D-modules in characteristic p

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Lecture 1. ① Motivation for D-modules

Ex. $\left(\partial_x - \frac{\partial f}{\partial x}\right) g = 0$, $f \in \mathbb{C}[x]$, $g \in \mathcal{O}(\mathbb{A}^1)^{hol}$ ⟵ holomorphic functions on cpx plane

$$g = c \cdot e^f \leftarrow \text{not algebraic}$$

Idea. To study certain alg. geom. object which "underlies" this equation

Ex. $D_{\mathbb{A}^1} = \mathbb{C}\langle x, \partial_x \rangle / [\partial_x, x] = 1$

1) $\mathcal{O}(\mathbb{A}^1)^{hol}$ is a $D_{\mathbb{A}^1}$-module (left)
$x \longmapsto$ mult. by x
$\partial_x \longmapsto$ differentiation

2) $D_{\mathbb{A}^1} / D_{\mathbb{A}^1}\left(\partial_x - \frac{\partial f}{\partial x}\right) =: M_f$

3) "Solutions" of $\left(\partial_x - \frac{\partial f}{\partial x}\right) g = 0 \longleftrightarrow \mathrm{Hom}_{D_{\mathbb{A}^1}}\left(M_f, \mathcal{O}(\mathbb{A}^1)^{hol}\right)$

$$\varphi(1) \longleftarrow\!\!\shortmid\ \varphi$$

② D-modules (global construction)

X smooth scheme $/\mathbb{C}$. $\mathcal{O}_X$ str. sheaf, T_X tangent bundle
$T_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ ⟵ derivation

Def. 1) $D_X = \langle \mathcal{O}_X, T_X \rangle /$ $f_1 \circ f_2 = f_1 \cdot f_2$
$f \circ v = f v$ ($f \in \mathcal{O}_X$, $v \in T_X$, $fv \in T_X$)
$v_1 \circ v_2 - v_2 \circ v_1 = [v_1, v_2] \in T_X$
$f \circ v - v \circ f = v(f) \in \mathcal{O}_X$

U affine, can just consider global sections

2) A $\underline{\mathcal{D}\text{-module}}$ on X is a sheaf of $\mathcal{D}_X$-modules which is $\mathcal{O}_X$-quasi-coherent.

$\mathcal{D}_X \hookrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$

not true over char. p

$\underline{\text{Rmk}}$. Let $\mathcal{E}$ be a $\mathcal{D}_X$-module $\rightsquigarrow$ $\mathcal{T}_X \underset{\mathbb{C}}{\otimes} \mathcal{E} \longrightarrow \mathcal{E}$ $\rightsquigarrow$ $\mathcal{T}_X \xrightarrow{\nabla} \mathcal{E}nd_{\mathbb{C}}(\mathcal{E})$

$\mathcal{O}_X$ $\mathcal{O}_X$

$\mathcal{O}_X$-linear

$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X$

linear w.r.t. this action

not linear w.r.t. this action

s.t. the image commutes w/ mult. by functions

$\nabla(fe) = e \otimes df + f\nabla(e)$

$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X \xrightarrow{\nabla} \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^2_X \longrightarrow \cdots$

$[\nabla(v_1), \nabla(v_2)] = \nabla([v_1, v_2])$

$\nabla^2 = 0$

If $\mathcal{E}$: v. bundle (loc. free $\mathcal{O}_X$-module of finite rank)

then $(\mathcal{E}, \nabla)$ is called a bundle w/ flat connection.

$\underline{\text{Ex}}$. $\mathcal{E} = \mathcal{O}_X$, $\mathcal{O}_X \longrightarrow \mathcal{O}_X \underset{\mathcal{O}_X}{\otimes} \Omega^1_X$

any connection is given by $f \longmapsto df + f \cdot \omega$

some diff. 1-form on X

has a connection $d : \mathcal{O}_X \longrightarrow \Omega^1_X$

$f \longmapsto df$

$(\mathcal{O}_X, d + \omega)$

flatness $\longleftrightarrow$ $d\omega = 0$

$\Rightarrow$ flat conn. on $\mathcal{O}_X$ $\longleftrightarrow$ closed 1-forms

$$\underbrace{\omega_1 \sim \omega_2}_{(\mathcal{O}_X, d+\omega_1) \simeq (\mathcal{O}_X, d+\omega_2)} \quad \text{iff} \quad \exists\ f \in \Gamma(X, \mathcal{O}_X)^\times \text{ s.t. } \omega_1 = \omega_2 + d\log f$$

Ex. $\pi : X \to S$ smooth projective map,

$$\mathcal{H}^i(\pi) := R^i\pi_{*,dR}\,\mathcal{O}_X = R^i\pi_*\left(\Omega^\bullet_{X/S,dR}\right)$$

$$\forall s \in S(\mathbb{C}), \quad \mathcal{H}^i(\pi)_s = H^i_{dR}(X_s)$$

Can be endowed w/ a nat'l <u>Gauß-Manin</u> connection.

③ <u>Higgs bundles</u>

$D_X = \langle \mathcal{O}_X, \mathcal{T}_X \rangle / \cdots$

$$F^{\leq n} D_X = \mathrm{Im}\left(\left(\mathcal{O}_X \oplus \mathcal{T}_X \oplus \mathcal{T}_X^{\otimes 2} \oplus \cdots \oplus \mathcal{T}^{\otimes n}\right) \to D_X\right) \qquad F^{\leq i} D_X \cdot F^{\leq j} D_X \subset F^{\leq i+j} D_X$$

<u>Prop.</u> $\mathrm{gr}^\bullet_F D_X = \mathrm{Sym}^\bullet_{\mathcal{O}_X} \mathcal{T}_X =: H_X$

<u>Rmk</u>: $q : T^*X \to X$, then $H_X = q_*\mathcal{O}_{T^*X}$ $\qquad \mathrm{Tot}(\mathcal{E}) = \mathrm{Spec}_X \mathrm{Sym}^\bullet_{\mathcal{O}_X} \mathcal{E}^\vee$

total space of Ω^1_X

q is affine, $\Rightarrow$ q.coh. H_X-modules on X $\longleftrightarrow$ q.coh. sheaves on T^*X.

<u>Def</u>. A Higgs bundle (sheaf) on X is an H_X-module which is q. coh. $\mathcal{O}_X$-mod.

Rmk. $T_X \longrightarrow H_X$, $\forall$ Higgs sheaf $\mathcal{E}$, we have $\mathcal{E} \otimes_{\mathcal{O}_X} T_X \longrightarrow \mathcal{E}$

$\mathcal{O}_X$-linear

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega_X^1 \qquad\qquad T_X \xrightarrow{\theta} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$$

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\theta} \mathcal{E} \otimes \Omega_X^2 \to \cdots \qquad\qquad [\theta(v_1), \theta(v_2)] = 0$$

$$\theta^2 = 0 \qquad\qquad \forall v_1, v_2$$

<u>Simpson's correspondence</u> ($X/\mathbb{C}$ projective)

$$\left\{\begin{array}{l}\text{Semisimple} \\ \text{bundles w/} \\ \text{flat connection} \\ \text{(of rank } n)\end{array}\right\} \underset{\text{(isom. of alg. vars)}}{\overset{1:1}{\longleftrightarrow}} \left\{\begin{array}{l}\text{semistable Higgs} \\ \text{bundles on } X \text{ w/ all} \\ \text{vanishing Chern classes} \\ \text{(of rank } n)\end{array}\right\}$$

Fix $x \in X(\mathbb{C})$ $\updownarrow$ Riemann-Hilbert correspondence (isom. of analytic vars)

$$\left\{\begin{array}{l}\text{semisimple rep'ns} \\ \text{of } \pi_1(X(\mathbb{C}), x) \text{ of rank } n\end{array}\right\}$$

Rmk $(\mathcal{E}, \nabla) \overset{\text{Simpson}}{\longleftrightarrow} (\mathcal{E}', \theta')$, $\mathcal{E}$ is not necessarily isomorphic to $\mathcal{E}'$, but their underlying topological (even C^∞ bundles) are the same

Ex. 1) $(\mathcal{O}_X, d) \longleftrightarrow (\mathcal{O}_X, 0)$

2) $\mathcal{H}^i(\pi) \longleftarrow$ they have Hodge filtration $\rightsquigarrow$ $\nabla: F^j\mathcal{H}^i(\pi) \longrightarrow F^{j-1}\mathcal{H}^i(\pi) \otimes \Omega_X^1$

$\theta: gr_F^{\bullet}\mathcal{H}^i(\pi) \longrightarrow gr_F^{\bullet-1}\mathcal{H}^i(\pi) \otimes \Omega_X^1$

$$\overset{\shortparallel}{R^i\pi_*(\Omega_{X/S}, dR)} \supset R^i\pi_*(\Omega^{\geq j}_{X/S}, dR) =: F^j\mathcal{H}^i(\pi)$$

$gr_F^{\cdot} \mathcal{H}^i(\pi)$ ⟵ this is the corresponding Higgs bundle to ∇_{GM}

Rmk. $(\mathcal{E}, \nabla) \xleftrightarrow{\text{Simpson}} (\mathcal{E}', \theta)$

then $\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\nabla} \cdots$ & $\mathcal{E}' \xrightarrow{\theta} \mathcal{E}' \otimes \Omega_X^1 \xrightarrow{\theta} \to \cdots$

are quasi-isom. after taking $R\Gamma(X, -)$

For $(\mathcal{O}_X, d) \longleftrightarrow (\mathcal{O}_X, 0)$, get Hodge-to-de Rham degeneration.

<u>Thm</u> (Barannikov-Kontsevich, Sabbah)

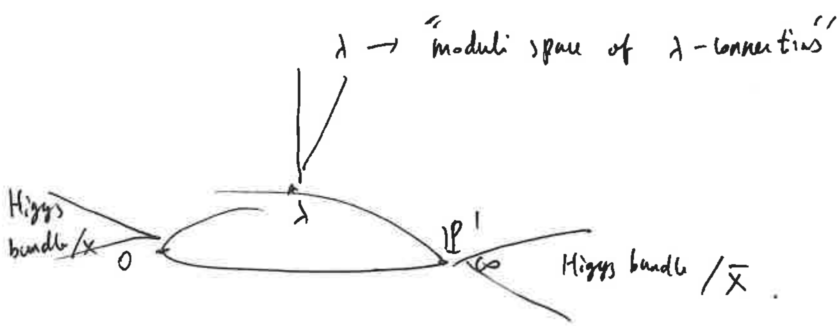
Let $f : X \to \mathbb{A}^1$ be a proper map $/ \mathbb{C}$

" As if $(\mathcal{O}_X, d + df) \xleftrightarrow{\text{Simpson}} (\mathcal{O}_X, df)$ "

(I) $0 \to \mathcal{O}_X \xrightarrow{d + df} \Omega_X^1 \xrightarrow{d+df} \Omega_X^2 \to \cdots$

(II) $0 \to \mathcal{O}_X \xrightarrow{\wedge df} \Omega_X^1 \xrightarrow{\wedge df} \Omega_X^2 \to \cdots$

(I) is q. isom. to (II). after taking $R\Gamma(X, -)$



$\lambda df \otimes e + 6 \nabla(e)$

(in char. p, at ∞ will have Higgs bundles over $X^{(1)}$)

Lecture 2

Reminder

D-modules	Higgs modules
• $D_X = \langle \mathcal{O}_X, \mathcal{T}_X \rangle / \text{rel'n}$	• $H_X = \mathrm{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{T}_X$
• $\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$	• $\mathcal{E} \xrightarrow{\theta} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$
$\nabla(fe) = df \otimes e + f \nabla(e)$	$\theta(fe) = f\theta(e)$
• $\mathcal{T}_X \xrightarrow{\nabla} \mathcal{E}nd_{\mathbb{C}}(\mathcal{E})$	• $\mathcal{T}_X \xrightarrow{\theta} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$
$[\nabla(v), f] = v(f)$	$[\theta(v_1), \theta(v_2)] = 0$.
$[\nabla(v_1), \nabla(v_2)] = \nabla[v_1, v_2]$	

Today: char p

(last time: we had Barannikov - Kontsevich thm $\quad X$ smooth projective

$$R\Gamma(X, \mathcal{O}_X \xrightarrow{d+df} \Omega^1_X \xrightarrow{d+df} \cdots) \simeq R\Gamma(X, \mathcal{O}_X \xrightarrow{df} \Omega^1_X \xrightarrow{df} \cdots))$$

I. Deligne - Illusie method

Let k be a perfect field of char. p, Let X/k be a smooth scheme

$$\Omega^{\bullet}_{X,dR} = \left[\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \to \Omega^i_X \to \cdots \to \Omega^{\dim X}_X \to 0\right]$$

is linear over $(\mathcal{O}_X)^p$: $d(f^p \omega) = f^p d\omega$

$F : X \to X^{(1)}$, relative Frobenius

$$\begin{tikzcd} X \arrow[r, dashed, "F_X"] \arrow[rr, bend left, "\mathrm{Frob}_X"] \arrow[dr, bend right] & X^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ & \mathrm{Spec}\, k \arrow[r, "\mathrm{Frob}_k"] & \mathrm{Spec}\, k \end{tikzcd}$$

$$F_*(\Omega^\bullet_{X, dR}) \in D^b Coh(X^{(1)}) \qquad (F \text{ finite})$$

$\uparrow$

complex of coherent sheaves on $X^{(1)}$.

Prop (Cartier isomorphism) $\mathcal{H}^\bullet(F_*(\Omega^\bullet_{X,dR})) \simeq \Lambda^\bullet_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}}$

Ex. $X = \mathbb{A}^1$, then $k[x] \xrightarrow{d} k[x]\,dx$ as $k[x^p]$-modules

$$x^m \longmapsto m x^{m-1} dx$$

$$\mathcal{H}^0(-) = k[x^p] \quad , \mathcal{H}^1(-) = k[x^p] \cdot x^{p-1} dx$$

Ex 2

$W_2(k)$

$\uparrow$

p-typical Witt vectors

$k = \mathbb{F}_p$, $W_2(k) = \mathbb{Z}/p^2$

Fix a lifting $(\tilde{X}, \tilde{F})$ of (X, F) to $W_2(k)$

- $\tilde{X}$ flat over $W_2(k)$
- $\tilde{X}_k \simeq X$

$$\tilde{F} : \tilde{X} \to \tilde{X}^{(1)} = \tilde{X} \times_{W_2(k)} W_2(k) \nearrow_{\mathrm{Frob}_k}$$

$$\tilde{F} \times_{W_2(k)} k = F$$

$F^* \Omega^1_{X^{(1)}} \xrightarrow{0} \Omega^1_X$

$f\,dg \longmapsto f^p dg^p = 0$

$\tilde{F}^* \Omega^1_{\tilde{X}^{(1)}} \xrightarrow{d\tilde{F}} \Omega^1_{\tilde{X}}$

0 mod p

$\leadsto$

eg. $\mathbb{A}^1 = \operatorname{Spec} k[x]$

$W_2(k)[x] + \tilde{F} : x \mapsto x^p$.

$\frac{d\tilde{F}}{p} : F^* \Omega^1_{X^{(1)}} \longrightarrow \Omega^1_X$ not zero.

$\leadsto$

$\Omega^1_{X^{(1)}} \longrightarrow F_* \Omega^1_X \quad \leadsto \quad \Lambda^\bullet_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}} \longrightarrow \mathcal{H}^\bullet(F_* \Omega^\bullet_{X, dR})$

Q: Is $F_* \Omega^\bullet_{X,dR}$ q. isom. to $\bigoplus \Omega^i_{X^{(1)}}[-i]$? $\dim H^*_{dR}(X) = \sum_{i+j=n} \dim H^i(X^{(1)}, \Omega^j_{X^{(1)}})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

A: No, not always. (there are X s.t. Hodge-to-de Rham SS not degenerate) $\dim H^i(X, \Omega^j_X)$

$$\tau^{\leq \bullet} F_* \Omega^\bullet_{X,dR} \overset{\text{ass. gr.}}{\rightsquigarrow} \bigoplus_i \Omega^i_{X^{(1)}}[-i], \quad F_* \Omega^{\geq \bullet}_{X,dR} \overset{\text{ass}}{\rightsquigarrow} \bigoplus \Omega^i_X[-i]$$

$X \to X^{(1)}$

$$R\Gamma(X^{(1)}, \underset{\substack{\shortparallel \\ F_*(\mathcal{F})}}{RF_*(\mathcal{F})}) = R\Gamma(X, \mathcal{F})$$

<u>Thm</u> (Deligne - Illusie)

Fix a lifting $\tilde{X}$ to $W_2(k)$. Then there is a nat'l q. isom

$$\tau^{\leq p-1} F_* \Omega^\bullet_{X,dR} \simeq \bigoplus_{i=0}^{p-1} \Omega^i_{X^{(1)}}[-i] \quad \text{in } D^b\mathrm{Coh}(X^{(1)}).$$

$$X/\mathbb{C} \overset{\text{spread out}}{\rightsquigarrow} X_R/R \rightsquigarrow X_s \quad s \in \operatorname{Spec} R \text{ of char } p.$$
R ⊂ f.g. $\mathbb{Z}$-alg.

If $p-1 > \dim X$, $\square$

II. Azumaya algebra

<u>Def</u>. A coherent sheaf of algebras A on X is called an <u>Azumaya alg.</u> if $\exists\ U \xrightarrow{f} X$ (étale cover) and an isom. $f^* A \simeq \mathrm{Mat}_{n \times n}(\mathcal{O}_U)$ for some n.

n is called the <u>rank of A</u>.

<u>Rmks</u> 1) It is enough to have an fpqc cover like this.

2) $\underline{\mathrm{Aut}}(\mathrm{Mat}_{n\times n}) = PGL_n \rightsquigarrow$ Azumaya algebras of rk $n \longleftrightarrow PGL_n$-torsors in étale top.

Ex. 1) $X = \operatorname{Spec} k$, k not alg. closed. Take D central division alg (e.g. $\mathbb{H}/\mathbb{R}$)

Artin-Wedderburn
$\rightsquigarrow D \otimes_k \bar{k} \simeq \mathrm{Mat}_{n\times n}(\bar{k}) \Rightarrow D$ defines an Azumaya algebra

2) Take any vec. bundle $\mathcal{E}$, and $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. Taking $U \xrightarrow{f} X$ trivializing $\mathcal{E}$,

$\rightsquigarrow f^*\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus \operatorname{rk}\mathcal{E}}) \simeq \mathrm{Mat}_{\operatorname{rk}\mathcal{E}\times\operatorname{rk}\mathcal{E}}(\mathcal{O}_U)$

Def. An Azumaya algebra $\mathcal{A}$ is called split if $\exists\ \mathcal{E}$ s.t. $\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

Prop. Let $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$, then there is an equiv. of cat.

$$\mathcal{A}\text{-}\mathrm{Mod}^{qcoh} \simeq \mathcal{O}_X\text{-}\mathrm{Mod}^{qcoh}$$

Idea of proof: "$\leftarrow$" $\quad \underset{\mathcal{A}}{\circlearrowright}\ \mathcal{E}\otimes\mathcal{F} \longleftarrow\!\shortmid \mathcal{F}$

"$\rightarrow$" $\quad \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, -)$

$Z(\mathcal{D}_X)$

center ; define $\mathcal{D}_X$ as before

$\langle \mathcal{O}_X, \mathcal{T}_X\rangle / \text{rel'n.}$

f^p commutes w/ any $v \in \mathcal{T}_X$

$[v, f^p] = v(f^p) = 0$

$\mathcal{T}_{X^{(1)}} \longrightarrow F_* \mathcal{D}_X$

Ex. $X = \mathbb{A}^1$. $\mathcal{D}_{\mathbb{A}^1} = k\langle x, \partial_x\rangle / ([\partial_x, x] = 1)$

$Z(\mathcal{D}_{\mathbb{A}^1}) = k[x^p, \partial_x^p]$
$\quad\quad \wr\!\wr$
$\mathcal{O}(T^*(\mathbb{A}^1)^{(1)})$

Thm 1) $Z(F_* \mathcal{D}_X) \simeq q_* \mathcal{O}_{T^*X^{(1)}}$

$q : T^*X^{(1)} \longrightarrow X^{(1)}$
$\quad \wr\!\wr$
$\operatorname{Spec}_{X^{(1)}} \operatorname{Sym}_{\mathcal{O}_{X^{(1)}}} \mathcal{T}_{X^{(1)}}$

$\rightsquigarrow$ can asso. to $F_* \mathcal{D}_X$ an algebra on $T^*X^{(1)}$

2) $F_* D_X$ is an Azumaya algebra over $T^*X^{(1)}$.

Ideal situation. D_X is a split Azumaya alg. (Never True).

Thm (Ogus-Vologodsky) Fix $\tilde{X}$ a lifting of X to $W_2(k)$, then

$$D_X \text{ splits on } \begin{array}{ccc} \widehat{T^*_{PD}X^{(1)}} & \longrightarrow & T^*X^{(1)} \\ \| & & \| \\ \widehat{\Gamma^*_{\mathcal{O}_{X^{(1)}}}\mathcal{T}_{X^{(1)}}} & & \mathrm{Sym}_{\mathcal{O}_{X^{(1)}}}\mathcal{T}_{X^{(1)}} \end{array}$$

Lecture 3.

① p-curvature map

② Azumaya property

$D_X = \langle \mathcal{O}_X, \mathcal{T}_X \rangle / \text{rel'n}$

Locally. $X \xrightarrow[\text{ét}]{\pi} \mathbb{A}^n$, $\mathcal{T}_X \simeq \pi^* \mathcal{T}_{\mathbb{A}^n} \simeq \mathcal{O}_X^{\oplus n}$

pick coord. $z_1, \cdots, z_m$,

$\partial_{z_1}, \cdots, \partial_{z_m}$, $\quad \partial_{z_i}(z_j) = \delta_{ij}$

then get $\partial_i \in \mathcal{T}_X$ → those vector fields that project to ∂_{z_i}.

$+ \; x_i = \pi^* z_i$, then $\partial_i(x_j) = \delta_{ij}$

For such chart $\Gamma(X, D_X) = \bigoplus_{I=(i_1,\cdots,i_n)} \mathcal{O}_X \partial^I$, $\partial^I = \partial_1^{i_1} \cdots \partial_n^{i_n}$

Reminder $\quad D_{X,\le n} \subset D_X \quad , \quad D_{X,\le n}\cdot D_{X,\le m} \subset D_{X,\le n+m}$

$\uparrow$

diff. ops of order $\le n$

"$\mathcal{O}_{T^*X}$" $= q_*\mathcal{O}_{T^*X}$

$\rightsquigarrow \mathrm{gr}_\bullet D_X \simeq \mathrm{Sym}_{\mathcal{O}_X} \mathcal{T}_X \leftarrow$

$\uparrow$ a sheaf over X.

$q: T^*X \to X$

T^*X is a symplectic variety , $\omega \in \Gamma(T^*X, \Omega^2_{T^*X})$, $d\omega = 0$

everywhere nondegenerate

$\omega|_\xi$ gives a skew-symmetric pairing on $T_\xi(T^*X)$

$\xi \in T^*X$

and it is non-degenerate.

gives a Poisson bracket $\{\cdot,\cdot\}$

on $\mathcal{O}_{T^*X}$ which is non-degenerate

$\{\cdot,\cdot\}: \Lambda^2_k \mathcal{O}_{T^*X} \to \mathcal{O}_{T^*X}$

$\{-,-\}$ is a k-linear Lie bracket on $\mathcal{O}_{T^*X}$ s.t.

$$\{f_1 f_2, g\} = f_2\{f_1, g\} + f_1\{f_2, g\}.$$

$\tilde{\omega}: \Lambda^2 \mathcal{T}_{T^*X} \to \mathcal{O}_{T^*X} \rightsquigarrow \mathcal{T}_{T^*X} \xrightarrow{\sim} \Omega^1_{T^*X}$

$v \mapsto \tilde{\omega}(v, -)$

$\{f, g\} = \omega(df \wedge dg)$

$\{f \in \mathcal{O}_{T^*X} : \{f,g\} = 0, \forall g \in \mathcal{O}_{T^*X}\}$

$\|$

- if ω is non-degenerate, then the Poisson center is given by $\ker d \subset \mathcal{O}_{T^*X}$.

$\downarrow$

$(\mathcal{O}_{T^*X})^p$

$\mathrm{gr}^\bullet D_X$

liftings to $D_{X,\le n_1}$, $D_{X,\le n_2}$

$\{f_1, f_2\} := [\tilde{f_1}, \tilde{f_2}] \in D_{X,\le n_1+n_2-1}$ $\leftarrow$ since $\mathrm{gr}_\bullet$ is commutative

mod $D_{X,\le n_1+n_2-2}$

$\mathrm{gr}_{n_1} D_X \quad \mathrm{gr}_{n_2} D_X \quad \mathrm{gr}_{n_1+n_2-1} D_X$

Lemma It is the bracket above. In particular, it is non-degenerate.

(and Poisson center is given by $(\mathcal{O}_{T^*X})^p$)

Example $X = \mathbb{A}^1$

$$D_X = k\langle x, \partial_x \rangle / [\partial_x, x] = 1, \quad gr^{\cdot} D_X = k[\overset{\deg 0}{x}, \overset{\deg 1}{y}]$$

y ← image of ∂_x

$\{x, x\} = [x, x] = 0$

$\{y, y\} = [\partial_x, \partial_x] = 0$ $\qquad \omega = -dx \wedge dy$

$\{x, y\} = [x, \partial_x] = -1$

1. **p-curvature**

Observation $\partial \in Der_k(\mathcal{O}_X)$, then $\partial^p \in End_k(\mathcal{O}_X)$ is also a derivation.

$\partial(fg) = f\partial(g) + g\partial(f) \rightsquigarrow \partial^p(fg) = f\partial^p g + g\partial^p f$.

$-^{[p]} : Der_k(\mathcal{O}_{X^{(1)}}) \longrightarrow F_* Der_k(\mathcal{O}_X)$

Construction. Consider a map $\iota : T_{X^{(1)}} \longrightarrow F_{X*} D_X$

sending $v^{(1)}$ to $\underbrace{v^p - v^{[p]}}_{F_* D_{X, \le p}}$.

Property: $\iota(v)$ acts by 0 on $\mathcal{O}_X$.

Ex $X = \mathbb{A}^1$, $\partial_x^{[p]} = 0$, $(x\partial_x)^{[p]} = x^p \partial_x$

Prop $\iota : T_{X^{(1)}} \to (F_X)_* \mathcal{D}_X$ is $\mathcal{O}_{X^{(1)}}$ - linear, and factors through $Z((F_X)_* \mathcal{D}_X)$.

Lemma $\mathcal{D}_{X, \le p-1} \hookrightarrow \mathrm{End}_k(\mathcal{O}_X)$

Pf. locally can write any $\underset{\mathcal{D}_{X,\le p-1}}{\overset{\cap}{D}}$ as $\sum_{I = (i_1, \dots, i_n)} f_I \partial^I$

find a monomial x^I s.t. $D \cdot x^I \neq 0$

Proof of the proposition. 1) Linearity $\iota(v+v') - \iota(v) - \iota(v')$

$\iota(f^{(1)} v) - f^p \iota(v)$ $\in \mathcal{D}_{X, \le p-1}$ + they act by 0 on $\mathcal{O}_X$

$\Rightarrow$ they are 0.

2) Centrality $[f, \iota(v)] \in \mathcal{D}_{X, \le p-1}$

$[v', \iota(v)] = \{v', v^p\}$ mod $\mathcal{D}_{X, \le p-1}$ + act by 0 on $\mathcal{O}_X$

$\Rightarrow$ they are 0.

$\Rightarrow \iota(T_{X^{(1)}}) \subset Z(F_* \mathcal{D}_X)$

Lecture 4. k perfect field of char p.

Reminder. $\iota : T_{X^{(1)}} \to Z((F_X)_* \mathcal{D}_X)$

$v^{(1)} \longmapsto v^p - v^{[p]}$

$\theta : T_X \to \mathrm{End}_k(\mathcal{O}_X)$, $v \in T_X \rightsquigarrow \theta(v) \rightsquigarrow \theta(v)^p \in \mathrm{End}_k(\mathcal{O}_X) \rightsquigarrow v^{[p]} \in T_X$.

is again a derivation $\theta(v^{[p]}) = \theta(v)^p$

$\rightsquigarrow \underset{\deg p}{v^p} - \underset{\deg 1}{v^{[p]}} \in \mathcal{D}_X$

In char 0,

$\tilde{\sigma}$ is an embedding $\longleftrightarrow$ In char p $\tilde{\sigma}|_{D_{X,\leq p-1}}$ is an embedding $(p \to \infty)$

2. Description of the center

1) $\mathcal{O}_{X^{(1)}} \overset{\mathcal{O}_{X^{(1)}}\text{-linear}}{\subset} Z(F_* D_X)$

$\wr\!\!\!\supset$

$F_*(\mathcal{O}_X^p)$ $\nearrow$

$\mathcal{O}_X^p \subset \mathcal{O}_X$ subsheaf of k-vec.sp.

$[f^p, \underset{\underset{\mathcal{T}_X}{\Uparrow}}{v}] = 0$

2) $\mathcal{T}_{X^{(1)}} \xrightarrow{\iota} Z((F_X)_* D_X)$

$\mathcal{O}_{X^{(1)}}$-linear map

$\rightsquigarrow \mathrm{Sym}(\iota) : \mathrm{Sym}_{\mathcal{O}_{X^{(1)}}} \mathcal{T}_{X^{(1)}} \longrightarrow Z(F_* D_X)$

Prop. the map $\mathrm{Sym}(\iota)$ is an isom.

Ex. $X = \mathbb{A}^1$, then $D_X = k\langle x, \partial_x\rangle / [\partial_x, x] = 1$

$Z(D_X) = k[x^p, \partial_x^p]$, $\partial_x^p = \iota(\partial_x)$

(for $\mathbb{G}_m$, $(x\partial_x)^p - x\partial_x = \iota(x\partial_x)$)

Proof $\mathrm{Sym}_{\mathcal{O}_{X^{(1)}}} \mathcal{T}_{X^{(1)}} \longrightarrow D_X$

$\mathrm{Fil}_{\leq pk} := \bigoplus_{i=0}^{k} \mathrm{Sym}^i_{\mathcal{O}_{X^{(1)}}} \mathcal{T}_{X^{(1)}} \longrightarrow D_{X,\leq pk}$ $\underset{\text{assoc. graded}}{\overset{\text{passing to}}{\rightsquigarrow}}$ $\bigoplus_{i=0}^{\infty} \mathrm{Sym}^i_{\mathcal{O}_X^{(1)}} \mathcal{T}_{X^{(1)}} \xrightarrow{\text{deg } pi} \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i_{\mathcal{O}_X} \mathcal{T}_X$

of $\mathrm{Sym}(\iota)$

$\cong \quad f^{(1)} \mapsto f^p \quad \cong$

Injectivity of $\mathrm{Sym}(\iota)$ follows from inj. of F_{T^*X}.

$\mathcal{O}_{T^*X^{(1)}} \xrightarrow{F_{T^*X}} F_* \mathcal{O}_{T^*X}$

Rmk. $Z(\mathcal{D}_X) \underset{\text{in } gr_*}{\overset{\text{image}}{\rightsquigarrow}} Z_p(\mathcal{D}_X)$ ← Poisson center

$\downarrow$ $\qquad\qquad\qquad$ $\downarrow$

$f \in \mathcal{D}_{X,\leq n} \rightsquigarrow f \in gr_n \mathcal{D}_X$

Poisson center in $\mathcal{O}_{T^*X}$ is $F_{T^*X}(\mathcal{O}_{T^*X^{(1)}})$.

Cor of the proof.

$(F_X)_* \mathcal{D}_X$ is loc. free over $\mathcal{O}_{T^*X^{(1)}}$ of rank $p^{2\dim X}$.

Pf. follows from $F_*(\mathcal{O}_{T^*X})$ being a loc. free $\mathcal{O}_{T^*X^{(1)}}$-module of rank $p^{2\dim X} = p^{\dim T^*X}$.

(If X smooth of dim d, $F_*\mathcal{O}_X$ is of rank p^d over $\mathcal{O}_{X^{(1)}}$)

3. Azumaya property

Desire.

$Z(F_* \mathcal{D}_X) \simeq q_*^{(1)} \mathcal{O}_{T^*X^{(1)}}$

$\wr$

$\mathcal{D}_X$ defines a certain sheaf of alg. on $T^*X^{(1)}$.

(use equiv. $QCoh(T^*X^{(1)})$ and $\underset{\text{is loc}}{}$ $Mod\ QCoh(q_*^{(1)} \mathcal{O}_{T^*X^{(1)}})$).

$q^{(1)}: T^*X^{(1)} \to X^{(1)}$

F_X - affine

$\underset{\text{on } X}{\mathcal{D}_X} \rightsquigarrow \underset{\text{on } X^{(1)}}{(F_X)_* \mathcal{D}_X} \overset{\text{loc}}{\rightsquigarrow} [\mathcal{D}_X]$ in $QCoh(T^*X^{(1)})$

$\uparrow$ alg.

Want: $\mathcal{D}_X$ is an Azumaya alg. on $T^*X^{(1)}$

Recall. $\mathcal{A}$ on X is an Azumaya alg. if $\exists\ Y \xrightarrow[\pi]{\text{f. flat}} X$ s.t. $\pi^*\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E}_Y)$

some vec. bdle on Y.

$rk(\mathcal{E}_Y)$ is the rank of $\mathcal{A}$.

Consider $T^{*,(1)}X := X \times_{X^{(1)}} T^*X^{(1)}$ ← total space of $F_X^* \Omega^1_{X^{(1)}}$

$$\begin{tikzcd} & T^{*,(1)}X \arrow[dl, "q'"'] \arrow[dr, "\pi"] & \\ X & & T^*X^{(1)} \end{tikzcd}$$

Subsheaf of alg. $\subset \mathcal{D}_X$

$$\mathcal{A}_X = (q')_* \mathcal{O}_{T^{*,(1)}X} = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i_{\mathcal{O}_X} F_X^* \mathcal{T}_{X^{(1)}}$$

$\uparrow$ $\mathrm{Alg}(\mathrm{QCoh}(X))$ $\qquad \wr\wr \quad F_X^{-1}(\mathcal{O}_{T^*X}) \otimes_{F_X^{-1}(\mathcal{O}_{X^{(1)}})} \mathcal{O}_X$

$$F^{-1}(\mathcal{O}_{T^*X^{(1)}}) \overset{\text{centr}}{\subset} \mathcal{D}_X.$$

$$\big(F_* \mathcal{A}_X = F_* \mathcal{O}_X \cdot \underbrace{\mathcal{O}_{T^*X^{(1)}}}_{= Z(F_*(\mathcal{D}_X))} \subset F_*(\mathcal{D}_X)\big)$$

$\mathcal{D}_X \supset \mathcal{A}_X$. Consider $\mathcal{D}_X$ as a right $\mathcal{A}_X$-module, denote this by $(\mathcal{D}_X)_{\mathcal{A}_X}$

There is a nat'l map $\underbrace{\mathcal{D}_X \otimes_{Z(\mathcal{D}_X)} \mathcal{A}_X}_{\pi^* \mathcal{D}_X} \xrightarrow{\varphi} \mathrm{End}_{\mathcal{A}_X}((\mathcal{D}_X)_{\mathcal{A}_X})$

$(\mathcal{D}_X)_{\mathcal{A}_X}$ defines a vec. bdle on $T^{*,(1)}X$

$\mathrm{End}_{\mathcal{A}_X}((\mathcal{D}_X)_{\mathcal{A}_X})$: split Azumaya alg. on $T^*X^{(1)}$.

$\mathcal{D}_X \curvearrowright (\mathcal{D}_X)_{\mathcal{A}_X}$ by left multiplication
$\mathcal{A}_X \curvearrowright (\mathcal{D}_X)_{\mathcal{A}_X}$ from the right
$\Big\}$ agree on $Z(\mathcal{D}_X)$

$$\pi^* \mathcal{D}_X \xrightarrow{\varphi} \mathrm{End}(\mathcal{E})$$

$\mathcal{E}$: vec. bdle on $T^{*,(1)}X$

<u>Thm</u> φ is an isom.

<u>Cor.</u> $\mathcal{D}_X$ is an Azumaya alg. (of rank $p^{\dim X}$).

<u>Ex.</u> $X = \mathbb{A}^1$, $k\langle x, \partial x\rangle / [\partial x, x] = 1$ over $k[x, \partial_x^p]$ splits.

<u>Lecture 5</u>. Missing.

Lecture 6.

Last time. proved that $\mathcal{D}_X$ defines an Azumaya algebra on $T^*X^{(1)}$

(strictly speaking, $(F_X)_* \mathcal{D}_X$ consider as a sheaf on $\mathrm{Spec}_{\mathcal{O}_{X^{(1)}}} Z(F_* \mathcal{D}_X) \simeq T^*X^{(1)}$)

$$Z(F_* \mathcal{D}_X) = \mathrm{Sym}_{\mathcal{O}_{X^{(1)}}} \mathcal{T}_{X^{(1)}}$$

$f: X \to Y$

$\mathcal{D}_X \quad \mathcal{D}_Y$

Problem. there is no nat'l map $T^*X \to T^*Y$. (there is a map $TX \to TY$)

Instead, there is a nat'l correspondence.

$$\begin{tikzcd} X \times_Y T^*Y \arrow[r, hook] \arrow[rr, bend left, "q_X"] \arrow[rr, bend right, "q_Y"'] & T^*X \times_Y T^*Y \arrow[r] \arrow[dr] & T^*X \\ & & T^*Y \end{tikzcd}$$

pullback of Azumaya algebras are Morita equivalent.

If X is étale, then $X \times_Y T^*Y \simeq T^*X$ $\quad (f^*\Omega^1_Y \simeq \Omega^1_X)$

$$\rightsquigarrow \quad \begin{tikzcd} T^*X \arrow[r, "\text{“}df\text{''}"] \arrow[d] & T^*Y \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We have two Azumaya algebras on $T^*X^{(1)}$: $\mathcal{D}_X$ and $(df^{(1)})^* \mathcal{D}_Y$

Fact: they are isomorphic.

Ex. f is an open embedding $U \xrightarrow{f} X$, $\quad \mathcal{D}_U \simeq (df^{(1)})^* \mathcal{D}_X$

Rmk $\mathcal{D}_X$ is not split unless X is a union of points

($\nexists$ a vect. bundle $\mathcal{E}$ on $T^*X^{(1)}$ s.t. $\mathcal{D}_X \simeq \mathcal{E}nd_{\mathcal{O}_{T^*X^{(1)}}}(\mathcal{E})$).

any such alg. Zariski locally on X has zero divisors

2. Milne's map

Recall $\underline{\mathrm{Aut}}(\mathrm{Mat}_n) = PGL_n$ ← as a group scheme

(sheaf of alg)

Azumaya alg on X of rank n $\longleftrightarrow$ $H^1_{\acute{e}t}(X, PGL_n)$

$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$ SES in étale top

lands in n-torsion

$$\rightsquigarrow H^1_{\acute{e}t}(X, GL_n) \to H^1_{\acute{e}t}(X, PGL_n) \to H^2_{\acute{e}t}(X, \mathbb{G}_m) =: Br(X)$$

pointed sets $\qquad H^2_{\acute{e}t}(X, \mu_n)$

$\mathbb{G}_m$-gerbe on X

vec. bdle $\mathcal{E} \longmapsto \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$

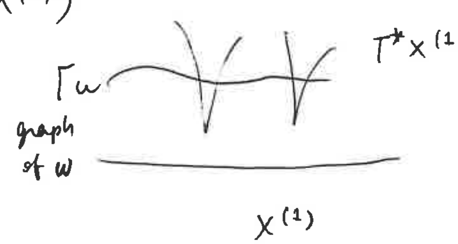
(stacks over X, locally isom. to $(B\mathbb{G}_m)_X$. equiv., "$B\mathbb{G}_m$-torsors")

<u>In our situation</u> $[D_X] \in Br(T^*X^{(1)})$ (it in fact is a p-torsion class)

D_X - Azumaya alg over $T^*X^{(1)}$

get a map $H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \xrightarrow{c_X} Br(X^{(1)})$

"Milne map" $\qquad \omega \longmapsto i_\omega^*[D_X]$



$\omega \in H^0(X^{(1)}, \Omega^1_{X^{(1)}})$

$i_\omega : X^{(1)} \to T^*X^{(1)}$

$\mathbb{F}_p(1) \simeq \mathcal{O}_X^\times/(\mathcal{O}_X^\times)^p\,[-1]$

$0 \to \mathbb{F}_p(1) \to \Omega^1_{X,\mathrm{log}} \to \Omega^1_{X^{(1)}} \to 0$

$u: X_{fl} \to X_{\acute{e}t}$, $Ru_*\mu_p \simeq (\mathcal{O}_X^\times/\mathcal{O}_X^{\times p})[-1]$

Prop. c_X is a homomorphism of abelian gps.

Proof. $T^*X^{(1)}$ is a vector group scheme over $X^{(1)}$.

$$(T^*X^{(1)}) \times_{X^{(1)}} (T^*X^{(1)}) \underset{p_2}{\overset{m}{\underset{p_1}{\longrightarrow}}} T^*X^{(1)}$$

addition

two projections

$$m^*[D_X] = p_1^*[D_X] + p_2^*[D_X] \in Br\left((T^*X^{(1)}) \times_{X^{(1)}} (T^*X^{(1)})\right)$$

Given $w_1, w_2 \in H^0(X^{(1)}, \Omega^1_{X^{(1)}})$, we can consider $X^{(1)} \xrightarrow[i_{w_1} \times i_{w_2}]{} T^*X^{(1)} \times_{X^{(1)}} T^*X^{(1)}$

$(i_{w_1,w_2})^*$ (equality) $\rightsquigarrow$ $i_{w_1+w_2}^*[D_X] = i_{w_1}^*[D_X] + i_{w_2}^*[D_X]$

Rmk A Azumaya alg.

$A \otimes_{O_X} A^{op}$ is split $\Rightarrow [A^{op}] = -[A]$ in $Br(X)$

$\cong$

$End_{O_X}(A)$

WT split

$m^* D_X \otimes_{O(T^*X^{(1)} \times_{X^{(1)}} T^*X^{(1)})} p_1^* D_X^{op} \otimes p_2^* D_X^{op}$: splitting bundle will be given by $D_X \otimes_{O_X} D_X$
same

$\circlearrowleft D_X^{op}$ $\circlearrowleft D_X^{op}$

1) Given M, N two left D_X-modules

can form $M \otimes_{O_X} N$ where $v(m \otimes n) = v(m) \otimes n + m \otimes v(n)$.

Rmk

If A is an Azumaya alg. of rank n and $A \xrightarrow{\varphi} End(\underset{rk = n}{\mathcal{E}})$ is a hom., then φ is an isom.

$D_X \curvearrowright D_X \otimes_{O_X} D_X$

$\uparrow$

$Z(D_X) \to Z(D_X) \otimes_{O_{X^{(1)}}} Z(D_X)$ $\xrightarrow[\text{over } X^{(1)}]{\text{rel. spectrum}}$ $T^*X^{(1)} \times_{X^{(1)}} T^*X^{(1)}$

$$m^* \mathcal{D}_X \simeq \mathcal{D}_X \otimes_{Z(\mathcal{D}_X)} \left(Z(\mathcal{D}_X) \otimes_{\mathcal{O}_{X^{(1)}}} Z(\mathcal{D}_X) \right)$$

$$\begin{array}{ccc} \pi^{-1}(y) & X & \\ & \downarrow \pi & \text{proper} \\ y \in & Y & \end{array}$$

$R^1\pi_* \mathcal{O}_X = R^2\pi_* \mathcal{O}_X = 0$, then $\mathcal{O}_{X,\omega}$, $\forall \omega \in H^0(X^{(1)}, \Omega^1_{X^{(1)}})$,

then restriction to any étale nbhd of $\pi^{-1}(y)$ splits,

$D^b(U(\mathfrak{g})_\lambda\text{-Mod}) \longleftrightarrow D^b_{\pi^{-1}(y)}(\mathrm{Coh}(T^* G/B))$ $\quad T^*G/B \xrightarrow{\pi} \mathcal{N}$ Springer map

2. $[\mathcal{D}_X] = [\mathcal{D}_{T^*X,\eta}]$

$\uparrow$

$\mathrm{Br}(T^*X^{(1)})$ $\qquad \eta \in H^0(T^*X^{(1)}, \Omega^1_{T^*(T^*X^{(1)})})$

canonical diff 1-form

---

3. <u>Frobenius descent</u> $\quad \mathcal{D}_{X,0} = i_0^* \mathcal{D}_X$ a split Azumaya alg.

$$\begin{array}{ccc} \mathcal{D}_X & \curvearrowright & \mathcal{O}_X \\ \cup & & \cup \\ Z(\mathcal{D}_X) & \to & \mathcal{O}_{X^{(1)}} \end{array}$$

$\mathcal{D}_{X,0} \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_{X^{(1)}}}(F_* \mathcal{O}_X)$ $\leftarrow \mathrm{rk} = p^{\dim X}$

<u>Rmk</u> $\quad \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})\text{-Mod}^{\mathrm{qcoh}} \underset{\mathrm{End}(\mathcal{E},-)}{\overset{\mathcal{E} \otimes_{\mathcal{O}_X} -}{\rightleftarrows}} \mathcal{O}_{X^{(1)}}\text{-Mod}^{\mathrm{qcoh}}$

$\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ $\quad F^*\mathcal{E} \leftarrow\!\shortmid \mathcal{E}$

$i_0^* F_* \mathcal{D}_X$

$\wr\!\wr$

$i_0^* \mathcal{D}_X\text{-Mod} \underset{\mathrm{End}_{\mathcal{D}_{X,0}}(F_* \mathcal{O}_X, -)}{\overset{F_* \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} -}{\rightleftarrows}} \mathcal{O}_{X^{(1)}}\text{-Mod}$ $\quad \mathrm{End}_{\mathcal{D}_X}(\mathcal{O}_X, -)$ $\leftarrow$ flat sections

$\mathcal{E} \mapsto \mathcal{E}^{\nabla = 0}$

$\uparrow \wr$

$\mathcal{D}_X \otimes_{A_X} \mathcal{O}_X\text{-Mod}^{\mathrm{qcoh}}$ on X $\longleftrightarrow$ sheaves w/ flat connection w/ p-curvature 0.

$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ s.t. $z(v)$ acts by 0 for any $v \in T_{X^{(1)}}$

$$F^*\mathcal{E} = F^{-1}(\mathcal{E}) \otimes_{F^{-1}(\mathcal{O}_{X^{(1)}})} \mathcal{O}_X$$

$$\nabla(e \otimes 1) = 0, \quad \forall e \in F^{-1}(\mathcal{E}).$$

Lecture 7 $\quad [\mathcal{D}_X] \in \mathrm{Br}(T^*X^{(1)})$

$$m^*[\mathcal{D}_X] = p_1^*[\mathcal{D}_X] + p_2^*[\mathcal{D}_X]$$

m: mult. / addition on $T^*X^{(1)}$

$$m^*\mathcal{D}_X \underset{\text{equiv.}}{\overset{\text{Morita}}{\sim}} p_1^*\mathcal{D}_X \otimes p_2^*\mathcal{D}_X$$

tensor str. on $\mathcal{D}_X$.

+ further coh. data

([OV], last section)

Idea $\quad G_X \longrightarrow (B^2\mathbb{G}_m)_X$

G_X: gp scheme

abelian gp scheme morphism

$$(B^2\mathbb{G}_m)(Y) = \tau^{\leq 2} R\Gamma_{\acute{e}t}(Y, \mathbb{G}_m)[2]$$

$X = pt$ $\qquad$ simplicial abelian gp

$$\pi_0(B^2\mathbb{G}_m(Y)) = H^2_{\acute{e}t}(Y, \mathbb{G}_m) = \mathrm{Br}(Y)$$

$$\pi_1(--) = H^1_{\acute{e}t}(Y, \mathbb{G}_m)$$

$$\pi_2(--) = \mathbb{G}_m(Y)$$

① PD-nbhd envelope

Let $\mathcal{E}$ be a vec. bdle over X,

$k \in \mathbb{Z}_{\geq 0}$

$$\mathcal{E}^{\otimes_{\mathcal{O}_X} k} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{E} \circlearrowleft \text{Sym gp}$$

coinv

$$\mathrm{Sym}^k_{\mathcal{O}_X}(\mathcal{E}) = (--)/gx - x, \; g \in \mathfrak{S}_k$$

$$\bigoplus_{\sum i_j = k} \mathcal{O}_X \, e_1^{i_1} \cdots e_r^{i_r}, \quad r = \mathrm{rank}\,\mathcal{E}$$

$x \mapsto \sum_{g \in \mathfrak{S}_k} g.x$ $\qquad$ $[x] \mapsfrom x$

inv.

$$(\mathcal{E} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{E})^{\mathfrak{S}_k} =: \Gamma^k_{\mathcal{O}_X}\mathcal{E}$$

$$\bigoplus_{\sum i_j = k} \mathcal{O}_X \, e_1^{[i_1]} \cdots e_r^{[i_r]}$$

$x^{[k]} \in \Gamma^k(\ldots)$

$x \otimes \cdots \otimes x \in \Gamma$

$\mathrm{Sym}^k \otimes \mathrm{Sym}^l \longrightarrow \mathrm{Sym}^{k+l}$

$[x] \; [y] \longmapsto [x \otimes y]$ well-defined

$\mathfrak{S}_k \times \mathfrak{S}_l \to \mathfrak{S}_{k+l}$

$\Gamma^k \otimes \Gamma^l \longrightarrow \Gamma^{k+l}$

$$x \;, y \longmapsto \sum_{g \in \mathfrak{S}_{k+l}/\mathfrak{S}_k \times \mathfrak{S}_l} g(x \otimes y)$$

$\Gamma^1 \otimes \Gamma^1 \longrightarrow \Gamma^2$

$x \quad y \longmapsto x \otimes y + y \otimes x$

$\mathrm{Sym}^*_{\mathcal{O}_X} \mathcal{E}$ becomes an alg

makes $\Gamma^*_{\mathcal{O}_X} \mathcal{E}$ an alg

$\mathrm{Sym}^*_{\mathcal{O}_X} \mathcal{E} \longrightarrow \Gamma^*_{\mathcal{O}_X} \mathcal{E}$ alg. homomorphism

$$\underbrace{\Gamma^1 \otimes \Gamma^1 \cdots \otimes \Gamma^1}_{p} \longrightarrow \Gamma^p$$

$x \otimes x \cdots \otimes x \longmapsto 0$

$$0 \longrightarrow (\mathcal{E})^p \longrightarrow \mathrm{Sym}^p(\mathcal{E}) \longrightarrow \Gamma^p(\mathcal{E}) \longrightarrow (\mathcal{E})^p \longrightarrow 0$$

$\mathcal{E} = \mathcal{O}_X \cdot e$, then $\Gamma^*_{\mathcal{O}_X} \mathcal{E} = \mathcal{O}_X [e_1, e_2, \cdots] \big/ e_i^p = 0$

$e_i = e^{[p^i]}$

1) $\mathrm{Sym}^\cdot$, $\Gamma^\cdot$ are functorial.

$\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ induces homomorphism of algebras $\mathrm{Sym}^\cdot(\mathcal{E}_1) \longrightarrow \mathrm{Sym}^\cdot(\mathcal{E}_2)$

$\Gamma^\cdot(\mathcal{E}_1) \longrightarrow \Gamma^\cdot(\mathcal{E}_2)$

2) $$\mathrm{Sym}^n(\mathcal{E}_1 \oplus \mathcal{E}_2) = \bigoplus_{i+j=n} \mathrm{Sym}^i(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \mathrm{Sym}^j(\mathcal{E}_2)$$

$$\Gamma^n(\mathcal{E}_1 \oplus \mathcal{E}_2) = \bigoplus_{i+j=n} \Gamma^i(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \Gamma^j(\mathcal{E}_2)$$

$\rightsquigarrow$

$$\mathrm{Sym}^*(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq \mathrm{Sym}^*(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \mathrm{Sym}^*(\mathcal{E}_2)$$

$$\Gamma^*(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq \Gamma^*(\mathcal{E}_1) \otimes_{\mathcal{O}_X} \Gamma^*(\mathcal{E}_2)$$

Any $\mathcal{E} \in VBun(X)$ is a coalg in $VBun(X)$ $\quad \mathcal{E} \xrightarrow{\Delta} \mathcal{E} \oplus \mathcal{E}$ defines a comm. coalg.

$$\begin{array}{ccc} & \mathrm{Sym}^* \swarrow \qquad \searrow \Gamma^* & \\ \mathrm{Sym}^*(\mathcal{E}) \xrightarrow{\Delta} \mathrm{Sym}^*(\mathcal{E}) \otimes \mathrm{Sym}^*(\mathcal{E}) & & \Gamma^*(\mathcal{E}) \xrightarrow{\Delta} \Gamma^*(\mathcal{E}) \otimes \Gamma^*(\mathcal{E}) \end{array}$$

Hopf algs

$$\Delta(e_1^{i_1} \cdots e_2^{i_2}) = \Delta(e_1)^{i_1} \cdots \Delta(e_2)^{i_2}$$

$$\Delta(e_1)^k = (e_1 \otimes 1 \oplus 1 \otimes e_1)^k$$

$$\Delta(e_1^k) = \sum_{i+j=k} \underbrace{\binom{k}{i}}_{\text{appear in mult. of } \Gamma^*(\mathcal{E})} e_1^i \otimes e_1^j$$

"dual"

$$\Delta(e_1^{[i_1]} \cdots e_2^{[i_2]}) = \Delta(e_1^{[i_1]}) \cdots \Delta(e_2^{[i_2]})$$

$$\Delta(e_1^{[k]}) = \sum_{i+j=k} e_1^{[i]} \otimes e_1^{[j]}$$

<u>Notation</u> Let $\mathcal{E}$ be a v.b. over X, then we can consider two gp schemes over X

$$E := \mathrm{Spec}_X(\mathrm{Sym}^*_{\mathcal{O}_X} \mathcal{E}^\vee) \longleftarrow E_{PD} = \mathrm{Spec}_X(\Gamma^*_{\mathcal{O}_X} \mathcal{E}^\vee)$$

total space of $\mathcal{E}$ $\qquad$ (or $E^\#$)

$$E_k = \mathrm{Spec}_X(\mathrm{Sym}^*_{\mathcal{O}_X}\mathcal{E}^\vee / \mathrm{Sym}^{\geq k+1}_{\mathcal{O}_X} \mathcal{E}^\vee) \longleftarrow E_{[k]} = \mathrm{Spec}_X(\Gamma^*_{\mathcal{O}_X}(\mathcal{E}^\vee) / \Gamma^{\geq k+1}_{\mathcal{O}_X}(\mathcal{E}^\vee))$$

k-th formal nbhd $\qquad$ k-th PD nbhd

also

$$E_{p-1} \simeq E_{[p-1]}$$

Formal analogue: $E^\wedge := \mathrm{colim}_{k \geq 0} E_k \qquad E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$, $E_0 = X$

$$= \text{"}\mathrm{Spf}_X \widehat{\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee}\text{"}$$

$$\hat{E_{PD}} = \operatorname{colim}_{k \geq 0} E_{[k]}$$

$$\| $$

$$\text{"}\operatorname{Spf}\left(\widehat{\Gamma^*_{\mathcal{O}_X} \mathcal{E}^\vee}\right)\text{"} \qquad \mathcal{O}(\hat{E_{PD}})$$

$$//$$

Let's say X is affine, $\Gamma(\hat{E_{PD}}, \mathcal{O})$ - top ring

there is a homomorphism of sheaves of abel gps

$$\mathcal{E}^\vee \longrightarrow \mathcal{O}^\times(\hat{E_{PD}})$$

$$v \longmapsto \exp(v) = 1 + v + v^{[2]} + v^{[3]} + \cdots$$

$\mathcal{A}$ $\quad E_1 \simeq E_{[1]}$ splits over $\hat{E_{PD}}$ $\Leftrightarrow$ splits over $E_1 \simeq E_{[1]}$.

Lecture 8

Last time: given a vec. bdle $\mathcal{E}$ on X,

$$\begin{tikzcd}
E = \operatorname{Spec}_X \operatorname{Sym}^\bullet_{\mathcal{O}_X} \mathcal{E}^\vee \arrow[d, hook] & & E_{PD} = \operatorname{Spec}_X \Gamma^*_{\mathcal{O}_X} \mathcal{E}^\vee \arrow[ll] \arrow[d, hook] \\
E_k = \operatorname{Spec}_X \left(\operatorname{Sym}^\bullet_{\mathcal{O}_X} \mathcal{E}^\vee / \operatorname{Sym}^{\geq k+1}_{\mathcal{O}_X} \mathcal{E}^\vee\right) \arrow[d, hook] \arrow[u, bend left, hook] & & E_{[k]} = \operatorname{Spec}_X \left(\Gamma^*_{\mathcal{O}_X} \mathcal{E}^\vee / \Gamma^{\geq k+1}_{\mathcal{O}_X} \mathcal{E}^\vee\right) \arrow[ll, "j_k"'] \arrow[d, hook] \arrow[u, bend right, hook] \\
E^\wedge = \operatorname{colim}_k E_k & & \hat{E_{PD}} = \operatorname{colim}_k E_{[k]} \arrow[ll]
\end{tikzcd}$$

j_k is an isom. for $k \leq p-1$

Theorem (Ogus – Vologodsky) Let $\mathcal{A}$ be an Azumaya alg. on E, endowed w/ a sym. monoidal str., ($\mu : E \times E \to E$, $\mu^* \mathcal{A} \overset{m.e.}{\sim} p_1^* \mathcal{A} \otimes p_2^* \mathcal{A}$ + associativity & commutativity constraints)

+ $\mathcal{A}$ should be equiv. to $\mathcal{O}_E$ locally on X.

Then if $\mathcal{A}$ splits on E_1, then it also splits on $\hat{E_{PD}}$.

(together w/ tensor str.)

$$E \xrightarrow[\text{of abelian gp objects}]{\text{homo.}} (B^2 \mathbb{G}_m)_X$$

<u>Lemma</u>. $E_{[k]} = S^k_X E_1$

$$\left(Y \xrightarrow{\text{affine scheme}} X = \mathrm{Spec}_X R \right.$$

$$\underbrace{\left(Y \underset{X}{\times} \cdots \underset{X}{\times} Y \right)}_{k} \Big/ \mathfrak{S}_k \quad \left. \text{(categorical quotient}\right)$$

$$= \mathrm{Spec}_X \left(R \underset{\mathcal{O}_X}{\otimes} \cdots \underset{\mathcal{O}_X}{\otimes} R \right)^{\mathfrak{S}_k}$$

<u>Proof</u>

$$\begin{tikzcd}
E_{PD} & E_1 \arrow[l, hook', "e_1"'] \\
 & E_1 \times \cdots \times E_1 \arrow[ul, "e_{1,1}+e_{1,2}+\cdots"'] \arrow[dl] \\
S^k_X E_1 \arrow[uu]
\end{tikzcd}$$

factors through $E_{[k]}$.

$$\Gamma^*_{\mathcal{O}_X} \mathcal{E}^\vee \ni e^{[n]} \xrightarrow{m_k^*} \sum_{i_1+\cdots+i_k=n} e^{[i_1]} \otimes \cdots \otimes e^{[i_k]}$$

$$E_{[k]} = \mathrm{Spec}_X (\mathcal{O}_X \oplus \mathcal{E}^\vee \oplus \Gamma^2 \mathcal{E}^\vee \oplus \cdots \oplus \Gamma^k \mathcal{E}^\vee)$$

$$S^k_X E_{[1]} = \mathrm{Spec}_X \left(\left((\mathcal{O} \oplus \mathcal{E}^\vee)^{\otimes k} \right)^{\mathfrak{S}_k} \right)$$

$$= \mathrm{Spec}_X \left(\underline{\Gamma^k_{\mathcal{O}_X} (\mathcal{O}_X \oplus \mathcal{E}^\vee)} \right) \simeq \bigoplus_{i=1}^{k} \Gamma^k (\mathcal{E}^\vee)$$

$$\begin{tikzcd}
E \arrow[r] & (B^2 \mathbb{G}_m)_X \\
E_1 \arrow[u, hook] \arrow[ur] \arrow[r] & S^\infty_X E_1 \arrow[u, dashed]
\end{tikzcd}$$

$$\bigcup S^k_X E_1 \simeq E^{\wedge}_{PD} \quad \rightsquigarrow \text{" all things gen. by } E_1 \text{"}$$

D_X an Azumaya alg on $T^*X^{(1)}$, $\mathcal{E}$ is $\Omega^1_{X^{(1)}}$ over $X^{(1)}$.

(*) $A \curvearrowright M_1$

$M_k = M_1 \circledast \cdots \circledast M_1$

Rmk. $q_1 : E_1 \to X$

$(q_1)_* \mathcal{O}_{E_1} \simeq \mathcal{O}_X \oplus \mathcal{E}^\vee$

A w/ tensor str.

$[i_1^* A] \in Br(E_1) = H^2_{et}(E_1, \mathbb{G}_m) = H^2_{et}(X, (q_1)_* \mathbb{G}_m)$

$i_0^* A$ is canonically split. $\mu^* A \sim p_1^* A \oplus p_2^* A$

$$0 \to (1+\mathcal{E}^\vee) \to (q_1)_* \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0$$

$(1+\mathcal{E}^\vee) \simeq \mathcal{E}^\vee$; $\hookleftarrow$ étale sheaves on X

$$H^2_{et}(X, \mathcal{E}^\vee) \to H^2_{et}(\quad) \to H^2_{et}(X, \mathbb{G}_m)$$

$\lambda \mapsfrom$; $[i_1^* A] \longmapsto 0$

$\exp : \mathcal{E}^\vee \to \mathcal{O}_{E^{\wedge}_{PD}}$

$[i_{PD}^* A] = \exp([i_1^* A]) \in Br(\mathcal{O}_{E^{\wedge}_{PD}})$

2. D-modules

D_X defines an Azumaya alg. over $T^*X^{(1)}$, + it has a nat'l tensor str.

given by $D_X \underset{\mathcal{O}_X}{\otimes} D_X$ (vec. bundle on $T^*X^{(1)} \underset{X^{(1)}}{\times} T^*X^{(1)}$)

$\rightsquigarrow$ a splitting module for $\mu^* D_X \otimes p_1^* D_X^{op} \otimes p_2^* D_X^{op}$

$$E = T^*X^{(1)}$$

Goal: To show that D_X splits over E_1

$$E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow T^*X^{(1)}$$

$E_0 \cong X^{(0)}$, 0 section

$D_X \curvearrowright \mathcal{O}_X$

$Z(D_X) = \mathrm{Sym}^{\cdot}_{\mathcal{O}_{X^{(1)}}} T_{X^{(1)}}$

defining an action of $i_0^* D_X$ on $F_* \mathcal{O}_X$

Need to construct a D_X-module M_1 s.t. when we restrict the action to $Z(D_X)$, it is a vec. bundle on E_1 of rank $p^{\dim X}$.

Let $I = (T_{X^{(1)}}) \subset Z(D_X)$,

$$0 \to I\mathcal{O}(E_1) \to \mathcal{O}(E_1) \to \mathcal{O}(E_0)/I \to 0$$

$$0 \to T_{X^{(1)}} \to \mathcal{O}(E_1) \to \mathcal{O}_{X^{(1)}} \to 0$$

(with $\mathcal{O}(E_0)/I = \mathcal{O}_{X^{(1)}}$)

$$F_*\mathcal{O}_X \underset{\mathcal{O}_{X^{(1)}}}{\otimes} T_{X^{(1)}}$$

$$F^* T_{X^{(1)}} \to M_1 \to F^* \mathcal{O}_{X^{(1)}}$$

torsor of liftings of Frobenius

3. Frobenius lifts

Fix $\tilde{X}/W_2(k)$ a lift of X/k, $\widetilde{X^{(1)}} = \tilde{X} \times_{W_2(k) \xrightarrow{\text{Frob}}} W_2(k)$

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed] & \tilde{X}^{(1)} \\ X \arrow[u, hook] \arrow[r, "F_X"] & X^{(1)} \arrow[u] \end{tikzcd}$$

obstruction to lifting F_X (to $\tilde{F}_X : \tilde{X} \to \tilde{X}^{(1)}$) is a class $obs(F_X) \in H^1(X, F^*T_{X^{(1)}}) = Ext^1(\mathcal{O}_X, F^*T_{X^{(1)}})$

gives an ext'n $0 \to F^*T_{X^{(1)}} \to M_1 \to \mathcal{O}_X \to 0$

2) another incarnation: $\mathcal{H}om(-, \mathcal{O}_X)$

$N = (M_1^\vee)$

$$0 \to \mathcal{O}_X \to N \to F^*\Omega^1_{X^{(1)}} \to 0$$

Take $\Omega^\bullet_{X,dR} = \mathcal{O}_X \to \Omega^1_X \xrightarrow{d} \cdots$

$$\begin{tikzcd} 0 \arrow[r] & F_*\mathcal{O}_X/\mathcal{O}_{X^{(1)}} \arrow[r] & F_*\Omega^1_{X,d} \arrow[r] & \Omega^1_{X^{(1)}} \arrow[r] & 0 \\ & F_*\mathcal{O}_X \arrow[u] \arrow[r] & & \Omega^1_{X^{(1)}} & \end{tikzcd}$$