# Representation theory of p-adic groups

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## Lecture 1

$p$ prime. $\underset{<\infty}{\underline{F / \mathbb{Q}_p}}$ or $F = \mathbb{F}_q((t))$

$F \supset \mathcal{O} \ni \varpi \qquad \mathcal{O}/\varpi\mathcal{O} \simeq \mathbb{F}_q \quad , q = |\mathbb{F}_q| = p^?$

$\mathrm{val}: F \longrightarrow \mathbb{Z} \cup \{\infty\}$ , $\underset{\text{"p-adic gp"}}{\underset{\uparrow}{G}} = \underline{G}(F)$ , $\underline{G}$ conn'd reductive gp $/F$

**Def.** A $\underline{\text{smooth}}$ rep of $G$ is a pair $(\pi, V)$ consisting of

$V$: a $\mathbb{C}$-vec. sp.

$\pi$: $G \longrightarrow \mathrm{Aut}(V)$ gp hom.

s.t. $\forall v \in V$, $\exists$ $K \subset G$ cpt open s.t. $\pi(K)v = v$.

**Rmk.** a basis of open nbhds of $1 \in GL_n(F)$ is given by $1 + \varpi^N \mathrm{Mat}_{n\times n}(\mathcal{O})$.

**E.g.** $V = \mathbb{C}$, $\underset{\text{triv}}{\underset{\shortparallel}{\pi}} : G \to 1 \in \mathbb{C}^\times = \mathrm{Aut}(\mathbb{C})$ (trivial rep)

**Def.** Let $P = M \ltimes N \subset G$ be a parabolic subgp, and a smooth rep $(\sigma, V_\sigma)$ of $M$, then the parabolic induction $(\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G V_\sigma)$ is the following smooth rep:

$$\mathrm{Ind}_P^G V_\sigma = \Big\{ f: G \to V_\sigma : \underset{m\in M,\, n\in N,\, g\in G}{f(mng) = \sigma(m) f(g)},\ \exists K_f \subset G \text{ cpt open subgp s.t. } f(gk) = f(g),\ \forall k \in K_f \Big\}$$

e.g. $G = GL_2(F)$

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = B = \underset{\substack{\shortparallel \\ M = T}}{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$(\sigma, V_\sigma) = (\text{triv}, \mathbb{C})$

$$\mathrm{Ind}_B^G \mathbb{C} = \{ f : \underset{\substack{\shortparallel \\ \mathbb{P}^1(F)}}{B \backslash G} \to \mathbb{C} \}$$

$f$ locally constant

$$\underset{g \in G}{(\mathrm{Ind}_P^G \sigma)(g)} : \begin{array}{ccc} \mathrm{Ind}_P^G V_\sigma & \longrightarrow & \mathrm{Ind}_P^G V_\sigma \\ [x \mapsto f(x)] & \longmapsto & [x \mapsto f(xg)] \end{array}$$

in char $\ell$: two notions
sub (cuspidal)
subqt (supercusp.)

**Def.** A (super)cuspidal rep $(\pi, V)$ of $G$ is an irred. (smooth) rep s.t.

$(\pi, V) \not\hookrightarrow (\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G V_\sigma)$ for $P = M \ltimes N \subsetneq G$ and

irred. rep. $(\sigma, V_\sigma)$ of $M$.

**Fact.** Let $(\pi, V)$ be an irred. rep of $G$, then $\exists\ P = M \ltimes N \subset G$ and a supercuspidal rep

$(\sigma, V_\sigma)$ of $M$ s.t. $(\pi, V) \hookrightarrow (\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G V_\sigma)$.

Question 1: How to construct (all) supercuspidal reps?

Bernstein decomposition:

$$\underset{\text{smooth reps of } G}{\mathrm{Rep}(G)} = \prod_{\{(M, \sigma)\}/\sim} \overbrace{\mathrm{Rep}(G)_{[M, \sigma]}}^{\text{Bernstein block}}$$

$M$: Levi, $\sigma$: s.c. rep. of $M$

$(M, \sigma) \sim (gMg^{-1}, \sigma(g^{-1} \_ g) \otimes \chi)$

for $g \in G$ and some $\boxed{\text{unramified}}$ char. $\chi : gMg^{-1} \longrightarrow \mathbb{C}^\times$

trivial on all cpt subgps

$\mathrm{Rep}(G)_{[M,\sigma]}$ consists of all reps of $G$, all of whose irred subqts are contained in

$\mathrm{Ind}_{P'}^{G} V_{\sigma'}$ w/ $P' = M'N'$

$(M', \sigma') \sim (M, \sigma)$ for some $P', M', \sigma'$.

<u>Example</u> $G = SL_2(F)$

(a) $M = G$, $\mathrm{Rep}(G)_{[G,\sigma]} = \{\sigma, \sigma \oplus \sigma, \sigma \oplus \sigma \oplus \sigma, \dots\}$

$\mathrm{Hom}_G(\sigma, \sigma) \simeq \mathbb{C}$.

(b) $M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = T$, $\mathrm{Rep}(G)_{[T, \mathrm{triv}]}$ ← principal block

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Steinberg rep.

$1 \to \mathrm{triv} \to \mathrm{Ind}_B^G \mathrm{triv} \longrightarrow St \to 1$

<u>Question 2</u>. Understand $\mathrm{Rep}(G)_{[M,\sigma]}$.

<u>Rough answer to 2</u> (Morris '93, Bushnell-Kutzko '98, Yu 2001, Kim-Yu 2017, F. 2021

Kim 2007, F. 2021, Adler-F.-Mishra-Ohara 08/2024)

(a) $\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathrm{Mod}-\left(\mathcal{H}_{aff}(W_{aff}, q) \rtimes \mathbb{C}[\Omega, \mu]\right)$

(b) (AFMO) $\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathrm{Rep}(G^0)_{[M^0, \sigma^0]}$

$(p \nmid |W|)$

$\sigma^0$ is of depth zero

i.e. corresponds to rep of finite gp of Lie type

How about Q1 (construction of s.c. reps)

Folklore conj.

Every supercuspidal rep is of the form

$c\text{-ind}_K^G \rho$, where $K \subset G$ is cpt-mod-center open subgp

$\|$ $\rho$ is an irred rep of $K$ ($\Rightarrow \dim V_\rho < \infty$)

$\left\{ f : G \to V_\rho : \begin{array}{l} f(kg) = \rho(k) f(g), \quad k \in K, g \in G \\ f \text{ cptly supported} \end{array} \right\}$

known w/ explicit $(K, \rho)$ :

$GL_n$; classical gps, $p \neq 2$ ; inner forms of $GL_n$

$G$ splits over a tame ext $E/F$ and $p \nmid |\text{Weyl gp of } G|$.

Bruhat-Tits theory

Suppose $G$ split,

Moy-Prasad filtration

Def. A BT triple is a triple $(T, \{X_\alpha\}_{\alpha \in \Phi(G,T)}, x_{BT})$

(1) $T \subset G$ is a split max'l torus

(2) $X_\alpha \in \underbrace{\mathrm{Lie}(G)_\alpha}_{\substack{\text{one dim'l subspace on which} \\ T \text{ acts via } \alpha}} - \{0\}$ s.t. $\{X_\alpha\}$ form a Chevalley system

(3) $x_{BT} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$

# Lecture 2

$G$ split . BT triple

$$(T, \{X_\alpha\}_{\alpha \in \Phi(G,T)}, x_{BT})$$

$T$: max split torus

$X_\alpha \in (\mathrm{Lie}\, G)_\alpha - \{0\}$

$$x_{BT} \in \underbrace{X_*(T)}_{\mathrm{Hom}(\mathbb{G}_m, T)} \otimes_{\mathbb{Z}} \mathbb{R}$$

## Moy-Prasad filtration of $T$.

$$T_0 = \{t \in T : \mathrm{val}(\chi(t)) = 0,\ \forall \chi \in X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)\}$$

$= $ max'l cpt subgp of $T$.

e.g. $G = SL_2(F) \supset T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ , $T_0 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathcal{O}^\times \right\}$

$$T_r = \{t \in T_0 : \mathrm{val}(\chi(t) - 1) \geq r,\ \forall \chi \in X^*(T)\}$$

e.g. $T_r = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t = 1 + \varpi^{\lceil r \rceil} \mathcal{O} \right\}$

## Moy-Prasad filtration for root gps $U_\alpha \subset G$

$x_\alpha : F \xrightarrow{\sim} U_\alpha \subset G$

$$\mathrm{Lie}(x_\alpha) : F \xrightarrow{\sim} \mathrm{Lie}\, U_\alpha = (\mathrm{Lie}\, G)_\alpha$$

$$1 \longmapsto X_\alpha$$

e.g. $x_\alpha : * \mapsto \begin{array}{cc} & j \\ i & \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array}$ $U_\alpha$

$r \in \mathbb{R}_{\geq 0}$.

$$U_{\alpha,x,r} := x_\alpha\left(\varpi^{\lceil r-\alpha(x_{BT})\rceil}\mathcal{O}\right)$$

$$\begin{array}{c} x_{BT} \in \mathrm{Hom}(\mathbb{G}_m, T) \otimes \mathbb{R} \\ \times \\ \alpha \in \mathrm{Hom}(T, \mathbb{G}_m) \\ \downarrow \\ \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \otimes \mathbb{R} \\ \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} \otimes \mathbb{R} \simeq \mathbb{R} \end{array}$$

e.g. $G = SL_2(F)$.

(a) $X_1 := \left(T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \left\{\underset{X_\alpha}{\underset{\parallel}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}}, \underset{X_{-\alpha}}{\underset{\parallel}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}}\right\}, x_{BT} = 0 \in X_*(T) \otimes \mathbb{R}\right)$

$$\alpha : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t^2$$

$U_\alpha = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, $x_\alpha : x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U_\alpha$

$$U_{\alpha,x,r} = \begin{pmatrix} 1 & \varpi^{\lceil r \rceil}\mathcal{O} \\ 0 & 1 \end{pmatrix}$$

$$U_{-\alpha,x,r} = \begin{pmatrix} 1 & 0 \\ \varpi^{\lceil r \rceil}\mathcal{O} & 1 \end{pmatrix}$$

(b) $X_2 = \left(T, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}, x_{BT} = \frac{1}{4}\alpha^\vee\right)$ $\quad \alpha^\vee : t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$,

then $\alpha(\alpha^\vee) = 2 \quad \rightsquigarrow \quad \alpha(x_{BT}) = \frac{1}{4} \cdot 2 = \frac{1}{2}$

$$U_{\alpha,x,r} = \begin{pmatrix} 1 & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ 0 & 1 \end{pmatrix}, \qquad U_{-\alpha,x,r} = \begin{pmatrix} 1 & 0 \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1 \end{pmatrix}$$

e.g. $U_{\alpha,x,0} = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix}$, $U_{-\alpha,x,0} = \begin{pmatrix} 1 & 0 \\ \varpi\mathcal{O} & 1 \end{pmatrix}$

<u>Moy-Prasad filtration</u>

$r \in \mathbb{R}_{\geq 0}$, $\quad G_{x,r} := \langle T_r, U_{\alpha,x,r} \rangle_{\forall \alpha \in \Phi(G,T)}$.

eg. (a) $G_{x_1,0} = SL_2(\mathcal{O})$

$r>0$, $G_{x_1,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r\rceil}\mathcal{O} \\ \varpi^{\lceil r\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}_{\det=1}$

(b) $G_{x_2,0} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}_{\det=1}$

$G_{x_2,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}$

$G_{x,r+} \left(= G_{x,r+\text{"}\varepsilon\text{"}}\right) = \bigcup_{s>r} G_{x,s}$

$G_{x,0}$ called parahoric subgp

<u>Good properties</u>

(i) $G_{x,0} \triangleright G_{x,r}$

(ii) $G_{x,0}/G_{x,0+} \simeq \mathbb{F}_q$-points of a reductive gp

eg. (a) $G_{x_1,0}/G_{x_1,0+} \simeq SL_2(\mathbb{F}_q)$

(b) $G_{x_2,0}/G_{x_2,0+} \simeq \begin{pmatrix} \mathbb{F}_q & 0 \\ 0 & \mathbb{F}_q \end{pmatrix}_{\det=1}$

(iii) $[G_{x,r}, G_{x,s}] \subset G_{x,r+s}$

($\Rightarrow$ $\underset{r>0}{G_{x,r}/G_{x,r+}}$ can be viewed as an $\mathbb{F}_q$-vec. sp.)

## Bruhat-Tits building (non-traditional def)

Def The (reduced) Bruhat-Tits building $\mathcal{B}(G,F)$ is as a set

$$\{\text{BT triples}\}/\sim$$

$X_1 \sim X_2 \iff G_{X_1,r} = G_{X_2,r}, \ \forall r \geq 0.$

$\rightsquigarrow$ can write $G_{X,r}$ for $X \in \mathcal{B}(G,F)$

Properties.

(a) $G$ acts on $\mathcal{B}(G,F)$ such that for $X \in \mathcal{B}(G,F)$, $g \in G$, $G_{gX,r} = g\, G_{X,r}\, g^{-1}$

(b) $\mathcal{B}(G,F)$ can be equipped w/ a polysimplicial structure, s.t.

for $X, Y \in \mathcal{B}(G,F)$, $X$ and $Y$ are in the interior of the same polysimplex

$\iff G_{X,0} = G_{Y,0}$.

Fix $T$,

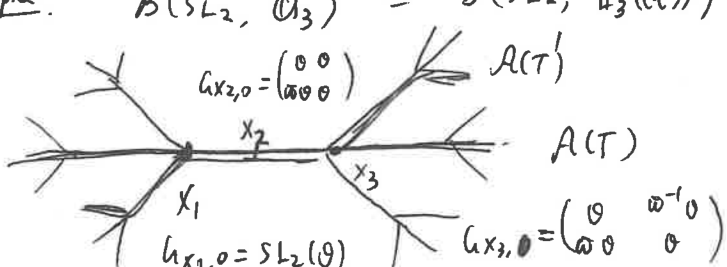
$$\mathcal{A}(T,F) = \{(\overset{\text{fixed}}{T}, \{x_\alpha\}, x_{BT})\}/\sim$$

$$= \{(\overset{\text{fix}}{T}, \overset{\text{fix}}{\{X_\alpha\}}, x_{BT})\}/\sim$$

$(T, \{x_\alpha\}, x_{BT,1}) \sim (T, \{X_\alpha\}, x_{BT,2}) \iff x_{BT,1} - x_{BT,2} \in X_*(Z(G)) \otimes \mathbb{R}$

$\rightsquigarrow$ we can equip $A(T,F)$ w/ structure of an affine space over $X_*(T) \otimes \mathbb{R} / X_*(Z(G)) \otimes \mathbb{R}$.

example. $\mathcal{B}(SL_2, \mathcal{O}_3) = \mathcal{B}(SL_2, \mathbb{F}_3((t))) = \mathcal{B}(PGL_2, \mathcal{O}_3)$



**Def** Let $(\pi, V)$ be an irred (smooth) rep. of $G$. The <u>depth</u> of $(\pi, V)$ is the smallest non-negative real no. s.t.

$$V^{G_{x,r+}} \neq 0 \quad \text{for some } x \in \mathcal{B}(G, F)$$

$\uparrow$

$G_{x,r+}$ - fixed vectors

**Thm** (Moy–Prasad, '94/96) Let $x \in \mathcal{B}(G,F)$ be a vertex.

$G_x := \mathrm{Stab}_G(x) = \{g \in G : gx = x\}$. Let $(\rho, V_\rho)$ be an irrep. of $G_x$ s.t.

(i) $\rho|_{G_{x,0+}} = \mathbb{1}_{V_\rho}$

(ii) $\rho|_{G_{x,0}}$ is a cuspidal rep of $G_{x,0}/G_{x,0+}$

Then $\mathrm{cind}_{G_x}^{G} \rho$ is an (irred) s.c. rep. of $G$ of depth 0.

All depth 0 s.c. reps arise in this way. if $G$ simply conn'd, $G_x = G_{x,0}$

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**Lecture 3** Construction of s.c. reps à la Yu (+ twist by F.– Kaletha – Spice)

Input: (i) $G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^{n-1} \subset G^n = G$ tame twisted Levi subgps

i.e. $\underline{G}^i \times E \subset \underline{G}^{i+1} \times E$ Levi subgp for some $E|F$ tame.

remove this for types s.t. $Z(G^0)/Z(G)$ is anisotropic, i.e. cpt

$\boxed{G \text{ splits over a tame ext'n.}}$

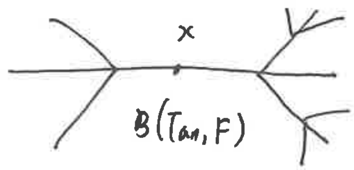
eg. $G = SL_2(\mathbb{Q}_p)$, $p \neq 2$, $n = 1$, $G^0 = T_{an} = \left\{\begin{pmatrix} a & b \\ pb & a \end{pmatrix} \in SL_2(\mathbb{Q}_p)\right\}$

over $E = \mathbb{Q}_p(\sqrt{p})$, $\underline{G}^0_E = \left\{\begin{pmatrix} a & b \\ pb & a \end{pmatrix} \in SL_2(\mathbb{Q}_p(\sqrt{p}))\right\} \sim \left\{\begin{pmatrix} a+b\sqrt{p} & 0 \\ 0 & a-b\sqrt{p} \end{pmatrix}\right\} \simeq \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$

(ii) $x \in B(G^0, F) \subset B(G^1, F) \subset \cdots \subset B(G, F)$

s.t. $x$ is a vertex in $B(G^0, F)$ $\longrightarrow$ remove this for types

e.g.




(iii) $0 < r_0 < r_1 < r_2 < \cdots < r_{n-1}$

(iv) $\phi_i$ $(0 \le i \le n-1)$ $\underbrace{\text{a}}_{(G^{i+1}, G^i)\text{-generic}}$ character of $G^i$ of depth $r_i$

e.g. $\begin{pmatrix} a & b \\ pb & a \end{pmatrix} \mapsto e^{\frac{2\pi i}{p} \cdot (2ab)}$ , $r = \frac{1}{2}$

(v) $\rho^0$ irrep of $G^0_x$ s.t. $\rho^0|_{G^0_{x,0+}}$ is trivial, $\rho^0|_{G^0_{x,0}}$ is cuspidal rep of $G^0_{x,0}/G^0_{x,0+}$.

<u>Construction</u>. $\widetilde{K} = G^0_x \, G^1_{x, r_0/2} \, G^2_{x, r_1/2} \cdots G^n_{x, \frac{r_{n-1}}{2}}$

$\widetilde{\rho} = \rho^0 \otimes \kappa$ rep'n of $\widetilde{K}$

$\rho^0 : \widetilde{K} \to \widetilde{K} / G^0_{x,0+} G^1_{x, r_0/2} \cdots \simeq G^0_x / G^0_{x,0+} \xrightarrow{\rho^0} \mathrm{End}(V_{\rho^0})$

$\kappa = \kappa^{nt} \otimes \varepsilon_{FKS} \leftarrow \varepsilon_{FKS} : \widetilde{K} \to \widetilde{K} / G^0_{x,0+} G^1_{x, r_0/2} \cdots \simeq G^0_x / G^0_{x,0+} \to \{\pm 1\} \subset \mathbb{C}^\times$

not twisted

(Fintzen - Kaletha - Spice).

built from $\{\phi_i\}$ via theory of Heisenberg - Weil rep.

Thm (Yu 2001 (Fintzen 2021) $p \neq 2$, Fintzen - Schwein 2025, $p=2$, $q \neq 2$)

$c\text{-ind}_K^G \tilde{\rho}$ is $\underset{\text{key}}{\underline{\text{irreducible}}}$ supercuspidal rep.

Thm (Kim 2007, $p \gg 0$, char $F = 0$, Fintzen 2021)

If $p \nmid |\text{Weyl gp of } G|$, then <u>all</u> sup. cusp. reps arise in this way.

(Hakim - Murnaghan (Kaletha)) $\rightsquigarrow$ which data give same output

same for $\overline{\mathbb{F}}_\ell$ - reps if $\ell \neq p$.

$M \subset G$ Levi subgp $(\sigma, V_\sigma)$ a sup. cusp. rep of $M$

Def. A pair $\underline{(K, \rho)}$ consisting of a cpt open subgp $K \subset G$, and an irrep $(\rho, V_\rho)$ of $K$ is an $[M, \sigma]$ - type, if for all irrep $(\pi, V)$ of $G$, TFAE

(i) $\pi \in \text{Rep}(G)_{[M,\sigma]}$

(ii) $\rho \hookrightarrow \pi|_K$, i.e. $\text{Hom}_K(\rho, \pi) \neq 0$

$$\text{Hom}_K(\rho, \pi) \cong \text{Hom}_G(\text{cind}_K^G \rho, \pi)$$

Example. $G = SL_2(F)$, $T = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$

$(\underset{\substack{\uparrow \\ \text{Iwahori}}}{I_w}, \text{triv})$ is a $[T, \text{triv}]$ - type

Thm (Bushnell-Kutzko 1998)

If $(K,\rho)$ is an $[M,\sigma]$-type, then $\mathrm{Rep}(G)_{[M,\sigma]} \simeq \text{Mod-}\mathcal{H}(G,K,\rho)$.

$$\mathcal{H}(G,K,\rho) = \{ f : G \to \mathrm{End}(V_\rho) : f(kgk') = \rho(k) f(g) \rho(k'),\ k,k' \in K,\ g \in G \}$$

f cptly supported

+ convolution

$$\begin{matrix} \mathcal{H}(G,K,\rho) \\ \wr\!\wr \\ \mathrm{End}_G(\mathrm{cind}_K^G \rho) \end{matrix}$$

Examples. 1) $G = SL_2(F)$

(a) $[M = G,\ \sigma = \mathrm{cind}_K^G \rho]$

$$\mathcal{H}(G,\tilde{K},\rho) \simeq \underset{\substack{\wr\wr \\ \mathbb{C}}}{\mathrm{End}(\mathrm{cind}_{\tilde{K}}^G \rho)}$$

$(\tilde{K},\rho)$ is a type for

(b) $[M = T,\ \sigma = \mathrm{triv}]$

$$\mathcal{H}(G, I_w, \mathrm{triv}) = \{ f : \underbrace{I_w \backslash G / I_w}_{N(T)/T_0 = W_{aff} = \langle s_0, s_1 : s_i^2 = 1 \rangle} \to \mathbb{C} : f \text{ cptly supported} \}$$

$$= \bigoplus_{w \in W_{aff}} \mathbb{C} \cdot T_w \quad \text{w/ relations}$$

$$\left.\begin{array}{l} \bullet\ T_w = T_{s_{i_1}} \cdots T_{s_{i_n}} \\ \quad w = s_{i_1} \cdots s_{i_n},\quad s_{i_j} \in \{s_0, s_1\} \\ \bullet\ T_{s_i}^2 = q\, T_1 + (q-1)\, T_{s_i} \end{array}\right\} \quad H_{aff}(W_{aff}, q_s)$$

Thm (Kim-Yu 2017, Fintzen 2021) A construction (G tame.) analogous to Yu's construction (but w/ more general input) yields an $[M,\sigma]$-type.

If $p \nmid |\text{Weyl gp of } G|$, then $\forall\, [M,\sigma]$, there exists a type as above.

Fix an input $(G^0 \subset \cdots \subset G, x, \{r_i\}, \{\phi_i\}, \rho^0) \rightsquigarrow$ type $(K,\rho)$.

$$\begin{array}{ccl} x \in \mathcal{B}(G^0, F) & \subset \cdots & \\ \parallel \quad \cup & & \\ x \in \mathcal{B}(M^0, F) & \text{vertex} & \end{array} \qquad M = \mathrm{Cent}(Z_{split}(M^0)).$$

Fact. $\mathrm{Supp}(\mathcal{H}(G,K,\rho)) = K\,\mathrm{supp}(\mathcal{H}(G^0, \underset{K\cap G^0}{\overset{\parallel}{K^0}}, \rho^0))\,K$

Prop (Adler-Fintzen-Mishra-Ohara 08/2024)

$\exists$ subgp $N^\heartsuit \subset N_{G^0}(M^0, (M^0_x)_{cpt})$ s.t.

$${}_{K^0}\backslash^{\mathrm{Supp}(G^0, K^0, \rho^0)}/_{K^0} \cong \underbrace{N^\heartsuit / N^\heartsuit \cap (M^0_x)_{cpt}}_{\text{group } =: W^\heartsuit}$$

Thm (AFMO, 2024)

$\exists$ a rep $\tilde{\kappa} : N^\heartsuit(K\cap M) \to \mathrm{End}(V_\kappa)$ s.t. $\tilde{\kappa}|_{K\cap M} = \kappa$ and

$$I : \mathcal{H}(G^0, K^0, \rho^0) \xrightarrow{\sim} \mathcal{H}(G,K,\rho)$$

given by the following: If $\varphi \in \mathcal{H}(G^0, K^0, \rho^0)$ is supp. on $K^0 n K^0$ w/ $n \in N^\heartsuit$, then $I(\varphi)$ is supp. on $KnK$ and $I(\varphi)(n) = d_n \cdot \varphi(n) \otimes \tilde{\kappa}(n) \in \mathrm{End}(V_\rho)$, $\rho \simeq \rho^0 \otimes \kappa$.

$$d_n = \sqrt{\frac{|K^\circ / n K^\circ n^{-1} \cap K^\circ|}{|K / n K n^{-1} \cap K|}}$$

Cor. $\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathrm{Rep}(G^\circ)_{[M^\circ, \sigma^\circ]}$

corresponding to $(K, \rho)$

Thm (AFMO)

$$W^\heartsuit \simeq W(\rho)_{\mathrm{aff}} \rtimes \Omega(\rho)$$

$$H(G, K, \rho) \simeq H_{\mathrm{aff}}(W(\rho)_{\mathrm{aff}}, \{q_s\}) \rtimes \mathbb{C}[\Omega(\rho), \mu]$$

for some 2-cocycle $\mu$, and some $q_s \in \mathbb{Q}_{>1}$, $s \in$ set of simple reflections of $W(\rho)_{\mathrm{aff}}$.

$(\pi, V) \rightsquigarrow$ $r^1$ depth, $\phi^1$, $G_1$, $r^2$, $\phi^2$, $G_2$, ---

$G_{x,r^1}/G_{x,r^1+} \curvearrowright V^{G_{x,r^1+}}$ $\qquad G_1 \supset G_2 \supset \cdots$

$\Downarrow$

$\phi^1$ very nice

$\int$

$X^1 \in \mathrm{Lie}(G)^*$

$\mathrm{Cent}(G^1) =: G_1$.