Singular Supports in equal and mixed characteristics

Takeshi Saito

Lecture 1

Geometric case : Beilinson

mixed char. : far from complete

k field, X smooth scheme $/k$, T^*X cotangent bundle $/X$, Ω^1_X

Λ finite field $/\mathbb{F}_\ell$, $\ell \neq \operatorname{char} k$. $D^b_{(c)}(X_{\text{ét}}; \Lambda)$

sheaf = obj. of D^b_c

$C \subset E$ closed subset of a vector bundle is called <u>conical</u> if C is stable under $\mathbb{G}_m$- action.

$SS\, \mathcal{F} = C \subset T^*X$ closed conical subset

$\updownarrow$ ← not so direct

$\mathcal{F}$

Format for the definition: we say $\mathcal{F}$ is <u>micro-supported on C</u> if a certain functorial property on C $\Rightarrow$ the corresponding property for $\mathcal{F}$.

<u>Def</u>. If the smallest C on which $\mathcal{F}$ is <u>micro. supp</u> on C exits, then we define

$SS(\mathcal{F}) := \text{smallest } C.$

$\mathcal{F}$ m.s. on C, $C \subset C'$ $\Rightarrow$ $\mathcal{F}$ m.s. on C'.

Problem : $\mathcal{F}$ m.s. on C, C', $\xrightarrow{?}$ $\mathcal{F}$ m.s. on $C \cap C'$.

Beilinson proved the existence by reduction to $X = \mathbb{P}^n$.

Closed conical subset $C^{\subset T^*X}$ is determined by its base $B = C \cap X \subset X \leftarrow \mathrm{supp}\,\mathcal{F}$

and projectivisation $\mathbb{P}(C) \subset \mathbb{P}(T^*X)$

$\mathbb{P}^n$, $\mathbb{P}^{n\vee} = \{$ hyperplanes in $\mathbb{P}^n \}$

$$\begin{array}{ccl} & \mathbb{P}(T^*\mathbb{P}^n) \supset \mathbb{P}(SS\,\mathcal{F}) & \\ & \| \quad\quad \text{family of} & \\ \mathbb{P}^n \xleftarrow{\;q\;} & Q \quad \text{universal hyperplanes} & \\ & \downarrow q^\vee & \\ & \mathbb{P}^{n\vee} & \end{array}$$

naive Radon transform

$$R\,\mathcal{F} = (R)\,q^\vee_*\, q^*\mathcal{F}$$

---

1. Beilinson's original definition.

1.1 local acyclicity.

<u>Def'n</u> 1.1 $f : X \to Y$ morphism of schemes, $\mathcal{F}$ sheaf on X.

We say that f is $\mathcal{F}$-acyclic if the following condition is satisfied:

Let $t \to s$ be a specialisation of geometric points of Y,

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & X \times_Y Y_{(s)} & \xleftarrow{j} & X_t \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & Y_{(s)} & \longleftarrow & t \\ & & \uparrow & & \\ & & \text{strict localisation} & & \end{array}$$

(*) $\mathcal{F}_{X_s} \xrightarrow{\sim} i^* j_* \mathcal{F}_{X_t}$ is an isomorphism.

We say f is universally (ULA) $\mathcal{F}$-acyclic if for every $Y' \xrightarrow{g} Y$, the base change

$f' : X' = X \times_Y Y' \to Y'$ is $g'^* \mathcal{F}$-acyclic.

(*): $\forall\, x \to X_s$ geom. pt, $\mathcal{F}_x \xrightarrow{\sim} R\Gamma(\underbrace{X_{(x)} \times_{Y_{(s)}} t}_{\text{Milnor fibre}}, \mathcal{F})$

1.2

__Examples__ 1. $f: X \to Y$ smooth, $\mathcal{F}$ locally constant $\Rightarrow$ f is $\mathcal{F}$-acyclic

$\forall q$, $\mathcal{H}^q(\mathcal{F})$ is locally constant

"local acyclicity of smooth morphism"

2. $f: X \to \operatorname{Spec} k$, f is of finite type $\Rightarrow$ $\forall \mathcal{F}$, f univ. $\mathcal{F}$-acyclic

"generic local acyclicity"

3. If $f: X \to Y$ $\mathcal{F}$-acyclic, & if $g: Y \to Z$ smooth

$\Rightarrow$ $g \circ f : X \to Z$ is $\mathcal{F}$-acyclic.

generalisation of 1. (Illusie)

4. $f = 1_X : X \to X$, is $\mathcal{F}$-acyclic $\Leftrightarrow$ $\mathcal{F}$ is locally constant.

5. If $f: X \to Y$ proper, $g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$ is $\mathcal{F}$-acyclic, $\Rightarrow$ g is $f_* \mathcal{F}$-acyclic.

(proper base change thm)

Definition 1.3 X smooth $/k$, $\mathcal{F}$ sheaf on X. $C \subset T^*X$ closed conical subset

We say $\mathcal{F}$ is micro-supp. on C if the following condition is satisfied:

Let $h : W \to X$, $f : W \to Y$ be morphisms of smooth schemes $/k$.

If h is C-transversal & if f is $\underbrace{h^\circ C\text{-acyclic}}_{\text{transversal in Beilinson}}$, then f is $\underbrace{h^*\mathcal{F}}_{\text{(univ.)}}$-acyclic.

$C \subset E$ closed conical subset of vector bundle $/X$

base $B = \{x \in X : C_x \neq \emptyset\} = C \cap X \subset X$ closed subset.

$\uparrow$ Supp $\mathcal{F}$.

$\cup$ $\qquad\qquad$ $\cup$ closed

Support $S = \{x \in X : C_x \not\subset \{0\}\} = $ Image of $\mathbb{P}(C)$ by $\mathbb{P}(E) \to X$

Def 1.4 X smooth $/k$, $C \subset T^*X$ closed conical.

$h : W \to X$ morphism of smooth sch. $/k$ $\qquad$ ($\Leftrightarrow$) $\;\subset$ 0-section

We say h is C-transversal if the support of $\underbrace{h^*C}_{W \times_X C} \cap \operatorname{Ker}(T^*X \times_X W \to T^*W)$ is empty.

$$\begin{tikzcd} T^*X \times_X W \arrow[r] & T^*X \\ h^*C \arrow[u, hook] \arrow[r] & C \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Eg. 1.5 1. $Z \subset X$ closed immersion / smooth $/k$

$C = T^*_Z X \subset T^*X$ conormal bundle.

Then $h : W \to X$ is C-transversal $\Leftrightarrow$ h is transversal to $Z \to X$

i.e.
$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d] \\ W \arrow[r] & X \end{tikzcd}$$
$\hookleftarrow$ smooth $/k$,

$\operatorname{codim}_W U = \operatorname{codim}_X Z$

2. If h is smooth $\Rightarrow$ h is C-transversal, $\forall C$

3. $C \subset T^*_X X$, $\Rightarrow$ $\forall h$ is C-transversal.
$\uparrow$
0-section

4. If h is C-transversal & $C' \subset C$ $\Rightarrow$ h is C'-transversal

Lemma. 1.6 Let $E \xrightarrow{f} F$ be a linear morphism of vector bundles $/X$,
$C \subset E$ closed conical subset. Suppose $\operatorname{Supp}\big(C \cap \operatorname{Ker}(E \to F)\big) = \phi$
$\Rightarrow$ $E \to F$ is finite on C & $f(C) \subset F$ is a closed conical subset.

$A \to B$ graded rings
$\cup$
J graded ideal $\Rightarrow$ B/J is finite over A

$B/(J + A_{\geq 1} B)$ is
finite over $A/A_{\geq 1}$.

$$T^*X \times_X W \to T^*W$$
$$\cup$$
$$h^*C$$

$h^\circ C :=$ image of $h^*C \subset T^*W$ is a closed conical subset.

Def 1.7 $f: X \to Y$ morphism of smooth sch. $/k$

$C \subset T^*X$ closed conical subset. We say f is C-acyclic if the support of the inverse image of C by $T^*Y \times_Y X \to \underset{\underset{C}{\cup}}{T^*X}$ is empty.

Eg. 1.8

1. $C \subset$ 0-section & f smooth $\Rightarrow$ f is C-acyclic
2. $X \to \operatorname{Spec} k$, is C-acyclic for $\forall C$.
3. If $X \to Y$ C-acyclic & $Y \to Z$ smooth $\Rightarrow$ $X \to Z$ C-acyclic.
4. $f = 1_X$ is C-acyclic $\Leftrightarrow$ $C \subset$ 0-section
5. If f is C-acyclic $\Rightarrow$ f is smooth on a nbhd of the base of C.

$(h,f): W \to X \times Y$ is $C \times T^*Y$-transversal $\Leftrightarrow$ h: C-transversal, f: $h^\circ C$-acyclic

$\Uparrow$ call (h,f) C-acyclic

Eg. 1.9 1. Every sheaf $\mathcal{F}$ on X is micro supported on T^*X

$C = T^*X$, (h,f) is T^*X-acyclic

$\Leftrightarrow$ (h,f) is smooth

$\Rightarrow$ we may assume $W = \mathbb{A}^n \times X \times Y$

because local acyclicity is étale local

$$\begin{tikzcd}
\mathcal{F} \arrow[r, phantom, "\curvearrowright"] & X \arrow[d, "\text{ULA}"'] & X \times Y \arrow[l] \arrow[d] & W \arrow[l] \arrow[d, "h^*\mathcal{F}\text{-acyclic}"] \arrow[ll, bend right, "h"'] \\
& \operatorname{Spec} k & Y \arrow[l] & Y \times \mathbb{A}^n \arrow[l, "\text{smooth}"]
\end{tikzcd}$$

Want: $h^*\mathcal{F}$-acyclic.

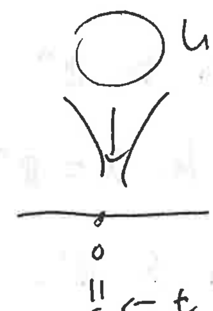
Example 1.2.3

2. If $\mathcal{F}$ is m.s. on C, then $\operatorname{supp}\mathcal{F} \subset$ base of $C = B$.

$U = X - B$. WTS $\mathcal{F}|_U = 0$

$$X \xleftarrow{h} U \xrightarrow{f} \mathbb{A}^1$$

$\uparrow$ constant 0



(h,f) is C-acyclic $\Rightarrow$ f is $\mathcal{F}|_U$-acyclic

$\mathcal{F}_{X_s} \rightleftarrows i_* j^* \mathcal{F}_{X_t} = 0$

3. Assume $\mathcal{F}$ is m.s. on C, $U \subset X$ open,

$\mathcal{F}|_U$ is m.s. on $C' \subset T^*U \subset T^*X$

$\Rightarrow$ $\mathcal{F}$ is m.s. on $C|_{X-U} \cup \overline{C'} = C_1$

Assume (h,f) is C_1-acyclic.

$$\begin{array}{c} U \subset \\ X \xleftarrow{h} W \xrightarrow{f} Y \\ \supset X-U \end{array}$$

On the inverse image of U, (h,f) is C'-acyclic

$\Rightarrow$ f is $h^*\mathcal{F}$-acyclic

On a nbhd V of the inverse image of $X-U$, (h,f) is C-acyclic.

$f|_V$ is $h^*\mathcal{F}$-acyclic

4. $\mathcal{F} = 0 \iff \mathcal{F}$ is micro supp. on ϕ

5. $\mathcal{F}$ is locally constant $\iff$ $\mathcal{F}$ is micro supp. on the 0-section.

$\Rightarrow$ (h,f) is $T^*_X X$-acyclic. $\Rightarrow$ f is smooth $\Rightarrow$ f is $h^*\mathcal{F}$-acyclic.

$\Leftarrow \quad (1_X, 1_X)$ is $T^*_X X$- acyclic $\Rightarrow 1_X$ is $\mathcal{F}$- acyclic

$\Rightarrow \mathcal{F}$ is locally constant.

2. Proof of existence

- reduction to $\mathbb{P}^n$
- Proof for $X = \mathbb{P}^n$

2.1 Reduction to $\mathbb{P}^n$.

- reduction to affine schemes (local)
- reduction to $\mathbb{A}^n$ (closed immersion)
- reduction to $\mathbb{P}^n$ (local)

local: Prop 2.1. X smooth $/k$.

1. Let $U \subset X$ be an open subset. If $\mathcal{F}$ is micro supp. on $C \subset T^*X$ then $\mathcal{F}|_U$ is micro supp on $C|_U$.

If $C = SS(\mathcal{F}) \Rightarrow SS(\mathcal{F}|_U) = C|_U$.

2. Let (U_i) be open covering of X, $C \subset T^*X$,

If $\mathcal{F}|_{U_i}$ are micro supp. on $C|_{U_i}$, $\forall i$, then $\mathcal{F}$ is micro supp. on C.

If $C_i = SS(\mathcal{F}|_{U_i})$, then $C = \bigcup C_i$ satisfies $C|_{U_i} = C_i$ & $C = SS(\mathcal{F})$.

Suppose $C = SS(\mathcal{F})$ — $\mathcal{F}|_U$ is m.s. on $C|_U$

— If $\mathcal{F}|_U$ is m.s. on $C' \Rightarrow C' \supset C|_U$.

$\mathcal{F}$ is m.s. on $C|_{X-U} \cup \overline{C'} \supset C$

restiction to U $C' \supset C|_U$

<u>Closed immersion</u> $i: X \to P$ closed immersion of smooth sch. $/k$

$C \subset T^*X$ closed conical subset.

$$\begin{tikzcd}
i_\circ C \subset T^*P & \\
T^*P|_X \arrow[u, phantom, "\cup"] \arrow[r] & T^*X \\
i_\circ C \arrow[u, phantom, "\cup"] \arrow[r, "\urcorner"] & C \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

<u>Example</u> 2.2 If $C = T^*_X X$, $\Rightarrow i_\circ C = T^*_X P$

If $C \subset T^*X$ and if $h: W \to P$ is $i_\circ C$- transversal, then

h is transversal to $X \to P$ on a nbhd of Base of C.

smooth $/k$ - on a nbhd of the inv. image of base of C

$$\begin{tikzcd}
V' \arrow[d, phantom, "\cap"] & & \\
V \arrow[r, "h'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \supset B \arrow[d] \\
W \arrow[r, "h"] & P
\end{tikzcd}$$

$h'|_{V'}$ is C- transversal.

<u>Lecture 2</u>. X/k smooth. $\mathcal{F}$. $SS(\mathcal{F}) \subseteq T^*X$

$\mathcal{F}$: micro- supp. on C : If $(h: W \to X,\ f: W \to Y)$ is C- acyclic,

$((h,f): W \to X \times Y$ is $C \times T^*Y$- transversal$)$

$\Rightarrow$ f is $h^*\mathcal{F}$- acyclic.

Proof of existence of SS

- reduction to $\mathbb{P}^n$ local, closed immersion
- $\mathbb{P}^n$ case : Radon transform

Closed immersion

Prop 2.3 $i: X \to P$ closed immersion of smooth schemes $/k$, $\mathcal{F}$ a sheaf on X

1. If $\mathcal{F}$ is micro-supp. on $C \subset T^*X$, then $i_* \mathcal{F}$ is m.s. on $i_\circ C$.

$$\begin{tikzcd} T^*P \supset T^*P|_X \arrow[r] & T^*X \\ i_\circ C \arrow[u] \arrow[r, phantom, "\urcorner"] & C \arrow[u, phantom, "\cup"] \arrow[l] \end{tikzcd}$$

2. Let $C_P \subset T^*P$ be a closed conical subset and let $C \subset T^*X$ be the closure of the image of $C_P|_X$ by $T^*P|_X \to T^*X$.

If $i_* \mathcal{F}$ is micro. supp. on C_P, then $\mathcal{F}$ is micro-supp. on C.

Further, if $C_P = SS(i_* \mathcal{F})$, then $C = SS(\mathcal{F})$.

Prof 1. Assume (h, f) is $i_\circ C$-acyclic, want : f is $h^* i_* \mathcal{F}$-acyclic.

$$\begin{tikzcd} X \arrow[d, "i"'] & V \arrow[l, "h'"'] \arrow[d, "i'"] \arrow[dr, "f'"] \arrow[dl, phantom, "\llcorner"] & \\ P & W \arrow[l, "h"] \arrow[r, "f"] & Y \end{tikzcd}$$

After shrinking W, we may assume that h is transversal to $X \to P$. i.e. V smooth $/k$, $\mathrm{codim}_W V = \mathrm{codim}_P X$. Further, (h', f') is C-acyclic. $\Rightarrow$ f' is $h'^* \mathcal{F}$-acyclic.

$\Rightarrow$ f is $i'_* h'^* \mathcal{F}$-acyclic

$i'_* h'^* \mathcal{F} \cong h^* i_* \mathcal{F}$

2. Assume (h, f) is C-acyclic.

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[dr] & X \arrow[r, "i"] & P \\ & X \times W \arrow[u] \arrow[r] & P \times W \arrow[u] \end{tikzcd}$$

smooth

We may assume that h is a (closed) immersion.

$$\begin{tikzcd} X \times Y \arrow[d] & W \arrow[l, "{(h,f)}"'] \arrow[d] \\ P \times Y & V \arrow[l, "{(\tilde h, \tilde f)}"] \end{tikzcd}$$

closed immersion

transversal

$$C \times_{T^*Y} \ker\left((T^*X \times T^*Y)|_W \to T^*W\right)$$
$$\wr\!\wr$$
$$C_P \times_{T^*Y} \ker\left(\left((T^*P \times T^*Y)|_V \to T^*V\right)\right)\Big|_W$$

C-acyclicity $\Rightarrow$ C_P-acyclicity on a nbhd of W

$\Rightarrow$ $\tilde f$ is $\tilde h^* i_* \mathcal{F}$-acyclic

$\Rightarrow$ f is $h^* \mathcal{F}$-acyclic.

$\mathcal{F}$ is micro supp. on C ✓

If $\mathcal{F}$ is micro-supp. on C', then $C' \supset C$.

$i_* \mathcal{F}$ micro-supp. on $i_\circ C' \supset C_P$

$\Downarrow$

$C' \supset C$

2.2 Radon transform.

$\mathbb{P} = \mathbb{P}^n / k$. $\qquad \mathbb{P}^\vee = \{\text{hyperplanes in } \mathbb{P}\}$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & & & \\ \mathbb{P} & \mathbb{P}\times\mathbb{P}^\vee \arrow[l, "q"'] & Q \arrow[l, phantom, "\supset"] \arrow[d, "q^\vee"] & \\ & & \mathbb{P}^\vee & \end{tikzcd}$$

Q = univ. family of hyperplanes

<u>Naive Radon transform</u> $R\mathcal{F} = (R)\, q^\vee_* \, q^* \mathcal{F}$

<u>almost</u> equiv. of category

$$R^\vee \mathcal{G} = q_* \, q^{\vee *} \mathcal{G}$$

<u>Prop 2.4</u> We have a distinguished Δ $\quad (n \neq 0)$

$$\bigoplus_{s=0}^{n-2} p^* p_* \mathcal{F}(-s)[-2s] \to R^\vee R \mathcal{F} \to \mathcal{F}(-(n-1))[-2(n-1)] \to$$

$p : \mathbb{P} \to \operatorname{Spec} k$

<u>Proof</u>

$$\begin{tikzcd} \mathbb{P} & Q \arrow[l, "q"'] \arrow[d, "q^\vee"'] & Q \times Q \arrow[l] \arrow[d, "p^\vee"] \arrow[dl, phantom, "\lrcorner"] \\ & \mathbb{P}^\vee & Q \arrow[l, "q^\vee"'] \arrow[d, "q"] \\ & & \mathbb{P} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P} & \mathbb{P}\times\mathbb{P} \arrow[l, "p_1"'] \arrow[d, "p_2"] & Q \times_{\mathbb{P}^\vee} Q \arrow[l, "q\times q"'] \\ & \mathbb{P} & \end{tikzcd}$$

$$R^\vee R \mathcal{F} = p_{2*}\left(p_1^* \mathcal{F} \otimes (q\times q)_* \Lambda\right)$$

$$\begin{tikzcd} \mathbb{P}\times\mathbb{P}^\vee\times\mathbb{P} \arrow[d] & Q \times_{\mathbb{P}^\vee} Q \arrow[l, phantom, "\supset"] \arrow[dl, "q\times q"'] \arrow[d, equal] \\ \mathbb{P} \overset{\Delta}{\subset} \mathbb{P}\times\mathbb{P} & \{(x,H,y) : x,y \in H\} \end{tikzcd}$$

On Δ, the fibre is the hyperplanes in P containing x

Outside Δ, fibre is a linear subspace of codim 2, containing x, y.

Up to degree $2(n-2)$, $H^s(\text{fibre}) = H^s(\mathbb{P}^\vee)$ everywhere

$2(n-1)$, the support of direct image $\Delta = H^{2(n-1)}(\mathbb{P}^\vee)$

Otherwise 0

$$(p\times p)^* \tau_{\leq 2(n-2)} p^\vee_* \Lambda \to (q\times q)_* \Lambda \to \delta_* \Lambda(-(n-1))[-2(n-1)] \to$$

$$\begin{tikzcd} \mathbb{P}\times\mathbb{P} \arrow[r, "p\times p"] & \operatorname{Spec} k \\ \mathbb{P} \arrow[u, "\delta"] & \end{tikzcd} \qquad p^\vee : \mathbb{P}^\vee \to \operatorname{Spec} k$$

$$\begin{tikzcd} \mathbb{P} \arrow[d] & \mathbb{P}\times\mathbb{P} \arrow[l] \arrow[d] \\ \operatorname{Spec} k & \mathbb{P} \arrow[l] \end{tikzcd}$$

<u>Corollary 2.6</u> We have (1) $\Rightarrow$ (2)

(1) $\mathcal{F}$ is micro. supp. on C

(2) $R^\vee R \mathcal{F}$ is micro supp. on $C^+ = C \cup \underbrace{T^*_{\mathbb{P}}\mathbb{P}}_{\text{0-section}}$

2.3 Proof of existence

$h^\circ C$, $g_\circ C$ g proper.

$g: X' \to X$ proper morphism of smooth schemes $/k$.

$C \subset T^*X'$ closed conical

$$\begin{tikzcd} T^*X' & T^*X \times_X X' \arrow[l] \arrow[r, "\text{proper}"] & T^*X \\ C \arrow[u, phantom, "\cup"] & \square \arrow[l] \arrow[r] \arrow[u, phantom, "\cup"] & g_\circ C \arrow[u, phantom, "\cup"] \end{tikzcd}$$
closed conical subset.

Prop 2.8 X smooth $/k$, $\mathcal{F}$ on X, micro-supp. on C

1. If $h: W \to X$ is C-transversal, $\Rightarrow h^*\mathcal{F}$ is micro-supp. on $h^\circ C$

2. If $g: X \to X'$ proper, $\Rightarrow g_*\mathcal{F}$ is microsupp. on $g_\circ C$.

Proof 1 (g, f) $g: V \to W$, $f: V \to Y$ is $h^\circ C$-acyclic

$\Rightarrow hg: V \to X$, f is C-acyclic

$\Rightarrow f$ is $(hg)^*\mathcal{F}$-acyclic.

$C \subset T^*\mathbb{P}$. We call $C^\vee = q_\circ^\vee q^\circ C \subset T^*\mathbb{P}^\vee$

Legendre transform

Corollary 2.9 We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) $\mathcal{F}$ is micro supp. on C

(2) $R\mathcal{F}$ is micro supp. on $C^\vee$ $(R\mathcal{F} = q^\vee_* q^*\mathcal{F})$

(3) $\mathcal{F}$ is micro supp on $C^{\vee\vee +} = (C^\vee)^\vee \cup T^*_{\mathbb{P}}\mathbb{P}$

(2) $\Rightarrow$ (3): $R^\vee R\mathcal{F}$ is micro supp on $C^{\vee\vee}$.

$$\begin{array}{ccc} \mathbb{P}(C) & & \mathbb{P}(C^\vee) \\ \cap & & \cap \\ Q = \mathbb{P}(T^*\mathbb{P}) & = & \mathbb{P}(T^*\mathbb{P}^\vee) \end{array}$$

$\mathbb{P} = \mathbb{P}(V) = \{\text{lines in } V\}$

$0 \to \Omega^1_{\mathbb{P}/k}(1) \to \mathcal{O}_{\mathbb{P}} \otimes V^\vee \to \mathcal{O}(1) \to 0$

$$\begin{array}{lcl} L & \subset & T^*\mathbb{P} \times_{\mathbb{P}} Q \qquad , \quad Q \subset \mathbb{P} \times \mathbb{P}^\vee \\ \uparrow\wr & & \quad\ \uparrow pr_1 \\ L_Q & = & \ker\left((T^*\mathbb{P} \times T^*\mathbb{P}^\vee)|_Q \to T^*Q \right) = T^*_Q(\mathbb{P} \times \mathbb{P}^\vee) \end{array}$$

$$\cup$$

$$= \left(T^*\mathbb{P} \times_{\mathbb{P}} Q\right) \cap \left(T^*\mathbb{P}^\vee \times_{\mathbb{P}^\vee} Q\right)$$

Prop 2.10 $\quad C \subset T^*\mathbb{P}$ closed conical subset

$$E = \mathbb{P}(C) \subset \mathbb{P}(T^*\mathbb{P}) = Q \qquad\qquad \mathbb{P} \xleftarrow{q} Q \xrightarrow{q^\vee} \mathbb{P}^\vee$$

1. E is the complement of the largest open subset where $(q, q^\vee)$ is C-acyclic

2. $C^\vee$ is equal to the image of $L_Q|_E$ by the second proj. up to the 0-section. $\qquad \mathbb{P}(C) = \mathbb{P}(C^\vee)$

3. $C^{\vee\vee} \subset C^+ = C \cup T^*_{\mathbb{P}}\mathbb{P} \qquad \mathbb{P}(C) = \mathbb{P}(C^\vee) = \mathbb{P}(C^{\vee\vee})$

Proof 2 $\quad C$-acyclicity : $\quad C \times T^*\mathbb{P}^\vee$- transversality

$$\mathrm{supp}\left((C \times T^*\mathbb{P}^\vee)|_Q \cap \ker\left((T^*\mathbb{P} \times T^*\mathbb{P}^\vee)|_Q \to T^*Q \right) \right) = \phi$$

$$\uparrow \qquad\qquad\qquad\qquad \underset{\overset{\Downarrow}{L}}{\overset{\|}{L_Q}} \subset T^*\mathbb{P} \times_{\mathbb{P}} Q$$

complement of the open subset where the intersection $\subset$ 0-section

1. $\Leftrightarrow$ $\mathrm{supp} = E$.

2. $C^\vee = q^\vee_\circ q^\circ C$

$$\begin{array}{ccccccc} q^\circ C = C \times_{\mathbb{P}} Q & \subset & T^*\mathbb{P} \times_{\mathbb{P}} Q & \hookrightarrow & T^*Q & & \\ \cup & & \cup & & & & \\ \text{up to 0-section } L_Q|_E & \longrightarrow & L_Q & \subset & T^*\mathbb{P}^\vee \times_{\mathbb{P}^\vee} Q & \longrightarrow & T^*\mathbb{P}^\vee \supset C^\vee \end{array}$$

Prop 2.11 We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)

(1) $\mathcal{F}$ is micro supp on C

(2) $q^\vee$ is univ. $q^*\mathcal{F}$- acyclic outside $E = \mathbb{P}(C)$

(3) $R\mathcal{F}$ is micro supp on $C^{\vee +}$

(4) $\mathcal{F}$ is micro supp. on C^+.

Cor 2.12 $\mathcal{F}$ on $\mathbb{P}$, $E \subset Q = \mathbb{P}(T^*\mathbb{P})$ be the complement of the largest open where $q^\vee$ is univ. $q^*\mathcal{F}$- acyclic, then the closed conical subset C s.t. the base of $C = \operatorname{Supp}\mathcal{F}$, $\underset{\mathbb{P}(T^*\mathbb{P})}{\mathbb{P}(C) = E \cap}$ is the $SS(\mathcal{F})$.

$\mathcal{F}$ is micro supp. on C; If $\mathcal{F}$ micro supp. on $C' \Rightarrow C' \supset C$.

(2) $\Rightarrow$ (4)

$\mathcal{F}$ is micro supp on C^+ $\quad$ $\mathcal{F}|_{(\mathbb{P}-B)} = 0$

$\Rightarrow \mathcal{F}$ is micro supp. on $C^+|_B \cup \Phi = C$

(1) $\Rightarrow$ (2) $\quad \mathbb{P}(C') \supset \mathbb{P}(C) = E$

$B' \supset \operatorname{Supp}\mathcal{F} = B. \quad \Rightarrow C' \supset C$

Proof (1) $\Rightarrow$ (2) $\quad (q, q^\vee)$ is C- acyclic outside E (prop 2.10.1)

$\mathcal{F}$ is micro supp on $C \Rightarrow q^\vee$ is univ $q^*\mathcal{F}$- acyclic.

(2) $\Rightarrow$ (3) Assume (h, f) is $C^{\vee +}$- acyclic. Want f is $h^* R\mathcal{F}$ - acyclic.

$$\begin{tikzcd}
\mathbb{P} & Q \arrow[l, "q"'] \arrow[d, "q^\vee"] & Q_W \arrow[l, "h'"'] \arrow[d, "q^\vee_W"] \arrow[dr, "f'"] & \\
& \mathbb{P}^\vee & W \arrow[l, "h"'] \arrow[r, "f"] & Y
\end{tikzcd}$$

$q^\circ C$

C

$$h^* R F = h^* q^\vee_* q^* F \simeq \underbrace{q^\vee_{W*}}_{\text{proper}} h'^* q^* F$$

Suffices to show f' is $(qh')^* F$- acyclic

Outside E, $q^\vee$ is univ. $q^* F$- acyclic.

Outside $E_W = E \times_Q Q_W$, $q^\vee_W$ is $h'^* q^* F$- acyclic.

Since (h, f) is $\underset{\substack{\cap \\ (C^{\vee +})}}{T^*_{\mathbb{P}^\vee} \mathbb{P}^\vee}$- acyclic, f is smooth

by Example 1.23 $f' = f \circ q_W$ is $h'^* q^* F$- acyclic.

On a nbhd of E_W:

(h, f) is $\overset{q^\vee_\circ q^\circ C}{\overset{\shortparallel}{C^\vee}}$- acyclic.

(h', f') is $q^\circ C$- acyclic. (by skipped Lemma 2.7)

$\Updownarrow$

(qh', f') is C- acyclic

$$Q_W \overset{\text{codim } 1}{\subset} \mathbb{P} \times W$$

conormal bundle L_W

is the pull-back of L_Q

Supp. of $(C \times T^* W)|_{Q_W} \cap L_W$

$= (T^* \mathbb{P} \times T^* W)|_{L E_W}$

$\Rightarrow (qh', f')$ is $T^* \mathbb{P}$ - acyclic on a nbhd of E_W

Since $\mathcal{F}$ is micro supp. on $T^*\mathbb{P}$, f' is $(gh')^*\mathcal{F}$-acyclic.

Lecture 3. Goal: transplant Beilinson's work to mixed characteristic.

Obstacles: – we don't have cotangent bundle of the correct rank. FW-cotangent bundle

– we don't have sufficiently many second morphisms f.

3. Variations C-transversality

micro supp. (h, f) C-acyclicity

$\mathcal{F}$-acyclicity

$\mathcal{F}$-transversality ?

3.1 $\mathcal{F}$-transversality

$h: W \to X$ separated of finite type.

$\mathcal{F}$-sheaf on X, $D^b_c(X;\Lambda)$ $\Lambda/\mathbb{F}_\ell$, ℓ invertible on X.

$$c_{h,\mathcal{F}}: h^*\mathcal{F} \otimes h^!\Lambda \longrightarrow h^!\mathcal{F} \overset{\text{adjunction}}{\Longleftrightarrow} h_!(h^*\mathcal{F} \otimes h^!\Lambda) \to \mathcal{F}$$

$$h_!(h^*\mathcal{F} \otimes h^!\Lambda) = \mathcal{F} \otimes h_!h^!\Lambda \xrightarrow{\text{adjunction}} \mathcal{F}$$

Def 3.1 Assume $h: W \to X$ is a sep. morphism of f.t. of smooth schemes / k field

$\mathcal{F}$-sheaf on X. We say h is $\mathcal{F}$-transversal if

$c_{h,\mathcal{F}}: h^*\mathcal{F} \otimes h^!\Lambda \xrightarrow{\sim} h^!\mathcal{F}$ is an isom.

Example 3.2 1. Let $Z \hookrightarrow X$ be closed subscheme sm. $/k$

If $h: W \to X$ is transversal to $i: Z \to X$, then h is $i_*\Lambda$-transversal.

2. If $h: W \to X$ is smooth, then h is $\mathcal{F}$-transversal for every $\mathcal{F}$.

(Poincaré duality)

3. If $\mathcal{F}$ is locally constant, $\forall\, h$ is $\mathcal{F}$-transversal.

Relation between transversality and acyclicity /k

$C \subset T^*X$,

$$\begin{tikzcd} X \arrow[d, "f"'] & X_y = W \arrow[l, "h"'] \arrow[d] \\ Y & y \arrow[l] \end{tikzcd}$$

smooth

f is C-acyclic, h is C-transversal.

$$\begin{tikzcd} T^*Y \times_Y X \arrow[r] & T^*X \\ ? \arrow[u, hook] \arrow[r] & C \arrow[u, phantom, "\cup"] \\ \text{0-section} \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

, $h^*C \cap \ker(T^*X \times_X W \to T^*W)$

$\subset$ 0-section

f is C-acyclic on a nbhd of W $\Leftrightarrow$ h is C-transversal.

Lemma 3-3 Let
$$\begin{tikzcd} X \arrow[d, "f"'] & W \arrow[l, "h"'] \arrow[d, "g"] \arrow[dl, phantom, "\lrcorner"] \\ Y & V \arrow[l, "i"'] \end{tikzcd}$$
be a cartesian diagram of smooth schemes /k & assume f is smooth.

1. If f is C-acyclic, $\Rightarrow$ h is C-transversal & g is $h^\circ C$-acyclic.

2. If f is univ. $\mathcal{F}$-acyclic $\Rightarrow$ h is $\mathcal{F}$-transversal & g is $h^*\mathcal{F}$-acyclic.

Pf 1. $$\underbrace{\ker\left(T^*X \times_X W \times_W T^*V \times_V W \to T^*W\right)}_{= \,\mathrm{Im}\left(T^*Y \times_Y W\right)} \cap \left(h^*C \times_W h^*TV\right) \subset \text{0-section}.$$

2. We may assume $i: V \to X$ is a closed immersion.

$$W \overset{h}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X - W$$

$$\begin{tikzcd}
\mathcal{F}\otimes h_! h^! \Lambda \arrow[r] \arrow[d] & \mathcal{F}\otimes \Lambda \arrow[r] \arrow[d, equal] & \mathcal{F}\otimes j_* j^* \Lambda \arrow[r, "+1"] \arrow[d] & {} \\
h_! h^! \mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & j_* j^* \mathcal{F} \arrow[r, "+1"] & {}
\end{tikzcd}$$

$c_{h,\mathcal{F}}$ isom. $\Leftrightarrow$ isom. $\Leftrightarrow$ isom.

Prop 3.4. Let $f: X \to Y$ be a morphism of noetherian schemes, $\mathcal{F} \in D(X)$. TFAE

(1) f is $\mathcal{F}$-acyclic

(2) $\forall$
$$\begin{tikzcd}
X \arrow[d, "f"'] & U \arrow[l, "h"'] \arrow[d, "g"] \arrow[dl, phantom, "\llcorner"] \\
Y & V \arrow[l, "j"]
\end{tikzcd}$$
s.t. horizontal arrows are immersions,

& $\forall$ constructible sheaf $\mathcal{G}$ on V

$$\mathcal{F}\otimes f^* j_* \mathcal{G} \xrightarrow{\sim} h_*(h^*\mathcal{F}\otimes g^*\mathcal{G})$$

Cor 3.5. Let
$$\begin{tikzcd}
X \arrow[d, "f"'] & W \arrow[l, "h"'] \arrow[d, "g"] \arrow[dl, phantom, "\llcorner"] \\
Y & V \arrow[l, "i"]
\end{tikzcd}$$
be a Cartesian diagram of schemes s.t. f is smooth & i is an immersion.

$\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on $\mathcal{G}$, Assume f is $\mathcal{F}$-acyclic, & i is $\mathcal{G}$-trans.

$\Rightarrow$ h is $\mathcal{F}\otimes f^*\mathcal{G}$-transversal.

3.2 Equivalent def'ns

Prop 3.6 X smooth $/k$, $\mathcal{F}$ sheaf on X, $C \subset T^*X$, TFAE

(1) $\mathcal{F}$ is micro supp. on C

(2) Let (h, f) be a pair of morphisms $h: W \to X$, $f: W \to Y$ of smooth schemes $/k$, s.t. h is C-transversal, f is $h^\circ C$-acyclic, then h is $\mathcal{F}$-transversal & f is $h^*\mathcal{F}$-acyclic.

(3) (h, f) s.t. $(h,f): W \to X \times Y$ is $C \times T^*Y$- transversal, then for any sheaf $\mathcal{G}$ on Y, (h, f) is $\mathcal{F} \boxtimes \mathcal{G}$- transversal.

Proof (2) $\Rightarrow$ (1) clear

(1) $\Rightarrow$ (2) h C- trans. $\Rightarrow$ h $\mathcal{F}$ - trans.

By $W \to W \times X \to X$, we may assume h is an immersion.

We may assume to have

$$\begin{tikzcd} X \arrow[d, "f"'] & W \arrow[l, hook'] \arrow[d] \\ Y & y \arrow[l, hook'] \end{tikzcd}$$ cartesian.

Smooth

h C- trans. $\to$ f C- acyclic (on a nbhd of W)

$\mathcal{F}$ m.s. on $C \to$ $\Rightarrow$ f $\mathcal{F}$- acyclic

$\xRightarrow{\text{Lemma 3.3}}$ h $\mathcal{F}$ - transversal.

(2) $\Leftrightarrow$ (3). $(h, f): W \xrightarrow{(1_W, f)} W \times Y \xrightarrow{h \times 1_Y} X \times Y$

Compare (h) h $\mathcal{F}$ - trans. & $h \times 1_Y$ $\mathcal{F} \boxtimes \mathcal{G}$- trans. $\forall \mathcal{G} \in D(Y)$

(f) f $h^*\mathcal{F}$- acyclic & $(1, f)$ $h^*\mathcal{F} \boxtimes \mathcal{G}$ - trans. $\forall \mathcal{G} \in D(Y)$

(h): $p: W \to P = \operatorname{Spec} k$

h C- transversal $\Leftrightarrow$ (h, p) $C \times T^*P$ - transversal

(2) $\Downarrow$ $\Downarrow$ (3)

h $\mathcal{F}$ - transversal $\Leftarrow$ (h, p) $\mathcal{F} \boxtimes \mathcal{G}$ - trans. $\forall \mathcal{G}$

Lemma 3.8. Let $h: W \to X$ $\mathcal{F}$-trans. $\Rightarrow \forall Y^{sm/k}, \forall \mathcal{G}$ on Y

$h \times 1: W \times Y \to X \times Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.

Pf. Cor 3.5.

$$\begin{tikzcd} W \times Y \arrow[r] \arrow[d] & X \times Y \arrow[r] \arrow[d] & Y \quad \mathcal{G} \arrow[d, "\leftarrow \text{univ. } \mathcal{G}\text{-acyclic}"] \\ W \arrow[r, "h"] & X \arrow[r] & P \end{tikzcd}$$

$\mathcal{F}$ on X

$h \times 1$ is

$f^* \mathcal{F} \otimes pr_2^* \mathcal{G}$ - acyclic

$\quad \shortparallel$

$\mathcal{F} \boxtimes \mathcal{G}$

Prop 3.9 (cf. Braverman-Gaitsgory) Let $f: X \to Y$ be a morphism of schemes of f.t. / excellent regular noeth. sch. S

$\mathcal{F}$ sheaf on X. Assume $X \to S$ is universally $\mathcal{F}$-acyclic, $Y \to S$ smooth, TFAE:

(1) f is $\mathcal{F}$-acyclic

(2) $\forall$ constructible $\mathcal{G}$ on Y, $(1, f): X \to X \times_S Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.

Prop 3.10 X sm/k, $\mathcal{F} \in D(X)$, $C \subset T^*X$.

(1) $\mathcal{F}$ is microsupp. on C

(2) The support of $\mathcal{F} \subset$ base of C, let $h: W \to X$ morphism of smooth schemes /k, such that h is C-transversal, then h is $\mathcal{F}$-transversal.

We have (1) $\Rightarrow$ (2). If k is perfect, (1) $\Leftrightarrow$ (2).

Pf. (1) $\Rightarrow$ (2) done

(2) $\Rightarrow$ (1) for k perfect.

Assuming (h, f) C-acyclic, want to prove that f is $h^* \mathcal{F}$-acyclic.

We may assume $h = 1_X$. also f is smooth.

Lemma 3.11 $f: X \to Y$ smooth morphisms of sch. of f.t. $/k$

$\mathcal{F}$ sheaf on X. Assume $\forall$ cartesian diagrams

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "p'"'] \arrow[d, "f'"] & W \arrow[l, "h"'] \arrow[d, "g"] \\ Y & Y' \arrow[l, "p"'] & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

s.t. i closed immersion,

Y', Z smooth / finite ext'n of k, $p: Y' \to Y$ proper & gen. finite on Y

then h is $p'^*\mathcal{F}$ - transversal

$\Rightarrow f$ is $\mathcal{F}$ - acyclic.

Proof. omitted.

Pf of Prop 3.10 cont'd. C-acyclicity of $f \Rightarrow$ p', hp' C- trans. 3.3

$\overset{(2)}{\Rightarrow}$ p', hp', $\mathcal{F}$ - trans.

$\overset{\text{transitivity}}{\Rightarrow}$ h: $p'^*\mathcal{F}$ - trans

$\overset{3.11}{\Rightarrow}$ f is $\mathcal{F}$ - acyclic.

4. Mixed characteristic.

$S = \operatorname{Spec} \mathcal{O}_K$, $\mathcal{O}_K$ DVR mixed char, res field k perfect

X regular scheme of f.t. $/S$

We expect to have a cotangent bundle s.t. at $x \in X_k$ closed pt, the fibre is m_x/m_x^2.

We will define a locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$ - module $F\Omega_X$ of rank $= \dim X$

at $\forall x \in X_k$,

exact

$$0 \to F^*(m_x/m_x^2) \to F\Omega_{X,x} \otimes k(x) \to F^*(\Omega^1_{k(x)/\mathbb{F}_p}) \to 0$$

F^* base change by the absolute Frobenius

4.1 Frobenius - Witt cotangent bundle

<u>Def 4.1</u>. p prime number.

1. $P \in \mathbb{Z}[X,Y]$, $\quad P = \sum_{i=1}^{p-1} \frac{(p-1)!}{i!\,(p-i)!} X^i Y^{p-i} \quad = \left(\frac{(X+Y)^p - X^p - Y^p}{p}\right)$

2. A ring, M A-module. We say a mapping $w: A \to M$ is Frobenius - Witt (FW) derivation, if $\forall a, b \in A$, $w(a+b) = w(a) + w(b) - P(a,b)\,w(p)$
$$w(ab) = a^p w(b) + b^p w(a).$$

3. $F\Omega_A$: A-module representing the functor

$$A\text{-mod} \to \text{Set}$$
$$M \mapsto \{\text{FW-derivations } A \to M\}$$

$w: A \to M$ FW-der. $\Rightarrow$ $W(n) = \frac{n - n^p}{p} w(p)$

$$a \in A, \quad w(na) = n \cdot w(a) + a^p w(n)$$

$$(n^p - n) w(a) = 0. \qquad \forall a \in A, \forall n \in \mathbb{Z}$$

$\Rightarrow$ If $A / \mathbb{Z}_{(p)}$ $\Rightarrow$ $p \cdot F\Omega_A = 0$.

$F\Omega_A$, A/pA-module

If A is regular of f.t. over $\mathcal{O}_K$ $\Rightarrow$ $F\Omega_A$ locally free of finite rank$/A/pA$

rk $=\dim A$

$F\Omega_X$ locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-mod.

FT^*X associated vector bundle on $X_{\mathbb{F}_p} \subset X$ FW cotangent bundle

If $X \to Y$ smooth, $X, Y/S$ reg. f.t.

$$0 \to FT^*Y \times_Y X \to FT^*X \to F^*(T^*X/Y|_{X_{\mathbb{F}_p}}) \to 0$$

$$\Omega^1_{X/Y}$$

exact

4.2 Singular support.

Def 4.2 X reg. of. f.t. $/S = \operatorname{Spec} \mathcal{O}_K$. $\mathcal{F}$ sheaf on X, $C \subset FT^*X$.

We say $\mathcal{F}$ is micro supp on C if the following condition is satisfied:

- (the supp of $\mathcal{F}$) $\cap X_{\mathbb{F}_p} \subset$ base of C.
- Let $h: W \to X$ morphism of reg. schemes of f.t. $/S$

If h is C-trans $\Rightarrow$ h is $\mathcal{F}$-trans on a nbhd of $W_{\mathbb{F}_p}$

$h^*C \cap \ker(FT^*X \times_X W \to FT^*W) \subset$ 0-section.

Lecture 4

Def 4.2 X/S reg f.t. $\mathcal{F}$, $C \subset FT^*X$.
$S = \mathrm{Spec}\, \mathcal{O}_K$

We say $\mathcal{F}$ is m.s. on C if
- $\mathrm{supp}(\mathcal{F})|_{X_{\mathbb{F}_p}} \subset$ base of C
- Let $h: W \to X$ sep. of f.t., W reg $/S$.

If h is C-transversal $\Rightarrow$ h is $\mathcal{F}$-trans. on a nbhd of $W_{\mathbb{F}_p}$.

Example 4.3. 1. Every sheaf on X is m.s. on FT^*X

2. $\mathcal{F}$ is m.s. on $\underset{\text{0-section}}{FT^*_X X}$ $\Leftrightarrow$ $\mathcal{F}$ is loc. const. on a nbhd of $X_{\mathbb{F}_p}$

$\Leftarrow$ easy

$\Rightarrow$ less easy

Open problems 1. Existence of SS. Reduction to $\mathbb{P}^n$ is OK.

Radon transform $(q, q^\vee)$

going to introduce a relative version w/ (h, f)

2. Dimension of SS. geom. case: Beilinson: $SS = \bigcup_a C_a$, $\dim C_a = \dim X$.

$$\begin{array}{c} X \\ \downarrow \\ c \in C \text{ curve}/k \end{array}$$
closed pt

$SS|_{T^*X \times_C c}$ irred. comp. $\dim \begin{cases} n \\ n-1 \end{cases}$, $n = \dim X$

3. Description at codim 1 generic pt

$D \subset X$ regular $\overset{\text{irred.}}{\vee}$ divisor , $j : U = X - D \hookrightarrow X$, $\mathcal{F} = j_! \mathcal{G}$

$\cap$

$X_{\mathbb{F}_p}$

loc. const. on U

$\xi \in D$ generic pt

rep. of $G_K = \mathrm{Gal}(\bar{K} | K)$

$K = \mathrm{Frac}\ \hat{\mathcal{O}}_{X,\xi}$

geom. case X/k smooth , $G_K \curvearrowright V$

$\uparrow$ char p

$\uparrow$ upper numbering filtration

Abbes - S.

$F = k(\xi)$, res. field of K

$$V = \bigoplus_{r \geq 1} V^{(r)} \quad , V^{(1)} = V^{P = G_K^{1+}}$$

wild inertia

For simplicity, assume $V = V^{(r)}$, $r > 1$ wild case

$$V^{(r)} = \bigoplus_{\chi} \chi^{n_\chi} \qquad \mathrm{Gr}^r G_K = G_K^r / G_K^{r+}$$

abelian and killed by p

$$\mathrm{Hom}_{\mathbb{F}_p}(\mathrm{Gr}^r G_K, \mathbb{F}_p) \xhookrightarrow{\text{char}} \mathrm{Hom}_{\bar{F}}\left(m_{\bar{K}}^r / m_{\bar{K}}^{r+}, \Omega^1_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \bar{F}\right)$$

$\bar{F}$-v.s. of dim 1

$$m_{\bar{K}}^r = \{ a \in \bar{K} : \mathrm{val}\ a \geq r \}$$

$r+ \qquad\qquad > r$

$$SS(j_! \mathcal{G})_{\bar{\xi}} = \bigcup_{\chi} \mathrm{Im}(\mathrm{char}\ \chi) \subset T^* X_{\bar{\xi}}$$

In mixed char., replace $\Omega^1_{X,\xi} \otimes \bar{F}$ by $F\Omega_{X,\xi} \otimes \bar{F}$.

4. $CC = \sum r_a C_a$

Virtually, $K_0(X_{\bar{e}t}, \Lambda)$

Assume rank $\mathcal{F}|_{X_k}$ rank function $= 0$ constant.

$\dim C_a = n = \dim X$

$\dim C = n$, $\dim h^\circ C \neq n$

- compatibility w/ pull-back by properly transversal morphism.

$\uparrow$ dim is fine

- — " — push forward by proper morphism

- $X = S = \operatorname{Spec} O_K$, $CC(\mathcal{F}) = (\operatorname{rank} \mathcal{F}_x - \operatorname{Sw} \mathcal{F}) \cdot \underset{\substack{\uparrow \\ 1-pt}}{FT^*S}$

$\Rightarrow$ conductor formula

$\dim X = 2$, $\underset{\substack{\uparrow \\ rk\ 1}}{[\mathcal{G}]} - [\Lambda]$ Ooe defined a candidate of CC, proved conductor formula

Def'n 4.4 X reg of f.t. $/ S = \operatorname{Spec} O_K$, $\mathcal{F}$ sheaf on X, $C \subset FT^*X$

We say $\mathcal{F}$ is S-micro supported on C if the following condition is satisfied:
Let $h: W \to X$, $f: W \to Y$ (sep) morphisms of reg schemes of f.t. $/S$ & assume Y is smooth $/S$. If (h,f) is C-acyclic over S, then

(h, f) is $\mathcal{F}$- acyclic over S on a nbhd of $W_{\mathbb{F}_p}$.

Heuristic observation. Pretend S has a base field k $(= \mathbb{F}_1)$

$f_0 : W \to Y_0$, Y_0 smooth $/k$

(h, f_0) $\mathcal{F}$- acyclic if $(h, f_0) : W \to X \times_k Y_0$ is $\mathcal{F} \boxtimes \mathcal{G}_0$ - transversal $\forall \mathcal{G}_0$ on Y_0.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cong$

If one can determine which $\mathcal{G}$ comes from Y_0, $X \times_S Y$ if $Y = Y_0 \times_k S$

can make analogous definition.

$\mathcal{G}$ pull-back of $\mathcal{G}_0$ to Y

$\mathcal{F} \boxtimes \mathcal{G}$ - transversal

If $\mathcal{G}$ comes from Y_0, then its micro-supp should also come from Y_0

$$\begin{array}{ccc} FT^*S \times_S Y & \longrightarrow & FT^*Y \\ & & \cup \\ & & C' \end{array}$$

on which $\mathcal{G}$ is m.s.

We say $Y \to S$ is C'-acyclic if the inverse image of C' in $FT^*S \times_S Y$ is a subset of the 0-section.

<u>Def 4.5</u> 1. Y smooth $/S$, $\mathcal{G}$ sheaf on Y

We say $\mathcal{G}$ is S-acyclic if there exists $C' \subset FT^*Y$ on which $\mathcal{G}$ is micro-supported, and such that $Y \to S$ is C'-acyclic.

$\Rightarrow Y \to S$ $\mathcal{G}$-acyclic

2. Let $h : W \to X$, $f : W \to Y$, $\mathcal{F}$, C be as in 4.4.

We say (h, f) is $\mathcal{F}$-acyclic over S if for every $\mathcal{G}/Y$

which is S-acyclic, $(h,f): W \to X \times_S Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.

Def 4.6 X, C as in 4.4

1. $h: W \to X$, $f: W \to Y$ as in 4.4

We say (h,f) is C-acyclic over S if

$$(h^*C \times_W f^* FT^*Y) \cap \ker\Big((FT^*X \times_X W) \times_W (FT^*Y \times_Y W) \to FT^*W \Big)$$
$$\subset \ker\Big((FT^*X \times_X W) \times_W (FT^*Y \times_Y W) \to FT^*(X \times_S Y) \times_{X \times_S Y} W \Big)$$

2. We say C is S-saturated if it is stable under the action of $FT^*S \times_S X$.

Example 4.7 1. If (h,f) is C-acyclic & $Y \to S$ is C'-acyclic

$\Rightarrow$ $(h,f): W \to X \times_S Y$ is $pr_1^\circ C + pr_2^\circ C'$-transversal

2. Assume $h: W \to X$ is C-transversal, $f: W \to Y = S$, then if C is S-saturated $\Rightarrow$ (h,f) is C-acyclic $/S$.

Expectation 4.8. $\mathcal{F}, \mathcal{G} / X$. Assume $\mathcal{F}$ is m.s. on C, $\mathcal{G}$ m.s. on C'.

If $\mathrm{supp}(C \cap C') = \phi$, $\to$ $\mathcal{F} \otimes \mathcal{G}$ is m.s. on $\overline{C + C'}$.

More weakly, $\mathcal{F}/X$, $\mathcal{G}/Y$ sm. over S, $\mathcal{F}$ m.s. on C, $\mathcal{G}$ m.s. on $\underset{\substack{\uparrow \\ S\text{-acyclic}}}{C'}$.

then $\mathcal{F} \boxtimes \mathcal{G}$ is m.s. on $pr_1^\circ C + pr_2^\circ C'$.

[OK in geom. case]

Lemma 4.9 1. Assume $\mathcal{F}$ is m.s. on C. If Expectation 4.8 holds, then $\mathcal{F}$ is S-m.s. on C.

2. Assume $\mathcal{F}$ is S-m.s. on C. If C is S-saturated, then $\mathcal{F}$ is m.s. on C.

Proof 1. Assume (h, f) C-acyclic $/S$. Want (h, f) $\mathcal{F}$-acyclic $/S$

$\mathcal{G}$ on Y $\underline{S\text{-acyclic}}$, $(h,f): W \to X \times_S Y$ $\underline{\mathcal{F} \boxtimes \mathcal{G}\text{-}}$ transversal.

m.s. on C' s.t. $\quad$ m.s. on

$Y \to S$ is C'-acyclic $\quad$ $pr_1^\circ C + pr_2^\circ C'$-trans.

2. Assume $h: W \to X$ C-trans. Want h is $\mathcal{F}$-trans.

$f: W \to S$ $(= Y)$ $\quad$ $(h, f): W \to X \times_S S = X$ is C-acyclic$/S$

If SS etc exist.

$$SS_S \mathcal{F} \subset SS\mathcal{F} \subset SS_S^{sat} \mathcal{F}$$

If $FT^*S \times_S X \to FT^*X$ is 0, equal

Thm. $SS_S^{\underline{sat}} \mathcal{F}$ exists

Pf. Beilinson reduce to $X = \mathbb{P}^n_S$

$$FT^*X \qquad F^*(T^*(X/S)|_{\mathbb{F}_p})$$

$$0 \to FT^*S \times_S X \to FT^*X \to \overset{\int}{-\shortparallel-} \to 0 \qquad \text{exact}$$

5. Proof of Prop 3.9 (cf. Braverman - Gaitsgory)

Prop 3.9 $\quad f: X \to Y$ morphism of finite type over an excellent regular noetherian scheme.

$\mathcal{F}$ sheaf on X. Assume $X \to S$ is univ. $\mathcal{F}$-acyclic & $Y \to S$ smooth

TFAE (1) f is $\mathcal{F}$-acyclic

(2) $\forall\ \mathcal{G}$ on Y, $\gamma = (1_X, f): X \to X \times_S Y$ is $\mathcal{F} \boxtimes \mathcal{G}$ - transversal.

Pf In (2), $\mathcal{G} = i_* L$, $i: V \xrightarrow{\text{regular}} Y$ immersion, L loc. const

$$\begin{tikzcd}
X \times_Y V = W \arrow[r, "\gamma'"] \arrow[d, "h"] & X \times_S V \arrow[r, "pr_2"] \arrow[d] & V \arrow[d, "i"] \\
X \arrow[r, "\gamma"] & X \times_S Y \arrow[r, "pr_2"] & Y
\end{tikzcd}$$

Prop 3.4 $f: X \to Y$ morphism (of f.t.) $\mathcal{F}$ sheaf on X, TFAE

(1) f is $\mathcal{F}$-acyclic

(2) $\forall$
$$\begin{tikzcd}
X \arrow[d, "f"] & U \arrow[l, "h"'] \arrow[d, "g"] \\
Y & V \arrow[l, "i"']
\end{tikzcd}$$
s.t. i is an immersion.

$\forall\ \mathcal{G}$ on V, $\mathcal{F} \otimes f^* \underbrace{i_* \mathcal{G}} \xrightarrow{\sim} h_* (h^* \mathcal{F} \otimes g^* \mathcal{G})$.

$$\mathcal{F} \otimes f^* i_* \mathcal{G} = \gamma^*(\mathcal{F} \boxtimes i_* L) \simeq h_* (\gamma'^* (\mathcal{F} \boxtimes L)) = h_*(h^*\mathcal{F} \otimes g^*\mathcal{G}), \quad L \text{ loc. const}$$

(1') $\forall\ i: V \to Y$ immersion, L/V loc. const.

(3.2) $\gamma^*(\mathcal{F} \boxtimes i_* L) \to h_* (\gamma'^*(\mathcal{F} \boxtimes L))$ is an isom

(2') —— " —— " —— $c: \gamma^*(\mathcal{F} \boxtimes i_* L) \otimes \gamma^! \Lambda \to \gamma^! (\mathcal{F} \boxtimes i_* L)$ — " —
(3.3)

$$\begin{tikzcd}
\gamma^*(\mathcal{F} \boxtimes i_*\mathcal{L}) \otimes \gamma^!\Lambda \arrow[r, "c_\gamma"] \arrow[d, "(A)"'] \arrow[dr, phantom, "\mathrm{III}"] & \gamma^!(\mathcal{F} \boxtimes i_*\mathcal{L}) \arrow[d, "\wr", "\text{proper base change}"'] \\
h_*\left(\gamma'^*(\mathcal{F} \boxtimes \mathcal{L}) \otimes h^*\gamma^!\Lambda\right) \arrow[r, "\text{“} h_*\varepsilon_{\gamma'} \text{”}"'] & h_*\left(\gamma'^!(\mathcal{F} \boxtimes \mathcal{L})\right)
\end{tikzcd} \qquad (3.4)$$

$$\gamma'^*(\mathcal{F} \boxtimes \mathcal{L}) \otimes h^*\gamma^!\Lambda \to \gamma'^!(\mathcal{F} \boxtimes \mathcal{L}) \qquad (3.5)$$

<u>Claim</u> TFAE

(1°) (3.5) & (A) are isom.

(2°) (3.3) is an isom.

(3°) all arrows in (3.4) are isom.

(1°) ⇒ (3°), ✓

(2°) ⇒ (1°) by adjunction (3.5) isom. hence (A)

(3°) ⇒ (2°) . ✓

(2°) ⇒ (1°) implies (2') ⇒ (1').

(1') ⇒ (2'). Assume <u>(1')</u>, want (3.3) to be an isom.

(A) isom.

enough to prove (3.5)

First, i open & $\mathcal{L} = \Lambda$. transitivity of transversality, γ is $\mathcal{F} \boxtimes \Lambda$ -transversal.

(3.3) is an isom.

Next i closed immersion, $\mathcal{L} = i^! \Delta$ distinguished Δ.

Last i general. i closed immersion. $\mathcal{L} =$. ✓.