Logarithmic Geometry and Hodge Theory

Michael Barz

$$\begin{array}{c} X \\ \downarrow \\ S \end{array} \quad \text{smooth projective}$$

$$S \xrightarrow{\text{period map}} \text{moduli of PHS}$$

Ex. $\Delta^* = \{ q \in \mathbb{C} : q \neq 0, \ |q| < 1 \}$

$$E_q = \mathbb{C}^\times / q^{\mathbb{Z}}$$

$$\begin{array}{c} E \\ \downarrow \\ \Delta^* \end{array} \quad \text{family of elliptic curve}$$

$$\Delta^* \longrightarrow PSL_2(\mathbb{Z}) \backslash \mathcal{H}$$
$$q \longmapsto \frac{1}{2\pi i} \log(q)$$

As $q \to 0$, E_q becomes a nodal curve.

Q. In what sense can a <u>singular</u> variety carry a PHS?

Can we extend period maps?

A. Kato-Usui: Yes! Something called "logarithmic PHS"

"Log Smoothness"

Ex. $xy = t$

$$\begin{array}{ccc} \mathbb{A}^2_{x,y} & & (x,y) \\ \downarrow t & & \downarrow \\ \mathbb{A}^1_t & & t = xy \end{array}$$

$\Omega^1_f = \langle dx, dy : x\,dy + y\,dx = 0 \rangle$

$dt \longmapsto d(xy) = x\,dy + y\,dx$

At $(0,0)$, relation is trivial!

<u>Def'n</u> If X is a variety, and D a reduced effective divisor, we can define

$\Omega^1_X(\log D)$ = "1-forms on X w/ logarithmic poles along D"

$$\Omega^1_{\mathbb{A}^1_t}(\log\,(t=0)) = \frac{dt}{t}\,\mathbb{C}[t]$$

$$\frac{dt}{t} = d(\log t)$$

$$\Omega^1_{\mathbb{A}^2}\left(\log \begin{matrix} x=0 \\ y=0 \end{matrix}\right) = \left\langle \frac{dx}{x}, \frac{dy}{y} \right\rangle$$

$$\Omega^1_f(\log) = \left\langle \frac{dx}{x}, \frac{dy}{y} : \frac{dx}{x} + \frac{dy}{y} = 0 \right\rangle$$

$$\frac{dt}{t} \longmapsto \frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y}$$

<u>Def'n</u>. A <u>log structure</u> on a scheme X is a sheaf of comm. monoids $\mathcal{M}_X$ on $X_{\text{ét}}$ and a map $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow[\sim]{\alpha} \mathcal{O}_X^\times$ is an isom. of sheaves of monoids.

<u>Monoid</u>: A set M w/ an assoc., unital binary operation.

<u>Ex</u>. $(\mathbb{N}, +)$, $(k^\times, \cdot)$, $(k, \cdot)$. If M is a monoid, there's a group M^{gp} assoc. to it. Eg. $(\mathbb{N},+) \rightsquigarrow (\mathbb{Z},+)$

<u>Ex</u>. $\mathbb{A}^1_t$ w/ log pole along $t=0$. $\mathcal{M}(U) = \{ f \in \mathcal{O}_X(U) : f \text{ only vanishes along } t=0 \}$

_Ex._ If X a var. and D a reduced effective divisor,

$$\mathcal{M}(U) = \{ f \in \mathcal{O}_X(U) : f \text{ only vanishes along } D \}$$

Def'n A morphism of log schemes $(X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$

= the data of a map $f : X \longrightarrow Y$ of schemes, and a map $f^{-1}\mathcal{M}_Y \longrightarrow \mathcal{M}_X$

of sheaves of monoids, s.t.

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r] \arrow[d, "f^{-1}\alpha_Y"'] \arrow[dr, phantom, "///"] & \mathcal{M}_X \arrow[d, "\alpha_X"] \\ f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

Def'n If $f : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$ a morphism of log schemes,

$$\Omega^1_f(\log) = \left(\Omega^1_f \oplus \left(\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp} \right) \right) / \sim$$

$$f \otimes s \approx f \, d\log(s)$$

$$\sim : \left[\begin{array}{l} \alpha(t) \otimes t = dt \\ 1 \otimes s = 0 \quad \text{for } s \in \operatorname{im}\left(f^{-1}\mathcal{M}_Y \to \mathcal{M}_X\right) \end{array} \right.$$

Def A monoid M is _integral_ if $M \to M^{gp}$ is injective.

Def. A log scheme $(X, \mathcal{M}_X)$ is _fine_ if $\mathcal{M}_X$ is étale locally integral & $\mathcal{M}_X / \mathcal{O}_X^\times$ is finitely gen. as a monoid.

Log smoothness

A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of log schemes is _log smooth_ if

1) f is loc. of finite pres.

2) X, Y are fine

3) if $T' \overset{i}{\hookrightarrow} T$ is a square zero ext'n, N a fine log str. on T, and $N' = i^{-1} N$, then in any diagram

$$\begin{tikzcd} (T', N') \arrow[r] \arrow[d] & (X, \mathcal{M}_X) \arrow[d] \\ (T, N) \arrow[r] \arrow[ur, dashed] & (Y, \mathcal{M}_Y) \end{tikzcd}$$

étale loc. on T, we can solve this lifting problem.

Thus Every nodal curve X/k can be given a log structure $\mathcal{M}_X$ so that

$$(X, \mathcal{M}_X) \longrightarrow (\operatorname{Spec} k, k^\times) \quad \text{is log smooth.}$$

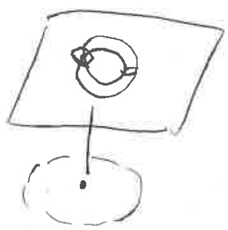
Moreover, any log smooth curve over $(\operatorname{Spec} k, k^\times)$ is nodal.

Def'n let X be a log $\mathbb{C}$-analytic space

$$X^{\log} = \left\{ (x, h) : x \in X,\ h : \mathcal{M}_{X,x} \longrightarrow S^1 \ \text{s.t.}\ h(\alpha^{-1}(f)) = \frac{f(x)}{|f(x)|} \right\}$$

Example

$E \downarrow \Delta$



$E^{\log} \downarrow \Delta^{\log} = S^1 \times [0, 1)$

Let X be a log $\mathbb{C}$-analytic space. A log VPHS on X is

- a local system $\mathcal{H}_{\mathbb{Z}}$ on $X^{\log}$
- $Q : \mathcal{H}_{\mathbb{Q}} \otimes \mathcal{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ ...
- filtration $F^\bullet$ on $\mathcal{H}_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{X^{\log}}$ obeying

1) Griffiths transversality
2) $F^\bullet \mathcal{H}_{\mathcal{O}}$ is pulled back from X
3) if $x \in X$, and $y = (x, h) \in X^{\log}$, and $s : \mathcal{O}_{X^{\log}, y} \longrightarrow \mathbb{C}$ extends $ev : \mathcal{O}_{X,x} \to \mathbb{C}$

then if $|\exp(s(\log f_i))| \ll 1$, then $(\mathcal{H}_{\mathbb{Z},y}, Q_y, F^\bullet(s))$ is a PHS

f_i gen. $\mathcal{M}_{X,x}/\mathcal{O}_{X,x}^\times$; $F^\bullet(s) = \mathbb{C} \otimes_{\mathcal{O}_{X,y}^{\log}} (F_y^\bullet)$