Cherednik algebras and Hilbert schemes

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Thm [Levasseur – Stafford] $/\mathbb{C}$

$$\left(\mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g})\, \mathrm{ad}\, \mathfrak{g} \right)^{G} \simeq \mathcal{D}(\mathfrak{h})^{W}$$

$G = SL_n$, $V = \mathbb{C}^n$, $GL_n \curvearrowright \mathfrak{g} \times V$

$\rightsquigarrow \tau : \mathfrak{gl}_n \to \mathcal{D}(\mathfrak{g} \times V)$

$c \in \mathbb{C}$, $\tau_c = \tau - c\,\mathrm{tr}$

$$\left(\frac{\mathcal{D}(\mathfrak{g} \times V)}{\mathcal{D}(\mathfrak{g} \times V)\, \tau_c(\mathfrak{gl}_n)} \right)^{GL_n} \simeq \cancel{\mathcal{D}(\mathfrak{h})^{W}}$$

$e H_c e$

Def. The rat'l Cherednik algebra H_c is the subalg. of $\mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$ gen. by

W, $x_i - x_{i+1}$, $y_i - y_{i+1}$, $i = 1, \cdots, n-1$

$$y_i = \frac{\partial}{\partial x_i} - c \sum_{j \neq i} \frac{1 - (ij)}{x_i - x_j}$$

Eg $c = 0$, $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes W$

$e = \frac{1}{|W|} \sum_{w \in W} w$, $e H_0 e = \mathcal{D}(\mathfrak{h})^{W}$

$e H_c e$ spherical RCA

$\mathcal{O}(H_c)$: category $\mathcal{O}$ of H_c

Ψ

$\Delta_c(\tau)$, $\tau \vdash n$

$\wr\!\wr$ $\qquad \pi_\tau \in \mathrm{Irrep}(\mathfrak{S}_n)$

$H_c \otimes_{S(\mathfrak{h}) \rtimes W} \pi_\tau \twoheadrightarrow L_c(\tau)$ simple

Thm (Berest - Etingof - Ginzburg)

Only when $c = \frac{m}{n}$, $(m,n)=1$, H_c has finite-dim. reps

When $c=\frac{m}{n}$, $m>0$, $(m,n)=1$, the only simple f.d. module is $L_c(\mathrm{triv})$.

Def (Gorsky - Etingof - Losev) We call $M \in \mathcal{O}(H_c)$ of minimal support if there is no subset of $\mathrm{supp}(M) \subset \mathfrak{h}$ of smaller dim than the supp. of some $N \in \mathcal{O}(H_c)$.

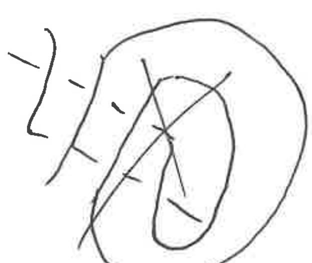
Thm (Wilcox) When $c=\frac{m}{n}$, $m>0$, The simple minimal supported modules are

$L_c(n_0\tau)$, $d=\gcd(m,n)$, $n_0=\frac{n}{d}$, $\tau\vdash d$. of $\mathrm{supp} = W\cdot \mathfrak{h}_d$

$$\mathfrak{h}_d = \left\{ \begin{array}{c} x_1 = \cdots = x_{n_0} \\ \vdots \\ x_{(d-1)n_0+1} = \cdots \end{array} \right\}$$

Torus ~~knots~~ links.

$$T_{m,n} = \{x^m = y^n\} \cap S^3_\varepsilon$$



2 linked unknots

$$h = \frac{1}{2}\sum_{i=1}^{n} (x_iy_i + y_ix_i) \curvearrowright L_c(\lambda) = \bigoplus_{k\in\mathbb{Z}} L_c(\lambda)(k)$$

$$\mathrm{ch}_q\, L_c(\lambda) = \sum_k q^k \dim L_c(\lambda)(k)$$

Thm (Gorsky–Oblomkov–Rasmussen–Shende, coprime

Gorsky–Etingof–Losev, $m>0$)

$$\sum_{\mu \vdash d} \dim \pi_\mu \sum_i a^{2i} \mathrm{ch}_q \left(\mathrm{Hom}_{B_n} \left(\Lambda^i \mathfrak{h}, L_c(n \circ \mu) \right) \right)$$

$$\parallel$$

$$\mathrm{HOMFLY}_{a,q}(T_{m,n})$$

Conj. (GORS) $\exists$ filtrat'n on $L_c(n \circ \mu)$ s.t.

$$\cdots \quad \mathrm{ch}_{q,t}\left(\cdots \ \mathrm{gr}^F L_c(n \circ \mu) \right)$$

$$\parallel$$

$$\dim_{a,q,t} \mathrm{HHH}(T_{m,n})$$

Thm [M.] Conj. holds $(m,n)=1$ w.r.t. F^{Hodge}

$(F^{\mathrm{ind}} = F^{\mathrm{Hodge}} = F^{\mathrm{alg}})$

Rmk m,n - symmetry

q,t - symmetry

Proof strategy:

Construct $\mathcal{F}_c \in \mathrm{Coh}^{\mathbb{G}_m \times \mathbb{G}_m}(\mathrm{Hilb}^n(\mathbb{A}^2))$ whose equiv. K-theory class computes LHS.

Hamiltonian reduction $\quad \mathbb{H}_c : (\mathcal{D}(\mathfrak{g} \times V), \mathfrak{gl}_n)\text{-mod} \longrightarrow eH_c e\text{-mod}$

[Gan–Ginzburg] $\quad M \longmapsto M^{\tau_c(\mathfrak{gl}_n)}$

$\quad ? \longmapsto eL_c(n \circ \mu)$

$$\mathfrak{h}_d \supset \mathfrak{h}_d^{\circ} = \left\{ \left(\begin{array}{cc|c} \begin{matrix} x_1 & & \\ & \ddots & \\ & & x_1 \end{matrix} & & \\ \hline & \begin{matrix} x_2 & & \\ & \ddots & \\ & & x_2 \end{matrix} & \\ \hline & & \ddots \end{array} \right) : x_i \neq x_j, \ i \neq j \right\}$$

$$Y_d = \mathfrak{g}_{reg} \times_{\mathfrak{h}/\!/W} \mathfrak{h}_d^{\circ}$$

$$\pi_1(Y_d) = Br_d \times \mathbb{Z}/n_0\mathbb{Z}, \quad n_0 = \frac{n}{d}.$$

Take $\mathcal{L}_c$ = local system on Y_d corresponding to $c \in (\mathbb{Z}/n_0\mathbb{Z})^{\times}$.

$\mathcal{M}_c$ = minimal ext. of $\mathcal{L}_c$ to $\mathfrak{g} \times \mathfrak{h}_d$.

$$p : \mathfrak{g} \times \mathfrak{h}_d \to \mathfrak{g}$$

$$p_+ M = \bigoplus_{\tau \vdash d} \pi_{\tau} \otimes N_c(\tau)$$

Ex $d = n$, $Y_d = \tilde{\mathfrak{g}}_{rs}$, $\mathcal{M}_c$ – HC D module

$p_+ M$: Springer D-module

[Hotta – Kashiwara]

$d = 1$, $\mathcal{M}_c$ cuspidal char. D-module [Lusztig].

Thm [Calaque – Enriques – Etingof]

$$\mathbb{H}_c(N_c(\tau)) = eL_c(n_0\tau)$$

$\mathcal{M}_c$: Hodge module F^H

$\pi_* gr^H \mathcal{M}_c$ $\qquad \pi : T^*(\mathfrak{g} \times \mathfrak{h}_d) \to T^*\mathfrak{g}$

is supp. on $\{[x,y] = 0\}$

$\rightsquigarrow \mathcal{F}_c \in Coh(Hilb^n(\mathbb{C}^2))$

$$\| $$

$$\{(x,y,v) : x,y \in \mathfrak{g},\ v \in V,\ [x,y] = 0,\ \mathbb{C}[x,y]v = V\}/GL_n.$$

$$[\mathcal{F}_c] = \,? \in K^{\mathbb{C}^\times \times \mathbb{C}^\times}(Hilb^n(\mathbb{C}^2))$$

Elliptic Hall algebra:

$$A = \frac{\mathbb{C}(q,t)\langle P_{m,n} : (m,n) \in \mathbb{Z}^2 \setminus 0 \rangle}{[P_{m,n}, P_{m',n'}] = P_{m+m',n+n'} + \cdots}$$

$$A \curvearrowright \bigoplus_n K^{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes_{\mathbb{C}[q^\pm, t^\pm]} \mathbb{C}(q,t)$$

$$\wr\!\!\mid$$

$$\mathbb{C}(q,t)[z_1, z_2, \cdots]^{\mathfrak{S}_\infty}$$

Thm. $K(\mathcal{F}_c) = P^d_{m_0,n_0} \cdot 1$ $\qquad m_0 = \frac{m}{d},\ n_0 = \frac{n}{d}$

$\displaystyle\bigoplus_{\substack{\mu \vdash d \\ \dim \pi_\mu}} \mathrm{ch}_{q,t}(L_c(n_0\mu))$ $\qquad HHH(T_{m,n}),\ (m,n) = 1$ $\quad$ [Mellit]

when $m > 0$, conjecture of Wilson.

$\pi_* \mathrm{gr}^H M_c = ?$

$(m,n) = 1$

$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & * \\ & 0 \end{pmatrix} \right\} \supset \mathfrak{b} \supset \mathfrak{n}$

$$\begin{tikzcd}
G/B & G\times^B(\mathfrak{n}\times\mathfrak{n}) \arrow[l, "q"'] \arrow[dl, "p"'] \arrow[d, "{[-,-]}"] & Z^{DG} \arrow[l] \arrow[d] \\
T^*\mathfrak{g} & G\times^B \mathfrak{b} & G/B \arrow[l]
\end{tikzcd}$$

$$\pi_* \mathrm{gr}^H M_c = p_*(q^* \mathcal{L}_{\lambda_c} \otimes \mathcal{O}_{Z^{DG}})$$

$$\lambda_c = (\lceil c \rceil, \lceil 2c \rceil - \lceil c \rceil, \cdots)$$

For general $m > 0$, take blocks

convolution product of coprime case