Étale cohomology of v-stacks (II)

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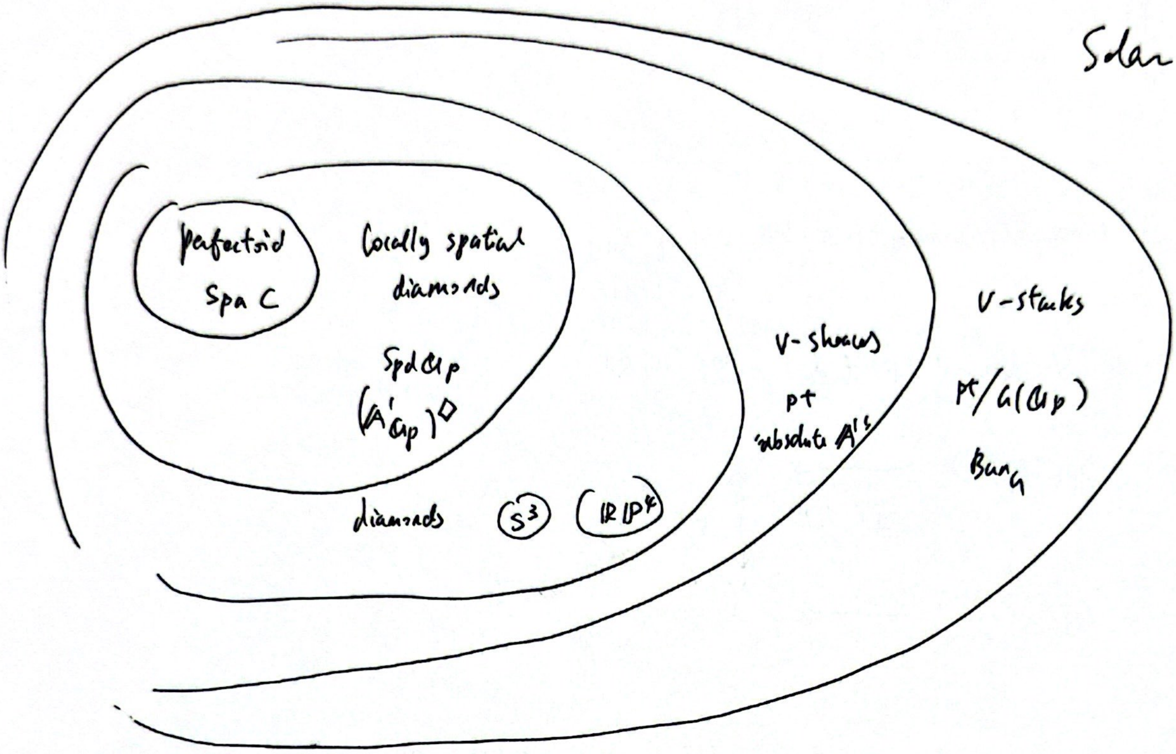
Solar system



<u>Last time:</u> defined $D_{\acute{e}t}(X, \Lambda)$ for small v-stack X for finite ring Λ w/ $p \in \Lambda^\times$

By construction, get for free:

- $f^*: D_{\acute{e}t}(Y) \longrightarrow D_{\acute{e}t}(X)$ for $f: X \to Y$
- $\otimes$ tensor product (derived)

$\Rightarrow$ Right adjoints
- $Rf_* : D_{\acute{e}t}(X) \longrightarrow D_{\acute{e}t}(Y)$
- $R\mathcal{H}om(-,-)$

<u>Next:</u> proper pushforward $Rf_!$

For schemes: pick compactification

$$\begin{tikzcd} X \arrow[r, hook, "\text{open}", "j"'] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd} \qquad Rf_! := R\bar{f}_* \circ j_!$$

For adic spaces: $\exists$ canonical compactifications (if any)

<u>Valuative criterion</u>: $\forall$ affinoid perf'd $\mathrm{Spa}(R, R^+)$

$$\begin{tikzcd} \mathrm{Spa}(R, R^\circ) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spa}(R, R^+) \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

Def. Let X be a v-sheaf. Define $\bar{X}$ by $\bar{X}(R, R^+) := X(R, R^\circ)$, v-sheaf

For $f: X \to Y$ v-sheaves, define

$$\bar{X}^{/Y}(R, R^+) := X(R, R^\circ) \times_{Y(R, R^\circ)} Y(R, R^+)$$

then $\bar{X}^{/Y} \to Y$ automatically satisfies valuative criterion.

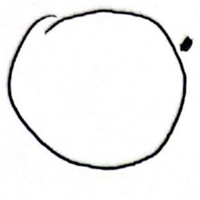
<u>Warning</u>: $\bar{X}^{/Y}$ may depart the class of locally spatial diamonds

$X \to \bar{X}^{/Y}$ may not be open.

<u>Ex</u>. $\mathbb{B}_C = \mathrm{Spa}(C\langle T\rangle, \underbrace{\mathcal{O}_C\langle T\rangle}_{\text{largest } R^+})$ closed unit disc $/C$

$$\begin{tikzcd} \mathbb{B}_C \arrow[d] \\ \mathrm{Spa}\, C \end{tikzcd}$$

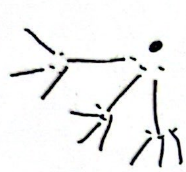
$$\overline{\mathbb{B}}_C^{/C} = \mathrm{Spa}\,(C\langle T\rangle, \underbrace{\mathcal{O}_C + \mathfrak{m}_{\mathcal{O}_C}\langle T\rangle}_{\text{smallest } R^+})$$



• radius $1+\varepsilon$.

$$\sum a_n T^n \mapsto \sup_n |a_n|(1+\varepsilon)^n$$

$\mathbb{B}_C \hookrightarrow \mathbb{P}^1$



Ex. $\overline{(\mathrm{Spa}\, R, R^+)}^{/C} = \mathrm{Spa}\,(R, \mathcal{O}_C + R^{\circ\circ})$

v-stacks

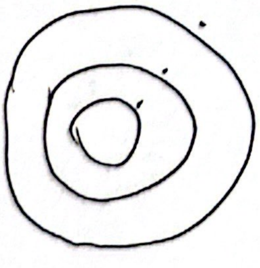
Def. $f : X \to Y$ is partially proper if separated + satisfies valuative criterion

f is proper if partially proper + quasicompact.

Rmk Partial properness is often easy, quasicompactness usually difficult.

Ex. $\mathbb{A}^1$ is partially proper

$\mathrm{Spa}\,(R, R^+) \longmapsto R$



Facts: • If $f: X \to Y$ q.c. and separated, then $\bar{f}^{/Y}: \bar{X}^{/Y} \to Y$ is proper.

• $X \hookrightarrow \bar{X}^{/Y}$ may not be open (but true in "nature")

It is open if X has any compactification $/Y$.

In this case, we say f is compactifiable.

Def. If $f: X \to Y$ quasicompact, separated, compactifiable, then

$$Rf_! := \underbrace{R\bar{f}^{/Y}_*}_{\text{proper}} \circ j_!.$$

Thm. If $f: X \to Y$ v-stacks is

• compactifiable

• representable in spatial diamonds

• of "finite geometric dimension"

"shriekable"

$\Rightarrow$ finite cohom. dim.

then $Rf_!$ has right adjoint $Rf^!$.

Pf is application of adjoint functor theorems.

In particular, get

• relative dualizing sheaf $\mathbb{D}_f = f^!(\Lambda)$

• relative Verdier duality on $D_{\acute{e}t}(X)$

$$\mathbb{D}_f(\mathcal{F}) := R\mathcal{H}om(\mathcal{F}, \mathbb{D}_f)$$

⚡ It's not a biduality

Cohomological Smoothness

No direct geometric def'n of smoothness for perfectoid rings.

Instead ask for all the cohomological shadows of smoothness.

Def. Let $\ell \neq p$. Consider shriekable $f: X \to Y$. Say f is ℓ-cohomologically smooth if

$$f^!(-) \simeq D_f \otimes f^* : D_{ét}(Y; \mathbb{F}_\ell) \to D_{ét}(X; \mathbb{F}_\ell),$$

where D_f is invertible ($\Leftrightarrow$ v-locally $\mathbb{F}_\ell[n]$)

and the same holds after any base change.

(if true, necessarily $D_f \simeq \mathbb{D}_f$)

Say f is coh. smooth if ℓ-coh. smooth, $\forall \ell \neq p$.

Good formal properties

(i) $X \xrightarrow{f} Y \xrightarrow{g} Z$, f, g coh. smooth $\Rightarrow$ $g \circ f$ coh. smooth

(ii) $X \xrightarrow{f} \!\!\!\!\rightarrow Y \xrightarrow{g} Z$, f coh. smooth, $g \circ f$ coh. smooth $\Rightarrow$ g coh. smooth

(iii) coh. smoothness can be checked v-locally on target.

Prop. coh. smooth $\Rightarrow$ f is open.

Difficulty is in producing examples.

Ex. S = profinite set

$\underline{S} \times \mathrm{Spa}\, C = \mathrm{Spa}\, \mathrm{Cont}(S, C) \xrightarrow{f} \mathrm{Spa}\, C$

f is proétale, proper, but NOT coh. smooth.

$$R\Gamma(S, f^! \Lambda) = R\mathrm{Hom}(Rf_! \Lambda, \Lambda) \simeq R\mathrm{Hom}(Rf_* \Lambda, \Lambda)$$

$$= \mathrm{Cont}(S, \Lambda)^\vee \qquad \text{"}\Lambda\text{-valued measures on } S\text{"}$$

local system on S = fin. proj. modules / $\mathrm{Cont}(S, \Lambda)$

Ex. Étale $\Rightarrow$ coh. smooth, $f^! = f^*$.

Prop. $\mathbb{B} \xrightarrow{f} pt$ is coh. smooth, and $Rf^! \Lambda = \Lambda[2](1)$.

Proof uses Huber's results on Poincaré duality for smooth curves / alg. closed field C.

However, still need to study arbitrary base changes.

By v-descent, can focus on str. tot. disc. "= union of points" to reduce to Huber (again)

Prop. Let $f: X \xrightarrow{K\text{-equiv.}} Y$ coh. smooth, free $\circlearrowleft$ K pro-p, $\circlearrowleft$ triv. locally spatial diamonds, then $f/_K: X/\underline{K} \to Y$ is coh. smooth.

Proof deferred.

Ex. C perfectoid / $\mathbb{Q}_p$. $(\mathbb{A}^1_C)^\diamond \downarrow (\mathrm{Spa}\, C)^\diamond$ is coh. smooth.

Cover $\mathbb{A}^1_C$ by $\mathbb{B}_C$, then by $\mathbb{T}_C = \mathrm{Spa}(C\langle T^{\pm 1}\rangle)$

$$\begin{array}{ll} \tilde{\mathbb{T}}^1 & \mathrm{Spa}(C\langle T^{\pm 1/p^\infty}\rangle) \\ \mathbb{Z}_p \downarrow & \\ \mathbb{T}^1 & \end{array}$$

$(\tilde{\mathbb{T}}^1_C)^\flat = \mathrm{Spa}\, C^\flat \langle T^{\pm 1/p^\infty}\rangle \longrightarrow \mathrm{Spa}\, C^\flat$

is the tire over $C^\flat \overset{\text{open}}{\subset} \mathbb{B}_{C^\flat}$.

$$\Rightarrow \underbrace{(\tilde{\mathbb{T}}^1_C)^\flat / \mathbb{Z}_p}_{(\mathbb{T}^1_C)^\diamond} \xrightarrow{\text{coh. smooth}} \mathrm{Spa}\, C^\flat$$

Ex. $X/\mathbb{Q}_p$ smooth rigid variety

$\Rightarrow X^\diamond \longrightarrow (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond$ coh. smooth b/c X locally étale $/\mathbb{A}^n$.

Ex. $\mathrm{Gr}_{G,\nu}$ is coh. smooth (it's an affine fibration over a partial flag variety)

Ex. $(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond \longrightarrow \mathrm{pt}$ is coh. smooth.

$(\mathrm{Spa}\,\mathbb{Q}_p^{cyc})^\diamond = \mathrm{Spa}\,\mathbb{F}_p((t^{1/p^\infty}))$ is perf'd ball $\Rightarrow$ coh. smooth.

$\mathbb{Z}_p^\times \curvearrowright$; $\downarrow$ pt

$(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond = (\text{perf'd ball})/\mathbb{Z}_p^\times \xleftarrow[\text{finite étale}]{} ((\text{perf'd ball})/(1+p\mathbb{Z}_p))$

$\Rightarrow X^\diamond_{FF}$, Div^1_X are coh. smooth.

Extend def'n of coh. smoothness to v-stacks:

$\mathfrak{X}$ is coh. smooth /pt if

$$\begin{tikzcd} X \arrow[d, "\substack{\text{dd} \\ \text{coh. smooth}}"'] \arrow[dr, dashed, "\substack{\text{dd} \\ \text{coh. smooth}}"] & \\ \mathfrak{X} \arrow[r] & Y \end{tikzcd}$$

old: f shriekable

Ex. $[pt/G(\mathbb{Q}_p)] \longrightarrow pt$ is coh. smooth, G smooth alg gp / $\mathbb{Q}_p$

↯ $pt \to [pt/G(\mathbb{Q}_p)]$ not coh. smooth

$G(\mathbb{Z}_p) \overset{\text{free}}{\curvearrowright} G^{\diamond}_{\mathbb{Q}_p}$ coh. smooth

$$\begin{tikzcd} G^{\diamond}_{\mathbb{Q}_p} \arrow[d] \\ pt \end{tikzcd} \Rightarrow \begin{tikzcd} {[G^{\diamond}_{\mathbb{Q}_p}/G(\mathbb{Z}_p)]} \arrow[r, "\text{étale}"] \arrow[d, "\text{coh. smooth}"'] & {[G^{\diamond}_{\mathbb{Q}_p}/G(\mathbb{Q}_p)]} \arrow[dl, "\text{coh. smooth}"] \\ pt & \end{tikzcd}$$

$$\begin{tikzcd} {[G^{\diamond}_{\mathbb{Q}_p}/G(\mathbb{Q}_p)]} \arrow[r, "\text{coh. smooth}"] & pt/G(\mathbb{Q}_p) \\ G^{\diamond}_{\mathbb{Q}_p} \arrow[u] \arrow[r, "\text{coh. smooth}"'] \arrow[ur, phantom, "\urcorner", very near start] & pt \arrow[u] \end{tikzcd}$$

$\Rightarrow pt/G(\mathbb{Q}_p)$ coh. smooth.

Artin stacks Analogy

Schemes $\rightsquigarrow$ alg. spaces $\rightsquigarrow$ Artin stacks

perfectoids $\rightsquigarrow$ loc.-spatial diamonds $\rightsquigarrow$ Artin v-stacks

Def. A small v-stack $\mathfrak{X}$ is Artin if

- $\Delta_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable in locally spatial diamonds
- $\exists$ coh. smooth surjection $X \twoheadrightarrow \mathfrak{X}$ w/ X a locally spatial diamond.

Thm. Bun_G is a coh. smooth Artin v-stack ("dim 0").

Sketch

$$\begin{tikzcd} & G(\mathbb{Q}_p) \backslash \mathrm{Gr}_{G,\mu} \arrow[d, "\pi_\mu"] \\ S \arrow[r, "\mathcal{E}"] & \mathrm{Bun}_G \end{tikzcd}$$

is coh. smooth

fiber of π_μ is $\{\underbrace{\mathcal{E}_0}_{\substack{\text{fiberwise}\\ \text{trivializable}}} \overset{\mu}{\dashrightarrow} \mathcal{E}\} \overset{\text{open}}{\subset} \underbrace{\{\mathcal{E}' \overset{\mu}{\dashrightarrow} \mathcal{E}\}}_{\mathrm{Gr}_{G,\mu}}$

$\coprod_\mu G(\mathbb{Q}_p) \backslash \mathrm{Gr}_{G,\mu} \twoheadrightarrow \mathrm{Bun}_G$ surjective (Beauville-Laszlo uniformization) $\square$.

Relation to representation theory

Bun_2 $\qquad GL_2(\mathbb{Q}_p) \qquad [\mathbb{Q}_p^\times \times_{\mathbb{Q}_p} H^0(\mathcal{O}(2))] \qquad \cdots$

$$\bullet \rightsquigarrow \bullet \rightsquigarrow \bullet \rightsquigarrow$$

$\mathcal{O}^2 \qquad \mathcal{O}(1) \oplus \mathcal{O}(-1) \qquad \mathcal{O}(2) \oplus \mathcal{O}(-2)$

$D^\times \qquad [\mathbb{Q}_p \times_{\mathbb{Q}_p^\times} H^0(\mathcal{O}(1))]$

$$\bullet \rightsquigarrow \bullet \rightsquigarrow \bullet$$

$\mathcal{O}(1/2) \qquad \mathcal{O} \oplus \mathcal{O}(1)$

$b \in B(G) \rightsquigarrow$ locally closed stratum Bun_G^b

$\mathrm{Bun}_G^b = [pt / \mathcal{G}_b]$ where

$$1 \to \text{"unipotent group diamond"} \to \mathcal{G}_b \to G_b(\mathbb{Q}_p) \to 1$$

$b \in G(\breve{\mathbb{Q}}_p)$

Prop $D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) = D_{\acute{e}t}(pt/G_b(\mathbb{Q}_p), \Lambda) = D(\mathrm{Rep}_\Lambda^{sm}(G_b(\mathbb{Q}_p)))$

Thm Semiorthogonal decomposition of $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ into $D(\mathrm{Rep}_\Lambda^{sm}(G_b(\mathbb{Q}_p)))$.