Real group, symmetric varieties and Langlands duality

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Real & Symmetric correspondence

$G/\mathbb{C}$

$$\left[\begin{array}{ll} \eta: G \to G & \text{anti-holo. invol'n} \\ \eta^2 = Id & G^\eta = G_{\mathbb{R}} \text{ real form} \end{array}\right] \longleftrightarrow \left[\begin{array}{ll} \theta: G \to G & \text{alg./holo. Inv.} \\ \theta^2 = Id, & \underbrace{K = G^\theta}_{\text{Sym. subgp}}, \ \underbrace{X = K\backslash G}_{\text{Sym. var.}} \end{array}\right]$$

	Real	Symmetric
Cartan	$G_{\mathbb{R}}$	$X = K\backslash G$
Harish-Chandra	$G_{\mathbb{R}}$-rep	$(\mathfrak{g}, K)$-mod, $\mathfrak{g} = \mathrm{Lie}\, G$
Matsuki	$G_{\mathbb{R}}$-orbits on G/B	K-orbits on G/B
MUV	$D(G_{\mathbb{R}}\backslash G/B)$	$D(K\backslash G/B)$
	Real Langlands for $\mathbb{P}^1$ $\mathrm{Bun}_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}})$ $\longleftrightarrow$	Relative Satake $L^+G \curvearrowright LX$
	real Langlands for twistor $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ $\mathrm{Bun}_{..}(\widetilde{\mathbb{P}}^1_{\mathbb{R}})$	Twisted relative Satake $L^+G \curvearrowright L\widetilde{X}$

Complex Satake and complex Langlands on $\mathbb{P}^1$

$Gr = LG/L^+G$

$L^+G \curvearrowright Gr \rightsquigarrow D(L^+G\backslash Gr)$

complex Satake cat.

$$Perv(L^+G\backslash Gr) \xrightarrow[\sim]{\text{geom. Satake}} Rep\, \check{G} \quad , \quad \check{G}/\mathbb{C} \quad \text{cpx dual gp of } G$$

$Bun_G(\mathbb{P}^1)$ ← moduli stack of G-bundles on $\mathbb{P}^1$

Thm (V. Lafforgue, R. Bezrukavnikov) $D(L^+G\backslash Gr) \xrightarrow[\sim]{q_* p^!} D(Bun_G(\mathbb{P}^1))$

$$\begin{tikzcd} & G\backslash Gr \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ L^+G\backslash Gr & & G[t^{-1}]\backslash Gr \arrow[r, "\text{uniformization}", "\sim"'] & Bun_G(\mathbb{P}^1) \end{tikzcd}$$

Cor. (V. Lafforgue, R.B. – M.F.)

$$\begin{tikzcd} D(L^+G\backslash Gr) \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & D(Bun_G(\mathbb{P}^1)) \arrow[d, phantom, "\wr"] \\ QCoh(\check{\mathfrak{g}}[2]/\check{G}) \arrow[r, "\sim"] & QCoh(Loc_{\check{G}}(\mathbb{P}^1)) \end{tikzcd}$$

$L^+G \curvearrowright$

$LX = X((t))$ formal loop space of X.

$D(L^+G\backslash LX)$ ← relative Satake cat.

$\eta : \mathbb{P}^1 \to \mathbb{P}^1$, $\eta(z) = \bar{z}$ be the cpx conj.

$(\mathbb{P}^1)^\eta = \mathbb{P}^1_{\mathbb{R}}$ $\leftarrow$ real proj. line.

$\mathrm{Bun}_G(\mathbb{P}^1) \circlearrowleft \eta$ $\qquad$ $\mathrm{Bun}_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}}) = (\mathrm{Bun}_G(\mathbb{P}^1))^\eta$

real form

Thm. (D. Nadler) $\quad G_{\mathbb{R}} \longleftrightarrow X = K \backslash G$

$$D(L^+G \backslash LX) \xrightarrow[\sim]{q_* p^!} D(\mathrm{Bun}_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}}))$$

Pf: ① Assume K is conn'd.

$(G_{\mathbb{R}})$

$K_{\mathbb{R}} = G_{\mathbb{R}} \cap K$

$$\begin{array}{ccc} & L_{poly} K_{\mathbb{R}} \backslash Gr & \\ \swarrow^{p} & & \searrow^{q} \\ L^+G \backslash LX \simeq LK \backslash Gr & & L_{poly} G_{\mathbb{R}} \backslash Gr \end{array}$$

is real unif.

$\mathrm{Bun}_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}})$

, $L_{poly} G_{\mathbb{R}} = \left\{ \begin{array}{ccc} \gamma: \mathbb{C}^\times & \to & G \\ \cup & & \cup \\ S^1 & \to & G_{\mathbb{R}} \end{array} \right\}$

② A construction of Matsuki-Morse flow on Gr:

LK-orbits on Gr $\longleftrightarrow$ $L_{poly} G_{\mathbb{R}}$-orbits on Gr

Applications

Real $\Rightarrow$ Symmetric $\qquad$ $L^+G_{\mathbb{R}} \circlearrowleft$

$$Gr_{\mathbb{R}} = G_{\mathbb{R}}((t)) / G_{\mathbb{R}}[[t]] \leftarrow \text{real affine } Gr.$$

$$= LG_{\mathbb{R}} / L^+G_{\mathbb{R}}$$

$D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$ ← real Satake cat.

Thm (Nadler, –) The real Hecke action at the real point $0 \in \mathbb{P}^1_{\mathbb{R}}$ define an equiv.

$$D(Bun_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}})) \xrightarrow{\sim} D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$$

s.t. the composition equiv.

$$D(L^+G \backslash LX) \xrightarrow{\sim} D(Bun_{G_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}})) \xrightarrow{\sim} D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})$$

is t-exact and compatible w/ fusion and complex Hecke action.

$$\begin{array}{ccc}
D(L^+G \backslash LX) & \xrightarrow{\sim} & D(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}}) \\
\cup & & \cup \\
Perv(L^+G \backslash LX) & \xrightarrow{\sim} & Perv(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}}) \\
\cup & & \cup \\
Perv(L^+G \backslash LX)_0 & \xrightarrow{\sim} & Perv(L^+G_{\mathbb{R}} \backslash Gr_{\mathbb{R}})_0 \\
\wr\| & & \wr\| \\
Rep(\check{G}_{sph}) & \xrightarrow{\sim} & Rep(\check{G}_{real}) \\
\uparrow & & \uparrow \\
\text{Gaitsgory–Nadler dual gp} & & \text{Nadler's dual group}
\end{array}$$

$\check{G}_{real}$ is computed by Nadler using the concrete geometry of $Gr_{\mathbb{R}}$.

Cor. $\check{G}_{sph} \simeq \check{G}_{real}$. In particular, the Weyl gp & root datum of $\check{G}_{sph}$ is the same as those from the structure theory of X.

Proof.

Symmetric $\Rightarrow$ Real

Thm (Lingfei-Yi, –)

① The IC-complex $IC(\mathcal{O}, \mathcal{L})$ of a L^+G-equiv. local system of a L^+G-orbit

$\mathcal{O} \subset LX$ is pointwise pure

② The dg Ext algebra $RHom(IC_0, IC_0 * \mathrm{Rep}\,\check{G}_{sph})$ is formal.

Pf It follows from the construction of conical "Morse-Springer" slices for L^+G-orbit closure in LX.

Cor. The real dg Ext algebra

$$\underset{D((L^+G_{\mathbb{R}})\backslash Gr_{\mathbb{R}})}{RHom}(IC, IC_0 * \mathrm{Rep}\,\check{G}_{real})$$ is formal.

This result was used in the proof of real / relative Satake equiv. for the split rank

Symmetric pairs

$X = Sp_{2n} \backslash GL_{2n}$, $SO_{2n-1} \backslash SO_{2n}$, $F_4 \backslash E_6$

$\updownarrow$ $\updownarrow$ $\updownarrow$

$\mathbb{P}^{n-1}(\mathbb{H})$ S^{2n-2} $\mathbb{O}\mathbb{P}^2$

quaternion octonion proj. plane