A}_M^\times) \twoheadrightarrow \{\pm 1\}$, $\|x\| = \prod_v |x_v|_v$

3) θ unramified except at p, and the primes that ramified in $M|\mathbb{Q}$

4) if $v|p$, then $\#\theta(\mathcal{O}_{M,v}^\times) = p-1$.

$$\theta : U = \mathbb{C}^\times \times \prod_{\substack{v \text{ unramified} \\ \text{over } \mathbb{Q}}} \mathcal{O}_{M,v}^\times \times \prod_{\substack{v \text{ ramified} \\ \text{over } \mathbb{Q}}} (1 + \overset{\text{uniformizer}}{\pi_v} \mathcal{O}_{M,v}) \longrightarrow M(\zeta_{p-1})^\times$$

satisfying 3), 4)

trivial everywhere except at $v|p$.

$v|p$, $\theta|_{\mathcal{O}_{M,v}^\times} = \chi$ of order $p-1$

$\theta|_{\mathcal{O}_{M,\bar{v}}^\times} = \chi^{-1}$

extend θ to $U\mathbb{A}^\times$ satisfying 2), 3), 4) — OK if agree on $U \cap \mathbb{A}^\times$

$$= \mathbb{R}^\times \times \prod_{\substack{v \text{ unramified} \\ \text{in } M|\mathbb{Q}}} \mathbb{Z}_v^\times \times \prod_{v \text{ ram.}} (1 + v\mathbb{Z}_v)$$

θ trivial as is $\|x\|^{-1} x_\infty \delta_{M|\mathbb{Q}}(x)$.

$\alpha \in M^\times \cap U\mathbb{A}^\times$, $\alpha / \overset{\text{cpx conj}}{c}\alpha \in M^\times \cap U$
$= \{\pm 1\} \cap U = \{1\}$

$\Rightarrow \alpha \in \mathbb{Q}^\times$, $\theta|_{\mathbb{Q}^\times} = \mathrm{Id}$ $\therefore$ θ extends to $M^\times U\mathbb{A}^\times$ satisfying 1) – 4)

$M \times U\mathbb{A}^\times$

$\cap$ finite index

$\mathbb{A}_M^\times$

$\therefore$ extend θ to $\mathbb{A}_M^\times$ (possibly make N bigger)

Choose ℓ' s.t. $\ell' \nmid 2\ell(p-1)p$ at which M is unramified and ℓ' splits completely in N

Choose $\lambda' | \ell'$ a prime of N

$$\theta_{\lambda'} : G_M \cong \mathbb{A}_M^\times / M^\times M_\infty^\times \longrightarrow N_{\lambda'}^\times \cong \mathbb{Z}_{\ell'}^\times$$

$$x \longmapsto \theta(x)\, x_{\lambda'}^{-1}|_M$$

<u>$\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'}$</u>

$$\det \mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} = \varepsilon_{\ell'}^{-1}$$

$$\Big\updownarrow$$

$$\theta_{\lambda'}\Big|_{\mathbb{A}^\times} \delta_{M|\mathbb{Q}}$$

i.e. $x \longmapsto \|x\|^{-1} x_\infty x_{\ell'}^{-1}$

$$\Big\updownarrow$$

$$\varepsilon_{\ell'}^{-1}$$

$\Delta \overset{2}{\trianglelefteq} T$ $\gamma \in T - \Delta$

$\det \mathrm{Ind}_\Delta^T \chi$

$= (\chi \circ \mathrm{tr})\, \delta_{T/\Delta}$ $\delta_{T/\Delta} : T/\Delta \xrightarrow{\sim} \{\pm 1\}$

$\mathrm{tr} : T^{ab} \longrightarrow \Delta^{ab}$

$\varepsilon \longmapsto \varepsilon\gamma\varepsilon\gamma^{-1}$

<u>Lecture 15</u> <u>Thm</u>. $\ell > 3$ prime. $\bar{r} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$, $S = \{2, \ell\}$,

- $\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$
- $\bar{r}|_{G_{\mathbb{Q}_\ell}}$ is FL w/ HT nos $\{0,1\}$
- $\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell^{-1} \end{pmatrix} \otimes \delta$, δ nr

Then $\exists\ F|\mathbb{Q}$ totally real field finite Galois,

w/ $F \cap \overline{\mathbb{Q}}^{\ker \bar{r}}(\zeta_\ell) = \mathbb{Q}$ s.t. $\bar{r}$ is automorphic over F w/ wt 0

and level $U_0(S_1)$ for some $S_1 \not\ni \ell$.

Pf. $M|\mathbb{Q}$ imag. quad field, $2, \ell$ unramified in M, some prime $p > 3$ ramified.

$\theta : \mathbb{A}_M^\times \longrightarrow N^\times$ $\quad N \supset M$ finite

ℓ' splits completely in N, $\lambda' | \ell'$ prime of N

$\ell' \neq \ell$, M and θ nr @ ℓ' and $\ell' \nmid (p-1)$

$$\begin{tikzcd} \theta_{\lambda'} : \mathbb{A}_M^\times / M^\times M_\infty^\times \arrow[r] & \mathbb{Z}_{\ell'}^\times \\ G_M^{ab} \arrow[u, phantom, "\wr"] & \\ x \arrow[r, mapsto] & \theta(x)\, x_{\lambda'}|_M^{-1} \end{tikzcd}$$

$\overline{\theta_{\lambda'}} = \theta_{\lambda}' \bmod \ell'$

det $\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} = \varepsilon_\ell^{-1}$

$\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'}$ non-ramified at $2, \ell$ $\leftarrow$ only ramified @ ℓ', p, primes ramified in $M|\mathbb{Q}$.

$\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \bar{\theta}_\lambda' \Big|_{G_{\mathbb{Q}(\zeta_{\ell'})}}$ is absolutely irred.

$p = v\bar{v}$, $\bar{\theta}_\lambda' \big|_{I_v}$ order $p-1$, $\bar{\theta}_\lambda' \big|_{I_{\bar{v}}} = \bar{\theta}_\lambda' \big|_{I_v}^{-1}$

$\bar{\theta}_\lambda' \big|_{G_{M(\zeta_{\ell'})}} \neq \bar{\theta}_\lambda' \big|^{c}_{G_{M(\zeta_{\ell'})}}$ because $I_v \leftarrow I_p \rightarrow I_{\bar{v}}$ not on inertia above p

$\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \theta_{\lambda'}|_{G_{\mathbb{Q}_{\ell'}}}$ is FL w/ HT no's $\{0,1\}$.

$T/\mathbb{Q}$ moduli space of elliptic curves E w/ $\left[\begin{array}{l} E[\ell]^\vee \cong \bar{r} \\ E[\ell']^\vee \cong \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \bar{\theta}_{\lambda'} \end{array}\right.$

Weil pairing

$\Updownarrow$

det pairing

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = xv - yu.$$

$T(\mathbb{C}) \cong \Gamma(\ell\ell') \backslash \mathfrak{H}$ $\Rightarrow$ T is geom. conn'd.

$\{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv I_2 \ (\ell\ell')\}$

$T(\mathbb{R}) \neq \phi$, $\bar{r}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\left(\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta_{\lambda'}}\right)(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$

<u>Claim</u> $\exists\ L/\mathbb{Q}_\ell$ unramified w/ $T(L) \neq \phi$

$\exists\ L'/\mathbb{Q}_{\ell'}$ unramified w/ $T(L') \neq \phi$.

<u>Lemma</u> Suppose ℓ is an odd prime. $\bar{r} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_\ell)$ FL w/ HT no's $\{0,1\}$ $\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$.

then $\exists\ L/\mathbb{Q}_\ell$ unramified and E/L an elliptic curve w/ good red'n

and $E[\ell]^\vee \cong \bar{r}$ w/ Weil $\longleftrightarrow$ det. (make further un ext so $E[\ell']^\vee$ trivial & $\mathrm{Ind}\,\bar{\theta}_{\lambda'}$ trivial)

Pf. <u>Case 1</u> $\bar{\tau}$ irreducible. $\bar{\tau} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \phi$. $\phi|_{I_{\mathbb{Q}_{\ell^2}}} : I_{\mathbb{Q}_{\ell^2}} \to I^{ab}_{\mathbb{Q}_{\ell^2}|\mathbb{Q}_{\ell^2}}$

$$\begin{tikzcd} I^{ab}_{\mathbb{Q}_{\ell^2}|\mathbb{Q}_{\ell^2}} \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z}_{\ell^2}^\times \arrow[d] \\ \mathbb{F}_{\ell^2}^\times \end{tikzcd}$$

$E|\mathbb{Q}_{\ell^2}$ any elliptic curve w/ good supersingular reduction.

$\phi(\ell) = -1$

$$\begin{tikzcd} E[\ell] \arrow[d, "\text{alter by elt of } \mathbb{F}_{\ell^2}^\times"'] & \cong \bar{\tau} \\ & \mathbb{F}_{\ell^2}^\times \subset GL_2(\mathbb{F}_\ell) \arrow[u, bend left] \end{tikzcd}$$

which commutes w/ Gal. action to get alternating pairings to match. $N_{\mathbb{F}_{\ell^2}|\mathbb{F}_\ell}(\mathbb{F}_{\ell^2}^\times) = \mathbb{F}_\ell^\times$

<u>Case 2</u>. $\bar{\tau}$ reducible. $\bar{\tau} \sim \begin{pmatrix} \bar{\chi} & * \\ 0 & \bar{\chi}^{-1}\bar{\varepsilon}_\ell^{-1} \end{pmatrix}$ $\bar{\chi}$ unramified

$*$ peu ramifiée

Choose alternating pairing $\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = xv - yu$

$$* \in H^1(G_{\mathbb{Q}_\ell}, \mathbb{F}_\ell(\bar{\chi}^2 \bar{\varepsilon}_\ell)).$$

well-defined by pairing choice.

Choose n and $\bar{E}|\mathbb{F}_{\ell^n}$ s.t. $\bar{\chi}|_{G_{\mathbb{F}_{\ell^n}}} = 1$, $\bar{E}|\mathbb{F}_{\ell^n}$ ordinary elliptic curve. w/ a point of order ℓ.

<u>Serre-Tate</u>: $T_\ell \bar{E} \cong \mathbb{Z}_\ell(\psi)$, $\psi : G_{\mathbb{F}_{\ell^n}} \to (1+\ell\mathbb{Z}_\ell) \subset \mathbb{Z}_\ell^\times$

lifts of $\bar{E}$ to $\mathbb{Z}_{\ell^n}$ $\overset{\text{elts of}}{\longleftrightarrow}$ $H^1(G_{\mathbb{Q}_\ell}, \mathbb{Z}_\ell(\psi^2 \varepsilon_\ell))$

E_x $\quad T_\ell E_x \sim \begin{pmatrix} \psi^{-1}\varepsilon_\ell & * \\ 0 & \psi \end{pmatrix}$ s.t. Weil pairing is $\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$

$= xv - yu.$

$$\xrightarrow{\ell} H^1(G_{\mathbb{Q}_\ell}, \mathbb{Z}_\ell(\psi^2\varepsilon_\ell)) \longrightarrow H^1(G_{\mathbb{Q}_\ell}, \mathbb{F}_\ell(\varepsilon_\ell)) \cong \mathbb{Q}_{\ell^n}^\times / (\mathbb{Q}_{\ell^n}^\times)^\ell$$

$$\cup$$

$$\to H^2(G_{\mathbb{Q}_{\ell^n}}, \mathbb{Z}_\ell(\psi^2\varepsilon_\ell))[\ell] \to 0 \qquad \begin{pmatrix} \text{the class} \\ \text{of } \bar{\imath}(G_{\mathbb{Q}_{\ell^n}}) \end{pmatrix} \in \mathbb{Z}_{\ell^n}^\times / (\mathbb{Z}_{\ell^n}^\times)^\ell$$

$$0 \searrow$$

<u>STP</u> $\mathbb{Z}_{\ell^n}^\times / (\mathbb{Z}_{\ell^n}^\times)^\ell \longrightarrow H^1(G_{\mathbb{Q}_{\ell^n}}, \mathbb{F}_\ell(\varepsilon_\ell)) \longrightarrow H^2(G_{\mathbb{Q}_{\ell^n}}, \mathbb{Z}_{\ell^n}(\psi^2\varepsilon_\ell))[\ell]$

is zero

<u>Local Tate duality</u>

$$\begin{tikzcd}
 & 0 & & \\
 & \operatorname{Hom}(I_{\mathbb{Q}_{\ell^n}}, \mathbb{F}_\ell)^{G_{\mathbb{F}_{\ell^n}}} \arrow[u] \arrow[r] & \mathbb{F}_\ell \arrow[r] & 0 \\
H^0(G_{\mathbb{Q}_\ell}; \mathbb{Q}_\ell/\mathbb{Z}_\ell)(\psi^{-2})/\ell \arrow[ur] \arrow[r] & \operatorname{Hom}(G_{\mathbb{Q}_{\ell^n}}, \mathbb{F}_\ell) \arrow[u] & & \\
0\,? \arrow[u] & \operatorname{Hom}(G_{\mathbb{F}_{\ell^n}}, \mathbb{F}_\ell) \arrow[u] & & \\
 & 0 \arrow[u] & &
\end{tikzcd}$$

$$\begin{array}{c} \Downarrow \\ x \end{array}$$

$(\psi^{-2}(\sigma) - 1)\left(\frac{x}{\ell}\right)$

$= 0$ if $\sigma \in I_{\mathbb{Q}_{\ell^n}}$

<u>Prop</u> K a no. field, $K^{\text{aux}} | K$ finite Galois

S fin set of places of K.

$T | K$ geom. conn'd curve (variety), for $v \in S$, suppose
smooth

$L_v' | K_v$ is a finite Galois ext'n, and $\Omega_v \subset T(L_v')$ is $\operatorname{Gal}(L_v' | K_v)$-inv't
$\ne \emptyset$ open for v-adic top.

Then $\exists$ $L|K$ a finite Galois ext'n, w/ $L \cap K^{\text{avoid}} = K$

and $\rho \in T(L)$ s.t. $w|v \in S \Rightarrow L'_v \cong L_w$

and $\rho \in T(L_w) \cong T(L'_v)$ is in Ω_v

Lecture 16   Thm $\ell > 3$, $\bar{r}: G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$, $S = \{2, \ell\}$

$\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$, $\bar{r}|_{G_{\mathbb{Q}_\ell}}$ FL, HT no's $\{0,1\}$.

$\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta$. δ ur.

$\Rightarrow$ $\exists$ $F|\mathbb{Q}$ a finite Galois totally real ext'n w/ $F \cap \bar{\mathbb{Q}}^{\ker \bar{r}}(\zeta_\ell) = \mathbb{Q}$
and 2 & ℓ unramified in F
s.t. $\bar{r}|_{G_F}$ is automorphic w/ wt 0 and U_0-level prime to ℓ.

Pf. $M|\mathbb{Q}$ imaginary quad. $\theta_{\lambda'}: G_M \to \mathbb{Z}_{\ell'}^\times$, $\exists \ell' \neq \ell$ $p \neq \ell, \ell', 2$, FL

ℓ' splits in M

2, ℓ unramified in M

FL $HT_\tau(\theta_{\lambda'}) = \{0\}$

$HT_{\bar{\tau}}(\theta_{\lambda'}) = \{1\}$.

ramified only at ℓ', p, + primes that ramify in $M|\mathbb{Q}$.

$\det \operatorname{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} = \varepsilon_{\ell'}^{-1}$

$\operatorname{Ind}_{G_M}^{G_\mathbb{Q}} \overline{\theta_{\lambda'}}\Big|_{G_{\mathbb{Q}(\zeta_{\ell'})}}$ irred

T moduli space of E + $i \left(\begin{array}{l} E[\ell] \cong \bar{r} \\ E[\ell'] \cong \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta}_{\lambda'} \end{array} \right) \Big]$ taking Weil pairing to fixed pairing on RHS.

$\uparrow$

geom. conn'd,
smooth

$T(\mathbb{R}) \neq \phi$. $\exists\ L \,|\, \mathbb{Q}_\ell$ unramified, and $(E, i) \in T(L)$, E good red'n.

$L' \,|\, \mathbb{Q}_{\ell'}$ - - - - - - $(E', i') \in T(L')$, E' good red'n.

$\exists\ L'' \,|\, \mathbb{Q}_2$ unramified w/ $T(L'') \neq \phi$.

$q \in \mathbb{Q}_2^\times$

$v(q) > 0$

$E_q \,|\, \mathbb{Q}_2$

$E_q(\overline{\mathbb{Q}}_2) \cong \overline{\mathbb{Q}}_2^\times / q^{\mathbb{Z}}$, look for q such that the image of q in

$E_q[\ell] \cong \bar{r}|_{G_{\mathbb{Q}_2}} \otimes \delta^{-1}$

$E_q[\ell']$ unramified. $\underset{\mathbb{Z}/\ell\mathbb{Z}.}{\overset{V_2}{\wr\wr}}$ $\mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^\ell \cong H^1(G_{\mathbb{Q}_2}, \mathbb{F}_\ell(\bar{\epsilon}_\ell))$

$\hookrightarrow *$.

Choose L'' s.t. on $G_{L''}$, $\delta = 1$, $q \in (\mathbb{Q}_2^\times)^{\ell'}$.

$E_q[\ell']$ and $\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta}_{\lambda'}$ are both trivial.

$T/\mathbb{Q}$ $\mathbb{Q}^{av} = \overline{\mathbb{Q}}^{\ker \bar{r} \times \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta}_{\lambda'}} (\zeta_{\ell\ell'})$

$S = \{\infty, 2, \ell, \ell', p,$ primes that ramify in $M \,|\, \mathbb{Q}\}$

$L_v' \,|\, \mathbb{Q}_v$ finite Galois.

$\Omega_v \overset{\text{open}}{\subset} T(L_v')$

$\neq$

ϕ

$v = \infty$, $L'_\infty = \mathbb{R}$, $\Omega_v = T(\mathbb{R})$

$v = 2$, $L'_2 = L''$, $\Omega_2 = T(L'_2)$.

$v = \ell$, $L'_\ell = L$, $\Omega_\ell =$ good red'n locus of $T(L'_\ell)$

$v = \ell'$, $L'_{\ell'} = L'$. $\Omega_{\ell'} =$ - —— —$T(L'_{\ell'})$

$v = p$

or v ramified in M $\Big]$ $L'_v \,|\, \mathbb{Q}_v$ finite Galois

s.t. $\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} \Big|_{G_{L'_v}}$ unramified $\qquad \Omega_v = T(L'_v)$

and $T(L'_v) \neq \phi$

$F \,|\, \mathbb{Q}$ finite Galois, $\quad F \cap \mathbb{Q}^{av} = \mathbb{Q}$

- F totally real.
- $2, \ell, \ell'$ unramified in F.
- $\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} \Big|_{G_F}$ is unramified away from ℓ'

and E/F good reduction at ℓ & ℓ'.

$E[\ell]^\vee \cong \bar{r}|_{G_F}$.

$E[\ell']^\vee \cong \mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \overline{\theta}_{\lambda'} \Rightarrow$ the action of I_v for $v \nmid \ell'$ on $T_{\ell'}E$ is unipotent.

$GL_2(\mathbb{Z}_{\ell'})$ has no ℓ' torsion $(\ell' > 3)$

(otherwise $\mathbb{Q}(\zeta_{\ell'})^\times$
$\cap$
$GL_2(\mathbb{Q}_{\ell'})$)

$\therefore$ E has good red'n or multiplicative reduction everywhere.

$\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} \Big|_{G_F}$ is automorphic of wt 0 and level 1

from π unramified everywhere.

$\therefore E[\ell']^\vee \cong \mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \overline{\theta}_{\lambda'} \Big|_{G_F \hookleftarrow G_{F(\zeta_{\ell'})}}$ is aut. of wt 0 and level 1.

$$\begin{tikzcd}
(T_{l'}E)^\vee \text{ — is automorphic of wt } 0 \text{ and } U_0\text{-level prime to } l. & \\
\text{automorphy lifting thm.} \arrow[u] & \pi \arrow[ul, leftrightarrow] \arrow[d, leftrightarrow] \\
& (T_l E)^\vee
\end{tikzcd}$$

$$\left(\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \bar{\theta}_{\lambda'}\right)\left(G_{F(\zeta_{l'})}\right)$$

$$\overset{\text{by linearly disjtness}}{=} \left(\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \bar{\theta}_{\lambda'}\right)\left(G_{\mathbb{Q}(\zeta_{l'})}\right) \text{ irreducible.}$$

$(T_l E)^\vee$ is automorphic of wt 0 and U_0-level prime to l.

$E[l]^\vee \cong \bar{r}|_{G_F}$ is auto. of wt 0 and U_0-level prime to l.

<u>Thm</u>. Suppose $l>3$, $\bar{r}: G_{\mathbb{Q},S} \to GL_2(\mathbb{F}_l)$, $S=\{2,l\}$

- $\det \bar{r} = \bar{\varepsilon}_l^{-1}$, • $\bar{r}|_{G_{\mathbb{Q}_l}}$ is FL, HT no's $\{0,1\}$.
- $\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_l^{-1} \end{pmatrix} \otimes \delta$, δ nr.
- $\bar{r}$ is irreducible.

Then $\exists\, L|\mathbb{Q}_l$ finite and $r: G_{\mathbb{Q},S} \to GL_2(\mathcal{O}_L)$ cts w/

$(r \bmod \lambda_L) \cong \bar{r}$

$\det r = \varepsilon_l^{-1}$

$r|_{G_{\mathbb{Q}_l}}$ FL, HT no's $\{0,1\}$

$r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_l^{-1} \end{pmatrix} \delta$, δ nr

$R^{univ}_{\bar r}|_{G_F}$ finite / $\mathbb{Z}_\ell$ $\Rightarrow$ Such an r exists.

$\Uparrow_{\text{ALT}}$

Moreover, $r|_{G_F}$ is automorphic of wt 0 and U_0-level prime to ℓ

$\Uparrow$ ALT $\quad$ $\uparrow$ as in last theorem

need to check: $\bar r|_{G_{F(\zeta_\ell)}}$ irred.

$\Uparrow$

$\bar r|_{G_{\mathbb{Q}(\zeta_\ell)}}$ irred.

$\Uparrow$

$\bar r|_{G_E}$ irred.

$$\begin{tikzcd}
\mathbb{Q}(\zeta_\ell) \arrow[d, dash, "\frac{\ell-1}{2}"] \\
E \arrow[d, dash, "2"] \\
\mathbb{Q}
\end{tikzcd}$$

if not, $\bar r \cong \mathrm{Ind}_{G_E}^{G_{\mathbb{Q}}} \bar\phi$, $\bar\phi : G_E \longrightarrow \mathbb{F}_{\ell^2}^\times$

$$\bar\phi|_{I_{E_\ell}} : I_{E_\ell^{ab}/E_\ell} \cong \mathcal{O}_{E_\ell}^\times \twoheadrightarrow \mathbb{F}_\ell^\times \longrightarrow \mathbb{F}_{\ell^2}^\times$$

$$\bar\phi^\tau|_{I_{E_\ell}} = \bar\phi|_{I_{E_\ell}}$$

$\Rightarrow \#(\mathrm{Proj}\,\bar r)(I_{\mathbb{Q}_\ell}) \leq 2.$

$\geq \ell-1 \longleftarrow$ from FL condition.

$\bar r$ FL, HT $= \{0,1\}$ $\Rightarrow \bar r|_{I_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix}$ or has order ℓ^2-1. $\square$

If $\underbrace{F > E}_{\text{soluble}} > \mathbb{Q}$ $\Rightarrow r|_{G_E}$ is automorphic.

Lecture 17

$\ell > 3$, $\quad \bar{r} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$

$S = \{2, \ell\}$.

$\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$

$r|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell^{-1} \end{pmatrix} \otimes \delta$, δ nr

$r|_{G_{\mathbb{Q}_\ell}}$ is FL, HT $\{0, 1\}$

irreducible

$\exists\, L|\mathbb{Q}_\ell$ finite, $r : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathcal{O}_L)$

$r \bmod \lambda_L \cong \bar{r}$

$\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_\ell^{-1} \end{pmatrix} \otimes \delta$, δ nr

$\bar{r}|_{G_{\mathbb{Q}_\ell}}$ is FT, HT nos $\{0, 1\}$

$\exists\, F|\mathbb{Q}$ finite Galois totally real

$\forall\, F \supset E \supset \mathbb{Q}$ soluble $\qquad r|_{G_E}$ automorphic.

$$\begin{tikzcd} \overline{\mathbb{Q}}_\ell \arrow[rr, "i", "\sim"'] \arrow[dr, "j"'] & & \overline{\mathbb{Q}}_3 \arrow[dl, equal] \\ & \mathbb{C} & \end{tikzcd}$$

$\pi_E \qquad r_{\ell, j^{-1}}(\pi_E) \cong r|_{G_E}$

$r_{3, ij^{-1}}(\pi_E) =: r'_E$

$r'_E : G_{E, \{2,3\}} \longrightarrow GL_2(\overline{\mathbb{Q}}_3)$ ss

s.t. $\det r'_E = \varepsilon_3^{-1}$

$v|2$, $r'_E|_{G_{E_v}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_3^{-1} \end{pmatrix} \otimes \delta$, δ nr

$v|3$, $r'_E|_{G_{E_v}}$ is FL, HT nos $\{0, 1\}$

$v \nmid 6\ell$, $\quad \operatorname{tr} r'_E(\mathrm{Frob}_v) = i \operatorname{tr} r|_{G_E}(\mathrm{Frob}_v)$

$\bar{r}|_{G_F}$ irreducible.

$\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$,

$$\begin{tikzcd}
 & F \arrow[dl, dash, "\text{soluble}"'] \arrow[dr, dash, "\sigma\,\mathrm{Gal}(F|E)\sigma^{-1}"] & \\
E \arrow[dr, dash] & & \sigma E \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$r'_{\sigma E} \simeq r'_E \circ \mathrm{conj}_{\sigma^{-1}}$$

r'_E is irreducible: if not, $r'_E \cong \chi_1 \oplus \chi_2$

FL above 3.

<u>Fact</u>. If E is a totally real field, and $\chi : G_E \longrightarrow \overline{\mathbb{Q}}_p^{\times}$ is a cts (de Rham) FL character, then $\chi = \varepsilon_p^{n} \cdot \underset{\uparrow}{\psi}$ finite order.

$\chi_1 = \varepsilon_3^{n_1}\psi_1, \quad \chi_2 = \varepsilon_3^{n_2}\psi_2$

$$\underset{\substack{\uparrow \\ \text{irred.}}}{r}\Big|^{ss}_{G_E} \cong \varepsilon_\ell^{n_1}\, i^{-1}(\psi_1) \oplus \varepsilon_\ell^{n_2}\, i^{-1}(\psi_2)$$

traces equal on all but finitely many Frobenius elts.

$$\operatorname{tr} r|^{ss}_{G_E}(\mathrm{Frob}_v) = i^{-1} \operatorname{tr} r'_E(\mathrm{Frob}_v)$$

$$= i^{-1}\left(\# k(v)^{-n_1}\psi_1(\mathrm{Frob}_v) + \# k(v)^{-n_2}\psi_2(\mathrm{Frob}_v)\right)$$

#.

For any
\# field F

ℓ prime. $\quad \underline{GR}_{\ell,F}$ = cat. of ss cts linear rep. of G_F on a fin. dim'l $\overline{\mathbb{Q}}_\ell$ vector space.

has $\otimes$: in char. 0 the tensor product of two ss reps of any gp is ss.

V, W irred, $\quad \mathrm{Hom}_{\underline{GR}_{\ell,F}}(V,W) = \begin{cases} \overline{\mathbb{Q}}_\ell, & \text{if } V \cong W \\ 0, & V \not\cong W \end{cases}$

I = index set for iso. classes of irred. reps in $\underline{GR}_{\ell,F}$

$i \longleftrightarrow V_i$

$V \cong \bigoplus_{i \in I} V_i^{\oplus n_i}$, $\quad n_i$ unique.

$E|F$ finite ext'n, $\quad \mathrm{res}_{E|F} : \underline{GR}_{\ell,F} \to \underline{GR}_{\ell,E}$

Fact. If V is a char. 0 rep. of a gp Γ and $D \subset \Gamma$ has finite index,

V ss $|\ \Gamma \iff V$ ss $|\ D$

$\mathrm{ind}_{E|F} : \underline{GR}_{\ell,E} \longrightarrow \underline{GR}_{\ell,F}$.

$\mathrm{Rep}_{F,\ell}$ free ab. gp. basis $[V_i]$ $\quad i \in I$.

$V \in \underline{GR}_{F,\ell}$, $\quad [V] \in \mathrm{Rep}_{F,\ell}$. $\quad \left[\bigoplus_{i \in I} V_i^{\oplus n_i}\right] = \sum_{i \in I} n_i [V_i]$

$\sigma \in G_F$, $\quad \mathrm{tr}_\sigma : \mathrm{Rep}_{F,\ell} \longrightarrow \overline{\mathbb{Q}}_\ell$, $\quad [V] \mapsto \mathrm{tr}\, \sigma|_V$

$$\mathrm{Rep}_{F,\ell} \hookrightarrow \prod_{\sigma \in G_F} \overline{\mathbb{Q}}_\ell$$

$$A \longmapsto (\mathrm{tr}_\sigma A)$$

induced ring structure s.t. $[V][W] = [V \otimes W]$.

$$(-,-)_{F,\ell} : \mathrm{Rep}_{F,\ell} \times \mathrm{Rep}_{F,\ell} \longrightarrow \mathbb{Z}$$

$$([V],[W]) = \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{G_F}(V,W)$$

If $A \in \mathrm{Rep}_{F,\ell}$ and $(A,A)_{F,i} = 1$, then $A = \pm[V_i]$, some $i \in I$.

$$\left(\sum n_i [V_i], \sum n_i [V_i]\right) = \sum n_i^2$$

$\sigma \in G_{\mathbb{Q}}$, $\mathrm{conj}_{\sigma *} : \mathrm{Rep}_{F,\ell} \longrightarrow \mathrm{Rep}_{\sigma F,\ell}$

$$[(V,r)] \longmapsto [(V, r \circ \mathrm{conj}_{\sigma^{-1}})]$$

$\mathrm{conj}_{\sigma^{-1}} : G_{\sigma F} \to G_F$

$\mathrm{tr}_\tau \circ \mathrm{conj}_\sigma = \mathrm{tr}_{\sigma^{-1}\tau\sigma}$; $(\mathrm{conj}_\sigma A, \mathrm{conj}_\sigma B)_{\sigma F,\ell} = (A,B)_{F,\ell}$.

$F'|F$, $\mathrm{res}_{F'|F} : \mathrm{Rep}_{F,\ell} \longrightarrow \mathrm{Rep}_{F',\ell}$, $\mathrm{tr}_\tau \circ \mathrm{res}_{F'|F} = \mathrm{tr}_\tau$

$$[V] \longmapsto [V|_{G_{F'}}]$$

$$\mathrm{res}_{\sigma F'|\sigma F} \circ \mathrm{conj}_\sigma = \mathrm{conj}_\sigma \circ \mathrm{res}_{F'|F}$$

$\mathrm{Ind}_{F'|F} : \mathrm{Rep}_{F',\ell} \longrightarrow \mathrm{Rep}_{F,\ell}$

$$[V] \longmapsto \left[\mathrm{Ind}_{G_{F'}}^{G_F} V\right]$$

$$tr_\sigma \, \mathrm{Ind}_{F'|F} = \sum_{\substack{\tau\in G_F|G_{F'} \\ \tau\sigma\tau^{-1}\in G_{F'}}} tr_{\tau\sigma\tau^{-1}}$$

$$ind_{F'|F}\left(A \; res_{F'|F} B\right) = \left(ind_{F'|F} A\right) B$$

$$\left(\mathrm{Ind}_{F'|F} A, B\right)_{F,\ell} = \left(A, res_{F'|F} B\right)_{F',\ell}$$

Frobenius reciprocity

$$\begin{tikzcd}
F'(\sigma F'') \arrow[rr, "\sigma^{-1}"] \arrow[d, dash] & & (\sigma^{-1}F')F'' \arrow[d, dash] \\
F' \arrow[dr, dash] & & F'' \arrow[dl, dash] \\
& F &
\end{tikzcd}$$

$$res_{F''|F} \; ind_{F'|F} A = \sum_{\sigma\in G_{F'}\backslash G_F/G_{F''}} ind_{(\sigma^{-1}F')F''|F''} \circ conj_\sigma^{-1} \cdot res_{F'(\sigma F'')|F'} A$$

Mackey's formula

Go back to our $F, E, \ell, 3$.

(Brauer's thm)

$$\underset{Gal(F|Q)}{\left[triv_{G_Q}\right]} = \sum_{\substack{F\supset E\supset Q \\ F|E \text{ soluble}}} n_E \left(ind_{E|Q} \psi_E\right]$$

$$n_E \in \mathbb{Z}$$

$$\psi_E : G_E \longrightarrow \overline{Q}_\ell^\times$$

$$\begin{tikzcd}
G_E \arrow[r] \arrow[d] & \overline{Q}_\ell^\times \\
Gal(F|E) \arrow[ur] &
\end{tikzcd}$$

finite order

$$A_\ell := [r] = \sum_E n_E \; ind_{E|Q}\left(\left[\psi_E \otimes r|_{G_E}\right]\right)$$

$$A_3 := \sum_E n_E \; ind_{E|Q}\left(\left[i\psi_E\right]\left[r'_E\right]\right) \in \mathrm{Rep}_{Q,3}$$

$(A_\ell, A_\ell)_{\mathbb{Q},\ell} = 1$, will imply $(A_3, A_3)_{\mathbb{Q},3} = 1$.

tr $A_\ell = 2$ — tr $A_3 = 2$

$\Rightarrow A_3 = [r']$ check $\text{res}_{E/\mathbb{Q}} [r'] = [r'_E]$

$\searrow$ all local info. we want about r'

Lecture 18. $\ell > 3$, $S = \{2, \ell\}$

$\bar{r} : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F}_\ell)$

$\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$

$\bar{r}|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\varepsilon}_\ell^{-1} \end{pmatrix} \otimes \delta$, δ nr

$\bar{r}|_{G_{\mathbb{Q}_\ell}}$ FL w/ HT no's $\{0, 1\}$ $\Bigg] \circledast$

- $\bar{r}$ is automorphic over some finite Galois totally real ext'n $F/\mathbb{Q}$ unramified @ ℓ and 2, w/ $F \cap \bar{\mathbb{Q}}^{\ker \bar{r}}(\zeta_\ell) = \mathbb{Q}$

$\Downarrow$

$R^{\text{univ}}_{\bar{r}}$ finite $/ \mathbb{Z}_\ell$ $\Rightarrow \exists\ r : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathcal{O})$, lifting $\bar{r}$

$L/\mathbb{Q}_\ell$ finite, $\mathcal{O} = \mathcal{O}_L$ $\det r = \varepsilon_\ell^{-1}$

$\circledast$ still hold

also $r|_{G_F}$ automorphic.

$\underbrace{F \supset E \supset \mathbb{Q}}$ soluble, $r|_{G_E}$ automorphic.

$i : \overline{\mathbb{Q}}_\ell \Rightarrow \overline{\mathbb{Q}}_3$, $r'_E : G_E \longrightarrow GL_2(\overline{\mathbb{Q}}_3)$

- unramified outside 6
- $\det r'_E = \varepsilon_3^{-1}$
- ⊛ w/ ℓ replaced by 3
- irreducible.
- $r'_{\sigma E} = r'_E \circ \mathrm{conj}_{\sigma^{-1}}$
- $E \supset E'$, $r'_{E'}|_{G_E} = r'_E$.
- If $v \nmid 6\ell$, then $\mathrm{tr}\, r'_E(\mathrm{Frob}_v) = \mathrm{tr}\, r|_{G_E}(\mathrm{Frob}_v)$

  det det

$\underset{\overset{\uparrow}{\text{no field}}}{K}$, $\underset{\overset{\uparrow}{\text{prime}}}{p}$ cat. of SS cts reps of G_K over $\overline{\mathbb{Q}}_p$.

$\mathrm{Rep}_{K,p}$ Grothendieck gp (ring)

$\bigoplus_{i \in I} \mathbb{Z}[V_i]$ V_i irreds.

$([V],[W]) = \dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_{G_K}(V,W)$

$\mathrm{ind}_{K'|K}$, $\mathrm{res}_{K'|K}$.

$\sigma \in G_K$, $\mathrm{tr}_\sigma : \mathrm{Rep}_{K,p} \longrightarrow \overline{\mathbb{Q}}_p$

Frobenius reciprocity , Mackey's formula.

Brauer: $\left[\mathrm{triv}_{\mathrm{Gal}(F|\mathbb{Q})}\right] = \sum_{\substack{\mathbb{Q} \subset E \subset F \\ F|E \text{ soluble}}} \underset{\overset{\uparrow}{\mathbb{Z}}}{n_E} \left[\mathrm{ind}_{E|\mathbb{Q}} \psi_E\right]$ $\psi_E : \mathrm{Gal}(F|E) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

$$\Rightarrow [\rho] = \sum_E n_E \cdot \mathrm{Ind}_{E|\mathbb{Q}}\left([\psi_E]\ \mathrm{res}_{F|E}[\rho]\right)$$

$$A_3 = \sum_E n_E \cdot \mathrm{Ind}_{E|\mathbb{Q}}\left([i\psi_E][\rho'_E]\right)$$

If $(A_3, A_3) = 1$, then $A_3 = [\rho']$ some $\rho' : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_3)$.
$\mathrm{tr}\, A_3 = 2$

$$(A_3, A_3) = \sum_{E,E'} n_E n_{E'} \left(\mathrm{Ind}_{E|\mathbb{Q}}([i\psi_E][\rho'_E]),\ \mathrm{Ind}_{E'|\mathbb{Q}}([i\psi_{E'}][\rho'_{E'}])\right)_{\mathbb{Q},3}$$

$$= \sum_{E,E'} n_E n_{E'} \left([i\psi_E][\rho'_E],\ \mathrm{res}_{E|\mathbb{Q}}\mathrm{Ind}_{E'|\mathbb{Q}}\left([i\psi_{E'}][\rho'_{E'}]\right)\right)_{E,3}$$

$$= \sum_{E,E'} n_E n_{E'} \sum_{\sigma \in G_{E'}\backslash G_{\mathbb{Q}}/G_E} \left([i\psi_E][\rho'_E],\ \mathrm{ind}_{(\sigma^{-1}E')E|E}\ \mathrm{conj}_{\sigma}^{-1}\ \mathrm{res}_{E'(\sigma E)|E'}[i\psi_{E'}][\rho'_{E'}]\right)_{E,3}$$

$$= \sum_{E,E'} \sum_{\sigma \in G_{E'}\backslash G_{\mathbb{Q}}/G_E} n_E n_{E'} \Big(\underbrace{\left(\mathrm{res}_{(\sigma^{-1}E')E|E}[i\psi_E]\right)\rho'_{(\sigma^{-1}E')E}}_{\text{irreducible}},\ \underbrace{\left(\mathrm{conj}_{\sigma}^{-1}\cdot\left(\mathrm{res}_{E'(\sigma E)|E'}[i\psi_{E'}]\right)\right)\rho'_{(\sigma^{-1}E')E}}_{\text{irreducible}}\Big)_{(\sigma^{-1}E')E,3}$$

$$= \sum_{E,E'} \sum_{\sigma \in G_{E'}\backslash G_{\mathbb{Q}}/G_E} n_E n_{E'} \begin{cases} 1 & \text{if } (i\psi_E)(\tau)\ \mathrm{tr}\,\rho'_{(\sigma^{-1}E')E}(\tau) \\ & = (i\psi_{E'})(\sigma\tau\sigma^{-1})\ \mathrm{tr}\,\rho'_{(\sigma^{-1}E')E}(\tau),\ \forall \tau \in G_{(\sigma^{-1}E')E} \\ 0 & \text{, otherwise} \end{cases}$$

$\uparrow$ or even for Frob_v for all $v \nmid 6\ell$. a place of $(\sigma^{-1}E')E$

$$\overset{=}{\uparrow} \quad ([\rho],[\rho])_{\mathbb{Q},\ell} = 1 \qquad \psi_E(\tau)\,\mathrm{tr}\,\rho(\tau) \qquad \Longleftrightarrow$$

Same calculating replacing ρ'_E by $\rho|_{G_E}$ everywhere

$$= \psi_{E'}(\sigma\tau\sigma^{-1})\,\mathrm{tr}\,\rho(\tau)$$

$\forall \tau = \mathrm{Frob}_v$, $v \nmid 6\ell$ a place of $(\sigma^{-1}E')E$

Similarly, $\operatorname{tr}_1 A_3 = \sum_E n_E \operatorname{tr}_1 \operatorname{Ind}_{E|\mathbb{Q}} [i\varphi_E][r'_E]$

$$= \sum_E n_E [E:\mathbb{Q}] \cdot 2 = \operatorname{tr}_1 [r] = 2$$

$\therefore$ $A_3 = [r']$, $\dim r' = 2$

- $\operatorname{res}_{E|\mathbb{Q}} A_3 = \sum_{E'} n_{E'} \operatorname{res}_{E|\mathbb{Q}} \operatorname{Ind}_{E'|\mathbb{Q}} [i\varphi_{E'}][r'_{E'}]$

$F \supset E$
soluble

$$= \sum_{E'} \sum_{\sigma \in G_{E'} \backslash G_{\mathbb{Q}} / G_E} n_{E'} \operatorname{Ind}_{(\sigma^{-1}E')E|E} \underbrace{\operatorname{conj}_\sigma^{-1} \operatorname{res}_{E'(\sigma E)|E'} [i\varphi_{E'}][r'_{E'}]}_{\substack{\shortparallel \\ \left(\operatorname{conj}_\sigma^{-1} \operatorname{res}_{E'(\sigma E)|E'} [i\varphi_{E'}]\right)[r'_{(\sigma^{-1}E')E}] \\ \shortparallel \\ \operatorname{res}_{(\sigma^{-1}E')E} [r'_E]}}$$

$$= \left(\sum_{E'} \sum_{\sigma \in G_{E'} \backslash G_{\mathbb{Q}} / G_E} n_{E'} \operatorname{Ind}_{(\sigma^{-1}E')E|E} \operatorname{conj}_\sigma^{-1} \operatorname{res}_{E'(\sigma E)|E'} [i\varphi_{E'}] \right) [r'_E]$$

$$= \left(\operatorname{res}_{E|\mathbb{Q}} [\mathrm{triv}] \right) [r'_E] = [r'_E].$$

$\uparrow$
Same
argument

$$\therefore\ r'|_{G_E} = r'_E, \quad \forall\, F \supset E \text{ soluble}.$$

$$\begin{array}{ccccccc} \overline{\mathbb{Q}} & \supset & F & \supset & F^{\mathrm{Gal}(F|\mathbb{Q})_w} & \supset & \mathbb{Q} \\ \tilde{w} & & w & & & & v \end{array} \qquad r'|_{G_F}$$

$$\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})_{\tilde{w}} \in \mathrm{Gal}(\overline{\mathbb{Q}}|F^{\mathrm{Gal}(F|\mathbb{Q})_w})$$

$$r'|_{G_{\mathbb{Q},\tilde{w}}} = r'_{F\,\mathrm{Gal}(F|\mathbb{Q})_w}\Big|_{G_{\mathbb{Q},\tilde{w}}}$$

$\therefore$ r' is unramified outside 6.

$v \nmid 6\ell$, $\underset{\det}{\mathrm{tr}}\, r'(\mathrm{Frob}_v) = \underset{\det}{\mathrm{tr}}\, r'_{F\,\mathrm{Gal}(F|\mathbb{Q})_w}(\mathrm{Frob}_v) = \underset{\det}{\mathrm{tr}}\, r|_{G_F\,\mathrm{Gal}(F|\mathbb{Q})_w}(\mathrm{Frob}_v)$

$= \underset{\det}{\mathrm{tr}}\, r(\mathrm{Frob}_v)$

- $v=2$. $r'|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_3^{-1} \end{pmatrix} \otimes \delta$, δ non ramified.

$v=3$ $r'|_{G_{\mathbb{Q}_3}}$ FL, HT nos $\{0,1\}$.

$\det r' = \varepsilon_3^{-1}$

$\Rightarrow$ r' reducible: a contradiction. Finish proof of FLT!

Lecture 19 Thm. $F|\mathbb{Q}$ totally real no field, $\ell > 3$ a prime, nr. in F

$(\Rightarrow [F(\zeta_\ell):F] > 2)$

S a finite set of places of F containing all places above ℓ.

$r: G_{F,S} \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell)$ cts

s.t. 1) $\det r = \varepsilon_\ell^{-1}$

2) $v \in S$, $v \nmid \ell$, $r|_{I_{F_v}}$ unipotent

3) $v | \ell$, r is FL w/ HT nos $\{0,1\}$.

4) $\bar{r}$ (reduction of r, $G_{F,S} \to GL_2(\overline{\mathbb{F}}_\ell)$) satisfies

$i: \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

a) $\bar{r}|_{G_{F(\zeta_\ell)}}$ is irreducible

b) $\bar{r} = \bar{r}_{i,\ell}(\pi_0)$ for

Some cuspidal auto. rep. π_0 of $GL_2(\mathbb{A}_F)$ which is algebraic w/ wts

$HC_\tau(\pi_\infty) = \{0,-1\}$, $\forall \tau$. and if $v \notin S$ or $v | \ell$, then $\pi_{0,v}$ unramified

and if $v \in S$, $v \nmid \ell$, then $\pi_{0,v}^{I_w} \neq 0$, $I_w = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : v | c \right\}$

$\uparrow$
$GL_2(\mathcal{O}_{F,v})$

require $\chi_{\pi_0} = \|\cdot\|^{-1}$

$\uparrow$ central character

⊛

Then $r = r_{\ell,\iota}(\pi)$ for some π, satisfying ⊛.

Pf. WLOG. • $[F:\mathbb{Q}]$ even.

• If $v \in S$ and $v \nmid \ell$, then $q_v = \# k(v) \equiv 1 \pmod{\ell}$

$F^{av} = \bar{F}^{\ker \bar{r}}(\zeta_\ell)$

Prop. Suppose K is a no. field and S a finite set of places of K and for $v \in S$, suppose $L_v' | K_v$ is a finite Galois ext'n. and suppose $K^{av} | K$ is a finite Galois ext'n. Then $\exists$ a finite soluble Galois ext'n $L|K$ s.t. if $w | v \in S$, $L_v \cong L_v'$ as a K_v-algebra, and $L \cap K^{av} = K$.

Pf. Exercise in class field theory.

$\Pi = \pi^\infty \otimes \pi_\infty$.

$\cong \bigoplus_{\pi} \pi^\infty$ cusp. auto. w/ $\chi_\pi = \|\cdot\|^{-1}$, $HC_\tau(\pi_\infty) = \{0,-1\}$, $\forall \tau$.

$$\mathcal{A}_0\left(GL_2(F)\backslash GL_2(\mathbb{A}_F)\right)_0 = \left\{ \begin{array}{l} \varphi \in \mathcal{A}_0(GL_2(F)\backslash GL_2(\mathbb{A}_F)) : \\ \bullet \text{ for } v|\infty,\ \mathfrak{z}_v \text{ acts w/ HC param. } \{0,-1\} \\ \bullet\ \varphi(gu) = \prod_{v|\infty} j^{-2}(u_v)\varphi(g),\ u = \prod u_v \in \prod_{v|\infty} SO(2) \\ \bullet\ \varphi(gz) = \varphi(g)\|z\|^{-1} \text{ for } z \in \mathbb{A}_F^\times. \end{array} \right\}$$

$\circlearrowleft$ $GL_2(\mathbb{A}_F^\infty)$

$\bigoplus \pi$

$j^{-2} : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto (a+bi)^2$

$a^2+b^2=1$

V is smooth rep. of $GL_2(F_v)$

$$T_v: \underset{\underset{V}{\uparrow}}{x} \longmapsto \int_{GL_2(F_v)} \underset{\uparrow}{\mathrm{char}}_{GL_2(\mathcal{O}_{F,v})\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix} GL_2(\mathcal{O}_{F,v})}(g)\, gx \, d\mu_g$$

characteristic function

ϖ_v uniformizer in $\mathcal{O}_{F,v}$ $\qquad \mu_g(GL_2(\mathcal{O}_{F,v})) = 1$

$$\begin{tikzcd} T_v: V \arrow[r] & V^{GL_2(\mathcal{O}_{F,v})} \\ V^{GL_2(\mathcal{O}_{F,v})} \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$Iw_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,v}) : \begin{matrix} c \equiv 0 \ (v) \\ a \equiv d \ (v) \end{matrix} \right\}$$

$a \in F_v^\times \cap \mathcal{O}_{F,v}$, $\quad U_a : x \longmapsto \int_{GL_2(F_v)} \mathrm{char}_{Iw_1 \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} Iw_1}(g)\, gx \, d\mu'_g.$

$\mu'_g(Iw_1) = 1.$

$U_a: V^{Iw_1} \longrightarrow V^{Iw_1}$

$\underline{L}$: Let π be an irred. smooth rep. of $GL_2(F_v)$

1) $\pi^{GL_2(\mathcal{O}_{F,v})} \neq 0 \iff \mathrm{rec}(\pi)$ is unramified.

(π unramified) $\Longrightarrow$ $\pi \in$ some unramified principal series.

In this case, $\dim \pi^{GL_2(\mathcal{O}_{F,v})} = 1$.

and if $\mathrm{rec}(\pi)(\mathrm{Frob}_v)$ has eigenvalues α, β, then

T_v acts on $\pi^{GL_2(\mathcal{O}_{F,v})}$ by $q_v^{1/2}(\alpha+\beta)$.

2) $\pi^{Iw_1} \neq 0 \iff$ either

a) $rec(\pi) = (\chi_1 \oplus \chi_2, \underset{N}{\overset{\shortparallel}{0}})$ w/

χ_i tamely ramified, and $\chi_1\chi_2$ unramified, and $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$

or b) $rec(\pi) = (\chi \oplus \chi|\cdot|, 0)$

χ tamely ramified, χ^2 unramified

or c) $rec(\pi) = \left(\begin{pmatrix} \chi|\cdot| & 0 \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

χ tamely ramified, χ^2 unramified.

$rec(n\text{-}Ind(\chi_\alpha \times \chi_\beta))(Frob_v) \sim \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

$\varphi : GL_2(F_v) \to \mathbb{C}$

$\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_\alpha(a)\chi_\beta(d)|a/d|^{1/2}\varphi(g)$

Moreover, In case a) $\dim \pi^{Iw_1} = 2$ and we can find a basis so each U_a acts as

$$\begin{pmatrix} \chi_2(a)|a|^{-\frac{1}{2}} & * \\ 0 & \underbrace{\chi_1(a)|a|^{-\frac{1}{2}}}_{\chi_1(Art\, a)} \end{pmatrix}, \quad a \in F_v^\times \cap \mathcal{O}_{F,v}$$

In case b), $\dim \pi^{Iw_1} = 1$, U_a acts as $\chi(a)|a|^{-1/2}$

In case c), $\dim \pi^{Iw_1} = 1$, U_a acts as $\chi(a)|a|^{1/2}$.

Lecture 20. F totally real, $[F:\mathbb{Q}]$ even

$$\mathcal{A}_0(GL_2(F)\backslash GL_2(\mathbb{A}_F))_0 = \left\{ \begin{array}{l} \rho \in \mathcal{A}_0(GL_2(F)\backslash GL_2(\mathbb{A}_F)) : \\ 1)\ \text{for } v|\infty,\ \mathfrak{z}_v \text{ acts w/ HC params } \{0,-1\}, \\ 2)\ \varphi(gu) = \prod_{v|\infty} J^{-2}(u_v)\varphi(g),\ \forall u = \prod u_v \in \prod_{v|\infty} SO(2) \\ 3)\ \varphi(gz) = \|z\|^{-2}\varphi(g),\ \forall z \in \mathbb{A}_F^\times \end{array} \right\}$$

$J^{-2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (a+bi)^2$ $\quad \overset{\circlearrowright}{GL_2(\mathbb{A}_F^\infty)}$

$$\cong \bigoplus_{\substack{\pi \text{ cuspidal} \\ \text{auto}}} \pi^{\infty}$$

$$\pi^{\infty} = \bigotimes_{v \nmid \infty}{}' \pi_v$$

π_v is ∞-dim'l.

π_∞ has HC params $\{0,-1\}$ @ all $v|\infty$

$\chi_\pi = \|\cdot\|^{-1}$

$D|F$ be the quat. alg over F, ramified exactly @ the ∞ places of F.

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong (\mathbb{H})^{[F:\mathbb{Q}]}, \quad D \otimes_{\mathbb{Q}} \mathbb{A}^{\infty} \cong M_{2\times 2}(\mathbb{A}_F^{\infty})$$

$$A(D^\times \backslash (D\otimes \mathbb{A}^\infty)^\times, \mathbb{C})_0 = \left\{ \varphi : D^\times \backslash (D\otimes\mathbb{A}^\infty)^\times / \mathbb{A}_F^\times \to \mathbb{C} : \varphi \text{ locally constant} \right\}$$

not canonical!

JL

as $GL_2(\mathbb{A}_F^\infty)$-module

N.B. $D^\times/F^\times \hookrightarrow (D\times\mathbb{A}^\infty)^\times/\mathbb{A}_F^\times$

image discrete + cocompact

$$\cong A_0(GL_2(F)\backslash GL_2(\mathbb{A}_F))_0 \otimes \|\det\|^{1/2}$$

$$\oplus \bigoplus_{\substack{\chi : \mathbb{A}_F^\times / \overline{F^\times \times (F_\infty^\times)^0} \to \mathbb{C}^\times \\ \text{cts}, \ \chi^2 = 1}} \chi \circ (\text{reduced norm})$$

$$i : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$$

$$\underset{i}{\cong} A(D^\times \backslash (D\otimes\mathbb{A}^\infty)^\times, \overline{\mathbb{Q}}_\ell)_0 \leftarrow \text{defined same way}$$

semisimple.

$$\cong$$

$$\bigoplus_{\substack{\pi \text{ cusp. auto} \\ HC_\tau(\pi_\infty) = \{0,-1\} \\ \chi_\pi = \|\cdot\|^{-1}}} i(\pi^\infty \otimes \|\det\|^{1/2}) \oplus \bigoplus_{\substack{\chi : \mathbb{A}_F^\times / \overline{F^\times \times (F_\infty^\times)^0} \to \overline{\mathbb{Q}}_\ell^\times \\ \chi^2 = 1}} \chi \circ (\text{reduced norm})$$

$\pi \subset \mathcal{A}\left(D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times, \overline{\mathbb{Q}}_\ell\right)_0$ irreducible

$\rightsquigarrow$

$r(\pi) : G_F \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell)$ cts rep.

If $\pi = i\left(\pi'^\infty \otimes \|\det\|^{1/2}\right)$, then $r(\pi) = r_{\ell,i}(\pi')$.

If $\pi = \chi \circ$ reduced norm, then $r(\pi) = \chi \circ \mathrm{Art}^{-1} \oplus (\chi \circ \mathrm{Art}^{-1})\, \varepsilon_\ell^{-1}$

$$\mathrm{Art} : \mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^0} \xrightarrow{\sim} \mathrm{Gal}(F^{ab}|F)$$

1) $\det r(\pi) = \varepsilon_\ell^{-1}$

2) $v \nmid \ell$ and π_v is unramified, then $\mathrm{tr}\, r(\pi)(\mathrm{Frob}_v) = $ eigenvalue of T_v

3) $v | \ell$, and π_v is unramified, then $r(\pi)|_{G_{F_v}}$ on $\pi_v^{GL_2(\mathcal{O}_{F,v})}$ is FL w/ HT nos $\{0,1\}$.

4) $v \nmid \ell$ and $\pi_v^{Iw_v^1} \neq 0$, $Iw_v^1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,v}) : \begin{matrix} c \equiv 0 \ (v) \\ a \equiv d \ (v) \end{matrix} \right\}$

then one of the following holds:

a) $\dim \pi_v^{Iw_v^1} = 2 \Rightarrow r(\pi)|_{G_{F_v}}$ factors through $G_{F_v}^{ab}$

and $r(\pi)|_{G_{F_v}}(\mathrm{Art}_{F_v} a)$ has the same eigenvalues as U_a has $\pi_v^{Iw_v^1}$

$a \in F_v^\times \cap \mathcal{O}_{F,v}$

b) $\left.\begin{matrix} \dim \pi_v^{Iw_v^1} = 1, \\ \pi_v \ \infty\text{-dim'l} \end{matrix}\right] \Rightarrow r(\pi)|_{G_{F_v}} \sim \begin{pmatrix} \psi & * \\ 0 & \psi \varepsilon_\ell^{-1} \end{pmatrix}$ where $\psi(\mathrm{Art}_{F_v} a) = $ eigenvalue of U_a on 1-dim'l space $\pi_v^{Iw_v^1}$

(and ψ finite order) $a \in F_v^\times \cap \mathcal{O}_{F,v}$

c) $\left.\begin{array}{l}\dim \pi_v^{Iw_v'} = 1 \\ \pi_v \text{ 1-dim'l}\end{array}\right] \Rightarrow r(\pi)|_{G_{F_v}} \cong \psi \oplus \psi\varepsilon_\ell$ where $\psi(\mathrm{Art}_{F_v} a)$ $\quad a \in F_v^\times \cap \mathcal{O}_{F,v}$

$= \text{eigenvalue of } U_a \text{ on } \pi_v^{Iw_v'}$

(and $\psi = \varepsilon_\ell^{-1} \cdot$ finite order character)

$L | \mathbb{Q}_\ell$ finite, $\quad \mathcal{O} = \mathcal{O}_L$, $\lambda \lhd \mathcal{O}$, $\quad \mathcal{O}/\lambda = \mathbb{F}$.

R a finite set of places of F, containing no place above ℓ.

$v \in R$, $\quad \underset{\text{subgp}}{k(v)^\times \supset \Delta_v} \xrightarrow{\chi_v} \mathcal{O}^\times$

$$\Delta = \prod_{v\in R} \Delta_v, \quad \chi = \prod \chi_v$$

$$U_\Delta(R) = \left\{ u \in GL_2(\hat{\mathcal{O}}_F) : \forall v \in R, \ u_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \text{ w/ } \begin{array}{l} c_v \equiv 0 \ (v) \\ a_v \equiv d_v (v) \\ a_v \in \Delta_v \end{array} \right\}$$

$$\begin{tikzcd} u \arrow[r, mapsto] \arrow[d, "\chi"'] & \prod_{v\in R} \chi_v(a_v) \\ \mathcal{O}^\times & \end{tikzcd}$$

If $\Delta = \prod_{v\in R} k(v)^\times$, $\quad U_0(R)$ for $U_\Delta(R)$

$\Delta = \{1\}$, $\qquad U_1(R)$ for $U_\Delta(R)$.

A an $\mathcal{O}$-algebra.

$$S(U_0(R), \chi, A) = \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / (\mathbb{A}_F^\infty)^\times \to A : \begin{array}{l}\varphi(gu) = \chi(u)\varphi(g) \\ \forall u \in U_\Delta(R)\end{array} \right\}$$

$$S(U_0(R), \chi, \overline{\mathbb{Q}}_\ell) \subset \mathcal{A}(D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times, \overline{\mathbb{Q}}_\ell)_0 = \bigoplus \pi$$

$\qquad \|$

$$\bigoplus \pi^{U_\Delta(R), \chi}$$

$$S(U_\Delta(R), \chi, A) = \bigoplus_{g \in D^\times \backslash (D\otimes \mathbb{A}^\infty)^\times / (\mathbb{A}_F^\infty)^\times U_\Delta(R)} A(\chi)^{g^{-1}D^\times g \cap ((\mathbb{A}_F^\infty)^\times U_\Delta(R))}$$

$\wedge$ finite free A-mod

$\downarrow$

either (0) or $\cong A$.

$$g = \overset{D^\times}{\delta}\, g\, \overset{(\mathbb{A}_F^\infty)^\times}{z}\, u \in U_\Delta(R)$$

$\varphi(g) = \varphi(g)\, \chi(u)$

$$0 \to F^\times \to (\mathbb{A}_F^\infty)^\times U_\Delta(R) \cap g^{-1}Dg \twoheadrightarrow \Sigma(g) \subset D^\times/F^\times$$

$$x \longmapsto gxg^{-1}$$

Compact + discrete $\Rightarrow$ finite

$\delta \in D^\times$ s.t. $\delta^m \in F^\times$,

$\Leftrightarrow$ $\delta^m = (\delta^m)^*$ — main involution

reduced tr $\delta = \delta + \delta^*$

$\Leftrightarrow$ $(\delta/\delta^*)^m = 1$ $\Rightarrow$ $[F(\delta/\delta^*) : F] \leq 2$

$\therefore$ $\ell \nmid |\Sigma(g)|$

$[F(\zeta_\ell) : F] > 2$

$\nearrow$

ℓ unramified in F, $\ell > 3$.

$A \to B$, $S(U_\Delta(R), \chi, A) \otimes_A B \xrightarrow{\sim} S(U_\Delta(R), \chi, B)$.

Lecture 21

F, D, $L \mid \mathbb{Q}_\ell$, $\mathcal{O}$, λ, $\mathbb{F}$. ℓ nr in F, $\ell > 3$.

$A \mid \mathcal{O}$. R finite set of finite places of F not dividing ℓ.

$$v \in R, \quad \underbrace{\Delta_v \subset k(v)^\times \to \mathcal{O}^\times}_{\chi_v}, \quad \Delta = \prod_{v \in R} \Delta_v, \quad \chi = \prod_{v \in R} \chi_v.$$

$$U_\Delta(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) : \begin{matrix} v \mid c, & \forall v \in R \\ (a/d) \bmod v \in \Delta_v \end{matrix} \right\} \xrightarrow{\chi} \mathcal{O}^\times$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \prod_{v \in R} \chi_v\left(\frac{a}{d} \bmod v\right)$$

finite free A-module

$$S(U_\Delta(R), A)_\chi = \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / (\mathbb{A}_F^\infty)^\times \longrightarrow A : \varphi(gu) = \chi(u)\varphi(g), \ \forall u \in U_\Delta(R) \right\}$$

$$\cong \bigoplus_{g \in D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / \mathbb{A}_F^{\infty,\times} U_\Delta(R)} A(\chi)^{\underbrace{((\mathbb{A}_F^\infty)^\times U_\Delta(R) \cap g^{-1} D^\times g)/F^\times}_{\text{finite gp order prime to } \ell}}$$

$\uparrow$ finite

$$S(U_\Delta(R), \bar{\mathbb{Q}}_\ell) = \overset{GL_2(\mathbb{A}_F^\infty)}{\overset{\curvearrowright}{\mathcal{A}}}\left(D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times\right)_0^{U_\Delta(R), \chi}$$

$$= \bigoplus_{\pi} \pi^{U_\Delta(R), \chi}$$

Lemma 1. ~~If $\Delta_1 \subset \Delta_2 \subset \prod_{v \in R} k(v)^\times$ w/ $|\Delta_2/\Delta_1|$ a power of ℓ. χ char. of Δ_2,~~

~~then $S(U_{\Delta_1}(R), A)_\chi$ is a free $A[\Delta_2/\Delta_1]$-module.~~ ~~$\delta \in \Delta_2$, lifts to a $u_\delta \in U_{\Delta_2}(R)$~~

~~$\delta\varphi = \varphi(\cdot\, u_\delta)$.~~

Lemma. Suppose $R = R_1 \sqcup R_2$, and $\Delta_1 \subset \Delta_2$, $\#(\Delta_2/\Delta_1) =$ power of ℓ.

$v \in R_1$, $\Delta_{1,v} = \Delta_{2,v}$, $\chi : \Delta_2 \to \mathcal{O}^\times$, $v \in R_2$, $\chi_v = 1$.

then $S(U_{\Delta_1}(R), A)_\chi$ is a free $A[\Delta_2/\Delta_1]$-module.

$$\delta \qquad U_\delta \in \prod_{v \in R_2} U_R(\Delta_2)_v \text{ lifting } \delta$$

$$\delta\varphi = \varphi(\_ \cdot U_\delta) \qquad U_\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad v \in R_2$$

$a/d \bmod v = \delta \bmod \Delta_1 \in \Delta_2/\Delta_1$

Pf. $$S(U_{\Delta_1}(R), A) = \bigoplus_{g \in D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / \mathbb{A}_F^{\infty,\times} U_{\Delta_2}(R)} \bigoplus_{h \in D^\times \backslash D^\times g \mathbb{A}_F^{\infty,\times} U_{\Delta_2}(R) / \mathbb{A}_F^{\infty,\times} U_{\Delta_1}(R)} A(\chi)^{h^{-1} D^\times h \cap (\mathbb{A}_F^\infty)^\times U_{\Delta_1}(R) / F^\times}$$

$h = gu$, $u \in U_{\Delta_2}(R)$

$$u^{-1}\left(g^{-1} D^\times g \cap (\mathbb{A}_F^\infty)^\times U_{\Delta_1}(R)\right) / F^\times$$

$\Vert$ $|\Delta_2/\Delta_1|$ power of ℓ

$$g^{-1} D^\times g \cap (\mathbb{A}_F^\infty)^\times U_{\Delta_2}(R) / F^\times$$

sum becomes $$\bigoplus_{u \in \mathbb{A}_F^{\infty,\times} U_{\Delta_2}(R) / (g^{-1} D^\times g \cap U_{\Delta_2}(R)) \cdot U_{\Delta_1}(R) (\mathbb{A}_F^\infty)^\times}$$

$$= \bigoplus_{g \in D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / \mathbb{A}_F^{\infty,\times} U_{\Delta_2}(R)} \bigoplus_{u \in \mathbb{A}_F^{\infty,\times} U_{\Delta_2}(R) / \underbrace{(g^{-1} D^\times g \cap U_{\Delta_2}(R))}_{\text{this modulo } F^\times \text{ has order prime to } \ell} U_{\Delta_1}(R) (\mathbb{A}_F^\infty)^\times} \cdots$$

$$= \bigoplus_{g \in D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / \mathbb{A}_F^{\infty,\times} U_{\Delta_2}(R)} \bigoplus_{\delta \in \Delta_2/\Delta_1} A(\chi)^{g^{-1} D^\times g \cap U_{\Delta_2}(R) (\mathbb{A}_F^\infty)^\times}$$

action of $\Delta_2/\Delta_1 = U_{O_2}(R)/U_{O_1}(R)$ is just translation in this sum.

$S(U_\Delta(R), A)_x$ A noetherian. (A will be $\mathcal{O}, \mathbb{F}, L, \overline{\mathbb{Q}}_\ell$)

$\circlearrowleft$

T_v, $v \notin R \cup \{v|\ell\}$

$\mathbb{T}(U_\Delta(R), A)_x$ = A-subalg. of

- $\mathrm{End}_A(S(U_\Delta(R), A)_x)$ gen. by T_v for $v \nmid \ell$, $v \notin R$.

Char $U_\Delta(R)\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}U_\Delta(R)$ $d\mu$

$\mu(U_\Delta(R)) = 1$.

- $\mathbb{T}(U_\Delta(R), A)_x$ commutative
- finite / A
- A torsion-free.

$$\mathbb{T}(U_\Delta(R), \mathcal{O})_x \otimes_{\mathcal{O}} A \twoheadrightarrow \mathbb{T}(U_\Delta(R), A)$$

kernel is killed by a power of ℓ.

$$0 \to \mathbb{T}(U_\Delta(R), \mathcal{O})_x \hookrightarrow \mathcal{O}^{\oplus n^2} \longrightarrow \underset{\substack{\wr\\ \text{free} \oplus \ell\text{-torsion}}}{Q} \longrightarrow 0$$

$n = \mathrm{rk}_{\mathcal{O}} S(U_\Delta(R), \mathcal{O})_x$

$$\underset{\substack{\uparrow\\ \ell\text{-torsion}}}{\mathrm{Tor}_1^{\mathcal{O}}(Q, A)} \longrightarrow \mathbb{T}(U_\Delta(R), \mathcal{O})_x \otimes A \underset{\substack{\uparrow\\ \text{image } \mathbb{T}(U_\Delta(R), A)_x}}{\longrightarrow} A^{\oplus n^2}$$

$\therefore$ injective if $A = L, \overline{\mathbb{Q}}_\ell$. kernel nilpotent if $A = \mathbb{F}$.

$$\mathbb{T}(U_\Delta(R),\mathcal{O})_\chi = \bigoplus_{m \text{ max'l}} \mathbb{T}(U_\Delta(R),\mathcal{O})_{\chi, m}.$$

$\uparrow$

finite $/\mathcal{O}$

$$S(U_\Delta(R),A)_\chi = \bigoplus_m S(U_\Delta(A),A)_{\chi, m}$$

Lecture 22. F totally real, even degree, ℓ un. in F, $\ell > 3$

$L | \mathbb{Q}_\ell$ finite

$\mathcal{O} = \mathcal{O}_L$

$\mathcal{O}/\lambda = \mathbb{F}$

A noetherian an $\mathcal{O}$-alg.

$\mathbb{F}, \mathcal{O}, \ell, \overline{L}$

$D | F$ ram. @ ∞ places.

R finite set of finite places of F

$U_\Delta(R) \subset GL_2(\hat{\mathcal{O}}_F)$ $\twoheadrightarrow \Delta$, $\quad \Delta = \prod_{v \in R} \Delta_v$, $\Delta_v \subset k(v)^\times$

$\chi = \prod \chi_v$, $\quad \chi_v : \Delta_v \to \mathcal{O}^\times$

finite free $/A$

$$S(U_\Delta(R),A)_\chi = \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / (\mathbb{A}_F^\infty)^\times \longrightarrow A : \varphi(gu) = \chi(u)\varphi(g),\ u \in U_\Delta(R) \right\}$$

$\circlearrowleft$

T_v, $\quad v \notin R \cup \{v | \ell\}$

$\mathbb{T}(U_\Delta(R),A)_\chi$ = A-alg. in $\mathrm{End}_A(S(U_\Delta(R),A)_\chi)$ gen. by T_v.

$\uparrow$

finite torsion free $/A$

$$S(U_\Delta(R),A)_\chi = S(U_\Delta(R),\mathcal{O})_\chi \otimes_{\mathcal{O}} A$$

$$\mathbb{T}(U_\Delta(R),\mathcal{O})_\chi \otimes_{\mathcal{O}} A \twoheadrightarrow \mathbb{T}(U_\Delta(R),A)$$ w/ nilpotent kernel

iso if $\ell^{-1} \in A$.

$$\mathbb{T}(U_\Delta(R),\mathcal{O})_\chi = \bigoplus_{\substack{m \\ \text{max'l}}} \mathbb{T}(U_\Delta(R),\mathcal{O})_{\chi, m}$$

$S \qquad\qquad S$

$\mathfrak{m} \triangleleft \mathbb{T}(U_\Delta(R), \mathcal{O})_\chi$ max'l.

$$(0) \neq S(U_\Delta(R), \mathcal{O})_{\chi, \mathfrak{m}} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell = \bigoplus_{\pi} \boxed{\pi_{\mathfrak{m}}^{U_\Delta(R), \chi}} \longrightarrow \text{one nonzero, say } \pi$$

$\pi \subset \mathcal{A}(D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / (\mathbb{A}_F^\infty)^\times, \bar{\mathbb{Q}}_\ell)$ irred.

$$\begin{tikzcd} r(\pi): G_F \arrow[r] \arrow[dr] & GL_2(\bar{\mathbb{Q}}_\ell) \\ & GL_2(\mathcal{O}_{\bar{\mathbb{Q}}_\ell}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\bar{r}(\pi): G_F \longrightarrow GL_2(\bar{\mathbb{F}}_\ell)$

unramified outside $R \cup \{v | \ell\}$

$v \notin R \cup \{v|\ell\}$

$\operatorname{tr} \bar{r}(\pi)(\mathrm{Frob}\, v)$

$= \text{eigenvalue of } T_v \text{ on } \pi^{U_\Delta(R), \chi}$

$= \text{eigenvalue on } \pi_v^{GL_2(\mathcal{O}_{F,v})}$

$$\begin{tikzcd} \mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi, \mathfrak{m}} \arrow[r] \arrow[d] & \mathcal{O}_{\bar{\mathbb{Q}}_\ell} \arrow[r, two heads] & \bar{\mathbb{F}}_\ell \\ k(\mathfrak{m}) \arrow[urr, dashed] & & \end{tikzcd}$$

from π

$T_v \longmapsto \operatorname{tr} \bar{r}(\pi)(\mathrm{Frob}\, v)$

(Chebotarev $\Rightarrow$ $\operatorname{tr} \bar{r}(\pi)$ valued in $k(\mathfrak{m})$)

FACT. If $r: \Gamma \longrightarrow GL_2(\bar{k})$ semisimple and $\operatorname{tr} r$ is valued in k, and if $\mathrm{Br}(k') = 0$, $\forall\, k'/k$ finite, then r is conjugate to a rep. $\Gamma \longrightarrow GL_2(k)$.

$\therefore$ semisimplify + conjugate, get $\bar{r}_{\mathfrak{m}}: G_F \longrightarrow GL_2(k(\mathfrak{m}))$

characterizes $\bar{r}_{\mathfrak{m}}$ completely $\longleftarrow$ unramified outside $R \cup \{v|\ell\}$

$\det \bar{r}_{\mathfrak{m}} = \bar{\varepsilon}_\ell^{-1}$, $\operatorname{tr} \bar{r}_{\mathfrak{m}}(\mathrm{Frob}_v) = T_v \bmod \mathfrak{m}$

$v \notin R \cup \{v|\ell\}$

__Def__. $\mathfrak{m}$ is called __non-Eisenstein__ if $\bar{r}_{\mathfrak{m}}$ is absolutely irreducible.

$$\begin{tikzcd}
\prod_{\pi} \bar{\mathbb{Q}}_\ell & & \mathbb{T}(U_\Delta(R),\mathcal{O})_{\chi,\mathfrak{m}} \arrow[ll] \arrow[r, hook] \arrow[dl] & \prod_{\pi} {}_{k(\mathfrak{m})} \mathcal{O}_{L_\pi} \arrow[dll, bend left] =: \tilde{\mathbb{T}} = \{(a_\pi) \in \prod \mathcal{O}_{L_\pi} : \\
& \prod_{\pi} \mathcal{O}_{L_\pi} \arrow[ul, dash, bend right] & & a_\pi \bmod \text{max'l ideal in } k(\mathfrak{m}) \text{ and indep. of } \pi\}
\end{tikzcd}$$

π

$\pi_{\mathfrak{m}}^{U_\Delta(R),\chi} \neq 0$

for some L_π / L finite

Choose L_π s.t. $r(\pi)$ factors

$$\begin{tikzcd}
r(\pi) : G_F \arrow[r] \arrow[dr] & GL_2(\bar{\mathbb{Q}}_\ell) \\
& \cup \\
& GL_2(\mathcal{O}_{L_\pi})
\end{tikzcd}$$

WLOG $r(\pi) \bmod \lambda_{L_\pi} = \bar{r}_{\mathfrak{m}}$

$$\begin{tikzcd}
\prod r(\pi) : G_F \arrow[r] \arrow[dr, dashed] & GL_2(\tilde{\mathbb{T}}) \subset GL_2\left(\prod_{\pi} \bar{\mathbb{Q}}_\ell\right) \\
& \cup \\
& GL_2(\mathbb{T}(U_\Delta(R),\mathcal{O})_{\chi,\mathfrak{m}})
\end{tikzcd}$$

Carayol:

if $\mathfrak{m}$ non-Eisenstein, then up to conj. $r_{\mathfrak{m}} : G_F \to GL_2(\mathbb{T}(U_\Delta(R),\mathcal{O})_{\chi,\mathfrak{m}})$

$r_{\mathfrak{m}} \bmod \mathfrak{m} = \bar{r}_{\mathfrak{m}}$

$\det r_{\mathfrak{m}} = \varepsilon_\ell^{-1}$

$r_{\mathfrak{m}}$ unramified outside $R \cup \{v|\ell\}$. and for $v \notin R \cup \{v|\ell\}$, $\operatorname{tr} r_{\mathfrak{m}}(\mathrm{Frob}_v) = T_v$.

$r_{\mathfrak{m}}$ is FL @ v for all $v|\ell$ w/ HT nos $\{0,1\}$.

If $v \in R$ and $\sigma \in I_{F_v}$, then $r_{\mathfrak{m}}(\sigma)$ has char. poly. $(X - \chi_v^{Gal}(\sigma))(X - \chi_v^{Gal}(\sigma)^{-1})$

$$\begin{tikzcd}
I_{F_v} \arrow[r, two heads] \arrow[d, phantom, "\bigcup"] & I_{F_v^{ab}|F_v} & \mathcal{O}_{F_v}^{\times} \arrow[l, hook', "\mathrm{Art}"', "\sim"] \arrow[d] & \\
I' = \text{preimage of } \Delta_v \arrow[drrr, bend right, "\chi_v^{\mathrm{Gal}}"'] & & k(v)^{\times} \arrow[d, phantom, "\bigcup"] & \\
 & & \Delta_v \arrow[r, "\lambda_v"] & \mathcal{O}^{\times}
\end{tikzcd}$$

Rmk. If S is a finite set of places of F, then

$\mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi, m} \leftarrow$ non-Eisenstein is gen. over $\mathcal{O}$ by T_v for

$v \notin S \cup R \cup \{v|\ell\}$.

Pf. the subalg. gen. by these T_v

contains $\mathrm{tr}\, r_m(\mathrm{Frob}_v)$ for $v \notin S \cup R \cup \{v|\ell\}$.

$\therefore$ by Chebotarev contains $\mathrm{tr}\, r_m(\sigma)$, $\forall \sigma \in G_F$

$v \in S - (R \cup \{v|\ell\})$ contains $T_v = \mathrm{tr}\, r_m(\mathrm{Frob}_v)$

Focus on one m.

For convenience, extend $\mathcal{O}$ s.t. $k(m) = \mathbb{F}$

and all eigenvalues of $\bar{r}_m(\sigma)$ for $\sigma \in G_F$ are in $\mathbb{F}$.

$v \in R$, $a \in F_v^{\times} \cap \mathcal{O}_{F_v}$, $\sigma \in G_{F_v}$, s.t. $\sigma|_{F_v^{ab}} = \mathrm{Art}(a)$

ASSUME. $X^2 - \mathrm{tr}\,\bar{r}_m(\sigma)X + \det \bar{r}_m(\sigma)$ has roots $\alpha_v, \beta_v \in \mathbb{E}$

w/ $\alpha_v/\beta_v \neq 1, |a|^{\pm 1}$.

will describe $r_m|_{G_{F_v}}$

Lecture 23. $F|\mathbb{Q}$ totally real w/ even degree, D/F a quaternion alg. split at ∞.

R a finite set of finite places, $k(v)^\times \supset \Delta_v \xrightarrow{\chi_v} \mathcal{O}^\times$, $\forall v \in R$,

$L|\mathbb{Q}_\ell$, $\mathcal{O}/\lambda = \mathbb{E}$.

For every $\mathcal{O}$-algebra A, have

$$S(U_\Delta(R), A)_\chi = \{\varphi : \varphi(\delta g z u) = \chi(u)\varphi(g),\ \delta \in D^\times,\ z \in \mathbb{A}_F^{\infty,\times},\ u \in U_\Delta(R)\}$$

Hecke algebra $\mathbb{T}(U_\Delta(R), \mathcal{O})_\chi = \prod_{m} \mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi, m}$ finite torsion-free over $\mathcal{O}$.

For each m, we were able to construct a torsion Galois rep.

$$\bar{r}_m : G_F \longrightarrow GL_2(k(m))$$

unramified away from $R \cup \{v|\ell\}$ satisfying $\mathrm{tr}\,\bar{r}_m(\mathrm{Frob}_v) = T_v$ and $\det \bar{r}_m = \varepsilon_\ell^{-1}$. We said m is *non-Eisenstein* if $\bar{r}_m$ is absolutely irreducible.

In this case, we were able to lift it to

$$r_m : G_F \longrightarrow GL_2\left(\mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi, m}\right)$$

satisfying the properties

- $\det r_m = \varepsilon_\ell^{-1}$
- unramified away from $R \cup \{v | \ell\}$
- FL w/ HT no's $\{0, 1\}$ at $v | \ell$
- for $v \in R$, tamely ramified at v w/ the property that if $\sigma \in I_{F_v}$ maps to $a \in \Delta_v$ then its char. poly. is $\mathrm{char}_{r_m(\sigma)}(T) = (T - \chi_v^{Gal}(\sigma))(T - \chi_v^{Gal}(\sigma)^{-1})$

We also saw that $\mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi, m}$ is gen. by T_v for $v \notin R \cup \{v | \ell\} \cup S$ for any finite set S of finite places.

For simplicity, we extend L so that $k(m) = \mathbb{F}$, and for all $\sigma \in G_F$ the eigenvalues of $\bar{r}_m(\sigma)$ are also contained in $\mathbb{F}$.

For $v \in R$, suppose there exists $a \in F_v^\times \cap \mathcal{O}_{F,v}$ and $\sigma \in G_{F_v}$ s.t. $\sigma \mapsto \mathrm{Art}(\sigma) \in G_{F_v}^{ab}$ and the polynomial $X^2 - \mathrm{tr}\, \bar{r}_m(\sigma) X + \det \bar{r}_m(\sigma)$ has roots $\alpha_v, \beta_v \in \mathbb{F}$ w/ $\alpha_v / \beta_v \notin \{1, |a|_v^{\pm 1}\}$. For every $\pi \subset \mathcal{A}(D^\times \backslash (D \otimes \mathbb{A}^\infty_F)^\times)_0$ s.t.

$$0 \neq \pi_m^{U_\Delta(R), \chi} \subset S(U_\Delta(R), \overline{\mathbb{Q}}_\ell)_{\chi, m},$$

recall there were three possibilities for π and $\pi_v^{U_\Delta(R)_v, \chi_v}$.

Because of this eigenvalue condition, we see that

- π is infinite-dim'l.
- $\pi_v^{U_\Delta(R)_v, \chi_v}$ is 2-dim'l

and consequently $r_\pi|_{G_{F_v}}$ factors through $G_{F_v}^{ab}$. On the other hand, we have

$$r_m : G_F \to GL_2(\mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi,m}) \subset GL_2(\tilde{\mathbb{T}}) \subset GL_2\Big(\prod_{\pi_m^{U_\Delta(R),\chi} \neq 0} \overline{\mathbb{Q}_\ell}\Big)$$

and hence $r_m|_{G_{F_v}}$ factors through $G_{F_v}^{ab}$. By Hensel's lemma, it follows that

$$X^2 - \mathrm{tr}\, r_m(\sigma) X + \det r_m(\sigma) = (X - A_v)(X - B_v)$$ for elts $A_v, B_v \in \mathbb{T}(U_\Delta(R), \mathcal{O})_{\chi,m}$ lifting α_v, β_v.

Consider $e_{\alpha_v} = \dfrac{r_m(\sigma) - B_v}{A_v - B_v}$, $e_{\beta_v} = \dfrac{r_m(\sigma) - A_v}{B_v - A_v}$

These satisfy $e_{\alpha_v} + e_{\beta_v} = 1$ and $e_{\alpha_v} e_{\beta_v} = 0$. Hence we can decompose

$$\mathbb{T}(U_\Delta(R), \mathcal{O})^{\oplus 2}_{\chi,m} = e_{\alpha_v} \mathbb{T}^2 \oplus e_{\beta_v} \mathbb{T}^2.$$

where the fact that $r_m|_{G_{F_v}}$ factors through $G_{F_v}^{ab}$ implies that both components are preserved by all $\tau \in G_{F_v}$. This shows that there are two characters.

$$\chi_{\alpha_v}, \chi_{\beta_v} : G_{F_v} \longrightarrow \mathbb{T}(U_\Delta(R), \mathcal{O})^\times_{\chi,m}$$

s.t. $r_m|_{G_{F_v}} \sim \begin{pmatrix} \chi_{\alpha_v} & 0 \\ 0 & \chi_{\beta_v} \end{pmatrix}$.

We can also do a similar thing on the automorphic forms side. Recall for $b \in F_v^\times \cap \mathcal{O}_{F,v}$, there is an operator U_b acting on $S(U_\Delta(R), \mathcal{O})_{\chi, m}$. This satisfies the polynomial

$$(X - \chi_{\alpha_v}(\mathrm{Art}(b)))(X - \chi_{\beta_v}(\mathrm{Art}(b))).$$

This is because we can reduce to $\overline{\mathbb{Q}}_\ell$ - coeff, and those where $\pi_m^{U_\Delta(R), \chi} \neq 0$ and then $\pi_v^{Iw_v^1}$ is 2-dim'l where the U_b action has the same char. poly. as $r_\pi(\mathrm{Art}(b))$.

We can then construct the idempotents $\tilde{e}_{\alpha_v} = \frac{U_a - B_v}{A_v - B_v}$, $\tilde{e}_{\beta_v} = \frac{U_a - A_v}{B_v - A_v}$.

Then we can define $S(U_\Delta(R), \mathcal{O})_{\chi, m} = \tilde{e}_{\alpha_v} S(U_\Delta(R), \mathcal{O}) \oplus \tilde{e}_{\beta_0} S(U_\Delta(R), \mathcal{O})_{\chi, m}$

where both summands are preserved by U_b for all $b \in F_v^\times \cap \mathcal{O}_{F,v}$, and U_b acts on each component by $\chi_{\alpha_v}(\mathrm{Art}(b))$ and $\chi_{\beta_v}(\mathrm{Art}(b))$.

Choosing Taylor-Wiles primes

Let R be a finite set of finite places, where for all $v \in R$, we have $q_v = |k(v)| \equiv 1 \pmod{\ell}$. This is the places where r can be ramified.

We choose an auxiliary set of primes Q, disjt from R, s.t. $q_v \equiv 1 \pmod{\ell}$ for all $v \in Q$.

- For $v \in R$, we will set $\Delta_v = k(v)^\times$ for $v \in R$, because we can ensure that the ramifications are unipotent. To make the argument work, we will allow χ_v to be any ℓ-power order.

- For $v \in Q$, we will set $\Delta_v = \ker(k(v)^\times \to k(v)^\times_\ell)$, where $k(v)^\times_\ell$ is the max'l ℓ-power order quotient. We will have $\chi_v = 1$.

<u>Lecture 24</u> F, D , $L|\mathbb{Q}_\ell$, $\mathcal{O}$, λ, $\mathbb{F}$

R finite set of finite places of F not containing any $v|\ell$.

$v \in R$, $\Delta_v = k(v)^\times$, $\chi_v : \Delta_v \to \mathcal{O}^\times$ ℓ-power order

$q_v = \# k(v) \equiv 1 \ (\ell)$ $\quad \chi = \prod_{v \in R} \chi_v$

$$\begin{tikzcd} \mathbb{T}(\mathcal{U}_\Delta(R), \mathbb{F}) = \mathbb{T}(\mathcal{U}_\Delta(R), \mathbb{F})_\chi & \mathbb{T}(\mathcal{U}_\Delta(R), \mathcal{O})_\chi^{\vartriangle \mathfrak{m}} \arrow[l, two heads] \end{tikzcd} \quad \forall \chi$$

$\forall \chi$

$\triangledown$
$\mathfrak{m}$ max'l ideal.

- non-Eisenstein

$\bar{r} = \bar{r}_\mathfrak{m}$

- $k(\mathfrak{m}) = \mathbb{F}$, $\bar{r}_\mathfrak{m}(\sigma)$ has eigenvalues in $\mathbb{F}$, $\forall \sigma \in G_F$

achieved by soluble base change

- $\bar{r}_\mathfrak{m}$ is unramified away from ℓ.
- $\bar{r}_\mathfrak{m}|_{G_{F_v}} = 1$, $\forall v \in R$

Q a finite set of auxiliary primes

- disjt from $R \cup \{v|\ell\}$
- $q_v \equiv 1 \ (\ell)$, $\forall v \in Q$
- $\bar{r}_\mathfrak{m}(\mathrm{Frob}_v)$ has <u>distinct</u> e vals α_v, β_v, $\forall v \in Q$.

$\Delta_v = \ker(k(v)^\times \to$ max'l ℓ-power quotient$)$, $\chi_v = 1$.

$(\Delta_v^\circ = k(v)^\times)$

$$\Delta_Q = \prod_{v \in R \cup Q} \Delta_v, \qquad \chi_Q = \chi$$

$$\begin{tikzcd}
\mathbb{T}(U_{\Delta_\emptyset}(R), \mathcal{O})_{x,m} & & \mathbb{T}(U_{\Delta_Q}(R\cup Q), \mathcal{O})_x \arrow[ll, two heads] \arrow[d] & \triangleright\, m_Q \\
 & & \mathbb{T}(U_{\Delta_Q^\circ}(R\cup Q), \mathcal{O})_x \arrow[ull] & \triangleright\, m_Q^\circ
\end{tikzcd}$$

$\uparrow$ generated by T_v

$v \notin R \cup Q \cup \{v | \ell\}$

as m non-Eisenstein.

m_Q – premage of m

max'l ideal.

<u>Notation</u>,

$$\mathbb{T}\left(x, Q; \begin{smallmatrix}\mathcal{O} \\ \mathbb{F}\end{smallmatrix}\right) = \mathbb{T}\left(U_{\Delta_Q}(R \cup Q), \begin{smallmatrix}\mathcal{O} \\ \mathbb{F}\end{smallmatrix}\right)_{x, m_Q}$$

$$\circlearrowright \qquad\qquad\qquad \circlearrowright$$

$$S\left(x, Q; \begin{smallmatrix}\mathcal{O} \\ \mathbb{F}\end{smallmatrix}\right) = \left(\prod_{v \in Q} \tilde{e}_{\alpha_v}\right) S\left(U_{\Delta_Q}(R \cup Q), \begin{smallmatrix}\mathcal{O} \\ \mathbb{E}\end{smallmatrix}\right)_{x, m_Q}$$

$$r_{Q,x}^{\text{mod}} = r_{m_Q} : G_F \longrightarrow GL_2(\mathbb{T}(x, Q; \mathcal{O}))$$

lifting $\overline{r}_m$:

- det $r_{Q,x}^{\text{mod}} = \varepsilon_\ell^{-1}$
- unramified away from $Q \cup R \cup \{v | \ell\}$.
- FL w/ HT no's $\{0, 1\}$, at every $v|\ell$
- If $v \in R$, and $\sigma \in I_{F_v}$, then

  $r_{Q,x}^{\text{mod}}(\sigma)$ has char. poly $(X - \chi_v^{\text{Gal}}(\sigma))(X - \chi_v^{\text{Gal}}(\sigma)^{-1})$

$$\begin{tikzcd}
 & \chi_v^{\text{Gal}} : I_{F_v} \arrow[r, two heads] & I_{F_v^{ab}|F_v} \\
k(v)^\times & & \mathcal{O}_{F_v}^\times \arrow[ll, two heads] \arrow[u, "s"] \\
\mathcal{O}^\times \arrow[from=2-1, "\chi_v"] & &
\end{tikzcd}$$

- If $v \in Q$, then $r^{\mathrm{mod}}_{Q,\chi}\big|_{G_{F_v}} = \chi = \chi_{\alpha_v} \oplus \chi_{\beta_v}$

$$\chi_{\alpha_v/\beta_v} : G_{F_v} \to \mathbb{T}(\chi, Q; \mathcal{O})^\times$$

$H_v = k(v)^\times/\Delta_v$

$$(\chi_{\alpha_v/\beta_v} \bmod m_Q)(\mathrm{Frob}_v) = \alpha_v/\beta_v.$$

$H_Q = \prod_{v \in Q} H_v$

$a \in F_v^\times \cap \mathcal{O}_{F_v}$, $\quad U_a = \chi_{\alpha_v}(\mathrm{Art}\,a) \in \mathrm{End}(S(\chi, Q; \mathcal{O}))$

<u>Lemma</u> $S(\chi, Q; \mathcal{O})$ is a finite free $\mathcal{O}[H_Q]$-module and if $\mathfrak{a}_Q \triangleleft \mathcal{O}[H_Q]$ is the augmentation ideal, then ⊛ $S(\chi, Q; \mathcal{O})/\mathfrak{a}_Q S(\chi, Q; \mathcal{O}) \xrightarrow{\sim} S(\chi, \phi, \mathcal{O})$

Moreover, the action of $\mathcal{O}[H_Q]$ factors through $\mathcal{O}[H_Q] \to \mathbb{T}(\chi, Q; \mathcal{O})$

from $\prod(\chi_{\alpha_v} \circ \mathrm{Art})$

$$H_Q \cong U_{\Delta_Q^0}(R \cup Q)/U_{\Delta_Q}(R \cup Q)$$

$h \in H_v$ acts by $U_{\tilde h}$, $\tilde h \in \mathcal{O}_{F,v}^\times$ lifts h

$$\begin{pmatrix} \tilde h & 0 \\ 0 & 1 \end{pmatrix} \in U_{\Delta_Q^0}(R \cup Q)$$

⊛ M finite free $/\mathcal{O}[H_Q]$, then $\mathrm{tr}_{H_Q} = \sum_{h \in H_Q} h : M/\mathfrak{a}_Q M \xrightarrow{\sim} M^{H_Q}$

reduce to $M = \mathcal{O}[H_Q]$

<u>Need</u> $S(\chi, \phi; \mathcal{O}) \xrightarrow{\sim} S(\chi, Q; \mathcal{O})^{H_Q}$

$S(\widehat{U_{\Delta_Q}}(R \cup Q), \mathcal{O})_{\chi, m_Q} \nearrow \quad \prod_{v \in Q} \tilde e_{\alpha_v} = e$

$$\begin{tikzcd} S(\chi, \phi, \mathcal{O}) \arrow[r, bend left, "e"] & S(\chi, Q, \mathcal{O})^{H_Q} \arrow[l, bend left] \end{tikzcd}$$

$$t = \sum_{u \in U_\Delta(R)/U_{\Delta_Q^0}(R \cup Q)} u$$

$$\rotatebox{90}{=}$$

$$\prod_{v \in Q} GL_2(\mathcal{O}_{F_v})/Iw_v .$$

Claim A $\quad \underset{\rotatebox{90}{=}}{te} : S(\chi, \phi, \mathcal{O}) \xrightarrow{\sim} S(\chi, \phi, \mathcal{O})$

$$\prod_{v \in Q} \underbrace{tr_{GL_2(\mathcal{O}_{F_v})/Iw_v}}_{\text{sum of } q_v+1 \text{ terms}} \frac{U_{\pi_v} - B_v}{A_v - B_v} = \frac{q_v \overset{\overset{A_v+B_v}{\rotatebox{90}{=}}}{T_v} - (q_v+1) B_v}{A_v - B_v}$$

$A_v - B_v \quad A_v = \chi_{\alpha_v}(\mathrm{Frob}_v) \quad B_v = \chi_{\beta_v}(\mathrm{Frob}_v)$

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad = \frac{qA_v - B_v}{A_v - B_v} \in \text{Unit}$$

mod $m = 1$.

Claim B $\quad \mathrm{rk}\, S(\chi, \phi, \mathcal{O}) = \mathrm{rk}\, S(\chi, Q, \mathcal{O})^{H_Q}$

Claim A + B $\Rightarrow$ e is an isom.

Lecture 25 $\quad F, D, \quad R \sqcup Q \sqcup \{v | \ell\}$

$v \in R \Rightarrow q_v \equiv 1 \ (\ell)$, $\Delta_v = k(v)^\times$, $\chi_v : k(v)^\times \to \mathcal{O}^\times$ ℓ-power order

$L/\mathbb{Q}_\ell$, $\quad \mathcal{O}, \lambda, \mathbb{F}$. $\quad U_\Delta(R)_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,v}) : v | c \right\}$

$$\begin{tikzcd} U_\Delta(R)_v \arrow[d, "\chi_v"'] & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[d, mapsto] \\ \mathcal{O}^\times & \chi_v\left(\frac{a}{d}\right) \end{tikzcd}$$

$\chi = \prod \chi_v$

$\mathbb{T}(U_\Delta(R), \mathbb{F})_\chi \rhd m$, non-Eisenstein, $\bar{r}_m$ irred., unramified outside ℓ,

indep. of χ

$\det \bar{r}_m = \varepsilon_\ell^{-1}, \qquad v \in R, \quad \bar{r}_m \big|_{G_{F_v}} = 1.$

$v \in Q \Rightarrow \qquad q_v \equiv 1 \ (\ell)$

$\qquad \bar{r}_m(\mathrm{Frob}_v)$ has <u>distinct</u> evals α_v, β_v.

$\qquad \Delta_v = \ker\left(k(v)^\times \longrightarrow \text{max'l } \ell\text{-power order quotient}\right)$

$\qquad \chi_v = 1$

$$\mathbb{T}\left(U_{\Delta_Q}(R \cup Q), \overset{\mathcal{O}}{\mathbb{F}}\right)_\chi$$

$$\mathbb{T}(\chi, Q, \underset{\mathbb{F}}{\mathcal{O}}) = \mathbb{T}\left(U_{\Delta_Q}(R \cup Q), \underset{\mathbb{F}}{\mathcal{O}}\right)_{\chi, m}$$

$$S\left(\chi, Q, \overset{\mathcal{O}}{\mathbb{F}}\right) = \left(\prod_{v \in Q} \tilde{e}_{\alpha_v}\right) S\left(U_{\Delta_Q}(R \cup Q), \overset{\mathcal{O}}{\mathbb{F}}\right)_{\chi, m}$$

$$\tilde{e}_{\alpha_v} = \frac{U_{\pi_v} - \textcircled{B_v}}{A_v - B_v} \sim \text{roots of } X^2 - \mathrm{tr}\, r_m(\sigma) X + \det r_m(\sigma)$$

$$\sigma \in G_{F_v} \longmapsto \Big(\mathrm{Art}(\pi_v) \qquad \text{in } \mathbb{T}(\chi, Q, \mathcal{O}) \text{ lifting } \beta_v$$

lifting α_v

$$H_Q = \prod_{v \in Q} k(v)^\times / \Delta_v = \text{max'l } \ell\text{-power order qt of } \prod_{v \in Q} k(v)^\times.$$

$r_Q^{\mathrm{mod}} : G_F \longrightarrow GL_2(\mathbb{T}(\chi, Q, \mathcal{O})) \qquad (\chi_{\alpha_v} \bmod m)(\mathrm{Frob}_v) = \alpha_v$

$v \in Q, \ r_Q^{\mathrm{mod}}\big|_{G_{F_v}} \sim \begin{pmatrix} \chi_{\alpha_v} & 0 \\ 0 & \chi_{\beta_v} \end{pmatrix}$

$S(\chi, Q, \mathcal{O})$ is finite free $/\mathcal{O}[H_Q] \rhd \overset{\text{augmentation}}{\underset{\downarrow}{}} \mathfrak{a}_Q$ ideal

$U_\Delta(R)$

$\cap$

$U_{\Delta_Q^0}(R \cup Q)$

$\Delta \quad H_Q$

$U_{\Delta_Q}(R \cup Q)$

$$\begin{tikzcd}
\mathrm{tr}_{H_Q} : S(x, Q, \mathcal{O})/\mathfrak{a}_Q \arrow[r, "\sim"] & S(x, Q, \mathcal{O})^{H_Q} \\
& \parallel \\
& \left(\prod_{v \in Q} \tilde{e}_{\alpha_v}\right) S(U_{\Delta_Q^\circ}(R \cup Q), \mathcal{O})_{x, m} \arrow[dd, bend left, "\mathrm{tr} = t"] \\
& \wr \\
& S(x, \phi, \mathcal{O}) \arrow[uu, bend left, "e = \prod_{v \in Q} \tilde{e}_{\alpha_v}"]
\end{tikzcd}$$

$t \circ e : S(x, \phi, \mathcal{O}) \circlearrowleft$ is an automorphism

$\Rightarrow$ e is injective.

Claim. $\mathrm{rk}_{\mathcal{O}} S(x, \phi, \mathcal{O}) = \mathrm{rk}_{\mathcal{O}}\left(\left(\prod \tilde{e}_{\alpha_v}\right) S(U_{\Delta_Q^\circ}(R \cup Q), \mathcal{O})_{x, m}\right)$

check after $\otimes_{\mathcal{O}} \overline{\mathbb{Q}}_\ell$.

$$\sum_\pi \dim_{\overline{\mathbb{Q}}_\ell} \pi_m^{U_\Delta(R), x} = \sum_\pi \dim_{\overline{\mathbb{Q}}_\ell} \left(\prod \tilde{e}_{\alpha_v}\right) \pi_m^{U_{\Delta_Q^\circ}(Q \cup R), x}$$

$$\pi_m^{Q, U_\Delta(R), x} \otimes \bigotimes_{v \in Q} \pi_v^{GL_2(\mathcal{O}_{F_v})} \qquad\qquad \pi_m^{Q, U_\Delta(R), x} \otimes \bigotimes_{v \in Q} \left(\tilde{e}_{\alpha_v} \pi_v^{Iw_v}\right)$$

$$= \dim_{\overline{\mathbb{Q}}_\ell} \pi_m^{Q, U_\Delta(R, x} = \dim \pi_m^{Q, U_\Delta(R), x} \prod_{v \in Q} \dim \underbrace{\tilde{e}_{\alpha_v} \underbrace{\pi_v^{Iw_v}}_{2 \text{ dim'l}}}_{1 \text{ dim'l}}$$

Lemma. If M and N are finite free $\mathcal{O}$-modules of the same rank,

and if $e : M \to N$, $t : N \to M$ s.t. $t \circ e$ an automorphism of M. then t & e are isom.

Pf. $\square$

$$\begin{tikzcd} R^{univ}_{\chi,Q} \arrow[d, "\text{surjective}"', "{tr\, r^{univ}(Frob_v) \mapsto T_v}"'] \\ \mathbb{T}(\chi, Q, \mathcal{O}) \end{tikzcd}$$

$R^{univ}_{\chi,Q}$ = universal deformation ring for lifts r of $\bar{r}_m$ to complete local noetherian $\mathcal{O}$-alg. s.t.

1) r unramified outside $R \cup Q \cup \{v | \ell\}$

2) $\det r = \varepsilon_\ell^{-1}$

3) if $v | \ell$, then $r|_{G_{F_v}}$ is FL

4) if $v \in R$ and $\sigma \in I_{F_v}$, then

$r(\sigma)$ has char. poly. $(X - \chi_v^{Gal}(\sigma))(X - \chi_v^{Gal}(\sigma)^{-1})$

$v \in Q$, $\varphi \in G_{F_v}$ lifting $Frob_v$

$\Downarrow$

$v \in Q$, $r^{univ}(I_{F_v})$ is a pro-ℓ-group.

$\bar{r}_m(\varphi)$ has distinct evals $\alpha_v, \beta_v \in \mathbb{F}$

$r^{univ}(\varphi)$ has distinct evals $A_v, B_v \in R^{univ}_{\chi,Q}$.

$$\sim \begin{pmatrix} A_v & 0 \\ 0 & B_v \end{pmatrix}$$

$$\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} + \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$$

$$\|$$

$$I_{F_v} \twoheadrightarrow \underset{\langle \sigma \rangle}{\overset{\|}{\mathbb{Z}_\ell(1)}} \qquad r^{univ}(\sigma) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$x-1, y, z, w-1 \in m$

$I = \langle y, z \rangle$

$$r^{univ}(\varphi)^{-1}\, r^{univ}(\sigma)\, r^{univ}(\varphi) = r^{univ}(\sigma)^{q_v}$$

$$\begin{pmatrix} x & B_v/A_v\, y \\ A_v/B_v\, z & w \end{pmatrix} \equiv \begin{pmatrix} x^{q_v} & 0 \\ 0 & y^{\varepsilon_v} \end{pmatrix} + \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \bmod mI$$

$$\boxed{\begin{matrix} \left(B_v/A_v - 1\right) y \\ \left(A_v/B_v - 1\right) z \end{matrix}} \in mI \qquad \therefore\ y, z \in mI, \quad \text{i.e. } I = mI$$

units

$\Rightarrow I = (0)$ Nakayama

$$\begin{tikzcd}
\mathcal{O}[H_Q] \arrow[r] \arrow[dr, bend right, "\prod_{v \in Q} \chi_{d_v} \circ \mathrm{Art}"'] & R^{univ}_{\chi, Q} \arrow[d] \\
& \Pi(\chi, Q, \mathcal{O})
\end{tikzcd}$$

$$\begin{tikzcd}
R^{univ}_{\chi, Q} / a_Q \arrow[r, "\sim"] \arrow[dr] & R^{univ}_{\chi, \phi} \arrow[d] \\
& R
\end{tikzcd}$$

disjt from $R \cup \{v | \ell\}$

any Q s.t. $v \in Q \Rightarrow q_v \equiv 1 (\ell)$

evals of $\bar{r}_m(\mathrm{Frob}_v)$ distinct

$$\begin{tikzcd}
a_H \triangleleft \mathcal{O}[H_Q] \arrow[r] & R^{univ}_{\chi, Q} \arrow[d] \arrow[r, phantom, "\curvearrowright"] & S(\chi, Q, \mathcal{O}) \arrow[d, "tr"] \\
& R^{univ}_{\chi, \phi} \arrow[r, phantom, "\curvearrowright"] & S(\chi, \phi, \mathcal{O})
\end{tikzcd}$$

mod out by a_H

finite free $/\mathcal{O}[\Delta_Q]$

$\rightarrow$ kernel of action of $R^{univ}_{\chi, \phi}$ on $S(\chi, \phi, \mathcal{O})$ is nilpotent.

Lecture 26 & 27. I was in Tucson for AZ Winter School.