p- divisible groups

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§1. p-div. gp / ring

Setting.
- S a scheme
- G is comm. gp on $(Sch_S)_{fppf}$
- p prime number.
- $G[p^n] = \ker(G \xrightarrow{p^n} G)$

Def. G is called a p-div. gp if

(1) G is of p-torsion (i.e. $G = \varinjlim_n G[p^n]$)

(2) G is p-div. (i.e. $G \xrightarrow{p \cdot id} G$ is an epi)

(3) $G[p]$ is a finite, locally free gp scheme.

Lemma G as above
(1) $G[p^n] = (G[p^{n+1}])[p^n]$ (only need G comm. gp)

(2) $\forall\, 0 \le i \le n$, $p^{n-i} : G[p^n] \to G[p^i]$ is an epi

(3) $\forall\, n \geq 2$, $G[p^n]$ is a truncated p-div. gp of level n

(i.e. locally free, finite, comm. gp scheme /S. which is flat $\mathbb{Z}/p^n\mathbb{Z}$-mod)

(4) There exists a locally const. function h on S (called the height) s.t. the rank of $G[p^n]$ at $s \in S$ is $p^{nh(s)}$.

Cor. Def $\Leftrightarrow$ $\{G_n\}$ a system of comm. gp schemes

s.t. ① G_n is finite & locally free

② $G_n = G_{n+1}[p^n]$

③ $\exists\, h$ locally const function s.t. $\operatorname{order}(G_n|_s) = p^{nh(s)}$

$\left(G = \varinjlim G_n \text{ is a } p\text{-div. gp} \right)$

<u>Sketch of proof</u>

(1) (2) by def.

(3) <u>Claim 1</u>. $G[p^n]$ is a flat $\mathbb{Z}/p^n\mathbb{Z}$-mod

$\Leftrightarrow \ker(p^{n-i}) = \operatorname{im}(p^i)$, $\forall\, i = 0, 1, \cdots, n$

<u>Claim 2</u>. If $G[p^n]$ is a flat $\mathbb{Z}/p^n\mathbb{Z}$-module,

then $G[p^n]$ finite locally free $/S$

$\Leftrightarrow G[p^n][p]$ is finite locally free $/S$

$(\Rightarrow G[p^i] = G[p^n][p^i]$ is finite locally free for all $1 \leq i \leq n)$

Proof of claim 1. "$\Rightarrow$" elementary

"$\Leftarrow$" $\forall M$ sheaf of $\mathbb{F}_p$-module

WTS $\mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(M, G[p^n]) = 0$

$$E_{ij}^2 = \mathrm{Tor}_i^{\mathbb{Z}/p\mathbb{Z}}\left(M, \mathrm{Tor}_j^{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, G[p^n])\right)$$

$$\Rightarrow \mathrm{Tor}_{i+j}^{\mathbb{Z}/p^n\mathbb{Z}}(M, G[p^n])$$

$$\left(M \otimes_{\mathbb{Z}/p^n\mathbb{Z}} G[p^n] = M \otimes_{\mathbb{Z}/p\mathbb{Z}} \left(\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} G[p^n]\right)\right)$$

$$\begin{tikzcd}
\rightsquigarrow 0 & \mathrm{Tor}_1^{\mathbb{Z}/p\mathbb{Z}}(M, G/pG) \arrow[l] \arrow[d, phantom, "\shortparallel"] & \mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(M, G[p^n]) \arrow[l] \\
& 0 & M \otimes_{\mathbb{Z}/p\mathbb{Z}} \underbrace{\mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, G[p^n])}_{\substack{\shortparallel \\ 0}} \arrow[u]
\end{tikzcd}$$

by hypothesis

Proof of claim 2. "$\Leftarrow$"

$$0 \to G[p] \to G[p^2] \to G[p] \to 0$$

$$0 \to G[p] \to G[p^3] \to G[p^2] \to 0$$

$$\cdots \qquad \square$$

(4) $\mathrm{rank}\,(G[p]_s) = p^{h(s)}$

<u>Examples</u>. (1) A/S abelian scheme (comm., proper, smooth $/S$, geom. conn'd fibers)

$A[p^\infty] := \varinjlim_n A[p^n]$ is a p-div. gp of height $2\dim_S A$.

<u>Reason</u>
$$\begin{tikzcd} A \arrow[rr, "p"] \arrow[dr] & & A \arrow[dl] \\ & S & \end{tikzcd}$$
is faithfully flat

$\boxed{\text{fiber flatness criterion}}$ EGA (IV, 11.3.11)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & S & \end{tikzcd}$$
of f.p.; g lfp., h lfp.

f is flat & g flat $\Leftrightarrow$ h is flat over $f(X)$ and f_s is flat. $\forall s \in S$.

Check over each $s \in S$.

$\rightsquigarrow$ reduce to $S = \mathrm{Spec}\,k$

p is locally free of deg $p^{2\dim A}$.

(3)
$$\begin{tikzcd} A[p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d, "e"] \\ A \arrow[r, "p"'] & A \end{tikzcd}$$

p proper

$A[p]$ proper, quasi-finite $\Rightarrow$ $A[p]$ is finite

$\downarrow$

$A[p]$ flat $\longrightarrow$ $A[p]$ is locally free, finite.

<u>Example</u> T torus $/S$

$T[p^\infty] := \varinjlim_n T[p^n]$ is p-div. gp of height $= \dim_S T$

Pf. reduce to $T \cong \mathbb{G}_m^{\dim_S T}$

Fact $\{p\text{-div. gps}/S\} \hookrightarrow \{\text{abelian sheaves}/S_{fppf}\}$

closed under taking quotients & extensions

(but it is not closed under taking subobjects)

Def. Let G be a p-div. gp

Its Cartier dual $G^\vee := \varinjlim_n G[p^n]^\vee$ w/ transition maps (dual of $\cdot p$)

This is a p-div. gp.

Example. (1) $(A[p^\infty])^\vee \cong A^\vee[p^\infty]$ (Thm)

(2) $\mu_{p^\infty} = \varinjlim_n G_m[p^n]$ $ht = 1$

$$\begin{array}{c} \mu_{p^\infty} \\ \updownarrow \text{dual} \\ \mathbb{Q}_p/\mathbb{Z}_p \end{array}$$

§ (Quasi-) isogeny

Def An isogeny $f: G \to H$ is a surjection of fppf sheaves, s.t. $\ker(f)$ is rep. by a finite, locally free gp scheme $/S$.

Notation. $\underline{\mathrm{Hom}}(G, H) := \{$ sheaf of maps between G & $H\}$

Def. A quasi-isogeny $\rho : G \dashrightarrow H$ is a global section

$\rho \in \underline{\mathrm{Hom}}(G, H) \otimes \mathbb{Q}$ s.t. Zariski locally on S,

$p^n \rho$ is an isogeny for some $n \in \mathbb{Z}_{\geq 0}$.

Def A polarization λ on a p-div. gp G is a q-isog. $\lambda : G \dashrightarrow G^\vee$.

s.t. $\lambda^\vee = -\lambda$.

We call λ a principal polarization if λ is an isom.

Example A abelian scheme $\rightsquigarrow$ $A[p^\infty]$ p-div. gp

$$A_1 \xrightarrow[\text{(q isog.)}]{\text{isog.}} A_2 \quad \rightsquigarrow \quad A_1[p^\infty] \xrightarrow[\text{(q. isog.)}]{\text{isog.}} A_2[p^\infty]$$

$$\begin{matrix} \lambda_A : A \longrightarrow A^\vee \\ \lambda_A^\vee = \lambda_A \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} \lambda : A[p^\infty] \longrightarrow A^\vee[p^\infty] \\ (\lambda^\vee = -\lambda) \end{matrix}$$

$$\begin{tikzcd} A[p^n] \times A[p^n] \arrow[r, "\mathrm{id} \times \lambda"] \arrow[d, "\mathrm{swap}"] & A[p^n] \times A^\vee[p^n] \arrow[r, "e_n"] \arrow[d, "\vee"] & \mathbb{G}_m[p^n] \arrow[d, equal] \\ A[p^n] \times A[p^n] \arrow[r] & A^\vee[p^n] \times A[p^n] \arrow[r, "e_n \circ \mathrm{swap}"] & \mathbb{G}_m[p^n] \end{tikzcd}$$

§. Dieudonné theory / perfect field.

G finite comm. gp scheme / perfect field k, $\mathrm{char}(k) = p > 0$.

e.g. $\mu_{p^n} = \mathbb{G}_m[p^n]$

$\alpha_{p^n} = \ker(\mathbb{G}_a \xrightarrow{\mathrm{Fr}^n} \mathbb{G}_a)$

$G = G^{\mathrm{con}} \times G^{\text{ét}}$

Def. Dieudonné ring $D_k := W(k)\{F, V\} / \left(\begin{array}{ll} FV = VF = p, & \\ Fc = \mathrm{Fr}(c)\cdot F & c \in W(k) \\ \mathrm{Fr}(c)\cdot V = V\cdot c & \end{array} \right)$

e.g. $D_k / D_k V^n \cong \mathrm{End}'_{k\text{-gp}}(W_n)$

Thm. $\exists$ $\left\{ \begin{array}{l} \text{finite comm. gp} / k \\ G \text{ of order } p^r \end{array} \right\} \xrightarrow{M} \left\{ \begin{array}{l} \text{left } D_k\text{-modules } M(G) \\ \text{w/ } \mathrm{length}_{W(k)} M(G) = r \end{array} \right\}$

s.t. (1) M is an anti-equiv.

(2) functorial in k $\quad M(G_{k'}) \cong M(G_k) \otimes_{W(k)} W(k')$

(3) $\mathrm{Fr}: G \to G^{(p)} \leftrightarrow \mathrm{Fr}^*(M(G)) \xrightarrow{F} M(G)$

$V: G^{(p)} \to G \leftrightarrow M(G) \xrightarrow{V} \mathrm{Fr}^*(M(G))$

(4) $G^\vee \longleftrightarrow$ $W(k)[\frac{1}{p}]/W(k)$ - dual of $M(G)$

$$M(G^\vee) \cong \mathrm{Hom}_{W(k)}(M(G),\ W(k)[\tfrac{1}{p}]/W(k))$$

F action by $V^\vee$

V action by $F^\vee$

(5) $G = G^{con.}$ connected, $\Leftrightarrow$ $F \curvearrowright M(G)$ nilpotent

G étale $\Leftrightarrow$ $F \curvearrowright M(G)$ isom.

G unipotent $\Leftrightarrow$ $G^\vee$ is conn'd $\Leftrightarrow$ $V \curvearrowright M(G)$ is nilpotent

$G = G^{mult} \overset{def}{\Leftrightarrow} G^\vee$ is ét $\Leftrightarrow$ $V \curvearrowright M(G)$ is isom.

(6) $\{p\text{-div. gps } G\} \xrightarrow[\text{anti-equiv}]{} \{(M, F, V) :$ M free $W(k)$-mod. of finite rk, F, V $\}$
$VF = p$, $FV = p$
as above

<u>Example</u> (order $= p$)

$\mathrm{length}_{W(k)}(M) = 1 \Rightarrow M \cong k$ as $W(k)$-mod.

$W(k) \twoheadrightarrow k \curvearrowright M$ and $\dim_k M = 1$

$M = k \cdot e$, $\quad Fe = \alpha e$, $Ve = \beta e$

$$\begin{cases} FV(e) = F_2(\beta)\cdot\alpha\cdot e = pe = 0 \\ VF(e) = pe \end{cases}$$

$\Leftrightarrow \alpha\beta = 0$

$e \rightsquigarrow \lambda e$

$F(\lambda e) = (\lambda^{p-1}\alpha)\cdot(\lambda e)$

$V(\lambda e) = (\lambda^{1/p-1}\beta)\cdot(\lambda e)$

$$M/\cong \longleftrightarrow \{(\alpha,\beta)\in k\times k : \alpha\beta = 0\} \Big/ \underset{\lambda\in k^\times}{(\alpha,\beta)\sim(\lambda^{p-1}\alpha,\ \lambda^{1/p-1}\beta)}$$

<u>Class 1</u> $\alpha = 0$, $\beta \neq 0$, $k^\times/(k^\times)^{p-1}$

<u>Class 2</u> $\alpha \neq 0$, $\beta = 0$. $k^\times/(k^\times)^{p-1}$

<u>Class 3</u> $\alpha,\beta = 0$, $(0,0)$

$k = \mathbb{F}_p$, $(2p-1)$- isom. classes

$k = \overline{\mathbb{F}_p}$, $(0,0) \longleftrightarrow \alpha_p$

$(1,0) \longleftrightarrow \mathbb{Z}/p\mathbb{Z}$ étale

$(0,1) \longleftrightarrow \mu_p$ mult.

Example (order p^2) conn'd unip. p-torsion

$k = \bar{k}$

$\dim_k = 2$ v.s. V, F nilp.

① $F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\longleftrightarrow$ α_{p^2}

$V = 0.$

② $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\longleftrightarrow$ $\alpha_{p^2}^\vee$

$F = 0$

③ $F = V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\longleftrightarrow$ $E[p]$

$\subset$ super singular elliptic curve.

④ $V = F = 0.$ $\alpha_p \times \alpha_p$

Example $M(\alpha_{p^n}) = D_k / (D_k F^n + D_k V)$

§. Dieudonné theory / semi-perfect rings

Def. A ring R (in char p) is called semi-perfect, if

$F_R : R \twoheadrightarrow R$ is surj.

Def. $f: R \to S$ is called an isogeny if f is surj. & $\exists\, n$ s.t. $F_R^n(\ker(f)) = 0.$

Rmk "Be isogeneous" is an equiv. condition

$$\begin{tikzcd} R \arrow[r, two heads] & R/(\ker f) \arrow[r, "\sim"] \arrow[d, "F_2^n"'] & S \arrow[dl, "g"] \\ & R & \end{tikzcd}$$

Def/Prop. R semi-perfect ring

$R^\flat := \varprojlim_{F_2} R$ inverse limit topology

Then $R^\flat$ is a Huber ring $\Leftrightarrow$ $R \overset{\text{iso.}}{\sim} S = \overset{\text{perfect}}{T} / J$ f.g. ideal.

In this case, we call f-semiperfect ring.

Pf. $J := \ker (R^\flat \to R)$

$\{F_2^n(J)\}$

Thm (Fountaine) R semi-perfect

$\exists$ a universal p-adically complete PD-thickening

$A_{cris}(R)$ of R
$\circlearrowleft$
F_2

Fact (1) R is f-semi perfect

$W(R^\flat) \hookrightarrow A_{cris}(R)$ is injective.

(2)

$$\begin{tikzcd}
\{p\text{-div. gps}/k\} \arrow[r, "D"] & \{\text{Dieudonné crystals}/R\} \arrow[d, "R\ f\text{-semi-perfect}"', "\text{evaluate at } A_{cris}(R)"] \\
& \left\{\begin{array}{c}\text{finite proj. module} \\ \text{over } A_{cris}(R) \\ \text{w/ } F, V\end{array}\right\}
\end{tikzcd}$$

Thm. passing to isogeny,

$$\{p\text{-div. gps}/R\}/_{\text{isog.}} \to \left\{\begin{array}{c}\text{f. proj. } B^+_{cris}(R)\text{-modules} \\ \cdots\end{array}\right\}$$

is fully faithful.