Topological cyclic homology

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Lecture 2

Motivation: trace methods

R ring $\rightsquigarrow$ algebraic K-theory groups

$$K_*(R) \quad , * \geq 0$$

Theoretically important, but very **hard** to compute. e.g. $K_*(\mathbb{Z})$

Def $\quad K_0(R) = \{\text{iso. classes of f.g. proj. } R\text{-modules}\}^{gp}$

$$\oplus$$

higher K-groups are defined as the homotopy groups of the space obtained by group completing the cat. of f.g. proj. R-modules.

Idea of trace methods. Approximate the groups by more algebraic invariants.

$$\begin{tikzcd}
K_*(R) \arrow[dr, bend left, "\text{cyctr}"] \arrow[ddrr, "\text{trace}"] & & \\
& TC_*(R) \arrow[r] \arrow[d] & HC^-_*(R) \arrow[d] \\
& THH_*(R) \arrow[r] & HH_*(R)
\end{tikzcd}$$

Meta theorem. The map cyctr is often (close to) an isom.

<u>relative K-groups</u> $K_*(R,I)$, $I \subset R$ ideal

$TC_*(R,I)$

<u>K-groups w/ coefficients</u> $K_*(R;\mathbb{Z}_p)$

$TC_*(R;\mathbb{Z}_p)$

<u>Theorem</u>. (1) Dundas-Goodwillie-McCarthy: if $I \subset R$ nilpotent, then

$$\text{cyctr}: K_*(R,I) \xrightarrow{\sim} TC_*(R,I) \text{ is an isom.}$$

(2) Clausen-Mathew-Morrow: if R is comm. and I-complete, then

$$K_*(R,I;\mathbb{Z}_p) \xrightarrow{\sim} TC_*(R,I;\mathbb{Z}_p) \quad \text{ise}$$

(3) ——||—— : If R p-complete, then

$$TC_*(R;\mathbb{Z}_p) \simeq K_*^{et}(R,\mathbb{Z}_p)$$

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k field, R k-alg. P f.g. projective right R-module,

$$\mathrm{Hom}_R(P,P) \simeq P \otimes_R \mathrm{Hom}_R(P,R) \;\cancel{\xrightarrow{ev}}\; R$$

$$x \otimes \varphi \longmapsto \varphi(x)$$

$$x \otimes r\varphi \longmapsto r\varphi(x)$$

$$xr \otimes \varphi \longmapsto \varphi(xr) = \varphi(x)\cdot r$$

$$R/[R,R] = R/(xy \sim yx)$$

$$\mathrm{Hom}_R(p,p) \xrightarrow{tr} R/[R,R]$$

Left R-module $\cong$ right R^{op}-module

R-R-bimodule $\cong$ $R \otimes_k R^{op}$-module

$$R/[R,R] \cong R \otimes_{R \otimes_k R^{op}} R$$

$$HH(R/k) \cong R \otimes^L_{R \otimes_k R^{op}} R$$

$$tr: K_*(R) \longrightarrow HH_*(R/k)$$

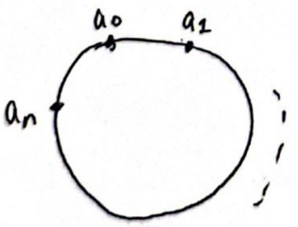
<u>Lecture 3</u>. <u>Classical Hochschild homology</u>

k field, R k-algebra

<u>Def</u>. $HH_*(R/k)$ is homology of

$$HH(R/k) = (\cdots \xrightarrow{\partial} R \otimes_k R \xrightarrow{\partial} R)$$

$$\partial(a_0 \otimes \cdots \otimes a_n) = a_0 a_1 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n \cdots$$
$$+ (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$



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Example $R=k$

$$HH(k/k) = (\cdots \xrightarrow{0} k \xrightarrow{1} k \xrightarrow{0} k)$$

$$HH_*(k/k) = \begin{cases} k & * = 0 \\ 0, & \text{otherwise} \end{cases}$$

R commutative, $HH_0(R/k) = R$

$$HH_1(R/k) = R^{\otimes_k 2}/\mathrm{im}\,\partial \cong \Omega^1_{R/k}$$

$\Omega^1_{R/k}$ gen. as R-module dx, $x \in R$

- $d(x+y) = dx + dy$
- $d(xy) = xdy + ydx$
- $dx = 0$, $x \in k$

$$d: R \longrightarrow \Omega^1_{R/k}$$

$$x \otimes y \longleftrightarrow x\,dy$$

Lemma R comm. k-alg, then

$HH_*(R/k)$ has structure of a strictly graded comm. ring. ($x^2 = 0$, $|x| = \text{odd}$)

only matters when char $k = 2$

Proof $HH(R/k)$ comes from

$$\cdots R \otimes_k R \otimes_k R \rightrightarrows R \otimes_k R \rightrightarrows R$$

Eilenberg-Zilber. $\square$

<u>Definition</u>. $\Omega^*_{R/k} = \Lambda^*_R \Omega^1_{R/k} = \bigoplus_{k\geq 0} \Lambda^k_R \Omega^1_{R/k}$

$$= \frac{\mathrm{Sym}_R\, \Omega^1_{R/k}[1]}{x^2=0 \text{ for } x\in\Omega^1_{R/k}}$$

Get a map $\Omega^*_{R/k} \longrightarrow HH_*(R/k)$

<u>Thm</u> (Hochschild- Kostant- Rosenberg) If R/k has cotangent complex concentrated in deg 0, $\Omega^*_{R/k} \longrightarrow HH_*(R/k)$ is iso.

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k arbitrary comm. ring ($k=\mathbb{Z}$)

R dga over k which is k-flat

$$HH(R/k) = \underset{\text{cpx}}{\text{total}}\left(\cdots \to R\otimes_k R \to R\right)$$

R not k-flat, replace by a k-flat resolution R^b

$$HH(R/k) := HH(R^b/k)$$

<u>Prop</u>. For a dga R, $HH(R/k) \simeq R\otimes^L_{R\otimes^L_k R^{op}} R$

<u>Proof</u>. replace R by flat resolution, consider the cpx

$$\left(\cdots \to R\otimes_k R\otimes_k R \to R\otimes_k R\right) \simeq R \qquad \text{(Bar cpx)}$$

Apply $-\otimes_{R\otimes_k R^{op}} R$, total cpx computes $R\otimes^L_{R\otimes^L_k R^{op}} R$ and $HH(R/k)$. $\square$

<u>Lemma</u> If R is a dga coming from a simplicial comm. ring, then $HH(R/k)$ also comes from a simplicial comm. k-alg.

In particular, $HH_*(R/k)$ form a strictly graded comm. k-alg.

<u>Proof</u>. replace by flat simplicial comm. k-alg.

$HH(R_n/k)$ gives a <u>bisimplicial</u> comm. k-alg

Total cpx $\simeq$ cpx assoc. to diagonal $\square$

<u>Lemma</u> A, B dga's, then $HH(A \otimes^L B/k) \simeq HH(A/k) \otimes^L_k HH(B/k)$

<u>Proof</u>. $HH(A/k)$, $HH(B/k)$ simplicial dgas.

So $HH(A/k) \otimes^L_k HH(B/k)$ bisimplicial dga

diagonal is $HH(A \otimes_k B/k)$. $\square$

<u>Proof of HKR</u> for polynomial rings

- $R = k[x]$, $\Omega^*_{k[x]/k} = k[x] \otimes \Lambda(dx)$

$$HH(k[x]/k) = k[x] \otimes^L_{k[x] \otimes_k k[x]^{op}} k[x]$$

$$= k[x] \otimes^L_{k[a,b]} k[x]$$

Resolution $k[a,b] \xrightarrow{\cdot (a-b)} k[a,b]$

- $R = k[x_1, \cdots, x_n]$. Both sides commute w/ tensor products.

- $R =$ poly. in arbitrarily many generators.

  both sides commute w/ filtered colimits.

Prop $HH_*(\mathbb{F}_p/\mathbb{Z}) = $ "$\mathbb{F}_p\{1, x, \frac{x^2}{2}, \frac{x^3}{3}, \cdots\}$"

$$= \mathbb{F}_p[x_1, x_2, \cdots] \big/ x_i x_j = \binom{i+j}{i} x_{i+j}$$

$$= \mathbb{F}_p\langle x\rangle .$$

Proof Compute $\mathbb{F}_p \otimes^L_{\mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathbb{F}_p^{op}} \mathbb{F}_p$

$$\mathbb{F}_p \simeq \mathbb{Z}[\varepsilon]/\varepsilon^2, \quad \partial\varepsilon = p, \quad |\varepsilon| = 1$$

$$A = \mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathbb{F}_p^{op} \simeq \mathbb{F}_p[\varepsilon]/\varepsilon^2, \quad \partial\varepsilon = 0$$

Let $A\langle x\rangle = \dfrac{A[x_1, x_2, \cdots]}{\left(x_i x_j = \binom{i+j}{i} x_{i+j}\right)}$, $\quad \begin{matrix}\partial x_i = \varepsilon x_{i-1} \\ (\partial x_1 = \varepsilon)\end{matrix}$, $\quad |x_i| = 2i$

homology concentrated in deg 0, $\simeq \mathbb{F}_p$.

$A\langle x\rangle \otimes_A \mathbb{F}_p = \mathbb{F}_p\langle x\rangle$

Appendix. Simplicial comm. rings

Def. $sCRing = Fun(\Delta^{op}, cRing)$

$$\left(\cdots \ R_2 \ \substack{\rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow} \ R_1 \ \substack{\rightarrow \\ \leftarrow \\ \rightarrow} \ R_0 \right)$$

Thm. (Eilenberg - Zilber)

$SAb \longrightarrow Ch$ is lax sym. monoidal

(m,n)-shuffle : $(\mu,\nu) : \underset{\overset{\parallel}{(\{0,1,\dots,m+n\},\leq)}}{[m+n]} \longrightarrow [m] \times [n]$

- μ, ν both surjective
- μ, ν "jump" at disjt positions.
- $\mathrm{sign}(\mu,\nu) = \mathrm{sign}(\sigma)$, σ permutation induced by (μ,ν)

Prop. $R \in$ scRing, then $R_m \otimes R_n \xrightarrow{\bullet} R_{m+n}$

$$x \cdot y = \sum_{(\mu,\nu)} \mathrm{sign}(\mu,\nu)\, s_\mu(x)\, s_\nu(y)$$

gives R the str. of a $\overset{\text{strict}}{\vee}$ cdga.

Proof.
- Associativity : both $(xy)z$, $x(yz)$ given by $\sum_{(\lambda,\mu,\nu)} \mathrm{sign}(\lambda,\mu,\nu)\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z)$
- commutativity, same summands, up to $\dfrac{\mathrm{sign}(\mu,\nu)}{\mathrm{sign}(\nu,\mu)} = (-1)^{mn}$
- strictness: in x^2, the summands corresp. to (μ,ν) and (ν,μ) cancel. $x^2 = 0$
- derivation, leaving out an index from (m,n) shuffle (μ,ν)

  gives either $(m-1,n)$- shuffle (μ',ν), $(m,n-1)$- shuffle (μ,ν').

Prop. $R \in$ scRing,

$$\gamma_k : R_n \longrightarrow R_{nk} \qquad \text{for } n, k \geq 1$$

- $\gamma_1(x) = x$
- $\gamma_k(x)\gamma_\ell(x) = \binom{k+\ell}{k} \gamma_{k+\ell}(x)$ $\qquad x^k = k!\, \gamma_k(x)$
- $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$
- $\gamma_k(xy) = x^k \gamma_k(y)$
- $\gamma_k(\gamma_\ell(x)) = \frac{(k\ell)!}{k!\,(\ell!)^k} \gamma_{k\ell}(x)$

Proof. x odd, $\gamma_1(x) = x$, $\gamma_k(x) = 0$, $k \geq 2$

x even:

$$\underbrace{x \cdot x \cdot \cdots \cdot x}_{k} = \sum_{(\mu_1, \cdots, \mu_k)} \operatorname{sign}(\mu_1, \cdots, \mu_k)\, S_{\mu_1}(x) \cdots S_{\mu_k}(x)$$

$\sigma \in \mathfrak{S}_k$, $(\mu_1, \cdots, \mu_k)$ and $(\mu_{\sigma(1)}, \cdots \mu_{\sigma(k)})$ give same summand.

$$x^k = k!\, \gamma_k(x) \qquad \text{(by definition)}$$

Prop $R \in$ scRing

- $d\gamma_k(x) = \gamma_{k-1}(x) \cdot dx$ $\quad$ if x even
- γ_k sends boundaries to boundaries
- γ_k well-defined divided power str. on $H_*(R)$.

Proof Say x is boundary, Then $x = dy$ for y w/ $d_i y = 0$, $i > 0$
$= d_0 y$

define $$\gamma_k'(y) = \sum_{\substack{(\mu_1, \cdots, \mu_k) \\ \mathfrak{S}_k\text{-rep}}} \operatorname{sign}(\mu_1, \cdots, \mu_k)\, s_{\mu_1'}(y) \cdots s_{\mu_k'}(y)$$

$$\mu_i : [kn] \to [n]$$

$$\mu_i' : [1+kn] \to [1+n]$$

$$d_0 \gamma_k'(y) = \gamma_k(x)$$

$$d_i \gamma_k'(y) = 0, \quad i > 0$$

i.e. $d\gamma_k'(y) = \gamma_k(x)$

Lecture 4. The Connes operator on HH

k comm. ring, R k alg.

$$HH_*(R/k) = H_*(HH(R/k))$$

$$HH(R/k) = (\cdots \to R \overset{L}{\underset{k}{\otimes}} R \xrightarrow{\partial} R) \in D(k)$$

HKR theorem: If $L_{R/k}$ flat dimension 0, then $HH_*(R/k) \cong \Omega^*_{R/k}$.

$$d : \Omega^*_{R/k} \to \Omega^{*+1}_{R/k}, \quad R \xrightarrow{d} \Omega^1_{R/k}$$

Def. R an assoc. k-alg. We define a k-linear map

$$B : HH(R/k)_n \longrightarrow HH(R/k)_{n+1}$$

$$x_0 \otimes \cdots \otimes x_n \longmapsto \sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)} \left(1 \otimes x_\sigma + (-1)^n x_\sigma \otimes 1\right),$$ $x_\sigma = x_{\sigma(0)} \otimes \cdots \otimes x_{\sigma(n)}$

C_{n+1} = gp of cyclic permutations of $\{0, \cdots, n\}$

Ex. • $B^2 = 0$

• $dB + Bd = 0$

The operator B equips $HH(R/k)$ w/ the str. of a DG-module over the DGA

$$A = \mathbb{Z}[b]/b^2, \quad |b| = 1, \quad \partial = 0$$

$$= H_*(\mathbb{T}, \mathbb{Z}), \quad \mathbb{T} = U(1)$$

$\Rightarrow$ $HH_*(R/k)$ is a graded module over $\mathbb{Z}[b]/b^2$

In particular, there is an operator $B: HH_*(R/k) \to HH_{*+1}(R/k)$ w/ $B^2 = 0$.

Fact. If R is comm., then B is a derivation, i.e. it satisfies the graded Leibniz rule.

$$B(xy) = B(x)\cdot y + (-1)^{|x|} x B(y)$$

Warning: This is NOT true on the nose on $HH(R/k)$.

Proposition. The map $\Omega^*_{R/k} \to HH_*(R/k)$ sends de Rham differential d to the Connes operator B, i.e. the square

$$\begin{tikzcd} \Omega^*_{R/k} \arrow[r] \arrow[d, "d"'] & HH_*(R/k) \arrow[d, "B"] \\ \Omega^{*+1}_{R/k} \arrow[r] & HH_{*+1}(R/k) \end{tikzcd}$$

Proof. The map $\Omega^*_{R/k} \to HH_*(R/k)$ is determined by its effect in deg 0 & 1.

deg 0: $R \to HH_0(R/k) = R$

deg 1: $\Omega^1_{R/k} \longrightarrow HH_1(R/k) = \dfrac{R \otimes_k R}{\sim}$

$x\,dy \longmapsto [x \otimes y]$

Thus in order to check the statement, it is enough to check that

$$\begin{tikzcd} R = \Omega^0_{R/k} \arrow[r] \arrow[d, "d"] & HH_0(R/k) \arrow[d, "B"] \\ \Omega^1_{R/k} \arrow[r, shift left] & HH_1(R/k) \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\shortparallel"] & {[r]} \arrow[d, mapsto] \\ dr & {[1 \otimes r + r \otimes 1]} \arrow[l, mapsto] \end{tikzcd}$$

<u>Example.</u> For $HH_*(\mathbb{F}_p/\mathbb{Z}) \simeq \mathbb{F}_p\langle x \rangle$, $\quad |x| = 2$

the Connes operator acts trivially for degree reasons.

<u>Question.</u> Does this mean that it is "trivial" on $HH(\mathbb{F}_p/\mathbb{Z})$?

We consider the (derived) mod p reduction

$$\begin{aligned} HH(\mathbb{F}_p) /\!\!/ p &= HH(\mathbb{F}_p) \otimes^L_{\mathbb{Z}} \mathbb{F}_p \\ &\simeq HH(\mathbb{F}_p) \otimes_{\mathbb{Z}} \left(\Lambda_{\mathbb{Z}}(\varepsilon),\ |\varepsilon| = 1,\ \partial\varepsilon = p\right) \\ &\simeq HH(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon) \end{aligned}$$

no differential since $\partial\varepsilon = p = 0$

$$\begin{aligned} H_*(HH(\mathbb{F}_p) /\!\!/ p) &\simeq HH_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon) \\ &\simeq \mathbb{F}_p\langle x \rangle \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon) \end{aligned}$$

$|\varepsilon| = 1$
$|x| = 2$

Prop. We have

$$B(x^{[n]}) = 0, \quad B(\varepsilon) = x.$$

$$\left(\Rightarrow B(\varepsilon x^{[n]}) = x \cdot x^{[n]} = (n+1)\, x^{[n+1]}\right)$$

Proof. The map $HH(\mathbb{F}_p) \longrightarrow HH(\mathbb{F}_p)/\!/p$ is compatible w/ B,

thus we get $B(x^{[n]}) = 0$ in $H_*(HH(\mathbb{F}_p)/\!/p)$.

The way $HH_*(\mathbb{F}_p) = \mathbb{F}_p\langle x\rangle$ is computed is by replacing $\mathbb{F}_p$ by $(\Lambda_{\mathbb{Z}}(\varepsilon);\ \partial\varepsilon = p)$

$$\cdots \xrightarrow{\partial} \Lambda_{\mathbb{Z}}(\varepsilon) \otimes \Lambda_{\mathbb{Z}}(\varepsilon) \xrightarrow{\partial} \Lambda_{\mathbb{Z}}(\varepsilon)$$

$$\begin{tikzcd}
 & {} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \cdots \arrow[r] & \mathbb{Z}(\varepsilon\otimes\varepsilon) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdot \arrow[r] \arrow[d] & {} & \mathbb{Z}(\varepsilon\otimes 1)\oplus\mathbb{Z}(1\otimes\varepsilon) \arrow[r] \arrow[d] & \mathbb{Z}\varepsilon \arrow[d] \\
\mathbb{Z} \arrow[rr, "1"'] & & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z}
\end{tikzcd}$$

The class x is given by $1\otimes\varepsilon - \varepsilon\otimes 1$

We have $B(\varepsilon) = 1\otimes\varepsilon - \varepsilon\otimes 1$

After mod p reduction, we replace all $\mathbb{Z}$'s by $\mathbb{F}_p$. then ε is a cycle, and $B([\varepsilon]) = [x]$.

$\square$.

Lecture 5. Periodic and cyclic homology

R a k-alg. $HH(R/k)$, $B: HH(R/k) \to HH(R/k)[-1]$

$\Rightarrow HH(R/k)$ is a DG-module over the DGA $A = \left(k[b]/b^2,\ |b|=1,\ \partial=0\right)$

$\Rightarrow HH(R/k) \in \mathrm{DGMod}_A \to \mathrm{DGMod}_A[\mathrm{qiso}^{-1}] \simeq \mathrm{Mod}_A(Dk)$

Def. (1) The cyclic homology of R is

$$HC_*(R/k) = H_*\left(k \overset{L}{\underset{A}{\otimes}} HH(R/k)\right) = \mathrm{Tor}^A_*(k, HH(R/k))$$

(2) The negative cyclic homology of R is

$$HC^-_*(R/k) = H_*(R\mathrm{Hom}_*(k, HH(R/k))) = \mathrm{Ext}^{-*}_A(k, HH(R/k))$$

This is a module over $\mathrm{Ext}^*_A(k,k) \cong k[t]$, $|t| = -2$

(3) The periodic homology of R is

$$HP_*(R/k) = HC^-_*(R/k)[t^{-1}].$$

Prop. For any k-alg. R, we have $HC^-(R/k) \cong \left(HH(R/k)[\![t]\!],\ \partial + tB\right)$

$|t| = -2$

here $\partial + tB$ is defined as

$x t^i \mapsto (\partial x) t^i + (Bx) t^{i+1}$. $x \in HH(R/k)$

$$HP(R/k) \simeq \left(HH(R/k)((t)),\ \partial + tB\right)$$

Proof. Resolve k as an A-algebra by $C = A\{x_0, x_1, \cdots\}$. $|x_k| = 2k$.

$\partial(x_k) = b \cdot x_{k-1}$

This is a coalgebra w/ $\Delta(X_k) = \sum_{i+j=k} X_i \otimes X_j$

$\Rightarrow \underline{R\mathrm{Hom}}_A(k, HH(R/k)) = \underline{\mathrm{Hom}}_A(C, HH(R/k)) \simeq (HH(R)[[t]], \partial + tB)$

The formula for HP follows by localizing at t.

Rmk. This formula works more generally for any object $H \in DMod_A$ to give $R\mathrm{Hom}_A(k, H)$.

- A is a Hopf algebra: $\varepsilon : A \to k$, $\varepsilon(b) = 0$

$\Delta : A \to A \otimes A$, $\Delta(b) = 1 \otimes b + b \otimes 1$

$\Rightarrow \underline{R\mathrm{Hom}}_A(k, -)$ is a lax sym mon. functor

as such it is given by $(H[[t]], \partial + tB)$

$\Rightarrow$ C is an algebra object in $DGMod_A$.

i.e. C is a DGA and B satisfies the Leibniz rule, then

$\underline{R\mathrm{Hom}}_A(k, C)$ is a DGA.

Problem Even if R is commutative, B is NOT on the nose a derivation on $HH(R)$.

but this is true up to chain homotopy.

But this is enough to deduce that $(HH(R)[[t]], \partial + tB)$ has a product up to chain homotopy, i.e. that

$HC_*^-(R)$ and $HP_*(R)$ are graded comm. algebras.

Def. Let R be a comm. k-alg. The de Rham cohomology of R relative to k is defined as $H^*_{dR}(R/k) = H_*(\Omega^*_{R/k}, d)$

Rmk. H^*_{dR} can be defined for schemes X over k.

$\boxed{\text{Thm A}}$ Assume that $\mathbb{Q} \subset k$, and $\mathbb{L}_{R/k}$ is flat dim 0

Then there are natural isomorphisms $HP_*(R/k) \simeq H^*_{dR}(R/k)((t))$, $|t| = -2$.

$$HC^-_n(R/k) = Z^n_{dR}(R/k) \oplus \prod_{i \geq 1} H^{n+2i}_{dR}(R/k)$$

Rmk. • For a scheme X/k w/ $\mathbb{L}_{X/k}$ is flat dim 0, we have $HP_*(X/k) \simeq H^*_{dR}(X/k)((t))$.

• False if char k is not 0.

Consider
$$\mu: \underset{\substack{\shortparallel \\ R^{\otimes(n+1)}}}{HH(R/k)_n} \longrightarrow \Omega^n_{R/k}$$

$$r_0 \otimes r_1 \otimes \cdots \otimes r_n \longmapsto \frac{1}{n!} r_0\, dr_1 \cdots dr_n$$

Lemma. This is a map of CDGA's and A-linear

$$(HH(R/k), \partial, B) \longrightarrow (\Omega^*_{R/k}, 0, d)$$

Proof
$$\mu(\partial(r_0 \otimes \cdots \otimes r_n)) = \mu\begin{pmatrix} r_0 r_1 \otimes \cdots \otimes r_n \\ - r_0 \otimes r_1 r_2 \otimes \cdots \otimes r_n \\ \pm r_n r_0 \otimes r_1 \otimes \cdots \otimes r_{n-1} \end{pmatrix} = \begin{matrix} r_0 r_1\, dr_2 \cdots dr_n \\ - r_0\, d(r_1 r_2)\, dr_3 \cdots dr_n \\ \vdots \end{matrix} = 0$$

$$\mu(B(r_0 \otimes \cdots \otimes r_n)) = \sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)} \mu(1 \otimes r_0 \pm r_\sigma \otimes 1) = \frac{1}{(n+1)!} \sum_{\sigma \in C_{n+1}} \mathrm{sgn}(\sigma)\, dr_{\sigma(0)} \cdots dr_{\sigma(n)}$$

$$= \frac{1}{n!} dr_0 \otimes \cdots \otimes dr_n = d\left(\frac{1}{n!} r_0\, dr_1 \cdots dr_n\right) = d(\mu(r_0 \otimes \cdots \otimes r_n))$$

Cor If R has flat $\mathbb{L}_{R/k}$ and $\mathbb{Q} \subset k$, then

$$(HH(R/k), \partial, B) \simeq (\Omega^*_{R/k}, 0, d)$$

Pf. We always have that the composition

$$\Omega^*_{R/k} \to HH_*(R/k) \xrightarrow{H_*(\mu)} \Omega^*_{R/k}$$ is the identity, so the statement

follows from HKR. □

For Theorem A, we get

$$\begin{aligned} HP_*(R/k) &= H_*(HH(R/k)((t)), \partial + tB) \\ &\simeq H_*(\Omega^*_{R/k}((t)), td) \\ &\simeq H^*_{dR}(R/k)((t)). \end{aligned}$$

$$HC^-_*(R/k) \simeq H_*(\Omega^*_{R/k}[[t]], td)$$

Construction. If k is arbitrary, then we have the Postnikov filtration

$\tau_{\geq \bullet} HH(R/k)$ on $HH(R/k)$ and leads to a filtration on $HP(R/k)$.

Concretely, $(\tau_{\geq \bullet} HH(R/k)((t)), \partial + tB)$ and leads to a multiplicative, conditionally

convergent spectral sequence

$$E_2 = HH_*(R/k)((t)) \Longrightarrow HP_*(R/k)$$

w/ E_3-page given by $H_*(HH_*(R/k), B)((t)) \Rightarrow HP_*(R/k)$

If R has flat cotangent cpx, then this is $E_3 = H^*_{dR}(R/k)((t)) \Rightarrow HP_*(R/k)$

Def R comm. ring, we define the <u>divided power series algebra</u>

$R\langle\langle x\rangle\rangle$ as the completion of $R\langle x\rangle$ at the filtration given by the divided powers of x.

Prop. For $R = \mathbb{F}_p$, we have

$$HC^-_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p[t]\langle\langle x\rangle\rangle / xt-p, \quad |x|=2,\ |t|=-2,$$

$$HP_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p[t^{\pm 1}]\langle\langle x\rangle\rangle / xt-p \cong \left(\mathbb{Z}_p\langle\langle y\rangle\rangle / y-p\right)[t^{\pm 1}], \quad |y|=0$$

Rmk. $HP_0(\mathbb{F}_p/\mathbb{Z}) \cong HC^-_0(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p\langle\langle y\rangle\rangle / y-p$

is obtained by adjoining divided powers of p to $\mathbb{Z}_p$

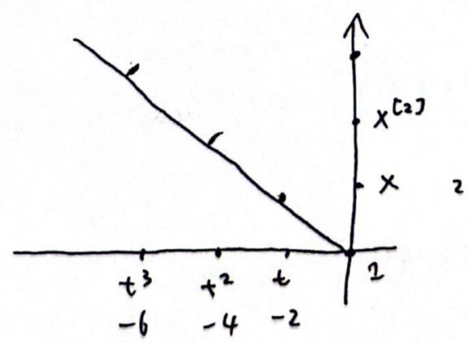
$\rightarrow$ get torsion $y^{[p]} - p^p/p!$, $p!$ - torsion

- In fact, we have $\mathbb{Z}_p\langle y\rangle / y-p \cong \mathbb{Z}_p\langle z\rangle / z$, $z = y-p$

- In fact, $HP_*(\mathbb{F}_p/\mathbb{Z})$ is 2-periodic derived de Rham coh. of $\mathbb{F}_p$ relative to $\mathbb{Z}$.

Proof - Recall that $HH_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{F}_p\langle x\rangle$.

- SS $\mathbb{F}_p\langle x\rangle[t] \Rightarrow HC^-_*(\mathbb{F}_p/\mathbb{Z})$



$\bullet = \mathbb{F}_p$

Collapses (parity vanishing)

Thus $HC^-_*(\mathbb{F}_p/\mathbb{Z})$ has a filtration w/ assoc. graded given by $\mathbb{F}_p\langle x\rangle[t]$.

" ..

• The connective cover of $HC^-(\mathbb{F}_p/\mathbb{Z})$ can be represented by a simplicial comm. ring

Thus it admits divided powers in pos. degree homotopy gps.

In particular, every choice of $x, t \in HC^-_*(\mathbb{F}_p/\mathbb{Z})$ gives a map

$$\mathbb{Z}\langle x\rangle[t] \longrightarrow HC^-_*(\mathbb{F}_p/\mathbb{Z})$$

If we had x, t s.t. $xt = p$ in $HC^-_*(\mathbb{F}_p/\mathbb{Z})$, then this map induces an isom. on assoc. graded

of the map $\mathbb{Z}\langle x\rangle[t]/xt-p \longrightarrow HC^-_*(\mathbb{F}_p/\mathbb{Z})$

Back to the computation of $HH(\mathbb{F}_p/\mathbb{Z})$.

$$\mathbb{F}_p \simeq (\Lambda_{\mathbb{Z}}(\varepsilon), \partial\varepsilon = 0).$$

In $HH(\mathbb{F}_p/\mathbb{Z})$, $X = B\varepsilon$, $\partial X = 0$, $BX = 0$

We get directly that in $HH(\mathbb{F}_p/\mathbb{Z})$, we have X cycle represents $X \in HH_*(\mathbb{F}_p/\mathbb{Z})$

$$HC^-(\mathbb{F}_p) = (HH(\mathbb{F}_p/\mathbb{Z})[[t]], \partial + tB)$$

$$(\partial + tB)\varepsilon = p + tX \quad \Rightarrow \quad p = tX \text{ in } HC^-_*(\mathbb{F}_p/\mathbb{Z})$$

<u>Rmk</u>. One can also deduce this computation using that $(HH(\mathbb{F}_p/\mathbb{Z}), \partial, B)$

$$\simeq \left(\mathbb{Z}[\varepsilon]/\varepsilon^2\langle X\rangle, \begin{matrix}\partial\varepsilon = p \\ \partial X = 0\end{matrix}, \begin{matrix}B\varepsilon = X \\ BX = 0\end{matrix}\right)$$

• One can construct LES

$$\cdots \to HP_{*+1}(R/k) \to HC_{*-1}(R) \longrightarrow HC^-_*(R/k) \longrightarrow HP_*(R/k) \to HC_{*-2}(R) \longrightarrow \cdots$$

Lecture 6. HKR and the cotangent complex

$\mathcal{C}$ 1-cat., generated by cpt projective objects.

(K compact projective if $\mathrm{Hom}(K,-)$ preserves filtered colimits (compact)

and $\mathrm{Hom}(K,-)$ preserves reflexive coequalizers.

$\bullet \rightleftarrows \bullet$)

generated: $\mathrm{Hom}(K,-)$ detects isomorphisms)

$$\mathrm{Anim}(\mathcal{C}) = \mathrm{Fun}^{\pi}((\mathcal{C}^{cp})^{op}, \mathcal{S})$$

- $\mathcal{C}^{cp} \subset \mathrm{Anim}(\mathcal{C})$ full subcat.

  $\mathrm{Ind}(\mathcal{C}^{cp}) \subset \mathrm{Anim}(\mathcal{C})$

- general objects can be represented as geometric realizations of simplicial diagrams w/ entries in $\mathrm{Ind}(\mathcal{C}^{cp})$.

- $\mathrm{Map}_{\mathrm{Anim}(\mathcal{C})}(X, \operatorname{colim}_{j \in \Delta^{op}} Y_j) = \operatorname{colim}_{j \in \Delta^{op}} \mathrm{Map}_{\mathrm{Anim}(\mathcal{C})}(X, Y_j)$

  X cpt proj.

Example. $D(\mathcal{A})_{\geq 0} = \mathrm{Anim}(\mathcal{A})$

Nonadditive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, get $\mathcal{A} \longrightarrow \mathcal{B} \to \mathrm{Anim}(\mathcal{B})$

extends to a filtered colim + geom. realization preserving functor $\mathrm{Anim}(\mathcal{A}) \to \mathrm{Anim}(\mathcal{B})$.

Example. $\Lambda^n_R : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$, get $L\Lambda^n_R : D(R)_{\geq 0} \longrightarrow D(R)_{\geq 0}$.

Computed by representing objects by simplicial diagrams of projective modules, and then levelwise applying Λ^n_R.

e.g. $R = \mathbb{Z}$, $L\Lambda^2_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = ?$

$$\mathbb{Z}/n \simeq \begin{array}{c} \mathbb{Z} \\ \downarrow n \\ \mathbb{Z} \end{array} \overset{\text{Dold-Kan}}{\simeq} \begin{array}{c} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \\ \downarrow\downarrow\downarrow \\ \mathbb{Z}\oplus\mathbb{Z} \\ \downarrow\downarrow \\ \mathbb{Z} \end{array} \overset{\Lambda^2_{\mathbb{Z}}}{\rightsquigarrow} \begin{array}{c} \downarrow\downarrow\downarrow \\ \Lambda^2_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z}) \\ \downarrow\downarrow \\ \Lambda^2_{\mathbb{Z}}(\mathbb{Z}) \end{array} \quad \begin{array}{c} \downarrow\downarrow\downarrow \\ \mathbb{Z} \\ \downarrow\downarrow \\ 0 \end{array} \simeq \mathbb{Z}/n[1]$$

$$\mathrm{Anim}(\mathrm{CRing}) = \mathrm{Fun}^{\Pi}\left((\mathrm{CRing}^{cp})^{op}, \mathcal{S}\right)$$

$$\simeq \mathrm{Fun}^{\Pi}\left((\mathrm{Poly}^{f.g.})^{op}, \mathcal{S}\right)$$

$\uparrow$

f.g. poly. rings $/\mathbb{Z}$

$$\mathrm{Ind}(\mathrm{CRing}^{cp}) = \text{Poly. rings } /\mathbb{Z}$$

So think of objects as represented by simplicial diagrams of polynomial rings,

$$\mathrm{Map}_{\mathrm{Anim}(\mathrm{CRing})}\left(\mathbb{Z}[x], \operatorname{colim}_{\Delta^{op}} Y_i\right) \simeq \operatorname{colim}_{\Delta^{op}} Y_i.$$

$$\mathrm{Fun}^{\text{geom. + filtered}}(\mathrm{Anim}(\mathrm{cRing}), \mathcal{D}) = \mathrm{Fun}(\mathrm{Poly}^{f.g.}, \mathcal{D})$$

$$= \mathrm{Fun}^{\text{filtered}}(\mathrm{Poly}, \mathcal{D})$$

Example. The functor $HH(-/\mathbb{Z}) : \mathrm{Poly} \longrightarrow D(\mathbb{Z})_{\geq 0}$ commutes w/ filtered colimits, and so it extends to a functor

$$LHH : \mathrm{Anim}(\mathrm{cRing}) \longrightarrow D(\mathbb{Z})_{\geq 0}.$$

on $\mathrm{cRing} \to \mathrm{Anim}(\mathrm{cRing})$ this agrees w/ $HH(R/\mathbb{Z})$

Construction. For $C \in D(\mathbb{Z})$, have $\tau_{\geq n} C \longrightarrow C$ w/

- isom on H_i, $i \geq n$
- $H_i(\tau_{\geq n} C) = 0$ for $i < n$

$$\tau_{\geq n} : D(\mathbb{Z}) \longrightarrow D(\mathbb{Z})$$

$$\mathrm{Cofib}\left(\tau_{\geq n+1} C \longrightarrow \tau_{\geq n} C\right) \simeq (H_n C)[n]$$

For polynomial ring R,

$$\begin{tikzcd} \vdots \arrow[d] & \\ \tau_{\geq n+1} HH(R/\mathbb{Z}) \arrow[d] & \\ \tau_{\geq n} HH(R/\mathbb{Z}) \arrow[r] \arrow[d] & \Omega^n_{R/\mathbb{Z}}[n] \\ \vdots \arrow[d] & \\ HH(R/\mathbb{Z}) \simeq \tau_{\geq 0} HH(R/\mathbb{Z}) & \end{tikzcd}$$

Def. We define $F^n_{HKR} HH(-/\mathbb{Z}) : \mathrm{Anim}(\mathrm{cRing}) \longrightarrow D(\mathbb{Z})_{\geq 0}$

as geom. realization preserving extension of $\tau_{\geq n} HH(-/\mathbb{Z})$ from Poly

Have $F^{n+1}_{HKR} HH(R/\mathbb{Z}) \longrightarrow F^n_{HKR} HH(R/\mathbb{Z}) \longrightarrow L\Omega^n_{R/\mathbb{Z}}[n]$

Lemma. $F^n_{HKR} HH(R/\mathbb{Z}) \in D(\mathbb{Z})_{\geq n}$.

Lemma. If $F: \mathrm{cRing} \to \mathrm{Ab}$ commutes w/ reflexive coequalizers, then

$H_0(LF(R)) = F(R)$.

Proof Resolve R as $R_0 \leftleftarrows R_1 \leftleftarrows R_2 \cdots$, then

$R = \mathrm{coeq}(R_1 \rightrightarrows R_0)$ in Set, cRing.

$LF(R) \simeq$ total cpx of $F(R_0) \leftleftarrows F(R_1) \leftleftarrows \cdots$

$H_0 LF(R) = \mathrm{coeq}(F(R_1) \rightrightarrows F(R_0))$.

Thm (HKR, v2) If $L\Omega^n_{R/\mathbb{Z}}$ is concentrated in degree 0 for each n, then HKR holds for R.

Proof Long exact sequence on H_k associated to $F_{HKR} HH^{n+1} \to F_{HKR} HH^n \to L\Omega^n_{R/\mathbb{Z}}[n]$

shows: • $H_n F^n_{HKR} \xrightarrow{\sim} \Omega^n_{R/\mathbb{Z}}$

• $H_i F^{n+1}_{HKR} \xrightarrow{\sim} H_i F^n_{HKR}$ is isom. for $i > n$.

$\Omega^n_{R/\mathbb{Z}} \simeq H_n F^n_{HKR} \xrightarrow{\sim} H_n F^0_{HKR} = HH_n(R/\mathbb{Z})$.

$L\Omega^n_{R/k}$ agrees w the value on R of

$$\mathrm{An}(\mathrm{cRing}_{/R}) \longrightarrow D(R)_{\geq 0} \longrightarrow D(\mathbb{Z})_{\geq 0} \qquad \boxed{\text{BLACKBOX}}$$

$$A \in \mathrm{Poly}_{/R} \longmapsto R \otimes_A \Omega^n_{A/\mathbb{Z}}$$

$$R \otimes_A \Omega^n_{A/\mathbb{Z}} \cong \Lambda^n_R \big(\underbrace{R \otimes_A \Omega^1_{A/\mathbb{Z}}}_{\text{cp. in } \mathrm{Mod}_R}\big)$$

$$L\Omega^n_{R/\mathbb{Z}} = L\Lambda^n_R \left(L\Omega^1_{R/\mathbb{Z}}\right)$$

Prop. If $L\Omega^1_{R/\mathbb{Z}}$ is concentrated in deg 0, and $\Omega^1_{R/\mathbb{Z}}$ is a <u>flat</u> R-module, then $L\Omega^n_{R/\mathbb{Z}}$ is also concentrated in deg 0.

Proof. If $\Omega^1_{R/\mathbb{Z}}$ is proj, then $L\Lambda^n_R(\Omega^1_{R/\mathbb{Z}})$ is just $\Lambda^n_R \Omega^1_{R/\mathbb{Z}}$ in deg 0.

In general, use Lazard's theorem. $\square$

Theorem (HKR) If $L\Omega^1_{R/\mathbb{Z}}$ is concentrated in deg 0, $\Omega^1_{R/k}$ flat R-module, then

$$HH_n(R/\mathbb{Z}) \simeq \Omega^n_{R/\mathbb{Z}}.$$

<u>Digression: The cotangent complex and obstruction theory</u>

Defined $L\Omega^1_{-/k}$ by non-abelian deriving $\Omega^1_{-/k}: k\text{-Alg} \longrightarrow \mathrm{Ab}$

to $\mathrm{Ani}(k\text{-Alg}) \longrightarrow D(\mathbb{Z})_{\geq 0}$

<u>Lemma</u>. The following agree:

(1) $L\Omega^1_{-/k} : \mathrm{Ani}(k\text{-}\mathrm{Alg}) \longrightarrow D(\mathbb{Z})_{\geq 0}$, evaluated on R

(2) $L\left(A \mapsto R \underset{A}{\otimes} \Omega^1_{A/k}\right) : \mathrm{Ani}(k\text{-}\mathrm{Alg}/R) \longrightarrow D(R)_{\geq 0}$, evaluated on $R \xrightarrow{id} R$.

write $R = \operatorname{colim}_{i \in \Delta^{op}} A_i$, w/ A_i Poly.

<u>Proof</u>. $(2) \simeq \operatorname{colim}_{i \in \Delta^{op}} R \underset{A_i}{\otimes} \Omega^1_{A_i/k} \simeq \operatorname{colim}_{i \in \Delta^{op}} \left(\operatorname{colim}_{j \in \Delta^{op}} A_j\right) \overset{L}{\underset{A_i}{\otimes}} \Omega^1_{A_i/k}$

$$\simeq \operatorname{colim}_{i \in \Delta^{op}} \left(\operatorname{colim}_{j \in \Delta^{op}_{i/}} A_j\right) \overset{L}{\underset{A_i}{\otimes}} \Omega^1_{A_i/k} \qquad (\Delta^{op} \text{ is sifted})$$

$$\simeq \operatorname{colim}_{i \to j \in (\Delta^{op})^{\Delta^1}} A_j \overset{L}{\underset{A_i}{\otimes}} \Omega^1_{A_i/k}$$

$$\simeq \operatorname{colim}_{i \in \Delta^{op}} \Omega^1_{A_i/k} \simeq (1).$$

The functor $\mathrm{Ani}(k\text{-}\mathrm{Alg}/R) \longrightarrow D(R)_{\geq 0}$ commutes w/ <u>all</u> colimits.

<u>Example</u>. If $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ regular sequence, then

$$\begin{tikzcd} k[f_1, \dots, f_m] \arrow[r] \arrow[d] & k[x_1, \dots, x_n] \arrow[d] \\ k \arrow[r] & R \end{tikzcd}$$

is a pushout in $\mathrm{Anim}(k\text{-}\mathrm{Alg}/R)$

$L\left(R \underset{\bullet}{\otimes} \Omega^1_{\bullet/k}\right)$ takes this to pushout in $D(R)_{\geq 0}$

$$\begin{tikzcd} R\{df_1, \dots df_m\} \arrow[r, "\text{(Jacobi matrix)}"] \arrow[d] & R\{dx_1, \dots dx_n\} \arrow[d] \\ 0 \arrow[r] & L\Omega^1_{R/k} \end{tikzcd}$$

In this case, $L\Omega^1_{R/k}$ has $H_0 = $ coker of Jacobi matrix,

$H_1 = $ kernel.

<u>Def</u>: $L\Omega^1_{R/k}$ is called the <u>cotangent complex</u> of R/k, and denoted $\mathbb{L}_{R/k}$.

Given a comm ring S (k-alg) and an S-module M, have split square-zero ext'n $S \oplus M$

(k-alg) lifts

$$\begin{tikzcd} & S \oplus M \arrow[d] \\ R \arrow[r, "\varphi"'] \arrow[ur, dashed] & S \end{tikzcd}$$

corresponds to φ-linear (k-linear) derivations $R \to M$.

i.e. R-module maps $\Omega^1_{R/\mathbb{Z}} \to \varphi_* M$
(R/k)

<u>Prop.</u> – There's a functor $D(S)_{\geq 0} \to \mathrm{Anim}(k\text{-Alg})/S$

on projectives: $P \longmapsto S \oplus P$

– We have equivalences of mapping spaces

$$\mathrm{Map}_{D(R)_{\geq 0}}(\mathbb{L}_{R/k}, \varphi_* M) \simeq \mathrm{fib}_\varphi \left(\begin{array}{c} \mathrm{Map}_{\mathrm{Anim}(k\text{-Alg})}(R, S \oplus M) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Anim}(k\text{-Alg})}(R, S) \end{array} \right)$$

Given a pair $\tilde{R} \to R$, but I satisfies $I^2 = 0$,

call this a (non-split) square-zero extension.

then $\mathbb{L}_{\tilde{R}/R}$ has $H_0 = 0$ and $H_1 = \tilde{R} \otimes_R I = I$

we have a tautological map of R-modules $\mathbb{L}_{\tilde{R}/R} \to I[1]$ iso. on H_1.

Proof

corresponding to a map of animated $\tilde{R}$-algebras

$$R \xrightarrow{\delta} R \oplus I[1]$$

Prop.
$$\begin{tikzcd} \tilde{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "s"] \\ R \arrow[r, "\delta"] & R \oplus I[1] \end{tikzcd}$$
is a pullback of animated $\tilde{R}$-algebras

<u>Take away</u>: (non-split) square-zero ext'ns $\tilde{R} \to R$ of k-algs along I corresponds to

maps $\mathbb{L}_{R/k} \longrightarrow I[1]$. ($\rightsquigarrow R \to R \oplus I[1]$

$\rightsquigarrow$ pullback)

Maps $S \longrightarrow \tilde{R}$ lifting a given $S \to R$ correspond

$$\begin{tikzcd} & \tilde{R} \arrow[r] \arrow[d] & R \arrow[d, "s"] \\ S \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & R \arrow[r, "\delta"] & R \oplus I[1] \end{tikzcd}$$

to lifts

$$\begin{tikzcd} & R \arrow[d, "s"] \\ S \arrow[ur, dashed] \arrow[r] \arrow[dr] & R \oplus I[1] \arrow[d] \\ & R \end{tikzcd} \quad id$$

exists iff $\mathbb{L}_{S/k} \longrightarrow I[1]$ is nullhomotopic.

and lifts are in correspondence to choice of nullhomotopies.

Those form a <u>torsor</u> over $\pi_1 \operatorname{Map}(\mathbb{L}_{S/k}, I[1])$

Altogether: • $\tilde{R}$ are classified by $\pi_0 \operatorname{Map}_{D(R)}(\mathbb{L}_{R/k}, I[1])$

• For $S \to R$, there is an obstruction in $\pi_0 \operatorname{Map}_{D(S)}(\mathbb{L}_{S/k}, I[1])$

for the existence of lifts $S \to \tilde{R}$

- lifts (if they exist) are parametrized by $\pi_1 \mathrm{Map}_{D(S)}(\mathbb{L}_{S/k}, I[1])$

$$= \pi_0 \mathrm{Map}_{D(S)}(\mathbb{L}_{S/k}, I).$$

Ex. Assume S is a k-alg. TFAE

1) For every (non-split) square-zero ext'n $\tilde{R} \to R$ and every map $S \to R$, there is a lift

2) $H_1 \mathbb{L}_{S/k} = 0$ and $H_0 \mathbb{L}_{S/k}$ is a proj. S-module

"formally smooth"

Theorem. If $R_1/\mathbb{F}_p$ is perfect ($x \mapsto x^p$ is an automorphism), then there exists flat $\mathbb{Z}/p^n$-algebras R_n, unique up to isom., w/ $R_1 \simeq R_n \otimes_{\mathbb{Z}/p^n} \mathbb{F}_p$.

In particular, $R_{n-1} \simeq R_n \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^{n-1}$, get a tower

$$\begin{pmatrix} \vdots \\ \downarrow \\ R_n \\ \downarrow \\ R_{n-1} \\ \downarrow \\ \vdots \end{pmatrix}$$

$\lim R_n =: R$ is the unique flat, p-complete $\mathbb{Z}_p$-alg.

w/ $R_1 \simeq R \otimes_{\mathbb{Z}_p} \mathbb{F}_p = R/p$.

Characterize R_n as a non-split square-zero ext'n of R_{n-1} by R_1:

tensor $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0$ over $\mathbb{Z}/p^n$ w/ R_n,

get $0 \to R_1 \to R_n \to R_{n-1} \to 0$

classified by $\pi_0 \mathrm{Map}_{D(R_{n-1})}(\mathbb{L}_{R_{n-1}/\mathbb{Z}}, R_1[1])$

Bu...

also wants to somehow be compatible w/ element of

$$\pi_0 \, \mathrm{Map}_{D(\mathbb{Z}/p^{n-1})} (\mathbb{L}_{(\mathbb{Z}/p^{n-1})/\mathbb{Z}}, \mathbb{L}_{\mathbb{Z}/p}[1]) \quad \text{classifying } \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}.$$

<u>Lemma</u>. isomorphism classes of such R_n are a <u>torsor</u> over

$$\pi_0 \, \mathrm{Map}_{D(R_{n-1})} (\mathbb{L}_{R_{n-1}/\mathbb{Z}/p^{n-1}}, R_1[1])$$

$$\simeq \pi_0 \, \mathrm{Map}_{D(R_1)} (\mathbb{L}_{R_1/\mathbb{F}_p}, R_1[1])$$

<u>Lemma</u>. If $R_1/\mathbb{F}_p$ is perfect, $\mathbb{L}_{R_1/\mathbb{F}_p} \simeq 0$.

<u>Proof</u>. $\varphi: A \to A$, $x \mapsto x^p$ induces 0 on $\Omega^1_{A/\mathbb{F}_p}$

by resolving R_1 polynomially, see that φ acts by 0 on $\mathbb{L}_{R_1/\mathbb{F}_p}$

It also acts by an isomorphism. $\square$

The R from the Thm is called <u>Witt vectors</u> of R_0, $W(R_0)$.

<u>Example</u>. $\mathbb{F}_{p^n}/\mathbb{F}_p$ lifts to $\mathbb{Z}_p$-alg. $W(\mathbb{F}_{p^n})$, w/ $W(\mathbb{F}_{p^n})/p \simeq \mathbb{F}_{p^n}$.

(e.g. write $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/(f(x))$, take $\mathbb{Z}_p[x]/(\tilde{f}(x))$)

<u>Digression: Hochschild homology of schemes</u>

X scheme over k, Define $HH(X/k)$:
(1) Extend from the affine case
(2) Generalize $HH(-/k)$ to dg cats $/k$ and define $HH(X/k) = HH(\mathrm{Perf}(X)/k)$

"Non-comm. approach"

Def. For a scheme, we define

$$HH(X/k) = \lim_{\substack{U \subset X \\ U \text{ open affine}}} HH(\mathcal{O}(U)/k) \in D(k)$$

The limit is over the functor $HH(\mathcal{O}(-)/k) : \{\text{affine opens in } X\}^{op} \longrightarrow D(k)$

$$HH(\mathrm{Spec}(R)/k) = \lim_{U \subset \mathrm{Spec}(R)} HH(\mathcal{O}(U)/k) \simeq HH(\mathcal{O}(\mathrm{Spec}\, R)/k) = HH(R/k).$$

Example $\mathbb{P}^1_k$ scheme, it has an open cover consisting of

$(\mathbb{A}^1_k)^+ = \mathrm{Spec}(k[x])$, $(\mathbb{A}^1_k)^- = \mathrm{Spec}(k[y])$

$(\mathbb{A}^1_k)^+ \cap (\mathbb{A}^1_k)^- = \mathbb{G}_m = \mathrm{Spec}(k[x, x^{-1}])$

Descent / Mayer-Vietoris sequence ?

Thm For any pair $U, V \subset X$ open subscheme of X s.t. $X = U \cup V$, the square

$$\begin{tikzcd}
HH(X/k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & HH(U/k) \arrow[d, "i^*"] \\
HH(V/k) \arrow[r, "j^*"'] & HH(U \cap V/k)
\end{tikzcd}$$

is a pullback in $D(k)$

Rmk. We will see that $HH(-/k)$ satisfies even flat (= fpqc) descent.

Corollary. In the situation of the theorem, we get a LES

$$\cdots \to HH_n(X) \to HH_n(U) \oplus HH_n(V) \xrightarrow{(i^*, j^*)} HH_n(U \cap V) \to HH_{n-1}(X) \to \cdots$$

Lemma. For any comm. R and $x \in R$, we have

$$HH(R[x^{-1}]/k) \cong HH(R/k) \underset{R}{\otimes} R[x^{-1}] = HH(R/k)[x^{-1}]$$

$$\begin{tikzcd} HH(\mathbb{P}^1/k) \arrow[r] \arrow[d] & HH(k[x]/k) \arrow[d] & x \arrow[d, mapsto] \\ HH(k[y]/k) \arrow[r] & HH(k[x^{\pm 1}]/k) & x \end{tikzcd}$$

$$y \longmapsto x^{-1}$$

MV sequence:

$$(f\,dx, g\,dy) \longmapsto f(x)dx + g(x^{-1})\left(-\frac{1}{x^2}\right)dx$$

$$0 \to HH_1(\mathbb{P}^1/k) \to k[x]dx \oplus k[y]dy \to k[x^{\pm 1}]dx \to$$

$$\to HH_0(\mathbb{P}^1/k) \to k[x] \oplus k[y] \longrightarrow k[x^{\pm}] \to$$

$$(f, g) \longmapsto f(x) + g(x^{-1})$$

$$\to HH_{-1}(\mathbb{P}^1/k) \to 0$$

- $HH_{-1}(\mathbb{P}^1/k) = 0$
- $HH_0(\mathbb{P}^1/k) = k \oplus k$
- $HH_1(\mathbb{P}^1/k) = 0$

Proof of thm. $U, V \subset X$

$$\begin{tikzcd} \lim\limits_{A \subset X} HH(A) \cong HH(X) \arrow[r] \arrow[d] & HH(U) = \lim\limits_{A \subset X} HH(A \cap U) \arrow[d] \\ \lim\limits_{A \subset X} HH(A \cap V) \simeq HH(V) \arrow[r] & HH(U \cap V) = \lim\limits_{A \subset X} HH(A \cap U \cap V) \end{tikzcd}$$

WLOG: we can assume that X is affine, and U, V are affine, and are standard open

Thus we reduce to the case

$$\begin{tikzcd} HH(R/k) \arrow[r] \arrow[d] & HH(R[x^{-1}]/k) \arrow[d] \\ HH(R[y^{-1}]/k) \arrow[r] & HH(R[x^{-1},y^{-1}]/k) \end{tikzcd}$$

s.t. $1 = \alpha x + \beta y$ for some $\alpha, \beta \in R$.

In this case, we get

$$\begin{tikzcd} HH(R/k) \arrow[r] \arrow[d] & HH(R/k) \otimes_R R[x^{-1}] \arrow[d] \\ HH(R/k) \otimes_R R[y^{-1}] \arrow[r] & HH(R/k) \otimes_R R[x^{-1},y^{-1}] \end{tikzcd}$$

To see that this is a pullback, it suffices to prove that

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R[x^{-1}] \arrow[d] \\ R[y^{-1}] \arrow[r] & R[x^{-1},y^{-1}] \end{tikzcd}$$

is a pullback.

This is true because $R[x^{-1}]/\!/R \simeq R[x^{-1},y^{-1}]/R[y^{-1}]$.

this is because on $R/\!/y$, x acts invertible. $R/y \to R/y[x^{-1}]$.

<u>Recall</u>. If R has flat cotangent cpx $\mathbb{L}_{R/k}$ (e.g. R is smooth), then we have that $HH_*(R/k) \simeq \Omega^*_{R/k}$.

In general we had a filtration $F^*_{HKR} HH(R/k)$ whose n-th assoc graded is given by

$$F^{n+1}_{HKR} HH(R/k) \to F^n_{HKR} HH(R/k) \to L\Omega^n_{R/k}[n] = (L\Lambda^n)\circ(\mathbb{L}_{R/k})[n]$$

This filtration is complete in the sense that

$$\varprojlim F^n_{HKR} HH(R/k) = 0.$$

limit in $\mathcal{D}(k)$.

We define a similar filtration $HH(X/k)$ as

$$F^n_{HKR}(X/k) = \lim_{\substack{U \subset X \\ \text{affine open}}} F^n_{HKR}\Big(HH(\mathcal{O}(U)/k)\Big).$$

Prop. This defines a complete filtration on $HH(X/k)$, i.e.

$\varprojlim_n F^n_{HKR}(X/k) = 0$, and the n-th assoc. graded is given by

$$\lim_{U \subset X} L\Omega^n_{U/k}[n] = R\Gamma(X, L\Omega^n_{\mathcal{O}/k}[n])$$

(the derived Hodge cohomology of X).

In particular, we get a spectral sequence

$$R\Gamma^k(X, L\Omega^n_{\mathcal{O}/k}[n]) \Rightarrow HH_*(X/k)$$

Corollary: If $\mathbb{Q} \subset k$, then

$$HH(X/k) \simeq \prod R\Gamma(X, L\Omega^n_{\mathcal{O}/k}[n])$$

Proof.
- True if $X = k[x_1, \dots, x_n]$
- Also true for affine X by non-abelian derivation.
- The general case follows.

Lecture 7. Hochschild homology in ∞-categories.

Def. $\mathrm{Ass}^{\otimes}_{act}$ w/

- objects : finite sets
- morphisms : a map $f: S \to T$ together w/ a total ordering on each $f^{-1}(t) \subset S$.

Sym. monoidal wrt. $\amalg$.

Write $\langle 1 \rangle$ one elt. set

In $\mathrm{Ass}^{\otimes}_{act}$, $\langle 1 \rangle$ is an assoc. alg. object.

$$\langle 1 \rangle \amalg \langle 1 \rangle \to \langle 1 \rangle \qquad \phi \to \langle 1 \rangle$$

$$\begin{tikzcd} \langle 1 \rangle \amalg \langle 1 \rangle \amalg \langle 1 \rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \langle 1 \rangle \amalg \langle 1 \rangle \arrow[d] \\ \langle 1 \rangle \amalg \langle 1 \rangle \arrow[r] & \langle 1 \rangle \end{tikzcd}$$

Def. An assoc. algebra in a sym. monoidal ∞-cat. $\mathcal{C}$ is given by a sym. mon. functor

$$N(\mathrm{Ass}^{\otimes}_{act}) \to \mathcal{C}$$

The underlying object is the value at $\langle 1 \rangle$.

Def. For a finite nonempty totally ordered set S, $\mathrm{Cut}(S) = \{(S_0, S_1) : S_0, S_1 \subset S,\ S_0 < S_1,\ S_0 \amalg S_1 = S\}$

$$\mathrm{Cut}^{cyc}(S) = \mathrm{Cut}(S) / (S, \phi) \sim (\phi, S)$$

the functors $\mathrm{Cut}: \Delta^{op} \to \mathrm{Set}$ and $\mathrm{Cut}^{cyc}: \Delta^{op} \to \mathrm{Set}$ agree w/ the simplicial sets Δ^1 & $S^1 = \Delta^1 / \partial\Delta^1$.

Cut^{cyc} even defines a functor $\Delta^{op} \to \mathrm{Ass}^{\otimes}_{act}$.

for $f: S \to T$ map of totally ordered sets, and a cut $(S_0, S_1) \in \mathrm{Cut}^{cyc}(S)$,

the preimage $(f^*)^{-1}(S_0, S_1) \subset \mathrm{Cut}^{cyc}(T)$ is

- If (S_0, S_1) is nontrivial, $(f^*)^{-1}(S_0, S_1)$ is given by all cuts "between" $f(S_0)$, $f(S_1)$.

- If (S_0, S_1) is trivial, $(f^*)^{-1}(S_0, S_1)$ is given by all cuts "outside" of $f(S)$.

<u>Def</u>. For an algebra $A : \mathrm{Ass}^{\otimes}_{act} \to \mathcal{C}$

$$HH(A/\mathcal{C}) := \operatorname*{colim}_{\Delta^{op}} (A \circ \mathrm{Cut}^{cyc})$$

<u>Lemma</u>. For an ordinary ring (or dga), have an alg. A in $D(\mathbb{Z})$. Then

$$HH(A/D(\mathbb{Z})) \simeq HH(A/\mathbb{Z})$$

<u>Proof</u>. The functor $Ch(\mathbb{Z}) \to D(\mathbb{Z})$ preserves tensor products of K-flat cpxes.

So if we take our dga to be K-flat, the functor $Ch(\mathbb{Z}) \to D(\mathbb{Z})$ preserves all tensor products in cyclic bar complex.

Then use that in $D(\mathbb{Z})$, a colimit of a simplicial diagram is computed by a total complex.

Def For a <u>ring spectrum</u> A (i.e. assoc alg. in Sp),

$$THH(A) := HH(A/Sp).$$

For ordinary ring R, have Eilenberg–MacLane spectrum HR.

This is canonically a ring spectrum.

$$THH(R) := THH(HR)$$

Informally,

$$\left| \quad \cdots \quad \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; HR \otimes HR \rightrightarrows HR \quad \right|$$

Example. $THH(\mathbb{S}) = \mathbb{S}$

Definition. $\mathrm{LMod}^{\otimes}_{\mathrm{cat}}$

- obj: finite sets w/ elts labelled or "coloured" by $\{a, m\}$ (algebra, module).

  (i.e. $S \to \{a, m\}$)

- maps: maps $S \to T$ together w/ total ordering on each preimage.

  and
  - preimage of a-coloured elt are completely a-coloured
  - preimages of m-coloured elt contain precisely one m-coloured elt, the maximum.

  ($\mathrm{RMod}^{\otimes}_{\mathrm{cat}}$: minimum)

A left module in $\mathcal{C}$ is a sym. mon. functor $\mathrm{LMod}^{\otimes}_{\mathrm{cat}} \to \mathcal{C}$

Define a cat. $\mathrm{LRMod}^{\otimes}_{\mathrm{cat}}$

- objects: finite set w/ "colours" $\{r, a, \ell\}$
- maps: $f: S \to T$ w/ ordering on preimages.

- preimages of a-coloured elts are a-coloured

  r-coloured elts : r at min, and rest a-coloured

  ℓ-coloured — ℓ max, — —

A sym. mon. functor $\mathrm{LRMod}^{\otimes}_{act} \to \mathcal{C}$ is a pair of $\mathrm{LMod}^{\otimes}_{act} \to \mathcal{C}$,

$\mathrm{RMod}^{\otimes}_{act} \to \mathcal{C}$

which agree on $\mathrm{Ass}^{\otimes}_{act}$.

Def. $N \otimes_A M$ is the colimit of the composite $\Delta^{op} \xrightarrow{\mathrm{Cut}(-)} \mathrm{LRMod}^{\otimes}_{act} \xrightarrow{(N,A,M)} \mathcal{C}$

colour (ϕ, S) : r

(S, ϕ) : ℓ

others : a

— Can also do bimodules, where we have colours $\{a, m\}$, preimage of m-coloured elts have precisely one m-coloured elt. (anywhere!)

Cut^{cyc} factors through $\mathrm{BMod}^{\otimes}_{act}$, so there is a version $HH(A/\mathcal{C}; M)$

$$\left| \cdots \substack{\to\\\to\\\to\\\to} A \otimes A \otimes M \substack{\to\\\to\\\to} A \otimes M \substack{\to\\\to} M \right|$$

Def. $\mathrm{Comm}^{\otimes}_{act} = \mathrm{Fin}$

Def. A comm. alg. in an ∞-cat. $\mathcal{C}$ is a sym mon. functor $N(\mathrm{Comm}^{\otimes}_{act}) \to \mathcal{C}$

$$\mathrm{Comm}^{\otimes}_{act} = \mathrm{Fin}$$

Lemma. For a comm. alg. A, $HH(A/\mathcal{C})$ again has a comm. alg. str.

Functor $H : D(\mathbb{Z}) \to Sp \rightsquigarrow \pi_\bullet H(C) = H_*(C)$ lax sym monoidal

preserves colimits. So we get a canonical map

$$THH(HR) \to H(HH(R))$$

In particular, $THH_*(HR) \to HH_*(R)$

H canonically factors through an <u>equivalence</u>

$$D(\mathbb{Z}) \to Mod(H\mathbb{Z}) \quad \text{which is symmetric monoidal.}$$

So $H(HH(R)) = HH(HR/Mod(H\mathbb{Z}))$

$$= \left| \cdots \overset{\to}{\underset{\to}{\to}} HR \underset{H\mathbb{Z}}{\otimes} HR \rightrightarrows HR \right|$$

$$= THH(HR/H\mathbb{Z})$$

(while $THH(HR) = THH(HR/\mathbb{S})$)

<u>Example.</u> If R is a $\mathbb{Q}$-alg., $THH_*(R) \to HH_*(R)$ is an iso.

This follows from $\mathbb{S} \otimes H\mathbb{Q} \simeq H\mathbb{Z} \otimes H\mathbb{Q}$.

i.e. $\mathbb{S} \otimes HR \simeq H\mathbb{Z} \otimes HR$ for a $\mathbb{Q}$-algebra R.

thus $HR \underset{\mathbb{S}}{\otimes} HR \simeq HR \underset{H\mathbb{Z}}{\otimes} HR$ $\square$.

<u>Prop.</u> For a classical ring R, $THH_i(R) \to HH_i(R)$ is an iso. for $i \le 2$,

surjective for $i = 3$.

Proof Fiber of $THH(HR) \longrightarrow THH(HR/H\mathbb{Z})$

geometric realization of a simplicial diagram of form

$$\begin{array}{rl} & \vdots \\ & \downarrow\downarrow\downarrow \\ \text{2 connective } \longrightarrow & \mathrm{fib}\,(HR \otimes_{\mathbb{S}} HR \to HR \otimes_{H\mathbb{Z}} HR) \\ \text{(analysis of } \mathbb{S} \to H\mathbb{Z}) & \downarrow\downarrow \\ 0 = & \mathrm{fib}\,(HR \to HR) \end{array}$$

Realization will be 3-connective. $\square$

For $\mathbb{F}_p$, see that $THH_2(\mathbb{F}_p) \simeq HH_2(\mathbb{F}_p) \simeq \mathbb{F}_p$ w/ generator "x".

Thm (Bökstedt)

$$THH_*(\mathbb{F}_p) = \mathbb{F}_p[x]$$

Note that $\mathbb{F}_p[x] \to \mathbb{F}_p\langle x\rangle$ is zero in deg. $\geq p$.

Lecture 8 Bökstedt periodicity.

Thm (Bökstedt) $THH_*(\mathbb{F}_p) = \mathbb{F}_p[x]$, $|x| = 2$

Thm (Bökstedt, v2) $THH(\mathbb{F}_p)$ is, as $H\mathbb{F}_p - \mathbb{E}_1$-algebra, free on one generator, of deg 2

i.e. $THH(\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3$

$\Omega \Sigma S^2$ is the free $\mathbb{E}_1$-algebra on pointed sphere S^2.

$$\mathrm{Map}_{\mathbb{E}_1/H\mathbb{F}_p}(H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3, R) \simeq \mathrm{Map}_{\mathbb{E}_1}(\Omega S^3, \Omega^\infty R) \simeq \mathrm{Map}_{S_*}(S^2, \Omega^\infty R)$$

$$\simeq \mathrm{Map}_{Sp}(\Sigma^\infty_+ S^2, R) \simeq \mathrm{Map}_{H\mathbb{F}_p}(\Sigma^2 H\mathbb{F}_p, R)$$

<u>Proof of equivalence</u>. The element $x \in THH_2(\mathbb{F}_p)$ defines an $\mathbb{E}_1$-map

$$H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3 \longrightarrow THH(\mathbb{F}_p)$$

this map is an equiv. if and only if $\pi_* THH(\mathbb{F}_p) = \mathbb{F}_p[x]$.

$$\pi_*(H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3) \simeq H_*(\Omega S^3; \mathbb{F}_p) \simeq \mathbb{F}_p[x] \qquad \square$$

<u>Lemma</u>. $THH(R) \simeq R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R$

<u>Proof sketch</u>. Write $R \simeq R \otimes_R R = \left| R \otimes_{\mathbb{S}} R \leftleftarrows R \otimes_{\mathbb{S}} R \otimes_{\mathbb{S}} R \,\substack{\leftarrow\\ \leftarrow \\ \leftarrow}\, \cdots \right|$

$$R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R = \underset{\Delta^{op}}{\mathrm{colim}}\; R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R^{\otimes s+1} = THH(R) \qquad \square$$

Need to understand $H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$.

<u>Thm</u> (Milnor) As algebra, $\pi_*(H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p)$ is

$p = 2$: $\mathbb{F}_2[\xi_1, \xi_2, \cdots]$, $|\xi_i| = 2^i - 1$

p odd: $\Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \cdots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \cdots]$, $|\tau_i| = 2p^i - 1$

$|\xi_i| = 2p^i - 2$.

View $H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$ as $H\mathbb{F}_p$ $\mathbb{E}_\infty$-algebra via "inclusion" of the right factor.

h.. ..

Lemma. As an $\mathbb{E}_2$- $H\mathbb{F}_p$-algebra, $H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$ is free on one generator of deg 1,

i.e. $H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3 \xrightarrow{\sim} H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$.

Proof of Bökstedt's thm

$$\begin{aligned}
THH(\mathbb{F}_p) &\simeq H\mathbb{F}_p \otimes_{H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p} H\mathbb{F}_p \\
&\simeq H\mathbb{F}_p \otimes_{H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3} H\mathbb{F}_p \\
&\simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \mathrm{Bar}(*, \Omega^2 S^3, *) \\
&\simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3 \qquad \square
\end{aligned}$$

Remark Bökstedt's thm & our lemma are <u>equivalent</u>, using that a map

$A \to B$ of conn'd $H\mathbb{F}_p$-algebras is an equiv. if and only if

$H\mathbb{F}_p \otimes_A H\mathbb{F}_p \to H\mathbb{F}_p \otimes_B H\mathbb{F}_p$ is an equiv.

applied to the map $H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3 \to H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$

<u>p odd</u>. For an $\mathbb{E}_\infty$ - $H\mathbb{F}_p$-algebra, there are Dyer-Lashof operations.

$i \in \mathbb{Z}$, $Q^i : \pi_n(R) \to \pi_{n+2(p-1)i}(R)$

$\beta Q^i : \pi_n(R) \to \pi_{n+2(p-1)i-1}(R)$

For $|x|=n$, $Q^{\frac{n}{2}}x = x^p$, $Q^i x = 0$ for $i < \frac{n}{2}$

$\beta Q^i x = 0$ for $i \leq \frac{n}{2}$

$Q^{\frac{n}{2}}x$ for n even, and $Q^{\frac{n+1}{2}}x$, $\beta Q^{\frac{n+1}{2}}x$ for n odd

are already defined for an $\mathbb{E}_2$-algebra.

<u>Thm.</u> (Dyer-Lashof, $p=2$ Araki-Kudo)

$$H_*(\Omega^2 S^3; \mathbb{F}_p) = \Lambda(a, Q^1 a, Q^p Q^1 a, \cdots)$$
$$\left(= \pi_*(H\mathbb{F}_p \otimes \Omega^2 S^3)\right) \otimes \mathbb{F}_p[(\beta Q^1)a, (\beta Q^p)Q^1 a, \cdots], \quad |a| = 1$$

<u>Thm.</u> (Steinberg)

On $\pi_*(H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p)$ we have
$$\tau_i = Q^{p^{i-1}} Q^{p^{i-2}} \cdots Q^1 \tau_0$$
$$\xi_i = (\beta Q^{p^{i-1}}) Q^{p^{i-2}} \cdots Q^1 \tau_0$$

<u>Proof of lemma.</u> We have

$$\pi_1(H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p) \xrightarrow{\sim} \pi_1(H\mathbb{F}_p \underset{H\mathbb{Z}}{\otimes} H\mathbb{F}_p) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p$$

connectivity

Get a map of $\mathbb{E}_2$-$H\mathbb{F}_p$-algebras

$$H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3 \longrightarrow H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p$$

iso. on π_1. Both sides are generated in the same way by $\mathbb{E}_2$-DL-operations

So it's an isom.

Lecture 9. Properties of THH

- $R \in \mathrm{Alg}(Sp) \supset \mathrm{Alg}(Ab)$

$$HR \leftarrow\!\shortmid R$$

$$THH(R) = \operatorname*{colim}_{\Delta^{op}} \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow}, R \otimes_{\mathbb{S}} R \rightrightarrows R \right) \in Sp$$

- If R is a k-alg, where $k \in CAlg(Sp)$

$$THH(R/k) = \operatorname*{colim}_{\Delta^{op}} \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_k R \rightrightarrows R \right)$$

$R \in \mathrm{Alg}(\mathrm{Mod}_k)$

$$= HH(R/\mathrm{Mod}_k)$$

$$HH(R) \simeq THH(R/\mathbb{Z}) = \operatorname*{colim}_{\Delta^{op}} \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow}, R \otimes_{\mathbb{Z}} R \rightrightarrows R \right)$$

$$D(\mathbb{Z}) \simeq \mathrm{Mod}_{H\mathbb{Z}}$$

Prop. $THH : \mathrm{Alg}(Sp) \longrightarrow Sp$ is sym. monoidal functor.

$$THH(A \otimes_{\mathbb{S}} B) \simeq THH(A) \otimes_{\mathbb{S}} THH(B)$$

More generally, $HH(-/\mathcal{C}) : \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$ is sym. monoidal.

Proof. Consider $U : \mathrm{Alg}(Sp) \longrightarrow Sp$ as an obj. in the sym. monoidal ∞-cat.

$\mathrm{Fun}^{\otimes}(\mathrm{Alg}(Sp), Sp)$. in fact, it is an algebra in there.

$$THH(-) \simeq HH\left(U / \mathrm{Fun}^{\otimes}(\mathrm{Alg}(Sp), Sp)\right)$$

Cor. If R is an $\mathbb{E}_n$-ring spectrum, then $\mathrm{THH}(R)$ is $\mathbb{E}_{n-1}$.

Proof. $\mathrm{Alg}_{\mathbb{E}_n}(Sp) \simeq \mathrm{Alg}_{\mathbb{E}_{n-1}}\left(\mathrm{Alg}_{\mathbb{E}_1}(Sp)\right) \xrightarrow{\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{THH})} \mathrm{Alg}_{\mathbb{E}_{n-1}}(Sp)$ $\square$

In particular, for $n=\infty$, we have that $\mathrm{THH}(R)$ is also $\mathbb{E}_\infty$.

Recall. $A, B \in \mathrm{CAlg}(Sp)$, $A \otimes_{\mathbb{S}} B$ is the coproduct in $\mathrm{CAlg}(Sp)$.

Prop. For $R \in \mathrm{CAlg}(Sp)$, $\mathrm{THH}(R) \simeq \underset{S^1}{\mathrm{colim}}^{\mathrm{CAlg}(Sp)}(R) = R^{\otimes S^1}$

diagram $S^1 \xrightarrow{pr} pt \xrightarrow{R} \mathrm{CAlg}(Sp)$

$$\boxed{A \otimes_{\mathbb{S}} A = \underset{\{0,1\}}{\mathrm{colim}}\, A = A^{\otimes 2}}$$

Warning: This colimit $\underset{S^1}{\mathrm{colim}}^{\mathrm{CAlg}(Sp)} R$ is *different* from the colimit $\underset{S^1}{\mathrm{colim}}^{Sp} R$, which is given by

$$\underset{\Delta^{op}}{\mathrm{colim}}\left(\cdots R\otimes R\otimes R \rightrightarrows R\otimes R \rightrightarrows R\right) \simeq R\otimes S^1 \simeq R \otimes \Sigma_+ S^1$$
$$\simeq R \oplus \Sigma R$$

Base change formulas for THH

$$\mathrm{THH}(A \otimes_{\mathbb{S}} B) \simeq \mathrm{THH}(A) \otimes_{\mathbb{S}} \mathrm{THH}(B)$$

Prop. (1) For any lax sym. monoidal functor $F: \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Alg}(\mathcal{C})$, we get a natural map

$$\mathrm{HH}(FA/\mathcal{D}) \longrightarrow F(\mathrm{HH}(A/\mathcal{C}))$$

(2) If F is strong sym. monoidal and preserves geom. realizations, then this is an equiv.

Proof.

$$\begin{tikzcd}
\operatorname{colim}_{\Delta^{op}} \left(\cdots \Rrightarrow FA \otimes_e FA \rightrightarrows FA \right) \arrow[dd, bend right] & \\
& F\left(\operatorname{colim}_{\Delta^{op}} \left(\cdots \Rrightarrow A \otimes_e A \rightrightarrows A \right) \right) \\
\operatorname{colim}_{\Delta^{op}} \left(\cdots \Rrightarrow F(A \otimes A) \rightrightarrows FA \right) \arrow[ur] &
\end{tikzcd}$$

Examples. (1) $H : D(\mathbb{Z}) \longrightarrow Sp$

get $H\left(HH(R/\mathbb{Z})\right) \longleftarrow THH(R)$

(2) If $k \to k'$ map of comm. ring spectra,

$- \otimes_k k' : \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_{k'}$, we get

$$THH(R/k) \otimes_k k' \xrightarrow{\sim} THH(R \otimes_k k'/k')$$

A comm. $\mathbb{F}_p$-algebra K is called <u>perfect</u> if

$\varphi : K \longrightarrow k,\ k \longmapsto k^p$ is an isom.

Examples include finite fields $\mathbb{F}_{p^n}$ or $\mathbb{F}_p[x^{1/p^\infty}] = \operatorname{colim}\left(\mathbb{F}_p[x] \xrightarrow{\varphi} \mathbb{F}_p[x] \longrightarrow \cdots \right)$

We have $Hk \longrightarrow THH(k)$, $k \longrightarrow THH_*(k)$

& $THH(\mathbb{F}_p) \longrightarrow THH(k)$, $\mathbb{F}_p[x] \longrightarrow THH_*(k)$

$\Rightarrow Hk \otimes_{H\mathbb{F}_p} THH(\mathbb{F}_p) \longrightarrow THH(k)$, $k[x] \longrightarrow THH_*(k)$

Thm (Bökstedt periodicity for perfect rings)

For any perfect $\mathbb{F}_p$-algebra k, the map $k[x] \to THH_*(k)$ is an isom,

equivalently, $THH(k)$ is the free $\mathbb{E}_1$-alg. on x over k.

Proof. We use that if k perfect, there is a comm. ring spectrum

$\mathbb{S}_{W(k)}$ called "spherical Witt vectors" s.t.

$\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} H\mathbb{F}_p \simeq k$ (lift of k along $\mathbb{S} \to H\mathbb{F}_p$)

$$THH(k) = THH(\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} H\mathbb{F}_p) \simeq THH(\mathbb{S}_{W(k)}) \otimes_{\mathbb{S}} THH(\mathbb{F}_p)$$

$$\simeq (THH(\mathbb{S}_{W(k)}) \otimes_{\mathbb{S}} H\mathbb{F}_p) \otimes_{H\mathbb{F}_p} THH(\mathbb{F}_p)$$

$$\simeq THH(k/H\mathbb{F}_p) \otimes_{H\mathbb{F}_p} THH(\mathbb{F}_p)$$

it suffices to prove that $THH(k/H\mathbb{F}_p) = HH(k/\mathbb{F}_p)$ is equiv. to k.

i.e. $HH_*(k/\mathbb{F}_p)$ for $* \geq 1$. $\square$

If k is an $\mathbb{E}_\infty$-ring, $THH(k) = k^{\otimes S^1}$

We get a map $THH(k) \to k$ of $\mathbb{E}_\infty$-rings

$$THH(k) = k^{\otimes S^1} = \operatorname{colim}_{S^1}^{CAlg(Sp)} k \longrightarrow \operatorname{colim}_{pt}^{CAlg(Sp)} k = k$$

$$S^1 \to pt$$

D...

s.t. $$\begin{tikzcd} k \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{THH}(k) \arrow[r] & k \end{tikzcd}$$

Rmk. Such a retract $\mathrm{THH}(k) \to k$ does generally $\underline{\underline{\text{NOT}}}$ exist if k is only $\mathbb{E}_n$ for $n<\infty$.

Thm k comm. ring spectrum, and R an assoc. k-alg. then

$$\mathrm{THH}(R/k) \simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k$$

"Proof" $\operatorname{colim}_{\Delta^{op}} \left(\cdots \substack{\to\\\to\\\to} R \otimes_k R \rightrightarrows R \right)$

$$\simeq \operatorname{colim}_{\Delta^{op}} \left(\cdots \substack{\to\\\to\\\to} R \otimes_{\mathbb{S}} R \otimes_{k \otimes_{\mathbb{S}} k} k \rightrightarrows R \right)$$

$$\simeq \operatorname{colim}_{\Delta^{op}} \left[\left(\cdots \substack{\to\\\to\\\to} R \otimes_{\mathbb{S}} R \rightrightarrows R \right) \otimes_{\left(\cdots \substack{\to\\\to\\\to} k \otimes_{\mathbb{S}} k \rightrightarrows k \right)} \left(\cdots k \substack{\to\\\to\\\to} k \rightrightarrows k \right) \right]$$

$$\simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k. \qquad \square.$$

Lecture 10. The circle action on THH

Group action on object of 1-category $\mathcal{C}$

$\mathrm{Fun}(BG, \mathcal{C})$ $\quad$ BG : cat. w/ one object, endomorphisms G.

For an ∞-cat. $\mathcal{C}$, $\mathrm{Fun}(N(BG), \mathcal{C})$.

$N(BG)$ Kan cpx, w/ $\Omega N(BG) \simeq G$

So it coincides w/ BG, $B : \mathrm{Alg}^{gr}_{\mathbb{E}_1}(S) \xrightarrow{\sim} S_*^{cn}$

Example. $B\mathbb{Z} \simeq S^1$.

Def. For a grouplike $\mathbb{E}_1$-alg. G in S, and an ∞-cat. $\mathcal{C}$, the cat. of objects of $\mathcal{C}$ w/ G-action as $\mathrm{Fun}(BG, \mathcal{C})$.

Def. $X_{hG} = \operatorname{colim}_{BG} X$ homotopy orbit

$X^{hG} = \lim_{BG} X$ homotopy fixed pt

Special case: in S, if X has the trivial G-action, then

$$X_{hG} \simeq X \times BG$$
$$X^{hG} \simeq \mathrm{Map}(BG, X)$$

Prop. $\mathrm{Fun}(BS^1, D(\mathbb{Z})) \simeq \mathrm{Mod}_A(D(\mathbb{Z}))$, $A = \Lambda_{\mathbb{Z}}(\varepsilon)$, $|\varepsilon| = 1$

Proof sketch: $\mathrm{Fun}(BS^1, D(\mathbb{Z})) \simeq \mathrm{Mod}_{C_*(S^1)}$

Have maps

$$\begin{tikzcd} \mathrm{Free}_{\mathbb{E}_1}(\varepsilon) \arrow[r] \arrow[dr] & C_*(S^1) \\ & A \end{tikzcd}$$

Both factors through

$$\begin{tikzcd} \tau_{\leq 1} \mathrm{Free}_{\mathbb{E}_1}(\varepsilon) \arrow[r, "\sim"] \arrow[dr, "\sim"] & C_*(S^1) \\ & A \end{tikzcd}$$

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which are multiplicative since $\tau_{\le 1}$ is lax monoidal on $D(\mathbb{Z})_{\ge 0}$. $\square$

Rmk. This equiv. is not compatible w/ sym. monoidal structures.

Theorem. On $HH(R/\mathcal{C})$, have a nat'l S^1-action

$$\left(HH : Alg_{\mathbb{E}_1}(\mathcal{C}) \longrightarrow Fun(BS^1, \mathcal{C}) \right).$$

(which agrees w/ the S^1-action on $HH(R/\mathbb{Z})$ obtained from Connes operator)

Def. Λ_∞ (paracyclic cat.) has

- objects : totally ordered sets w/ $\mathbb{Z}$-action equivalent to $\frac{1}{n}\mathbb{Z}$
- morphisms : equivariant order-preserving maps.

Have an $S^1 (\simeq B\mathbb{Z})$-action on Λ_∞.

On 1-categories. $B\mathbb{Z} \times \Lambda_\infty \longrightarrow \Lambda_\infty$

on objects: $* \times (\Lambda_\infty)_0 \longrightarrow (\Lambda_\infty)_0$ trivially

on morphisms: $\mathbb{Z} \times Hom_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z}) \longrightarrow Hom_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z})$

Def. Λ (cyclic category)

- objects: same as Λ_∞
- morphisms: $Hom_\Lambda(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z}) = Hom_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z})/\mathbb{Z}$

Lemma. $N(\Lambda) = N(\Lambda_\infty)_{hS^1}$

Lemma $\mathrm{Fun}(N(\Delta), \mathcal{C}) \simeq \mathrm{Fun}(N(\Lambda_\infty), \mathcal{C})^{hS^1}$.

Proof. Need:
$$\mathrm{Map}(D, \mathrm{Fun}(N(\Delta), \mathcal{C})) = \mathrm{Map}(D, \mathrm{Fun}(N(\Delta_\infty), \mathcal{C}))^{hS^1}$$

$$\Leftrightarrow \quad \mathrm{Map}(N(\Delta), \mathrm{Fun}(D, \mathcal{C})) = \mathrm{Map}(N(\Lambda_\infty), \mathrm{Fun}(D, \mathcal{C}))^{hS^1}$$

Def. A *cyclic object* in $\mathcal{C}$ is a functor $N(\Lambda^{op}) \longrightarrow \mathcal{C}$.

Have a functor $\Delta \longrightarrow \Lambda_\infty$

$$[n-1] \mapsto \mathbb{Z} \times [n-1] \qquad \text{(lexicographic)}$$

$$\cdots \amalg [n-1] \amalg [n-1] \amalg \cdots$$

$$\simeq \tfrac{1}{n}\mathbb{Z}$$

The *underlying simplicial object* of a cyclic object is identified w/

$$\mathrm{Fun}(N(\Lambda^{op}), \mathcal{C}) \to \mathrm{Fun}(N(\Lambda_\infty^{op}), \mathcal{C}) \to \mathrm{Fun}(N(\Delta^{op}), \mathcal{C})$$

Lemma. The diagram

$$\begin{tikzcd} \mathrm{Fun}(\Lambda_\infty^{op}, \mathcal{C}) \arrow[r, "\mathrm{colim}_{\Lambda_\infty^{op}}"] \arrow[d, "\mathrm{res}"] & \mathcal{C} \\ \mathrm{Fun}(\Delta^{op}, \mathcal{C}) \arrow[ur, "\mathrm{colim}_{\Delta^{op}}"'] & \end{tikzcd}$$

commutes.

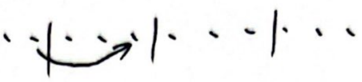
Lemma. For a cyclic object X, $\operatorname{colim}_{\Lambda_\infty^{op}} X \simeq \operatorname{colim}_{\Delta^{op}} X$ carries an S^1-action.

Proof. $\mathrm{Fun}(N(\Lambda^{op}), \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(N(\Lambda_\infty^{op}), \mathcal{C})^{hS^1} \xrightarrow{\mathrm{colim}^{hS^1}} \mathcal{C}^{hS^1} \simeq \mathrm{Fun}(BS^1, \mathcal{C})$

Proof of theorem.

Let $\mathrm{Cut}^{\mathbb{Z}}: \Lambda_\infty^{op} \to \mathrm{Ass}_{act}^{\otimes}$

$S \longmapsto$ "set of $\mathbb{Z}$-equivariant cuts"



$$S = \cdots \amalg S_{-1} \amalg S_0 \amalg S_1 \amalg \cdots$$

preimage of $(S_i) \in \mathrm{Cut}^{\mathbb{Z}}(S)$ in $\mathrm{Cut}^{\mathbb{Z}}(T)$ under f^*, for $S \to T$

consisting of all cuts "between" $f(S_0), f(S_1)$.

This functor is $\mathbb{Z}$ invariant on morphisms, so it factors through $\Lambda \to \mathrm{Ass}_{act}^{\otimes}$

Restriction to Δ^{op} is precisely $\mathrm{Cut}^{cyc}: \Delta^{op} \to \mathrm{Ass}_{act}^{\otimes}$.

Digression: THH of the integers.

Thm. (Bökstedt) We have an isom.

$$THH_*(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}/n, & * = 2n-1 \\ 0, & \text{else} \end{cases}$$

In fact, $THH_*(\mathbb{Z})$ is the homology of the DGA

$$THH_*(\mathbb{Z}) \simeq H_*\left(\underset{|x|=2,\ |e|=1}{\mathbb{Z}[x] \otimes \Lambda(e)},\ \partial x = e,\ \partial e = 0 \right)$$

$$\cdots \xrightarrow{0} \mathbb{Z}x^2 \xrightarrow{\cdot 2} \mathbb{Z}ex \xrightarrow{0} \mathbb{Z}x \xrightarrow{\cdot 1} \mathbb{Z}e \xrightarrow{0} \mathbb{Z} \to 0 \to 0$$

Rmk. One can show that in fact, $THH(\mathbb{Z})$ is as an $\mathbb{E}_1$-alg. over $H\mathbb{Z}$ given as

$$H(\mathbb{Z}[x]\otimes\Lambda(e),\partial)$$

As a consequence of this, we get that

$$THH(\mathbb{Z})/p = THH(\mathbb{Z})\underset{H\mathbb{Z}}{\otimes}H\mathbb{F}_p$$

is isomorphic on homotopy groups to the homology of the DGA $\left(\mathbb{F}_p[x]\otimes\Lambda(e),\ \begin{matrix}\partial x = e\\ \partial e = 0\end{matrix}\right)$

Def. For a ring (spectrum) R, we define

- $THH(R;\mathbb{Z}_p) = THH(R)^\wedge_p$ ← p-adic completion

- $THH(R;\mathbb{Q}) = THH(R)_\mathbb{Q}$ rationalization

- $THH(R;\mathbb{Q}_p) = THH(R,\mathbb{Z}_p)_\mathbb{Q}$

$$\begin{tikzcd}
THH(R) \arrow[r] \arrow[d] & \prod_{p\text{ prime}} THH(R,\mathbb{Z}_p) \arrow[d] \\
THH(R;\mathbb{Q}) \arrow[r] & \left(\prod_{p\text{ prime}} THH(R;\mathbb{Z}_p)\right)_\mathbb{Q}
\end{tikzcd}$$

Lemma: We have ① $THH(R;\mathbb{Z}_p) = THH(R^\wedge_p;\mathbb{Z}_p) = HH(R^\wedge_p/\mathbb{S}^\wedge_p)$

② $THH(R;\mathbb{Q}) = THH(R_\mathbb{Q}/\mathbb{Q}) = HH(R_\mathbb{Q}/\mathbb{Q})$

Proof

① $THH(R;\mathbb{Z}_p) = \left(\operatorname{colim}_{\Delta^{op}} \left(\cdots \substack{\to\\\to\\\to} R \otimes_{\mathbb{S}} R \rightrightarrows R \right) \right)^{\wedge}_p$

$\simeq \left(\operatorname{colim}_{\Delta^{op}} \left(\cdots \substack{\to\\\to\\\to} (R \otimes_{\mathbb{S}} R)^{\wedge}_p \rightrightarrows R^{\wedge}_p \right) \right)^{\wedge}_p$

$\simeq \left(\operatorname{colim}_{\Delta^{op}} \left(\cdots \substack{\to\\\to\\\to} R^{\wedge}_p \otimes_{\mathbb{S}} R^{\wedge}_p \rightrightarrows R^{\wedge}_p \right) \right)^{\wedge}_p = THH(R^{\wedge}_p;\mathbb{Z}_p)$

② $THH(R;\mathbb{Q}) = THH(R) \otimes_{\mathbb{S}} H\mathbb{Q}$

$\simeq THH(R \otimes_{\mathbb{S}} H\mathbb{Q} / H\mathbb{Q}) \simeq THH(R_{\mathbb{Q}}/H\mathbb{Q})$ $\square$

Example. $THH(\mathbb{Z};\mathbb{Q}) = HH(\mathbb{Q}/\mathbb{Q}) = H\mathbb{Q}$

Thm 2. $THH_*(\mathbb{Z};\mathbb{Z}_p) \simeq \begin{cases} \mathbb{Z}_p, & * = 0 \\ \mathbb{Z}_p/n\mathbb{Z}_p, & * = 2n-1 \\ 0, & \text{else} \end{cases} \simeq H_*\left(\mathbb{Z}_p[x] \otimes \Lambda(e), \begin{array}{l} \partial e = 0 \\ \partial x = e \end{array} \right).$

Proof of Thm 1 from Thm 2.

From theorem 2, we get that $THH(\mathbb{Z};\mathbb{Q}_p) \simeq H\mathbb{Q}_p$

Thus
$$\begin{tikzcd} THH(\mathbb{Z}) \arrow[r] \arrow[d] & \prod_{p \text{ primes}} THH(\mathbb{Z};\mathbb{Z}_p) \arrow[d] \\ H\mathbb{Q} \arrow[r] & H\left(\left(\prod_p \mathbb{Z}_p \right)_{\mathbb{Q}} \right) \end{tikzcd}$$

$\Rightarrow THH_0(\mathbb{Z}) = \mathbb{Z}$, $THH_*(\mathbb{Z}) = \prod_{p \text{ primes}} THH_*(\mathbb{Z};\mathbb{Z}_p)$, $* > 0$ $= \prod_{p \text{ prime}} \mathbb{Z}_p/n\mathbb{Z}_p = \mathbb{Z}/n\mathbb{Z}$

$* = 2n-1$

<u>Def</u> We define a comm. ring spectrum ($= \mathbb{E}_\infty$)

$$\mathbb{S}[z] = \mathbb{S}[\mathbb{N}] = \Sigma^\infty_+ \mathbb{N}$$, where $\mathbb{N}$ is considered as a comm. alg. in $(\mathcal{S}, \times)$.

To give a discrete comm. ring R the str. of an $\mathbb{S}[z]$-algebra is equiv. to giving an element $\pi \in R$. $\quad (z \mapsto \pi)$

$$\begin{tikzcd} \mathbb{S}[z] \arrow[r] \arrow[dr, bend right] & H\mathbb{Z}[z] \arrow[d] \\ & HR \end{tikzcd}$$

Consider $\mathbb{Z}$ as an $\mathbb{S}[z]$-algebra by sending z to p.

<u>Warning</u>: If we consider a comm. ring spectrum R instead of a discrete ring, then an $\mathbb{E}_\infty$-map $\mathbb{S}[z] \to R$ is <u>NOT</u> the same as an elt $z \in \pi_0(R)$.

In other words, $\mathbb{S}[z]$ is <u>NOT</u> the free $\mathbb{E}_\infty$-alg. on a single generator.

($\mathbb{S}[z]$ is free as an $\mathbb{E}_1$-algebra though).

<u>Thm</u> (relative version of Bökstedt periodicity)

We have that $\mathrm{THH}_*(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \simeq \mathbb{Z}_p[x]$, $|x| = 2$

<u>Proof</u>. We have $\mathrm{THH}(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p)/p \simeq \mathrm{THH}(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \otimes_{H\mathbb{Z}} H\mathbb{F}_p$

$$\simeq \mathrm{THH}(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathbb{S}[z]} \mathbb{S}$$

$$\simeq THH(\mathbb{Z} \otimes_{\mathbb{S}[z]} \mathbb{S}/\mathbb{S}, \mathbb{Z}_p) \simeq THH(\mathbb{F}_p, \mathbb{Z}_p) = THH(\mathbb{F}_p)$$

We study the LES $\;THH_*(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{p} THH_*(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p) \to THH_*(\mathbb{F}_p) \to THH_{*+1}(\mathbb{F}_p)$

There exists an elt $x \in THH_2(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p)$ lifting $x \in THH_2(\mathbb{F}_p)$

Thus we get a map $\quad \mathbb{Z}_p[x] \longrightarrow THH_*(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p)$

which is an isom. after mod p reduction $\Rightarrow$ isom. (derived p-complete, p-torsion free)

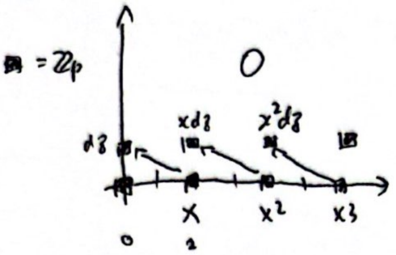
$$
\begin{aligned}
THH(R/\mathbb{S}[z]) &= THH(R) \otimes_{THH(\mathbb{S}[z])} \mathbb{S}[z] \\
&= THH(R) \otimes_{H\mathbb{Z} \otimes_{\mathbb{S}} THH(\mathbb{S}[z])} (H\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S}[z]) \\
&\simeq THH(R) \otimes_{HH(\mathbb{Z}[z]/\mathbb{Z})} \mathbb{Z}[z]
\end{aligned}
$$

Prop There is a convergent, multiplicative spectral sequence

$$
\begin{array}{c}
THH_n(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^m_{\mathbb{Z}[z]/\mathbb{Z}} \Rightarrow THH_{n+m}(R) \\
\wr\wr \\
THH_*(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}} \Lambda(dz)
\end{array}
$$

<u>Proof of thm 2</u>. SS takes the following form.

$$\mathbb{Z}_p[x] \otimes_{\mathbb{Z}_p} \Lambda(dz) \implies THH_*(\mathbb{Z};\mathbb{Z}_p)$$



There can only be a d_2 differential, and this by the Leibniz rule, determined by $d_2(x) = \alpha\cdot dz \in \mathbb{Z}_p\cdot dz$

We get that $THH_1(\mathbb{Z};\mathbb{Z}_p) = \mathbb{Z}_p/\alpha\,\mathbb{Z}_p$

$$\begin{gathered} \| \\ HH_1(\mathbb{Z}/\mathbb{Z};\mathbb{Z}_p) = 0 \end{gathered}$$

$\Rightarrow \alpha$ unit. $\square$

Lecture 11. Negative topological cyclic homology.

Def $TC^-(R) := THH(R)^{hS^1}$

Def. $\mathcal{C}$ stable if $\mathcal{C}$ has finite limits & finite colimits, has a zero object,

and
$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array}$$
is pushout $\Leftrightarrow$ pullback.

Lemma. If $\mathcal{C}$ is stable, there is a unique lift

$$\begin{tikzcd} & Sp \arrow[d, "\Omega^\infty"] \\ \mathcal{C}^{op}\times\mathcal{C} \arrow[ur, dashed, "map"] \arrow[r, "Map"'] & S \end{tikzcd}$$

such that map is exact in both arguments.

Proof sketch. In $\mathcal{C}$, $Y \simeq \Omega\Sigma Y \simeq \Omega^2\Sigma^2 Y \simeq \cdots$

so $Map(X,Y) \simeq \Omega Map(X,\Sigma Y) \simeq \Omega^2 Map(X,\Sigma^2 Y) \simeq \cdots$ defines a spectrum.

Lemma. I small ∞-cat. then

- $\mathrm{Fun}(I, Sp)$ is stable
- $\lim_I F \simeq \mathrm{map}_{\mathrm{Fun}(I,Sp)}(\mathrm{const}\,\mathbb{S}, F)$

Prop. $HC^-(R) \simeq HH(R)^{hS^1}$

Proof. Identify $D(\mathbb{Z})$ w/ $\mathrm{Mod}_{H\mathbb{Z}}$. So:

$$
\begin{aligned}
HH(R)^{hS^1} &\simeq \mathrm{map}_{\mathrm{Fun}(BS^1, Sp)}(\mathbb{S}, HH(R)) \\
&\simeq \mathrm{map}_{\mathrm{Fun}(BS^1, \mathrm{Mod}_{H\mathbb{Z}})}(H\mathbb{Z}, HH(R)) \\
&\simeq \mathrm{map}_{\mathrm{Fun}(BS^1, D(\mathbb{Z}))}(\mathbb{Z}, HH(R)) \\
&\simeq \mathrm{map}_{\mathrm{Mod}(C_*(S^1))}(\mathbb{Z}, HH(R)) \\
&\simeq \mathrm{map}_{\mathrm{Mod}(\underset{\Lambda(\varepsilon)}{\overset{\shortparallel}{A}})}(\mathbb{Z}, HH(R)) \\
&\simeq R\mathrm{Hom}_A(\mathbb{Z}, HH(R)) \simeq HC^-(R).
\end{aligned}
$$

$\square$

Lemma. $(-)^{hS^1}: \mathrm{Fun}(BS^1, Sp) \to Sp$ is lax sym monoidal, and the S^1-action on $THH(R)$ (or $HH(R)$) is compatible w/ the product str. (if R is $\mathbb{E}_\infty$).

$\Rightarrow$ There is an $\mathbb{E}_\infty$-str. on $TC^-(R)$ if R is $\mathbb{E}_\infty$ (and compatibly on $HC^-(R)$).

Lemma. Let $BS^1 \to Sp$ be an S^1-action on HA

then $\pi_*(HA)^{hS^1} = A[t]$ w/ $|t| = -2$.

$A \otimes \mathbb{Z}[t]$

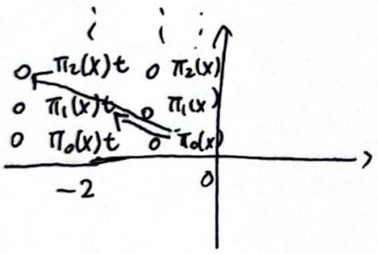
Proof. The full subcat. of Sp on all Eilenberg-MacLane spectra (in degree 0) is a 1-category: $\mathrm{Map}(HA, HB) \simeq \mathrm{Hom}(A, B)$ (discrete)

So a functor $BS^1 \to Sp$, $* \to HA$ is constant!

$HA^{hS^1} \simeq \mathrm{map}_{Sp}(\Sigma^\infty_+ BS^1, HA)$ has homotopy gps $H^*(BS^1; A)$. $\square$

Theorem (HFPSS) There is a multiplicative, conditionally convergent spectral sequence.

$$\pi_*(X)[t] \Rightarrow \pi_*(X^{hS^1}).$$



Proof sketch, Have tower of spectra w/ S^1-action.

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
\tau_{\geq 2} X \arrow[d] & & (\tau_{\geq 2} X)^{hS^1} \arrow[d] \\
\tau_{\geq 1} X \arrow[r] \arrow[d] & \Sigma H\pi_1(X) \arrow[r, squiggly, "(-)^{hS^1}"] & (\tau_{\geq 1} X)^{hS^1} \arrow[r] \arrow[d] & (H\pi_1 X)^{hS^1} \\
\tau_{\geq 0} X \arrow[r] \arrow[d] & H\pi_0 X & (\tau_{\geq 0} X)^{hS^1} \arrow[r] \arrow[d, dash] & (H\pi_0 X)^{hS^1} \\
\vdots & & {} &
\end{tikzcd}$$

$\square$

An S^1-action on X gives a map $\Sigma^\infty_+ S^1 \otimes X \to X$

The canonical map $\Sigma^\infty_+ S^1 \to \Sigma^\infty_+ pt \simeq \mathbb{S}$ splits, so we have a splitting

$$\Sigma^\infty_+ S^1 \simeq \mathbb{S} \oplus \Sigma^\infty S^1 \simeq \mathbb{S}^0 \oplus \mathbb{S}^1$$

we get a map $\Sigma X \simeq \mathbb{S}^1 \otimes X \to X$, i.e. a map $b: \pi_n(X) \to \pi_{n+1}(X)$

Lemma. In the HFPSS, the differential d_2 is obtained by

$$d_2\, \alpha = b\alpha \cdot t \qquad (\alpha \in \pi_*(X))$$
$$d_2\, t = \eta t^2$$

$TC^-(\mathbb{F}_p)$

$$\begin{array}{cccc}
 & 0 & 0 & 0 \\
\cdots & \mathbb{F}_p t x^2 & 0 & \mathbb{F}_p \cdot x^2 \\
\cdots & 0 & 0 & 0 \\
\cdots & \mathbb{F}_p t x & 0 & \mathbb{F}_p \cdot x \\
 & 0 & 0 & 0 \\
\cdots & \mathbb{F}_p t & 0 & \mathbb{F}_p
\end{array}$$

$TC^-_{2k+1}(\mathbb{F}_p) = 0$

1) $TC^-_{2k}(\mathbb{F}_p)$ are complete filtered ab. groups w/ assoc. graded sequence of $\mathbb{F}_p$'s.

Choose $\tilde{t} \in TC^-_{-2}$, $\tilde{x} \in TC^-_2$ lifts of t, x.

2) $TC^-_0(\mathbb{F}_p) \twoheadrightarrow \mathbb{F}_p$ w/ kernel generated as ideal by $\tilde{t}\tilde{x}$. So there is a relation $p = a\tilde{t}\tilde{x}$.

3) Have a map $THH(\mathbb{F}_p) \to HH(\mathbb{F}_p)$ iso. on $\tau_{\le 2}$. So we get

$$(\tau_{\le 2} THH(\mathbb{F}_p))^{hS^1} \xrightarrow{\sim} (\tau_{\le 2} HH(\mathbb{F}_p))^{hS^1}$$. Saw in SS for HC^-: $\tilde{t}\tilde{x} = p$.

$\Rightarrow a$ is a unit

We can modify our choice of $\tilde{x}$ for $a=1$, so $\tilde{t}\tilde{x}=p$

Get map $\mathbb{Z}_p[\tilde{t},\tilde{x}]/(p-\tilde{t}\cdot\tilde{x}) \longrightarrow TC^-_*(\mathbb{F}_p)$

<u>Theorem</u>. $TC^-_*(\mathbb{F}_p) \simeq \mathbb{Z}_p[\tilde{x},\tilde{t}]/(\tilde{x}\tilde{t}-p)$

i.e. $TC^-_{2k}(\mathbb{F}_p) \simeq \mathbb{Z}_p$, gen. by $\tilde{x}^k$ if $k \geq 0$

$\tilde{t}^{-k}$ if $k \leq 0$.

<u>Lecture 12</u>. <u>Topological periodic homology</u>

<u>Def</u> $TP(R) = THH(R)^{tS^1}$.

Since $(-)^{tG}$ is lax sym. monoidal in a way compatible w/ $(-)^{hG}$

so $TP(R)$ is a $TC^-(R)$-algebra if R is comm.

<u>Prop.</u> For spectrum X w/ G-action, there is a multiplicative conditionally convergent spectral sequence

$$\pi_p\left((H\pi_q(X))^{tG}\right) \Rightarrow \pi_{p+q}\left(X^{tG}\right)$$

<u>Proof sketch</u>: Take Whitehead filtration $\tau_{\geq \bullet}X$ on X

Apply $(-)^{tG}$. $\square$

For $H\mathbb{Z}$, $G=S^1$, consider the cofiber sequence

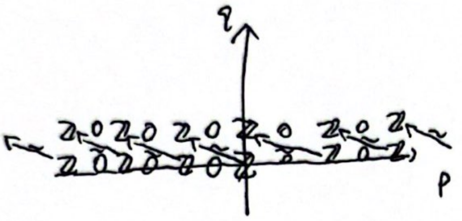
$$\Sigma H\mathbb{Z}_{hS^1} \simeq \left(S^{\mathrm{ad}\,G}\otimes H\mathbb{Z}\right)_{hG} \longrightarrow H\mathbb{Z}^{hS^1} \longrightarrow H\mathbb{Z}^{tS^1}$$

$$\pi_* H\mathbb{Z}^{tS^1} = \begin{cases} \mathbb{Z} & \text{for } * \text{ even} \\ 0 & \text{else} \end{cases}$$

Lemma $\pi_* H\mathbb{Z}^{tS^1} \simeq \mathbb{Z}[t^{\pm 1}]$.

Proof Consider the Tate SS for $H\mathbb{Z} \otimes \Sigma^\infty_+ S^1$

We know $(H\mathbb{Z} \otimes \Sigma^\infty_+ S^1)^{tS^1} = 0$



Fix generators x_i of $\pi_{2i} H\mathbb{Z}^{tS^1}$, $x_0 = 1$ & $x_{-i} = t^i$ $(i \geq 0)$

$\sigma_0, \sigma_1 \in H_*(S^1)$

Generator in degree $(2i, j)$ is $x_i \cdot \sigma_j$

$d_2(\sigma_0) = \pm t \cdot \sigma_1$, so $d_2(x_i \cdot \sigma_0) = \pm x_i \cdot t\sigma_1$

also $\pm x_{i-1} \cdot \sigma_1$

$x_i \cdot t = \pm x_{i-1}$. Choose x_i s.t. all signs are $+$.

So get $\pi_{2i} H\mathbb{Z}^{tS^1} = \mathbb{Z}[t^{\pm 1}]$. $\square$

Rmk Same argument w/ $HA \otimes \Sigma^\infty_+ S^1$ (as $H\mathbb{Z}$-mod),

get that $\pi_* HA^{tS^1} = A[t^{\pm 1}]$.

Prop. If X is an obj. of $\mathrm{Fun}(BS^1, \mathrm{Mod}_{H\mathbb{Z}})$

(equivalently, a module over $H\mathbb{Z}^{triv}$ in $\mathrm{Fun}(BS^1, Sp)$)

then $X^{tS^1} \simeq X^{hS^1} \otimes_{H\mathbb{Z}^{hS^1}} H\mathbb{Z}^{tS^1}$

$\pi_* X^{tS^1} \simeq \pi_* X^{hS^1}[t^{-1}]$

<u>Proof sketch</u>: We have $H\mathbb{Z}^{tS^1} \simeq \mathrm{colim}\left(H\mathbb{Z}^{hS^1} \xrightarrow{t} \Sigma^2 H\mathbb{Z}^{hS^1} \xrightarrow{t} \cdots\right)$

as $H\mathbb{Z}^{hS^1}$-modules.

So we need to show $X^{tS^1} = \mathrm{colim}\left(X^{hS^1} \xrightarrow{t} \Sigma^2 X^{hS^1} \xrightarrow{t} \cdots\right)$

- Works for X Eilenberg-Maclane
- Both sides are compatible w/ fiber sequences.
- To extend to <u>all</u> X, need some connectivity argument. (□)

Prop. $HP(R) = HH(R)^{tS^1}$.

<u>Proof</u>. Since $HH(R)$ is an $H\mathbb{Z}$-module, $HH(R)^{tS^1}$ is obtained from $HH(R)^{hS^1} = HC^-(R)$ by inverting t. So it coincides w/ $HP(R)$. □

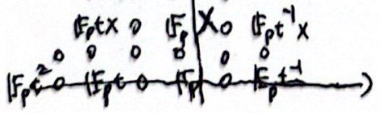
Prop. The Tate ss for tS^1 takes the form

$$\pi_*(X)[t^{\pm 1}] \Rightarrow \pi_*(X^{tS^1})$$

$$THH_*(R)[t^{\pm 1}] \Rightarrow TP_*(R)$$

Thm. $\pi_*(TP(\mathbb{F}_p)) = \mathbb{Z}_p[t^{\pm 1}]$

<u>Proof</u>. The Tate ss is a periodized version of the HFPSS and degenerates.



$$\pi_{-2k} TP(\mathbb{F}_p) = \pi_{2k} TC^-(\mathbb{F}_p) \sim \mathbb{Z}_p \cdot \tilde{t}$$

Can also choose a representative in $\pi_2 TP(\mathbb{F}_p)$ of t^{-1}, $\widetilde{t^{-1}}$

$\tilde{t} \cdot \widetilde{t^{-1}}$ is a unit $\Rightarrow$ $\tilde{t}$ invertible, and $\tilde{t}^{-k}$ generates $\pi_{2k}(TP(\mathbb{F}_p))$

$tx = p$
$\Rightarrow x = pt^{-1}$.

Lecture 13. The cyclotomic structure

R ring (spectrum)

$\rightarrow THH(R) \in Sp^{BS^1}$

$\rightarrow TC^-(R) = THH(R)^{hS^1}$, $TP(R) = THH(R)^{tS^1}$

holds in any sym. monoidal ∞-cat. in place of Sp.

Construction. Assume we have a fiber sequence $X \to Y \to Z$

of pointed conn'd spaces, which we write as

$$BH \to BG \to BG/H$$

Then there is a functor $\begin{array}{c} B(G/H) \longrightarrow \mathcal{S} \\ pt \longmapsto BH \end{array}$ which classifies BG,

i.e. G/H acts on BH s.t. $(BH)_{hG/H} \simeq BG$.

Consequence: For any ∞-cat. $\mathcal{C}$, we have $Fun(BG, \mathcal{C}) \simeq Fun(BH, \mathcal{C})^{hG/H}$

(With some additional work), this implies that the functors

$$(-)^{hH}, (-)^{tH} : Fun(BH, \mathcal{C}) \to \mathcal{C} \quad \text{induce functors}$$

$$(-)^{hH}, (-)^{tH} : Fun(BG, \mathcal{C}) \simeq Fun(BH, \mathcal{C})^{hG/H} \to Fun(pt, \mathcal{C})^{hG/H}$$
$$= Fun(BG/H, \mathcal{C})$$

i.e. for any normal subgp $H \trianglelefteq G$, if

X has a G-action, then X^{hH} & X^{tH} have "residual" G/H-actions

s.t. $(X^{hH})^{hG/H} \simeq X^{hG}$.

<u>Example</u>. Assume $G = \mathbb{T} = U(1) = S^1$, and $H = C_p \subset \mathbb{T}$ cyclic group of order p.

embedded as roots of unity in $\mathbb{T}$.

Then for $X \in \mathcal{C}^{B\mathbb{T}}$, we have that X^{hC_p} and X^{tC_p} have residual $\mathbb{T}/C_p$-action

We identify
$$\begin{aligned} \mathbb{T}/C_p &\xrightarrow{\sim} \mathbb{T} \\ z &\longmapsto z^p \end{aligned}$$

<u>Def</u>. A <u>cyclotomic structure</u> on a spectrum $X \in Sp^{B\mathbb{T}}$ w/ $\mathbb{T}$-action is given by maps $\varphi_p : X \longrightarrow X^{tC_p}$ that are $\mathbb{T}$-equivariant for every prime p. We refer to φ_p as the <u>cyclotomic Frobenius</u>.

- A <u>cyclotomic spectrum</u> is an spectrum w/ $\mathbb{T}$-action and cyclotomic structure.

<u>Thm</u>. For every ring spectrum R, $THH(R)$ admits a nat'l cyclotomic structure.
$$THH(R) \xrightarrow{\varphi_p} THH(R)^{tC_p}.$$

<u>Remark</u>. - The cyclotomic str. on $THH(R)$ is induced from the Tate diagonal.
$$\Delta_p : Id \to T_p \text{ on spectrum.}$$

- It does <u>not</u> exist for $HH(R/\mathcal{C})$ where $\mathcal{C}$ is an arbitrary stable sym. mon. ∞-cat.

One can for example prove that in $D(\mathbb{Z})$, there is no map $X \to (X \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} X)^{tp}$.

Therefore, $HH(R/\mathbb{Z})$ does not admit any sort of cyclotomic structure.

<u>First construction for an $\mathbb{E}_\infty$-ring.</u>

<u>Induction</u>. $\mathcal{C}$ an ∞-cat. w/ all small colimits, then we have a forgetful functor

for any group G $\quad \mathcal{C}^{BG} \to \mathcal{C}$

<u>Prop</u> This functor has a left adjoint, given by $c \in \mathcal{C} \mapsto \operatorname{colim}_G (\mathrm{const}_c) = c \otimes G$

w/ G-action on G.

<u>Proof</u>: $\mathrm{Map}_{\mathcal{C}^{BG}}(c \otimes G, d) \simeq \mathrm{Map}_{\mathcal{C}}(c \otimes G, d)^{hG}$

$$\simeq \mathrm{Map}_{\mathcal{S}}(G, \mathrm{Map}_{\mathcal{C}}(c,d))^{hG} \simeq \mathrm{Map}_{\mathcal{C}}(c,d).$$

<u>Example</u> – $\mathcal{C} = Sp$, G finite, then $\bigoplus_{g \in G} c$ is the "free" object.

– $G = \mathbb{T}$, the free $\mathbb{T}$-object is $c \otimes \mathbb{T} = c \oplus c[1]$.

<u>Example</u> $\mathcal{C} = CAlg(Sp)$, G finite. $R \in CAlg(Sp)$.

$R^{\otimes G} = \operatorname{colim}_G (R) = \underbrace{R \otimes \cdots \otimes R}_{G \text{ many factors}}$ w/ permutation G-action.

Prop For $R \in \mathrm{CAlg}(\mathrm{Sp})$, we have that $R \to \mathrm{THH}(R)$ exhibits $\mathrm{THH}(R)$ as the free $\mathbb{E}_\infty$-ring w/ S^1-action under R.

Proof. By our general formula, the free obj. is given by

$$\operatorname{colim}_{S^1} (R) = R^{\otimes S^1}$$

We have seen that this colimit is $\mathrm{THH}(R)$ in lecture 9.

The fact that this S^1-action agrees w/ the previously constructed S^1-action on $\mathrm{THH}(R)$ is skipped. $\square$

Construction: • For any $R \in \mathrm{CAlg}(\mathrm{Sp})$, we have the map $R \to \mathrm{THH}(R)$

• The target has an $\mathbb{T}$-action, in particular, we get an induced C_p-action.

Thus we get a unique ext'n to a C_p-equiv. map $\underbrace{R \otimes \cdots \otimes R}_{p\text{-times}} \longrightarrow \mathrm{THH}(R)$.

$$\to (R \otimes \cdots \otimes R)^{tC_p} \to \mathrm{THH}(R)^{tC_p}.$$

- The Tate diagonal $R \to (R \otimes \cdots \otimes R)^{tC_p}$ of $\mathrm{CAlg}(\mathrm{Sp})$

Prop. There is a unique $\mathbb{T}$-equiv. map $\mathrm{THH}(R) \xrightarrow{\varphi_p} \mathrm{THH}(R)^{tC_p}$ of $\mathbb{E}_\infty$-rings.

fitting into the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\Delta_p"'] & \mathrm{THH}(R) \arrow[d, dashed, "\exists!\, \varphi_p"] \\ (R \otimes \cdots \otimes R)^{tC_p} \arrow[r] & \mathrm{THH}(R)^{tC_p} \end{tikzcd}$$

$\square$

<u>The case of $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$.</u> Recall that $\mathrm{THH}(R) = \operatorname*{colim}_{[n] \in \Delta^{op}} R^{\otimes(n+1)}$

$$= \operatorname*{colim}_{\bullet \in \Delta^{op}} (\mathrm{THH}(R)_\bullet)$$

where $\mathrm{THH}(R)_\bullet : \Lambda \to \mathrm{Sp}$

$$\begin{tikzcd}
\mathrm{THH}(R) & \cdots \arrow[r, phantom, "\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}"] & R\otimes R\otimes R \arrow[loop above, "C_3"] \arrow[r, phantom, "\substack{\rightarrow\\ \rightarrow\\ \rightarrow}"] \arrow[d, "\Delta_p"] & R\otimes R \arrow[loop above, "C_2"] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\Delta_p"] & R \arrow[d, "\Delta_p"] \\
(\mathrm{sd}_p \mathrm{THH}(R))^{tC_p} & \cdots \arrow[r, phantom, "\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}"] & (R^{\otimes 3p})^{tC_p} \arrow[loop below, "C_3"] \arrow[r, phantom, "\substack{\rightarrow\\ \rightarrow\\ \rightarrow}"] & (R^{\otimes 2p})^{tC_p} \arrow[loop below, "C_2"] \arrow[r, shift left] \arrow[r, shift right] & (R^{\otimes p})^{tC_p}
\end{tikzcd}$$

The cyclotomic str. on THH(R) arises from a map of cyclic objects as above

The main computation comes from the way of viewing the target as a cyclic object.

As a result, we get a map of spectra w/ $\mathbb{T}$-action

$$\begin{aligned}
\mathrm{THH}(R) = \operatorname*{colim}_{\Delta^{op}} (\mathrm{THH}(R)_\bullet) \to \operatorname*{colim}_{\Delta^{op}} & \left((\mathrm{sd}_p \mathrm{THH}(R))^{tC_p} \right) \\
& \quad \wr\!\wr \\
\operatorname*{colim}_{[n] \in \Delta^{op}} & \left((R^{\otimes p(n+1)})^{tC_p} \right) \\
& \quad \downarrow \\
& \left(\operatorname*{colim}_{[n] \in \Delta^{op}} (R^{\otimes p(n+1)}) \right)^{tC_p} \\
& \quad \wr\!\wr \\
& \quad \mathrm{THH}(R)^{tC_p}
\end{aligned}$$

Lecture 14. The definition of TC

For every prime no. p, a $\mathbb{T}$-equiv. map $THH(R) \to THH(R)^{tC_p}$

Def. A cyclotomic spectrum is a spectrum w/ $\mathbb{T}$-action $X \in Sp^{B\mathbb{T}}$ together w/ $\mathbb{T}$-equiv. maps $\varphi_p : X \to X^{tC_p}$ for every prime p.

The ∞-cat. of cyclotomic spectra is defined as the pullback

$$\begin{tikzcd} CycSp \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p (Sp^{B\mathbb{T}})^{\Delta^1} \arrow[d, "\Pi(ev_0, ev_1)"] \\ Sp^{B\mathbb{T}} \arrow[r, "(id, (-)^{tC_p})"] & \prod_p Sp^{B\mathbb{T}} \times Sp^{B\mathbb{T}} \end{tikzcd}$$

In particular, a map of cyclotomic spectra (X, φ_p) to (Y, φ_p') is given by

(1) $\mathbb{T}$-equiv map $X \xrightarrow{f} Y$

(2) A $\mathbb{T}$-equiv. homotopy filling the square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_p"'] \arrow[dr, phantom, "\Swarrow"] & Y \arrow[d, "\varphi_p'"] \\ X^{tC_p} \arrow[r, "f^{tC_p}"'] & Y^{tC_p} \end{tikzcd}$$
for every prime p.

The mapping space is given as the equalizer:

$$(*) \quad Map_{CycSp}(X, Y) \longrightarrow Map_{Sp^{B\mathbb{T}}}(X, Y) \underset{(\varphi_p')_*}{\overset{(\varphi_p)^*}{\rightrightarrows}} Map_{Sp^{B\mathbb{T}}}(X, Y^{tC_p}) \qquad \left(\prod_p\right)$$

Prop. (1) CycSp is a stable ∞-cat. and $(*)$ holds for mapping spectra.

(2) CycSp has all limits and colimits.

(3) CycSp has a sym. monoidal str. given by $(X, \varphi_p) \otimes (Y, \varphi_p') = (X \otimes_{\mathbb{S}} Y, X \otimes_{\mathbb{S}} Y \to X^{tC_p} \otimes_{\mathbb{S}} Y^{tC_p} \to (X \otimes_{\mathbb{S}} Y)^{tC_p})$

(4) A comm. alg. in CycSp is given by

(1) $X \in \mathrm{CAlg}(\mathrm{Sp})^{B\mathbb{T}}$

(2) a $\mathbb{T}$-equiv. map of comm. algs $\varphi_p : X \longrightarrow X^{tC_p}$ for any p.

In particular, $\mathrm{THH}(R)$ for R an $\mathbb{E}_\infty$-ring is a comm. alg. in CycSp.

<u>Recall</u>. $TC^-(R) = \mathrm{THH}(R)^{hS^1} = \mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}^{\mathrm{triv}}, \mathrm{THH}(R))$

We will give a similar description for TC.

<u>Example</u> $\mathbb{S}^{\mathrm{triv}}$ is canonically a cyclotomic spectrum ($\mathbb{E}_\infty$)

- underlying spectrum w/ $\mathbb{T}$-action is $\mathbb{S}^{\mathrm{triv}}$

- for every prime, the map $\varphi_p : \mathbb{S} \to \mathbb{S}^{tC_p}$ is the unit.

i.e. the map $\mathbb{S} \longrightarrow \mathbb{S}^{hC_p} \xrightarrow{\mathrm{can}} \mathbb{S}^{tC_p}$

pullback along $BC_p \to pt$

<u>Def</u>. For a ring spectrum R, we define $TC(R) = \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, \mathrm{THH}(R))$

- If R is commutative, then this is an $\mathbb{E}_\infty$-ring spectrum

We will write $\pi_* TC(R) = TC_*(R)$

- More generally, for any cyclotomic spectrum X, we write

$TC(X) = \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X)$.

$TC(R) = TC(\mathrm{THH}(R))$.

Explicitly evaluate this

$$\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X) \simeq \mathrm{Eq}\left(\mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}^{\mathrm{triv}}, X) \rightrightarrows \prod_{p \text{ prime}} \mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}^{\mathrm{triv}}, X^{tC_p})\right)$$

$$\simeq \mathrm{Eq}\left(X^{h\mathbb{T}} \rightrightarrows \prod_p (X^{tC_p})^{h\mathbb{T}}\right)$$

$$\simeq \mathrm{fib}\left(X^{h\mathbb{T}} \xrightarrow{\varphi_p^{h\mathbb{T}} - \mathrm{can}^{h\mathbb{T}}} \prod_p (X^{tC_p})^{h\mathbb{T}}\right), \quad \varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$$

$$\mathrm{can}^{h\mathbb{T}} : X^{h\mathbb{T}} \simeq (X^{hC_p})^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$$

<u>Thm</u>. Assume that X is bounded below w/ $\mathbb{T}$-action. Then the canonical map

$X^{t\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$ exhibits the RHS as the p-completion of LHS.

Moreover, If X is p-complete, then so is $X^{t\mathbb{T}}$.

<u>As a result</u>. If $X \in \mathrm{CycSp}$ w/ underlying spectrum bounded below, then

$$\varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}} = (X^{t\mathbb{T}})^\wedge_p$$

$$\varphi : X^{h\mathbb{T}} \to (X^{t\mathbb{T}})^\wedge := \prod_p (X^{t\mathbb{T}})^\wedge_p$$

<u>Cor</u>. We have that if $X \in \mathrm{CycSp}$ bounded below, then

$$TC(X) \simeq \mathrm{fib}\left(X^{h\mathbb{T}} \xrightarrow{\varphi - \mathrm{can}} (X^{t\mathbb{T}})^\wedge\right)$$

$$\simeq \mathrm{Eq}\left(X^{h\mathbb{T}} \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} (X^{t\mathbb{T}})^\wedge\right)$$

$$\boxed{TC(R) \simeq \mathrm{Eq}\left(TC^-(R) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} TP(R)^\wedge\right)}$$ R connective ring spectrum

For X a qcqs scheme, we define $\mathrm{THH}(X) = \lim\limits_{\substack{U \subset X \\ \text{affine open}}} \mathrm{THH}(\mathcal{O}_U)$

$$TC^-(X) := \mathrm{THH}(X)^{h\mathbb{T}} = \lim_{\substack{U \subset X \\ \text{affine open}}} TC^-(\mathcal{O}_U)$$

$$TP(X) := \mathrm{THH}(X)^{t\mathbb{T}} = \lim_{\substack{U \subset X \\ \text{affine open}}} TP(\mathcal{O}_U)$$

$$TC(X) := TC(\mathrm{THH}(X)) \simeq \lim_{\substack{U \subset X \\ \text{affine open}}} TC(\mathcal{O}_U)$$

$$= \mathrm{Eq}\left(TC^-(X) \rightrightarrows TP(X)^\wedge \right)$$

Remark: One can show, that for $X \in \mathrm{CycSp}$ n-connective, then $TC(X)^\wedge_p = TC(X;\mathbb{Z}_p)$ is $(n-1)$-connective.

In particular, $TC(R;\mathbb{Z}_p)$ for R ring (or conn. ring spectra) is (-1)-connective.

Thm. X bounded below, $\mathbb{T}$-action.

$$(X^{t\mathbb{T}})^\wedge_p \simeq (X^{tC_p})^{h\mathbb{T}} \quad (*)$$

Proof. Both sides of $(*)$ are exact in X, and commute w/ the Postnikov tower.

$\Rightarrow$ reduce to $X = HA$.

- Resolving A by a 2-term cpx of free group, we can assume that A is free.

Now check that HA^{tC_p} is p-torsion $\Rightarrow$ p-complete $\Rightarrow (X^{tC_p})^{h\mathbb{T}}$ is p-complete.

- Check that this is an equiv. after mod p

$\Rightarrow$ Assume A is $\mathbb{F}_p$-v.s. $\quad A = \bigoplus \mathbb{F}_p$

- Both sides of $(*)$ commutes w/ $\bigoplus \mathbb{F}_p$ (need spectral seq. arg.)

$\Rightarrow$ reduce to $A = \mathbb{F}_p$.

$\pi_*\left(H\mathbb{F}_p^{tT}\right) \simeq \mathbb{F}_p[t^{\pm 1}]$, $|t| = -2$

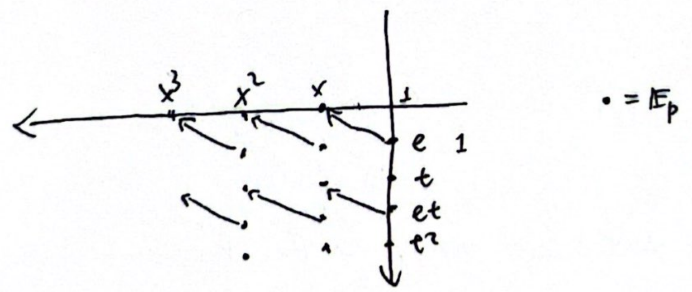
WTS $\pi_*\left(\left(\left(H\mathbb{F}_p\right)^{tC_p}\right)^{hT}\right) \simeq \mathbb{F}_p[t^{\pm 1}]$

Recall. p odd $\quad \pi_*\left(H\mathbb{F}_p^{tC_p}\right) \simeq \mathbb{F}_p[t^{\pm 1}] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(e)$, $\quad |t| = -2$, $|e| = -1$

HFPSS to compute $\pi_*\left(\left(H\mathbb{F}_p^{tC_p}\right)^{hT}\right) \Leftarrow \mathbb{F}_p[t^{\pm 1}] \otimes \Lambda_{\mathbb{F}_p}(e) \otimes \mathbb{F}_p[x]$

As a warm up, Compute HFPSS for $\left(\left(H\mathbb{F}_p\right)^{hC_p}\right)^{hT} \simeq H\mathbb{F}_p^{hT}$

$\mathbb{F}_p[t] \otimes \Lambda_{\mathbb{F}_p}(e) \otimes \mathbb{F}_p[x] \Rightarrow \mathbb{F}_p[t]$.



$\square$.

Lecture 15. TC of $\mathbb{F}_p$.

Recall R ring (or connective ring spectrum)

$$TC(R) = \mathrm{Eq}\left(TC^-(R) \underset{can}{\overset{\varphi}{\rightrightarrows}} TP(R)^\wedge\right)$$

Where
- can : $TC^-(R) = THH(R)^{h\mathbb{T}} \to THH(R)^{t\mathbb{T}} = TP(R) \to TP(R)^\wedge$
- φ : $TC^-(R) \to TP(R) = \prod_{p \in \mathcal{P}} \left(THH(R)^{tC_p} \right)^{h\mathbb{T}}$

φ is given by $\left(\varphi_p^{h\mathbb{T}}\right)_{p \in \mathcal{P}}$.

Same for $X \in \mathrm{CycSp}$ w/ $X \in \mathrm{Sp}$ bounded below.

Example $R = \mathbb{F}_p$,

- $TC^-_*(\mathbb{F}_p) = \mathbb{Z}_p[x,t]/(xt-p)$, $|x| = 2$, $|t| = -2$
- $TP_*(\mathbb{F}_p) = TC^-_*(\mathbb{F}_p)[t^{-1}] = \mathbb{Z}_p[t^{\pm 1}] \Rightarrow TP(\mathbb{F}_p) = TP(\mathbb{F}_p)^\wedge$
- can, φ : $TC^-_*(\mathbb{F}_p) \to TP_*(\mathbb{F}_p)$ are ring maps.

  $\mathrm{can}(t) = t$

  $\mathrm{can}(x) = pt^{-1}$

Lemma There exists a unit $\alpha \in \mathbb{Z}_p^\times$ s.t. $\varphi(x) = \alpha t^{-1}$, $\varphi(t) = \alpha^{-1} p t$.

Remark With some more work, one can show that in fact $\alpha = 1$.

Proof. We have the following comm. diagram

$$\begin{tikzcd}
TC^-(\mathbb{F}_p) \arrow[r, "\varphi"] \arrow[d] & TP(\mathbb{F}_p) \arrow[r] \arrow[d] & (\mathbb{F}_p)^{t\mathbb{T}} \arrow[d] & \text{induced from } THH(\mathbb{F}_p) \to \mathbb{F}_p \arrow[l] \\
THH(\mathbb{F}_p) \arrow[r, "\varphi_p"'] & THH(\mathbb{F}_p)^{tC_p} \arrow[r] & (\mathbb{F}_p)^{tC_p} & \simeq \mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(e)
\end{tikzcd}$$

$p=2$ $\mathbb{F}_p[s^{\pm 1}]$

$\varphi : \mathbb{Z}_p[x,t]/(xt-p) \to \mathbb{Z}_p[t^{\pm 1}]$

We have $\varphi(x) = \alpha t^{-1}$, $\varphi(t) = \mu t$

Now chasing $t \in TC^-_{-2}(\mathbb{F}_p)$ through the diagram clockwise, it is zero.

The counterclockwise composition is $[\mu] \in \mathbb{F}_p$

Thus $[\mu] = 0$, i.e. $\mu = p\mu'$.

Then $p = \varphi(p) = \varphi(xt) = \varphi(x)\varphi(t) = \alpha\mu = p\alpha\mu'$

$\Rightarrow \alpha\mu' = 1$, i.e. α is a unit.

<u>Now</u> compute $TC_*(\mathbb{F}_p)$ by the LES assoc. w/ $TC \to TC^- \xrightarrow{\varphi - can} TP$

$$\to TC_2(\mathbb{F}_p) \to \mathbb{Z}_p \cdot x \to \mathbb{Z}_p t^{-1} \to \qquad x \mapsto \alpha t^{-1} - t^{-1}$$

$$\to TC_1(\mathbb{F}_p) \to 0 \to 0 \to$$

$$\to TC_0(\mathbb{F}_p) \to \mathbb{Z}_p \xrightarrow[0]{\varphi - can} \mathbb{Z}_p \to$$

$$\to TC_{-1}(\mathbb{F}_p) \to 0 \to 0 \to$$

$$\to TC_{-2}(\mathbb{F}_p) \to \mathbb{Z}_p t \to \mathbb{Z}_p t \to$$

$$t \mapsto \alpha^{-1}pt - t = (\alpha^{-1}p - id)$$

All the non-deg zero maps are of the form (up to a unit) $id - p\beta : \mathbb{Z}_p \to \mathbb{Z}_p$

<u>Theorem</u>. We have $TC_*(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0, -1 \\ 0, & \text{else} \end{cases}$

<u>Remark</u>: – As a ring, $TC_*(\mathbb{F}_p) = \mathbb{Z}_p[\varepsilon]/\varepsilon^2$, $|\varepsilon| = -1$.

One can show that $TC(\mathbb{F}_p) \simeq H\mathbb{Z}_p^{S^1}$ as an $\mathbb{E}_\infty$-ring.

– Quillen has shown that the p-completion of algebraic K-theory of $\mathbb{F}_p$ is

$$K_*(\mathbb{F}_p; \mathbb{Z}_p) = \pi_*\left((K\mathbb{F}_p)^\wedge_p\right) = \begin{cases} \mathbb{Z}_p, & * = 0 \\ 0, & \text{else} \end{cases} \Rightarrow K(\mathbb{F}_p, \mathbb{Z}_p) \simeq \tau_{\geq 0} TC(\mathbb{F}_p)$$

Cor. There is an $\mathbb{E}_\infty$-map $\quad H\mathbb{Z} \to TC(\mathbb{F}_p)$

• $TC(A)$ is an $H\mathbb{Z}$-module spectrum for each $\mathbb{F}_p$-algebra A.

Proof. – Connective cover is a map $H\mathbb{Z}_p \to TC(\mathbb{F}_p)$

– $TC(A)$ is a module over $TC(\mathbb{F}_p)$ since A is an $\mathbb{F}_p$-module in $Alg(Ab) \hookrightarrow Alg(Sp)$

Prop. We have an adjunction

$$(-)^{triv} : Sp \rightleftarrows CycSp : TC$$

X^{triv} : trivial $\mathbb{T}$-action, $\varphi : X \to X^{hC_p} \xrightarrow{can} X^{tC_p}$.

The left adjoint is symmetric monoidal, the right adjoint is lax sym. monoidal.

Cor. • We have a map of cyclotomic $\mathbb{E}_\infty$-rings $\quad H\mathbb{Z}^{triv} \to THH(\mathbb{F}_p)$.

• For each $\mathbb{F}_p$-alg. A, $THH(A)$ is naturally a $H\mathbb{Z}^{triv}$-module.

Warning ↯ The map $H\mathbb{Z} \xrightarrow{f} THH(\mathbb{F}_p)$ is <u>different</u> from the "obvious" map

$g : H\mathbb{Z} \to H\mathbb{F}_p \xrightarrow{i} THH(\mathbb{F}_p)$ $\qquad$ even as maps of spectra.

To see this, we show that postcomposing the two maps w/

$$v : THH(\mathbb{F}_p) \xrightarrow{\varphi_p} THH(\mathbb{F}_p)^{tC_p} \to (\mathbb{F}_p)^{tC_p} \quad \text{are different,}$$

i.e. $\quad v \circ f \neq v \circ g$.

(1) For the map f, consider the diagram

$$\begin{tikzcd}
H\mathbb{Z} \arrow[r, "triv"] \arrow[d] \arrow[ddr, bend right] & H\mathbb{Z}^{tC_p} \arrow[d] \arrow[dd, bend left, "\pi^{tC_p}"] \\
THH(\mathbb{F}_p) \arrow[r, "\varphi"] & THH(\mathbb{F}_p)^{tC_p} \arrow[d] \\
& \mathbb{F}_p^{tC_p}
\end{tikzcd}$$

Thus, $\tau\circ f$ is equiv. to the composite $H\mathbb{Z} \xrightarrow{\pi} H\mathbb{F}_p \xrightarrow{\text{triv}} H\mathbb{F}_p{}^{tC_p}$

(2) Consider the diagram.

$$\begin{tikzcd}
H\mathbb{F}_p \arrow[r, "\Delta_p"] \arrow[d] \arrow[ddr, bend right, "(*)"'] & (H\mathbb{F}_p \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} H\mathbb{F}_p)^{tC_p} \arrow[d] \\
THH(\mathbb{F}_p) \arrow[r, "\varphi"'] & THH(\mathbb{F}_p)^{tC_p} \arrow[d] \\
& H\mathbb{F}_p{}^{tC_p}
\end{tikzcd}$$

The $\mathbb{E}_\infty$-Frobenius.

Thus, τg is equiv. to the composite $H\mathbb{Z} \xrightarrow{\pi} H\mathbb{F}_p \xrightarrow{\mathbb{E}_\infty\text{-Frob}} H\mathbb{F}_p{}^{tC_p}$

<u>Theorem</u>. The maps $H\mathbb{F}_p \xrightarrow{\text{triv}} (H\mathbb{F}_p)^{tC_p}$ and the $\mathbb{E}_\infty$-Frobenius $H\mathbb{F}_p \to (H\mathbb{F}_p)^{tC_p}$ differ by all the Steenrod operations.

<u>Cor</u>. $\tau\circ f \not\simeq \tau\circ g \Rightarrow f \not\simeq g$.

<u>Def</u>. We define the ∞-cat. of cyclotomic chain complexes $\mathrm{Cyc}\,D\mathbb{Z}$ as the pullback

$$\begin{tikzcd}
\mathrm{Cyc}\,D\mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{p\in P}((D\mathbb{Z})^{B\mathbb{T}})^{\Delta^1} \arrow[d, "\Pi(s\times t)"] \\
(D\mathbb{Z})^{B\mathbb{T}} \arrow[r, "(id,(-)^{tC_p})_{p\in P}"'] & \prod_{p\in P} (D\mathbb{Z})^{B\mathbb{T}} \times (D\mathbb{Z})^{B\mathbb{T}}
\end{tikzcd}$$

$$\boxed{C \in (D\mathbb{Z})^{B\mathbb{T}},\quad C \to C^{tC_p},\ \forall p}$$

Prop. $\mathrm{Cyc}(D\mathbb{Z}) \simeq \mathrm{Mod}_{H\mathbb{Z}^{triv}}(\mathrm{Cyc}\,Sp)$

Cor. For an $\mathbb{F}_p$-algebra A, $THH(A)$ is naturally an obj. of $\mathrm{Cyc}\,D\mathbb{Z}$.

Lecture 16. TC of perfect rings

Recall. $TC_*(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0, 1 \\ 0, & \text{else} \end{cases}$

k throughout a perfect $\mathbb{F}_p$-algebra, i.e. $\varphi: k \to k$ isomorphism.

Example.
- $k = \mathbb{F}_{p^n}$ finite field
- $k = \prod k_i$, k_i perfect $\mathbb{F}_p$-algebras
- $k = C^0(X; \mathbb{F}_p)$ cts maps from some top. space X to $\mathbb{F}_p$
- $k = \mathbb{F}_p[x^{1/p^\infty}]$.

We have $THH_*(k) = k[x]$

Recollection on Witt vectors

There exists comm. rings $W(k)$ called p-typical Witt vectors of k, w/ the following properties.

(1) $W(k)$ is p-torsion free

(2) $W(k)$ is p-complete

(3) $W(k)/p \simeq k$

Moreover, $W(k)$ is uniquely characterized by 1,2,3.

Theorem Let R be any derived p-complete ring, then

$$\mathrm{Hom}_{\mathrm{CRing}}(W(k), R) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CRing}}(k, R/p) \quad \text{is a bijection.}$$

Proof - If R is p-complete, i.e. $R = \varprojlim R/p^n$, then we use obstruction theory to inductively lift maps through the square zero ext'n $R/p^{n+1} \to R/p^n$

- If R is derived p-complete, then $R = \varprojlim R/p^n$ as animated comm. rings, thus as in the first step, we can reduce to $R/\!/p$.

Then as a last step again use that $R/\!/p \to R/p$ is a square zero ext'n.

Cor. We have an equiv. of cats

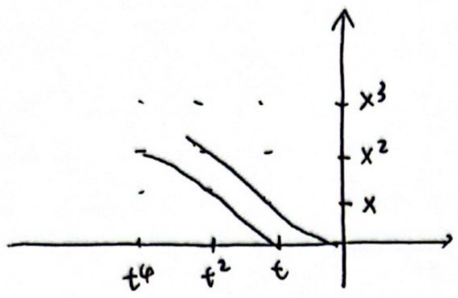
$$\left\{\begin{array}{l} p\text{-torsion free, } p\text{-complete rings} \\ \text{w/ perfect mod } p \text{ reduction} \end{array}\right\} \underset{W}{\overset{/p}{\rightleftarrows}} \left\{\text{Perfect } \mathbb{F}_p\text{-algs}\right\}$$

Cor. There is a ring isom.

$F : W(k) \longrightarrow W(k)$ ~ "Witt vector Frobenius" s.t. $F/p : k \to k$ is the Frobenius $\varphi : x \mapsto x^p$.

Prop. For a perfect $\mathbb{F}_p$-alg. k, we have nat'l isoms
$$TC^-_*(k) \simeq W(k)[x,t]/_{xt-p}$$
$$TP_*(k) \simeq W(k)[t^{\pm 1}]$$

Proof.
Consider the HFPSS for $THH(k)^{h\mathbb{T}}$.



$\cdot = k$

There is a map of SS from the HFPSS from $THH(\mathbb{F}_p)^{hT}$ induced from $\mathbb{F}_p \to k$.

Thus $xt = p$ in $TC^-_0(k)$.

But xt is a nonzero divisor in $TC^-_0(k)$, thus it is p-torsion free, and also

p-adically complete w/ mod p reduction k $\Rightarrow$ $TC^-_0(k) \simeq W(k)$

The rest follows since we have the map of SS. $\square$

can, φ: $TC^-(k) \longrightarrow TP(k)$

- $\mathrm{can}(x) = pt^{-1}$, $\mathrm{can}(t) = t$, $\mathrm{can}(r) = r$, $r \in W(k)$
- $\varphi(x) = t^{-1}$, $\varphi(t) = pt$

Rmk. For any connective ring spectrum, the map $\pi_0\,\mathrm{can}: TC^-_0(R) \to TP_0(R)$ is an iso.

Thus we can consider $\pi_0\varphi$ as an endomorphism of $TC^-_0(R) = TP_0(R)$.

Prop. We have that $\pi_0\varphi: Wk \to Wk$ is the Witt vector Frobenius F.

Proof. Consider the diagram

$$\begin{tikzcd} & TC^-(k) \arrow[r, "\varphi"] \arrow[d] & TP(k) \arrow[d] & \\ k \arrow[r] & THH(k) \arrow[r, "\varphi"] & THH(k)^{tC_p} \arrow[r] & k^{tC_p} \end{tikzcd}$$

This diagram induces on π_0 the following:

$$\begin{tikzcd}
 & W(k) \arrow[r, "\pi_0\varphi"] \arrow[d, "/p"] & W(k) \arrow[d] \arrow[dr, "/p"] & \\
k \arrow[r, "id"] \arrow[rrr, bend right] & k \arrow[r] & ? \arrow[r] & k
\end{tikzcd}$$

The composite $k \to THH(k) \to THH(k)^{tC_p} \to k^{tC_p}$ is the $\mathbb{E}_\infty$-Frobenius of k. $\square$

Now $TC_*(k)$ can be computed by the LES assoc w/ $TC(k) \to TC^-(k) \xrightarrow{can-\varphi} TP(k)$

The map $can - \varphi : TC^-_*(k) \to TP_*(k)$ is given as follows

in deg $2n$, $n \geq 0$:

$$W(k)x^n \to W(k)t^{-n}$$
$$\alpha x^n \mapsto (\alpha p^n - F(\alpha))t^{-n}$$

in deg $-2n$, $n \geq 0$: $W(k)t^n \to W(k)t^n$

$$\alpha t^n \mapsto (\alpha - F(\alpha)p^n)t^n$$

These maps are isos for $n \neq 0$.

For $n = 0$, it is given by $id - F$.

<u>Thm</u>. For k a perfect $\mathbb{F}_p$-alg, we have

$$TC_*(k) = \begin{cases} W(k)^F = \ker(id - F), & * = 0 \\ W(k)_F = \operatorname{coker}(id - F), & * = -1 \\ 0, & \text{else} \end{cases}$$

<u>Facts</u> ① $W(k)^F \simeq W(k^\varphi)$, $k^\varphi = \ker(id - \varphi) \subset k$

Cor. k a field, then $TC_0(k) = \mathbb{Z}_p$

Proof. $k^\varphi = \{x \in k : x^p = x\} = \mathbb{F}_p$.

② We have $k^\varphi = C^0(X; \mathbb{F}_p)$ for a pro-finite space X. ("Stone duality")

In fact, X is the underlying top. space of $\mathrm{Spec}(k^\varphi)$.

It is also the space of connected components of $\mathrm{Spec}(k)$.

③ $W(k^\varphi) = W(C^0(X; \mathbb{F}_p)) = \underset{\substack{\uparrow \\ \text{cts maps}}}{C^0(X; \mathbb{Z}_p)} \simeq C^0(\mathrm{Spec}(k^\varphi); \mathbb{Z}_p)$

$\simeq C^0(\mathrm{Spec}(k); \mathbb{Z}_p)$

Cor. For k a perfect $\mathbb{F}_p$-alg, we get $TC_0(k) = C^0(\mathrm{Spec}(k); \mathbb{Z}_p) = C^0(\mathrm{Spec}(k^\varphi); \mathbb{Z}_p)$

④ $TC_{-1}(k) = W(k)_F$ is p-torsion free. and it's mod p-reduction is

$k_\varphi = \mathrm{coker}(id - \varphi)$

Thus $TC_{-1}(k)$ is p-completely free of rank the dim. of k_φ:

$TC_{-1}(k) = \left(\bigoplus_S \mathbb{Z}_p\right)^\wedge_p$ where $k_\varphi = \bigoplus_S \mathbb{F}_p$.

Cor. If $k_\varphi = 0$, eg. k alg closed field, then $TC_{-1}(k) = 0$.

Example. $TC_*(\overline{\mathbb{F}}_p) = H\mathbb{Z}_p$

⑤ Say $k^\varphi = \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{i \text{ times}}$, then k_φ is a k^φ-module

can be described as $k_\varphi = V_1 \times \cdots \times V_i$ for $V_1, \cdots, V_i \in \mathrm{Vect}_{\mathbb{F}_p}$.

$\Rightarrow \quad W(k)_F = \widetilde{V}_1 \times \cdots \times \widetilde{V}_i$ as a $W(k)^F = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ - module, where $\widetilde{V}_k$ is a p-complete, p-torsion free lift of V_i to $\mathbb{Z}_p$.

Lecture 17 Frobenius lifts and group rings.

Def. A p-cyclotomic spectrum w/ Frobenius lift is a p-cyclotomic spectrum X, w/ a $\mathbb{T}$-equiv. factorization $X \xrightarrow{\psi_p} X^{hC_p} \xrightarrow{triv} X^{tC_p}$.

Example. X^{triv} admits a Frobenius lift since $\varphi_p : X \xrightarrow{\psi_p} X^{hC_p} \longrightarrow X^{tC_p}$

$$\mathrm{map}(\Sigma_+^\infty pt, X) \to \mathrm{map}(\Sigma_+^\infty BC_p, X)$$

If X has a Frobenius lift, $\varphi : X^{h\mathbb{T}} \to X^{t\mathbb{T}}$ factors

$$\begin{tikzcd} X^{h\mathbb{T}} \arrow[r, "(\psi_p)^{h\mathbb{T}}"] \arrow[dr, "\psi"'] & (X^{hC_p})^{h\mathbb{T}} \arrow[r, "\mathrm{triv}^{h\mathbb{T}}"] & (X^{tC_p})^{h\mathbb{T}} \\ & X^{h\mathbb{T}} \arrow[u, phantom, "\wr"] \arrow[r, "can"] & X^{t\mathbb{T}} \arrow[u, phantom, "\wr"] \end{tikzcd}$$

(X p-complete)

So $\varphi - \mathrm{can} : X^{h\mathbb{T}} \to X^{t\mathbb{T}}$ factors as

$$X^{h\mathbb{T}} \xrightarrow{\psi - id} X^{h\mathbb{T}} \xrightarrow{can} X^{t\mathbb{T}}.$$

Theorem: X p-complete bounded below p-cyclotomic spectrum w/ Frobenius lift. then

$$\begin{tikzcd} TC(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Sigma X_{h\mathbb{T}} \arrow[d, "tr"] \\ X \arrow[r, "\psi_p - id"] & X \end{tikzcd}$$

Proof sketch.

$$\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] & \Sigma X_{h\mathbb{T}} \arrow[r] \arrow[d, "Nm"] & 0 \arrow[d] \\
X^{h\mathbb{T}} \arrow[r, "\varphi - id"] \arrow[d] & X^{h\mathbb{T}} \arrow[r, "can"'] \arrow[d] & X^{t\mathbb{T}} \\
X \arrow[r, "\varphi_p - id"'] & X &
\end{tikzcd}$$

pullback as well.

Cor. $TC(X^{triv}) = X \oplus \Big(X \otimes \underbrace{fib\left(\Sigma \mathbb{S}_{h\mathbb{T}} \xrightarrow{tr} \mathbb{S}\right)}_{\Sigma \mathbb{CP}^{\infty}_{-1}}\Big)$

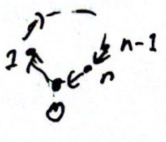
Let G be an $\mathbb{E}_1$- group in $\mathcal{S}$ (equivalently, $G \simeq \Omega Y$)

We write $\mathbb{S}[G] = \Sigma^{\infty}_{+} G$.

Lemma. $THH(\mathbb{S}[G]) = \Sigma^{\infty}_{+} LBG$

Proof. $THH(\mathbb{S}[G]) = \Sigma^{\infty}_{+} |Bar^{cyc}(G)|$

We compare $Bar^{cyc}(G)$ w/ the cyclic object

$Fun(S_\bullet, BG)$, $S_n =$ 

$Fun(S_n, BG)$ is NOT $G^{\times n}$, but $(G^{\times n})_{h(G^{\times n})}$.

So $|Fun(S_\bullet, BG)|$ is obtained from $|Bar^{cyc}(G)|$ by taking orbits under

$\left| \cdots \ G\times G\times G \rightrightarrows G\times G \rightrightarrows G \right| \simeq pt.$

projections

Now $\mathrm{Fun}(S_n, BG) = \mathrm{Map}(|S_n|, BG) \simeq \mathrm{Map}(S^1, BG) \simeq LBG$.

Prop (1) The $\mathbb{T}$-action on $THH(\mathbb{S}[G]) = \Sigma^\infty_+ LBG$ comes from the $\mathbb{T}$-action on LBG given by rotating loops.

(2) The Frobenius $\Sigma^\infty_+ LBG \longrightarrow \Sigma^\infty_+ LBG^{tC_p}$ admits a lift $\Sigma^\infty_+ LBG \to \Sigma^\infty_+ LBG^{hC_p}$ coming from the C_p-invariant map $\underbrace{LBG \longrightarrow LBG}$ precomposing with $\deg_p : S^1 \to S^1$.

(Tate diagonal $\mathbb{S}[G] \to (\mathbb{S}[G]^{\otimes p})^{tC_p}$ lifts through the diagonal $G \to G^{\times p}$) $(\square)$

Prop. For G an $\mathbb{E}_1$-group w/ $\pi_0 G$ a p-group.

$$\begin{tikzcd} \Sigma^\infty_+ LBG \arrow[r, "\varphi_p - id"] \arrow[d, "ev_1"] & \Sigma^\infty_+ LBG \arrow[d, "ev_1"] \\ \Sigma^\infty_+ BG \arrow[r, "id - id = 0"] & \Sigma^\infty_+ BG \end{tikzcd}$$

is a pullback after p-completion

Proof Cofibers of horizontal maps are $(\Sigma^\infty_+ LBG)_{h\mathbb{N}} \simeq \Sigma^\infty_+ (\varphi_p^{-1} LBG)_{h\mathbb{Z}}$

$(\Sigma^\infty_+ BG)_{h\mathbb{N}} \simeq \Sigma^\infty_+ (BG)_{h\mathbb{Z}}$

It's sufficient to check that $\varphi_p^{-1} LBG \to BG$ is an $\mathbb{F}_p$-homology equivalence.

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d] & G \arrow[r, "p"] \arrow[d] & G \arrow[r] \arrow[d] & {} \\ LBG \arrow[r] \arrow[d, "ev_1"] & LBG \arrow[r] \arrow[d, "ev_1"] & LBG \arrow[r] \arrow[d, "ev_1"] & \cdots \\ BG \arrow[r, "id"] & BG \arrow[r, "id"] & BG \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} p^{-1} G \simeq p^{-1} G_0 \arrow[d] \\ \varphi_p^{-1} LBG \arrow[d] \\ BG \end{tikzcd}$$

Now

$H_*(G_0;\mathbb{F}_p)$ is a connected Hopf algebra, and p induces

$$H_*(G_0) \xrightarrow{\Delta} H_*(G_0)^{\otimes p} \xrightarrow{\mu} H_*(G_0)$$

This map is degreewise nilpotent. $\square$

<u>Cor</u>. For G an $\mathbb{E}_1$-group w/ $\pi_0 G$ a p-group,

$$\begin{tikzcd}
TC(\mathbb{S}[G];\mathbb{Z}_p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \left(\Sigma(\Sigma^\infty_+ LBG)_{h\mathbb{T}}\right)^\wedge_p \arrow[d] \\
(\Sigma^\infty_+ BG)^\wedge_p \arrow[r, "0"] & (\Sigma^\infty_+ BG)^\wedge_p
\end{tikzcd}$$

i.e. $$TC(\mathbb{S}[G];\mathbb{Z}_p) \simeq \left(\Sigma^\infty_+ BG \oplus \mathrm{fib}\left(\Sigma(\Sigma^\infty_+ LBG)_{h\mathbb{T}} \xrightarrow{tr} \Sigma^\infty_+ LBG \xrightarrow{ev_1} \Sigma^\infty_+ BG\right)\right)^\wedge_p.$$