Twistor $\mathcal{D}$-modules

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Lecture 1

Motivation: Kashiwara's conj. (mid 90's)

$f: X \to Y$ a morphism between proj. var. (or cpx Kähler mflds)

$\mathcal{F}$ a ss $\mathbb{C}$-perverse sheaf on X, i.e.

$\mathcal{F} = \bigoplus_i IC(Z_i, V_i)$ $\left[\begin{array}{l} Z_i \subset X \text{ irred.} \\ V_i \text{ ss loc. sys. on } \overset{\text{smooth}}{Z_i^\circ} \subset Z_i \end{array}\right.$

Conj. (now a theorem) $Rf_* \mathcal{F} \simeq \bigoplus_k {}^pR^k f_* \mathcal{F}[k]$ and each ${}^pR^k f_* \mathcal{F}$ is perverse semisimple on Y, and relative HLT holds.

Ex. X, Y and f smooth, and $\mathcal{F}$ a ss loc. sys on X. then $Rf_* \mathcal{F} \simeq \bigoplus_k R^k f_* \mathcal{F}[k]$ and each $R^k f_* \mathcal{F}$ is a ss loc. sys. on Y.

Idea (Simpson) There is more str. hidden on a ss loc. sys. E.g. if $Y = pt$, the Conj. reduces to HLT, but there is a stronger statement:

Thm (Simpson) In the example, the graded vec. sp. $(H^\bullet(X, \mathcal{F}), L_\omega)$ underlies a polarizable sl_2-twistor str.

Meta-theorem (Simpson) If the words "mixed Hodge str." (resp. "variation of mixed Hodge str.") are replaced by the words "mixed twistor str." (resp.

"variation of mixed twistor str.") in the hypotheses and conclusions of any theorem in Hodge theory, then one obtains a true statement. The proof of the new statement will be analogous to the proof of the old statement.

Corlette - Simpson correspondence $\quad X$ smooth proj. or Kähler

$$\left\{ \begin{array}{l} (V, \nabla) \text{ simple} \\ \text{flat bundle on } X \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} (E, \theta) \text{ stable} \\ \text{Higgs bundle on } X \\ \text{w/ all } c_i = 0 \end{array} \right\}$$

$$\Updownarrow \text{ easy}$$

$$\left\{ \begin{array}{l} \mathcal{V} \text{ simple} \\ \mathbb{C}\text{- local sys. on } X \end{array} \right\} \qquad H^\bullet(X, \mathcal{V})$$

- (Deligne) $\exists$ a family $(V^{(z)}, \nabla^{(z)})$ paremetrizes by $z \in \mathbb{C}^*$ which degenerates to (E, θ) when $z \to 0$. $\quad z = \lambda :=$ <u>twistor parameter</u>
- $H^k_{dR}(X, (V^{(z)}, \nabla^{(z)})) = H^k(X, \mathcal{V}^{(z)})$ degenerates to

$$H^k_{Dol}(X, (E, \theta)) := H^k(X, (E \otimes \Omega^\bullet_X, \wedge\theta))$$

$\rightsquigarrow$ notion of a <u>twistor structure</u>.

Complex Hodge structures (reminder)

A <u>polarized complex Hodge str.</u> of weight $w \in \mathbb{Z}$ consists of

- a finite-dim'l cpx vec. sp. H
- a (Hodge) decomp. $H = \bigoplus_{p \in \mathbb{Z}} H^{p, w-p}$
- a positive definite Hermitian pairing $h: H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ s.t. the decomp. is h-orthogonal.

Drawback. These data do not vary holomorphically.

Filtrations. $F'^{p}H = \bigoplus_{p' \geq p} H^{p', w-p'}$, $F''^{p}H = \bigoplus_{q' \geq q} H^{w-q', q'}$

$H^{p, w-p} = F'^{p}H \cap F''^{w-p}H \rightsquigarrow$ w-opposite filtrations

Sesquilinear pairings $S : H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}$ Hermitian Sesquilinear pairing s.t.

the direct sum decomp. is S-orthogonal.

Positivity: $h = \bigoplus_{p} (-1)^{w-p} S|_{H^{p,w-p}}$ is positive definite (in particular, S is

nondeg.)

Abelian. The cat. of polarizable Hodge structures of weight w is <u>abelian</u> and any

bi-filtered morphism (w.r.t. $F'^{\bullet}$, $F''^{\bullet}$) is strict.

Def. of a twistor str. (twistor str., purity, polarization, Tate object & Tate twist, etc.)

Twistor conjugation.

- Projective line $\mathbb{P}^1$ w/ fixed cpx charts $\mathbb{C}_z$, $\mathbb{C}_{z'}$ w/ coord. z, z' and $z' = z^{-1}$ on $\mathbb{C}^*_z$.
- Δ_z : the closed disc of radius 1, w/ boundary $\mathbb{S}$.
- $\mathcal{O}_{\mathbb{S}} = \mathcal{O}_{\mathbb{C}^*}|_{\mathbb{S}}$: sheaf of cpx valued <u>real analytic funcs</u> on $\mathbb{S}$
- Anti-holomorphic involution of $\mathbb{P}^1$, i.e., hol. map $\mathbb{P}^1 \longrightarrow \overline{\mathbb{P}^1}$.

$\sigma : z \mapsto -1/\overline{z}$, $\sigma \circ \sigma = Id_{\mathbb{P}^1}$

σ exchanges 0 and ∞, and $\sigma|_{\mathbb{S}} = -Id_{\mathbb{S}}$ (no fixed pt)

- For $U \subset \mathbb{P}^1$ open, $\overline{U} = \overline{\sigma(U)}$

- For $\varphi \in \mathcal{O}(U)$, set $\overline{\varphi} = \sigma^* \overline{\varphi} \in \mathcal{O}(\overline{U})$

$\overline{\varphi}(z) = \overline{\varphi(-1/\bar{z})}$ is holomorphic on $\overline{U}$. E.g. $\bar{z} = -1/z = -z'$

$\varphi(z) = \sum_{n \geq 0} a_n z^n$ on Δ_z, then $\overline{\varphi}(z') = \sum_{n \geq 0} (-1)^n \bar{a}_n z'^n$ is holomorphic

on $\overline{\Delta_z} = \Delta_{z'}$.

<u>Def</u> A <u>twistor str.</u> $\mathcal{T}$ is a holomorphic vec. bundle on $\mathbb{P}^1$

$\rightsquigarrow$ Full (not abelian) subcat. of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{O}_{\mathbb{P}^1})$

A morphism is <u>strict</u> if its cokernel is a <u>twistor structure</u>.

Naive conjugate twistor str.: $\overline{\mathcal{T}}$. conj. vect. bund. on $\overline{\mathbb{P}^1}$.

Conj. twistor str. $\overline{\mathcal{T}} = \sigma^* \overline{\mathcal{T}}$: holom. vect. bundle on $\mathbb{P}^1$.

Dual twistor structure $\mathcal{T}^\vee$: Dual bundle.

Hermitian dual twistor str. $\mathcal{T}^*$: $\sigma^* \overline{\mathcal{T}^\vee} = \overline{\mathcal{T}^\vee}$.

The Tate twistor str. For $\ell \in \mathbb{Z}$,

$$\mathbb{T}(\ell) = \mathcal{O}_{\mathbb{P}^1}(-\ell\{0\} - \ell\{\infty\}) \subset \mathcal{O}_{\mathbb{P}^1}(*(0 \cup \infty))$$

Then $\mathbb{T}(\ell) \simeq \mathcal{O}_{\mathbb{P}^1}(-2\ell)$ is pure of weight -2ℓ. Can. sect. (z^ℓ, z'^ℓ)

Pure / mixed twistor str.

<u>Def.</u> A twistor str. $\mathcal{T}$ is <u>pure of weight $w \in \mathbb{Z}$</u> if it is isom. to $\mathcal{O}_{\mathbb{P}^1}(w)^2$.

<u>Exer.</u> If $\mathcal{T}$ is pure of wt w, $\mathcal{T}^*$ is pure of weight ??

Abelianity The full subcat. of $\mathrm{Mod}_{coh}(\mathcal{O}_{\mathbb{P}^1})$ consisting of pure twistor str. of weight w is abelian.

Exer. $\mathcal{T}, \mathcal{T}'$ pure of weight w, w', any morphism $\mathcal{T} \to \mathcal{T}'$ is zero if $w > w'$.

Tate twist. $\mathcal{T}(\ell) = \mathcal{T} \otimes \mathbb{T}(\ell)$.

$\mathcal{T}$ of weight $w \iff \mathcal{T}(\ell)$ of weight $w - 2\ell$.

Reduction to weight 0. Set $\mathcal{U}(0,\ell) = \mathcal{O}_{\mathbb{P}^1}(-\ell\{\infty\}) =$ "$\mathbb{T}(\ell/2)$"

Then $\mathcal{T}$ is pure of wt $w \Rightarrow \mathcal{T} \otimes \mathcal{U}(0,w)$ is pure of weight 0.

(We have a half Tate twist in this cat.)

Def. A mixed twistor str. is a W-filtered ~~twistor~~ twistor str. $(\mathcal{T}, W_\bullet\mathcal{T})$ s.t.

$\mathrm{gr}^W_\ell \mathcal{T}$ is pure of weight ℓ for each $\ell \in \mathbb{Z}$.

Example. $\mathcal{T} = \mathcal{O}^2_{\mathbb{P}^1} = W_1(\mathcal{T})$, $W_{-1}\mathcal{T} = \mathcal{O}_{\mathbb{P}^1}(-1)$.

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$$

Polarization of a pure twistor str.

Def A sesquilinear pairing of weight w on a twistor str. $\mathcal{T}$ is an $\mathcal{O}_{\mathbb{P}^1}$-linear pairing

$$S: \mathcal{T} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \overline{\mathcal{T}} \to \mathbb{T}(-w)$$

equiv., $S: \mathcal{T} \to \mathcal{T}^*(-w)$.

- $S^*: \mathcal{T}(w) \to \mathcal{T}^* \rightsquigarrow S^*: \mathcal{T} \to \mathcal{T}^*(-w)$
- S is Hermitian if $S^* = S$.
- S is non-deg. if $S: \mathcal{T} \xrightarrow{\sim} \mathcal{T}^*(-w)$ is an isom.

- A <u>pre-polarization of weight w</u> is a <u>non-deg. Hermitian sesquilinear pairing</u> of weight w.

<u>Example</u>. $\mathcal{U}(0,\ell) \xrightarrow{\sim} \mathcal{U}(0,\ell)^* \otimes \mathbb{T}(\ell)$. So S induces a pre-polarization of weight $\underline{0}$.

$$S_0 : (\mathcal{T} \otimes \mathcal{U}(0,w)) \otimes \overline{(\mathcal{T} \otimes \mathcal{U}(0,w))} \longrightarrow \mathbb{I}(0) = \mathcal{O}_{\mathbb{P}^1}$$

<u>Def</u>. A polarization of a pure twistor str. $\mathcal{T}$ of weight w is a pre-polarization of weight w s.t.
$$H^0(S_0) : \underbrace{H^0(\mathbb{P}^1, \mathcal{T} \otimes \mathcal{U}(0,w))}_{\cong\ \mathbb{C}^2} \otimes \overline{H^0(\mathbb{P}^1, \mathcal{T} \otimes \mathcal{U}(0,w))} \longrightarrow \mathbb{C}$$
is positive definite.

What does a polarized twistor str. look like?

The cat. of polarized twistor str. of weight w (the morphism being the morphisms of twistor str.) is equiv. to the cat. of $\mathbb{C}$-vec. sp. w/ a positive def. Hermitian form (the morphisms being all linear maps)

<u>Functor</u>: $(\mathcal{T}, S) \mapsto (H^0(\mathbb{P}^1, \mathcal{T} \otimes \mathcal{U}(0,w)), H^0(S_0))$

<u>Cor</u>. Let $\mathcal{T}_1$ be a subtwistor str. (pure of weight w) of the polarized twistor str. $(\mathcal{T}, S)$ (the weight is fixed)

Then S induces a polarization on the subtwistor, which is a direct summand of $(\mathcal{T}, S)$.

<u>Motivation</u>.

- Variation of $\mathcal{T}$ parametrized by a cpx mfd X: $C^\infty_X \mathcal{O}_{\mathbb{P}^1}$-bundle

 Bad for coherence properties.

Proof

$\rightsquigarrow$ Start from $\mathcal{M}', \mathcal{M}''$ on the chart $\mathbb{C}_z$, and $\mathcal{T}$ obtained by gluing $\mathcal{M}'$ and $\sigma^* \mathcal{M}'' = \overline{\mathcal{M}''}$. Then $\mathcal{M}', \mathcal{M}''$ vary holomorphically, analogous to $F' H, F'' H$ in Hodge theory.

But near singularities of the variation, the gluing can degenerate badly.

$\rightsquigarrow$ Replace the gluing by a <u>pairing</u> that could be degenerate.

i.e. start from $\mathcal{M}', \mathcal{M}''$ and "glue" $\mathcal{M}'$ w/ $\overline{\mathcal{M}''}^{\vee} = \mathcal{M}''^{*}$.

$$\mathcal{M}' \otimes \overline{\mathcal{M}''} \rightarrow \cdots$$

<u>Def. of a triple</u>

- A <u>triple</u> $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ consists of coherent sheaves $\mathcal{M}', \mathcal{M}''$ on some nbhd of Δ_z and of a sesquilinear pairing $C : \mathcal{M}'|_{\mathbb{S}} \otimes_{\mathcal{O}_{\mathbb{S}}} \overline{\mathcal{M}''}|_{\mathbb{S}} \longrightarrow \mathcal{O}_{\mathbb{S}}$
- A morphism $\varphi : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ is a pair (φ', φ''), w/
  - $\varphi' : \mathcal{M}_1' \rightarrow \mathcal{M}_2'$ (covariant)
  - $\varphi'' : \mathcal{M}_2'' \rightarrow \mathcal{M}_1''$ (contravariant)

  s.t. $C_1(m_1', \overline{\varphi''(m_2'')}) = C_2(\varphi'(m_1'), \overline{m_2''})$ on $\mathbb{S}$

- $\rightsquigarrow$ abelian cat. of triples

Example. $\ker \varphi = (\ker \varphi', \operatorname{coker} \varphi'', C_1)$

well-defined since $C_1(\ker \varphi', \overline{\operatorname{Im} \varphi''}) = 0$.

Hermitian dual triple $\mathcal{T}^*$: $(\mathcal{M}', \mathcal{M}'', C)^* = (\mathcal{M}'', \mathcal{M}', C^*)$ w/

$$C^*(m'', \overline{m'}) = \overline{C(m', \overline{m''})}$$

$$\varphi = (\varphi', \varphi'') : \mathcal{T}_1 \to \mathcal{T}_2, \quad \varphi^* = (\varphi'', \varphi') : \mathcal{T}_2^* \to \mathcal{T}_1^*$$

$$\varphi' : \mathcal{M}_1' \to \mathcal{M}_2', \quad \varphi'' : \mathcal{M}_2'' \to \mathcal{M}_1''.$$

Check $C_2^*(m_2'', \overline{\varphi'(m_1')}) \overset{?}{=} C_1^*(\varphi''(m_2''), \overline{m_1'})$ on $\mathbf{S}$. ✓

The Tate triple
$$\begin{cases} \mathbb{T}(\ell) = (z^\ell \mathcal{O}_{\mathbb{C}_z}, z^{-\ell}\mathcal{O}_{\mathbb{C}_z}, t_\ell) \\ t_\ell(z^\ell \cdot 1, \overline{z^{-\ell}\cdot 1}) = (-1)^\ell z^{2\ell} \qquad (z^\ell \cdot \overline{z^{-\ell}} \text{ on } \mathbf{S}) \end{cases}$$

Half-Tate triple
$$\begin{cases} \mathcal{U}(0,\ell) = (\mathcal{O}_{\mathbb{C}_z}, z^{-\ell}\mathcal{O}_{\mathbb{C}_z}, u_\ell) \\ u_\ell(1, \overline{z^\ell \cdot 1}) := \overline{z^{-\ell}} = (-1)^\ell z^\ell \end{cases}$$

Pre-polarization of weight w : $S : \mathcal{T} \xrightarrow{\sim} \mathcal{T}^*(-w)$, i.e. if $\underline{w=0}$,

$$S = (S', S'') : (\mathcal{M}', \mathcal{M}'', C) \xrightarrow{\sim} (\mathcal{M}'', \mathcal{M}', C^*)$$

$$S' : \mathcal{M}' \xrightarrow{\sim} \mathcal{M}'', \quad S'' : \mathcal{M}' \xrightarrow{\sim} \mathcal{M}'', \quad S'' = S'.$$

$$C(m', \overline{S''(n')}) = C^*(S'(m'), \overline{n'}) = \overline{C(n', \overline{S'(m')})}$$

i.e. the pairing $(m', n') \mapsto C(m', \overline{S'(n')})$ is twistor - Hermitian.

<u>Twistor str. as triples</u>. A triple $\mathcal{T}$ defines a twistor str. if $\mathcal{M}', \mathcal{M}''$ are vec. bundles in the nbhd of Δ_z and C defines a gluing between $\mathcal{M}'$ and $\mathcal{M}''^* = \overline{\mathcal{M}''}^\vee$, i.e. is a non-deg. pairing.

- $\rightsquigarrow$ pure twistor str. of pure weight w as a triple $(\mathcal{M}', \mathcal{M}'', C)$.

- Assume $w=0$. Any such $(\mathcal{T}, S)$ is isom. to $(\mathcal{T}_0, S_0)$ w/

$\mathcal{T}_0 = (\mathcal{M}', \mathcal{M}', C_0(\cdot, \bar{\cdot}))$ w/ $C_0^* = C_0$ and $S_0 = (\mathrm{Id}, \mathrm{Id})$).

s.t. $\exists$ an orthonormal basis of $\mathcal{M}'$, i.e. a frame ε of $\mathcal{M}'$ s.t. $\varepsilon|_S$ is orthonormal

w.r.t. $C_0(\cdot, \bar{\cdot})$

- If w is arbitrary.
  - Start from $(\mathcal{T}_0, S_0)$ polarized of wt 0,
  - Set $\mathcal{T}_w = (\mathcal{M}', z^w \mathcal{M}', C_w)$ w/ $C_w(m', \overline{z^w n'}) = \overline{z^w}\, C_0(m', \overline{n'})$
  - Check $S_w = (\mathrm{Id}, \mathrm{Id})$ is a pre-polar. of weight w of $\mathcal{T}_w$: ✓

$\rightsquigarrow$ <u>Def.</u> $(\mathcal{T}, S)$ pol. pure of wt w iff isom. to some $(\mathcal{T}_w, (\mathrm{Id}, \mathrm{Id}))$

Hodge str. as triples

Recall $F'^{p}(H) = \bigoplus_{p' \geq p} H^{p, w-p}$, $F''^{q}(H) = \bigoplus_{q' \geq q} H^{w-q, q}$

Define $F''^{p}(H^*)$ s.t. $(H^*)^{p,q} = (H^{-q,-p})^*$

Set $\mathcal{M}' = \bigoplus_p F'^{p}(H)\, z^{-p} \subset \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} H$ $\qquad$ $\mathbb{C}[z]$-module

$\mathcal{M}'' = \bigoplus_q F''^{q}(H^*)\, z^{-q} \subset \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} H^*$

So $\mathcal{M}' = \bigoplus_p \left[H^{p, w-p} z^{-p}\, \mathbb{C}[z] \right]$ (finite sum)

$\mathcal{M}'' = \bigoplus_q \left[(H^*)^{q, -w-q} z^{-q}\, \mathbb{C}[z] \right]$

$= \bigoplus_q \left[(H^{p, w-p})^* z^{w-p}\, \mathbb{C}[z] \right]$ (finite sum)

$\overline{\mathcal{M}''} = \bigoplus_p \left[(H^{p,w-p})^\vee (z^{-1})^{w-p}\, \mathbb{C}[z^{-1}] \right]$ (finite sum)

- $\mathcal{M}'|_S \simeq \mathcal{O}_S \otimes_{\mathbb{C}} H$, $\overline{\mathcal{M}''}|_S \simeq \mathcal{O}_S \otimes_{\mathbb{C}} H^\vee$
- C induced by $\langle \cdot, \cdot \rangle : H \otimes H^\vee \to \mathbb{C}$.

. <u>Polarization when $w=0$</u>. A polarization of H is a Hermitian isom. $S : H \xrightarrow{\sim} H^*$ which induces for each p isom. $H^{p,-p} \xrightarrow{\sim} (H^{p,-p})^*$ s.t. there exists a basis ε_p satisfying $\langle \varepsilon_p, \overline{(-1)^p S \varepsilon_p} \rangle = Id$.

<u>Integrable twistor str.</u>

<u>Motivation</u>. get more numerical invts.

- Irregular Hodge theory [Deligne, Sabbah-Yu)
- Exponential mixed Hodge structures (Kontsevich -Soibelman, Fresán - Sabbah - Yu, Y. Qin)

<u>Def</u> An <u>int. twistor str.</u> is a pair $(\mathcal{T}, \nabla)$ where $\mathcal{T}$ is a twistor str. (vect. bdle on $\mathbb{P}^1$) and ∇ is a meromorphic conn'n w/ a pole of <u>order ≤ 2</u> at 0 & ∞, and no other pole.

An <u>integrable triple</u> is a triple $((\mathcal{M}', \nabla), (\mathcal{M}'', \nabla), C)$ where ∇ has a pole of order ≤ 2 at 0 and no other pole, and C is compatible w/ ∇, i.e.

$$z\partial_z C(m', \overline{m''}) = C(\nabla_{z\partial_z} m', \overline{m''}) - C(m', \overline{\nabla_{z\partial_z} m''}). \quad (\overline{z\partial_z} = -z\partial_z)$$

<u>Irregular Hodge no.</u> Given $(\mathcal{M}', \nabla)$ in an integrable triple, consider the Deligne meromorphic extension at ∞.

$\rightsquigarrow (\widetilde{\mathcal{M}'}, \widetilde{\nabla})$ on $\mathbb{P}^1$ w/ a regular sing. at ∞

$\rightsquigarrow$ Deligne's filtration $V^\bullet \widetilde{\mathcal{M}'}$: on $V^\gamma \widetilde{\mathcal{M}'}$, $\widetilde{\nabla}$ has a log pole at ∞ w/ residue having real part in $[\gamma, \gamma+1)$

$\rightsquigarrow$ Birkhoff – Grothendieck decomp. $V^\circ \widetilde{\mathcal{M}}' \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$,

$a_1 \geq a_2 \geq \cdots$

- $v_\gamma = \#\{i : a_i \geq 0\}$, $\nu_\gamma = v_\gamma - v_{>\gamma}$.

$\rightsquigarrow$ Irregular Hodge filtration: jumping no. γ w/ multiplicity ν_γ.

Lecture 2. Variations of twistor structure

Harmonic flat bundles

- X cpx mfd,
- (H, D) : flat C^∞ bundle on X, $D = \overset{(1,0)}{D'} + \overset{(0,1)}{D''}$, $(V, \nabla) = (\ker D'', D')$ — flat holomorphic bundle
- h: a Hermitian metric on H

Lemma. $\exists!$ connection $D_E = D_E' + D_E''$ on H and $\exists!$ C^∞-linear morphism $\theta = \theta' + \theta''$: $\mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$

s.t.

(1) D_E is compatible w/ h, i.e.

$$d' h(u, \bar{v}) = h(D_E' u, \bar{v}) + h(u, \overline{D_E'' v}), \quad d'' h(u, \bar{v}) = h(D_E'' u, \bar{v}) + h(u, \overline{D_E' v})$$

(2) θ is self-adjoint w.r.t. to h, i.e. $h(\theta u, \bar{v}) = h(u, \overline{\theta v})$, i.e.

$$h(\theta' u, \bar{v}) = h(u, \overline{\theta'' v}), \quad h(\theta'' u, \bar{v}) = h(u, \overline{\theta' v})$$

(3) $D = D_E + \theta$, i.e. $D' = D_E' + \theta'$, $D'' = D_E'' + \theta''$.

Def. (H, D, h) is harmonic if $(D_E'' + \theta')^2 = 0$.

$\Rightarrow$ a lot of identities, e.g. $(D_E'')^2 = 0$, $D_E''(\theta') = 0$, $\theta' \wedge \theta' = 0$.

$\rightsquigarrow$ $(E, \theta) = (\ker D_E'', \theta')$ holomorphic Higgs bundle.

$\forall z \neq 0$, $\left(V^{(z)}, \nabla^{(z)}\right) = \left(\ker\left(D_E'' + z\,\theta_E''\right),\ z D_E' + \theta_E'\right)$ holom. bundle w/ flat z-connection.

Trivial example If h is compatible w/ D, i.e. h is a flat Hermitian metric, then

$D_E = D$, $\theta = 0$, $E = V$

Recall.

Theorem. If X is smooth, compact Kähler, then

(Corlette) (H, D, h) harmonic $\Rightarrow$ (V, ∇) semisimple

(Simpson) (V, ∇) semisimple $\rightarrow$ $\exists$ harmonic metric. (almost unique)

Flat z-connection

(V, ∇, h) holom. flat bundle w/ harmonic metric

$\rightsquigarrow$ $(\mathcal{V}, \nabla^{(z)})$ holom. bundle on $X \times \mathbb{C}_z = \mathcal{X}$ w/ a flat z-connection.

Def. $\nabla^{(z)}$ is a flat z-connection on the vec-bundle $\mathcal{V}$ on $\mathcal{X}$, i.e.

- is $\mathcal{O}_{\mathbb{C}_z}$-linear
- satisfies the z-Leibniz rule for $\varphi \in \mathcal{O}_{\mathcal{X}}$:

$$\nabla^{(z)}\left(\varphi(x,z)\cdot m\right) = z\, d_X\varphi \otimes m + \varphi \cdot \nabla^{(z)}(m)$$

- and (flatness) $\left(\nabla^{(z)}\right)^2 = 0$.

Rmk When restricted to $X \times \mathbb{C}_z^*$, $\frac{1}{z}\nabla^{(z)}$ is a flat relative connection on $\mathcal{V}$.

Twistor conjugation. For $\varphi \in \Gamma(X \times U, \mathcal{O}_{X \times \Delta_z})$, define

$$\overline{\varphi} = \sigma^* \overline{\varphi} \in \Gamma(\overline{X} \times \overline{U}, \mathcal{O}_{\overline{X} \times \overline{\Delta_z}}).$$

If $\mathcal{V}$ is $\mathcal{O}_{X \times \Delta_z}$-coherent w/ a flat z-connection $\nabla^{(z)}$, then $(\overline{\mathcal{V}}, \overline{\nabla}^{(z)})$

is $\mathcal{O}_{\overline{X} \times \overline{\Delta_z}}$-coherent w/ flat $\overline{z}$-connection.

Smooth twistor structures

Def. A smooth triple $\mathcal{T}$ is a triple $((\mathcal{H}', \nabla^{(z)}), (\mathcal{H}'', \nabla^{(z)}), C)$

where $(\mathcal{H}', \nabla^{(z)}), (\mathcal{H}'', \nabla^{(z)})$ are holomorphic vec. bundle on $\mathcal{X}$ w/ flat z-conn.

and C is a sesquilinear pairing

$$C: \mathcal{H}'|_{\mathbb{S}} \otimes \overline{\mathcal{H}''}|_{\mathbb{S}} \to \mathcal{C}^{\infty, an}_{X \times \mathbb{S}} \qquad \mathbb{S} = \mathbb{C}^* \text{ or } \mathbb{S} = S^1$$

Compatible w/ the connections:

$$\begin{cases} z\, d'_X\, C(m', \overline{m''}) = C(\nabla^{(z)} m', \overline{m''}) \\ \overline{z}\, d''_X\, C(m', \overline{m''}) = C(m', \overline{\nabla^{(z)} m''}) \end{cases}$$

s.t. for each $x_0 \in X$, $i^*_{x_0}(\mathcal{H}', \mathcal{H}'', C)$ is a twistor structure on $\{x_0\} \times \mathbb{P}^1$.

i.e. C_{x_0} is non-degenerate for any x_0, i.e. C is non-degenerate.

$\rightsquigarrow$ vec. bundle on $X \times \mathbb{P}^1$

C^∞ for X, hol. for $\mathbb{P}^1$

- A <u>pre-polarization</u> S of weight w on a smooth twistor structure

$\mathcal{T} = ((\mathcal{H}', \nabla^{(z)}), (\mathcal{H}'', \nabla^{(z)}), C)$ is an isom. $S : \mathcal{T} \xrightarrow{\sim} \mathcal{T}^*(-w)$ s.t. $S^* = S$.

- $\mathcal{T}$ is <u>pure of weight w</u> if each $\mathcal{T}_{x_0}$ is pure of weight w.

- S is a <u>polarization</u> of $\mathcal{T}$ if each S_{x_0} is a polarization of $\mathcal{T}_{x_0}$.

<u>Example on the punctured disc.</u>

- Δ_t^* punctured disk w/ coord. t
- $(V, \nabla) = (\mathcal{O}_{\Delta^*}, d + d(1/t))$
- $(V^{(z)}, \nabla^{(z)}) = (\mathcal{O}_{\Delta^* \times \mathbb{C}_z}, zd + d(1/t))$
- $C(1, \bar{1}) = \exp(1/tz) \cdot \overline{\exp(1/tz)} = \exp\left(1/tz - z/\bar{t}\right)$

For fixed $z \in \mathbb{C}^*$, this extends as a distribution on Δ_t iff $z \in \mathbb{S}$.

$\rightsquigarrow$ For the general def'n of a wild twistor $\mathcal{D}$-modules, one needs to restrict C to $\mathbb{S}$.

But for regular twistor $\mathcal{D}$-modules, one can assume that C is defined on

$X \times \mathbb{C}_z^*$ and takes values in $\mathcal{C}^{\infty, an}_{X \times \mathbb{C}^*}$.

<u>Lemma</u> Equivalence between

- smooth polarized twistor structure of weight 0 on X
- flat holomorphic bundle (V, ∇) on X w/ a harmonic metric h.
- holomorphic Higgs bundle (E, θ) w/ a harmonic metric h.

Harmonic flat bundle $\Rightarrow$ pol. var. of pure twistor structure, $w=0$

Start (H, D'_V, D''_V, h) flat C^∞-bundle w/ harmonic metric.

$\rightsquigarrow$ (H, D''_E, θ, h) Higgs bundle w/ harmonic metric

also get D'_E, θ', θ''.

$\mathcal{H} := \mathcal{C}^{\infty, an}_{\mathcal{X}} \otimes_{\mathcal{C}^\infty_X} H.$

Convenient to set $\Omega^1_{\mathcal{X}} := z^{-1} \Omega^1_{X \times \mathbb{C}_z / \mathbb{C}_z}$. corresp. to Griffiths transversality

Define $D'_{\mathcal{H}} = D'_E + z^{-1}\theta'_E : \mathcal{H} \to \Omega^1_{\mathcal{X}} \otimes \mathcal{H}$

On the other hand. $D''_{\mathcal{H}} = D''_E + z\,\theta''_E : \mathcal{H} \to \Omega^1_{\bar{X} \times \mathbb{C}_z / \mathbb{C}_z} \otimes \mathcal{H}$

- $(\mathcal{H}, D'_{\mathcal{H}}, D''_{\mathcal{H}})|_{z=1} = (H, D'_V, D''_V)$
- $(\mathcal{H}, D'_{\mathcal{H}}, D''_{\mathcal{H}})|_{z=0} := (\mathcal{H}|_{z=0}, \mathrm{Res}_{z=0} D'_{\mathcal{H}}, D''_{\mathcal{H}}|_{z=0}) = (H, \theta'_E, D''_E)$

Check $(D''_{\mathcal{H}})^2 = 0$, and set $\mathcal{H}' = \ker D''_{\mathcal{H}}$, equipped w/

$$D'_{\mathcal{H}} : \mathcal{H}' \to \Omega^1_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{H}'.$$

$\rightsquigarrow$ $z D'_{\mathcal{H}}$ is a flat z-connection.

- Set $\mathcal{H}'' = \mathcal{H}'$ and $S = (\mathrm{Id}, \mathrm{Id})$

<u>Definition of C</u>.

Regard h as a C^∞_X-linear morphism $H \otimes_{\mathcal{C}^\infty_X} \overline{H} \to C^\infty_X$

On $\overline{H}$, consider $D'_{\overline{E}} := \overline{D''_E}$, $D''_{\overline{E}} := \overline{D'_E}$, $\theta'_{\overline{E}} := \overline{\theta''_E}$, $\theta''_{\overline{E}} := \overline{\theta'_E}$

$\rightsquigarrow$ For local sections u, v of H,

$$d' h(u, \bar{v}) = h(D'_E u, \bar{v}) + h(u, D'_{\bar{E}} \bar{v}), \quad h(\theta'_E u, \bar{v}) = h(u, \theta'_{\bar{E}} \bar{v})$$

Extend h by $\mathcal{C}^{\infty,an}_X|_{\mathcal{S}}$ - linearity as $h_{\mathcal{S}} : \mathcal{H}|_{\mathcal{S}} \otimes_{\mathcal{C}^{\infty,an}_{\mathcal{X}}|_{\mathcal{S}}} \overline{\mathcal{H}}|_{\mathcal{S}} \to \mathcal{C}^{\infty,an}_{\mathcal{X}}|_{\mathcal{S}}$

- Set $D'_{\overline{\mathcal{H}}} = \overline{D''_{\mathcal{H}}}$, $D''_{\overline{\mathcal{H}}} = \overline{D'_{\mathcal{H}}}$, then

$$d'_X h_{\mathcal{S}} (1\otimes u, \overline{1\otimes v}) = h_{\mathcal{S}}\left(D'_{\mathcal{H}}(1\otimes u), \overline{1\otimes v}\right) + h_{\mathcal{S}}\left(1\otimes u, D'_{\overline{\mathcal{H}}}(\overline{1\otimes v})\right)$$

$$d''_X h_{\mathcal{S}} (1\otimes u, \overline{1\otimes v}) = h_{\mathcal{S}}\left(D''_{\mathcal{H}}(1\otimes u), \overline{1\otimes v}\right) + h_{\mathcal{S}}\left(1\otimes u, D''_{\overline{\mathcal{H}}}(\overline{1\otimes v})\right)$$

- Set $C = h_{\mathcal{S}}|_{\mathcal{H}'|_{\mathcal{S}}} : \mathcal{H}'|_{\mathcal{S}} \otimes_{\mathcal{O}_{\mathcal{S}}} \overline{\mathcal{H}'}|_{\mathcal{S}} \longrightarrow \mathcal{C}^{\infty,an}_{\mathcal{X}}|_{\mathcal{S}}$

Various checks

...

The Hodge – Simpson theorem

Pushforward of a smooth twistor structure by the constant map.

- X cpt cpx mfd, $f : X \to pt$

- $\mathcal{T} = ((\mathcal{H}', D'_{\mathcal{H}}), (\mathcal{H}', D'_{\mathcal{H}}), C)$ smooth triple on X pure, polar. $S = (\mathrm{Id}, \mathrm{Id})$, wt 0.

- $\mathcal{E}^{p,q}_{\mathcal{X}} = z^{-p} \cdot \mathcal{C}^{\infty,an}_{X \times \mathbb{C}_z} \otimes \mathcal{E}^{p,q}_X$

De Rham complex $^p DR(\mathcal{H}', D'_{\mathcal{H}}) = (\Omega^\bullet_{\mathcal{X}} \otimes \mathcal{H}', D'_{\mathcal{H}})[n]$ ($n = \dim X$)

$\simeq (\mathcal{E}^\bullet_{\mathcal{X}} \otimes \mathcal{H}, D_{\mathcal{H}})[n]$ Dolbeault lemme

Pushforward $T f_*^{(j)} \mathcal{T}$:

$$\left(H^{j+n}(X, \Omega_{\mathcal{X}}^{\bullet} \otimes \mathcal{H}', D'_{\mathcal{H}}), H^{-j+n}(X, \Omega_{\mathcal{X}}^{\bullet} \otimes \mathcal{H}', D'_{\mathcal{H}}), C_j \right)$$

$$\wr\!\mid$$

$$\left(H^{j+n}(X, \mathcal{E}_{\mathcal{X}}^{\bullet} \otimes \mathcal{H}, D_{\mathcal{H}}), H^{-j+n}(X, \mathcal{E}_{\mathcal{X}}^{\bullet} \otimes \mathcal{H}, D_{\mathcal{H}}), C_j \right)$$

C_j naturally deduced from C w/ suitable constants.

Thm X cpt Kähler mfd, ω = Kähler class, $L_\omega = \omega \wedge$, $f: X \to pt$ constant map

$(\mathcal{T}, S)$ pure polarized smooth twistor str., of wt 0 on X, then

$\left(\bigoplus_j T f_*^{(j)} (\mathcal{T}, S), L_\omega \right)$ is a polarized sl_2-twistor structure.

Sketch of the proof

Will show that $H^{j+n}(X, \mathcal{E}_{\mathcal{X}}^{\bullet} \otimes \mathcal{H}, D_{\mathcal{H}})$ is strict, i.e. locally $\mathcal{O}_{\mathfrak{z}}$-free.

- Start from (H, D_V, h) harmonic $\rightsquigarrow$ $D'_E, D''_E, \theta'_E, \theta''_E$
- Set $\mathcal{D}_0 = D''_E + \theta'_E$, $D_\infty = D'_E + \theta''_E$ (harmonic $\Rightarrow$ $D_0^2 = 0$, $D_\infty^2 = 0$)
- Set $D_{\mathfrak{z}} = D_0 + \mathfrak{z} D_\infty$ $(= \mathfrak{z} D'_H + D''_H)$, $D_{\mathfrak{z}}^2 = 0$, $\forall \mathfrak{z}$
- Kähler identities $\Delta_{D_V} = 2\Delta_0 = 2\Delta_\infty$, $\Delta_{D_{\mathfrak{z}_0}} = (1 + |\mathfrak{z}_0|^2) \Delta_{D_0}$

and L_ω commutes w/ them.

$\Rightarrow$ The spaces $\mathrm{Harm}^j_{\mathfrak{z}_0}(H)$ of $\Delta_{D_{\mathfrak{z}_0}}$-harmonic sections is <u>indep. of $\mathfrak{z}_0$</u>, and

$\mathrm{Harm}^{\bullet}_{\mathfrak{z}_0}(H)$ = cohom. of $\Gamma(X, (\mathcal{E}_X^{\bullet} \otimes H, D_{\mathfrak{z}_0}))$.

$\Rightarrow H^{j+n}(X, \mathcal{E}_{\mathcal{X}}^{\bullet} \otimes \mathcal{H}, D_{\mathcal{H}}) \simeq \mathcal{O}_{\mathbb{C}_{\mathfrak{z}}} \otimes_{\mathbb{C}} \mathrm{Harm}^{j+n}_{\mathfrak{z}_0}(H)$.

The remaining part of the proof is analogous to that in Hodge theory.

Lecture 3. $\mathcal{R}$-modules, $\mathcal{R}$-triples

X cpx mfd. $\mathcal{X} = X \times \mathbb{C}_z$, $\pi: \mathcal{X} \to X$

- Basic objects in the theory of Hodge modules: holonomic D_X-modules M w/ a coherent filtration $F_\bullet M$.
(good)

$\rightsquigarrow$ Rees module $R_F M = \bigoplus_p F_p M z^p$ coherent graded module over the graded Rees ring $R_F D_X$.

$F_\bullet D_X$: order filtration

- $R_F D_X = \bigoplus_p F_p D_X \cdot z^p$. Locally, $R_F D_X = \mathcal{O}_X[z]\langle z\partial_{x_1}, \cdots, z\partial_{x_n}\rangle$

Set $\eth_{x_i} = z\partial_{x_i}$, so that $[\eth_{x_i}, \varphi] = z\,\partial\varphi/\partial x_i$.

For twistor theory, relax the grading and analytify w.r.t. z.

$\rightsquigarrow$ locally $\mathcal{R}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}\langle \eth_{x_1}, \cdots, \eth_{x_n}\rangle$

Recall.

Left D_X-module $\iff$ $\mathcal{O}_X$-module w/ a flat conn'n $\nabla: M \to \Omega^1_X \otimes M$, $\nabla^2 = 0$

Similarly,

left $\mathcal{R}_{\mathcal{X}}$-module $\iff$ $\mathcal{O}_{\mathcal{X}}$-module w/ a $\underbrace{\text{flat } z\text{-connection } \nabla^{(z)}: \mathcal{M} \to \Omega^1_{\mathcal{X}} \otimes \mathcal{M}}_{\mathcal{O}_{\mathbb{C}^*_z} \text{ linear morphism}}$, $(\nabla^{(z)})^2 = 0$.

$\Omega^1_{\mathcal{X}} := z^{-1}\pi^*\Omega^1_X$

Recall $gr^F D_X \simeq \mathcal{O}_X[TX]$

Similarly, $gr^F \mathcal{R}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}[\pi^* TX]$

Recall

$R_F \mathcal{D}_X / (z-1) R_F \mathcal{D}_X \simeq \mathcal{D}_X$ and $R_F \mathcal{D}_X / z R_F \mathcal{D}_X \simeq gr^F \mathcal{D}_X \simeq \mathcal{O}_X[TX]$

Similarly,

$\mathcal{R}_{\mathcal{X}} / (z-1) \mathcal{R}_{\mathcal{X}} \simeq \mathcal{D}_X$, $\mathcal{R}_{\mathcal{X}} / z \mathcal{R}_{\mathcal{X}} \simeq \mathcal{O}_X[TX]$

$\Rightarrow$ for an $\mathcal{R}_{\mathcal{X}}$-module $\mathcal{M}$,

$\mathcal{M}/(z-1)\mathcal{M}$ is a $\mathcal{D}_X$-module, i.e. an $\mathcal{O}_X$-module w/ flat connection.

$\mathcal{M}/z\mathcal{M}$ is an $\mathcal{O}_X[TX]$-module, i.e. an $\mathcal{O}_X$-module w/ a Higgs field

$\theta: E \to \Omega^1_X \otimes E$ w/ $\theta \wedge \theta = 0$.

<u>Example</u>. If $\mathcal{M} = (R_F M)^{an}$, $\mathcal{M}/(z-1)\mathcal{M} = M$ and $\mathcal{M}/z\mathcal{M} = gr^F M$.

<u>Characteristic variety</u>.

$\mathrm{Char}\, M = \mathrm{Supp}\, gr^F M$.

- $\mathcal{M}$ coherent $\mathcal{R}_{\mathcal{X}}$-module. <u>Locally on $\mathcal{X}$</u>, $\exists$ coherent $F_\bullet \mathcal{R}_{\mathcal{X}}$-filtration.

$\rightsquigarrow$ $\mathrm{Char}\, \mathcal{M}$ = supp of $gr^F \mathcal{M}$ in $(T^*X) \times \mathbb{C}_z$, conic w.r.t. the $\mathbb{C}^*$-action on T^*X.

$\rightsquigarrow$ Various subsets of T^*X attached to $\mathcal{M}$:

- $\forall z_0 \in \mathbb{C}^*_z$, $Li^*_{z_0} \mathcal{M} := [\mathcal{M} \xrightarrow{z-z_0} \mathcal{M}]$ is a cpx of $\mathcal{D}_X$-modules

$\rightsquigarrow$ $\mathrm{Char}(Li^*_{z_0} \mathcal{M})$:

$\forall z_0 \in \mathbb{C}^*_z$, $\mathrm{Char}\, \mathcal{M} \cap (T^*X \times \{z_0\})$

- $\Sigma(\mathcal{M}) = \mathrm{supp}(\mathcal{M}/z\mathcal{M})$ possibly <u>not</u> conic.

Relation: $\forall z_0 \in \mathbb{C}^*_z$, $\mathrm{Char}(Li^*_{z_0}\mathcal{M}) \subset \mathrm{Char}\,\mathcal{M} \cap (T^*X \times \{z_0\})$

<u>Example</u> $X = \mathbb{C}_x$, $\mathcal{M} = \mathcal{R}_{\mathcal{X}} / \mathcal{R}_{\mathcal{X}}(x\partial_x - \alpha(z))$, then (holomorphic function of z)

$$\mathrm{Char}\,\mathcal{M} = \{x\xi = 0\} \times \mathbb{C}_z$$

$$\mathrm{Char}(Li^*_{z_0}\mathcal{M}) = \{x\xi = 0\}, \quad z_0 \in \mathbb{C}^*_z$$

$$\Sigma(\mathcal{M}) = \{x\xi - \alpha(0)\} \qquad (\text{not conic})$$

<u>Def</u> An $\mathcal{R}_{\mathcal{X}}$-module is <u>strict</u> if it has no $\mathcal{O}_{\mathbb{C}_z}$-torsion. A morphism is <u>strict</u> if its coker. is strict.

<u>Example</u>. If $\mathcal{M}$ is a graded $R_F\mathcal{D}_X$-module, $\mathcal{M}$ is <u>strict</u> iff $\exists$ an F-filtration $F_\bullet M$ w/ $M = \mathcal{M}/(z-1)\mathcal{M}$ s.t. $\mathcal{M} = R_F M$.

<u>Thm</u> ($\Leftarrow$ Gabber's involutivity thm) $\mathcal{M}$ <u>strict</u> coherent $\mathcal{R}_{\mathcal{X}}$-module. Then $\Sigma(\mathcal{M})$ and $\mathrm{Char}(Li^*_{z_0}\mathcal{M})$ $(z_0 \in \mathbb{C}^*_z)$ are involutive in T^*X, and $\mathrm{Char}\,\mathcal{M} \subset T^*X \times \mathbb{C}_z$ is involutive w.r.t. the Poisson bracket $z\{\cdot,\cdot\}$.

<u>Def</u>. A coherent $\mathcal{R}_{\mathcal{X}}$-module is <u>holonomic</u> if $\exists \Lambda \subset T^*X$ Lagrangian (conic) s.t. $\mathrm{Char}\,\mathcal{M} \subset \Lambda \times \mathbb{C}_z$.

<u>Prop</u>.
- $\mathcal{M}$ holonomic $\Rightarrow$ $Li^*_{z_0}\mathcal{M}$ is $\mathcal{D}_X$-holonomic for each $z_0 \in \mathbb{C}^*_z$.
- $\mathcal{M}$ holonomic and <u>strict</u> $\Rightarrow$ $\Sigma(\mathcal{M})$ Lagrangian

Proof $\Sigma(\mathcal{M})$ involutive $\Rightarrow$ $\dim X \le \dim \Sigma(\mathcal{M})$

- A coherent F-filtration on $\mathcal{M}$ induces a coherent F-filtration on $\mathcal{M}/z\mathcal{M}$

and dimension preserved by grading, so $\dim \Sigma(\mathcal{M}) = \dim \operatorname{supp} gr^F(\mathcal{M}/z\mathcal{M})$

- Note

$$gr_k^F \mathcal{M} / z\, gr_k^F \mathcal{M} = \frac{F_k \mathcal{M}}{z F_k \mathcal{M} + F_{k-1}\mathcal{M}} \longrightarrow \frac{F_k \mathcal{M}}{(F_k\mathcal{M} \cap z\mathcal{M}) + F_{k-1}\mathcal{M}} = gr_k^F(\mathcal{M}/z\mathcal{M})$$

Hence

$$\dim \Sigma(\mathcal{M}) = \dim \operatorname{Supp} gr^F(\mathcal{M}/z\mathcal{M}) \le \dim \operatorname{Supp}(gr^F\mathcal{M}/z\, gr^F\mathcal{M}) \le \dim \Lambda = \dim X . \quad \square$$

Recall If M is a coherent $\mathcal{D}_X$, $\operatorname{Char} M \subset \underbrace{T_X^* X}_{\text{zero section of } T^*X}$

is locally isom. to $(\mathcal{O}_X^n, d)$ (Cauchy Theorem)

Thm $\mathcal{M}$ strict coh. $\mathcal{R}_{\mathcal{X}}$-mod. w/ $\operatorname{Char} \mathcal{M} \subset T_X^* X \times \mathbb{C}_z$, then

- $\mathcal{M}$ is $\mathcal{O}_{\mathcal{X}}$-coherent.
- $\mathcal{M}|_{X \times \mathbb{C}_z^*}$ is locally isom. to $(\mathcal{O}^2_{X\times\mathbb{C}_z^*}, zd)$
- $\exists\, Z \subset X$ closed analytic nowhere dense s.t. $\mathcal{M}|_{\mathcal{X}\setminus\mathcal{Z}}$ is locally $\mathcal{O}_{\mathcal{X}}$-free.

$Z \times \{0\}$ where $\mathcal{M}$ is $\mathcal{O}_{\mathcal{X}}$-coh. but possibly <u>not</u> locally free.

$\varpi : \Sigma(\mathcal{M}) \to X$ may not be flat

Functors on $\mathcal{R}_{\mathscr{X}}$-modules

De Rham cpx

Recall

- $\Omega^1_{\mathscr{X}} = z^{-1}\Omega^1_{X\times\mathbb{C}^*/\mathbb{C}^*}$ and $\Omega^k_{\mathscr{X}} = \Lambda^k\Omega^1_{\mathscr{X}}$

- $\mathcal{M}$ left $\mathcal{R}_{\mathscr{X}}$-mod $\Leftrightarrow$ $\mathcal{O}_{\mathscr{X}}$-mod. w/ flat $\nabla^{(z)}: \mathcal{M} \to \Omega^1_{\mathscr{X}} \otimes \mathcal{M}$

$$DR(\mathcal{M}) := \left[\mathcal{M} \xrightarrow{\nabla^{(z)}} \Omega^1_X \otimes \mathcal{M} \to \cdots \xrightarrow{\nabla^{(z)}} \Omega^n_{\mathscr{X}} \otimes \mathcal{M}\right]$$

$${}^pDR(\mathcal{M}) = DR(\mathcal{M})[\dim X]$$

Side-changing. $\omega_{\mathscr{X}} = \Omega^n_{\mathscr{X}}$ $(n = \dim X)$ is a right $\mathcal{R}_{\mathscr{X}}$-module

$\mathcal{M}$ left $\mathcal{R}_{\mathscr{X}}$-module $\Leftrightarrow$ $\omega_{\mathscr{X}} \otimes \mathcal{M}$ right $\mathcal{R}_{\mathscr{X}}$-module.

Pushforward & pullback by $f: X \to Y$.

- Transfer module. $$\mathcal{R}_{\mathscr{X}\to\mathscr{Y}} := \mathcal{O}_{\mathscr{X}} \underset{f^{-1}\mathcal{O}_{\mathscr{Y}}}{\otimes} f^{-1}\mathcal{R}_{\mathscr{Y}}$$

as a left $\mathcal{R}_{\mathscr{X}}$-module and a right $f^{-1}\mathcal{R}_{\mathscr{Y}}$-module

- Spencer resolution: $Sp^{\bullet}_{X\to Y}(\mathcal{R}_{\mathscr{X}}) \xrightarrow{\sim} \mathcal{R}_{\mathscr{X}\to\mathscr{Y}}$.

locally free $\mathcal{R}_{\mathscr{X}}$-mod.

Pullback of $\mathcal{N} \in \mathrm{Mod}^{\mathrm{left}}(\mathcal{R}_{\mathscr{Y}})$

$$Rf^*\mathcal{N} := \mathcal{R}_{\mathscr{X}\to\mathscr{Y}} \overset{L}{\underset{f^{-1}\mathcal{R}_{\mathscr{Y}}}{\otimes}} f^{-1}\mathcal{N}.$$

Pushforward of $\mathcal{M} \in \mathrm{Mod}^{\mathrm{right}}(\mathcal{R}_{\mathscr{X}})$

$${}^pf_*\mathcal{M} := Rf_*\left(\mathcal{M} \overset{L}{\underset{\mathcal{R}_{\mathscr{X}}}{\otimes}} \mathcal{R}_{\mathscr{X}\to\mathscr{Y}}\right) \simeq Rf_*\left(\mathcal{M} \underset{\mathcal{R}_{\mathscr{X}}}{\otimes} Sp^{\bullet}_{\mathscr{X}\to\mathscr{Y}}\right)$$

Thm (Kashiwara's estimate) $\mathcal{M}$ good holonomic $\mathcal{R}_{\mathcal{X}}$-module and $f: X \to Y$ proper (on $\operatorname{Supp} \mathcal{M}$),

good ← having coh. filtration

then ${}^R f_* \mathcal{M}$ is <u>holonomic</u> & $\operatorname{Char}({}^R f_* \mathcal{M}) \subset f\left[(T^* f)^{-1} \operatorname{Char} \mathcal{M}\right]$

$$\bigcap$$

$$f\left((T^*f)^{-1}(\Lambda)\right) \times \mathbb{C}_z$$

<u>Sesquilinear pairings</u>

Target: $\mathcal{D}b_{X \times \mathbb{S}/\mathbb{S}}$ distributions on $X \times \mathbb{S}$ which are holomorphic w.r.t. z, i.e. annihilated by $\overline{\partial_z}$.

$\rightsquigarrow$ can restrict to any $z \in \mathbb{S}$ and get a distribution in $\mathcal{D}b_X$.

$\rightsquigarrow$ acted on by $\mathcal{R}_{\mathcal{X}} \underset{\mathcal{O}_{\mathbb{S}}}{\otimes} \overline{\mathcal{R}_{\mathcal{X}}} =: \mathcal{R}_{\mathcal{X},\overline{\mathcal{X}}}$.

<u>Examples</u> ($\dim X = 1$)

- $|x|^{2\alpha(z)}$ w/ $\alpha: \mathbb{S} \to \mathbb{C}$ holomorphic (dist. w/ bounded growth at the origin)
- $\exp(z/x) \cdot \overline{\exp(z/x)} = \exp(z/x - 1/z\bar{x})$ defines a distribution only if we restrict z to $\mathbb{S}$ because for $z \in \mathbb{S}$, $z/x - 1/z\bar{x} = z/x - \overline{z/x}$ is purely imaginary (Example used in wild twistor theory)

<u>Def</u>. $\mathcal{M}', \mathcal{M}''$: left $\mathcal{R}_{\mathcal{X}}$-modules. A sesquilinear pairing C is an $\mathcal{R}_{\mathcal{X},\overline{\mathcal{X}}}$-linear morphism $C: \mathcal{M}'|_{\mathbb{S}} \underset{\mathcal{O}_{\mathbb{S}}}{\otimes} \overline{\mathcal{M}''}|_{\mathbb{S}} \longrightarrow \mathcal{D}b_{X \times \mathbb{S}/\mathbb{S}}$.

Example. If Char $\mathcal{M}'$, Char $\mathcal{M}''$ $\subset (T^*_X X) \times \mathbb{C}_z$, then C takes values in

$\mathcal{C}^{\infty,an}_{X\times S}$: for $(x_0, z_0) \in X \times S$, choose local frames e' of $\mathcal{M}'_{x_0,z_0}$ and

e'' of $\mathcal{M}''_{x_0,-z_0}$ s.t. $\bar{\partial} e' = 0$, $\bar{\partial} e'' = 0$. Then $C(e_k, \overline{e_\ell}) \in \mathcal{D}b_{X\times S/S, (x_0,z_0)}$

killed by d_X', d_X''.

Category $\mathcal{R}$-Triples (X)

Objects $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$, and C is a sesquilinear pairing.

no coherence assumption.
holonomy

Morphisms. pairs (φ', φ'') of $\mathcal{R}_{\mathcal{X}}$-linear morphisms

$\varphi' : \mathcal{M}' \longrightarrow \mathcal{M}'$, $\varphi'' : \mathcal{M}'' \longrightarrow \mathcal{M}''$ + compatibility w/ C

Tate twist. $\mathcal{T}(\ell) = (z^\ell \mathcal{M}', z^{-\ell}\mathcal{M}'', t_\ell C)$

Hermitian dual $\mathcal{T}^* = (\mathcal{M}'', \mathcal{M}', C^*)$. $C^*(m'', \overline{m'}) := \overline{C(m', \overline{m''})}$

Pre-polarization of weight w. Hermitian iso. $S : \mathcal{T} \Rightarrow \mathcal{T}^*(-w)$.

Pushforward in the category $\mathcal{R}$-triples (X)

$f : X \longrightarrow Y \rightsquigarrow f : \mathcal{X} \longrightarrow \mathcal{Y}$, $\mathcal{T}$ = right triple

Simpler to work w/ right $\mathcal{R}_{\mathcal{X}}$-modules $\rightsquigarrow$ C values in

$\mathfrak{C}_{X\times S/S} := z^{-n} \pi^{-1} \mathcal{E}^{2n}_X \otimes_{\pi^{-1} C^\infty_X} \mathcal{D}b_{X\times S/S}$

- $\mathcal{R}_{\mathcal{X}\to\mathcal{Y}} := \mathcal{O}_{\mathcal{X}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{R}_{\mathcal{Y}}$ & ${}^{R}f_{*}^{(j)} \mathcal{M} = \mathcal{H}^{j}\left(Rf_{*}\left(\mathcal{M} \otimes^{L}_{\mathcal{R}_{\mathcal{X}}} \mathcal{R}_{\mathcal{X}\to\mathcal{Y}}\right)\right)$

- Resolution $\mathcal{R}_{\mathcal{X}\to\mathcal{Y}} \simeq Sp_{\mathcal{X}\to\mathcal{Y}} := Sp(\mathcal{R}_{\mathcal{X}}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{R}_{\mathcal{X}\to\mathcal{Y}}$

(omit $|_{\mathbf{S}}$): $\left(\mathcal{M}' \otimes_{\mathcal{R}_{\mathcal{X}}} Sp_{\mathcal{X}\to\mathcal{Y}}\right) \otimes_{\mathcal{O}_{\mathbf{S}}} \overline{\left(\mathcal{M}'' \otimes_{\mathcal{R}_{\mathcal{X}}} Sp_{\mathcal{X}\to\mathcal{Y}}\right)}$

$$\to \left(\mathcal{M}' \otimes_{\mathcal{O}_{\mathbf{S}}} \mathcal{M}''\right) \otimes_{\mathcal{R}_{\mathcal{X},\overline{\mathcal{X}}}} Sp_{\mathcal{X},\overline{\mathcal{X}}\to\mathcal{Y},\overline{\mathcal{Y}}}$$

$$\xrightarrow{C} \mathfrak{C}_{X\times\mathbf{S}/\mathbf{S}} \otimes_{\mathcal{R}_{\mathcal{X},\overline{\mathcal{X}}}} Sp_{\mathcal{X},\overline{\mathcal{X}}\to\mathcal{Y},\overline{\mathcal{Y}}}$$

Apply Rf_{*} $\rightsquigarrow$ ${}^{R}f_{*}\mathcal{M}'|_{\mathbf{S}} \otimes_{\mathcal{O}_{\mathbf{S}}} \overline{{}^{P}f_{*}\mathcal{M}''|_{\mathbf{S}}} \longrightarrow {}^{R}f_{*}\, \mathfrak{C}_{X\times\mathbf{S}/\mathbf{S}}$.

<u>Integration of currents</u>: ${}^{R}f_{*}\,\mathfrak{C}_{X\times\mathbf{S}/\mathbf{S}} \xrightarrow{\int_f} \mathfrak{C}_{Y\times\mathbf{S}/\mathbf{S}}$

Define ${}^{T}f_{*}^{(j,-j)}C : {}^{R}f_{*}^{(j)}\mathcal{M}'|_{\mathbf{S}} \otimes_{\mathcal{O}_{\mathbf{S}}} \overline{{}^{R}f_{*}^{(-j)}\mathcal{M}''|_{\mathbf{S}}} \to \mathfrak{C}_{Y\times\mathbf{S}/\mathbf{S}}$

as the composition $\cdot$ (multiplied by $(-1)^{j(j-1)/2}$)

<u>Example</u>: pushforward by a closed inclusion.

- $g: X \to \mathbb{C}$, $i: X \hookrightarrow Y = X\times\mathbb{C}_t$ graph inclusion
- ${}^{R}i_{*}\mathcal{M} \simeq \bigoplus_{k\geq 0} i_{*}\mathcal{M}\otimes\eth_t^k$
- $\left({}^{T}i_{*}C\right)\left(m'\otimes\eth_t^k, \overline{\left(m''\otimes\eth_t^{\ell}\right)}\right) = \left[\int_i C(m',\overline{m''})\right]\cdot\eth_t^k\,\overline{\eth_t^{\ell}}$

Lecture 4 Specialization of $\mathcal{R}$-modules.

Specialization of holonomic $\mathcal{D}$-modules.

For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, restriction to $x_0 \in X$ has various cohomologies if x_0 is a singular point of $\mathcal{M}$.

$\rightsquigarrow$ replace restriction w/ iteration of nearby/vanishing cycles along functions vanishing at x_0.

The V-filtration along a function.

$$x \longmapsto (x, g(x) = t)$$

- $g: X \to \mathbb{C}$ holom. func., $i: X \hookrightarrow X \times \mathbb{C}_t$ graph inclusion

- for $k \in \mathbb{Z}$, $\nearrow$ filtration

$$V_k(\mathcal{D}_{X\times\mathbb{C}}) = \{ P \in \mathcal{D}_{X\times\mathbb{C}} : \forall j \in \mathbb{Z}_{\geq 0},\ P \cdot (t^j \mathcal{O}_{X\times\mathbb{C}}) \subset (t^{j-k} \mathcal{O}_{X\times\mathbb{C}}) \}$$

E.g. $t \in V_{-1}(\mathcal{D}_{X\times\mathbb{C}})$, $\partial_t \in V_1(\mathcal{D}_{X\times\mathbb{C}})$, $P(x, t, \partial_x, t\partial_t) \in V_0(\mathcal{D}_{X\times\mathbb{C}})$

$\rightsquigarrow$ notion of coherent $\nearrow$ V-filtration indexed by $\mathbb{Z}$ on any coherent $\mathcal{D}_{X\times\mathbb{C}}$-module.

— Given a finite set $A \subset [-1, 0)$, notion of coherent V-filtration indexed by $A + \mathbb{Z}$ on a coherent $\mathcal{D}_{X\times\mathbb{C}}$-module: <u>nested</u> family $({}^{(a)}U_\bullet)_{a\in A}$ of coherent V-filtrations indexed by $\mathbb{Z}$, i.e. s.t. $\forall a, b \in A$, $\forall k, l \in \mathbb{Z}$, $a + k \leq b + l \Rightarrow {}^{(a)}U_k \subset {}^{(b)}U_l$

$U_\bullet$ indexed by $A + \mathbb{Z} \subset \mathbb{R}$

$U_{a+k} = {}^{(a)}U_k$.

Def. A coherent right D_X-module M is specializable along g, if $\forall$ local section m (of ${}^D i_* M$), $\exists\ b_m(s) \in \mathbb{C}[s] \setminus \{0\}$ s.t. $m \cdot b_m(t\partial_t) \in m \cdot V_{-1} D_{X\times\mathbb{C}}$.

$\rightsquigarrow$ $\exists\ A$ finite, $\exists!$ coherent $\nearrow$ V-filt'n $V_\bullet({}^D i_* M)$ indexed by $A+\mathbb{Z}$ s.t. $\forall a \in A$, $\forall k \in \mathbb{Z}$, $t\partial_t$ acts on $gr^V_{a+k}({}^D i_* M)$ w/ eigenvalues $\alpha \in \mathbb{C}$ having finite order and s.t. $Re(\alpha) = a+k$. (called the Kashiwara–Malgrange filtration).

$$\Rightarrow \forall k \geq 0, \left[\begin{array}{lll} t^k : gr^V_a({}^D i_* M) \xrightarrow{\sim} gr^V_{a-k}({}^D i_* M), & \forall a \in [-1,0) \\ \partial_t^k : gr^V_a({}^D i_* M) \xrightarrow{\sim} gr^V_{a+k}({}^D i_* M), & \forall a \in (-1,0] \end{array} \right.$$

Def. (Nearby / vanishing cycles along (g)) For $\alpha \in \mathbb{C}$,

$$\Psi_{g,\alpha} M = \ker (t\partial_t - \alpha)^N \text{ on } gr^V_{Re\,\alpha} M, \quad N \gg 0.$$
$$\circlearrowleft\ t\partial_t =: N$$

$$\Rightarrow \forall k \geq 1, \left[\begin{array}{lll} t^k : \Psi_{g,\alpha}(M) \xrightarrow{\sim} \Psi_{g,\alpha-k}(M), & \text{if } \alpha \neq 0 \\ \partial_t^k : \Psi_{g,\alpha}(M) \xrightarrow{\sim} \Psi_{g,\alpha+k}(M) & \text{if } \alpha \neq -1 \end{array} \right.$$

$\rightsquigarrow$ Enough to consider α w/ $Re\,\alpha \in [-1,0)$ (nearby cycles) or $\alpha = 0$ (unipotent vanishing cycles).

An extension of the construction

One can recover the various $\Psi_{g,\alpha}(M)$ from other V-filtrations indexed by $\mathbb{R}$.

Lemma. Let $U_\bullet({}^D i_* M)$ be a coherent V-filtration indexed by $A+\mathbb{Z}$ w/ $A \subset \mathbb{R}$ finite. Assume that for each $a \in A+\mathbb{Z}$, $t\partial_t$ has a minimal poly. on $gr^U_a({}^D i_* M)$ w/ roots in $R(a) \subset \mathbb{C}$ satisfying: $\forall a_0 \in A + \mathbb{Z}$, $\forall \alpha_1, \alpha_2 \in R([a_0, a_0+1))$,

$\alpha_1 \neq \alpha_2 \Rightarrow \alpha_1 - \alpha_2 \notin \mathbb{Z}$. Then

$$\Psi_{g,\alpha}(M) = \ker\left((t\partial_t - \alpha)^N : gr_a^u \to gr_a^u\right) \quad \text{if } \alpha \in R(a).$$

<u>Theorem</u>. If M is holonomic, then M is specializable along any (g), and $\forall \alpha \in \mathbb{C}$,

$\underset{\circlearrowright N}{\Psi_{g,\alpha}(M)}$ is holonomic and supported on $g^{-1}(0)$.

<u>Rmk</u>. $\{[\alpha] \in \mathbb{C}/\mathbb{Z}\}$ classifies rank one bundles w/ holom. conn'n on Δ_x^*.

$$d - \alpha\frac{dx}{x} \simeq d - (\alpha + k)\frac{dx}{x}.$$

An <u>illuminating computation</u>.

- (E, θ, h): harmonic Higgs bundle of rk 1 on Δ_x^*. Classification?
- ε any holomorphic basis of E w/
$$\theta\varepsilon = \varphi(x)\,\varepsilon\, dx \quad , \varphi(x) \text{ holom. on } \Delta^*.$$
- Set $\varphi(x) = \partial_x \psi(x) + \frac{\alpha}{x}$ w/ $\psi \in \mathcal{O}(\Delta^*)$ w/o constant term and $\alpha \in \mathbb{C}$.
- Set $\|\varepsilon\|_h = \exp(\eta(x))$ w/ η real and C^∞ on Δ^*.
- (E, θ, h) harmonic $\Leftrightarrow$ η is harmonic on Δ^*.

  Write $\eta(x) = \operatorname{Re}\gamma(x) - a\log|x|$ w/ γ holom. on Δ^* and $a \in \mathbb{R}$.

Replace ε by $e = \exp(-\gamma(x)) \cdot \varepsilon$, so that $\eta(x) = -a\log|x|$ w/ $a \in \mathbb{R}$, and

$\|e\|_h = |x|^{-a}$.

- Set $u = (a, \alpha) \in \mathbb{R} \times \mathbb{C}$. (V, ∇) flat bundle assoc. to (E, θ, h).

Set $v = |x|^{-2\bar{\alpha}} \exp(\varphi - \bar{\varphi}) \cdot e$, then v is a holom. basis of V. and

$\|v\|_h = |x|^{-a - 2\operatorname{Re}\alpha}$. Furthermore, set $e(1,u) = -a + 2i \operatorname{Im} \alpha$.

$$\theta e = \left(x \partial_x \varphi + \alpha\right) \frac{dx}{x} \otimes e, \qquad \nabla v = \left(2x \partial_x \varphi + e(1,u)\right) \frac{dx}{x} \otimes v$$

More generally, for any $z \in \mathbb{C}$ fixed, set

$$p(z,u) = a + 2\operatorname{Re}(\bar{\alpha} z), \qquad e(z,u) = \alpha - a z - \bar{\alpha} z^2$$

Note $\overline{e(z,u)/z} = e(z,u)/z$.

<u>Lemma</u>. $\{(E,\theta,h) \text{ harmonic}\} \simeq \{(\varphi, u \bmod \mathbb{Z} \times \{0\}) \text{ as above}\}$

<u>Pf</u>. ε' other holom. basis of E

$\rightsquigarrow e'$ w/ $\|e'\|_h = |x|^{-a'}$, $a' \in \mathbb{R}$

$\rightsquigarrow$ a pair (φ', u')

Then $e' = \mu(x) e$ w/ $\mu(x)$ holom. and moderate growth, hence merom., hence

$a' - a \in \mathbb{Z}$.

The Higgs field has the same expression in bases e and e', implying $\varphi = \varphi'$ and

$\alpha = \alpha'$. $\square$

Set $v^{(z)} = e^{\bar{z}\varphi - z\bar{\varphi}} |x|^{-2\bar{\alpha} z} \cdot e$

$\Rightarrow v^{(z)}$ is a holom. basis of $V^{(z)}$ w/ $\|v^{(z)}\|_h = |x|^{-p(z,u)}$ and

$$\nabla^{(z)} v^{(z)} = \left((1+|z|^2) x \partial_x \varphi + e(z,u)\right) \frac{dx}{x} \otimes v^{(z)}.$$

<u>Def</u>. The harmonic Higgs bundle (E,θ,h) or flat bundle (V,∇,h) is

- <u>regular</u> if φ is holomorphic on Δ: $\varphi \in \mathcal{O}(\Delta)$
- <u>wild</u> if φ is meromorphic on Δ: $\varphi \in \mathcal{O}(\Delta)[\frac{1}{x}]$.

- not considered if φ has an essential sing. at the origin.

The case where $\alpha = i\alpha''$ is purely imaginary. This is the case involved in Kashiwara's conjecture.

- $a \leftrightarrow \|v\|_h$, $\quad \alpha \leftrightarrow \mathrm{Res}_{x=0}\,\theta$.
- $e(\mathfrak{z},u)/\mathfrak{z} = -a + i\left(\mathfrak{z}+\frac{1}{\mathfrak{z}}\right)\alpha''$, $\quad p(\mathfrak{z},u) = a + 2\alpha''\,\mathrm{Im}\,\mathfrak{z}$.

Strictly specializable $\mathcal{R}_{\mathcal{X}}$-module

Define the $\nearrow$ filtration $V_\bullet(\mathcal{R}_{\mathcal{X}\times\mathbb{C}_t})$ so that

$t \in V_{-1}(\mathcal{R}_{\mathcal{X}\times\mathbb{C}_t})$, $\partial_t \in V_1(\mathcal{R}_{\mathcal{X}\times\mathbb{C}_t})$ · $P(x,t,\mathfrak{z},\eth_x,t\eth_t) \in V_0(D_{\mathcal{X}\times\mathbb{C}_t})$

$\rightsquigarrow$ given $g: X \to \mathbb{C}_t$, $i: X \hookrightarrow X\times\mathbb{C}_t$ notion of coherent V-filtration on ${}^{R}i_{*}\mathcal{M}$ indexed by $A+\mathbb{Z}$.

Bernstein relation. $\mathcal{M}$ coh. over $\mathcal{R}_{\mathcal{X}}$. Say $\mathcal{M}$ is <u>specializable</u> along (g) at $(x_0,\mathfrak{z}_0)\in X\times\mathbb{C}_{\mathfrak{z}}$ if

- for any local section $m \in \mathcal{M}_{x_0,\mathfrak{z}_0}$, $\exists$ a minimal finite set $\mathcal{U}_m \subset \mathbb{R}\times\mathbb{C}$

and for each $u \in \mathcal{U}_m$, an integer $\nu_{m,u}$ s.t.

(Bernstein rel.) $\quad m\cdot\prod_{u\in\mathcal{U}_m}\left(t\eth_t + e(\mathfrak{z},u)\right)^{\nu_{m,u}} \in m\cdot V_{-1}(\mathcal{R}_{\mathcal{X}\times\mathbb{C}_t,(x_0,0,\mathfrak{z}_0)})$

$\rightsquigarrow$ Filtration by the parabolic order: $\forall c\in\mathbb{R}$

$\widetilde{V}_c({}^{R}i_{*}\mathcal{M}_{(x_0,0,\mathfrak{z}_0)}) = \{m : p(\mathfrak{z}_0,u)\le c,\ \forall u\in\mathcal{U}_m\}$

Possibly <u>not</u> coherent V-filt'n.

• But $\exists$ a (possibly not unique) coherent V-filt'n $U_\bullet^{(z_0)}(R i_* \mathcal{M})$ indexed by $\mathbb{Z}$, defined in the nbhd of $(x_0, 0, z_0)$, $\exists\, \mathcal{U}$ finite and, $\forall u \in \mathcal{U}$, $\nu_u \in \mathbb{Z}_{\geq 0}$, s.t.

$$\forall k \in \mathbb{Z},\quad U_k^{(z_0)}(R i_* \mathcal{M}) \cdot \prod_{u \in \mathcal{U}} (t\partial_t + e(z,u) + kz)^{\nu_u} \subset U_{k-1}^{(z_0)}(R i_* \mathcal{M}).$$

$\rightsquigarrow \exists$ a coh. V-filtration indexed by the parabolic order at $(x_0, 0, z_0)$: for $c \in \mathbb{R}$

$$U_c^{(z_0)}(R i_* \mathcal{M}) \cdot \prod_{\substack{u \in \mathcal{U} + \mathbb{Z}(1,0) \\ p(z_0,u) = c}} (t\partial_t + e(z,u))^{\nu_u} \subset U_{<c}^{(z_0)}(R i_* \mathcal{M}).$$

Strict specializability

Prop. Assume $\exists\, U_\bullet^{(z_0)}(R i_* \mathcal{M})$ s.t. each $gr_c^{U^{(z_0)}}(R i_* \mathcal{M})$ is strict (no $\mathcal{O}_{\mathbb{C}_z}$-torsion). Then this filtration is unique. It is called the Kashiwara-Malgrange filtration at $(x_0, 0, z_0)$, and is denoted by $V_\bullet^{(z_0)}(R i_* \mathcal{M}_{x_0})$. It is then equal to the filtration by the parabolic order.

Sketch of proof. $U_\bullet$, $U'_\bullet$ coherent V-filt'n,

$\Rightarrow \exists\, k \geq 0, \forall d,\ U_{d-k} \subset U'_d \subset U_{d+k} \subset U'_{d+2k}$

Let $m \in U'_d$. Assume $m \in U_c \setminus U_{<c}$ for some $c \in (d, d+k]$.

Then $\exists\ b_{U'}(s)$ (resp. $b_U(s)$) w/ roots $-e(z,u)$ for $u \in \mathbb{R} \times \mathbb{C}$ s.t.

$p(z_0, u) \leq d$ (resp. $p(z_0,u) = c$), and

$$m \cdot b_{U'}(t\partial_t) \in U'_{<c-k} \subset U_{<c} \quad \text{and} \quad m \cdot b_U(t\partial_t) \in U_{<c}.$$

Note that $b_U(s)$ and $b_{U'}(s)$ have no common root since

$p(z_0, u_1) \neq p(z_0, u_2) \Rightarrow u_1 \neq u_2 \Rightarrow e(z,u_1) \neq e(z,u_2)$

$\rightsquigarrow \exists\ p(z) \in \mathbb{C}[z] \setminus 0$ s.t. $p(z)[m] = 0 \in gr_c^U$

But gr_c^U strict $\Rightarrow m \in U_{<c} \rightsquigarrow$ contrad. $\square$

$\Rightarrow \ gr_c^{V^{(z_0)}}(R i_* \mathcal{M})$ decomposes as

$$\bigoplus_{\substack{u \in \mathcal{U} + \mathbb{Z}(1,0) \\ p(z_0,u)=c}} \psi_{g,u}^{(z_0)}(\mathcal{M}) \quad , \quad \psi_{g,u}^{(z_0)}(\mathcal{M}) := \ker\left(t\partial_t + e(z,u)\right)^N, \ N \gg 0$$

$$N \circlearrowright$$

<u>Def</u>. We say that $\mathcal{M}$ is <u>strictly specializable</u> along (g) if

- has a Kashiwara-Malgrange filt'n in the nbhd of any $(x_0, z_0) \in X \times \mathbb{C}_z$,

and

- $\forall k \geq 1$, $\left[\begin{array}{l} t^k : \ gr_c^{V^{(z_0)}}(R i_* \mathcal{M}) \xrightarrow{\sim} gr_{c-k}^{V^{(z_0)}}(R i_* \mathcal{M}), \quad \text{if } c<0 \\ \partial_t^k : \ gr_c^{V^{(z_0)}}(R i_* \mathcal{M}) \xrightarrow{\sim} gr_{c+k}^{V^{(z_0)}}(R i_* \mathcal{M}) \quad \text{if } c>-1 \end{array}\right.$

<u>Some consequences</u>. Assume that $\mathcal{M}$ is <u>strictly spec. along (g)</u>.

- For any $z_0 \in \mathbb{C}_z$, the KM filt'n $V_\bullet^{(z_0)}(R i_* \mathcal{M})$ is defined <u>globally</u> on X (but depends on z_0). (Note that $V_\bullet^{(z_0)}(R i_* \mathcal{M}) = R i_* \mathcal{M}$ on $X \times \mathbb{C}_t^* \times nb(z_0)$)

- $V_\bullet^{(z)}(R i_* \mathcal{M})$ is locally constant w.r.t. z: $\forall z_0$, $\forall c \in \mathbb{R}$, $\forall \varepsilon > 0$,
  $\exists \eta(z_0, c, \varepsilon)$ s.t. $|z - z_0| < \eta \Rightarrow$

$$V_{c-\varepsilon}^{(z)}(R i_* \mathcal{M}) = V_{c-\varepsilon}^{(z_0)}(R i_* \mathcal{M}) \quad \text{and} \quad V_{c+\varepsilon}^{(z)}(R i_* \mathcal{M}) = V_{c+\varepsilon}^{(z_0)}(R i_* \mathcal{M}).$$

- Nilpotent endomorphism $N = (t\partial_t + e(z,u))$ acting on $\Psi_{g,u}^{(z_0)}(\mathcal{M})$.

- For u fixed and z varying, the various $\Psi_{g,u}^{(z)}(\mathcal{M})$ glue as a coherent $\mathcal{R}_{\mathcal{X}}$-mod. $\Psi_{g,u}(\mathcal{M})$ on X supp. on $g^{-1}(0)$.

Def. The nearby / vanishing cycle quiver is the quiver

$$\begin{tikzcd} \Psi_{t,(-1,0)} \mathcal{M} \arrow[r, bend left, "\mathrm{can} = \partial_t"] & \Phi_{t,(0,0)} \mathcal{M} \arrow[l, bend left, "\mathrm{var} = t"] \end{tikzcd}$$

w/ $\mathrm{var} \circ \mathrm{can} = \mathrm{N}$ and $\mathrm{can} \circ \mathrm{var} = \mathrm{N}$ on the respective modules.

__First properties__

__Prop__

$g : X \to \mathbb{C}$ holom. func. and $\mathcal{M}$ strictly spec. $\mathcal{R}_{\mathcal{X}}$-mod. along (g)

(1) If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are strictly spec. along (g).

(2) If $\mathcal{M}$ is supp. on $g^{-1}(0)$, then, __locally__ $V^{(z_0)}_{<0}({}^R i_{g*} \mathcal{M}) = 0$ and ${}^R i_{g*} \mathcal{M} = V^{(z_0)}_0 \mathcal{M}$.

__Cor__ (strict Kashiwara's equiv.)

If g is smooth, denote $i : g^{-1}(0) \hookrightarrow X$. Then functor ${}^R i_*$ induces an equiv. between the cat. of coherent __strict__ $\mathcal{R}_{g^{-1}(0)}$-modules and the full subcat. of __strict spec.__ modules consisting of objects supp. on $g^{-1}(0)$. An inverse functor is $\Phi_{g,0}$.

__Restriction to $z = z_0 \neq 0$__

If $z_0 \neq 0$, $i^*_{z_0} \mathcal{M}$ is a coherent D_X-module which is specializable along (g): any loc. section satisfies a Bernstein equation.

__Question__: Relation between $\Psi_{g, e(z_0,u)/z_0}(i^*_{z_0} \mathcal{M})$ and $i^*_{z_0}(\Psi_{g,u} \mathcal{M})$?

__Related question__: Is $i^*_{z_0}(V^{(z_0)}_\bullet(\mathcal{M}))$ a V-filt'n that enables to compute $\Psi_{g,\bullet}(i^*_{z_0} \mathcal{M})$?

Prop. There exists an open dense subset of $\mathbb{C}_z$ s.t. for any z_0 in this subset, $i_{z_0}^*(V_\bullet^{(z_0)}(\mathcal{M}))$ is a filt'n computing $\Psi_{g,\bullet}(i_{z_0}^*\mathcal{M})$.

Cor. For any z_0 in this open dense subset and any $u \in \mathcal{U} + \mathbb{Z}(1,0)$, we have
$\Psi_{g, e(z_0,u)/z_0}(i_{z_0}^*\mathcal{M}) = i_{z_0}^*(\Psi_{g,u}\mathcal{M})$.

Example the purely imaginary case: Assume $\mathcal{U} \subset \mathbb{R} \times i\mathbb{R}$, write $u = a + i\alpha''$, and recall $e(z,u)/z = -a + i(z + 1/z)\alpha''$, $p(z,u) = a + 2\alpha'' \operatorname{Im} z$.

Say $z_0 \in \mathbb{C}_z$ is singular w.r.t. $\mathcal{U}$ if $\exists\, u_1 \neq u_2 \in \mathcal{U} + \mathbb{Z}(1,0)$ s.t. $e(z_0,u_1) = e(z_0,u_2)$

Note $\operatorname{Sing}(\mathcal{U}) \subset i\mathbb{R}_z$ and is discrete in $i\mathbb{R}_z^*$.

For $z_0 \notin \operatorname{Sing}(\mathcal{U})$, $\Psi_{g, e(z_0,u)/z_0}(i_{z_0}^*\mathcal{M}) = i_{z_0}^*(\Psi_{g,u}\mathcal{M})$.

Regularity

Reminder for D_X-modules: Various def'ns of regularity, all known to be equiv.

$\rightsquigarrow$ R-H correspondence: $\mathrm{Mod}_{\mathrm{hol reg}}(D_X) \xrightarrow[\sim]{\mathrm{PDR}} \mathrm{Perv}_{\mathbb{C}}(X)$.

Regularity along a function for $\mathcal{R}_{\mathcal{X}}$-modules. Let $g: X \to \mathbb{C}_t$ be a holom. func., $i: X \hookrightarrow X \times \mathbb{C}_t$ the graph inclusion and $\mathcal{R}_{\mathcal{X} \times \mathbb{C}_t/\mathbb{C}_t}$ be the corresponding sheaf of relative diff. ops. We say that an $\mathcal{R}_{\mathcal{X}}$-module $\mathcal{M}$ which is strictly spec. along (g) is regular along (g) if $\forall z_0 \in \mathbb{C}_z$, some (equiv., any) term $V_c^{(z_0)}(\mathcal{R}_{i_*}\mathcal{M})$ of its V-filtration along (g) is $\mathcal{R}_{\mathcal{X} \times \mathbb{C}_t/\mathbb{C}_t}$-coherent.

Case of $\mathcal{D}_X$-mod. Let M be a holonomic $\mathcal{D}_X$-module w/ supp. Z of dim d.

Then M is reg when one of the following conditions is satisfied.

(Reg_0): $d=0$

(Reg_d): $d \geq 1$ and for any germ $g: (X, x_0) \to (\mathbb{C}, 0)$ of holom. func. on X.

(1) M is regular along (g)

(2) if $\dim(g^{-1}(0) \cap Z) \leq d-1$, the holonomic $\mathcal{D}_X$-mods $\Psi_g M$ and $\Phi_{g,1} M$ satisfy (Reg_{d-1}).

Pushforward and Specialization

Thm. Assume

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$ g holom func. f proper (or proper on $\mathrm{Supp}\, M$)

- M f-good and strictly spec. along $(g \circ f)$

- ${}^R f_*^{(j)} (\Psi_{g\circ f, u} M)$ is strict for any u and any j.

Then ${}^R f_*^{(j)} M$ is strictly spec. along (g), and

$${}^R f_*^{(j)} (\Psi_{g\circ f, u} M) \simeq \Psi_{g,u} ({}^R f_*^{(j)} M).$$

Furthermore, if M is regular along (g), then so are ${}^R f_*^{(j)} M$ $(j \in \mathbb{Z})$.

Idea of the proof. General statement of V-filtred complexes, essentially due to M. Saito

Prop. Let $(\mathcal{N}^\cdot, U_\cdot \mathcal{N}^\cdot)$ be a V-filtered complex of $\mathcal{R}_{\mathcal{Y}\times\mathbb{C}}$-modules.

We assume that

$\exists$ Bernstein poly.

(1) the graded complex $gr^U \mathcal{N}^\cdot$ is strict and monodromic,

(2) there exists k_0 s.t. for all $k \leq k_0$ and all j, right mult. by t induces an isom. $t: U_k \mathcal{N}^j \xrightarrow{\sim} U_{k-1} \mathcal{N}^j$.

(3) there exists $j_0 \in \mathbb{Z}$ s.t. for all $j \geq j_0$ and any k, one has $\mathcal{H}^j(U_k \mathcal{N}^\cdot) = 0$

Then for any j, k, the morphism $\mathcal{H}^j(U_k \mathcal{N}^\cdot) \to \mathcal{H}^j(\mathcal{N}^\cdot)$ is injective.

Moreover, the filtration $U_\cdot \mathcal{H}^j(\mathcal{N}^\cdot)$ defined by

$$U_k \mathcal{H}^j(\mathcal{N}^\cdot) = \mathrm{Im}\left(\mathcal{H}^j(U_k \mathcal{N}^\cdot) \to \mathcal{H}^j(\mathcal{N}^\cdot)\right)$$

satisfies $gr^U \mathcal{H}^j(\mathcal{N}^\cdot) = \mathcal{H}^j(gr^U \mathcal{N}^\cdot)$.

Sketch of proof. 3 steps.

- First prove a formal analogue.

$$\widehat{U_k \mathcal{N}^\cdot} := \varprojlim_{\ell \geq 0} U_k \mathcal{N}^\cdot / U_{k-\ell} \mathcal{N}^\cdot \quad \text{and} \quad \widehat{\mathcal{N}^\cdot} = \varinjlim_k \widehat{U_k \mathcal{N}^\cdot}.$$

For this, use the Bernstein relation.

- Show vanishing of the t-torsion of $\mathcal{H}^j(U_k \mathcal{N}^\cdot)$ for $k \leq k_0$ & all j.
- Lift the first step to the non-formal setting.

Lecture 5. Specialization of $\mathcal{R}$-triples.

Specialization of sesquilinear pairings

Goal:

- $\mathcal{M}', \mathcal{M}''$ coh. right $\mathcal{R}_{\mathcal{X}}$-modules which are strictly specializable along (g),

$g: X \to \mathbb{C}$

- $C : \mathcal{M}'|_{\mathcal{S}} \otimes_{\mathcal{O}_{\mathcal{S}}} \overline{\mathcal{M}''|_{\mathcal{S}}} \longrightarrow \mathfrak{C}_{X \times S / S}$ a sesquilinear pairing

- To define for each $u \in \mathbb{R} \times \mathbb{C}$ a sesquilinear pairing

$$\Psi_{g,u} C : \Psi_{g,u} \mathcal{M}'|_{\mathcal{S}} \otimes_{\mathcal{O}_{\mathcal{S}}} \overline{\Psi_{g,u} \mathcal{M}''} \longrightarrow \mathfrak{C}_{X \times S / S}$$

supp. on $g^{-1}(0)$

s.t. on the suitable modules,

- $\Psi_{g,u} C(\cdot, \overline{N \cdot}) = -z^2 \Psi_{g,u} C(N \cdot, \overline{\cdot})$, $N = t\eth_t + e(z,u)$
- $\Psi_{g,u}(C^*) = (\Psi_{g,u} C)^*$ $N : \Psi_{g,u}(\mathcal{M}) \to \Psi_{g,u}(\mathcal{M})(-1)$

Idea: use of Mellin transform. Fix $(x_0, z_0) \in X \times S$.

For local sections m', m'' of $\mathcal{M}', \mathcal{M}''$ in a nb (x_0, z_0) in $X \times S$ the current $C(m', \overline{m''})$ has some finite order p on $\mathrm{nb}_{X \times S}(x_0, z_0)$.

For $2\operatorname{Re} s > p$, the function $|g|^{2s}$ is C^p.

$\Rightarrow$ for any such s, $C(m', \overline{m''}) \cdot |g|^{2s}$ is a section of $\mathfrak{C}_{X \times S | S}$ on $\mathrm{nb}_{X \times S}(x_0, z_0)$.

Moreover, $\forall\, h \in C_c^\infty(\mathrm{nb}_{X \times S}(x_0, z_0))$, the function $s \mapsto \langle C(m', \overline{m''}) \cdot |g|^{2s}, h \rangle$ is holom. on the half-plane $\{2\operatorname{Re} s > p\}$.

First, try to extend it as a meromorphic function on $\mathbb{C}_s$.

Prop. • $\mathcal{M}', \mathcal{M}''$ strictly spec. along (g),

• C a sesquilinear pairing.

• Then, $\forall\ (x_0, z_0) \in X \times S$, $\exists$ a finite set $\mathcal{U} \subset \mathbb{R} \times \mathbb{C}$ s.t.

$\forall\ m' \in \mathcal{M}'_{(x_0, z_0)}$, $m'' \in \mathcal{M}''_{(x_0, z_0)}$, $\exists\ L \geq 0$ s.t. the correspondence

$$C_c^\infty(\text{nb}(x_0, z_0)) \ni h \longmapsto \left(\prod_{u \in \mathcal{U}} \Gamma(s + e(z, u)/z)^{-L} \right) \left\langle C(m', \overline{m''}) \cdot |g|^{2s}, h \right\rangle$$

defines for any s a current in $\Gamma(\text{nb}(x_0, z_0), \mathfrak{C}_{X \times S | S})$ which is <u>holomorphic</u> w.r.t. s.

<u>Sketch of proof</u>. Use Bernstein relation in the form

$$m' g^s \cdot b_{m'}(s, z) = m' \cdot g^{s+1} P(x, z, s, \eth_x)$$

and similarly w/ m'' so that $b_{m'}(s,z)\, C(m', m'') \cdot |g|^{2s}$ is holomorphic on $\{2 \operatorname{Re} s > p-1\}$, and similarly w/ m''.

$\Rightarrow$ can take $\mathcal{U} = \mathcal{U}(m') \cap \mathcal{U}(m'')$. $\square$

<u>Link w/ nearby / vanishing cycles</u>. More convenient to work w/ ${}^R i_* \mathcal{M}'$, ${}^R i_* \mathcal{M}''$, ${}^T i_* C$.

<u>Prop.</u> Let $(x_0, z_0) \in X \times S$ and $c', c'' \in \mathbb{R}$.

• $\exists\ L \geq 0$ and a finite set $\mathcal{U} \subset \mathbb{R} \times \mathbb{C}$ satisfying $\forall\ u \in \mathcal{U}$,

$\Psi_{g,u} \mathcal{M}'_{(x_0, 0, z_0)} \neq 0$, $\Psi_{g,u} \mathcal{M}''_{(x_0, 0, -z_0)} \neq 0$, $p(z_0, u) \leq c'$, $p(-z_0, u) \leq c''$, s.t.

• $\forall\ m' \in V_{c'}^{(z_0)}({}^R i_* \mathcal{M}')$,

• $\forall\ m'' \in V_{c''}^{(-z_0)}({}^R i_* \mathcal{M}'')$, and

• for any <u>non-negative</u> cutoff function $\chi(t)$, the correspondence

$$C_c^\infty(nb(x_0,z_0)) \ni h(x,z) \mapsto \left(\prod_{u\in\mathcal{U}} \Gamma(s+e(z,u)/z)^{-L}\right) \left\langle (T_{i_x}^* c)(m',\overline{m''})\cdot |t|^{2s},\ h(x,z)\,\chi(t)\right\rangle$$

defines a current which is holom. w.r.t. s.

- If $[m'] \in \Psi_{g,u'}\mathcal{M}'_{x_0,z_0} \subset gr^{V^{(z_0)}}_{c'}\mathcal{M}'_{(x_0,z_0)}$ and $[m''] \in \Psi_{g,u''}\mathcal{M}''_{(x_0,-z_0)}$, then $\left(\prod \Gamma^{-L}\right)$ can be indexed by the subset $\mathcal{U}(m',m'')$ s.t. $u\in\mathcal{U}(m',m'') \iff p(z_0,u)+p(-z_0,u) < c'+c''$ or $u=u'=u''$.

- If $u'=u''=u_0$, then the polar coeffs of the merom. current $h(x,z) \mapsto \left\langle (T_{i_x}^* c)(m',\overline{m''})\cdot |t|^{2s},\ h(x,z)\,\chi(t)\right\rangle$ along $s=-e(z,u_0)/z$ only depend on $[m']$, $[m'']$ and do not depend on the cutoff χ. Furthermore, they define currents in $\mathfrak{C}_{X\times S|S}$ supported on $g^{-1}(0)\times S$.

<u>Sketch of proof of the last point.</u>

Poles are along $s+e(z,u-(n,0))/z$ for $n\in\mathbb{Z}_{\geq 0}$ and $u\in\mathcal{U}(m',m'')$.

If $u\neq u_0$ or if $u=u_0$ and $n\geq 1$, we have

$$p(z_0,u-(n,0)) + p(\sigma(z_0),u-(n,0)) < p(z,u_0) + p(\sigma(z_0),u_0)$$

after the second point, hence a similar inequality for any z in the nbhd of z_0.

<u>Note</u>. $p(z,u)+p(\sigma(z),u) = -2\,\mathrm{Re}(e(z,u)/z)$.

<u>Conclusion</u> If $[m'],[m''] \in \Psi_{g,u_0}\mathcal{M}_{(x_0,z_0)}$, the graphs of the functions

$z\mapsto -e(z,u-(n,0))/z$ for $u\in\mathcal{U}(m',m'')$, $n\in\mathbb{Z}_{\geq 0}$ w/ either $u\neq u_0$ or $u=u_0$ & $n\geq 1$.

do not intersect the graph of $z \mapsto -e(z,u_0)/z$ in the nbhd of z_0.

<u>Def</u>. $\Psi_{g,u} M'|_S \otimes_{\mathcal{O}_S} \overline{\Psi_{g,u} M''|_S} \xrightarrow{\Psi_{g,u} C} \mathfrak{C}_{X\times S|S}$

$$([m'], \overline{[m'']}) \longmapsto \operatorname*{Res}_{s=-e(z,u)/z} \left((\Gamma_{i_g} C)(m', \overline{m''}) \cdot |t|^{2s}, \cdot \chi(t) \right)$$

<u>Some properties</u>.

- $\Psi_{g,u} C(N[m'], \overline{[m'']}) = -z^2\, \Psi_{g,u} C([m'], \overline{N[m'']})$

Since $N = t\partial_t + e(z,u)$ and $\overline{e(z,u)/z} = e(z,u)/z$, enough to prove

$$\Psi_{g,u} C([m' t\partial_t], \overline{[m'']}) = -z^2\, \Psi_{g,u} C([m'], \overline{[m'' t\partial_t]})$$

$$\Uparrow$$

$$t\partial_t (|t|^{2s}) = -z^2\, \overline{t\partial_t} (|t|^{2s}) \quad \text{if } s \gg 0$$

$$\Uparrow$$

$$t\partial_t (|t|^{2s}) = \bar t \partial_{\bar t} (|t|^{2s}), \quad \text{if } s \gg 0.$$

- $\Psi_{g,u}(C^*) = (\Psi_{g,u} C)^*$: check directly.

<u>Rmk</u> If M' or M'' is supp on $g^{-1}(0)$, then $\Psi_{g,u} C = 0$ for any u.

$\rightsquigarrow$ Need to modify the defin of $\Psi_{g,u} C$ if $u \in \mathbb{N} \times \{0\}$

<u>Specialization of triples</u>.

<u>Def</u>. • A triple $\mathcal{T} = (M', M'', C)$ is strictly spec. if M', M'' are so.

- $\Psi_{g,u} \mathcal{T} := (\Psi_{g,u} M', \Psi_{g,u} M'', C)$.
- Such a $\mathcal{T}$ is <u>regular</u> along g if M', M'' are so.

Some properties

- $\Psi_{g,u}(\tau^*) \simeq (\Psi_{g,u}\tau)^*$.

- $N : \Psi_{g,u}\tau \longrightarrow \Psi_{g,u}\tau(-1)$

- $\rightsquigarrow \exists$ monodromy weight filtration $M_\bullet(\Psi_{g,u}\tau)$

- $\longrightarrow gr^M_\bullet(\Psi_{g,u}\tau)$ is an <u>sl_2-triple</u>.

<u>The middle extension quiver</u>. We set $\Phi_{g,(0,0)}\tau = \mathrm{Im}\, N(\frac{1}{2}) \subset \Psi_{g,(-1,0)}\tau(-\frac{1}{2})$

In such a way, we get a quiver $\left(\Psi_{g,(-1,0)}\tau,\ \Phi_{g,(0,0)}\tau,\ can,\ var\right)$

as

$$\Psi_{t,(-1,0)}\tau \underset{var = Id}{\overset{can = N}{\rightleftarrows}} \mathrm{Im}\, N = \Phi_{t,(0,0)}\tau(-\tfrac{1}{2})$$

<u>Pushforward and specialization of triples</u>.

- Let $\tau = (\mathcal{M}', \mathcal{M}'', C)$ be a triple on X. same setting as before (most crucially, f proper)

Then $\forall\, u \in \mathbb{R} \times \mathbb{C}$, ${}^R f_*^{(j)}(\Psi_{g\circ f,u}\tau) \simeq \Psi_{g,u}({}^R f_*^{(j)}\tau)$.

<u>Sketch of proof</u>.

- Need to show ${}^R f_*^{(j,-j)}(\Psi_{g\circ f,u}C) = \Psi_{g,u}({}^R f_*^{(j,-j)}C)$.

- Use the computation w/ lifting of local sections to $V_\bullet^{(g_0)}$ and the property that $V_\bullet^{(g_0)}$ commutes w/ ${}^R f_*^{(j)}$.

Non characteristic pullback

$Y \subset X$ smooth submfd,

$\mathcal{M}$ holonomic $\mathcal{R}_{\mathcal{X}}$-module w/ $\mathrm{Char}\, \mathcal{M} \subset \Lambda \times \mathbb{C}_z$.

- $\mathcal{M}$ is <u>non-characteristic</u> along Y if $T_Y^* X \cap \Lambda \subset T_X^* X$ for a minimal such Λ.
- $\mathcal{M}$ is <u>strictly nonchar.</u> along Y if it is nonchar. along Y and the complex $\mathcal{O}_{\mathcal{Y}} \overset{L}{\underset{\mathcal{O}_{\mathcal{X}}}{\otimes}} \mathcal{M}$ is <u>strict</u>, i.e. each of its cohomology module is strict.

<u>Some properties</u>. Assume $\mathcal{M}$ is strictly nonchar. along Y. Then

- $\mathrm{Tor}^j_{\mathcal{O}_{\mathcal{X}}}(\mathcal{M}, \mathcal{O}_{\mathcal{Y}}) = 0$ for $j \neq 0$ (well-known for $\mathcal{D}$-modules and use strictness to pass to $\mathcal{R}$-modules).
- If $Y = \{t=0\}$ for some local coord. t, $\mathcal{M}$ is strict spec. along Y and $\forall z_0 \in \mathbb{C}$, $V_\cdot^{(z_0)}$ is indexed by $\mathbb{Z}$ and
$$V_k^{(z_0)} \mathcal{M} = \begin{cases} \mathcal{M} \cdot t^{-k+1} & \text{if } k \leq -1 \\ \mathcal{M} & \text{if } k \geq -1 \end{cases}$$

$\rightsquigarrow$ ${}^{R}i^* \mathcal{M} = V_{-1}\mathcal{M} / V_{-2}\mathcal{M} \cdot t$

$\rightsquigarrow$ $\mathcal{M}$ regular along Y.

Sesquilinear pairing between strictly nonchar. $\mathcal{R}$-modules

$Y = \{t=0\} \subset X$: smooth hypersurfaces, $j : U = X \setminus Y \hookrightarrow X$: the open inclusion.

For a sesquilinear pairing C between strictly non-char. $\mathcal{M}', \mathcal{M}''$ along Y, define

${}^{T}i^* C = \Psi_{t,-1} C$.

Prop (Uniqueness across a non char. divisor)

Assume $\mathcal{M}', \mathcal{M}''$ strictly non char. along Y.

Then, given any sesquilinear pairing $C^\circ : j^*\mathcal{M}'|_S \otimes_{\mathcal{O}_S} \overline{j^*\mathcal{M}''|_S} \to \mathfrak{C}_{U\times S}|_S$

$\exists$ at most one sesquilinear pairing $C : \mathcal{M}'|_S \otimes_{\mathcal{O}_S} \overline{\mathcal{M}''|_S} \to \mathfrak{C}_{X\times S}|_S$

which extends C°.

Pf. Need to show $C^\circ = 0 \Rightarrow C = 0$, So assume $C(m', \overline{m''})$ supported on Y.

$\Rightarrow C := C(m', \overline{m''}) = \sum_{k,\ell \le p} C_{k,\ell}\, \delta_{t=0}\, \partial_t^k \partial_{\bar t}^\ell$

$\leadsto$ if η vanishes at order $\geq p+1$ along $t=0$, then $\langle C, \eta\rangle = 0$, hence $C_{k,\ell}(\eta) = 0$.

— Use Bernstein equation to show that all $C_{k,\ell}$ vanish. $\square$

Lecture 6. Monodromy filtration and sl_2 structures

The Lefschetz decomposition

A k-linear abelian cat. (k = some field)

(H, N) an object of A w/ a nilpotent endomorphism.

Lemma (Jacobson-Morozov) $\exists!$ $\nearrow$ exhaustive filtration of H indexed by $\mathbb{Z}$, called the monodromy filtration relative to N and denoted by $M_\bullet(N)H$, or simply $M_\bullet H$ s.t.

(a) $\forall \ell \in \mathbb{Z}$, $N(M_\ell H) \subset M_{\ell-2} H$,

(b) $\forall \ell \geq 1$, N^ℓ induces an isom. $gr_\ell^M H \xrightarrow{\sim} gr_{-\ell}^M H$.

Explicit formula: $M_\ell(N) = \sum_{k \geq 0, -\ell} N^k (\ker N^{\ell+1+2k})$

Def. $sl_2 = \langle X, Y, H \rangle$, $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$,

$w = e^X e^{-Y} e^X = e^{-Y} e^X e^{-Y} \in SL_2$ Weyl element

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Given $(H, N) \rightsquigarrow (gr_\bullet^M H, gr N)$ is an sl_2-rep'n in A, by setting

$H = \ell \, Id$ on $gr_\ell^M H$ and $Y = gr N$. The action of X is defined uniquely.

Lefschetz decomposition.

For an sl_2-rep'n $H = \bigoplus_\ell H_\ell$ and $\ell \geq 0$, set

$$P_\ell(H) := \ker(Y^{\ell+1} : H_\ell \longrightarrow H_{-\ell-2}) = \ker(X : H_\ell \longrightarrow H_{\ell+2})$$

$$H_\ell \xrightarrow[Y^\ell]{\sim} H_{-\ell} \xrightarrow[Y]{} H_{-\ell-2}$$

Then, for $\ell \geq 0$, $H_\ell = \bigoplus_{j \geq 0} Y^j P_{\ell+2j}(H)$ and $H_{-\ell} = \bigoplus_{j \geq 0} Y^{\ell+j} P_{\ell+2j}(H)$

Lefschetz quiver.

Lefschetz quiver (H, G, c, v) w/ H, G objects in A, and $c : H \to G$, $v : G \to H$

s.t. $c \circ v =: N_G$ and $v \circ c =: N_H$ are nilpotent.

$\rightsquigarrow$ $v \circ N_G = N_H \circ v$, $c \circ N_H = N_G \circ c$.

$\longrightarrow$ Abelian cat. Objects denoted by $H \underset{v}{\overset{c}{\rightleftarrows}} G$

Def. (H, G, c, v): Lefschetz quiver in A.

- middle ext'n: If c is epi & v mono.
- punctual support: If $H = 0$
- (H, G, c, v) is S-decomposable if $=$ middle ext. $\oplus$ punct. supp.

Lemma (H, G, c, v) is <u>S-decomposable</u> $\iff$ $G = \mathrm{Im}\, c \oplus \ker v$.

$$\rightsquigarrow \; H \underset{v}{\overset{c}{\rightleftarrows}} G \simeq H \underset{v}{\overset{c}{\rightleftarrows}} \mathrm{Im}\, c \;\oplus\; 0 \underset{0}{\overset{0}{\rightleftarrows}} \ker v.$$

Prop For an <u>S-decomp. Letschetz quiver</u> (H, G, c, v),

$$c(M_\ell(N_H)) \subset M_{\ell-1}(N_G) \quad \text{and} \quad v(M_\ell(N_G)) \subset M_{\ell-1}(N_H).$$

(Use the explicit expression of $M_\bullet$).

<u>$\mathfrak{sl}_2$-quiver</u>. (H, G, c, v) w/ H, G $\mathfrak{sl}_2$-rep'n in $\mathcal{A}$ and $c: H \to G$, $v: G \to H$

w/ $c: H_k \to G_{k-1}$, and $v: G_k \to H_{k-1}$, for each $k \in \mathbb{Z}$.

s.t. $c \circ v = Y_G$ and $v \circ c = Y_H$.

Note : c, v commute w/ Y, but not morphisms of $\mathfrak{sl}_2$-rep'n in $\mathcal{A}$ since they do not commute w/ H (nor w/ X).

- Both $c: H_k \to G_{k-1}$ and $v: G_k \to H_{k-1}$ are <u>epi for $k \le 0$</u> and <u>mono for $k \ge 1$</u>

Prop. (H, G, c, v) : $\mathfrak{sl}_2$-quiver in $\mathcal{A}$.

- middle ext'n if <u>c is epi & v is mono</u>.
- punctual support: if $H = 0$.
- (H, G, c, v) is S-decomp. if $=$ middle ext. $\oplus$ punct. supp.

Prop. If (H, G, c, v) is an S-decomp. Lefschetz quiver in $\mathcal{A}$ w/ nilp. endo. N_H, N_G, then the M-graded quiver $(\mathrm{gr}^M_\bullet H, \mathrm{gr}^M_\bullet G, \mathrm{gr}\, c, \mathrm{gr}\, v)$ is an S-decomp. $\mathfrak{sl}_2$-quiver.

Def'n An sl_2-quiver (H, G, c, v) satisfies the <u>weak Lefschetz property</u> if v is an isom. for $k \le -1$ (and an epi for $k=0$).

<u>Justification of term.</u>: X projective, Y a generic hyperplane section. Then

(weak Lefschetz) $H_k(Y;\mathbb{Z}) \longrightarrow H_k(X;\mathbb{Z})$ is $\begin{cases} \text{iso. if } k < \dim Y \\ \text{onto if } k = \dim Y \end{cases}$.

Prop. (1) If (H, G, c, v) is S-decomp., it satisfies the weak Lefschetz property.

(2) If (H, G, c, v) satisfies the weak Lefschetz property, then $v: G_{-1} \xrightarrow{\sim} H_{-2}$ is an iso.

$\rightsquigarrow$ $P_0(H) = \ker(Y: H_0 \to H_{-2}) = \ker(c: H_0 \to G_{-1})$.

<u>sl_2-twistor structures</u>

Def'n. Polarized sl_2-twistor str. centred at w:

- $\mathcal{T} = \bigoplus_j \mathcal{T}_j$, $\mathcal{T}_j$ pure twistor str. of weight $w+j$ $(\simeq \mathcal{O}(w+j)^2)$
- $X: \mathcal{T}_j \longrightarrow \mathcal{T}_{j+2}(1)$, $Y: \mathcal{T}_j \longrightarrow \mathcal{T}_{j-2}(-1)$, $H = \bigoplus_j j\,\mathrm{Id}_{\mathcal{T}_j}$.
- $\mathcal{T}^* = \bigoplus_j (\mathcal{T}^*)_j$ w/ $(\mathcal{T}^*)_j = (\mathcal{T}_{-j})^*$ centred at $-w$.
- Action of sl_2 on $\mathcal{T}^*$: X acts as X^*, Y as Y^* and H as $-H^*$.
- Tate twist: $\mathcal{T}(\ell) = \bigoplus \mathcal{T}_j(\ell)$ centred at $w-2\ell$.
- Pre-polarization $\mathcal{S}: \mathcal{T} \xrightarrow{\sim} \mathcal{T}^*(-w)$ commuting w/ the sl_2-actions, in particular $\mathcal{S}_j: \mathcal{T}_j \xrightarrow{\sim} (\mathcal{T}_{-j})^*(-w+j)$
- $\mathcal{S}$ is a <u>polarization</u> if for each $j \in \mathbb{Z}$ $\begin{cases} \mathcal{S} \circ w \text{ is a polarization of wt } w+j \text{ of } \mathcal{T}_j \quad (w: \text{Weyl elt}) \\ \text{equivtly, } (-1)^j \mathcal{S} \circ Y^j \text{ is a polarization of wt } w+j \text{ of } P_j(\mathcal{T}). \end{cases}$

$\mathfrak{sl}_2$-twistor quiver w/ central weight w

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left, "c"] & \mathcal{T}' \arrow[l, bend left, "v"] \end{tikzcd}$$

(-1)

w/
- $\mathcal{T}$: $\mathfrak{sl}_2$ - twistor str. w/ central wt $w-1$
- $\mathcal{T}'$: $\mathfrak{sl}_2$ - twistor str. w/ central wt w
- $c: \mathcal{T} \to \mathcal{T}'$, $v: \mathcal{T}' \to \mathcal{T}(-1)$

Polarization

- $(\mathcal{T}, \mathcal{T}', c, v)^* := (\mathcal{T}^*(1), \mathcal{T}'^*, -v^*, -c^*)$ centred at $-w$
- Pre-polarization: (S, S') w/
  - S = pre-pol. of $\mathcal{T}$ of wt $w-1$,
  - S' = pre-pol. of $\mathcal{T}'$ of wt w
  - comm. diagrams

$$\begin{tikzcd} \mathcal{T} \arrow[r, "S"] \arrow[d, "c"'] & \mathcal{T}^*(-w+1) = \mathcal{T}^*(1)(-w) \arrow[d, "-v^*"] \\ \mathcal{T}' \arrow[r, "S'"] & \mathcal{T}'^*(-w) \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{T}' \arrow[r, "S'"] \arrow[d, "v"'] & \mathcal{T}'^*(-w) \arrow[d, "-c^*"] \\ \mathcal{T}(-1) \arrow[r, "S"] & \mathcal{T}^*(-w) \end{tikzcd}$$

- Polarization: S, S' are polarization of $\mathcal{T}, \mathcal{T}'$.

Three Theorems

Thm. If $(\mathcal{T}, \mathcal{T}', c, v)$ is a middle ext'n $\mathfrak{sl}_2$-quiver of central wt w, and if $\mathcal{T}$ is a polarizable $\mathfrak{sl}_2$ - twistor str., then $(\mathcal{T}, \mathcal{T}', c, v)$ is polarizable.

Thm. Let $(\mathcal{T}, \mathcal{T}', c, v)$ be a polarizable $\mathfrak{sl}_2$ - twistor quiver of central wt w. Then the $\mathfrak{sl}_2$-twistor str. $\mathcal{T}'$ decomposes as $\mathcal{T}' = \operatorname{Im} c \oplus \operatorname{Ker} v$ in the cat. of $\mathfrak{sl}_2$- twistor - str., and $(\mathcal{T}, \mathcal{T}', c, v)$ is S-decomp.

Thm. Let $(\mathcal{T},\mathcal{T}',c,v)$ be an sl_2-twistor quiver of central wt w s.t.

(1) $(\mathcal{T},\mathcal{T}',c,v)$ satisfies the weak Lefschetz property.

(2) there exists a pre-polarization (S,S') of $(\mathcal{T},\mathcal{T}',c,v)$ s.t. S' is a polarization of $\mathcal{T}'$ and $P_0 S$ is a polarization of $P_0\mathcal{T}$.

Then S is a polarization of $\mathcal{T}$ and $(\mathcal{T},\mathcal{T}',c,v)$ is S-decomposable.

Proofs by reduction to wt 0 by "half Tate twist's".

Polarized sl_2-rep'n.

- H finite dim'l vec. space w/ an sl_2-rep'n $\rightsquigarrow$ $H=\bigoplus_\ell H_\ell$
- $S: H \to H^*$ $\leftrightarrow$ $S: H\otimes\overline{H}\to\mathbb{C}$ non deg. s.t.

$$S(Xx,\bar y) = -S(x,\overline{Xy})$$
$$S(Yx,\bar y) = -S(x,\overline{Yy})$$
$$S(Hx,\bar y) = -S(x,\overline{Hy})$$

i.e. S induces $S_\ell: H_\ell\otimes\overline{H_{-\ell}}\to\mathbb{C}$ non deg. $\forall \ell$ and $\bigoplus_\ell H_\ell$ is S-orthogonal.

$\rightsquigarrow$ for $x,y\in P_\ell(H)$,

$$P_\ell S(x,\bar y) := S_\ell(x,\overline{Y^\ell y}) = (-1)^\ell S_{-\ell}(Y^\ell x,\bar y).$$

Def'n. S is a polarization of H if S is Hermitian and

$h(x,\bar y) := S(wx,\bar y) = S(x,\overline{w^{-1}y})$ (Hermitian) positive definite.

Note $h(Xx,\bar y) = h(x,\overline{Yy})$, $h(Hx,\bar y) = h(x,\overline{Hy})$ and $\bigoplus_\ell H_\ell$ is h-orthogonal.

Equiv. def'n. S is a polarization of H iff S is Herm. and $\forall \ell\geq 0$, $P_\ell S$ is (Hermitian) pos. def. on $P_\ell(H)$.

Proof: ($\Rightarrow$) Note w/ $P_\ell(H) = (-1)^\ell \frac{Y^\ell}{\ell!}$, so for $0 \neq x \in P_\ell(H)$,

$$0 < h(x,\bar{x}) = S(wx,\bar{x}) = \frac{(-1)^\ell}{\ell!} S(Y^\ell x, \bar{x}).$$

($\Leftarrow$) For $0 \neq x \in H_\ell$, set $x = \sum_{j \geq 0} Y^j x_{\ell+2j}$, $x_{\ell+2j} \in P_{\ell+2j}(H)$.

Then
$$S(wx,\bar{x}) = \sum_{j,k \geq 0} S(wY^j x_{\ell+2j}, \overline{Y^k x_{\ell+2k}})$$
$$= \sum_{j,k \geq 0} \frac{(-1)^{\ell+j}}{(\ell+j)!} S(Y^{\ell+j} x_{\ell+2j}, \overline{Y^k x_{\ell+2k}})$$
$$= \sum_{j \geq 0} \frac{(-1)^{\ell+j}}{(\ell+j)!} S(Y^{\ell+j} x_{\ell+2j}, \overline{Y^j x_{\ell+2j}})$$
$$= \sum_{j \geq 0} \frac{1}{(\ell+j)!} S(x_{\ell+2j}, \overline{Y^{\ell+2j} x_{\ell+2j}})$$
$$= \sum_{j \geq 0} \frac{1}{(\ell+j)!} P_{\ell+2j} S(x_{\ell+2j}, \overline{x_{\ell+2j}}) > 0.$$

Polarization of Im Y (first thm)

Set $G = \mathrm{Im}\, Y$ w/ induced sl_2-rep'n.

- $G_\ell = Y(H_{\ell+1}) \subset H_{\ell-1} \cong \begin{cases} H_{\ell+1} & \ell \geq 0 \\ H_{\ell-1}, & \ell \leq 0 \end{cases}$
- $P_\ell(G) \cong Y(P_{\ell+1}(H))$.

Def. For $x, y \in G$, set $x = Yx_1$, $y = Yy_1$ and $S_G(x,\bar{y}) := S(x_1,\bar{y}) = -S(x,\bar{y}_1)$

(indep. of choice)

Positivity: For $\ell \geq 0$ and $x \in P_\ell(G)$, can choose $x_1 \in P_{\ell+1}(H)$

and $S_G(x, \overline{Y^\ell x}) = S(x_1, \overline{Y^\ell x}) = S(x_1, \overline{Y^{\ell+1} x_1}) > 0$.

S - decomposition (Second thm)

Assume (H, G, c, v) an sl_2- quiver

$c: H_\ell \to G_{\ell-1}$, $v: G_\ell \to H_{\ell-1}$, $v \circ c = Y_H$, $c \circ v = Y_G$

- S_H, S_G: polarizations s.t. $\forall \ell$, $S_{H,\ell}(vx, \bar{y}) = S_{G,\ell+1}(x, \overline{cy}) : G_{\ell+1} \otimes \overline{H_{-\ell}} \to \mathbb{C}$.

Then (H, G, c, v) is S-decomposable, i.e.

$$(H, G, c, v) \simeq (H, \mathrm{Im}\, c, c, v) \oplus (0, \ker v, 0, 0).$$

Idea: to play w/ Lefschetz decomposition and positivity

Note. • c, v injective for $\ell \geq 1$

• $\begin{cases} c(P_\ell H) \subset \ker(Y_G : G_{\ell-1} \to G_{-2\ell-1}) \\ v(P_\ell G) \subset \ker(Y_H : H_{\ell-1} \to H_{-2\ell-1}) \end{cases}$

Lefschetz decomp. $\Rightarrow$ $\begin{cases} c(P_\ell H) \subset Y_G(P_{\ell+1} G) & \text{if } \ell = 0 \\ c(P_\ell H) \subset P_{\ell-1} G \oplus Y_G(P_{\ell+1} G) & \text{if } \ell \geq 1 \end{cases}$

$\begin{cases} v(P_\ell G) \subset Y_H(P_{\ell+1} H) & \text{if } \ell = 0 \\ v(P_\ell G) \subset P_{\ell-1} H \oplus Y_H(P_{\ell+1} H) & \text{if } \ell \geq 1 \end{cases}$

First step, positivity $\Rightarrow$ $v(P_\ell G) \cap P_{\ell-1} H = 0$. We prove $y \in P_\ell G$ and

$vy \in P_{\ell-1} H \Rightarrow y = 0$. By contradiction, assume $y \neq 0$.

- Positivity implies $S_G(y, \overline{Y_G^\ell y}) > 0$ and $S_H(vy, \overline{Y_H^{\ell-1} vy}) \geq 0$
- Then $0 \leq S_H(vy, \overline{Y_H^{\ell-1} vy}) = S_H(vy, \overline{v Y_G^{\ell-1} y}) = S_G(cvy, \overline{Y_G^{\ell-1} y})$

$= S_G(Y_G y, \overline{Y_G^{\ell-1} y}) = - S_G(y, \overline{Y_G^\ell y}) < 0$ ▨

<u>2nd step</u>: Playing w/ Lefschetz decomp. and positivity

$\Rightarrow c(P_\ell H) \subset P_{\ell-1} G$ if $\ell \geq 0$, and $=$ if $\ell \geq 2$.

<u>3rd step, end of proof</u>

- C in 2nd step $\Rightarrow$ c compatible w/ Lefschetz decomp.
- Similar argument $\Rightarrow$ v also.
- Then proving $G = \mathrm{Im}\, c \oplus \ker v$ is obtained by checking on each $P_\ell G$ for $\ell \geq 0$.
- But for $\ell \geq 1$, Step 2 $\Rightarrow$ $P_\ell G = c(P_{\ell+1} H)$.

  Moreover, $\ker(v|_{P_\ell G}) = 0$, so assertion ok.

  Remains the case $\ell = 0$, done directly.

<u>Weak Lefschetz (third theorem)</u>.

Assume $\cdot\ (H, G, c, v)$ an sl_2-quiver

$c: H_\ell \to G_{\ell-1}$, $v: G_\ell \to H_{\ell-1}$, $v \circ c = Y_H$, $c \circ v = Y_G$

- S_H, S_G: pre-polarizations s.t. $\forall \ell$, $S_{H,\ell}(x, \overline{vy}) = S_{G,\ell-1}(cx, \overline{y}) : H_\ell \otimes \overline{G_{-\ell+1}} \to \mathbb{C}$.

  S_G and $P_0(S_H)$ are polarizations

  Then S_H is a polarization ($\Rightarrow (H, G, c, v)$ S-decomp.)

<u>Claim</u>. $c(P_\ell H) \subset P_{\ell-1} G$ if $\ell \geq 1$ and $c(P_0 H) = 0$.

$$0 = Y_H^{\ell+1}(P_\ell H) = v Y_G^\ell c(P_\ell H)$$

But $Y_G^\ell c(P_\ell H) \subset G_{-\ell-1}$, so $WL \Rightarrow Y_G^\ell c(P_\ell H) = 0$, i.e. $c(P_\ell H) \subset P_{\ell-1} G$.

<u>End of the proof</u>. $\ell \geq 1$ and $0 \neq x \in P_\ell H$.

$S_H(x, \overline{Y_H^\ell x}) = S_H(x, \overline{v Y_G^{\ell-1} cx}) = S_G(cx, \overline{Y_G^{\ell-1} cx}) > 0$ since $cx \in P_{\ell-1} G$ after the claim. $\square$

Differential polarized sl_2-twistor structure

Given a polarized sl_2-twistor str. $(\mathcal{T}, S)$ of central wt w. A differential is a morphism $d: \mathcal{T} \to \mathcal{T}(-1)$ s.t.

- $d \circ d = 0$
- d is self-adjoint wrt. S
- $[H, d] = -d$ (i.e. $d: \mathcal{T}_\ell \to \mathcal{T}_{\ell-1}$) and $[d, Y] = 0$.

<u>Thm</u> $(\mathcal{T}, S, d)$ as above. Then $(\ker d / \operatorname{Im} d, H, Y, S)$ is a polarized sl_2-twistor str.

of central wt w.

<u>Proof by reduction to weight 0</u>:

- Assume (H, S) polarized sl_2 str. $d: H \to H$ s.t.
  - $d \circ d = 0$, $S(d\cdot, \bar{\cdot}) = S(\cdot, \overline{d\cdot})$, $[d, H] = -d$, $[d, Y] = 0$
- Then $\ker d / \operatorname{Im} d$ is of the same kind.

Positivity of $h \Rightarrow$ can use harmonic theory for h, d.

$d^* = h$-adjoint of d and $\Delta = dd^* + d^*d$

- $[d^*, H] = d^* \Rightarrow [\Delta, H] = 0$. i.e. Δ preserves grading.

In a way compatible w/ the grading,

$\ker d / \operatorname{Im} d \simeq \ker d \cap \ker d^* = \ker \Delta$, $H = \ker \Delta \overset{\perp}{\oplus} \operatorname{Im} \Delta$.

- Main question: Δ commutes w/ Y. i.e. $\Delta \in P_0(\operatorname{End} H)$.

This reduces to the understanding of the Lefschetz decomposition of $(\mathbb{C}d^* \oplus \mathbb{C}d)^{\otimes 2}$.

"Clebsch-Gordan formulae"

Lecture 7. Decomposability w.r.t. the support.

$\mathcal{M}$: holonomic $\mathcal{R}_{\mathcal{X}}$-module $\overline{T^*_{Z_i^\circ}X}$, Z_i° smooth part of Z_i

$\mathrm{Char}\,\mathcal{M} \subset \Lambda \times \mathbb{C}_z$, $\Lambda = \bigcup_i T^*_{Z_i}X$ w/ Z_i irred. closed analytic in X.

Assume that the family Z_i is minimal w.r.t. this property

Question: Is it possible to decompose $\mathcal{M}$ as $\bigoplus_i \mathcal{M}_i$ s.t.

$\mathrm{Supp}\,\mathcal{M}_i \subset Z_i$ or $\mathcal{M}_i = 0$. and the same for any coherent submodule or quotient module of $\mathcal{M}_i$?

Analogous question for regular holonomic $\mathcal{D}_X$-modules or perverse sheaves: F a constr. cpx:

Is F isom. to the direct sum of some $IC(L_i)$ where L_i is a local system on a Zariski open subset Z_i° of Z_i?

$\rightsquigarrow$ Notion of S(upport) decomposability.

However, strictness is also involved in the definition. (strict S-decomp.)

Idea (M. Saito). Analyze this property w.r.t. any holomorphic function.

The main prop.

Prop $g: X \to \mathbb{C}$ holom. funct. and $\mathcal{M}$ strictly spec. $\mathcal{R}_{\mathcal{X}}$-module along (g).

(a) TFAE
(1) $\mathrm{var}: \Psi_{g,(0,0)}\mathcal{M} \to \Psi_{g,(-1,0)}\mathcal{M}$ is injective
(2) $\mathcal{M}$ has no proper sub-$\mathcal{R}_{\mathcal{X}}$-module supp. on $g^{-1}(0)$
(3) $\mathcal{M}$ has no proper strictly spec. submodule supp. on $g^{-1}(0)$.

(b) If $\mathrm{can}: \Psi_{g,(-1,0)}\mathcal{M} \to \Psi_{g,(0,0)}\mathcal{M}$ is onto, then $\mathcal{M}$ has no proper quotient having a KM filtration and supp. on $g^{-1}(0)$.

(c) TFAE (1') $\Psi_{g,(0,0)}\mathcal{M} = \mathrm{Im}\,\mathrm{can} \oplus \ker \mathrm{can}$

(2') $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ w/ $\mathcal{M}_1$ satisfying (a) and (b) and $\mathcal{M}_2$ supp. on $g^{-1}(0)$.

[Note: $\mathcal{M}_1$, $\mathcal{M}_2$ are strictly spec.]

Strict S-decomposability

Def. An $\mathcal{R}_{\mathcal{X}}$-module $\mathcal{M}$ is

- strictly S-decomp. along (g) if it is strictly specializable along (g) and satisfies the equiv. conditions (c);
- strictly S-decomp. at $x_0 \in X$ if for any analytic germ $g: (X, x_0) \to (\mathbb{C}, 0)$, $\mathcal{M}$ is strictly S-decomp. along (g) in some nbhd of x_0;
- strictly S-decomp. if it is strictly S-decomposable at all pts $x_0 \in X$.

Prop / Def. Assume $\mathcal{M}$ holonomic strictly S-dec. and $\mathrm{supp}\,\mathcal{M} \subset Z \subset X$ equidim'l closed analytic subset. Say that $\mathcal{M}$ has pure support Z if one of the equiv. cond'ns holds:

(1) near any $x_0 \in X$, $\nexists \neq 0$ coh. submodule w/ supp. of codim ≥ 1 in Z.

(2) near any $x_0 \in X$, $\nexists \neq 0$ morphism $\mathcal{M} \to \mathcal{N}$ w/ $\mathcal{N}$ strictly S-dec. at x_0 and w/ image of codim ≥ 1 in Z

(3) For any germ $g: (X, x_0) \to (\mathbb{C}, 0)$, both (a), (b) hold, i.e. var is injective and can is onto.

Proof that (3) $\Rightarrow$ (2) If $\varphi: \mathcal{M} \to \mathcal{N}$ w/ $\mathrm{Im}\,\varphi \subset g^{-1}(0)$ and $\mathcal{N}$ strictly S-decomp. at x_0. Then $\mathrm{Im}\,\varphi$ has a KM filtration, so (b) $\Rightarrow$ $\mathrm{Im}\,\varphi = 0$.

Proof that (2) $\Rightarrow$ (3): Fix g and consider the decomp. $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ as in (c) along g.

The projection $\mathcal{M} \to \mathcal{M}_2$ must have image 0, so $\mathcal{M} = \mathcal{M}_1$. $\square$

Thm. Assume that $\mathcal{M}$ is holonomic and strictly S-decomp. Let $(Z_i)_{i\in I}$ be a minimal family of irred. closed analytic subsets of X, s.t. $\operatorname{Char} \mathcal{M} \subset \bigcup_i (T^*_{Z_i} X) \times \mathbb{C}_z$. Then there exists a unique decomp. $\mathcal{M} = \bigoplus_i \mathcal{M}_i$, where $\mathcal{M}_i = 0$ or has pure support Z_i.

Idea of proof. Argue locally and glue various local decompositions according to uniqueness. Locally, choose suitable germs of holomorphic functions to separate the various local irred. components, and apply the def'n. $\square$

Some Corollaries

Cor. $\mathcal{M}, \mathcal{N}$ holonomic and strictly S-dec. (Z_i) family of pure components for both. Then

(1) any morphism $\mathcal{M}_{Z_i} \to \mathcal{N}_{Z_j}$ vanishes if $Z_i \neq Z_j$;

(2) $\mathcal{M}, \mathcal{N}$ are strict.

(3) any sesquilinear pairing $C: \mathcal{M}_{Z_i}|_S \otimes_{\mathcal{O}_S} \overline{\mathcal{N}_{Z_j}|_S} \to \mathfrak{C}_{X\times S|S}$ vanishes if $Z_i \neq Z_j$.

Proof (1) The image is supp. on $Z_i \cap Z_j$. If $Z_i \cap Z_j \neq Z_j$, the image is zero since $\mathcal{N}_{Z_j}$ has pure supp. Z_j, so $Z_j \subset Z_i$. If the codim is ≥ 1, then the morphism is zero by def. of pure support. $\square$

(2) local question near any $x_0 \in X$, can assume that $\mathcal{M}$ has pure support Z (irreducible near x_0).

• $\exists\, Z^\circ \subset Z$ smooth open dense s.t. $\mathcal{M}|_{Z^\circ}$ is strict:

By Kashiwara's equiv., can reduce to $Z^\circ = X$ and Char $\mathcal{M} \subset T$ and Char $\mathcal{M} \subset (T^*_X X) \times \mathbb{C}_z$.

$\rightsquigarrow$ can assume $\mathcal{M}$ is $\mathcal{O}_{\mathcal{X}}$ - coherent and $\exists$ open dense X° s.t. $\mathcal{M}$ is $\mathcal{O}_{\mathcal{X}^\circ}$ - loc. free

• m: local section of $\mathcal{M}_{x_0}$ killed by $p(z)$. $p \in \mathbb{C}[z] \setminus \{0\}$

$\Rightarrow$ $R_{\mathcal{X}} \cdot m \subset \mathcal{M}$ supp. in Codim ≥ 1 in Z.

Z = pure supp. of $\mathcal{M}$ $\Rightarrow$ $m = 0$

(3) Local question on $X \times S$ $\rightsquigarrow$ fix $(x_0, z_0) \in \mathcal{X}$.

• Assume e.g. $Z_i \not\subset Z_j$ $\rightsquigarrow$ $\exists g$ s.t. $g \equiv 0$ on Z_j and $g \not\equiv 0$ on Z_i.

Can assume g is a local coord. t.

• Consider C as a morphism $\mathcal{M}_{Z_i}|_S \longrightarrow \mathcal{H}om_{\overline{R_{\mathcal{X}}|_S}}(\overline{\mathcal{N}_{Z_j}|_S}, \mathfrak{C}_{X \times S|S})$ ($R_{\mathcal{X}}$ - linear)

Fix local $R_{\mathcal{X}}$ - generators $n_1, \dots, n_\ell$ of $\mathcal{N}_{Z_j, (x_0, z_0)}$.

Since $V^{(z_0)}_{<0}(\mathcal{N}_{Z_j, (x_0, z_0)}) = 0$, there exists $q \geq 0$ s.t. $t^q n_k = 0$ for all $k = 1, \dots, \ell$

Let $m \in \mathcal{M}_{Z_i, (x_0, z_0)}$, and let p = max ord $[C(m)(\overline{n_k})]$ on nb(x_0, z_0).

Note $t^{p+1+q} / \bar{t}^q$ is C^p $\rightsquigarrow$ $\forall\, k = 1, \dots, \ell$.

$$C(m)(\overline{n_k}) \cdot t^{p+1+q} = C(m)(\overline{n_k})\, \bar{t}^q \cdot \frac{t^{p+1+q}}{\bar{t}^q} = 0$$

$\rightarrow$ $t^{p+1+q}\, C(m) \equiv 0$

• Apply this to generators of $\mathcal{M}_{Z_i, (x_0, z_0)}$ $\rightsquigarrow$ all local sections of $C(\mathcal{M}_{Z_i, (x_0, z_0)})$

killed by some t^N.

• $\mathcal{M}_{Z_i}$ has pure supp. Z_i $\Rightarrow$ $V^{(z_0)}_{<0} \mathcal{M}_{Z_i, (x_0, z_0)}$ $R_{\mathcal{X}}$ - generates $\mathcal{M}_{Z_i, (x_0, z_0)}$.

$\rightsquigarrow$ enough to show $C(V^{(z_0)}_{<0} \mathcal{M}_{Z_i, (x_0, z_0)}) = 0$.

Dagger b

(a) Show $C(V_k^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)}) = 0$ for $k \ll 0$.

For z_0 fixed, $\exists\, k \ll 0$ s.t. $t: V_k^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)} \xrightarrow{\sim} V_{k-1}^{(z_0)} \mathcal{M}_{Z_i,z_0}$.

$\to t: C(V_k^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)}) \xrightarrow{\sim} C(V_{k-1}^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)})$, hence t acts injectively on

$C(V_k^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)})$. But t is nilpotent. ※

(b) Let $k<0$ s.t. $C(V_{k-1}^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)}) = 0$, and $m \in V_k^{(z_0)} \mathcal{M}_{Z_i\,(x_0,z_0)}$

$\exists\, b(s) = \prod_{u:\, \beta(z_0,u) \in (k-1,k]} (s + e(z,u))^L$ s.t. $m \cdot b(t\partial_t) \in V_{k-1}^{(z_0)} \mathcal{M}_{Z_i,(x_0,z_0)}$.

hence $C(m) \cdot b(t\partial_t) = 0$.

$\exists\, N$ s.t. $C(m)\, t^{N+1} = 0$. Let $B(s) = \prod_{\ell=0}^{N} (s - \ell z)$

$\rightsquigarrow C(m) \cdot B(t\partial_t) = 0$. $\left(C(m)\, t^{N+1}\, \partial_t^{N+1} = 0 \right)$

$b(s)$ & $B(s)$ have no common root $\Rightarrow \exists\, p(z) \in \mathbb{C}[z] \setminus \{0\}$ s.t.

$C(m) \cdot p(z) = 0 \Rightarrow C(m) = 0$.

Proof of the main proposition

Set $\mathcal{N} = R i_{g*} \mathcal{M}$ on $\mathcal{X} \times \mathbb{C}$.

Proof of (a1) $\Leftrightarrow$ (a2): Enough to show that the maps

$$\begin{tikzcd}
 & \ker(t: V_0\mathcal{N} \to V_{-1}\mathcal{N}) \arrow[dl] \arrow[dr] & \\
\ker(t: \mathcal{N} \to \mathcal{N}) & & \ker(t: \mathrm{gr}_0^V \mathcal{N} \to \mathrm{gr}_{-1}^V \mathcal{N})
\end{tikzcd}$$

are isomorphisms.

- Right one: clear since $t: V_{<0}\mathcal{N} \to V_{<-1}\mathcal{N}$ is an isom.
- Left one: use that t is injective on $\mathrm{gr}^V_c \mathcal{N}$ for $c\neq 0$.

Proof of (a2) $\Leftarrow$ (a3) ($\Rightarrow$ clear)

- $\mathcal{T}$: t-torsion submodule of $\mathcal{N}$
- $\mathcal{T}'$: submod. gen. by $\mathcal{T}_0 := \ker[t: \mathcal{N} \to \mathcal{N}]$

Claim. $\mathcal{T}'$ is strictly spec.

(a3) $\Rightarrow$ $\mathcal{T}'=0$, hence $t: \mathcal{N}\to\mathcal{N}$ is injective, so $\mathcal{T}=0$.

- $\mathcal{T}'$ is $\mathcal{R}_{\mathcal{X}\times\mathbb{C}}$-coherent:

  Note $\mathcal{T}_0 := \ker(t: \mathrm{gr}^V_0\mathcal{N} \to \mathrm{gr}^V_{-1}\mathcal{N})$

  = kernel of a morphism between $\mathcal{R}_{\mathcal{X}}$-coh. modules

  $\Rightarrow$ $\mathcal{T}_0$ is $\mathcal{R}_{\mathcal{X}}$-coh.

  $\Rightarrow$ $\mathcal{T}'$ is $\mathcal{R}_{\mathcal{X}\times\mathbb{C}}$-coh.

- $\mathcal{T}'$ is strictly spec.: • $\mathcal{T}_0 := \ker(t: \mathrm{gr}^V_0 \mathcal{N} \to \mathrm{gr}^V_{-1}\mathcal{N}) \Rightarrow \mathcal{T}_0$ strict
- $U_\bullet\mathcal{T}'$: filt'n induced by $V_\bullet\mathcal{N}$ on $\mathcal{T}'$. Then $U_{<0}\mathcal{T}'=0$ and $\mathrm{gr}^U_c\mathcal{T}'=0$ for $c\notin\mathbb{Z}_{\geq 0}$.
- $\forall k\geq 0$, $\mathcal{T}_0 + \mathcal{T}_0\eth_t + \cdots + \mathcal{T}_0\eth_t^k =: U'_k\mathcal{T}'$: induction on k to show $U_k = U'_k$.

Inclusion $\supset$ clear.

Inclusion $\subset$: $(x_0,z_0)\in\mathcal{X}$, m in $U_k\mathcal{T}'_{(x_0,z_0)}$ and $U'_\ell\mathcal{T}'_{(x_0,z_0)}$.

$\ell>k \Rightarrow m\in\mathcal{T}'_{(x_0,z_0)}\cap V_{\ell-1}\mathcal{N}_{(x_0,z_0)}$. hence $t^\ell m\in V_{-1}\mathcal{N}_{(x_0,z_0)}\cap\mathcal{T}'_{(x_0,z_0)}=0$

Set $m = m_0 + m_1\eth_t + \cdots + m_\ell\eth_t^\ell$, w/ $m_j t=0$ $(j=0,\dots,\ell)$

Then $m_\ell t^\ell \eth_t^\ell = 0$.

Note $t^\ell \partial_t^\ell m_\ell = \prod_{j=0}^{\ell}(\partial_t t - j\eth)\cdot m_\ell = (-1)^\ell \ell!\, \eth^\ell m_\ell$ and $\mathcal{T}_0$ is strict

$\Rightarrow m_\ell = 0 \rightsquigarrow m \in U_{\ell-1}\,\mathcal{T}'_{(x_0,\eth_0)}$.

$\Rightarrow \partial_t : \mathrm{gr}^U_k \mathcal{T}' \to \mathrm{gr}^U_{k+1}\mathcal{T}'$ onto for $k \geq 0$

- $\mathrm{gr}^U_i \mathcal{T}' \subset \mathrm{gr}^V_i \mathcal{N} \Rightarrow$ is strict $\mathrm{gr}^U_i \mathcal{T}'$

$\rightsquigarrow \partial_t : \mathrm{gr}^U_k \mathcal{T}' \to \mathrm{gr}^U_{k+1}\mathcal{T}'$ injective for $k \geq 0$.

$\Rightarrow \mathcal{T}'$ strictly spec. and $U_\bullet \mathcal{T}' = V_\bullet \mathcal{T}'$.

Lecture 8 Polarizable twistor D-modules.

Motivations:

- To extend the notion of harmonic flat bundle in order to include singularities
- Idea: to mimic M. Saito's def'n of polarizable Hodge modules

$\rightsquigarrow$ define the category $\mathrm{pTM}(X,w)$ of polarizable pure twistor D-modules of wt w on X by induction on the dimension of the (pure) support, as a subcat. of triples on X.

Drawback. It is difficult to show that a given triple belongs to $\mathrm{pTM}(X)$, because it requires infinitely many conditions.

Advantage. can prove stability of various functors, so that once we know one object of $\mathrm{pTM}(X)$, we deduce infinitely many of them by applying functors.

E.g. We prove that a harmonic flat bundle on X corresponds to an object of $\mathrm{pTM}(X)$.

Rmk. We will focus on regular twistor D-modules, since we aim at proving Kashiwara's conj. for perverse sheaves (and not for arbitrary holonomic D-mods possibly w/ irreg. sing.)

X cpx mfd, $w \in \mathbb{Z}$, $d \in \mathbb{Z}_{\geq 0}$

<u>Def</u> The cat. $TM_{\leq d}(X, w)$ is the full subcat. of $\mathcal{R}$-Triples(X) for which the objects are triples $(\mathcal{M}', \mathcal{M}'', C)$ satisfying:

(HSD) $\mathcal{M}', \mathcal{M}''$ are holonomic, strictly S-decomposable, and have supp. of dim $\leq d$.

(REG) $\forall\, U \subset X$ open and $\forall\, g: U \to \mathbb{C}$ holom., the restrictions $\mathcal{M}'|_U, \mathcal{M}''|_U$ are regular along $\{g=0\}$.

($TM_{>0}$) $\forall\, U \subset X$ open, $\forall\, g: U \to \mathbb{C}$ holom, $\forall\, u \in (\mathbb{R} \times \mathbb{C}) \setminus (\mathbb{Z}_{\geq 0} \times \{0\})$ and $\forall\, l \in \mathbb{Z}_{\geq 0}$, the triple $gr_l^M \Psi_{g,u}(\mathcal{M}', \mathcal{M}'', C) := (gr_l^M \Psi_{g,u}(\mathcal{M}'), gr_{-l}^M \Psi_{g,u}(\mathcal{M}''), gr_l^M \Psi_{g,u} C)$ is an object of $TM_{\leq d-1}(U, w+l)$.

(TM_0) $\forall$ zero-dim'l pure component $\{x_0\}$ of $\mathcal{M}'$ or $\mathcal{M}''$, we have

$$(\mathcal{M}'_{\{x_0\}}, \mathcal{M}''_{\{x_0\}}, C_{\{x_0\}}) = {}^T i_{\{x_0\}*} (\mathcal{H}', \mathcal{H}'', C_0)$$

where $(\mathcal{H}', \mathcal{H}'', C)$ is a twistor str. of dim 0 and wt w.

<u>Polarization</u>

A <u>polarization</u> of an object $\mathcal{T}$ of $TM_{\leq d}(X, w)$ is a pre-polarization $\mathcal{S}: \mathcal{T} \xrightarrow{\sim} \mathcal{T}^*(-w)$ of wt w s.t.

($PTM_{>0}$): $\forall\, U \subset X$ open, $\forall\, g: U \to \mathbb{C}$, $\forall\, u \in (\mathbb{R} \times \mathbb{C}) \setminus \{\mathbb{Z}_{\geq 0} \times \{0\}\}$ and $\forall\, l \in \mathbb{Z}_{\geq 0}$, the morphism $(-1)^l P_l \Psi_{g,u} \mathcal{S}$ induces a polarization of $P_l \Psi_{g,u} \mathcal{T}$.

(PTM_0) for any zero-dim'l strict component $\{x_0\}$ of $\mathcal{M}'$ or $\mathcal{M}''$, we have $\mathcal{S} = {}^T i_{\{x_0\}*} \mathcal{S}_0$ where $\mathcal{S}_0$ is a polarization of the zero-dim'l twistor structure $(\mathcal{H}', \mathcal{H}'', C_0)$.

First properties

Strictness All objects occurring in the def'n of $TM(X,w)$ together w/ $\Psi_{g,(k,0)}\mathcal{M}$ for all g

and $k \in \mathbb{Z}_{\geq 0}$ are strict.

Proof. By strict S-decomp. for $\mathcal{M}$

By def'n for $\Psi_{g,u}\mathcal{M}$ w/ $u \neq (k,0)$ w/ $k \in \mathbb{Z}_{\geq 0}$

Remains $\Psi_{g,(k,0)}\mathcal{M}$. Can assume that $\mathcal{M}$ has pure support Z.

- If $g \equiv 0$ on $\mathrm{Supp}\,\mathcal{M}$, Kashiwara's equiv. + strictness of $\mathcal{M}$

  $\Rightarrow$ strictness of $\Psi_{g,(0,0)}\mathcal{M}$ and then of $\Psi_{g,(k,0)}\mathcal{M}$ for $k \in \mathbb{Z}_{\geq 0}$

  ($\eth_t^k$ induces isom)

- If $g \not\equiv 0$ on $\mathrm{Supp}\,\mathcal{M}$, then var is injective $\Rightarrow$ strictness of $\Psi_{g,(0,0)}\mathcal{M}$ and then

  of $\Psi_{g,(k,0)}\mathcal{M}$ for $k \in \mathbb{Z}_{\geq 0}$.

Rmk. In fact, all $gr_\ell^{M}\Psi_{g,u}\mathcal{M}$ are strict.

Locality: The property of being an object of $TM(X,w)$ or of the subcat. $pTM(X,w)$

is local on X.

Kashiwara's equiv $i: X \hookrightarrow X'$ locally closed submfd. Then ${}^{T}i_*$ induces an equiv.

$TM(X,w) \xrightarrow{\sim} TM_X(X',w)$.

Generic structure. If $\mathcal{T} \in TM(X,w)$ has pure support Z, then on a smooth open dense subset

$Z^\circ \subset Z$, $\mathcal{T}$ corresponds by Kashiwara's equiv. to a variation of pure twistor str. of wt w.

Proof. Can choose Z° s.t. $\mathrm{Char}\,\mathcal{M}', \mathrm{Char}\,\mathcal{M}'' \subset T^*_{Z^\circ}X \times \mathbb{C}_z$.

- Kashiwara's equiv. $\Rightarrow$ can assume $Z^\circ = X$ and $\mathrm{Char}\,\mathcal{M}', \mathrm{Char}\,\mathcal{M}'' \subset T^*_X X \times \mathbb{C}_z$.

- Up to restricting to a dense open set, can assume $\mathcal{M}', \mathcal{M}''$ locally $\mathcal{O}_{\mathcal{X}}$-free.

- Iterating $\Psi_{t_i,(-1,0)}$ for local coord. $t_1, \cdots, t_n$, reach a pure twistor str. of wt w. □

Stability by direct summand. If $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ in $\mathcal{R}$-Triples(X) and $\mathcal{T} \in \mathrm{TM}(X,w)$, then so are $\mathcal{T}_1$ and $\mathcal{T}_2$.

Proof. Holonomy, strict specializability, S-decomposability and regularity along (g) are stable by direct summand.

$\rightsquigarrow$ argue by induction on $\dim \mathrm{Supp}\,\mathcal{T}$ to reduce to $\dim X = 0$.

$\rightsquigarrow$ use that a direct summand in $\mathrm{Mod}_{coh}(\mathbb{P}^1)$ of a trivial holom. vector bundle is of the same kind. □

Decomp. wrt pure support. The cat. $\mathrm{TM}(X,w)$ is the direct sum of the full subcats $\mathrm{TM}_Z(X,w)$ for $Z \subset X$ closed analytic irred.

Proof. For $\mathcal{T}$ S-decomp., we know $\mathcal{T} = \oplus \mathcal{T}_i$. $\mathcal{M}_i', \mathcal{M}_i''$ having pure supp. Z_i are zero.

But $\mathcal{M}_i'' = 0 \Rightarrow \mathcal{M}_i' = 0$. proof by induction on $\dim Z_i$ as follows:

- If $\dim Z_i = 0$, use that C is non-deg.

$\rightsquigarrow$ On a smooth open dense Z_i°, Kashiwara's equiv. $\Rightarrow$ variation of pure twistor str.

hence $\mathcal{M}_i' = 0$ on Z_i° $\rightsquigarrow$ Z_i = pure support $\Rightarrow \mathcal{M}_i' = 0$.

Conclusion: $\mathcal{M}_i', \mathcal{M}_i''$ have the same pure support Z_i.

Lastly, morphisms are decomposable w.r.t the pure supp. decomposition.

The abelianity theorem

Thm $TM(X,w)$ is abelian. all morphisms are strict and strictly specializable along any germ $g:(X,x_0)\to(\mathbb{C},0)$.

Proof. Induction on $\dim \mathrm{Supp}\,\mathcal{T}$. Introduce

- $W\mathcal{R}$-Triples(X): objects are objects of $\mathcal{R}$-Triples(X) equipped w/ a finite filtration $W_\bullet$; morphisms are morphisms in $\mathcal{R}$-Triples(X) that preserve the filtrations W.
- $WTM(X,w)$: full subcat. of $W\mathcal{R}$-Triples(X), objects $(\mathcal{T},W_\bullet)$ s.t. $\forall \ell\in\mathbb{Z}$, $gr_\ell^W\mathcal{T}\in TM(X,w+\ell)$

To prove by induction on d:

(a$_d$) $TM_{\le d}(X,w)$ abelian, any morphism is strict and strictly spec. ($\Rightarrow$ any object is strict ?)

(b$_d$) $WTM_{\le d}(X,w)$ abelian, any morphism is strict and strictly compatible w/ W.

Proof of (a$_0$): By (TM_0), follows from abelianity of pure twistor str.

Proof of (a$_d$) $\Rightarrow$ (b$_d$): Not difficult. e.g. if each graded morphism $gr_\ell^W: gr_\ell^W\mathcal{T}_1\to gr_\ell^W\mathcal{T}_2$ is strict, then φ is strict.

Proof of (b$_{d-1}$) $\Rightarrow$ (a$_d$) $(d\ge 1)$: local question. $\mathcal{T}_1,\mathcal{T}_2\in TM(X,w)$ w/ pure support Z irred. and $\varphi=(\varphi',\varphi'')\ :\mathcal{T}_1\to\mathcal{T}_2$.

To show: $\ker\varphi$, $\mathrm{coker}\,\varphi$ are strictly spec., S-decomposable and are zero or have pure supp. Z.

Let $g:X\to\mathbb{C}$, $g\not\equiv 0$ on Z. By the graph inclusion and Kashiwara's equiv's can assume g is a local coord. t.

- By def'n $\Psi_{t,u}\mathcal{T}_i\in WTM(X,w)$ for $u\notin\{0\}\times\mathbb{Z}_{\ge 0}$.
- $(b_{d-1})\Rightarrow \Psi_{t,u}\varphi$ is strict for $u\notin\{0\}\times\mathbb{Z}_{\ge 0}$.

ETS. (i) $\Psi_{t,u}\varphi$ is strict for $u=(0,0)$.

(ii) can is onto for $\ker\varphi'$, $\ker\varphi''$.

(iii) var is injective for $\operatorname{coker}\varphi'$, $\operatorname{coker}\varphi''$.

Proof of (i). • var injective for M', M''.

$\rightarrow \Psi_{t,(0,0)}\varphi' = \Psi_{t,(-1,0)}\varphi'|_{\operatorname{Im} N}$. Same for φ''.

• $(B_{d-1}) \Rightarrow N$ strict and $\operatorname{Im} N \in \mathrm{WTM}_{\leq d-1}(X,w)$

• $(B_{d-1}) \Rightarrow \Psi_{t,(-1,0)}\varphi'|_{\operatorname{Im} N}$ is strict. Same for φ''. $\square$

Proof of (ii) & (iii): Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Psi_{t,-1}\ker\varphi' \arrow[r] \arrow[d, "\mathrm{can}"] & \Psi_{t,-1}M_1' \arrow[r, "\Psi_{t,-1}\varphi'"] \arrow[d, "\mathrm{can}"] \arrow[dd, bend right=60, "N_1"'] & \Psi_{t,-1}M_2' \arrow[r] \arrow[d, "\mathrm{can}"] \arrow[dd, bend right=60, "N_2"'] & \Psi_{t,-1}\operatorname{coker}\varphi' \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \Psi_{t,0}\ker\varphi' \arrow[r] \arrow[d, "\mathrm{var}"] & \Psi_{t,0}M_1' \arrow[r, "\Psi_{t,0}\varphi'"] \arrow[d, "\mathrm{var}"] & \Psi_{t,0}M_2' \arrow[r] \arrow[d, "\mathrm{var}"] & \Psi_{t,0}\operatorname{coker}\varphi' \arrow[r] \arrow[d, "\mathrm{var}"] & 0 \\
0 \arrow[r] & \Psi_{t,-1}\ker\varphi' \arrow[r] & \Psi_{t,-1}M_1' \arrow[r] & \Psi_{t,-1}M_2' \arrow[r] & \Psi_{t,-1}\operatorname{coker}\varphi' \arrow[r] & 0
\end{tikzcd}$$

• $(B_{d-1}) \Rightarrow \begin{cases} \Psi_{t,-1}\ker\varphi' = \ker\Psi_{t,-1}\varphi' \\ \Psi_{t,-1}\operatorname{coker}\varphi' = \operatorname{coker}\Psi_{t,-1}\varphi' \end{cases}$

Need to show: can onto and var injective.

$\Leftrightarrow$ to show $\operatorname{Im} N_i \cap \ker\Psi_{t,-1}\varphi' = N_i(\ker\Psi_{t,-1}\varphi')$, $i=1,2$

Apply the next lemma, due to (B_{d-1}), and using that $M(N_1)$, $M(N_2)$ are equal (up to shift by w) to the weight filtrations.

Lemma. (E_1,N_1), (E_2,N_2) two $\mathbb{Z}$-modules w/ nilp. endomorphisms. Assume that $\lambda: (E_1,N_1) \to (E_2,N_2)$ is strictly compatible w/ $M(N_1)$, $M(N_2)$, then $\operatorname{Im} N_i \cap \ker\lambda = N_i(\ker\lambda)$, $i=1,2$.

Morphisms. Seen in the proof:

- Let $\mathcal{T}_1, \mathcal{T}_2 \in TM(X,w)$ w/ pure support Z irred.,
- Let $\varphi: \mathcal{T}_1 \to \mathcal{T}_2$ be a morphism.
- Then $\ker\varphi$ and $\operatorname{coker}\varphi$ in $TM(X,w)$ have pure support Z (or are zero)

Cor Assume φ generically an isom. Then φ is an isom. $\square$

Cor. If $w_1 > w_2$, $\nexists$ a non zero morphism $TM(X,w_1) \ni \mathcal{T}_1 \xrightarrow{\varphi} \mathcal{T}_2 \in TM(X,w_2)$.

Proof. Can assume same pure support Z. Result known for smooth twistor str.

$\Rightarrow \varphi$ generically 0 $\Rightarrow \varphi'(\mathcal{M}_1') = 0$.

Apply the same argument to $\varphi^*: \mathcal{T}_2^* \to \mathcal{T}_1^*$, since $-w_2 > -w_1$ $\Rightarrow \varphi''(\mathcal{M}_2'') = 0$

Conclusion: $\varphi = 0$. $\square$

The semi-simplicity thm.

Thm. Let $\mathcal{T} \in pTM(X,w)$ and let S be a polarization. Let $\mathcal{T}_1$ be a subobject of $\mathcal{T}$ in $TM(X,w)$, then S induces a polarization S_1 of $\mathcal{T}_1$ and $\mathcal{T}_1$ is a direct summand of $\mathcal{T}$.

Cor. The cat. $pTM(X,w)$ is abelian and semi-simple. Any morphism between simple objects is zero or an isom.

Proof of Thm. Can assume $\mathcal{T}$ has pure support Z irred., $w=0$ and $S = (Id, Id)$, so that $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$, $C^* = C$ and $\mathcal{T} = \mathcal{T}^*$.

(i) The results hold if $\dim X = 0$.

(ii) The results hold for a variation of polarized pure twistor str. of wt 0.

(iii). Consider the exact seq. in $TM(X,0)$, defining $\mathcal{T}_2^*$ as $\mathcal{T}/\mathcal{T}_1$.

$$0 \to \mathcal{T}_1 \to \mathcal{T} \to \mathcal{T}_2^* \to 0$$
$$0 \leftarrow \mathcal{T}_1^* \leftarrow \mathcal{T} \leftarrow \mathcal{T}_2 \leftarrow 0.$$

(iv) $\to$ $\exists$ a morphism $\varphi : \tau_1 \oplus \tau_2 \to \tau$ in $TM(X,0)$ whose ker and coker have supp. in codim ≥ 1 in Z due to Kashiwara's equiv. and (ii).

(v) But ker φ and coker φ have pure supp. Z or are zero.

(vi) $\Rightarrow$ ker $\varphi = 0$ and coker $\varphi = 0$

(vii) $\Rightarrow$ $\tau = \tau_1 \oplus \tau_2$

(viii) The polarization property holds by induction, starting from (i).

Lecture 9. Kashiwara's Conjecture

- $f : X \to Y$ morphism between sm. proj. vars (or cpt Kähler mfd)
- $\mathcal{F}$ semisimple $\mathbb{C}$-perverse sheaf on X, i.e.

$$\mathcal{F} = \bigoplus_i IC(Z_i, \mathcal{V}_i), \quad \left[\begin{array}{l} Z_i \subset X \text{ irreducible} \\ \mathcal{V}_i \text{ s.s. loc. sys. on } Z_i^{\circ} \subset Z_i \end{array}\right.$$

Conj. (now a theorem) $Rf_* \mathcal{F} \simeq \bigoplus_k {}^pR^k f_* \mathcal{F}[k]$ and each ${}^pR^k f_* \mathcal{F}$ is perverse semi-simple on Y and relative HLT holds.

hard Lefschetz

L relative ample l.b., $\quad c_1(L)^k : {}^pR^{-k} f_* \xrightarrow{\sim} {}^pR^k f_*$

RH corresp. $\longleftrightarrow$ Reg hol. $\mathcal{D}$-mod.

Original conj. : any s.s. hol. $\mathcal{D}$-mod. $\qquad$ var. of tw. structure (harmonic flat bundle)

$\updownarrow$ next time.

s.s. p.s. on X $\xrightarrow{?}$ polarized smooth regular twistor $\mathcal{D}$-module on X

Corlette + Simpson (lisse case)

T. Mochizuki (NCD)

conj. of Kashiwara regular case

decomposition thm

s.s. p.s. on Y $\longleftarrow$ polarized regular twistor $\mathcal{D}$-module on Y

Simpson + Hamm-Lê DT.

Purely imaginary twistor $\mathcal{D}$-modules

Motivation

- X smooth proj.
- $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$: polarizable pure twistor $\mathcal{D}$-module
- $M = i^*_{z=1} \mathcal{M}' =: \Xi_{DR}(\mathcal{T})$: associated reg. holonomic $\mathcal{D}_X$-mod.

In general, one expects some <u>semi-stability</u> property. But would like a <u>semi-simplicity</u> property.

<u>Example 1</u> No singularity.

- $M = (V, \nabla)$ flat bundle on X
- (V, ∇) stable $\overset{\text{def}}{\Longleftrightarrow}$ $\forall (V', \nabla|_{V'}) \subset (V, \nabla)$, $\dfrac{\deg V'}{\operatorname{rk} V'} < \dfrac{\deg V}{\operatorname{rk} V}$. (slope)

But ∇ flat $\Rightarrow$ $\deg V = 0$, $\deg V' = 0$

So (V, ∇) stable $\Leftrightarrow$ (V, ∇) simple $\Leftrightarrow$ V^∇ simple loc. syst.

<u>Example 2.</u> curves (Simpson, 1990)

X sm. proj. curve, D reduced divisor on X

$M = M(*D)$: flat merom. bundle on X (loc. free $\mathcal{O}_X(*D)$-module) w/ flat conn.' having reg. sing.

$M^\bullet$: decreasing exhaustive filtration by loc. free $\mathcal{O}_X$-modules indexed by $B + \mathbb{Z}$ for some finite set $B \subset \mathbb{R}$ s.t. $\forall b \in B + \mathbb{Z}$,

$$M^{b+k} = \mathcal{O}_X(-kD) \otimes M^b \quad (\Rightarrow M = \mathcal{O}_X(*D) \otimes M^b)$$

and $\operatorname{gr}^b M$ supp. on D

$\nabla : M^b \to \Omega^1_X \otimes M^{b-1}$ (log. pole on each M^b, possible because M has reg. sing.)

<u>Parabolic degree</u>: $\operatorname{par.deg}(M, M^\bullet) := \deg M^0 + \sum_{b \in [0,1)} b \dim \operatorname{gr}^b M$

Ex Kashiwara–Malgrange (decreasing) filtration $V^\cdot M$ s.t. eigenvalues of $\mathrm{Res}\nabla$ on $V^b M$ have real part in $[b, b+1)$.

Deligne lattices

- Residue formula $\Rightarrow$ $\deg V^0 M = -\sum \text{eig. Res}\,\nabla|_{V^0 M}$

$$= -\sum \mathrm{Re}\left(\text{eig. Res}\,\nabla|_{V^0 M}\right)$$

$$= -\sum_{b\in[0,1)} b \dim \mathrm{gr}_V^b M$$

Conclusion: $\text{par.deg}(M, V^\cdot) = 0$.

Def $(M, M^\cdot)$ is stable if $\forall N \subset M$ $(N \neq 0)$ preserved by the conn. ∇, equipped w/ $N^\cdot = N \cap M^\cdot$, $\dfrac{\text{par.deg}(N,N^\cdot)}{\mathrm{rk}\,N} < \dfrac{\text{par.deg}(M,M^\cdot)}{\mathrm{rk}\,M}$

Lemma $(M, V^\cdot)$ is stable $\Leftrightarrow$ $(M, V^\cdot)$ is simple

Proof Note that if $N \subset M$ is $\neq 0$ and preserved by ∇, then $N \cap V^\cdot M = V^\cdot N$

Hence, slope $(N, V^\cdot) = 0$ cannot be $<$ slope $(M, V^\cdot) = 0$.

Parabolic filtration assoc. w/ a metric

- $j: X^* = X \setminus D \hookrightarrow X$
- V holom. bundle on X^*
- h: Hermitian metric on $H := \mathcal{C}^\infty_{X^*} \otimes V$

Set $M \subset j_* V$ loc. holom. sections whose h-norm has moderate growth along D.
$|t|^{-N}$

- h is said to be moderate if M is $\mathcal{O}_X(*D)$-coherent (i.e., loc. free of finite rk)

$\rightsquigarrow$ parabolic filtration: for $x_0 \in D$ w/ local coord. t,

$$P^b_{h,x_0} := \{ v \in (j_* V)_{x_0} : \lim_{t\to 0} |t|^{b+\varepsilon} \|v(t)\|_h = 0,\ \forall \varepsilon > 0 \text{ small} \}$$

On (V, ∇), h is said to be tame if

- h-norm of any flat section has moderate growth near D.

- if h is harmonic, this is equiv. (Simpson's main estimate, 1990) to the eigenvalues of the Higgs field at each puncture $x_0 \in D$ w/ loc. coord. having coeff. bounded by $C/|t|$.

Thm (Simpson, 1990) Assume that h is tame and harmonic on (V, ∇). then h is moderate and each term of the parabolic filt'n is $\mathcal{O}_X$-locally free and ∇ has log poles on it.

Thm A filtered reg. sing. merom. bundle w/ conn. $(M, M^{\bullet})$ is poly-stable, each summand being of parabolic deg. zero, iff $\exists$ a tame harmonic metric h on $(V, \nabla) := j^* M$.

Then $M^{\bullet} = P_h^{\bullet}$.

Furthermore, $P_h^{\bullet} = V^{\bullet}$ iff eigenvalues of the residue of the Higgs field are purely imaginary

More precisely.

- $u = (a, \alpha) \in \mathbb{R} \times \mathbb{C}$
- v holom. section in V_α s.t. $\operatorname{Res} \nabla([v]) = e(1,u)[v]$.
- then $\|v\|_h = |t|^{-a-2\operatorname{Re}\alpha}$ and $e(1,u) := -a + 2i \operatorname{Im} \alpha$
- Eigenvalues of the Higgs field : $e(0,u) = \alpha$
- $P_h^{\bullet} = V^{\bullet} \Rightarrow \operatorname{Re}(\alpha) = 0$.

Cor. A reg. merom. bundle w/ conn. M is semisimple iff $\exists$ a "purely imaginary" tame harmonic metric..

$\rightsquigarrow$ One can consider the cat. of purely imaginary twistor $\mathcal{D}$-modules by making more specific the cond. of strict. spec.: always assume that $u \in \mathbb{R} \times (i\mathbb{R})$.

$\rightsquigarrow$ From now on, assume implicitly twistor $\mathcal{D}$-modules are purely imaginary.

Semisimplicity of the assoc. D-module

<u>Thm</u> Let X be sm. proj. and $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be a (purely imag.) polarizable pure twistor D-mod. Then the reg. holom. $\mathcal{D}_X$-mod. $M = \Xi_{DR}(\mathcal{M}')$ is semisimple.

Can assume that $\mathcal{T}$ has wt 0 and pure supp. an irred. closed subvar. $Z \subset X$.

Need to prove

(1) M has pure supp. Z

(2) $\exists$ a smooth Zar. open subset $Z^o \subset Z$ and a s.s. flat bundle (V, ∇) on Z^o s.t. $M|_{Z^o} = {}^D i_* (V, \nabla)$

<u>Note</u>: $\Leftrightarrow$ ${}^p DR(M) \simeq Ri_* IC(Z, V^\nabla) [\dim Z]$.

<u>Point (1)</u>. restriction to $z = 1$:

- If $\mathcal{T}$ is purely imaginary, then $\forall x_o \in X$, and $\forall g : nb(x_o) \to \mathbb{C}$ holom. func., singularities of z w.r.t. $(\mathcal{M}', g)$ are contained in $i\mathbb{R}$ (hence $z = 1$ is not singular).

$\rightsquigarrow$ The (can, var) quiver of $\mathcal{M}'$ restricts at $z = 1$ to the (can, var) quiver of M.

$\rightsquigarrow$ $\mathcal{M}'$ S-dec. along (g) at x_o $\Rightarrow$ M S-dec. along (g) at x_o

$\rightsquigarrow$ $\mathcal{M}'$ has pure supp. Z $\Rightarrow$ M has pure supp. Z.

<u>Point (2)</u> Semisimplicity of (V, ∇): by induction on $\dim Z$.

The case $\dim Z \geq 2$

- Assume $X \subset \mathbb{P}^N$
- Assume $\mathrm{Char}\, \mathcal{M}' \subset \Lambda \times \mathbb{C}_z$ w/ $\Lambda = T^*_Z X \cup \bigcup_i T^*_{Z_i} X$ w/ $Z_i \subset Z$.
- Choose a hyperplane H in $\mathbb{P}^N$ which is non-char. w.r.t. Λ.

$\rightsquigarrow$ $\mathcal{M}'$ is strictly nonchar. along H since it is strictly spec. along H.

(Zariski–Lefschetz thm)

Recall. $\pi_1(Z^o \cap H) \to \pi_1(Z^o)$ is onto (Thm. of Hamm–Lê, 1985)

$\rightsquigarrow (V,\nabla)|_{Z^o \cap H}$ s.s. $\Rightarrow (V,\nabla)$ s.s.

But $T i_H^* \mathcal{T} \in pTM(X,0)$ because locally $T i_H^* \mathcal{T} = \psi_{t,(-1,0)} \mathcal{T} = gr_0^M \psi_{t,(-1,0)} \mathcal{T}$

if $H = \{t=0\}$. So $(V,\nabla)|_{Z^o \cap H}$ is s.s. by induction.

<u>The case $\dim Z = 1$ w/ Z sing.</u> We reduce to the case where Z is a smooth proj. curve.

Let $\nu : \tilde{Z} \to Z$ be the normalization, so that $\tilde{Z}$ is smooth proj.

- $\mathcal{T}|_{Z^o} \equiv$ flat bundle (V,∇) w/ harmonic metric h.

$\rightsquigarrow$ Harmonic Higgs bundle on Z^o

<u>Claim.</u> On $\tilde{Z}$, the eigenvalues of the Higgs field have a pole of order at most 1, and the eigenvalues of the residues of the Higgs field are purely imaginary.

Then apply Simpson's thm on $\tilde{Z} \Rightarrow (V,\nabla)$ is s.s.

<u>Proof of the claim</u>

- We do not know yet that (V,∇) extends to an object of $pTM(\tilde{Z},0)$.
- Let us choose a finite mor. $\pi : Z \to \mathbb{P}^1$ (by projecting from a pencil in $\mathbb{P}^N$ w/ base not intersecting Z).
- Can assume $\pi|_{Z^o}$ is étale.

- - - . - -