Hecke algebras at roots of unity

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Lecture 1.

G finite gp
$\shortparallel$
$\underline{G}(\mathbb{F}_q)$
$\uparrow$
Conn'd reductive alg gp over $\overline{\mathbb{F}_q}$.

$q = p^f$, p prime. $\mathbb{F}_q$ finite field

$F: \underline{G} \longrightarrow \underline{G}$, $\underline{G}^F = \underline{G}(\mathbb{F}_q)$

AIM, K alg. closed field, $\mathrm{Irr}_K(G) = ?$

Classification $\{L^\lambda : \lambda \in \Lambda\}$

Dimension formulas $\dim L^\lambda = ?$

Character values $\mathrm{trace}(g : L^\lambda) = ?$

- char $K = 0$, best-known
- Char $K = p$,
- char $K = \ell \neq p$

(I) Elementary approach, induction of rep'ns from "nat'l" subgps (Borel, parabolic)

(II) Algebraic geometry / perverse sheaves

$\rightsquigarrow$ use $\underline{G}$.

(I) based on Hom functors

A fin-dim. assoc. alg. $/K$, M fin. dim'l A-module

$$E := \mathrm{End}_A(M)^{opp}$$

$$F : A\text{-mod} \longrightarrow E\text{-mod}$$
$$V \longmapsto \mathrm{Hom}_A(M, V)$$

<u>Thm</u> (Green ~1978. Cabanes, Linckelmann, ...)

<u>Assume</u> (A) $\left\{\begin{array}{c}\text{composition factors}\\ \text{of } \mathrm{soc}(M)\end{array}\right\} = \left\{\begin{array}{c}\text{composition factors}\\ \text{of } M/\mathrm{Rad}(M)\end{array}\right\}$

(B) E is symmetric, or more generally, self-injective.

Then $\{V \in \mathrm{Irr}(A) : V \text{ occurs in (A)}\}/_{\text{iso.}} \xrightarrow{\sim} \mathrm{Irr}(E)$

$$V \longmapsto F(V)$$

<u>Example 1</u> (Green, Sawada, Tinberg, Curtis, Richen)

char $k = p$, B Borel subgp $\subset G$, $B = \underset{\substack{\uparrow\\ \text{unipotent radical}}}{U}\, T$, $U \in \mathrm{Syl}_p(G)$.

$\mathrm{Irr}_k(U) = \{\underset{\substack{\uparrow\\ \text{trivial } U\text{-mod}}}{K_U}\}$ $\rightsquigarrow$ $V \in \mathrm{Irr}_k(G)$ $V|_U$ has K_U as a submodule and a factor module

Frobenius reciprocity, V is a submodule & a factor module of $\mathrm{Ind}_U^G(K_U)$.

Let $A = KG$ gp alg, $M = \mathrm{Ind}_U^G(K_U)$

Then (A) is satisfied where the sets consist of all of $\mathrm{Irr}_K(G)$.

$E = \mathrm{End}_A(\mathrm{Ind}_U^G(K_U))^{op}$ Schur basis

$\rightsquigarrow$ explicit presentation of E. $\to$ E self-injective

$\rightsquigarrow$ classification of $\mathrm{Irr}(E)$, all 1-dim'l.

Thm. $\mathrm{Irr}_K(G) \xleftrightarrow{1:1} \mathrm{Irr}(E)$

Cor'ts: G simple, simply conn'd, $\Rightarrow |\mathrm{Irr}_K(G)| = |\mathrm{Irr}(E)| = q^{\mathrm{rk}\, G}$.

$p \gg 0$, Lusztig's formula for $\dim V$

Now assume that $\mathrm{char}\, K = \ell > 0$, $\ell \neq p$

Above argument does not work. $\exists\, V \in \mathrm{Irr}_K(G)$ s.t. $V|_U$ does not contain K_U.

Consider parabolic subgroup $P \subset G$

$P = U_P \cdot L$

Harish-Chandra induction

$R_L^G : KL\text{-mod} \longrightarrow KG\text{-mod}$

${}^*R_L^G : KG\text{-mod} \longrightarrow KL\text{-mod}$ Harish-Chandra restriction

$V \in KL\text{-mod}$, $\rightsquigarrow$ $\tilde{V} \in KP\text{-mod}$ $\rightsquigarrow$ $R_L^G(V) = \mathrm{Ind}_P^G(\tilde{V})$

(U_P acts trivially)

$V \in KG\text{-mod}$, ${}^*R_L^G(V) = \mathrm{Fix}_{U_P}(V)$

<u>Def</u> $Y \in \mathrm{Irr}_K(G)$ is called "cuspidal" if

${}^*R_L^G(Y) = \{0\}$ for all $L \subsetneq G$.

Same def'n applyies to $\mathrm{Irr}_K(L)$

Consider pairs (L, X) where L = Levi of some parabolic subgp of G,

$X \in \mathrm{Irr}_K(L)$ cuspidal rep. of L

$\mathrm{Irr}_K(G \mid (L, X)) = \{$ all $Y \in \mathrm{Irr}_K(G)$ s.t. X is a composition factor of ${}^*R_L^G(Y)$ $\}$

$\rightsquigarrow$ $\mathrm{Irr}_K(G) = \coprod_{(L,X)/\sim} \mathrm{Irr}_K(G \mid (L, X))$. Harish-Chandra series above (L, X).

Fix (L, X) and set $M = R_L^G(X)$, $A = KG$, $E = \mathrm{End}_{KG}(R_L^G(X))^{op}$

(A) ✓ char $K = 0$ obvious (since M is semisimple)

char $K = \ell > 0$, Hiss ~ 1993

(B) E symmetric algebra $\exists$ $\tau : E \to K$ trace map s.t. assoc. bilinear form $(a, b) := \tau(ab)$ is non-degenerate.

char $k=0$: Howlett – Lehrer

char $k=\ell>0$, G. – Hiss – Malle ~ 1996

Hom functor theorem $\Rightarrow$

$$\{V \in \mathrm{Irr}(A) : V \text{ occurs in } (A)\} \xrightarrow{\sim} \mathrm{Irr}_k(E)$$

present situation:

$$\mathrm{Irr}_k(G \mid (L,X)) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}_K(E)$$

<u>Two problems</u>.

(I) Find cuspidal rep's of all Levis of G

(II) Determine $\mathrm{Irr}\,(\underbrace{\mathrm{End}_{kG}(R_L^G(X))^{op}}_{\text{Hecke alg.}})$

$\ell=0$: a) solved by Lusztig

b) Howlett – Lehrer + Tits deformation argument

$$\mathrm{Irr}\,(\mathrm{End}_{kG}(R_L^G(X))^{op}) \overset{1-1}{\underset{\sim}{\longleftrightarrow}} \mathrm{Irr}_k(W(L,X))$$

$W(L,X)$ = stabilizer of X in $N_G(L)/L$.

$\uparrow$

close to a finite Coxeter gp. (in most cases)

$\ell > 0$. Classification of cuspidal is open

$G = GL_n(\mathbb{F}_q)$. Dipper - James ~1980/90

every cusp. irrep. in char $\ell > 0$ lifts to a cusp. irrep. in char. 0

not true for other types of groups

$$\mathrm{Irr}\left(\underbrace{\mathrm{End}_{kG}\left(R_L^G(X)\right)}_{\text{in general non-semisimple}}\right) = ?$$

So Tits deformation argument cannot apply.

Will see: $\exists$ nat'l injection

$$\mathrm{Irr}\left(\mathrm{End}_{kG}\left(R_L^G(X)\right)\right) \hookrightarrow \mathrm{Irr}_k\left(W(L,X)\right).$$

Lecture 2 $\quad G = \underline{G}(\mathbb{F}_q)$, $q = p^b$, k alg closed, char $k = \ell \geq 0$, $\ell \nmid p$

$$\mathrm{Irr}_k(G) = \coprod_{(L,X)/\sim} \mathrm{Irr}_k(G \mid (L,X))$$

(Levi, cuspidal)

$\mathrm{Irr}_k(G \mid (L,X))$

$= \{ Y \in \mathrm{Irr}_k(G) : Y \text{ is a submodule}$

" (or factor module) of $R_L^G(X) \}$

$$\mathrm{Irr}\left(\underbrace{\mathrm{End}_{kG}\left(R_L^G(X)\right)}_{\text{Hecke alg.}}\right)$$

$W(L,X)$ = Stabilizer of X in $N_G(L)/L$

Mackey formula + Frobenius reciprocity + def. of cuspidal.

$\mathrm{End}_{KG}(R_L^G(X)) = \mathrm{Hom}_{KG}(R_L^G(X), R_L^G(X))$ has dim. $|W(L,X)|$

get standard basis $\{T_w : w \in W(L,X)\}$

get presentation in terms of generators & relations

<u>Special case</u>: $(L,X) = (T, K_T)$, $B = U \cdot T$

$\uparrow$
trivial

$W(L,X) = W$ Weyl gp of G.

W Coxeter gp $= \langle s \in S : s^2 = 1, \; (st)^{m_{st}} = 1 \; (s \neq t) \rangle$

<u>Iwahori - Matsumoto (~1960's)</u>

The mult. of the std. basis $\{T_w : w \in W\}$ of $\mathcal{H} = \mathrm{End}_{KG}(R_T^G(K_T))$

is given by

For $w \in W$, write $w = s_1 \cdots s_\ell$ $(s_i \in S)$ reduced

Then $T_w = T_{s_1} \cdots T_{s_\ell}$

$w \in W$
$s \in S$,
$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (q^{c_s} 1_K) T_{sw} + (q^{c_s} - 1) 1_K T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

where $q^{c_s} = |BsB| / |B|$

$(c_s \in \mathbb{Z}_{\geq 0})$

If G is of "split type", then $c_s = 1$, $\forall s \in S$.

<u>Example of non-split type</u> $\quad G = GU_n(\mathbb{F}_q)$

Weyl gp $\quad W = W(B_m)$ where $m = \begin{cases} \frac{n-1}{2}, & n \text{ odd} \\ \frac{n}{2} & n \text{ even} \end{cases}$

$c = \begin{cases} 3, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$

Important observation: the structure constants of $\mathcal{H}$ are given by polynomials in q

$$T_x T_y = \sum_{z \in W} g_{xyz}(q)\, 1_k \cdot T_z$$
$(x, y \in W)$

where $g_{xyz} \in \mathbb{Z}[x]$ are independent of p, q.

Can define "generic" algebra $H = H_A(W, S, \{c_s\})$ where $A = \mathbb{Z}[v, v^{-1}]$

basis T_x, T_y and mult. given by $T_x T_y = \sum_{z \in W} g_{xyz}(v^2) T_z$

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ v^{2c_s} T_{sw} + (v^{2c_s} - 1) T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

Consider ring hom. $\theta : A \to K$

$v \mapsto q^{1/2} 1_K$

then $\mathcal{H} = K \otimes_A H$

<u>Hence</u>: Study one object, $H = H_A(W, S, \{c_s\})$

and rep'ns of $K \otimes_A H$ for various $\theta : A \to k$ (k field)

Convenient framework: "cellular algebras"

<u>General def.</u> H assoc alg over an integral domain A

H finitely gen. and free over A.

"Cell datum" for H $(\Lambda, M, C, *)$ where

Λ finite set, endowed w/ a partial order $\leq$

$\{M(\lambda) : \lambda \in \Lambda\}$ $M(\lambda)$ finite set

$\{C^\lambda_{S,T} : \lambda \in \Lambda, S, T \in M(\lambda)\}$ is an A-basis of H

$*: H \to H$ is an anti-invol'n, s.t. $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$

$$h \cdot C^{\lambda}_{S,T} = \sum_{S' \in M(\lambda)} r_h(S,S')\, C^{\lambda}_{S',T} + A\text{-linear combination of terms } C^{\mu}_{U,V}$$

$(h \in H)$ for $\mu < \lambda$.

$r_h(S,S')$ does not depend on T

<u>In particular</u> $H(\leq \lambda) = \langle C^{\mu}_{U,V} : \mu \in \Lambda,\ \mu \leq \lambda,\ U,V \in M(\mu) \rangle_A$

two-sided ideal

basis $\{C^{\lambda}_{S,T}\}$ of H adapted to left, right and two-sided ideal str. of H

everything defined over A.

Given $\lambda \in \Lambda$, let $W(\lambda)$ be a free A-module w/ basis $\{C_S : S \in M(\lambda)\}$

Left action of H on $W(\lambda)$: $h \cdot C_S = \sum_{S' \in M(\lambda)} r_h(S',S)\, C_{S'}$

Bilinear form $g^{\lambda} : W(\lambda) \times W(\lambda) \longrightarrow A$

$(C_S, C_T) \longmapsto r_h(S,S)$ where $h = C^{\lambda}_{S,T}$

Then $g^{\lambda}(h \cdot C_S, C_T) = g^{\lambda}(C_S, h^* C_T)$

Let $\theta : A \to k$ be a ring hom. into a field k, $H_k = k \otimes_A H$

$\{C^{\lambda}_{S,T} \otimes 1 : \lambda \in \Lambda,\ S,T \in M(\lambda)\}$ basis of H_k.

$W_k(\lambda) = k \otimes_A W(\lambda)$ H_k-module

$g^{\lambda}_k : W_k(\lambda) \times W_k(\lambda) \longrightarrow k$ invariant bilinear form

Invariance of g^λ_k $\rightsquigarrow$ $\operatorname{rad} g^\lambda_k \subset W_k(\lambda)$ H_k-submodule

Define $L^\lambda := W_k(\lambda) / \operatorname{rad}(g^\lambda_k)$

<u>Thm</u> (Graham–Lehrer)

$$\operatorname{Irr}(H_k) = \{L^\lambda : \lambda \in \Lambda^\circ_k\} \quad \text{where } \Lambda^\circ_k = \{\lambda \in \Lambda : g^\lambda_k \neq 0\}$$

Back to generic Hecke alg: $H = H_A(W, S, \{c_s\})$, $A = \mathbb{Z}[v, v^{-1}]$

Std basis $\{T_w : w \in W\}$

$\tilde{T}_s := v^{-c_s} T_s$ $(s \in S)$ $\qquad \tilde{T}_w = \tilde{T}_{s_1} \cdots \tilde{T}_{s_\ell}$ if $w = s_1 \cdots s_\ell$ reduced

$\{\tilde{T}_w : w \in W\}$ A-basis of H

Ring involution $\overline{\sum_{w \in W} a_w \tilde{T}_w} := \sum_{w \in W} a_w|_{v \to v^{-1}} \tilde{T}_{w^{-1}}^{-1}$

<u>Thm</u> (Kazhdan–Lusztig, Lusztig) For each $w \in W$, there is a unique $C_w \in H$ s.t.

$\overline{C_w} = C_w$ and $C_w = (-1)^{\ell(w)} \tilde{T}_w$ + $v\mathbb{Z}[v]$-combination of $\tilde{T}_y$ $(y < w)$

$\{C_w : w \in W\}$ is an A-basis of H.

<u>Example</u> $W = \mathfrak{S}_n$, $c_s = 1$, $\forall s \in S$.

$$\mathfrak{S}_n \xleftrightarrow{1-1} \{(S,T) : S, T \text{ standard tableaux of the same shape}\} = \coprod_{\lambda \vdash n} T(\lambda) \times T(\lambda)$$

set of std tableaux of shape λ

$\Lambda = \{\lambda \vdash n\}$

$M(\lambda) = T(\lambda)$

$\leq$ = dominance order

$T_w^* = T_{w^{-1}}$

$C^\lambda_{S,T} = C_w$ where $w \longleftrightarrow (S,T)$

$\rightsquigarrow$ cell datum for H

<u>Lecture 3.</u> W finite Weyl gp, generating set S

$H = H_A(W, S, \{c_s\})$ generic Iwahori-Hecke alg. over $A = \mathbb{Z}[v, v^{-1}]$

$\theta : A \longrightarrow k$ ring hom., k field, $\mathcal{H}_k = k \otimes_A H_A$ specialized over k

$\mathrm{Irr}(\mathcal{H}_k) = ?$

<u>Main Application</u> $G = \underline{G}(\mathbb{F}_q)$ finite gp of Lie type, Weyl gp W,

$q^{c_s} = |BsB| / |B|$, K alg. closed field of char $\ell \geq 0$, $\ell \nmid q$.

$\theta : A \longrightarrow K$
$v \longmapsto q^{1/2} \cdot 1_K$.

$$\mathrm{Irr}(\mathcal{H}_K) \xleftrightarrow{1-1} \mathrm{Irr}_K(G \mid (T, K_T))$$
$$\| $$
$$\{ Y \in \mathrm{Irr}_K(G) : Y \text{ is a factor (or a sub) module of } R_T^G(K_T) \}$$
$$= K[G/B]$$

Have seen $G = GL_n(\mathbb{F}_q)$, $W = \mathfrak{S}_n$, $c_s = 1$, $\forall s \in S$

H cellular algebra, $\{C_w\}$ is a cellular basis.

NOT TRUE in general, but ...

In general case, consider only G of split type, $c_s = 1$, $\forall s \in S$.

$H = H_A(W, S, \{c_s = 1\})$, $\{C_w\}$ KL basis

$$C_x C_y = \sum_{z \in W} \underbrace{h_{xyz}}_{\in A} C_z$$

Define function $a : W \longrightarrow \mathbb{Z}_{\geq 0}$ by

$$\underset{(z \in W)}{a(z)} = \min\{i \geq 0 : v^i h_{xyz} \in \mathbb{Z}[v], \forall x, y \in W\}$$

Lusztig's a-function

Given $i \geq 0$. define $H^{\geq i} = \langle C_w : a(w) \geq i \rangle_A \subset H$.

two-sided ideal.

Filtrat'n. $H = H^{\geq 0} \supset H^{\geq 1} \supset \dots \supset H^{\geq N} = \{0\}$

Specialize $v \mapsto 1$, $A \to \mathbb{C}$, specialized alg. $\mathbb{C}[W]$

$$\rightsquigarrow \mathbb{C}[W] = \mathbb{C}[W]^{\geq 0} \supset \mathbb{C}[W]^{\geq 1} \supset \dots \supset \mathbb{C}[W]^{\geq N} = \{0\}$$

chain of 2-sided ideals

Consider $\mathrm{Irr}_{\mathbb{C}}(W) = \{E^\lambda : \lambda \in \Lambda\}$

Given $\lambda \in \Lambda$, $\exists !\ i \geq 0$ s.t. E^λ is a composition factor of $\mathbb{C}[W]^{\geq i} / \mathbb{C}[W]^{\geq i+1}$

Define $a_\lambda = i$

Get $\mathrm{Irr}_{\mathbb{C}}(W) \longrightarrow \mathbb{Z}_{\geq 0}$

$E^\lambda \longmapsto a_\lambda$

Partial order on Λ: $\lambda \leq \mu \overset{\text{def}}{\iff} \lambda = \mu$ or $a_\mu < a_\lambda$.

$W = \mathfrak{S}_n$, $\Lambda = \{\lambda \vdash n\}$, $\lambda \leq \mu \Rightarrow a_\mu < a_\lambda$.

<u>Bad prime</u>

A_n	/	G_2, F_4, E_6,	2, 3
B_n, C_n, D_n	2	E_8	2, 3, 5

replace $A = \mathbb{Z}[v, v^{-1}]$ by $A = R[v, v^{-1}]$ where $R \subset \mathbb{C}$ subring in which bad primes are invertible.

<u>Thm</u> (G ~ 2007) Under the above conditions, H is a cellular alg, where

Λ as above, $\mathrm{Irr}_{\mathbb{C}}(W) = \{E^\lambda : \lambda \in \Lambda\}$, $\leq$ as above (using a-invariants)

$M(\lambda)$ = basis of E^λ . $C^\lambda_{S,T}$ = certain $\mathbb{Z}$-linear combination of C_w's where $a(w) = a_\lambda$

$\theta : A \longrightarrow k$ specialization. $\rightsquigarrow W_k(\lambda)$, $g^\lambda_k : W_k(\lambda) \times W_k(\lambda) \longrightarrow k$

$L^\lambda_k = W_k(\lambda) / \mathrm{rad}(g^\lambda_k)$

$\mathrm{Irr}(H_k) = \{ L^\mu_k : \mu \in \Lambda^\circ_k \}$ where $\Lambda^\circ_k = \{ \lambda \in \Lambda : g^\lambda_k \neq 0 \}$

Semisimple case (Graham-Lehrer)

H_k semisimple $\Leftrightarrow$ g^λ_k non-degenerate $\forall \lambda \in \Lambda$ $\Leftrightarrow$ $L^\lambda_k = W_k(\lambda)$, $\forall \lambda \in \Lambda$

$\Leftrightarrow \mathrm{Irr}(H_k) = \{ W_k(\lambda) : \lambda \in \Lambda \}$

In particular, in this case. $\mathrm{Irr}(H_k) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}_{\mathbb{C}}(W)$

$W_k(\lambda) \longleftrightarrow E^\lambda$

NOW: non-semisimple case, arising from group of Lie type context.

$G = \underline{G}(\mathbb{F}_q)$, char $k = \ell > 0$, $\ell \nmid q$.

$\{ Y \in \mathrm{Irr}_k(G) : \mathrm{Fix}_B(Y) \neq 0 \} \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(H_k) \overset{1-1}{\longleftrightarrow} \Lambda^\circ_k \subset \Lambda$

$e = \min \{ i \geq 1 : 1 + q + q^2 + \dots + q^{i-1} \equiv 0 \mod \ell \}$

Consider another specialization $\theta_e : A \to \mathbb{C}$

$v \mapsto \zeta_{2e}$

root of unity of order $2e$

H_e = specialized alg. over $\mathbb{C}$

basis $\{ T_w : w \in W \}$, $T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w) \\ \zeta_e T_{sw} + (\zeta_e - 1) T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$

Theory of cellular alg. $\mathrm{Irr}(H_{\mathbb{C}}) = \{ L_e^{\mu} : \mu \in \Lambda_e^{\circ} \}$

where $\Lambda_e^{\circ} = \{ \lambda \in \Lambda : g_e^{\lambda} \neq 0 \}$

<u>Question</u>. Relation between $\mathrm{Irr}(H_{\mathbb{C}})$ and $\mathrm{Irr}(H_k)$

over $\mathbb{C}$ par. ζ_e | over k, char $k = \ell$ par. $q\, 1_k$

<u>James' conjecture</u> (James ~ 1990 for $W = \mathfrak{S}_n$)

$$P_W = \sum_{w \in W} u^{\ell(w)} = \prod_{1 \le i \le |S|} \frac{u^{d_i} - 1}{u - 1} \qquad \{d_i\} \text{ degrees of } W.$$

Poincare polynomial $\quad (u = v^e)$

A_{n-1}	$2,3,4,\dots,n$	G_2	$2,6$
B_n, C_n	$2,4,6,\dots,2n$	F_4	$2,6,8,12$
D_n	$2,4,6,\dots,2(n-1), n$	E_6	$2,5,6,8,9,12$
		E_7	$2,6,8,10,12,14,18$
		E_8	$2,8,12,14,18,20,24,30$.

$$|W| = d_1 d_2 \cdots d_{|S|}.$$

<u>General version of James' conjecture</u>

Assume ℓ is not bad for W, and that $e \cdot \ell$ does not divide any degree of W.

Then $\dim L_k^{\lambda} = \dim L_e^{\lambda}, \quad \forall \lambda \in \Lambda$

In particular, $\Lambda_k^{\circ} = \Lambda_e^{\circ}$.

Application

$$\{ Y \in \mathrm{Irr}_K(G) : \mathrm{Fix}_B(Y) \neq \{0\} \} \xleftrightarrow{1-1} \Lambda^\circ_\ell$$

under above condition

Remarks (a) Statement of James' conjecture is true if $\ell >> 0$.

The real issue is to find bound from when on it's true.

James' formulation for $W = \mathfrak{S}_n$: $e\ell > n$

b) Known $\Lambda^\circ_k \subset \Lambda^\circ_\ell$ and $\dim L^\mu_k \leq \dim L^\mu_\ell$, $\forall \mu \in \Lambda^\circ_k$.

So, in order to show that $\Lambda^\circ_k = \Lambda^\circ_\ell$, it's enough to show that

$|\Lambda^\circ_k| = |\Lambda^\circ_\ell|$.

Actually, this is known to be true. G - Rouquier ~ 1997 (general argument)

$\Lambda^\circ_k = \Lambda^\circ_\ell$ known to hold whenever ℓ is not a bad prime.

c) Determination of $\Lambda^\circ_\ell \subset \Lambda$ in all cases.

$W = \mathfrak{S}_n$, $\Lambda = \{\lambda \vdash n\}$. Dipper - James ~ 1980

$\Lambda^\circ_\ell = \{\lambda \vdash n : \lambda \text{ e-regular}\}$.

$W = B_n, D_n$, $\Lambda = \{$certain pairs of partitions of $n\}$. Jacon ~ 2003

$W =$ exceptional type. $\rightsquigarrow$ Λ°_ℓ are known, James' conj. verified (G.-Müller)