Geometric Langlands for projective curves

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Lecture 1. Setup. Fix k a field ($k = \bar{k}$) "geometry field"

Fix X/k a smooth projective curve of genus g.

Fix a "sheaf theory" w/ coefficient field $e \supset \mathbb{Q}$

Example. (i) $k = \mathbb{C}$, e any char. 0 field.

(Betti) $\mathrm{Shv}(Y) = \{$sheaves of e-vec. sps on $Y(\mathbb{C})$ $\leftarrow$ top. space w/ cpx topology$\}$

$\cup$

$\mathrm{Lisse}(Y) = \{$locally constant sheaves$\}$

(ii) Fix ℓ prime, $\ell \neq \mathrm{char}(k)$, $e = \bar{\mathbb{Q}}_\ell$.

(ℓ-adic / étale) $\mathrm{Shv}(Y) = \ell$-adic sheaves on Y

$\cup$

$\mathrm{Lisse}(Y) =$ lisse sheaves

(iii) (de Rham) char $k = 0$, $e = k$. $\mathrm{Shv}(Y) = D\text{-mod}(Y)$

if Y is smooth & affine,

$D\text{-mod}(Y) = \{$mods over $\mathrm{Diff}(Y)\}$

$\cup$

$\mathrm{Lisse}(Y) = \{$(ind-) vector bundles w/ flat $\nabla\}$

Rmk. $k = \mathbb{C}$, Y smooth projective var., $\exists$ an equiv. $\mathrm{Lisse}^{dR}(Y) \simeq \mathrm{Lisse}^{Betti}(Y; \mathbb{C})$

$(\mathcal{E}, \nabla) \mapsto \{\ker(\nabla : \mathcal{E}^{an} \to \mathcal{E}^{an} \otimes \Omega^1)\}$

(really $dR(\mathcal{E}^{an}, \nabla)$)

Geometric class field theory = abelian "GL"

Input: σ rank 1 local system on X.

Output: χ_σ sheaves on $\mathrm{Bun}_{\mathbb{G}_m} = \{\mathcal{L} \text{ l.b. on } X\}$

Reminder: X sm. proj. curve $\rightsquigarrow$ $\mathrm{Jac}(X)$ Jacobian of X.

This is an abelian var. parametrizing l.b. on X of deg 0 w/ equ. a triv. at a fixed pt $x \in X(k)$.

If $k = \mathbb{C}$, $\mathcal{O}$ = hol. funcs,

$$0 \to \mathbb{Z}(1) \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$$

$$\begin{array}{ccc} H^1(\mathcal{O}^\times) & \xrightarrow{\deg} & H^2(\mathbb{Z}(1)) \\ \parallel & & \wr\!\wr \\ \{\text{l.b. on } X\} & & \mathbb{Z} \end{array}$$

$$\underset{\substack{\uparrow \\ \text{cpx analytically}}}{\mathrm{Jac}(X) \simeq} H^1(X, \mathcal{O}) / H^1(X, \mathbb{Z}(1)) = \mathbb{C}^g / \mathbb{Z}^{2g} \simeq \prod S^1$$

$$\mathrm{Bun}_{\mathbb{G}_m} \simeq \mathrm{Jac}(X) \times \underset{\substack{\uparrow \\ \deg}}{\mathbb{Z}} \times \underset{\substack{\uparrow \\ \text{autom. of } \mathcal{L} = k^\times}}{B\mathbb{G}_m}$$

$x \in X \rightsquigarrow \mathcal{O}(x)$ l.b. whose sections are mer. functions w/ pole of order ≤ 1 at x.
(no other poles allowed)

$\mathcal{E}$ v.b. on X, $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(x)$.

What we want:

- $\chi_\sigma|_{\mathcal{L}(x)} \simeq \chi_\sigma|_{\mathcal{L}} \otimes \sigma_x$, $\forall \mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m}$, $x \in X$ (Hecke property)
- $\chi_\sigma|_{\mathcal{O}_X = \text{triv}} \simeq e$ (normalization)

<u>Construction 1</u> (Lazy, Betti)

$$\sigma \Leftrightarrow \pi_1(X, x_0) \to e^{\times} \quad (\Leftrightarrow) \quad \pi_1^{ab} \to e^{\times}$$

$$\pi_1^{ab} \;\| \; H_1(X;\mathbb{Z}) \simeq H^1(X,\mathbb{Z})$$

Poincaré duality

<u>On the other hand</u>, $\mathrm{Jac}(X) \simeq H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$

$$\Rightarrow \pi_1(\mathrm{Jac}\,X) \simeq H^1(X,\mathbb{Z})$$

$X_\sigma|_{\mathrm{Jac}\,X}$ will be the local system attached to $H^1(X,\mathbb{Z}) \to e^{\times}$ via the left board.

<u>Better construction</u> (Deligne)

<u>Idea</u>: Need to tell $X_\sigma|_{\mathcal{L}}$.

Write $\mathcal{L} \simeq \mathcal{O}(D)$ for some divisor $D = \sum n_i x_i$, $x_i \in X$, $n_i \in \mathbb{Z}$

Our axioms $\to$ $X_\sigma|_{\mathcal{L}} \simeq \bigotimes_i \sigma_{x_i}^{\otimes n_i}$

So we see X_σ is "overdetermined".

<u>Q</u>. Why is this isom. indep. of the choice of D?

$\mathrm{Sym}^d(X) = \{$moduli space of effective divisors of deg d on $X\}$

$$= \{(\mathcal{L}, s) : \underbrace{\mathcal{L} \in \mathrm{Bun}^d_{\mathbb{G}_m}}_{\text{deg } d \text{ l.b.}},\ s \in \Gamma(\mathcal{L}), s \neq 0\}$$

Calculation, $X = \mathbb{P}^1 \Rightarrow \mathrm{Sym}^d X = \mathbb{P}^d = \{\sum_{i=0}^{d} a_i t^i \neq 0\}/\mathbb{G}_m$

divisor of zeroes of f $\leftarrow\!\shortmid f$

$AJ_d : \mathrm{Sym}^d X \to \mathrm{Bun}^d_{\mathbb{G}_m}$ map forgetting s,

fiber at $\mathcal{L}$ is $\Gamma(\mathcal{L}) \setminus 0$.

When $d > 2g-2$, Riemann-Roch $\Rightarrow \dim \Gamma(\mathcal{L}) = d + (1-g)$

$\Rightarrow AJ_d$ is smooth

Our vec. spaces $\Gamma(\mathcal{L}) \setminus 0$ are simply connected

<u>Notation</u> $\sigma^{(d)} \in \mathrm{Lisse}(\mathrm{Sym}^d X)$

$\underbrace{\mathrm{add}_{d*} (\sigma \boxtimes \cdots \boxtimes \sigma)^{\mathfrak{S}_d}}_{d \text{ times}}$ $\qquad \mathrm{add}_d : X^d \to \mathrm{Sym}^d X$

lives on X^d $\qquad \mathfrak{S}_d =$ symmetric gp.

Axioms $\Rightarrow$ $\chi_\sigma |_{\mathrm{Sym}^d X} \simeq \sigma^{(d)}$

But geometry $\to$ for $d \gg 0$, $\sigma^{(d)}$ must descend to $\mathrm{Bun}^d_{\mathbb{G}_m}$.

defines $\chi_\sigma |_{\coprod_{d \gg 0} \mathrm{Bun}^d_{\mathbb{G}_m}}$, then easily extend to all of $\mathrm{Bun}_{\mathbb{G}_m}$

$\mathrm{add}_* (\sigma^{\boxtimes d})$ fiber at D:

$\bigoplus \otimes \sigma_{x_i}$	$d=2$
all ways of writing $D = \sum x_i$ $(x_1, \ldots, x_n)$	$D = x+y$, $x \neq y$
	Seeing $\sigma_x \otimes \sigma_y \oplus \sigma_y \otimes \sigma_x$ (swapped by $\mathfrak{S}_2$)
	$D = 2x$
	See : $\sigma_x^{\otimes 2}$

Invariants for $\mathfrak{S}_d$ removes the redundancy : $\rightsquigarrow \sigma^{(d)}|_D = \bigotimes \sigma_{x_i}^{\otimes n_i}$, $D = \sum n_i x_i$

<u>Now</u> start w/ non-abelian theory

$G = PGL_2$

<u>Input</u>. σ SL_2 - local sys on X (σ rk 2 loc. sys w/ $\wedge^2\sigma \simeq$ triv)

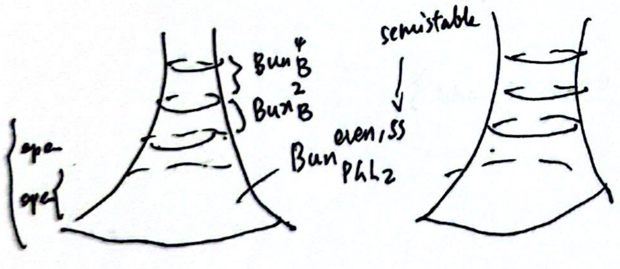
<u>Output</u>. sheaf $\mathcal{F}_\sigma$ on Bun_{PGL_2}

$\mathcal{F}_\sigma$: no longer lisse, but perverse

Bun_{PGL_2}: moduli stack of PGL_2 - torsors on X

A PGL_2 - bundle on X = a rk 2 v.b. on X modulo ambiguity of tensoring by line bundles.

Picture for Bun_{PGL_2}



$Bun^{even}_{PGL_2}$ $\qquad$ $Bun^{odd}_{PGL_2}$

$\deg(\mathcal{E} \otimes \mathcal{L}) = \deg \mathcal{E} + 2\deg \mathcal{L}$

$\Rightarrow \deg \mathcal{E} \bmod 2$ well-defined

$B = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} = \mathbb{G}_m \ltimes \mathbb{G}_a$ Borel of PGL_2

$Bun_B = \{ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O}_X \to 0 \}$

$Bun_B^d = $ ${}^{\iota}$pt where $\deg \mathcal{L} = d$.

Picture says:

$d > 1$, $\mathrm{Bun}_B^d \longrightarrow \mathrm{Bun}_{PGL_2}$ is locally closed

$\mathrm{Bun}_G^{ss} = \{ \mathcal{E} \text{ s.t. } \forall\, \mathcal{L} \subset \mathcal{E} \Rightarrow \deg \mathcal{L} \leq \frac{1}{2} \deg \mathcal{E} \}$

<u>Note</u> $\mathrm{Bun}_{PGL_2}^{even} = \bigcup_{n \geq 0} U_n$ $\quad U_n =$ union of first n terms in picture

U_n is open, $U_n \neq U_{n+1}$.

<u>Summary</u>. (Observation) $\mathrm{Bun}_{PGL_2}^{even}$ is not quasi-cpt

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<u>Lecture 2</u>. <u>Last time</u>: Given an irred. SL_2 local sys σ on X, want a corresponding sheaf $\mathcal{F}_\sigma$ on Bun_{PGL_2} w/ some unspecified properties.

- $\mathcal{F}_\sigma$ should be perverse
- $\mathcal{F}_\sigma \big|_{\mathrm{Bun}_{PGL_2}^{\varepsilon}}$ should be irreducible perverse. $\varepsilon \in \{\text{even}, \text{odd}\}$.
- $\mathcal{F}_\sigma$ should be <u>cuspidal</u>
  counterpart to σ being irreducible

$G = PGL_2$

<u>Def</u> $CT_* : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_{G_m})$

$\quad \| $

$\quad q_* p^!$

$\mathrm{Bun}_B = \{ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0 \}$

$$\begin{tikzcd} & \mathrm{Bun}_B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_{G_m} \end{tikzcd}$$

"$\mathcal{E}$" $\quad$ "$\mathcal{L}$"

<u>Variant</u>

CT_*^n:

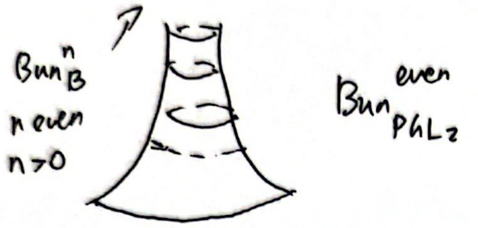
$$\begin{tikzcd} & \mathrm{Bun}_B^n \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_{G_m}^n \end{tikzcd}$$

Def $\mathcal{F} \in Shv(Bun_G)$ is cuspidal if $CT_*(\mathcal{F}) = 0$. $\Leftrightarrow$ $CT_*^n(\mathcal{F}) = 0$, $\forall n$

Philosophy. cuspidal $\leftrightarrow$ σ is irreducible

Rmk. For general reductive gps, use all proper parabolics & their Levi quotients.

Last time: Drew Philosophy, cuspidal $\Rightarrow$ vanishing "at ∞".



Claim: $\mathcal{F}$ cuspidal $\Rightarrow$ its !-restriction to Bun_B^n is zero, $\underbrace{n \gg 0}_{\text{depending only on } g = \text{genus}}$.

$Bun_B^n = \{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0\}$

$\searrow q_n$

$Bun_{\mathbb{G}_m}^n$

$\deg \mathcal{L} > 2g - 2$

RR $\Rightarrow$ $H^1(\mathcal{L}) = 0$

$\Rightarrow$ any ext'n as above splits (non-canonically)

So the fiber of q_n at $\mathcal{L}$ is $\mathbb{B}H^0(\mathcal{L})$

In general, see the fiber $H^1(\mathcal{L}) \times \mathbb{B}H^0(\mathcal{L})$, or more canonically $R\Gamma(\mathcal{L})[1]$.

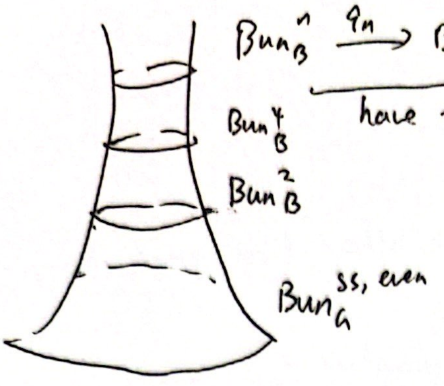
Because $\mathbb{G}_a$ is contractible, $Shv(\mathbb{B}\mathbb{G}_a) = Vect$, get an equivalence by pullback / pushforward from / to the point.

For us, $n > 2g-2$, $q_{n*}: Shv(Bun_B^n) \to Shv(Bun_T^n)$ is an equiv.

So $CT_*^n(\mathcal{F}) = 0 \Rightarrow q_{n*}\, p_n^!(\mathcal{F}) = 0 \overset{n \gg 0}{\Longrightarrow} p_n^!(\mathcal{F}) = 0$.

$\mathrm{Bun}_B^n \xrightarrow{q_n} \mathrm{Bun}_T^n$

have the same sheaves $n >> 0$



<u>Def</u>: $\underset{\underset{q_! \, p^*}{\shortparallel}}{CT_!} : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_T)$ & $\underset{\underset{q_{n!} \, p_n^*}{\shortparallel}}{CT_!^n}$.

Formally, if $\mathcal{F}$ is $\overset{\text{eg.}}{\text{constructible}}$, $CT_*(\mathcal{F}) = \mathbb{D}\, CT_!\, \mathbb{D}(\mathcal{F})$, $\mathbb{D}$ = Verdier duality

<u>Thm</u> (Drinfeld - Gaitsgory) $CT_*^n \simeq \underbrace{\mathrm{inv}}_{} \circ CT_!^{-n}$ "2nd adjointness"

$$\begin{gathered} \uparrow \\ \mathrm{Bun}_{\mathbb{G}_m}^n \xrightarrow{\sim} \mathrm{Bun}_{\mathbb{G}_m}^{-n} \\ \mathcal{L} \longmapsto \mathcal{L}^\vee \end{gathered}$$

<u>Cor.</u> $\mathcal{F}$ cuspidal $\Rightarrow$ $*$-restriction to Bun_B^n vanish for $n >> 0$.

<u>Pf</u> $CT_*^{-n}(\mathcal{F}) = 0 \overset{DG}{\Rightarrow} CT_!^n(\mathcal{F}) = 0, \ \forall n$

$n >> 0$, apply the same analysis as before.

Q. How does $\mathrm{Bun}_B^n \xrightarrow{p_n} \mathrm{Bun}_G$ look if $n << 0$?

<u>Claim</u>: for $n << 0$, $\mathrm{Bun}_B^n \to \mathrm{Bun}_G$ is smooth.

<u>Pf</u>. Like in Lin's talk, tangent space to Bun_H at P_H is $H^1(X, \mathfrak{h}_{P_H})$

$$H^1(X, \mathfrak{b}_{P_B}) \xrightarrow{\text{diff'l for } p} H^1(X, \mathfrak{g}_P) \longrightarrow H^1(X, (\mathfrak{g}/\mathfrak{b})_P) \longrightarrow 0$$

$H^1(X, \mathfrak{b}_{P_B})$: tangent to Bun_B. $H^1(X, \mathfrak{g}_P)$: tangent to Bun_G.

$$\underset{\mathfrak{b}}{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}} \subset \underset{\mathfrak{g}}{\begin{pmatrix} * & * \\ * & * \end{pmatrix}}$$

For us, $(\mathfrak{g}/\mathfrak{b})_{P_B} = \mathcal{L}^{-1}$ $\quad (\mathfrak{g}/\mathfrak{b} = \mathfrak{n}^-)$

Rmk DG theorem $\Rightarrow$ if $\mathcal{F}$ is constructible & cuspidal, $\mathbb{D}\mathcal{F}$ is also cuspidal

Summary.

finite union of strata

$\mathcal{U}$

$\cap$

Bun_G

$\mathcal{F}$ is both $!$ and $*$ extended from $\mathcal{U}$,

i.e., "cleanly" extended.

$\longrightarrow$

Turn more to $\mathcal{F}_\sigma$: motivation from theory of modular forms ("automorphic forms for PGL_2")

Reminder: a (holomorphic) modular form (of wt k and level 1) is a sum

$f(q) = \sum_{n \geq 0} a_n q^n$ converging for $|q| < 1$, s.t. $\forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$, $q = e^{2\pi i \tau}$, $\mathrm{Im}\,\tau > 0$,

$$f(g \cdot \tau) = (c\tau + d)^k f(\tau).$$

Langlands conjectures $\quad \sigma : \text{"}\mathrm{Gal}_{\mathbb{Q}}\text{"} \xrightarrow{\text{irreducible (unramified)}} SL_2\text{"}\mathbb{C}\text{"}$ $\quad$ (σ odd)

Expect. $\exists\, f_\sigma(q)$ a modular form w/ the following forms:

$$f_\sigma(q) = \underbrace{a_0}_{0} + \overbrace{a_1}^{1} q + a_2 q^2 + a_3 q^3 + \cdots \qquad a_p = \mathrm{tr}(\sigma(\mathrm{Fr}_p)) \text{ for } p \text{ prime}$$

$a_0 = 0$: cuspidality $\quad a_1 = 1$: normalization

$\mathrm{Fr}_p \in \mathrm{Gal}\,\mathbb{Q}$

Frob. conj. class of p.

$a_{nm} = a_n a_m$ if $(n,m)=1$.

V std SL_2-rep'n

$a_{p^{n+1}} = a_p a_{p^n} - p^{k-1} a_{p^{n-1}}$

$V \otimes \mathrm{Sym}^{n-1} V \simeq \mathrm{Sym}^n V \oplus \mathrm{Sym}^{n-2} V$

<u>Langlands Conjecture.</u> $f_\sigma(q) = \sum_n a_n q^n$ is a modular form.

<u>Goal.</u> Analogues of a_n's for sheaves on Bun_{PGL_2}

$a_n(f) \leftarrow$ nth Fourier Coefficient of modular form f, $n>0$

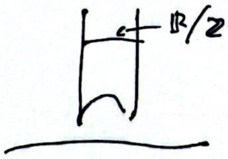
Replace n by an effective divisor D on X

$n = p_1^{r_1} \cdots p_k^{r_k} \rightsquigarrow$ divisor $\sum r_i [p_i]$ on $\mathrm{Spec}\,\mathbb{Z}$

$a_n(f)$ number replaced by a vector space $\mathrm{coeff}_D(\mathcal{F}) \in \mathrm{Vect}$

"Dth Fourier coefficient of $\mathcal{F}$"

$$a_n(f) = \int_{\mathbb{R}/\mathbb{Z}} f(\tau)\, e^{-2\pi i n \tau}\, d\tau$$



this $\mathbb{R} \longleftrightarrow$ $\underbrace{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}}_{N = \text{unip. radical of Borel.}} \in SL_2(\mathbb{R})$

In <u>our setup</u>, $D=0$ (analogous to $a_1(f)$)

$\mathrm{Bun}_N = \{0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0\}$

$\mathrm{Bun}_N^\Omega = \{0 \to \Omega_X \to \mathcal{E} \to \mathcal{O}_X \to 0\}$ $\quad \Omega$ = l.b. of 1-forms on X

$\mathrm{Bun}_N^\Omega = \mathbb{B} H^0 \Omega_X \times H^1 \Omega$

$$\begin{tikzcd} & \mathrm{Bun}_N^\Omega \arrow[dl, "p_N"'] \arrow[dr, "\psi"] & \\ \mathrm{Bun}_G & & \mathbb{A}^1 \simeq H^1(\Omega_X) \end{tikzcd}$$

Serre duality

$\mathrm{coeff}_0(\mathcal{F}) := C^\cdot(\mathrm{Bun}_N^\Omega, p_N^!(\mathcal{F}) \otimes \psi^!(\exp))$

What is exp?

$\exp \in Shv(\mathbb{A}^1)$

Example. D-mod. $\exp = (\mathcal{O}_{\mathbb{A}^1}, \nabla = d - dt)$

Example. In the l-adic context.

$$\begin{array}{ccccc} \mathbb{F}_p & \longrightarrow & \mathbb{G}_a & \xrightarrow{\pi} & \mathbb{G}_a/\mathbb{F}_p \overset{AS}{\simeq} \mathbb{G}_a \\ \cap & & & & t \longmapsto t^p - t \\ \mathbb{G}_a(\mathbb{F}_p) & & & & \end{array}$$

$\pi_*(\overline{\mathbb{Q}}_\ell) \circlearrowleft \mathbb{F}_p$

Fix $1 \neq \zeta_p$ a pth root of unity in $\overline{\mathbb{Q}}_\ell^\times$, I obtain a 1-dim'l summand $\boxed{\exp}$

In Betti setting, exp doesn't exist, there are tricks to avoid this.

Lecture 3. In no. theory, σ: "$\mathrm{Gal}_{\mathbb{Q}}$" $\longrightarrow$ SL_2 "$\mathbb{C}$" irred.

$f_\sigma(q) = \Sigma a_n q^n$, $a_0 = 0$, $a_1 = 1$. $a_p = tr(\sigma(\mathrm{Fr}_p))$ p prime

$\{a_p\}$ determine a_n's

Goal. imitate this in geometry.

modular forms $\rightsquigarrow Shv(Bun_G)$, $G = PGL_2$

$f \longmapsto a_0(f) \rightsquigarrow CT_* : Shv(Bun_G) \to Shv(Bun_T)$

$f \longmapsto a_1(f) \rightsquigarrow coeff_0 : Shv(Bun_G) \longrightarrow Vect$

Other Fourier coefficients:

indexing set	geometry
$\{a_n's\}$	D effective
$\uparrow$	divisor on X
indexed by	
$n \geq 1 \leftrightarrow$	
eff. divisors	
on Spec $\mathbb{Z}$	

Define $\mathrm{Bun}_N^{\Omega(-D)} = \{ 0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O} \to 0 \}$

There's a map $p_{N,D} : \mathrm{Bun}_N^{\Omega(-D)} \longrightarrow \mathrm{Bun}_G$

take the PGL_2 bundle for $\mathcal{E}$.

There's a map

$$\begin{tikzcd}
\mathrm{Bun}_N^{\Omega(-D)} \arrow[r] \arrow[rr, bend right, "\psi_D"'] & H^1(\Omega(-D)) \arrow[r] & H^1(\Omega) = \mathbb{A}^1
\end{tikzcd}$$

Serre duality

Define: $\mathrm{coeff}_D : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$

by the same formula:

$$\mathcal{F} \longmapsto \mathrm{coeff}_D(\mathcal{F}) := C^\bullet(\mathrm{Bun}_N^{\Omega(-D)}, p_{N,D}^! \mathcal{F} \otimes \psi_D^! \exp)$$

complex computing cohomology

$a_p = \mathrm{tr}(\mathrm{Frob}_p) \Longleftrightarrow$ if $D = x$, $\mathrm{coeff}_D(\mathcal{F}_\sigma) = \sigma_x$ fiber of σ at x

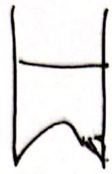
Expectation: $\exists$ canonical $\mathcal{F}_\sigma$ so that if $D = \sum n_i x_i$, $x_i \neq x_j$ for $i \neq j$.

then $\mathrm{coeff}_D(\mathcal{F}_\sigma) = \bigotimes_i \mathrm{coeff}_{n_i x_i}(\mathcal{F}_\sigma) = \bigotimes_i \mathrm{Sym}^{n_i}(\sigma_{x_i})$

$\Updownarrow$

multiplicativity + recursion for a_{p^n} from yesterday

f on $\mathbb{H}$



$$\int_{\mathrm{Im}(z)=y} f(z)\,dz = a_0(f)(y)$$

const term really func. on $\mathbb{R}^{>0} \rightsquigarrow \pm\hat{\mathbb{Z}}^\times \backslash \mathbb{A}^\times / \mathbb{G}^\times$

constant in holomorphic case, not in general

Want to refine coeff$_D$'s.

Can easily construct

$$\mathrm{coeff}_d : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Sym}^d X)$$

So $\forall D$, TFDC:

$$\begin{tikzcd} \mathrm{Shv}(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}_d"] \arrow[dr, "\mathrm{coeff}_D"'] & \mathrm{Shv}(\mathrm{Sym}^d X) \arrow[d, "!\text{-fiber at } D"] \\ & \mathrm{Vect} \end{tikzcd}$$

imitate construction, take D as a variable.

<u>Super naive assertion</u>:

$$\prod \mathrm{coeff}_d : \mathrm{Shv}(\mathrm{Bun}_G)_{cusp} \to \prod_{d\geq 0} \mathrm{Shv}(\mathrm{Sym}^d X) \text{ is fully faithful ?}$$

<u>NOT TRUE</u> b/c RHS splits up into factors $\mathbb{Z}^{\geq 0}$, while LHS only over even & odd.

<u>Claim</u>. $\mathcal{F}$ <u>cuspidal perverse sheaf</u> on Bun_G, can uniquely recover $\mathcal{F}$ from $\mathrm{coeff}_d(\mathcal{F})$ & $\mathrm{coeff}_{d+1}(\mathcal{F})$ for $d \gg 0$ depending on genus. Special to $PGL_2 = G$.

Fourier-Deligne transform:

in ℓ-adic, D-module settings

V vec. sp. $\qquad Shv(V) \overset{\text{Four}}{\simeq} Shv(V^\vee)$

$$\begin{tikzcd} & V \times V^\vee \arrow[dl, "pr_1"'] \arrow[d, "ev"] \arrow[dr, "pr_2"] & \\ V & \mathbb{A}^1 & V^\vee \end{tikzcd}$$

$$\text{Four}(\mathcal{F}) = pr_{2*}(pr_1^!(\mathcal{F}) \otimes ev^!(\exp))$$

$$(\text{Four}\, \mathcal{F})_\lambda = " \int_V \mathcal{F}_v \cdot \exp(\lambda(v))\, dv "$$

Rmk Also works for vb's over a base

$$\begin{tikzcd} E \arrow[dr] & & E^\vee \arrow[dl] \\ & S & \end{tikzcd} \qquad \text{Four}: Shv(E) \simeq Shv(E^\vee)$$

Rmk In Betti setting, can define Fourier for monodromic sheaves.

Rewrite coeff d using Fourier transforms and geometry of bundles.

$$\begin{tikzcd} E_d = Bun_B^{-d} \arrow[d, equal] \arrow[dr, "q_{-d}"] & \\ \{\mathcal{L} + 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0\} & Bun_T^{-d} = Bun_{\mathbb{G}_m}^{-d} \end{tikzcd}$$

fibers of q_{-d} are "$H^1(\mathcal{L})^\vee$" $\leftarrow$ really $R\Gamma(\mathcal{L})[1]$

$d>0 \Rightarrow H^0(\mathcal{L})=0 \quad (\deg \mathcal{L} = -d)$, then q_{-d} is a vec. bundle over $Bun_{\mathbb{G}_m}^{-d}$

the dual v.b. $E_d^\vee$ has fiber $H^1(\mathcal{L})^\vee$ at $\mathcal{L} \in Bun_{\mathbb{G}_m}^{-d}$

$\overset{\text{Serre}}{=} H^0(\mathcal{L}^\vee \otimes \Omega^1)$

$E_d^\vee = \{ \mathcal{L} : \deg \mathcal{L} = -d, \quad s : \mathcal{L} \to \Omega^1 \}$

$E_d^\vee \supset \{ s \neq 0 \} = E_d^\vee \setminus \underbrace{\text{zero section}}_{\mathrm{Bun}_{G_m}^{-d}}$

$$\begin{array}{ccc} \{s \neq 0\} & & \\ \wr\!\| & \Omega(-D) \subset \Omega & \\ & \uparrow s = \text{canonical} & \\ \mathrm{Sym}^{d-(2g-2)} X & D & \end{array}$$

$$\begin{tikzcd}[column sep=small] & \mathrm{Bun}_B^{-d} \arrow[dl, "p_{-d}"'] \arrow[dr, "\text{v.b.}"] & & \{\mathcal{L} \to \Omega^1\} \arrow[dl, "\text{dual v.b.}"] & \mathrm{Sym}^{d-(2g-2)} X \arrow[l, hook', "\overset{\circ}{j}"'] \\ \mathrm{Bun}_G & & \mathrm{Bun}_{G_m}^{-d} & & \end{tikzcd}$$

<u>Easy check</u>: $\mathrm{coeff}_{d-(2g-2)}(\mathcal{F})$ can be calculated via:

- $!$-pullback $\mathcal{F}$ along p_{-d}
- Fourier transform
- restriction along $\overset{\circ}{j}$

<u>Ops</u>! Forgot to say:

expect $\mathrm{coeff}_d(\mathcal{F}_\sigma) = \sigma^{(d)}$

$\sigma^{(d)} = \mathrm{add}_{d*}(\sigma^{\boxtimes d})^{S_d}$

$\mathrm{rk}\,\sigma > 1 \Rightarrow$ perverse, not lisse

Eg. $\mathrm{coeff}_1(\mathcal{F}_\sigma) = \sigma \in \mathrm{Shv}(X)$

Why can we recover $\underbrace{\mathcal{F}}_{\substack{\text{cuspidal} \\ \text{perverse}}}$ from $\mathrm{coeff}_d(\mathcal{F})$?

Last time $d \gg 0 \Rightarrow p_{-d}$ is smooth.

Fact. fibers of p_{-d} are connected or empty

General fact. $f: Y \to Z$ smooth map. w/ connected fibers

pullback on perverse sheaves is fully faithful (on abelian categories)

if fibers are sometimes empty, sometimes not, okay on irred. perv. sheaves w/ full support.

Summary. $p_{-d}^!$ doesn't lose information about our $\mathcal{F}_\sigma$.

Fourier transform is an equiv. $\Rightarrow$ okay.

Q: What about restriction along j?

Claim $\mathcal{F}$ cuspidal $\Rightarrow$ Four $(p_{-d}^! \mathcal{F}) \leftarrow$ $*$-extended along j.

This suffices for our purposes.

Claim $\Leftrightarrow$ Four $(p_{-d}^! \mathcal{F})$!-restrict to complement $Bun_{G_m}^{-d} \hookrightarrow \{L \to \Omega^{1}\}$ gives 0.

This fiber of Four $\Leftrightarrow$ $*$-direct image:

$$\begin{tikzcd} & Bun_B^{-d} \arrow[dl, "p_{-d}"'] \arrow[dr, "q_{-d}"] & \\ Bun_G & & Bun_{G_m}^{-d} \end{tikzcd}$$

result is $q_{-d*} p_{-d}^! (\mathcal{F}) = CT_*^{-d}(\mathcal{F}) = 0$. $\square$

cuspidality.

Thm (GL for PGL_2. Drinfeld, Laumon, Frenkel–Gaitsgory–Vilonen)

σ irred. SL_2 local system

$\exists\ \mathcal{F}_\sigma$ on Bun_{PGL_2}, perverse, irred. on each conn'd component.

$$\mathrm{coeff}_d(\mathcal{F}_\sigma) \simeq \sigma^{(d)}.$$

Also: $\mathcal{F}_\sigma$ is a Hecke eigensheaf.

Briefly (incomplete):

$$\mathrm{Hecke} = \{x \in X,\ \mathcal{E} \subsetneq \mathcal{E}' \subsetneq \mathcal{E}(x)\} \qquad \overset{\uparrow}{\underset{\downarrow}{}}\ \ell_x \subset \mathcal{E}(x)|_x$$

up to tensoring by $\mathcal{L}$

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times X \end{tikzcd}$$

$$\underset{\mathrm{std}}{H}(\mathcal{F}) := \overrightarrow{h}_* \overleftarrow{h}^!(\mathcal{F})$$

Eigen property: $\quad H_{\mathrm{std}}(\mathcal{F}_\sigma) \simeq \mathcal{F}_\sigma \boxtimes \sigma \qquad$ (up to shifts)

Lecture 4. Hecke functors

G reductive gp $/k$ (split, conn'd), $\check{G}$ dual reductive gp $/e$.

Reminder. G has a root datum, $\check{G}$ has the dual root datum.

Eg. $G = PGL_n \Rightarrow \check{G} = SL_n$.

$\mathrm{Bun}_G \leftarrow$ moduli of G-bundles

Examples

G	G-bundles
GL_n	rk n v.b. $\mathcal{E}$
SL_n	$\mathcal{E}$ + $\det \mathcal{E} \simeq \mathcal{O}_X$
PGL_n	$\mathcal{E}$ modulo $\mathcal{L} \otimes -$
O_n	$\mathcal{E}$ + sym. non-deg. form $\mathcal{E} \otimes \mathcal{E} \to \mathcal{O}$
Sp_n	same but alternating

<u>Key piece of structure</u>. $V \in \mathrm{Rep}\, G$, P_G G-bundle on S, $\rightsquigarrow V_{P_G} \in \mathrm{QCoh}(S)$

v.b. if V is finite dim'l.

$\{G\text{-bundles on } S\} \longleftrightarrow \{\mathrm{Rep}(G) \to \mathrm{QCoh}(S)$ right t-exact sym. monoidal $\}$

I also have $\check{G}$-local systems (or any H/e)

$$\left.\begin{array}{l} \mathrm{Rep}\,\check{G} \longrightarrow \mathrm{Shv}(X) \\ \Leftrightarrow \ \mathrm{Rep}\,\check{G} \longrightarrow \mathrm{Lisse}(X) \end{array}\right] \begin{array}{l}\text{right} \\ \text{t-exact}\end{array}$$

<u>Geom. Satake</u>: $V \in \mathrm{Rep}\,\check{G}$

$$H_V : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

<u>Example</u>. $G = PGL_2$, $\check{G} = SL_2$, $V = \mathrm{std}$, get Hecke functor from last lecture.

<u>In general</u>: $x \in X$

$$\mathrm{Rep}\,\check{G} \xrightarrow{\#} \mathrm{Shv}(\mathrm{Gr}_{G,x})^{L_x^+ G} = \mathrm{Shv}(L_x^+G \backslash \mathrm{Gr}_{G,x})$$

$$V \longmapsto S_V$$

$$L_x G = G(k((t_x)))$$ (Laurent series at x)

$$L_x^+ G = G(k[[t_x]]) \qquad \mathrm{Gr}_{G,x} = L_xG / L_x^+G$$

$$\mathrm{Hecke}_x = \{P_G, \tilde{P}_G \text{ on } X + \text{iso. } P_G|_{X \setminus x} \simeq \tilde{P}_G|_{X\setminus x}\}$$

$$\begin{tikzcd} & \mathrm{Hecke}_x \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[dd, "\pi"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \\ & \mathrm{Hecke}_x^{loc} & \end{tikzcd}$$

$$\mathrm{Hecke}_x^{loc} = \{P_G, \tilde{P}_G \text{ on } D_x = \mathrm{Spec}\, k[[t_x]], \text{ w iso. on } \mathring{D} = \mathrm{Spec}\, k((t_x))\}$$

$$= L_x^+G \backslash \mathrm{Gr}_{G,x}$$

$$H_{V,x}(\mathcal{F}) = \overrightarrow{h}_*(\overleftarrow{h}^!(\mathcal{F}) \overset{!}{\otimes} \pi^!(S_V))$$

$H_V(\mathcal{F})$ constructed by the same procedure, but you can vary $x \in X$.

$V_1, V_2 \in \operatorname{Rep} \check{G}$,

$$\begin{tikzcd} H_{V_1,V_2}: \operatorname{Shv}(\operatorname{Bun}_G) \arrow[r, "H_{V_2}"] \arrow[rr, bend right, "H_{V_1,V_2}"'] & \operatorname{Shv}(\operatorname{Bun}_G \times X) \arrow[r, "H_{V_1} \times \mathrm{id}_X"] & \operatorname{Shv}(\operatorname{Bun}_G \times X \times X) \end{tikzcd}$$

"Hecke functors commute": $H_{V_1,V_2} \underset{\text{canonically}}{\simeq} \underbrace{\mathrm{swap}}_{\text{two factors of } X\times X} \circ H_{V_2,V_1}$.

More generally, given a finite set I,

$V = \{V_i\}_{i\in I}$, $V_i \in \operatorname{Rep} \check{G}$.

$\exists\, H_V : \operatorname{Shv}(\operatorname{Bun}_G) \longrightarrow \operatorname{Shv}(\operatorname{Bun}_G \times X^I)$

<u>Q</u>: What acts on $\operatorname{Shv}(\operatorname{Bun}_G)$?

<u>Eg.</u> $H_{V,x}$, $x \in X$.

<u>Formal answer</u>.

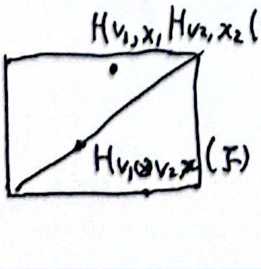
Fix a finite set I, $V \in \operatorname{Rep}(\check{G}^I)$, $V = \bigotimes_{i\in I} V_i$, $V_i \in \operatorname{Rep}\check{G}$.

$\mathcal{G} \in \operatorname{Shv}(X^I)$.

Get a functor:

$$\begin{tikzcd} \operatorname{Shv}(\operatorname{Bun}_G) \arrow[r, "H_V"] & \operatorname{Shv}(\operatorname{Bun}_G \times X^I) \arrow[r, "\overset{!}{\otimes} p_2^!(\mathcal{G})"] & \operatorname{Shv}(\operatorname{Bun}_G \times X^I) \arrow[d, "p_{1*}"] \\ & & \operatorname{Shv}(\operatorname{Bun}_G) \end{tikzcd}$$

$\mathcal{F} \in \operatorname{Shv}(\operatorname{Bun}_G)$, $I = \{1, 2\}$.



family of sheaves on Bun_G, indexed by X^2.

Use $\mathcal{G}$ as a measure on X^2 and integrate.

Eg. $I = *$, $\mathcal{G} = \delta_x$ $\rightsquigarrow$ $H_{V,x}$ from before.

$I \overset{\alpha}{\twoheadrightarrow} J$ (eg. $\{1,2\} \to *$)

$$\begin{tikzcd}
\mathrm{Rep}\,\check{G}^{I} \otimes \mathrm{Shv}(X^J) \arrow[r, "(\text{tensor along } \alpha) \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \Delta^{\alpha}_{*}"'] \arrow[dr, phantom, "\wr\wr"] & \mathrm{Rep}\,\check{G}^{J} \otimes \mathrm{Shv}(X^J) \arrow[d] \\
\mathrm{Rep}\,\check{G}^{I} \otimes \mathrm{Shv}(X^I) \arrow[r] & \mathrm{End}(\mathrm{Shv}(\mathrm{Bun}_G))
\end{tikzcd}$$

commutes.

Universal source for endofunctors:

$$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} := \operatorname*{colim}_{(I \to J) \in \mathrm{TwArr}} \mathrm{Rep}\,\check{G}^{I} \otimes \mathrm{Shv}(X^J)$$

colim means eg. $\mathrm{Rep}\,\check{G}^{I} \otimes \mathrm{Shv}(X^J) \xrightarrow{\mathrm{ins}_{I,J}} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$

$$\begin{tikzcd}
\mathrm{Rep}\,\check{G}^{I} \otimes \mathrm{Shv}(X^J) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{ins}_{I,J}"] & \mathrm{Rep}\,\check{G}^{J} \otimes \mathrm{Shv}(X^J) \arrow[d, "\mathrm{ins}_{J,J}"] \\
\mathrm{Rep}\,\check{G}^{I} \otimes \mathrm{Shv}(X^I) \arrow[r, "\mathrm{ins}_{I,J}"] & \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}
\end{tikzcd}$$

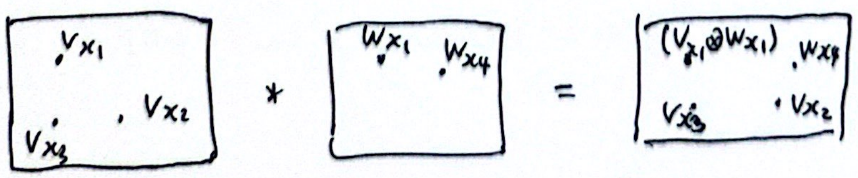
$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ is (symmetric) monoidal:

$V \in \mathrm{Rep}\,\check{G}^{I}$ $\qquad$ $W \in \mathrm{Rep}\,\check{G}^{J}$

$\mathcal{G} \in \mathrm{Shv}(X^I)$ $\qquad$ $\mathcal{H} \in \mathrm{Shv}(X^J)$

$$\mathrm{ins}_{I,I}(V \boxtimes \mathcal{G}) \underset{\substack{\uparrow \\ \text{mon. product}}}{*} \mathrm{ins}_{J,J}(W \boxtimes \mathcal{H}) = \mathrm{ins}_{I \sqcup J}(V \boxtimes W \boxtimes \mathcal{G} \boxtimes \mathcal{H})$$

Ultimate Hecke: $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ acts on $\mathrm{Shv}(\mathrm{Bun}_G)$ (by Hecke functors).



What can $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ do for us?

- Glue all the Fourier coefficients into one home.
- Full and correct (useful) def'n of Hecke eigensheaf.

Fix σ a $\check{G}$-local system on X.

<u>Construction</u>. obtain a symmetric monoidal functor

$$\mathrm{ev}_\sigma : \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \to \mathrm{Vect}$$

<u>Formal procedure</u>. Need to define

$$\mathrm{Rep}\,\check{G}^I \otimes \mathrm{Shv}(X^I) \longrightarrow \mathrm{Vect}, \quad \forall I \text{ subject to compats.}$$

σ $\check{G}$-loc. sys on X $\rightsquigarrow$ $\sigma^{\boxtimes I}$ $\check{G}^I$-loc. sys on X^I

$\rightsquigarrow \mathrm{Rep}(\check{G}^I) \longrightarrow \mathrm{Lisse}(X^I) \rightsquigarrow \mathrm{Rep}(\check{G}^I) \otimes \mathrm{Shv}(X^I) \longrightarrow \mathrm{Lisse}(X^I) \otimes \mathrm{Shv}(X^I)$

$\xrightarrow{\text{tensor}} \mathrm{Shv}(X^I)$

$\xrightarrow{\text{coh.}} \mathrm{Vect}.$

<u>State-sanctioned def'n of Hecke eigensheaf</u> w/ eigenvalue σ.

$\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ w/ extra structure

$$\Leftrightarrow \quad \begin{tikzcd} \mathrm{Vect} \arrow[r, "\mathcal{F}"] & \mathrm{Shv}(\mathrm{Bun}_G) \end{tikzcd}$$

w/ the extra structure of $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$-linearity.

$\circlearrowleft \mathrm{ev}_\sigma$ on Vect, $\circlearrowleft \mathrm{ev}_\sigma$ on $\mathrm{Shv}(\mathrm{Bun}_G)$

$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ $\qquad$ $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$

Amounts to: $V_1, \dots, V_n \in \mathrm{Rep}\,\check{G}$

$$H_V(\mathcal{F}) \simeq \mathcal{F} \boxtimes V_{1,\sigma} \boxtimes \cdots \boxtimes V_{n,\sigma}$$

$$\in \mathrm{Shv}(\mathrm{Bun}_G \times X^n)$$

Version of GL: σ does not come from a $\check{P}$-loc. sys, $\check{P} \subsetneq \check{G}$ parabolic

(GLC)

Thm σ irred. $\check{G}$-loc. sys, $\exists!$ $\mathcal{F}_\sigma$ Hecke eigensheaf

equipped w/ $\mathrm{coeff}_0(\mathcal{F}_\sigma) \simeq e$.

Thm (jt w/ Arinkin, Beraldo, Campbell, Chen, Færgeman, Gaitsgory, Lin, Rozenblyum)

except if char $k = p > 0$.

What's known about $\mathcal{F}_\sigma$:

- $\mathcal{F}_\sigma$ is perverse,
- $\mathcal{F}_\sigma$ is cuspidal
- $\mathcal{F}_\sigma$ is a semisimple perverse sheaf.
- Let $Z_{\check{G}} \subset \underbrace{S_\sigma}$

  autom. gp of σ.

  $S_\sigma / Z_{\check{G}}$ finite. trivial for $\check{G} = GL_n$, not in general. Schur's Lemma

  $\mathcal{F}_\sigma \simeq \bigoplus_{\rho \in \mathrm{Irrep}(S_\sigma)/\sim} \mathcal{F}_{\sigma,\rho}^{\dim \rho}$, $\mathcal{F}_{\sigma,\rho}$ simple perverse sheaf

  connected cpts of $\mathrm{Bun}_G \longleftrightarrow \mathrm{Irrep}(Z_{\check{G}})$

  collecting ρ's w/ same central char. ("L-packets")

  $\rightsquigarrow$ decomposition of $\mathcal{F}_\sigma$ according to connected cpts of Bun_G.

- $CC(\mathcal{F}_\sigma) = [\mathrm{Nilp}]$, $\mathrm{Nilp} \subset T^* \mathrm{Bun}_G$ global nilp. cone.
- Can use this to calculate gen. rank of $\mathcal{F}_\sigma$ is $\prod d_i^{(2d_i - 1)(g-1)}$, d_i's exponents of G, g = genus

Lecture 5.

Motto: $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ let us "glue" all the Whittaker coefficients of a sheaf on Bun_G.

Whittaker coeffs for G:

$\updownarrow$

Fourier

coeff$_0$: $\mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$

$$\mathrm{Bun}_N^{\Omega} := \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{\check{\rho}(\Omega_X^{1}) := 2\check{\rho}(\Omega^{1/2})\}$$

Rmk. $2\check{\rho} = \sum_{\substack{\check{\alpha} > 0 \\ \text{coroot}}} \check{\alpha}$

If I take the fiber over $\{\mathrm{triv}\} \in \mathrm{Bun}_T$, I get Bun_N.

For each simple root α_i, I have a map $N \to \mathbb{G}_a$

$$\begin{tikzcd} \mathrm{Bun}_N^{\Omega} \arrow[r] \arrow[drr, bend right, "\psi_i"'] & \mathrm{Bun}_{\mathbb{G}_a}^{\Omega} = R\Gamma(\Omega)[1] \arrow[dr, "\text{Serre}"] & \\ & & H^1(\Omega) = \mathbb{A}^1 \end{tikzcd}$$

$$\psi = \sum_{\alpha_i \text{ simple}} \psi_i$$

Use
$$\begin{tikzcd} & \mathrm{Bun}_N^{\Omega} \arrow[dl, "p_N"'] \arrow[dr, "\psi"] & \\ \mathrm{Bun}_G & & \mathbb{A}^1 \end{tikzcd}$$
$\rightsquigarrow$ coeff$_0$ as in the PGL_2-case.

$= \sum \check{\lambda}_i x_i$

Variant: D divisor on X w/ values in $\check{\Lambda}^+ \leftarrow$ dominant cowts of G

Can define $\check{\rho}(\Omega)(-D) \in \mathrm{Bun}_T$

&
$$\begin{tikzcd} & \mathrm{Bun}_N^{\Omega(-D)} \arrow[dl] \arrow[dr, "\psi_D"] & \\ \mathrm{Bun}_G & & \mathbb{A}^1 \end{tikzcd}$$
$\rightsquigarrow$ coeff$_D$

__Example.__ $G = PGL_2$, D $\mathbb{Z}^{\geq 0}$-valued divisor $\leftrightarrow$ effective divisor

get same coeff_D.

__Example.__ $G = \mathbb{G}_m$, D is an arbitrary divisor.

$\mathrm{coeff}_D : \mathrm{Shv}(\mathrm{Bun}_{\mathbb{G}_m}) \longrightarrow \mathrm{Vect}$ takes the !-fiber at $\mathcal{O}(D)$.

Casselman – Shalika (Frenkel – Gaitsgory – Vilonen)

__Thm__ $D = \sum \check{\lambda}_i x_i$, $\check{\lambda}_i \in X^+$, $x_i \neq x_j$, $i \neq j$.

comp. of Hecke functors $H_{V^{\check{\lambda}_i}, x_i}$

$$\mathrm{coeff}_D(\mathcal{F}) = \mathrm{coeff}_0\left(\circ_i H_{V^{\check{\lambda}_i}, x_i}(\mathcal{F})\right)$$

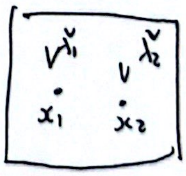
__In other words,__ $V^{\check{\lambda}_i} \in \mathrm{Rep}\,\check{G}$

$x_i \rightsquigarrow V^{\check{\lambda}_i} \otimes \delta_{x_i} \in \mathrm{Rep}\,\check{G} \otimes \mathrm{Shv}(X)$

I can tensor these over i $\rightsquigarrow$ object of $\mathrm{Rep}\,\check{G}^I \otimes \mathrm{Shv}(X^I) \longrightarrow \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$

Act by that Hecke functor, extract 0th Whit coeff, then I get coeff_D.

__Imagine__



$\mathcal{F}_\sigma \leftarrow$ normalized eigensheaf

$\mathrm{coeff}_0(\mathcal{F}_\sigma) \simeq e$

$$H_{\{V^{\check{\lambda}_i}, x_i\}}(\mathcal{F}_\sigma) \simeq \mathcal{F}_\sigma \otimes \bigotimes_i (V_\sigma^{\check{\lambda}_i})_{x_i} \quad \text{(Hecke property)}$$

__CS__ $\Rightarrow$ $\mathrm{coeff}_D(\mathcal{F}_\sigma) = \mathrm{coeff}_0(\mathcal{F}_\sigma) \otimes \bigotimes_i (V_\sigma^{\check{\lambda}_i})_{x_i} = \bigotimes_i V_{\sigma, x_i}^{\check{\lambda}_i}$

This is the sort of formula we had on Wednesday.

Fundamental pairing:

$$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \xrightarrow[\text{action}]{\text{Hecke}} \mathrm{Shv}(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}_0} \mathrm{Vect}$$

By CS, this pairing has the knowledge of all Whittaker coefficients, and "ptwise" exactly this info.

General constructions:

- $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \xrightarrow{\Gamma} \mathrm{Vect}$, $\Gamma = \mathrm{Hom}_{\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}}(\mathrm{triv}, -)$

unit object in our sym. monoidal str.

Categorical duality: Given $\mathcal{C}$ cat.,

$$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \otimes \mathcal{C} \xrightarrow{B} \mathrm{Vect} \iff \mathcal{C} \xrightarrow{F} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$$

dictionary. $\Gamma(F(\mathcal{F}) \otimes \mathcal{G}) = B(\mathcal{G} \boxtimes \mathcal{F})$, $\forall \mathcal{F} \in \mathcal{C}$, $\forall \mathcal{G} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$

Reinterpret our pairing as a functor $\mathrm{Shv}(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}^{\mathrm{Wh}}} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$

Best version of "q-expansion" for G.

Thm (Beraldo, Frenkel-Gaitsgory-Vilonen) $\mathrm{coeff}^{\mathrm{Wh}}$ is fully-faithful on $\mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ for $G = GL_n$ or PGL_n

PGL_2. imitate proof of Fourier inversion.

GL_n: imitate Piatetski-Shapiro & Shalika.

Categorical GL equiv. Assume dR setting

Another player. $LS_{\check{G}} = LS^{dR}_{\check{G}}$, $\check{G} = GL_n$: moduli stack of rk n v.b.'s on X w/ ∇

means

$\exists$ canonical sym. monoidal functor:

$$\mathrm{Loc} : \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(\mathrm{LS}^{dR}_{\check{G}})$$

$$V \in \mathrm{Rep}\,\check{G},\ x \in X \rightsquigarrow V \otimes \delta_x \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$$

$$x \in X \rightsquigarrow \mathrm{LS}^{dR}_{\check{G}}(X) \xrightarrow{ev_x} \mathrm{LS}^{dR}_{\check{G}}(\hat{D}_x) = \mathbb{B}\check{G}$$

$$V \in \mathrm{QCoh}(\mathbb{B}\check{G})$$

pullback to get a v.b. $ev_x^*(V) \leftarrow$ v.b. on $\mathrm{LS}^{dR}_{\check{G}}(X)$

$$\mathrm{Loc}(V \otimes \delta_x)$$

general construction is same idea.

<u>Thm</u> (Gaitsgory–Rozenblyum–) Loc has a fully faithful right adjoint.

In other words, get a fully faithful emb. $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \hookrightarrow \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$.

<u>Thm</u>. (Drinfeld–Gaitsgory). Action of $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ on $\mathrm{Dmod}(\mathrm{Bun}_G)$ factors over Loc.

<u>Cor</u> $\mathrm{coeff}^{\mathrm{Wh}}$ factors as:

$$\begin{tikzcd} \text{D-mod}(\mathrm{Bun}_G) \arrow[r, dashed, "\mathbb{L}_{G,\mathrm{coarse}}"] \arrow[dr, "\mathrm{coeff}^{\mathrm{Wh}}"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\ & \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \arrow[u, hook] \end{tikzcd}$$

<u>Idea of cat. GLC</u>:

$\mathbb{L}_{G,\mathrm{coarse}}$ is "almost" an equiv.

<u>Obstruction</u>. If G is non-abelian. $\underline{e}_{\mathrm{Bun}_G} \leftarrow$ const sheaf.

$$\mathbb{L}_{G,\mathrm{coarse}}(\underline{e}_{\mathrm{Bun}_G}) = 0.$$

Thm (ver. 2)

$\mathbb{L}_{G,\mathrm{coarse}}$ induces an equiv.

$$\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}} \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}).$$

Cor. coeff$^{\mathrm{ult}}$ is fully-faithful on cuspidal cat. for D-mods. for G.

Correction to GLC due to Arinkin - Gaitsgory.

$$\mathrm{QCoh}(\mathcal{Y}) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathcal{Y}))$$

Thomason - Trobaugh

$$\mathrm{IndCoh}(\mathcal{Y}) = \mathrm{Ind}(\mathrm{Coh}(\mathcal{Y}))$$

$$\mathrm{Perf}(\mathcal{Y}) \subset \mathrm{Coh}(\mathcal{Y})$$

objects in Coh w/ finite Tor/Ext amplitude

strict incl. if $\mathcal{Y}$ has singularities

In the case $\mathcal{Y} = \mathrm{LS}_{\check{G}}$, AG: defined a subcat

$$\mathrm{Perf}(\mathrm{LS}_{\check{G}}) \subset \mathrm{Coh}_{\mathrm{Nilp}} \subset \mathrm{Coh}(\mathrm{LS}_{\check{G}}) \quad \text{as follows:}$$

$\mathrm{Coh}_{\mathrm{Nilp}}$ is the subcat. of Coh obtained by taking $\mathcal{P} \in \mathrm{Perf}(\mathrm{LS}_{\check{M}})$

$$\begin{tikzcd} & \mathrm{LS}_{\check{P}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{LS}_{\check{G}} & & \mathrm{LS}_{\check{M}} \end{tikzcd}$$

proper

finite tor amplitude

Take $p_* q^*(\mathcal{P}) \in \mathrm{Coh}(\mathrm{LS}_{\check{G}})$

do this for all $\check{P}$ (including $\check{G}$).

<u>Small thm.</u>

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, dashed, "\exists!\, \mathbb{L}_G"] \arrow[dr, "\mathbb{L}_{G,\mathrm{coarse}}"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) = \mathrm{Ind}(\mathrm{Coh}_{\mathrm{Nilp}}) \arrow[d, "\Psi"] \\ & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

subject to small conditions

Ψ ← ind-ext'n of $\mathrm{Coh}_{\mathrm{Nilp}} \subset \mathrm{Coh} \subset \mathrm{QCoh}$

Technical words:

$\mathbb{L}_G$ is supposed to send cpt objects to objects bounded from below in the t-structure on RHS.

Thm (ver. 2) $\mathbb{L}_G$ induces an equiv. $\mathrm{Dmod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$

Rmk. • Pf uses particularities of dR setting:

(always) use KM localization at critical level to prove "geometric" statements about $\mathrm{Dmod}(\mathrm{Bun}_G)$

More seriously, use $\mathrm{LS}_{\check{G}}^{dR}$ has few global algebraic functions.

Still get Betti versions (& ℓ-adic / char. 0 base) "by Riemann-Hilbert" in same sense.

Rmk. The "concrete" things explained yesterday really use the categorical assertions.

σ irred. $\exists!\ \mathcal{F}_\sigma \in \mathrm{Shv}(\mathrm{Bun}_G)$ eigensheaf for σ w/ $\mathrm{coeff}_0(\mathcal{F}_\sigma) \simeq e$

s.t. • $\mathcal{F}_\sigma$ is perverse

• $\mathcal{F}_\sigma$ is semisimple, $\mathcal{F}_\sigma = \bigoplus_{\rho \in \mathrm{Irrep}(S_\sigma)} \mathcal{F}_{\sigma,\rho}^{\dim \rho}$ ← simple.

$S_\sigma = \mathrm{Aut}(\mathcal{F}_\sigma)$

• $\mathrm{CC}(\mathcal{F}_\sigma) = [\mathrm{Nilp}]$

$\mathcal{O}_{\mathrm{LS}_{\check{G}}} \longleftrightarrow \mathrm{Poinc}_!^{\mathrm{vac}}$ ← corep. coeff_0

"Poinc" $\longleftrightarrow e_{\mathrm{Bun}_G}$

↑

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