A local geometric Langlands for irreducible isoclinic formal connections

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§1. Recollection on supercuspidal rep'n

F n.a. local field. residue field k, char $k \neq 2$.

G conn'd reductive gp $/F$, splits after tame ext.

- A Yu datum consists of
  - a tuple of twisted Levi subgps $G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n$
  - $x \in \mathcal{B}(G_n)$ building.
  - $r_1 > r_2 > \cdots > r_n > 0$, real no.
  - ρ irred. rep. of $(G_n)_x$
  - $\phi_i : G_i(F) \longrightarrow \mathbb{C}^\times$ of depth r_i, $1 \leq i \leq n$ satisfying conditions.

From a Yu datum, Yu constructed a type $(J, \tilde{\Phi})$: $J \subset G(F)$ cpt open
$\tilde{\Phi}$: rep'n of J

Thm (Yu, Fintzen) $\text{c-ind}_J^{G(F)} \tilde{\Phi}$ is irr., thus s.c. (supercuspidal)

Thm (Kim, Fintzen) If $p \nmid |W|$, Yu's construction exhausts all s.c.

Rmk Hakim–Murnaghan: fibers of $\{\text{Yu data}\} \to \text{s.c.}$

§. Toral supercuspidal rep'ns

For simplicity, G split, simply-conn, simple. Consider $n=1$, Yu datum:

$G = G_0 \supset G_1 = S$

$\phi : S \longrightarrow \mathbb{C}^\times$ of depth $\underline{r = r_1}$.

where S is an elliptic max'l torus

$x \in \mathcal{B}(S)$, $\rho = 1$

Assume $G_{x, r/2} = G_{x, r/2+}$. $J = S_{0+} G_{x, r/2}$

$\tilde{\phi}: J \to \mathbb{C}^\times$ by first proj. to S_{0+}, then take ϕ. s.t.

$\phi|_{S_r/S_{r+}} \in (S_r/S_{r+})^* \hookrightarrow \mathfrak{g}_F$ is reg. s.s.

Let $E|F$ be ext. where S_E splits.

$$\mathfrak{g}_F = \mathfrak{s}_F \oplus \left(\bigoplus_{\alpha \in \Phi(G_E, S_E)} \mathfrak{g}_{\alpha_E} \right) \to \mathfrak{s}_F$$

<u>Adler</u>: $\text{c-Ind}_J^{G(F)} \tilde{\phi}$ is irr.

- $G_{x, r/2} \neq G_{x, r/2+}$, use $G_{x, r/2} \supset G_{x, r/2, m} \supset G_{x, r/2+}$.

<u>Kaletha</u>: proposes L-packets

<u>Def.</u> A <u>toral</u> s.c. parameter of gen. depth $r > 0$, is a discrete Langlands parameter

$\varphi: W_F \to {}^L G$ s.t. $(I_F^{r+} \subset I_F^r \subset P_F = I_F^{0+} \subset I. \subset W_F)$

(i) $C_{\hat{G}}(\varphi(I_F^r))$ is a max. torus.

(ii) $\varphi(I_F^{r+}) = 1$.

<u>Kaletha</u>, construct L-packets for such parameters consisting <u>toral s.c.</u> reps.

simple s.c.	$\subset$ epipelagic	$\subset$ toral	$\subset$ regular s.c.	$\subset$	s.c.
Gross–Reeder	Reeder–Yu	Adler	Kaletha		Yu

<u>Q</u>: Is Kaletha's construction compatible w/ local Langlands?

A: Yes for toral, in de Rham of Frenkel-Gaitsgory.

§ Local geometric Langlands

- $k = \mathbb{C}$, $F = \mathbb{C}((t)) \supset \mathcal{O}_F = \mathbb{C}[[t]]$

$\check{G}/\mathbb{C}$ adjoint type. $\check{G}((t))$, $\check{\mathfrak{g}}((t))$.

$D^\times = \operatorname{Spec} F \subset D = \operatorname{Spec} \mathcal{O}_F$

A (de Rham) $\check{G}$-local system on $D^\times$ is a $\check{G}((t))$-gauge equiv. class of

$d + \check{\mathfrak{g}}((t))\, dt$

Thm (Babbitt-Varadarajan) $\forall\, \nabla \in d + \check{\mathfrak{g}}((t))\, dt$, $\exists\, g \in \check{G}(\bar{F})$ $\left(g \in \check{G}((t^{1/d}))\right)$

s.t. $\mathrm{Ad}_g \nabla = d + \left(D_1 t^{-r_1} + D_2 t^{-r_2} + \cdots + D_k t^{-r_k} + D_{k+1}\right)\frac{dt}{t}$ $\quad (*)$

"canonical form".

$D_1, D_2, \cdots, D_k, D_{k+1} \in \check{\mathfrak{g}}$ mutually commutes

nonzero (if exists)

$D_1, \cdots, D_k$ semisimple

$r_1 > r_2 > \cdots > r_k > 0$ ratl no.

Here $(D_1, \cdots, D_k, \exp(2\pi i D_{k+1}))$ is unique up to $\check{G}$-conj.

Def (Jakob-Yu) ∇ is isoclinic if in $(*)$, D_1 is reg. s.s.

- Oper forms

Let $\{p_{-1}, 2\check{\rho}, p_1\}$ be principal $\mathfrak{sl}_2$-triple.

$\check{\mathfrak{g}}^{p_1} = \bigoplus_{i=1}^{n} \mathbb{C} p_i$, $[\check{\rho}, p_i] = (d_i - 1) p_i$

Def An oper (canonical form) is an eqn of the form

$$\nabla = d + \left(p_{-1} + \sum_{i=1}^{n} v_i(t)\, p_i\right) dt \qquad v_i(t) \in \mathbb{C}((t)).$$

$$\mathrm{Op}_{\check{G}}(D^\times) \simeq \mathbb{C}((t))^n, \quad \nabla \longleftrightarrow (v_i(t))_i$$

$$p : \mathrm{Op}_{\check{G}}(D^\times) \longrightarrow \mathrm{Loc}_{\check{G}}(D^\times) = \left(d + \check{\mathfrak{g}}((t))\, dt\right) / \check{G}((t)).$$

Thm (Frenkel–Zhu) p is surj.

Feigin–Frenkel isom.

$\hat{\mathfrak{g}} = \mathfrak{g}((t)) + \mathbb{C}\mathbb{1}$ affine Kac–Moody alg at critical level.

$\tilde{\mathcal{U}}(\hat{\mathfrak{g}})$ = completed $\mathcal{U}(\hat{\mathfrak{g}}) / (\mathbb{1} - 1)$

$Z \subset \tilde{\mathcal{U}}(\hat{\mathfrak{g}})$

Thm (Feigin–Frenkel) $\mathrm{Fun}\, \mathrm{Op}_{\check{G}}(D^\times) \simeq Z$.

· Frenkel – Gaitsgory.

$\forall\, \nabla \in \mathrm{Loc}_{\check{G}}(D^\times)$, take $\forall\, \chi \in p^{-1}(\nabla) \subset \mathrm{Op}_{\check{G}}(D^\times)$,

$\Leftrightarrow \chi : Z \to \mathbb{C}$

· $\hat{\mathfrak{g}}\text{-mod}_\chi = \{\tilde{\mathcal{U}}(\hat{\mathfrak{g}})\text{-mod on which } Z \text{ acts by } \chi\}$

Conj. $\hat{\mathfrak{g}}\text{-mod}_\chi$ depends only on ∇.

$G((t)) \curvearrowright \hat{\mathfrak{g}}\text{-mod}_\chi \rightsquigarrow G((t)) \curvearrowright K_0(\hat{\mathfrak{g}}\text{-mod}_\chi)$

§ Isoclinic irr. v.s. toral s.c.

Let $(\mathfrak{j}, \tilde{\varphi})$ be the Lie alg. of (J, Φ) for toral s.c.
$\mathfrak{j} \subset \mathfrak{g}[[t]]$

$$\mathrm{Vac}_{J,\tilde\phi} = \mathrm{Ind}_{J+\mathbb{C}\mathbb{1}}^{\hat{\mathfrak g}} \tilde\phi \in \hat{\mathfrak g}\text{-mod}$$

$$Z_{J,\tilde\phi} = \mathrm{Im}\,(Z \longrightarrow \mathrm{End}_{\hat{\mathfrak g}} \mathrm{Vac}_{J,\tilde\phi})$$

$$\mathrm{Op}_{J,\tilde\phi} = \mathrm{Spec}\ Z_{J,\tilde\phi} \hookrightarrow \mathrm{Spec}\ Z = \mathrm{Op}_{\check G}(D^\times)$$

Thm. (Y.) (i) $\mathrm{Op}_{J,\tilde\phi} = p^{-1}(\nabla_{J,\tilde\phi})$

(ii) Have explicit formula for one $\nabla_X \in p^{-1}(\nabla_{J,\tilde\phi})$ from $(J,\tilde\phi)$.

(iii) $\{\nabla_{J,\tilde\phi} : (J,\tilde\phi) \text{ toral s.c.}\} = \{\text{irr. isoclinic}\}$

<u>Evidence to FG</u> (heuristic)

For $X \in p^{-1}(\nabla_{J,\tilde\phi})$, $\mathrm{of}\ \mathrm{Vac}_{J,\tilde\phi}/\ker X \in \hat{\mathfrak g}\text{-mod}_X$ is $\ker\tilde\phi$ - integrable $\quad \begin{matrix} J \\ \cup \end{matrix}$

$$\underbrace{\text{c-ind}_J^{G(F)}\tilde\phi}_{\text{irr.}} \simeq \underbrace{K_0(\hat{\mathfrak g}\text{-mod}_X)}_{\text{irr}}$$

§ <u>Global: Airy $\check G$-conn.</u>

Airy $\check G$-conn: $\check G$-conn. on $\mathbb{A}^1_{\mathbb{C}}$ that is isoclinic w/ slope $r = 1 + \frac{1}{h}$ at ∞

h = Coxeter no.

<u>Thm</u> (Jakob - Kamgarpour - Yi.) If $\phi|_{S_{r/2}/S_r} = 0$,

$\exists!$ $(J, \tilde\phi^*(d\text{-}dt))$ - equiv. irr. D-mod $A_{\tilde\phi}$ on Bun_G, $G(\mathcal{O}_\infty) = \ker\tilde\phi$

Thm. (Y.) Eigenvalue of $A_{\tilde\phi}$ is an Airy $\check G$-conn.