On some chiral constructions

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ZOE: Come and I'll peel off.
BLOOM: (Feeling his occiput dubiously with the unparalleled embarrassment of a harassed peddler *gauging the symmetry* of her peeled pears.) Somebody would be dreadfully jealous if she knew.

«*Ulysses*», James Joyce

Abstract

The term "chiral" originates from physics. One visualized example is the polarization of light. In the setting of $2d$ conformal field theory, chirality more or less means holomorphicity. To describe a full $2d$ CFT, one needs a complex variable z and its conjugate $\bar{z}$. The chiral half consists of quantum fields that only depend on z, i.e. holomorphic in z. The algebraic structure underlying $2d$ holomorphic CFT is vertex algebra (chiral algebra).

In this note, we describe two chiralizations of classical mathematical structures. Namely, we explain the chiral version of BRST reduction (Hamiltonian reduction) and de Rham complex. Hopefully one will see the close relation between them.

This note is meant to serve as a survey in this field. We claim no originality of results stated in this paper.

Contents

1 Vertex algebras

"I am an old man, and I know that a definition cannot be so complicated."
I.M. Gelfand (after a talk on vertex algebras in his Rutgers seminar)

1.1 Vertex algebras

We first review some basic definitions and facts about vertex algebras, where we mainly follow [Kac98], [BK03] and [FBZ04].

Definition 1 (Vertex superalgebra)**.** By a *vertex superalgebra*, we mean a quadruple $(\mathbb{V}, |0\rangle, T, Y)$, where

- $\mathbb{V} = \mathbb{V}^0 \oplus \mathbb{V}^1$ is a superspace;
- $|0\rangle \in \mathbb{V}^0$ is an even vector called the *vacuum state*;
- $T \in (\operatorname{End} \mathbb{V})^0$ is an even operator called the *translation operator*;
- There exists a *state-field correspondence*, viz. we have an even linear map

$$Y : \mathbb{V} \to \operatorname{End} \mathbb{V}[\![z, z^{-1}]\!], a \mapsto a(z) = Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

  satisfying the following anxioms:

  - For each $v \in \mathbb{V}$, $a_{(n)} v = 0$ for $n \gg 0$;
  - $Y(|0\rangle, z) = \mathrm{id}_{\mathbb{V}}$;
  - $Y(a, z)|0\rangle \in \mathbb{V}[\![z]\!]$ and $\lim_{z \to 0} Y(a, z)|0\rangle = a$ for all $a \in \mathbb{V}$;
  - $Y(Ta, z) = \partial_z Y(a, z)$ for all $a \in \mathbb{V}$;
  - (Locality) For $a, b \in \mathbb{V}$, there exists some $N_{a,b} \in \mathbb{Z}_+$ such that

$$(z - w)^{N_{a,b}} [Y(a, z), Y(b, w)] = 0.$$

Definition 2 (Conformal vertex algebra)**.** An even vector L in a vertex superalgebra $(\mathbb{V}, |0\rangle, T, Y)$ is called a *conformal vector* (of Virasoro central charge c), if the corresponding field $Y(L, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies

- The endomorphisms L_n satisfy the Virasoro commutation relations

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} c.$$

- $L_{-1} = T$.
- The operator L_0 is diagonalizable on $\mathbb{V}$.

The quintuple $(\mathbb{V}, |0\rangle, T, Y)$ is called a *conformal vertex superalgebra* of Virasoro central charge c.

The quantum field $Y(L, z)$ corresponds to a conformal vector L is called the *energy-momentum field* of the vertex algebra $\mathbb{V}$; L_0 is called the *energy operator*. The eigenvalue decomposition

$$\mathbb{V} = \bigoplus_{\Delta \in \mathbb{C}} \mathbb{V}[\Delta]$$

is called the *energy decomposition*. Here

$$\mathbb{V}[\Delta] = \{a \in \mathbb{V} : L_0 a = \Delta a\},$$

elements from $\mathbb{V}[\Delta]$ have *conformal weight* Δ.

From now on, we simply drop the prefix "super" if everything is even. We will also use $\mathbb{V}$ alone to denote a vertex algebra if no confusion arises.

One consequence of the above definition is the *Borcherds identity*, which was originally used by Richard Borcherds in his definition of vertex algebras.

Proposition 1 (Borcherds identity)**.** For $a, b \in \mathbb{V}$, we have

$$[a_{(m)}, b_{(n)}] = \sum_{i \geq 0} \binom{m}{i} (a_{(i)} b)_{(m+n-i)};$$
$$(a_{(m)} b)_{(n)} = \sum_{j \geq 0} (-1)^j \binom{m}{j} (a_{(m-j)} b_{(n+j)} - (-1)^m b_{(m+n-j)} a_{(j)}).$$

As usual, we define morphism between vertex algebras, vertex subalgebra and vertex algebra ideal.

Definition 3 (Morphism between vertex algebras)**.** A morphism of vertex superalgebra ρ between vertex superalgebras

$$(\mathbb{V}, |0\rangle, T, Y) \to (\mathbb{V}', |0\rangle', T', Y')$$

is a even linear map $\rho \colon \mathbb{V} \to \mathbb{V}'$ mapping $|0\rangle$ to $|0\rangle'$, intertwining the translation operators and satisfying

$$\rho(Y(a, z)b) = Y(\rho(a), z)\rho(b).$$

Moreover, suppose both $\mathbb{V}$ and $\mathbb{V}'$ are conformal vertex algebras. We say ρ is a morphism of conformal vertex algebras if it is a morphism of vertex algebras and sends the energy-momentum field of $\mathbb{V}$ to that of $\mathbb{V}'$.

Definition 4 (Vertex subalgebra)**.** $\mathbb{V}' \subset \mathbb{V}$ is a vertex subalgebra if it is T-invariant, contains the vacuum state, and satisfies $a(z)b \in \mathbb{V}'((z))$ for all $a, b \in \mathbb{V}'$.

Definition 5 (Vertex algebra ideal)**.** $\mathbb{J} \subset \mathbb{V}$ is a vertex algebra ideal if it is T-invariant, and satisfies $a(z)b \in \mathbb{J}((z))$ for all $a \in \mathbb{J}, b \in \mathbb{V}$.

Definition 6 (Normally ordered product)**.** For $a, b \in V$ of parity $p(a), p(b)$ respectively, we define the normally ordered product of fields $a(z), b(z)$ to be

$$:a(z)b(z): = a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z) a(z)_-.$$

Here for a formal power series $f(z) = \sum_{n \in \mathbb{Z}} f_n z^n$, we write

$$f(z)_+ = \sum_{n \geq 0} f_n z^n, f(z)_- = \sum_{n < 0} f_n z^n.$$

For several fields, we follow the convention that the normally ordered product is read from right to left. For example,

$$:a(z)b(z)c(z): = :a(z):b(z)c(z)::.$$

We can also define normally ordered product of states $a_1, \cdots, a_N$ in $\mathbb{V}$. Namely,

$$:a_1 \cdots a_N:(z) = :a_1(z) \cdots a_N(z):.$$

Lemma 1 (Tensor product)**.** For two vertex superalgebras $(\mathbb{V}_1, |0\rangle_1, T_1, Y_1)$ and $(\mathbb{V}_2, |0\rangle_2, T_2, Y_2)$, the data $(\mathbb{V}_1 \otimes \mathbb{V}_2, |0\rangle_1 \otimes |0\rangle_2, T_1 \otimes 1 + 1 \otimes T_2, Y)$, where

$$Y(a_1 \otimes a_2, z) = Y_1(a_1, z) \otimes Y_2(a_2, z)$$

defines a new vertex superalgebra, called the *tensor product* of $\mathbb{V}_1$ and $\mathbb{V}_2$.

Moreover, if both $(\mathbb{V}_1, |0\rangle_1, T_1, Y_1, L_1)$ and $(\mathbb{V}_2, |0\rangle_2, T_2, Y_2, L_2)$ are conformal vertex algebras, then $\mathbb{V}_1 \otimes \mathbb{V}_2$ is also a conformal vertex algebra, with conformal vector given by

$$L = L_1 \otimes |0\rangle_2 + |0\rangle_1 \otimes L_2.$$

1.2 Operator product expansion

The definition of vertex algebra is very complicated, but its main usage is to formulate *operator product expansions* (OPE) arising in physics literature. There are three versions, but they are essentially the same.

Proposition 2 (OPE in physics literature)**.** Let $\mathbb{V}$ be a vertex algebra. For $a, b \in \mathbb{V}$, we have

$$a(z)b(w) = \sum_{n \geq 0} \frac{(a_{(n)}b)(w)}{(z-w)^{n+1}} + :a(z)b(w):.$$

The right hand side makes sense because for $n \gg 0$, $a_{(n)}b = 0$. We write

$$a(z)b(w) \sim \sum_{n \geq 0} \frac{(a_{(n)}b)(w)}{(z-w)^{n+1}}.$$

The right hand side is the singular part of $a(z)b(w)$ along $z = w$.

Example 1 (OPE of the energy-momentum field)**.** In the definition of conformal vector, the Virasoro commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}c$$

is equivalent to an OPE

$$L(z)L(w) \sim \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}.$$

Proposition 3 (Singular part of OPE)**.** Let $\mathbb{V}$ be a vertex algebra. For $a, b \in \mathbb{V}$, we have

$$[a(z), b(w)] = \sum_{n \geq 0} \frac{1}{n!}(a_{(n)}b)(w)\partial_w^n \delta(z-w).$$

The right hand side makes sense because for $n \gg 0$, $a_{(n)}b = 0$. This formula is called the *singular part* of the OPE.

The formal Fourier transform of the singular part of the OPE gives rise to the λ-bracket.

Definition 7 (λ-bracket)**.** For $a, b \in \mathbb{V}$, we define their λ*-bracket* to be

$$[a_\lambda b] = \sum_{n\geq 0} \frac{\lambda^n}{n!} a_{(n)} b \in \mathbb{V}[\![\lambda]\!].$$

In fact, $[a_\lambda b] \in \mathbb{V}[\lambda]$ because for $n \gg 0$, $a_{(n)}b = 0$.

We have some formulae of λ-bracket that are useful in computations.

Proposition 4. For $a, b \in \mathbb{V}$ with parity $p(a), p(b)$ respectively and $c \in \mathbb{V}$, we have

$$\begin{aligned}
&[(Ta)_\lambda b] = -\lambda[a_\lambda b], [a_\lambda (Tb)] = (\lambda + T)[a_\lambda b];\\
&[b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-T} b];\\
&[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]].
\end{aligned}$$

Proposition 5 (Non-commutative Wick formula)**.** For $a, b \in \mathbb{V}$ with parity $p(a), p(b)$ respectively and $c \in \mathbb{V}$, we have

$$\begin{aligned}
&[a_\lambda {:}bc{:}] = {:}[a_\lambda b]c{:} + (-1)^{p(a)p(b)} {:}b[a_\lambda c]{:} + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu;\\
&[{:}ab{:}_\lambda c] = {:}(e^{T\partial_\lambda} a)[b_\lambda c]{:} + (-1)^{p(a)p(b)} {:}(e^{T\partial_\lambda} b)[a_\lambda c]{:} + (-1)^{p(a)p(b)} \int_0^\lambda [b_\mu [a_{\lambda-\mu} c]] d\mu.
\end{aligned}$$

Proposition 6 (λ-bracket under tensor product)**.** For two vertex superalgebras $\mathbb{V}_1$ and $\mathbb{V}_2$, the λ-bracket in the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2$ has the form

$$[(a_1 \otimes a_2)_\lambda (b_1 \otimes b_2)] = \sum_{n\geq 0} \frac{\lambda^n}{n!} \sum_{s+t=n-1} a_{1(s)} b_1 \otimes a_{2(t)} b_2.$$

1.3 Examples of vertex algebras

Example 2 (Affinization of finite-dimensional Lie algebra)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, equipped with an invariant symmetric bilinear form $(\cdot,\cdot)$. Here invariance means that $([x, y], z) = (x, [y, z])$ for $x, y, z \in \mathfrak{g}$. The *loop algebra* of $\mathfrak{g}$ is

$$L\mathfrak{g} = \mathfrak{g}(\!(t)\!) = \mathfrak{g} \otimes \mathbb{C}(\!(t)\!)$$

with Lie brackets defined by

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t).$$

The *affinization* $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ is a central extension of $L\mathfrak{g}$. Namely, $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ as a vector space and the commutation relations are given by $[K, \cdot] = 0$ (i.e. K is a central element in $\hat{\mathfrak{g}}$) and

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\mathrm{Res}_{t=0}\, f dg)(A, B)K.$$

Inside $\hat{\mathfrak{g}}$, we have the "positive" Lie subalgebra $\mathfrak{g}[\![t]\!] \oplus \mathbb{C}K$. For a scalar $k \in \mathbb{C}$, one has a one-dimensional representation $\mathbb{C}_k = \mathbb{C}v_k$ of $\mathfrak{g}[\![t]\!] \oplus \mathbb{C}K$ on which $\mathfrak{g}[\![t]\!]$ acts by 0 and K acts by k. The *vacuum representation of level* k is the induced representation $V^k(\mathfrak{g})$ of $\hat{\mathfrak{g}}$:

$$V^k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[\![t]\!]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_k = U\hat{\mathfrak{g}} \otimes_{U(\mathfrak{g}[\![t]\!]\oplus\mathbb{C}K)} \mathbb{C}_k.$$

For any $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we write

$$A_n = A \otimes t^n \in L\mathfrak{g}$$

and $A(z)$ is the field

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}.$$

Let $\{J^a\}_{1\leq a\leq \dim\mathfrak{g}}$ be an ordered basis of $\mathfrak{g}$, then by PBW theorem,

$$V^k(\mathfrak{g}) \simeq U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])$$

as a vector space and has a basis of monomials of the form

$$J^{a_1}_{n_1}\cdots J^{a_m}_{n_m} v_k$$

where $n_1 \leq n_2 \leq \cdots \leq n_m < 0$ and if $n_i = n_{i+1}$, then $a_i \leq a_{i+1}$. Here by slightly abusing of notation, we denote still by v_k the image of $1\otimes v_k$ in $V^k(\mathfrak{g}) = U\hat{\mathfrak{g}} \otimes_{U(\mathfrak{g}[\![t]\!]\oplus\mathbb{C}K)} \mathbb{C}v_k$.

We now describe the vertex algebra structure on $V^k(\mathfrak{g})$.

- The vacuum state is $|0\rangle = v_k$.
- The translation operator T sends v_k to 0 and should satisfy commutation relations

$$[T, J^a_n] = -nJ^a_{n-1}.$$

- The vertex operators are given by

$$Y(J^{a_1}_{n_1}\cdots J^{a_m}_{n_m} v_k, z) = \frac{1}{(-n_1-1)!\cdots(-n_m-1)!} {:}\partial_z^{-n_1-1} J^{a_1}(z)\cdots\partial_z^{-n_m-1}J^{a_m}(z){:}.$$

  In particular, $Y(v_k, z) = \mathrm{id}$ and

$$Y(J^a_{-1}v_k, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}.$$

Example 3 (Fermionic Fock space)**.** Consider the topological vector space $\mathbb{C}(\!(t)\!) \oplus \mathbb{C}(\!(t)\!)dt$ equipped with the residue pairing. Let $\mathcal{C}\ell$ be the Clifford algebra associated to it. Precisely, it has *odd* topological generators $\psi_n = t^n, \varphi_n = t^{n-1}dt, n \in \mathbb{Z}$, satisfying anti-commutation relations

$$[\psi_n, \psi_m] = [\varphi_n, \varphi_m] = 0, [\psi_n, \varphi_m] = \delta_{n,-m}.$$

Here $[\cdot,\cdot]$ is the super bracket. The *fermionic Fock space* (or the *bc-system*) $\mathcal{F}$ is a representation of $\mathcal{C}\ell$, generated by a vector $|0\rangle$, such that

$$\psi_n|0\rangle = 0, n \geq 0, \varphi_n|0\rangle = 0, n > 0.$$

$\mathcal{F}$ has a basis consisting of monomials of the form

$$\psi_{n_1}\cdots\psi_{n_k}\varphi_{m_1}\cdots\varphi_{m_\ell}|0\rangle,$$

where $n_1 < n_2 < \cdots < n_k < 0$ and $m_1 < m_2 < \cdots < m_\ell \leq 0$.

We now describe the conformal vertex superalgebra structure on $\mathcal{F}$.

- The vacuum state is $|0\rangle$.
- The parity of $|0\rangle$ is 0, and the parity on $\mathcal{F}$ is compatible with the action of $\mathcal{C}\ell$.
- The translation operator T sends $|0\rangle$ to 0 and should satisfy commutation relations

$$[T, \psi_n] = -n\psi_{n-1}, [T, \varphi_n] = -(n-1)\varphi_{n-1}.$$

- The fields corresponding to $\psi_{-1}|0\rangle$ and $\varphi_0|0\rangle$ are given by

$$Y(\psi_{-1}|0\rangle, z) = \psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}, Y(\varphi_0|0\rangle, z) = \varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^{-n}$$

  respectively. In general,

$$\begin{aligned} &Y(\psi_{n_1}\cdots\psi_{n_k}\varphi_{m_1}\cdots\varphi_{m_\ell}|0\rangle, z) \\ =& \prod_{i=1}^{k}\frac{1}{(-n_i-1)!}\prod_{j=1}^{\ell}\frac{1}{(-m_j)!}{:}\partial_z^{-n_1-1}\psi(z)\cdots\partial_z^{-n_k-1}\psi(z)\partial_z^{-m_1}\varphi(z)\cdots\partial_z^{-m_\ell}\varphi(z){:}. \end{aligned}$$

- The energy-momentum field is

$$L(z) = {:}\partial_z\varphi(z)\psi(z){:}.$$

  The Virasoro central charge is -2.

We can generalize the above construction to higher dimension. Namely, let U be a finite-dimensional vector space with an ordered basis $\{v^i\}_{i\in I}$. Take the dual basis $\{v^{i^*}\}$ of U^*. The complete topological vector space $U(\!(t)\!) \oplus U^*(\!(t)\!)dt$ is still equipped with the natural symmetric bilinear form given by the pairing between U and U^* and the residue pairing. So we can still define the Clifford algebra $\mathcal{C}\ell_U$ corresponding to this bilinear form and we have the corresponding fermionic Fock space $\mathcal{F}_U$.

More precisely, $\mathcal{C}\ell_U$ has odd topological generators $\psi^i_n = v^i \otimes t^n, \varphi^i_n = v^{i^*} \otimes t^{n-1}dt, i \in I, n \in \mathbb{Z}$, satisfying anti-commutation relations

$$[\psi^i_n, \psi^j_m] = [\varphi^i_n, \varphi^j_m] = 0, [\psi^i_n, \varphi^j_m] = \delta_{i,j}\delta_{n,-m}.$$

$\mathcal{F}_U$ is a representation of $\mathcal{C}\ell_U$, generated by a vector $|0\rangle$, such that

$$\psi^i_n|0\rangle = 0, n \geq 0, \varphi^i_n|0\rangle = 0, n > 0.$$

The vertex superalgebra structure on $\mathcal{F}_U$ is isomorphic to the tensor product of $|I|$ copies of fermionic Fock space $\mathcal{F}$. In particular, we have

$$Y(\psi^i_{-1}|0\rangle, z) = \psi^i(z) = \sum_{n\in\mathbb{Z}} \psi^i_n z^{-n-1}, Y(\varphi^i_0|0\rangle, z) = \varphi^i(z) = \sum_{n\in\mathbb{Z}} \varphi^i_n z^{-n}.$$

We have OPEs

$$[\psi^i(z), \psi^j(w)] = [\varphi^i(z), \varphi^j(w)] = 0, [\psi^i(z), \varphi^j(w)] = \delta_{i,j}\delta(z-w).$$

The energy-momentum field is given by

$$L(z) = \sum_{i\in I} {:}\partial_z\varphi^i(z)\psi^i(z){:}.$$

Correspondingly, the Virasoro central charge is $-2|I|$.

There are important $\mathbb{Z}$-gradings on $\mathcal{C}\ell_U$ and $\mathcal{F}_U$, called the (fermionic) *charge gradation*, denoted by $\clubsuit$. To be specific, we set $\clubsuit\,|0\rangle = 0, \clubsuit\,\varphi_n^i = 1, \clubsuit\,\psi_n^i = -1$. In physics, φ_n^i's are called *creation operators* in the sense that they increase the charge number by 1 while ψ_n^i's are called *annihilation operators* in the sense that they decrease the charge number by 1. The subspace of $\mathcal{F}_U$ consisting of elements of charge q is denoted by $\mathcal{F}_U^q$.

2 Chiral BRST reduction

Bechi–Rouet–Stora–Tyutin reduction, abbreviated as *BRST reduction*, is roughly speaking an infinite-dimensional cohomology theory. It was firstly used by physicists as a tool for gauge fixing and removing non-physical terms in quantum field theory (see *Faddeev–Popov ghosts*). Mathematically, this is a machinery for infinite-dimensional Hamiltonian reduction [KS87]. Borya Feigin used BRST reduction, in the name of *semi-infinite homology*, to study infinite-dimensional Lie algebras like Kac–Moody algebras and the Virasoro algebra [Fei84]. His joint work with Ed Frenkel provides a conceptual definition of W-algebras [FF90], which was firstly studied by physicists using OPEs.

In this section, we describe a slight generalization of BRST reduction, called *chiral BRST reduction*. This is a BRST reduction for vertex algebras while the usual BRST reduction works for Lie algebras. Note that Sasha Beilinson and Volodya Drinfeld have defined such reduction for *chiral algebras* [BD04]. While they use modern language in algebraic geometry, our exposition here is much more elementary.

No doubt, everything in this section is well known to experts, but we fail to put an earliest reference. Essentially, we are following [FF90], [FBZ04] and [Ara17]. The exposition of classical BRST reduction in [BD91] is also very enlightening.

2.1 Chiral BRST reduction

Let $\mathfrak{n}$ be a finite-dimensional Lie algebra, $\rho\colon \mathfrak{n} \to \mathrm{End}(U)$ a finite-dimensional representation. Unlike the finite-dimensional case, we cannot define a $L\mathfrak{n} = \mathfrak{n}(\!(t)\!)$ representation on the fermionic Fock space $\mathcal{F}_U$ in general. Instead, we should replace $L\mathfrak{n}$ by some central extension of it.

Define an invariant symmetric bilinear form on $\mathfrak{n}$ by the formula

$$(x, y) = \mathrm{Tr}_U(\rho(x)\rho(y)).$$

Denote by $\hat{\mathfrak{n}}$ the corresponding affinization of $\mathfrak{n}$ (see Example 2), $V^k(\mathfrak{n}) = \mathrm{Ind}_{\mathfrak{n}[\![t]\!]\oplus\mathbb{C}K}^{\hat{\mathfrak{n}}} \mathbb{C}_k$ the corresponding vertex algebra at level k. In this section, the level k is not important, so we simply write $V(\mathfrak{n})$ instead of $V^k(\mathfrak{n})$.

Fix an ordered basis $\{x^\alpha\}$ of $\mathfrak{n}$. Denote by (f_{ij}^α) the representation matrix of $\rho(x^\alpha)$ under the basis $\{v^i\}$ of U, viz. $\rho(x^\alpha)\cdot v^j = \sum_i f_{ij}^\alpha v^i$. Define quantum fields $\tilde{x}^\alpha(z)$ from the fermionic Fock space $\mathcal{F}_U$ by

$$\tilde{x}^\alpha(z) = \sum_{n\in\mathbb{Z}} \tilde{x}_n^\alpha z^{-n-1} = \sum_{i,j} f_{ij}^\alpha {:}\psi^i(z)\varphi^j(z){:}.$$

Lemma 2. The assignment $x^\alpha \otimes t^n \mapsto \tilde{x}_n^\alpha$ and $K \mapsto \mathrm{id}$ defines a representation of $\hat{\mathfrak{n}}$ on $\mathcal{F}_U$. The corresponding map of $\hat{\mathfrak{n}}$-modules $\tilde{\rho}\colon V(\mathfrak{n}) \to \mathcal{F}_U$, sending the vacuum vector to the vacuum vector, is a morphism of vertex superalgebras.

Now suppose we are given a vertex algebra $\mathbb{V}$, together with a morphism of vertex algebras $L\colon V(\mathfrak{n}) \to \mathbb{V}$. Let $U = \mathfrak{n}$ and consider the adjoint representation of $\mathfrak{n}$ on itself. We choose the same ordered basis $\{x^\alpha = \psi^\alpha\}$ for $\mathfrak{n}$ and U. Define $\mathbb{U}$ to be the tensor product $\mathbb{V} \otimes \mathcal{F}_{\mathfrak{n}}$, together with a morphism of vertex superalgebras

$$\mathcal{L} = L \otimes |0\rangle_{\mathcal{F}_{\mathfrak{n}}} + |0\rangle_{\mathbb{V}} \otimes \tilde{\rho} \colon V(\mathfrak{n}) \to \mathbb{V} \otimes \mathcal{F}_{\mathfrak{n}}.$$

Moreover, $\mathbb{U}^\bullet = \mathbb{V} \otimes \mathcal{F}_{\mathfrak{n}}^\bullet$ inherits the charge gradation from $\mathcal{F}_{\mathfrak{n}}$.

Note that $\mathfrak{n}$ can be viewed as a subspace of $V(\mathfrak{n})$ through the map

$$\mathfrak{n} \to V(\mathfrak{n}), x \mapsto x_{-1}|0\rangle_{V(\mathfrak{n})}.$$

We can also view $U = \mathfrak{n}$ as a subspace of $\mathcal{F}_{\mathfrak{n}}$ through the map

$$U = \mathfrak{n} \to \mathcal{F}_{\mathfrak{n}}, \psi \mapsto \psi_{-1}|0\rangle_{\mathcal{F}_{\mathfrak{n}}},$$

and view $U^* = \mathfrak{n}^*$ as a subspace of $\mathcal{F}_{\mathfrak{n}}$ through the map

$$U^* = \mathfrak{n}^* \to \mathcal{F}_{\mathfrak{n}}, \varphi \mapsto \psi_0|0\rangle_{\mathcal{F}_{\mathfrak{n}}}.$$

For brevity, we will make such identifications below.

Theorem 1. There exists a unique element $Q \in \mathbb{U}^1$ such that

$$[Q_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = \mathcal{L}(x) \text{ for } x = \psi \in \mathfrak{n}.$$

We have $[Q_\lambda Q] = 0$.

Proof. Uniqueness. We need the following lemma, whose proof is based on explicit calculations of OPEs.

Lemma 3. For $P \in \mathbb{U}$, if $[P_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = 0$ for all $\psi \in \mathfrak{n}$, then $P \in \mathbb{U}^{\leq 0}$.

Now suppose both Q and Q' satisfy the desired properties, then $P = Q - Q' \in \mathbb{U}^1$ and $[P_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = 0$ for all $\psi \in \mathfrak{n}$. By Lemma 3, $P \in \mathbb{U}^{\leq 0}$. Therefore $P = 0$ because $\mathbb{U}^{\leq 0} \cap \mathbb{U}^1 = 0$.
Existence. We write down Q explicitly. Let

$$Q^0 = \sum_\alpha L(x^\alpha) \otimes \varphi^\alpha.$$

We can compute that

$$\begin{aligned}
[{Q^0}_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] &= \sum_\alpha [(L(x^\alpha) \otimes \varphi^\alpha)_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] \\
&= \sum_\alpha \sum_{n\geq 0} \frac{\lambda^n}{n!} \sum_{s+t=n-1} (L(x^\alpha)_{(s)}|0\rangle_{\mathbb{V}} \otimes (\varphi^\alpha)_{(t)}\psi \\
&= \sum_\alpha \sum_{n\geq 0} \frac{\lambda^n}{n!} \sum_{s+t=n-1} L((x^\alpha)_{(s)}|0\rangle_{V(\mathfrak{n})}) \otimes \varphi^\alpha_{t+1}\psi \\
&= \sum_\alpha \sum_{n\geq 0} \frac{\lambda^n}{n!} \sum_{s+t=n-1} L(x^\alpha_s|0\rangle_{V(\mathfrak{n})}) \otimes (\delta_{t,0}\varphi^\alpha(\psi) - \varphi_{-1}\psi^\alpha_{t+1})|0\rangle_{\mathcal{F}_{\mathfrak{n}}}.
\end{aligned}$$

Notice that $x^\alpha_s|0\rangle_{V(\mathfrak{n})} = 0$ unless $s \leq -1$, so $s + t = n - 1 \geq -1$ implies that $t \geq 0$. If $t > 0$, then $\delta_{t,0} = 0$ and $\psi_{-1}\varphi^\alpha_{t+1}|0\rangle_{\mathcal{F}_{\mathfrak{n}}} = 0$. So the only nonzero term in the summation above is $s = -1, t = 0$. Now we conclude that

$$[{Q^0}_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = \sum_\alpha L(x^\alpha) \otimes \varphi^\alpha(\psi)|0\rangle_{\mathcal{F}_{\mathfrak{n}}} = L(x) \otimes |0\rangle_{\mathcal{F}_{\mathfrak{n}}}.$$

Next we define

$$Q^1 = -\frac{1}{2}\sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha\beta}|0\rangle_{\mathbb{V}} \otimes {:}\varphi^\alpha\varphi^\beta\psi^\gamma{:}.$$

Here $c_\gamma^{\alpha\beta}$'s are the structure constants of $\mathfrak{n}$, viz.

$$[x^\alpha, x^\beta] = \sum_\gamma c_\gamma^{\alpha\beta} x^\gamma.$$

Noticing that $(|0\rangle_{\mathbb{V}})_{(s)} = \delta_{s,-1}\,\mathrm{id}_{\mathbb{V}}$, we can see that

$$\begin{aligned}
&[(|0\rangle_{\mathbb{V}} \otimes {:}\varphi^\alpha\varphi^\beta\psi^\gamma{:})_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] \\
&= \sum_{n\geq 0}\frac{\lambda^n}{n!}|0\rangle_{\mathbb{V}} \otimes ({:}\varphi^\alpha\varphi^\beta\psi^\gamma{:})_{(n)}\psi \\
&= |0\rangle_{\mathbb{V}} \otimes [{:}\varphi^\alpha\varphi^\beta\psi^\gamma{:}_\lambda\psi].
\end{aligned}$$

Now by non-commutative Wick formula, we can compute that

$$[{:}\varphi^\alpha\varphi^\beta\psi^\gamma{:}_\lambda\psi] = {:}(e^{T\partial_\lambda}\varphi^\alpha)[{:}\varphi^\beta\psi^\gamma{:}_\lambda\psi]{:} + {:}(e^{T\partial_\lambda}{:}\varphi^\beta\psi^\gamma{:})[\varphi^\alpha{}_\lambda\psi]{:} + \int_0^\lambda [{:}\varphi^\beta\psi^\gamma{:}_\mu[\varphi^\alpha{}_{\lambda-\mu}\psi]]d\mu.$$

By the OPE $[\varphi^\alpha(z), \psi(w)] = \delta(z-w)\varphi^\alpha(\psi)\,\mathrm{id}_{\mathcal{F}_\mathfrak{n}}$, we see that $[\varphi^\alpha{}_{\lambda-\mu}\psi] = \varphi^\alpha(\psi)|0\rangle_{\mathcal{F}_\mathfrak{n}}$, so the last integral vanishes. Similarly, we find $[\varphi^\alpha{}_\lambda\psi] = \varphi^\alpha(\psi)|0\rangle_{\mathcal{F}_\mathfrak{n}}$ and so

$${:}(e^{T\partial_\lambda}{:}\varphi^\beta\psi^\gamma{:})[\varphi^\alpha{}_\lambda\psi]{:} = \varphi^\alpha(\psi){:}\varphi^\beta\psi^\gamma{:}.$$

Using the non-commutative Wick formula again, we see that

$$[{:}\varphi^\beta\psi^\gamma{:}_\lambda\psi] = {:}(e^{T\partial_\lambda}\varphi^\beta)[\psi^\gamma{}_\lambda\psi]{:} - {:}(e^{T\partial_\lambda}\psi^\gamma)[\varphi^\beta{}_\lambda\psi]{:} - \int_0^\lambda [\varphi^\beta{}_\mu[\psi^\gamma{}_{\lambda-\mu}\psi]]d\mu.$$

By the OPEs, we see the first and the third term vanish, and the middle term is

$$-{:}(e^{T\partial_\lambda}\psi^\gamma)[\varphi^\beta{}_\lambda\psi]{:} = -\varphi^\beta(\psi)\psi^\gamma.$$

So we conclude that

$$[{:}\varphi^\alpha\varphi^\beta\psi^\gamma{:}_\lambda\psi] = -\varphi^\beta(\psi){:}\varphi^\alpha\psi^\gamma{:} + \varphi^\alpha(\psi){:}\varphi^\beta\psi^\gamma{:}.$$

Since $c_\gamma^{\alpha\beta}$ is skew-symmetric in α and β, we see that

$$\begin{aligned}
[Q^1{}_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] &= -\frac{1}{2}\sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha\beta}[(|0\rangle_{\mathbb{V}} \otimes {:}\varphi^\alpha\varphi^\beta\psi^\gamma{:})_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] \\
&= -\frac{1}{2}\sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha\beta}|0\rangle_{\mathbb{V}} \otimes (-\varphi^\beta(\psi){:}\varphi^\alpha\psi^\gamma{:} + \varphi^\alpha(\psi){:}\varphi^\beta\psi^\gamma{:}) \\
&= \sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha\beta}|0\rangle_{\mathbb{V}} \otimes \varphi^\beta(\psi){:}\varphi^\alpha\psi^\gamma{:} \\
&= |0\rangle_{\mathbb{V}} \otimes \tilde{\rho}(x).
\end{aligned}$$

Set

$$Q = Q^0 + Q^1 = \sum_{\alpha} L(x^\alpha) \otimes \varphi^\alpha - \frac{1}{2} \sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha\beta} |0\rangle_{\mathbb{V}} \otimes :\varphi^\alpha \varphi^\beta \psi^\gamma:.$$

From its precise form, we see that $Q \in \mathbb{U}^1$. By the computations above, we see that for $x = \psi \in \mathfrak{n}$,

$$[Q_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = [Q^0{}_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] + [Q^1{}_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = L(x) \otimes |0\rangle_{\mathcal{F}_\mathfrak{n}} + |0\rangle_{\mathbb{V}} \otimes \tilde{\rho}(x) = \mathcal{L}(x).$$

Note that the condition $[Q_\lambda(|0\rangle_{\mathbb{V}} \otimes \psi)] = \mathcal{L}(x)$ can also be viewed as an OPE

$$[Q(z), \mathrm{id}_{\mathbb{V}} \otimes \psi(w)] = \mathcal{L}(x)(w)\delta(z-w).$$

Moreover, by the construction of $\tilde{\rho}$, we see that for $x = \psi, x' = \psi' \in \mathfrak{n}$, we have an OPE

$$[\tilde{\rho}(x)(z), \psi'(w)] = [\psi, \psi'](w)\delta(z-w).$$

Now for any $x = \psi, x' = \psi' \in \mathfrak{n}$, we can compute that

$$\begin{aligned}
&[[[Q(z), Q(z')], \mathrm{id}_{\mathbb{V}} \otimes \psi(w)], \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')] \\
&= -[[[Q(z'), \mathrm{id}_{\mathbb{V}} \otimes \psi(w)], Q(z)] + [[Q(z), \mathrm{id}_{\mathbb{V}} \otimes \psi(w)], Q(z')], \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')] \\
&= -[[\mathcal{L}(x)(w)\delta(z'-w), Q(z)] + [\mathcal{L}(x)(w)\delta(z-w), Q(z')], \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')] \\
&= [[Q(z), \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')], \mathcal{L}(x)(w)\delta(z'-w)] + [[\mathcal{L}(x)(w)\delta(z'-w), \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')], Q(z)] \\
&+ [[Q(z'), \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')], \mathcal{L}(x)(w)\delta(z-w)] + [[\mathcal{L}(x)(w)\delta(z-w). \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')], Q(z')] \\
&= [\mathcal{L}(x')(w')\delta(z-w'), \mathcal{L}(x)(w)\delta(z'-w)] \\
&+ [\delta(z'-w)[L(x)(w) \otimes \mathrm{id}_{\mathcal{F}_\mathfrak{n}} + \mathrm{id}_{\mathbb{V}} \otimes \tilde{\rho}(x)(w), \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')], Q(z)] \\
&+ [\mathcal{L}(x')(w')\delta(z'-w'), \mathcal{L}(x)(w)\delta(z-w)] \\
&+ [\delta(z-w)[L(x)(w) \otimes \mathrm{id}_{\mathcal{F}_\mathfrak{n}} + \mathrm{id}_{\mathbb{V}} \otimes \tilde{\rho}(x)(w), \mathrm{id}_{\mathbb{V}} \otimes \psi'(w')], Q(z')] \\
&= \delta(z-w')\delta(z'-w)\delta(w-w')\mathcal{L}([x', x])(w') + \delta(z'-w)[\mathrm{id}_{\mathbb{V}} \otimes [\tilde{\rho}(x)(w), \psi'(w')], Q(z)] \\
&+ \delta(z'-w')\delta(z-w)\delta(w'-w)\mathcal{L}([x', x])(w') + \delta(z-w)[\mathrm{id}_{\mathbb{V}} \otimes [\tilde{\rho}(x)(w), \psi'(w')], Q(z')] \\
&= \delta(z-w')\delta(z'-w)\delta(w-w')\mathcal{L}([x', x])(w') + \delta(z'-w)\delta(w-w')[\mathrm{id}_{\mathbb{V}} \otimes [\psi, \psi'](w'), Q(z)] \\
&+ \delta(z'-w')\delta(z-w)\delta(w'-w)\mathcal{L}([x', x])(w') + \delta(z-w)\delta(w-w')[\mathrm{id}_{\mathbb{V}} \otimes [\psi, \psi'](w'), Q(z')] \\
&= \delta(z-w')\delta(z'-w)\delta(w-w')\mathcal{L}([x', x])(w') + \delta(z'-w)\delta(w-w')\delta(z-w')\mathcal{L}([x, x'])(w') \\
&+ \delta(z'-w')\delta(z-w)\delta(w'-w)\mathcal{L}([x', x])(w') + \delta(z-w)\delta(w-w')\delta(z'-w')\mathcal{L}([x, x'])(w') \\
&= 0 + 0 = 0.
\end{aligned}$$

In other words,

$$[[[Q_\lambda Q]_\mu(|0\rangle_{\mathbb{V}} \otimes \psi)]_\nu(|0\rangle_{\mathbb{V}} \otimes \psi')] = 0 \text{ for all } \psi, \psi' \in \mathfrak{n}.$$

Applying Lemma 3 twice, we conclude that $[Q_\lambda Q] = 0$. □

Remark 1. We may call Q the *chiral BRST charge*. We prove the existence directly by writing down an explicit formula of Q. It is possible to make some conceptual remarks, see [Akm93] and [Vor93].

Since Q is odd, $[Q_\lambda Q] = 0$ implies that $Q_{(0)}^2 = 0$. Now we have a cochain complex $(\mathbb{U}^\bullet = \mathbb{V} \otimes \mathcal{F}_\mathfrak{n}^\bullet, Q_{(0)})$.

Definition 8 (Chiral BRST reduction)**.** The cohomology $H^*(\mathbb{U}^\bullet, Q_{(0)})$ is called the *chiral BRST reduction* of $\mathbb{V}$.

Lemma 4. The chiral BRST reduction $H^*(\mathbb{U}^\bullet, Q_{(0)})$ carries a standard structure of vertex algebra.

Proof. By Borcherds identity, one has the key formula

$$[Q_{(0)}, u_{(m)}] = (Q_{(0)}u)_{(m)} \text{ for any } u \in \mathbb{U}.$$

Using this it is easy to derive that

$$Z = \{u \in \mathbb{U} : Q_{(0)}u = 0\}$$

is a vertex subalgebra of $\mathbb{U}$, and

$$B = Q_{(0)}\mathbb{U} \subset Z$$

is an ideal of Z. So the quotient $H^*(\mathbb{U}^\bullet, Q_{(0)}) = Z/B$ carries a standard structure of vertex algebra. ☐

2.2 $\mathcal{W}$-algebras

$\mathcal{W}$-algebras are certain generalizations of affine Lie algebras and the Virasoro algebra. They are symmetries for certain enhanced conformal field theories. They are (generally) no longer Lie algebras, so the language of vertex algebra is necessary. We briefly discuss in this section the construction of W-algebras using a special chiral BRST reduction, called the *quantum Drinfeld–Sokolov reduction*.

Let $\mathfrak{g}$ be a simple Lie algebra. We fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$

Here $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ (resp. $\mathfrak{n}_-$) is the corresponding upper (resp. lower) nilpotent subalgebra. Let $\{x^\alpha\}_{\alpha\in\Delta_+}$ be the basis of $\mathfrak{n}$ consisting of Chevalley generators, indexed by positive roots Δ_+. Let $\varkappa$ be the normalized invariant bilinear form on $\mathfrak{g}$. Namely,

$$\varkappa = \frac{1}{2h^\vee} \times \text{Killing form}.$$

Here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Since n is nilpotent, $\varkappa$ restricts to 0 on $\mathfrak{n}$. Let $\hat{\mathfrak{g}}$ (resp. $\hat{\mathfrak{n}}$) be the affinization of $\mathfrak{g}$ (resp. $\mathfrak{n}$) with respect to $\varkappa$, and $V^k(\mathfrak{g})$ (resp. $V^k(\mathfrak{n})$) the corresponding vertex algebra at level k. By construction, $V^k(\mathfrak{n})$ is a vertex subalgebra of $V^k(\mathfrak{g})$, viz. the inclusion $\iota\colon V^k(\mathfrak{n}) \to V^k(\mathfrak{g})$ is a morphism between vertex algebras.

There is a distinguished character of $\mathfrak{n}$, called the *Drinfeld–Sokolov reduction.*

Definition 9 (Drinfeld–Sokolov character)**.** The functional χ defined on $\mathfrak{n}$ by

$$\chi(x^\alpha) = \begin{cases} 1, \alpha \text{ is simple;} \\ 0, \text{otherwise.} \end{cases}$$

is a character of $\mathfrak{n}$, called the *Drinfeld–Sokolov character.*

It can be checked that the character $\chi\colon \mathfrak{n} \to \mathbb{C}$ extends to a morphism between vertex algebras (still denoted by χ)

$$\chi\colon V^k(\mathfrak{n}) \to V^k(\mathfrak{g}), x(z) \mapsto \chi(x)\,\mathrm{id} \;\text{ for } x \in \mathfrak{n}.$$

Now we have adequate data to define a chiral BRST reduction. Echoing with notations in previous section, we set $\mathbb{V} = V^k(\mathfrak{g})$, together with a morphism $L = \iota + \chi\colon V^k(\mathfrak{n}) \to V^k(\mathfrak{g}) = \mathbb{V}$.

Definition 10 ($\mathcal{W}$-algebra)**.** The chiral BRST reduction $H^*(V^k(\mathfrak{g}) \otimes \mathcal{F}_{\mathfrak{n}})$ of $V^k(\mathfrak{g})$ with above data is called the (principal) W-algebra associated to $\mathfrak{g}$ at level k, denoted by $\mathcal{W}^k(\mathfrak{g})$.

Remark 2 (Functoriality)**.** An important consequence of this conceptual construction of $\mathcal{W}$-algebra is that we have functoriality. Namely, given a $V^k(\mathfrak{g})$-module M (there are many different definitions of modules over a vertex algebra. Here we follow the convention of [Kac98]), the operator $Q_{(0)}$ still acts on the $V^k(\mathfrak{g})\otimes\mathcal{F}_{\mathfrak{n}}$-module $M\otimes\mathcal{F}_{\mathfrak{n}}$. Now the cohomology $H^*(M\otimes\mathcal{F}_{\mathfrak{n}}, Q_{(0)})$ becomes a $\mathcal{W}^k(\mathfrak{g}) = H^*(V^k(\mathfrak{g}) \otimes \mathcal{F}_{\mathfrak{n}})$-module. In this sense, the quantum Drinfeld–Sokolov reduction defines a functor

$$V^k(\mathfrak{g})\text{-}\mathsf{Mod} \to \mathcal{W}^k(\mathfrak{g})\text{-}\mathsf{Mod}, M \mapsto H^*(M \otimes \mathcal{F}_{\mathfrak{n}}, Q_{(0)}).$$

Example 4 (Virasoro algebra)**.** We calculate the simplest case $\mathcal{W}^k(\mathfrak{sl}_2), k \neq -2, 0$ (as a physicist) explicitly. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}_2$. The vertex algebra $V^k(\mathfrak{sl}_2)$ is generated by quantum fields $e(z), h(z), f(z)$ satisfying OPEs

$$\begin{aligned}
h(z)e(z) &\sim \frac{2e(z)}{z-w};\\
h(z)f(z) &\sim -\frac{2f(z)}{z-w};\\
h(z)h(w) &\sim \frac{2k}{(z-w)^2};\\
e(z)f(w) &\sim \frac{k}{(z-w)^2} + \frac{h(w)}{z-w};\\
e(z)e(w) &\sim 0;\\
f(z)f(w) &\sim 0.
\end{aligned}$$

The vertex algebra $\mathcal{F}_{\mathbb{C}e}$ is generated by fermionic quantum fields (ghosts) $\psi(z), \varphi(z)$ satisfying OPEs

$$\begin{aligned}
\psi(z)\varphi(w) &\sim \frac{1}{z-w};\\
\psi(z)\psi(w) &\sim 0;\\
\varphi(z)\varphi(w) &\sim 0.
\end{aligned}$$

In this case, the Drinfeld–Sokolov character is simply given by

$$\chi = \varphi_1 = \int \varphi(z)dz.$$

So the BRST differential is

$$d = \int e(z)\varphi(z) + \varphi(z)dz.$$

We decompose it into $d = d_0 + d_1$, where

$$d_0 = \int e(z)\varphi(z)dz, d_1 = \int \varphi(z)dz.$$

From the OPEs

$$(e(z)\varphi(z))(e(w)\varphi(w)) \sim 0, \varphi(z)\varphi(w) \sim 0,$$

we see that $d_0^2 = d_1^2 = 0$. Now we can introduce a bigrading on $C^\bullet = V^k(\mathfrak{sl}_2) \otimes \mathcal{F}^\bullet_{\mathbb{C}e}$ such that d_0 has degree $(1,0)$, d_1 has degree $(0,1)$ and $C^i = \bigoplus_{p+q=i} C^{p,q}$, and use the associated spectral sequence to compute the cohomology $H^*(C^\bullet, d)$. Explicitly, the bigrading of quantum fields are given by

$$\deg e(z) = (1,1), \deg h(z) = (0,0), \deg f(z) = (-1,-1),$$
$$\deg \psi(z) = (0,-1), \deg \varphi(z) = (0,1).$$

The E_1 page is $E_1^{p,q} = H^p(C^{\bullet,q}, d_0)$. Let $\tilde{h}(z) = h(z) + 2{:}\psi(z)\varphi(z){:}$. We can calculate that

$$\begin{aligned}
&\tilde{h}(z)(e(w)\varphi(w)) \sim 0;\\
&\tilde{h}(z)\tilde{h}(w) \sim \frac{2(k+2)}{(z-w)^2};\\
&\varphi(z)(e(w)\varphi(w)) \sim 0.
\end{aligned}$$

Claim. By explicit calculations, we will see that the algebra of cohomology of $C^\bullet$ with respect to $d_0 = \int e(z)\varphi(z)dz$ is generated by fields $\tilde{h}(z)$ and $\varphi(z)$.

Now it follows that $E_1^{P,q} = 0$ unless $p = 0, q \geq 0$. So the cohomology $H^*(C^\bullet, d)$ equals to the cohomology of $\left(\bigoplus_{q\geq 0} E_1^{0,q}, d_1 = \int \varphi(z)dz\right)$.

Set $\nu = \sqrt{2(k+2)}$ and $J(z) = \tilde{h}(z)/\nu$. $J(z)$ is a free bosonic field, viz.

$$J(z)J(w) \sim \frac{1}{(z-w)^2}.$$

On the cohomology space, we have

$$0 = d_0 f(z) = h(z)\varphi(z) + k\partial_z\varphi(z) = \nu\left({:}J(z)\varphi(z){:} + \frac{1}{2}\nu\partial_z\varphi(z)\right),$$

so

$${:}J(z)\varphi(z){:} = -\frac{1}{2}\nu\partial_z\varphi(z).$$

We have

$$J(z)\varphi(w) = -\frac{2}{\nu}\frac{1}{z-w}\varphi(w) - \frac{1}{2}\nu\partial_w\varphi(w) + \cdots.$$

This coincides with the first two terms of the operator product of $J(z)$ with the vertex operator

$$V(-2/\nu, z) = {:}\exp\left(-(2/\nu)\int J(z)\right){:}.$$

Claim. The relations between $J(z)$ and $\varphi(z)$ coincides with the ones between $J(z)$ and $V(-2/\nu, z) = {:}\exp\left(-(2/\nu)\int J(z)\right){:}$.

Using this, we see that $E_1^{0,\bullet}$ is generated by the scalar bosonic field $J(z)$ and $V(-2/\nu, z)$ with relations

$$V(-2/\nu, z)V(-2/\nu, w) \sim 0.$$

The cohomology with respect to $d_1 = \int \varphi(z)dz$ coincides with the centralizer of $\int V(-2/\nu, z)dz$ in the bosonic free field (see Section 3.1 below). From results in conformal field theory, we know that it is a Virasoro algebra with central charge

$$c = 1 - 12\left(\frac{1}{2}\nu - \frac{1}{\nu}\right)^2 = 1 - \frac{6(k+1)^2}{k+2}.$$

Remark 3. The calculation above is the prototype for *Wakimoto free field realization*. The operator $V(-2/\nu, z)$ is an example of the *screening operator*. We use some claims that are not proved here. A rigorous calculation and generalization to general $\mathcal{W}$-algebra can be found in [FBZ04]. Its application to the *geometric Langlands program* can be found in [Fre07].

3 Chiral de Rham complex

For a complex (analytic or algebraic) manifold X, there is the standard de Rham complex $(\Omega_X, d_{\mathrm{DR}})$, that is fundamental in complex (differential or algebraic) geometry. Motivated by string theory, Fyodor Malikov, Vadim Schechtman and Arkady Vaintrob [MSV99] constructed the *chiral de Rham complex* $(\Omega_X^{\mathrm{ch}}, d_{\mathrm{DR}}^{\mathrm{ch}})$. It is a complex of sheaves of vertex algebras. There is a canonical embedding

$$(\Omega_X, d_{\mathrm{DR}}) \hookrightarrow (\Omega_X^{\mathrm{ch}}, d_{\mathrm{DR}}^{\mathrm{ch}}).$$

This embedding is a quasi-isomorphism, but still Ω_X^{ch} carries a lot more information that cannot be read from Ω_X. For example, Ω_X^{ch} admits $\mathcal{N} = 2$ supersymmetry when X is Calabi–Yau.

We try to give a concise exposition of the story in this section.

3.1 Fock spaces

In the previous section, we have described fermionic Fock space. This construction has a bosonic counterpart.

Example 5 (Bosonic fock space)**.** We view $\mathbb{C}((t))$ as a commutative Lie algebra, endowed with a canonical invariant symmetric bilinear form

$$(f, g) = -\operatorname{Res}_{t=0} f dg.$$

Correspondingly, we denote by $\mathcal{H}$ the affinization of $\mathbb{C}((t))$ and by $\mathcal{B}$ the vacuum representation of level 1. $\mathcal{B}$ is called the *bosonic Fock space* (or the *$\beta\gamma$-system*).

To echo with the construction of fermionic fields in previous section, we can define $\mathcal{H}$ and $\mathcal{B}$ alternatively as follows. Namely, $\mathcal{H}$ has a basis $a_n, b_n, n \in \mathbb{Z}$ and 1. The commutation relations are given by

$$[a_n, a_m] = [b_n, b_m] = 0, [a_n, b_m] = \delta_{n,-m}.$$

$\mathcal{B}$ is a representation of $\mathcal{H}$, generated by a vector $|0\rangle$, such that

$$a_n|0\rangle = 0, n \geq 0, b_n|0\rangle = 0, n > 0.$$

$\mathcal{B}$ has a basis consisting of monomials of the form

$$a_{n_1} \cdots a_{n_k} b_{m_1} \cdots b_{m_\ell}|0\rangle,$$

where $n_1 < n_2 < \cdots < n_k < 0$ and $m_1 < m_2 < \cdots < m_\ell \leq 0$.

The vertex algebra structure on $\mathcal{B}$ is determined by

- The vacuum state is $|0\rangle$.
- The translation operator T sends $|0\rangle$ to 0 and should satisfy commutation relations

$$[T, a_n] = -na_{n-1}, [T, b_n] = -(n-1)b_{n-1}.$$

- The fields corresponding to $a_{-1}|0\rangle$ and $b_0|0\rangle$ are given by

$$Y(a_{-1}|0\rangle, z) = a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, Y(b_0|0\rangle, z) = b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n}$$

respectively. In general,

$$Y(a_{n_1}\cdots a_{n_k} b_{m_1}\cdots b_{m_\ell}|0\rangle, z)$$
$$= \prod_{i=1}^{k} \frac{1}{(-n_i-1)!} \prod_{j=1}^{\ell} \frac{1}{(-m_j)!} :\partial_z^{-n_1-1} a(z)\cdots \partial_z^{-n_k-1} a(z) \partial_z^{-m_1} b(z) \cdots \partial_z^{-m_\ell} b(z):.$$

- The energy-momentum field is

$$L(z) = :\partial_z b(z) a(z):.$$

The Virasoro central charge is 2.

As usual, this construction can be generalized to $\mathbb{C}^N$ by taking tensor product. Namely, $\mathcal{H}_{\mathbb{C}^N}$ has generators $a_n^i, b_n^i, i = 1, \cdots, N, n \in \mathbb{Z}$ and 1, with relations

$$[a_n^i, a_m^j] = [b_n^i, b_m^j] = 0, [a_n^i, b_m^j] = \delta_{i,j}\delta_{n,-m}.$$

$\mathcal{B}_{\mathbb{C}^N}$ is a representation of $\mathcal{H}_{\mathbb{C}^N}$, generated by a vector $|0\rangle$, such that

$$a_n^i|0\rangle = 0, n \geq 0, b_n^i|0\rangle = 0, n > 0.$$

The vertex algebra structure on $\mathcal{B}_{\mathbb{C}^N}$ is isomorphic to the tensor product of N copies of bosonic Fock space $\mathcal{B}$. In particular, we have

$$Y(a_{-1}^i|0\rangle, z) = a^i(z) = \sum_{n\in\mathbb{Z}} a_n^i z^{-n-1}, Y(b_0^i|0\rangle, z) = b^i(z) = \sum_{n\in\mathbb{Z}} b_n^i z^{-n}.$$

Here we write $a^i = a_{-1}^i|0\rangle, b^i = b_0^i|0\rangle$ for brevity. We have OPEs

$$[a^i(z), a^j(w)] = [b^i(z), b^j(w)] = 0, [a^i(z), b^j(w)] = \delta_{i,j}\delta(z-w).$$

The energy-momentum field is given by

$$L(z) = \sum_{i=1}^{N} :\partial_z b^i(z) a^i(z):.$$

Correspondingly, the Virasoro central charge is $2N$.

3.2 Prototype: chiral de Rham algebra of an affine space

We apply our standard strategy to construct a sheaf on a complex variety X:

- Firstly, we construct for the affine space $\mathbb{C}^N$.
- Secondly, we localize the above construction.
- Finally, we glue local sheaves together.

We do the first step in this section.

Definition 11. The tensor product $\Omega_N = \mathcal{B}_N \otimes \mathcal{F}_N$ is the *chiral de Rham algebra* of the affine space $\mathbb{C}^N$. Here $\mathcal{B}_N = \mathcal{B}_{\mathbb{C}^N}, \mathcal{F}_N = \mathcal{F}_{\mathbb{C}^N}$.

The conformal vertex algebra Ω_N admits $\mathcal{N} = 2$ supersymmetry in the following sense:

Proposition 7 ($\mathcal{N} = 2$ supersymmetry)**.** The Virasoro central charge of Ω_N is 0. Denote by L the standard conformal vector in $\Omega_N = \mathcal{B}_N \otimes \mathcal{F}_N$. There exists an even element J of conformal weight 1, an odd element Q of conformal weight 1 and an odd element G of conformal weight 2, such that we have OPEs

$$\begin{aligned}
L(z)L(w) &\sim \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w};\\
J(z)J(w) &\sim \frac{N}{(z-w)^2};\\
L(z)J(w) &\sim -\frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w};\\
G(z)G(w) &\sim 0;\\
L(z)G(w) &\sim \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w};\\
J(z)G(w) &\sim -\frac{G(w)}{z-w};\\
Q(z)Q(w) &\sim 0;\\
L(z)Q(w) &\sim \frac{Q(w)}{(z-w)^2} + \frac{\partial_w Q(w)}{z-w};\\
J(z)Q(w) &\sim \frac{Q(w)}{z-w};\\
Q(z)G(w) &\sim \frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w)}{z-w}.
\end{aligned}$$

Proof. The Virasoro central charge of Ω_N is 0 because the Virasoro central charge of $\mathcal{B}_N$ (resp. $\mathcal{F}_N$) is $2N$ (resp. $-2N$). We write down the fields $L(z), J(z), G(z), Q(z)$ explicitly.

$$\begin{aligned}
L(z) &= \sum_{i=1}^{N} \left({:}\partial_z b^i(z) a^i(z){:} \otimes \mathrm{id}_{\mathcal{F}_N} + \mathrm{id}_{\mathcal{B}_N} \otimes {:}\partial_z \varphi^i(z) \psi^i(z){:} \right);\\
J(z) &= \sum_{i=1}^{N} \mathrm{id}_{\mathcal{B}_N} \otimes {:}\varphi^i(z)\psi^i(z){:};\\
G(z) &= \sum_{i=1}^{N} {:}(\mathrm{id}_{\mathcal{B}_N} \otimes \psi^i(z))(\partial_z b^i(z) \otimes \mathrm{id}_{\mathcal{F}_N}){:};\\
Q(z) &= \sum_{i=1}^{N} {:}(a^i(z) \otimes \mathrm{id}_{\mathcal{F}_N})(\mathrm{id}_{\mathcal{B}_N} \otimes \varphi^i(z)){:}.
\end{aligned}$$

Translating the OPEs into the language of λ-bracket, we can calculate the desired formulae explicitly using non-commutative Wick formula. □

Remark 4. The construction of Q coincides with the chiral BRST reduction described in the previous section. As usual, Ω_N has the fermionic charge gradation

$$\Omega_N = \bigoplus_{q\in\mathbb{Z}} \Omega_N^q.$$

The operator $d = Q_{(0)} = \sum_{i=1}^N \sum_{n\in\mathbb{Z}} {:}a_n^i \varphi_{-n}^i{:}$ defines a differential on Ω_N.

Let $\left(\Omega(\mathbb{C}^N) = \bigoplus_{p=0}^N \Omega^p(\mathbb{C}^N), d_{\mathrm{DR}}\right)$ be the usual algebraic de Rham complex of $\mathbb{C}^N$. We identify the coordinate functions on $\mathbb{C}^N$ with letters $b_0^1, \cdots, b_0^N$, and their differentials with odd variables $\varphi_0^1, \cdots, \varphi_0^N$. Then we have identifications of super commutative DG algebras

$$\Omega(\mathbb{C}^N) \simeq \mathbb{C}[b_0^1, \cdots, b_0^N] \otimes \bigwedge(\varphi_0^1, \cdots, \varphi_0^N),$$

where the de Rham differential d_{DR} is represented by

$$d_{\mathrm{DR}} = \sum_{i=1}^N a_0^i \otimes \varphi_0^i,$$

here $a_0^i = \partial_{b_0^i}$.

We can identify the vector space Ω_N with the space of polynomials in even variables a_n^i, b_m^i ($n < 0, m \leq 0$) and odd variables ψ_n^i, φ_m^i ($n < 0, m \leq 0$). The following theorem justify the assertion that Ω_N is the chiralization of $\Omega(\mathbb{C}^N)$.

Proposition 8. The obvious inclusion $\iota\colon (\Omega(\mathbb{C}^N), d_{\mathrm{DR}}) \hookrightarrow (\Omega_N, d)$ is a quasi-isomorphism.

Proof. $\Omega(\mathbb{C}^N)$ is the subspace of conformal weight 0 in Ω_N. From the OPE

$$L(z)G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w};$$
$$Q(z)G(w) \sim \frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w)}{z-w},$$

one can derive that $[G_0, d] = -L_0$ and $[G_0, L_0] = 0$. So G_0 gives a homotopy to 0 for the operator d on all subcomplexes of non-zero conformal weight. Therefore, all cohomologies live in the conformal weight zero subspace, that is to say, live in $\Omega(\mathbb{C}^N)$. □

3.3 Completion and Zariski localization

Denote by A_N the polynomial algebra $\mathbb{C}[b^1, \cdots, b^N]$. It is identified with the subspace $\mathcal{B}_N[0] \subset \mathcal{B}_N$ of conformal weight 0. The space $\mathcal{B}_N$ has an obvious A_N-module structure. Let $\widehat{A_N} = \mathbb{C}[\![b^1, \cdots, b^N]\!]$ be the completion of A_N with respect to the augmentation ideal $I = (b^1, \cdots, b^N)$. Set

$$\widehat{\mathcal{B}_N} = \widehat{A_N} \otimes_{A_N} \mathcal{B}_N.$$

We can define a conformal vertex algebra structure on $\widehat{\mathcal{B}_N}$.

Lemma 5 (Completion)**.** Every element $c \in \widehat{\mathcal{B}_N}$ is a limit of elements $c_i \in \mathcal{B}_N$ (in the I-adic topology). Regarding the fields $c_i(z)$ as elements in $\mathrm{End}(\widehat{\mathcal{B}_N})[\![z, z^{-1}]\!]$, the limit of the fields $c_i(z)$ exists, and will be denoted by $c(z)$. This map $c \mapsto c(z)$ defines the state-field correspondence on $\widehat{\mathcal{B}_N}$, and gives rise to a conformal vertex algebra structure on $\widehat{\mathcal{B}_N}$.

Similarly, one can construct Zariski localization. For $0 \neq f \in A_N$, denote by $A_{N,f}$ the localization of A_N at f. Set

$$\mathcal{B}_{N,f} = A_{N,f} \otimes \mathcal{B}_N.$$

Lemma 6 (Zariski localization)**.** $\mathcal{B}_{N,f}$ admits a standard structure of conformal vertex algebra.

Proof. Let $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n}$ be the quantum field corresponding to f. Hopefully, one wants to define the field corresponding to f^{-1} as

$$\begin{aligned} f(z)^{-1} &= (f_0 + f_{-1}z + f_1 z^{-1} + f_{-2}z^2 + f_2 z^{-2} + \cdots)^{-1} \\ &= f_0^{-1}(1 + f_0^{-1}(f_{-1}z + f_1 z^{-1} + f_{-2}z^2 + f_2 z^{-2} + \cdots))^{-1} \\ &= f_0^{-1}(1 + f_0^{-2}(2f_{-1}f_1 + 2f_{-2}f_2 + \cdots) + \cdots). \end{aligned}$$

The last equality is the formal geometric series expansion. In the right hand side, the coefficient of each power of z is an infinite sum, but it is well-defined as an operator on $A_{N,f}$. We only need to invert $f_0 = f$. Using the reconstruction theorem à la Victor Kac, it is easy to show that the above data determine the conformal vertex algebra structure on $\mathcal{B}_{N,f}$. □

Now we arrive at the juncture to sheafify $\mathcal{B}_N$. Let X denote the affine space $\mathrm{Spec}(A_N)$. We have an $\mathcal{O}_X$-quasicoherent sheaf $\mathcal{O}_X^{\mathrm{ch}}$ corresponding to the A_N-module $\mathcal{B}_N$. We have seen that for each $0 \neq f \in A_N$, the space $V_{N,f} = \Gamma(U_f, \mathcal{O}_X^{\mathrm{ch}})$ admits a standard structure of conformal vertex algebra, where $U_f = \mathrm{Spec}(A_{N,f}) \subset X$. If $U_f \subset U_g$, the restriction map $V_{N,g} \to V_{N,f}$ is a morphism of conformal vertex algebras. Now applying standard argument in sheaf theory, we see that $\mathcal{O}_X^{\mathrm{ch}}$ is a sheaf of conformal vertex algebras.

Similarly, we can construct the completion

$$\widehat{\Omega_N} = \widehat{A_N} \otimes_{A_N} \Omega_N$$

and the Zariski localization

$$\Omega_{N,f} = A_{N,f} \otimes_{A_N} \Omega_N.$$

We have a $\mathcal{O}_X$-quasicoherent sheaf Ω_X^{ch} of conformal vertex algebras corresponding to the A_N-module Ω_N.

3.4 How to glue chiral de Rham sheaves

In this section, we explain how to glue chiral de Rham sheaves. We work in the formal setting to deal with coordinate transformations. It is straightforward to work in the smooth/analytic category.

Let X be the formal scheme $\mathrm{Spf}\,\mathbb{C}[\![b^1, \cdots, b^N]\!]$. Consider the formal $N|N$-dimensional superscheme $\tilde{X} = \Pi TX$. $\tilde{X}$ has the same underlying space as X, but the structure sheaf of $\tilde{X}$ coincides with the sheaf of de Rham differential forms on X. On $\tilde{X}$, we have N even coordinates $b^1, \cdots, b^N$ and N odd ones $\varphi^1 = db^1, \cdots, \varphi^N = db^N$.

We have assigned to this superscheme $\tilde{X}$ a conformal vertex superalgebra $\widehat{\Omega_N}$, generated by even fields $a^i(z), b^i(z)$ and odd fields (ghosts) $\psi^i(z), \varphi^i(z)$, with OPEs

$$\begin{aligned}
&a^i(z)b^j(w) \sim \frac{\delta_{i,j}}{z-w};\\
&a^i(z)a^j(w) \sim 0, b^i(z)b^j(w) \sim 0;\\
&\varphi^i(z)\psi^j(w) \sim \frac{\delta_{i,j}}{z-w};\\
&\varphi^i(z)\varphi^j(w) \sim 0, \psi^i(z)\psi^j(w) \sim 0;\\
&b^i(z)\varphi^j(w) \sim 0, b^i(z)\psi^j(w) \sim 0,\\
&a^i(z)\varphi^j(w) \sim 0, a^i(z)\psi^j(w) \sim 0.
\end{aligned}$$

Intuitively, we may think of a^i (resp. ψ^i) as the vector field ∂_{b^i} (resp. ∂_{φ^i}).

Consider an invertible coordinate transformation

$$\tilde{b}^i = g^i(b^1, \cdots, b^N), b^i = f^i(\tilde{b}^1, \cdots, \tilde{b}^N),$$

here $g^i \in \mathbb{C}[\![b^1, \cdots, b^N]\!], f^i \in [\![\tilde{b}^1, \cdots, \tilde{b}^N]\!]$. It will induce a transformation of the odd coordinates $\varphi^i = db^i$ by the following rules:

$$\tilde{\varphi}^i = \sum_{j=1}^N \frac{\partial g^i}{\partial b^j}\varphi^j, \varphi^i = \sum_{j=1}^N \frac{\partial f^i}{\partial \tilde{b}^j}\tilde{\varphi}^j.$$

Correspondingly, the vector fields will transform as

$$\begin{aligned}
\partial_{\tilde{b}^i} &= \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{b^j} + \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^\ell}(g(b))\frac{\partial g^\ell}{\partial b^r}\varphi^r\partial_{\varphi^k};\\
\partial_{\tilde{\varphi}^i} &= \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{\varphi^j}.
\end{aligned}$$

Motivated by these formulae, we define

$$\begin{aligned}
&\tilde{b}^i(z) = g^i(b)(z);\\
&\tilde{\varphi}^i(z) = \left(\frac{\partial g^i}{\partial b^j}\varphi^j\right)(z);\\
&\tilde{a}^i(z) = \left(a^j\frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\right)(z) + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^\ell}(g(b))\frac{\partial g^\ell}{\partial b^r}\varphi^r\psi^k\right)(z);\\
&\tilde{\psi}^i(z) = \left(\frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\psi^j\right)(z).
\end{aligned}$$

Proposition 9. The fields

$$\tilde{a}^i(z), \tilde{b}^i(z), \tilde{\psi}^i(z), \tilde{\varphi}^i(z)$$

share the same OPEs with

$$a^i(z), b^i(z), \psi^i(z), \varphi^i(z).$$

Corollary 1. For each automorphism $g = (g^1, \cdots, g^N)$ of $\mathbb{C}[\![b^1, \cdots, b^N]\!]$, the above formulae determine a morphism of conformal vertex algebras

$$\tilde{g}\colon \widehat{\Omega_N} \to \widehat{\Omega_N}.$$

This defines a group homomorphism $G_N \to \mathrm{Aut}(\widehat{\Omega_N})$, where G_N is the automorphism group of $\mathbb{C}[\![b^1, \cdots, b^N]\!]$.

Proof. We check that the assignment $g \mapsto \tilde{g}$ is indeed a group homomorphism. By the reconstruction theorem à la Kac, it suffices to check on generators. This follows from a direct computation. It seems to be tedious, but an interesting phenomenon called *cancellation of anomalies* will happen.

Let $'b^i = g_1^i(b), '' b^i = g_2^i('b)$ be two successive coordinate transformations, f_j the inverse to g_j. We now show $\widetilde{g_2 g_1}(a^i) = \tilde{g}_2\tilde{g}_1(a^i)$ as an example (there are no anomalies for the generators b, ψ and φ). In the coordinates $'a$, etc., the element $''a^i_{-1}|0\rangle$ has the form

$$''a^i_{-1}|0\rangle = \left('a^j_{-1}\frac{\partial f_2^j}{\partial'' b^i}(g_2('b_0)) - \frac{\partial^2 f_2^k}{\partial'' b^i \partial'' b^\ell}(g_2('b_0))\frac{\partial g_2^\ell}{\partial' b^r}('b_0)'{\psi_{-1}}^k{}'\varphi_0^r \right)|0\rangle.$$

Further expressed in the coordinates a, etc., we get

$$\begin{aligned}\tilde{g}_2\tilde{g}_1(a^i) = &\left[a^p\frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z) + \frac{\partial^2 f_1^p}{\partial' b^j\partial' b^q}(g_1(b))\frac{\partial g_1^q}{\partial b^s}\psi^p\varphi^s(z)\right]_{-1}\frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b))(z)_0|0\rangle \\ &- \frac{\partial^2 f_2^k}{\partial'' b^i\partial'' b^\ell}(g_2 g_1(b))\frac{\partial g_2^\ell}{\partial' b^r}(g_1(b))(z)_0\left[\frac{\partial f_1^p}{\partial' b^k}(g_1(b))\psi^p(z)\right]_{-1}\left[\frac{\partial g_1^r}{\partial b^q}\varphi^q(z)\right]_0|0\rangle.\end{aligned}$$

The above expression is equal to $\widetilde{g_2 g_1}(a^i)$ plus two anomalous terms:

$$a_0^p\left[\frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z)\right]_{-1}\frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b_0))|0\rangle = \left[\frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z)\right]_{-1}\frac{\partial}{\partial b_0^p}\frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b_0))|0\rangle$$

coming from the first summand, and

$$\begin{aligned}&-\frac{\partial^2 f_2^k}{\partial'' b^i\partial'' b^\ell}(g_2 g_1(b_0))\frac{\partial g_2^\ell}{\partial' b^r}(g_1(b_0))\left[\frac{\partial f_1^p}{\partial' b^k}(g_1(b))(z)\right]_{-1}\frac{\partial g_1^r}{\partial b^q}(b_0)\psi_0^p\varphi_0^q|0\rangle \\ =&-\frac{\partial^2 f_2^k}{\partial'' b^i\partial'' b^\ell}(g_2 g_1(b_0))\frac{\partial g_2^\ell}{\partial' b^r}(g_1(b_0))\left[\frac{\partial f_1^p}{\partial' b^k}(g_1(b))(z)\right]_{-1}\frac{\partial g_1^r}{\partial b^p}(b_0)|0\rangle\end{aligned}$$

coming form the second. These two terms mysteriously cancel out. □

The above results allow us to define the sheaf of conformal vertex algebras Ω_X^{ch} for each complex manifold X, by gluing the prototype of sheaves over $\mathbb{C}^N$.

Remark 5. In the algebraic situation, standard arguments in *formal geometry* à la Israel Moiseevich Gelfand and David Kazhdan ensure the existence of our sheaves, cf. [GK71].

Remark 6. One might want to construct a chiral structure sheaf $\mathcal{O}_X^{\mathrm{ch}}$ for a general complex manifold X. In general, you will get an anomaly while transforming coordinates, so it is not possible to do so.

3.5 $\mathcal{N} = 2$ supersymmetry

Under a coordinate transformation

$$\tilde{b}^i = g^i(b^1, \cdots, b^N), b^i = f^i(\tilde{b}^1, \cdots, \tilde{b}^N),$$

the fields $L(z), J(z), G(z), Q(z)$ will transform as follows.

Proposition 10. We have

$$\begin{aligned}
\tilde{L}(z) &= L(z);\\
\tilde{J}(z) &= J(z) + \partial_z \left(\operatorname{tr}\log\left(\frac{\partial g^i}{\partial b^j}(z)\right)\right);\\
\tilde{G}(z) &= G(z);\\
\tilde{Q}(z) &= Q(z) + \left(\frac{\partial}{\partial \tilde{b}^r}\left(\operatorname{tr}\log\left(\frac{\partial f^i}{\partial \tilde{b}^j}\right)\right)\tilde{\varphi}^r(z)\right).
\end{aligned}$$

Now assume X is a Calabi–Yau manifold, then these fields $L(z), J(z), G(z), Q(z)$ are globally well-defined, so Ω_X^{ch} admits $\mathcal{N}=2$ supersymmetry.

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