# Affine grassmannian & Satake category

$k = \mathbb{C}, \overline{\mathbb{F}_q}, \mathbb{F}_q$

## §1. Ind-schemes

prestack : $k\text{-Alg} \longrightarrow \text{Grpds}$

**Def.** An ind-scheme $X$ is a presheaf $k\text{-Alg} \longrightarrow \text{Sets}$ s.t.

$X \cong \operatorname{colim}_n X_n$, $X_n$ is a qcqs scheme,

transition $i_n : X_n \longrightarrow X_{n+1}$ is a closed immersion.

**Def.** Ind-scheme $X$ is reduced, if $\exists\ X \cong \operatorname{colim} X_n$, $X_n$ reduced.

**Def.** (P) be a property of morphisms of schemes, stable under base change,

then we say $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of prestacks has (P), if $f$ is representable

& its base change to any scheme has (P).

**eg.** (P) = proper, closed immersion, open immersion.

**Def.**
$$\begin{tikzcd} \text{Sch} \arrow[r, "|\cdot|"] \arrow[d, hook] & \text{Top} \\ \text{PreStk} \arrow[ur, dashed, "|\cdot|"'] & \end{tikzcd}$$
left Kan ext'n.

$X \cong \operatorname{colim}_n X_n$, $|X| \cong \operatorname{colim}_n |X_n|$.

**Lemma.** If $X = \varinjlim_n X_n$, $X_n$ noetherian, $X_n \longrightarrow X_{n+1}$ injective on conn'd components,

then $\forall\ Y$ conn'd comp. of $X$, $Y \longrightarrow X$ is open & closed immersion.

## §2. Loop groups

$X$ a scheme over $k$

**Def.** Loop space $LX$ is the presheaf $R \mapsto X(R((t)))$.

Arc space $L^+X$ is the presheaf $R \mapsto X(R[[t]])$

**Prop.** $X$ affine scheme of f.t. over $k$

(1) $L^+X$ is an affine scheme (of $\infty$-type)

If $X$ is conn'd / reduced / formally smooth,

then so is $L^+X$.

(2) $LX$ is an ind-scheme, ind-affine over $k$.

(3) $L^+X \longrightarrow LX$ is a closed immersion.

e.g. $X = \mathbb{G}_m$

(1) $L^+\mathbb{G}_m \cong \mathbb{G}_m \times \mathbb{A}^\omega$

$$\underset{(a_0 + a_1 t + a_2 t^2 + \cdots)}{R[[t]]^\times} \mapsto (a_0), \ (a_1, a_2, a_3, \cdots)$$

(2) $L\mathbb{G}_m \cong \mathbb{Z} \times \mathbb{G}_m \times \mathbb{A}^\omega \times \hat{\mathbb{A}}^\infty_0$, $\hat{\mathbb{A}}^\infty_0 = \varinjlim_n \operatorname{Spec} k[x_1, \cdots, x_n]/(x_1^n, \cdots, x_n^n)$

$\operatorname{Spec} R$ conn'd.

$$t^k (\underset{\substack{\uparrow \\ R^\times}}{a_0} + \underbrace{a_1 t + a_2 t^2 + \cdots}_{\mathbb{A}^\omega}) (1 + \underbrace{a_{-1} t^{-1} + a_{-2} t^{-2} + \cdots + a_{-n} t^{-n}}_{\text{nilpotent}})$$

**Def.** The affine grassmannian $\mathrm{Gr}_G$ of an affine $\overset{\text{smooth}}{\vee}$ gp scheme $G$ to be the fppf sheafifications of $[LG/L^+G]_{\acute{e}t}$.

**Lemma** $\mathrm{Gr}_G$ is the presheaf

$$R \mapsto \{ \mathcal{E} \text{ a } G\text{-torsor on } D_R = \mathrm{Spec}\, R[[t]],\ \mathcal{E}|_{D_R^\times} \overset{\beta}{\underset{\sim}{\to}} \mathcal{E}^{\mathrm{triv}}|_{D_R^\times} \}$$

$D_R^\times = \mathrm{Spec}\, R((t))$.

**Prop.** $\mathrm{Gr}_G$ is an ind-scheme, ind of ft.

If $G$ is reductive, then $\mathrm{Gr}_G$ is ind-proper.

**eg.** $\mathrm{Gr}_{\mathbb{G}_m} \cong \mathbb{Z} \times \hat{\mathbb{A}}^\infty_0$.

**Sketch of Pf.** **Step 1.** reduce to $G = GL_n$

**Facts:** (1) $\exists$ $k$-linear faithful rep'n $\rho: G \to GL_n$ s.t.

$GL_n/G$ is quasi-affine.

If $G$ is reductive, can require $GL_n/G$ to be affine.

(2) $\mathrm{Gr}_G \xrightarrow{\tilde{\rho}} \mathrm{Gr}_{GL_n}$, $\tilde{\rho}$ is locally closed immersion if $GL_n/G$ is quasi-affine

closed immersion affine.

Step 2. For $GL_n$. (only for $k$-pts)

$$Gr_{GL_n}(k) = \{ \text{lattices in } k((t))^n \}$$

$\Lambda$ f. proj. $k[[t]]$ submod. $L$ of $k((t))^n$ s.t.

$$L \underset{k[[t]]}{\otimes} k((t)) = k((t))^n$$

$$Gr^N = \{ t^{-N}\Lambda_0 \subset \Lambda \subset t^N \Lambda_0 \}$$

$\Lambda_0 = k[[t]]^{\oplus n}$

$Gr = \bigcup_{N \geq 1} Gr^N$, $\quad Gr^N \hookrightarrow Gr(2nN)$

$\Lambda \longmapsto \Lambda / t^{-N}\Lambda_0$

image = subspaces stable under $t$-action.

## §3. Subvar. of $Gr$.

1. Big cell.

<u>Def</u>. $L^-G : R \mapsto G(R[t^{-1}])$

$L^-G$ is ind-gp-scheme ind-of. f. type

eg. $G = G_m$, $\quad L^-G \cong \hat{\mathbb{A}}_0^\infty \times G_m$.

<u>Def</u>. $L^{--}G = \ker\left(L^-G \xrightarrow{t^{-1} \mapsto 0} G\right)$. <u>eg.</u> $L^{--}G_m \cong \hat{\mathbb{A}}_0^\infty$

**Prop.** $\forall\, x \in \mathrm{Gr}(k)$, the orbit map

$$\begin{aligned} L^{--}G &\longrightarrow \mathrm{Gr} \\ g &\longmapsto g\cdot x \end{aligned}$$ is an open immersion.

$$L^{--}G \times L^{+}G \xhookrightarrow{\text{big cell}} LG$$

2. Schubert varieties

Recall. Cartan decomp.

$$G(k((t))) = \coprod_{\lambda \in X_*(T)_+} G(k[\![t]\!])\, t^{\lambda}\, G(k[\![t]\!])$$

$$\rightsquigarrow \mathrm{Gr}_{red} = \coprod_{\lambda \in X_*(T)_+} \mathrm{Gr}_{\lambda}$$

- $\mathrm{Gr}_{\lambda}$ smooth of dim $\langle 2\rho, \lambda\rangle$
  $\rho$ ← half sum of $\Phi_+$

**Def.** For $\lambda, \mu \in X_*(T)_+$, we say $\lambda \geq \mu$ if $\lambda - \mu = \sum a_n \mu_n$, $a_n \in \mathbb{Z}_{\geq 0}$, $\mu_n \in \Phi_+^{\vee}$.

$$\mathrm{Gr}_{\leq\lambda} := \coprod_{\mu\leq\lambda} \mathrm{Gr}_{\mu}.$$

**Fact.** $\overline{\mathrm{Gr}_{\lambda}} := \mathrm{Gr}_{\leq\lambda}$ called Schubert var.

**Def.** $\lambda \in X_*(T)_+$ is called $\begin{cases} \text{even} & \text{if } \langle 2\rho, \lambda\rangle \text{ is even} \\ \text{odd} & \text{if } \langle 2\rho, \lambda\rangle \text{ is odd.} \end{cases}$

**Prop.** $\pi_0(\mathrm{Gr}_G) \cong X_*(T)/\mathbb{Z}\Phi^\vee$

For $c \in X_*(T)/\mathbb{Z}\Phi^\vee$, $\mathrm{Gr}^c$ connected comp.

**Cor.** $\mathrm{Gr}^c$ closed & open in $\mathrm{Gr}_G$.

**Prop** $\mathrm{Gr}_\lambda \subset \mathrm{Gr}_c \iff \lambda \in c$

**Cor.** Parity of $L^+G$-orbits on each conn'd component is constant.

**Def.** We call $\lambda \in X_*(T)$ minuscule, if $\langle \alpha, \lambda \rangle = 0$ or $1$ for any $\alpha \in \Phi_+$.

$X_*(T)_+^{min}$ = the set of minuscule positive coweights.

**Prop** $X_*(T)_+^{min} \longrightarrow X_*(T)_+ \twoheadrightarrow X_*(T)/\mathbb{Z}\Phi^\vee$ is a bijection.

& minimal in each coset.

**Cor.** $\lambda \in X_*(T)_+^{min}$, $\mathrm{Gr}_{\leq\lambda} = \mathrm{Gr}_\lambda$ smooth

$$\cong G/P_\lambda$$

<u>Semi-infinite orbits</u> Assume $G$ split.

$$\begin{tikzcd} & \mathrm{Gr}_{B,red} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Gr}_{T,red} & & \mathrm{Gr}_{G,red} \end{tikzcd}$$

Prop.

**Fact.** 1) $q$ is locally closed immersion on each conn'd comp.

2) $p$ is bijective on $\pi_0$

$$(\mathrm{Gr}_{\mathbb{G}_m})_{red} \cong \mathbb{Z}$$

$$(\mathrm{Gr}_T)_{red} \cong X_*(T)$$

**Def.** $\lambda \in X_*(T)$

$$S_\lambda := q(p^{-1}(\lambda)) \xrightarrow{\text{locally closed}} \mathrm{Gr}_{G, red}$$

**Rmk.** $B \hookleftarrow U$

$S_\lambda = LU$-orbit through $t^\lambda$

$(S_\lambda \cong L^{--}B)$

**Rmk.** $\mathrm{Gr} = \coprod_{\lambda \in X_*(T)} S_\lambda$

$$G(k((t))) = \coprod_{\lambda \in X_*(T)} U(k((t)))\, t^\lambda\, G(k[[t]]) \quad \text{Iwasawa decomp.}$$

## §4. Beilinson - Drinfeld grassmannian

$X$ a smooth proj. curve $/ k = \overline{\mathbb{F}_q}, \mathbb{C}$

$I$ finite set

**Def.** $\mathrm{Gr}_{X^I}$ presheaf

$$R \mapsto \{ (x_i)_{i \in I} \in X(R)^I,\ \mathcal{E} \text{ a } G\text{-torsor on } X_R,\ \beta: \mathcal{E}|_{X_R - \cup \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}^{triv}|_{X_R - \cup \Gamma_{x_i}} \}$$

**Thm.** $\begin{array}{c} Gr_{X^I} \\ \downarrow \ell_I \\ X^I \end{array}$ are ind-proper ind-scheme.

"remembering legs"

**Example** $|I| = \{1\}$. $Gr_X = \{ x \in X, \ \mathcal{E} \overset{\Gamma_x}{\dashrightarrow} \mathcal{E}^{triv} \}$

$$Gr = \{ \mathcal{E}|_{D^\times} \overset{\sim}{\dashrightarrow} \mathcal{E}^{triv}|_{D^\times} \}$$

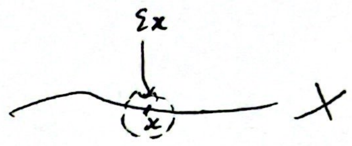
Fix $x \in X(k)$

**Thm.** $Gr_G$ is isom. to the presheaf $Gr_G'$.

$$R \mapsto \{ \mathcal{E} : G\text{-torsor on } X_R, \ \mathcal{E}|_{X_R - \{x\}} \xrightarrow{\sim} \mathcal{E}^{triv}|_{X_R - \{x\}} \}$$

**Pf.** $Gr_G' \longrightarrow Gr_G$

$(\mathcal{E}, \beta) \mapsto (\mathcal{E}|_{\Gamma_{\hat{x}}}, \beta|_{\Gamma_{\hat{x}}})$




$\mathcal{E}^{triv}|_{X - \{x\}}$

$(\mathcal{E}_x, \mathcal{E}^{triv}|_{X - \{x\}}) \rightsquigarrow \mathcal{E}$ on $X$.

**Cor.** $\ell : Gr_X \longrightarrow X$ fibers of $\ell$ are $Gr_G$

Globally $Gr_X \neq X \times Gr_G$

$Aut^+(D) =$ auto. gp of formal disc $D$ preserving origin

$Aut^+(D) = Aut^{++}(D) \rtimes \mathbb{G}_m$

$\mathrm{Conn}: X \longrightarrow B\mathrm{Aut}^{+}(D)$ corresponding to

$\hat{X} \longrightarrow X$ $\quad \mathrm{Aut}^{+}(D)$ - torsor

$\mathrm{Aut}^{+}(D) \curvearrowright \mathrm{Gr}_G$ , $g \cdot (\mathcal{E}, \beta) = (g^* \mathcal{E}, g^* \beta)$

$$\begin{tikzcd} \mathrm{Gr}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ \mathrm{Gr}_G / \mathrm{Aut}^{+}(D) \arrow[r] & B\mathrm{Aut}^{+}(D) \end{tikzcd}$$

i.e. $\quad \mathrm{Gr}_X \cong \hat{X} \overset{\mathrm{Aut}^{+}(D)}{\times} \mathrm{Gr}_G$

<u>Example</u>. $|I| = 2$

$$\ell : \mathrm{Gr}_{X^2} \longrightarrow X^2$$

(1) over $x_1 = x_2$, the fiber $= \mathrm{Gr}_G$

(2) over $x_1 \neq x_2$, the fiber $= \mathrm{Gr}_G \times \mathrm{Gr}_G$

<u>Lemma</u>

$$\begin{tikzcd} (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U \arrow[r] \arrow[d] \arrow[loop above, "\mathfrak{S}_2"] & \mathrm{Gr}_{X^2} \arrow[d] \arrow[loop above, "\mathfrak{S}_2"] & \mathrm{Gr}_X \arrow[l] \arrow[d] \arrow[loop above, "\mathrm{id}"] \\ U \arrow[r, hook, "j", "\text{open}"'] & X^2 & X \arrow[l, hook', "\Delta"', "\text{closed}"] \end{tikzcd}$$

<u>Rmk</u>. Such phenomenon generalizes to general finite sets $|I|$.

## §5. Satake cat.

$X$ scheme of f type$/k$

$\mathrm{Shv}^c(X) \supset \mathrm{Perv}(X)$

Let $G$ be an affine gp $\curvearrowright X$

$G \times X \overset{pr}{\underset{act}{\rightrightarrows}} X$

**Def.** $\mathrm{Perv}_G(X)$ $G$-equiv. obj. in $\mathrm{Perv}(X)$, i.e.

(1) $\mathcal{F} \in \mathrm{Perv}(X)$

(2) $\theta : pr^* \mathcal{F} \xrightarrow{\sim} act^* \mathcal{F}$

satisfying 1) $e^* \theta = id$, 2) cocycle condition on $G \times G \times X$.

**Prop.** $\mathrm{Perv}_G(X) \xrightarrow{\mathrm{Oblv}} \mathrm{Perv}(X)$ is fully faithful if $G$ is conn'd. smooth.

**Cor.** $N \overset{\text{closed}}{\triangleleft} G$, $N$ conn'd, $N \curvearrowright X$ trivially, then

$\mathrm{Perv}_G(X) \xrightarrow{\sim} \mathrm{Perv}_{G/N}(X)$.

**Now** $X = \varinjlim (\dashrightarrow X_n \xrightarrow{i_n} X_{n+1} \hookrightarrow \cdots)$ ($X_n$ scheme of f. type)

$\circlearrowleft$

$\mathcal{G}$ pro-alg. gp scheme

$\mathcal{G} \twoheadrightarrow G_n \curvearrowright X_n$ (can arrange $G_n \overset{qt}{\twoheadleftarrow} G_{n+1}$

$\mathrm{Perv}_{\mathcal{G}}(X_n) := \mathrm{Perv}_{G_n}(X_n)$, $\mathrm{Perv}_{\mathcal{G}}(X) := \varinjlim_n \mathrm{Perv}_{\mathcal{G}}(X_n)$

$\rightsquigarrow L^+G \curvearrowright Gr_G$

$$\mathrm{Sat}^{loc}_G := \mathrm{Perv}_{L^+G}(Gr_G)$$

<u>Prop.</u> $\mathrm{Sat}^{loc}_G$ is semisimple, irred. obj. $IC_\mu$.

$\uparrow$ need coeff $= \overline{\mathbb{Q}_\ell}$ (not true for $\mathbb{Z}_\ell$)

<u>Def.</u> $Gr \tilde{\times} Gr : R \mapsto \mathcal{E}_1, \mathcal{E}_2$ $G$-torsors on $D_R$

$$\mathcal{E}_1 \overset{D_R^\times}{\underset{\beta_1}{\xrightarrow{\sim}}} \mathcal{E}_2 \overset{D_R^\times}{\underset{\beta_2}{\xrightarrow{\sim}}} \mathcal{E}^{triv}$$

$$(m, pr) : Gr \tilde{\times} Gr \longrightarrow Gr \times Gr$$

$$(\mathcal{E}_1 \xrightarrow{\beta_1} \mathcal{E}_2 \xrightarrow{\beta_2} \mathcal{E}_{triv}) \mapsto \begin{pmatrix} \mathcal{E}_1 \xrightarrow{\beta_2 \circ \beta_1} \mathcal{E}_{triv} \\ \mathcal{E}_2 \xrightarrow{\beta_2} \mathcal{E}_{triv} \end{pmatrix}$$

<u>Convolution prod.</u>

$$\tilde{X} \underset{\pi}{\xrightarrow{G\text{-torsor}}} X$$

integral case: need perverse truncation

$\mathcal{F} \in \mathrm{Perv}(X)$ $\qquad \pi^*[\dim G]\mathcal{F} \boxtimes \mathcal{G} \in \mathrm{Perv}_G(\tilde{X} \times Y)$

$\mathcal{G} \in \mathrm{Perv}_G(Y)$

$$\underset{\text{descent}}{\overset{\text{smooth}}{\Longrightarrow}} \mathcal{F} \tilde{\boxtimes} \mathcal{G} \in \mathrm{Perv}(X \tilde{\times} Y)$$

$$X \tilde{\times} Y = \tilde{X} \overset{G}{\times} Y$$

<u>Rmk.</u> True for $D^b_c$.

$\rightsquigarrow * : \ \mathrm{Sat}^{loc}_G \times \mathrm{Sat}^{loc}_G \xrightarrow{\widetilde{\boxtimes}} \mathrm{Perv}_{L^+G}(Gr \tilde{\times} Gr) \xrightarrow{m_!} \mathrm{Sat}^{loc}_G$

<u>Thm</u>. $A, B \in \mathrm{Sat}^{loc}_G$, $A * B$ is perverse.

## §6. Semi-local Satake cat.

$$\mathrm{Sat}^{sloc}_G \subset \mathrm{Perv}_{(L^+G)_{X^I}}(Gr_{X^I})$$

those $\mathcal{F}$ ULA wrt. $t_I : Gr_{X^I} \to X^I$.

ULA $\Rightarrow$

1) $I \twoheadrightarrow J$, $\Delta(\Phi) : Gr_J \to Gr_I$

$$\Delta(\Phi)^*[-] : \mathrm{Sat}^{sloc}_I \to \mathrm{Sat}^{sloc}_J$$

2) $|I| = 2$, $\quad Gr_X \tilde{\times} Gr_X \xrightarrow{m_2} Gr_{X^2}$

$$\mathcal{F} \boxtimes \mathcal{G} := m_{2!}(\mathcal{F} \widetilde{\boxtimes} \mathcal{G})$$

$$\rightsquigarrow \boxtimes : \mathrm{Sat}^{sloc}_{G,\{\cdot\}} \times \mathrm{Sat}^{sloc}_{G,\{\cdot\}} \to \mathrm{Sat}^{sloc}_{\{+,*\}}$$

3) $\mathcal{F} \boxtimes \mathcal{G} = j_{!*}(\mathcal{F} \boxtimes \mathcal{G}|_U)$

4) $\Delta : X \to X^2 \rightsquigarrow \Delta : Gr_X \to Gr_{X^2}$

$$\circledast : \Delta^*[-1] \circ \boxtimes : \mathrm{Sat}^{sloc}_{G,\{\cdot\}} \times \mathrm{Sat}^{sloc}_{G,\{\cdot\}} \to \mathrm{Sat}^{sloc}_{G,\{\cdot\}}$$