Twistor theory and vertex algebras

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Every harmonic function on $\mathbb{R}^2$ is $\mathrm{Re}\, f(z)$, f holomorphic.

<u>Better</u>. A complex-valued harmonic func. is $\dfrac{\text{Holomorphic} + \text{Anti-Holomorphic}}{\text{constants}}$.

On $\mathbb{R}^4$, There are many complex structures.

Complex str. compatible w/ orientations are $SO(4)/U(2)$

<u>Recall</u>. $$\mathrm{Spin}(4) = SU(2) \times SU(2)$$
$$\cup$$
$$\widetilde{U(2)} = SU(2) \times U(1)$$

We see, complex str. on $\mathbb{R}^4$, compatible w/ orientation is $SU(2)/U(1) \simeq \mathbb{CP}^1$.

Penrose Transform:

Every $\mathbb{C}$-valued harmonic function on $\mathbb{R}^4$ is an $\int$ of hol. func.,

$\int$ over complex structures.

If $z \in \mathbb{CP}^1$ gives a cpx str. on $\mathbb{R}^4$. As z varies,

$\mathbb{R}^4 \times \mathbb{CP}^1$ becomes a rank 2 holomorphic vec. bundle on $\mathbb{CP}^1$.

As a cpx mfd, it is $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$, called $\mathbb{PT}$ (projective twistor space)

Give $\mathbb{R}^4 = \mathbb{C}^2$ coord. u_1, u_2,

(hol. in some reference cpx str.)

$\mathbb{PT}$ has coord. v_1, v_2, z

z on $\mathbb{CP}^1$, v_1, v_2 on $\mathcal{O}(1)$-fibers

Related to $\mathbb{R}^4$ by $v_1 = u_1 + z\bar{u}_2$, $v_2 = u_2 - z\bar{u}_1$.

If $x \in \mathbb{R}^4$, there is a corresponding $\mathbb{CP}^1$ in $\mathbb{PT} = \mathbb{R}^4 \times \mathbb{CP}^1$. Call this $\mathbb{CP}^1_x$.

<u>Thm</u> (Penrose) There is an isom. between

1) $\mathbb{C}$- valued harmonic functions on $\mathbb{R}^4$

2) $\check{H}^1(\mathbb{PT}, \mathcal{O}(-2))$

If $\alpha \in \check{H}^1(\mathbb{PT}, \mathcal{O}(-2))$, $\alpha|_{\mathbb{CP}_x} \in \check{H}^1(\mathbb{CP}^1_x, \mathcal{O}(-2))$

$$= \check{H}^1(\mathbb{CP}^1_x, K_{\mathbb{CP}^1_x})$$

The harmonic function is $f(x) = \int_{\mathbb{CP}^1_x} \alpha$

<u>Proof that f is harmonic</u>: Write $\mathbb{PT} = U \cup V$, U locus $z \neq \infty$
V locus $z \neq 0$

U, V both $\mathbb{C}^3$, $U \cap V$ $\mathbb{C}^2 \times \mathbb{C}^\times$

$\check{H}^1(\mathbb{PT}, \mathcal{O}(-2)) = \{\Gamma(v_1, v_2, z)\,dz,\ v_i \in \mathbb{C},\ z \in \mathbb{C}^\times\} /$ those Γ that extends over 0 or ∞.

$$f(u, \bar{u}) = \oint_{|z|=1} \Gamma(u_1 + z\bar{u}_2,\ u_2 - z\bar{u}_1,\ z)\,dz$$

$\Gamma(u_1 + z\bar{u}_2,\ u_2 - z\bar{u}_1,\ z)$ is harmonic.

1) Any func. on $\mathbb{R}^4$ which is hol. in <u>some</u> cpx str. is harmonic.

2). Explicitly check.

$\Rightarrow$ f is harmonic, as it is an $\int$ of harmonic things.

<u>How to prove isom</u>: compute character as rep'n of $SO(4)$.

Proof

How do we know this is surj.?

A basis for harmonic func. is $e^{p\cdot x}$ where $p \in \mathbb{C}^4$ satisfies $p\cdot p = 0$.

Fact. p is null $\Leftrightarrow$ the linear function $x \mapsto p\cdot x$ is holomorphic in SOME cpx str.

$$\frac{\text{Null vector}}{\mathbb{C}^\times} = \text{Quadric in } \mathbb{CP}^3 = \mathbb{CP}^1 \times \mathbb{CP}^1$$

One $\mathbb{CP}^1$ = cpx str., other = linear hol. funcs in that cpx. str.

$e^{p\cdot x}$ comes from a class in $H^1(\mathbb{PT}, \mathcal{O}(-2))$ which is like

$$\delta_{z=z_0}\, e^{\lambda_1 v_1 + \lambda_2 v_2}, \qquad \delta_{z=z_0} = \bar\partial\left(\frac{dz}{z-z_0}\right).$$

Ward correspondence.

There is an isom. (of stacks) between

1) holomorphic vec. bundles V on $\mathbb{PT}$ s.t. $\forall x \in \mathbb{R}^4$, $V|_{\mathbb{CP}^1_x}$ is trivial.

2) Bundles on $\mathbb{R}^4$ w/ connections satisfy self-dual Yang-Mills (SDYM) equations.

To prove this, it's useful to replace $\mathbb{R}^4$ by $\mathbb{C}^4$.

$\mathbb{C}^4 = \{\mathbb{CP}^1\text{'s in } \mathbb{PT}, \text{ linearly embedded}\}$

$\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$, lines in $\mathbb{CP}^3 = Gr(2,4)$

$\mathbb{C}^4 \subset Gr(2,4)$ is the big cell of lines that don't hit ∞.

More concretely, $\mathbb{PT} = \mathcal{O}(1)^2 \to \mathbb{CP}^1$, sections is $H^0(\mathbb{CP}^1, \mathcal{O}(1)^2) = \mathbb{C}^4$.

$x \in \mathbb{C}^4 \rightsquigarrow \mathbb{CP}^1_x \subset \mathbb{PT}$. $\mathbb{C}^4$ has a cpx linear inner product.

Lemma. $x \in \mathbb{C}^4$ is null $\Leftrightarrow$ $\mathbb{CP}^1_x$ intersects the zero section.

$x \in \mathbb{C}^4$ gives 2 sections of $\mathcal{O}(1)$

$x_1 + z x_2$

$x_3 + z x_4$

The condition that these have a common zero is $\det\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 0$.

Ex. Rank One

$H^1(\mathbb{P}\mathbb{T}, \mathcal{O})$ = Sol'ns of SDYM abelian case

$\uparrow$ $\qquad\qquad\qquad$ $\uparrow$

$H^1(\mathbb{P}\mathbb{T}, \mathcal{O}(-2))$ = Harmonic functions

$\uparrow$ $\qquad\qquad\qquad$ $\uparrow$

$H^0(\mathbb{P}\mathbb{T}, \mathcal{O}_{z=0} \oplus \mathcal{O}_{z=\infty})$ = Hol + Anti-Hol. funcs

If φ is a func. set $A = \bar\partial\varphi$ satisfies SDYM $\Leftrightarrow$ φ is harmonic

SDYM eq'ns are
$$\left[\begin{array}{l} F^{0,2} = 0 \\ F^{2,0} = 0 \\ \omega \wedge F^{1,1} = 0 \end{array}\right.$$

$$\omega \wedge \partial\bar\partial\varphi = \Delta\varphi$$

Complex bundles on $\mathbb{C}^4$ w/ a cplx analytic conn'.

Can satisfy SDYM.

For each $p \in \mathbb{C}^4$. $\{\text{null planes } \mathbb{C}^2 \subset \mathbb{C}^4\} = \mathbb{CP}^1 \amalg \mathbb{CP}^1$ has 2 components.

SDYM $\Leftrightarrow$ connection is flat for planes in one of components.

$\Lambda^2\mathbb{C}^4 = so(4, \mathbb{C}) = sl_2 \oplus sl_2$. SD means components of F in one of the sl_2's $= 0$.

If V is a hol. bundle on $\mathbb{PT}$, $V|_{\mathbb{CP}^1}$ is trivial, define a new bundle $W(V)$ on $\mathbb{C}^4$ by $W(V)_x = H^0(\mathbb{CP}^1_x, V)$

$$\begin{tikzcd} & Fl(1,2,4) \arrow[dl] \arrow[dr] & \\ \mathbb{CP}^3 & & Gr(2,4) \end{tikzcd}$$

We need to give $W(V)$ a connection.

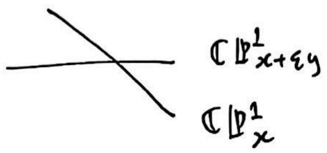
We need to give an iso. $W(V)_x \simeq W(V)_{x+\varepsilon y}$.

$T_x \mathbb{C}^4$ is spanned by null vectors

$= T(\text{null cone})$

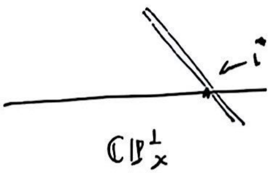
Suffices to give an iso $W(V)_x \simeq W(V)_{x+\varepsilon y}$ where y is null.

y is null $\iff$ $\mathbb{CP}^1_{x+\varepsilon y} \cap \mathbb{CP}^1_x$ is non-empty.



If $i = \mathbb{CP}^1_x \cap \mathbb{CP}^1_{x+\varepsilon y}$, $H^0(\mathbb{CP}^1_x, V) = V_i = H^0(\mathbb{CP}^1_{x+\varepsilon y}, V)$.

SDYM $\iff$ connection is flat on certain null planes.



The space of $\mathbb{CP}^1$'s in $\mathbb{PT}$ which intersect i $= H^0(\mathbb{CP}^1, \mathcal{O}(1-i) \oplus \mathcal{O}(1-i))$

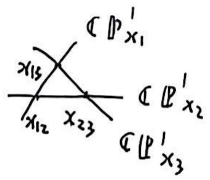
$= \mathbb{C}^2 \subset \mathbb{C}^4$

It is a null plane, as any two intersect.

On this $\mathbb{C}^2$, conn. is flat $\iff$ SDYM.

This is obvious!. $\quad H^0(\mathbb{CP}^1, \mathcal{O}(1) \otimes \mathbb{C}^2) = H^0(\mathbb{CP}^1, \mathcal{O}(1)) \otimes \mathbb{C}^2$

In $\mathbb{PT}$, consider 3 $\mathbb{CP}^1$'s that form a triangle:



Parallel transport around a null triangle

$$
= \text{map} \quad
\begin{tikzcd}
H^0(\mathbb{CP}^1_{x_1}, E) \arrow[r] & E_{x_{12}} & H^0(\mathbb{CP}^1_{x_2}, E) \arrow[l] \arrow[d] \\
E_{x_{13}} \arrow[u] & H^0(\mathbb{CP}^1_{x_3}, E) \arrow[l] \arrow[r] & E_{x_{23}}
\end{tikzcd}
$$

How to build bundles on $\mathbb{P}\mathbb{T}$?

ADHM construction : bundle is the cohomology of a complex of bundles that looks like

$$
\begin{tikzcd}
 & \mathcal{O}^k \arrow[dr, "V_2+Y(z)"] & \\
\mathcal{O}(-1)^k \arrow[ur, "V_1+X(z)"] \arrow[r, "V_2+Y(z)"'] \arrow[dr, "I(z)"'] & \mathcal{O}^k \arrow[r, "-V_1-X(z)"'] & \mathcal{O}(1)^k \\
 & \mathcal{O}^N \arrow[ur, "J(z)"'] & \\
-1 & 0 & 1
\end{tikzcd}
$$

I, J, X, Y are order 1 polys in z. This is a complex if

$$J(z) I(z) + [X(z), Y(z)] = 0.$$

$\hookrightarrow$ 3 eqn's , coeffs of $1, z, z^2$.

$H^0(-)$ is, in good cases, a rank N bundle on $\mathbb{P}\mathbb{T}$, w/ $c_2 = k$.

$\xrightarrow{\text{Ward}}$ A sol'n to SDYM eqn. for $GL(N, \mathbb{C})$.

If we want a $U(N)$ - connection, we need reality conditions.

$I = I_0 + z I_1$, J, X, Y . $I_0 = J_1^\dagger$, $X_0 = Y_1^\dagger$, $\cdots$

Before imposing reality

2 $N \times K$ matrices I_0, I_1

2 $K \times N$ matrices J_0, J_1

4 $K \times K$ matrices X_0, X_1

3 complex eqns

After imposing reality

1 $N \times K$ complex mat

1 $K \times N$

2 $K \times K$

3 real eqns.