

Introduction to Fontaine's thm

Def. $A \rightarrow S$ abelian scheme : smooth proper scheme over S

which is a group object over S s.t. fibers are conn'd.

\Rightarrow all fibers are abelian varieties (over \mathbb{C} : $\simeq (\mathbb{G}/\Lambda)$)

group of points is abelian.

Compact complex Lie group is abelian.

$A[n] = \ker(A \xrightarrow{x^n} A)$ finite over S of degree n^{2g} , $g = \dim A_S$

Etale group scheme over S away from points of char. dividing n .

Thm. $\#A \rightarrow \text{Spec } \mathbb{Z}$ abelian scheme

Or. The only curve over \mathbb{Z} is $\mathbb{P}_{\mathbb{Z}}^1$

($f: C \rightarrow \text{Spec } \mathbb{Z}$ rel dim 1 smooth proper w/ conn'd fibers)

Proof $C \rightsquigarrow \text{Jac}(C) := \text{Pic}^0_{C/S}$ (def. theory ab $\in H^2(X, \mathcal{O})$)

dimension = genus of C

If $g > 0$, contradicts Fontaine's thm.

$g=0$ = "conics".

C/K genus 0 curve, Riemann-Roch W_C^{-2} has 3 sections

$C \hookrightarrow \mathbb{P}_K^2$ as a smooth conic.

$C \mid \alpha$ good reduction at all primes.

Say that C/\mathbb{Q}_p has good reduction at p if there is some $\overset{\text{smooth}}{\ell}/\mathbb{Z}_p$ and an isom.

$$\ell_{C_{\mathbb{Q}_p}} \cong C_{\mathbb{Q}_p}.$$

Upshot: "good reduction is unique" unless $g=0$.

$g=0$: want to show C/\mathbb{Q} , $C \cong \mathbb{P}_{\mathbb{Q}}^1$.

fact. $C \cong \mathbb{P}^1 \Leftrightarrow C(\mathbb{Q}) \neq \emptyset$

Idea, Hasse principle says suffices to show $C(\mathbb{Q}_p) \neq \emptyset$

use good reduction at p .

$$\begin{array}{ccc} \ell & \downarrow & \\ \mathbb{Q}_p & \xrightarrow{\text{smooth}} & \text{has an } \text{Spur } \mathbb{Z}_p \\ \text{Spur } \mathbb{F}_p & & \mathbb{F}_p\text{-point. (Chevalley-Waring thm)} \end{array}$$

use smoothness to lift to a \mathbb{Z}_p -point of ℓ . thus $C(\mathbb{Q}_p) \neq \emptyset$

If $K \neq \mathbb{Q}$, \exists conics w/ good reduction $\ncong \mathbb{P}^1$ @ all finite primes

[e.g. $K = \mathbb{Q}(\sqrt{2})$, and $\{x^2 + y^2 + z^2 = 0\}$ in \mathbb{P}_K^2 : good reduction at all finite places]

$C(\mathbb{Q}_p) \neq \emptyset$, $\left(\frac{-1, -1}{K}\right) = K<i, j>/\langle i^2 + 1, j^2 + 1, ij + ji \rangle$ split at all primes

$$0 \rightarrow Br(K) \rightarrow \bigoplus Br(K_v) \xrightarrow{\sum \text{inv}} \mathbb{Q}/\mathbb{Z} \rightarrow 0 \quad (\text{but not so}).$$

conics \hookrightarrow 2-torsion elts.



Proof of Fontaine's Thm for $g=1$

$p \in A/K$ dim 1, curve of genus 1, cubic in \mathbb{P}_K^2 , $\mathcal{O}(3P_0)$, $E \hookrightarrow \mathbb{P}_K^2$.

$$\text{Claim. } 3y^2 + a_1 zxy + a_3 z^2y = x^3 + a_2 zx^2 + a_4 z^2x + a_6 z^3 \quad (*)$$

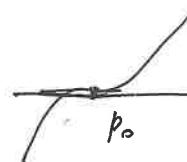
Same as choice of words on \mathbb{P}^2 s.t.

$$(1) P_0 \mapsto [0:1:0]$$

(2) tangent line to E at P_0 is line $\{z=0\}$

(3) $\{z=0\}$ is a flex (tangent of order 3 at P_0)

$$2P_0 + p' \sim H|_E \sim 3P_0 \quad P_0 \sim p' \Rightarrow P_0 = p'$$



\Rightarrow tangent at P_0 is a flex line.

Abstract Weierstrass Models

$$x \mapsto u^2x, y \mapsto u^3y$$

Let equations are modified by $a_i \mapsto u^i a_i$ over any B

then new (x, y) satisfies equation for $u^i a_i$.

$$\begin{aligned} \mathbb{P}_B \left(\underbrace{\mathcal{O}_3 \oplus L^{-2}x \oplus L^{-3}y}_{\mathcal{E}} \right) & \quad 3 \in H^0(B, \mathcal{E}) \\ & \quad x \in H^0(B, \mathcal{E} \otimes L^2) \quad \mathcal{E} = \pi_* \mathcal{O}(1) \\ & \quad y \in H^0(B, \mathcal{E} \otimes L^3) \end{aligned}$$

$$a_i \in H^0(B, L^{\otimes i})$$

$$w \in \mathbb{P}(\mathcal{E}) \quad (\text{but out by } (*) \in H^0(\mathbb{P}, \mathcal{O}(3) \otimes \pi^* L^6))$$

$$\text{Prop. } w \xrightarrow{s} w' \quad \text{then claim: (1) } \varphi: L \xrightarrow{\sim} L' \\ \text{(2) } \psi: \mathcal{O} \oplus L^{-2} \oplus L^{-3} \xrightarrow{\sim} \mathcal{O} \oplus L'^{-2} \oplus L'^{-3}$$

that preserves the flag

$$\mathcal{O}_3 \subset \mathcal{O}_3 \oplus L^{-2}y \subset \mathcal{E}$$

$$\begin{pmatrix} 1 & a & b \\ & \varphi^2 & c \\ & & \varphi^3 \end{pmatrix}$$

$$\text{Proof } \frac{dx}{y} \mapsto u^{-1} \frac{dx}{y}$$

$$\omega_{W/B} = \omega_{\mathbb{P}/B} \otimes N_{W/\mathbb{P}}$$

$$0 \rightarrow \mathcal{L}_{\mathbb{P}/B} \rightarrow \mathcal{E}(-1) \rightarrow \mathcal{O} \rightarrow 0$$

$$\det \mathcal{L}_{\mathbb{P}/B} = \det \mathcal{E} \otimes \mathcal{O}(-3) \\ = \mathcal{O}(-3) \otimes L^{-5}$$

$$\mathcal{N} = \mathcal{O}(3) \otimes L^6 \sim \omega_{W/B} \otimes \mathbb{I}^*$$

$$\omega = \mathbb{I}$$

$$\mathcal{E} \simeq \pi_* \mathcal{O}(3s)$$

$$\omega \rightarrow \mathbb{P}(\pi_* \mathcal{O}(3s))$$

$\Delta \in H^0(B, \mathcal{L}^{\otimes 12})$ whose zeroes are the singular fibers.
integral poly. in a_i

For R a DVR, $K = \text{Frac}(R)$ and $E|K$. Consider all integral Weierstrass models $/R$.

$a \in R$, $\Delta \in R$, ask to find $\min \text{val}(\Delta)$.

$\min = 0 \iff \text{good reduction.}$

Thm. \min Weierstrass models $/ \hat{\mathcal{O}_{C,x}}$ are unique

If C is a Dedekind scheme, $\exists! W \rightarrow C$ whose completion at $x \in C$ is minimal

Weierstrass. * But $L \neq 0$ in general.

Upshot: K no. field, \mathcal{O}_K ring of integers. If $\text{cl}(\mathcal{O}_K) = 1$ (e.g. \mathbb{Z})

$L = \emptyset \Rightarrow$ Weierstraß eq'n w/ $a_i \in \mathbb{Z}$

s.t. $\Delta(a_i)$ is minimal at all p .

Part. If $\{\$ good reduction at every $P \Rightarrow \Delta = \pm 1$

