Complex analytic vanishing cycles for formal schemes

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Algebraic geometry.

R henselian DVR (e.g. $\mathcal{O}_{\mathbb{C},0}$)

$K = \mathrm{Frac}(R)$, $\tilde{R} = R/m$ residue field

Assume $\tilde{R}$ alg. closed.

$$G = \mathrm{Gal}(K^a|K) \qquad \begin{tikzcd} \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(R) & \mathrm{Spec}(\tilde{R}) \arrow[l] \\ \mathrm{Spec}(K^a) \arrow[u] \arrow[ur] & & \end{tikzcd}$$

$\mathfrak{X} \longrightarrow \mathrm{Spec}(R)$,

$$\begin{tikzcd} F & \mathfrak{X}_\eta \arrow[r, hook, "j"] & \mathfrak{X} & \mathfrak{X}_s \arrow[l, "i"'] \\ \bar{F} & \mathfrak{X}_\eta \otimes_K K^a = \mathfrak{X}_{\bar{\eta}} \arrow[u] \arrow[ur, "\bar{j}"'] & & \end{tikzcd}$$

sheaves

$$\Psi_\eta : \mathfrak{X}_\eta^\sim \longrightarrow \mathfrak{X}_s(G)^\sim$$

$$F \longmapsto i^* \bar{j}_* \bar{F}$$

<u>Fact</u>. if F is constructible $\mathbb{Z}/n\mathbb{Z}$-mod, $(n, \mathrm{char}\,\tilde{R}) = 1$

$\Rightarrow$ $R^q \Psi_\eta(F)$ are constructible.

Formal geometry

k non-arch. field w/ discrete valuation

$k^\circ = \{a \in k : |a| \le 1\}$

$\cup$

$k^{\circ\circ} = \{a \in k : |a| < 1\}$. Assume $\tilde{k} = k^\circ/k^{\circ\circ}$ - alg. closed.

Def. A special formal scheme $/k^\circ$ is a locally finite union of open

$$\mathrm{Spf}(A), \qquad A = k^\circ\{T_1,\dots,T_m\}[[S_1,\dots,S_n]]/\mathfrak{a}$$

$$(w, S_1,\dots,S_m), \quad w \text{ generator of } k^{\circ\circ}.$$

$\mathfrak{X}$ special formal scheme $/k^\circ$ $\rightsquigarrow$ $\mathfrak{X}_s = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\underset{\text{an ideal of definition}}{\mathcal{I}})$

$\rightsquigarrow$ $\mathfrak{X}_\eta$ - k-analytic space

e.g. if $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{X}_\eta = \{x \in \mathbb{A}^{m+n} : |T_i(x)| \le 1,\ |S_j(x)| < 1,\ f(x) = 0,\ \forall f \in \mathfrak{a}\}$

$\psi_\eta : \mathfrak{X}_\eta^\sim \longrightarrow \mathfrak{X}_s(a)^\sim$.

Example. $\mathcal{X}$ scheme of finite type $/k^\circ$

$\hat{\mathcal{X}}$; $\mathcal{Y} \subset \mathcal{X}_s$ $\rightsquigarrow$ $\hat{\mathcal{X}}_{/\mathcal{Y}}$.

Facts. (i) $\mathcal{X}$ scheme of finite type $/k^\circ$,

$\mathcal{Y} \subset \mathcal{X}_s$ closed subscheme

$\mathcal{F}$ constructible $\mathbb{Z}/n\mathbb{Z}$-mod on $\mathcal{X}_\eta$, $(n, \mathrm{char}\,\tilde{k}) = 1$.

$\Rightarrow$ $R\Psi_\eta(\mathcal{F})|_{\mathcal{Y}} \cong R\Psi_\eta(\hat{\mathcal{F}}_{/\mathcal{Y}})$ $\rightsquigarrow$ $\mathrm{Aut}(\hat{\mathfrak{X}}_{/\mathcal{Y}})$ - action on $R\Psi_\eta(\mathcal{F})|_{\mathcal{Y}}$.

$$(\hat{\mathcal{X}}_{/\mathcal{Y}})_\eta \longrightarrow \mathcal{X}_\eta$$

$$\hat{\mathcal{F}}_{/\mathcal{Y}} \qquad\qquad \mathcal{F}$$

(ii) $\mathfrak{X}$ - special formal scheme, $\mathcal{F}$ constructible on $\mathfrak{X}_\eta$, $\Rightarrow$ $R^q\Psi_\eta(\mathcal{F})$ are constructible.

Λ - finite G-module $\rightsquigarrow$ $\Lambda_{\mathfrak{X}_\eta}$

e.g. $\varphi: \mathfrak{Y} \to \mathfrak{X}$

$$\theta_\eta(\varphi, \Lambda) : \varphi_s^*\left(R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})\right) \longrightarrow R\Psi_\eta\left(\Lambda_{\mathfrak{Y}_\eta}\right)$$

(iii) (Continuity) given $\mathfrak{X}, \mathfrak{Y}, \Lambda$, $\exists$ an ideal of def. of $\mathfrak{Y}$,

$$\forall \varphi, \psi : \mathfrak{Y} \to \mathfrak{X},\ \varphi \equiv \psi (I) \Rightarrow \theta_\eta(\varphi, \Lambda) = \theta_\eta(\psi, \Lambda).$$

<u>Complex geometry.</u>

$$\begin{tikzcd} D^* = D \setminus \{0\} \arrow[r, hook] & D & \{0\} \arrow[l] \\ \overline{D^*} \arrow[u] \arrow[ur, bend right] & \end{tikzcd} \qquad \circlearrowleft \Pi = \pi_1(D^*)$$

$X \xrightarrow{f} D$, $Y \subset f^{-1}(0)$

$$\begin{tikzcd} \mathcal{F} & X_\eta \arrow[r, hook, "j"] & X & Y \arrow[l, hook', "i"'] \\ \bar{\mathcal{F}} & X \times_{D^*} \overline{D^*} = X_{\bar\eta} \arrow[u] \arrow[ur, "\bar{j}"'] & & \end{tikzcd}$$

$$\Psi_\eta : X_{\tilde\eta} \longrightarrow Y(\Pi)^\sim$$

$$\mathcal{F} \longmapsto i^*(\bar{j}_* \mathcal{F})$$

$(\mathbb{C}^n, 0) \xrightarrow{f} (\mathbb{C}, 0)$

$B \ni 0$, $B \xrightarrow{f} \mathbb{C}$

$R^q\Psi_\eta(\mathcal{F})_0$

$(B, b) \xrightarrow{f} (\mathbb{C}, 0)$ $\qquad X_\eta = X \otimes_{\mathcal{O}_{\mathbb{C},0}} K$.

X scheme of finite type $/ \mathcal{O}_{B,b}$

$$\rightsquigarrow \begin{tikzcd} X^h \arrow[r, "v"] \arrow[rr, bend right] & B \arrow[r, "f"] & \mathbb{C} \end{tikzcd} \qquad X^h_\eta = g^{-1}(\mathbb{C}^*)$$

$Y = X^h_s = v^{-1}(b) \overset{g}{\subset} g^{-1}(0)$

Facts (i) $\mathcal{F}$ constr. étale $\mathbb{Z}/n\mathbb{Z}$-mod on X_η,

$$R\Psi_\eta(\mathcal{F})\Big|_{X_s}^h \Rightarrow R\Psi_\eta(\mathcal{F}^h)\Big|_{X_s^h}$$

(ii) Λ - finitely gen. abelian gp Π - module.

$\rightsquigarrow \Lambda_{X_\eta^h}$. $R^q\Psi_\eta(\Lambda_{X_\eta^h})$ are algebraically constructible on X_s^h.

Y scheme of finite type $/\mathbb{C}$

Def. $\mathcal{F}$ on Y^h is (algebraically) constructible if $\exists\ Y = Y_0 \supset Y_1 \supset \dots \supset Y_n = \phi$ Zariski closed

s.t. $\mathcal{F}\big|_{Y_i^h \setminus Y_{i+1}^h}$ is loc. constant w/ finitely gen. stalks.

$D_c(\Pi\text{-Mod}) = \{\Lambda^\cdot \in D(\Pi\text{-Mod}) : H^q(\Lambda^\cdot) \text{ are fin. gen. } /\mathbb{Z}\}$

Y scheme of finite type$/\mathbb{C}$ $\rightsquigarrow D_c(Y^h)$.

K n.a. field $/\mathbb{C}$ w/ discrete valuation, $\underset{\text{residue field}}{\tilde{K}} \simeq \mathbb{C}$. $(\Rightarrow K \simeq \mathbb{C}((z)))$

$$G = \mathrm{Gal}(K^a|K) = \varprojlim \mu_n = \hat{\mathbb{Z}}$$
$$\cup$$
$$\Pi = \left\langle \left(e^{\frac{2\pi i}{n}}\right)_{n\geq 1} \right\rangle = \mathbb{Z}$$

Thm. Given a quasi-compact special formal scheme $\mathfrak{X}/K^0$ One can construct an exact functor

$$D_c^b(\Pi\text{-Mod}) \longrightarrow D_c^b(\mathfrak{X}_s^h(\Pi)), \quad \Lambda^\cdot \mapsto \underset{\substack{\uparrow \\ \text{just a notation}}}{R\Psi_\eta^h}(\Lambda^\cdot_{\mathfrak{X}_\eta})$$

s.t. the following is true.

(i) $\mathfrak{X} \mapsto R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$ is functorial in $\mathfrak{X}$, i.e. $\forall \varphi: \mathfrak{Y} \to \mathfrak{X}$,

$$\rightsquigarrow \theta^h_\eta(\varphi, \Lambda^\cdot) : \varphi_s^{h*}\left(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})\right) \to R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{Y}_\eta})$$

(ii) (Comparison w/ formal geom.)

if $\Lambda^\cdot \in D^b_c(\mathbb{Z}/n\mathbb{Z}[G]\text{-Mod})$, $\Rightarrow R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \xrightarrow{\sim} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$

(iii) given a generator $\omega \in K^{\circ\circ}$. $\left(\mathcal{O}_{C,0} \to K^\circ,\ z \mapsto \omega,\ \hat{\mathcal{O}}_{C,0} \xrightarrow{\sim} K^\circ \right)$

$(B, b) \to (C, 0)$

X scheme / $\mathcal{O}_{B,b}$

$\hat{X}$ special formal scheme / K°. $R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}^h_\eta}) \xrightarrow{\sim} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$

(iv) given a subscheme $Y \subset \mathfrak{X}_s$, $\mathfrak{X}$ q-cpt special formal scheme / K°

$$R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})\big|_Y \xrightarrow{\sim} R\Psi^h_\eta(\Lambda^\cdot_{(\mathfrak{X}/Y)_\eta})$$

(v) $R\Psi^h_\eta(\mathbb{Z}_{\mathfrak{X}_\eta}) \otimes^L_{\mathbb{Z}} \Lambda^\cdot \xrightarrow{\sim} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$

(vi) (continuity) Given $\mathfrak{X}$ w/ reg (reg-smooth) $\mathfrak{X}_\eta$, $\exists\, n \geq 1$ s.t.

$\forall \mathfrak{Y}$ w/ regular $\mathfrak{Y}_\eta$, $\forall \varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ s.t. $\varphi \equiv \psi \pmod{I^n}$, I an ideal of def'n of $\mathfrak{Y}$,

$\forall \Lambda^\cdot$, $\theta^h_\eta(\varphi, \Lambda^\cdot) = \theta^h_\eta(\psi, \Lambda^\cdot)$.

X cpt strictly K-analytic space. ($\forall X$, fix $\mathfrak{X}$)

$\exists$ a formal model $\mathfrak{X}$ of X ($\Leftarrow$ Raynaud), a formal scheme of finite type / K°

$X = \mathfrak{X}_\eta$. Λ is a π-mod. f.g. / $\mathbb{Z}$ (e.g. $\Lambda = \mathbb{Z}$)

Proo…

$$H^q(\bar{X};\Lambda) := R^q\Gamma(\mathfrak{X}_s^h,\ R\Psi_\eta^h(\Lambda_{\mathfrak{X}_\eta})) \curvearrowleft \pi \qquad \left(\bar{X} = X \hat{\otimes}_k \widehat{k^a}\right)$$

<u>Fact.</u> (i) $X \mapsto H^q(\bar{X};\Lambda)$ is functorial in X

(ii) if Λ is finite then $H^q(\bar{X};\Lambda) = H^q_{\text{ét}}(\bar{X};\Lambda)$

$\forall$ prime ℓ, $\quad H^q(\bar{X};\mathbb{Z}) \otimes_{\mathbb{Z}} \substack{\mathbb{Z}_\ell \\ \mathbb{Q}_\ell} \xrightarrow{\sim} H^q_{\text{ét}}\left(\bar{X}; \substack{\mathbb{Z}_\ell \\ \mathbb{Q}_\ell}\right)$

Y proper scheme over K, $X = Y^{an}$, $\quad$ RHS $= H^q_{\text{ét}}\left(\bar{Y}, \substack{\mathbb{Z}_\ell \\ \mathbb{Q}_\ell}\right)$

<u>Conj.</u> Hodge str. on $H^q(\bar{X};\mathbb{Z})$; $\quad$ limit Hodge structure.