Mixed Hodge theory: Some intuitions

Pierre Deligne

ℓ-adic

X $\quad X_{\bar{k}}$

k $\quad \bar{k}$

$\ell \neq p$, $\quad H^i(X_{\bar{k}}; \mathbb{Z}_\ell)$

X smooth $/k$ char 0

$H^i_{dR}(X)$

$H_{\text{crys}}(X)$ $\quad k$ perfect char p

$/\mathbb{C}$ $\quad H^*(X(\mathbb{C}); \mathbb{Z})$

X proj. sm.

H^1 $\quad \operatorname{Pic}^\circ(X)$

$k = \mathbb{C}$ $\quad X$ proj. smooth

$$H^i(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}$$

?

$$W, \quad \operatorname{Gr}^W_i(H)_{\mathbb{C}} = \bigoplus_{p+q=i} H^{pq}$$

F

X proj. non singular, $\quad \underline{\mathbb{C}} \longrightarrow \Omega^*$

$$E_1^{pq} = H^q(X, \Omega^p) \Rightarrow H^{p+q}(X(\mathbb{C}), \mathbb{C})$$

$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

ℓ-adic

$H^i(X_{\bar{k}}, \mathbb{Q}_\ell)$ $\quad \operatorname{Gal}(\bar{k}|k)$

<u>pure</u> weight i

<u>OK</u>

$X \hookrightarrow \bar{X}$

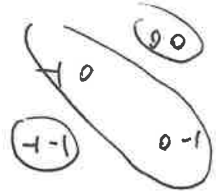
W increasing filtration

$\operatorname{Gr}^W_i(H^P)$ $\quad H^1 \quad \operatorname{Pic}^\circ$

$T \to * \to A$

$* / \mathbb{Q}_p, \quad H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes \mathbb{C}_p = \oplus H^i_{(p)}$

F of $H^i \otimes \mathbb{C}$



X proj. smooth   purity $H^i = \bigoplus_{p+q=i} H^{p,q}$ $\longleftrightarrow$ Weil conj.

Gal- action comes for free by transport of str.

$$\begin{pmatrix} X & \\ k & \bar{k} \end{pmatrix} \quad H^i(X, \mathbb{Q}_\ell)$$
$$\updownarrow \qquad\qquad \updownarrow$$
$$\begin{pmatrix} X' & \\ k' & \bar{k'} \end{pmatrix} \quad H^i(X', \mathbb{Q}_\ell)$$

Algebraic cycles

Hodge side

class in $H^{pp}_{\mathbb{Z}}(X) = \mathrm{Hom}(\mathbb{Z}_{(0,0)}, H^{2p}(X, \mathbb{Z})(p))$

class in $\mathrm{Ext}^1(\mathbb{Z}_{(0,0)}, H^{2p-1}(X, \mathbb{Z})(p))$

$H^{00}_{\mathbb{Z}} = H_{\mathbb{Z}} \cap F^0$

$\ker \to H_{\mathbb{Z}} \oplus F^0 \to H \otimes \mathbb{C} \to \mathrm{coker}$

Z codim p $\begin{matrix} X \\ k \quad \bar{k} \end{matrix}$ $H^{2p}(X_{\bar{k}}, \mathbb{Q}_\ell(p))$

$$\begin{matrix} X & & X_{\bar{k}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & & \mathrm{Spec}(\bar{k}) \end{matrix}$$

$H^{2p}(X_{\acute{e}t}, \mathbb{Q}_\ell(p))$

$\oplus H^i(-)$

$$R\Gamma(X, \mathbb{Q}_\ell(p)) = R\Gamma(\mathrm{Spec}(k), R\Gamma(X_{\bar{k}}, \mathbb{Q}_\ell(p)))$$

$$E^2_{pq} = H^p(\mathrm{Gal}(\bar{k}|k), H^q(X_{\bar{k}}, \mathbb{Q}_\ell(n)))$$

$$\Rightarrow H^{p+q}(X, \mathbb{Q}_\ell(n)).$$

degenerate $H^i(\mathrm{Gal}, H^{2n-i}(X_{\bar{e}t}, \mathbb{Q}_\ell(n)))$

Monodromy

mixed polarizable Variation of Hodge str.

$\mathcal{F}_{\mathbb{Z}}$, F^P on $\mathcal{F}_{\mathbb{Z}} \otimes \mathbb{C}$, ψ

$S/\mathbb{C}$ $(-1,0)$, $(0,-1)$

$\mathcal{F}$ smooth ℓ-adic sheaf

pure

S smooth $/\mathbb{F}_q$

mixed

$\mathbb{Z}(1)$ $(-1,-1)$

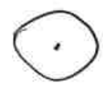
T^* variation of mixed Hodge

$N: V_t \to V_t$

$(-1,-1)$

$\bar{\mathbb{F}}_q$

$(m/m^2)^\vee$



t

ηt

$\mathrm{Spa}\left(\mathbb{F}_q((t))\right)$

$\mathcal{F}$ W_N $\mathrm{Gr}^i \overset{\sim}{\to} \mathrm{Gr}^{-i}$

t $\sqrt[\ell^n]{t}$ μ_{ℓ^n}

$I \to \quad \to \mathrm{Gal}(\bar{\mathbb{F}}_q | \mathbb{F}_q)$

~~quasi~~-unipotent $\mathbb{Z}_\ell(1)$

$(-1,0)$ $(0,-1)$

$\mathcal{F}$

W

A $\mathcal{F}$

$A^{an} =$ $\mathcal{F}_t$

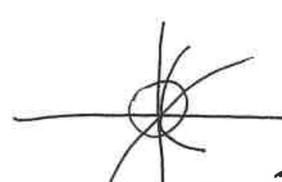
$$\frac{T \to \quad \to A_\circ}{\Gamma}$$

D^{*n}

T_i,

$\log T_i = N_i$

$\sum \lambda_i N_i$, $\lambda_i \in \mathbb{Q}_{>0}$ / $\mathbb{R}$



$\mathbb{Z}_\ell(1)^2 \to * \to \mathrm{Gal}(\ )$