Equivalence (II)

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Goal: To show

There exists an integer d s.t.

- $\forall \mathcal{F} \in Shv_c(I\backslash LG/I)$, if $H^i(\Psi(\mathcal{F})) = 0$ for $i \in [-d,d]$, then we have ${}^pH^0(\mathcal{F}) = 0$

(i.e. $\forall \mathcal{F} \in Shv_c(I\backslash LG/I)$, ${}^pH^0(\mathcal{F}) \neq 0$, then $\exists\ k \in [-d,d]$, s.t. $H^k(\Psi(\mathcal{F})) \neq 0$)

$$\begin{tikzcd}
Perf(St_{Nilp}/\check{G}) \arrow[r, "\Phi_{perf}"] \arrow[d] & Shv_c \arrow[dl, "\Psi"] \\
Coh(St_{Nilp}/\check{G}) &
\end{tikzcd}$$

if $H^i(\Psi(\mathcal{F}))=0$ $\forall i \in [-d,d]$

Lusztig's thm: $\lambda \in X_*(T)_+$ large enough, we only need to show that

$$\mathcal{F} * J_\lambda \text{ concentrated in deg. } [d - 2\dim \tilde{\mathfrak{g}}, +\infty) \cup (-\infty, -d + 2\dim \tilde{\mathfrak{g}}]$$

$$\left(\Rightarrow \mathcal{F} * J_\lambda * J_{-\lambda} \quad [d - 2\dim\tilde{\mathfrak{g}} - 2\dim G/B, +\infty) \cup (-\infty, -d + 2\dim\tilde{\mathfrak{g}} + 2\dim G/B] \right)$$

$$!\text{-supp}(\mathcal{F} * J_\lambda) \subset \underbrace{!\text{-supp}(\mathcal{F}) * !\text{-supp}(J_\lambda)}_{\substack{\| \\ S\cdot\lambda}}$$

$S\cdot\lambda$ finite

can require $S\cdot\lambda \subset W_{fin} \cdot X_*(T)_+ \Rightarrow$ each element in $S\cdot\lambda$ is minimal length rep. in W_{ext}/W_{fin}

Lemma. $\forall\, w \in W_{ext}$,

$$RHom(\Delta_w, \mathcal{F} * J_\lambda) = 0 \quad \text{or} \quad RHom(\Delta_w, \mathcal{F} * J_\lambda) = RHom(\Delta_w * \Xi, \mathcal{F} * J_\lambda)$$

Pf. $w \notin S \cdot \lambda$ ✓

$w \in S \cdot \lambda$, RHS $= RHom(Av_! \Delta_w^{IW}, \mathcal{F} * J_\lambda)$

$Av_! \Delta_w^{IW}$ has a filtration w/ graded $\bigoplus_{w' \in w W_{fin}} \Delta_{w'}$. □

$$RHom(\Delta_w * \Xi, \mathcal{F} * J_\lambda)$$

$$\simeq RHom(Av_* \Delta_w, Av_* (\mathcal{F} * J_\lambda))$$

$$\simeq RHom(\Phi_{AB}^{-1}(\Delta_w^{IW}), \Phi_{AB}^{-1}(Av_* (\mathcal{F} * J_\lambda)))$$

① Δ_w^{IW}

Φ_{AB}^{-1} : $Perv \longrightarrow Coh^{[0, \dim \tilde{g}]}$

② $H^i(\Phi(\mathcal{F})) = 0$, $\forall\, i \in [-d, d] \Rightarrow Av_* (\mathcal{F} * J_\lambda) \in (-\infty, -d) \cup (d, +\infty)$

③ $Coh(\tilde{N}/\check{h})$ has coh. dim $\dim \tilde{N}$.

$\Rightarrow Ext^i(\Delta_w, \mathcal{F} * J_\lambda) = 0$, $\forall\, i \in (-\infty, -d + 2\dim \tilde{g}) \cup (d - 2\dim \tilde{g}, +\infty)$.

$\rightarrow \mathcal{F} * J_\lambda$ is concentrated in ↑

$$
\begin{tikzcd}
\mathrm{Perf}\left(\tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} / \check{H}\right) \arrow[rr, "\Phi_{perf} = \Phi_{diag}(-) * \tilde{\Xi} * \Phi_{diag}(-)"] \arrow[d, "i"'] & & \mathrm{Shv}_c(I \backslash LG / I) \arrow[dll, "\Psi"] \\
\mathrm{Coh}\left(\tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} / \check{H}\right) & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Shv}_c(I \backslash LG / I) \arrow[r] \arrow[dr, "\Psi'"'] & \mathrm{End}\left(\mathrm{Shv}_c\left(I \backslash LG / (I^u, \psi)\right)\right) \simeq \mathrm{End}\left(\mathrm{Coh}\left(\tilde{\mathcal{N}} / \check{H}\right)\right) \\
& \mathrm{End}_{\mathrm{Perf}(\check{\mathfrak{g}}/\check{H})}\left(\mathrm{Coh}(\tilde{\mathcal{N}}/\check{H})\right) \simeq \mathrm{Coh}\left(\tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} / \check{H}\right) \arrow[u]
\end{tikzcd}
$$

<u>Thm</u> (BZFN) $\mathrm{QCoh}(X) = \mathrm{Ind}\,\mathrm{Perf}(X)$, $\mathrm{QCoh}(Y) = \mathrm{Ind}\,\mathrm{Perf}(Y)$, $\pi : X \to Y$ proper

$$
\begin{tikzcd}
\Rightarrow \ \mathrm{QCoh}(X \times_Y X) \arrow[r, "\sim"] & \mathrm{Fun}_{\mathrm{QCoh}(Y)}\left(\mathrm{QCoh}(X), \mathrm{QCoh}(X)\right). \\
\mathrm{Coh}(X \times_Y X) \arrow[r, "\sim"] \arrow[u] & \mathrm{Fun}_{\mathrm{Perf}(Y)}\left(\mathrm{Perf}(X), \mathrm{Coh}(X)\right) \arrow[u, hook]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Perf} \arrow[r, "\Phi_{perf}"] \arrow[d] & \mathrm{Shv}_c \arrow[dl, "\Psi"'] \arrow[dl, bend left, "\Psi'"] \arrow[dl, phantom, "\simeq"] \\
\mathrm{Coh} &
\end{tikzcd}
$$
monoidal

To show $\Psi \simeq \Psi'$:

check $\Phi_{perf} \dashv \Psi'$.

Exercise:

$$\begin{tikzcd}
\mathrm{Perf}(X \times_Y X) \arrow[r, hook, "i"] & \mathrm{Coh}(X \times_Y X) \\
\mathrm{Perf}(X) \otimes_{\mathrm{Perf}(Y)} \mathrm{Perf}(X) \arrow[u, "\mathrm{BZFN}", "\cong"'] \arrow[ur] &
\end{tikzcd}$$

$$(M, N) \longmapsto \Delta_* M * \mathcal{O} * \Delta_* N$$