Wall-crossing for invariants of equivariant CY3 categories

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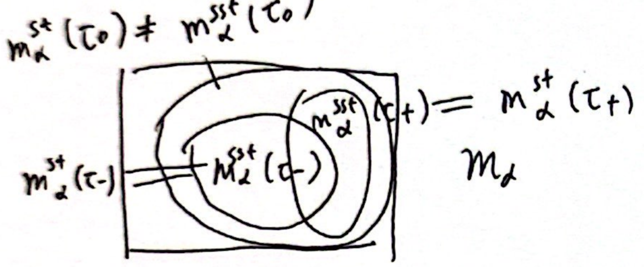
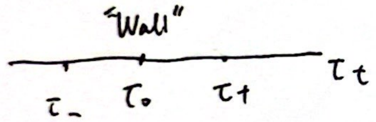
$\mathcal{A}$ abelian cat. e.g. $\mathrm{Coh}(X)$ X smooth proj. var.

$\bigsqcup_\alpha \mathfrak{M}_\alpha = \mathfrak{M}$ moduli stack of objects in $\mathcal{A}$

$(\tau_t)_t$ family of stability conditions on $\mathcal{A}$.





Assume $\mathfrak{M} \circlearrowleft T = (\mathbb{C}^\times)^2$

has a T-equiv. symmetric obstruction theory.

perfect $\nearrow$ $\mathbb{E}_{\mathfrak{M}} \to \mathbb{L}_{\mathfrak{M}}$ in $D_{\mathrm{QCoh},T}(\mathfrak{M})$

$h^1 \xrightarrow{\sim} h^1$ $\leftarrow$ "automorphisms"

$h^0 \xrightarrow{\sim} h^0$ "deformations"

$h^{-1} \twoheadrightarrow h^{-1}$ "obstructions"

and $\mathbb{E}_{\mathfrak{M}} \xrightarrow{\sim} \overset{\overset{\text{wt of } T}{\swarrow}}{\kappa \otimes} \mathbb{E}^\vee_{\mathfrak{M}}[1]$

e.g. for $\mathrm{Coh}(X)$, standard obstruction theory is $h^i(\mathbb{E}_{\mathfrak{M}})|_{[\mathcal{E}]} = \mathrm{Ext}^{1-i}_X(\mathcal{E},\mathcal{E})^\vee$

If $T \curvearrowright X$ is eq. CY3, $K_X \simeq \kappa \otimes \mathcal{O}_X$

Serre duality $\Rightarrow$ eq. sym. $\mathbb{F}$.

Can define "universal enumerative invts".

$$Z_\alpha(\tau) := \chi\left(\mathcal{M}^{sst}_\alpha(\tau), \hat{\mathcal{O}}^{vir} \otimes -\right)$$

if $\mathcal{M}^{sst}_\alpha(\tau) = \mathcal{M}^{st}_\alpha(\tau)$ — T-equiv. K-homology of $\mathcal{M}$.

(Analogue of $\int_{[\mathcal{M}^{sst}_\alpha(\tau)]^{vir}} -$)

<u>Thm</u> [KLT'25, based off Joyce]

Assume $\mathcal{M}$ is graded monoidal.

"monoidal" $\oplus : \mathcal{M}_\alpha \times \mathcal{M}_\beta \longrightarrow \mathcal{M}_{\alpha+\beta}$

"grading" $\Psi : [pt/\mathbb{C}^\times] \times \mathcal{M}_\alpha \to \mathcal{M}_\alpha$

induces a "degree" on sheaves

and assume $\mathbb{F}_\mathcal{M}$ "bilinear of degree ± 1".

Assume .... (minor technical condition)

Then
$$Z_\alpha(\tau_1) = \sum_{\substack{n>0 \\ \alpha=\alpha_1+\dots+\alpha_n \\ \tau_0(\alpha_i)=\tau_0(\alpha)}} \theta(\alpha_1,\dots,\alpha_n,\tau_0,\tau_1) \left[\left[\cdots\left[\left[Z_{\alpha_1}(\tau_0), Z_{\alpha_2}(\tau_0)\right], Z_{\alpha_3}(\tau_0)\right]\cdots\right] Z_{\alpha_n}(\tau_0)\right]$$

"universal coefficients" of Joyce

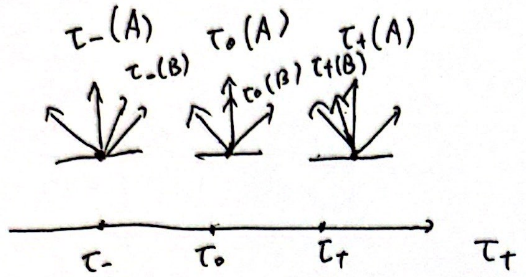
Lie bracket on K-homology

some generalization of univ. enumerative invts eg. $[\phi, \psi](\theta) = [\mathrm{ch}\, \mathbb{E}_{\alpha,\beta}]_k \, \phi(\theta)\, \psi(\theta)$

on M_α on M_β

<u>Simplest case</u> two sets A & B of classes α, β s.t. $\alpha = \alpha' + \beta_1 + \dots + \beta_n$

$\alpha, \alpha' \in A$, $\beta_i \in B$



$$\sum_{\alpha\in A} Q^\alpha Z_\alpha(\tau_+) = \exp\left(\left[\sum_{\beta\in B} Q^\beta Z_\beta(\tau_0), -\right]\right) \sum_{\alpha\in A} Q^\alpha Z_\alpha(\tau_-)$$

<u>Concrete application</u>: DT/PT vertex correspondence [Pandharipande - Thomas, Nekrasov - Okounkov]

on $X = (\mathbb{P}^1)^3$

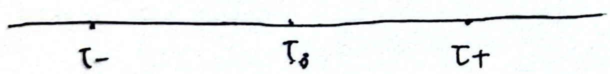
DT $I = [\mathcal{O}_X \xrightarrow{s} J]$

coker $s = 0$

$\dim J = 1$

$J = [$

PT

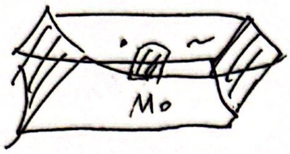


Fix $I|_{D_i} = \lambda_i$ (prescribed)

$$V^{DT}_{\lambda_1,\lambda_2,\lambda_3}(\theta) = \left[\begin{array}{c}\text{sth indep.}\\ \text{of } \lambda_1,\lambda_2,\lambda_3\end{array}\right] V^{PT}_{\lambda_1,\lambda_2,\lambda_3}(\theta)$$

$$\Rightarrow \frac{V^{DT}_{\lambda_1,\lambda_2,\lambda_3}(\theta)}{V^{PT}_{\lambda_1,\lambda_2,\lambda_3}(\theta)} = \frac{V^{DT}_{\phi\phi\phi}(\theta)}{V^{PT}_{\phi\phi\phi}(\theta)}$$

<u>Proof idea</u>. Suppose $\mathbb{M}$ is a master space.



$\mathbb{M} \circlearrowleft \mathbb{C}^\times \times T$ "proper smooth scheme"

$$\mathbb{M}^{\mathbb{C}^\times} = \underset{\{f_-=0\}}{\underset{\|}{M_-}} \sqcup \underset{\{f_+=0\}}{\underset{\|}{M_+}} \sqcup M_0$$

$\mathbb{C}^\times$-wt $\quad -1 \quad +1$

$$\chi(\mathbb{M}, F) = \chi\left(M_-, \frac{F|_{M_-}}{\mathrm{euler}^K(N_{M_-|\mathbb{M}})}\right)$$
$$+ \chi\left(M_+, \frac{F|_{M_+}}{\mathrm{euler}^K(N_{M_+|\mathbb{M}})}\right)$$
$$+ \chi\left(M_0, \frac{F|_{M_0}}{\mathrm{euler}^K(N_{M_0|\mathbb{M}})}\right)$$

Taking residue.
$$0 = \chi(M_-, F|_{M_-}) - \chi(M_+, F|_{M_+}) + \overbrace{\mathrm{res}_s\, \chi\left(M_0, \frac{F|_{M_0}}{\mathrm{euler}^K(N_{M_0|\mathbb{M}})}\right)}^{\substack{\text{Lie bracket} \\ \| \\ \text{operat'n on } Z_{\alpha_1}(\tau_0),\, Z_{\alpha_2}(\tau_0)}}$$

<u>Fact</u>: for Joyce & us, $\mathbb{M}$ is such that $M_\pm = \mathfrak{M}^{sst}_\alpha(\tau_\pm)$

$$M_0 = \coprod_{\substack{\alpha=\alpha_1+\alpha_2 \\ \tau_0(\alpha_1)=\tau_0(\alpha_2)}} \mathfrak{M}^{sst}_{\alpha_1}(\tau_0) \times \mathfrak{M}^{sst}_{\alpha_2}(\tau_0)$$

F smooth map $\mathbb{M} \xrightarrow{\pi} \mathfrak{M}$
has a sym. O.T.

Symmetric pullback of O.Ts
1. Smooth pullback.

is an O.T. for $\mathbb{M}$. <u>not</u> symmetric

$$\begin{tikzcd} \mathbb{L}_\pi[-1] \arrow[r, dashed, "\delta"] \arrow[d, equal] & \pi^*\mathbb{E}_{\mathfrak{M}} \arrow[r] \arrow[d] & \mathrm{Cone}(\delta) \arrow[r, "+1"] \arrow[d] & {} \\ \mathbb{L}_\pi[-1] \arrow[r] & \pi^*\mathbb{L}_{\mathfrak{M}} \arrow[r] & \mathbb{L}_{\mathbb{M}} \arrow[r, "+1"] & {} \end{tikzcd}$$

Pro... ll

2. "Dual" smooth pullback

$$\begin{tikzcd}
\mathbb{L}_\pi[-1] \arrow[r, "\xi"] \arrow[d, equal] & \mathrm{cone}(\delta)^\vee[3]\otimes\kappa \arrow[r] \arrow[d] & \mathbb{E}_M \arrow[d] & \\
\mathbb{L}_\pi[-1] \arrow[r, "\delta"] \arrow[d] & \pi^*\mathbb{E}_M \arrow[r] \arrow[d, "\delta^\vee[1]\otimes\kappa"] & \mathrm{cone}(\delta) \arrow[r, "+1"] \arrow[d, "\xi"] & {} \\
0 \arrow[r] & \mathbb{L}_\pi^\vee[2]\otimes\kappa \arrow[r, equal] & \mathbb{L}_\pi^\vee[2]\otimes\kappa \arrow[r, "+1"] & {}
\end{tikzcd}$$

is O.T. for M

is sym if $\xi = \xi$.

Assume $\delta^\vee[1]\circ\delta = 0$

Thm [KLT]

option 1 : work affine locally (étale)

assumption hold by cohom. vanishing

"almost perfect obs theory" $\rightsquigarrow$ gives obstruction sheaf $h^{-1}(\mathbb{E}) \Rightarrow \hat{\mathcal{O}}^{vir}_M$

Jouanolou's device

Option 2 : pullback to a very big affine bundle $a: A \longrightarrow M$

assumption hold on A by construction

$a^*: K_T(M) \Rightarrow K_T(A) \rightarrow \hat{\mathcal{O}}^{vir}_M = a^*\hat{\mathcal{O}}^{vir}_A$