Classification (cont'd)

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Steps : Started w/ X/k smooth proj. surf.

(1) $X \to \dots \to X^{min}$, X^{min} minimal, no (-1)-curves

is unique if $\kappa(X) \geq 2$

(2) otherwise X ruled (e.g. $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$)
non-unique

Thm (Castelnuovo) X is rat'l iff $q = \dim \mathrm{Pic}^0_X = 0$ and $P_2(X) = \dim H^0(X, \omega_X^{\otimes 2}) = 0$

(3) $K_{X^{min}}$ nef $\iff$ X not ruled and $K_{X^{min}}$ nef $\Rightarrow$ $|nK_{X^{min}}|$ no basepoints, $n \gg 0$

(4) $\kappa(X) = 2$ general type

$\kappa(X) = 1$ $X^{min} \to X^{can}$ $p_g = 1$ fibration.

$\kappa(X) = 0$ ω_X is torsion $nK_{X^{min}} \equiv 0$

Numerical invariants

$h^{p,q} = \dim H^q(X, \Omega^p_X)$

in char $\neq 0$, Hodge symmetry $h^{p,q} = h^{q,p}$ fails

$h^{1,0} \neq h^{0,1}$ for some examples

$H^2(X, \mathcal{O}) \underset{\text{duality}}{\overset{\text{Serre}}{=}} H^0(X, \omega_X)^\vee = H^0(X, \Omega^2_X)^\vee$ $\quad h^{2,0} = h^{0,2}$ ✓

Pic^0_X might not be a smooth group scheme (e.g. μ_p)

$H^1(X,\mathcal{O}) = T_0 \operatorname{Pic}^\circ_X$, $q \le h^{0,1}$ (eq iff $\operatorname{Pic}^\circ_X$ smooth)

obstructions $\in H^2(X,\mathcal{O})$ if $h^{0,2}$ $(= h^{2,0}) = 0$, then $\operatorname{Pic}^\circ_X$ smooth.

Chern classes $c_i = c_i(T_X) \in CH^\bullet(X)$

need cohomology (Weil) $H^i(X;K) = H^i_{et}(X, \mathbb{Q}_\ell)$

- valued in finite-dim'l vec. sp. over field K char $=0$
- cycle class map $CH^i \xrightarrow[\gamma]{} H^{2i}(X;K)$ lots of axioms
- Poincaré duality, Kunneth formula, Lefschetz fixed point.

$c_2 = \sum_{i=0}^{4} (-1)^i \dim_K H^i(X,K)$

$\mathcal{E}$ v.b. rk $= 2$

$c_{2-k+1}(\mathcal{E}) = \left[\text{locus where } k \text{ general sections are dependent}\right]$

$c_2(T_X) = [\text{zero locus of tangent field}]$

$X \xrightarrow{\sim} \Delta \subset X \times X$

$[\Delta]^2 = \sum (-1)^i \dim_K H^i(X,K)$

$[\Delta]^2 = [\Delta] \cap c_2(\mathcal{N}_{\Delta|X\times X}) = c_2(T_X)$

Noether's formula. $12\,\chi(\mathcal{O}_X) = c_1^2 + c_2$ GRR $\chi(\mathcal{F}) = \int_X Td(\cancel{X})\, ch(\mathcal{F})$

$b_1 = \dim_K H^1(X,K) = 2q$

<u>idea</u>. "universal coeff. thm" for ℓ prime $\gg 0$, $H^1(X, \mathbb{Z}/\ell^n\mathbb{Z}) \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{b_1}$

$(\ell,p)=1$

idea: this is isom. to $(\operatorname{Pic}^\circ_X)_{red}[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{2q}$, $2q = b_1$

$p_g = \dim H^0(X, \omega_X) = h^{2,0} = h^{0,2}$,

$\chi(\mathcal{O}) = 1 - h^{0,1} + p_g$ — $12(1 - h^{0,1} + p_g) = c_1^2 + c_2$

— $c_2 = 1 - 2q + b_2 - 2q + 1 = 2(1-2q) + b_2$

Case X minimal and $\kappa(X) = 0$, $nK_X \equiv 0$. $\Rightarrow c_1^2 = 0$. also $p_g \in \{0, 1\}$.

$\rightarrow \quad 10 + 12p_g = 12h^{0,1} - 2b_1 + 2b_2 = 8h^{0,1} + 2(2h^{0,1} - b_1) + b_2$

$\Delta = 2h^{0,1} - b_1 = 2(h^{0,1} - q)$, $h^{0,1} \geq q$, $\Delta \geq 0$

$h^{0,1}, b_1 \in \mathbb{Z}_{\geq 0}$, $\Delta = 2h^{0,1} - b_1 \geq 0$, $p_g \in \{0, 1\}$.

$b_2 = 10 + 12p_g - 8h^{0,1} - 2\Delta \geq 0$

	def invariants			not invariant			
b_2	b_1	c_2	$\chi(\mathcal{O}_X)$	$h^{0,1}$	$h^{2,0} = p_g$	Δ	
22	0	24	2	0	1	0	(K3)
6	4	0	0	2	1	0	(abelian surface)
10	0	12	1	0 1	0 1	0 2	(Enriques)
2	2	0	0	1 2	0 1	0 2	(hyperelliptic)
14	2	12	1	1	1	0	DNE

Prop. if X minimal $nK_X \equiv 0$ and $c_2 \neq 0$, then finitely many finite étale covers

$\pi_1^{ét}(X)$ is finite

Proof $\pi : X' \to X$, then X' minimal, $nK_{X'} = 0$,

$\pi^* \omega_X \simeq \omega_{X'}$, so X' appears on list

$c_2(X') = \pi^* c_2(X)$, $c_2(X') = (\deg \pi) c_2(X)$

Cor - $K3$ simply conn'd

- Enriques can have $\pi_1 = \mathbb{Z}/2$ covered by $K3$.

- DNE does not exist. b/c has finite π_1 but nontrivial Pic^0_X.

Thm. no Enriques over $\mathrm{Spec}\, \mathbb{Z}$.

Def. X/k a smooth proj. ~~variety~~ surface is Enriques if

1) K_X numerically trivial, $nK_X \equiv 0$ ~~(hence $K_X = 0$)~~

2) $b_2 = 10$.

Lemma. - $\mathrm{Num}_{X_{\bar{k}}} \simeq \mathbb{Z}^{10}$ w/ some Galois action

if char $k \neq 2$, $\mathrm{Pic}\, X_{\bar{k}} \simeq \mathbb{Z}^{10} \oplus \underbrace{(\mathbb{Z}/2\mathbb{Z})}_{\omega_X}$

Def a family of Enriques surface $f : \mathcal{X} \to S$ smooth proj. w/ fibers Enriques.

($\mathcal{X}$ - alg. space + proper)

"Proof": (1) $X = \mathcal{X}_{\bar{\mathbb{Q}}}$ $\mathrm{Pic}_X = (\mathbb{Z}/2) \oplus \mathbb{Z}^{10}$ has a trivial Galois action

(prop 5.5)

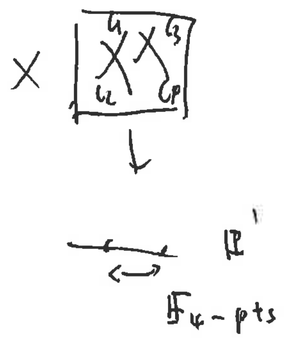
$\mathrm{Pic}_{\mathcal{X}/\mathbb{Z}} = (\mathbb{Z}/2 \oplus \mathbb{Z}^{10})_{\mathrm{Spec}\, \mathbb{Z}}$

(2) $|\mathcal{X}(\mathbb{F}_p)| = 1 + 10p + p^2$, $p = 2$, get 25 points

(3) $X_2 := \mathcal{X}_{\mathbb{F}_2}$, $X_2 \longrightarrow \mathbb{P}^1$ defined over $\mathbb{F}_2$.

$X_2(\mathbb{F}_2)$ lies over $\{0, 1, \infty\} = \mathbb{P}^1(\mathbb{F}_2)$.

(4) prove that the only geometrically reducible fibers lie over $\{0, 1, \infty\}$.



$C_1 + C_2 \longleftrightarrow C_3 + C_4$

but Gal acts trivially on Num

$C_1 + C_2 \underset{num}{\sim} C_3 + C_4$

$C_1 \longleftrightarrow C_3$ but can prove that only rel's b/w fibers is $\sum C_i \sim \sum C_j$.

(5) have fibration w/ ≤ 3 reducible fibers, 25 points in those fibers

(6) only case left "exceptional".

(7) exceptional Enriques do not lift to $W_2(\mathbb{F}_2)$. X