Universal ~~ ~~ monodromic affine Hecke categories

Jeremy Taylor

$G/\mathbb{C}$ reductive gp. $\mathring{I} \subset I$ prounipotent radical of Iwahori

$\widetilde{Fl} := G((t))/\mathring{I}$ enhanced affine flags

a T-torsor over $Fl := G((t))/I$

$\mathcal{H} = Shv_{(I)}(\widetilde{Fl})$ weakly I-constructible work in analytic topology, allow infinite dim'l stalks.

$\check{G}/k$, Langlands dual gp, char $k = 0$.

$\tilde{\check{G}} = \check{G} \overset{\check{B}}{\times} \check{B}$, $St := \tilde{\check{G}} \underset{\check{G}}{\times} \tilde{\check{G}}$

<u>Thm</u> (G.-Dhillon)

There is a monoidal equiv. $\mathcal{H} \simeq \mathrm{IndCoh}_{\check{G}}(St)$

This is a family of equivalences over $\check{T} \times \check{T}$

Fiber at $(1,1)$ recovers Bezrukavnikov's equivalence.

Torus case: — Betti $Shv_{loc.\,const}(T \times \Lambda) \simeq QCoh(\check{T} \times pt/\check{T})$

— de Rham $DMod(T \times \Lambda) \simeq QCoh(\check{\mathfrak{t}}/\check{\Lambda} \times pt/\check{T})$

The big tilting

$$\mathcal{H} \simeq \mathrm{IndCoh}_{\check{G}}(St)$$

Ξ tilting $\qquad\qquad\qquad$ $\mathcal{O}$ structure sheaf

(supported on G/U)

<u>Thm</u> (T.) There exists $\Xi \in Shv_{(B)}(G/U)$ satisfying

(1) $\mathrm{Hom}(\Xi, -) : Shv_{(B)}(G/U) \longrightarrow \mathrm{QCoh}(\check{T} \times \check{T})$

is monoidal

(2) the ! and $*$- restriction to any B-orbit is free over $\check{T}$ and concentrated

in perverse degree 0.

(3) $\mathrm{End}(\Xi) \simeq \mathcal{O}(St)^{\check{G}}$

$\left(\simeq \mathcal{O}(\check{T} \times_{\check{T}/\!/W} \check{T}) \text{ if } Z(G) \text{ conn'd} \right)$

The Iwahori-Whittaker category

By monoidality, define $\quad {}_{\times}\mathcal{H} := \mathrm{QCoh}(\check{T}) \underset{Shv_{(B)}(G/U)}{\otimes} \mathcal{H}$

where $Shv_{(B)}(G/U) \xrightarrow{\mathrm{Hom}(\Xi,-)} \mathrm{QCoh}(\check{T} \times \check{T}) \curvearrowright \mathrm{QCoh}(\check{T})$

Want to match

$$ {}_{\times}\mathcal{H}_{\times} \curvearrowright {}_{\times}\mathcal{H} \curvearrowleft \mathcal{H} \longleftrightarrow \mathrm{QCoh}_{\check{G}}(\check{G}) \curvearrowright \mathrm{QCoh}_{\check{G}}(\tilde{\check{G}}) \curvearrowleft \mathrm{IndCoh}_{\check{G}}(St)$$

Universal monodromic Arkhipov - Bezrukavnikov equivalence

Construct a monoidal functor

$$
\begin{tikzcd}
\mathrm{QCoh}_{\check{G}}(\tilde{\check{\mathfrak{g}}}) \arrow[r] \arrow[rr, bend left] & \mathcal{H} \arrow[r] & {}_{\chi}\mathcal{H} \\
\mathrm{Free}_{\check{G}}(\tilde{\check{\mathfrak{g}}}) \arrow[u, hook] \arrow[r] \arrow[dr, "\text{restriction}"'] & \mathrm{Perv}^{\mathrm{Waki}}_{(I)}(\widetilde{Fl}) \arrow[u] \arrow[d, "gr"] & \\
& \mathrm{Free}_{\check{T}}(\check{\mathfrak{t}}) &
\end{tikzcd}
$$

Two approaches to fully faithfulness

(1) AB localize in nilpotent directions

$$
\begin{tikzcd}
\mathrm{Shv}_{(I,\chi)}(Fl) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{QCoh}(\check{N}/\check{B}) \arrow[d] \\
/\langle IC_w : \ell(w)>0 \rangle \arrow[r, phantom, "\simeq"] & \mathrm{QCoh}(\check{N}^{reg}/\check{B})
\end{tikzcd}
$$

(2) localize in semisimple directions

Suffices to prove $\forall \mu, \lambda \in \Lambda^+$,

$$
\begin{tikzcd}
\mathrm{Hom}_{\check{B}/\check{B}}(\mathcal{O}(-\mu), V_\lambda \otimes \mathcal{O}) \arrow[r, "\sim"] \arrow[d, "i^*", "\text{restric}"'] & \mathrm{Hom}(\Delta_{-\mu}, Z_\lambda) \arrow[dl, "gr"] \\
\mathrm{Hom}_{\check{T}/\check{T}}(\mathcal{O}(-\mu), V_\lambda \otimes \mathcal{O}) &
\end{tikzcd}
$$

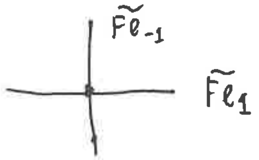
Prove that images of i^* and gr coincide, both characterized by order vanishing condition along walls in $\check{T}$.

Example. $G = PGL(2)$, $\Lambda = \mathbb{Z}$, $\mathcal{O}(\check{T}) = k[x^{\pm 1}]$

$$\mathcal{H} \simeq \operatorname{Ind} \operatorname{Coh}_{\check{G}}(St)$$

$$0 \to \Delta_{-1} \to Z_1 \to \nabla_1 \to 0, \quad 0 \to \mathcal{O}(-1) \to V_1 \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

supp. on $\qquad\qquad$ Supp. on diagonal



ext'n is classified by $\operatorname{Ext}^1(\nabla_1, \Delta_{-1}) \simeq k[x^{\pm 1}]/x^2-1$.

$$\operatorname{Hom}(\nabla_1, Z_1) \xrightarrow{gr} \operatorname{Hom}(\nabla_1, \nabla_1)$$

$$\to \operatorname{Ext}^1(\nabla_1, \Delta_{-1}) = k[x^{\pm 1}]/(x^2-1)$$

$\operatorname{image}(gr) = $ ideal gen. by (x^2-1).

Universal monodromic Bezrukavnikov equivalence.

The commuting actions $\quad {}_\chi\mathcal{H}_\chi \curvearrowright {}_\chi\mathcal{H} \curvearrowleft \mathcal{H}$

$$\begin{array}{cc} {}_\chi\mathcal{H}_\chi & {}_\chi\mathcal{H} \\ \wr\wr & \wr\wr \\ \operatorname{QCoh}_{\check{G}}(\check{\mathfrak{g}}) & \operatorname{QCoh}_{\check{G}}(\tilde{\check{\mathfrak{g}}}) \end{array}$$

gives a functor $\mathcal{H} \xrightarrow{\iota^!} \operatorname{End}_{{}_\chi\mathcal{H}_\chi}({}_\chi\mathcal{H}) \simeq \operatorname{QCoh}_{\check{G}}(St)$

Claim. (1) $\iota^!$ admits a fully faithful left adjoint $\iota_!$

(2) $\ker \iota^! = \mathcal{H}^{\leq -\infty}$ (example $w_{\widetilde{F\ell}}$)

(3) $\iota^!$ induces an equiv. $\mathcal{H}^{cpt} \xrightarrow{\sim} \mathrm{Coh}_{\check{n}}(St)$.

Claim (1) follows by a general observation of Ben-Zvi – Gunningham – Orem,
Beraldo – Lin – Reeves

The modules ${}_x\mathcal{H}$ and $\mathcal{H}_x$ are dual to each other

- $u : {}_x\mathcal{H}_x \longrightarrow {}_x\mathcal{H} \otimes_{\mathcal{H}} \mathcal{H}_x$

  $c : \mathcal{H}_x \otimes_{{}_x\mathcal{H}_x} {}_x\mathcal{H} \longrightarrow \mathcal{H}$ preserve compactness

- $u^R : {}_x\mathcal{H} \otimes_{\mathcal{H}} \mathcal{H}_x \longrightarrow {}_x\mathcal{H}_x$ fully faithful

Triangle identities and $uu^R \simeq id$