Automorphic forms and the Langlands program

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Lecture 1

Goal: Learn something

Start with $N \in \mathbb{Z}_{\geq 1}$, $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

a Dirichlet character (i.e. a group homomorphism)

Attached to χ is

$$\begin{tikzcd}
\rho_\chi : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \arrow[r] \arrow[dr] & GL_1(\mathbb{C}) & \\
& \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times \arrow[u, "\chi"'] & \\
& \zeta_N \mapsto \zeta_N^n \longleftarrow\!\shortmid n &
\end{tikzcd}$$

Next: GL_2 story.

Say f is a cuspidal modular form, & an eigenform for the Hecke ops T_p.

Say $T_p f = \lambda_p f$, $\lambda_p \in \mathbb{C}$.

Turns out that the subfield of $\mathbb{C}$ genrated by the λ_p is a number field $E_f \subset \mathbb{C}$.

If $\ell \in \mathbb{Z}$ is a prime number & $\lambda | \ell$ is a prime of E_f, then following a suggestion of Serre, Deligne constructed

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow GL_2(\overline{E_{f,\lambda}})$$

ρ_f is "attached to f" in some way.

f is a modular form: f has a level $N \geq 1$
weight $k \geq 1$
& character χ

Turns out that ρ_f is unramified outside $N\ell$.

& if p is prime, $p \nmid N\ell$, then $\rho_f(\mathrm{Frob}_p)$ has char. poly.

$$X^2 - \lambda_p X + p^{k-1}\chi(p)$$

(Chebotarev density $\Rightarrow$ $\exists \leq$ one semisimple ρ_f w/ this property).

A word on Deligne's construction:

Deligne constructs ρ_f using étale cohomology (non-trivial coefficients)
(& then using trivial coefficients)
(all for $k \geq 2$; for $k=1$, Deligne + Serre 1974)

Questions arising from Deligne's construction:

If $p \mid N\ell$, what does ρ_f look like locally at p?

Case 1) $p \mid N$, $p \neq \ell$, then the answer is given by the so-called local Langlands correspondence.

Case 2) $p = \ell$. Then we should use the p-adic local Langlands correspondence.

Easier variant of this: instead of asking for ρ_f, could instead ask for $\overline{\rho_f} : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow GL_2(\underbrace{\mathbb{F}_\ell}_{\text{res. field of } \overline{E}_\lambda \text{ at } \lambda})$.

($\bar{\rho}_f \subset$ ℓ-torsion of an appropriate ab. var.)

Q) Are ρ_χ & ρ_f special cases of a general story?

Thm (Harris, Lan, Taylor, Thorne, 2013, Scholze)

E totally real or CM number field.

π: a cuspidal automorphic repⁿ of $GL_n(\mathbb{A}_E)$

Assume that π_∞ is "cohomological"

(this is an "algebraicity" assumption)

Then $\exists\ \rho_\pi : Gal(\bar{E}|E) \to GL_n(\bar{\mathbb{Q}}_\ell)$ attached to π in some canonical way. (analogue of giving char. poly. of $\rho_f(Frob_p)$)

Seen: $\begin{pmatrix} \text{algebraic, or analytic gadget} \\ x, f, \pi \end{pmatrix} \overset{\text{technical}}{\underset{\text{machinery}}{\rightsquigarrow}} \left(\text{rep}^{\text{n}}\text{s of Galois groups}\right)$

Interesting question: Can we classify the image?

i.e. say ρ: Galois group $\to GL_n$(field), Is $\rho \cong$ to a repⁿ coming from an algebraic or analytic gadget?

1-dim'l case:

Say $K|\mathbb{Q}$ is a finite Galois extⁿ. & $\rho: Gal(K|\mathbb{Q}) \to GL_1(\mathbb{C})$

Is $\rho \cong \rho_\chi$, χ a Dirichlet character?

By replacing K by a subfield if necessary, we can assume

$\rho : \mathrm{Gal}(K|\mathbb{Q}) \hookrightarrow \mathbb{C}^\times$ is injective, hence $\mathrm{Gal}(K|\mathbb{Q})$ is abelian.

$$\begin{tikzcd}
\chi : (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{C}^\times \\
\mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}) \arrow[ur, "\rho_\chi"'] &
\end{tikzcd}$$

ρ_χ gives rise to $\mathrm{Gal}(L|\mathbb{Q}) \xhookrightarrow{\rho_\chi} \mathbb{C}^\times$ for some $L \subset \mathbb{Q}(\zeta_N)$.

So question becomes: If K is a number field, Galois over $\mathbb{Q}$, with abelian Galois group, does there exists $N \geq 1$ s.t. $K \subset \mathbb{Q}(\zeta_N)$?

Answer Yes (Kronecker - Weber thm)

("explicit case of global class field theory")

i.e. $\forall \rho : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to GL_1(\mathbb{C})$ cts, $\exists \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ s.t.

$\rho \cong \rho_\chi$.

GL_2. If f is a cuspidal modular eigenform as before, then ρ_f has the following properties:

1) ρ_f is abs. irred.

2) ρ_f is "odd", i.e. $\det \rho_f(\text{cpx conjugation}) = -1$.

3) ρ_f is unramified outside a finite set of primes & ρ_f is potentially semistable at ℓ.

$\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{Q}}_\ell)$

Condition in p-adic Hodge theory.

In the early 1990s, Fontaine & Mazur asked if $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to GL_2(\bar{\mathbb{Q}}_\ell)$ satisfied (1). (2). (3), then was $\rho \cong \rho_f$ some f?

This conjecture is now basically known, by work of

Kisin "The Fontaine-Mazur conj. for GL_2"

& Emerton "Local-global compatibilities in the p-adic Langlands prog. for GL_2"

GL_n case: Is $\rho : \mathrm{Gal}(\bar{E}|E) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$

+ assumptions $\Rightarrow \rho \cong \rho_\pi$ as in HLTT?

Barnet-Lamb, Gee, Geraghty, Taylor, prove this in many cases.

Lecture 2.

First thing:

Part 1: the local Langlands Correspondence for GL_n/K

(K = finite ext. of $\mathbb{Q}_p$) is, vaguely speaking, a canonical bijection

$$\begin{pmatrix} \text{Certain (typically } \infty\text{-dim'l)} \\ \text{irred. } \mathbb{C}\text{-rep}^{\text{n}}\text{s} \\ \text{of } GL_n(K) \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Certain} \\ n\text{-dim'l complex} \\ \text{rep}^{\text{n}}\text{s of a group} \\ \text{related to } \mathrm{Gal}(\bar{K}|K) \end{pmatrix}$$

$n=1$: this is "local class field theory"

[remark: first proofs were global]

$n>1$: local Langlands conjectures for GL_n/K are a 2000 theorem of Harris + Taylor [Proofs are global]

<u>Infinite Galois groups</u>.

<u>Reminder of finite case</u>. $L|k$ finite field ext,

$L|k$ is Galois if $L|K$ is normal & separable.

then $Gal(L|K)$ = field aut. of L fixing K pointwise, finite group of size $\dim_K L$.

There's an inclusion-reversing correspondance

$$\left(\text{Subgps } H \subset Gal(L|k)\right) \longleftrightarrow \left(\begin{matrix}\text{fields } M: \\ k \subset M \subset L\end{matrix}\right)$$

$$Gal(L|M) = \left(\begin{matrix} g \in Gal(L|k) \\ g|_M = id \end{matrix}\right) \longleftarrow\!\shortmid M$$

Now say K is a field, & $L|K$ is an algebraic extn, possibly infinite degree.

We say $L|K$ is <u>Galois</u> if it's normal & separable

Set $Gal(L|K)$ = field aut. $\varphi: L \to L$ s.t. $\varphi|_K = id: K \to K$.

$\varphi \in \mathrm{Gal}(L|k)$ is determined by $\varphi(\lambda)$, $\lambda \in L$.

If $\lambda \in L$, then $\exists M$, $L \supset M \supset K$ s.t. $M|k$ is finite & Galois

& $\lambda \in M$.

In particular, $\mathrm{Gal}(L|k) \longrightarrow \mathrm{Gal}(M|k)$

& $\varphi(\lambda)$ is determined by image of φ in $\mathrm{Gal}(M|k)$.

In particular, φ is determined by $\varphi|_M$. $\forall M$, $L \supset \underbrace{M \supset K}_{\text{finite}}$ Galois

$$\mathrm{Gal}(L|k) \hookrightarrow \prod_{\substack{L \supset M \supset k \\ M|k \text{ finite Galois}}} \mathrm{Gal}(M|k)$$

We've shown $\mathrm{Gal}(L|k) = \varprojlim_{M \text{ as above}} \mathrm{Gal}(M|k)$

The gps $\mathrm{Gal}(M|k)$ are finite groups. Given them all the discrete top.

Put the product topology on $\prod_{M \text{ as above}} \mathrm{Gal}(M|k)$

$\mathrm{Gal}(L|k)$ turns out to be a closed subspace of this product.

Give it subspace topology.

If $L|k$ is Galois & we equip $\mathrm{Gal}(L|K)$ with the subspace topology

$$\left(\begin{matrix}\underline{\text{Closed}} \text{ subgps} \\ \text{of } \mathrm{Gal}(L|K)\end{matrix}\right) \longleftrightarrow \left(\text{Fields } M : L \supset M \supset K\right)$$

$$\mathrm{Gal}(L|M) \longleftarrow\!\shortmid\ M$$

Examples

0) $K = \mathbb{Q}$, $L = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n}) \subset \mathbb{C}$, p prime. Set $L_n = \mathbb{Q}(\zeta_{p^n})$

Known: $\mathrm{Gal}(L_n | \mathbb{Q}) = (\mathbb{Z}/p^n\mathbb{Z})^\times$

$$\mathrm{Gal}(L|\mathbb{Q}) \hookrightarrow \prod_{n \geq 1} (\mathbb{Z}/p^n\mathbb{Z})^\times \qquad \text{not surjective:}$$

$$\mathbb{Q} \subset L_1 \subset L_2 \subset L_3 \subset \cdots \subset L_n \subset \cdots$$

More precisely, if $\varphi \longleftrightarrow (\varphi_n)$, then $\varphi_m = \varphi_n \mod p^m$ $(m \leq n)$

In particular, $\mathrm{Gal}(L|\mathbb{Q}) = \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$

1) K finite, $L = \overline{K}$.

Say $\# K = q$, $L =$ union of $\mathbb{F}_{q^n}$

NB $\mathbb{F}_q \subset \mathbb{F}_{q^2} \not\subset \mathbb{F}_{q^3}$

$L_n = \mathbb{F}_{q^n}$, then $L_n \subset L_m \iff n$ divides m.

Reminder. $L_n | K$ is Galois & gen. by Frob_q.

$$\begin{aligned} \mathrm{Gal}(\overline{K}|K) &\hookrightarrow \prod \mathbb{Z}/n\mathbb{Z} \\ g &\longmapsto (g_n^{\psi}) \end{aligned}$$

(g_n) is in the image of $\mathrm{Gal}(\overline{K}|K) \iff g_n \mod m = g_m$, $\forall m | n$.

$$\mathrm{Gal}(\overline{K}|K) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

2) Local fields.

Let's stick to the case $K | \mathbb{Q}_p$ finite.

Choose an alg. closure $\bar{K}$ of K.

Want to understand $\mathrm{Gal}(\bar{K}|K)$.

We fail to do this, but will get some scraps.

$\mathbb{Q}_p \supset \mathbb{Z}_p$ = p-adic integers

$v: \mathbb{Q}_p \longrightarrow \mathbb{Z} \cup \{\infty\}$ normalized valuation

$v(p^n u) = n$ if $u \in \mathbb{Z}_p^\times$ is a unit

All the same for general K.

$K \supset \mathcal{O}_K$ = integers of K, $\mathcal{O}_K$ = ring

$\mathcal{O}_K \supset \mathfrak{p}_K$ = maximal ideal, $\mathfrak{p}_K = (\pi_K)$ principal.

$\exists\ v: K^\times \twoheadrightarrow \mathbb{Z}$, $v(\pi_K) = 1$, $v(\pi_K^n u) = n$, $u \in \mathcal{O}_K^\times$.

Say $L|K$ is an alg. extension. (possibly infinite)

v_K extends to

$$\begin{tikzcd} L^\times \arrow[r] & \mathbb{Q} \\ K^\times \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \mathbb{Z} \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

$$L \supset \underset{\{0\} \cup \{\alpha \in L : v(\alpha) \geq 0\}}{\underset{\parallel}{\mathcal{O}_L}} \supset \mathfrak{p}_L = \text{maximal ideal.}$$

$\mathcal{O}_L / \mathfrak{p}_L = k_L$ = residue field = alg. extension of $k_K = \mathcal{O}_K / \mathfrak{p}_K$ = fin field.

If $L|K$ is Galois, then we get a map

$$\mathrm{Gal}(L|K) \longrightarrow \mathrm{Gal}(k_L|k_K)$$ & this is surjective.

& not injective in general.

We say $L|K$ is unramified, if the natural map

$\mathrm{Gal}(L|K) \to \mathrm{Gal}(k_L|k_K)$ is injective (equivalently, bijective)

Set up: $K \,|\, \mathbb{Q}_p$ finite

$$K \supset \mathcal{O}_K \supset \mathfrak{p}_K = (\pi_K)$$

$\nwarrow$ uniformizer

TFAE. 1) $\mathfrak{p}_L = \pi_K \mathcal{O}_L$

2) $v_K(L^\times) = \mathbb{Z}$.

3) $L|K$ is unramified.

Composition of 2 unramified extns of K is unram.

If $L|K$ is algebraic, $\exists!$ maxm'l unramified subextn $L \supset M \supset K$

$$\mathrm{Gal}(M|K) = \mathrm{Gal}(k_M|k_K) = \text{cyclic or pro-cyclic}$$

$M|K$ unramified & maximal.

Lecture 3. Recall $K \,|\, \mathbb{Q}_p$ finite

Say $L|K$ algebraic, normal, so Galois.

$$\mathrm{Gal}(L|K) \twoheadrightarrow \mathrm{Gal}(k_L|k_K).$$

Define $I_{L|K}$ = kernel of this map.

$I_{L|K}$ = <u>inertia subgroup</u> of $\mathrm{Gal}(L|K)$

$I_{L|K} \subset \mathrm{Gal}(L|K)$, quotient = $\mathrm{Gal}(k_L|k_K)$

k_K finite, $\mathrm{Gal}(k_L|k_K)$ is topologically gen. by 1 elt.

To understand $\mathrm{Gal}(L|K)$, need to focus on $I_{L|K}$.

Clear, $I_{L|K}$ is a closed subgroup of $\mathrm{Gal}(L|K)$

Fund. thm of Galois theory: $I_{L/K} \longleftrightarrow \underbrace{L \supset}_{I_{L|K}} \underbrace{M \supset K}_{\text{Gal gp} = \mathrm{Gal}(k_L|k_K)}$

M = union of all subfields of L unramified over K.

Special interesting case: $L = \overline{K}$.

$$\begin{tikzcd} \overline{K} \arrow[d, dash, "I_{\overline{K}|K}"] \\ K^{nr} \arrow[d, dash, "\hat{\mathbb{Z}} = \mathrm{Gal}(\overline{k}_K|k_K)"] \\ K \end{tikzcd}$$

Example. if $K = \mathbb{Q}_p$, then $K^{nr} = \bigcup_{\substack{m \geq 1 \\ p \nmid m}} \mathbb{Q}_p(\zeta_m)$

Now say $L|K$ as usual (Galois) & assume $I_{L|K}$ is <u>finite</u>.
(e.g. $L|K$ finite)

Know $I_{L|K} \triangleleft \mathrm{Gal}(L|K)$.

Put a filtration on $I_{L|K}$.

If $\sigma \in I_{L|K}$, then
$$\sigma : L \longrightarrow L$$
$$\sigma : \mathcal{O}_L \longrightarrow \mathcal{O}_L$$
$$\sigma : \mathfrak{p}_L \rightarrow \mathfrak{p}_L$$

Note that because $I_{L|K}$ is finite, we have that $\mathfrak{p}_L = (\pi_L)$ is principal.

& $v_L : L \twoheadrightarrow \mathbb{Z}$ discrete valuation satisfies

$v_L = (\# I_{L|K}) v_K$ on $K^\times$.

If $i \geq 1$, define $I_{L|K,i} = \left\{ \sigma \in I_{L|K} : \frac{\sigma(\pi_L)}{\pi_L} \in 1 + \mathfrak{p}_L^i \right\}$

Set $I_{L|K,0} = I_{L|K}$.

Check (not too hard)

$I_{L|K} = I_{L|K,0} \supset I_{L|K,1} \supset I_{L|K,2} \supset \cdots$ are all subgroups of $I_{L|K}$.

$I_{L|K,i} \triangleleft \mathrm{Gal}(L|K)$.

Furthermore, if $i \gg 0$, then $I_{L|K,i} = \{1\}$

Note that $I_{L|K} / I_{L|K,1} \hookrightarrow k_L^\times$
$$\sigma \longmapsto \sigma(\pi_L)/\pi_L$$

& in particular, $\left(I_{L|K} / I_{L|K,1} \right)$ is cyclic of order prime to p.

Note also that if $i \geq 1$,

$$I_{L|K,i} / I_{L|K,i+1} \hookrightarrow \mathfrak{p}_L^i / \mathfrak{p}_L^{i+1} \left(\cong (k_L, +) \right), \sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L} - 1$$

& in particular,

$$I_{L|K,i} / I_{L|K,i+1} \cong \mathbb{Z}/p^n\mathbb{Z} \text{ has order a power of } p.$$

Upshot: $I_{L|K,1}$ is the unique Sylow p-subgroup of $I_{L|K}$.

(in particular, $I_{L|K}$ is solvable).

We say $L|K$ is <u>tamely ramified</u> if $I_{L|K,1} = \{1\}$.

[Note: unramified ext^s are tamely ramified!]

If $I_{L|K,1} \neq 1$, say $L|K$ is wildly ramified.

We're really interested in case $L = \overline{K}$.

$I_{L|K}$ is not finite here. The "lower numbering" $I_{L|K,r}$ does not behave well w.r.t. extensions of L.

($L'|L|K$, $L'|K$ & $L|K$ Galois, $I_{L'|K}$ finite, $I_{L'|K} \twoheadrightarrow I_{L|K}$,

but $I_{L'|K,i}$ doesn't become identified w/ $I_{L|K,i}$).

Crazy fix: Introduce a relabelling of filtration.

$L|K$ Galois, $I_{L|K}$ finite.

Set $g_i = \# I_{L|K,i}$, $g_0 \geqslant g_1 \geqslant \dots \geqslant g_M = 1$, $M >> 0$.

Define $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$,

φ is piecewise linear & continuous, φ is linear on $(i, i+1)$. $\varphi(0) = 0$,

& on $(i, i+1)$, φ has slope $\frac{g_{i+1}}{g_0}$

$\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a strictly increasing bijection.

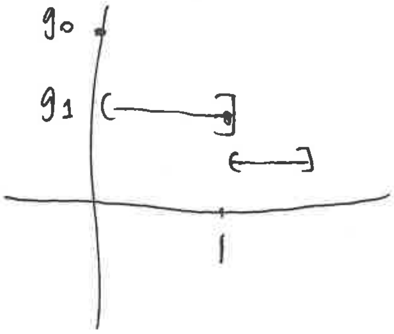
If $v \in \mathbb{R}$, $v \geqslant 0$, define $I_{L|K, v} = I_{L|K, \lceil v \rceil}$.

<u>Defn</u> (upper numbering) $u \in \mathbb{R}_{\geqslant 0}$, $I^u_{L|K} := I_{L|K, \varphi^{-1}(u)}$.

<u>Prop^n</u> (Pf in Serre local fields) If $L'|L|K$ all Galois, $I_{L'|K}$ finite,

then $I^u_{L|K} = \mathrm{Im}(I^u_{L'|K})$ via obvious map.

Graph $v \longmapsto \# I_{L|K, v}$ jumps @ $v \in \mathbb{Z}_{\geqslant 0}$



graph $u \longmapsto \# I^u_{L|K}$, jumps may not be in $\mathbb{Z}$, but are in $\mathbb{Q}$.

[Thm (Hasse-Arf): if $L|K$ is abelian, then jumps in $I^u_{L|K}$ are at integers!]

If $L|K$ is any Galois extⁿ, can define $\overset{u\in\mathbb{R}}{I^u_{L|K}}$ by glueing $I^u_{M|K}$ for $M|K$ algebraic, $I_{M|K}$ finite.

($I_{L|K}$ finite)

Recall I define $L|K$ to be <u>tamely ramified</u> if $I_{L|K,1}=\{1\}$.

This is $\Leftrightarrow$ $I_{L|K,\varepsilon}=\{1\}$, $\forall \varepsilon>0$

This is $\Leftrightarrow$ $I^{\delta}_{L|K}=\{1\}$, $\forall \delta>0$.

Last defn is good for any Galois $L|K$.

Compositum of 2 tame extns is tame

$\therefore$ $L|K$ contains a maximal tamely ramified extension.

$$\begin{tikzcd}
L \arrow[d, dash, "\text{) wildly ramified, Galois gp} = \text{pro-}p."] \\
K^2 \arrow[d, dash] \arrow[dd, phantom, "\text{max'l tamely ramified.}", shift left=12ex] \\
K^1 \arrow[d, dash, "\text{) max'l unramified}"] \\
K
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{K} \arrow[d, dash] \\
K^t \arrow[d, dash] & \leftarrow \text{max. tamely ramified extn.} \\
K^{nr} \arrow[d, dash] & \text{What is } K^t\text{?} \\
K
\end{tikzcd}$$

If $L|K$ finite Galois, then

$$\begin{tikzcd}
L \arrow[d, dash] \\
K_2 \arrow[d, dash, "\text{) Gal. gp } I_{L|K}/I_{L|K,1} \cong \text{cyclic order } m,\ p\nmid m"] \\
K_1 \arrow[d, dash] \\
K
\end{tikzcd}$$

Kummer theory (see Birch in C-F)

Note that $K^{nr}\supset\mu_m$ = gp of mth roots of unity in $\bar{K}$ ($p\nmid m$)

If $K_2 \mid K_{nr}$ is Galois, w/ gp $\cong \mathbb{Z}/m\mathbb{Z}$,

then k_2 must be $K_{nr}(\sqrt[m]{\alpha})$, some $\alpha \in K_{nr}$.

Not too hard to check that in fact, K_2 must be $K_{nr}(\sqrt[m]{\pi_K})$

Can check now that $K_t = \bigcup_{\substack{m \geq 1 \\ p \nmid m}} K_{nr}(\sqrt[m]{\pi_K})$

Note $\mathrm{Gal}(K_{nr}(\sqrt[m]{\pi_K}) \mid K_{nr}) = \mu_m$ via map

$$\sigma \longmapsto \frac{\sigma(\sqrt[m]{\pi_K})}{\sqrt[m]{\pi_K}}$$

So $\mathrm{Gal}(K^t \mid K^{nr}) = \varprojlim_{p \nmid m} \mu_m \underset{\uparrow}{\cong} \varprojlim_{p \nmid m} \mathbb{Z}/m\mathbb{Z} = \prod_{\ell \neq p} \mathbb{Z}_\ell$

Non-canonical.

$$\begin{tikzcd}
\bar{K} \arrow[d, dash, "\text{pro-}p"] \\
K^t \arrow[d, dash, "\cong \prod_{\ell \neq p} \mathbb{Z}_\ell"] \\
K^{nr} \arrow[d, dash, "= \hat{\mathbb{Z}} = \prod_{\ell} \mathbb{Z}_\ell"] \\
K
\end{tikzcd}$$

Ex. Can now understand $\mathrm{Gal}(K^t \mid K)$: (pro)-cyclic sub

(pro)-cyclic quotient, generated by a canonical generator Frob.

$\mathrm{Gal}(K^{nr} \mid K) = \mathrm{Gal}(k_{K^{nr}} \mid k_K)$

Frob (defn) $\longleftrightarrow$ $x \overset{\psi}{\longmapsto} x^q$, $q = |k_K|$

If we lift Frob to $\mathrm{Gal}(K^t|K)$, then it acts by conjugation on the normal subgroup $\mathrm{Gal}(K^t|K^{nr})$

$$\sigma \mapsto \mathrm{Frob}\cdot\sigma\cdot\mathrm{Frob}^{-1}$$

$$\mathrm{Gal}(K^t|K^{nr}) = \varprojlim_m \mu_m(\bar{k})$$

__Exercise.__ check that the map induced by Frob is $\zeta \mapsto \zeta^q$

This is the glue, telling us what $\mathrm{Gal}(K^t|K)$ is.

__Lecture 4.__ We've just seen an attempt to analyse the group $\mathrm{Gal}(\bar{K}|K)$ via an explicit attack on the inertia group.

Obstacle: Sylow p-subgp & inertia gp is hard.

Here's another approach: let's try to understand the abelianization of $\mathrm{Gal}(\bar{K}|K)$

__Def.__ $K|\mathbb{Q}_p$ finite. Recall

$$\begin{tikzcd} 1 \arrow[r] & I_{\bar{K}|K} \arrow[r] & \mathrm{Gal}(\bar{K}|K) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\ & & & \text{gen by Frob.} \arrow[u] & \end{tikzcd}$$

$\mathrm{Frob} \in \hat{\mathbb{Z}}$. $(\mathrm{Frob})^{\mathbb{Z}} = \{\ldots, \mathrm{Frob}^{-2}, \mathrm{Frob}^{-1}, 1, \mathrm{Frob}, \mathrm{Frob}^2, \ldots\} = \mathbb{Z} \subset \hat{\mathbb{Z}}$

$$\begin{tikzcd} & & \mathrm{Frob} & 1 & \\ 1 \arrow[r] & I_{\bar{K}|K} \arrow[r] \arrow[d, equal] & W_K \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & I_{\bar{K}|K} \arrow[r] & \mathrm{Gal}(\bar{K}|K) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \end{tikzcd}$$

Formally, $W_K = \{ g \in \mathrm{Gal}(\bar{K}|K) : \mathrm{Im}(g)$ in $\hat{\mathbb{Z}} = \mathrm{Gal}(K^{nr}|K)$

is in $(\mathrm{Frob})^{\mathbb{Z}} \}$

Topologise W_K thus:

$I_{\bar{K}|K}$ is <u>open</u> in W_K w/ usual topology

(so $W_K / I_{\bar{K}|K} = \mathbb{Z}$ w/ discrete topology)

In particular, W_K does <u>NOT</u> have subspace topology.

$$\begin{tikzcd}
 & W_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{Z} \arrow[d] & \text{discrete gp} \\
\text{in cat of top gps} \arrow[ur, dash] & \mathrm{Gal}(\bar{K}|K) \arrow[r] & \hat{\mathbb{Z}} & \text{profinite gp}
\end{tikzcd}$$

<u>Define</u>. for G a topological grp, the subgp G^c to be the topological closure of the subgp of G generated by $ghg^{-1}h^{-1} : g, h \in G$.

$$G/G^c = \text{max'l abelian Hausdorff quotient of } G.$$

$$\begin{matrix} G/G^c \\ \parallel \\ G^{ab} \end{matrix}$$

<u>Main thm of LCFT</u>. $K | \mathbb{Q}_p$ finite, then $\exists$ canonical iso.

$$r_K : K^\times \xrightarrow{\sim} W_K^{ab}.$$

Here's a big list of properties of this iso..

$$\begin{tikzcd} r_K : K^\times \arrow[r, "\sim"] & W_K^{ab} \\ \cup & \cup \\ \mathcal{O}_K^\times \arrow[r, "\sim"] & \text{Image of } I_{\bar{K}|K} \\ \cup & \cup \\ i \geq 1,\ 1+\mathfrak{p}_K^i \arrow[r, "\sim"] & \text{Image of } I_{\bar{K}|K}^i \end{tikzcd}$$

& $r_K(\pi_K) \in \mathrm{Frob}^{-1} \cdot \mathrm{Im}(I_{\bar{K}|K})$

<u>Confusing remark</u>: If X & Y are 2 abelian groups,

& if $\varphi : X \to Y$ is a canonical isomorphism,

then $\psi : X \to Y$, $\psi(x) = \varphi(x)^{-1}$ is also an isom.

So there are in fact <u>two</u> canonical isos $K^\times \to W_K^{ab}$.

Can tell them apart: the one we will use identifies a uniformizer $\pi_K \in K^\times$ w/ <u>the inverse</u> of Frob.

<u>Def</u> (Deligne) <u>Geometric Frobenius</u> = Frob^{-1}.

Arithmetic Frobenius = Frob.

Do as Deligne: uniformizer $\longleftrightarrow$ geometric Frobenius.

More properties of r_K: If $L|K$ is finite, then

$$\begin{tikzcd}
\mathrm{Gal}(\bar{K}|L) \arrow[r, hook] & \mathrm{Gal}(\bar{K}|K) \\
W_L \arrow[u, phantom, "\cup"] \arrow[r, hook] \arrow[d] & W_K \arrow[u, phantom, "\cup"] \arrow[d] \\
W_L^{ab} \arrow[r] & W_K^{ab}
\end{tikzcd}$$

$$\begin{tikzcd}
L^{\times} \arrow[r, "r_L"] \arrow[d, "N_{L|K}"'] & W_L^{ab} \arrow[d] \\
K^{\times} \arrow[r, "r_K"] & W_K^{ab}
\end{tikzcd}$$

$\exists$ Verlagerung : $H \subset G$ finite index

Transfer $\exists\, V : G^{ab} \longrightarrow H^{ab}$

"norm"

$$g \longmapsto \prod_{i=1}^{r} \gamma_i g \gamma_i^{-1}$$

γ_i = set of coset reps.

Next property of r_K:

$$\begin{tikzcd}
L^{\times} \arrow[r, "r_L"] & W_L^{ab} \\
K^{\times} \arrow[u, hook] \arrow[r, "r_K"] & W_K \arrow[u, "\text{Transfer}"']
\end{tikzcd}$$

Final thing LCFT tells you:

If $L|k$ is finite & Galois, then $W_L \underset{\substack{\text{finite index} \\ \text{normal subgp}}}{\subset} W_K$, $W_K/W_L = \mathrm{Gal}(L|K)$.

$$\underbrace{W_L^c}_{\text{closure of } [W_L, W_L]} \overset{\text{characteristic subgp}}{\subset} W_L \Rightarrow W_L^c \trianglelefteq W_K.$$

Define $W_{L|K} = W_K / W_L^c$.

$$1 \longrightarrow \underset{\substack{\| \\ W_L^{ab}}}{L^{\times}} \longrightarrow W_{L|K} \longrightarrow \mathrm{Gal}(L|K) \longrightarrow 1.$$

This extn ~~then~~ gives rise to an element of $H^2(\mathrm{Gal}(L|K), L^\times)$.

This element is called $\alpha_{L|K}$.

Turns out $H^2(\mathrm{Gal}(L|K), L^\times)$ is cyclic order $n = [L:K]$

& $\alpha_{L|K}$ is a canonical generator.

$\alpha_{L|K}$ is called the fundamental class.

There are lots of cohomology gps that one can now compute using $\alpha_{L|K}$ $(\cup \alpha_{L|K})$

Upshot: we now understand Gal gp of max'l abelian extn of K.

$L|K$ <u>abelian</u> $:=$ $L|K$ Galois & $\mathrm{Gal}(L|K)$ abelian.

Composition of abelian extns is abelian.

$\exists$ max'l abelian ext^n $K^{ab} \subset \bar{K}$.

$$\begin{tikzcd}
 & \bar{K} \arrow[d, dash] \\
 & K^{ab} \arrow[d, dash] \\
K^{nr} \arrow[ur, dash] \arrow[dr, dash] & \\
 & K
\end{tikzcd}$$

) Gal gp $= \mathrm{Gal}(\bar{K}|K)^{ab}$

$$\begin{tikzcd}
I_{K^{ab}|K} \arrow[r] & \mathrm{Gal}(K^{ab}|K) \arrow[r] & \mathrm{Gal}(K^{nr}|K) \arrow[d, equal] \\
 & & \hat{\mathbb{Z}} \\
I_{K^{ab}|K} \arrow[uu, equal] \arrow[r] & W_K^{ab} \arrow[uu, hook] \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[u, hook] \\
 & K^\times &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gal}(\bar{K}|K)^{ab} \arrow[d, equal] & \\
\therefore\ \mathrm{Gal}(K^{ab}|K) & \\
\cong \mathcal{O}_K^\times \times & \hat{\mathbb{Z}}. \arrow[d, equal] \\
I_{K^{ab}|K} \arrow[u] & \mathrm{Gal}(K^{nr}|K)
\end{tikzcd}$$

Lecture 5

Working towards statement of LLC.

LLC for GL_n/K ($K|\mathbb{Q}_p$ finite)

vaguely speaking, say $\begin{pmatrix} n\text{-dim'l reps} \\ \text{of "Galois gp"} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{reps of} \\ GL_n(K) \end{pmatrix}$.

LHS: This "Galois gp" is a "Weil - Deligne group"

$$\begin{tikzcd} \text{"Weil - Deligne group"} \arrow[d, leftrightarrow] \\ \text{Weil gp } W_K \end{tikzcd}$$

Upshot: need to start talking about rep'ns of W_K.

Recall:

$$\begin{tikzcd} 1 \arrow[r] & I_{\bar{K}|K} \arrow[r] & \mathrm{Gal}(\bar{K}|K) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\ 1 \arrow[r] & I_{\bar{K}|K} \arrow[r] \arrow[u, equal] & W_K \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

Let E be a field, put discrete topology on E, & on $GL_n(E)$, $n \in \mathbb{Z}_{\geq 0}$ fixed.

Let's consider $\rho: W_K \longrightarrow GL_n(E)$ a continuous gp. hom.

Continuity $\Leftrightarrow$ $\ker \rho$ open.

$\rho(I_{\bar{K}|K})$ = cts image of compact space

$\Rightarrow$ compact subset of discrete set $GL_n(E)$

$\Rightarrow$ finite.

Hence can use the theory of lower numbering on $\rho(I_{\bar{K}|K})$

$$\begin{tikzcd}
\bar{K} \arrow[d, dash, "I_{\bar{K}|K}"'] \arrow[dr, dash] & \\
K^{nr} \arrow[r, dash, "\rho(I_{\bar{K}|K})"'] & L = L(\rho)
\end{tikzcd}$$

$$\rho(I_{\bar{K}|K}) = I_{L|K} \supset I_{L|K,1} \supset \cdots$$

<u>Define</u> $f(\rho)$ = conductor of ρ

$$= \sum_{i=0}^{\infty} \frac{1}{[I_{L|K} : I_{L|K,i}]} \dim\left(V/V^{I_{L|K,i}}\right)$$

where $\rho : W_K \to GL_n(E) = \mathrm{Aut}_E(V)$, $V = E^n$

[notation: $H \to \mathrm{Aut}(V)$, $V^H = \{v \in V : hv = v, \forall h \in H\}$

Note: sum is finite, as $i \gg 0 \Rightarrow I_{L|K,i} = \{1\}$

$$\Rightarrow \dim\left(V/V^{I_{L|K,i}}\right) = 0.$$

<u>Claim</u>: $f(\rho) \in \mathbb{Q}_{\geq 0}$

<u>Easy</u>. $f(\rho) = 0 \iff \rho$ is unramified $\left(\iff \rho(I_{\bar{K}|K}) = \{1\}\right)$.

<u>Rmk</u>. $f(\rho) \in \mathbb{Z}_{\geq 0}$.

Note: $f(\rho) = \dim\left(V/V^{I_{L|K}}\right) + \sum_{i \geq 1} \cdots$

Examples. Recall $r_K :$

$$\begin{tikzcd} K^\times \arrow[r, "\sim"] & W_K^{ab} \\ & W_K \arrow[u, two heads] \arrow[ul, squiggly, two heads] \end{tikzcd}$$

We have $v_K : K^\times \longrightarrow \mathbb{Z}$

$\pi_K \longmapsto 1$

Let's introduce a norm on K: $\|\lambda\| = (\text{small real number})^{v(\lambda)}$

Interlude: If K is any field complete w.r.t. non-trivial non-arch. norm, can set up a good theory of rigid geometry, e.g. $K = \mathbb{C}((t))$: $\underset{= n}{v(t^n + \text{higher terms})}$

Norm on K.

$|f| = \varepsilon^{v(f)}$, $0 < \varepsilon < 1$, $\varepsilon \in \mathbb{R}$. Any ε will do.

In our situation, $K/\mathbb{Q}_p$ finite, there is a canonical norm!

It's because the residue field k_K is finite $\Rightarrow$ K is locally compact $\Rightarrow$ $\exists$ additive Haar measure on K. Call it μ.

$\mu(\text{open set}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. $\mu(\mathcal{O}_K) < \infty$, say $\mu(\mathcal{O}_K) = 1$.

What is $\mu(\mathfrak{p}_K)$? $\mu(X) = \mu(X + a)$, $\mu(\mathcal{O}_K) = q\,\mu(\mathfrak{p}_K) \Rightarrow \mu(\mathfrak{p}_K) = \frac{1}{q_K}$

Cute idea: if $a \in K^\times$, then $\|a\| =$ factor by which multiplication by a scales Haar measure.

eg. if $K = \mathbb{R}$, $X = [0, 1]$, $\mu(X) = 1$,

say $\lambda \in \mathbb{R}^\times$, define $\|\lambda\|$ by $\|\lambda\| = \mu(\lambda X)$.

<u>Fun exercise</u>. $K = \mathbb{C}$, $\|z\| = ?$

Back to $K | \mathbb{Q}_p$ finite, $\|\pi_K\| = \mu(\pi_K \mathcal{O}_K) = \mu(\mathfrak{p}_K) = \frac{1}{q}$

Upshot: there is a natural norm on K.

$\|\cdot\| : K \to \mathbb{R}_{\geq 0}$, $\|0\| = 0$, $\|\lambda\| = (\frac{1}{q})^{v(\lambda)}$, $\lambda \neq 0$

$\|\lambda\| = q^{-v_K(\lambda)}$, $v_K : K^\times \to \mathbb{Z}$.

<u>Rk</u> : $\|\cdot\| : K \to \mathbb{Q}_{\geq 0}$

Have
$$\begin{tikzcd}
W_K \arrow[r, two heads] \arrow[drr, bend right, "\|\cdot\|"'] & W_K^{ab} \arrow[r, equal, "r_K"] & K^\times \arrow[d, "\|\cdot\|"] \\
 & & \mathbb{Q}_{>0}
\end{tikzcd}$$

This defines $\|\cdot\| : W_K \to \mathbb{Q}_{>0}$

- Gives us an example! $E = \mathbb{Q}$ (or any field of char. 0)

$\|\cdot\| : W_K \to GL_1(E)$.

More generally, $\|\cdot\|^m$: $m \in \mathbb{Z}$ are all reps of W_K.

<u>Ex</u>. $f(\|\cdot\|^n) = 0$.

What is $\|\widetilde{\text{Frob}}\|$? $\widetilde{\text{Frob}} \in W_K$ lifts $\text{Frob} \in \hat{\mathbb{Z}}$.

$r_K : K^\times \longrightarrow W_K^{ab}$

$$\begin{tikzcd}
\mathcal{O}_K^\times \pi_K \arrow[r] \arrow[d, mapsto] & \text{Frob}^{-1}(I_{\bar{K}|K}) \arrow[dl] \\
\frac{1}{q} &
\end{tikzcd}$$

$\|\widetilde{\text{Frob}}\| = q$.

Weil-Deligne rep^{n}s.

A <u>Weil-Deligne rep^n</u> is a pair (ρ_0, N),

Here $\rho_0 : W_K \longrightarrow \text{Aut}_E(V) \cong GL_n(E)$ is a cts rep as before (E field, discrete top. & char(E)=0)

& $N : V \to V$ is an E-linear nilpotent endomorphism.

s.t. $\forall \sigma \in W_K$, $\rho_0(\sigma) N \rho_0(\sigma)^{-1} = \|\sigma\| \cdot N$

<u>Examples</u>. Let ρ_0 be a cts rep. as before & $N=0$.

Example w/ $N \neq 0$? $E = \mathbb{Q}$, $V = \langle e_1, e_0 \rangle_{\mathbb{Q}}$,

$$\rho_0(\sigma) = \begin{pmatrix} \|\sigma\| & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

<u>Ex</u>. $V = \mathbb{Q}^n$, basis $e_0, e_1, \cdots, e_{n-1}$, $\rho_0(\sigma) e_i = \|\sigma\|^i e_i$, $N e_i = e_{i+1}$.

$St(n)$ "Steinberg"

We say a Weil-Deligne rep^n (ρ_0, N) is F-semisimple if

$\rho_0(\widetilde{\mathrm{Frob}})$ is a semisimple matrix (i.e. diagonalizable over $\overline{E}$).

— indep. of choice of $\widetilde{\mathrm{Frob}}$.

One side of LLC bijection for GL_n:

$$\begin{pmatrix} \text{n-dim'l F-semisimple} \\ \text{Weil-Deligne reps} \\ \text{of } W_K \text{ up to iso.} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{next} \\ \text{time} \end{pmatrix}$$

Lecture 6. Representations of $GL_n(K)$

Set up E a field, Discrete top.

V/E : vector space (∞-dim'l fine)

$K \mid \mathbb{Q}_p$ finite, $\pi : GL_n(K) \longrightarrow \mathrm{Aut}_E(V)$, a group hom.

want a sensible notion of continuity.

Say π is smooth if $\forall v \in V$, $\mathrm{Stab}_\pi(v) := \{g \in GL_n(K) : gv = v\}$ is open.

Say that a smooth π is smooth-admissible or just admissible, if $\forall U \subset GL_n(K)$ open, V^U is finite dim.

(If I say "π admissible", this implies π smooth)

Tricky but true: π irred & smooth $\Rightarrow$ π admissible.

basis of open nbhd of 1 in $GL_n(K)$ is $\{g \in GL_n(\mathcal{O}_K) : g \equiv \mathrm{Id} \bmod \varpi_K^m\}$

π is <u>irred.</u> if only 2 $GL_n(K)$- invt subspaces, namely 0 & V.

Very algebraic defn.

<u>Vague statement of local Langlands conjectures for GL_n</u>

LCFT : $K^\times \xrightarrow{\sim} W_K^{ab}$

Want to understand : all of W_K.

Langlands' re-interpretation of LCFT:

$$\left(\text{Irred. 1-dim'l rep}^{\text{n}}\text{s of } \underset{\substack{\shortparallel \\ GL_1(K)}}{K^\times}\right) \longleftrightarrow \left(\text{Irred. 1-dim'l reps of } W_K\right)$$

Local Langlands conj. for GL_n: $\exists$ "canonical" bijection.

$$\begin{pmatrix}\text{Irred. adm. rep}^{\text{n}}\text{s of} \\ GL_n(K)\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{F-semisimple } n\text{-dim'l} \\ \text{Weil-Deligne reps of } W_K\end{pmatrix}$$

<u>Next time</u> : $n=2$.

<u>Lecture 7</u>. Can interpret the word "canonical" as meaning "the bijection satisfies a big list of nice properties",

e.g. - $\exists$ notion of duality on both sides

- L-functions & ε factors etc.

Kevin's understanding of the history:

LLC for GL_n: The big list of nice properties that the bijection must satisfy became sufficiently long that it became a theorem that there was ≤ 1 bijection satisfying all the properties on the list.

Turns out $\exists \geq 1$ bijection satisfying these properties:

In function field case, thm of Laumon, Rapoport, Stuhler
p-adic fields: 2000, Harris - Taylor.
] proofs are global.

Two obvious observations:

1) Brilliant generalization of local class field theory.

2) Completely pointless as it relates 2 completely uninteresting sets.

- we have seen neither Weil-Deligne reps nor smooth-admissible reps in other branches of mathematics.

Plan today: begin to check LLC is useful.

1) $n=1$.

2) Weil-Deligne reps showing up "in nature".

3) Examples of π's

<u>Rmk</u>. If G is any connected reductive group / K,

there's a local Langlands correspondence for G

$$\begin{pmatrix} \text{Certain Weil-Deligne} \\ \text{reps } (\rho, N): W_K \to {}^L G(\mathbb{C}) \end{pmatrix} \xleftarrow[\text{finite fibers}]{\text{surjection,}} \begin{pmatrix} \text{Smooth irred. adm.} \\ \text{reps of } G(K) \end{pmatrix}$$

$\uparrow$ L-group

satisfying a big list of natural properties (Borel, Corvalli's) which AFAIK do not yet uniquely characterize this so-called "canonical" surjection.

Fibres called "L-packets".

<u>LLC for $n=1$</u>. LHS: 1-dim'l Weil-Deligne reps

$$(\rho_0, N): W_K \longrightarrow GL_1(\mathbb{C})$$

$N = 1\times 1$ nilpotent mat. $\Rightarrow N = 0$.

$\rho_0: W_K \longrightarrow GL_1(\mathbb{C})$, kernel of ρ_0 factors through W_K^{ab}.

LHS = 1-dim'l cts $\mathbb{C}$-reps of W_K^{ab}

RHS: π = Smooth admissible irred. rep. of $K^\times$.

adm. + irred. $\Rightarrow$ dim π is finite $\to$ dim $\pi = 1$.

$\left(U \subset GL_1(K) \text{ cpt open} \Rightarrow U \triangleleft GL_1(K) \right)$

RHS = $\underset{\substack{\uparrow \\ \text{sm. adm.}}}{\text{cts}}$ gp homs $K^\times \longrightarrow \mathbb{C}^\times$

LCFT $W_K^{ab} = K^\times$ $\Rightarrow$ LLC for GL_1.

Source of Weil-Deligne reps:

<u>l-adic representations</u>.

$K | \mathbb{Q}_p$ finite, say $\rho : Gal(\bar{K}|K) \longrightarrow GL_n(\mathbb{Q}_\ell)$ is a continuous repⁿ.

Here ℓ is prime, $\ell \neq p$. $\mathbb{Q}_\ell$ has ℓ-adic topology, $GL_n(\mathbb{Q}_\ell)$ - ℓ-adic top.

Note 1: these show up in nature.

Eg. - ℓ-adic Tate module of an elliptic curve E/K.

- ℓ-adic étale cohomology of algebraic variety, $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$, X/K alg var.

- ℓ-adic deformations of examples. give new examples

Remark. If $E|K$ is an elliptic curve w/ split multiplicative reduction, then $E(\bar{K}) \cong \bar{K}^\times / q^{\mathbb{Z}}$, $q \in K$, $|q| < 1$

$\rightsquigarrow$ explicit calculation of Tate module

$$T_\ell E \Big|_{I_{\bar{K}|K}} = \begin{pmatrix} \text{cyclo} & \overset{\curvearrowright}{*} \\ 0 & \text{id} \end{pmatrix} \text{can be non trivial \& infinite}$$

Recall $\ell \neq p$. If ρ is an ℓ-adic repⁿ as above, $\rho(I_{\bar{K}|K})$ can be infinite, but it can't be too bad: $\rho(I^{\varepsilon}_{\bar{K}|K})$ is finite if $\varepsilon > 0$.
$\quad \llcorner$ pro-p

& recall $\mathrm{Gal}(K^t | K^{nr}) \simeq \prod_{\substack{r \text{ prime} \\ r \neq p}} \mathbb{Z}_r$.

The part we should be worrying about is the $\mathbb{Z}_\ell$ part.

Fix $t : \mathrm{Gal}(K^t / K^{nr}) \twoheadrightarrow \mathbb{Z}_\ell$.

Also fix $\varphi \in \mathrm{Gal}(\bar{K}|K)$ lifting $\mathrm{Frob} \in \mathrm{Gal}(K^{nr}|K)$

Prop (Grothendieck) If $\rho : \mathrm{Gal}(\bar{K}|K) \to GL_n(E)$, $E = \mathbb{Q}_\ell$, is a cont. ℓ-adic rep^n, then $\exists!\ (\rho_0, N) : W_K \to GL_n(E)$

$\llcorner$ E discrete topology.

s.t. $\rho(\varphi^m \sigma) = \rho_0(\varphi^m \sigma) \exp(N t(\sigma))$.

$\sigma \in I_{\bar{K}|K}$, $m \in \mathbb{Z}$

Rk: Tate curve example, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$\ulcorner$ Final few remarks: If (ρ_0, N) arises this way, eigenvalues of $\rho_0(\varphi)$ will be ℓ-adic units. (so not all (ρ_0, N) arise in this way.)

Conversely, if (ρ_0, N) given, & eigenvalues of ρ_0 are ℓ-adic units, (ρ_0, N) comes from a ρ. $\lrcorner$

Smooth admissible reps of $GL_n(K)$: Last thing on $n=1$:

If $\pi : K^\times \to \mathbb{C}^\times$ is smooth adm. (irred.), Define $f(\pi)=0$ if $\pi|_{\mathcal{O}_K^\times}=1$

& $f(\pi)=r\geq 1$ if $r=$ smallest positive integer s.t. $\pi|_{1+\mathfrak{p}_K^r\mathcal{O}_K}=1$.

LLC $n=1$: If $\rho_0=(\rho_0,\underset{\overset{\|}{0}}{N})\longleftrightarrow\pi$, then we want $f(\rho_0)=f(\pi)$.

<u>$n=2$</u> Cool construction of a π.

Say $\chi_1,\chi_2: K^\times\longrightarrow\mathbb{C}^\times$ continuous.

Define $I(\chi_1,\chi_2)=\Big\{\varphi: GL_2(K)\longrightarrow\mathbb{C};\ \varphi$ locally constant,

$$\varphi\left(\begin{pmatrix}a&b\\0&d\end{pmatrix}g\right)=\chi_1(a)\chi_2(d)\,\|a/d\|^{\frac{1}{2}}\,\varphi(g)\Big\}$$

"$\mathrm{Ind}_B^G\ \chi_1\otimes\chi_2$"

<u>Define</u> $\pi: GL_2(K)\longrightarrow\mathrm{Aut}_{\mathbb{C}}(I(\chi_1,\chi_2))$

$$g\in GL_2(K),\ \big(\pi(g)\varphi\big)(h)=\varphi(hg)$$

<u>Lecture 8</u>. $\pi: GL_2(K)\curvearrowright I(\chi_1,\chi_2)$. Is it smooth, adm., irred.?

<u>Really useful Lemma</u>, Let $B(K)=\left\{\begin{pmatrix}a&b\\0&d\end{pmatrix}\in GL_2(K)\right\}$, then

$$GL_2(K)=B(K)\,GL_2(\mathcal{O}_K)$$

<u>Rmk</u>. $\varphi: GL_2(K)\longrightarrow\mathbb{C}$ loc. const. $\Rightarrow\varphi$ is cts w.r.t. discrete top. on $\mathbb{C}$

$\varphi(GL_2(\mathcal{O}_K))=$ finite, $\varphi(B(K))$ controlled by def. of $I(\chi_1,\chi_2)$.

Recall $I(\chi_1, \chi_2) = \{ \varphi : GL_2(k) \to \mathbb{C}$ cts

$\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d) \|a/d\|^{\frac{1}{2}} \varphi(g) \}$

Exercise. $I(\chi_1, \chi_2)$ is smooth & admissible

Dumb observation: $\|a/d\|^{\frac{1}{2}} = \frac{\|a\|^{\frac{1}{2}}}{\|d\|^{\frac{1}{2}}}$

Set $\widetilde{\chi_1} : k^\times \to \mathbb{C}^\times$, $\widetilde{\chi_1}(x) = \chi_1(x)\|x\|^{\frac{1}{2}}$

$\widetilde{\chi_2} : k^\times \to \mathbb{C}^\times$, $\widetilde{\chi_2}(x) = \chi_2(x)\|x\|^{-\frac{1}{2}}$.

Q) Is $I(\chi_1, \chi_2)$ irreducible ?

Easy : No.

$I^{\text{naive}}(\chi_1, \chi_2) = \{ \varphi : GL_2(k) \to \mathbb{C}$ l. const. ,

$\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)\varphi(g) \}$.

$I^{\text{naive}}(\chi_1, \chi_2) = I(\chi_1 \|\cdot\|^{-\frac{1}{2}}, \chi_2 \|\cdot\|^{\frac{1}{2}})$.

Note if $\chi_1 = \chi_2$ = trivial rep. $(\mathrm{Ind}(\chi), \rho) = (\chi, \mathrm{res}\,\rho)$

Turns out $\exists$ another class of examples where $I(\chi_1, \chi_2)$ is not irred ;

answer is something like : $\chi_1/\chi_2 = \|\cdot\|^2$ or possibly $\|\cdot\|^{-2}$.

What has happened here: there's a duality

$\exists$ natural pairing $I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \to \mathbb{C}$

involving an integral on G & on $B = \begin{pmatrix} * & x \\ 0 & * \end{pmatrix}$.

At some point, need to change a left Haar measure on B to a right Haar measure

- measures differ by a fudge factor $\|a/d\|$

This is where $\|a/d\|^{\frac{1}{2}}$ comes from: makes dual of $I(\chi_1, \chi_2) = I(\chi_1^{-1}, \chi_2^{-1})$.

Turns out that $I(\chi_1, \chi_2)$ is irreducible if $\chi_1 / \chi_2 \neq \|\cdot\|^{\pm 1}$

Ex. If $\chi_1 / \chi_2 = \|\cdot\|^{-1}$, then

$$0 \to \text{1-dim'l rep of } GL_2(k) \to I(\chi_1, \chi_2) \longrightarrow S(\chi_1, \chi_2) \to 0$$

$$g \mapsto \left(\chi_1 \times \|\cdot\|_k^{\frac{1}{2}}\right)(\det(g))$$

Fact. $S(\chi_1, \chi_2 = (\chi_1 \times \|\cdot\|))$ is irred.

If $\chi_1/\chi_2 = \|\cdot\|^{+1}$, then $\exists$ exact sequence

$$0 \to S(\chi_2, \chi_1) \to I(\chi_1, \chi_2) \to \left(\chi_2 \times \|\cdot\|^{\frac{1}{2}}\right) \circ \det \to 0$$

[Because $I(\chi_1, \chi_2)^\vee = I(\chi_1^{-1}, \chi_2^{-1})$]

Now let's take$^{\varepsilon}$ about a completely different construction:

§1 of Jacquet – Langlands;

|

Smooth + irred

$\Rightarrow$ adm.

Observation of Weil: If k is any field, Weil constructed a presentation of $SL_2(k)$.

$$\underbrace{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

generators + explicit obvious rel^{n}s

Upshot: can construct reps of $SL_2(k)$ by giving explicit actions of

$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on a v.s., + check rel^{n}s.

Weil observed that we can use eg. v.s. of L^2 functions on k $k \to \mathbb{C}$

$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$: $f(x) \mapsto f(tx)$.

$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$: $f(x) \mapsto f(u+x)$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: Fourier transform! explicit integral.

→ another source of reps of $SL_2(k)$ (sometimes $GL_2(K)$).

Fact. If $L|K$ is a quadratic extension & if $\chi : L^\times \to \mathbb{C}^\times$ admissible & if $\chi \neq \chi \circ \sigma$ ($1 \neq \sigma \in Gal(L|K)$), then Jacquet-Langlands construct an irred. ∞-dim'l rep^n $BC_L^K(\chi)$ of $GL_2(K)$

Thm 4.6 of JL, p72. [L^2-funcs on L + Fourier transform]

Facts

$I(\chi_1, \chi_2)$, $S(\chi, \chi \times \|\cdot\|)$, $BC_L^k(\psi)$ are all ∞-dim'l sm. adm.

$I(\chi_1, \chi_2)$ irred. if $\chi_1 / \chi_2 \neq \|\cdot\|^{\pm 1}$

$S(\chi, \chi\|\cdot\|)$ & $BC_L^k(\psi)$ irred.

Only isomorphism between these reps.

$$I(\chi_1, \chi_2) \simeq I(\chi_2, \chi_1), \quad \chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$$

Amazing: If $\mathrm{char}(k_K) = p > 2$, these are all of the ∞-dim'l irred. adm. reps of $GL_2(K)$.

Only f.d. irred. adm. reps of $GL_2(K)$ are 1-dim'l & of form

$$\chi \circ \det, \quad \chi : K^\times \to \mathbb{C}^\times.$$

Lecture 9. Recall we've seen the following examples of smooth irred. adm. rep^{n}s of $GL_2(K)$: $I(\chi_1, \chi_2) = \{\varphi : GL_2(K) \to \mathbb{C}, \dots\}$, $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$

$S(\chi, \chi \times \|\cdot\|)$ (a sub / quotient of I)

$\chi \circ \det$ (1-dim'l)

$BC_L^k(\psi)$ ($\psi : L^\times \to \mathbb{C}^\times$ adm., $\psi \neq \psi \circ c$).

Fact: If $p \neq 2$, this is all the smooth irred. reps of $GL_2(K)$, $p = \mathrm{char}(k_K)$.

Conductors

Stick to π irred. adm. rep. of $GL_2(K)$, assume $\dim(\pi) = \infty$.

[$G = GL_n/K$, or more generally, G any conn'd reductive gp,

& π = smooth adm. rep of $G(K)$ there's a notion

" π is generic ".

If $G = GL_2/K$, π generic $\Leftrightarrow$ $\dim(\pi) = \infty$]

Thm of Casselman (Antwerp proceedings) For $n \geq 0$, define

$$U_1(p_K^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^n \right\}.$$

The $U_1(p_K^n)$ are all cpt & open.

$\therefore$ $d(\pi, n) = \dim \pi^{U_1(p_K^n)} < \infty$ (admissibility).

Casselman showed $\exists\, f(\pi) \in \mathbb{Z}_{\geq 0}$ st. $d(\pi, n) = \max(0, 1 + n - f(\pi))$ $\forall\, n \geq 0$. [Using Whittaker model]

Exercise of unvalued difficulty: check this for $I(\chi_1, \chi_2)$, $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$

Exercises that are definitely possible (assume Casselman).

1) $f(I(\chi_1, \chi_2)) = f(\chi_1) + f(\chi_2)$

2) $f(S(\chi, \chi\|\cdot\|)) = \begin{cases} 1, & f(\chi) = 0 \\ 2f(\chi), & f(\chi) > 0. \end{cases}$

Exercise

1) (Schur's Lemma) π = irred. adm. rep. of $GL_2(K)$ $(GL_n(K))$,

then $\exists$ adm. char. $\chi_\pi : K^\times \longrightarrow \mathbb{C}^\times$.

s.t. $\forall \lambda \in K^\times = Z(GL_n(K))$, λ acts on π via the scalar $\chi_\pi(\lambda)$.

$\lambda \longmapsto \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$

χ_π = central character of π.

2) $\chi_{I(\chi_1, \chi_2)} = \chi_1 \chi_2$

3) $\chi_{S(\chi_1, \chi_2)} = \chi_1 \chi_2$

4) $\chi_{\varphi \circ \det} = \varphi^2$, $\varphi : K^\times \longrightarrow \mathbb{C}^\times$

LLC for GL_2/K . (LLC for GL_1 assumed)

π's	Notation	ρ's
$I(\chi_1, \chi_2)$	$\pi_1 : K^\times \longrightarrow \mathbb{C}^\times$ $\updownarrow$ $\rho_1 : W_K \longrightarrow \mathbb{C}^\times$	$\begin{pmatrix} \rho_0 = \rho_1 \oplus \rho_2 \\ N = 0 \end{pmatrix}$.
$S(\chi_1, \chi_1\|\cdot\|)$		$\left(\rho_0 = \begin{pmatrix} \|\cdot\|\rho_1 & 0 \\ 0 & \rho_1 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$.
$\chi \circ \det$		$\begin{pmatrix} \rho_1 \cdot \|\cdot\|^{\frac{1}{2}} & 0 \\ 0 & \rho_1 \cdot \|\cdot\|^{-\frac{1}{2}} \end{pmatrix}$, $N = 0$
$BC_L^K(\psi)$, $\psi : L^\times \to \mathbb{C}^\times$, $\psi \leftrightarrow (\sigma : W_L \to \mathbb{C}^\times)$,		$(\mathrm{Ind}_{W_L}^{W_K} \sigma, N = 0)$.

$(p=2)$

extra stuff

$(p=2)$

extra stuff.

/

e.g. turns out ∃ S_4 extension of $\mathbb{Q}_2$.

$$\begin{tikzcd} W_{\mathbb{Q}_2} \arrow[r, two heads] \arrow[drr, dashed] & S_4 \arrow[r, hook] & PGL_2(\mathbb{C}) \\ & & GL_2(\mathbb{C}) \arrow[u] \end{tikzcd}$$

For GL_2/K, this is actually how LLC were proved.

Bernstein - Zelevinsky

⌈ GL_n/K: Use rep^n theory technique to reduce the problem to matching

irred. (ρ_0, N)'s w/ super cuspidal π's (e.g. $BC_L^K(\psi)$).

Matching done via a global argument.

(

uses number fields ⌋

(ρ_0, N) def^ns: Say (ρ_0, N) = WD rep^n.

$$f(\rho_0, N) := f(\rho_0) + \dim\left(V^{I_{F|K}} / (\ker N)^{I_{F|K}} \right)$$

$(=0$ if $N=0)$

Note also $\det(\rho_0) : W_K \to \mathbb{C}^\times$

Ex. To the extend that this is possible for you, check that

if $\pi \longleftrightarrow (\rho_0, N)$ via LLC GL_2/K,

then $f(\pi) = f(\rho_0, N)$.

& $\chi_\pi \longleftrightarrow \det \rho_0$ via LCFT.

Ex. Check that if $p > 2$, then already listed all the F-semisimple 2-dim'l WD-repns of W_K.

[Tate "Number theory background" 2.2.5.2 will help]

(irred, not-induced rep of W_K has dim $= p^d$.

Finish LLC story by talking explicitly about $f(\pi)=0$ case

– the "unramified" case.

Turns out that $(\pi \longleftrightarrow (\rho_0, N))$

If $f(\pi) = 0 = f(\rho_0, N)$,

then $\pi = I(\chi_1, \chi_2)$, $\chi_1, \chi_2 : K^\times \longrightarrow K^\times/\mathcal{O}_K^\times \longrightarrow \mathbb{C}^\times$ (& $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$)

or $\pi = \chi \circ \det$,
$$\begin{tikzcd} K^\times \arrow[r, "\chi"] \arrow[dr] & \mathbb{C}^\times \\ & K^\times/\mathcal{O}_K^\times \arrow[u] \end{tikzcd}$$

& on W_K side. $\rho = \rho_1 \oplus \rho_2$, $\rho_i : W_K \to W_K/I_{\bar{K}|K} \longrightarrow \mathbb{C}^\times$ & $N = 0$.

Say π is ∞-dim'l, & $f(\pi) = 0$ $\left(\therefore \pi = I(\chi_1, \chi_2), \cdots \right)$

$f(\pi) = 0 \Rightarrow \pi^{GL_2(\mathcal{O}_K)}$ has dim $\underline{1}$.

[General case: G/K conn'd reductive. Assume G is unramified (eg. GL_n), π = smooth adm. irred. rep. of $G(K)$.

We say π is unramified, if $\exists$ hyperspecial max'l cpt subgroup $H \subset G(K)$ s.t. $\pi^H \neq 0$. (eg. $H = GL_n(\mathcal{O}_K) \subset GL_n(K)$).]

Want to do calculations w/ π.

$\pi^{GL_2(\mathcal{O}_K)}$ = concrete place to start

— not $GL_2(K)$-invt though.

<u>Trick.</u> Use Hecke operators.

$G = GL_2(K)$ (or any locally cpt totally disconn'd top. gp)

If π = adm. rep. of G, & if $U, V \subset G$ are cpt open subgps, (eg. $U = U_1(\mathfrak{p}_K^n)$ or $GL_2(\mathcal{O}_K)$ ...) & if $g \in G$.

then $\exists$ Hecke operator

$$[UgV] : \pi^V \longrightarrow \pi^U.$$

defined thus: write $\underset{\substack{(\text{cpt subset}\\ \text{of } G)}}{UgV} = \coprod_{i=1}^{r} g_i V$ (finite, V open)

$$\& \ [UgV] \underset{\underset{\pi^V}{\uparrow}}{x} := \sum_{i=1}^{r} g_i x .$$

- Some kind of averaging / trace process.

$\underline{\text{Ex}}$. $[UgV]x \in \pi^U$ & is indep. of choices of g_i.

Back to $GL_2(k) \curvearrowright \pi$, $f(\pi) = 0$.

$U = V = GL_2(\mathcal{O}_k)$.

$\underline{\text{Defn}}$: $T = \left[U \begin{pmatrix} \pi_k & 0 \\ 0 & 1 \end{pmatrix} V \right] : \pi^{GL_2(\mathcal{O}_k)} \circlearrowleft$

$S = \left[U \begin{pmatrix} \pi_k & 0 \\ 0 & \pi_k \end{pmatrix} V \right] : \pi^{GL_2(\mathcal{O}_k)} \circlearrowleft$

$\dim \pi^{GL_2(\mathcal{O}_k)} = 1$, T acts via scalar $t \in \mathbb{C}$

S ———— $s \in \mathbb{C}$.

<u>Perfect exercise</u>

If $\pi = I(\chi_1, \chi_2)$, $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$, $f(\pi) = 0$, then

$t = \sqrt{q_k}\,(\alpha + \beta)$, $s = \chi_\pi(\pi_k) = \alpha\beta$

w/ $\alpha = \chi_1(\pi_k)$, $\beta = \chi_2(\pi_k)$.

Also do 1-dim'l unram. case.

As a consequence, show that if π = adm. irred. rep. of $GL_2(k)$ & $f(\pi) = 0$, (assume $\pi = I(\chi_1, \chi_2)$ or 1-dim'l), then $\pi \longleftrightarrow (\rho_0, N)$, where $N = 0$, $\rho_0 : W_K \to W_K / I_{E|K} \cong \underset{\substack{\Psi \\ \text{Frob}}}{\mathbb{Z}} \longrightarrow GL_2(\mathbb{C})$

w/ $\rho_0(\text{Frob})$ having char. poly $X^2 - \frac{t}{\sqrt{q_K}} X + S$.

More ambitious: $G = GL_n$, $$T_i = \left[GL_n(\mathcal{O}_K) \begin{pmatrix} \overbrace{\begin{matrix} \pi_K & & \\ & \pi_K & \\ & & \ddots \;\, \pi_K \end{matrix}}^{i} & \\ & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix} GL_n(\mathcal{O}_K) \right]$$

T_i eigenval. t_i.

What is char. poly. of $\rho_0(\text{Frob})$?

[If $G = G(K)$, G/K unramified & if π is an unramified rep^n of G, then Langlands' re-interpretation of the Satake isomorphism associates to π a semisimple conj. class in ${}^L G(\mathbb{C})$]

$\pi \longrightarrow \rho_0$: $\rho_0(\text{Frob})$ = this conj. class.

Lecture 10 Part 2: The global Langlands correspondence

In this part, K will be a number field, i.e. a fin. extn of $\mathbb{Q}$

Start by talking about structure of $\mathrm{Gal}(\underset{\text{fin. Galois extn}}{L} | K)$

& in particular, its relationship to local Galois groups.

Taking limits, we get structure on $\mathrm{Gal}(E|K)$

Global analogue of Weil-Deligne rep may be "representation of global Langlands group".

However, we still have ℓ-adic representations of $Gal(\bar{K}|K)$

- working defn of "ρ" side
- π side: automorphic representations

Uncheckable conjecture: all π's for $GL_n/K \longleftrightarrow$ n-dim'l reps of global Langlands gp
(Checkable for GL_1?)

$(\text{Langlands gp})^{ab} = K^\times \backslash \mathbb{A}_K^\times$

Checkable conjecture:

$$\begin{tikzcd}
 & \text{algebraic auto. reps } \pi \text{ of } GL_n/K \arrow[dl] \arrow[d, leftrightarrow] \arrow[r, leftrightarrow] & \text{"nice" } \left(\begin{array}{l} \rho: Gal(\bar{K}|K) \to GL_n(\bar{\mathbb{Q}}_\ell) \\ \text{unram. outside finite set,} \\ \text{de Rham} \end{array}\right) \arrow[dl, leftrightarrow] \\
\left(\begin{array}{l}\text{p-adic auto} \\ \text{reps of } GL_n/K\end{array}\right) \arrow[rr, dashed, leftrightarrow] & \text{Motives} & (\rho\text{'s to } GL_n(\bar{\mathbb{Q}}_p), \text{ drop de Rham?})
\end{tikzcd}$$

$n=1$: will find that "uncheckable conj" = Global CFT.

Galois gp: K finite ext. of $\mathbb{Q}$, $K \supset \mathcal{O}_K$ = alg. integers in K

e.g. $K = \mathbb{Q}$, $\mathcal{O}_K = \mathbb{Z}$.

Choose $0 \neq \mathfrak{p}$ a prime ideal (hence maximal) of $\mathcal{O}_K$,

$\mathcal{O}_K/\mathfrak{p} = k_\mathfrak{p}$ = residue field = finite field.

Can complete K at $\mathfrak{p}$, eg. can define

$$\mathcal{O}_{K\mathfrak{p}} := \varprojlim \mathcal{O}_K/\mathfrak{p}^n, \quad \& \quad K_{\mathfrak{p}} = \mathrm{Frac}(\mathcal{O}_{K\mathfrak{p}})$$

Another approach: $\mathfrak{p}$ fixed, if $\lambda \in K^\times$, then $\lambda \mathcal{O}_K$ = fractional ideal of K.

$\therefore$ factors as $\mathfrak{p}^{v_\mathfrak{p}(\lambda)} \times$ (other prime ideals to various powers)

$v_\mathfrak{p} : K^\times \twoheadrightarrow \mathbb{Z}$, can define $\|\lambda\|_\mathfrak{p}$, norm on K,

$\|0\|_\mathfrak{p} = 0$, $\quad \|\lambda\|_\mathfrak{p} = (q_\mathfrak{p})^{-v_\mathfrak{p}(\lambda)}$

$\nwarrow \# k_\mathfrak{p}$

norm on K $\longrightarrow$ metric $d(x,y) = \|x-y\|_\mathfrak{p}$

Complete K wrt. this metric & get $K_\mathfrak{p}$ = local field.

$K_\mathfrak{p}$ = finite extn of $\mathbb{Q}_p$, where $\mathfrak{p} \cap \mathbb{Z} = (p)$.

Now say $L|K$ is a finite Galois extn, get finite Galois gp $\mathrm{Gal}(L|K)$.

Say $\mathfrak{p} \subset \mathcal{O}_K$ as above., $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$

$\mathfrak{p} \rightsquigarrow \mathfrak{p}\mathcal{O}_L$ = nonzero ideal of $\mathcal{O}_L$

$\mathfrak{p}\mathcal{O}_L$ factors into primes of L: $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\dots\mathfrak{P}_g^{e_g}$,

$\mathfrak{P}_i$: prime ideals of $\mathcal{O}_L$.

$\mathrm{Gal}(L|K)$ acts on L $\therefore$ acts on $\mathcal{O}_L$.

Acts trivially on K & hence on $\mathfrak{p}$

$\therefore$ fixes the ideal $\mathfrak{p}\mathcal{O}_L$ (as a set, not pointwise)

If $\sigma \in \mathrm{Gal}(L|K)$, then $\sigma(\mathfrak{p}_i)$ is a prime ideal of $\mathcal{O}_L$

$\mathfrak{p}_i$ divides $\mathfrak{p}$ $\therefore$ $\sigma(\mathfrak{p}_i)$ divides $\sigma(\mathfrak{p}) = \mathfrak{p}$.

X, Y objects in math defined by axioms w/ structures

$i : X \to Y$ isomorphism

$(*) =$ calculation of some kind in X "transport de structure"

$i(*) =$ calculation in Y

Thinking clearly about that trivial observation in the particular case where $X = Y$ & $i \neq$ identity can sometimes really help.

Eg. $X = Y = L$, $\sigma = i \in \mathrm{Gal}(L|K)$.

In particular, $\mathrm{Gal}(L|K)$ acts on $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_g\}$.

Fact: action is transitive.

Cor: all e_i are the same.

Cor. $L_{\mathfrak{p}_1} \cong L_{\mathfrak{p}_2} \cong \cdots \cong L_{\mathfrak{p}_g}$

Set-up: $L|K$ finite Galois, $\mathfrak{p}$ as before, $\mathfrak{p}\mathcal{O}_L = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\ldots\mathfrak{p}_g^{e_g}$.

Set $\tilde{P} = P_1$, fixed choice of prime of L

Define $D_{\tilde{\mathfrak{p}}} := \{ \sigma \in \mathrm{Gal}(L|k) : \sigma(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}} \}$

$$\mathrm{Gal}(L|k)/D_{\tilde{\mathfrak{p}}} = \{ P_1, \dots, P_g \}.$$

If $\sigma \in D_{\tilde{\mathfrak{p}}}$, $\sigma : L \to L$, $\sigma : \mathcal{O}_L \to \mathcal{O}_L$, $\sigma : \tilde{\mathfrak{p}} \to \tilde{\mathfrak{p}}$

By transport de structure, get $\sigma : L_{\tilde{\mathfrak{p}}} \to L_{\tilde{\mathfrak{p}}}$ fixing $k_{\mathfrak{p}}$.

Turns out that $L_{\mathfrak{p}} | k_{\mathfrak{p}}$ is Galois,

$$\begin{tikzcd} D_{\tilde{\mathfrak{p}}} \cong \mathrm{Gal}(L_{\tilde{\mathfrak{p}}} | k_{\mathfrak{p}}) \arrow[d, hook] & \leftarrow \text{local Galois gp} \\ \mathrm{Gal}(L|k) & \end{tikzcd}$$

- $\mathrm{Gal}(L|k)$, choose $\mathfrak{p}$ of $\mathcal{O}_k$, choose $\tilde{\mathfrak{p}} | \mathfrak{p}\mathcal{O}_L$

$\rightsquigarrow$ $\mathrm{Gal}(L|k) \supset D_{\tilde{\mathfrak{p}}} = \mathrm{Gal}(L_{\tilde{\mathfrak{p}}} | k_{\mathfrak{p}})$

$\mathrm{Gal}(L_{\tilde{\mathfrak{p}}} | k_{\mathfrak{p}}) \supset$ Inertia subgp

<u>Global fact.</u> If $\mathfrak{p} \nmid$ discriminant of $L|k$, then this inertia subgp is trivial. $L_{\tilde{\mathfrak{p}}}$ is an unramified extn of $k_{\mathfrak{p}}$, $\therefore \exists\, \mathrm{Frob}_{\tilde{\mathfrak{p}}} \in D_{\tilde{\mathfrak{p}}}$

$$D_{\tilde{\mathfrak{p}}} \subset \mathrm{Gal}(L|k).$$

Slightly annoying thing: $\mathrm{Frob}_{\tilde{\mathfrak{p}}}$ depends not only on $\mathfrak{p}$, but a choice of $\tilde{\mathfrak{p}} | \mathfrak{p}\mathcal{O}_L$

Say $\tilde{\mathfrak{p}}'$ another choice, By transitivity, $\exists\ \sigma \in \mathrm{Gal}(L|K)$, $\sigma(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}}'$.

Transport of structure: $D_{\tilde{\mathfrak{p}}'} = \sigma D_{\tilde{\mathfrak{p}}} \sigma^{-1}$ & $\mathrm{Frob}_{\tilde{\mathfrak{p}}'} = \sigma \,\mathrm{Frob}_{\tilde{\mathfrak{p}}}\, \sigma^{-1}$

Upshot: can define $\mathrm{Frob}_{\mathfrak{p}}$ = conj. class of $\mathrm{Frob}_{\tilde{\mathfrak{p}}}$.

$$= \{ \mathrm{Frob}_{\tilde{\mathfrak{p}}'} : \tilde{\mathfrak{p}}' \mid \mathcal{O}_L \mathfrak{p} \}$$

Works for all $\mathfrak{p} \nmid \mathrm{disc}(L|K)$.

Lecture 11

$L|K$ finite Galois, $K \supset \mathcal{O}_K \supset \mathfrak{p}$ = non-zero prime ideal

$L \supset \mathcal{O}_L \supset \mathfrak{p}\mathcal{O}_L$ = probably not prime

Factorize $\tilde{\mathfrak{p}}^e \times$ other stuff. $\mathrm{Gal}(L|K) \supset D_{\tilde{\mathfrak{p}}/\mathfrak{p}} = \{\sigma \in \mathrm{Gal}(L|K) : \sigma(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}}\}$

Fact. If $\sigma \in D_{\tilde{\mathfrak{p}}/\mathfrak{p}}$, then $\sigma : L \to L$ σ fixes K ptwise

$\sigma : \mathcal{O}_L \to \mathcal{O}_L$ σ fixes $\mathcal{O}_K$ "

$\sigma : \tilde{\mathfrak{p}} \to \tilde{\mathfrak{p}}$ σ fixes $\mathfrak{p}$

$\sigma : \mathcal{O}_L/\tilde{\mathfrak{p}}^n \circlearrowleft$ $\rightsquigarrow$ $\sigma : L_{\tilde{\mathfrak{p}}} \to L_{\tilde{\mathfrak{p}}}$, fixes $K_{\mathfrak{p}}$.

$$\sigma \in \mathrm{Gal}(L_{\tilde{\mathfrak{p}}} | K_{\mathfrak{p}}) = D_{\tilde{\mathfrak{p}}|\mathfrak{p}} \supset I_{\tilde{\mathfrak{p}}|\mathfrak{p}} = I_{L_{\tilde{\mathfrak{p}}}|K_{\mathfrak{p}}}.$$

Miracle: $0 \neq \Delta = \mathrm{disc}(L|K) \subset \mathcal{O}_K$, & $\mathfrak{p} \nmid \Delta \Rightarrow I_{\tilde{\mathfrak{p}}|\mathfrak{p}} = \{1\}$, $\forall\ \tilde{\mathfrak{p}}|\mathfrak{p}$.

$\therefore$ For all $\mathfrak{p} \notin$ finite set $S = \{$primes of $\mathcal{O}_K$ dividing $\Delta\}$, get $\forall\ \tilde{\mathfrak{p}}|\mathfrak{p}$, a cyclic group $D_{\tilde{\mathfrak{p}}|\mathfrak{p}} = \langle \mathrm{Frob}_{\mathfrak{p}} \rangle = D_{\tilde{\mathfrak{p}}|\mathfrak{p}} / I_{\tilde{\mathfrak{p}}|\mathfrak{p}}$

$= \mathrm{Gal}(k_{\tilde{\mathfrak{p}}} | k_{\mathfrak{p}})$.

$\tilde{\mathfrak{p}}$ upstairs. get element $\mathrm{Frob}_{\tilde{\mathfrak{p}}} \in \mathrm{Gal}(L|K)$.

$\mathfrak{p}$ downstairs : get a bunch of conjugate elements $\{\mathrm{Frob}_{\tilde{\mathfrak{p}}} : \tilde{\mathfrak{p}} | \mathfrak{p}\}$.

get a conjugacy class $\mathrm{Frob}_{\mathfrak{p}}$.

Fact. Given $L|K$ as above, every conjugacy class C in $\mathrm{Gal}(L|K)$ turns out to equal $\mathrm{Frob}_{\mathfrak{p}}$ for ∞ly many primes $\mathfrak{p}$.

In fact, the density of primes $\mathfrak{p}$ s.t. $\mathrm{Frob}_{\mathfrak{p}} = C$ is $\#\left(C/\mathrm{Gal}(L|K)\right)$

Variant for infinite extensions: K number field. Fix an algebraic closure $\bar{K}$ of K.

$\mathrm{Gal}(\bar{K}|K)$ is ramified at every prime of K.

Let S = finite set. of max. ideals of $\mathcal{O}_K$

If $\underbrace{K \subset L_1}_{\text{finite}} \subset \bar{K}$, $\underbrace{K \subset L_2}_{\text{finite}} \subset \bar{K}$, & if L_1, L_2 unramified outside S

(i.e. $\mathfrak{p} \mid \mathrm{disc}(L_i|K) \Rightarrow \mathfrak{p} \in S$), then $L_1 L_2$ is unram. outside S too.

Define $K^S = \bigcup_{\substack{L|K \text{ finite (Galois)} \\ \text{unramified outside } S}} L$

[ex. $S = \phi$, $K = \mathbb{Q} \Rightarrow K^S = \mathbb{Q}$

Crazy example. $K = \mathbb{Q}\left(\sqrt{-2\times 3\times 5 \cdots}\right)$
$\forall p < 10^6$,

$S = \phi$, $K^S | K$ infinite! [Golod-Shafarevich]]

$K = \mathbb{Q}$, $S = \{p\}$. p fixed prime.

$K^S \supset \mathbb{Q}(\zeta_p)$ = spl. field of $X^p - 1$.

& in fact $K^S \supset \mathbb{Q}(\zeta_{p^n})$, $\forall n \geq 1$.

$K = \mathbb{Q}$, $N \in \mathbb{Z}_{\geq 1}$, $S = \{p : p \text{ prime}, p | N\}$

$\mathbb{Q}(\zeta_N) | \mathbb{Q}$ unramified outside S.

$\mathrm{Gal}(\mathbb{Q}(\zeta_N) | \mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$

If $p \in \mathbb{Z}$ is prime, $p \notin S$ (i.e. $p \nmid N$), then Frob_p = a canonical conjugacy class in $(\mathbb{Z}/N\mathbb{Z})^\times$

$\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times$
$\sigma \longmapsto t$, $\sigma(\zeta) = \zeta^t$, $\forall \zeta, \zeta^N = 1$.

$\therefore$ on residue field, $\sigma(\zeta) = \zeta^t$. $\mathrm{Frob}_p(x) = x^p$ on res. field. $\therefore t = p$ works.

$K^S | K$: if $K = \mathbb{Q}$, $S = \{p\}$, then $K^S \supset \mathbb{Q}(\zeta_{p^n})$, $\forall n \geq 1$

$\therefore K^S \supset \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n}) =: \mathbb{Q}(\zeta_{p^\infty})$

$\therefore$ $\mathrm{Gal}(K^S|K) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})|\mathbb{Q}) = \varprojlim \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})|\mathbb{Q})$

$K = \mathbb{Q}$, $S = \{p\}$.

$$= \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times.$$

If r is a prime number, $r \neq p$, then $\mathrm{Frob}_r \in (\mathbb{Z}/p^n\mathbb{Z})^\times$

$$= \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})|\mathbb{Q})$$

is just $r \in (\mathbb{Z}/p^n\mathbb{Z})^\times$.

Take $\varprojlim$ & get $\mathrm{Frob}_r \in \mathbb{Z}_p^\times$

General story: if K is a number field, S = finite set of finite places of K.

$$\mathrm{Gal}(K^S|K) = \varprojlim_{\substack{L|K \\ \text{finite, unram. outside } S \\ \text{Galois}}} \mathrm{Gal}(L|K)$$

$\forall\ \mathfrak{p} \notin S$, get conj. class. $\mathrm{Frob}_{\mathfrak{p}, L|K} \underset{\text{conj. class}}{\subset} \mathrm{Gal}(L|K)$

These glue together to produce a conj. class

$$\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}, K^S|K} \subset \mathrm{Gal}(K^S|K) \quad \text{a conj. class.}$$

Chebotarev density for finite ext^{n}s said

$$\begin{aligned} \{\mathfrak{p} \notin S\} &\longrightarrow \{\text{conj. classes in } \mathrm{Gal}(L|K)\} \\ \mathfrak{p} &\longmapsto \mathrm{Frob}_{\mathfrak{p}} \end{aligned}$$

is surjective.

Cor. If $L|K$ is infinite, Galois, unramified outside S, then the union of the conj. classes $\{\mathrm{Frob}_\mathfrak{p} : \mathfrak{p} \notin S\}$ is a dense subset of $\mathrm{Gal}(L|K)$.

Cor. If $F : \mathrm{Gal}(L|K) \to X$ is continuous, & constant on conj. classes then we may well be able to recover F from the data of $F(\mathrm{Frob}_\mathfrak{p})$, $\mathfrak{p} \notin S$.

Brauer-Nesbitt thm: G group, E field, Recall $\rho : G \to GL_n(E)$ is said to be semisimple if $\rho = \bigoplus$ irred. rep^{n}s.

If $\rho_1, \rho_2 : G \to GL_n(E)$ are 2 semisimple rep^{n}s, & if $\forall g \in G$, char. poly. of $\rho_1(g)$ = char. poly. of $\rho_2(g)$, then $\rho_1 \cong \rho_2$.

⌈ $\mathcal{Y}$: analogue for general $H \hookrightarrow GL_n(E)$ fails ⌋.

Rmk. If $\mathrm{char}(E) = 0$, then $\underset{\text{semisimple}}{\mathrm{trace}\ \rho_1 = \mathrm{trace}\ \rho_2} \Rightarrow \rho_1 \cong \rho_2$.

as can compute char. poly. $(\rho(g))$ from trace $\rho(g^i)$, $0 \le i \le n$

if $\div n!$ is ok.

⌈ non-example, $G = \mathbb{Z}/3\mathbb{Z}$, $E = \overline{\mathbb{F}}_2$, $n=2$, find non-iso. ρ_1, ρ_2 semisimple reducible s.t. trace ρ_1 = trace ρ_2 ⌋ (both 0)

Upshot: If $\rho : \mathrm{Gal}(K^S|K) \to GL_n(E)$, $E|\mathbb{Q}_\ell$ finite ext^n.

(ℓ-adic topology.

is a continuous semisimple rep.,

& if I happen to know char. poly. of $\rho(\mathrm{Frob}_p)$.

$\forall p \notin S$, ⌈ Rmk: this is a well-defined poly. $F_p \in E[X]$ ⌋

then ρ is uniquely determined by this data.

Example: $K = \mathbb{Q}$, $S = \{p\}$, $L = \mathbb{Q}(\zeta_{p^\infty})$.

$$\mathrm{Gal}(L|K) = \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times = GL_1(\mathbb{Z}_p) \subset GL_1(\mathbb{Q}_p)$$

Then

$$\begin{tikzcd}
 & \mathrm{Gal}(\mathbb{Q}^S|\mathbb{Q}) \ni (\mathrm{Frob}_r) \arrow[d] & \\
\rho : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \arrow[ur] \arrow[r, hook, two heads] & \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})|\mathbb{Q}) \ni (\mathrm{Frob}_r) \arrow[d, equal] & r \neq p. \\
\mathrm{Frob}_r \in \arrow[d, equal] & \mathbb{Z}_p^\times \arrow[r] & GL_1(\mathbb{Q}_p) \\
r & &
\end{tikzcd}$$

ρ is called the p-adic cyclotomic character. & by (BN + Č), ρ is determined by the fact that $\rho(\mathrm{Frob}_r) = r$, $\forall\, r \neq p$.

Let's call the p-adic cyclotomic character ω_p. (non-standard terminology)

Quite a confusing thing: Let p & ℓ be 2 different primes.

$$\begin{aligned}
S = \{p, \ell\},\quad \omega_p &: \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^S|\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})|\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p^\times \\
\omega_\ell &: \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^S|\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})|\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_\ell^\times
\end{aligned}$$

Note : Frob_r, $r \notin S$ $\overset{\cup \, \mathrm{Frob}_r}{}$ give a dense subset of $\mathrm{Gal}(\mathbb{Q}^S | \mathbb{Q})$.

$\omega_p(\mathrm{Frob}_r) = r$, $\omega_\ell(\mathrm{Frob}_r) = r$.

Want Brauer - Nesbitt to imply $\omega_p = \omega_\ell$. No good !

B-N is about 2 representations to $GL_n(E)$.

One of our E is $\mathbb{Q}_p$, other is $\mathbb{Q}_\ell$.

In fact, ω_p & ω_ℓ couldn't be more non-isomorphic !

$$\begin{tikzcd}[column sep=small]
\mathbb{Q}(\zeta_{p^\infty}) \arrow[rr, phantom, "\text{totally disjoint}"] & & \mathbb{Q}(\zeta_{\ell^\infty}) \\
& \mathbb{Q} \arrow[ul, dash, "\ker \omega_p"] \arrow[ur, dash, "\ker \omega_\ell"'] &
\end{tikzcd}$$

Lecture 12. Another weird example.

K number field, ($K = \mathbb{Q}$ fine) . $E|K$ elliptic curve,

$S_0 =$ finite set of finite $\overset{\frown \text{ max. ideals of } \mathcal{O}_K}{\text{places}}$ of K. where E has bad reduction.

ℓ prime number. $E[\ell^n](\overline{K}) \circlearrowleft \mathrm{Gal}(\overline{K}|K)$

$$\varprojlim_n : \mathrm{Gal}(\overline{K}|K) \longrightarrow GL_2(\mathbb{Z}_\ell) \quad (\text{well-defined up to conjugation})$$
$$= T_\ell E.$$

Fact. If $\mathfrak{p} \notin S_0$, $\mathfrak{p} \nmid \ell$, **Fact**. $\rho_{E,\ell}$ factors through $\mathrm{Gal}(K^{S_0 \cup \{\mathfrak{p} | \ell\}} | K)$

$$\text{char poly}\left(\rho_{E,\ell}(\mathrm{Frob}_\mathfrak{p})\right) = X^2 - a_\mathfrak{p} X + (N\mathfrak{p}) \in \mathbb{Q}[X] \hookrightarrow \mathbb{Q}_\ell[X].$$

$$a_\mathfrak{p} = 1 + N\mathfrak{p} - \# \tilde{E}(k_\mathfrak{p}).$$

" Trace $\rho_{E,\ell}(\mathrm{Frob}_\mathfrak{p})$ is $a_\mathfrak{p}$, <u>independent</u> of ℓ".

$\rho_{E,\ell} \not\cong \rho_{E,\ell'}$ if $\ell \neq \ell'$.

$\uparrow$ wildly ramified at ℓ

wild inertia

much better behaved at ℓ.

$\rho_{E,\ell'}$ (wild inertia @ ℓ) = finite.

ℓ-adic representations

K number field. E: finite ext. of $\mathbb{Q}_\ell$

S finite set of max. ideals of $\mathcal{O}_K$.

If $\rho : \mathrm{Gal}(K^s|K) \longrightarrow GL_n(E)$ is continuous w.r.t. ℓ-adic topology on RHS
profinite top. on LHS.

We call ρ an **ℓ-adic rep.** of $\mathrm{Gal}(\bar{K}|K)$. (Say ρ is unramified outside S).

We say ρ is <u>rational over E_0</u> $(E_0 \subset E)$, if $\forall\ \mathfrak{p} \notin S$, char. poly. of $\rho(\mathrm{Frob}_\mathfrak{p}) \in E[X]$ is in $E_0[X]$.

<u>Eg:</u> $\rho = \omega_\ell$ cyclo. char. $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to GL_1(\mathbb{Q}_\ell)$

$\rho(\mathrm{Frob}_r) = r$, $\forall \neq \ell$

$\therefore$ ρ is rational over $\mathbb{Q}$.

Tate module of elliptic curve, rational | $\mathbb{Q}$.

$\rho = H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell)$. ℓ-adic étale coh. of smooth proper alg. var., rational over $\mathbb{Q}$ (perhaps a famous theorem of Deligne).

⌈ Another defn: ρ is <u>pure of weight w</u>, if ρ is rational over some number field E_0, & $\forall\, i : \overline{E_0} \hookrightarrow \mathbb{C}$, $\forall$ eigenvalues α of $\rho(\mathrm{Frob}\,\mathfrak{p})$,

$|i(\alpha)| = q_{\mathfrak{p}}^{-w/2}$, $q_{\mathfrak{p}} = \# k_{\mathfrak{p}} = \#(\mathcal{O}_K/\mathfrak{p})$ ⌋

Deligne: $H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell)$ pure of weight i. X smooth proper.

Ex. cyclo. char. pure wt -2 $\left(H^2(\mathbb{P}^1_{\bar{K}}, \mathbb{Q}_\ell) = \omega_\ell^{-1} \right)$.

Ex. T_ℓ (Ell. curve) pure wt -1

$\Updownarrow$

roots of $x^2 - a_p x + N\mathfrak{p}$ are complex conjugate

$\hookrightarrow$ $|a_p| \le 2\sqrt{N\mathfrak{p}}$.

Now ℓ will vary! <u>Setup</u>: K number field, S_0 = finite set of finite places.

Also, given the following data, $\forall\, \mathfrak{p} \notin S_0$, a polynomial $F_{\mathfrak{p}}(x) \in E_0[x]$ (E_0 number field)

Say also that $\forall$ max. ideal $\lambda \subset \mathcal{O}_{E_0}$, we have an $(E_0)_\lambda$ = finite ext. of $\mathbb{Q}_\ell$, $\lambda | \ell$. we have an ℓ-adic rep^n $\rho_\lambda : \mathrm{Gal}(K^{S_0 \cup \{\mathfrak{p}|\ell\}}|K) \longrightarrow GL_n(\overline{(E_0)_\lambda})$.

We say that ρ_λ is a <u>compatible system</u> of λ-adic rep^{n}s. if $\forall \lambda$, $(\lambda | \ell)$. $\forall \mathfrak{p} \notin S_0$, s.t. $\mathfrak{p} \nmid \ell$, $\rho_\lambda(\mathrm{Frob}_\mathfrak{p})$ has char. poly. $F_\mathfrak{p}(X)$. [indep. of λ].

<u>Examples</u>. cyclo. char. $F_p(X) = X - p$.

$T_\ell E$, $\forall \ell$, $E_0 = \mathbb{Q}$, $F_p(X) = X^2 - a_p X + Np$.

$H^i_{\acute{e}t}(\cdot -)$ known to be a compatible system.

[Cool generalization: use local Langlands, ρ_λ as above are <u>strongly compatible</u>

if $\forall \mathfrak{p} \in S_0$, $\mathfrak{p} | p$, $\forall \lambda$ ~~of~~ E_0, $\lambda \nmid p$,

$$\begin{tikzcd} \rho_\lambda \big|_{\mathrm{Gal}(\bar{K}_\mathfrak{p} | k_\mathfrak{p})} \arrow[r, "\text{Grothendieck}"] & \text{Weil-Deligne reps} \arrow[d, "LL"] \\ & \pi : GL_n(K_\mathfrak{p}) \to \infty\text{-dim'l space} \\ & \pi = \pi(\rho_\lambda, \mathfrak{p}) \arrow[u] \end{tikzcd}$$

Strongly compatible if this π is indep. of λ.

Unknown for étale cohomology of smooth proj. var.]

<u>Global class field theory</u>. i.e. what is $\mathrm{Gal}(\bar{k}|k)^{ab}$?

Answer involves adeles.

Infinite places of k: K number field, degree d over $\mathbb{Q}$, $[K:\mathbb{Q}] = d$.

There are d field embeddings $k \xrightarrow{\sigma} \mathbb{C}$. They're of 2 kinds:

1) $\mathrm{Im}(\sigma) = \sigma(k) \subset \mathbb{R}$. Let $r_1 = \#$ of such $\sigma = \#$ of real embeddings

2) $\mathrm{Im}(\sigma) \not\subseteq \mathbb{R}$, then $c \circ \sigma$ is another different field embedding $K \to \mathbb{C}$

$c: \mathbb{C} \to \mathbb{C}$ cx conj.

Upshot: the non-real σ's come in pairs $\sigma, c\sigma$

[σ, $c\sigma$ induce same norm on K].

Say there are $2r_2$ such maps.

$r_1 + 2r_2 =$ total # of σ's $= d$.

Eg. $K = \mathbb{Q}(\sqrt[3]{2})$, $r_1 = r_2 = 1$.

An <u>infinite place</u> v of K is either a real place $v = \sigma : K \to \mathbb{R}$

or a complex place $v = \{\sigma, c\sigma\}$. $\sigma : K \to \mathbb{C}$, $\mathrm{im}(\sigma) \not\subseteq \mathbb{R}$.

<u>Def</u>. $K_\infty = \prod_{v \text{ infinite}} K_v$, where if v is real, $K_v = \mathbb{R} \overset{\sigma}{\supset} K$

& if $v \leftrightarrow \{\sigma, c\sigma\}$ is complex, $K_v \cong \mathbb{C}$, & $K \overset{\sigma}{\hookrightarrow} \mathbb{C} = K_v$

<u>Rk</u>. $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$.

$K_\infty \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} =$ ring.

<u>Note</u>. $K_\infty^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ is not in general connected.

$(K_\infty^\times)^\circ \cong \mathbb{R}_{>0}^{r_1} \times (\mathbb{C}^\times)^{r_2}$

What's coming: $\mathbb{A}_K = \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}} \times K_\infty$.

Lecture 13. Weird infinite place defn.

$$\sigma : K \longrightarrow \mathbb{C}$$

equiv. rel. $\sigma \sim \sigma$, $\sigma \sim c\circ\sigma$ Equiv. classes = infinite places of K.

$$K_\infty = \bigoplus_{v \text{ an inf. place}} K_v = \prod_{v \mid \infty} K_v = K \otimes_{\mathbb{Q}} \mathbb{R}$$

Example. $K = \mathbb{Q}(\sqrt[3]{2})$. $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{Q}[x]/(x^3-2) \otimes_{\mathbb{Q}} \mathbb{R}$

$$= \mathbb{R}[x]/(x^3-2)$$

Roots of x^3-2 in $\mathbb{C}$, $\alpha = \sqrt[3]{2} \in \mathbb{R}_{>0}$, $w, \bar{w}$ other 2 roots. $w = \alpha e^{2\pi i/3}$, $\bar{w} = \alpha e^{-2\pi i/3}$

$$K_\infty = \mathbb{R}[x] \Big/ \underset{\nwarrow}{(x-\alpha)}\underset{\nearrow}{(x^2+\alpha x+\alpha^2)} \underset{CRT}{=} \underset{\underset{\mathbb{R}}{\parallel}}{\frac{\mathbb{R}[x]}{(x-\alpha)}} \times \underset{\underset{\mathbb{C}}{\wr\wr}}{\frac{\mathbb{R}[x]}{(x^2+\alpha x+\alpha^2)}}$$

irred. & coprime

$\sigma : K \longrightarrow \mathbb{C}$
$\sigma(\sqrt[3]{2}) = w$
$c \circ \sigma(\sqrt[3]{2}) = \bar{w}$

Def. (The adeles). K number field, $K \supset \mathcal{O}_K$
$\hookleftarrow$ infinitely many max'l ideals $\mathfrak{p}$
= "finite places".

$$K, \mathfrak{p} \rightsquigarrow \begin{matrix} K_{\mathfrak{p}} \\ \cup \\ \mathcal{O}_{K,\mathfrak{p}} \end{matrix}$$

Define. $\mathbb{A}_{K,f} :=$ finite adeles of K.

$$\mathbb{A}_{K,f} \subset \prod_{\mathfrak{p} \text{ finite place}} K_{\mathfrak{p}} \qquad (\text{RHS too big})$$

$$\mathbb{A}_{K,f} = \{ (x_{\mathfrak{p}})_{\mathfrak{p} \text{ finite place}} \in \prod_{\mathfrak{p}} K_{\mathfrak{p}} \text{ s.t. } x_{\mathfrak{p}} \in \mathcal{O}_{K,\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p} \}.$$

$$= \{ (x_{\mathfrak{p}}) \in \prod_{\mathfrak{p}} K_{\mathfrak{p}} \text{ s.t. } \exists S \text{ finite set of "bad" } \mathfrak{p} \text{ w/ the property that } x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}, \forall \mathfrak{p} \notin S \}$$

$\mathbb{A}_{K,f}$ is clearly a ring.

$K \longrightarrow \mathbb{A}_{K,f}$: diagonal embedding

$\overset{\Psi}{\lambda} = a/b$, $a,b \in \mathcal{O}_K$, $b \neq 0$ $\qquad S = \{\mathfrak{p} \text{ of } K : \mathfrak{p} \mid b\} = \{\mathfrak{p} : v_{\mathfrak{p}}(\lambda) < 0\}$ FINITE.

Topologize $\mathbb{A}_{K,f}$ by saying that the subring $\prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}$ is open w/ the usual topology.

Notation: $\mathbb{A}_{K,f}$ is sometimes written $\mathbb{A}_{K,f} = \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}}$ ← restricted product.

Def. The adeles of K are $\mathbb{A}_{K,f} \times K_\infty = \prod_{v \text{ all places}}{}' K_v$ (product topology)

Rmk Just as $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} = K \otimes_{\mathbb{Q}} \mathbb{Q}_\infty$, we have $\mathbb{A}_{K,f} = K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},f}$

& $\mathbb{A}_K = K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}$

$$\mathbb{Q} \ni \lambda \mapsto (\lambda,\lambda,\lambda,\cdots) \hookrightarrow \mathbb{A}_{\mathbb{Q},f}$$

Lemme. $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} + \prod_p \mathbb{Z}_p$. i.e. $\forall\, (x_p) \in \mathbb{A}_{\mathbb{Q},f}$.

$\forall$ p prime, $x_p \in \mathbb{Q}_p$, $x_p \in \mathbb{Z}_p$ for almost all p.

$\exists\, \lambda \in \mathbb{Q}$, & $\mu \in \prod_p \mathbb{Z}_p$, s.t. $x = \lambda + \mu$.

Ex. $\mathbb{A}_{K,f} = K + \prod_{\mathfrak{p}} \mathcal{O}_{K,\mathfrak{p}}$.

Turns out we're actually interested in $\mathbb{A}_K^\times$ = units of $\mathbb{A}_K$ = ideles of K

Non-example: $x_p = p$, $x = (x_p) \in \mathbb{A}_{\mathbb{Q},f}$, $\frac{1}{x} \in \prod \mathbb{Q}_p$, $\notin \mathbb{A}_{\mathbb{Q},f}$.

Can check $\mathbb{A}_{K,f}^\times = \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}}^\times$

$= (x_{\mathfrak{p}}) \in \prod_{\mathfrak{p}} K_{\mathfrak{p}}^\times$ s.t. $x_{\mathfrak{p}} \in \mathcal{O}_{K,\mathfrak{p}}^\times$ for almost all $\mathfrak{p}$.

& hence $\mathbb{A}_K^\times = \prod_v{}' K_v^\times$

Topology on $\mathbb{A}_{K,f}^\times$: $\prod_{\mathfrak{p}} \mathcal{O}_{K,\mathfrak{p}}^\times$ is open w/ usual topology.

Global class field theory.

$$\begin{tikzcd} \mathrm{Gal}(\bar{K}|K) \arrow[d, phantom, "\cup"] \\ \mathrm{Gal}(\bar{K}|K)^c \end{tikzcd} \qquad \begin{tikzcd} \bar{K} \arrow[dd, dash] \arrow[dr, dash] & \\ & K^{ab} \arrow[dl, dash, "\mathrm{Gal}(\bar{K}|K)^{ab}"] \\ K & \end{tikzcd}$$

$$\frac{\mathrm{Gal}(\bar{K}|K)}{\mathrm{Gal}(\bar{K}|K)^c} = \mathrm{Gal}(\bar{K}|K)^{ab}$$

= max. Hausdorff abelian quot.

More prosaically, choose $\bar{K} \supset K$, then

$$K^{ab} = \bigcup_{\substack{K \subset L \subset \bar{K} \\ L|K \text{ finite} \\ \text{Galois, } Gal(L|K) \\ \text{abelian}}} L$$

eg. $K = \mathbb{Q}$, $K^{ab} \supset \mathbb{Q}(\zeta_N)$, $\forall N \geq 1$.

$$\Rightarrow \quad K^{ab} \supset \bigcup \mathbb{Q}(\zeta_N).$$

<u>Thm (GCFT)</u>

$$K^\times \backslash \mathbb{A}_K^\times \xrightarrow{r_K} Gal(K^{ab}|K)$$

r_K cts gp hom. ("global Artin map").

$[K \xhookrightarrow{\text{diag}} \mathbb{A}_K \quad \therefore \quad \mathbb{A}_K^\times \supset K^\times]$

<u>Rk</u>. r_K can't be an iso. RHS = profinite

LHS $\supset K_\infty^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$

Note $(K_\infty^\times)^\circ \cong (\mathbb{R}_{>0})^{r_1} \times (\mathbb{C}^\times)^{r_2}$ must be in the kernel of r_K

as its image is connected in a totally disconnected group.

$\ker r_K$ = closed & contains image C_K of $(K_\infty^\times)^\circ$ in $K^\times \backslash \mathbb{A}_K^\times$.

<u>Thm</u>. $\ker r_K = \overline{C_K}$ ← topological closure of C_K.

Rmk. $K = \mathbb{Q}$ or im quad. field, C_k is closed, NOT in general.

Properties of r_K: $\forall\, \mathfrak{p}$ finite place,

$$\begin{tikzcd} k^\times \backslash \mathbb{A}_k^\times \arrow[r, "r_K"] & \mathrm{Gal}(\bar{k}|k)^{ab} \\ k_\mathfrak{p}^\times \arrow[r, "r_{k_\mathfrak{p}}"] \arrow[u, "\text{cpt at } \mathfrak{p}"] & \mathrm{Gal}(\bar{k}_\mathfrak{p}|k_\mathfrak{p})^{ab} \arrow[u, "\text{well-defined as Gal are abelian}"'] \end{tikzcd}$$

If $L|K$ is a finite ext,

$$\begin{tikzcd} L^\times \backslash \mathbb{A}_L^\times \arrow[r, "r_L"] \arrow[d, "N"'] & \mathrm{Gal}(L^{ab}|L) \arrow[d] \\ k^\times \backslash \mathbb{A}_k^\times \arrow[r, "r_K"] & \mathrm{Gal}(k^{ab}|k) \end{tikzcd}$$

Lecture 14 Remark : $r_K : k^\times \backslash \mathbb{A}_k^\times \twoheadrightarrow \mathrm{Gal}(k^{ab}|k)$

Know kernel $\therefore$ we know the group $\mathrm{Gal}(k^{ab}|k)$.

In general however, we don't know k^{ab} !

$k = \mathbb{Q}$ or im quad, k^{ab} known

$\underline{k = \mathbb{Q}}$, Let's analyse $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$.

Lemma. $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \right)$.

i.e. if $(x_v) \in \mathbb{A}_\mathbb{Q}^\times$, $v \in \{2, 3, 5, \cdots, \infty\}$ then $\exists\, \lambda \in \mathbb{Q}^\times$ s.t. $x_p/\lambda \in \mathbb{Z}_p^\times$, $\forall p$, & $x_\infty/\lambda > 0$.

Rmk 1) K a number field, harder to push though!

$x_\mathfrak{p}$ problematic, $v_\mathfrak{p}(x_\mathfrak{p}) = n$.

Set $\lambda \in K^\times$, $\mathcal{O}_K \lambda = \mathfrak{p}^n$.

- but what if $\mathfrak{p}^n \neq$ principal?

Ex. $K^\times \backslash \mathbb{A}_K^\times \Big/ \left(\prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}^\times \times K_\infty^\times\right)$ = class gp of K.

$K_\infty^\times \rightsquigarrow (K_\infty^\times)^\circ$: narrow class gp.

In fact, we showed $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \times \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right)$

Cor. $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times = \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$.

Cor. $\ker r_\mathbb{Q} = \mathbb{R}_{>0}$, & $\mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q})^{ab} = \prod_p \mathbb{Z}_p^\times = \hat{\mathbb{Z}}^\times$

$\mathrm{Gal}(\mathbb{Q}^{ab}|\mathbb{Q}) = \hat{\mathbb{Z}}^\times = \varprojlim (\mathbb{Z}/N\mathbb{Z})^\times$.

$||$

$\varprojlim_{\substack{L|\mathbb{Q} \\ \text{Galois, abelian}}} \mathrm{Gal}(L|\mathbb{Q})$

$\mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$

$\rightsquigarrow$ looks like $\mathbb{Q}^{ab} = \bigcup \mathbb{Q}(\zeta_N)$. True.

Def. K number field. A Größencharacter (Grossencharacter, Hecke character$^{\text{GC}}$) is a cts gp hom. $K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$.

Leak: GCs = automorphic rep^{n}s for GL_1/K. $\rightsquigarrow$ GL_n/K, $\varphi : GL_n(K) \backslash GL_n(\mathbb{A}_K) \to \mathbb{C}^\times$

Examples $K = \mathbb{Q}$, $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times = \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$.

Exercise The cts gp homs $\mathbb{R}_{>0} \longrightarrow \mathbb{C}^\times$ are all of the form $x \longmapsto \overset{\exp(x \log s)}{\overset{\shortparallel}{x^s}}$, $s \in \mathbb{C}$

$\mathbb{C}^\times$: non-small subgp property.

$\hat{\mathbb{Z}}^\times \xrightarrow{\alpha} \mathbb{C}^\times$ α cts gp hom.

Take $U \subset \mathbb{C}^\times$, open disc, center 1, radius 0.1 (small)

$\alpha^{-1}(U)$ open.

$\hat{\mathbb{Z}}^\times = \varprojlim (\mathbb{Z}/N\mathbb{Z})^\times$, $\Rightarrow$ $\alpha^{-1}(U) \supset \underset{(\hat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times)}{\ker} =: K_N$ for some N.

$\alpha(K_N) \subset U$ & is a group $\therefore \alpha(K_N) = \{1\}$.

$\therefore$ α factors through

$$\begin{tikzcd}
\hat{\mathbb{Z}}^\times \arrow[rr, "\alpha"] \arrow[dr] & & \mathbb{C}^\times \\
& (\mathbb{Z}/N\mathbb{Z})^\times \arrow[ur, "\chi"'] &
\end{tikzcd}$$

χ ← Dirichlet char.

Upshot. a GC for $\mathbb{Q}$ is $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \longrightarrow \mathbb{C}^\times$

& for each GC, $\exists$ pair (χ, s), χ Dirichlet char, $s \in \mathbb{C}$

s.t. the GC on $\hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$, factors as

$$\begin{tikzcd}
\hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \arrow[d] & \\
(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{R}_{>0} \arrow[r] & \mathbb{C}^\times . \\
(n, x) \arrow[r, mapsto] & \chi(n) \times x^s
\end{tikzcd}$$

Cor Set of GCs for $\mathbb{Q}$ has the structure of a Riemann surface.

Fix Dir. char. χ, $\mathbb{C} \hookrightarrow$ set of GCs
$s \longmapsto (\chi, s)$

$\uparrow$ The $\mathbb{C}$-eigencurve for $GL_1/\mathbb{Q}$.

Tate's thesis takes a GC ψ, & defines $L(\psi) \in \mathbb{C} \cup \{\infty\}$.

L funcs on RS of all GCs. L has mero ext. to all of the $\mathbb{C}$-eigencurves

Tate checks that the restriction of L to the copy of $\mathbb{C}$ attached to χ

is $L(\chi, s)$, $\quad L(\chi,s) = L(\psi)$, ψ = GC attached to (χ, s).

Generalization to K.

Recall for $K_{\tilde{\mathfrak{p}}} | \mathbb{Q}_p$ finite, there's a canonical norm

$\|\pi_{\tilde{\mathfrak{p}}}\| = \frac{1}{q}$, $\quad q = \# k_{\tilde{\mathfrak{p}}}$.

<u>Fact</u> (a def) $\|x\| = \prod_v \|x_v\|_v$
= finite.

This canonical norm trick extends to $\mathbb{A}_K$

$\ulcorner$ $\exists$ Haar measure on $\mathbb{A}_K$ & mult. by $x \in \mathbb{A}_K^\times$ scalars Haar measure by $\|x\| \in \mathbb{R}_{>0}$ $\lrcorner$

$$\begin{array}{l} \|\cdot\| : \mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}. \\ \qquad\quad \cup \\ \qquad\quad K^\times \end{array}$$

$\|\cdot\| : K^\times \longrightarrow \mathbb{R}_{>0}$ turns out to be trivial.

$\ulcorner K = \mathbb{Q}$, $\quad \|\tfrac{3}{4}\| = \prod_p \|3/4\|_p \times |3/4|_\infty = \frac{1}{3} \times 4 \times \frac{3}{4} = 1.$ $\lrcorner$

Upshot: $\|\cdot\| : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}$ (restricts to $\|\cdot\|_{\tilde{\mathfrak{p}}}$ on $K_{\tilde{\mathfrak{p}}}$, usual norm on $\mathbb{R}^\times$, $|x+yi| = x^2+y^2$ on $\mathbb{C}$).

Hence set of all GC's for GL_1/K also becomes a RS = $\underset{\substack{\text{infinite} \\ \text{disjoint union}}}{\coprod} \mathbb{C}$

$\psi_1, \psi_2 : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ are in the same conn. component

$\Leftrightarrow$ $\psi_1/\psi_2 = \|\cdot\|^s$ for some s. $\quad$ holo. nbhd of ψ_1 is $\psi_1 \times \|\cdot\|^s$, $s \in B(0, \varepsilon)$.

Tate's thesis: defines one meromorphic func. on this RS & proves func. eqn.

Say $\psi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \longrightarrow \mathbb{C}^\times$. $\psi = (\chi, s)$.

$\chi = 1$, $s = \sqrt{-e} \in \mathbb{C}$, is there a Galois repⁿ attached to ψ? <u>NO</u>.

<u>Idea</u>. $\exists$ global Langlands gp $L_{\mathbb{Q}}$ (L_K for general K),

$(L_K)^{ab} = K^\times \backslash \mathbb{A}_K^\times$ (global Langlands group <u>will</u> have this property).

<u>Thm</u>. $\exists$ canonical bijection

$$\left\{ \begin{matrix} \text{automorphic rep}^{\text{n}}\text{s} \\ \text{of } GL_1 / K \end{matrix} \right\} \hookrightarrow \left(\begin{matrix} \text{1-dim'l rep}^{\text{n}}\text{s} \\ \text{of } L_K \end{matrix} \right)$$

$$\psi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times \qquad L_K \to L_K^{ab} = K^\times \backslash \mathbb{A}_K^\times \xrightarrow{\psi} \mathbb{C}^\times$$

... If $s \in \mathbb{Z}$, life would be better.

$K = \mathbb{Q}$, $\psi = (\chi, s)$

$$\chi : \underset{\displaystyle \overset{\shortparallel}{Gal(\mathbb{Q}(\zeta_N) | \mathbb{Q})}}{(\mathbb{Z}/N\mathbb{Z})^\times} \longrightarrow E_0^\times \quad , E_0 = \mathbb{Q}(\zeta_m) \subset \mathbb{C}$$

$\chi \rightsquigarrow \chi_\lambda : Gal(\overline{\mathbb{Q}} | \mathbb{Q}) \to GL_1((E_0)_\lambda)$, compatible system of λ-adic Galois repⁿs.

$s = 1$, $\chi = \text{triv}$, $\psi = \|\cdot\|^1$, $\|\pi_{\mathfrak{p}}\| = 1/q_{\mathfrak{p}}$.

<u>Lecture 15</u>. Reminder of a definition

K number field, E_0 number field, S: finite set of max. ideals of $\mathcal{O}_K$.

A <u>compatible system</u> of λ-adic Galois representations is $\forall \lambda$ finite place of E_0, a rep^n

$$\rho_\lambda : \mathrm{Gal}(\bar{k}|k) \longrightarrow GL_n\left(\overline{(E_0)_\lambda}\right),$$ λ running through finite places of E_0

& $\forall\, \tilde{\mathfrak{p}}$ finite places of k, $\tilde{\mathfrak{p}} \notin S$, a polynomial $F_{\tilde{\mathfrak{p}}}(X) \in E_0[X]$ monic degree n,

s.t. $\forall \lambda$, & $\forall\, \tilde{\mathfrak{p}} \nmid \ell$ $(\lambda | \ell)$, $\tilde{\mathfrak{p}} \notin S$, ρ_λ unramified at $\tilde{\mathfrak{p}}$,

$\rho_\lambda(\mathrm{Frob}\,\tilde{\mathfrak{p}})$ has char. poly. $F_{\tilde{\mathfrak{p}}}[X]$ (indep. of λ !!!)

<u>2 examples</u>

1) $$\begin{tikzcd} \chi : (\mathbb{Z}/N\mathbb{Z})^\times \arrow[rr] \arrow[dr, "\text{finite gp}"'] & & \mathbb{C}^\times \\ & \mathbb{Q}(\zeta_d)^\times \arrow[ur] & \end{tikzcd}$$, $d = \varphi(N)$, $E_0 = \mathbb{Q}(\zeta_d)$

$K = \mathbb{Q}$, $S = \{\text{primes } p \mid N\}$,

$$\begin{tikzcd} & \mathrm{Frob}_p & & \\ \rho_\lambda : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \arrow[r, two heads] & \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}) \arrow[u, phantom, sloped, "\in"] \arrow[d, equal] & & \\ & (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\chi"] & E_0^\times \arrow[r] & (E_{0,\lambda})^\times \arrow[d, equal] \\ & p \arrow[u, phantom, sloped, "\in"] & & GL_1(E_{0,\lambda}) \end{tikzcd}$$

<u>Easily checked</u>: compatible system

$F_p(X) = X - \chi(p)$

$p \nmid N$

2) $n \in \mathbb{Z}$, e.g. $n = 1$.

$$\begin{tikzcd} & \bigcup \mathbb{Q}(\zeta_{\ell^n}) \arrow[d, equal] & & \\ K = \mathbb{Q},\ E_0 = \mathbb{Q},\ w_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \arrow[r, two heads] & \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})|\mathbb{Q}) \arrow[r, equal] & \mathbb{Z}_\ell^\times \arrow[r, hook] & GL_1(\mathbb{Q}_\ell) \end{tikzcd}$$

$S = \emptyset$, $F_p(X) = X - p$.

If $p \neq \ell$, $w_\ell(\mathrm{Frob}_p) = p$ has char. poly. $X - p$, indep. of ℓ !!

3) "$T_\ell E$, $E/\mathbb{Q}$"

<u>GC's.</u> K number field, $K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ cts gp hom

Last time $K = \mathbb{Q}$, $\mathbb{A}_K^\times = \mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$.

$$\begin{tikzcd} \hat{\mathbb{Z}}^\times \arrow[d] \\ (\mathbb{Z}/N\mathbb{Z})^\times \end{tikzcd}$$

$(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$, $s : \mathbb{R}_{>0} \longrightarrow \mathbb{C}^\times$, (χ, s) GC.

$x \longmapsto x^s$

<u>$K = \mathbb{Q}(i)$</u> $K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$

$$\mathbb{A}_K = \mathbb{A}_{K,f} \times K_\infty$$
$$K_\infty \cong \mathbb{C}$$

$$\mathbb{A}_{K,f}^\times = \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}}^\times$$

$\curvearrowleft$ $(x_{\mathfrak{p}} : \mathfrak{p} \triangleleft K,\ v_{\mathfrak{p}}(x_{\mathfrak{p}}) = 0, \forall \mathfrak{p} \notin$ finite set$)$

Note the following simple construction:

Given $x = (x_{\mathfrak{p}}) \in \mathbb{A}_{K,f}^\times$, define $n_{\mathfrak{p}} \in \mathbb{Z}$ by $n_{\mathfrak{p}} = v_{\mathfrak{p}}(x_{\mathfrak{p}})$

$n_{\mathfrak{p}}$: completely unrelated integers. $n_{\mathfrak{p}} = 0$ for all but finitely many $\mathfrak{p}$.

Define a (fractional) ideal $I(x)$, $I \subset K$, $I(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$ FINITE PROD.

(e.g. if all $n_{\mathfrak{p}} \geq 0$, $I(x) =$ non-zero ideal in $\mathcal{O}_K$).

$\mathbb{A}_{K,f}^\times \longrightarrow$ fractional ideals of K

$K^\times \longrightarrow$ principal fractional ideals

<u>Back to $\mathbb{Q}(i)$</u>

$$\mathbb{A}_K^\times = \mathbb{A}_{K,f}^\times \times \overset{\mathbb{C}^\times}{\overset{\text{is}}{K_\infty^\times}}$$

<u>Claim</u>. $\mathbb{A}_K^\times = K^\times \cdot \left(\prod_p \mathcal{O}_{K_p}^\times \times K_\infty^\times\right)$ "K has class number 1"

(works $\forall K$ class no 1)

<u>Pf</u>. $x \in \mathbb{A}_K^\times$, $x = x_f \times x_\infty$

$($

$I(x_f) =$ fractional ideal $= (\lambda)$.

$x_f/\lambda = (x_p/\lambda)_{p \text{ of } K}$, $v_p(x_p/\lambda) = 0$ ($v_p(\lambda) = v_p(x_p)$ BY DEF)

$\therefore x_f/\lambda \in \prod_p \mathcal{O}_{K_p}^\times$

Consequence of claim, if K is a number field w/ class number 1 (e.g. $\mathbb{Q}(i)$),

to give $\psi\big|_{\prod \mathcal{O}_{K_p}^\times \times K_\infty^\times} : \prod \mathcal{O}_{K_p}^\times \times K_\infty^\times \longrightarrow \mathbb{C}^\times$

s.t. ψ is trivial on $K^\times \cap \left(\prod_p \mathcal{O}_{K_p}^\times \times K_\infty^\times\right)$.

$\overset{\psi}{\lambda} \longmapsto (\lambda, \lambda, \dots)$ λ unit, $\lambda \in \mathcal{O}_K^\times$

$K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}(i)$, $\mathcal{O}_K^\times = \{\pm 1, \pm i\}$

HC for $\mathbb{Q}(i)$, $\mathbb{Z}[i] \ni n \neq 0$,

$(\mathbb{Z}[i]/n)^\times \xrightarrow{\chi} \mathbb{C}^\times$, χ gives $\prod_p \mathcal{O}_{K_p}^\times = (\hat{\mathcal{O}}_K)^\times \twoheadrightarrow (\mathcal{O}_K/n)^\times \xrightarrow{\chi} \mathbb{C}^\times$

$$\begin{tikzcd}
K_\infty \cong \mathbb{C}^\times \arrow[r] & \mathbb{C}^\times \\
\{re^{i\theta} \in \mathbb{R}_{>0} \times S^1\} \arrow[u, equal] \arrow[ur] &
\end{tikzcd}$$

$$S^1 = \{z : |z| = 1\} \longrightarrow \mathbb{C}^\times$$

$$z \longmapsto z^n, \; n \in \mathbb{Z}$$

$\mathbb{R}_{>0} \ni x \longmapsto x^s \in \mathbb{C}^\times$, $s \in \mathbb{C}$.

$re^{i\theta} \longmapsto r^s e^{in\theta}$

Upshot: choose $s \in \mathbb{C}$, $n \in \mathbb{Z}$, get $K_\infty^\times \cong \mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$

Upshot: Given $\chi : (\mathbb{Z}[i]/N)^\times \longrightarrow \mathbb{C}^\times$, $n \in \mathbb{Z}$, $s \in \mathbb{C}$

get $\psi_0 : \prod_p \mathcal{O}_{K_p}^\times \times K_\infty^\times \longrightarrow \mathbb{C}^\times$

Problem. ψ_0 does not extend to a GC, as maybe ψ_0 is not trivial in $\mathcal{O}_K^\times = \{\pm 1, \pm i\}$.

Fix: consider $(\psi_0)^4$: $\mathcal{O}_K^\times$ now in kernel.

$(\psi_0)^4$ will extend to $\psi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$

Worked example 3. $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{2}]^\times = \pm 1 \times \langle (1+\sqrt{2}) \rangle$

K class no 1, $\mathbb{A}_K^\times = K^\times \cdot \left(\prod_p \mathcal{O}_{K_p}^\times \times K_\infty^\times \right)$,

Char of $\prod_p \mathcal{O}_{K_p}^\times \twoheadrightarrow (\mathbb{Z}[\sqrt{2}]/m)^\times \xrightarrow{\chi} \mathbb{C}^\times$

$K_\infty^\times = \mathbb{R}^\times \times \mathbb{R}^\times$, $\chi_\infty : K_\infty^\times \longrightarrow \mathbb{C}^\times$

$\chi_\infty(x_1, x_2) = x_1^{s_1} \times x_2^{s_2}$

$\psi_0 : \prod \mathcal{O}_{K_p}^\times \times K_\infty^\times \longrightarrow \mathbb{C}^\times$

$\psi_0(x_f, x_\infty) = \chi(x_f)\,\psi_0(x_\infty)$.

ψ_0 extends to a GC $\iff$ $\psi_0\big|_{\mathcal{O}_k^\times} = 1$. ← finite no. of eqns

$k = \mathbb{Q}(\sqrt{2})$, $\mathcal{O}_k^\times = \pm 1 \times \langle 1+\sqrt{2} \rangle$ $\quad\psi_0(1+\sqrt{2})$ = trivial.

$$|1+\sqrt{2}|^{s_1} \times \underset{\underset{|1+\sqrt{2}|^{-s_2}}{\|}}{|1-\sqrt{2}|^{s_2}} = \text{root of unity}$$

$$s_1 - s_2 \in \frac{\mathbb{Q}\,\pi i}{\log(1+\sqrt{2})}$$

discrete set

<u>Upshot</u>: If $\chi : (\mathbb{Z}[\sqrt{2}]/m)^\times \xrightarrow{\chi} \mathbb{C}^\times$

$\psi_\infty : \mathbb{R}^\times \times \mathbb{R}^\times \to \mathbb{C}^\times$

$\psi_\infty(x_1, x_2) = |x_1|^s\,|x_2|^s\,\mathrm{sgn}(x_1)^{0 \text{ or } 1}\,\mathrm{sgn}(x_2)^{0 \text{ or } 1}$

then $\psi_0 : \prod_p \mathcal{O}_{k_p}^\times \times k_\infty^\times \to \mathbb{C}^\times$ to some finite power extends to some

$\psi : k^\times \backslash \mathbb{A}_k^\times \to \mathbb{C}^\times$.

Tate's $\mathbb{C}$-analytic eigencurve is 1-dim'l again.

Lecture 16 Compatible systems of 1-d Galois rep^{n}s.

examples: $K = \mathbb{Q}$.

ex 1) Dirichlet char. χ

ex 2) (cyclo. char.)^n

ex 3) product.

GC of $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times = \underset{\chi}{\hat{\mathbb{Z}}^\times} \times \underset{s \in \mathbb{C}}{\mathbb{R}_{>0}}$ "There are too many of these"

<u>Def^n</u>. A GC ψ (general k) for $k^\times \backslash \mathbb{A}_k^\times$ is said to be <u>algebraic</u> if when restricted to $(k_\infty^\times)^\circ$, ψ looks like

$$\begin{tikzcd} (k_\infty^\times)^\circ \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{C}^\times \\ (\mathbb{R}_{>0})^{r_1} \times \mathbb{C}^{r_2} & \end{tikzcd}$$

$$\psi(x_1, x_2, \dots, x_{r_1}, z_1, z_2, \dots, z_{r_2}) = x_1^{n_1} \dots x_{r_1}^{n_{r_1}} z_1^{n_{r_1+1}} (\overline{z_1})^{n_{r_1+2}} \dots$$

$n_i \in \mathbb{Z}$

Eg. $\|\cdot\| : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \mathbb{C}^\times$ is algebraic, all the $n_i = 1$.

$k = \mathbb{Q}$, $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \longrightarrow \mathbb{C}^\times$

$$\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times = \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \qquad \mathbb{R}_{>0} \xrightarrow{x \mapsto x^s}$$

algebraic $\Leftrightarrow$ $s \in \mathbb{Z}$

Philosophy:

χ : GC for k

($=$ automorphic rep^n for GL_1/k) $\leftrightarrow$ 1-dim'l rep^n of global Langlands gp L_k

$\uparrow$ meaningless object

Thm (Weil) If χ is an <u>algebraic</u> GC, then $\exists$ compatible system of λ-adic rep^{n}s attached to χ. And converse!

Idea: $\chi : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \mathbb{C}^\times$, $\psi|_{(k_\infty^\times)^\circ} : x \mapsto x^n$ sort of thing.

Want: $\mathrm{Gal}(\bar{k}|k)^{ab} \longrightarrow GL_1(\overline{\mathbb{Q}_\ell})$

$$\mathrm{Gal}(\bar{k}|k)^{ab} = k^\times \backslash \mathbb{A}_k^\times / \overline{(k_\infty^\times)^\circ}$$

Proof. Given $\chi : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \mathbb{C}^\times$ algebraic.

Say $\chi \big|_{(k_\infty^\times)^\circ} (x_\infty) = \prod_{v \text{ real}} x_v^{n_v} \times \prod_{\substack{v \leftrightarrow \{\sigma, c\sigma\} \\ \text{Complex}}} (\sigma x_v)^{n_{v,1}} (\overline{\sigma x_v})^{n_{v,2}}$, $n_v \in \mathbb{Z}$

Define $\chi_0 : \mathbb{A}_k^\times \longrightarrow \mathbb{C}^\times$ by

$$\chi_0(x) = \chi(x) \Bigg/ \Bigg(\prod_{v \text{ real}} x_v^{n_v} \times \prod_{\substack{v \leftrightarrow \{\sigma, c\sigma\} \\ \text{complex}}} (\sigma x_v)^{n_{v,1}} (\overline{\sigma x_v})^{n_{v,2}} \Bigg)$$

χ_0 : trivial on $(k_\infty^\times)^\circ$, χ_0 now non-trivial on $k^\times$ ☹.

$\chi_0(\lambda) = \prod_{\sigma : k \to \mathbb{C}} \sigma(\lambda)^{n_\sigma}$. On the other hand, χ_0 is trivial on "cont. part" of $\mathbb{A}_k^\times$. One can check that $\mathrm{Im}(\chi_0) \subset E_0$, E_0 : number field $\subset \mathbb{C}$.

$$k^\times \backslash \mathbb{A}_k^\times / \overline{(k_\infty^\times)^\circ} = \mathrm{Gal}(\bar{k}|k)^{ab}$$

Now say λ is a finite place of E_0.

$\chi_0 \big|_{k^\times} : \lambda \longmapsto \prod \sigma(\lambda)^{n_\sigma} \in E_0 \subset E_{0,\lambda}$, $n_\sigma \in \mathbb{Z}$

$$\begin{aligned} \mathbb{Q}_\ell^\times &\longrightarrow \mathbb{Q}_\ell^\times \\ x &\longmapsto x^n \end{aligned}$$

Claim. $\chi_0 \big|_{k^\times} : k^\times \longrightarrow (E_{0,\lambda})^\times$ extends to a cts hom. $\chi_\ell : (k \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \longrightarrow (E_{0,\lambda})^\times$

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \prod_{\mathfrak{p}|\ell} K_{\mathfrak{p}}.$$

Define $\psi_\lambda : \mathbb{A}_K^\times \longrightarrow (E_{0,\lambda})^\times$

$$\psi_\lambda(x) = \frac{\chi_0(x)}{\chi_\ell(\underset{\curvearrowleft (x_\mathfrak{p} : \mathfrak{p}|\ell)}{x_\ell})}.$$

ψ_λ now very complicated at $K_\mathfrak{p}$, $\mathfrak{p}|\ell$.

"$\psi_\lambda = \chi$" on $K_\mathfrak{p}^\times$, $\forall\, \mathfrak{p} \nmid \ell$.

Check: ψ_λ trivial at $(K_\infty^\times)^0$, trivial on $K^\times$

$\therefore$ extends to cts
$$\begin{tikzcd} K^\times \backslash \mathbb{A}_K^\times / \overline{(K_\infty^\times)^0} \arrow[r] \arrow[d, equal] & (E_{0,\lambda})^\times \\ \mathrm{Gal}(\bar{K}|K)^{ab} & \end{tikzcd}$$
$\square$

$F_\mathfrak{p}(X) = X - \chi_0(\pi_{K_\mathfrak{p}})$ compatible system.

Converse: If ψ_λ = compatible system, then to show it comes from an alg. HC, we may have to deal w/ the following Q:

E_0 number field,

$s \in \mathbb{C}$,

$$2^s, 3^s, 5^s, 7^s, \dots \in E_0 \xRightarrow{\text{Waldschmidt}} s \in \mathbb{Z}$$

Big picture:

$$\begin{tikzcd}
\begin{matrix}\text{automorphic rep}\\ \text{for } GL_1/K\end{matrix} \arrow[r, leftrightarrow] \arrow[d, phantom, "\cup"] & \begin{matrix}\text{1-dim'l cpx rep}^{ns}\\ \text{of } L_K\end{matrix} \\
\begin{matrix}\text{algebraic auto. reps}\\ \text{for } GL_1/K\end{matrix} \arrow[r, leftrightarrow, "\text{Weil}", "\text{Waldschmidt}"'] \arrow[d, phantom, "\cap"] & \begin{matrix}\text{Compatible systems of}\\ \text{1-dim'l } \ell\text{-adic rep}^{ns} \text{ of } Gal(\bar{K}|K)\end{matrix} \arrow[d, phantom, "\cap"] \\
\begin{matrix}\text{p-adic auto. rep}^{ns} \text{ for}\\ GL_1/K\end{matrix} \arrow[r, leftrightarrow, "\begin{matrix}\uparrow\\ \text{Check p-adic}\\ \text{global Langlands}\\ \text{conj for } GL_1\end{matrix}"'] & \begin{matrix}\text{cts p-adic reps}\\ Gal(\bar{K}|K) \to GL_1(\bar{\mathbb{Q}}_p)\end{matrix}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{matrix}\text{auto. reps}\\ GL_n/K\end{matrix} \arrow[r, leftrightarrow] \arrow[d, phantom, "\cup"] & \begin{matrix}\text{n-dim'l } \mathbb{C}\text{-reps}\\ \text{of } L_K \text{ no det}\end{matrix} \\
\begin{matrix}\text{alg. auto.}\\ \text{reps } GL_n/K\end{matrix} \arrow[r, leftrightarrow, "\begin{matrix}\circledast\\ \text{L. Clozel (1990)}\end{matrix}"'] \arrow[d, phantom, "\cap"] & \begin{matrix}\text{Compatible systems of}\\ \text{n-dim'l } \ell\text{-adic rep}^{ns}\end{matrix} \arrow[d, phantom, "\cap"] \\
\begin{matrix}\text{no det} \quad \text{p-adic auto. rep}^{ns}\\ GL_n/K\end{matrix} \arrow[r, leftrightarrow] & \begin{matrix}\text{cts p-adic Galois}\\ Gal(\bar{K}|K) \to GL_n(\mathbb{C}_p)\end{matrix}
\end{tikzcd}$$

Fontaine - Mazur. $E|\mathbb{Q}_p$ finite

If $\rho: Gal(\bar{K}|K) \to GL_n(E)$ is cts, semisimple, unram. outside a finite set of places & potentially semistable. $\Rightarrow$ Hodge-Tate.

$\Rightarrow$ bunch of p-adic numbers are in $\mathbb{Z}$

$\xRightarrow{\text{BIG CONJ}}$ ρ comes from a motive.

$\Rightarrow$ ρ is part of a compatible system of l-adic rep^{n}s.

$K = \mathbb{Q}$, $\quad l = p$.

$$\begin{tikzcd}
\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \arrow[r, "\chi"] & \mathbb{C}_p^\times \\
& \mathbb{Q}_p^\times \\
& \overline{\mathbb{Q}}_p^\times \\
\chi|_{\mathbb{Z}_p^\times} \arrow[r, hook] & \mathbb{Q}_p^\times \arrow[uuul]
\end{tikzcd}$$

$\chi|_{\mathbb{Z}_p^\times}$ cts gp hom. $\mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times$ $\quad$ = "weight"

$\supset \{ x \mapsto x^n \,;\, n \in \mathbb{Z} \}$

$\frac{d}{dx} \chi|_{x=1} \in \mathbb{C}_p$. $\qquad x \mapsto x^n \times \chi(x)$

χ FINITE order char.

Algebraic number in $\mathbb{Z}$

Lecture 17.

Last time: "web of modularity" was stated.

Algebraic auto. reps for G/K $\longleftrightarrow$ compatible systems of semisimple l-adic Galois rep^{n}s

$$\mathrm{Gal}(\overline{K}|K) \longrightarrow {}^L G(\overline{\mathbb{Q}}_p).$$

$G = GL_n$, ${}^L G = GL_n$.

General G: subtleties $\quad$ * More than one notion of algebraicity (C-algebraic, L-algebraic)

* $\longleftrightarrow$ is not a bijection for general G.

For several reasons.

LHS: local & global L-packets (does not exist for GL_n)

(π's in an L-packet $\longrightarrow$ same ρ).

Different global Langlands parameters on RHS might be isomorphic everywhere locally.

$\longleftrightarrow$: "correspondence".

One way of thinking about it.

π for G algebraic $\longrightarrow$ ρ_π : defined up to some Tate-Shafarevich gp.
(= 1 for GL_n, Brauer-Nesbitt)

— Upshot: subtle issues for general G

Conclusion: forget general G when talking about Langlands correspondence.

Stick to GL_n.

GL_n: you can choose

Langlands Corvallis: dreaming of motives

Clozel Ann Arbor: concrete conjecture + statement of actual thm.

π : auto repⁿ of $GL_n | K$, s.t. ① K totally real or CM

② Strong self-duality condition on π, $\Rightarrow \exists\, \rho_\pi$ compatible system.

③ Strong algebraicity cond. (cohomological)

Clozel's pf: Step 1: Find appropriate Shimura varieties. "Eichler-Shimura" $\mathrm{Frob}_p \longleftrightarrow$ Hecke action

Step 2: Relate cohomology of this variety to automorphic forms $\square$

Fast forward to 2013: Harris - Lan - Taylor - Thorne:

Remove the self-duality condition. Scholze: 2nd pf.

Idea: given π, $\pi \oplus \pi^\vee$ is self-dual. Take limits of coh. of SVs to get ρ.

$$\begin{tikzcd} & \pi \oplus \pi^\vee \arrow[dl, dash] \arrow[dr, dash] & \\ GL_n & & GL_n \end{tikzcd}$$

$$\underbrace{GL_n \qquad GL_n}_{GL_{2n}}$$

Genesis of these ideas:

Weil's construction $\chi : G_{\mathbb{C}} \longrightarrow \rho_\chi$ 1-dim'l ℓ-adic reps.

Eichler - Shimura $(k=2)$

Deligne $(k>2)$ If f is a weight k modular eigenform,

Deligne - Serre $(k=1)$ then $\exists$ compatible system of 2-dim'l Galois rep^n of $\mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q})$.

$\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\bar{\mathbb{Q}})$ finite image

For Deligne's thm to fit into this picture,

$$\Delta = q \prod (1-q^n)^{24} = \sum \tau(n) q^n, \quad \tau(2)\tau(3) = \tau(6), \ldots$$

$$\tau(p) \equiv 1 + p^{11} \pmod{691}$$

$\rho_{\Delta, 691}$, mod 691 = triv. $\oplus (\text{cyclo.})^{11}$

What isn't an automorphic rep^n?

Local Langlands conjectures: $K | \mathbb{Q}_p$ finite. F-semisimple

ALL smooth adm. irred. rep^n of $GL_n(K)$ $\longleftrightarrow$ n-dim'l Weil-Deligne reps

Global Langlands conjectures are about automorphic rep^ns of $GL_n(\mathbb{A}_K)$

An auto. rep^n is by def^n an irred. rep^n of $GL_n(\mathbb{A}_K)$. $\mathbb{A}_K = \prod_v' K_v$

$$GL_n(\mathbb{A}_K) = \prod_v' GL_n(K_v)$$

G, H finite. V irred. rep^n of $G \times H$.

Fact. $V = \underset{\text{irred. of } G}{\underset{\curvearrowright}{V_1}} \otimes V_2 \curvearrowleft$ irred. rep of H

So maybe if π is a nice well-behaved irred. rep of $GL_n(\mathbb{A}_K)$, maybe

$\pi = \otimes' \pi_v$, π_v = irred. adm. rep. of $GL_n(K_v)$

This is indeed true. $\boxed{\text{Flath, Corvallis}}$

Idea:

$$\begin{tikzcd}
\pi \arrow[r, leftrightarrow, "GL"] \arrow[d, phantom, "\shortparallel"] & \rho = \text{compatible system} \\
\otimes \pi_v & \\
\pi_p \arrow[r, leftrightarrow, "LL"] \arrow[d, phantom, "\circlearrowleft"] & \text{Local WD rep.} \\
GL_n(K_p) & \text{prop. of Grothendieck.} \\
& \rho|\, Gal(\overline{K_p}|K_p) \to GL_n(\overline{\mathbb{Q}}_\ell) \\
& \rightsquigarrow \text{WD rep. } WD(\rho) \arrow[uuul, bend left, "LLC"]
\end{tikzcd}$$

Consequence.

Def^n of an automorphic rep^n of GL_n/K CANNOT BE "an arbitrary smooth adm. irred. rep. of $GL_n(\mathbb{A}_K)$"

Why not?

GL_1. I'm going to guess that an automorphic rep. of GL_1/K is just a rep. of $\mathbb{A}_K^\times = \prod' K_v^\times$.

1-dim'l $\mathbb{C}$-rep of $\mathbb{A}_\mathbb{Q}$

$K = \mathbb{Q}$, rep^n of $\mathbb{Q}_2^\times$, $\mathbb{Z}_2^\times \to 1$, $2 \mapsto 7$

rep^n of $\mathbb{Q}_3^\times$, $\mathbb{Z}_3^\times \mapsto 1$, $3 \mapsto 7$

... $\mathbb{Q}_p^\times$, $\forall p < 100$, $\mathbb{Z}_p^\times \to 1$, $p \mapsto 7$

$p \geq 100$, $\mathbb{Q}_p^\times \mapsto 1$, $\mathbb{R}^\times \mapsto 1$.

$\pi_p, \forall p$ $\quad$ π_∞.

$\pi = \otimes \pi_v$ RANDOM REP OF $GL_1(\mathbb{A}_\mathbb{Q})$

Say π is an auto. rep. of GL_1.

$\pi \longrightarrow \rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to GL_1(\mathbb{Q}_\ell)$

$\pi \to \rho_\ell$: If it respects the local Langlands correspondence, then

$\rho_\ell(\mathrm{Frob}_p) = 1, \quad \forall p \geq 100 \Rightarrow \rho_\ell$ = trivial 1-d rep (Chebotarev)

$\Rightarrow \rho_\ell(\mathrm{Frob}_2) = 1 \neq 7$.

Upshot: def'n of an aut. rep. of $GL_1(\mathbb{A}_K)$ CANNOT BE "take any irrep + smoothness" STARK CONTRAST TO LOCAL CASE.

Why is our π no good? $\pi : \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$

We were looking at $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$

$\mathbb{Q}^\times \longrightarrow \mathbb{A}_{\mathbb{Q}}^\times$, Our crazy π: Is it trivial on $\mathbb{Q}^\times$?

$$\pi(2) = \pi(\underset{\underset{\mathbb{Q}_2}{\uparrow}}{2}, \underset{\underset{\mathbb{Q}_3}{\uparrow}}{2}, 2, 2, \cdots, \underset{\underset{\mathbb{R}}{\uparrow}}{2}) = 7 \times 1 \times 1 \times \cdots \times 1 \times \cdots = 7 \neq 1$$

$\mathbb{A}_{k,f}^\times \longrightarrow$ fractional ideals

See in the literature - Hecke characters

χ (ideals prime to n) $\longrightarrow \mathbb{C}^\times$

s.t. if $\alpha \equiv 1 \pmod{n}$, then $\chi(\alpha) = \alpha^m$ or something.

G finite gp.

Want all irred. rep. of G, can find them in group ring $\mathbb{C}[G] \cong \bigoplus_{\substack{\pi \text{ irred.} \\ \text{rep. of } G}} \pi^{\dim \pi}$

If $H \subset G$ is a subgp, can look instead at $\mathbb{C}[H\backslash G]$ = funcs $\underset{\underset{\{Hg : g \in G\}}{\shortparallel}}{H\backslash G} \longrightarrow \mathbb{C}$

$(g * \varphi)(r) = \varphi(rg)$ - well-defined action

$\mathbb{C}[H\backslash G] \cong \bigoplus_{\pi \in S} \pi^{m(\pi)} \subset \mathbb{C}[G]$

S probably $\overset{\text{now}}{\underline{\text{not}}}$ all irreps, $m(\pi) \leq \dim \pi$

Idea: $G = GL_n(\mathbb{A}_k)$, Maybe we want to consider functions on $GL_n(\mathbb{A}_k)$

Maybe we should focus on $\varphi : GL_n(k)\backslash GL_n(\mathbb{A}_k) \longrightarrow \mathbb{C}$

Take the set of all nice φ, call it $\mathcal{A}_0(GL_n(k)\backslash GL_n(\mathbb{A}_k))$

Define an action of $GL_n(\mathbb{A}_K)$: $(g * \varphi)(r) = \varphi(rg)$

eg. $n = 1$. A GC will be nice, $\therefore$ in $\mathcal{A}_0(G)$

A finite sum of completely different GCs will be nice too.

- maybe it'll be $\oplus \pi$, π irred. rep of $GL_n(\mathbb{A}_K)$

Maybe those π's are aut. rep's!

<u>Lecture 18</u> K number field, S finite set of finite places.

$Gal(K^S|K)$
ψ
$Frob_{\mathfrak{p}}$, $\mathfrak{p} \notin S$
conj. classes

If $Gal(K^S|K)$ happened to be a free group, freely generated by $Frob_{\mathfrak{p}}$, then we could define

$$\rho : Gal(K^S|K) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$$

<u>Wrong</u>

by choosing random matrices $M_{\mathfrak{p}} \in GL_n(\bar{\mathbb{Q}}_\ell)$, $\forall \mathfrak{p} \notin S$

Send $Frob_{\mathfrak{p}}$ to $M_{\mathfrak{p}}$.

The global Langlands conjectures would say " take any irred. rep π of $GL_n(\mathbb{A}_K)$

$\longrightarrow$ get ρ_π "

$$\pi = \otimes \pi_v, \quad \pi_{\mathfrak{p}} \text{ unram.} \longmapsto \rho(Frob_{\mathfrak{p}}) = M_{\mathfrak{p}}, \forall \mathfrak{p}.$$

Truth: All $Frob_{\mathfrak{p}}$'s are related in some vastly complex way which no human understands.

Cebotarev: $Frob_{\mathfrak{p}}$ are dense

Based on successes for GL_1, we will restrict to rep^{n}s π of $GL_n(\mathbb{A}_K)$

which show up in $\mathcal{A}_0(GL_n/K) = \{$nice functions $GL_n(K) \backslash GL_n(\mathbb{A}_K) \longrightarrow \mathbb{C}\}$

What's a nice function?

$n=1$: GCs were nice.

χ : GC, $\chi : GL_1(\mathbb{A}_K) \longrightarrow \mathbb{C}$

$$(g * \chi)(r) = \chi(rg) = \chi(g)\chi(r)$$

$\therefore g * \chi = \chi(g) \times \chi$. $\mathbb{C}\chi$ = 1-dim'l v.sp, $GL_1(\mathbb{A}_K)$ acts via χ.

Continuity. GCs locally cst @ finite places. $\chi(\hat{\mathcal{O}}_K^\times)$ = finite.

Smooth @ infinity. $x \mapsto x^s$

more than smooth. – not growing too fast

$$f(x) = x^s = \exp(s \log x)$$

$$\boxed{x f'(x) = s f(x)}$$

"nice" will mean $GL_n(K)$-inv't, locally cst @ finite places,

Smooth @ infinite places, differential equations?

boundedness

Interlude on differential equations

G : Lie group, $\mathfrak{g}$ = Lie algebra of G, $\mathfrak{g}$ is differential operators on G.

$\exp : \mathfrak{g} \longrightarrow G$ $X \in \mathfrak{g} \rightsquigarrow$ differential operator on (C^∞ funcs $G \to \mathbb{C}$)

$$(X * f)(g) = \frac{d}{dt}\left(f(g\cdot \exp(tX))\right)\Big|_{t=0}$$

Example. $G = GL_1(\mathbb{R}) = \mathbb{R}^\times$, $\underset{\overset{\Downarrow}{X=1}}{\mathfrak{g} = \mathbb{R}}$

$f : G \longrightarrow \mathbb{C}$,

$$(X * f)(g) = \frac{d}{dt}\left(f(g \cdot e^t)\right)\Big|_{t=0} = f'(g)\, g$$

Example $f(x) = x^s$, $\quad x f'(x) = s f(x)$, $\quad X * f = s \cdot f$.

$\mathfrak{g} \ni X$, $X \longrightarrow$ diff. op. on $f : G \to \mathbb{C}$

$X =$ left inv. diff. op.

$G = GL_2(\mathbb{R})$, $\quad \mathfrak{g} = M_2(\mathbb{R}) \ni E, F, H, Z$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Each of these give diff ops on $f : GL_2(\mathbb{R}) \to \mathbb{C}$.

$V = \{ C^\infty \text{ funcs } f : GL_2(\mathbb{R}) \to \mathbb{C} \}$.

$E, F, H, Z : V \longrightarrow V$.

These maps do not commute, $EF - FE = 2H$

$\therefore$ bad idea to ask for simultaneous eigenfunc.

Plan: find a bunch of diff ops that commute!

Start w/ Lie algebra $\mathfrak{g}$, $\rightsquigarrow$ enveloping algebra $U\mathfrak{g}$

$Z(U\mathfrak{g})$: a whole bunch of commuting diff. ops, for which we might hope that our nice funcs are simultaneous eigen funcs.

Harish-Chandra figured out what $Z(U(\mathfrak{g}\otimes\mathbb{C}))$ is: $U(\mathfrak{h}_{\mathbb{C}})^{W}$.

<u>Lecture 19</u>. <u>Goal</u>: trying to figure out what a nice function is.

G conn'd reductive $/K$, $\mathcal{A}(G)=\{\varphi: G(K)\backslash G(\mathbb{A}_K)\longrightarrow\mathbb{C}$ s.t. φ is nice $\}$,

$$\text{Reminder: } GL_1(\mathbb{A}_K)\underset{\substack{\uparrow\\ \text{finite index,}\\ \text{Class gp!}}}{\supset}\underset{\substack{\uparrow\\ \varphi\text{ trivial}}}{K^\times}\underset{\substack{\uparrow\\ \text{finite in practice}}}{\prod_p \mathcal{O}_{K_p}^\times}\underset{\substack{\uparrow\\ \text{real manifold.}}}{K_\infty^\times}$$

$$\overset{\mathbb{R}/\mathbb{Z},\ \mathbb{R},\ \dots}{\mathcal{O}_K^\times\backslash K_\infty^\times}$$

In fact, same is true for GL_n (pf for $GL_2/\mathbb{Q}$ later)

$$GL_n(\mathbb{A}_K)\underset{\substack{\text{finite}\\ \text{index}}}{\supset}\underset{\substack{\uparrow\\ \text{trivial}}}{GL_n(K)}\prod_p\underbrace{GL_n(\mathcal{O}_{K_p})}_{\text{finite}}GL_n(K_\infty).$$

$$GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})\longrightarrow\mathbb{C}$$

<u>Reminder</u>: $GL_1/\mathbb{Q}$, at ∞, φ is a fun. on $\mathbb{Z}^\times\backslash\mathbb{R}^\times=\mathbb{R}_{>0}$.

want $\mathcal{A}(GL_1/\mathbb{Q})\ni$ GCs.

Conclusion:
$$\begin{aligned}\mathbb{R}_{>0}&\longrightarrow\mathbb{C}^\times\\ x&\longmapsto x^s\end{aligned}$$
must be <u>nice</u>.

$\mathbb{R}_{>0}$: $G(\mathbb{R})$, $\mathfrak{g}$ = Lie algebra, $\mathfrak{g} \cong \mathbb{R} \ni$ basis vector D.

Saw last time $(Df)(x) = xf'(x)$.

eg. $f(x) = x^s$, $Df = sx^s = sf$. $\therefore (D-s)f = 0$.

Sum of 2 GC's needs to be nice.

$$\mathbb{R}_{>0} \longrightarrow \mathbb{C}$$
$$x \longmapsto \underset{= f(x)}{x^s + 7x^t}, \quad s, t \in \mathbb{C}.$$

$Df(x) = xf'(x) = sx^s + 7tx^t$, $(D-s)f = 7(t-s)x^t$

$\therefore (D-t)(D-s)f = 0$.

Abstractly, the algebra $\mathbb{C}[D]$ acts on C^∞ funcs $\mathbb{R}_{>0} \longrightarrow \mathbb{C}$

& for the sum of GCs f we just looked at $\exists\ 0 \neq \underset{\underset{(D-s)(D-t)}{\shortparallel}}{D'} \in \mathbb{C}[D]$ s.t. $D'f = 0$.

$\therefore$ if $I = \{D' \in \mathbb{C}[D] : D'f = 0\}$, then I is an ideal of $\mathbb{C}[D]$,

& in the cases we saw, I had <u>finite codimension</u>.

General : Lie alg. $\mathfrak{g}$ of $G(K_\infty)$, basis $e_1, e_2, \cdots, e_d$ of $\mathfrak{g}$.

$U(\mathfrak{g}_\mathbb{C})$ enveloping algebra, Harish-Chandra: $Z(U\mathfrak{g}_\mathbb{C}) = \mathrm{Sym}(\mathfrak{t}_\mathbb{C})^W$

$\uparrow$

bi-G-invt diff. ops = canonical source of (higher order) diff ops.

Example $GL_2/\mathbb{Q}$, H-C-N machine.

$$Z(Ug) = \mathbb{C}[\Delta, Z]$$
$$\uparrow \; \uparrow$$
2 diff ops.

$\mathfrak{g} = \text{Lie}(GL_2(\mathbb{R})) = M_2(\mathbb{R})$.

Standard basis for $\mathfrak{g}$: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Delta = H^2 + 2EF + 2FE$, then Δ commutes w/ E, F, H, Z.

$\Delta, Z \in Z(U(\mathfrak{g}_{\mathbb{C}}))$, turns out that center really is $\mathbb{C}[\Delta, Z]$.

Δ = some second order diff. op. on $\{f : GL_2^+(\mathbb{R}) \longrightarrow \mathbb{C}\}$.

Reminder: $GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc > 0 \right\}$ acts on $\mathbb{H}$ upper half plane.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$, transitively

Surjection $GL_2^+(\mathbb{R}) \twoheadrightarrow \mathbb{H}$
$$\gamma \longmapsto \gamma i = \frac{ai + b}{ci + d}$$

$\{\gamma \in GL_2^+(\mathbb{R}) : \gamma i = i\} = \underset{\substack{\uparrow \\ \text{center}}}{\mathbb{R}^\times} SO_2(\mathbb{R})$. $\mathbb{H} = GL_2^+(\mathbb{R}) / \mathbb{R}^\times SO_2(\mathbb{R})$

So now say $f : \mathbb{H} \longrightarrow \mathbb{C}$, let F be associated func. on $GL_2^+(\mathbb{R})$.

ΔF descends to a func. Δf on $\mathbb{H}$.

(Up to a constant), $\Delta f = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$

$GL_2/\mathbb{Q}$: our def'n of nice : $\exists\, \underbrace{I \subset \mathbb{C}[\Delta]}_{\text{finite codim}}$ s.t. $I \cdot f = 0$.

Here $Z(U(\mathfrak{g}_{\mathbb{C}})) = \mathbb{C}[\Delta, Z]$.

Upshot: looks like we're interested in funcs $f : \mathbb{H} \to \mathbb{C}$

s.t. $\Delta f = \lambda f$, $\lambda = $ cst.

$(Zf = \underset{\uparrow}{\mu} f)$

central character

Recall thm. of Deligne et al.

f : mod. form. eigenform $\longrightarrow$ $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to GL_2(\bar{\mathbb{Q}}_\ell)$

$\det \rho_f(\underset{\uparrow}{c}) = -1$

complex conjugation.

Now let $f(x)$ = random irred. cubic poly. / $\mathbb{Q}$, 3 real roots α, β, γ.

$K = \mathbb{Q}(\alpha, \beta, \gamma)$. chances are $\mathrm{Gal}(K|\mathbb{Q}) \cong S_3$.

$$\begin{tikzcd}
\text{Fix } \rho_0 : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \arrow[r, two heads] & \mathrm{Gal}(K|\mathbb{Q}) \arrow[d, phantom, "\cong"] & \\
& S_3 \arrow[r, "\text{irred 2d}"] & GL_2(\mathbb{Q})
\end{tikzcd}$$

$\det \rho_0(c) = \det \rho_0(1) = 1$

$\forall\, \ell$ prime, get $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_0} GL_2(\mathbb{Q}) \hookrightarrow GL_2(\mathbb{Q}_\ell)$

ρ_ℓ's should be attached to some π!

Maaß wrote down fnc: $\mathbb{H} \to \mathbb{C}$, not holomorphic,

invt under $\Gamma_1(N)$, $N = \mathrm{cond.}(\rho_0)$

$\Delta f = \lambda f$, $\lambda \neq 0$.

Defn. G connected reductive gp./ K number field,

H_∞ = thing that many people would call K_∞

= max. cpt subgp of $G(K_\infty)$.

eg. $G = GL_n/\mathbb{Q}$, $G(K_\infty) = GL_n(\mathbb{R})$, $H_\infty = O_n(\mathbb{R})$

A function $\varphi : G(K) \backslash G(\mathbb{A}_K) \to \mathbb{C}$ is called an automorphic form if

1) φ is smooth, i.e. if we write $\underset{\substack{\rotatebox{90}{$\in$}\\ (x,y)}}{G(\mathbb{A}_K)} = \underset{\substack{\rotatebox{90}{$\in$}\\ x}}{G(\mathbb{A}_{K,f})} \times \underset{\substack{\rotatebox{90}{$\in$}\\ y}}{G(K_\infty)}$

then for fixed x, φ is C^∞ w.r.t. y & for fixed y, φ is locally const.

2) φ is "well-behaved on cpt subgps" (admissibility)

i.e. 2a. $\exists\, U_f \subset G(\mathbb{A}_{K,f})$ cpt open s.t. $\varphi(gu) = \varphi(g)$, $\forall\, u \in U_f$.

2b. $\mathbb{C}$-v.space spanned by $g \mapsto \varphi(g h_\infty)$ is finite-dim'l as $h_\infty \in H_\infty$.

[e.g. φ could be trivial on H_∞! like $\varphi : \mathbb{H} \to \mathbb{C}$]

3) $\exists I \subset Z(U(\mathfrak{g}_\mathbb{C}))$, $\mathfrak{g} = \mathrm{Lie}(G(K_\infty))$

finite codim.

s.t. $I.(y \mapsto \varphi(x,y)) = 0, \ \forall x \in G(\mathbb{A}_{K,f})$.

4) Growth conditions $|\varphi(x,y)| \leq \text{const} \times \|y\|^N$

$\hat{}$ sensible norm on $G(K_\infty)$

<u>Lecture 20</u>. G/K conn'd, $H_\infty \subset G(K_\infty)$ max. cpt.

An <u>automorphic form</u> $\varphi: {}_{G(K)}\backslash G(\mathbb{A}_K) \to \mathbb{C}$ is a smooth, slowly increasing, H_∞-finite, $\mathfrak{z}$-finite fun.

$$\mathfrak{z} = Z(U(\mathfrak{g}_\mathbb{C})).$$

$\mathcal{A}(G) = \{\varphi : \text{auto. form for } G\} = \mathbb{C}\text{- vec. sp.}$

$G(\mathbb{A}_{K,f})$ acts on left on $\mathcal{A}(G)$:

$(g * \varphi)(\gamma) = \varphi(\gamma g)$.

Unfortunately, $G(K_\infty)$ does not act on $\mathcal{A}(G)$.

$g \in G(K_\infty) \Rightarrow gH_\infty g^{-1} \neq H_\infty$ in general.

However, H_∞ acts & $\mathfrak{g}_\mathbb{C}$ acts, $\mathfrak{g} = \mathrm{Lie}(G(K_\infty))$.

<u>Remark</u>. There's a second way of doing all this, where $\mathcal{A}(G) := L^2(G(K)\backslash G(\mathbb{A}_K), \psi)$

= Hilbert space

$G(K_\infty)$ acts!

There's something called a $(\mathfrak{g}, K)$-module

$(\mathfrak{g}, H_\infty)$-module

$$\left(\mathcal{A}(G) = (\mathfrak{g}_{\mathbb{C}}, H_\infty)\text{-module} \right).$$

So $\mathcal{A}(G) \curvearrowleft G(\mathbb{A}_{K,f}) \times (\mathfrak{g}_{\mathbb{C}}, H_\infty)$

An automorphic rep. π for G/K is an irred. subquotient of $\mathcal{A}(G)$.

I don't know what this means.

$$L^2(\mathbb{R}) \curvearrowleft \underset{r}{\overset{\in}{\mathbb{R}}} \qquad (r * f)(x) = f(x+r) \qquad \varphi_y : \mathbb{R} \longrightarrow \mathbb{C}^\times$$

$f : \mathbb{R} \to \mathbb{C}$

$x \longmapsto e^{ixy}$

$\boxed{\begin{array}{c} y \in \mathbb{R} \\ \text{fix} \end{array}}$

Fourier transform

$f = \int c(y)\, \varphi_y.$

$$r * \varphi_y = e^{iry}\, \varphi_y$$

$$|\varphi_y(x)| = 1, \ \forall x \Rightarrow \varphi_y \notin L^2.$$

Fix: Need 2 things.

1) $\varphi \in \mathcal{A}(G)$ is cuspidal if (tedious boundary conditions).

$$\int_{N(K)\backslash N(\mathbb{A}_K)} \varphi(xn)\, dn = 0$$

$MN = P$
$\subseteq$ max. proper parabolic of G

Comments: $G = GL_2/\mathbb{Q}$, max. proper parabolic = conjugate of $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$$G/B \cong \mathbb{P}^1_{\mathbb{Q}}$$

$$\left[f : \overset{\mathbb{R}_{>0} \to \mathbb{C}^\times}{x \longmapsto x^s \log x}, \quad (D-s)^2 f = 0. \right]$$

2) Say Z = center of G.

Fix $\psi : Z(K)\backslash Z(\mathbb{A}_K) \longrightarrow \mathbb{C}^\times$

Def. $\underset{\substack{\uparrow \\ \text{Cuspidal}}}{\mathcal{A}_0} (G, \underset{\substack{\uparrow \\ \text{center acting} \\ \text{via } \psi}}{\psi}) = \{ \varphi \in \mathcal{A}(G) : \varphi \text{ is cuspidal}, \ \varphi(gz) = \psi(z)\varphi(g)$

$\forall z \in Z(\mathbb{A}_K), g \in G(\mathbb{A}_K) \}$.

Def. A cuspidal automorphic rep. π of $G(\mathbb{A}_K)$ is an irred. subrep. of $\mathcal{A}_0(G, \psi)$ for some ψ.

Thm (Langlands). If π is an auto. repⁿ of G that is not cuspidal, then $\pi \cong \mathrm{Ind}_P^G \pi_0$, π_0 cuspidal on some smaller gp.

Example. $G = GL_1 \times GL_1$, then a cuspidal auto. rep. for G is a pair χ_1, χ_2 of GCs.

"$I(\chi_1, \chi_2)$" = non-cuspidal auto. rep. of GL_2.

Langlands: every auto. repⁿ of GL_2 is either cuspidal or built in this way

Global Langlands for GL_n/K

Cuspidal auto. repⁿs for GL_n/K $\xleftrightarrow{\text{Langlands reciprocity}}$ irred. n-dim'l reps of L_K. ↑

$\pi = \otimes \pi_p \otimes \pi_\infty$ $\quad \pi_p \xleftrightarrow{LLC} \rho_p : WD(K_p) \longrightarrow GL_n(\mathbb{C})$.

__Fact__: semisimple rep^n of a group $= \bigoplus$ irred. rep^{n}s.

$$\Downarrow \text{ philosophy}$$

auto. reps can be built $\leftsquigarrow$ Langlands functoriality

Hard analysis thm of Langlands $\to$ from cuspidal auto. reps on smaller gps.

In general, reciprocity is a philosophy. functoriality = concrete consequences that makes sense

__2nd example__. π auto. rep. of GL_2/K.

Philosophy: $\pi \rightsquigarrow \rho : L_K \longrightarrow GL_2(\mathbb{C})$.

$Sym^2(\rho) : L_K \longrightarrow GL_3(\mathbb{C})$.

Philosophy $\rightsquigarrow \exists$ $Sym^2(\pi)$ auto. rep. of GL_3/K.

$Sym^2(\pi)$ does exist - hard thm in functional analysis.

__Example of an automorphic form for $GL_2/\mathbb{Q}$__:

Reminder of notation: $f : \mathbb{H} \longrightarrow \mathbb{C}$ func.

$k \in \mathbb{Z}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$.

Define $f|_k \gamma : \mathbb{H} \longrightarrow \mathbb{C}$ by $(f|_k\gamma)(\tau) = (\det\gamma)^{k-1}(c\tau+d)^{-k} f(\gamma\tau)$

Say now f is a cuspidal modular form level N, wt k,

i.e. $f|_k\gamma = f$, $\forall$ $\gamma \in \Gamma_1(N) = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N$ + boundedness conditions

Say we have a cuspidal modular form f & a complex number s,

Let's build φ !

Need to define $\varphi : GL_2(\mathbb{A}_{\mathbb{Q}}) \longrightarrow \mathbb{C}$.

Recall: we proved. $*$ $GL_2(\mathbb{Q}_p) = B(\mathbb{Q}_p)\, GL_2(\mathbb{Z}_p)$

$$GL_2(\mathbb{Z}_p) = \{1\}\, GL_2(\mathbb{Z}_p)$$

$$\Rightarrow GL_2(\mathbb{A}_f) = B(\mathbb{A}_f)\, GL_2(\hat{\mathbb{Z}})$$

$$\mathbb{A}_f = \mathbb{Q} + \hat{\mathbb{Z}}, \qquad \begin{pmatrix} \mathbb{A}_f^\times & \mathbb{A}_f \\ 0 & \mathbb{A}_f^\times \end{pmatrix}$$

$$\mathbb{A}_f^\times = \mathbb{Q}^\times \cdot \hat{\mathbb{Z}}^\times$$

$$\Rightarrow GL_2(\mathbb{A}_f) = B(\mathbb{Q})\, GL_2(\hat{\mathbb{Z}}).$$

Last trick: $U_f = U_1(N) = \{ m \in GL_2(\hat{\mathbb{Z}}) : m \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N \}$.

then $GL_2(\hat{\mathbb{Z}}) = \bigsqcup \tilde{\gamma}_i U_f$, $\underset{\in GL_2(\mathbb{Q})}{\tilde{\gamma}_i}$ lifts γ_i cosets for $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subset GL_2(\mathbb{Z}/N\mathbb{Z})$

Hence $GL_2(\mathbb{A}_f) = GL_2(\mathbb{Q})\, U_1(N)$.

Hence $GL_2(\mathbb{A}) = GL_2(\mathbb{Q})\, U_1(N)\, GL_2^+(\mathbb{R})$.

Given f, s as before, define $\varphi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \longrightarrow \mathbb{C}$

$$\text{by } \varphi(\underset{\substack{\uparrow \\ GL_2(\mathbb{Q})}}{\gamma}\ \underset{\substack{\uparrow \\ U_1(N)}}{u}\ \underset{\substack{\uparrow \\ GL_2^+(\mathbb{R})}}{h}) = (f|_k h)(i) \times (\det h)^s$$

Claim $\varphi \in \mathcal{A}(G)$!

Let's check φ is well-defined.

Note $\gamma_1 u_1 h_1 = \gamma_2 u_2 h_2$

$$\Rightarrow \gamma_2^{-1}\gamma_1 = u_2 h_2 h_1^{-1} u_1^{-1} \in U_1(N)\, GL_2^+(\mathbb{R}) \cap GL_2(\mathbb{Q})$$

$\therefore$ $h_2 h_1^{-1} = \gamma_\infty \in \Gamma_1(N)$, $h_2 = \gamma_\infty h_1$.

$$\underset{\displaystyle f|_k h_2}{\underset{\shortparallel}{f|_k \gamma_\infty h_1}} = f|_k h_1$$

φ is $GL_2(\mathbb{Q})$-inv't, $U_1(N)$-finite.

H_∞-finite:
$$f|_k \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (i)$$
$$= (\sin\theta\, i + \cos\theta)^{-k} f(i)$$

$\Rightarrow$ φ is H_∞-finite.

Cauchy-Riemann $\Rightarrow$ eigenform for Δ

Z acts via formula including s.

$$\Delta\varphi = (k^2 - 2k)\varphi$$
$$Z\varphi = (2s + k - 2)\varphi.$$

$GL_2(\mathbb{A}_k)$- rep^n spanned by φ = automorphic rep^n attached to f.

f eigenform $\Rightarrow$ π irred.

↑ cuspidal auto. rep attached to f.