BRST reduction

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Abstract

This is a note on Bechi–Rouet–Stora–Tyutin (BRST) reduction for myself. We first present the finite-dimensional picture. There is a classical version and a quantum version. In some sense, the quantum picture is even more natural than the classical picture. Instead of a quantization process from classical picture to quantum picture, we derive the classical picture by taking semi-classical limit of the quantum picture. However, to follow some convention, we talk about classical situation firstly. Next we move to the infinite-dimensional case. This is the "true" BRST reduction for physiscists' need because the group of gauge transformations is typically infinite-dimensional. In mathematical literature, this infinite-dimensional BRST reduction is also called semi-infinite cohomology. They are essentially the same. Our exposition here follows some early work by Feigin, Frenkel, Garland, Zuckerman et al. But there exists another approach based on Tate's linear algebra that was first proposed by Beilinson. This approach has become essential to chiral algebra and semi-infinite homological algebra. I intend to write a note on Tate's linear algebra, so I will not include these discussions in this note.

Everything will be over $\mathbb{C}$ unless specifically stated.

1 Classical BRST reduction

We begin with classical BRST reduction in the finite dimensional setting. We present it in a way that allows a natural quantization, which will be discussed in the next section.

1.1

Let V be a finite-dimensional vector space, then there is a canonical non-degenerate bilinear form $(\cdot|\cdot)$ on $V \oplus V^*$ given by half of[1] the pairing between V and V^*. Let $\mathcal{C}\ell = \mathcal{C}\ell_V$ be the corrsponding *Clifford algebra*. We assign a grading to $\mathcal{C}\ell = \mathcal{C}\ell^\bullet$ so that V has degree -1 and V^* has degree 1. Now we see that $\mathcal{C}\ell$ is a unital superalgebra that is naturally[2] isomorphic to $\bigwedge^\bullet V \otimes \bigwedge^\bullet V^*$ as vector spaces. Fix a basis $\{x_\alpha\}_{\alpha\in\Lambda}$ of V and let $\{x^\alpha\}_{\alpha\in\Lambda}$ be the corresponding dual basis of V^*, then $\mathcal{C}\ell$ is generated by odd generators x_α, x^α with relations

$$[x_\alpha, x_\beta] = [x^\alpha, x^\beta] = 0, [x_\alpha, x^\beta] = \delta_\alpha^\beta.$$

We have an increasing filtration on $\mathcal{C}\ell$ by setting $\mathcal{C}\ell_p = \bigwedge^{\leq p} V \otimes \bigwedge^\bullet V^*$. We can check that

$$\mathcal{C}\ell_p \cdot \mathcal{C}\ell_q \subset \mathcal{C}\ell_{p+q}, [\mathcal{C}\ell_p, \mathcal{C}\ell_q] \subset \mathcal{C}\ell_{p+q-1}.$$

[1] By half, we mean $(x+\phi|y+\psi) = \frac{1}{2}(\phi(y)+\psi(x))$ for $x, y \in V, \phi, \psi \in V^*$. This is a good normalization in the sense that $[x,\phi] = x\phi + \phi x = \phi(x)$ in the Clifford algebra.

[2] Over a base field with characteristic not equal to 2, we can prove (not easy) that $\operatorname{gr}\mathcal{C}\ell = \bigwedge$, where the filtration of $\mathcal{C}\ell$ is the standard one.

Thus the associated graded algebra $\overline{\mathcal{C}\ell} = \mathrm{gr}\,\mathcal{C}\ell$ is naturally a graded commutative algebra, called the *classical Clifford algebra.* It is a Poisson algebra in the usual way. We have $\overline{\mathcal{C}\ell} = \bigwedge^{\bullet} V \otimes \bigwedge^{\bullet} V^*$ as a commutative superalgebra. The Poisson bracket $\{\cdot,\cdot\}$ vanishes on V and V^*, and for $v \in V, v^* \in V^*$ one has $\{v, v^*\} = v^*(v)$.

Now $\overline{\mathcal{C}\ell}$ has two gradings: one given by the grading on $\mathcal{C}\ell$ (denoted by upper indices) and one given by the filtration of $\mathcal{C}\ell$ (denoted by lower indices). We see that $\overline{\mathcal{C}\ell}_1^{-1} = V$ and $\overline{\mathcal{C}\ell}_0^1 = V^*$. The subspace $\overline{\mathcal{C}\ell}_1^0$ is a Lie subalgebra of $\overline{\mathcal{C}\ell}$. It normalizes $\overline{\mathcal{C}\ell}_1^{-1} = V$ and $\overline{\mathcal{C}\ell}_0^1 = V^*$ and the corresponding adjoint action identifies it with $\mathrm{End}(V) = \mathfrak{gl}(V)$ and $\mathrm{End}(V^*) = \mathfrak{gl}(V^*)$.

1.2

Lemma 1. Let $\overline{\mathcal{C}\ell}^V = \{w \in \overline{\mathcal{C}\ell} : \{v, w\} = 0, \forall v \in V\}$. Then

$$\overline{\mathcal{C}\ell}^V = \bigwedge V.$$

Proof. Pick a total order $<$ on Λ. Then $\{x_{i_1} \cdots x_{i_s} x^{j_1} \cdots x^{j_t} : i_1 < \cdots < i_s, j_1 < \cdots < j_t\}$ forms a basis of $\overline{\mathcal{C}\ell}$. For scalars $c^{i_1 \cdots i_s}_{j_1 \cdots j_t}$ all but a finite number equal to 0, one can compute that

$$\left\{ x_\alpha, \sum_{\substack{i_1<\cdots<i_s \\ j_1<\cdots<j_t}} c^{i_1\cdots i_s}_{j_1\cdots j_t} x_{i_1} \cdots x_{i_s} x^{j_1} \cdots x^{j_t} \right\}$$

$$= \sum_{\substack{i_1<\cdots<i_s \\ j_1<\cdots<j_t}} \pm c^{i_1\cdots i_s}_{j_1\cdots j_t} x_{i_1} \cdots x_{i_s} (\pm \delta^{j_1}_{\alpha} x^{j_2} \cdots x^{j_t} \pm \delta^{j_2}_{\alpha} x^{j_1} x^{j_3} \cdots x^{j_t} \pm \cdots \pm \delta^{j_t}_{\alpha} x^{j_1} \cdots x^{j_{t-1}}).$$

The lemma follows by comparing coefficients using the lexicographical order. □

1.3

Now we set $V = \mathfrak{n}$ to be a finite-dimensional Lie algebra, $\overline{R}$ a Poisson algebra, $\overline{L} \colon \mathfrak{n} \to \overline{R}$ a Lie algebra homomorphism. The adjoint action of $\mathfrak{n}$ yields a Lie algebra homomorphism $\overline{\rho} \colon \mathfrak{n} \to \overline{\mathcal{C}\ell}_1^0 \subset \overline{\mathcal{C}\ell}$. Identifying $\mathfrak{n}$ with $\overline{\mathcal{C}\ell}_1^{-1}$, we have $\{\overline{\rho}(x), y\} = [x, y]$ for $x, y \in \mathfrak{n}$. Set $\overline{\mathcal{A}} = \overline{\mathcal{C}\ell} \otimes \overline{R}$; $\overline{\mathcal{A}}^{\bullet} = \overline{\mathcal{C}\ell}^{\bullet} \otimes \overline{R}$ is a Poisson graded algebra. It carries an additional grading $\overline{\mathcal{A}}^{\bullet}_{(i)} = \overline{\mathcal{C}\ell}^{\bullet}_i \otimes \overline{R}$ compatible with the product (but not with the Poisson bracket). We have a morphism of Lie algebras

$$\overline{\mathcal{L}} \colon \mathfrak{n} \to \overline{\mathcal{A}}^0, n \mapsto \overline{\rho}(n) \otimes 1 + 1 \otimes \overline{L}(n).$$

For $x, y \in \mathfrak{n}$,

$$\{\overline{\mathcal{L}}(x), y\} = [x, y].$$

1.4

Lemma 2. There is a unique element $\overline{Q} \in \overline{\mathcal{A}}^1$ such that for any $n \in \mathfrak{n}$ one has $\{\overline{Q}, n\} = \overline{\mathcal{L}}(n)$. One has $\overline{Q} \in \overline{\mathcal{A}}^1_{(\leq 1)}$ and $\{\overline{Q}, \overline{Q}\} = 0$. Here we identify $\mathfrak{n}$ with $\overline{\mathcal{C}\ell}_1^{-1}$.

Proof. The uniqueness of $\overline{Q}$ follows from Lemma 1 because regarded as an $\bigwedge \mathfrak{n}$-module,

$$\overline{\mathcal{A}}^{\mathfrak{n}} = \bigwedge \mathfrak{n} \otimes \overline{R} \subset \overline{\mathcal{A}}^{\leq 0}.$$

Now we construct $\overline{Q}$ explicitly. Let $\overline{Q}_1$ be the image of

$$\frac{1}{2}\overline{\rho} \in \mathrm{Hom}(\mathfrak{n}, \overline{\mathcal{C}\ell}^0_1) = \mathfrak{n}^* \otimes \overline{\mathcal{C}\ell}^0_1 \subset \overline{\mathcal{A}}^1_{(0)} \otimes \overline{\mathcal{A}}^0_{(1)}$$

under the product map $\overline{\mathcal{A}}^1_{(0)} \otimes \overline{\mathcal{A}}^0_{(1)} \to \overline{\mathcal{A}}^1_{(1)}$ and $\overline{Q}_0$ be

$$\overline{L} \in \mathrm{Hom}(\mathfrak{n}, \overline{R}) = \mathfrak{n}^* \otimes \overline{R} \subset \overline{\mathcal{A}}^1_{(0)}.$$

Then $\overline{Q} = \overline{Q}_1 + \overline{Q}_0 \in \overline{\mathcal{A}}^1_{(\leq 1)}$ and we can see that[3]

$$\begin{aligned}\{\overline{Q}, n\} &= \{\overline{Q}_1, n\} + \{\overline{Q}_0, n\} \\ &= \overline{\rho}(n) \otimes 1 + 1 \otimes \overline{L}(n).\end{aligned}$$

For $n, n' \in \mathfrak{n}$,

$$\begin{aligned}\{n, \{n', \{\overline{Q}, \overline{Q}\}\}\} &= 2\{n, \{\{\overline{Q}, n'\}, \overline{Q}\}\} \\ &= 2\{n, \{\overline{\mathcal{L}}(n'), \overline{Q}\}\} \\ &= 2\{\overline{\mathcal{L}}(n'), \{\overline{Q}, n\}\} - 2\{\overline{Q}, \{\overline{\mathcal{L}}(n), n'\}\} \\ &= 2\{\overline{\mathcal{L}}(n'), \overline{\mathcal{L}}(n)\} - 2\overline{\mathcal{L}}([n, n']) = 0.\end{aligned}$$

Since $\{\overline{Q}, \overline{Q}\} \in \overline{\mathcal{A}}^2$, applying Lemma 1 twice, we see that $\{\overline{Q}, \overline{Q}\} = 0$. □

1.5

Under the basis $\{x_\alpha\}_{\alpha\in\Lambda}$ of $\mathfrak{n}$ and the dual basis $\{x^\alpha\}_{\alpha\in\Lambda}$ of $\mathfrak{n}^*$, we can compute $\overline{Q}$ explicitly. Let $c^\gamma_{\alpha\beta}$ be the structure constants of $\mathfrak{n}$, i.e.

$$[x_\alpha, x_\beta] = \sum_{\gamma\in\Lambda} c^\gamma_{\alpha\beta} x_\gamma \text{ for } \alpha, \beta \in \Lambda.$$

For $\alpha, \beta \in \Lambda$,

$$\{\overline{\rho}(x_\alpha), x_\beta\} = [x_\alpha, x_\beta] = \sum_{\gamma\in\Lambda} c^\gamma_{\alpha\beta} x_\gamma,$$

so

$$\overline{\rho} = -\sum_{\alpha,\beta,\gamma\in\Lambda} x^\alpha \otimes 1 \otimes c^\gamma_{\alpha\beta} x^\beta x_\gamma \otimes 1$$

and

$$\overline{Q}_1 = -\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Lambda} c^\gamma_{\alpha\beta} x^\alpha x^\beta x_\gamma \otimes 1.$$

On the other hand, it is easy to see that

$$\overline{Q}_0 = \overline{L} = \sum_{\alpha\in\Lambda} x^\alpha \otimes \overline{L}(x_\alpha).$$

In summary, we derive the usual form of $\overline{Q}$ in physics literature as

$$\overline{Q} = -\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Lambda} c^\gamma_{\alpha\beta} x^\alpha x^\beta x_\gamma \otimes 1 + \sum_{\alpha\in\Lambda} x^\alpha \otimes \overline{L}(x_\alpha).$$

[3] An explicit computation is given later.

1.6

Set $d = \{\overline{Q}, \cdot\}$. This is a derivation of $\overline{\mathcal{A}}^{\bullet}$ of degree 1 and square 0. So $\overline{\mathcal{A}}$ is a Poisson DG superalgebra; it is called the *BRST reduction* of $\overline{R}$. Its cohomology $H^{\bullet}(\overline{\mathcal{A}}, d)$ inherits a Poisson DG superalgebra structure.

The decomposition $\overline{Q} = \overline{Q}_1 + \overline{Q}_0$ leads to a decomposition of the differential $d = \{\overline{Q}, \cdot\}$ by the bigrading and we see that $\overline{\mathcal{A}}$ is the total complex of the bicomplex with bidifferentials $d' = \{\overline{Q}_1, \cdot\} \colon \overline{\mathcal{A}}^{i}_{(j)} \to \overline{\mathcal{A}}^{i+1}_{(j)}, d'' = \{\overline{Q}_0, \cdot\} \colon \overline{\mathcal{A}}^{i}_{(j)} \to \overline{\mathcal{A}}^{i+1}_{(j-1)}$. As we will see later, $(\overline{\mathcal{A}}^{\bullet}_{(-\bullet)}, d'')$ is the Koszul complex $P = \bigwedge^{-\bullet} \mathfrak{n} \otimes \overline{R}$ for $\overline{L} \colon \mathfrak{n} \to \overline{R}$ and $\overline{\mathcal{A}}$ is the Chevalley–Eilenberg complex of Lie algebra cochains of $\mathfrak{n}$ with coefficients in P.

1.7

Assume that $\overline{L}$ is *regular*, i.e. $H_i(P) = 0$ for $i \neq 0$. This means that the projection $P \to \overline{R}/\overline{RL}(\mathfrak{n})$ is a quasi-isomorphism. Hence $\overline{\mathcal{A}} \to C^{\bullet}(\mathfrak{n}, \overline{R}/\overline{RL}(\mathfrak{n}))$ is also a quasi-isomorphism. Here $C^{\bullet}(\mathfrak{n}, \overline{R}/\overline{RL}(\mathfrak{n}))$ is the Chevalley–Eilenberg complex of Lie algebra cochains of $\mathfrak{n}$ with coefficients in $\overline{R}/\overline{RL}(\mathfrak{n})$. Passing to cohomology, we get an isomorphism

$$H^{\bullet}\overline{\mathcal{A}} \simeq H^{\bullet}(\mathfrak{n}, \overline{R}/\overline{RL}(\mathfrak{n})).$$

Thus $H^{i}\overline{\mathcal{A}}$ vanishes for $i < 0$ and

$$H^{0}\overline{\mathcal{A}} \simeq (\overline{R}/\overline{RL}(\mathfrak{n}))^{\mathfrak{n}}$$

equals to the usual Hamiltonian reduction of $\overline{R}$ with respect to the Hamiltonian action $\overline{L}$ as described below.

1.8

We make a digression on Hamiltonian reduction[4], which will provide some motivations for previous constructions. Let $(M, \{\cdot, \cdot\})$ be a Poisson affine algebraic variety, then we have a Lie algebra homomorphism[5]

$$v \colon \mathbb{C}[M] \to \operatorname{Der} \mathbb{C}[M], f \mapsto \{f, \cdot\}.$$

Here $\operatorname{Der} \mathbb{C}[M] = \{D \in \operatorname{End}(\mathbb{C}[M]) : D(fg) = D(f)g + fD(g) \text{ for } f, g \in \mathbb{C}[M]\}$.

Let G be a reductive group acting on M by Poisson automorphisms. The G action induces a Lie algebra homomorphism $\phi \colon \mathfrak{g} \to \operatorname{Der} \mathbb{C}[M]$. A G-equivariant regular map $\mu \colon M \to \mathfrak{g}^*$ is a *moment map* for the G-action on M if the pull back $\mu^* \colon \mathfrak{g} \to \mathbb{C}[M]$ satisfies the commutative diagram

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[dl, "\mu^*"'] \arrow[d, "\phi"] \\
\mathbb{C}[M] \arrow[r, "v"] & \operatorname{Der} \mathbb{C}[M]
\end{tikzcd}$$

Let $\mathcal{O}$ be a closed coadjoint orbit of G, $I_{\mathcal{O}}$ be the ideal in $\mathbb{C}[\mathfrak{g}^*]$ such that $\mathbb{C}[\mathcal{O}] = \mathbb{C}[\mathfrak{g}^*]/I_{\mathcal{O}}$, and let $J_{\mathcal{O}}$ be the ideal in $\mathbb{C}[M]$ generated by $\mu^*(I_{\mathcal{O}})$. We observe that $\mathbb{C}[M]^G$ is a Poisson algebra and $J^G_{\mathcal{O}}$ is a Poisson ideal in $\mathbb{C}[M]^G$, so $A = \mathbb{C}[M]^G/J^G_{\mathcal{O}}$ is a Poisson algebra. Geometrically, $\operatorname{Spec} A$ is the categorical quotient $\mu^{-1}(\mathcal{O})/G$. The scheme $\mu^{-1}(\mathcal{O})/G$ is called the *Hamiltonian reduction* of M with respect to G along $\mathcal{O}$.

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[4] In physical language, we are constructing reduced phase spaces.

[5] In classical mechanics, one says that $v(f)$ is the Hamiltonian vector field corresponding to the Hamiltonian f.

1.9

When constructing the Hamiltonian reduction, we utilize the quotient and invariant of a Poisson algebra. In homological algebra, a convenient tool to deal with quotient (resp. invariant) is the Koszul complex (Chevalley–Eilenberg complex). To do both steps simultaneously, we construct a bicomplex and use certain spectral sequance to do the computation.

Let $-d_1$ be the composition of $\mathfrak{n}\otimes\overline{R} \xrightarrow{L\otimes\overline{R}} \overline{R}\otimes\overline{R} \xrightarrow{\text{product}} \overline{R}$. We can extend d_1 to a superderivation

$$d_1\colon \bigwedge^{\bullet}\mathfrak{n}\otimes\overline{R}\to\bigwedge^{\bullet-1}\mathfrak{n}\otimes\overline{R}.$$

In terms of commutative algebra, we have taken the Koszul resolution of the $\mathrm{Sym}^{\bullet}\,\mathfrak{n}$-module $\overline{R}$. H_0 of this Koszul complex is just $\overline{R}/\overline{R}L(\mathfrak{n})$.

Since $\overline{R}$ is an $\mathfrak{n}$-module, we have the Chevalley–Eilenberg complex

$$d_2\colon \bigwedge^{\bullet}\mathfrak{n}^*\otimes\overline{R}\to\bigwedge^{\bullet+1}\mathfrak{n}^*\otimes\overline{R}.$$

H^0 of this Chevalley–Eilenberg complex is just $\overline{R}^{\mathfrak{n}}$.

One can check that $d_1^2=0, d_2^2=0, d_1d_2+d_2d_1=0$. Thus we get a bicomplex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots & \bigwedge^p\mathfrak{n}\otimes\bigwedge^{q+1}\mathfrak{n}^*\otimes\mathcal{P} \arrow[l, "d_1"'] \arrow[u, "d_2"] & \bigwedge^{p+1}\mathfrak{n}\otimes\bigwedge^{q+1}\mathfrak{n}^*\otimes\mathcal{P} \arrow[l, "d_1"'] \arrow[u, "d_2"] & \cdots \arrow[l, "d_1"'] \\
\cdots & \bigwedge^p\mathfrak{n}\otimes\bigwedge^{q}\mathfrak{n}^*\otimes\mathcal{P} \arrow[l, "d_1"'] \arrow[u, "d_2"] & \bigwedge^{p+1}\mathfrak{n}\otimes\bigwedge^{q}\mathfrak{n}^*\otimes\mathcal{P} \arrow[l, "d_1"'] \arrow[u, "d_2"] & \cdots \arrow[l, "d_1"'] \\
 & \vdots \arrow[u, "d_2"] & \vdots \arrow[u, "d_2"] &
\end{tikzcd}$$

Echoes with previous notations, $\frac{1}{2}d_2=d', d_1=d''$. The total differential is $d=d'+d''=\frac{1}{2}d_2+d_1$. Under the identification $\bigwedge\mathfrak{n}\otimes\bigwedge\mathfrak{n}^*\simeq\bigwedge(\mathfrak{n}\oplus\mathfrak{n}^*)$, d coincides with the Chevalley–Eilenberg differential of the complex $\bigwedge^{\bullet}(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes\overline{R}$. Here $\mathfrak{n}\oplus\mathfrak{n}^*$ is a Lie algebra with commutation relations

$$\begin{aligned}
&[x,y]_{\mathfrak{n}\oplus\mathfrak{n}^*}=[x,y]_{\mathfrak{n}} \text{ for } x,y\in\mathfrak{n},\\
&[\phi,\psi]=0 \text{ for } \phi,\psi\in\mathfrak{n}^*,\\
&[x,\phi]\in n^* \text{ and } [x,\phi](y)=-\phi([x,y]) \text{ for } x,y\in\mathfrak{n},\phi\in\mathfrak{n}^*.
\end{aligned}$$

The bilinear form $(\cdot|\cdot)$ given by half of the pairing between $\mathfrak{n}$ and $\mathfrak{n}^*$ is symmetric and non-degenerate, which allows us to make the identification $(\mathfrak{n}\oplus\mathfrak{n}^*)^*\simeq\mathfrak{n}\oplus\mathfrak{n}^*$. We use $\bigwedge(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes\overline{R}$ rather than $\bigwedge\mathfrak{n}\otimes\bigwedge\mathfrak{n}^*\otimes\overline{R}$ because the first one allows quantization in a seemingly easier way, which will be described in the next section.

2 Quantum BRST reduction

We quantize the constructions above.

2.1

Let V be a finite-dimensional vector space. A natural quantization of $\overline{\mathcal{C}\ell}_V$ is $\mathcal{C}\ell_V$ in the sense that $\operatorname{gr}\mathcal{C}\ell = \overline{\mathcal{C}\ell}$ with respect to the standard filtration of a Clifford algebra.

Now we fix $V = \mathfrak{n}$ to be a finite-dimensional Lie algebra. Let R be an associative algebra (quantization of $\overline{R}$), $L\colon \mathfrak{n} \to R$ a Lie algebra homomorphism. Set $\mathcal{A}^\bullet = \mathcal{C}\ell^\bullet \otimes R$. $\mathcal{A}$ is an associative graded algebra. We have a Lie algebra homomorphism

$$\mathcal{L} = \rho + L\colon \mathfrak{n} \to \mathcal{A}^0, n \mapsto \rho(n) + L(n).$$

Below we make the identification of $\mathfrak{n}$ with $\mathcal{C}\ell_1^{-1} \subset \mathcal{A}^{-1}$.

2.2

Lemma 3. There exists a unique element $Q \in \mathcal{A}^1$ such that for any $n \in \mathfrak{n}$, one has $[Q, n] = \mathcal{L}(n)$. In fact, $Q \in \mathcal{C}\ell_1^1 \otimes R$ and $Q^2 = 0$.

Proof. The uniqueness follows from a lemma similar to Lemma 1. In fact, Q is given by exactly the same formula

$$Q = -\frac{1}{2} \sum_{\alpha,\beta,\gamma\in\Lambda} c^\gamma_{\alpha\beta} x^\alpha x^\beta x_\gamma \otimes 1 + \sum_{\alpha\in\Lambda} x^\alpha \otimes L(x_\alpha).$$

□

Since Q is odd, $d = \operatorname{ad} Q$ is a superderivation. Thus $\mathcal{A}$ is an associative DG superalgebra; it is called the *BRST reduction* of R.

2.3

Let $C(\mathfrak{n}, R)$ be the Chevalley–Eilenberg DG algebra of Lie algebra cochains of $\mathfrak{n}$ with coefficients in R (with respect to the adjoint action of $L\colon \mathfrak{n} \to R$). As a graded algebra, it equals to $\bigwedge^\bullet \mathfrak{n}^* \otimes R$, and is a subalgebra of $\mathcal{A}^\bullet$. By our construction of d, we have the following lemma:

Lemma 4. $C(\mathfrak{n}, R)$ is a DG subalgebra of $\mathcal{A}$, i.e. the differentials are compatible.

Therefore, we have a morphism of graded algebras

$$H^\bullet(\mathfrak{n}, R) \to H^\bullet\mathcal{A}.$$

In particular, we get a morphism $\mathfrak{Z} \to H^0\mathcal{A}$, where $\mathfrak{Z} \subset R^\mathfrak{n} = H^0(\mathfrak{n}, R)$ is the center of R.

2.4

One application of the quantum BRST reduction in representation theory is the BRST realization of the center of a semisimple Lie algebra.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Let $R = U\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$, equipped with the standard filtration. Fix a principal nilpotent element e of $\mathfrak{g}$ in $\mathfrak{b}$, i.e. $\dim \mathfrak{g}^e = \operatorname{rank} \mathfrak{g}$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. By the Jacobson–Morozov theorem, we can find a $\mathfrak{sl}_2$-triple $\{e, h_0, f\}$ inside $\mathfrak{g}$. Let $(\cdot|\cdot) = \frac{1}{2h^\vee} \times$ Killing form be the normalized invariant bilinear form on $\mathfrak{g}$, here $h^\vee$ is the dual Coxeter number. We have a character χ of $\mathfrak{n}$ defined by

$$\chi(x) = (f|x) \text{ for } x \in \mathfrak{n}.$$

Then we have a Lie algebra homomorphism $L\colon \mathfrak{n} \to R = U\mathfrak{g}, x \mapsto x - \chi(x)$. These data $(\mathfrak{n}, R, L)$ allow us to define the BRST reduction $\mathcal{A}$ of $R = U\mathfrak{g}$. By discussions before, we have an algebra homomorphism $\mathfrak{Z} \to H^0\mathcal{A}$, where $\mathfrak{Z} = Z\mathfrak{g}$ is the center of $U\mathfrak{g}$.

The theorem by Kostant asserts the following:

Theorem 1. We have $H^i\mathcal{A} = 0$ for $i \neq 0$. At degree 0, the map $Z\mathfrak{g} \to H^0\mathcal{A}$ defined above is an isomorphism.

One proof uses Kazhdan filtration and a certain spectral sequence. This is also closely related to the Whittaker model.

3 Infinite-dimensional case

In reality, a system that a physicist concerns is usually infinite-dimensional, so it is important to develop an infinide-dimensional version of the BRST reduction. The basic ideas are similar to the finite-dimensional case, but we need to do some modifications. Instead of using the Clifford algebra itself, we should pass to its Fock module[6].

3.1

Let V be an infinite-dimensional vector space with a countable basis $\{\cdots, \underline{-2}, \underline{-1}, \underline{0}, \underline{1}, \underline{2}, \cdots\}$. An infinite wedge product $\underline{i_1} \wedge \underline{i_2} \wedge \cdots$ is *admissible*, if

$$i_1 > i_2 > \cdots \text{ and } i_n = i_{n-1} - 1 \text{ for } n \gg 0.$$

An admissible infinite wedge product is also called a *semi-infinite form*. Let $F = \bigwedge^{\infty/2} V$ be the vector space with a basis consisting of all semi-infinite forms. In physics literature, F is often called the *fermionic Fock space*.

Let $|m\rangle = \underline{m} \wedge \underline{m-1} \wedge \cdots$ be the *vacuum vector* with *charge* m. A semi-infinite form has charge m if it differs from $|m\rangle$ only at a finite number of places. Then we have the *charge decomposition*

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$$

where $F^{(m)}$ denotes the linear span of all semi-infinite forms of charge m.

3.2

The j^{th} fermion's creation operator ψ_j^* is defined by

$$\psi_j^*(\underline{i_1} \wedge \underline{i_2} \wedge \cdots) = \begin{cases} 0 \text{ if } j = i_s \text{ for some } s, \\ (-1)^s \underline{i_1} \wedge \cdots \wedge \underline{i_s} \wedge \underline{j} \wedge \underline{i_{s+1}} \wedge \cdots \text{ if } i_s > j > i_{s+1}. \end{cases}$$

The j^{th} fermion's annilation operator ψ_j is defined by

$$\psi_j(\underline{i_1} \wedge \underline{i_2} \wedge \cdots) = \begin{cases} 0 \text{ if } j \neq i_s \text{ for all } s, \\ (-1)^{s+1} \underline{i_1} \wedge \underline{i_2} \wedge \cdots \wedge \underline{i_{s-1}} \wedge \underline{i_{s+1}} \wedge \cdots \text{ if } j = i_s. \end{cases}$$

[6]Physicists call this "fixing vacuum".

We can check that these operators satisfy relations

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}, \psi_i\psi_j + \psi_j\psi_i = 0, \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0.$$

Therefore they generate a Clifford algebra $\mathcal{C}\ell$. It is clear that F is irreducible viewed as a $\mathcal{C}\ell$-module.

The following formulae are useful in computations:

$$[\psi_i^*\psi_j, \psi_k] = \delta_{kj}\psi_i^*, [\psi_i^*\psi_j, \psi_k] = -\delta_{ki}\psi_j.$$

3.3

Let $\mathfrak{g}$ be an infinite-dimensional tamely $\mathbb{Z}$-graded Lie algebra, i.e. $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is a $\mathbb{Z}$-graded Lie algebra with $\dim \mathfrak{g} = \infty, \dim \mathfrak{g}_n < \infty$. The restricted dual to $\mathfrak{g}$ is $\mathfrak{g}^* = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n^*$. Moreover, we assume the Lie algebra cohomology $H^2(\mathfrak{g}, \mathbb{C}) = 0$, i.e. $\mathfrak{g}$ has no nontrivial central extensions[7]. Examples include the Virasoro algebra and affine Lie algebras.

We still have the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, whence we derive again the Clifford algebra $\mathcal{C}\ell = \mathcal{C}\ell_{\mathfrak{g}} = \mathcal{C}\ell(\mathfrak{g} \oplus \mathfrak{g}^*)$. Let $\{e_i\}_{i\in\mathbb{Z}}$ be a homogeneous basis of $\mathfrak{g}$ with the property that $\deg e_j \geq \deg e_i$ for $j \geq i$. Denote by $\{e_i^*\}_{i\in\mathbb{Z}}$ the corresponding dual basis of $\mathfrak{g}^*$. We have the fermionic Fock space $F = \bigwedge^{\infty/2} \mathfrak{g}^*$. Here e_i^* is identified with $\underline{i}$ in previous notation. Then previous construction of creation and annilation operators makes F a $\mathcal{C}\ell$-module by a linear map ψ sending e_i^* to ψ_i^*, e_i to ψ_i.

3.4

Denote by ad (resp. ad*) the adjoint (resp. coadjoint) action of $\mathfrak{g}$ on $\mathfrak{g}$ (resp. $\mathfrak{g}^*$). In the finite-dimensional case, $\bigwedge^\bullet \mathfrak{n}^*$ is an $\mathfrak{n}$-module. We also want to make $\bigwedge^{\infty/2} \mathfrak{g}^*$ a $\mathfrak{g}$-module. For $x \in \mathfrak{g}_n, n \neq 0$, the action $\rho(x)$ is defined in the natural way:

$$\rho(x)e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots = \sum_{k\geq 1} e_{i_1}^* \wedge \cdots \wedge \operatorname{ad}^* x(e_{i_k}^*) \wedge \cdots .$$

The right hand sum is in fact a finite sum by the definition of semi-infinite forms. Moreover, we can check that for $y \in \mathfrak{g}, y^* \in \mathfrak{g}^*$,

$$[\rho(x), \psi(y)] = \psi(\operatorname{ad} x(y)), [\rho(x), \psi(y^*)] = \psi(\operatorname{ad}^* x(y^*)). \tag{1}$$

However, this definition does not hold for $\mathfrak{g}_0$ because an infinity may arise. To subtract the infinity, we first fix a vacuum

$$\omega_0 = e_{i_0}^* \wedge e_{i_0-1}^* \wedge e_{i_0-2}^* \wedge \cdots .$$

Here $e_{i_0}^* \in \mathfrak{g}_m^*$ for some m and $e_{i_0+1}^* \in \mathfrak{g}_{m+1}^*$. Then for an element $\beta \in \mathfrak{g}_0^*$, we define the action of $x \in \mathfrak{g}_0$ on the vacuum by

$$\rho(x)\omega_0 = \beta(x)\omega_0.$$

It can be extended to an action on $\bigwedge^{\infty/2} \mathfrak{g}^*$ by the same relations 1.

We define the Wick normal ordering by

$$:\psi_i\psi_j^*: = \begin{cases} \psi_i\psi_j^*, i \leq i_0 \\ -\psi_j^*\psi_i, i > i_0 \end{cases}.$$

[7]We do not require this in the finite-dimensional case. This is one of the basic differences between the finite and infinite dimensional cases. We can treat it more systematically using Tate's central extension.

Then the action ρ can be described explicitly:

$$\rho(x) = \sum_{i\in\mathbb{Z}} {:}\psi(e_i)\psi(\mathrm{ad}^* x(e_i^*)){:} + \beta(x) \text{ for } x \in \mathfrak{g}.$$

3.5

Lemma 5. We can suitably choose β so that $[\rho(x), \rho(y)] = \rho([x, y])$ for $x, y \in \mathfrak{g}$. Through this ρ, we make $\bigwedge^{\infty/2} \mathfrak{g}^*$ a $\mathfrak{g}$-module.

Proof. First notice that $[\rho(x), \rho(y)] - \rho([x, y])$ is always a scalar operator for $x, y \in \mathfrak{g}$. Denote by $\gamma \in \bigwedge^2 \mathfrak{g}^*$ the 2-form

$$\gamma(x, y) = [\rho(x), \rho(y)] - \rho([x, y]).$$

Observe that $\gamma(x, y) = 0$ for $x \in \mathfrak{g}_m, y \in \mathfrak{g}_n$ with $m + n \neq 0$. We check that γ is a 2-cocycle. This is equivalent to say that

$$\sum_{\mathrm{cyc}} \gamma([x, y], z) = 0 \text{ for } x, y, z \in \mathfrak{g}. \tag{2}$$

Since $[[\rho(x), \rho(y)] - \rho([x, y]), \rho(z)] = [\gamma(x, y), \rho(z)] = 0$, we can compute that

$$\begin{aligned}\sum_{\mathrm{cyc}} \gamma([x, y], z) &= \sum_{\mathrm{cyc}} \{[\rho([x, y]), \rho(z)] - \rho([[x, y], z])\} \\ &= \sum_{\mathrm{cyc}} [[\rho(x), \rho(y)], \rho(z)] - \rho\left(\sum_{\mathrm{cyc}} [[x, y], z]\right) = 0.\end{aligned}$$

Now since we have the assumption that $H^2(\mathfrak{g}, \mathbb{C}) = 0$, we can find a 1-form β' such that $\gamma = \partial\beta'$, where ∂ is the Chevalley–Eilenberg differential of $\mathfrak{g}$. Now we can substitute β by $\beta - \beta'$, and then γ vanishes. □

From now on we fix one such β and thus we fix a $\mathfrak{g}$-module structure for $\bigwedge^{\infty/2} \mathfrak{g}^*$.

3.6

We fix the charge of the vacuum ω_0 to be 0. For $x \in \mathfrak{g}$, the annilation operator $\psi(x)$ decreases the charge by 1 and for $x^* \in \mathfrak{g}^*$, the creation operator $\psi(x^*)$ increases the charge by 1. This charge decomposition gives a grading to $\bigwedge^{\infty/2+\bullet} \mathfrak{g}^*$, denoted by charge. For each $m \in \mathbb{Z}$, the subspace $\bigwedge^{\infty/2+m} \mathfrak{g}^*$ of charge m is a $\mathfrak{g}$-submodule.

Let $\mathcal{O}$ be the full subcategory of $\mathfrak{g}$-Mod consisting of modules that are locally $\bigoplus_{n>0} \mathfrak{g}_n$-finite.

Now let (V, L) be a $\mathfrak{g}$-module from $\mathcal{O}$. Then $\bigwedge^{\infty/2+\bullet} \mathfrak{g}^* \otimes V$ is a $\mathfrak{g}$-module by tensor product $\mathcal{L} = \rho \otimes 1 + 1 \otimes L$.

Theorem 2. There exists a unique linear map $d\colon \bigwedge^{\infty/2+\bullet} \mathfrak{g}^* \otimes V \to \bigwedge^{\infty+\bullet+1} \mathfrak{g}^* \otimes V$ of charge 1 such that for any $x \in \mathfrak{g}$, one has $\mathcal{L}(x) = d\psi(x) + \psi(x)d$. Moreover, one has $d^2 = 0$.

Proof. Uniqueness follows from a lemma similar to Lemma 1. As for existence, one can write down d explicitly:

$$d = \sum_{i\in\mathbb{Z}} \psi(e_i^*) \otimes L(e_i) + \frac{1}{2} \sum_{i,j\in\mathbb{Z}} {:}\psi([e_i, e_j])\psi(e_i^*)\psi(e_j^*){:} \otimes 1 + \psi(\beta) \otimes 1.$$

Here the Wick normal ordering is defined by

$$:\psi_i\psi_j^*\psi_k^*: = \begin{cases} \psi_i\psi_j^*\psi_k^*, i \le i_0 \\ \psi_j^*\psi_k^*\psi_i, i > i_0 \end{cases}.$$

$\square$

Now we have a cochain complex $(\bigwedge^{\infty/2+\bullet} \mathfrak{g}^* \otimes V, d)$, its cohomology $H^{\infty/2+\bullet}(\mathfrak{g}, V)$ is called the *BRST reduction* of V or the *semi-infinite cohomology* of $\mathfrak{g}$ with coefficients in V.

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