Prismatic Dieudonné theory

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Lecture 1

For the entire course, fix a prime $p>2$.

Thm (Néron - Ogg - Shafarevich) Let $K|\mathbb{Q}_p$ be a finite ext. An abelian var. A/K has good reduction iff $T_\ell(A)$ is unramified, where $\ell \neq p$.

But what about $\ell = p$? First we reduce this to a question on p-div. gps, after noting that A has good reduction iff $G = A[p^\infty]$ has good reduction. It turns out that G has good reduction if and only if $T_p G$ is crystalline, which was conj. by Fontaine and proved later.

In fact, Breuil-Kisin classified p-div. gps over $\mathcal{O}_K$, in the following sense: There is a fully faithful functor

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \longrightarrow \{\text{Breuil-Kisin modules}\}$$

and so we can define an equiv.

$$BT_{/\mathcal{O}_K} \simeq \{\text{subcat. of BK-modules}\}.$$

This picture has generalized by Bhatt-Scholze, where they defined a notion of a prismatic F-crystal. Then what we have is an equiv:

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathrm{Gal}_K) \simeq \{\text{prismatic } F\text{-crystals} / \mathrm{Spf}\, \mathcal{O}_K\}$$

Here, note that on the left hand side there is no Hodge-Tate condition. This has also been generalized to smooth p-adic formal schemes by Guo-Reinecke.

The idea is to try to construct a functor from BT/R to the cat. of prismatic F-crystals over Spf R. This will be an adaptation of the ideas for crystalline Dieudonné theory. Namely, we will take $\mathrm{Ext}^1(\underline{G}, \mathcal{O})$ over the prismatic site. However, this functor is not fully faithful in many cases.

<u>Thm</u> (Anschütz - Le Bras) If R is quasi-syntomic, then this functor induces an equiv. between BT/R and the cat. of admissible prismatic F-crystals on Spf R.

If $pR = 0$, then this recovers crystalline Dieudonné theory, so we cannot expect this to be an equiv. in general.

<u>Ex</u>. If R is integral perfectoid, then it is quasi-syntomic. If R is the completion of a locally complete intersection ring, then it is quasi-syntomic.

The problem is that the F-crystal forgets some information. We have to look at sth called prismatic F-gauges. This roughly carries the extra information of the Nygaard filtration. This filtration is a bit difficult to see from Bhatt-Scholze's description.

Thm (Madad) Assuming that R is quasi-syntomic, there is an equiv.

$$BT_{/R} \simeq \left\{ \begin{array}{c} \text{locally free prismatic } F\text{-gauges} / \mathrm{Spf}\, \mathcal{O}_K \\ \text{w/ Hodge-Tate weights } 0, 1 \end{array} \right\}$$

This now generalizes to p-adic formal schemes that are locally of finite type, due to Gardner - Madapusi. But we won't discuss this.

Here is an outline of the proof of Anschütz - Le Bras's result. First we check that both sides are stacks over the quasi-syntomic site. At this point, we note that the quasi-syntomic site has a nice set of generators, called quasi-regular semi-perfectoid rings. On these rings, both sides can be made explicit.

Rmk. We will use the stacky approach. There is a stack X^{Δ} s.t. q.coh sheaves on X^{Δ} are the same as prismatic F-crystals, and prismatic F-gauges are the same as q.coh. sheaves on X^{syn}.

Lecture 2. Usually, prismatic F-crystal means an obj. of $D_{qc}(X^{\Delta})$, but in this class we will use the more classical notion of a prismatic F-crystal in vector bundles, i.e. a locally free sheaf on the prismatic site.

Rmk. For general p-adic formal scheme $\mathfrak{X}$, this object $\mathfrak{X}^{\Delta}$ itself may be derived, but in the quasi-syntomic case, this derivedness does not occur.

Def. Let S be a scheme and let $I \subset \mathcal{O}_S$ be a coherent sheaf of ideals.

A <u>divided power str.</u> on I is a collection of maps $\gamma_n : I \to I$ satisfying cond'ns.

- $\gamma_0(x) = 1$
- $\gamma_1(x) = x$
- $\gamma_n(x)\gamma_m(x) = \binom{n+m}{n} \gamma_{n+m}(x)$
- $\gamma_n(ax) = a^n \gamma_n(x)$
- $\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y)$
- $\gamma_n(\gamma_m(x)) = \dfrac{(nm)!}{n!\,(m!)^n} \gamma_{nm}(x)$

In this case, we say that (S, I, γ) is a divided power scheme, and write

$$S_0 = V(I) \hookrightarrow S.$$

We will often (but not always) assume that I consists of locally nilp. sections, where $S_0 \hookrightarrow S$ is a thickening. In this case, we call (S_0, S, γ) a <u>pd-thickening</u>

<u>Example</u> The most important example is $S = \operatorname{Spec} \mathbb{Z}_p$ and $I = (p)$, $\gamma_n(p) = \frac{p^n}{n!}$.

Fix a base pd-scheme (S, I, γ).

<u>Def.</u> Let X/S_0 be a scheme. A pd-thickening over X to (S, I, γ) is a scheme $U \to X$ together w/ a pd-thickening (U, T, δ) fitting into the diagram

$$\begin{tikzcd} T \arrow[dd] & U \arrow[l] \arrow[d] \\ & X \arrow[d] \\ S & S_0 \arrow[l] \end{tikzcd}$$

where γ and δ are compatible in $T \to S$.

The big crystalline site $CRIS(X/S)$ is the cat. of this divided power thickenings, w/ the Zar. top., meaning the covering maps should look like $(U', T') \to (U, T)$ where $T' \to T$ is an open immersion and $U' \simeq U \times_T T'$.

We can also think of a sheaf $\mathcal{F}$ on $CRIS(X/S)$ as the data of, for all $(U, T, \delta) \in CRIS(X/S)$, a sheaf $\mathcal{F}_T$ on T, together w/ for each $f: (U, T, \delta) \to (U', T', \delta')$ a comparison map $c_f : f^{-1} \mathcal{F}_{T'} \to \mathcal{F}_T$ s.t.

- some cocycle conditions hold
- if f is an open embedding, then c_f is an isom.

We call the corresponding topos $(X/S)_{CRIS}$.

Ex. There is the str. sheaf $\mathcal{O}_X$, sending $(U, T, \delta) \mapsto H^0(T, \mathcal{O}_T)$. For every big Zariski sheaf $\mathcal{F}$ on X_{ZAR}, we also have the sheaf $(U, T, \delta) \mapsto \mathcal{F}(U)$, which we just call $\underline{\mathcal{F}} \in (X/S)_{CRIS}$. Then there is also the example

$$I_{X/S} := \ker(\mathcal{O}_{X/S} \twoheadrightarrow \underline{\mathbb{G}_a})$$

Def. A crystal in $\mathcal{O}_{X/S}$-modules is a sheaf $\mathcal{F}$ of $\mathcal{O}_{X/S}$-modules, s.t.

- for all (U, T, δ), the sheaf $\mathcal{F}_T$ is a q. coh. $\mathcal{O}_T$-module.
- for all f, the map $c_f : f^* \mathcal{F}_{T'} \xrightarrow{\sim} \mathcal{F}_T$ is an isom.

Thm Assume that $(S_0, S, \gamma) = (\mathrm{Spec}\, \mathbb{F}_p, \mathrm{Spec}\, \mathbb{Z}_p, \gamma)$. If G is a p-div gp over X, the Dieudonné module $\mathbb{D}(G) = \mathcal{E}xt^1_{X/S}(\underline{G}, \mathcal{O}_{X/S})$ is a crystal of $\mathcal{O}_{X/S}$-modules, locally free of rank equal to the height $h = ht(G)$.

Moreover, there are maps $V: \mathbb{D}(G) \to \mathbb{D}(G)^\sigma$, $F: \mathbb{D}(G)^\sigma \to \mathbb{D}(G)$ induced by $V_{X/S}: G^{(p)} \to G$ and $F_{X/S}: G \to G^{(p)}$.

We will first reduce to the case of $G_n = G[p^n]$, a finite flat gp scheme. Now there is this idea of Raynaud, which is to embed locally on X this G_n into an abelian scheme, so that we have

$$0 \to G_n \to A^0 \to A^1 \to 0$$

Then we reduce to understanding this $\mathrm{Ext}^i(\underline{A}, \mathcal{O}_{X/S})$. This method will also tell us that if $G = A[p^\infty]$, then we have $\mathbb{D}(G) = H^1_{crys}(A, \mathcal{O})$

Lecture 4. Deligne spectral sequence $\Rightarrow$ Thm. $f: A \to X$ abelian scheme.

1) $\mathcal{H}om_{X/S}(\underline{A}, \mathcal{O}^{crys}_{X/S}) = 0$

2) $\mathcal{E}xt^1_{X/S}(\underline{A}, \mathcal{O}^{crys}_{X/S}) = R^1 f_{CRIS *}(\mathcal{O}^{crys}_{A/S})$

locally free crystal $\mathbb{D}(A) =$

3) $\mathcal{E}xt^2_{X/S}(\underline{A}, \mathcal{O}^{crys}_{X/S}) = 0$

Goal: Explain how $R^1 f_{CRIS *}(\mathcal{O}^{crys}_{A/S})$ gives a crystal in $\mathcal{O}^{crys}_{X/S}$-mod which is locally free.

relate this to de Rham cohomology of A/X.

<u>de. Rham complex</u> $[\mathcal{O}_A \xrightarrow{\nabla} \Omega^1_{A/X} \xrightarrow{\nabla} \Omega^2_{A/X} \xrightarrow{\nabla} \cdots]$

$$\mathcal{H}^n_{dR} := R^n f_* (\Omega^\bullet_{A/X})$$

<u>Prop</u>. $\forall n \geq 0$, $\mathcal{H}^n_{dR}$ is finite locally free $/X$, formation commutes w/ base change.

$\mathcal{H}^1_{dR}$ is of rk $2d$, $d =$ rel. dim A/X

$\Lambda^n \mathcal{H}^1_{dR} \to \mathcal{H}^n_{dR}$ is an isom. for $d \mid n$.

<u>Prop</u> $R^1 f_{CRIS*} (\mathcal{O}^{crys}_{A/S})$ is a crystal of $\mathcal{O}_{X/S}$-mod, locally free of rk $2d$.

<u>Sketch</u>: Show $R^1 f_{CRIS*} (\mathcal{O}^{crys}_{A/S})$ is a crystal, work locally on X

reduce to case: $(U, T, \delta) \in CRIS(X/S)$, T affine, $U \hookrightarrow T$ is nil immersion

$f' : A' \to T$

$$\begin{array}{ccc} A_U & & \\ \| & & \\ A \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

① $\exists$ abelian scheme A'/T lifting $A \times_X U$

② Have isomorphisms

$R^\bullet f_{CRIS*} (\mathcal{O}^{crys}_{A/S})(U, T, \delta) \xrightarrow{\sim} R^\bullet f_{A_U/T\,*} (\mathcal{O}_{A_U/T})$

true for general smooth lift over T (evaluate on small site)

$\xrightarrow[\sim]{b} R^\bullet f'_* (\Omega^\bullet_{A'/T})$

as $\mathcal{O}_T$-mod on Zariski site of T

Conclusion. $\mathbb{D}(A)$ is locally free of rk $2d$.

For p div gp $G = \varinjlim_n \underbrace{G[p^n]}_{\substack{\shortparallel \\ G_n}}$

$0 \to G_n \to A_0 \to A_1 \to 0$ locally embed G_n into abelian scheme,

$\Rightarrow \mathcal{E}xt^1_{X/S}(\underline{G_n}, \mathcal{O}_{X/S}^{crys})$ is also a crystal in $\mathcal{O}_{X/S}^{crys}$-mod

Prop. $\mathbb{D}(G) = \mathcal{E}xt^1(\underline{G}, \mathcal{O}_{X/S}^{crys})$

this is locally free, crystal of $\mathcal{O}_{X/S}^{crys}$-mod, of rk $h = ht(G)$

If $(U, T, \delta) \in CRIS(X/S)$ s.t. $p^n \mathcal{O}_T = 0$,

$$\mathcal{E}xt^1_{X/S}(\underline{G}, \mathcal{O}_{X/S}^{crys})_{(U,T,\delta)} = \mathcal{E}xt^1_{X/S}(\underline{G_n}, \mathcal{O}_{X/S}^{crys})_{(U,T,\delta)}$$

is locally free of rk h over $\mathcal{O}_T$

$0 \to \underline{G_n} \to \underline{G} \xrightarrow{p^n} \underline{G} \to 0$ assoc. LES

Fact. $\mathcal{H}om_{X/S}(\underline{G}, \mathcal{O}_{X/S}^{crys}) = 0$. [exact sequence implies p^n injective]

and X is p-nilpotent

Lecture 3

Thm Let G be a finite flat gp scheme $/S$. Then Zariski-locally on S, there exists an abelian scheme A and a closed embedding $G \hookrightarrow A$ (This is even true for a Jacobian of a smooth curve.)

Sketch: We may reduce to the case when $S = \operatorname{Spec} R$, R is a noetherian local ring. The point is that there exists a proj. bdle $P = \mathbb{P}(\mathcal{E})$ s.t. the Cartier dual $G^\vee$ acts freely on $P - Z$ for some Z of codim at least 2. This is constructed as $\mathcal{E} = \mathcal{O}_{(G^\vee)}^{\otimes n}$.

Now when we take the quotient $Q = P/G^\vee$, this is projective and smooth away from this codim 2 locus. By Bertini's theorem, we have X/S a smooth relative curve contained in Q, disjt away from the image of Z. By construction, there is a $G^\vee$-torsor P'/X. This gives a closed embedding $G \to \operatorname{Pic}(X/S)$.

This lands in $\operatorname{Pic}(X/S)^\circ$ because the quotient is $\mathbb{Z}$ and G is torsion.

(A reference is Berthelot - Breen - Messing.)

Anyway, once we have this theorem, we locally have a SES

$$0 \to G \to A^0 \to A^1 \to 0.$$

Thm. (Deligne) Let $\mathcal{X}$ be a general topos. For $G \in \mathcal{X}$ an abelian gp. $\exists$ a functorial (in G) resolution

$$C(G)_\bullet = (\cdots \to \mathbb{Z}[G^3] \oplus \mathbb{Z}[G^2] \to \mathbb{Z}[G^2] \to \mathbb{Z}[G]) \text{ of } G$$

The chain cpx is somewhat complicated.. For example, the first differential is

$d_1 : [x,y] \mapsto -[x] + [x+y] - [y]$, and the second differential d_2 is the sum of $[x,y] \mapsto [x,y] - [y,x]$; $[x,y,z] \mapsto -[y,z] + [x+y,z] - [x,y+z] + [x,y]$.

This result now gives a SS

$$E_1^{ij} = \mathrm{Ext}^j_{Ab(\mathcal{X})}(C(G)_i, F) \Rightarrow \mathrm{Ext}^{i+j}_{Ab(\mathcal{X})}(C(G)_\bullet, F) = \mathrm{Ext}^{i+j}_{Ab(\mathcal{X})}(G, F)$$

for any $F \in Ab(\mathcal{X})$.

We may apply this to $G = \underline{A}$ for A/X an abelian scheme, and $F = \mathcal{O}^{crys}_{X/S}$.

If we pretend we have the Künneth formula and such, we can write out the SS as

$$M' \longrightarrow M'^{\oplus 2} \oplus M^{\otimes 2} \longrightarrow M'^{\oplus 2} \oplus M^{\otimes 2} \oplus M'^{\oplus 3} \oplus \cdots$$

$$M \longrightarrow M^{\oplus 2} \longrightarrow M^{\oplus 2} \oplus M^{\oplus 3}$$

$$\mathcal{O}^{crys}_{X/S} \longrightarrow \mathcal{O}^{crys}_{X/S} \longrightarrow (\mathcal{O}^{crys}_{X/S})^{\oplus 2}$$

where we write $M = R^1 f_{CRYS *}(\mathcal{O}^{crys}_{A/S})$, $M' = \wedge^2 M$.

Lecture 5. If E is a sheaf in $\mathcal{O}^{crys}_{X/S}$-mod in fppf topology on $CRIS(X/S)$ s.t.

(1) $\forall (U,T,\delta)$, the assoc. Zariski sheaf E_T on T is q.coh.;

(2) $\forall$ fppf covering $(U',T',\delta') \xrightarrow{f} (U,T,\delta)$

$$\begin{array}{ccc} U' & \hookrightarrow & T' \\ \downarrow & & \downarrow \text{fppf} \\ U & \hookrightarrow & T \end{array}$$

$f^* E_T \xrightarrow{\sim} E_{T'}$ isom. of sheaf of $\mathcal{O}_{T'}$-mod.

$$\alpha_* : \text{Sheaves on } CRIS(X/S)_{fppf} \longrightarrow \text{Sheaves on } CRIS(X/S)_{Zar}$$

Prop $R^i \alpha_* (E) = 0$ for $i > 0$.

idea: sheafify $H^i(T'/T, E_T)$, $i>0$. $T' \to T$ both affine

Prop. If G abelian sheaf on $CRIS(X/S)_{fppf}$, E as before,

Have: $R\mathcal{H}om_{X/S,\,fppf}(\underline{G}, E) = \alpha^*\left(R\mathcal{H}om_{X/S,\,Zar}(\underline{G}, E)\right)$

This follows from deriving

$$\mathcal{H}om_{X/S,\,fppf}(\alpha^*(\underline{G}), -) \simeq \alpha^* \mathcal{H}om_{X/S,\,Zar}(\underline{G}, \alpha_*(-))$$

Upshot: $\mathcal{E}xt^i_{X/S}(\underline{G}, \mathcal{O}^{crys}_{X/S})$ independent of topology when viewed as presheaf on $CRIS(X/S)$

Prop. (1) $\mathcal{E}xt^2_{X/S}(\underline{G}, \mathcal{O}) = 0$

G p-div. gp / X

(2) $\mathcal{E}xt^1_{X/S}(\underline{G}, \mathcal{O})_{(U,T,\delta)} = \mathcal{E}xt^1_{X/S}(\underline{G_n}, \mathcal{O})_{(U,T,\delta)}$

when $p^n \mathcal{O}_T = 0$

and $\mathcal{E}xt^1_{X/S}(\underline{G_n}, \mathcal{O})_{(U,T,\delta)}$ is locally free of rk $h / \mathcal{O}_T$.

Consider.

$$\begin{tikzcd}
0 \arrow[r] & \underline{G_n} \arrow[r] \arrow[d] & \underline{G} \arrow[r, "p^n"] \arrow[d, "\wr"] & \underline{G} \arrow[r] \arrow[d, "p^m"] & 0 \\
0 \arrow[r] & \underline{G}_{n+m} \arrow[r] & \underline{G} \arrow[r, "p^{n+m}"] & \underline{G} \arrow[r] & 0
\end{tikzcd}$$

$m \geq n$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}xt^1(\underline{G}, \mathcal{O}) \arrow[r] \arrow[d, equal] & \mathcal{E}xt^1(\underline{G}_{n+m}, \mathcal{O}) \arrow[r] \arrow[d] & \mathcal{E}xt^2(\underline{G}, \mathcal{O}) \arrow[r] \arrow[d, "p^m = 0"] & 0 \\
0 \arrow[r] & \mathcal{E}xt^1(\underline{G}, \mathcal{O}) \arrow[r] & \mathcal{E}xt^1(\underline{G}_n, \mathcal{O}) \arrow[r] & \mathcal{E}xt^2(\underline{G}, \mathcal{O}) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \mathcal{E}xt^1(\underline{G}, \mathcal{O}) \simeq \mathrm{Im}\left(\mathcal{E}xt^1(\underline{G}_{n+m}, \mathcal{O}) \to \mathcal{E}xt^1(\underline{G}_n, \mathcal{O})\right)$

$\underset{\text{surj.}}{\simeq} \mathcal{E}xt^1(\underline{G_n}, \mathcal{O})$

and also get $\mathcal{E}xt^2(\underline{G}, \mathcal{O}) = 0$

Claim have SES of fin. flat gp schemes

$$0 \to G' \to G \to G'' \to 0$$

get

$$\mathcal{E}xt^1(G'', \mathcal{O}) \to \mathcal{E}xt^1(G, \mathcal{O}) \to \mathcal{E}xt^1(G', \mathcal{O}) \to 0 \quad \text{is exact}$$

embed G into A abelian scheme

$$\begin{tikzcd} \mathcal{E}xt^1(G,\mathcal{O}) \arrow[rr, "(*)"] & & \mathcal{E}xt^1(G',\mathcal{O}) \\ & \mathcal{E}xt^1(\underline{A},\mathcal{O}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\leftarrow$ this is surj. $\to$ $(*)$ surj.

Claim $\mathbb{D}(G_n) = \mathcal{E}xt^1(\underline{G}_n, \mathcal{O})$ is locally gen. by $h = ht(G)$ sections.

If claim is true, consider

$$0 \to \underline{G}_n \to A[p^n] \to H \to 0$$

know that

$\mathbb{D}(A[p^n])$ locally free of rk $2g$

know: H is also a truncated p-div gp, ht $2g-h$

Sketch of pf of claim: Let $e: S \to G$ unit section

$G = G_n$, $\tau^{\leq 1} R\mathcal{H}om_S(G^\vee, \mathbb{G}_a) \simeq$ Lie $G = R\mathcal{H}om_{\mathcal{O}_S}(\text{coLie}\, G, \mathcal{O}_S)$

$\text{coLie}\, G = e^* \mathbb{L}_{G/S}$, $\omega_G = \mathcal{H}^0(\text{coLie}\, G)$, $\nu_G = \mathcal{H}^1(\text{Lie}\, G)$

rk of $\nu_{G^\vee} = \dim G^\vee = h - \dim G$

Recall. have exact seq.

$$0 \to I_{X/S} \to \mathcal{O}_{X/S} \to \underline{G}_a \to 0 \quad \text{sheaves on } CRIS(X/S)$$

$w_G \leftarrow$ if G is BT_n, then this is loc. free, rk dim G $\qquad$ $\nu_{G^\vee}$, $G^\vee$ = Cartier dual.

$$\mathcal{E}xt^1(\underline{G}, I_{X/S}) \to \mathcal{E}xt^1(\underline{G}, \mathcal{O}) \to \mathcal{E}xt^1(\underline{G}, \underline{G}_a)$$

$$\to \mathcal{E}xt^2(\underline{G}, I_{X/S}) \to \cdots$$

$\mathcal{E}xt^2(\underline{G}, I_{X/S}) = 0 \leftarrow$ deduce from $\mathcal{E}xt^2(\underline{G}, \mathcal{O})$

Lecture 6.

Prismatic site.

Def. δ-ring: (A, δ)

$\delta : A \to A$ map of sets

s.t.
1) $\delta(1) = 0 = \delta(0)$
2) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$
3) $\delta(x+y) = \delta(x) + \delta(y) + \dfrac{x^p+y^p-(x+y)^p}{p}$

map of δ-rings: $(A, \delta) \to (A', \delta')$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowright"] & A' \arrow[d, "\delta'"] \\ A \arrow[r] & A' \end{tikzcd}$$

• $\varphi(x) = x^p + p\,\delta(x)$ is a ring homomomorphism lifting Frobenius. A/p

• If A is p-torsion free, there is a bijection between δ-structures on A and lifts of Frobenius.

e.g. $\overset{A}{\overset{\shortparallel}{W(k)}}$ has a unique Frobenius lift φ, define $\delta(x) = \dfrac{\varphi(x) - x^p}{p}$.

$k = \mathbb{F}_q$

$q = p^n$

Def. $d \in A$ is <u>distinguished</u> if $\delta(d)$ is a unit in A.

<u>Observe</u>, • $A = W(K)$, take $d = p$, p is distinguished, $\delta(p) = 1 - p^{p-1}$ clearly unit.

• In general, p is distinguished in any <u>p-local</u> δ-ring

$$\underset{\substack{\shortparallel \\ \text{Jacobson radical}}}{p \in \mathrm{rad}(A)}$$

• $A = \mathbb{Z}_p[\![q-1]\!]$, δ-str. induced by $\varphi(q) = q^p$.

$[p]_q := \dfrac{q^p - 1}{q-1} = 1 + q + \cdots + q^{p-1}$ is distinguished

$\mathbb{Z}_p[\![q-1]\!] \longrightarrow \mathbb{Z}_p$ also respects δ-structures.

$[p]_q \longmapsto p$

• $K \mid \mathbb{Q}_p$ fin. ext., k = res. field of $\mathcal{O}_K$.

$W(k)[\![u]\!]$, δ-str., $\varphi(u) = u^p$

$\Downarrow$

$E(u)$ Eisenstein polynomial for π uniformizer in $\mathcal{O}_K$, $E(u)$ is distinguished.

$W(k)[\![u]\!] \longrightarrow W(k)$

$E(u) \longmapsto p\cdot$ unit

Lemma. $d \in A$, assume $p, d \in \mathrm{Rad}(A)$, then d is distinguished iff $p \in (d, \varphi(d))$.

$\downarrow$

geometrically,

$\mathrm{Spec}(A/(d))$ $\mathrm{Spec}(A/(\varphi(d)))$

intersects only on mod p fiber

($\Rightarrow$) clear

($\Leftarrow$)

Def. A prism is a pair (A, I)

$I \subset A$ ideal, A δ-ring s.t.

1) I is finite locally free of rk 1

(usually assume that $I = (f)$ principal)

2) A is derived (p, I)-complete

3) $p \in I + \varphi(I)A$.

(this holds if I is gen. by distinguished element).

recall. if A is not noetherian, $A \to \hat{A}_m$ may not be flat.

If we look at derived completion, then preserves flatness.

Observe: M A-mod, complete w.r.t. (f), then observe

$$R\varprojlim\left(\cdots M \xrightarrow{f} M \xrightarrow{f} M\right) \in D(A\text{-mod})$$

$= R\mathrm{Hom}_A(A_f, M[0])$

(view M as cpx $M[0]$)

$M = \varprojlim_n M/f^n M$. if interchange limit $\underset{\times f}{R\varprojlim}\left(\cdots M/f^n M \xrightarrow{f} M/f^n M\right) = 0$

$\Rightarrow R\varprojlim\left(\cdots M \xrightarrow{f} M\right) = 0$

If K complex, say K is derived f-complete if

$$R\varprojlim(\cdots \xrightarrow{f} K \xrightarrow{f} K) = 0.$$

"$R\mathrm{Hom}_A(A_f, K)$"

If $I = (f_1, \cdots, f_n)$, say derived I-complete if

$$R\varprojlim(\cdots \xrightarrow{f_i} K \xrightarrow{f_i} K) = 0 \quad \text{for all } i.$$

Prop. If M is derived I-complete, then $M \longrightarrow \varprojlim M/I^nM$ is surjective.

if further $\bigcap I^n M = (0)$, then derived I-complete $\Rightarrow$ classical I-complete.

Prop. If M has bounded f^∞-torsion, i.e. $M[f^\infty] = M[f^N]$ for some large N,

derived (f)-completion $=$ classical (f)-completion.

Lecture 7

$I = (f_1, \cdots, f_n)$

Prop If K is derived f_i-complete for all f_i, then it is f-complete for all $f \in I$.

Follows from: The set of $f \in A$ s.t. K is derived f-complete is a radical ideal.

Pf. K derived f-complete.

$g \in A$, $\quad R\mathrm{Hom}_A(A_{fg}, -) \simeq R\mathrm{Hom}_A(A_g, R\mathrm{Hom}_A(A_f, -))$

$\Rightarrow K$ derived fg-complete

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{fg(f+g)} \to 0$$

$\Rightarrow$ If f, g lie in set, so does $f+g$.

radical: $A_{f^n} = A_f$. $\square$

Derived completion: of K.

resolved using Koszul complex.

$$\hat{K} = R\varprojlim_m K \otimes^{\mathbb{L}}_{\mathbb{Z}[x_1,\dots,x_n]} \mathbb{Z}[x_1,\dots,x_n]/(x_1^m,\dots,x_n^m)$$

where x_i acts via f_i

this is derived I-complete

if K is derived I-complete, $K \xrightarrow{\sim} \hat{K}$.

Prop. (derived Nakayama) K derived I-complete, then

$$K = 0 \iff K \otimes^{\mathbb{L}}_A A/I = 0.$$

Prop. A is derived I complete $\Rightarrow I \subset \mathrm{rad}(A)$.

Pf $u \in 1+I$, $M = A/(u)$, $M \otimes^{\mathbb{L}}_A A/I = 0$

derived Nakayama $\Rightarrow M = 0$

Def. (A, I), A δ-ring,

(1) I finite loc. free of rk 1

(2) A is derived (p, I)-complete

(3) $p \in I + \varphi(I)A$.

this condition holds if $I = (d)$, where d is distinguished.

In fact, $\exists$ faithfully flat $A \to A'$ s.t. $IA' = (d')$, d' distinguished elt in A'.

A' is an ind-(Zariski localization) of A (i.e. colimit of localizations of A)

Prism (A, I) is bounded if A/I has bounded p^∞-torsion.

perfect if $\varphi : A \to A$ isomorphism

crystalline if $I = (p)$.

<u>Lemma</u>. If (A, I) is bounded, then A is classically (p, I)-complete.

$$A \simeq R\varprojlim_n R\varprojlim_m \mathrm{Kos}(A; I^n, p^m)$$

locally $I^n = (f)$

f non-zero divisor

$$\simeq R\varprojlim_n R\varprojlim_m \mathrm{Kos}(A/I^n ; p^m)$$

A/I has bounded p^∞-torsion

$\Rightarrow A/I^n$ has bounded p^∞-torsion

$$\simeq R\varprojlim_n R\varprojlim_m A/(I^n, p^m)$$

$$\simeq R\varprojlim A/(I^n, p^m)$$

Take H^0, $A = \varprojlim A/(I^n, p^m)$.

<u>Examples</u>.
- $(\mathbb{Z}_p, (p))$ bounded, crystalline,

  $(A, (p))$ bounded, crystalline

  A p-complete, p-torsion free δ-ring
- $(\mathbb{Z}_p[[q-1]], [p]_q)$ also bounded prism [q-de Rham prism]
- $(W(K)[[u]], E(u))$ also bounded, Breuil-Kisin prism
- $(W(\mathcal{O}_C^\flat) = A_{\inf}(\mathcal{O}_C), \ker\theta)$ also bounded, $A_{\inf}$-prism

  C alg. closed p-complete nonarch. char. 0

$f : (A, I) \to (B, J)$

s.t. $f : A \to B$ map of δ-rings.

$f(I) \subset J$

<u>Lemma</u>. (Rigidity) We in fact have $I \otimes_A B \simeq J$.

(Sketch) Pass to faithfully cover of A, B.

reduce to $(A, (d)), (B, (e))$

$\Rightarrow$ $d = ef$ for some $f \in B$.

$\delta(d) = e^p \delta(f) + f^p \delta(e) + p\delta(e)\delta(f)$

$\Rightarrow f$ unit $\Rightarrow$ $(d) = (e)$.

<u>Lecture 8</u>. $M = A/(u)$ is derived I-complete.

$u \in 1 + I$

<u>Prop</u>. The cat. of derived I-complete complexes is closed under taking fibers & cofibers.

<u>Def</u>. M A-module is I-completely (faithfully) flat if M/IM is (faithfully) flat$/_{A/I}$ and $\mathrm{Tor}_i^A(A/I, M) = 0$, $\forall i > 0$.

$K = M[0]$. $\left(K \otimes_A^L A/I\right)$ Tor amplitude $[0,0]$, and cohom. in deg 0 is a (faithfully) flat A/I-mod

- If M is flat A-module, then it is I-completely flat.

Lemma If M is a flat A-complex, then derived I-completion is I-completely flat.

Def. A derived I-complete A-alg. R is I-completely étale

(I-completely smooth, I-completely ind-smooth)

if $R \otimes_A^{\mathbb{L}} A/I$ has Tor amplitude $[0,0]$,

in deg 0, given by étale (smooth, ind-smooth) A/I-algebra.

Def : $(A,I) \to (B,J)$ is (faithfully) flat if $A \to B$ is (p,I)-completely (faithfully) flat.

Thm (A,I) bounded prism

(1) If $M \in D(A)$ is derived (p,I)-complete, (p,I)-completely flat complex, then M is an (honest) A-mod, classically (p,I)-complete.

(2) The cat. of (faithfully) flat maps $(A,I) \to (B,J)$ identifies w/ the cat. of (derived) (p,I)-complete (faithfully) flat δ-A-algebras B.

(2) follows from rigidity Lemma.

Thm. Equivalence of categories

$$\{(A,I) \text{ perfect prism}\} \simeq \{\text{(integral) perfectoid } R\}$$

$$(A,I) \longmapsto A/I$$

$$(A_{\inf}(R), \ker\theta) \longleftarrow R$$

R is p-complete $\mathbb{Z}_p$-alg.

- $\varphi: R/pR \to R/pR$ surj.
- $\exists\, \varpi \in R$ s.t. $\varpi^p = pu$, u unit
- $\ker\theta$ is principal.

in particular, $I = (\xi)$ is principal, (A,I) is bounded.

Prismatic site

(relative)

Fix bounded prism (A, I)

Let X smooth p-adic formal scheme over A/I.

$(X/A)_{\Delta}$: obj.

$$\begin{tikzcd} \mathrm{Spf}(B/IB) \arrow[r] \arrow[d, "f"'] & \mathrm{Spf}(B) \arrow[dd, "g"] \\ X \arrow[d] & \\ \mathrm{Spf}(A/I) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

- (B, IB) bounded prism
- $f: \mathrm{Spf}(B/IB) \to X$
- $(A, I) \to (B, IB)$ map of prisms

Topology

Covers $(B, IB) \longrightarrow (C, IC)$ which is faithfully flat map of prisms

Prop. The functors on $(X/A)_{\Delta}$ satisfying

$\mathcal{O}_{\Delta} : (B, IB) \longmapsto B$ prismatic structure sheaf

$\overline{\mathcal{O}}_{\Delta} : (B, IB) \longmapsto B/IB$ reduced prismatic structure sheaf

Claim. $\mathcal{O}_{\Delta}$, $\overline{\mathcal{O}}_{\Delta}$ are sheaves on $(X/A)_{\Delta}$.

Check sheaf condition:

$(B, IB) \longrightarrow (C, IC)$ faithfully flat.

(derived) Čech nerve

faithfully flat $\Rightarrow$ this is just (usual) Čech nerve. (p, I)-complete ✓

Lecture 9

flat A-complex : I-completely flat complex.

$$K \overset{\mathbb{L}}{\otimes} A/I \xrightarrow{\sim} \hat{K} \overset{\mathbb{L}}{\otimes} A/I$$

Recall: X smooth p-adic formal scheme

$\mathcal{O}_{\Delta}$ = prismatic structure sheaf $\quad (B, IB) \longmapsto B$

$\overline{\mathcal{O}}_{\Delta}$ = reduced prismatic structure sheaf. $\quad (B, IB) \longmapsto B/IB$

Def. $\Delta_{X/A} := R\Gamma\left((X/A)_{\Delta}, \mathcal{O}_{\Delta}\right)$ prismatic complex

$\Delta_{X/A} \overset{\mathbb{L}}{\otimes}_A A/I = \overline{\Delta}_{X/A} := R\Gamma\left((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}\right)$ Hodge-Tate complex

$\exists$ semilinear map $\varphi : \Delta_{X/A} \longrightarrow \Delta_{X/A}$ induced by Frob on $\mathcal{O}_{\Delta}$

$\Delta_{X/A}$ this is derived (p, I)-complete comm. alg. obj. in $D(A)$

In general, take left Kan ext'n from smooth case:

$L\Delta_{X/A}$

$L\overline{\Delta}_{X/A}$

For simplicity, $X = \mathrm{Spf}\ R$, R A/I-alg.

<u>idea</u>. functor is defined on poly. A/I-alg.

take a resolution of R in (infinite rank) poly. A/I alg:

$$\cdots \to A/I[A/I[R]^\wedge]^\wedge \to \underbrace{A/I[R]^\wedge}_{\text{p-complete, give p-adic top.}} \to R$$

$\rightsquigarrow$ take derived colimit.

Simplify $X = \mathrm{Spf}\ R$, R A/I-alg.

Let $(E^\bullet, d)$ be a diff. graded algebra,

$$(\cdots \to E^{i-1} \xrightarrow{d} E^i \xrightarrow{d} E^{i+1} \to \cdots)$$

satisfying (1) $a \cdot b = (-1)^{\deg a \cdot \deg b}\, b \cdot a$

(2) $E^i = 0$ for $i < 0$

e.g. de Rham complex

$$\left(\hat{\Omega}^\bullet_{R/(A/I)}, d\right)$$

Smoothness: taking p-completion of Ω^i

= derived p-completion of $\Omega^\bullet$

- For A-mod M, $n \in \mathbb{Z}$, define

$$M\{n\} = M \otimes_A (I/I^2)^{\otimes n}$$

Breuil-Kisin twist

Bockstein differential:

$$0 \to I^{n+1}/I^{n+2} \to I^n/I^{n+2} \to I^n/I^{n+1} \to 0$$

given $K \in D(A)$, get a map

$$\beta_I^n : H^n\left(K \otimes_A^{\mathbb{L}} I^n/I^{n+1}\right) \longrightarrow H^{n+1}\left(K \otimes_A^{\mathbb{L}} I^{n+1}/I^{n+2}\right)$$

$K = \overline{\Delta}_{R/A}$, get $H^n(\overline{\Delta}_{R/A})\{n\} \xrightarrow{\beta_I} H^{n+1}(\overline{\Delta}_{R/A})\{n+1\}$

Summarize: $\left(H^\bullet(\overline{\Delta}_{R/A})\{\bullet\}, \beta_I^\bullet\right)$ is a dga

How to construct $(\hat{\Omega}^\bullet_{R/(A/I)}, d^\bullet) \longrightarrow \left(H^\bullet(\overline{\Delta}_{R/A})\{\bullet\}, \beta_I^\bullet\right)$

idea: for $\bullet = 0$, just get map $R \longrightarrow H^0(\overline{\Delta}_{R/A})$

$\bullet = 1$, use universal property of Ω^1.

$$R \longrightarrow H^0(\overline{\Delta}_{R/A}) \xrightarrow{\beta_I} H^1(\overline{\Delta}_{R/A})\{1\}$$ is a A-linear derivation.

Upshot: this is a quasi-isomorphism.

Lecture 10 Def. M A/I-module, $M\{n\} := M \otimes_{A/I} I^n/I^{n+1}$

$$0 \longrightarrow I^{n+1}/I^{n+2} \longrightarrow I^n/I^{n+2} \longrightarrow I^n/I^{n+1} \longrightarrow 0$$

Apply $\Delta_{R/A} \otimes_A^{\mathbb{L}} -$

$$\Delta_{R/A} \otimes_A^{\mathbb{L}} I^{n+1}/I^{n+2} \simeq \left(\Delta_{R/A} \otimes_A^{\mathbb{L}} A/I\right) \otimes_{A/I}^{\mathbb{L}} \left(I^{n+1}/I^{n+2}\right)$$

$$\simeq \overline{\Delta}_{R/A} \otimes_{A/I} I^{n+1}/I^{n+2}$$

$\exists$ map $\eta_R : (\hat{\Omega}_{R/A}, d) \longrightarrow (H^\bullet(\bar{\Delta}_{R/A})\{\bullet\}, \beta_I)$

Thm. This is an isomorphism of dga.

First step to prove Thm: look at case where $I = (p)$.

and A p-complete, p-torsion free.

In this case, have crystalline comparison

Thm. $(\phi_A^* \Delta_{R/A})^\wedge \simeq R\Gamma_{crys}(R/A)$ in $D(A)$.

$\uparrow$ p-completed

Compatible w/ Frobenius.

Simplify $A = \mathbb{Z}_p$, $\phi_A = id$

idea: compare Čech-Alexander complex of both sides.

$P \to R$ surjection, P ind-smooth A-alg.

Let $P^\bullet$ = Čech nerve $A \to P$

$$\left(\underset{P^0}{\underset{\parallel}{P}} \rightrightarrows \underset{P^1}{\underset{\parallel}{P \otimes_A P}} \substack{\rightarrow\\\rightarrow\\\rightarrow} \underset{P^2}{\underset{\parallel}{P \otimes_A P \otimes_A P}} \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

Crystalline. $P^n \xrightarrow{\Sigma} P \to R$, let J^n = kernel of this map. (adjoin $\gamma_n(x) = \frac{x^n}{n!}$ for all $x \in J$, take p-adic completion)

$C^\bullet_{crys}(R/A) := (D_{J^0}(P^0) \rightrightarrows D_{J^1}(P^1) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots)$. $D_{J^i}(P^i)$ = divided power envelope

$R\Gamma_{crys} := \mathrm{Tot}\,(\mathrm{Crys}\,(R/A))$

<u>Prismatic</u>. replace w/ prismatic envelope $P^i\{\frac{J^i}{p}\}^\wedge = C$

prismatic envelope: Let $(A, I) \xrightarrow{f} (B, J)$ map of δ-rings. $f(I) \subset J$.

(A, I) bounded prism, B (p, I)-completely flat δ A-alg.

then $\exists$ universal (C, IC) s.t. $(B, J) \to (C, IC)$ ← bounded prism over (A, I)

$\left(\text{Notation: } C = B\{\frac{J}{I}\}^\wedge\right)$

<u>eg.</u> $A = \mathbb{Z}_p$, $I = (p)$, $B = \mathbb{Z}_p[X]^\wedge$, $J = (p, X)$.

adjoin $\frac{x}{p}$ to B. in the cat. of δ-rings.

how to adjoin y: $B\{y\} = B[y_0, y_1, y_2, \cdots]^\wedge$ this gives δ-ring structure

δ sends y_i to y_{i+1}

$B\{\frac{x}{p}\}$ defined as pushout in δ-rings

$$\begin{tikzcd} B\{t\} \arrow[r, "t \mapsto x"] \arrow[d, "t \mapsto pt"'] & B \arrow[d] \\ B\{t\} \arrow[r] & B\{\frac{x}{p}\} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Lecture 11

Let $\mathbb{Z}_p\{x\} = \mathbb{Z}_p[x_0 = x, x_1, x_2, \cdots]$ w $\delta(x_i) = x_{i+1}$, $B = \mathbb{Z}_p\{x\}^\wedge$, $J = (p, x)$

Claim $B\{\frac{\phi(x)}{p}\}^\wedge$ = p-completion of smallest δ-ring in $B[\frac{1}{p}]$ which contains $\frac{\phi(x)}{p}$

$\phi(x) = x^p + px_1$

$B\{\frac{\phi(x)}{p}\}^\wedge = C$

C = pd-envelope $D_{(x)}(\mathbb{Z}_p\{x\}^\wedge)$

(B, J) prismatic envelope $B\{\frac{x}{p}\}^\wedge$

If base prism is $(\mathbb{Z}_p, (p))$

$X = \mathrm{Spf}\, R$,

Crystalline comparison $\phi_A^*(\Delta_{R/A})^\wedge \simeq R\Gamma_{crys}(R/A)$

HT-comparison

$\eta_R : \Omega^\bullet_{R/(A/p)} \longrightarrow H^\bullet(\bar{\Delta}_{R/A})$

apply $\phi^*_{A/p}$ on both sides

LHS $\Omega^\bullet_{R^{(1)}/(A/p)}$

$\uparrow$

pull back along Frobenius of A/p

RHS $\phi_A^* H^\bullet_{crys}(R/A)$

Def: A ring $R/\mathbb{Z}_p$ is quasi-syntomic, if

(1) R is derived p-complete, bounded p^∞-torsion

(2) cotangent complex $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$.

i.e. if $K = \mathbb{L}_{R/\mathbb{Z}_p} \overset{L}{\underset{R}{\otimes}} R/p$, then $K \overset{L}{\underset{K/p}{\otimes}} M$ has cohomology only in deg $[-1, 0]$

for all R/p-mod M.

Similarly, $R \to R'$ of derived p-complete rings w/ bounded p^∞-torsion.

then f is quasi-syntomic if $R \to R'$ is p-completely flat, and

$\mathbb{L}_{R'/R}$ has p-complete Tor-amplitudes in $[-1, 0]$.

$R \to R/I$ regular embedding

$\mathbb{L}_{(R/I)/R} = I/I^2[1]$.

Let QSyn = cat. of quasi-syntomic rings.

Eg. p-completion of any noetherian locally complete intersection ring is in QSyn.

$\mathbb{L}_{\mathbb{Z}_p[x_1, \dots x_n]/\mathbb{Z}_p}$

$\simeq \Lambda^1_{\mathbb{Z}_p[x_1, \dots x_n]/\mathbb{Z}_p}[0]$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\mathbb{L}_{C/B} \to \mathbb{L}_{C/A} \to g^* \mathbb{L}_{B/A}$$

Lecture 12 Hodge-Tate comparison

$X = \mathrm{Spf}\, R$

$A/I \to R$

$P^0 \twoheadrightarrow R$

$P^i = P^0 \hat{\otimes}_A P^0 \hat{\otimes}_A \cdots \hat{\otimes}_A P^0$

compare : Čech complex $\{P^\bullet\}$

crystalline : $D_{J^0}(P^0) \rightrightarrows D_{J^1}(P^1) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$

prismatic : $P^0\{\frac{J^0}{p}\} \rightrightarrows P^1\{\frac{J^1}{p}\} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$

Claim. $D_{J^i}(P^i) \simeq P^i\left\{\frac{\phi(J^i)}{p}\right\}$, ϕ: rel. Frob. on $P^i / A/(p)$.

e.g. $P^0 = \mathbb{Z}_p\{x\}^\wedge$

$\mathbb{Z}_p[x = x_0, x_1, x_2, \cdots]^\wedge \qquad \delta(x_i) = x_{i+1}$.

$J^0 = (p, x_1, x_2, \cdots)$

$D_{J^0}(P^0) \simeq P^0\left\{\frac{\phi(J^0)}{p}\right\}$

LHS $\quad P^0[\frac{1}{p}]$ $\cup$ completion of ring gen. by $\left(P^0, \frac{x_i^n}{n!}\right) = D$

RHS. smallest δ-ring $= C$ contains P^0, $\frac{\phi(p)}{p}$, $\underbrace{\frac{\phi(x_i)}{p}}_{\text{adjoin } \frac{x_i^p}{p}}$.

$$\delta\left(\frac{x_i^p}{p}\right) = \frac{(x_i^p + p\,\delta(x_i))^p}{p^2} - \frac{x_i^{p^2}}{p^{p+1}}.$$

$D \subset C$

to show $C \subset D$: exhibit δ-str. on D.

$D \subset P^0[\frac{1}{p}] \circlearrowleft \phi$

ϕ preserves D.

$\boxed{\begin{array}{l} (A, I) \text{ bounded prism}, I \supset (p) \\ A \to B \text{ derived } IB\text{-complete} \\ \qquad IB\text{-completely étale} \\ \Rightarrow \exists \text{ unique } \delta\text{-str on } B. \end{array}}$

Claim. $P^0\{\frac{J^0}{p}\} \rightrightarrows P^1\{\frac{J^1}{p}\} \overrightarrow{\underrightarrow{\rightarrow}} \cdots$ are homotopic.

and $P^0\left\{\frac{\phi(J^0)}{p}\right\} \rightrightarrows P^1\left\{\frac{\phi(J^1)}{p}\right\} \overrightarrow{\underrightarrow{\rightarrow}} \cdots$

Case where $A = \mathbb{Z}_p$. ϕ_{P^i} : $P^\bullet \to P^\bullet$ this is a quasiisom. in $D(A)$.

b/c $A \to P^\bullet$ is quasiisom.

$$\left(\phi_A^* \Delta_{R/A}\right)^\wedge \simeq R\Gamma_{crys}(R/A) \quad \text{in } D(A).$$

$$\eta_R : \left(\hat{\Omega}^\bullet_{R/(A/p)}, d\right) \longrightarrow \left(H^\bullet(\bar{\Delta}_{R/A})\{\bullet\}, \beta_I\right)$$

ϕ_A^* on both sides

crystalline comparison + de Rham - crystalline

$$\left(\hat{\Omega}^\bullet_{R^{(1)}/(A/p)}, d\right) \xrightarrow{\phi_A^* \eta_R} \left(H^\bullet(\Omega^\bullet_{R/(A/p)})\{\bullet\}, \beta_I\right)$$

$$R^{(1)} = R \underset{A/p, \phi_A}{\otimes} A/p$$

compare w/ Cartier isomorphism in deg 1.

Lecture 13

R a $\mathbb{Z}_p$-algebra is <u>quasi-syntomic</u> if

1) R is derived p-complete, w/ bounded p^∞-torsion

2) cotangent complex $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1,0]$

$$M \overset{\mathbb{L}}{\underset{R/p}{\otimes}} \left(\mathbb{L}_{R/\mathbb{Z}_p} \overset{\mathbb{L}}{\underset{R}{\otimes}} R/p\right) \text{ has cohom. only in deg } [-1,0]$$

M any R/p-mod

<u>QSyn</u>
cat. of quasi-syntomic rings

Ex. • p-completion of any lci ring flat over $\mathbb{Z}_p$

• Any integral perfectoid ring $R \in$ QSyn $\longrightarrow$ part (1) BMS 2 Topological Hochschild homology

$R[p^\infty] = R[p]$.

<u>part (2)</u>. $\mathbb{Z}_p \longrightarrow A_{inf}(R) \xrightarrow{\theta} R$

observe $\mathbb{Z}_p \longrightarrow A_{inf}(R)$ relatively perfect mod p, i.e. $A_{inf}(R)/p$ is a perfect $\mathbb{F}_p$-alg.

$$\mathbb{L}_{A_{inf}(R)/\mathbb{Z}_p} \overset{\mathbb{L}}{\underset{\mathbb{Z}_p}{\otimes}} \mathbb{Z}_p/p \simeq \mathbb{L}_{(A_{inf}(R)/p)/\mathbb{F}_p} \simeq 0$$

$\uparrow$

Frob acts as isomorphism

on $\mathbb{L}_{(A_{inf}(R)/p)/\mathbb{F}_p}$, but also as 0.

$\Rightarrow$ exact triangle

$A \to B \to C$

$$\Rightarrow \mathbb{L}_{R/\mathbb{Z}_p} \overset{\mathbb{L}}{\underset{R}{\otimes}} R/p \simeq \mathbb{L}_{R/A_{inf}(R)} \overset{\mathbb{L}}{\underset{R}{\otimes}} R/p$$

$$\mathbb{L}_{B/A} \overset{\mathbb{L}}{\underset{B}{\otimes}} C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

$$\simeq \left(\ker\theta/(\ker\theta)^2[1]\right) \overset{\mathbb{L}}{\underset{R}{\otimes}} R/p.$$

<u>Def</u>: $R \in \mathrm{QSyn}$ is quasi-regular semiperfectoid (qrsp) if $\exists$ integral perfectoid S and surjection $S \twoheadrightarrow R$.

<u>Prop</u>: $R \in \mathrm{QSyn}$, $\exists$ $\underbrace{\text{quasi-syntomic cover}}$ $R \to R'$ w/ R' qrsp.

quasi-syntomic morphism: map of derived p-complete rings w/
(cover)
bounded p^∞-torsion.

s.t. $S \to S'$ satisfies 1) p-completely (faithfully) flat

2) $\mathbb{L}_{S'/S}$ has p-complete

Tor amplitude $[-1, 0]$

<u>Pf</u>. Choose free derived p-complete alg. $F = \widehat{\mathbb{Z}_p[\{x_i\}_{i\in I}]}$ (set I may be infinite)

$$\downarrow$$

$$R$$

F^∞ = adjoin all p-power roots of p, x_i, take p-completion

integral perfectoid

$F \longrightarrow F^{\infty}$ is a quasisyntomic cover

Check this, look mod p, $\mathbb{F}_p[x_i^{1/p^\infty}, p^{\frac{1}{p^\infty}}] \leftarrow \mathbb{F}_p[x_i]$

this is free

$\mathbb{L}_{\mathbb{F}_p[x_i^{1/p^\infty}, p^{1/p^\infty}]/\mathbb{F}_p[x_i]}$ has Tor-amplitude in $[-1,0]$

because can write as colimit of lci rings

$$R' := R \hat{\otimes}_F F^{\infty} \simeq (R \overset{\mathbb{L}}{\underset{F}{\otimes}} F^{\infty})^{\wedge}$$

(p-complete flatness + bounded p^∞-torsion)

$(R)_{QSYN}$ big quasi-syntomic site

obj. = derived p-complete R-alg

covers = quasi-syntomic covers

Prop $\Rightarrow$ basis for quasi-syntomic top. given by qrsp alg.

to prove iso. of sheaves on $(R)_{QSYN}$, check its values on qrsp alg.

Prop Let (A, I) bounded prism.

$A/I \longrightarrow R$ quasisyntomic

then $\exists$ prism $(B, IB) \in (R/A)_{\Delta}$

s.t. $R \longrightarrow B/IB$ is p-completely faithfully flat.

In particular, $A/I \to R$ quasisyntomic cover, then

$$(A, I) \longrightarrow (B, IB) \quad \text{is faithfully flat map of prisms.}$$

Prop relates $(R)_{\Delta}$ w/ $(R)_{QSYN}$.

$$(A, I) \longmapsto A/I$$

Lecture 4

$R \in QSyn$,

$(R)_{QSYN}$ big quasi-syntomic site

obj: derived p-complete R-alg., w/ bounded p^{∞}-torsion

topology: covers: quasisyntomic covers

$(R)_{\Delta}$ absolute prismatic site

$(R/A)_{\Delta}$
(A, I) base prism

obj. bounded prisms (B, J)

w/ map $R \to B/J$

topology: gen. by p-completely faithfully flat maps of prisms.

Functor $u: (R)_{\Delta} \longrightarrow (R)_{QSYN}$

$$(A, I) \longmapsto A/I$$

Want: this is cocontinuous functor

need to show $\forall (A, I) \in (R)_{\Delta}$ and a quasi-syntomic cover of A/I,

$\exists$ covering in $(R)_{\Delta}$ of (A, I) which refines this quasi-syntomic cover.

Prop. Let (A, I) bounded prism, $A/I \to S$ quasi-syntomic.

then $\exists$ prism $(B, IB) \in (R/A)_{\Delta}$, s.t. $S \to B/IB$ is p-completely faithfully flat.

Pf. let $S' = A/I\langle X_j\rangle_{j\in J}$ s.t. S' surjects onto S.

$$S'' = A/I\langle X_j^{1/p^\infty}\rangle \hat{\otimes}_{A/I\langle X_j\rangle} S$$

$S \to S''$ is quasi-syntomic cover, since $A/I\langle X_j\rangle_{j\in J} \to A/I\langle X_j^{1/p^\infty}\rangle_{j\in J}$

is quasi-syntomic cover.

Composition $A/I \to S \to S''$ is also quasi-syntomic morphism.

$\left(\Omega^1_{S''/(A/I)}\right)^{\wedge p} = 0$. look mod p: want $\Omega^1_{(S''/p)/(A/(I,p))} = 0$

$\Rightarrow \left(\mathbb{L}_{S''/(A/I)}\right)^{\wedge p}$ has p-complete Tor-amplitude in deg -1.

$\mathbb{L}_{S''/(A/I)}^{\wedge p}[-1]$ is p-completely flat $\overset{\text{H-T comp.}}{\Longrightarrow}$ $\overline{\Delta}_{S''/A}$ is concentrated in deg 0

$\xrightarrow{\text{derived Nakayama}}$ $\boxed{\Delta_{S''/A}}$ is concentrated in deg 0. $\;\boxed{\Delta_{S''/A}} = B$

If we left Kan extend the Hodge-Tate comparison, get

Thm. For any p-complete A/I-alg. R, have increasing filtration on $\overline{\Delta}_{R/A}$,

$$gr_i(\overline{\Delta}_{R/A}) \simeq \left(\wedge^i \mathbb{L}_{R/(A/I)}\{-i\}[-i]\right)^\wedge$$

in smooth case, $gr_i(\overline{\Delta}_{R/A}) \simeq \hat{\Omega}^i_{R/(A/I)}\{-i\}[-i]$

Claim (B, IB) is the prism we want.

$S \to B/IB$ is p-completely faithfully flat

$S \to S'' \to B/IB$

$\uparrow$ quasisyntomic cover

$S'' \to B/IB$ is p-completely flat

(follows from HT comparison)

to see faithful flatness:

$S''/pS'' \to B/(I,p)B$ follows from

unwinding $\Delta_{S''/A}$.

(B, IB) is initial in $(S''/A)_\Delta$.

Lecture 10 S qrsp ring

Prop. S_Δ has an initial object Δ_S^{init}

obj. (A, I) bounded, $S \to A/I$.

Pf. Case I: $S = R$ perfectoid ring

$\exists$ perfect prism $(\underset{A}{\underbrace{A_{inf}(R)}}, \underset{I}{\underbrace{\ker \theta}})$, $A/I \simeq R$.

this is the initial object.

Idea why (A, I) initial, (B, J) prism, $A/I \to B/J$,

then perfectness of $A/p \Rightarrow \exists!$ lifts the map to $A \to B$. (calculation of cotangent complexes)

B is p-complete.

Claim: $A \xrightarrow{f} B$ is a map of δ-rings.

If B is perfect, then $A \to B$ lifts Frobenius mod p

$\therefore$ map of δ-rings

If B not perfect, then f factors as

$$A \to B^{\mathrm{perf}} \to B$$

$B^{\mathrm{perf}} = \varinjlim_{\phi} B$

this map will be map of δ-rings.

Can give this str. of a δ-ring

Case II. S qrsp, $R \twoheadrightarrow S$ where R perfectoid

$$J = \ker\left(A_{\inf}(R) \to R \twoheadrightarrow S\right)$$

Consider $A_{\inf}(R)\{\frac{J}{d}\}^{\wedge(p,d)}$, $\ker\theta = (d)$

prismatic envelope

this is the initial object.

Prop. $R \twoheadrightarrow S$, R perfectoid.

$\Delta_{S/A_{\inf}(R)}$ is discrete, and is equal to Δ_S^{init}. $\Big]$ Δ_S.

(in particular, independent to R)

discrete: look at $\wedge^i \mathbb{L}_{S/(A/I)}[-i]$ these are p-completely flat S-modules.

(same argument as last time)

derived Hodge-Tate comparison: $\Delta_{S/A}$ is discrete.

Claim. $\Delta_{S/A} = A_{inf}(R)\{\frac{J}{d}\}^{\wedge(p,d)}$.

| R is R'-alg, R, R' perfectoid, $\Delta_{R/A_{inf}(R')} = A_{inf}(R)$ |

Nygaard filtration on Δ_S:

$$N^{\geq i}(\Delta_S) = \{x \in \Delta_S : \phi(x) \in d^i \Delta_S\} \quad (d) = \ker\theta$$

Thm. Have canonical isom.

$$N^{\geq i}(\Delta_S)/N^{\geq i+1}(\Delta_S) \simeq \underbrace{\mathrm{Fil}_i^{conj}(\overline{\Delta}_S)\{i\}}_{\text{derived HT comparison}} \quad \text{for } i \geq 0.$$

$$\begin{aligned}\overline{\Delta}_S &= \Delta_S / d\Delta_S \\ &= \overline{\Delta}_{S/A} \\ &= \Delta_{S/A} \otimes^{\mathbb{L}}_A A/I\end{aligned}$$

$\overline{\Delta}_{S/A}$ conjugate filtration

Lecture 11. R qrsp ring

$(\Delta_R, (d))$

Nygaard filtration $N^{\geq i}(\Delta_R) = \{x \in \Delta_R : \phi(x) \in d^i \Delta_R\}$

Thm. There is a canonical isom.

$$N^{\geq i}(\Delta_R)/N^{\geq i+1}(\Delta_R) \simeq \mathrm{Fil}_i^{conj}(\overline{\Delta}_R)\{i\} \quad \text{for } i \geq 0.$$

HT comparison for R p-completely smooth:

$$\hat{\Omega}^i_{R/(A/I)} \simeq H^i(\overline{\Delta}_{R/A})\{i\}.$$

on $\overline{\Delta}_{R/A}$: canonical filtration

in general $K^\bullet$: $(\mathrm{Fil}_i K^\bullet)_j = \begin{cases} 0, & j>i \\ \ker(K^i \to K^{i+1}), & j=i \\ K^j, & j<i \end{cases}$

$$\mathrm{gr}_i \mathrm{Fil}_\bullet K^\bullet \simeq H^i(K^\bullet)[-i]$$

$$\mathrm{gr}_i \mathrm{Fil}_\bullet \overline{\Delta}_{R/A} \simeq H^i(\overline{\Delta}_{R/A})[-i] \simeq \hat{\Omega}^i_{R/(A/I)}\{-i\}[-i]$$

in R p-completely smooth case: conjugation filtration : canonical filtration.

Other cases: left Kan ext.

Upgraded HT comparison: $\mathrm{gr}_i \mathrm{Fil}^\bullet \overline{\Delta}_{R/A} \simeq \wedge^i \mathbb{L}_{R/(A/I)}[-i]$

When $i=0$, $\Delta_R / \mathcal{N}^{\geq 1}(\Delta_R) \simeq R$

R quasi-syntomic ring

$(R)_{QSYN}$ = big quasi-syntomic site

$(R)_{\Delta}$ = absolute prismatic site

$(R)_{qsyn}$ = small quasi-syntomic site

obj.: $R' \to R$ quasisyntomic

$v_* :$ $Shv((R)_\Delta) \xrightarrow{u_*} Shv((R)_{QSYN}) \xrightarrow{\varepsilon_*} Shv((R)_{qsyn})$

$\varepsilon_* F(R') = F(R')$ for $R' \in (R)_{qsyn}$.

$u : R_\Delta \to (R)_{QSYN}$

$(A, I) \mapsto A/I$ $\quad u$ cocontinuous

$u^{-1} : Shv((R)_{QSYN}) \longrightarrow Shv((R)_\Delta)$ $\quad , u^{-1} \dashv u_*$

ε_* has a left adjoint: $\varepsilon^\natural : Shv((R)_{qsyn}) \to Shv((R)_{QSYN})$

sheafification of colimit over $R' \in (R)_{qsyn}$ which map to $R'' \in (R)_{QSYN}$.

$v^\natural = u^{-1} \circ \varepsilon^\natural$.

<u>Define</u> $\mathcal{N}^{\geq 1} \mathcal{O}_\Delta : (B, I) \mapsto \varphi^{-1}(I)$.

$\mathcal{O}_\Delta$ = str. sheaf on $(R)_\Delta$.

$(B, I) \longmapsto B$

$I_\Delta : (B, I) \mapsto I$

sheaves on $(R)_\Delta$

<u>Def</u> $\mathcal{O}^{pris} = v_* \mathcal{O}_\Delta$

$\mathcal{N}^{\geq 1} \mathcal{O}^{pris} = v_* \mathcal{N}^{\geq 1} \mathcal{O}_\Delta$

$I^{pris} = v_* I_\Delta$

<u>Thm</u> Quotient sheaf $\mathcal{O}^{pris} / \mathcal{N}^{\geq 1} \mathcal{O}^{pris} \simeq \mathcal{O}_{\widehat{\ }}$ str. sheaf on $(R)_{qsyn}$.

Pf. basis for quasisyntomic topology gen. by qrsp. rings.

check that for qrsp. $\Delta_R / N^{\geq 1} \Delta_R \simeq R$.

$R \in QSYN$

Def. A prismatic crystal $/R$ is $\mathcal{O}_\Delta$-mod M on $(R)_\Delta$ s.t. for all maps

$(B, J) \to (C, JC) \in (R)_\Delta$,

$$M(B,J) \otimes_B C \xrightarrow{\sim} M(C, JC).$$

loc. free prismatic crystal are loc. free $\mathcal{O}_\Delta$-mod.

Prop. v_* and $v^*(-) := \mathcal{O}_\Delta \otimes_{v^{\natural}\mathcal{O}^{pris}} v^{\natural}(-)$

induces an equiv. between cat. of finite loc. free $\mathcal{O}_\Delta$-mod

and cat. of fin. loc. free $\mathcal{O}^{pris}$-mod.

Pf. $v_*(\mathcal{O}_\Delta) = \mathcal{O}^{pris}$, so $M \to v_* v^* M$ is an isom. (check locally on $(R)_{qsyn}$)

Lecture 12

Prop. R quasisyntomic, v_*, $v^*(-) = \mathcal{O}_\Delta \otimes_{v^{\natural}\mathcal{O}^{prism}} v^{\natural}(-)$

induces equiv. between cat. of fin. loc. free $\mathcal{O}_\Delta$-mod and fin. loc. free $\mathcal{O}^{prism}$-mod.

Pf. If M fin. loc. free $\mathcal{O}^{prism}$-mod, then canonical map $M \to v_* v^* M$ is an isom.

Check this locally on $(R)_{qsyn}$.

if N fin. loc. free $\mathcal{O}_\Delta$-mod, $v^* v_* N \xrightarrow{\sim} N$

1) reduce to R qrsp: have $(R)_{\Delta}/h_{R'} \simeq (R')_{\Delta}$, $(R)_{qsyn}/R' \simeq (R')_{qsyn}$

given $R \to R'$

$$h_{R'}(B,J) = \mathrm{Hom}_R(R', B/J)$$

look at the induced slice topoi.

2) (p,I)-completely faithfully flat descent

$\Rightarrow$ $\{$fin. loc. free $\mathcal{O}_{\Delta}$-mod on $R_{\Delta}\} \simeq \{$fin. loc. free Δ_R-mod$\}$

3) reduce to N a fin. free Δ_R-mod

roughly:

$\{$fin. locally free Δ_R-mod$\}$ $\{$fin. loc. free R-mod$\}$

$$M \longmapsto M \overset{L}{\underset{\Delta_R}{\otimes}} \Delta_R/(d)$$

loc on R

<u>Def</u> A prismatic Dieudonné crystal over R quasisyntomic:

fin. loc. free $\mathcal{O}^{pris}$-mod M

w/ φ-semilinear map φ_M

$1 \otimes \varphi_M : \varphi^* M \to M$ coker is killed by I^{pris}.

<u>admissible</u> if $M \xrightarrow{\varphi_M} M \to M/I^{pris}M$, $M/\ker$ is finite loc free $\mathcal{O}$-mod F_M

s.t. $(\mathcal{O}^{pris}/I^{pris}) \underset{\mathcal{O}}{\otimes} F_M \longrightarrow M/I^{pris}M$ is injective.

$\boxed{?}$

Prop. Let $DM^{(adm)}(R)$ (adm) prismatic Dieudonné crystals $/R$

Prop $DM^{(adm)}$ forms a stack over $Qsyn$, endowed w/ quasi-syntomic topology.

Pf. follows from def.

$\rightarrow$ allows us to reduce to R = qrsp.

Def A prismatic Dieudonné-module $/R$ is a fin. loc. free Δ_R-mod M

φ-semilinear φ_M, linearization has coker killed by I.

(admissible) $M \xrightarrow{\varphi_M} M \rightarrow M/IM$ $\quad$ $M/\ker$ is finite loc. free R-mod.

$(\Delta_R/I\Delta_R) \otimes_R F_M \rightarrow M/IM$ $\quad$ injective.

Prop Equiv. R qrsp

$$DM^{adm}(R) \xrightarrow{\sim} \{\text{adm. prismatic Dieudonné modules } /R\}$$

For G p-div. gp $/R$ quasi-syntomic

$$M_\Delta(R) = \mathcal{E}xt^1_{(R)_{qsyn}}(G, \mathcal{O}^{pris})$$

G is sheaf on $(R)_{QSYN}$.

Lemma Have canonical isom. $M_\Delta(R) \simeq v_*\left(\mathcal{E}xt^1_{(R)_\Delta}(u^{-1}(G), \mathcal{O}_\Delta)\right)$ $\quad$ $u^{-1}: Shv((R)_{QSYN}) \rightarrow Shv((R)_\Delta)$

Next week. R perfectoid, Scholze–Weinstein ...

Lecture 13.

Admissible crystal:

image $\overbrace{M \xrightarrow{\varphi_M} M \longrightarrow M/I^{\mathrm{pris}}M}^{F_M} \circlearrowleft \mathcal{O} = \mathcal{O}^{\mathrm{pris}}/N^{\geq 1}\mathcal{O}^{\mathrm{pris}}$

is finite loc. free $\mathcal{O}$-module $+$ $\mathcal{O}^{\mathrm{pris}}/I^{\mathrm{pris}} \otimes_{\mathcal{O}} F_M \longrightarrow M/I^{\mathrm{pris}}M$ injective

$$N^{\geq 1}\mathcal{O}^{\mathrm{pris}} = \ker\left(\mathcal{O}^{\mathrm{pris}} \xrightarrow{\varphi} \mathcal{O}^{\mathrm{pris}} \to \mathcal{O}^{\mathrm{pris}}/I^{\mathrm{pris}}\right) \qquad \begin{matrix} \text{is} \\ \varphi^* M/\varphi^*\mathrm{Fil}M \end{matrix}$$

(equivalently) $\mathrm{Fil}M = \varphi_M^{-1}(I^{\mathrm{pris}}M)$

$M/\mathrm{Fil}M$ is finite loc. free $\mathcal{O}$-mod.

(part of claim $(N^{\geq 1}\mathcal{O}^{\mathrm{pris}}).M \subset \mathrm{Fil}M$)

$+$ injective map $\Leftrightarrow \varphi_M^* \mathrm{Fil}M \xrightarrow{\sim} I^{\mathrm{pris}}M$ $\qquad$ $\left[\varphi^* M = M \otimes_{\mathcal{O}^{\mathrm{pris}},\varphi} \mathcal{O}^{\mathrm{pris}}\right]$

For G p-div. gp $/R$,

$$M_{\Delta}(G) = \mathcal{E}xt^1_{(R)_{qsyn}}(G, \mathcal{O}^{\mathrm{pris}})$$

Lemma. $M_{\Delta}(G) \simeq v_*\left(\mathcal{E}xt^1_{(R)_\Delta}(u^{-1}(G), \mathcal{O}_\Delta)\right)$

Sketch. adjunction $R\mathcal{H}om_{(R)_{QSYN}}(G, Ru_*\mathcal{O}_\Delta) \simeq Ru_*\left(R\mathcal{H}om_{(R)_\Delta}(u^{-1}(G), \mathcal{O}_\Delta)\right)$

w.r.t.

$u: \mathrm{Shv}(R)_\Delta \longrightarrow \mathrm{Shv}(R)_{QSYN}$.

$$u_*\mathcal{E}xt^1_{(R)_\Delta}(u^{-1}(G), \mathcal{O}_\Delta) \simeq \mathcal{E}xt^1_{(R)_{QSYN}}(G, u_*\mathcal{O}_\Delta)$$

This follows from the vanishing of

(1) $\mathcal{H}om_{(R)_{QSYN}}\left(G, R^1 u_*(\mathcal{O}_{\Delta})\right)$

(2) $\mathcal{H}om_{(R)_{\Delta}}\left(u^{-1}(G), \mathcal{O}_{\Delta}\right)$

how to show (1), (2): $\mathcal{O}_{\Delta}$ is p-complete and derived p-completion of G on $(R)_{QSYN}$

is $T_pG[1]$, $T_pG = \varprojlim G[p^n]$

[this is because multiplication by p^n on G is surjective

remains to show $\mathcal{E}_* \mathcal{E}xt^1_{(R)_{QSYN}}(G, u_*\mathcal{O}_{\Delta}) \simeq M_{\Delta}(G)$

<u>Step 1</u> Deligne resolution

$$C(G)_\bullet = (\cdots \to \mathbb{Z}[G^3] \to \mathbb{Z}[G^2] \to \mathbb{Z}[G]) \simeq G$$

suffices to show $\forall k \geq 0, \forall j \geq 1$

$$\mathcal{E}_* \mathcal{E}xt^k_{(R)_{QSYN}}(\mathbb{Z}[G^j], u_*\mathcal{O}_{\Delta}) \simeq \mathcal{E}xt^k_{(R)_{QSyn}}(\mathbb{Z}[G^j], \mathcal{O}^{pris}).$$

<u>Lecture 14</u> M $\mathcal{O}^{pris}$-module, $\mathrm{Fil}\, M := \varphi_M^{-1}(I^{pris}\cdot M)$

admissibility: $M/\mathrm{Fil}\, M$ is finite proj. $\mathcal{O}^{pris}/N^{\geq 1}\mathcal{O}^{pris}$-module.

$$\mathcal{O}^{pris}/N^{\geq 1}\mathcal{O}^{pris} \longrightarrow M/\mathrm{Fil}\, M \quad , \quad \mathrm{Fil}\, M \supset N^{\geq 1}\mathcal{O}^{pris}\cdot M$$

$\varphi^* \mathrm{Fil}\, M \simeq I^{\mathrm{pris}} \cdot M \iff \mathcal{O}^{\mathrm{pris}}/I^{\mathrm{pris}} \underset{\overset{\nwarrow}{\varphi}\ \mathcal{O}^{\mathrm{pris}}/N^{\geq 1}\mathcal{O}^{\mathrm{pris}}}{\otimes} M/\mathrm{Fil}\, M \xrightarrow{1\otimes\varphi_M} M/I^{\mathrm{pris}} M$ injective

<u>Claim</u> $\varphi^* M/\varphi^*\mathrm{Fil}\, M \simeq \mathcal{O}^{\mathrm{pris}}/I^{\mathrm{pris}} \underset{\overset{\nwarrow}{\varphi}\ \mathcal{O}^{\mathrm{pris}}/N^{\geq 1}\mathcal{O}^{\mathrm{pris}}}{\otimes} M/\mathrm{Fil}\, M$

$$\begin{array}{c} \varphi^* M/\varphi^*\mathrm{Fil}\, M \\ \| \\ \mathcal{O}^{\mathrm{pris}} \underset{\varphi, \mathcal{O}^{\mathrm{pris}}}{\otimes} M \Big/ \mathcal{O}^{\mathrm{pris}} \underset{\varphi, \mathcal{O}^{\mathrm{pris}}}{\otimes} \mathrm{Fil}\, M \end{array}$$

$$\begin{tikzcd}
0 \arrow[r] & \varphi^*\mathrm{Fil}\, M \arrow[r, "1\otimes\varphi_M"] \arrow[d] & I^{\mathrm{pris}} M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \varphi^* M \arrow[r] & M \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

condition on Dieudonné module: $K \cdot I^{\mathrm{pris}} = 0$

$\rightarrow$ apply Snake lemma.

$0 \to Q \to \varphi^* M/\varphi^*\mathrm{Fil}\, M \to M/I^{\mathrm{pris}} M$

Prop. $M_{\Delta}(G) \simeq v_* \mathcal{E}xt^1_{(R)_\Delta}(u^{-1}(G), \mathcal{O}_\Delta)$

$u_* \mathcal{E}xt^1_{(R)_\Delta}(u^{-1}(G), \mathcal{O}_\Delta) \overset{i_1}{\simeq} \mathcal{E}xt^1_{(R)_{\mathrm{QSYN}}}(G, u_* \mathcal{O}_\Delta)$

Want to show $\varepsilon_* \mathcal{E}xt^1_{(R)_{\mathrm{QSYN}}}(G, u_* \mathcal{O}_\Delta)$

$\overset{i_2}{\simeq} \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \underbrace{\varepsilon_* u_* \mathcal{O}_\Delta}_{\| \\ \mathcal{O}^{\mathrm{pris}}})$

$\boxed{\begin{array}{l} R \text{ qrsp}, \quad \Delta_R \\ 1\otimes\varphi_M \text{ is isomorphism} \\ \text{after inverting } I = (\xi) \end{array}}$

Step 1. apply Deligne resolution: have spectral sequences

$$\mathcal{E}xt^k_{(R)_{QSYN}}(\mathbb{Z}[G^{n_j}], u_*\mathcal{O}_\Delta) \Rightarrow \mathcal{E}xt^{k+j}_{(R)_{QSYN}}(G, u_*\mathcal{O}_\Delta)$$

$$\mathcal{E}xt^k_{(R)_{qsyn}}(\mathbb{Z}[G^{n_j}], \mathcal{O}^{pris}) \Rightarrow \mathcal{E}xt^{k+j}_{(R)_{qsyn}}(G, \mathcal{O}^{pris})$$

$$\varepsilon_* \mathcal{E}xt^k_{(R)_{QSYN}}(\mathbb{Z}[G^{n_j}], u_*\mathcal{O}_\Delta) \qquad H^*_{QSYN}(X \times_{\mathrm{Spf} R} G^{n_j}, u_*\mathcal{O}_\Delta)$$

sheafification of presheaf sending $X \in (R)_{qsyn}$

$$\big\downarrow (*)$$

$$\mathcal{E}xt^k_{(R)_{qsyn}}(\mathbb{Z}[G^{n_j}], \mathcal{O}^{pris}) \qquad H^k_{qsyn}(X \times_{\mathrm{Spf} R} G^{n_j}, \varepsilon_* u_*\mathcal{O}_\Delta)$$

ε_* restriction from QSYN to qsyn.

Claim $(*)$ is an isomorphism

both can be calculated using cover by qrsp rings (no higher cohomology)

If R is qrsp

$$M_\Delta(G) = \mathcal{M}_\Delta(G)(R)$$

Prop. Let R be perfectoid, then $DM^{adm}(R)$: cat of admissible Δ_R Dieudonné mod

$\simeq DM(R)$ cat. w/o adm. condition $\simeq BT(R)$. p-div. gps $/R$

<u>Recall</u>. Breuil-Kisin-Fargues modules $/R$

finite loc. free $A_{inf}(R)$-mod M.

w/ φ-linear $\varphi_M : M[\frac{1}{\xi}] \to M[\frac{1}{\varphi(\xi)}]$

$(\xi) = \ker\theta$. s.t. $M \subset \varphi_M(M) \subset \frac{1}{\varphi(\xi)} M$

<u>Thm</u> (Scholze-Weinstein) Have equivalence

(R perfectoid)

$$\{\text{BKF-modules}/R\} \simeq \{p\text{-div. gp}/R\}$$

$$M^{SW}(G) \longleftarrow G$$

<u>Lecture 15</u>

<u>Thm</u>. (Scholze-Weinstein) R perfectoid, have equivalence

$$\{\text{BKF-modules}/R\} \xrightarrow{\sim} \{p\text{-div. gp}/R\}$$

$$M^{SW}(G) \longleftarrow G$$

<u>Idea</u>. 1) $pR = 0$ ($\Rightarrow$ R is perfect)

2) $R = \mathcal{O}_C$. C = complete alg. closed ext'n of $\mathbb{Q}_p$

perfect valuation rings

In 1), $A_{inf}(R) = W(R)$, this is just the statement that have equivalence w/ Dieudonné module for perfect ring. [Gabber, reduce to result of Berthelot]

2) $R = \mathcal{O}_C$, Scholze - Weinstein

$R = V$ valuation ring such that $V[\frac{1}{p}] = C$.

Use that for both BKF-mod, p-div gp

$$\begin{tikzcd} \{p\text{-div}/V\} \arrow[r] \arrow[d] & \{p\text{-div}/\bar{V}\} \arrow[d] \\ \{p\text{-div}/\mathcal{O}_C\} \arrow[r] & \{p\text{-div}/\mathcal{O}_C/m\} \end{tikzcd}$$

$V = \bar{V} \otimes_{\mathcal{O}_C/m} \mathcal{O}_C$

G p-div gp $/R = V$

$M = M^{SW}(G)$, $\quad M \subset M \otimes_{A_{inf}(R)} A_{crys}(R/p)$ (since $A_{inf}(R) \to A_{crys}(R/p)$ injective) is the same as crystalline Dieudonné functor applied to $G \otimes_R R/p$ evaluated at $A_{crys}(R/p)$.

idea: apply v-descent to both sides to reduce to product of points

More explicitly, R general perfectoid, $G = p\text{-div}/R$.

$M^{SW}(G)$ is the largest submodule of $\mathbb{D}^{crys}(G \otimes_R R/p)(A_{crys}(R/p))$

such that for all maps $R \to V$ ← valuation ring, $V[\frac{1}{p}] = C$.

have image in $M^{SW}(G \otimes_R V) \subset \mathbb{D}^{crys}(G \otimes_R V/p)(A_{crys}(V/p))$

Compare prismatic Dieudonné module for
1) R perfect
2) R is $\mathcal{O}_C$

1) R is perfect, want to show that $M_{\Delta}(G)$ recovers $M_{crys}(G)$.

look at $M_{crys}(G)$ (sheaf on $(R)_{CRIS,pr}$) on pr-topology (Lau) pr morphism : Zariski locally of the form

(p^{th} root) $\qquad \operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$ $\qquad \Rightarrow$ faithfully flat

$\Rightarrow$ morphisms can be lifted over PD-thickening.

A p nilpotent

$$A' = \varinjlim B_i, \qquad B_0 = A, \quad B_{i+1} = B_i\left[(a_j)^{1/p}_{j\in J}\right], \; a_j \in B_i$$

$$= B_i[X_j]_{j\in J} / (X_j^p - a_j)$$

pr cover: collection of jointly surjective pr-morphisms.

functors: $\operatorname{Shv}(R)_{pr} \xrightarrow{(u^{crys})^{-1}} \operatorname{Shv}(R/\mathbb{Z}p)_{CRIS,pr} \xrightarrow{u_*^{crys}} \operatorname{Shv}(R)_{pr}$

$$u_*^{crys}(F)(U) = \Gamma((U/\mathbb{Z}p)_{CRIS,pr}, F)$$

$(u^{crys})^{-1}$ = left adjoint

this defines morphism of topoi.

Lecture 16. Last time: pr topology

Functors $\operatorname{Shv}(R)_{pr} \xrightarrow{(u^{crys})^{-1}} \operatorname{Shv}(R/\mathbb{Z}p)_{CRIS,pr} \xrightarrow{u_*^{crys}} \operatorname{Shv}(R)_{pr}$

$$u_*^{crys}(F)(U) = \Gamma((U/\mathbb{Z}p)_{CRIS,pr}, F)$$

$$(u^{crys})^{-1}(F)(U,T,\delta) = F(U).$$

$$\begin{array}{ccc} \operatorname{Spec} R' & \xrightarrow{\text{pr-morphism}} & \operatorname{Spec} R \\ \text{PD thickening} \downarrow & & \downarrow \text{PD thickening} \\ \operatorname{Spec} B' & \dashrightarrow & \operatorname{Spec} B \\ & \text{pr morphism} & \end{array}$$

Claim: this induces morphism of topoi.

$\longrightarrow$ need to use fact: lift PD thickenings over pr-morphisms

Thm. R quasisyntomic, $pR=0$, G/R p-div gp

$$\mathcal{M}_{\Delta}(G) \simeq \nu_*^{crys}\left(\mathcal{E}xt^1_{(R/\mathbb{Z}_p)_{crys, pr}}\left((u^{crys})^{-1}(G), \mathcal{O}_{crys}\right)\right)$$ as sheaves on $(R)_{qsyn, pr}$.

$$\nu_*^{crys}: \mathrm{Shv}(R/\mathbb{Z}_p)_{CRIS, pr} \xrightarrow{u_*^{crys}} \mathrm{Shv}(R)_{pr} \xrightarrow{\text{restriction}} \mathrm{Shv}(R)_{qsyn, pr}$$

Key idea: let $\mathcal{O}^{crys} := \nu_*^{crys}(\mathcal{O}_{crys})$

$\mathcal{J}^{crys} := \nu_*^{crys}(\mathcal{I}_{crys})$

R' quasi-syntomic R-alg

Then $\exists$ canonical isomorphism $\mathcal{O}^{crys}(R') \xrightarrow{\sim} \mathcal{O}^{pris}(R')$

$\mathcal{J}^{crys}(R') \xrightarrow{\sim} N^{\geq 1}\mathcal{O}^{pris}(R')$

reduce to R' qrs perfect, $\Delta_{R'} = A_{crys}(R')$.

$\ker(A_{crys}(R') \to R') \simeq (\varphi^{-1}(\xi))$.

Last time: pr topology.

$$R\mathcal{H}om_{(R)_{pr}}(G, Ru_*^{crys}(\mathcal{O}_{crys})) \simeq Ru_*^{crys}\left(R\mathcal{H}om_{(R/\mathbb{Z}_p)_{CRIS, pr}}\left((u^{crys})^{-1}(G), \mathcal{O}_{crys}\right)\right)$$

this vanishes on qrs perfect inputs

Suffices to show

$\mathcal{H}om(G, R^1 u_*^{crys}(\mathcal{O}_{crys})) = 0$

$\mathcal{H}om((u^{crys})^{-1}(G), \mathcal{O}_{crys}) = 0$.

Thm $\Rightarrow$ if R is perfect. evaluate both sides on $\Delta_R = W(R)$,

$\mathcal{M}_{\Delta}(R)(\Delta_R) = M_{\Delta}(R)$, RHS: contravariant Dieudonné module $\mathbb{D}(G)$

$$\mathbb{D}(G) = \mathrm{Hom}_{W(R)}\left(M^{SW}(G),\ W(R)\right).$$

Remains to show when $R = \mathcal{O}_C$. $M_{\Delta}(R) = \mathrm{Hom}_{A_{\inf}(R)}\left(M^{SW}(G),\ A_{\inf}(R)\right)$

__Case 1.__ $G = X[p^\infty]$, $f: X \to \mathrm{Spf}\ \mathcal{O}_C$, X formal abelian scheme. $X/\mathcal{O}_C$.

__Claim.__ $M_{\Delta}(G) = R^1 f_{\Delta *}(\mathcal{O}_{\Delta})$

$M^{SW}(G)$ is dual to $R^1 f_{\Delta *}(\mathcal{O}_{\Delta})$

(14.8.3 Berkeley lectures)

__Thm__ G p-div gp$/\mathcal{O}_C$.

$\exists$ formal abelian scheme $X[p^\infty] \simeq G \times \check{G}$.

__Lecture 17__

__Setup.__ Show equivalence for $R =$ perfectoid, compare w/ Scholze–Weinstein functor.

Already shown: 1) $pR = 0$

$\boxed{2)\ R = \mathcal{O}_C}$ __Step 1.__ G p-div gp, $G = X[p^\infty]$, X formal abelian scheme $/R$

__Step 2.__ __Thm__ $\forall$ p-div gp $G/\mathcal{O}_C$, $\exists$ formal abelian scheme s.t. $X[p^\infty] \simeq G \times G^\vee$.

__Idea of pf.__ __Step 1.__ Find H/k, $k = \mathcal{O}_C/\mathfrak{m}$ alg. closed, s.t. $H \otimes_k \mathcal{O}_C/p \xrightarrow{\text{isogeny}} G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$.

i.e. up to isogeny $G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$ is isoclinic.

Step 2. Find abelian var. A/k s.t. $A_k[p^\infty] \simeq H \times H^\vee$.

$\Rightarrow \exists$ $A'_{\mathcal{O}_C/p}[p^\infty] \simeq G_{\mathcal{O}_C/p} \times G^\vee_{\mathcal{O}_C/p}$.
abelian var.

Step 3. By Serre-Tate, lifting $A'_{\mathcal{O}_C/p}$ is equiv. to lifting $\underbrace{G_{\mathcal{O}_C/p} \times G^\vee_{\mathcal{O}_C/p}}_{\text{clearly lifts}}$.

Look at idempotent $e: X[p^\infty] \longrightarrow X[p^\infty]$ w/ kernel G.

$$\Rightarrow \quad M_\Delta(G)^\vee = \ker(M_\Delta(e)^\vee)$$

$$M^{SW}(G) = \ker(M^{SW}(e))$$

R general (integral) perfectoid ring want to construct

$\theta: W(R^\flat) \longrightarrow R$
$\ker\theta = (\xi)$
Frob on R/p surjective
i.e. semiperfect

$$\alpha_{R,G}: M_\Delta(G) \longrightarrow (M^{SW}(G))^\vee \text{ isom.}$$

Input:
[$M_\Delta(G)$ is a finite loc. free Δ_R-mod.] $\longrightarrow$ will show later.

$$\begin{array}{c} M_\Delta(G \otimes_R R/p) \\ \text{is} \end{array}$$

$$\alpha_{R/p,\, G\otimes_R R/p}: M_\Delta(G) \otimes_{\Delta_R} A_{crys}(R/p)$$

$$\begin{array}{ccc} M_\Delta(G) & & \\ \vdots & & \downarrow \wr \\ M^{SW}(G)^\vee & \subset & (M^{SW}(G))^\vee \otimes_{\Delta_R} A_{crys}(R/p) \\ & & \text{is} \\ & & M^{SW}(G \otimes_R R/p)^\vee \end{array}$$

Check: $\alpha(M_\Delta(G)) \subset M^{SW}(G)^\vee$ (*)

By construction of $M^{SW}(G)^\vee$, suffices to check for V = perfectoid val. ring $V[\frac{1}{p}] = C$.

- if V char p, already shown this.
- if V mixed char, $M^{SW}(G_V) = M^{SW}(G_{\mathcal{O}_C}) \times_{M^{SW}(G_k)} M^{SW}(G_{\bar{V}})$

$k = \mathcal{O}_C/\mathfrak{m}$

Construction of isom $\alpha_{\bar{V}}, \alpha_{\mathcal{O}_C}, \alpha_k$ and naturality in their respective setups

$\Rightarrow$ (*) holds. $\Rightarrow$ $\alpha_{R,G}$ is injective.

to check isomorphism, check after base change to a field C or k.

$M_\Delta : BT_R \xrightarrow{\sim} DM_R \xrightarrow{\sim} DM_R^{adm}$: admissible prismatic Dieudonné module

DM_R = prismatic Dieudonné module

check $\mathrm{Fil}\, M = \varphi_M^{-1}(I^{pris} M)$ gives admissible filtration

proof goes through windows in the sense of Zink.

Back to general setup. R quasi-syntomic

Thm. $M_\Delta(G) = \mathrm{Ext}^1_{(R)_{qsyn}}(G, \mathcal{O}^{pris})$ is an admissible prismatic Dieudonné crystal

Main Thm

$M_\Delta : BT_R \xrightarrow{\sim} DM_R^{adm}$ = adm. prismatic crystals

Lecture 18

Thm. R quasisyntomic.

$$\mathcal{M}_{\Delta} : BT \longrightarrow DM^{adm}$$

i.e. $\mathcal{M}_{\Delta}(G)$ is an adm. prismatic Dieudonné crystal.

Step 1. $X =$ ~~p-adic completion of~~ formal abelian scheme over R, $R/(A/I)$

(A, I) bounded prism

Prop. 1 $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \mathcal{O}_{\Delta})$ is loc. free $\mathcal{O}_{\Delta}$ of rk $2\dim(X)$

Prop 2 $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \overline{\mathcal{O}}_{\Delta})$ is loc free $\overline{\mathcal{O}}_{\Delta}$ of rk $2\dim(X)$

$\mathcal{E}xt^i_{(R)_{\Delta}}(\text{——}) = 0$ for $i = 0, 2$

Pf of Prop 2. ~~apply argument~~

Step 1. $H^1(X, \overline{\Delta}_{X/A})$ is finite loc. free A/I-mod of rk $2\dim(X)$

and $H^d(X, \overline{\Delta}_{X/A}) = \Lambda^d H^1(X, \overline{\Delta}_{X/A})$

$\rightsquigarrow$ apply similar argument as in Berthelot – Breen – Messing about $H^1(X, \hat{\Omega}_{X/(A/I)})$ to reduce to showing first part.

Conj. filtration: $E_2^{ij} = H^i(X, \hat{\Omega}^j_{X/(A/I)})\{-j\} \Rightarrow H^{i+j}(X, \overline{\Delta}_{X/A})$

$\hookrightarrow$ HT comparison: filtration on $\overline{\Delta}_{X/A}$, graded pieces are $\hat{\Omega}^d_{X/(A/I)}\{-j\}$.

Claim. differentials of the spectral sequence all vanish.

For $n \in \mathbb{Z}_{\geq 0}$, multiplication by n on X induces multiplication by n^{i+j} on $H^i(X, \hat{\Omega}^j_{X/(A/I)})\{-j\}$

$\Rightarrow$ differentials vanish $(p \neq 2)$

on E_r $(r \geq 2)$

$H^i(X, \hat{\Omega}^j_{X/(A/I)})$ loc. free $\Rightarrow$ $H^i(X, \bar{\Delta}_{X/A})$ loc. free (A/I)-mod

for all i

Step 2 Same argument using Deligne spectral sequence as in crystalline case:

Once we have Künneth $H^1(X \times X, \bar{\Delta}_{X\times X/A}) \simeq H^1(X, \bar{\Delta}_{X/A}) \oplus H^1(X, \bar{\Delta}_{X/A})$

but: Hodge-Tate decomp. $\Rightarrow$ $\bar{\Delta}_{X/A} \overset{\mathbb{L}}{\underset{A/I}{\otimes}} \bar{\Delta}_{X/A} \simeq \bar{\Delta}_{X\times X/A}$.

Pf of Prop 1 Look at $(B, J) \in (R)_{\Delta}$

$\rightarrow$ passing to faithfully flat cover: $J = (\xi)$

~~take~~ consider SES

$$0 \to \mathcal{O}_{\Delta}/(\xi)^n \to \mathcal{O}_{\Delta}/(\xi)^{n+1} \to \mathcal{O}_{\Delta}/(\xi) \to 0$$

$$\mathcal{O}_{\Delta}/(\xi) = \bar{\mathcal{O}}_{\Delta}$$

take $\mathrm{Ext}^i(u^{-1}(X)|_{(B,J)}, -)$.

$$0 \to \mathrm{Ext}^1(u^{-1}(X)|_{(B,J)}, \mathcal{O}_{\Delta}/(\xi)^n) \to \mathrm{Ext}^1(u^{-1}(X)|_{(B,J)}, \mathcal{O}_{\Delta}/(\xi)^{n+1}) \to \mathrm{Ext}^1(-, \bar{\mathcal{O}}_{\Delta}) \to 0$$

and $\mathrm{Ext}^1\left(u^{-1}(X)\middle|(B,J), \mathcal{O}_{\Delta}\right) = \varprojlim_n \mathrm{Ext}^1\left(u^{-1}(X)\middle|_{(B,J)}, \mathcal{O}_{\Delta}/\xi^n\right)$

proof in fact shows: $\mathrm{Ext}^1(-) \simeq H^1(X \times_{\mathrm{Spf}(R)} \mathrm{Spf}(B/J), \overline{\Delta}_{X/A})$

$\Rightarrow R^1 f_{\Delta,*} \mathcal{O}_{\overline{\Delta}} = \mathcal{M}_{\Delta}(G)$

For general p-div. gp G, argue using Raynaud's thm:

$$0 \to G[p^n] \to X \to X' \to 0$$

$\rightsquigarrow$ deduce that $\mathcal{M}_{\Delta}(G)$ prismatic crystal

remains to show $\mathcal{M}_{\Delta}(G)$ is loc. free of rk $ht(G) = h$

this is shown by; we show $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(G[p^n]), \mathcal{O}_{\Delta})$ is loc. gen. by h sections.

By derived Nakayama, suffices to check after base change along $(W(K), p)$. $B \to K \leftarrow$ perfect field.

<u>Lecture 19</u> Last time: $\mathcal{E}xt^1_{(R)_{qsyn}}(G, \mathcal{O}^{pris})$ (R quasi-syntomic)

$$BT(R) \longrightarrow DM^{adm}(R)$$

since DM^{adm} forms a stack under quasi-syntomic topology.

suffices to ~~show~~ show functor lands in $DM^{adm}(R)$ for R qrsp.

Lemma If (C, J) henselian, $\bar{G}$ p-div gp over C/J, then $\bar{G}$ lifts to G over C.

Key input: $BT_n^h = \{n\text{-truncated p-div gp of ht } h\}$ is smooth

$\Rightarrow \bar{G}[p^n]$ lifts. $BT_n^h \longrightarrow BT_{n-1}^h$ is also smooth

R qrsp, $S \twoheadrightarrow R$, S perfectoid, $J = \ker(S \twoheadrightarrow R)$,

$S' =$ p-completion of henselization of S at J

$(S', \ker(S' \twoheadrightarrow R))$ henselian

Claim: S' is perfectoid.

This is almost purity result: henselization is colimit along étale maps.

S' perfectoid, $S' \twoheadrightarrow R$, $(S', \ker(S' \twoheadrightarrow R))$ henselian

$\bar{G}$ p-div gp $/R$

G p-div gp $/S'$

Last time: $BT(S') \xrightarrow{\sim} DM^{adm}(S')$.

base change admissible. filtration on $M_\Delta(G)$ to $M_\Delta(\bar{G})$

$\mathrm{Ext}^1_{(R)_{qsyn}}(-, \mathcal{O}^{pris}) : BT(R) \longrightarrow DM^{adm}(R)$

For R quasisyntomic, want to show equivalence

both BT^h, $DM^{adm, h}$ defines stacks for quasi-syntomic top.

BT_n^h is smooth Artin stack (Grothendieck)
(over $\mathbb{Z}$)

reduces to show equivalence for $\underline{R = \text{qrsp}}$.

Prop $M_{\Delta}(G) \simeq \mathrm{Hom}_{(R)_{qsyn}}(T_pG, \mathcal{O}^{pris})$. $T_pG = \varprojlim G[p^n]$

Pf. Let $\tilde{G} = \varprojlim_{\times p} G$

Have exact sequence of sheaves on $(R)_{qsyn}$

$$0 \to T_pG \to \tilde{G} \to G \to 0$$

this is a quasisyntomic cover

$$R\mathrm{Hom}_{(R)_{qsyn}}(\tilde{G}, \mathcal{O}^{pris}) = 0$$

derived p-complete

$\tilde{G}$ is $\mathbb{Q}_p$-vector space, derived p-completion vanishes.

Goal: prove fully faithful.

Consider $\mathcal{R} : Sh_R^{op} \to D_R$ = Cat. of $\mathcal{O}^{pris}[F]$-modules

$$\mathcal{G} \mapsto \mathcal{H}om_{(R)_{qsyn}}(\mathcal{G}, \mathcal{O}^{pris})$$

this has left adjoint $\mathcal{L}: D_R \to Sh_R^{op}$

$$M \mapsto \mathcal{H}om_{\mathcal{O}^{pris}[F]}(M, \mathcal{O}^{pris})$$

$$\mathrm{Hom}_{\mathcal{O}^{pris}[F]}(M, \mathcal{H}om(\mathcal{G}, \mathcal{O}^{pris})) \simeq \mathrm{Hom}_{(R)_{qsyn}}(\mathcal{G}, \mathcal{H}om(M, \mathcal{O}^{pris}))$$

Since both sides $M \times \mathcal{G} \to \mathcal{O}^{pris}$

which are $\mathcal{O}^{pris}[F]$- linear in the first component.

Claim. $\mathcal{R}$ is fully faithful on subcat. of Sh_R^{op} spanned by T_pG for

G p-div gp.

want to show $\mathcal{L}\mathcal{R}\mathcal{F} \to \mathcal{F}$ adjunction if $\mathcal{F} = T_pG$ is an isom.

Lecture 20

R qrsp ring, $M_\Delta : BT(R) \to DM^{adm}(R)$ fully faithful.

$$M_\Delta(G) \simeq \mathrm{Hom}_{(R)_{qsyn}}(T_pG, \mathcal{O}^{pris})$$

$$\underset{\mathcal{F}}{\uparrow}$$

$$\mathcal{R}: Sh_R^{op} \to D_R = \text{cat. of } \mathcal{O}^{pris}[F]\text{-mod}$$

left adjoint $\mathcal{L}: D_R \to Sh_R^{op}$ taking $\mathcal{H}om_{\mathcal{O}^{pris}[F]}(-, \mathcal{O}^{pris})$.

adjunction $\mathcal{L}\mathcal{R}\mathcal{F} \to \mathcal{F}$.

$$Sh_R^{op} \longleftrightarrow Sh_R$$

$$(*) \quad \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}^{pris}[F]}\left(\mathcal{H}om_{(R)_{qsyn}}(\mathcal{F}, \mathcal{O}^{pris}), \mathcal{O}^{pris}\right)$$

<u>Lemma 1</u> $T_p \check{G} \simeq \ker\left(N^{\geq 1} M_{\Delta}(G) \xrightarrow{\frac{\varphi}{\xi} - 1} M_{\Delta}(G)\right) \quad (\Delta_R, (\xi))$

Cartier dual (under $\check{G}$)

$$\ker\left(N^{\geq 1} M_{\Delta}(G) \xrightarrow{\frac{\varphi}{\xi} - 1} M_{\Delta}(G)\right) = \mathcal{H}om_{(R)_{qsyn}}(T_p G, N^{\geq 1}\mathcal{O}^{pris}).$$

<u>Idea</u>. $T_p \mathbb{G}_m \simeq \ker\left(N^{\geq 1}\mathcal{O}^{pris} \xrightarrow{\frac{\varphi}{\xi} - 1} \mathcal{O}^{pris}\right)$ (Bhatt–Lurie, 7.5.6)

$$T_p \mathbb{G}_m = \varprojlim_n \mathbb{G}_m[p^n]$$

$$\Rightarrow M_{\Delta}(\mu_{p^\infty}) \simeq \mathcal{O}^{pris}, \quad \varphi_M = \xi \cdot \varphi_{\mathcal{O}^{pris}}$$

Given this: $\mathcal{H}om_{(R)_{qsyn}}(T_p G, -)$ on both sides.

$$\text{Lemma 1} \Rightarrow T_p \check{G} \simeq \left(N^{\geq 1} M_{\Delta}(G)\right)^{\varphi_{M(G)} = \xi}$$

$$\simeq \left(M_{\Delta}(G)\right)^{\varphi_{M(G)} = \xi}$$

<u>Lemma 2</u>. $M_{\Delta}(\check{G}) = M_{\Delta}(G)^{\vee} \otimes_{\mathcal{O}^{pris}} M_{\Delta}(\mu_{p^\infty})$

<u>Sketch of proof</u>. $\Phi_G : M_{\Delta}(G)^{\vee} \otimes_{\mathcal{O}^{pris}} \underset{\ell}{M_{\Delta}(\mu_{p^\infty})} \longrightarrow M_{\Delta}(\check{G})$

$$\Phi_G(\delta \otimes \ell)(\alpha) := (\delta \circ M_{\Delta}(\alpha))(\ell) \in \mathcal{O}^{pris}$$

Since both sides are locally free of same rank, suffices to show surjectivity.

Show this after base change $R \to k$, k perfect field

Compare w/ Crystalline Dieudonné functor.

Lemma 1 + Lemma 2 $\Rightarrow$ $T_pG \simeq (M_\Delta(\check{G}))^{\varphi_M = \xi} \simeq Hom_{\mathcal{O}^{pris}}(M_\Delta(G), \mathcal{O}^{pris})$

this isom is not necessarily the same as adjunction.

$\simeq \mathcal{L}(M_\Delta(G))$ $(*)$

$\simeq \mathcal{L}R(T_pG)$

compose $\underbrace{\mathcal{F} \to \mathcal{L}R\mathcal{F} \xrightarrow[\sim]{(*)} \mathcal{F}}$

endomorphism of T_pG.

WTS : resultant endomorphism of T_pG is an isomorphism.

endomorphism is functorial in G, ring R.

<u>Claim</u>: only need to check isom. for $G = \mathbb{Q}_p/\mathbb{Z}_p$.

$T_pG = Hom_{(R)_{qsyn}}(\mathbb{Q}_p/\mathbb{Z}_p ; G)$

$\ni \alpha$

$T_p(\alpha) : T_p(\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow T_pG$

$$\begin{array}{ccc} \mathbb{Z}_p & & \\ 1 & \longmapsto & \alpha \\ \updownarrow & & \updownarrow \\ n & \longmapsto & n\cdot\alpha \end{array}$$

$T_p(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathbb{Z}_p \circlearrowleft$ mult. by unit

Lecture 21. Last time, R qrsp ring

$$M_{\Delta} : BT_R \longrightarrow \underbrace{DM^{adm}(R)}_{\text{prismatic Dieudonné module } / \Delta_R}$$

Showed fully faithful

remains to show essential surjectivity.

Idea. We will construct $\tilde{S}$ perfectoid, $\tilde{S} \twoheadrightarrow R$ s.t.

$$\underbrace{DM^{adm}(\tilde{S})}_{\substack{\text{know} \\ BT(\tilde{S})}} \longrightarrow DM^{adm}(R) \text{ induced by base change is essentially surj.}$$

main strategy: $(\tilde{S}, \ker(\tilde{S} \twoheadrightarrow R))$ henselian

Reduction step: reduce to the case of R

$\exists S \overset{\text{Perfectoid}}{=} A/I \twoheadrightarrow R$, $(A, \underbrace{\ker(A \twoheadrightarrow R)}_{J})$ henselian ← last time: can arrange s.t. henselian property holds

(A, I) bounded prism

s.t. J is gen. by (a_j) that admit compatible systems of p-power roots in A.

Prop (André's Lemma) $\exists S_1$ perfectoid, $S \to S_1$ p-completely faithfully flat s.t.

S_1 is absolutely integrally closed (every monic poly. w/ coeff in S_1 has solution in S_1).

$$\begin{array}{ccc} S & \twoheadrightarrow & R \\ \downarrow & \ulcorner & \downarrow \\ S_1 & \to & R_1 \end{array} \leftarrow \text{this is a quasi syntomic cover}$$

Observe, if we have essential surj. for R_1, can do faithfully flat descent to get for R.

$$M \in DM^{adm}(R)$$

$G' =$ base change to R_1, $M_\Delta(G') = M_{R_1}$

can descend G' to R.

Pf of André's lemma. Start w/ S, $=A/I$, $A = A_{inf}(S)$. adjoin roots for all possible monic poly w/ coeffs in S

$S \to \tilde{S}$ quasisyntomic

$(A, I) \to (B, J)$ map of prisms

get $\tilde{S}_1 = B/J$, (B, J) bounded prism

w/ $\tilde{S} \to \tilde{S}_1$ p-completely faithfully flat.

$\tilde{S}_2$ (B', J') perfection of (B, J), $B' = ($$(p,J)$-completion of $\varinjlim_\phi B)$

B' perfect $\Rightarrow$ $\tilde{S}_2$ is perfectoid, any monic poly w/ coeff in S has root.

wrsp. $R = S/(a_j)$ a_j admit compatible system of p^n-roots, $S \twoheadrightarrow R$

$$S' = \left(S\langle X_j^{1/p^\infty}\rangle / (X_j)\right)^{\wedge p}, \quad \tilde{S} = S[\![X_j^{1/p^\infty}]\!] := \left(\varinjlim_{\substack{n, J' \subset J \\ \text{finite}}} S[\![X_j^{1/p^n}]\!] : j \in J'\right)^{\wedge p}$$

$X_j^{1/p^n} \mapsto a_j^{1/p^n}$, $S' \to R$

Step 1: $DM^{adm}(S') \to DM^{adm}(R)$ essentially surj.

Step 2: $DM^{adm}(\tilde{S}) \to DM^{adm}(S')$ essentially surj.

Lecture 22

Recall: R qrsp , $\underset{S \text{ perfectoid}}{S \twoheadrightarrow R}$, $(S, \ker(S \overset{J}{\twoheadrightarrow} R))$ henselian

$J = (a_j)$ a_j admits compatible system of p^n-roots

$$S' = \left(S\langle x_j^{1/p^\infty}\rangle/(x_j)\right)^{\wedge p}$$

$\tilde{S} = S[[x_j^{1/p^\infty}]]$ ← this is perfectoid $\qquad \tilde{S} \twoheadrightarrow S' \twoheadrightarrow R$

1) $DM^{adm}(S') \longrightarrow DM^{adm}(R)$ essentially surjective

2) $DM^{adm}(\tilde{S}) \longrightarrow DM^{adm}(S')$ essentially surjective

Prop 1. $DM^{adm}(S') \longrightarrow DM^{adm}(R)$ essentially surjective

Claim: $\Delta_{S'} \longrightarrow \Delta_R$ surjective

$(\Delta_{S'}, \ker(\Delta_{S'} \to \Delta_R))$ henselian

Surjectivity: first observe $\overline{\Delta}_{S'} \longrightarrow \overline{\Delta}_R$ surjective by Hodge-Tate comparison.

$$\overline{\Delta}_{S'} \cong \left(\bigoplus_i \wedge^i \mathbb{L}_{S'/S}[-1]\right)^{\wedge p} \to \left(\bigoplus_i \wedge^i \mathbb{L}_{R/S}[-1]\right)^{\wedge p}$$

$\underbrace{\mathbb{L}_{S'/S}[-1]}_{\text{discrete}} \longrightarrow \underbrace{\mathbb{L}_{R/S}[-1]}$ $\qquad$ Nakayama $\Rightarrow \Delta_{S'} \to \Delta_R$ surjective

$[x_j] \longmapsto \underset{J/J^2}{(a_j)}$

Henselian: $(\Delta_{S'}, \ker(\Delta_{S'} \to \Delta_R))$

henselian

Know: $(S', \ker(S' \twoheadrightarrow R))$ henselian

Lemma. $I \subset J \subset \overset{\Delta_{S'}}{\overset{\parallel}{A}}$ ideals

TFAE: (1) (A, I) henselian, $(A/I, J/I)$ henselian

(2) (A, J) henselian

$$\begin{tikzcd} \Delta_{S'} \arrow[r, two heads] \arrow[d, "\theta"'] \arrow[dr, dashed] & \Delta_R \arrow[d] \\ S' \arrow[r, two heads] & R \end{tikzcd}$$

$DM^{adm}(S') \longrightarrow DM^{adm}(R)$

need to show (M, φ_M) prismatic Dieudonné module over Δ_R w/ admissible filtration

(N, φ_N) ——— " ——— over $\Delta_{S'}$ w/ adm. filt'n

Prop Let $(M, \varphi_M, \mathrm{Fil}\, M) \in DM^{adm}(R)$ (4.1.22)

R general qrsp ring

$\exists$ finite proj. Δ_R-mods L, T. s.t. $M = L \oplus T$

$\mathrm{Fil}\, M = L \oplus N^{\geq 1}\Delta_R \cdot T$

Moreover, given any such L, T and φ-semilinear $\Phi: L \oplus T \to L \oplus T$

isom after inverting I exactly defines such $(M, \varphi_M, \mathrm{Fil}\, M)$

Lecture 23 $\quad R$ qrsp, $\quad S \twoheadrightarrow R$, $\ker = J$ $\quad (\alpha_j)$

S perfectoid, $A = A_{\inf}(S)$, $(\xi) = I$, $A/I \xrightarrow{\sim} S$

$\Delta_S \simeq A_{\inf}(S) = A$

$$S' = \left(S\langle x_j^{1/p^\infty}\rangle_{j\in J} / (x_j)\right)^{\wedge p}, \quad \tilde{S} = \left(S[[X_j^{1/p^\infty} : j\in J]]\right)^{\wedge p}$$

Prop 2. $DM^{adm}(\tilde{S}) \longrightarrow DM^{adm}(S')$ essentially surjective

$$\simeq \left\{\left(M, \mathrm{Fil}\,M, \varphi_M, \frac{\varphi_M}{\xi}\right)\right\}$$

Observe $(\tilde{S}, (x_j)_{j\in J})$ henselian

(henselian preserved under colimits)

$$\Delta_{\tilde{S}} = \left(\varinjlim_{n, J'\subset J} A[[x_j^{1/p^n}]]_{j\in J'}\right)^{\wedge(p,\xi)}$$

$$\Delta_{S'} \simeq \Delta_{\tilde{S}}\left\{\frac{x_j}{\xi}\right\}_{j\in J}^{\wedge(p,\xi)}$$

(roughly: $\Delta_{S'}$ is prismatic envelope of $\Delta_{\tilde{S}}$, ideal $(x_j)_{j\in J}$)

$$B = \left(\Delta_{\tilde{S}} / (x_j : j\in J)\right)^{\wedge(p,\xi)}$$

$(B, (\xi))$ prism

$S' \simeq B/(\xi)$

universal property of $\Delta_{S'}$: $\Delta_{S'} \xrightarrow{\alpha} B$, $x_j \mapsto 0$

Consider φ-admMod $= \left\{ \left(M, \mathrm{Fil}\,M, \varphi_M, \frac{\varphi_M}{\xi}\right) : M \text{ mod}/B \right\}$

$DM^{adm}(S')$ base change essentially surj.

Similar argument to last time: $DM^{adm}(\tilde{S}) \longrightarrow \varphi$-admMod$/B$ essentially surj.

input: $(\Delta_{\tilde{S}}, \ker(\Delta_{\tilde{S}} \to B))$ henselian

Calculation

Last thing to show:

$$DM^{adm}(S') \xrightarrow[\text{fully faithful}]{} \varphi\text{-adm Mod}/B$$

Key input: $\varphi(\ker(\alpha)) \subset \xi \cdot \Delta_{S'}$

$\frac{\varphi}{\xi}$ is top. nilpotent.

$$\begin{tikzcd} \ker(\alpha) \arrow[r] \arrow[d, "\text{factors}"'] & \Delta_{S'} \\ \ker(\alpha) \arrow[ur] & \end{tikzcd}$$

$J := \ker(\Delta_{S'} \to B)$

To show full faithfulness, given $M_1 \to M_2$ map in $DM^{adm}(S')$

$$\mathrm{Hom}_{\Delta_{S'}}(M_1, M_2) \simeq \mathrm{Hom}_B(M_1/J, M_2/J)$$

Want to show $M_1 \xrightarrow{\beta} JM_2$, then $\beta = 0$

Consider $(\varphi')^{\#}(\beta)((\varphi')^{A})^{-1} = \underbrace{u(\alpha)}_{9}$ b/c of top nilp.

R qrsp, $DM^{adm}(R)$, $(M, \mathrm{Fil}M, \varphi_M)$, $M = L \oplus T$

$\mathrm{Fil}M = L \oplus N^{\geq 1}\Delta_R \cdot T$, $(M, \mathrm{Fil}M)$ filtred module over $(\Delta_R, N^{\geq 1}\Delta_R)$.

$\mathrm{Fil}^{\bullet}\Delta_R$ filtred prismatization.

$\mathrm{Fil}^{\bullet}M$ filtred module

$\varphi: \mathrm{Fil}^{\bullet}M \to I^{\bullet}\Delta_R \otimes_{\Delta_R} M =: I^{\bullet}M$ s.t.

F-gauge: module over Rees$(\mathrm{Fil}^{\bullet}\Delta_R)$ linearization $\mathrm{Fil}^{\bullet}M \mathbin{\hat{\otimes}}_{\mathrm{Fil}^{\bullet}\Delta_R} I^{\bullet}\Delta_R \to I^{\bullet}M$ is isom.

$\mathrm{Fil}^{i}M = \mathrm{Fil}^{i-1}\Delta_R \cdot L \oplus \mathrm{Fil}^{i}\Delta_R \cdot T$