Number theory learning seminar, 1.21.2025

Introduction

R ring, A/R abelian var. $A[p^\infty]/R$ p-div. gp.

a quasi-isogeny between p-div. groups = isog. / p-power

p-power q-isog between abel. var.

= p-power isog. / p-power (eg. $\frac{id}{p}$)

Prop If A/R abel. var., $\mathbb{X}/R$ p-divisible gp,

$\alpha : A[p^\infty] \dashrightarrow \mathbb{X}$ q-isog.

$\Rightarrow \exists !$ B/R abelian var. + $A \overset{f}{\dashrightarrow} B$ p-power q-isog.

+ $B[p^\infty] \cong \mathbb{X}$ s.t. $f[p^\infty] = \alpha$.

Pf. $\alpha = \frac{\alpha'}{p^n}$ for some n, & α' is a p-power isog. $(\alpha' : A[p^\infty] \to \mathbb{X})$

define $B = A/\ker \alpha'$. $(\ker \alpha' \subset A[p^\infty] \subset A)$

$f: A \overset{p^{-n}}{\dashrightarrow} A \xrightarrow{q} B$.

$$f[p^\infty]: \underbrace{A[p^\infty] \overset{p^{-n}}{\dashrightarrow} A[p^\infty] \xrightarrow{\alpha'} \mathbb{X}}_{\alpha} \simeq B[p^\infty]$$ $\square$

$$\{\text{abel. var.}/R\} \xrightarrow{A \mapsto A[p^\infty]} \{\text{p-div. gp}/R\}$$

$$\begin{array}{ccc} \{\text{abel. var.}/R\} & \xrightarrow{A\mapsto A[p^\infty]} & \{p\text{-div. gp}/R\} \\ \downarrow & & \downarrow \\ \dfrac{\{\text{abel. var.}/R\}}{p\text{-power q-isog.}} & \longrightarrow & \dfrac{\{p\text{-div. gp}/R\}}{\text{q-isog.}} \end{array}$$

is a Cartesian diagram of groupoids.

(functor {Abel. var.} $\to$ cat. of triples $\{([A], \mathbb{X}, [A[p^\infty]] \dashrightarrow \mathbb{X})\}$ is an equiv.).

<u>Claim.</u> this also works w/ polarizations & endomorphisms.

<u>Rmk.</u> naive "principal polarization" on $\mathbb{X}$ would be

$\lambda : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ s.t. $\lambda^\vee = -\lambda$.

<u>correct def</u>. A <u>principal polarization</u> is a pair $(\mathbb{L}, \lambda)$ where $\mathbb{L}/R$ is a rk 1 $\mathbb{Z}_p$-local sys., and $\lambda : \mathbb{X} \to \mathbb{X}^\vee \otimes \mathbb{L}$ s.t. $\lambda^\vee \otimes \mathbb{L} = -\lambda$.

[another interpretation: $\mathbb{Z}_p^\times$ - orbit of $\lambda : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ s.t. $\lambda^\vee = -\lambda$].

$(A, \lambda) \mapsto (A[p^\infty], \mathbb{Z}_p, \lambda[p^\infty])$.

Q-isog. of p-div. group. $(\mathbb{X}, \mathbb{L}, \lambda) \dashrightarrow (\mathbb{X}', \mathbb{L}', \lambda')$

is a $\mathbb{X} \dashrightarrow \mathbb{X}'$, $\mathbb{L} \longleftrightarrow \mathbb{L}'$ s.t.

$$\begin{tikzcd} \mathbb{X} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\simeq"] & \mathbb{X}^\vee \otimes \mathbb{L} \\ \mathbb{X}' \arrow[r, dashed] & \mathbb{X}'^\vee \otimes \mathbb{L}' \arrow[u, dashed] \end{tikzcd}$$

Def. For p. pol. $(A, \lambda) \dashrightarrow (A', \lambda')$, a p-power q-isog. is a p-power q-isog. $f: A \to A'$ s.t.

$$\begin{tikzcd} A \arrow[r, dashed, "\lambda"] \arrow[d, dashed, "f"'] & A^\vee \\ A' \arrow[r, "\lambda'"] & A'^\vee \arrow[u, dashed, "f^\vee"'] \end{tikzcd}$$

commutes up to a multiple of $\mathrm{Hom}_{cts}(\mathrm{Spec}\, R, p^{\mathbb{Z}})$

Prop.

$$\begin{tikzcd} \{\text{p. pol. abel. var.}\} \arrow[r] \arrow[d] & \{\text{p. pol. p-div. grp}\} \arrow[d] \\ \{\quad\}/_{\text{p-power q-isog.}} \arrow[r] & \{\quad\}/_{\text{q-isog.}} \end{tikzcd}$$

is Cartesian.

Let's work over $\mathrm{Spf}\, \mathbb{Z}_p$, i.e. where p is nilpotent in R.

Prop (Serre - Tate) There is an equiv. of groupoids

$$\frac{\{\text{abel. var. }/R\}}{\text{p-power q-isog.}} \xrightarrow{\sim} \frac{\{\text{abel. var }/(R/pR)\}}{\text{p-power q-isog.}}$$

and $$\frac{\{\text{p-div. }/R\}}{\text{q-isog.}} \xrightarrow{\sim} \frac{\{\text{p-div. }/(R/pR)\}}{\text{q-isog}}$$

<u>Thm</u> (3K) Let $(\mathbb{G}, \mathbb{X})$ be a Shimura datum for a PEL datum of type AC, unramified at p,

$\mathbb{E}$ = global reflex field, $v|p$ a place, $E = \mathbb{E}_v$, $G = \mathbb{G}_{\mathbb{Q}_p}$,

$\mu : \mathbb{G}_{m, \overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ Hodge cocharacter,

$K = K_p K^p \subset \mathbb{G}(\mathbb{A}_{\mathbb{Q}}^{\infty})$ neat open compact, where K_p is the hyperspecial

$\mathcal{S}_K(\mathbb{G}, \mathbb{X}) / \mathrm{Spf}\, \mathcal{O}_E$ the integral model of Shimura var.

$\mathcal{S}_K(\mathbb{G}, \mathbb{X})^{\circ} / \mathrm{Spa}\, E$ its generic fiber (good red. loc.)

Then there exists a v-stack

$\mathrm{Igs}_{K^p}(\mathbb{G}, \mathbb{X})^{\circ}$ fitting in a Cartesian diag.

$$\begin{tikzcd} \mathcal{S}_K(\mathbb{G}, \mathbb{X})^{\circ, \diamond} \arrow[r, "\pi_{HT}"] \arrow[d] & \left[\underline{K_p} \backslash \mathcal{F}\ell_{G, \mu^{-1}}^{\diamond} \right] \arrow[d] \\ \mathrm{Igs}_{K^p}(\mathbb{G}, \mathbb{X})^{\circ} \arrow[r] & \mathrm{Bun}_{G, k_E} \end{tikzcd}$$

<u>Rmk.</u> Bottom row lives over k_E, top row lives over E.

In perfectoid geom., $\exists$ $\mathrm{Spd}\, E \to \mathrm{Spd}\, k_E$.

<u>Rmk.</u> $\mathcal{S}_K(\mathbb{G}, \mathbb{X})^{\circ}$ are <u>not</u> in $\mathrm{Sh}_K(\mathbb{G}, \mathbb{X})^{an}$ because $\mathcal{S}_K(\mathbb{G}, \mathbb{X}) \underset{\mathcal{O}_E}{\otimes} E$ is a disjt union of multiple copies

② can descend $\mathrm{Igs}_{K^p}(G, X)^\circ$ from k_E to $\mathbb{F}_p$

③ get rid of unramified hypotheses.

Rmk.

reality	fantasy
abelian var.	shtukas for $\mathbb{Z}$ w/ 1 leg
p-div. gp	Шмугхи for $\mathbb{Z}_p$ w/ 1 leg
p-div. gp/q-isog.	Шмугхи for $\mathbb{Q}_p$ w/ 0 leg
Igusa stacks	(moduli of) Шмугхи for $\mathbb{Z}[\frac{1}{p}]$ w/ 0 legs

$\rightsquigarrow$ Fiber product diagram is Beauville - Laszlo gluing of шмугхи.

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§ Igusa varieties

Recall over $\overline{\mathbb{F}}_p$ there are finitely many q-isog. classes of p-div. gps of height h.

$\mathbb{X}/\overline{\mathbb{F}}_p$ that appears as $A[p^\infty]$.

$$\rightsquigarrow \begin{tikzcd} S_{\mathbb{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}_{\mathbb{X}} \arrow[d] \\ \mathrm{Ig}_{\mathbb{X}, \breve{\mathbb{Z}}_p} \arrow[r] & \mathrm{Spf}\, \breve{\mathbb{Z}}_p \end{tikzcd} \qquad \breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$$

where ① $\mathcal{M}_{\mathbb{X}}$ parametrizes p-div. $\mathbb{Y}/R$ (w/ extra str.) + q. isog. $\mathbb{Y}_{R/pR} \dashrightarrow \mathbb{X}_{R/pR}$.

② $S_{\mathbb{X}}$ parametrizes abel. var. A/R (w/ extra str.) + q. isog. $A_{R/pR}[p^\infty] \dashrightarrow \mathbb{X}_{R/pR}$

D...

③ $Ig_{X,\breve{\mathbb{Z}}_p}$ parametrizes A/R (w/ extra str.) + isom. $A_{R/pR}[p^\infty] \cong X_{R/pR}$.

" S_X is a $\mathrm{Aut}^\circ(X)$-torsor over the corresponding Newton stratum of $S_K(G,X)$"

$$\text{Newton stratum} = \left(\underbrace{Ig_{X,\breve{\mathbb{Z}}_p}} \times_{\breve{\mathbb{Z}}_p} \mathcal{M}_X\right)/\mathrm{Aut}^\circ(X)$$

canonical lift of its mod p fiber $Ig_X/\overline{\mathbb{F}}_p$

"Igusa variety"

Thm (Harris–Taylor, Mantovan, Caraiani–Scholze)

The cohomology $H^*(Sh(G,X))$ is built out of $H_*\left(\mathrm{Aut}^\circ(X), H^*(Ig_X)\otimes H_c^*(\mathcal{M}_X)\right)$ over different q-isog. classes X.

Igusa stack glues together Ig_X into a single family over Bun_G

$$Igs_{K^p}(G,X)^\circ \xrightarrow{\overline{\pi}_{HT}} \mathrm{Bun}_{G,\mu^{-1},k_E}$$

Define. $\mathcal{F} = R\overline{\pi}_{HT,\overline{\mathbb{F}}_p,*}(\mathbb{Z}/\ell^n\mathbb{Z}) \in D_{\acute{e}t}(\mathrm{Bun}_{G,\mu^{-1},\overline{\mathbb{F}}_p}, \mathbb{Z}/\ell^n\mathbb{Z})$

Prop. There is a "Hecke operator" $T_\mu^{[1]} : D_{\acute{e}t}(\mathrm{Bun}_{G,\mu^{-1},\overline{\mathbb{F}}_p}, \mathbb{Z}/\ell^n\mathbb{Z})$

$$\to D_{\acute{e}t}\Big(\underbrace{\mathrm{Bun}^{[1]}_{G,\overline{\mathbb{F}}_p}}_{\cong\,[*/G(\mathbb{Q}_p)]} \times \underbrace{\mathrm{Div}^1_E}_{=\,[\mathrm{Spd}\,E/\phi^{\mathbb{Z}}]}, \mathbb{Z}/\ell^n\mathbb{Z}\Big)$$

$$\cong \widehat{D}(G(\mathbb{Q}_p)\times W_E, \mathbb{Z}/\ell^n\mathbb{Z})$$

(After choosing $\sqrt{q} \in \mathbb{Z}/\ell^n\mathbb{Z}$) constructed purely locally and it satisfies

$$T_\mu^{[1]}(\mathcal{F}[-d])(-\tfrac{d}{2}) = R\Gamma(S_{K^p}(G,X)_{\overline{\mathbb{Q}}_p}, \mathbb{Z}/\ell^n\mathbb{Z})$$

as $G(\mathbb{Q}_p) \times W_E$ - rep.