Deformations (a) : tangent and obstruction spaces

Martin Olsson

Lecture 1.

Week 1

1) basic def'n
2) examples
3) obstruction spaces
4) examples

Week 2

5) Picard categories
6) Picard stacks
7) truncated cotangent complex
8) Overview of cotangent complex.

Motivation.

$k = \bar{k}$

X/k scheme of finite type

$x \in X(k)$.

The tangent space of X at x is the dual of the k-vec. sp. $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the max. ideal.

Dual numbers

R ring, I R-module.

$R[I]$ ring of dual numbers

$R[I]$: as a group, $R \oplus I$

$$(r, i) \cdot (r', i') := (rr', r'i + ri')$$

$$\begin{tikzcd} (r,0) & R[I] \arrow[r, two heads, "\pi"] & R & (r,i) \mapsto r \\ r \arrow[u, mapsto] & R \arrow[u] \arrow[ur, "id"'] & & \end{tikzcd}$$

Remark 1. $R[I]$ is functorial in I:

$g : I \to J$ induces a map

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ R[I] \arrow[rr] \arrow[dr] & & R[J]. \arrow[dl] \\ & R & \end{tikzcd}$$

$(r, i) \mapsto (r, g(i))$

Rmk 2. $I = R$, write $R[\varepsilon]$ for $R[I]$

(really should be $R[\varepsilon]/(\varepsilon^2)$)

<u>Remark 3</u>. X top. space, $\mathcal{O}$ sheaf of rings on X, I $\mathcal{O}$-module, then can define $\mathcal{O}[I]$.

In particular, if X is a scheme, I q.coh. $\mathcal{O}_X$-module then get a ringed space

$$X[I] := (|X|, \mathcal{O}_X[I]).$$

<u>Exer</u>. Show $X[I]$ is a scheme.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, equal] & X[I] \arrow[d] \\ & X \end{tikzcd}$$

<u>Relationship w/ derivatives</u>

$A \to R$ ring homomorphism

M R-module

an <u>A-derivation from R to M</u> is an A-linear map $\partial : R \to M$

s.t. $\partial(xy) = x\partial y + y \partial x$

$\rightsquigarrow$ R-module $\mathrm{Der}_A(R, M)$

$A\text{-}\mathrm{Alg}/R$ = cat. of pairs (C, f)

- C A-algebra
- $f : C \to R$ map of A-algebras

$(C, f) \to (C', f')$ is an A-alg. morphism $g : C \to C'$ s.t.

$$\begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & C' \arrow[dl, "f'"] \\ & R & \end{tikzcd}.$$

<u>Lemma</u>. For any A-derivation $\partial : R \to I$, the induced map

$$R \to R[I], \quad x \mapsto x + \partial(x) = (x, \partial(x))$$

is a morphism in $A\text{-}\mathrm{Alg}/R$ and the induced map

$$\mathrm{Der}_A(R, I) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}\mathrm{Alg}/R}(R, R[I])$$ is bijective.

R is viewed as in $A\text{-}Alg/R$ by
$$\begin{tikzcd} R \arrow[r, "id"] & R \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

Pf $\quad R \xrightarrow{s} R[I]$ in $A\text{-}Alg/R$

$x \mapsto (x, \delta(x))$

- map of A-algebras $\iff$ $\delta(x) = 0$ if x is in the image of A
- comp. w/ mult. $\iff$ $x, y \in R$, $(xy, \delta(xy)) = (x, \delta(x)) \cdot (y, \delta(y)) = (xy, x\delta(y) + y\delta(x))$

$\iff$ $\delta(xy) = y\delta(x) + x\delta(y)$. $\square$

Remark. $(f : C \twoheadrightarrow R) \in A\text{-}Alg/R$ and that $I = \ker(f)$ is square-zero. Then any section $s : R \to C/A$ induces an isom.
$$\begin{tikzcd} R[I] \arrow[rr, "\sim", "{(r,i) \mapsto s(r)+i}"'] \arrow[dr] & & C \arrow[dl] \\ & R & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ I \arrow[r, "id"] \arrow[d] & I \arrow[d] \\ R[I] \arrow[r, "\sim"] \arrow[d] & C \arrow[d] \\ R \arrow[r, equal] \arrow[d] & R \arrow[d] \\ 0 & 0 \end{tikzcd}$$

Special case. $C = R \otimes_A R / J^2$, $J = \ker(R \otimes_A R \xrightarrow{\cdot} R)$

$$\begin{tikzcd} C \arrow[d, "f"] \\ R \end{tikzcd}$$

$I = J/J^2 = \Omega^1_{R/A}$

$s : R \to C$

$x \mapsto x \otimes 1$

s induces an isom. $R \otimes_A R / J^2 \simeq R[\Omega^1_{R/A}]$.

$\Rightarrow \mathrm{Der}_A(R, \Omega^1_{R/A}) \xrightarrow{\sim}$ sections of the diagonal map $R \otimes_A R / J^2 \to R$.

Q. What is the universal der. $R \xrightarrow{d} \Omega^1_{R/A}$?

$\Omega^1_{R/A} = J/J^2$, $\quad d : R \to \Omega^1_{R/A} = J/J^2$

$x \mapsto x \otimes 1 - 1 \otimes x$

$1 \otimes x = x \otimes 1 + (1 \otimes x - x \otimes 1)$

$$\begin{tikzcd} R[\Omega^1_{R/A}] \arrow[r] & R \otimes_A R / J^2 \\ R \arrow[u, "sd"] & \end{tikzcd}$$

$(x, 1 \otimes x - x \otimes 1) \mapsfrom 1 \otimes x$

$sd(x) = (x, dx)$

The tangent space of a functor

Mod_R = Cat. of f.g. R-modules

$H : \mathrm{Mod}_R \longrightarrow \mathrm{Set}$ functor

commutes w/ finite product $\left[H(I \times J) \xrightarrow{\sim} H(I) \times H(J) \right]$

Prop. H factors canonically

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[rr, "H"] \arrow[dr, "\bar{H}"'] & & \mathrm{Set} \\ & (R\text{-modules}) \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

Sketch of proof. additive structure: $I \times I \longrightarrow I$, $(i, j) \longmapsto i + j$

$$\begin{tikzcd} H(I) \times H(I) & H(I \times I) \arrow[l, "\sim"'] \arrow[r, "\Sigma"] & H(I) \\ \arrow[from=1-1, to=1-3, bend right, "+"'] \end{tikzcd}$$

Mult. structure: $f \in R$,

$$\cdot f : H(I) \xrightarrow{H(\times f)} H(I).$$

$\square$

$A \longrightarrow R$ ring homo.

$A\text{-}\mathrm{Alg}/R$ has finite products

$$\begin{tikzcd} C \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & R & \end{tikzcd} \qquad (C, f) \times (C', f') = \left(C \times_R C', \ \begin{array}{c} (x, y) \\ \mathrm{I} \\ f(x) = f'(y) \end{array} \right)$$

Lemma. The functor $\mathrm{Mod}_R \longrightarrow A\text{-}\mathrm{Alg}/R$, $I \longmapsto (R[I], \ \pi : R[I] \longrightarrow R)$ commutes w/ finite products.

Pf: $I, J \in \mathrm{Mod}_R$,

$$\begin{tikzcd} R[I \times J] \arrow[rr, "\sim"'] \arrow[dr] & & R[I] \times_R R[J] \arrow[dl] \\ & R & \end{tikzcd}$$

is an isom.

Cor. $F: A\text{-}Alg/R \to Set$ s.t. for $I, J \in Mod_R$, the map

$$F(R[I] \times_R R[J]) \xrightarrow{\sim} F(R[I]) \times F(R[J]) \quad \text{is an isom.}$$

then $\forall\, I \in Mod_R$, the set $F(R[I])$ has a canonical R-module str.

Reason: $F(R[I])$ is the image of I under

$$\begin{tikzcd} Mod_R \arrow[r] \arrow[rr, bend right, "\text{commutes w/ fin. products}"'] & A\text{-}Alg/R \arrow[r, "F"] & Set \end{tikzcd}$$

$I \longmapsto R[I]$

Def Let $F: A\text{-}Alg/R \to Set$ be a functor satisfying cond. in Cor.. then the tangent space of F, denoted T_F, is the R-module $F(R[\varepsilon])$.

Rmk. enough that F def. on full sub cat. $\mathcal{C} \subset A\text{-}Alg/R$ closed under fin. products and contains $R[I]$'s.

Lecture 2. $A \to R$

$A\text{-}Alg/R$: cat. of diagrams

$$\begin{tikzcd} C \arrow[r, "f"] & R \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

$F: A\text{-}Alg/R \to Set$ functor.

and if $\forall\, I, J \in Mod_R$, the nat'l map

$$F(R[I \oplus J]) \xrightarrow{\sim} F(R[I]) \times F(R[J]), \text{ then get tangent space } T_F.$$

[in fact, $\forall I$, $F(R[I])$ is an R-module and $T_F := F(R[\varepsilon])$].

$$+: F(R[\varepsilon]) \times F(R[\varepsilon]) \longrightarrow F(R[\varepsilon])$$

$$\cong \quad F\left(\frac{R[\varepsilon_1, \varepsilon_2]}{(\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2)}\right) \nearrow \quad \varepsilon_i \mapsto \varepsilon$$

$\times f$: $F(R[\varepsilon]) \to F(R[\varepsilon])$ is induced by $R[\varepsilon] \to R[\varepsilon]$

$a + b\cdot\varepsilon \mapsto a + fb\varepsilon$.

Problem 1. R ring, $X \xrightarrow{g} \operatorname{Spec} R$ separated, smooth.

Consider the functor $\mathrm{Def}_X : \mathrm{Alg}/R \longrightarrow \mathrm{Set}$

$$\mathrm{Alg}/R = (\mathbb{Z}\text{-}\mathrm{Alg}/R)$$

$\mathrm{Def}_X(C \xrightarrow{f} R)$ = set of isom. classes of cartesian diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & X_C \arrow[d, "g_C"] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

w/ g_C smooth.

morphism of diagrams : arrow $h: X_C' \to X_C$ s.t.

$$\begin{pmatrix} X \to X_C' \\ \downarrow \quad \downarrow \\ \operatorname{Spec} R \to \operatorname{Spec} C \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} X \to X_C \\ \downarrow \quad \downarrow \\ \operatorname{Spec} R \to \operatorname{Spec} C \end{pmatrix}$$

$$\begin{tikzcd} & & X_C' \arrow[dl, "h"'] \arrow[ddl, bend left] \\ X \arrow[r] \arrow[urr, bend left] \arrow[d] & X_C \arrow[d] & \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} C & \end{tikzcd}$$

✓ commutes

(*)

Rmk. If $C = R[I]$ for some R-module I, then any morphism h as in (1) is an isom.

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_R \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_{X_C} \arrow[r] \arrow[d, "h^*"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \otimes_R \mathcal{O}_X \arrow[r] & \mathcal{O}_{X_C'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

all taking place on $|X|$.

Prop: $\forall\, I, J \in \mathrm{Mod}_R$, $\mathrm{Def}_X(R[I \oplus J]) \xrightarrow{\sim} \mathrm{Def}_X(R[I]) \times \mathrm{Def}_X(R[J])$ is an isom.

Proof in Brian's lecture 3.

How to compute $T\mathrm{Def}_X$ or more generally the R-module $\mathrm{Def}_X(R[I])$?

Special case: X affine

Facts: (1) $\mathrm{Def}_X(R[I])$ consists of one element.

(2) For any deformation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}\, R \arrow[r] & \mathrm{Spec}\, R[I] \end{tikzcd}$$

the set of maps $h: X' \to X'$ as in $(*)$

is in canonical bijection w $H^0(X, T_X \otimes I)$.

Why is there a lifting?

$X = \mathrm{Spec}\, R[x_1, \dots, x_n]/(f_1, \dots, f_l)$,

In fact, $X[I] \to \mathrm{Spec}\, R[I]$ is a smooth lifting

Reason. $X = \mathrm{Spec}\, B$, and $X \hookrightarrow X[I]$ be a smooth lifting.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] \arrow[dd, "g"'] & X[I] \arrow[d, "\cong"] \arrow[dd, bend left] \\ & X' \arrow[d] \\ \mathrm{Spec}\, R \arrow[r] & \mathrm{Spec}\, R[I] \end{tikzcd}$$

$$\begin{tikzcd} B \\ R \arrow[u, "\text{smooth}"] \end{tikzcd} \Rightarrow \begin{tikzcd} B[I] & B \arrow[l] \\ R[I] \arrow[u, "\text{smooth}"] & R \arrow[l] \arrow[u] \end{tikzcd}$$

Recall:

$$\begin{tikzcd} T_0 \arrow[r, "f_0"] \arrow[d, "j"'] & Y \arrow[d] \\ T \arrow[r] \arrow[ur, dashed, "f"] & S \end{tikzcd}$$

j closed immersion def. by a square-zero ideal J then the set of arrows f filling in the diagram is a pseudo-torsor under $\mathrm{Hom}(f_0^* \Omega^1_{Y/S}, J)$.

pseudo-torsor: either no arrow exists or if an arrow exists, then there is a simply transitive action of $\mathrm{Hom}(f_0^* \Omega^1_{Y/S}, J)$ on the set of arrows.

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr] \arrow[ddr, bend right] & X' \arrow[d, dashed] \\
& X' \arrow[d] \\
& \operatorname{Spec} R[I]
\end{tikzcd}$$

$$\begin{tikzcd}
& T \arrow[rr, bend left, "(b_1, b_2)"] & & \\
T_0 \arrow[ur] \arrow[rr] & & Y \arrow[r] & Y \times_S Y \arrow[d] \\
& & & S
\end{tikzcd}$$

$Y = \operatorname{Spec} B$

$T = \operatorname{Spec} A$

$T_0 = \operatorname{Spec} A_0$

$S = \operatorname{Spec} V$

$$\begin{tikzcd}
& A \arrow[dl] & & \\
A_0 & B \arrow[l] & B \otimes_V B / I_\Delta^2 \arrow[l] \arrow[ul, bend right, "b_1 \otimes b_2"'] & \\
& & V \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
I_\Delta / I_\Delta^2 \arrow[r] \arrow[d, equal] & J \\
A_0 \otimes_B \Omega^1_{Y/S} &
\end{tikzcd}$$

For general $X \to \operatorname{Spec} R$, this also shows that $(X[I] \to \operatorname{Spec} R[I]) \in \mathrm{Def}_X(R[I])$

Choose a covering $X = \bigcup_i U_i$ w/ each U_i affine.

Choose for each i a smooth lifting $U_i' \to \operatorname{Spec} R[I]$ of U_i.

$\mathcal{U} = \{U_i\}$. Want to patch U_i' to a lifting of X.

$X \hookrightarrow X' \quad \longleftrightarrow \quad \mathcal{O}_{X'} \xrightarrow{I \otimes_R \mathcal{O}_X} \mathcal{O}_X \quad$ on $|X|$.

$U_i' \to \operatorname{Spec} R[I] \quad \longleftrightarrow \quad \mathcal{O}_{U_i'} \xrightarrow{I \otimes_R \mathcal{O}_{U_i}} \mathcal{O}_{U_i} \quad$ on $|U_i|$.

on $U_{ij} = U_i \cap U_j$ get a diagram

$$\begin{tikzcd}
& U_i'|_{U_{ij}} \arrow[d, dashed, "\exists"] \\
U_{ij} \arrow[ur] \arrow[r] & U_j'|_{U_{ij}}
\end{tikzcd}$$

two elements of $\mathrm{Def}_{U_{ij}}(R[I])$.

Pick $\forall i$ an isom. $\sigma_i : U_i' \xrightarrow{\sim} U_i[I]$.

Note: Any other choice of σ_i is given by composing w/ an automorphism of $U_i[I]$

$$\begin{gathered} U_i[I] \\ \Updownarrow \\ H^0(U_i, T_X \otimes I). \end{gathered}$$

~~Look at U_{ijk}:~~

$$\begin{tikzcd} & U_i'|_{U_{ijk}} \arrow[d, "\sigma_{ij}"] \\ U_{ijk} \arrow[ur] \arrow[r] \arrow[dr] & U_j'|_{U_{ijk}} \arrow[d] \\ & U_k'|_{U_{ijk}} \end{tikzcd}$$

How to specify X'.

There is an obstruction for the σ_i's to glue to an isom. $X' \to X[I]$

$$x_{ij} : \begin{tikzcd} U_{ij}[I] \arrow[r, "\sigma_j^{-1}"] \arrow[rr, bend right, "x_{ij}"'] & U_{ij}' \arrow[r, "\sigma_i"] & U_{ij}[I] \end{tikzcd}$$

$$x_{ij} \in H^0(U_{ij}, T_{U_{ij}} \otimes I)$$

Lemma. $x_{ik} = x_{ij} + x_{jk}$ in $H^0(U_{ijk}, T_X \otimes I)$

Pf.
$$\begin{tikzcd} U_{ijk}[I] \arrow[r, "\sigma_k^{-1}"] \arrow[rr, bend left, "x_{jk}"] \arrow[rrrr, bend right, "x_{ik}"'] & U_{ijk}' \arrow[r, "\sigma_j"] & U_{ijk}[I] \arrow[r, "\sigma_j^{-1}"] \arrow[rr, bend left, "x_{ij}"] & U_{ijk}' \arrow[r, "\sigma_i"] & U_{ijk}[I] \end{tikzcd}$$
commutes. $\square$

Cor. The $\{x_{ij}\}$ define a Čech cocycle.

$\rightsquigarrow [X'] \in \check{H}^1(X, T_X \otimes I) = H^1(X, T_X \otimes I)$

$[X'] = 0 \Rightarrow \exists\, \partial_i \in H^0(U_i, T_X \otimes I)$ s.t. $\sigma_i - \sigma_j = x_{ij} = \partial_j - \partial_i$

Thm. The map

$$\mathrm{Def}_X(R[I]) \xrightarrow{\sim} H^1(X, T_X \otimes J), \quad X' \mapsto [X']$$

is an R-module isom.

Lecture 3. Obstruction theories

$\pi : A' \to A$ surjection of rings, $I = \ker(\pi)$ square-zero ideal. (A-module)

$g : X \to \operatorname{Spec} A$ smooth, separated scheme.

Problem. Understand liftings

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\boxtimes"] & X' \arrow[d, "g'"] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd} \quad \leftarrow \text{smooth}$$

$$\mathrm{Def}_X : \mathrm{Alg}/A \longrightarrow \mathrm{Set}$$

$$(C \xrightarrow{f} A) \mapsto \left\{ \begin{tikzcd} X \arrow[r] \arrow[d] & X_C \arrow[d, "\text{smooth}"] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C \end{tikzcd} \right\} / \simeq$$

Yesterday: $T_{\mathrm{Def}_X} = H^1(X, T_{X/A})$

$(\mathrm{Def}_X(A')) = H^1(X, T_X \otimes I)$

When X is affine,

1) $\exists$ lifting $X' \to \operatorname{Spec} A'$

2) any two liftings are isomorphic

3) the group of automorphisms of any lifting $X' \to \operatorname{Spec} A'$ is canonically isom. to $H^0(X, T_X \otimes I)$.

For general X, if $X' \to \operatorname{Spec} A'$ is a smooth lifting, I get a bijection

$$\mathrm{Def}_X(A' \to A) \underset{\sim}{\xrightarrow{\varphi_{X'}}} H^1(X, T_X \otimes I).$$

$$[X''] \longmapsto [\{x_{ij}\}] = \mathrm{Isom}(X', X'')$$

[Yesterday the fixed lifting was $X[I] \to \operatorname{Spec} A[I]$]

Def of $\varphi_{X'}$

Cover $X = \bigcup_i U_i$, U_i affine, $X'' \in \mathrm{Def}_X(A')$, $\forall i$, $U_i \hookrightarrow U_i''$, $U_i \hookrightarrow U_i'$, $\sigma_i : U_i'' \to U_i'$; choose $\sigma_i : U_i'' \to U_i'$, $\forall i$.

This gives $\forall i,j$, $\begin{tikzcd} U'_{ij} \arrow[r, "\sigma_j^{-1}"] \arrow[rr, bend right, "x_{ij}"'] & U''_{ij} \arrow[r, "\sigma_i"] & U'_{ij} \end{tikzcd}$.

$x_{ij} \in H^0(U_{ij}, T_X \otimes I)$

Another way to say it: X', X'' get a sheaf $\underline{\mathrm{Isom}}(X', X'')$ on $|X|$.

$$(U \subset X) \longmapsto \left\{ \begin{tikzcd} & U'' \arrow[d, "h"] \\ U \arrow[ur] \arrow[r] & U' \end{tikzcd} \right\}$$

torsor under $T_X \otimes I$

$\updownarrow$

$H^1(X, T_X \otimes I)$

Q: When $\exists\, X' \to \operatorname{Spec} A'$?

Let $\mathcal{U} = \{U_i\}$ be a covering of X by affines. Fix liftings $U'_i \to \operatorname{Spec} A'$.

$\forall i,j$, choose an isom. $\varphi_{ji}: U'_i|_{U_{ij}} \to U'_j|_{U_{ij}}$.

$$\begin{tikzcd} U'_i|_{U_{ijk}} \arrow[rr, "\varphi_{ki}"] \arrow[dr, "\varphi_{ji}"'] & & U'_k|_{U_{ijk}} \\ & U'_j|_{U_{ijk}} \arrow[ur, "\varphi_{kj}"'] & \end{tikzcd}$$

$\partial_{ijk} := \varphi_{ki}^{-1} \circ (\varphi_{kj} \circ \varphi_{ji})$

$\rotatebox{90}{$\in$}$

$H^0(U_{ijk}, T_X \otimes I)$

Lemma (i) $\{\partial_{ijk}\}$ is a Čech 2-cocycle.

(ii) If φ'_{ji} is a second choice of isom's, $\rightsquigarrow \{\partial'_{ijk}\}$,

then $\{\partial_{ijk}\} - \{\partial'_{ijk}\}$ is a Čech boundary.

$\rightsquigarrow o(g) \in H^2(X, T_X \otimes I)$.

Prop. $\exists$ lifting $X' \xrightarrow{g'} \operatorname{Spec} A'$ of g iff $o(g) = 0$.

Summary a) $\exists$ an obstruction $o(g) \in H^2(X, T_X \otimes I)$ s.t. $o(g) = 0 \iff \mathrm{Def}_X(A' \to A) \neq \emptyset$.

b) If $o(g) = 0$, then the set of isom. classes of liftings form a torsor under $H^1(X, T_X \otimes I)$.

c) For any lifting of g, the gp of auto's is canonically isom. to $H^0(X, T_X \otimes I)$.

Λ ring, $\quad F : \Lambda\text{-Alg} \to \text{Set}$

<u>Def</u>. An obstruction theory for F consists of the following data.

(i) $\forall$ morphism $A \twoheadrightarrow A_0$ of Λ-algs w/ kernel a nilp. ideal and A_0 reduced, and $a \in F(A)$, a functor $\mathcal{O}_a : (\text{f. type } A_0\text{-modules}) \to (\text{f. type } A_0\text{-modules})$.

(ii) $\forall$ diagram $A' \twoheadrightarrow A \twoheadrightarrow A_0$ and $a \in F(A)$, where $A' \to A$ surj., $\ker(A' \to A) =: J$ annihilated by $\ker(A' \to A_0)$. a class $o(a) \in \mathcal{O}_a(J)$, which is zero $\Leftrightarrow$ a lifts to $F(A')$.

deformation situation

This should be functorial in the nat'l way.

<u>Example</u>. $X \overset{j}{\hookrightarrow} X'$ closed immersion defined by a square-zero ideal J.

L l.b. on X.

<u>Problem</u>. Understand liftings of L to L' on X'.

lifting of L to X'?

a pair (L', τ), L' l.b. on X', and $\tau : j^*L' \xrightarrow{\sim} L$ on X.

$(L', \tau) \simeq (L'', \varepsilon)$ if $\exists$ $\sigma : L' \xrightarrow{\sim} L''$ s.t.
$$\begin{tikzcd} j^*L' \arrow[rr, "\sigma", "\sim"'] \arrow[dr, "\tau"'] & & j^*L'' \arrow[dl, "\varepsilon"] \\ & L & \end{tikzcd}$$
commutes.

$g \mapsto 1+g$

$0 \to J \to \mathcal{O}_{X'}^{\times} \to \mathcal{O}_X^{\times} \to 0$ seq. of sheaves on $|X|$

$(1+g)(1+f) = 1 + (g+f) + \underset{0}{\cancel{gf}}$

$$0 \to H^0(J) \to H^0(\mathcal{O}^*_{X'}) \to H^0(\mathcal{O}^*_X) \to H^1(J) \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \xrightarrow{\partial} H^2(J)$$

$$[\mathcal{L}] \in \mathrm{Pic}(X)$$

If $(\mathcal{L}',\iota)$, $(\mathcal{L}',\varepsilon)$, then there need not exist isom. $(\mathcal{L}',\iota) \Rightarrow (\mathcal{L}',\varepsilon)$.

Assume $H^0(\mathcal{O}^\times_{X'}) \twoheadrightarrow H^0(\mathcal{O}^\times_X)$ surj.

<u>Prop</u> a) $\exists$ obstruction $o(\mathcal{L}) \in H^2(X,J)$ which is $0 \Leftrightarrow \exists\, (\mathcal{L}',\iota)$.

$$o(\mathcal{L}) = \partial[\mathcal{L}]$$

b) If $o(\mathcal{L}) = 0$, then the set of isom. classes of liftings $(\mathcal{L}',\iota)$ is a torsor under $H^1(X,J)$.

c) $\forall$ lifting, the gp of autom. is can. in bijection w/ $H^0(X,J)$.

<u>Lecture 4</u>. $A' \to A$ surj. map of rings, w/ square-zero kernel J.

$$\begin{array}{c} P' \\ \downarrow \\ \mathrm{Spec}\,A' \end{array} \quad \text{smooth scheme w/ reduction} \quad \begin{array}{c} P \\ \downarrow \\ \mathrm{Spec}\,A \end{array}$$

and
$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "\text{smooth}"'] & P \arrow[d] \\ & \mathrm{Spec}\,A \end{tikzcd}$$

<u>Problem</u>: Understand how we can lift to a diagram
$$\begin{tikzcd} X' \arrow[r] \arrow[dr, "\text{smooth}"'] & P' \arrow[d] \\ & \mathrm{Spec}\,A' \end{tikzcd}$$

$$\begin{tikzcd} J\otimes\mathcal{O}_X \arrow[r, dashed] & \mathcal{O}_{X'} \arrow[r, dashed] & \mathcal{O}_X \\ & j^{-1}\mathcal{O}_{P'} \arrow[u, dashed] \arrow[r] & j^{-1}\mathcal{O}_P \arrow[u] \\ J \arrow[r] & A' \arrow[u] \arrow[r] & A \arrow[u] \arrow[uu, bend right] \end{tikzcd}$$

on $|X|$.

local problem on $|X|$.

$\mathcal{L}$ sheaf on $|X|$ which to any open $U \subset X$ associates the set of diagrams

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "j"'] & U' \arrow[d, "j'"] \arrow[dd, bend left=60, "\text{smooth}"] \\ P \arrow[r] \arrow[dr] & P' \arrow[d] \\ & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd}$$

What are the global sections?

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "j"'] & U' \arrow[d, "j"] \\ P \arrow[r, hook, "i"] \arrow[d] & P' \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd}
\qquad
\begin{tikzcd} & U' \arrow[d, dashed] \arrow[dd, bend left] \\ U \arrow[ur] \arrow[r, "i \circ j"] \arrow[dr, bend right] & P' \arrow[d] \\ & \operatorname{Spec} A' \end{tikzcd}$$

The set of arrows filling in the diagram form a torsor under $\operatorname{Hom}((i \circ j)^* \Omega^1_{P'/A'}, J \otimes \mathcal{O}_U)$

$= \operatorname{Hom}(j^* \Omega^1_{P/A}, J \otimes \mathcal{O}_U)$

$= j^* T_{P/A} \otimes_A J$

There is an action of $j^* T_{P/A} \otimes J$ on $\mathcal{L}$.

conormal bundle $:= j^* I$, $I \subset \mathcal{O}_P$ ideal of X

$= I/I^2 = N^\vee$

$$0 \to I/I^2 \xrightarrow{d} j^* \Omega^1_{P/A} \to \Omega^1_{X/A} \to 0$$

$$0 \to T_{X/A} \to j^* T_{P/A} \to N \to 0$$

$$0 \to T_{X/A} \otimes J \to j^* T_{P/A} \otimes J \to N \otimes J \to 0$$

Claim. $T_{X/A} \otimes J$ acts trivially on $\mathcal{L}$.

a section $\partial \in T_{X/A} \otimes J(U)$ corresponds to a diagram

$$\begin{tikzcd} & U' \arrow[d, "\partial"] \arrow[ddd, bend left=60] \\ U \arrow[ur] \arrow[r, hook] \arrow[d] & U' \arrow[d, "j'"] \\ P \arrow[r] \arrow[d] & P' \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd}
\qquad
\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[rr, bend left] & U' \arrow[r, "\partial"] \arrow[d, "j' \circ \partial"] & U' \arrow[dl, bend left, "j'"] \\ P \arrow[r, hook] & P' \end{tikzcd}$$

Proof.

So we get an action of $\mathcal{N} \otimes J$ on $\mathcal{L}$.

<u>Prop</u> $\mathcal{L}$ is a torsor under $\mathcal{N} \otimes J$.

<u>torsor</u>. a) $\forall\ U \subset X$, $\exists$ covering $U = \bigcup U_i$ s.t. $\mathcal{L}(U_i) \neq \phi$

b) $\forall\ U \subset X$, either $\mathcal{L}(U) = \phi$, or the action of $\mathcal{N} \otimes J(U)$ on $\mathcal{L}(U)$ is simply transitive.

<u>Sketch of pf</u>

Check that if U is affine, then action of $\mathcal{N} \otimes J(U)$ on $\mathcal{L}(U)$ is simply transitive.

$$0 \to T_{X/A} \otimes J(U) \to j^* T_{P/A} \otimes J(U) \to \mathcal{N} \otimes J(U) \to 0$$

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[dd, bend left] \\ P \arrow[r] \arrow[d] & P' \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd}$$

<u>General fact</u>. If G is a sheaf of abelian groups, then the set of isom. classes of G-torsors on $|X|$ are in can. bijection w/ $H^1(X, G)$.

In particular, $\mathcal{L} \longleftrightarrow [\mathcal{L}] \in H^1(X, \mathcal{N} \otimes J)$.

In our case, choose a covering of $X = \bigcup_i U_i$ w/ U_i affines, and $s_i \in \mathcal{L}(U_i)$.

On $U_i \cap U_j$, get two sections $s_i|_{U_{ij}}, s_j|_{U_{ij}} \in \mathcal{L}(U_{ij})$.

Action of $\mathcal{N} \otimes J(U_{ij})$ on $\mathcal{L}(U_{ij})$ is simply transitive

$\Rightarrow \exists!\ x_{ij} \in \mathcal{N} \otimes J(U_{ij})$ s.t. $x_{ij} * s_i|_{U_{ij}} = s_j|_{U_{ij}}$

<u>Check</u> $\{x_{ij}\}$ is a Čech 1-cocycle. $\rightsquigarrow H^1(X, \mathcal{N} \otimes J)$.

$\mathcal{L}$ trivial $\Leftrightarrow$ $\mathcal{L}(X) \neq \phi$ $\Leftrightarrow$ $[\mathcal{L}] \in H^1(X, \mathcal{N} \otimes J)$ is zero.

<u>Summary</u> (i) $\exists$ canonical obstruction $o(j) \in H^1(X, \mathcal{N} \otimes J)$,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & X' \arrow[d, "j'"] \\ P \arrow[r] \arrow[d] & P' \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd} \quad g$$

whose vanishing is nec. suff. for existence of a lifting of j.

(ii) The set of liftings j' of j form a torsor under

$H^0(X, \mathcal{N} \otimes J)$ if $o(j) = 0$.

<u>Rmk</u>

$$0 \to T_{X/A} \to j^* T_{P/A} \to \mathcal{N} \to 0$$

induces

$$H^0(X, \mathcal{N} \otimes J) \to H^1(X, T_{X/A} \otimes J) \to H^1(X, j^* T_{P/A} \otimes J) \to$$

$$\to H^1(X, \mathcal{N} \otimes J) \xrightarrow{\delta} H^2(X, T_{X/A} \otimes J)$$

What is $\delta(o(j)) = ?$ $\quad o(g)$!

<u>Ex.</u> P smooth proper surface $/k$. $X \subset P$ smooth rat'l curve w/ $X \cdot X = -1$.

[Hartshorne, V.1.4.1] $\quad \deg \mathcal{N} = -1$

$H^1(X, \mathcal{N} \otimes J) = 0$, $\quad H^0(X, \mathcal{N} \otimes J) = 0$.

$$\begin{tikzcd} & & X[\varepsilon] \arrow[dl] & & \\ & P[\varepsilon] \arrow[dl] \arrow[d] & & P \arrow[ll] \arrow[d] & X \arrow[l] \arrow[d] \arrow[ull] \\ P \arrow[d] & \operatorname{Spec} k[\varepsilon] \arrow[dl] & & \operatorname{Spec} k \arrow[ll] \arrow[l, leftarrow] & \\ \operatorname{Spec} k & & & & \end{tikzcd}$$

Lecture 5

<u>Picard cat.</u> is a groupoid P together w/ the following extra str.

(a) A functor $+ : P\times P \longrightarrow P$

(b) An isom. of functors

$$\begin{tikzcd}
 & P\times P\times P \arrow[dl, "+\times 1"'] \arrow[dr, "1\times +"] & \\
P\times P \arrow[dr, "+"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & P\times P \arrow[dl, "+"] \\
 & P &
\end{tikzcd}$$

$\sigma_{x,y,z} : (x+y)+z \xrightarrow{\sim} x+(y+z)$

(c) A nat'l transf.

$$\begin{tikzcd}
P\times P \arrow[rr, "\mathrm{flip}"] \arrow[dr, "+"'] & \overset{\tau}{\Rightarrow} & P\times P \arrow[dl, "+"] \\
 & P &
\end{tikzcd}$$

$\tau_{x,y} : x+y \xrightarrow{\sim} y+x.$

(0) $\forall x\in P$, the functor $P\to P,\ y\mapsto x+y$ is an equiv.

(i) (Pentagon axiom)

$$\begin{tikzcd}
 & (x+y)+(z+w) \arrow[dl, dash, "\sigma_{x,y,z+w}"'] \arrow[dr, dash, "\sigma_{x+y,z,w}"] & \\
x+(y+(z+w)) \arrow[d, dash, "\sigma_{y,z,w}"'] & /// & ((x+y)+z)+w \arrow[d, dash, "\sigma_{x,y,z}"] \\
x+((y+z)+w) \arrow[rr, dash, "\sigma_{x,y+z,w}"] & & (x+(y+z))+w
\end{tikzcd}$$

(ii) $\tau_{x,x} = \mathrm{id},\ \forall x\in P$

(iii) $\forall x,y\in P,\ \tau_{x,y}\circ\tau_{y,x} = \mathrm{id}.$

(iv) (Hexagon axiom)

$$\begin{tikzcd}
x+(y+z) \arrow[r, dash, "\tau"] \arrow[d, dash, "\sigma"'] & x+(z+y) \arrow[d, dash, "\sigma"] \\
(x+y)+z \arrow[d, dash, "\tau"'] \arrow[r, phantom, "//"] & (x+z)+y \arrow[d, dash, "\tau"] \\
z+(x+y) \arrow[r, dash, "\sigma"] & (z+x)+y
\end{tikzcd}$$

Ex. X scheme

$\mathrm{Pic}(X)$ groupoid of line bundles on X

$$\otimes : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$$

Example. $f : X \to Y$ morphism of schemes

I qcoh $\mathcal{O}_X$-module.

An $\underline{I\text{-ext'n of } X \text{ over } Y}$ is a diagram
$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & X' \arrow[dl, "f'"] \\ Y & \end{tikzcd}$$

where j is square-zero, together w/ an isom. $I \xrightarrow[\sim]{\iota} \ker(\mathcal{O}_{X'} \to \mathcal{O}_X)$

Let $\underline{\mathrm{Exal}}_Y(X, I)$ = cat. of I-extensions of X over Y.

Rmk a)
$$\begin{tikzcd} I \arrow[r, hook] & \mathcal{O}_{X'} \arrow[r, dashed] & \mathcal{O}_X \\ & f^{-1}\mathcal{O}_Y \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

b) If $A \to B$ is a morphism of sheaves of algs on a top. space T, and I is a B-module, get cat $\underline{\mathrm{Exal}}_A(B, I)$.

Obs. $\underline{\mathrm{Exal}}_Y(X, I)$ is a groupoid.

$$\begin{tikzcd} & X_2' \arrow[d, "h"] \arrow[ddl, bend left] \\ X \arrow[ur, bend left] \arrow[r] \arrow[d] & X_1' \arrow[dl] \\ Y & \end{tikzcd}$$

$$\begin{tikzcd} \iota_2 : I \arrow[r, "\sim"] & \ker(\mathcal{O}_{X_2'} \to \mathcal{O}_X) \\ \iota_1 : I \arrow[r, "\sim"] \arrow[u, "\mathrm{id}"] & \ker(\mathcal{O}_{X_1'} \to \mathcal{O}_X) \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & \mathcal{O}_{X_2'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u, equal] & \mathcal{O}_{X_1'} \arrow[r] \arrow[u, "\wr"] & \mathcal{O}_X \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

c) If $U \subset X$, then there is a restriction functor

$$\underline{\mathrm{Exal}}_Y(X, I) \longrightarrow \underline{\mathrm{Exal}}_Y(U, I|_U).$$

<u>Rmk</u>. $u: I \to J$ map of $\mathcal{O}_X$-modules

Then there is a functor $u_*: \underline{\mathrm{Exal}}_Y(X, I) \to \underline{\mathrm{Exal}}_Y(X, J)$

$$X' \longmapsto X'_u$$

$$\begin{tikzcd} I \arrow[r, hook] & \mathcal{O}_{X'} \arrow[r, "\pi"] & \mathcal{O}_X \\ & f^{-1}\mathcal{O}_Y \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$\mathcal{O}_{X'_u} = \mathcal{O}_{X'} \oplus_I J$$

$$= \underset{\mathcal{O}_{X'}[J]}{\underset{\parallel}{(\mathcal{O}_{X'} \oplus J)}} \Big/ \{(i, -u(i)) : i \in I\}$$

$$\begin{tikzcd} & X'_u \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[ur, "J"] \arrow[r, "I"] \arrow[dr] & X' \arrow[d] & \\ & Y & \end{tikzcd}$$

<u>Lemma</u> If I & J are two q-coh $\mathcal{O}_X$-modules, then

$$(pr_{1*}, pr_{2*}): \underline{\mathrm{Exal}}_Y(X, I \oplus J) \xrightarrow{\sim} \underline{\mathrm{Exal}}_Y(X, I) \times \underline{\mathrm{Exal}}_Y(X, J)$$

is an equiv. of cats.

$\Sigma: I \oplus I \to I$ summation map

$$+: \underline{\mathrm{Exal}}_Y(X, I) \times \underline{\mathrm{Exal}}_Y(X, I) \xleftarrow{\sim} \underline{\mathrm{Exal}}_Y(X, I \oplus I) \xrightarrow{\Sigma_*} \underline{\mathrm{Exal}}_Y(X, I)$$

σ, τ

<u>Ex</u>. Let $f: A \to B$ be a homomorphism of abelian gps. Define P_f:

obj. = elts $x \in B$, morphism $x \to y$ is an elt $h \in A$ w/ $f(h) = y - x$.

$$+ : P_f \times P_f \longrightarrow P_f$$

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto] \arrow[d, "{(h,g)}"'] & x+y \arrow[d, "h+g"] \\
(x',y') \arrow[r, mapsto] & x'+y'
\end{tikzcd}$$

T top space (site)

A <u>Picard (pre)-stack</u> over T is a (pre) stack $\mathcal{P}$ in groupoids, $\mathcal{P} \to Op(T)$ w/ morphisms of stacks $(+, \sigma, \tau)$ s.t. $\forall\, U \subset T$, the fiber $(\mathcal{P}(U), +, \sigma, \tau)$ is a Picard cat.

<u>Ex</u> $\mathcal{P}ic(-)$ defines a Picard stack on $|X|$.

<u>Ex</u>. $\underline{\mathrm{Exal}}_Y(-, I)$ gives a Picard stack on $|X|$.

<u>Ex</u>. $f : A \to B$ homom. of sheaves of ab. gps on a top. space T, then get Picard prestack $\mathrm{pch}(A \to B)$

T top. space, $\mathcal{P}_1, \mathcal{P}_2$ Picard stack over T. A morphism $\mathcal{P}_1 \to \mathcal{P}_2$ is a pair $(F, \imath)$, where $F : \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of stacks, and $\imath : F(x+y) \xrightarrow{\sim} F(x) + F(y)$.

s.t.

$$\begin{tikzcd}
F(x+y) \arrow[r, "\imath"] \arrow[d, "F(\tau)"'] \arrow[dr, phantom, "\text{///}"] & F(x)+F(y) \arrow[d, "\tau"] \\
F(y+x) \arrow[r, "\imath"] & F(y)+F(x)
\end{tikzcd}
,\quad
\begin{tikzcd}
F((x+y)+z) \arrow[r, "\imath"] \arrow[d, "F(\sigma)"'] & F(x+y)+F(z) \arrow[r, "\imath"] \arrow[d, phantom, "\text{///}"] & (F(x)+F(y))+F(z) \arrow[d, "\sigma"] \\
F(x+(y+z)) \arrow[r, dash] & F(x)+F(y+z) \arrow[r, "\imath"] & F(x)+(F(y)+F(z))
\end{tikzcd}$$

$\rightsquigarrow$ Picard stack $\underline{\mathrm{HOM}}(\mathcal{P}_1, \mathcal{P}_2)$.

- identity element
- kernels
- $\otimes$

Lecture 6

Picard stacks

T top space

$(\mathcal{P}, +, \sigma, \tau)$

$K^\bullet \in C^{[-1,0]}(T)$

$\|$

$[K^{-1} \xrightarrow{d} K^0]$ $\rightsquigarrow$ $pch(K^\bullet)$ Picard prestack

$pch(K^\bullet)_U$: obj. $x \in K^0(U)$

mor. $x \to y$ is an elt $z \in K^{-1}(U)$

s.t. $d(z) = y - x$.

$\rightsquigarrow ch(K^\bullet)$ stackification of $pch(K^\bullet)$

If $\mathcal{P}$ is a Picard stack, then $HOM(ch(K), \mathcal{P}) \xrightarrow{\sim} HOM(pch(K), \mathcal{P})$

Rmk $pch(K) \to ch(K)$ is fully faithful.

Rmk $f: K_1^\bullet \to K_2^\bullet$, this induces a morphism of Picard stacks

$ch(f) : ch(K_1) \to ch(K_2)$.

- $f_1, f_2 : K_1^\bullet \to K_2^\bullet$ and a homotopy h between f_1 and f_2

$[\, h: K_1^0 \to K_2^{-1}$ s.t. $\forall x \in K_1^0$, $f_1(x) - f_2(x) = dh(x)$ and $f_1^{-1} - f_2^{-1} = hd \,]$

Then get an isom. of morphisms $ch(h): ch(f_1) \to ch(f_2)$

i.e. $\forall x \in pch(K_1^\bullet)$, an isom. $ch(f_1)(x) \xrightarrow{\sim} ch(f_2)(x)$.

$x \in K_1^0$, $z \in K_2^{-1}$ s.t. $dz = f_1(x) - f_2(x)$.

Lemma. If K^{-1} is flasque, then $pch(K)$ is a stack.

Pf. $pch(K) \xrightarrow{\pi} ch(K)$

Let $U \subset T$ open and $x \in ch(K)_U$.

Let $\mathcal{L}$ be the sheaf on U which to any open $V \subset U$ assoc. the set of pairs

(y, ℓ), $y \in K^0(V)$, and $\quad \ell : \pi(y) \to x|_V$ in $ch(K)_V$

Claim $\mathcal{L}$ is a $K^{-1}|_U$ - torsor.

Reason $(y', \ell') \in \mathcal{L}$,
$$\begin{tikzcd} \pi(y) \arrow[r, "\ell"] \arrow[rr, bend right, "\zeta \in K^{-1}"'] & x|_V \arrow[r, "\ell'^{-1}"] & \pi(y') \end{tikzcd}$$

$\mathcal{L}$ is classified by an elt $[\mathcal{L}] \in H^1(U, K^{-1}|_U) = 0 \quad \rightsquigarrow \mathcal{L}$ is trivial, has section.

$\square$

Observations a) the sheaf assoc. to the presheaf

$U \longmapsto$ the set of isom. classes in $ch(K^\cdot)_U$

$= \mathcal{H}^0(K^\cdot) = K^0 / \mathrm{Im}(K^{-1} \to K^0)$

b) What is the automorphism gp of an obj. $x \in ch(K^\cdot)_U$? $\quad \mathcal{H}^{-1}(K)$

$\left[x \in K^0(U), \quad \mathrm{Aut}(x) = \{ \zeta \in K^{-1}(U) : d\zeta = x - x = 0 \} \right]$

Cor If $f : K_1^\cdot \to K_2^\cdot$ is a q-isom., then $ch(f) : ch(K_1^\cdot) \xrightarrow{\sim} ch(K_2^\cdot)$ is an equiv.

$\tilde{C}^{[-1,0]}(T) \subset C^{[-1,0]}(T)$ be full subcat. of complexes $K^{-1} \to K^0$ w/ K^{-1} injective.

Thm. ch induces an equiv. of 2-cats

$\tilde{C}^{[-1,0]}(T) \xrightarrow{\sim} ($ Picard stacks over T $)$

Cor (Picard stacks, isom. classes of morphisms) $\simeq D^{[-1,0]}(T)$.

Lemma $f: \mathcal{X} \to \mathcal{Y}$ morphism of stacks, and $\bar{f}: X \to Y$ is the corresponding map on sheaves of isom. classes. Assume $\bar{f}$ is an isom. and $\forall U \subset T$ and $x \in \mathcal{X}_U$, the map of sheaves $\mathrm{Aut}_{\mathcal{X}}(x) \to \mathrm{Aut}_{\mathcal{Y}}(f(x))$ is an isom. Then f is an isom.

Pf. Given $x, y \in \mathcal{X}_U$, want

$$\underline{\mathrm{Isom}}_{\mathcal{X}}(x,y) \to \underline{\mathrm{Isom}}_{\mathcal{Y}}(f(x), f(y)) \quad \text{to be an isom.}$$

injectivity $\quad \alpha, \beta: x \to y \quad , \quad f(\alpha) = f(\beta): f(x) \to f(y).$

$$\alpha^{-1} \circ \beta \in \ker\left(\mathrm{Aut}_{\mathcal{X}}(x) \to \mathrm{Aut}_{\mathcal{Y}}(f(x))\right) \Rightarrow \alpha = \beta$$

Surjectivity. $\quad \sigma: f(x) \to f(y)$. Enough to show σ is in image locally.

$x, y \mapsto$ same thing in X. So locally $\exists\ \tau: x \to y$.

$$\sigma^{-1} \circ f(\tau): f(x) \circlearrowleft$$

Essential surj.

$y \in \mathcal{Y}_T$, $\exists$ covering $T = \bigcup_i U_i$ and (x_i, l_i) w/ $x_i \in \mathcal{X}_{U_i}$,

$l_i: f(x_i) \xrightarrow{\sim} y|_{U_i}$ in $\mathcal{Y}_{U_i}$

Then on U_{ij}, $\exists!$ isom. $\sigma_{ij}: x_i|_{U_{ij}} \xrightarrow{\sim} x_j|_{U_{ij}}$ s.t.

$$\begin{tikzcd} f(x_i)|_{U_{ij}} \arrow[r, "f(\sigma_{ij})"] \arrow[d, "l_i"'] \arrow[dr, phantom, "\cong"] & f(x_j)|_{U_{ij}} \arrow[dl, "l_j"] \\ y|_{U_{ij}} & \end{tikzcd}$$

Commutes

$\sigma_{jk} \circ \sigma_{ij}$, $\sigma_{ik}: x_i|_{U_{ijk}} \to x_k|_{U_{ijk}}$ are both equal to the unique morphism

$$\begin{tikzcd} f(x_i)|_{U_{ijk}} \arrow[rr, "f(-)"] \arrow[dr, "\ell_i"'] & & f(x_k)|_{U_{ijk}} \arrow[dl, "\ell_k"] \\ & y|_{U_{ijk}} & \end{tikzcd}$$

Lemma $\mathcal{P}$ Picard stack $/T$, $\{U_i\}$ collection of open subsets, $k_i \in \mathcal{P}(U_i)$, $\forall i$.

$K = \bigoplus_i \mathbb{Z}_{U_i}$ $\quad [\mathbb{Z}_{U_i} = j_! \mathbb{Z},\ j : U_i \hookrightarrow T]$

Then $\exists$ morphism $F : ch(0 \to K) \to \mathcal{P}$ and isom's $\sigma_i : F(1 \in \mathbb{Z}_{U_i}(U_i)) \xrightarrow{\sim} k_i$.

and the data $(F, \{\sigma_i\})$ is unique up to unique isom.

Lemma. Let $\mathcal{P}$ be a Picard stack over T, then $\exists\ K \in C^{[-1,0]}(T)$ and an isom.

$ch(K) \xrightarrow{\sim} \mathcal{P}$.

Ex. $\mathcal{P}ic(X) = ch(\mathcal{O}_X^\times \to 0)$.

Pf. Choose data

a) $\{U_i \subset T\}_{i \in I}$ $\qquad$ s.t. $\forall V \subset T$, $k \in \mathcal{P}_V$

b) $\forall i$, $k_i \in \mathcal{P}(U_i)$ $\qquad$ $\exists$ cov. $V = \bigcup V_j$ s.t.

$k|_{V_j} \simeq k_i$, some i w/ $V_j \subset U_i$

$K^0 = \bigoplus_i \mathbb{Z}_{U_i}$. $\qquad F : ch(0 \to K^0) \to \mathcal{P}$.

$K^{-1}(V) = \{(x, \ell) ; x \in K^0(V),\ \ell : F(0) \xrightarrow{\sim} F(x)\}$.

$K^{-1} \to K^0$ $\qquad (x, \ell) + (x', \ell') = (x + x', ?)$

$(x, \ell) \mapsto x$

$$\begin{tikzcd} F(0) \arrow[r, "\sim"] \arrow[drr, bend right, "?"'] & F(0) + F(0) \arrow[r, "\ell + \ell'"] & F(x) + F(x') \arrow[d, phantom, "\shortparallel"] \\ & & F(x+x') \end{tikzcd}$$

$x \to x'$ in $pch(K^{-1} \to K^0)$:

$(x' - x, \ell)$, $\ell : F(0) \xrightarrow{\sim} F(x' - x)$.

$\rightsquigarrow \underbrace{F(0) + F(x)}_{F(x)} \xrightarrow{\sim} \overbrace{F(x' - x) + F(x)}^{F(x')}$ $\qquad \rightsquigarrow pch(K^{-1} \to K^0) \to \mathcal{P}$.

Lecture 7

Theorem: $ch : \tilde{C}^{[-1,0]}(T) \xrightarrow{\sim} (\text{Picard stacks})$ is an equiv. of 2-cats

Lemma P Picard stack, then $\exists\ K \in C^{[-1,0]}(T)$ and an equiv. $ch(K) \xrightarrow{\sim} P$.

Lemma. $K, L \in C^{[-1,0]}(T)$, and let $F : ch(K) \to ch(L)$ be a morphism of Picard stacks, then $\exists$ a q-isom. $k : K' \to K$ and a morphism $l : K' \to L$ s.t. $F \simeq ch(l) \circ ch(k)^{-1}$

In particular. if $K \in \tilde{C}^{[-1,0]}(T)$, then any morphism $F : ch(K) \to ch(L)$ is isom. to $ch(f)$ for some $f : K \to L$.

Sketch of proof Choose data $\{(U_i, k_i, l_i, \sigma_i)\}_{i\in I}$ s.t.

a) $U_i \subset T$ open set

b) $k_i \in K^0(U_i)$, $l_i \in L^0(U_i)$, $\sigma_i : F(k_i) \xrightarrow{\sim} l_i$

c) the map $K'^0 := \bigoplus_{i\in I} \mathbb{Z}_{U_i} \to K^0$ is surjective

$$K'^{-1} := K^{-1} \times_{K^0} K'^0$$

$l : K' \to L$: $l^0 : K'^0 \to L^0$, $\mathbb{Z}_{U_i} \to L^0$ given by l_i

$l^{-1} : K'^{-1} \to L^{-1}$

$(v, (U_i, k_i, l_i, \sigma_i)) \in K'^{-1}$

$\mapsto$ the unique elt $t \in L^{-1}$ s.t.
$$\begin{tikzcd} F(0) \arrow[r, "F(v)"] \arrow[d, "s"', "\wr"] & F(k_i) \arrow[d, "\sigma_i"] \\ 0 \arrow[r, "t"] & l_i \end{tikzcd}$$

The σ_i's define an isom. $\sigma : F \xrightarrow{\sim} ch(l) \circ ch(k)^{-1}$. $\square$

Lemma $K_1, K_2 \in \tilde{C}^{[-1,0]}(T)$. For two morphisms of complexes $f_1, f_2 : K_1 \to K_2$ w/ assoc. morphisms $F_1, F_2 : ch(K_1) \to ch(K_2)$ and any isom. $H : F_1 \Rightarrow F_2$, $\exists!$ homotopy $h : K_1^0 \to K_2^{-1}$ s.t. $H = ch(h)$.

Idea. If $k \in K_1^0$ is a section,

$$F_1(k) \xrightarrow{H} F_2(k)$$

$$\Updownarrow$$

section $h(k) \in K_2^{-1}$ s.t. $dh(k) = f_2(k) - f_1(k)$. $\square$.

Preliminary def'n. Let $f: X \to S$ be a morphism of schemes. The truncated tangent complex, $\tau_{\le 1}\mathbb{T}_{X/S}[1] \in \tilde{C}^{[-1,0]}(|X|)$ is the complex w/

$$ch(\tau_{\le 1}\mathbb{T}_{X/S}[1]) \simeq \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X).$$

Problems

a) This doesn't see $\mathcal{O}_X$-module structure.

b) Not the full complex.

Prop. Let $j: X \hookrightarrow S$ be a closed immersion defined by an ideal I. Then $\tau_{\le 1}\mathbb{T}_{X/S}[1]$ is q-isom. to $N_{X/S}$, where $N_{X/S} := \mathcal{H}om(j^*I, \mathcal{O}_X)$.

Pf. $\underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$

$$\begin{tikzcd} X \arrow[r, hook, "\mathcal{O}_X"] \arrow[d, "j"] & X' \arrow[dl] \\ S & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_X & \mathcal{O}_{X'} \arrow[l, two heads] & \mathcal{O}_X\cdot\varepsilon \arrow[l, hook'] \\ & j^{-1}(\mathcal{O}_S/I^2) \arrow[ul] \arrow[u] & j^*I \arrow[l, hook'] \arrow[u, "\partial \in N_{X/S}"'] \end{tikzcd}$$

$\square$

Prop. Let $f: X \to S$ be a smooth morphism. Then $\tau_{\le 1}\mathbb{T}_{X/S}[1] \simeq \underbrace{T_{X/S}[1]}_{T_{X/S} \to 0}$

Pf. $\mathcal{H}^0(\tau_{\le 1}\mathbb{T}_{X/S}[1]) = 0$

$\mathcal{H}^{-1}(\tau_{\le 1}\mathbb{T}_{X/S}[1]) = T_{X/S}$

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d] \\ X' \arrow[r] \arrow[ur, dashed, "?"] & S \end{tikzcd}$$

$X[\mathcal{O}_X\cdot\varepsilon]$

$\square$

Prop. Suppose given a comm. diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & P \arrow[dl, "g"] \\ S & \end{tikzcd}$$

g smooth, j closed immersion

Then $\tau_{\le 1} \mathbb{L}_{X/S}[1] \simeq \left(j^* T_{P/S} \to \mathcal{N}_{X/P} \right)$

I ideal of X in P, $I/I^2 \xrightarrow{d} j^* \Omega^1_{P/S}$.

Proof. $\zeta : j^* I \to \mathcal{O}_X$ (section of $\mathcal{N}_{X/P}$).

$$\begin{tikzcd} j^* I \arrow[r, "\zeta"] \arrow[d] & \mathcal{O}_X \cdot \varepsilon \arrow[d, dashed] \arrow[ddr, bend left] & \\ j^{-1}(\mathcal{O}_P / I^2) \arrow[r, dashed] \arrow[drr, bend right] & \mathcal{O}_{X'} \arrow[dr, dashed] & \\ & & \mathcal{O}_X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, hook] \arrow[d] & X_\zeta \arrow[dl] \arrow[ddl, bend left] \\ P \arrow[d] & \\ S & \end{tikzcd}$$

non-commutative diagram!

$$\begin{tikzcd} & X_{\zeta'} \arrow[d, "h"] \arrow[dd, bend left, "f'"] \\ X \arrow[ur, "z'"] \arrow[r, "z"] \arrow[dr] & X_\zeta \arrow[d, "f"] \\ & P \end{tikzcd}$$

$\zeta, \zeta' \in \mathcal{N}_{X/P}$.

$f \circ h$, f'

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X_{\zeta'} \arrow[dl, dashed, shift left] \arrow[dl, dashed, shift right] \arrow[ddl, bend left] \\ P \arrow[d] & \\ S & \end{tikzcd}$$

$(f \circ h - f') \in j^* T_{P/S}$

Upshot. $\exists$ fully faithful functor

$\mathrm{pch}\left(j^* T_{P/S} \to \mathcal{N}_{X/P} \right)$

$\to \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$.

Claim

$\mathrm{ch}\left(j^* T_{P/S} \to \mathcal{N}_{X/P} \right) \xrightarrow{\sim} \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$ is an equiv. $\left[\begin{tikzcd} X \arrow[r, hook] \arrow[dr] \arrow[ddr, bend right] & X' \arrow[dl, dashed] \arrow[ddl, bend left] \\ & P \arrow[d] \\ & S \end{tikzcd} \right]$ $\square$

a) choice of factorization of f.

b) factorization need not exist

Replacement for factorization.

$f: X \longrightarrow S$

F: (sheaves of $f^{-1}\mathcal{O}_S$- algebras) $\xrightarrow{\text{forgetful}}$ (sheaves of sets)

F has a left adjoint $\Omega \longmapsto f^{-1}\mathcal{O}_S\{\Omega\}$.

Idea: $X = \operatorname{Spec} A$

$$\begin{array}{c} X = \operatorname{Spec} A \\ \downarrow \\ S = \operatorname{Spec} B \end{array}$$

Choose elts $f_i \in A$ s.t.

$$\begin{tikzcd} B[x_i] \arrow[r, "x_i \mapsto A"] & A \\ B \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$f^{-1}\mathcal{O}_S\{\Omega\}(U) = f^{-1}\mathcal{O}_S(U)[x_i]_{i \in \Omega(U)}$$

$$\begin{tikzcd} \mathcal{O}_X & f^{-1}\mathcal{O}_S\{\Omega\} \arrow[l, two heads] \\ f^{-1}\mathcal{O}_S \arrow[u] \arrow[ur] & \end{tikzcd}$$

How to choose Ω?

a) $\Omega = F(\mathcal{O}_X)$

b) Choose open sets $U_i \overset{j_i}{\subset} |X|$, sections $f_i \in \mathcal{O}_X(U_i)$

$\Omega = \coprod_i j_{i!}\{*\}$

Def'n The truncated tangent complex of f is the complex $\mathcal{H}om(\Omega^1_{f^{-1}\mathcal{O}_S\{F(\mathcal{O}_X)\}/f^{-1}\mathcal{O}_S}, \mathcal{O}_X)$ of $\mathcal{O}_X$-modules.

$\rightarrow \mathcal{H}om(I/I^2, \mathcal{O}_X)$

Lecture 8. $X \xrightarrow{f} S$ morphism of schemes

$$\begin{tikzcd} GF(\mathcal{O}_X) = f^{-1}\mathcal{O}_S\{F(\mathcal{O}_X)\} \arrow[r, two heads, "\pi"] & \mathcal{O}_X \\ f^{-1}\mathcal{O}_S \arrow[u] \arrow[ur] & \end{tikzcd}$$

$F: (f^{-1}\mathcal{O}_S\text{-alg}) \longrightarrow (\text{sheaves of sets})$, with G in the opposite direction.

$\Omega^1_{GF(\mathcal{O}_X)/f^{-1}\mathcal{O}_S} \otimes \mathcal{O}_X$

$\uparrow$

I/I^2, $\quad I = \ker(\pi)$

$\tau_{\geq -1} \mathbb{L}_{X/S}$

Thm. For any qcoh. $\mathcal{O}_X$-module M,

$$\mathrm{ch}\left(\tau_{\leq 0}\left(R\mathcal{H}om\left(\tau_{\geq -1}\mathbb{L}_{X/S}, M\right)[1]\right)\right) \simeq \underline{\mathrm{Exal}}_S(X, M).$$

$\tau_{\geq -1}\mathbb{L}_{X/S}$ truncated cotangent complex.

$\boxed{\mathbb{L}_{X/S}}$ full cotangent complex.

Given $n \geq 0$, $\underbrace{GF \cdots GF\, GF}_{(n+1) \text{ copies of } GF}(\mathcal{O}_X)$ $\quad A_n$ – $f^{-1}\mathcal{O}_S$-alg

– $A_n \twoheadrightarrow \mathcal{O}_X$

$A_\bullet$ simplicial $f^{-1}\mathcal{O}_S$-algebra.

$$\begin{tikzcd} A_{n+1} \arrow[d, shift left=2, "d_i"] \arrow[d] \arrow[d, shift right=2] \\ A_n \arrow[u, bend left, "s_j"] \end{tikzcd}$$

$GF \xrightarrow{a} \mathrm{id}$

$\mathrm{id} \xrightarrow{b} FG$

$$\begin{tikzcd} A_2 = \cancel{GF}GFGF(\mathcal{O}_X) \arrow[d, "d_0"] \\ A_1 = GFGF(\mathcal{O}_X) \end{tikzcd}$$

$\Omega_{A_\bullet/f^{-1}\mathcal{O}_S} \otimes \mathcal{O}_X$ simplicial $\mathcal{O}_X$-module

$$\begin{tikzcd} \tilde{L}_2 \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \tilde{L}_1 \arrow[d, shift right=1, "d_0"'] \arrow[d, shift left=1, "d_1"] \\ \tilde{L}_0 \end{tikzcd}$$

$$\cdots \to \tilde{L}_2 \xrightarrow{d_0 - d_1 + d_2} \tilde{L}_1 \xrightarrow{d_0 - d_1} \tilde{L}_0$$

$\mathbb{L}_{X/S}$ = cotangent complex.

<u>Rmk</u> This is an actual complex of flat $\mathcal{O}_X$-modules.

(i) $\mathcal{H}^i(\mathbb{L}_{X/S})$ is qcoh and coh. if S loc. noetherian and f is of finite type.

(ii)
$$(*)\quad \begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$
then there is a base change morphism
$$u^* \mathbb{L}_{X/Y} \longrightarrow \mathbb{L}_{X'/Y'}.$$

If $(*)$ is cartesian and tor. indep. (e.g., either f, v is flat), then is a q-isom.

and $f'^* \mathbb{L}_{Y'/Y} \oplus u^* \mathbb{L}_{X/Y} \longrightarrow \mathbb{L}_{X'/Y}$ is a q isom.

(iii) $X \xrightarrow{f} Y \xrightarrow{g} Z$, then there is a dist. triangle

$$f^* \mathbb{L}_{Y/Z} \longrightarrow \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \longrightarrow f^* \mathbb{L}_{Y/Z}[1]$$

(iv) $\tau_{\geq -1} \mathbb{L}_{X/Y}$ is equal to our earlier $\tau_{\geq -1} \mathbb{L}_{X/Y}$.

<u>Rmk</u>. a) $\mathcal{H}^0(\mathbb{L}_{X/Y}) = \Omega^0_{X/Y}$

b) If f is smooth, then $\mathbb{L}_{X/Y} \xrightarrow{\sim} \Omega^1_{X/Y}$ is a q-isom.

c) If $X \hookrightarrow Y$ is a closed immersion which is lci, then

$\mathbb{L}_{X/Y} = I/I^2[1]$.

<u>Thm</u> (Illusie) $\mathrm{ch}\left(\tau_{\geq -1}\left(R\mathcal{H}om(\mathbb{L}_{X/Y}, I)[1]\right)\right) \simeq \underline{\mathrm{Exal}}_Y(X, I)$.

$\Rightarrow$ $\mathrm{Ext}^1(\mathbb{L}_{X/Y}, I) \simeq \mathrm{Exal}_Y(X, I)$.

$\mathrm{Ext}^0(\mathbb{L}_{X/Y}, I) = \mathrm{Hom}(\Omega^1_{X/Y}, I)$ is the auto gp of any
$$\begin{array}{ccc} X & \overset{I}{\hookrightarrow} & X' \\ & \searrow \ \swarrow & \\ & Y & \end{array}$$

Problem.

$$\begin{tikzcd} X_0 \arrow[r, dashed, "i"] \arrow[d, "f_0"'] & X \arrow[d, dashed, "f"] \\ Y_0 \arrow[r, "j"] \arrow[d] & Y \arrow[dl] \\ S & \end{tikzcd}$$

j closed immersion defined by square-zero ideal J.

Fill in diagram as indicated, w/ i square-zero

s.t. $f_0^* J \xrightarrow{\sim} \ker(\mathcal{O}_X \to \mathcal{O}_{X_0})$.

Solution.

$X_0 \to Y_0 \to Y$

$$f_0^* \mathbb{L}_{Y_0/Y} \to \mathbb{L}_{X_0/Y} \to \mathbb{L}_{X_0/Y_0} \xrightarrow{+1}$$

$$0 \to \mathrm{Ext}^0(\mathbb{L}_{X_0/Y_0}, f_0^*J) \to \mathrm{Ext}^0(\mathbb{L}_{X_0/Y}, f_0^*J) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^0(f_0^*\mathbb{L}_{Y_0/Y}, f_0^*J)}$$

$$\to \mathrm{Ext}^1(\mathbb{L}_{X_0/Y_0}, f_0^*J) \to \underset{\substack{\| \\ \mathrm{Exal}_Y(X_0, f_0^*J)}}{\mathrm{Ext}^1(\mathbb{L}_{X_0/Y}, f_0^*J)} \to \underset{\substack{\| \\ \mathrm{Hom}(f_0^*J, f_0^*J)}}{\mathrm{Ext}^1(f_0^*\mathbb{L}_{Y_0/Y}, f_0^*J)}$$

$$\xrightarrow{\partial} \mathrm{Ext}^2(\mathbb{L}_{X_0/Y_0}, f_0^*J)$$

Thm (i) $\exists$ obstruction $o(f_0) = \partial(\mathrm{id}) \in \mathrm{Hom}(f_0^*J, f_0^*J) = \mathrm{Ext}^1(f_0^*\mathbb{L}_{Y_0/Y}, f_0^*J)$

whose vanishing is nec. suff. for a solution to the problem.

(ii) If $o(f_0) = 0$, then the set of isom. classes of solutions form a torsor under

$\mathrm{Ext}^1(\mathbb{L}_{X_0/Y_0}, f_0^*J)$.

(iii) $\mathrm{Aut} = \mathrm{Ext}^0(\mathbb{L}_{X_0/Y_0}, f_0^*J)$.

Problem.

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "f_0"'] \arrow[dd, bend right=60, "h_0"'] & X \arrow[d, dashed, "f"] \arrow[dd, bend left=60, "h"] \\ Y_0 \arrow[r, "j"] \arrow[d, "g_0"'] & Y \arrow[d, "g"] \\ Z_0 \arrow[r, "k"] & Z \end{tikzcd}$$

I

J

K

Find dotted arrows.

$g^*K \twoheadrightarrow J$.

Thm (Illusie) There is can. class $o(f_0) \in \mathrm{Ext}^1(f_0^* \mathbb{L}_{Y_0/Z_0}, I)$ s.t. f exists $\Leftrightarrow o(f_0) = 0$. If $o(f_0) = 0$, then the set of maps f is a torsor under $\mathrm{Ext}^0(f_0^* \mathbb{L}_{Y_0/Z_0}, I)$.

Sketch. $e(X) \in \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathbb{L}_{X_0/Z}, I)$.

$e(Y) \in \mathrm{Ext}^1_{\mathcal{O}_{Y_0}}(\mathbb{L}_{Y_0/Z}, J)$

$$\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathbb{L}_{X_0/Z}, I) \to \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* \mathbb{L}_{Y_0/Z}, I)$$
$$e(X) \longmapsto \zeta_X$$

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{O}_{Y_0}}(\mathbb{L}_{Y_0/Z}, J) & \to & \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* \mathbb{L}_{Y_0/Z}, f_0^* J) \\ & & \downarrow \\ e(Y) \longmapsto \zeta_Y & & \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* \mathbb{L}_{Y_0/Z}, I). \end{array}$$

Want. $\zeta_X = \zeta_Y$.

$$h_0^* \mathbb{L}_{Z_0/Z} \to f_0^* \mathbb{L}_{Y_0/Z} \to f_0^* \mathbb{L}_{Y_0/Z_0} \xrightarrow{+1}$$

$$\mathrm{Ext}^0(h_0^* \mathbb{L}_{Z_0/Z}, I) \to \mathrm{Ext}^1(f_0^* \mathbb{L}_{Y_0/Z_0}, I) \to \mathrm{Ext}^1(f_0^* \mathbb{L}_{Y_0/Z}, I)$$

$\mathrm{Ext}^0(h_0^* \mathbb{L}_{Z_0/Z}, I) = 0$

$o(f_0) = \zeta_X - \zeta_Y \in \mathrm{Ext}^1(f_0^* \mathbb{L}_{Y_0/Z_0}, I)$

$\mathrm{Ext}^1(f_0^* \mathbb{L}_{Y_0/Z}, I) \to \mathrm{Hom}(h_0^* K, I) = \mathrm{Ext}^1(h_0^* h_{Z_0/Z}, I)$