p-adic uniformization of Shimura varieties and applications

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Shimura varieties and applications

How to use $\begin{pmatrix} \text{rigid geom} \\ \text{RZ spaces} \end{pmatrix}$ to study LLC.

GLC. G conn. reductive gp $/\mathbb{Q}$, e.g. $GL(n)$, $GSp(2n)$, $SO(n)\cdots$

$\rightsquigarrow$

${}^LG$: L-group of G.

$$1 \to \underbrace{\boxed{\hat{G}}}_{\text{dual group}} \to {}^LG \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

G	$\hat{G}$
$GL(n)$	$GL(n)$
$GSp(2n)$	$GSpin(2n+1)$
	$n=2$, $GSpin(5) \simeq GSp(4)$
$GSp(4)$	$GSp(4)$
$SO(2n)$	$SO(2n)$
$SL(n)$	$PGL(n)$
$\uparrow$ simply conn'd	$\uparrow$ adjoint
$PGL(n)$	$SL(n)$

Expect. $\left\{ \begin{array}{l} \text{L-packets of autom.} \\ \text{rep. of } G(\mathbb{A}) \end{array} \right\}$ π
(algebraic)

$\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \ell\text{-adic rep'ns of } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \\ \text{into } {}^LG(\overline{\mathbb{Q}}_\ell) \end{array} \right\}$ ρ
(algebraic)

$$\begin{tikzcd} \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[dr, "\mathrm{Id}"'] & {}^LG(\overline{\mathbb{Q}}_\ell) \arrow[d] \\ & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{tikzcd}$$

Expect: Étale cohom. of Shimura var. realizes GLC.

Rmk $\exists$ precise conj.

<u>LLC</u> $K|\mathbb{Q}_p$, $[K:\mathbb{Q}_p]<\infty$, G conn. red. $/K$. $\ell\neq p$

$$\left\{\begin{array}{l}\text{L-packets of}\\ \text{irred. sm. rep}\\ \quad\text{of } G(K)\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\ell\text{-adic rep. of}\\ W_K \text{ into } {}^L G(\overline{\mathbb{Q}}_\ell)\end{array}\right\}$$

(Weil gp)

$G = GL(1)$, $GL(1,K) = K^\times \underset{LCFT}{\subset} \mathrm{Gal}(K^{ab}|K)$

(dense image) Shimura var. & Rigid geom.

proved for $GL(n)$ $\boxed{\text{Harris - Taylor}}$, Henniart

$SO(5)\approx$ $GSp(4)$ Gan - Takeda $\leftarrow$ use theta corresp. $\quad LLC\left[\begin{array}{l}GL(2)\times GL(2) \approx SO(4)\\ GL(4) \approx SO(6)\end{array}\right.$

some other gps / rep's. ($\exists$ Arthur's project)

classical groups

<u>Expect</u>. Étale cohomology of RZ spaces realizes LLC (& LJLC)

<u>p-adic uniformization</u>. $G/\mathbb{Q}$ + (some cond.)

$$\left[\begin{array}{ll} K_\infty \subset G(\mathbb{R}) \longrightarrow & D = G(\mathbb{R})/K_\infty \quad \text{Herm. sym. domain}\\ K_f \subset G(\mathbb{A}_f) & \end{array}\right.$$

<u>Sh. var.</u> $\boxed{Sh_{K_f}} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty\cdot K_f$

q-proj. var. $/\mathbb{C}$ $= G(\mathbb{Q})\backslash D\times G(\mathbb{A}_f)/K_f$

defined over E

$[E:\mathbb{Q}]<\infty$, reflex field.

In some cases, Sh var. are moduli sp. of abel. var. w/ add. str. (PEL type)

$\to$ also defined over $\mathrm{Spec}\,\mathcal{O}_E$.

$Sh_\infty := \varprojlim_{K_f} Sh_{K_f}$

Expect. $G(\mathbb{A}_f) \times \mathrm{Gal}(\bar{E}|E) \curvearrowright H^*_{et}(Sh_\infty \otimes_E \bar{E} ; \mathbb{Q}_\ell) = \varinjlim_{K_f} H^*_{et}(Sh_{K_f} \otimes_E \bar{E} ; \mathbb{Q}_\ell)$

$\hookrightarrow$ realize GLC.

<u>local story</u>. $\mathfrak{p} \subset \mathcal{O}_E$ max. ideal

$k(\mathfrak{p})$ residue field

$Sh \otimes_{\mathcal{O}_E} \overline{k(\mathfrak{p})} = \coprod_b Sh_b$ Newton polygon stratification

<u>Assume</u>: $\boxed{b = \text{basic}} \leftarrow$ (AV are most supersingular).

formal scheme

$\to \mathcal{M}$ RZ space (cf. Teruyoshi's lecture)

$\mathcal{M}_\infty \longrightarrow \boxed{\mathcal{M}^{rig}}$: étale Gal. covering, Gal gp "=" $G(\mathbb{Z}_p)$

rigid analytic sp.

Take $x \in Sh_b(\overline{k(\mathfrak{p})})$.

$\updownarrow$

(A, λ, η) abel var. $/k(\mathfrak{p})$.

<u>Define</u> $I(\mathbb{Q}) = \mathbb{Q}\text{isog}(A, \lambda, \eta)$

<u>Known</u>. I is an inner form of G

$I(\mathbb{A}_f^p) \simeq G(\mathbb{A}_f^p)$

$I(\mathbb{R})$ cpt mod center

(e.g. E s.s. ell. $/\bar{\mathbb{F}}_p$,

$\mathrm{End}(E) \otimes \mathbb{Q}$ quat. alg. ram. at p, ∞)

<u>Def</u>. $J(\mathbb{Q}_p) := I(\mathbb{Q}_p)$.

<u>Expect</u> $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_K \curvearrowright H^*_c(\mathcal{M}^{rig}_\infty, \mathbb{Q}_\ell)$

$\cup$

Subgp

$\hookrightarrow$ realizes LLC & LJLC.

$K_f = K_p K^p$

$K_p \subset G(\mathbb{Q}_p), \quad K^p \subset G(\mathbb{A}_f^p)$

<u>Thm</u> (RZ) $\left(\widehat{Sh}_{f,\infty}\right)^{rig} \otimes_{\mathcal{O}_{E,p}} \widehat{\mathcal{O}}_{E,p}^{un} \cong I(\mathbb{Q}) \backslash \mathcal{M}_\infty^{rig} \times G(\mathbb{A}_f^p) / K^p.$

$\mathcal{D}$ herm. sym. domain

$\mathcal{D} \xrightarrow{\text{unif}} Sh(\mathbb{C})$

$\mathcal{M}_\infty^{rig} \xrightarrow{\text{unif}} \left(\widehat{Sh}_{f,\infty}\right)^{rig}$

$Sh / \mathcal{O}_E$

Hochschild – Serre spectral seq. (Fargues)

$N = \dim(\text{Sh. var.})$

$E_2^{i,j} = \mathrm{Ext}^i_{\text{sm-}J(\mathbb{Q}_p)\text{-mod}}\left(H_c^{2N-j}(\mathcal{M}_\infty^{rig}, \mathbb{Q}_\ell(N)), \mathcal{A}\right)$ $\quad \mathcal{A}$ = space of autom. forms of $I(\mathbb{A})$, trivial rep. at ∞.

$H_c^{2N-j}(\mathcal{M}_\infty^{rig}, \mathbb{Q}_\ell(N)) \circlearrowleft G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_K$

$\mathcal{A} \circlearrowleft I(\mathbb{A}_f) = J(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$

$\Rightarrow H^{i+j}\left(\left(\widehat{Sh}_{f,\infty}\right)^{rig}, \mathbb{Q}_\ell\right) \circlearrowleft \underline{G(\mathbb{A}_f)} \times W_K$

$G(\mathbb{A}_f) = G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$

How to use the spec. seq. to relate coh. of Shimura var. & RZ spaces?

Combining
- geom. argument
- automorphic argument

$G = GSp(4)$ or $GU(1,2)$ $\longrightarrow$ have nice stratification on $Sh \otimes \overline{k(p)}$

Geometry (jt w/ Miede)

: all non-basic NP have étale part

$\longrightarrow$ prove supercuspidal part of $H^*(Sh_\infty)$ & $H^*((Sh^{1}_{b,\infty})^{rig})$ are the same.

Autom. rep (Arthur's conj.)

(cf. Imai–Miede)

Assume π autom. rep. of $G(\mathbb{A})$

$\tilde{\pi}$: " of $I(\mathbb{A})$ s.t. $\pi_{p'} = \tilde{\pi}_{p'}$ $(\forall p' \neq p, \infty)$

$\tilde{\pi}_\infty$: triv. rep.

- HS spec. seq deg. at E_2 (Schneider–Stuhler, IHES)
- π_p s.c.
- $\forall \tilde{\pi}'$ autom. rep. of $I(\mathbb{A})$,

$\forall p' \neq p$, $\tilde{\pi}'_{p'} = \tilde{\pi}_{p'} \Rightarrow \tilde{\pi} = \tilde{\pi}'$ (strong mult. one outside p).

$$H^k(Sh_\infty)[\pi_f] \underset{\text{HS}}{=} \sum_i \operatorname{Ext}^i_J \left(H_c^{2N-k+i}(\mathcal{M}_\infty), \tilde{\pi}_p\right) \otimes \tilde{\pi}^p \quad \text{in Groth. gp.}$$

If $\tilde{\pi}_p$: s.c. $= \operatorname{Hom}_J\left(H_c^{2N-k+i}(\mathcal{M}_\infty), \tilde{\pi}_p\right) \otimes \tilde{\pi}^p$.

If $H^k(Sh)$ realizes GLC, then $H^k(\mathcal{M}_\infty)$ realizes LLC & dual of LJLC.

More interesting case, (non tempered) π : Saito–Kurokawa type. $\rightsquigarrow$ $k = 2, 3, 4$

(k=2) $H^2(Sh_\infty)[\pi_f] = \operatorname{Hom}(H_c^4, \pi_f) + \underbrace{\cancel{\operatorname{Ext}^1_J(H_c^5, \pi_p)}}_{= 0}$

$\underline{k=4}$

$$H^4(Sh_\infty)[\pi_f] = \underset{\substack{\| \\ 0}}{\cancel{\mathrm{Hom}}} + \mathrm{Ext}^1_J(\underset{\substack{\nparallel \\ 0}}{H^3_c}, \tilde{\pi}_\rho)$$

$H^3_c(M_\infty)$ realizes LLC & LJLC (as usual)

$H^4_c(M_\infty)$ realizes LLC & Zel. dual of LJLC.