The affine Grassmannian as a presheaf quotient

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§ The presheaf quotient suffices

Fix a ring A (e.g. $A = \mathbb{C}$), a reductive A-gp G (e.g. $G = GL_n$)

a smooth affine A-gp scheme whose geom. A-fibers are conn'd and reductive

Thomason, Gille

Rmk. $\exists\, G \hookrightarrow GL_{n,A}$ iff $\overbrace{\mathrm{rad}(G)}^{\text{max'l central torus}}$ is isotrivial $\leftarrow$ splits over a finite étale cover of A.

automatic if A is normal

The loop group (resp. positive loop subgroup) of G is the functor

$$LG : \{A\text{-algs}\} \longrightarrow \{\text{groups}\}$$
$$B \longmapsto G(B((t))).$$
$$\cup$$
$$(\text{resp. } L^+G : \quad B \longmapsto G(B[[t]]))$$

The affine grassmannian $Gr_G : \{A\text{-alg's}\} \longrightarrow \{\text{pointed sets}\}$

is the étale sheafification of the presheaf quotient $LG/L^+G : B \mapsto G(B((t)))/G(B[[t]])$

Fact. L^+G is an affine A-scheme

LG is an ind-affine A-ind-scheme $= \varinjlim_n (X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots)$

Gr_G is an ind-projective A-ind scheme

(at least if $\exists\, G \hookrightarrow GL_{n,A}$)

Upshot: $Gr_G(-)$ commutes w/ filtered direct limits of rings

$X_0 = L^+G$

$G(B((t)))/G(B[[t]]) \leftleftarrows G(B\{t\}[\tfrac{1}{t}])/G(B\{t\})$

Étale type approx cf. Bouthier–Č.

$\rightsquigarrow$ Gr_G can be formed w/ Henselian loops instead

Rmk let $B\{t\} = B[t]^{\mathrm{hens}}_t \subset B[[t]]$, $B\{t\}[\tfrac{1}{t}] \subset B((t))$.

<u>Main Thm (Č)</u>

(a) Gr_G is the Zariski sheafification of the presheaf quotient LG/L^+G.

<u>Compare with</u>: if $P \subset G$ is a parabolic subgp, then G/P is the Zariski sheafification of $B \mapsto G(B)/P(B)$ (b/c parabolic subgps of the same type are conjugate Zariski-locally on A [SGA3])

(b) Gr_G is the presheaf quotient LG/L^+G if G is <u>totally isotropic</u> in the sense that Zariski locally on A it has a <u>proper</u> parabolic subgroup.

meets every factor of $G^{ad} = G/Z(G)$

(split $\to$ quasi-split $\Rightarrow$ totally isotropic)

<u>Rmk</u> Not know whether (b) fails for general reductive G. Not sure whether there is a more general class of G for which Thm holds.

<u>Eg. (of Main Thm (a))</u> For a valuation ring $\mathcal{O}$ w/ fraction field K and a reductive $\mathcal{O}$-gp H, $H(K((t))) = H(\mathcal{O}((t)))\, H(K[[t]])$ (valuative crit. of properness + Thm (a))

<u>Eg.</u> ($H = G_m$) $K((t))^\times = t^{\mathbb{Z}}\, K[[t]]^\times$.

<u>§ Reinterpretation in terms of torsors</u>

Alternative def'n / modular description of the affine Grassmannian

$Gr_G : B \mapsto \{(\mathcal{E}, \tau) : \mathcal{E} \text{ is a } G\text{-torsor} / B[[t]],\ \tau : \mathcal{E}|_{B((t))} \xrightarrow{\sim} G_{B((t))} \text{ trivialization}\}$

$U \longleftarrow$ subfunctor parametrizing $(\mathcal{E}, \tau)$ w/ $\mathcal{E}$ trivial

LG/L^+G (presheaf quotient)

By renaming B to A, Main Thm reduces to

<u>Main Thm'</u> Let G be a reductive gp / ring A. If either

(a) A is semilocal, or

(b) G is totally isotropic,

then no nontrivial G-torsor E / $A[\![t]\!]$ trivializes over $A(\!(t)\!)$.

and $\mathrm{Gr}_G(A) = G(A(\!(t)\!)) / G(A[\![t]\!])$

In case (b), also $G(A(\!(t)\!)) = G(A[t^{\pm 1}])\, G(A[\![t]\!])$, so that

$\mathrm{Gr}_G(A) = G(A[t^{\pm 1}]) / G(A[t])$.

<u>Rmk</u> Not clear how to prove directly even when $A = \mathbb{Z}$ and $G = GL_{n,\mathbb{Z}}$

<u>Compare w/ (Birkhoff decomposition)</u> for a reductive gp H over a field k, $S \subset H$ max'l k-split torus

$$H(k(\!(t)\!)) = \coprod_{\substack{\lambda \in X_*(S)^+ \\ \parallel \\ \mathbb{G}_m \xrightarrow{\lambda} S \subset H}} H(k[t^{-1}])\, t^\lambda\, H(k[\![t]\!])$$

§ Passage to $\mathbb{P}^1_A$

By patching $\mathcal{E}$ w/ the trivial G-torsor over $\mathbb{P}^1_A \setminus \{t=0\}$, reduce Main Thm' to

<u>Thm</u> (a) For a reductive gp G over a semilocal ring A, every G-torsor E over $\mathbb{P}^1_A$

satisfies $E|_{\{t=0\}} \simeq E|_{\{t=\infty\}}$ ($\Leftrightarrow$ up to iso., the G-torsor $s^*(E)$ doesn't depend on $s \in \mathbb{P}^1_A(A)$).

(b) For a totally isotropic reductive gp G over a ring A and a G-torsor E over $\mathbb{P}^1_A$ if $E|_{\{t=\infty\}}$ is trivial, then $E|_{\mathbb{A}^1_A}$ is trivial

The theorem is proved by ultimately reducing to $A = k$ field

<u>"Classical Thm"</u> For a reductive gp H over a field k, every $\{H\text{-torsors over } \mathbb{P}^1_k \text{ trivial at } \infty\} \simeq$ ~~[illegible]~~ reduces to a $\mathbb{G}_m$-torsor via some $\mathbb{G}_m \to H$.

To reduce Thm (b) to local A use:

<u>Lemma</u> (Gabber–Quillen patching) For a locally f.p. gp. algebraic space G over a ring A, a G-torsor over $\mathbb{A}^1_A$ descends to A iff it does so Zariski locally on A

The proof of Thm (a) uses the geometry of Bun_G : alg stack $/A$ parametrizing G-torsors over $\mathbb{P}^1_A$.

$$\mathrm{Bun}_G = \mathrm{Res}_{\mathbb{P}^1_A/A}(\mathbb{B}G)$$

<u>Eg.</u> For an A-gp. M of mult. type,

$$\mathrm{Bun}_M \simeq \underbrace{\mathbb{B}M}_{\text{"constant torsors"}} \times \underbrace{X_*(M)}_{}$$

α-inflation of $\mathcal{O}(1)$ on $\mathbb{P}^1_A$ $\longleftarrow\!\shortmid$ $\alpha : \mathbb{G}_{m,A} \to M$

(eg. $\mathrm{Pic}(\mathbb{P}^1_A) \simeq \mathrm{Pic}(A) \times \mathbb{Z}$)

if $G = M$, then both (a) and (b) follow from this.

For general reductive G, use

<u>Prop</u> $BG \hookrightarrow \mathrm{Bun}_G$ is an open immersion.

<u>Pf</u> using deformation theory. $\square$

Now we modify the torsor to make it trivial over $\mathbb{P}^1_{A/m}$. This is based on:

<u>Thm</u> (Bruhat - Tits) "unramified nature of the Whitehead gp $G(K)^+ \triangleleft G(K)$",

For a simply conn'd, totally isotropic gp. G over a henselian DVR V w/ $K = \mathrm{Frac}(V)$, every elt of $G(K)/G(V)$ is represented by a product of "elementary matrices". Fix opposite (proper) parabolics $P^{\pm} \subset G$, $U^{\pm} \triangleleft P^{\pm}$

every elt. is rep'd by $u_1 u_2 \dots u_n$, $u_i \in U^{\pm}(K)$

$$\prod LU^{\pm}(k) \twoheadrightarrow \mathrm{Gr}_G(k).$$

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§ The case of the B_{dR}^+ - affine grassmannian

<u>Fix</u> K = nonarch. local field

$\cup$

$\mathcal{O}_K$ = its ring of integers

Consider perfectoid $\mathcal{O}_K$ - alg pair (A, A^+)

$\rightsquigarrow$ tilt $(A^\flat, A^{\flat +})$, pseudouniformizer $\varpi^\flat \in A^\flat$

Have: $\theta : W_{\mathcal{O}_K}(A^{\flat+}) \twoheadrightarrow A^+$

$$[a] \longmapsto a^\sharp$$

$\ker\theta$ is a principal ideal, generated by a nonzero divisor $\xi \in W_{\mathcal{O}_K}(A^{\flat+})$.

Mixed char analogue of $A[[t]]$: $\quad B_{dR}^+(A) = W_{\mathcal{O}_K}(A^{\flat+})\left[\frac{1}{[\varpi^\flat]}\right]^{\wedge}_{\xi}$

of $A((t))$ $\quad B_{dR}(A) = B_{dR}^+(A)\left[\frac{1}{\xi}\right]$.

Eg. 1) $K|\mathbb{Q}_p$ finite, A: K-alg. $\rightsquigarrow$ $B_{dR}^+(A)$ and $B_{dR}(A)$ are K-algs

2) $K|\mathbb{Q}_p$ finite, A is an $(\mathcal{O}_K/\pi_K)$-alg $\rightsquigarrow$ $B_{dR}^+(A) \simeq W_{\mathcal{O}_K}(A)$,

$$B_{dR}(A) \simeq W_{\mathcal{O}_K}(A)\left[\tfrac{1}{p}\right]$$

3) $K \simeq \mathbb{F}_q((\zeta))$ $\rightsquigarrow$ $B_{dR}^+(A) \simeq A[[\pi_K - \zeta]]$

$$B_{dR}(A) \simeq A((\pi_K - \zeta)).$$

Fix a reductive gp $G/\mathcal{O}_K$ (resp. $/K$)

Have. loop gp $LG: (A, A^+) \mapsto G(B_{dR}(A))$

$\cup \qquad\qquad \cup$

positive loop subgp $L^+G: (A, A^+) \mapsto G(B_{dR}^+(A))$

(if G/K restrict to $(A, A^+)/K$)

Def. The B_{dR}^+-affine grassmannian $Gr_G^{B_{dR}^+}$ is the étale sheafification of the presheaf quotient LG/L^+G. ($\Rightarrow$ get a v-sheaf)

Thm (Č - Youcis) $\mathrm{Gr}_G^{B_{dR}^+}$ is the sheafification of the presheaf quotient LG/L^+G for the analytic topology on perfectoid $\mathcal{O}_K$-alg. pairs (A, A^+).

Main inputs: - Henselian invariance / algebraization for G-torsors in the style of Elkik (Bouthier - Č)

$x \in \mathrm{Spa}(A, A^+)$

$$\{G\text{-torsors} \mid (k(x), k(x)^+)\} \Leftrightarrow \{G\text{-torsors over shrinking nbhds of } x \in \mathrm{Spa}(A, A^+)\}$$

- Grothendieck - Serre conj. for DVRs.