Introduction to LLC and application of rigid geometry to LLC

Sug Woo Shin , Teruyoshi Yoshida
申哲宇　吉田輝義

Lecture 1 (Shin)

1. Intro and Motivation

Fix $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, $\mathbb{A} := \prod_v' \mathbb{Q}_v \subset (\mathbb{Q}_2 \times \mathbb{Q}_3 \times \cdots) \times \mathbb{R}$

rough picture $\quad n \geq 1$

$$\begin{Bmatrix} \text{automorphic} \\ \text{rep'ns of} \\ GL_n(\mathbb{A}) \end{Bmatrix} \overset{GLC}{\longleftrightarrow} \begin{Bmatrix} n\text{-dim'l } \ell\text{-adic reps} \\ \text{of } Gal(\overline{\mathbb{Q}}|\mathbb{Q}) \end{Bmatrix}$$

$$\text{localize at } p \downarrow \qquad \circlearrowright \qquad \downarrow \text{localize at } p$$

$$\begin{Bmatrix} \text{irred. smooth} \\ \text{rep'ns of } GL_n(\mathbb{Q}_p) \end{Bmatrix} \overset{LLC}{\longleftrightarrow} \begin{Bmatrix} n\text{-dim'l rep'ns of} \\ Gal(\overline{\mathbb{Q}_p}|\mathbb{Q}_p) \end{Bmatrix}$$

GLC: huge conjecture. (partially established)

LLC: Theorem by Harris-Taylor, Henniart 1999

<u>n=1</u>: $\begin{matrix} GLC \\ LLC \end{matrix}$ = class field theory.

<u>Example + Motivation</u> $(n=2)$

$$\begin{Bmatrix} \underline{\text{cuspidal}} \text{ automorphic} \\ \text{reps of } GL_2(\mathbb{A}) \end{Bmatrix} \overset{GLC}{\longleftrightarrow} \begin{Bmatrix} \underline{\text{irred}} \text{ reps of} \\ Gal(\overline{\mathbb{Q}}|\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}_\ell}) \end{Bmatrix}$$

$$\begin{array}{ccc} \cup & \overset{\pi_f}{\updownarrow} & \cup \\ & f & \\ \left\{ \begin{array}{l} \text{Cusp. forms of wt 2} \\ \text{w/ coeff} \in \mathbb{Q} \end{array} \right\} & \overset{\circledast}{\longleftrightarrow} & \left\{ \begin{array}{l} \text{Tate modules of} \\ \text{elliptic curves} / \mathbb{Q} \end{array} \right\} \end{array}$$

$\circledast$ Shimura - Taniyama Conj. (Thm) 1999

$$E|\mathbb{Q} \rightsquigarrow V_\ell E = \left(\varprojlim_n E(\overline{\mathbb{Q}})[\ell^n] \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

$$\simeq \mathbb{Q}_\ell^2 \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$$

"Unramified case" (at p)

$p \nmid$ level of f $\iff$ E has good reduction mod p

$\iff$ I_p acts trivially on $V_\ell E$ (Néron - Ogg - Shafarevich)

LLC in this case boils down to $\boxed{a_p(f) = a_p(E)} \rightsquigarrow \circledast$

$a_p(f)$: T_p - eigenvalue

$a_p(E)$: $1 + p - \# E(\mathbb{F}_p)$

$= \mathrm{tr}(\mathrm{Frob}_p \mid V_\ell E)$

ramified case.

$p \mid$ level, E has bad reduction at p

Q. $\exists$ analog of $\circledast$?

Can $V_\ell E|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}$ be described only in terms of purely local information about f ?

Ans LLC for GL_2 $\pi_f = \otimes'_v \pi_{f,v}$

$$p\text{-component of } \pi_f \longleftrightarrow V_\ell E|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}$$

$\{\text{irred. sm. rep. of } GL_2(\mathbb{Q}_p)\} \longleftrightarrow \{\text{2-dim local rep.}\}$

2. Basic Def.

K finite ext'n of $\mathbb{Q}_p$

$\cup$

$\mathcal{O}_K \ni \varpi$ uniformizer

$\mathcal{O}_K/(\varpi) \simeq \mathbb{F}_q$

1) smooth reps

$G = GL_n(K) \leftarrow$ p-adic topology

$$G \supset \underbrace{GL_n(\mathcal{O}_K) \supset 1 + \varpi M_n(\mathcal{O}_K) \supset 1 + \varpi^2 M_n(\mathcal{O}_K) \supset \cdots}_{\text{cpt open subgps (open basis)}}$$

$\uparrow$ loc. cpt

Rmk. Could work w/ K-pts of red. gps

$k = \mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$ (coeff) Consider (π, V) – V k-vec. sp. (often ∞-dim'l)

– $\pi : G \to GL_k(V)$

Def (π, V) is smooth if

$$V = \bigcup_{\substack{K \\ \text{open cpt}}} V^K .$$

$\left(\Leftrightarrow \forall v \in V, \; \begin{array}{c} G \longrightarrow V \\ g \mapsto \pi(g)v \end{array} \text{ is "smooth" (loc. const)} \right)$

Ex. n=1 $K^\times \longrightarrow GL_1(k)$

$\cup$

$\mathcal{O}_K^\times$ has finite image

In rep. theory, – construct all (irred) reps

– classify " " $\leftarrow$ LLC

Coarse classification

$$\left\{\begin{array}{l}\text{irred. sm.}\\ GL_n(K)\text{-reps}\end{array}\right\} \supset \left\{\begin{array}{l}\text{square}\\ \text{- integrable}\end{array}\right\} \supset \{\text{supercuspidal}\}$$

$k = \mathbb{C}$

Standard

- building blocks
- most mysterious

Can access s.c. reps using rigid geometry and prove LLC.

2) Weil-Deligne reps

$$1 \longrightarrow \underset{\text{inertia}}{I_K} \longrightarrow G_K = \mathrm{Gal}(\overline{K}|K) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q) \longrightarrow 1$$

$$G_K \xrightarrow{v} \mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q) \cong \hat{\mathbb{Z}} \ni 1 \mapsto \mathrm{Frob}_q = (x \mapsto x^q)^{-1} \quad \text{geom. Frob.}$$

Def (Weil group) $W_K := v^{-1}(\mathbb{Z})$

Def (WD-reps) It's a triple (ρ, V, N).

- V finite dim. k-vec. sp.
- $\rho: W_K \longrightarrow GL_k(V)$, $\rho(I_K)$ has finite image
- $N \in \mathrm{End}_k(V)$ s.t. $\forall\, \tau \in W_K$, $\rho(\tau) N \rho(\tau)^{-1} = q^{-v(\tau)} N$.

(Advice: forget N if you wish)

Def (ρ, V, N) is

- *Frob-semisimple* if $\rho(\tau)$ is s.s. $\forall\, \tau \in W_K$
- *irred.* if ρ is irred. and $N = 0$.

Coarse classification

$$\left\{\begin{matrix} n\text{-dim} \\ \text{WD-reps} \end{matrix}\right\} \supset \{\text{indecomposable}\} \supset \{\text{irred}\}$$

3. Statement of LLC

Thm $\exists!$ Bijection $\quad (k = \mathbb{C})$

$$\left\{\begin{matrix} \text{irred. smooth} \\ GL_n(K)\text{-reps} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} n\text{-dim Frob.-s.s.} \\ \text{WD-reps of } W_K \end{matrix}\right\}$$

s.t. • $\underline{n=1}$ It's given by composing w/ $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$

$$\left[\underset{\chi}{K^\times \to GL_1(\mathbb{C})} \longmapsto \chi \circ \mathrm{Art}_K^{-1} : W_K^{ab} \to GL_1(\mathbb{C}) \right]$$

• Compatible w/ duals, twists, base change, ...

• equality of local L-factors, ε-factors.

s.c. $\longleftrightarrow$ irred.

sqr-int. $\longleftrightarrow$ indecomposable

$\vdots$

ref Kudla's article on LLC (in "Motives" 1994)

R. Taylor "Gal. reps".

Lecture 2 (Shin) Introduction to Rapoport-Zink spaces

1. Motivation $\quad K$ fin. ext of $\mathbb{Q}_p$

Recall LLC for GL_n

$$\begin{matrix} \left\{\begin{matrix} \text{irred. smooth} \\ GL_n(K)\text{-rep'n} \end{matrix}\right\} & \underset{1-1}{\longleftrightarrow} & \left\{\begin{matrix} n\text{-dim'l WD-reps} \\ \text{of } W_K \end{matrix}\right\} \\ \cup & & \cup \\ \text{s.c.} & \longleftrightarrow & \text{irred.} \end{matrix}$$

Dagger

$\boxed{Q}$ How to establish it ?

1. Classify rep'ns on each side, and try to match.

2. (Geom. approach) Find a vec. sp.

$$\text{nice vec. sp.} \circlearrowleft GL_n(K) \times W_K$$

Fact. ℓ-adic étale cohomology

$$X : \text{geom. obj}/K \rightsquigarrow H_{\acute{e}t}(X \times_K \bar{K} ; \bar{\mathbb{Q}}_\ell) \quad \circlearrowleft Gal(\bar{K}|K)$$

Ans. Find a geom. obj./K w/ $GL_n(K)$-symmetry.

use moduli space e.g. RZ-space

In fact, a similar idea applies to GLC.

	geom. object	moduli spaces for	group action
LLC	RZ-spaces	BT groups	(local Weil) $\times G \times J$
GLC	Shimura Var.	Abelian Var.	(global Gal) $\times$ (adelic G)

p-adic groups

2. BT groups

p-divisible groups

Def. S scheme (e.g. p is nilp in S)

A $\boxed{\text{BT gp}/S}$ (of ht h) is $\Sigma = (\Sigma_1 \to \Sigma_2 \to \Sigma_3 \to \cdots)$ s.t.

(i) Σ_m is finite comm. gp scheme $/S$, locally free of rk p^{mh},

(ii) $\Sigma_m = \ker(p^m : \Sigma_{m+1} \to \Sigma_{m+1})$

(iii) $p : \Sigma \to \Sigma$ is onto. (fppf top.) (p-divisible)

Ex. $\mathbb{Q}_p/\mathbb{Z}_p = (\underset{p}{p^{-1}\mathbb{Z}_p/\mathbb{Z}_p} \to \underset{p^2}{p^{-2}\mathbb{Z}_p/\mathbb{Z}_p} \to \cdots)$ ht 1.

· $\mu_{p^\infty} = (\mu_p \to \mu_{p^2} \to \cdots)$ $\qquad \mu_{p^n} = \{x^{p^n} = 1\}$

ht 1

· A abelian scheme $/S$ — dim g

$\to A[p^\infty] = (A[p] \to A[p^2] \to \cdots)$ $\qquad$ ht $2g$

<u>Def</u>. A map $i: \Sigma \to \Sigma'$ is an <u>isogeny</u>

if (i) i is onto (fppf top.)

(ii) $\ker i$ is loc. free <u>fin</u> gp scheme.

ideal

v.s.

fractional ideal

<u>Eg</u>. $p^m: \Sigma \to \Sigma$ is an isog., $m \geq 0$.

<u>Def</u> $j: \Sigma \to \Sigma'$ is a <u>quasi-isogeny</u>

if $j \in \underline{\mathrm{Hom}}(\Sigma, \Sigma') \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\overset{\text{Zar.}}{\text{locally}}$ on S, $\exists\, n \geq 0$ s.t. $p^n \cdot j$ is isog.

<u>Ex</u>. $p^m: \Sigma \to \Sigma$ is q-isog., $\forall m \in \mathbb{Z}$

<u>Rmk</u>. $\mathrm{End}(\Sigma)$ is a $\mathbb{Z}_p$-alg.

<u>Dieudonné theory</u> (over $\overline{\mathbb{F}}_p$)

<u>idea</u>: Understand BT gps using linear alg

$\exists$ nat'l equiv. $\left(\text{BT gps}/\overline{\mathbb{F}}_p\right) \xrightarrow[\sim]{\mathbb{D}} \left(\begin{array}{l}\text{fin. free } W = W(\overline{\mathbb{F}}_p)\text{-mods} \\ \text{equipped w/ } F, V \text{ actions} \\ \text{s.t. } FV = VF = p\end{array}\right)$

$\left\{\begin{array}{l}\text{"simple"} \\ \text{BT gps}/\overline{\mathbb{F}}_p\end{array}\right\}\Big/_{\text{q-isog.}} \xleftrightarrow{1-1} \mathbb{Q} \cap [0,1]$

$\Sigma \longmapsto \lambda = \frac{s}{t}$ $\quad (F^t = p^s \text{ on } \mathbb{D}(\Sigma))$

$\mathbb{Q}_p/\mathbb{Z}_p \longmapsto 0$ $\qquad \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) = W$ & $F = 1,\ V = p$

$\mu_{p^\infty} \longmapsto 1$ $\qquad \mathbb{D}(\mu_{p^\infty}) = W$ & $F = p,\ V = 1$

Fact. $\Sigma \longleftrightarrow \lambda$, $\mathrm{End}(\Sigma) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ = (central) division alg. $/\mathbb{Q}_p$, invariant λ

3. RZ data (of EL type)

K finite ext'n of $\mathbb{Q}_p$. B fin dim'l div. alg. $/K$

V fin. gen. B-module. $G = GL_B(V)$. ← alg gp $/\mathbb{Q}_p$

$\mu : \mathbb{G}_m \longrightarrow G$ over $\overline{K}$ ("weights are 0 or 1")

Σ BT gp $/\overline{\mathbb{F}}_p$, s.t. $\mathbb{D}(\Sigma) \subset V$

$i : \mathcal{O}_B \longrightarrow \mathrm{End}(\Sigma)$ $\quad$ $\mathbb{D}(\Sigma)$ $\mathcal{O}_B$-lattice "of type μ"

$\mathrm{QIsog}(\Sigma, i)$

$J = (\mathrm{End}(\Sigma, i) \otimes \mathbb{Q}_p)^\times$ ← alg gp $/\mathbb{Q}_p$

Ex (Lubin–Tate case)

$B = K = \mathbb{Q}_p$ (for simplicity), $V = K^n = \mathbb{Q}_p^n$, $G = GL_n$, $\mu : \mathbb{G}_m \longrightarrow GL_n$, $a \longmapsto \begin{pmatrix} a & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$ $\quad$ $E = K$ ← reflex field

Σ ht n, slope $1/n$. $J = D_{1/n}^\times$, $D_{1/n}$ C.D.A. $/\mathbb{Q}_p$, inv. $1/n$.

4. RZ spaces

E = field of def'n of $G(\overline{\mathbb{Q}}_p)$-conj. class of μ (finite over $\mathbb{Q}_p$)

Goal. Search for vec. sp. $\curvearrowleft W_E \times G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$

Consider the functor

$$\mathcal{M}_0 : \begin{pmatrix} \text{schemes} / \mathcal{O}_{\hat{E}^{ur}} \\ \text{in which } p \text{ is nilp} \end{pmatrix} \longrightarrow (\text{Sets})$$

$$S \longmapsto \{(X, i, \rho)\}/\sim$$

X BT gp$/S$

$i : \mathcal{O}_B \longrightarrow \mathrm{End}(X)$

$\rho : \Sigma \underset{\overline{\mathbb{F}}_p}{\times} S/pS \longrightarrow X \underset{S}{\times} S/pS$ q. isogeny, compatible w/ $\mathcal{O}_B$-action

$(X, i, \rho) \sim (X', i', \rho')$ if $\exists\, X \simeq X'$ carrying (i, ρ) to (i', ρ')

<u>Thm</u> $\mathcal{M}_0$ is representable by a formal scheme $/\mathrm{Spf}\, \mathcal{O}_{\hat{E}^{ur}}$.

Miede will explain: $\mathcal{M}_0 \rightsquigarrow \mathcal{M}_0^{ad}/\hat{E}^{ur} \rightsquigarrow H_c(\mathcal{M}_0^{ad} \times_{\hat{E}^{ur}} \overline{\hat{E}^{ur}} ; \overline{\mathbb{Q}_\ell})$

adic space

$\circlearrowleft$

$\mathrm{Gal}(\overline{\hat{E}^{ur}} | \hat{E}^{ur}) = I_E$ ($\rightsquigarrow$ action of W_E)

<u>J-action</u>.

$\alpha \in J(\mathbb{Q}_p)$ acts on $\mathcal{M}_0$ as $(X, i, \rho) \mapsto (X, i, \rho \circ \alpha)$.

$\rightsquigarrow J \curvearrowright H_c(\mathcal{M}_0^{ad})$

<u>G-action</u>. Consider tower

$$\{\mathcal{M}_K^{ad}\}_{K \subset G(\mathbb{Z}_p)} = \left\{ \begin{array}{c} \text{tower} \\ \vdots \\ \mathcal{M}_K^{ad} \\ \downarrow \\ \mathcal{M}_0^{ad} = \mathcal{M}_{G(\mathbb{Z}_p)}^{ad} \end{array} \right\}$$

$\circlearrowleft$ $G(\mathbb{Q}_p)$ "Hecke action"

$$\boxed{H_c(\mathcal{M}^{ad})} := \varinjlim_K H_c(\mathcal{M}_K^{ad})$$

$\circlearrowleft$

$W_E \times G \times J$

LLC JL

Conj. (loosely)

(essentially true. in LT case, rep = s.c. rep)

Lecture 3 (Yoshida) Intro. to Lubin–Tate spaces

§1. Sheaves

A base ring (comm. w/ unit)

$(A\text{-alg})^\vee := \{\text{all functors } X : (A\text{-alg}) \to (\text{Sets})\}$

Def. X : fpqc (fppf / étale / Zariski) sheaf

$\iff$ – $X(R \times R') \simeq X(R) \times X(R')$ (canonical)

– $\forall \varphi : R \to R'$ faithfully flat mor. in $(A\text{-alg})$

$\left(+ \text{ fin. pres.} / + \text{ étale} / + R' \simeq \prod_{i=1}^{m} R[\frac{1}{f_i}],\ f_i \in R\right)$

$R \xrightarrow{\varphi} R' \underset{id \otimes \varphi}{\overset{\varphi \otimes id}{\rightrightarrows}} R' \otimes_R R'$ induces

$X(R) \xrightarrow[\sim]{F(\varphi)} \{x \in X(R') : X(\varphi \otimes id)(x) = X(id \otimes \varphi)(x)\}$

Ex. Affine schemes := representable functors

$X = \operatorname{Spec} B$, $R \mapsto \operatorname{Hom}_{A\text{-alg}}(B, R)$ for $B \in (A\text{-alg})$

Lemma [(half of) fpqc descent] Affine schemes are fpqc sheaves.

$\Rightarrow$ Full subcategories

$$\begin{array}{ccccccccc}
(\text{Affine schemes}/A) & \subset & \overbrace{(\text{fpqc sheaves}/A) \subset (\text{fppf sheaves}/A) \subset (\text{étale sheaves}/A) \subset (\text{Zar. sheaves}/A)}^{\text{closed under } \varinjlim} \\
\cap & & \cup \\
(\text{schemes}/A) & \subset & (\text{formal schemes}/A) & & \subset (A\text{-alg})^\vee.
\end{array}$$

schemes $/A$:= X s.t. "Zariski locally" affine schemes

NOT closed under $\varinjlim$.

but $\exists$ many interesting ind-schemes ($\varinjlim$ of schemes) e.g. Formal schemes $/A := X$ s.t. "Zariski locally"

$$\simeq \varinjlim_m \operatorname{Spec}(B/I^m) \quad , I \subset B \text{ ideal}$$

Ex. $A = \mathbb{Z}_p$, $\operatorname{Spf}(\mathbb{Z}_p, (p)) := \varinjlim_m \operatorname{Spec}(\mathbb{Z}/p^m)$

$$R \longmapsto \begin{cases} pt & \text{if } p \text{ nilpotent in } R \\ \phi & \text{if not} \end{cases}$$

nilp. elt in R

$$\operatorname{Spf}(\mathbb{Z}_p[\![T_1, \dots, T_n]\!], (p, T_1, \dots, T_n)) : R \longmapsto \begin{cases} (\operatorname{Nilp} R)^n & \text{if } p \text{ nilp in } R \\ \phi & \text{if not} \end{cases}$$

Rmk fpqc set theoretic difficulties $\to$ use fppf.

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§2. Gp schemes, etc.

Replace (sets) by (Abel gps) or ($\mathcal{O}$-mod) for a ring $\mathcal{O}$
$\mathbb{Z}$-mod $=$ (Abel gps)

$$\Rightarrow \text{ get } \begin{cases} \text{(comm) gp schemes / abelian sheaves} \\ \mathcal{O}\text{-mod schemes / sheaves of } \mathcal{O}\text{-mod's} \end{cases}$$

$(\text{fppf}/A)_{\mathcal{O}} := \{\text{all fppf sheaves of } \mathcal{O}\text{-mods}/A\} \subset (A\text{-alg})^{\vee}_{\mathcal{O}}$
$:= \{\text{all functors } (A\text{-alg}) \to (\mathcal{O}\text{-mod})\}$

$(\text{fppf}/A)_{\mathcal{O}}$: abelian category : Hom's are $\mathcal{O}$-mod.

$I \subset \mathcal{O}$ ideal, $X \in (A\text{-alg})^{\vee}_{\mathcal{O}}$, $\Rightarrow$ <u>I-torsion points</u> $X[I] \in (A\text{-alg})^{\vee}_{\mathcal{O}/I}$

$$R \mapsto \{x \in X(R) : I \cdot x = 0\}$$

Ex. (g-dim'l) abelian schemes $/A$

$=$ gp scheme $\mathcal{A}/A$ which is proper smooth $/A$

$\forall$ geom. fiber connected w/ const. dim g

(abel. var. if A field ; elliptic curve if $g=1$)

$F|\mathbb{Q}$ fin. $\mathcal{O}_F \subset F$: ring of integers , A : $\mathcal{O}_F$-alg.

$\mathcal{O}_F$-abel. scheme $/A$ $:=$ $\mathcal{A} \in (A\text{-alg})^\vee_{\mathcal{O}_F}$, $\mathcal{A}$ abel. scheme.

$I \subset \mathcal{O}_F$ ideal $\Rightarrow$ $\mathcal{A}[I] \in (A\text{-alg})^\vee_{\mathcal{O}_F/I}$: affine $\mathcal{O}_F/I$-mod scheme

$\mathcal{A}[I] = \operatorname{Spec} B$

s.t. $B \in (A\text{-alg})$ fin. loc. free A-mod of rk $N_{F|\mathbb{Q}}(I)^{2g/[F:\mathbb{Q}]}$.

$\mathfrak{p} \subset \mathcal{O}_F$ prime, $\mathcal{O}_\mathfrak{p} := \varprojlim_m \mathcal{O}_F/\mathfrak{p}^m$. $\rightarrow$ $\mathcal{G} := \mathcal{A}[\mathfrak{p}^\infty] \in (A\text{-alg})^\vee_{\mathcal{O}_\mathfrak{p}}$

$R \longmapsto \bigcup_{m \geq 1} \mathcal{A}[\mathfrak{p}^m](R)$

---- ind gp scheme $(= \varinjlim_m \mathcal{A}[\mathfrak{p}^m])$

$\in (\mathrm{fppf}/A)_{\mathcal{O}_F}$

Thm (Serre - Tate) k field, char p, $(A, \mathfrak{m})$ Artin local ring. $A/\mathfrak{m} \simeq k$.

$\mathcal{A}/k$: abel. var. , $\mathcal{G} := \mathcal{A}[p^\infty] \in (\mathrm{fppf}/k)_{\mathbb{Z}_p}$

$\Rightarrow$ $\{\text{Deformations of } \mathcal{A} \text{ to } A\} \xrightarrow[\sim]{\text{bij}} \{\text{Deform. of } \mathcal{G} \text{ to } A\}$

$\{\text{Deformations of } \mathcal{A} \text{ to } A\} = (\tilde{\mathcal{A}}/A,\ \mathcal{A} \simeq \tilde{\mathcal{A}} \times_A k)/_{\text{isom.}}$ $\quad \tilde{\mathcal{A}} \mapsto \tilde{\mathcal{A}}[p^\infty]$

$\rightarrow$ can study the local str. of PEL Shimura var. (moduli of AV) using $\mathcal{A}[p^\infty]$

§3. Barsotti-Tate groups

p prime, $K|\mathbb{Q}_p$ fin. $K \supset \mathcal{O} \supset \mathfrak{p} \ni \varpi$, $k = \mathcal{O}/\mathfrak{p} \simeq \mathbb{F}_q$

Def $A \in (\mathcal{O}\text{-alg})$ <u>Barsotti-Tate $\mathcal{O}$-mod</u> $/A$ is $\mathcal{G} \in (A\text{-alg})^\vee_{\mathcal{O}}$ s.t.

1) $\mathfrak{p}$-torsion, (i.e. $\mathcal{G}[\mathfrak{p}^\infty] = \mathcal{G}$)
2) $\mathfrak{p}$-divisible (i.e. $\varpi : \mathcal{G} \to \mathcal{G}$ is epi in $(\mathrm{fppf}/A)_{\mathcal{O}}$)

0) $\forall m \in \mathbb{Z}_{>0}$, $\mathcal{G}[\mathfrak{p}^m]$: affine $\mathcal{O}/\mathfrak{p}^m$-mod scheme $/A$ which is finite loc. free $/A$.

0), 1) $\Rightarrow$ $\mathcal{G}$ ind-scheme $\in (\mathrm{fppf}/A)_{\mathcal{O}}$

<u>$\mathcal{O}$-height</u> $h = \mathrm{ht}_{\mathcal{O}}(\mathcal{G}) \in \mathbb{Z}_{\geq 0}$ is defined by $\mathrm{rk}_A\, \mathcal{G}[\mathfrak{p}] = q^h$ (if $\mathrm{Spec}\, A$ conn'd).

<u>Isogeny</u>. $f : \mathcal{G} \to \mathcal{G}'$ mor in $(A\text{-alg})^\vee_{\mathcal{O}}$ s.t.

1) f epi in $(\mathrm{fppf}/A)_{\mathcal{O}}$
2) $\ker f$ fin. loc. free $\mathcal{O}$-mod. scheme $/A$

<u>$\mathcal{O}$-height of f</u> is h if $\mathrm{rk}_A(\ker f) = q^h$.

($\mathcal{O}$-height of $\mathcal{G}$ = $\mathcal{O}$-ht of ϖ)

<u>Quasi-isogeny</u>. $f \in \mathrm{Hom}(\mathcal{G}, \mathcal{G}') \otimes \mathbb{Q}$ s.t. $\exists\, m \in \mathbb{Z}_{\geq 0}$, $p^m f$ isog.

f isog. $\Rightarrow$ $\exists\, f' : \mathcal{G}' \to \mathcal{G}$, $ff' = p^m$ for $m \in \mathbb{Z}_{\geq 0}$.

f q-isog. $\Rightarrow$ $\exists$ isog. f' w/ $p^m f f' = p^{m'}$, $p^{m-m'} f' : \mathcal{G}' \xrightarrow{\text{q-isog.}} \mathcal{G}$ inverse of f.

If $A \in (k\text{-alg})$, $\Rightarrow$ $F : \mathcal{G} \longrightarrow \mathcal{G}^{(q)} := \mathcal{G} \times_{\mathrm{Spec}\, A} \underset{(x \mapsto x^q)^*}{\overset{\mathrm{Spec}\, A}{\uparrow}}$ relative q-th power Frobenius mor.

$\cdots$: <u>isogeny</u>

its $\mathcal{O}$-ht $d =: \dim \mathcal{G}$ dimension of $\mathcal{G}$.

If $A \in (\mathcal{O}\text{-nilp}) := (R \text{ s.t. } p \text{ is nilp. in } R) \subset (\mathcal{O}\text{-alg})$,

$$\underline{\omega}_{\mathcal{G}[p^m]} := e^* \Omega_{\mathcal{G}[p^m]/A} \qquad , \quad \mathcal{G}[p^m] = \operatorname{Spec} B$$

$e \in \mathcal{G}[p^m](A)$ ... identity

$$= \Omega_{B/A} \otimes_{B \nearrow e} A \qquad A \underset{\longrightarrow}{\overset{e}{\longleftarrow}} B$$

--- independent of m for $m \gg 0$ $=: \underline{\omega}_{\mathcal{G}}$

locally free A-mod. of rank $\dim \mathcal{G}$

$\operatorname{Lie} \mathcal{G} := \underline{\omega}_{\mathcal{G}}^{\vee}$

$\mathcal{O} \longrightarrow \operatorname{End}(\mathcal{G})$ induces $\mathcal{O} \curvearrowright \operatorname{Lie}(\mathcal{G})$

We say $\mathcal{G}$ <u>compatible</u> if this coincides w/ $\mathcal{O} \to A \curvearrowright \operatorname{Lie}(\mathcal{G})$

<u>Messing</u>. $\mathcal{G}[p]$ is radiciel $/A$ [We say $\mathcal{G}$ is <u>formal</u> in this case]

$\left(\begin{array}{l} \mathcal{G}[p] = \operatorname{Spec} B, \; e : B \to A \\ I = \ker(e) \subset B : \text{unipotent} \end{array} \right)$

$\Leftrightarrow \quad \mathcal{G} \simeq \operatorname{Spf} A[[T_1, \dots, T_d]]$ as fppf sheaf of sets ("formal Lie variety")

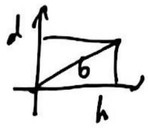
<u>Dieudonne theory</u> $\mathcal{G}/\bar{k}$ up to isog. $\overset{1:1}{\longleftrightarrow}$ K-isocrystals of slopes $\in [0,1]$

$\Uparrow$ $\overset{1:1}{\longleftrightarrow}$ Newton polygons b

<u>compatible BT $\mathcal{O}$-mods</u>

integral break points

b is <u>basic</u> (simple) if $(h,d) = 1$,



$J :=$ gp of self q-isogenies of $\mathcal{G}$ $=$ cent. div. alg. $/K$ w/ inv. $\frac{d}{h}$.

RZ-space for $\Sigma_6 / \bar{k}$: cptble BT $\mathcal{O}$-mod corresp. to 6.

$\mathcal{M} \in (\breve{\mathcal{O}}\text{-alg})^{\vee}$, $\breve{\mathcal{O}} := \widehat{\mathcal{O}^{ur}}$ ∴ CDVR w/ max. ideal $\mathfrak{p}$

$R \mapsto \mathcal{M}(R) = \{(\mathcal{G}, \rho)\}/\text{isom.}$ or ϕ if $\mathfrak{p}$ not nilp in R

$\mathcal{G}/R$: cptble BT $\mathcal{O}$-mod $/R$

$\rho : \Sigma_6 \times_{\bar{k}} R/\mathfrak{p} \xrightarrow{\text{q isog.}} \mathcal{G} \times_{\bar{k}} R/\mathfrak{p}$

Lecture 4 (Yoshida)

Complements / corrections

- $\exists$ q-isog. $\mathcal{G} \to \mathcal{G}' \Rightarrow ht_{\mathcal{O}}(\mathcal{G}) = ht_{\mathcal{O}}(\mathcal{G}')$ $\quad (\because \varpi f = f \varpi)$

$\mathcal{O}$-ht of q-isog. $f : \mathcal{G} \to \mathcal{G}'$:

$\varpi^m f$ isog $\Rightarrow ht_{\mathcal{O}}(f) = ht_{\mathcal{O}}(\varpi^m f) - \underbrace{ht_{\mathcal{O}}(\varpi^m)}_{= m\, ht_{\mathcal{O}}(\mathcal{G})}$

constant if Spec A connected

- formal scheme : X Zariski locally $\mathrm{Spf}(A, I) := \varinjlim_m \mathrm{Spec}\, A/I^m$

X_{red} : scheme. Zariski locally $(\mathrm{Spec}\, A/I)_{red} = \mathrm{Spec}\, A/\mathrm{rad}(I)$.

$X_{red}(A) = X(A)$ if A red.

- Messing $\left.\begin{array}{l}\mathcal{G}/R,\ \mathcal{G}[\mathfrak{p}] \text{ radiciel} \\ \mathfrak{p} \text{ nilp in } R\end{array}\right\} \Rightarrow \mathcal{G} \cong \mathrm{Spf}(A[[T_1, \dots, T_d]], (T_1, \dots, T_d))$

Zariski locally on R.

$K | \mathbb{Q}_p$ fin. $\quad K \supset \mathcal{O} \supset \mathfrak{p} \ni \varpi$, $k = \mathcal{O}/\mathfrak{p} \simeq \mathbb{F}_q$

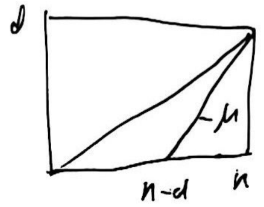
§4. RZ spaces

$n, d \in \mathbb{Z}_{>0}$, $\quad d \leq n$, $\quad (n, d) = 1$.

$\Sigma / \bar{k}$: compatible BT $\mathcal{O}$-mod of $\mathcal{O}$-height n.

dim d whose Dieudonné module is simple (single slope $\frac{d}{n}$)

--- unique up to isogeny



$\hat{\mathcal{O}} := \hat{\mathcal{O}}^{ur}$: CDVR w/ max. ideal $\mathfrak{p} := \mathfrak{p}\hat{\mathcal{O}}$

res. field $\bar{k}$

$(\hat{\mathcal{O}}\text{-nilp}) := (R : \mathfrak{p} \text{ nilpotent in } R) \subset (\hat{\mathcal{O}}\text{-alg})$.

$$\mathrm{Spf}(\hat{\mathcal{O}}, \mathfrak{p}) : R \mapsto \begin{cases} \{\cdot\} & (R \in (\hat{\mathcal{O}}\text{-nilp})) \\ \phi & (\text{o/w}) \end{cases}$$

Def **Rapoport-Zink space** for Σ is a functor $\mathcal{M} \in (\hat{\mathcal{O}}\text{-alg})^\vee$.

$$R \mapsto \mathcal{M}(R) = \begin{cases} \phi & \text{if } R \notin (\hat{\mathcal{O}}\text{-nilp}) \\ \{(\mathcal{G}, \rho)\}/_{\text{isom}} & \text{if } R \in (\hat{\mathcal{O}}\text{-nilp}) \end{cases}$$

$$\begin{cases} \mathcal{G} \text{ compatible BT } \mathcal{O}\text{-mod} / R \\ \rho : \Sigma \underset{\bar{k}}{\times} R/\mathfrak{p} \xrightarrow{\text{q-isog.}} \mathcal{G} \underset{R}{\times} R/\mathfrak{p}. \end{cases}$$

Rapoport-Zink. $\mathcal{M}$ is a formal scheme / $\mathrm{Spf}(\hat{\mathcal{O}}, \mathfrak{p})$.

Rmk. $\exists$ more general def'n.

$\mathcal{M} = \coprod_{h\in\mathbb{Z}} \mathcal{M}_h$: $\mathcal{M}_h :=$ locus where $ht_{\mathcal{O}}(\rho) = h$.

$J = \mathrm{Qisog}(\Sigma) = \{ \delta : \Sigma \to \Sigma \text{ q isog.} \}$, gp

$J = D^\times$, D/K : div. alg. of inv. $\frac{d}{n}$.

$J \curvearrowright \mathcal{M}$ by $\delta \in J$ acting as $\rho \mapsto \rho\circ\delta$. (hence $\mathcal{M}_h \xrightarrow[\sim]{\delta} \mathcal{M}_{h+ht(\delta)}$).

Recall. $L|K$ unram. of deg n $\Rightarrow$ $D = L[\Pi]$, $\Pi^n = \varpi$.

$v_D : D^\times \xrightarrow{Nrd} K^\times \xrightarrow{v} \mathbb{Z}$, $v_D(\Pi) = 1$

$D^\times = \mathcal{O}_D^\times \rtimes \Pi^{\mathbb{Z}}$, $\mathcal{O}_D$: max. order $\{v_D \geq 0\}$

$\mathcal{O}_D^\times := \ker v_D$

$\sigma := \mathrm{Frob}_{L|K}$ $\Rightarrow$ $\exists\, F \in D^\times$ w/ $v_D(F) = d$ s.t. $FxF^{-1} = \sigma(x)$ $(\forall x \in L)$.

$D^\times = J = \mathrm{Qisog}(\Sigma) \ni F : \Sigma \to \Sigma^{(q)} \simeq \Sigma$ (choose Σ/k)

rel. q-th power Frob.

$v_D = ht$, $d = ht(F)$ satisfies above.

$D = \mathrm{End}(\Sigma) \otimes \mathbb{Q} \supset \mathcal{O}_D \supset \mathrm{End}(\Sigma)$ ($\mathcal{O}$-Dieudonné theory)

$\overset{?}{=} \mathcal{O}_L[F]$ if $d \leq n-d$

$D^\times \supset \mathcal{O}_D^\times = \ker v_D = \{\text{q-isog. of } ht\ 0\}$

$\supset \mathrm{End}(\Sigma)^\times = \mathrm{Aut}(\Sigma)$.

Fact. $\mathcal{O}_D^\times = \mathrm{Aut}(\Sigma)$ if $d = 1$. In general not.

[Mieda: $\frac{d}{n} = \frac{2}{5}$, $1 + \varpi^{-1} F^3 \in \mathcal{O}_D^\times \setminus \mathrm{Aut}(\Sigma)$
$v_D = -5+6 = 1 \in \mathcal{O}_D \setminus \mathrm{End}(\Sigma)$]

This makes the $d=1$ case <u>very</u> simple.

§5. LT spaces

Def When $d=1$, RZ space $\mathcal{M}$ as above is called the Lubin-Tate space.

Lemma $\mathcal{M}_{0,red} = \operatorname{Spec} \bar{k}$.

[in general a scheme $/\operatorname{Spec} \bar{k}$ $= \operatorname{Spf}(\breve{\mathcal{O}}, p)_{red}$]

$\because$ $\mathcal{M}_{0,red}(\bar{k}) = \mathcal{M}_0(\bar{k}) = \{(G/\bar{k}, \rho)\}/\text{isom.}$

Fact implies

$$\left\{\begin{matrix} \text{q-isog.} \\ \text{ht } 0 \end{matrix} \ \Sigma \xrightarrow{\rho} G \right\} = \left\{ \text{isom. } \Sigma \xrightarrow{\rho} G \right\}$$

$\uparrow$ $\mathcal{O}_D^\times$ - torsor $\quad\quad$ $\uparrow$ $\operatorname{Aut}(\Sigma)$- torsor

$\rightarrow \mathcal{M}_0(\bar{k}) = \{\cdot\}$

A <u>reduced</u> scheme X is $\operatorname{Spec} \bar{k} \iff X(\bar{k}) = \{\cdot\}$. $\square$

As $\mathcal{M}_0$: (locally) formally of fin. presentation
(commutes w/ filtered $\varinjlim$)

$\mathcal{M}_0$ is determined by $\mathcal{M}_0|_{\mathcal{C}}$, where $\mathcal{C} = \left\{ \begin{matrix}\text{Artin. local} \\ (R, m)\end{matrix} \text{ w/ } \begin{matrix} \breve{\mathcal{O}} & \rightarrow & R \\ \downarrow & & \downarrow \\ \bar{k} & \xrightarrow{\sim} & R/m \end{matrix} \right\} \subset (\breve{\mathcal{O}}\text{-alg})$.

$\mathcal{M}_0|_{\mathcal{C}}: (R, m) \mapsto \{(G, \rho)\}/\text{isom.}$

$$\left\{\begin{array}{l} \mathcal{G}/R \\ \rho : \Sigma \times_{\bar{k}} \underbrace{R/\mathfrak{p}}_{} \xrightarrow[\text{ht } 0]{q\text{-isog}} \mathcal{G} \times_R R/\mathfrak{p} \end{array}\right\}$$

non-reduced $\bar{k}$-alg.

may not be isom.

Now use <u>Drinfeld's rigidity thm</u>: $R \in (\mathcal{O}\text{-nilp})$, $I \subset R$ nilp. ideal.

$\mathcal{G}, \mathcal{G}'$: BT $\mathcal{O}$-mod

$$\{q\text{-isog. } \mathcal{G} \to \mathcal{G}'\} \xrightarrow[\sim]{\text{bijective.}} \left\{\begin{array}{l} q\text{-isog.} \\ \mathcal{G} \times_R R/I \to \mathcal{G}' \times_R R/I \end{array}\right\}$$

(use rigidity for $I = m$)

$R \in \mathcal{C}$. $\left\{\Sigma \times_{\bar{k}} R/\mathfrak{p} \xrightarrow[\text{ht } 0]{q\text{-isog.}} \mathcal{G} \times_R R/\mathfrak{p}\right\} \xrightarrow{\sim} \left\{\Sigma \xrightarrow[\text{ht } 0]{q\text{-isog.}} \mathcal{G} \times_R \underset{\bar{k}}{\underset{\shortparallel}{R/m}}\right\}$

|| isom. (before)

<u>Prop</u> (classical def'n of LT space)

1) $\mathcal{M}_0 = \mathrm{Spf}\,(A, m)$, where (A, m) is a complete noeth. local $\hat{\mathcal{O}}$-alg.

pro-representing the deform. problem.

$$\mathcal{C}^\vee \ni \mathrm{Def}_\Sigma : (R, m) \longmapsto \{(\mathcal{G}, \rho)\}/\text{isom.}$$

$$\left[\begin{array}{l} \mathcal{G}/R \\ \rho : \Sigma \xrightarrow{\sim} \mathcal{G} \times_R R/m \end{array}\right.$$

2) Def_Σ is formally smooth (i.e. $R \twoheadrightarrow R'$ gives $\mathrm{Def}_\Sigma(R) \twoheadrightarrow \mathrm{Def}_\Sigma(R')$)

w/ $\mathrm{Def}_\Sigma(\bar{k}[t]/(t^2)) \simeq \bar{k}^{n-1}$.

This implies: $(A, m) \simeq (\hat{\mathcal{O}}[\![T_1, \dots, T_{n-1}]\!], (\mathfrak{p}, T_1, \dots, T_{n-1}))$.

3) As $D^\times = \mathcal{O}_D^\times \rtimes \pi^{\mathbb{Z}}$, choosing π gives $\pi^i : \mathcal{M}_0 \xrightarrow{\sim} \mathcal{M}_i$, $\forall i \in \mathbb{Z}$

hence $\mathcal{M} \simeq \coprod_{\mathbb{Z}} \mathrm{Spf}\,(A, m)$

2) proven by working w/ formal groups (Lubin–Tate, Drinfeld), or

crystalline Dieudonné theory.

$$\left(\begin{array}{lcc} \text{-- deform} & \mathrm{Fil} \subset & \mathbb{D}/p \\ & \wr\wr & \wr\wr \\ \text{Hodge fil.} & \bar{k}^{n-1} & \bar{k}^{n} \end{array}\right)$$

Rmk. Rigidity implies Serre–Tate.

§6. Level str.

$\Lambda \subset V$

$\Lambda := \mathcal{O}^n$, $V := K^n$, $G := \mathrm{Aut}_K(V) \simeq GL_n(K) \supset U_0 = \mathrm{Aut}_{\mathcal{O}}(\Lambda) \simeq GL_n(\mathcal{O})$

$$\mathcal{M}^{rig} = \coprod_{\mathbb{Z}} \left(\begin{array}{c} (n-1)\text{-dim'l} \\ \text{open poly disc of radius } 1 \\ \{|T_i| < 1,\ 1 \le i \le n-1\} \end{array} \right).$$

$\mathcal{G}/_{\mathcal{M}^{rig}}$ is an étale torsion $\mathcal{O}$-mod sheaf w/ stalk

$$\mathcal{G}_x \xleftarrow[\sim]{\eta} V/\Lambda \ (\simeq (K/\mathcal{O})^n) \quad \text{for } \forall x \in \mathcal{M}^{rig}(\bar{K}).$$

Note: $\mathrm{Aut}_{\mathcal{O}}(V/\Lambda) \simeq U_0$

Let $U \subset U_0$: open cpt.

$\mathcal{M}^{rig}_U :=$ (relative) moduli space of $\pi_1(\mathcal{M}^{rig}, x)$-invariant right U-orbits ηU (<u>level U str.</u>)

(choose $x \in \mathcal{M}_i$ for each i; the functor is indep. of x), up to can. isom.

Ex. $m \geq 1$, $U_m := 1 + p^m \mathrm{End}_{\mathcal{O}}(\Lambda) \triangleleft U_0$.

level U_m-str. $\longleftrightarrow$ $\eta_m : p^{-m}\Lambda/\Lambda \xrightarrow{\sim} \mathcal{G}[p^m]$.

$\Rightarrow \mathcal{M}^{rig}_{U_m}$: étale $U/U_m \simeq GL_n(\mathcal{O}/p^m)$ - covering of $\mathcal{M}^{rig}$

We have $U' \subset U \Rightarrow \mathcal{M}^{rig}_{U'} \to \mathcal{M}^{rig}_{U}$

$g : \mathcal{M}^{rig}_{U} \to \mathcal{M}^{rig}_{g^{-1}Ug}$ $\quad (g \in G)$

defined by $\eta U \mapsto \eta g (g^{-1} U g)$ $\leftarrow$ Hecke corresp.

$\to$ inverse system $\{\mathcal{M}^{rig}_{U}\}$ has action of $G \times J$.

- its ℓ-adic étale cohom. [= nearby cycles at $\mathcal{M}_{red}(\bar{k}) = \{\cdot\}$ $\underset{\mathbb{Z}}{\amalg}$]

realizes
| LLC (Harris-Taylor, Boyer, Dat)
| LJLC (Harris-Taylor, Strauch, Mieda)

- $\mathcal{M}^{rig}_{U_m}$ $(m \geq 1)$ has regular formal models (Drinfeld)

Lecture 5 (Yoshida) $K | \mathbb{Q}_p$, $K \supset \mathcal{O} \supset p \ni \varpi$, $k = \mathcal{O}/p \simeq \mathbb{F}_q$

$V = K^n \supset \mathcal{O}^n = \Lambda$, $G = Aut(V) \simeq GL_n(K)$. $U_0 = Aut(\Lambda) \simeq GL_n(\mathcal{O})$

§1. Iwahori LT space

fix a chain of lattices $\mathcal{L} : \Lambda = \Lambda_0 \subsetneq \Lambda_1 \subsetneq \Lambda_2 \subsetneq \cdots \subsetneq \Lambda_n = p^{-1}\Lambda_0$.

(lift of a flag in $p^{-1}\Lambda/\Lambda \simeq k^n$)

Iwahori subgp. $U_1 \subset U := \{g \in G : g\mathcal{L} = \mathcal{L}$, i.e. $g\Lambda_i = \Lambda_i\} \subset U_0$

$\Rightarrow \mathcal{M}^{rig}_{U} :=$ moduli of $0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_{n-1} \subsetneq \mathcal{G}[p]$

$\mathcal{G}[p] \simeq p^{-1}\Lambda/\Lambda \simeq k^n$

$\Leftrightarrow \mathcal{G} = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_{n-1} \twoheadrightarrow \mathcal{G}_n = \mathcal{G}/\mathcal{G}[p] \simeq \mathcal{G}$ chain of isog's of $\mathcal{O}$-ht 1

$\mathcal{G}_0 \xrightarrow{\varpi} \mathcal{G}$

$\mathcal{G}_i = \mathcal{G}/C_i$

naturally extends to

$\mathcal{M}_U :=$ moduli of ... over the formal scheme $\mathcal{M}$

$$\mathcal{M}_U \twoheadrightarrow \mathcal{M} \quad \text{fin. flat}$$

<u>Hecke corresp.</u>

$\mathbb{Z}[U\backslash G/U]$ acts on $\mathcal{M}_U^{rig}$ by algebraic corresp.

$$\begin{tikzcd} & \mathcal{M}^{rig}_{U\cap gUg^{-1}} \arrow[dl, "can"'] \arrow[dr, "g"] & \\ {[UgU]} : \mathcal{M}_U^{rig} \arrow[rr, dashed] & & \mathcal{M}_U^{rig} \end{tikzcd}$$

<u>Cohomology</u>: $\hat{K} := \mathrm{Frac}\, \hat{\mathcal{O}}$

(Mieda's lecture)

$$H^i(\mathcal{M}^{rig}_{\infty, \bar{\hat{K}}} ; \bar{\mathbb{Q}}_\ell) = \varinjlim_{U \text{ smaller}} H^i(\mathcal{M}^{rig}_{U, \bar{\hat{K}}}, \bar{\mathbb{Q}}_\ell) \underset{\text{of smooth}}{\circlearrowleft} G \times J$$

$$\wr$$

$$H^i(\mathcal{M}^{rig}_{U, \bar{\hat{K}}} ; \bar{\mathbb{Q}}_\ell) = H^i(\mathcal{M}_{\infty, \bar{\hat{K}}}, \bar{\mathbb{Q}}_\ell)^U \circlearrowleft \mathcal{H}_n := \bar{\mathbb{Q}}_\ell[U\backslash G/U]$$

<u>Geometry</u>. $\mathcal{M}_U = \coprod_{\mathbb{Z}} \mathrm{Spf}\, A_U$, A_U : complete local $\hat{\mathcal{O}}$-alg. fin. flat / $\hat{\mathcal{O}}[[T_1, \dots, T_{n-1}]]$.

$$\Downarrow$$

$$H^i(\mathcal{M}^{rig}_{U, \bar{\hat{K}}}, \bar{\mathbb{Q}}_\ell) = \prod_{\mathbb{Z}} (R\Psi \bar{\mathbb{Q}}_\ell)_x$$

closed pts

(fix a central char. $\Rightarrow$ concentrate on one $(R\Psi\bar{\mathbb{Q}}_\ell)_x$.)

<u>Fact</u>. (Taylor - Yoshida)

$$A_U \simeq \hat{\mathcal{O}}[[X_1, \dots, X_n]] / (\varpi - X_1 \cdots X_n)$$

where $X_i := \mathrm{Lie}(g_{i-1} \to g_i)$

$(R\Psi\mathbb{Q}_\ell)_x$ of Spf A_n is known (Rapoport - Zink, T. Saito)

§2. $R\Psi$ for semistable schemes

$n \geq 1$, $X/\mathcal{O}$ proper (strictly) semistable of rel. dim. $n-1$

--- Zariski locally étale over Spec $\mathcal{O}[X_1, \dots, X_n]/(\varpi - X_1 \cdots X_n)$.

($\Rightarrow$ X : reg. dim. n, flat $/\mathcal{O}$, $X_K := X \otimes_{\mathcal{O}} K$ smooth$/K$).

$Y := X \times_{\mathcal{O}} k = \bigcup_{i \in \Delta} D_i$, $\Delta = \{1, 2, \dots, t\}$, D_i : proper smooth $/k$, dim $n-1$

(SNC divisor on X)

- $1 \leq m \leq n$, $I \subset \Delta$, $|I| = m$.

$Y_I := \bigcap_{i \in I} Y_i$ ... proper sm. $/k$, dim $n-m$

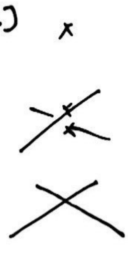
$$a_I : Y_I \underset{\text{closed immersion}}{\hookrightarrow} Y \xrightarrow{i} X \xleftarrow{j} X_K$$

$a_m := \coprod_{|I|=m} a_I$: $Y_m := \coprod_{|I|=m} Y_I \longrightarrow Y$ finite morphism.

$\mathbb{Q}_\ell^{[m]} := a_{m*}\mathbb{Q}_\ell$: sheaf $/Y$ --- constructible, constant of rk $\binom{s}{m}$ on

$Y_{(s)} := \bigcup_{|I|=s} Y_I \setminus \bigcup_{|I|>s} Y_I$

($m \leq s \leq n$)



$\mathbb{Q}_\ell^{[2]}$

$\mathbb{Q}_\ell^{[1]}$

$\mathbb{Q}_\ell$

Fact. • $i^* R^m j_* \mathbb{Q}_\ell = \mathbb{Q}_\ell^{[m]}(-m)$

• Using the resolution

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}_\ell \arrow[rr, "\partial^0"] \arrow[dr] & & \mathbb{Q}_\ell^{[1]} \arrow[rr, "\partial^1"] \arrow[dr] & & \mathbb{Q}_\ell^{[2]} \arrow[rr, "\partial^2"] \arrow[dr] & & \cdots \arrow[rr] & & \mathbb{Q}_\ell^{[n]} \arrow[r] & 0 \\
& & R^0\Psi\mathbb{Q}_\ell \arrow[ur] & & R^1\Psi\mathbb{Q}_\ell(1) \arrow[ur] & & R^2\Psi\mathbb{Q}_\ell(2) \arrow[ur] & & R^{n-1}\Psi\mathbb{Q}_\ell(n-1) \arrow[ur] & &
\end{tikzcd}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

= [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\mathrm{Sp}_m = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(1) \oplus \cdots \oplus \mathbb{Q}_\ell(m-1) \qquad \text{unip. rep'n}$$

$n=3$

$\mathbb{Q}_\ell^{[3]}[-2](-2)$		2
$\downarrow N$		
$\mathbb{Q}_\ell^{[2]}[-1](-1)$		1
$\mathbb{Q}_\ell^{[1]}$ $\downarrow N$	$\mathbb{Q}_\ell^{[3]}[-2](-1)$	0
$\mathbb{Q}_\ell^{[2]}[-1]$	$\downarrow N$	-1
	$\mathbb{Q}_\ell^{[3]}[-2]$	-2

§3. Affine Hecke modules

Iwahori – Bruhat decomp.

$$G = \coprod_{w \in W} UwU$$

W the extended affine Weyl gp

$$= \mathbb{Z}^n \rtimes \mathfrak{S}_n$$

$\mathcal{H}_n = \mathbb{Q}_\ell[U\backslash G/U]$ --- gen. by $\{S_1, \dots, S_{n-1}, T, T^{-1}\}$

$$S_i = U\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}U, \quad T = U\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \varpi & & & 0 \end{pmatrix}U$$

(in S_i the block is in columns i, $i+1$)

or by $\{S_1, \dots, S_{n-1}, \underbrace{T_1^{\pm}, \dots, T_n^{\pm}}\}$ ← gen. abelian subalg. in $\mathcal{H}_n$

$$T_i = U\begin{pmatrix} \varpi^{-1} & & & \\ & \ddots & & \\ & & \varpi^{-1} & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}U$$

(i entries ϖ^{-1})

$\mathcal{H}_m \otimes \mathcal{H}_{n-m} \xrightarrow{\sim} P_{m,n-m} \subset \mathcal{H}_n$ parabolic subalg.

$S_1, \dots, S_{m-1}, T_1, \dots, T_m \mapsto$ same thing

$S_1, \dots, S_{n-m-1}, T_1, \dots, T_{n-m} \mapsto S_{m+1}, \dots, S_{n-1}, T_{m+1}/T_m, \dots, T_n/T_m.$

$$P_{m,n-m} := \begin{pmatrix} \boxed{m} & * \\ 0 & \boxed{n-m} \end{pmatrix}.$$

$\mathcal{H}_n$: free $P_{m,n-m}$-mod. of rk $\binom{n}{m}$

$$\mathrm{Sp}_n : \begin{bmatrix} S_i \mapsto -1 \\ T \mapsto (-1)^{n-1} \\ (T_i \mapsto 1) \end{bmatrix}, \quad \mathbb{1}_n : \begin{bmatrix} S_i \mapsto q \\ T \mapsto 1 \\ (T_i \mapsto q^{i(n-i)}) \end{bmatrix}$$

--- 1-dim'l $\mathcal{H}_n$-modules obtained as U-inv. of Steinberg/trivial rep. of $G = GL_n(K)$

$$\mathrm{Ind}_{P_{m,n-m}}^{G}(\pi_1 \otimes \pi_2)^{\mathcal{U}} = \pi_1^{\mathcal{U}_m} \underline{\times} \pi_2^{\mathcal{U}_{n-m}}$$

(non-normalized)

defined as $\mathrm{Ind}_{P_{m,n-m}}^{\mathcal{H}_n}(\pi_1^{\mathcal{U}_m} \otimes \pi_2^{\mathcal{U}_{n-m}})$

$Sp_m \underline{\times} \mathbb{1}_{n-m}$ has rk $\binom{n}{m}$

$$0 \to \rho_m \to Sp_m \underline{\times} \mathbb{1}_{n-m} \to \rho_{m+1} \to 0$$

$$\updownarrow$$

$$0 \to \pi_m \to \mathrm{Ind}(Sp_m \times \mathbb{1}_{n-m}) \to \pi_{m+1} \to 0$$

$$\pi_{m+1} = Sp_m \boxplus \mathbb{1}_{n-m}$$

<u>Upshot</u>. $(R\psi\, \mathbb{Q}_\ell)_x = \bigoplus_{m=1}^{n} (Sp_m \underline{\times} \mathbb{1}_{n-m})[-(m-1)] \otimes Sp_m(-(m-1))$

"LLC"

This can be computed using (local) intersection theory.

<u>Q</u>: RZ space for $d \neq 1$?