Langlands Duality and Symmetric Varieties

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Lecture 1. (Joint work w/ D. Nadler, L. Yi)

Intro

$$\begin{tikzcd} & \check{G} \arrow[d, "\wr S"] & \text{Langlands dual gp} \\ G \arrow[ur] \arrow[r, "\text{geom. of } LG"'] & \check{G}_{geom} & \text{Tannakian dual gp} \end{tikzcd}$$

$\check{G} \simeq \check{G}_{geom}$ ← Satake equiv.

G: cpx reductive gp

Let $G \curvearrowright X$ ← any affine variety.

$$\begin{tikzcd} & \check{G}_X \subset \check{G} & \text{dual gp of } X \\ X \arrow[ur, "\text{comb. of } X"] \arrow[r, "\text{geom. of } LX"'] & \check{G}_{X, geom.} & \end{tikzcd}$$

• Nadler, Takeda	symmetric var.
• SV	spherical var.
• KS	general X

N ← symmetric var.

GN ← general X

Conj $\check{G}_X \simeq \check{G}_{X, geom.}$ $\Longleftrightarrow$ real gps $G_{\mathbb{R}}$

$$G \subset X = G/K \subset X = G/S \subset X$$

- G: reductive gp
- $X = G/K$: symmetric var. $K = G^{\theta}$, θ invol'n of G
- $X = G/S$: spherical variety, has an open B-orbit
- X: G-var.

Goal. Explain the proof of the Conj. when $X = G/K$ is a sym. variety.

L1: Langlands duality for G

L2: Symmetric varieties and their dual gp $\check{G}_X$

L3: Geometry of LX

L4: Construction of $\check{G}_{X,\text{geom}}$.

L5: $\check{G}_X \simeq \check{G}_{X,\text{geom}}$.

Rmk. [KS, N, T, SV]

$\check{G}_X \times SL_2 \subset \check{G}$

case by case construction.

G Langlands dual gps

$G/\mathbb{C}$ conn'd reductive gp $/\mathbb{C}$. Pick $T \subset G$ max. torus

$$\rightsquigarrow \Psi_G = (\underset{\substack{\uparrow \\ \text{roots}}}{X^*(T), \Phi}, \underset{\substack{\uparrow \\ \text{coroots}}}{X_*(T), \check{\Phi}}) \rightsquigarrow (\underset{\substack{\Uparrow \\ \text{root system}}}{V = \Phi \otimes_{\mathbb{Z}} \mathbb{R} \supset \Phi})$$

Def: An abstract root datum consists of a quadruple $\Psi = (X, \Phi, \check{X}, \check{\Phi})$

① X & $\check{X}$ are free ab. gp of same rank

together w/ a perfect pairing $\langle , \rangle : X \times \check{X} \to \mathbb{Z}$

② $\Phi \subset X$ roots $\quad \Phi \longleftrightarrow \check{\Phi}$

$\check{\Phi} \subset \check{X}$ coroots $\quad \alpha \longleftrightarrow \check{\alpha}$

③ $\langle \alpha, \check{\alpha} \rangle = 2$

④ $s_\alpha(\Phi) \subset \Phi$, $\quad s_\alpha(v) = v - \langle v, \check{\alpha} \rangle \alpha$

$s_{\check{\alpha}}(\check{\Phi}) \subset \check{\Phi}$, $\quad s_{\check{\alpha}}(w) = w - \langle w, \alpha \rangle \check{\alpha}$

Ψ is reduced if ⑤ $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$.

Thm. (Chevalley) ① $G_1 \simeq G_2 \iff \Psi_{G_1} \simeq \Psi_{G_2}$

② Let Ψ be a reduced root datum, then there exists a conn'd red. gp G s.t. Ψ_G is Ψ

Rmk. True for any G/k, $k=\bar{k}$.

Key observation If $\Psi = (X, \Phi, \check{X}, \check{\Phi})$, then $\check{\Psi} = (\check{X}, \check{\Phi}, X, \Phi)$ is again a (reduced) root datum.

Def. Let $\check{G}$ be the conn'd reductive gp $/\mathbb{C}$ w/ root datum $\check{\Psi}_G = (X_*(T), \check{\Phi}, X^*(T), \Phi)$

$\check{G}$: Langlands dual gp.

Ex. $G = SL_2 \supset T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times \right\}$

$\Psi_{SL_2} \simeq (\mathbb{Z}, \pm 2, \mathbb{Z}, \pm 1)$

$\alpha : \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \mapsto a^2$

$\check{\alpha} : a \longmapsto \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}$

$\check{\Psi}_{SL_2} = (\mathbb{Z}, \pm 1, \mathbb{Z}, \pm 2) = \Psi_{PGL_2}$.

$PGL_2 = GL_2 / Z \supset T = \left\{ \begin{bmatrix} a & \\ & 1 \end{bmatrix} \right\}$

$\alpha : \begin{bmatrix} a & \\ & 1 \end{bmatrix} \longmapsto a$, $\check{\alpha} : a \longmapsto \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} = \begin{bmatrix} a^2 & \\ & 1 \end{bmatrix}$

§ geometry of LG $F = \mathbb{C}((t))$, $\mathcal{O}_F = \mathbb{C}[[t]]$

$LG \supset L^+G$

Cartan decomposition $L^+G \times L^+G \curvearrowright LG = \coprod_{\lambda \in X_*(T)^+} LG^\lambda$, $LG^\lambda = L^+G \cdot t^\lambda \cdot L^+G$

<u>Closure rel'n</u> $\quad \overline{LG^\lambda} = \coprod_{\mu \le \lambda} LG^\mu . \qquad \mu \le \lambda \iff \lambda - \mu \in \mathbb{Z}_{\ge 0} \check{\Phi}^+$

<u>Affine grassmannian</u> $\quad Gr = LG/L^+G = \operatorname{colim}_{i \ge 0} \left(Gr^0 \subset Gr^1 \subset Gr^2 \subset \cdots \right)$

$Gr^0 = pt$, $\quad$ closed embedding

where Gr^i are L^+G- stable proj. varieties

Ex. $G = GL_n$, $\quad X_*(T)^+ = \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n \}$

Let $L_0 = \mathcal{O}_F^{\oplus n}$.

① $Gr = \{ \text{lattice } L \subset F^{\oplus n} \} \supset Gr^i = \{ t^i L_0 \subset L \subset t^{-i} L_0 \}$

$\qquad\qquad \downarrow \leftarrow$ closed embedding

$L \longmapsto \quad Gras(2ni) = \{ V \subset \mathbb{C}^{2ni} \}$

$V = L/t^i L_0 \hookrightarrow t^{-i} L_0 / t^i L_0 \simeq \mathbb{C}^{2ni}$

② $Gr^{(1,0,\dots,0)} = \overline{Gr^{(1,0,\dots,0)}} \simeq \mathbb{P}^{n-1}$

③ (Lusztig embedding) $\quad$ Let $\mathcal{N}_n \leftarrow n \times n$ nilpotent matrices / $\mathbb{C}$

$\phi : \mathcal{N}_n \hookrightarrow \overline{Gr^{(n,0,\dots,0)}}$

$M \longmapsto (t - M) \cdot L_0$

Fact: ϕ is an open embedding & has non-empty intersections w/ Gr^μ, $\mu \le (n, 0, \dots, 0)$

§. <u>Satake cat</u>. $\quad$ derived $\mathbb{H}$ cat. $\quad DSat_G \supset Sat_G \quad$ Satake cat. of G

$DSat_G = D^b_c(L^+G \backslash LG / L^+G) \supset Perv(L^+G \backslash LG / L^+G) = Sat_G$

equiv. derived cat.

$D^b_c(L^+G \backslash LG / L^+G) = D^b_c(L^+G \backslash Gr) = \operatorname{colim}_{i \ge 0} D^b_c(L^+G \backslash Gr^i)$

Convolution product.

$$\mathrm{Gr}\times\mathrm{Gr} \xleftarrow{p\times id} LG\times \mathrm{Gr} \xrightarrow{q} LG \overset{L^+G}{\times} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}$$

$\mathcal{F}_1, \mathcal{F}_2 \in D\mathrm{Sat}_G$, $\qquad \mathcal{F}_1 * \mathcal{F}_2 := m_! (\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)$

$(p\times id)^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq q^*(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)$

Thm (Lusztig, MV)

① Sat_G is a semisimple abelian cat. w/ irr. obj. IC^λ = IC-cpx of $\overline{\mathrm{Gr}^\lambda}$.

② convol'n $*$ is t-exact $\rightsquigarrow$ $(\mathrm{Sat}_G, *)$ ← semisimple abelian monoidal cat.

Proof sketch ① Parity vanishing.

$H^n(IC^\lambda) \neq 0 \rightarrow n \equiv \underset{\langle 2\rho, \lambda\rangle}{\underset{\shortparallel}{\dim \mathrm{Gr}^\lambda}} \mod 2$

② The map m is (stratified) semismall.

§ Tannakian dual gp $\check{G}_{geom}$.

Def. An abelian $\mathbb{C}$-linear cat. $\mathcal{C}$ equipped w/ the following data is called (neutral) tannakian cat. : ① A tensor product

$\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and $(\mathcal{F}_1\otimes\mathcal{F}_2)\otimes\mathcal{F}_3 \simeq \mathcal{F}_1\otimes(\mathcal{F}_2\otimes\mathcal{F}_3)$ + pentagon axiom

② Commutativity constraint: $c_{\mathcal{F}_1,\mathcal{F}_2} : \mathcal{F}_1\otimes\mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_2\otimes\mathcal{F}_1$ s.t.

$c_{\mathcal{F}_2,\mathcal{F}_1}\circ c_{\mathcal{F}_1,\mathcal{F}_2} = id$

③ Fiber functor: an exact faithful $\otimes$-functor

$$F : (\mathcal{C}, \otimes) \longrightarrow (\mathrm{Vect}/\mathbb{C}, \otimes)$$

④ Tensor unit $\mathbb{1} \in \mathcal{C}$

⑤ Rigidity: $\mathcal{F} \in \mathcal{C} \rightsquigarrow \check{\mathcal{F}} \in \mathcal{C}$

<u>Thm</u> (Deligne) Let $(\mathcal{C}, \otimes)$ be a Tannakian cat., then there exists an affine gp scheme H, called the tannakian dual gp of $\mathcal{C}$, s.t.

$$\begin{tikzcd}
(\mathcal{C}, \otimes) \arrow[rr, "\sim"] \arrow[dr, "F"'] & & (\mathrm{Rep}(H), \otimes) \arrow[dl, "\mathrm{For}"] \\
& \mathrm{Vect}/\mathbb{C} &
\end{tikzcd}$$

<u>Ex.</u> Let T be a torus,

$$\mathcal{C}_{X_\bullet(T)} \simeq \mathrm{Vect}^{X_\bullet(T)}, \quad c_{v,w} : v \otimes w \xrightarrow{\sim} w \otimes v$$
$$v_\lambda \otimes w_\mu \mapsto w_\mu \otimes v_\lambda$$

$$\begin{tikzcd}
\mathcal{C}_{X_\bullet(T)} \arrow[rr, "\simeq"] \arrow[dr] & & \mathrm{Rep}(T) \arrow[dl] \\
& \mathrm{Vect}/\mathbb{C} &
\end{tikzcd}$$

<u>Thm.</u> $(\mathrm{Sat}_G, *)$ is a tannakian cat w/ fiber functor given by $H^*(\mathrm{Gr}, -)$.

<u>Def.</u> The tannakian dual gp of G, denoted by $\check{G}_{geom}$, is the tannakian gp of $(\mathrm{Sat}_G, *)$

$$\begin{tikzcd}
(\mathrm{Sat}_G, *) \arrow[rr, "\simeq"] \arrow[dr, "F = H^*(\mathrm{Gr}, -)"'] & & (\mathrm{Rep}\, \check{G}_{geom}, \otimes) \arrow[dl, "\mathrm{For}"] \\
& \mathrm{Vect}/\mathbb{C} &
\end{tikzcd}$$

The most non-trivial part of the proof is the construction of $(\mathcal{F}_1, \mathcal{F}_2$:

$$\mathrm{Gr}^{(2)} = \{ (x, \mathcal{E}, \phi) : x \in \mathbb{A}^1,\ \mathcal{E} \in \mathrm{Bun}_G,\ \phi : \mathbb{C} \setminus \pm x \to \mathcal{E} \}$$

$$\begin{tikzcd} \mathrm{Gr}^{(2)} \arrow[d, "f"] \\ \mathbb{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gr} \times \mathrm{Gr} \times \mathbb{C}^\times \arrow[r, "j"] \arrow[d] & \mathrm{Gr}^{(2)} \arrow[d] & \mathrm{Gr} \arrow[l, "i"'] \arrow[d] \\ \mathbb{C}^\times \arrow[r] & \mathbb{C} & 0 \arrow[l] \end{tikzcd}$$

<u>Def</u> $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Sat}_G$,

① $\mathcal{F}_1 \underset{f}{*} \mathcal{F}_2 = \Psi_f(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1])$

nearby cycle.

② $\mathcal{F}_1 \underset{MV}{*} \mathcal{F}_2 = i^*[-1]\, j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1])$

<u>Thm</u> $\mathcal{F}_1 * \mathcal{F}_2 \overset{①}{\simeq} \mathcal{F}_1 \underset{f}{*} \mathcal{F}_2 \overset{②}{\simeq} \mathcal{F}_1 \underset{MV}{*} \mathcal{F}_2$. In particular, $*$ & $\underset{MV}{*}$ are t-exact.

<u>Proof sketch</u>. ① use global convolution diag.

$$\begin{tikzcd} \mathrm{Gr}^2 \times \mathbb{C}^\times \arrow[r] \arrow[d] & \widetilde{\mathrm{Gr}}_{glob} \arrow[d] & LG \overset{L^+G}{\times} \mathrm{Gr} \arrow[l] \arrow[d, "m"] \\ \mathrm{Gr}^2 \times \mathbb{C}^\times \arrow[r] & \mathrm{Gr}^{(2)} & \mathrm{Gr} \arrow[l] \end{tikzcd}$$

② ULA of $j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1])$ wrt. $\mathrm{Gr}^{(2)} \to \mathbb{C}$

$\Downarrow$

monodromy of Ψ_f is trivial

$$\begin{tikzcd} \mathrm{Gr}^{(2)} \arrow[d, "s"] & (x, \mathcal{E}, \phi) \arrow[d, mapsto] \\ \mathrm{Gr}^{(2)} & (-x, \mathcal{E}, \phi) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gr}^2 \times \mathbb{C}^\times \arrow[r] \arrow[d, "{(sw, -1)}"] & \mathrm{Gr}^{(2)} \arrow[d, "sw"] & \mathrm{Gr} \arrow[l] \arrow[d, equal, "id"] \\ \mathrm{Gr}^2 \times \mathbb{C}^\times \arrow[r] & \mathrm{Gr}^{(2)} & \mathrm{Gr} \arrow[l] \end{tikzcd}$$

$$i^* j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1]) \simeq i^* sw^* j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1])$$

$\rightsquigarrow$ fusion $\qquad \simeq i^* j_{!*}(\mathcal{F}_2 \boxtimes \mathcal{F}_1 \boxtimes \mathbb{C}[1])$

§ $\check{G} \simeq \check{G}_{geom}$.

① $\check{G}_{geom}$ is reductive $\Leftrightarrow$ Sat_G is s.s.

② The embedding $\check{T} \hookrightarrow \check{G}_{geom}$ comes from the theory of MV cycles:

$$T \subset B = TN \subset G, \qquad LN \curvearrowright Gr = \coprod_{\lambda \in X_*(T)} S^\lambda, \qquad S^\lambda = LN\, t^\lambda LG/L^+G$$

Thm (MV)

$$\begin{tikzcd}
\mathrm{Rep}\, \check{G}_{geom} \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Rep}\, \check{T} \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Vect} \arrow[d, phantom, "\wr"] \\
Sat_G \arrow[r] & \mathrm{Vect}^{X_*(T)} \arrow[r] & \mathrm{Vect}
\end{tikzcd}$$

$$\mathcal{F} \longmapsto \bigoplus_{\lambda \in X_*(T)} H_c^{\langle 2\rho, \lambda \rangle}(S^\lambda, \mathcal{F}) \simeq H^*(Gr, \mathcal{F}).$$

Lecture 2. § Symmetric variety.

$G/\mathbb{C}$, let $\theta: G \to G$ be an algebraic invol'n, $\theta^2 = Id$

$K = G^\theta$ ← symmetric subgp (maybe disconnected) $X = G/K$ symmetric variety.

Rmk. ① K might not be connected

② $K^\circ \subset K' \subset K$, $X' = G/K'$

③ $\tau: X \hookrightarrow G$ $\qquad \tau: X \xrightarrow{\sim} (G^{inv \circ \theta})^\circ \hookrightarrow G$

$gK \longmapsto g\theta(g)^{-1}$

Ex. $G = H \times H$, where $H/\mathbb{C}$ conn'd reductive, $\theta: H \times H \longrightarrow H \times H$

$(h_1, h_2) \mapsto (h_2, h_1)$

$G^\theta = H_\Delta \hookrightarrow G$

$X \simeq H \times H / H \simeq H$ (group case)

Ex. $G = GL_n$, $\theta(g) = {}^t g^{-1}$. $K = G^\theta = O_n$ ← orthogonal gp. disconn'd

$X = GL_n / O_n = \{ n\times n$ symmetric invertible matrices $\}$

Ex $G = GL_{2n}$,
$$J_n = \begin{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} & & & \\ & \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \end{bmatrix} \quad 2n$$

$2n$

$\theta(g) = J_n \, {}^t g^{-1} J_n^{-1}$, $K = G^\theta = Sp_{2n}$ ← sympl. gp., $X = GL_{2n}/Sp_{2n}$

Ex. (Linear Periods case) $G = GL_{p+q}$, $p \le q$,
$$I_{p,q} = \begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_p & 0 \\ 0 & \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}_q \end{bmatrix}$$

$\theta(g) = I_{p,q}\, g\, I_{p,q}^{-1}$, $K = G^\theta = GL_p \times GL_q \hookrightarrow GL_{p+q}$

$$\begin{bmatrix} \square & 0 \\ 0 & \square \end{bmatrix} \hookrightarrow \begin{bmatrix} \square \end{bmatrix}$$

§. Classifications

Fix (G, θ)

Def A torus $A \subset G$ is called θ-split if $\theta(a) = a^{-1}$.

A parabolic $P \subset G$ is called θ-split if $P \cap \theta(P)$ is Levi (P & $\theta(P)$ are opposite parabolic)

Thm (Cartan, ..., Springer ...)
$\mathbb{C}$ $\quad k = \bar{k}$, char $k \ne 2$

① The max'l θ-split torus are conjugate over K^0

② Let $A \subset G$ be a max'l θ-split torus. there exists a minimal θ-split parabolic such that $A \subset P$, $P \cap \theta(P) = Z_G(A)$. Any such P are conj. over K^0.

③ A Borel pair (T, B) is called <u>θ-split</u> if $A \subset T \subset B \subset P$ where $A \subset P$ are in ②.

Any θ-split Borel pair (T, B) are conj. over K°. Moreover, $\theta(T) = T$.

From now we fix $A \subset T \subset B \subset P$.

<u>Def.</u> (G, θ) is called <u>quasi-split</u> if $B = P$, it is called <u>split</u> if $A = T$.

<u>Ex</u> (split) $X = GL_n / O_n$. $\theta(g) = {}^t g^{-1}$

$$\begin{array}{ccccc} A = & T & \subset & B = P & , \ B \cap \theta(B) = T \\ & \parallel & & \parallel & \\ & \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} & & \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} & \end{array}$$

<u>Ex.</u> $X = H \times H / H$. $\theta(h_1, h_2) = (h_2, h_1)$

$$\begin{array}{ccccc} A & \subset T & \subset & B & = P \\ \cong & \parallel & & \parallel & \\ T_H & \to T_H \times T_H & & B_H \times B_H^{op} & \\ t & \mapsto (t, t^{-1}) & & & \end{array}$$

(quasi-split)

Ex. $GL_{p+q} / GL_p \times GL_q$, $p \le q$. $I'_{p,q} = \begin{bmatrix} & & \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_p \\ & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{q-p} & \\ \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_p & & \end{bmatrix}$

$\theta(g) = I'_{p,q} \, g \, (I'_{p,q})^{-1}$

$$\begin{array}{l} A \subset T \subset B \subset P \\ \parallel \\ \begin{bmatrix} t_1 & & & & & & & \\ & \ddots & & & & & & \\ & & t_p & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & t_p^{-1} & \\ & & & & & & & \ddots \\ & & & & & & & & t_1^{-1} \end{bmatrix} \subset \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \subset \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \subset \begin{bmatrix} * & & & & * \\ & \ddots & & & \\ & & \boxed{q-p} & & \\ & & & * & \\ 0 & & & & \ddots \ * \end{bmatrix} \end{array}$$

, quasi-split if $p = q$ or $p = q - 1$

~~Split iff $p = q = 1$.~~

Let $\Psi_G = (X^*(T), \Phi, X_*(T), \Phi^\vee)$ root datum of G
$\overset{\shortparallel}{\Psi}$

$$\underset{\uparrow \atop \text{Simple roots}}{\Delta} \subset \underset{\uparrow \atop \text{positive roots assoc. to } B}{\Phi^+} \subset \Phi$$

Since $\theta(T) = T \Rightarrow \theta \curvearrowright \Phi, \Phi^\vee$

we denote by $\theta_\Psi \curvearrowright \Psi_G$ the induced invol'n on Ψ.

<u>Thm</u> (Cartan, Araki, Springer, Satake, ...)

① There exits a subset $I \subset \Delta$ and an invol'n (might be trivial) $\tau : \Delta \to \Delta$

s.t. $\begin{cases} \theta(\alpha) = -w_I^\circ \circ \tau(\alpha), & \alpha \in \Delta \\ \tau(\alpha) = -w_I^\circ(\alpha), & \alpha \in I \end{cases}$ (here $w_I^\circ \in W$ is the longest elt in the parabolic $W_I \subset W$).

② The pair (G, θ) is determined by $(\Psi, \theta_\Psi, \tau, I \subset \Delta)$

<u>Rmk</u> ① $\alpha \in I \Rightarrow \theta(\alpha) = \alpha$ (but $\Leftarrow$ not true, e.g. $GL_{2n}/GL_n \times GL_n$)

② (G, θ) is quasi-split $\Leftrightarrow I = \phi$

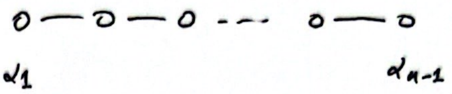
(resp. split) $\Leftrightarrow I = \phi$ and τ is trivial.

③ $(G, \theta = id) \Leftrightarrow I = \Delta$.

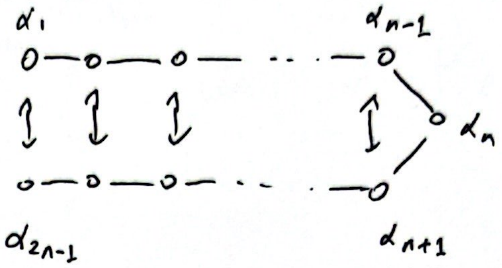
§ <u>Araki - Satake diagram</u>

Let Dyn be the Dynkin diagram of G. Given $(\Psi, \theta_\Psi, \tau, I \subset \Delta)$, we color black the vertices of I, and indicate the invol'n τ on the vertices $\Delta \setminus I$. The resulting diagram is called the Araki - Satake diagram.

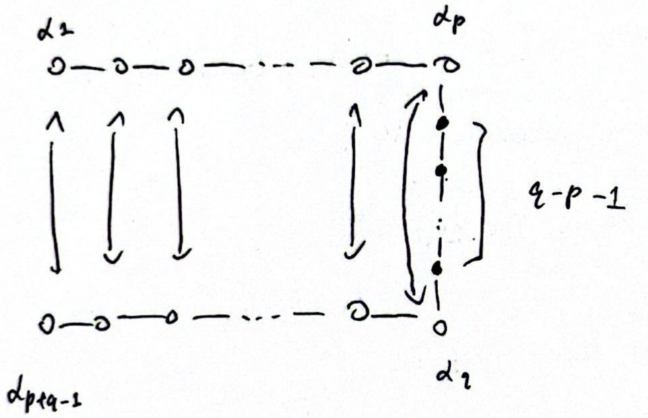
Ex. $X = GL_n / O_n$ split case.



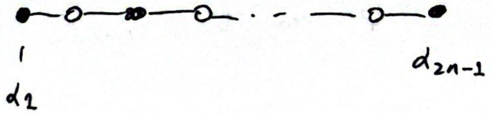
Ex. $GL_{2n} / GL_n \times GL_n$, quasi-split



Ex. $X = GL_{p+q} / GL_p \times GL_q$ $\quad p<q \quad , n = p+q$



Ex. $GL_{2n} \supset Sp_{2n}$



§ Dual gp of X.

$(G,\theta) \quad A \subset T \subset B \subset P$

$\rightsquigarrow \theta_\Psi \curvearrowright \Psi = (X^*(T), \Phi, X_*(T), \check{\Phi})$

$\theta_{\check{\Psi}} \curvearrowright \check{\Psi} = (X_*(T), \check{\Phi}, X^*(T), \Phi)$

$\theta \curvearrowright \check{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times \supset \check{T}_X = \{t \in \check{T} : \theta(t) = t^{-1}\}^0$

$r : X^*(\check{T}) \to X^*(\check{T}_X) \qquad r(\alpha) = \alpha|_{\check{T}_X} = \bar{\alpha}$

Def. $\check{\Phi}_X = r(\check{\Phi}) \setminus \{0\} \supset \check{\Delta}_X = r(\check{\Delta}) \setminus \{0\}$

Lemma $(\check{\Phi}_X \otimes_{\mathbb{Z}} \mathbb{R}, \check{\Delta}_X)$ is a possibly non-reduced root system w/ basis $\check{\Delta}_X$.

Let $\check{\Psi}_X = (X^*(\check{T}_X), \check{\Phi}_{X,red}, X_*(\check{T}_X), \Phi_{X,red})$ be the reduced root datum assoc. to $(\check{\Phi}_X \otimes \mathbb{R}, \check{\Delta}_X)$ by discarding the "short roots."

Def The dual gp of X is the reductive gp $\check{G}_X$ w/ root datum $\check{\Psi}_X$.

Thm. (Nadler, Takeda) There is an embedding

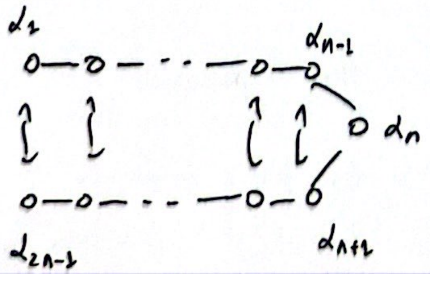
$$\begin{tikzcd} \check{G}_X \arrow[r, hook] & \check{G} \\ \check{T}_X \arrow[u, hook] \arrow[r] & \check{T} \arrow[u] \end{tikzcd}$$

Ex. (G,θ) split $\iff \check{G}_X = \check{G}$.

$(G, \theta = \text{triv}) \iff \check{G}_X = \{e\} \subset \check{G}$

Ex. $X = GL_n/O_n$ split $\iff \check{G}_X = GL_n \cong \check{G}$

Ex. $GL_{2n}/GL_n \times GL_n$, $\quad \check{T}_X = \{(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})\} \subset \check{T} = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}$



$\bar{\alpha}_1 = \bar{\alpha}_{2n-1} = \overline{e_1 - e_2}$

$\vdots$

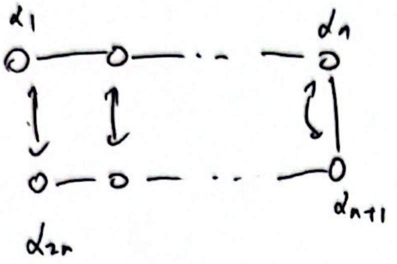
$\bar{\alpha}_{n-1} = \bar{\alpha}_{n+1} = \overline{e_{n-1} - e_n}$

$\bar{\alpha}_n = \overline{2e_n}$

$\to \check{\Psi}_X$ type o—o—⋯—o⇐o ($\bar{\alpha}_1 \; \bar{\alpha}_2 \; \dots \; \bar{\alpha}_{n-1} \; \bar{\alpha}_n$)

$\to \check{G}_X = Sp_{2n}$

Ex. $GL_{2n+1}/GL_{n+1}\times GL_n$, $\check{T}_X = \{(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1})\}$



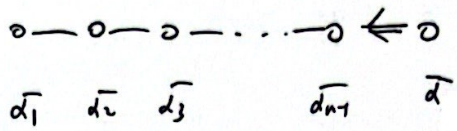
$\bar{\alpha_1} = \bar{\alpha_{2n}} = \overline{e_1 - e_2}$

$\vdots$

$\bar{\alpha}_{n-1} = \bar{\alpha}_{n+2} = \overline{e_{n-1} - e_n}$

$\bar{\alpha_n} = \bar{e_n}$ ← short root

$\bar{\alpha} = \overline{e_n - e_{n+2}} = 2\bar{e_n}$ ← long root



$\Rightarrow \check{G}_X = Sp_{2n}$

Ex. $X = GL_{2n}/Sp_{2n} \Rightarrow \check{G}_X = GL_n$

§ Embedding $\check{G}_X \hookrightarrow \check{G}$

Thm (Nadler) ① There exists a Levi subgp $\check{L}_1 \subset \check{G}$ and an invol'n $\check{\theta}_1 : \check{L}_1 \to \check{L}_1$ s.t. $\check{G}_X = (\check{L}_1)^{\check{\theta}_1, 0} \hookrightarrow \check{G}$

Lemma. Let $\check{L_0} = Z_{\check{G}}(2\rho_P)$ be the Levi subgp of centralizer of $2\rho_P$: sum of positive roots of the Levi $P \cap \theta(P) = Z_G(A)$. Let $\check{\Delta}_0 \subset \check{\Phi}_0$ be the set of roots & simple roots of $\check{L_0}$ w.r.t. $\check{B}_0 = \check{L_0} \cap \check{B}$

① $\check{L}_1 \subset \check{L_0}$. Moreover, $\check{L_0} = \check{L}_1 \iff \forall \check{\alpha} \in \check{\Delta}_0$, $\check{\alpha} - \theta(\check{\alpha}) \notin \check{\Phi}_0$

$\iff \theta_\varphi$ fixes a node in the component of Dyn of $\check{L_0}$ that it preserves

② In particular, if (G,θ) is quasi-split ($\check{L_0} = \check{G}$) and θ_φ fix a node in the component of Dyn that it preserves, then $\check{L_0} = \check{L_1} = \check{G}$ and $\check{G}_X = (\check{G})^{\check{\theta}, 0}$ is a sym sub gp.

Ex. $X = GL_{2n} / GL_n \times GL_n$, $\check{G} = \check{L}_1 = GL_{2n} \supset \check{G}_X = Sp_{2n}$

Ex. $X = GL_{2n+1} / GL_{n+1} \times GL_n$, $\check{G} = \check{L}_0 = GL_{2n+1} \supset \check{L}_1 = GL_{2n} \times GL_1 \supset \check{G}_X = Sp_{2n}$

Rmk $\check{L}_1 \neq \check{L}_0$ only appear in $GL_{p+q} / GL_p \times GL_q$, $p+q$ is odd.

§ Real groups

Let $\eta : G \to G$ be a conjugation, that is, an anti-holomorphic invol'n.

The fixed pt $G_{\mathbb{R}} = G^{\eta}$ is a real form. of G.

Ex $G = GL_n$, $\eta(g) = \bar{g}$, $G_{\mathbb{R}} = GL_{n,\mathbb{R}}$

Ex. $G = GL_n$, $\eta_c(g) = {}^t\bar{g}^{-1}$, $G_{\mathbb{R}} = U(n)$ $\leftarrow$ unitary gp, cpt real form

Thm ((Cartan) Let $\theta : G \to G$ be an invol'n. There exists a conjugation $\eta : G \to G$ s.t. $\eta_c = \theta \circ \eta = \eta \circ \theta$ and G^{η_c} is compact.

$(G, \theta)/\sim \;\longleftrightarrow\; (G, \eta)/\sim$ is a bijection.

$$\begin{tikzcd}[column sep=small]
 & GL_n = G \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
G_{\mathbb{R}} = GL_{n,\mathbb{R}} & O_n = K & U(n) \\
 & O_n(\mathbb{R}) = K_c \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

$\eta(g) = \bar{g}$

$\theta(g) = {}^t g^{-1}$

$\eta_c = \theta \circ \eta(g)$

$= ({}^t\bar{g}^{-1})$

Lecture 3. Rmk $I \subset \Delta$ is the simple roots of P $\Leftrightarrow$ $I = \phi$ iff $P = B$ is quasi-split

$\Leftrightarrow$ No black nodes in Araki - Satake diag.

Geometry of LX

Let X be an affine variety

$$\underset{\substack{\uparrow \\ \text{loop space}}}{LX} = X(\mathbb{C}((t))) \supset \underset{\substack{\uparrow \\ \text{arc space}}}{L^+X} = X(\mathbb{C}[[t]])$$

Ex $X = V$ vec. sp.

$$L^+V = V[[t]] = \lim_j \underbrace{V[[t]]/t^j V[[t]]}_{= L^+_j V} \leftarrow \text{finite dim vec. sp.}$$

$$\begin{array}{ccc} L^+_j V & \longrightarrow & L^+_{j'} V \\ \| & & \| \\ V[[t]]/t^j V[[t]] & \longrightarrow & V[[t]]/t^{j'} V[[t]] \end{array} \quad \text{if } j > j'$$

closed embeddings

$$LV = \operatorname*{colim}_{i \geq 0} \underset{\substack{\| \\ t^{-i}L^+V}}{L^i V} = \operatorname{colim}\big(\underset{\substack{\| \\ L^+V}}{L^0 V} \hookrightarrow \underset{\substack{\| \\ t^{-1}L^+V}}{L^1 V} \hookrightarrow \underset{\substack{\| \\ t^{-2}L^+V}}{L^2 V} \hookrightarrow \cdots\big)$$

Ex. (Group case) $X = G/\mathbb{C}$

$$LG \supset \underbrace{L^+G}, \quad L^+G = \lim_j L^+_j G = G\big(\mathbb{C}[[t]]/(t^{j+1})\big)$$

$$\begin{array}{ccc} LG & \longrightarrow & LG/L^+G = Gr = \operatorname*{colim}_{i \geq 0} Gr^i \\ \uparrow & & \uparrow \\ L^iG \rightarrow L^i_j G & \rightarrow & Gr^i \end{array}$$

$L^i_j G = LG/L^+G^j$

$$L^+G^j = \ker(L^+G \to L^+_j G)$$

$$L^iG = \lim_{j \geq 0} L^i_j G. \qquad LG = \operatorname{colim}_{i \geq 0} L^i G$$

Note that in those cases, the transition map in the limit $L^iG = \lim L^i_j G$,

$L^iV = \lim L^i_j V$ are smooth & affine.

<u>Thm</u> (Drinfeld) Assume X is smooth affine, then LX is "placid ind-scheme". That is, there is a presentation

$$LX \simeq \operatorname{colim}_{i \geq 0} L^iX = \operatorname{colim}\left(L^0X \to L^1X \to L^2X \to \cdots\right)$$

closed embeddings

étale (locally).

and $L^iX = \lim_{j \in J} L^i_j X$ where $L^i_j X$ are of f.t., and the transition map

$L^i_j X \to L^i_{j'} X$ are smooth affine.

Moreover, if $G \curvearrowright X$, one can find L^iX which is stable under the L^+G-action.

<u>Ex</u> $X = GL_n / O_n$. Let $P \subset \mathfrak{gl}_n$ be the space of $n \times n$ symmetric matrices.

$$\begin{array}{ccc} X \xrightarrow{\sim} & \overbrace{P^\circ}^{\text{invert. elts in } P} \subset & P \\ & \downarrow & \downarrow \det \\ & \mathbb{G}_m \subset & \mathbb{A}^1 \end{array}$$

$$LX = \{ \gamma \in LP : \det(\gamma) \in F^\times = \mathbb{C}((t))^\times \}$$

$$\coprod_{\ell \in \mathbb{Z}} \underbrace{LX_\ell}_{\text{Conn'd components}} = \{ \gamma \in LX : \operatorname{val}(\det(\gamma)) = \ell \}$$

$$\begin{array}{ccccc} L^iX_\ell & \to & L^iX & \to & t^{-i}L^+P \\ \downarrow & & \downarrow & & \downarrow \text{closed embedding} \\ LX_\ell & \to & LX & \to & LP \end{array}$$

$$LX_\ell = \operatorname{colim}_{i \geq 0} L^iX_\ell$$

Claim ① For any i, there exists $j_0 \gg 0$ s.t. $L^i X_e + t^j L^+ P = L^i X_e$, $j \geq j_0$.

② Let $L^i_j X_e = L^i X_e / t^j L^+ P$, $j \geq j_0$

Then $L^i X_e = \lim\limits_{j \geq j_0} L^i_j X_e$, $LX_e = \operatorname{colim} \lim L^i_j X_e$

$$\begin{tikzcd} L^+G \curvearrowright L^i X \arrow[d] & \\ L^+_j G \curvearrowright L^i_j X \end{tikzcd}$$

§ Spherical orbits on LX

$X = G/K$, $L^+G \curvearrowright LX$

We fix $A \subset T \subset B \subset P$. Recall $X \xrightarrow{\sim} (G^{\text{inv}\circ\theta})^\circ \subset G$
$x \mapsto x\theta(x)^{-1}$

$\cup$

A

$\Rightarrow X_\bullet(A)^+ \to LA \subset LX$
$(\lambda : \mathbb{G}_m \to A) \mapsto t^\lambda = \lambda(t)$

Thm Each L^+G-orbit in LX contains a unique elt of $X_\bullet(A)^+$.

Def We write $LX^\lambda = L^+G \cdot t^\lambda$ the corresponding orbit, $\lambda \in X_*(A)^+$.

Rmk L^+G-orbit are all ∞-dim'l.
$LX^0 = L^+X \leftarrow \infty$-dim.

The proof uses "wonderful compactification" of X. Assume G is adjoint. Then consider the Grass. $\mathrm{Grass}_d(\mathfrak{g})$ of $d = \dim(K)$ subspace in the adj. rep. $\mathfrak{g}$ of G

$G \curvearrowright \mathrm{Grass}_d(\mathfrak{g})$

$$X = G/K \xrightarrow{j} \mathrm{Grass}_d(\mathfrak{g}) \qquad g \longmapsto g(\mathrm{Lie}\,K)g^{-1}$$

We let $\bar{X}$ be the closure of $j(X)$, called the wonderful compactification of X

Ex. $X = PGL_2$. show $\bar{X} = \mathbb{P}^3$.

Let $\bar{A} \subset \bar{X}$ be the partial compactification char. by

$$\lim_{t \to 0} \lambda(t) \in \bar{A} \iff \lambda \in X_*(A)^+.$$

Let $P = LN$ ← Levi decomp. $L = P \cap \theta(P)$

<u>Fact</u>. The action map $N \times \bar{A} \hookrightarrow \bar{X}$ is an open embedding w/ non-empty intersection w/ $\bar{X} \setminus X$.

<u>Proof</u>. $X(F) \hookrightarrow \bar{X}(F) = \bar{X}(\mathcal{O}_F)$ ($\bar{X}$ is complete)

$$\gamma \longmapsto \bar{\gamma}$$

$$\begin{tikzcd} \mathrm{Spec}\, F \arrow[r] \arrow[d] & N \times \bar{A} \arrow[d, "\text{open}"] \\ \mathrm{Spec}\, \mathcal{O}_F \arrow[r, "\bar{\gamma}"'] \arrow[ur, dashed] & \bar{X} \end{tikzcd}$$

$\Rightarrow$ The fact and inf. lifting criterion of open embedding.

$\to$ We can assume $\bar{\gamma} \in \bar{A}(\mathcal{O}_F)$

Acting by $A(\mathcal{O}_F)$, we can assume $\bar{\gamma}$ rep. by $X_*(A)^+$

$$X_*(A) = A(F)/A(\mathcal{O}_F) \hookleftarrow \bar{A}(\mathcal{O}_F)/A(\mathcal{O}_F) \ni \bar{\gamma}$$
$$\bar{A}(\mathcal{O}_F)/A(\mathcal{O}_F) = X_*(A)^+$$

<u>Uniqueness</u>:

$$\begin{tikzcd} LX \arrow[r, hook] & LG \\ LX^\lambda \arrow[u, phantom, "\cup"] \arrow[r] & LG^\lambda = L^+G\, t^\lambda\, L^+G \arrow[u, phantom, "\cup"] \end{tikzcd}$$

§ Orbit closure

Let $\overline{LX^\lambda}$ be the orbit closure. of LX^λ. $\lambda \in X_\bullet(A)^+$

Consider the partial order $\mu \leq \lambda$ iff $\lambda - \mu \in \mathbb{Z}_{\geq 0} \check{\Phi}^+$.

Thm $\overline{LX^\lambda} = \coprod_{\mu \leq \lambda} LX^\mu$.

Sketch proof "$\subset$"
$$\begin{tikzcd}
LX \arrow[r, "i"] & LG \\
\coprod_{\mu\leq\lambda} LX^\mu \arrow[u] \arrow[r] & \overline{LG^\lambda} = \coprod_{\mu\leq\lambda} LG^\mu \arrow[u, hook, "\leftarrow \text{closed}"'] \\
LX^\lambda \arrow[u] \arrow[r] & LG^\lambda \arrow[u]
\end{tikzcd}$$

$\Rightarrow \overline{LX^\lambda} \subset \coprod_{\mu \leq \lambda} LX^\mu$

"$\supset$" Construct a curve C w/
$$\begin{tikzcd}
\{y\} \arrow[r] \arrow[d] & C \arrow[d] & C \setminus \{y\} \arrow[l] \arrow[d] \\
t^\mu \arrow[r] & \overline{LX^\lambda} & LX^\lambda \arrow[l]
\end{tikzcd}$$
$\leftarrow$ small rank cases

§ Relative Satake cat.

$LX = \mathrm{colim}\, L^i X$, $L^i X = \lim L^i_j X$ w/ smooth affine map $p: L^i_j X \to L^i_{j'} X$, $j > j'$

$p^{!*}: D^b_c(L^i_{j'} X) \to D^b_c(L^i_j X)$

$\quad \shortparallel$

$p^*[\dim p]$

$\quad \shortparallel$

$p^! (-\dim p)$

Define $D(L^i X) = \mathrm{colim}^{!*}\, D(L^i_j X)$

$\mathrm{Perv}(L^i X) = \mathrm{colim}^{!*}\, \mathrm{Perv}(L^i_j X)$ ext. by 0

Finally. we define $D(LX) = \mathrm{colim}\, (D(L^0 X) \to D(L^1 X) \to \cdots)$

$\qquad\qquad\qquad\quad \cup \qquad\qquad\qquad\qquad \cup$

$\mathrm{Perv}(LX) = \mathrm{colim}\, (\mathrm{Perv}(L^0 X) \to \mathrm{Perv}(L^1 X) \to \cdots)$

Rem ...

Assume $G \curvearrowright X$, we can define $D(L^+G\backslash LX) = \lim^{!*}$

$$\left(D(LX) \rightrightarrows D(LX^{[1]}) \substack{\rightarrow\\\rightarrow\\\rightarrow} D(LX^{[2]}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

$$L^+G\backslash LX = \operatorname{colim}\left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} LX^{[2]} \substack{\rightarrow\\\rightarrow\\\rightarrow} LX^{[1]} \rightrightarrows LX \right)$$

$$LX^{[n]} = LX \underset{L^+G\backslash LX}{\times} LX \times \cdots \underset{L^+G\backslash LX}{\times} LX \qquad LX \to L^+G\backslash LX$$

$\mathrm{For} : D(L^+G\backslash LX) \longrightarrow D(LX)$

$$\mathrm{Perv}(L^+G\backslash LX) \simeq \{ M \in D(L^+G\backslash LX) : \mathrm{For}(M) \in \mathrm{Perv}(LX) \}$$

§ IC-complexes

Assume $X = G/K$ is symmetric variety.

$$DSat_X = D(L^+G\backslash LX) \supset \mathrm{Perv}(L^+G\backslash LX) = Sat_X$$

Lemma. There is a placid presentation $LX \simeq \operatorname{colim} L^iX$ s.t.

① L^iX is L^+G-stable and contains finitely many L^+G-orbits

② For any $\lambda \in X_\bullet(A)^+$, we have $LX^\lambda \overset{j}{\hookrightarrow} \overline{LX^\lambda} \overset{s}{\hookrightarrow} L^iX$ for some i

finitely presented

The Lemma implies the singularities of $\overline{LX^\lambda}$ are in fact finite dim.

$$\begin{array}{ccccc} LX^\lambda & \to & \overline{LX^\lambda} & \to & L^iX \\ \downarrow & & \downarrow & & \downarrow \\ (LX^\lambda)_j & \to & (\overline{LX^\lambda})_j & \to & L^i_jX \end{array} \leftarrow \text{finite type.}$$

③ Let $\mathrm{Stab}_G(t^\lambda)$ be the stabilizer of t^λ under the action $g\cdot t^\lambda = g t^\lambda \theta(g)^{-1}$

Then $\mathrm{Rep}(\pi_0(\mathrm{Stab}_G(t^\lambda))) \simeq \mathrm{Perv}(L^+G\backslash LX^\lambda)$

$$\chi \longleftrightarrow \mathcal{L}_\chi$$

$IC_{\chi,\lambda} = j_{!*}(\mathcal{L}_\chi) \in \mathrm{Sat}_X$.

Moreover, $\{IC_{\chi,\lambda}\}$ are the irred. obj. in Sat_X.

---

§ Transversal slices $(k = k^{\circ})$

Affine Grass slice for Gr.

$$W^0_{\le\lambda} = \underbrace{L^{<0}G}_{=\ker(G[t^{-1}]\to G)} \cap \overline{LG^\lambda} \subset LG, \quad \lambda \in X_*(T)^+$$

Facts. The action map
$$m: W^0_{\le\lambda} \to \overline{\mathrm{Gr}^\lambda}, \quad w \longmapsto w t^\lambda$$
is an open embedding and non-empty intersection w/ Gr^μ, $\mu<\lambda$.

Assume $\lambda \in X_*(A)^+$, we define

$$\begin{tikzcd}
(W^0_{\le\lambda})^{\mathrm{inv}\circ\theta,\circ} = S^0_{\le\lambda} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \overline{LX^\lambda} \arrow[r] \arrow[d] & LX \arrow[d] \\
W^0_{\le\lambda} \arrow[r] & \overline{LG^\lambda} \arrow[r] \arrow[loop below, "\mathrm{inv}\circ\theta"] & LG
\end{tikzcd}$$

Thm (Yi-C.) ① The action map $L^+G \times S^0_{\le\lambda} \to \overline{LX^\lambda}$ is pro-smooth w/ open image, and the image has non-empty intersection w/ LX^μ, $\mu<\lambda$

② $\dim S^\circ_{\le\lambda} = \langle \lambda, \rho \rangle$

③ $S^\circ_{\le\lambda} \subset W^\circ_{\le\lambda}$ is a Lagrangian

Ex. $X = SL_n/SO_n$, $\lambda = (n-1,-1,\dots,-1)$

symmetric nilp. matrices

$$\begin{array}{ccccc} \mathcal{N}_n \cap \mathfrak{p} & \simeq & S^\circ_{\le\lambda} & \longrightarrow & \overline{L_X^\lambda} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N}_n & \simeq & W^\circ_{\le\lambda} & \longrightarrow & \overline{Gr^\lambda} \end{array}$$

$\mathrm{inv}\circ\theta(g) = \mathrm{inv}\,({}^t g^{-1}) = g^t$

$$M \longmapsto (1 - t^{-1}M)L_0$$

$n=2$, $\mathcal{N}_2 \cap \mathfrak{p} = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} : a^2 + b^2 = 0 \right\}$

$IC(L_X^\lambda)$ "$\simeq$" $IC(\mathcal{N}_n \cap \mathfrak{p})$

Thm (Yi-C.) $\mathcal{H}^n(IC_{X,\lambda}) = 0$ if $n + \langle \lambda, \rho \rangle$ is odd.

Def Relative Kostka–Foulkes polys for X:

$$P_{X',\lambda',X,\lambda}(q) = \sum_{i \in \mathbb{Z}} \left[\mathcal{L}_{X',\lambda'}, \mathcal{H}^{i-\langle\lambda,\rho\rangle}(IC_{X,\lambda})\big|_{L_X^{\lambda'}} \right] q^{i/2}$$

Cor. $P_{X',\lambda',X,\lambda} \in \mathbb{Z}_{\ge 0}[q]$ are poly. in q, w/ non-negative coefficients.

Moreover, there is "KLV" type algorithm to compute them.

Ex. $X = GL_{2n}/Sp_{2n}$. $X_\bullet(A)^+ = \{\lambda_1 \ge \dots \ge \lambda_n\} \subset \mathbb{Z}^n$

$P_{\lambda',\lambda} = P_{0,\lambda',0,\lambda}$ = KF poly. of $\check{G}_X = GL_n$, but w/ $q \to q^2$.

Lecture 4. Construction of Tannakian dual gp $\check{G}_{X,\mathrm{geom}}$ of X

$$X \rightsquigarrow DSat_X = D(L^+G \backslash LX)$$
$$\cup$$
$$Sat_X = Perv(L^+G \backslash LX)$$

§ Hecke action

There is an action of $DSat_G$ on $DSat_X$

$$DSat_G = D(L^+G \backslash LG / L^+G)$$

$$Gr \times LX \xleftarrow{q \times id} LG \times LX \xrightarrow{p} LG \overset{L^+G}{\times} LX \xrightarrow{m} LX$$

$\mathcal{F} \in DSat_G$

$\mathcal{M} \in DSat_X$

$$\mathcal{F} * \mathcal{M} = m_!(\mathcal{F} \tilde{\boxtimes} \mathcal{M}), \quad (q \times id)^*(\mathcal{F} \boxtimes \mathcal{M}) = p^*(\mathcal{F} \tilde{\boxtimes} \mathcal{M})$$

Rmk. Unlike the case of $DSat_G$, the Hecke action on $DSat_X$ is not t-exact.

Ex. $(G, \theta = id)$. $\rightsquigarrow$ $K = G$, $X = pt$

$Perv(L^+G \backslash LX) \simeq Vect^{\heartsuit}$

$\mathcal{F} * \mathcal{M} = H^*(Gr, \mathcal{F})$

Prop. The Hecke action $DSat_G \curvearrowright DSat_X$ is t-exact, i.e., $\mathcal{F} * \mathcal{M} \in Sat_X$ for $\mathcal{F} \in Sat_G$, $\mathcal{M} \in Sat_X$ $\Leftrightarrow$ X is quasi-split.

Ex. Group case $X = H \times H / H$, or $X = GL_{2n} / GL_n \times GL_n$ ...

semismallness of m

Pf. Use real gp...

Rmk. In the case when X is split. [SW] they prove a similar t-exactness results in the setting of "Jacquet functor".

§ Fusion product

For any $y \in \mathbb{C}$, we let $D_{\pm y}$ be the formal disk of $\{\pm y\} \subset \mathbb{C}$, and $\mathring{D}_{\pm y} = D_{\pm y} \setminus \{\pm y\}$ be formal punctured disk of $\pm y$.

Consider $LX^{(2)} = \left\{ (y, \nu) : \begin{array}{l} y \in \mathbb{C} \\ \nu : \mathring{D}_{\pm y} \to X \end{array} \right\} \xrightarrow{f} \mathbb{C}$

$$\begin{tikzcd} LX \times LX \times \mathbb{C}^\times \arrow[r, "j"] \arrow[d] & LX^{(2)} \arrow[d, "f"] & LX \arrow[l, "i"'] \arrow[d] \\ \mathbb{C}^\times \arrow[r] & \mathbb{C} & \{0\} \arrow[l] \end{tikzcd}$$

For any $\mathcal{F}_1, \mathcal{F}_2 \in DSat_X$, we define the fusion product $\mathcal{F}_1 \underset{f}{*} \mathcal{F}_2$ as the (unipotent) nearby cycles: $\mathcal{F}_1 \underset{f}{*} \mathcal{F}_2 := \Psi_f \left(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1] \right)$

$$= \underset{n \geq 1}{\operatorname{colim}}\ i^* j_* \left(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{L}_n \right)$$

where $\mathcal{L}_n$ is the unipotent local system on $\mathbb{C}^\times$ s.t. the generator $\pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$ acts on $\mathcal{L}_n = \mathbb{C}[x]/x^n$ by mult. of $(1-x)$.

$\mathcal{L}_1 \hookrightarrow \mathcal{L}_2 \hookrightarrow \mathcal{L}_3 \hookrightarrow \cdots$

$k[x]/x \xrightarrow{x} k[x]/x^2 \to \cdots$

Rmk. Need to argue that i^* and j_* are defined.

Lemma. The fusion product $\underset{f}{*}$ is t-exact.

$\rightsquigarrow (Sat_X, \underset{f}{*})$

§ Braidings

We have similar swap symmetry: $sw^{(2)} : LX^{(2)} \longrightarrow LX^{(2)}$

$$(y, \nu) \mapsto (-y, \nu)$$

$$\begin{tikzcd} LX \times LX \times \mathbb{C}^\times \arrow[r] \arrow[d, "sw \times (-1)"'] & LX^{(2)} \arrow[d, "sw^{(2)}", "\wr"'] & LX \arrow[l, hook'] \arrow[d, "\text{rotation } t \to -t"] \\ LX \times LX \times \mathbb{C}^\times \arrow[r] & LX^{(2)} & LX \arrow[l, hook'] \end{tikzcd}$$

$C_{\mathcal{F}_1, \mathcal{F}_2, n}: \quad i^* j_* (\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{L}_n) \simeq i^* j_* (\mathcal{F}_2 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{L}_n)$

$$\Rightarrow C_{\mathcal{F}_1, \mathcal{F}_2}: \quad \begin{array}{ccc} \mathcal{F}_1 \underset{t}{*} \mathcal{F}_2 & = \operatorname{colim}_n i^* j_* & (\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{L}_n) \\ & & \wr\!\!\! \downarrow s \\ \mathcal{F}_2 \underset{t}{*} \mathcal{F}_1 & = \operatorname{colim}_n i^* j_* & (\mathcal{F}_2 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{L}_n) \end{array}$$

Thm. $(\mathrm{Sat}_X, *_t, C = \{C_{\mathcal{F}_1, \mathcal{F}_2}\})$ is a braided tensor cat. w/ a fiber functor

$F: \mathrm{Sat}_X \longrightarrow \mathrm{sVect}_{\mathbb{C}}$ super vector spaces

$v_i \otimes v_j \longmapsto (-1)^{ij} v_j \otimes v_i$

Pf • $*_t$ is associative + ...

• construct the fiber functor F

Use real group

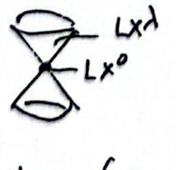
Rmks ① The fiber functor is not taking $H^*(LX, -)$

② Sat_X is not semisimple in general.

$X = SL_2/SO_2$, Let $\lambda \in X_*(A)^+$ s.t. $\langle \lambda, \rho \rangle = 1$.

$$j: LX^\lambda \hookrightarrow \overline{LX^\lambda} = LX^\lambda \amalg \underset{\substack{\| \\ L^+X}}{LX^0}$$

Claim $j_! \mathbb{C}[1] \in \mathrm{Sat}_X$ is a non-semisimple object.

(locally, )

③ The braiding $C_{\mathcal{F}_1, \mathcal{F}_2}$ might not be symmetric. that is, $C_{\mathcal{F}_2, \mathcal{F}_1} \circ C_{\mathcal{F}_1, \mathcal{F}_2} \neq \mathrm{id}$.
Related to the fact $\mathcal{F}_1 \underset{t}{*} \mathcal{F}_2 \neq i^*[-1] j_{!*}(-)$

(≠) $j_{!*}(-)$ is not ULA w.r.t. $LX^{(2)} \to \mathbb{C}$

$\Rightarrow$ monodromy of ψ_f is not trivial.

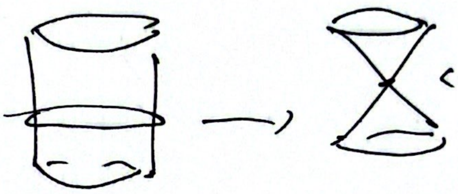
$\mathcal{L}_\lambda \circlearrowleft$ monodromy

<u>Ex</u> $X = GL_2/O_2$. $\lambda = (1,0) \in X_*(A)^+$, $A = T$

$$\begin{tikzcd} LX^\lambda \times LX^\lambda \times \mathbb{C}^\times \arrow[r] \arrow[d, "f"] & LX^{(2)} \arrow[d] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \end{tikzcd}$$

topologically look like

$$\begin{tikzcd} \mathbb{C}^\times \times \mathbb{C}^\times \arrow[r] \arrow[d] & \mathbb{C}^2 \arrow[d, "f"] & (x,y) \arrow[d, mapsto] \\ \mathbb{C}^\times \arrow[r] & \mathbb{C} & xy \end{tikzcd}$$


<- has non trivial vanishing cycle

$\psi_f(\mathbb{C}[-])$ has <u>non-trivial monodromy</u>.

④ $Sat_X \simeq sRep\left(\begin{array}{l}\text{quantum super gp}\\ \text{at a root of unity}\end{array}\right)$

§ Gaitsgory–Nadler cat.

<u>Def.</u> Let $Sat^\circ_X \subset Sat_X$ be the full subcat. w/ objects isom. to direct sum of perverse sheaves appearing in $IC_V * IC_{L^+X = L^\circ X}$, $V \in Rep\,\check{G}$.

We have perverse Hecke action $\overset{p}{*} : Rep\,\check{G} \times Sat_X \to Sat_X$

$$(\mathcal{F}, M) \longmapsto \bigoplus_{i \in \mathbb{Z}} {}^pH^i(\mathcal{F} * M)$$

$$\overset{p}{*} : Rep\,\check{G} \times Sat^\circ_X \to Sat^\circ_X$$

Lemma ① $\mathcal{F}_1, \mathcal{F}_2 \in Sat^{\circ}_X \Rightarrow \mathcal{F}_1 \ast_f \mathcal{F}_2 \in Sat^{\circ}_X$

② $\mathcal{F}_1 \ast_f \mathcal{F}_2 \simeq i^*[-1]\left(\bar{j}_{!*}\left(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1]\right)\right)$

Pf ① follows from i) $IC_{L^+X} \ast_f IC_{L^+X} \simeq IC_{L^+X}$

ii) $(IC_V * IC_{L^+X}) * (IC_W * IC_{L^+X}) \simeq IC_{V\otimes W} * IC_{L^+X}$

$$\begin{array}{c} L^+X^{(2)} \\ \downarrow \\ \mathbb{C} \end{array}$$

② follows from $L^+X^{(2)} \to \mathbb{C}$ is pro-smooth

$Gr^2 \times \mathbb{C}^\times \xhookrightarrow{\bar{j}} Gr^{(2)}$ $\bar{j}_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}[1])$ is ULA.

Cor. $c^2 = Id$ on Sat°_X and $(Sat^{\circ}_X, \ast_f)$ is a symmetric tensor semisimple abelian cat.

§ Central functor

We have a monoidal functor $F: Rep\,\check{G} \to Sat^{\circ}_X$

$V \mapsto IC_V \ast_f IC_{L^+X}$

Def. Let A be a monoidal cat., and $\mathcal{B}$ be a symmetric monoidal cat.,

A central functor $F: \mathcal{B} \to A$ is a monoidal functor together w/ functorial isom. $(*)$ $F(X)\otimes Y \xrightarrow{\sim} Y \otimes F(X)$, $X \in \mathcal{B}$, $Y \in A$ s.t.

① $F(X)\otimes F(Y') \simeq F(X\otimes X') \simeq F(X'\otimes X) \simeq F(X')\otimes F(X)$

② $F(X)\otimes Y_1 \otimes Y_2 \simeq Y_1 \otimes F(X) \otimes Y_2 \simeq Y_1 \otimes Y_2 \otimes F(X)$

③ $F(X_1 \otimes X_2) \otimes Y \simeq F(X_1) \otimes F(X_2) \otimes Y \simeq F(X_1) \otimes Y \otimes F(X_2)$

$\simeq Y \otimes F(X_1) \otimes F(X_2) \simeq Y \otimes F(X_1 \otimes X_2)$

Rmk. Let $Z(A) = \{ (X, \sigma = \{\sigma_Y\}) : \sigma_Y : X \otimes Y \xrightarrow{\sim} Y \otimes X \}$ be the Drinfeld center of A, which is a braided monoidal cat. Then F is central iff

$$\begin{tikzcd} B \arrow[r, "\tilde{F}"] \arrow[dr, "F"'] \arrow[r, bend left, "\text{braided tensor functor}"] & Z(A) \arrow[d, "\text{For}"] \\ & A \end{tikzcd}$$

Lemma (Bezrukavnikov) Consider $B = \operatorname{Rep} \check{G}$, and A is an abelian monoidal cat. ($\dim_A \operatorname{Hom}(X, Y) < \infty$) w/ exact $\otimes$-product. Let $F : \operatorname{Rep}\check{G} \to A$ be a central functor. Assume every object of A is isom. to subqt of $F(V)$, $V \in \operatorname{Rep} \check{G}$, then $\exists\, \check{H} \subset \check{G}$ s.t.

$A \simeq \operatorname{Rep} \check{H}$ and
$$\begin{tikzcd} F : \operatorname{Rep}\check{G} \arrow[r] \arrow[dr, "\text{Res}"'] & A \arrow[d, "\wr"] \\ & \operatorname{Rep}\check{H} \end{tikzcd}.$$

Thm $F : \operatorname{Rep}\check{G} \to \operatorname{Sat}^\circ_X$ is a central functor. Thus there's a subgp $\check{G}_{X,\mathrm{geom}} \subset \check{G}$

s.t.
$$\begin{tikzcd} \operatorname{Rep}\check{G} \arrow[r, "\text{res.}"] \arrow[dr, "F"'] & \operatorname{Rep}\check{G}_{X,\mathrm{geom}} \arrow[d, "\wr"] \\ & \operatorname{Sat}^\circ_X \end{tikzcd}.$$

Def The gp $\check{G}_{X,\mathrm{geom}}$ is called Tannakian dual gp of X. (GN - dual gp)

Rmk. ① $\check{G}_{X,\mathrm{geom}}$ is reductive $\Leftrightarrow$ $\operatorname{Sat}^\circ_X$ is semisimple.

Pro...

② should work for any smooth G-affine varieties.

Recall irr. objects in $\mathrm{Sat}_X \supset \mathrm{Sat}^\circ_X$ are parametrized by $IC_{\chi,\lambda}$, $\lambda \in X_\bullet(A)^+$, $\chi \in \mathrm{Rep}(\pi_0(\mathrm{Stab}_G(t^\lambda)))$

Ex. $X = SL_2/SO_2$, $X_\bullet(A)^+ = \{0,1,2,3,\dots\} = \mathbb{Z}_{\geq 0}$

λ 0   1   2   3   4 $\dots$

$g \cdot t^\lambda = g\, t^\lambda \theta(g)^{-1}$

$\mathrm{Stab}_{SL_2}(t^\lambda)$ SO_2 ± 1 ± 1 ± 1

Fact $\mathrm{Sat}^\circ_{SL_2/SO_2}$ has irr. objects

$\{IC_{0,0},\ IC_{0,2},\ IC_{0,4},\dots\}$

$IC_{0,\text{even}}$

$\mathrm{Rep}(PGL_2) \simeq \mathrm{Sat}^\circ_{SL_2/SO_2} \subset \mathrm{Sat}_{SL_2/SO_2}$

Lecture 5. $\check{G}_{X,\text{geom}} \simeq \check{G}_X$

Thm. There is tensor equiv. $(\mathrm{Sat}^\circ_X, *) \simeq (\mathrm{Rep}(\check{G}_X), \otimes)$

$$\begin{array}{ccc} (\mathrm{Sat}^\circ_X, *) & \simeq & (\mathrm{Rep}(\check{G}_X), \otimes) \\ \wr\!\wr & \nearrow\!\!\!\!\!\simeq & \\ (\mathrm{Rep}\,\check{G}_{X,\text{geom}}, \otimes) & & \end{array}$$

$\Rightarrow \check{G}_{X,\text{geom}} \simeq \check{G}_X$.

Two approaches

(Nearby cycle constructions)

[GN]: Vinberg degeneration of X to horospherical var.

(Identification of $\psi_{\check{G}_\text{geom}} = \psi_G$ is still incomplete using this approach.)

[CN]: Real gp: Degeneration of $\mathrm{Gr}_G \rightsquigarrow \mathrm{Gr}_{\mathbb{R}}$ real affine grass.

$$\begin{array}{ccl} \mathrm{Sat}_X^\circ & \subset & \mathrm{Perv}(L^+G\backslash LX) = \mathrm{Sat}_X \\ \| & & \quad\uparrow \\ \mathrm{Rep}\,\check{G} * IC_{L^+X} & & \\ X^\bullet(\check{T}_X)^+ & \hookrightarrow & X_\bullet(A)^+ \end{array}$$

$$\tau : T \longrightarrow T \qquad t \longmapsto t\theta(t)^{-1}$$

$$\Rightarrow \tau : T \to A \subset T \Rightarrow \check{T} \to \check{A} \xrightarrow{\text{surj. w/ finite kernel}} \check{T}_X \subset \check{T}$$

$$\begin{array}{ccccc} \Rightarrow X^\bullet(\check{T}_X) & \longrightarrow & X^\bullet(\check{A}) & = & X_\bullet(A) \\ \cup & & \cup & & \\ X^\bullet(\check{T}_X)^+ & \longrightarrow & X_\bullet(A)^+ & & \end{array}$$

Ex. (G,θ) is split, $T = A$

$$\tau : T \to A = T \qquad t \mapsto t^2 = t\theta(t)^{-1}, \quad \theta(t) = t^{-1}$$

$$\Rightarrow X^\bullet(\check{T}_X) \hookrightarrow X_\bullet(A) = X^\bullet(\check{A}) \qquad \lambda \longmapsto 2\cdot\lambda$$

Ex. $X = GL_{2n}/Sp_{2n}$. $\quad X^\bullet(\check{T}_X) \xrightarrow{\sim} X_\bullet(A)$

Fact. $IC_{x,\lambda} \in \mathrm{Sat}_X^\circ \Rightarrow$ $x = 0$, $\lambda \in X^\bullet(\check{T}_X) \subset X_\bullet(A)^+$

§. Construction of the fiber functor

More precisely, the embedding $\check{T}_X \hookrightarrow \check{G}_{X,\mathrm{geom}}$.

<u>Vinberg-degeneration</u>

$K \subset G$ (any subgp.) $\qquad \lambda : \mathbb{G}_m \to T$. consider $\tilde{K}^\circ \subset G \times \mathbb{C}^\times$

$$\tilde{K}^\circ = \{(g,t) : g \in \lambda(t) K \lambda(t)^{-1}\}$$

$$\mathbb{C}^\times = \mathbb{C}^\times$$

Let
$$\begin{tikzcd} \widetilde{K} \arrow[r, phantom, "\subset"] \arrow[d] & G\times\mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \end{tikzcd}$$
be the closure of $\widetilde{K}^{\circ}$ in $G\times\mathbb{C}$.

Fact. $\widetilde{K}\to\mathbb{C}$ is a smooth family of gps, called the Vinberg degeneration of K along the "direction" λ.

Consider the case $K=G^{\theta}\subset G$. Let $\lambda:\mathbb{G}_m\to A$ s.t. $\langle\lambda,\alpha\rangle<0$ for any

$\mathfrak{g}_\alpha\in \mathrm{Lie}\,N$ $\quad(P=LN)$, $\quad P$ minimal parabolic, $\quad L = MA$, $\quad M=Z_K(A)$

Fact:
$$\begin{tikzcd} K\times\mathbb{C}^\times \arrow[r, hook] \arrow[d] & \widetilde{K} \arrow[d] & MN \arrow[l] \arrow[d] \\ \mathbb{C}^\times \arrow[r] & \mathbb{C} & \{0\} \arrow[l] \end{tikzcd}$$

Ex. (Group case) $K=G\hookrightarrow G\times G$
$$\begin{tikzcd} G\times\mathbb{C}^\times \arrow[r, hook] \arrow[d] & G \arrow[d] & (N\times\bar{N})T_\Delta \arrow[l, hook'] \arrow[d] \\ \mathbb{C}^\times \arrow[r] & \mathbb{C} & \{0\} \arrow[l, hook'] \end{tikzcd}$$

Def. $\widetilde{X}=G\times\mathbb{C}/\widetilde{K}\to\mathbb{C}$ is called Vinberg deg. of X to G/MN.
$$\begin{tikzcd} X\times\mathbb{C}^\times \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] & G/MN \arrow[l] \arrow[d] \\ \mathbb{C}^\times \arrow[r] & \mathbb{C} & \{0\} \arrow[l] \end{tikzcd}$$
horospherical variety

$X_0=G/MN$

$$L\widetilde{X}=\{(y,\gamma)\ :\ y\in\mathbb{C},\ \gamma:\mathring{D}_x\to X_y\}$$
$\downarrow$
$\mathbb{C}$

X_y: fiber of $\widetilde{X}$ along y.

$$\begin{tikzcd} LX\times\mathbb{C}^\times \arrow[r] \arrow[d] & L\widetilde{X} \arrow[d] & LX_0 \arrow[l] \arrow[d] \\ \mathbb{C}^\times \arrow[r, hook] & \mathbb{C} & \{0\} \arrow[l] \end{tikzcd}$$

<u>Thm</u> (GN) ① $DSat_{X_0} = D(L^+G \backslash LX_0) \circlearrowleft DSat_G$

② $Sat_{X_0} = Perv(L^+G \backslash LX_0)$

$\cup \qquad \cup$

$Sat^{\circ}_{X_0} = Rep\, \check{G} \times^P \mathbb{C}_{L^+X_0}[--]$

③ $Sat^{\circ}_{X_0} \simeq Rep\, \check{T}_X$

<u>Rmk</u> ① $X_0 = G/MN$ is not affine. Then one needs to consider affine closure

$\overline{X_0} = \overline{G/MN}$ ← in general not smooth

Needs <u>global approach</u> to define and study $DSat_{X_0} \supset Sat_{X_0} \supset \dots$

(moduli space of quasi-maps)

④ The nearby cycle functor $\Psi_f : Sat_X \to Sat_{X_0}$ gives rise to a fiber functor

$$\begin{array}{ccc} \Psi_f : Sat_X & \longrightarrow & Sat_{X_0} \\ \cup & & \cup \\ Sat^{\circ}_X & \longrightarrow & Sat^{\circ}_{X_0} \\ \wr\wr & & \wr\wr \\ F : Rep\, \check{G}_{X,geom} & \longrightarrow & Rep\, \check{T}_X \end{array}$$

<u>Rmk.</u> $F = \Psi_f$ is very difficult to compute.

No MV-cycle thm for X at the moment

$(G, \theta) \qquad\qquad (G, \eta) \qquad\qquad \theta(g) = {}^t g^{-1} \qquad \eta(g) = \bar{g}$

$[\, X = SL_n/SO_n \longleftrightarrow SL_{n,\mathbb{R}} \,]$

$X \longleftrightarrow G_{\mathbb{R}}$

$LX \longleftrightarrow LG_{\mathbb{R}} = G_{\mathbb{R}}(\mathbb{R}((t)))$

$\cup$

$L^+G_{\mathbb{R}} = G_{\mathbb{R}}(\mathbb{R}[[t]])$

$Sat_X \qquad DSat_{\mathbb{R}} \doteq D(L^+G_{\mathbb{R}} \backslash LG_{\mathbb{R}} / L^+G_{\mathbb{R}}) = D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$

↑ real affine grass.

§. Real affine grass.

Facts ① Each $L^+G_{\mathbb{R}}$-orbit on $Gr_{\mathbb{R}}$ contains a unique element t^λ.

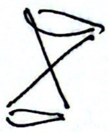
$$\lambda \in \underline{X_\bullet(A)^+} \qquad \left(t^\lambda = \lambda(t) \in (LA)^\eta \subset \underset{\substack{\shortparallel \\ Gr_{\mathbb{R}}}}{Gr^\eta} \right)$$

② $$\dim_{\mathbb{R}} \big(\underset{\substack{\shortparallel \\ L^+G_{\mathbb{R}}\, t^\lambda}}{Gr^\lambda_{\mathbb{R}}} \big) = \langle \lambda, 2\rho \rangle$$

③ $$Gr_{\mathbb{R}} = \mathrm{colim}\left(Gr^0_{\mathbb{R}} \hookrightarrow Gr^1_{\mathbb{R}} \hookrightarrow Gr^2_{\mathbb{R}} \hookrightarrow \cdots \right)$$

Ex $G_{\mathbb{R}} = SL_n(\mathbb{R})$, $\lambda = (n-1, -1, \dots, -1)$

$$\begin{tikzcd}
\mathcal{N}_n(\mathbb{R}) \arrow[r, hook, "\text{open}"'] \arrow[d] & \overline{Gr^\lambda_{\mathbb{R}}} \arrow[d] \\
W^\circ_{\le\lambda} = \mathcal{N}_n \arrow[r, "\text{open}"'] & \overline{Gr^\lambda}
\end{tikzcd}$$

$n=2$  $= \mathcal{N}_2(\mathbb{R})$

Have similar convolution product $*$ and fusion product on $DSat_{\mathbb{R}} = D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$

using:

$$Gr_{\mathbb{R}} \times Gr_{\mathbb{R}} \longleftarrow LG_{\mathbb{R}} \times Gr_{\mathbb{R}} \longrightarrow LG_{\mathbb{R}} \overset{L^+G_{\mathbb{R}}}{\times} Gr_{\mathbb{R}} \xrightarrow{m} Gr_{\mathbb{R}}$$

$$\begin{tikzcd}
Gr^2_{\mathbb{R}} \times \mathbb{R}_{>0} \arrow[r] \arrow[d] & Gr^2_{\mathbb{R}} \times \mathbb{R}^\times \arrow[r, "j"] \arrow[d] & Gr^{(2)}_{\mathbb{R}} \arrow[d] & Gr_{\mathbb{R}} \arrow[l, "i"'] \arrow[d] \\
\mathbb{R}_{>0} \arrow[r] & \mathbb{R}^\times \arrow[r] & \mathbb{R} & \{0\} \arrow[l]
\end{tikzcd}$$

$$\mathcal{F}_1 \underset{f}{*} \mathcal{F}_2 = i^* j_* (\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}_{\mathbb{R}_{>0}})$$

From now on we assume $K = K^\rho$ $\quad (G_{\mathbb{R}} = G_{\mathbb{R}}^\circ)$

$\Rightarrow \langle 2\rho, \lambda \rangle \in 2\mathbb{Z}$

Ex $\quad \lambda \in X_*(A)^+$

$\Rightarrow DSat_{\mathbb{R}} \supset Sat_{\mathbb{R}} = Perv(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$

Thm. (Taylor – Mocrats – C.) $(Sat_{\mathbb{R}}, *)$ is a braided tensor cat. w/ fiber functor

$$F : Sat_{\mathbb{R}} \to sVect_{\mathbb{C}}$$

$$\mathcal{F} \mapsto H^*(Gr_{\mathbb{R}}, \mathcal{F})$$

§ Hecke action and Satake equiv. for $G_{\mathbb{R}}$

BD

$Gr \times Gr \rightsquigarrow Gr$

$$-\eta \curvearrowright Gr^{(2)} = \{ \underset{(-\bar{y}, \eta(\mathcal{E}), \eta(\sigma))}{\overset{}{(y, \mathcal{E}, \sigma)}} : y \in \mathbb{C},\ \mathcal{E} \in Bun_G(\mathbb{C}),\ \sigma : \mathbb{C} \setminus \pm y \to \mathcal{E} \}$$

Real BD

$Gr \rightsquigarrow Gr_{\mathbb{R}}$

$$\begin{tikzcd}
Gr \times i\mathbb{R}^\times \arrow[r] \arrow[d] & (Gr^{(2)})^{-\eta} \arrow[d] & Gr_{\mathbb{R}} \arrow[l] \arrow[d] \\
i\mathbb{R} \setminus \{0\} \arrow[r] & i\mathbb{R} & \{0\} \arrow[l]
\end{tikzcd}$$

Consider $\Psi_{Gr_{\mathbb{R}}} : D(Gr) \to D(Gr_{\mathbb{R}})$

Fact: ① $\Psi_{\mathbb{R}}$ will induce a functor

$$\Psi_{\mathbb{R}} : DSat_G = D(L^+G \backslash Gr) \to DSat_{\mathbb{R}} = D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$$

Using the fact we can define the Hecke action $\mathrm{Rep}\,\check{G} \curvearrowright DSat_{\mathbb{R}}$, $V \in \mathrm{Rep}\,\check{G}$, $\mathcal{F} \in DSat_{\mathbb{R}}$

$$V * \mathcal{F} = \Psi_{\mathbb{R}}(\mathcal{I}(V)) * \mathcal{F}.$$

__Fact:__ The Hecke action is t-exact iff $G_{\mathbb{R}}$ is quasi-split

Intro the perverse Hecke action

i) $V \overset{p}{*} \mathcal{F} = \bigoplus_i {}^pH^i(V * \mathcal{F}) \in Sat_{\mathbb{R}}$

ii) $Sat^{\circ}_{\mathbb{R}} \subset Sat_{\mathbb{R}}$

$$\begin{gathered} Sat^{\circ}_{\mathbb{R}} \\ \| \\ \mathrm{Rep}\,\check{G} \overset{p}{*} \mathbb{C}_{Gr^{\circ}_{\mathbb{R}} = pt} \end{gathered}$$

__Thm__ (Nadler, thesis)

$$\begin{tikzcd} (Sat^{\circ}_{\mathbb{R}}, *) \arrow[r, "\simeq"] \arrow[dr, "H^*(Gr_{\mathbb{R}}, -)"'] & \mathrm{Rep}\,\check{G}_X \arrow[d, "For"] \\ & Vect_{\mathbb{C}} \end{tikzcd}$$

__Proof__ - The "same" as in the cpx gp case.

$$LN_{\mathbb{R}} \curvearrowright Gr_{\mathbb{R}} = \coprod_{\lambda \in X_*(A)} S^{\lambda}_{\mathbb{R}} \Rightarrow H^*(Gr_{\mathbb{R}}, \mathcal{F}) \simeq \bigoplus_{\lambda \in X_*(A)} H^*(S^{\lambda}_{\mathbb{R}}, \mathcal{F})$$

__§ Real - Symmetric equivalence__

__Thm__ (Nadler - C.) The real nearby cycles induce a t-exact tensor equiv. compatible w/ Hecke action.

$$\begin{tikzcd} DSat_X \simeq D(LK \backslash LG / L^+G) \arrow[r, "\Psi_{G_{\mathbb{R}}}", "\sim"'] & DSat_{\mathbb{R}} = D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}}) \\ Sat_X \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & Sat_{\mathbb{R}} \arrow[u, phantom, "\cup"] \\ Sat^{\circ}_X \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & Sat^{\circ}_{\mathbb{R}} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Cor $\check{G}_{X,\mathrm{geom}} \xrightarrow{\sim} \check{G}_X$

§. What next?

$$\begin{array}{ccccccc} \mathrm{Sat}_X & \supset \mathrm{Sat}_X^\circ & \simeq & \mathrm{Sat}_{\mathbb{R}}^\circ & \subset & \mathrm{Sat}_{\mathbb{R}} \end{array}$$

Symmetric varieties $\longleftrightarrow$ Real groups

$\nwarrow\searrow$ $\nearrow\swarrow$

Quantum Supergroups

$$\mathrm{SRep}\left(\begin{array}{c}\text{quantum supergp at a}\\ \text{root of unity}\end{array}\right) \supset \mathrm{Rep}\,\check{G}_X$$

Thm (Yi–C.) Assume $\langle \rho, \lambda \rangle$ is even and the stabilizers $\mathrm{Stab}_G(t^\lambda)$ are conn'd for all $\lambda \in X_\bullet(A)^+$. Then

$$\begin{array}{ccccc} \mathrm{Sat}_X^\circ & = & \mathrm{Sat}_X & \simeq & \mathrm{Rep}\,\check{G}_X \\ \wr\vert & & \wr\vert & & \\ \mathrm{Sat}_{\mathbb{R}}^\circ & = & \mathrm{Sat}_{\mathbb{R}} & & \end{array}$$

Ex $X = GL_{2n}/Sp_{2n}$. $Spin_{2n}/Spin_{2n-1}$, E_6/F_4, $SL_{p+q}/S(GL_p \times GL_q)$

Pf: parity vanishing.

Ex (Taylor–Macerato, C) $X = GL_2/O_2$. $\check{G}_X = GL_2$.

$$\begin{array}{ccccc} \mathrm{Rep}(GL_2) & \subset & \mathrm{Rep}\,U_{q=i}(GL_2) & \subset & \mathrm{sRep}(???) \\ \wr\vert & & \wr\vert & & \\ \mathrm{Sat}_X^\circ & \subset & \mathrm{Sat}_X^w & \subset & \mathrm{DSat}_X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Vect}_{\mathbb{C}} & \simeq & \mathrm{Vect}_{\mathbb{C}} & \subset & \mathrm{sVect}_{\mathbb{C}} \end{array}$$

Here $W = (1,0)$, Sat_X^W the cat. tensor generated by

$IC^1 = IC(LX^1)$ and IC_{det}.

$\not\simeq$

Sat_X^0

Conj.

$$\begin{array}{ccc} & \mathrm{DSat}_{G_{\mathbb{R}}} \simeq & \\ \mathrm{DSat}_X & \xrightarrow{\sim} & D(\mathrm{Bun}_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}})) \\ \wr\!\!| & & \wr\!\!| \\ \mathrm{QCoh}(\check{M}_X/\check{G}_X) & \xrightarrow[KD]{\sim} & \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}(\mathbb{P}^1_{\mathbb{R}})) \end{array}$$

real geom. Langlands

relative Langlands duality

some version of

<u>Ex</u> (Group case)

<u>Thm</u> (B–F)

$$\begin{array}{ccc} \mathrm{DSat}_G & \simeq & D(\mathrm{Bun}_G(\mathbb{P}^1)) \\ \wr\!\!| & & \wr\!\!| \\ \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G}) & \xrightarrow[\sim]{KD} & \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}(\mathbb{P}^1)) \end{array}$$

<u>Ex</u> $X = GL_{2n}/Sp_{2n}$, $\check{G}_X = GL_n$, $\check{M}_X = \mathfrak{gl}_n$