Serre – Tate theory and Igusa varieties

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Let $S' \twoheadrightarrow S$ be a surjection of rings in which p nilpotent,

$p^n S' = 0$

w/ nilpotent kernel $I \subset S'$.

$I^{\nu+1} = 0$

<u>Thm.</u> Equiv. of cats

(1) $\left(\text{p-div. gps up to isog.}/S' \right) \to \left(\text{p-div. gps up to isog}/S \right)$

$$G_{S'} \longmapsto G_{S'} \times_{S'} S$$

(2) $\left(\text{ab. schs up to p-power isog.}/S' \right) \to \left(\text{ab schs up to p-power isog.}/S \right)$

$$A_{S'} \longmapsto A_{S'} \times_{S'} S$$

<u>Pf</u>. (1) Let G be a p-div. gp$/S'$.

Its formal completion $\hat{G}$ along identity section is a <u>formal Lie gp.</u> ?

Let T' be an S'-alg. and $T = T' \otimes_{S'} S = T'/IT'$.

I nilpotent $\Rightarrow$ $\ker(G(T') \to G(T)) \xrightarrow{\sim} \underbrace{\ker(\hat{G}(T') \to \hat{G}(T))}_{\text{killed by } p^{m\nu}}$

In terms of coord. $x_1, \cdots, x_n$ of $\hat{G}$.

$\left([p^n].(x_1,\dots,x_r) \right)_i = p^n x_i + \left(\deg \geq 2 \text{ in } x_1,\dots,x_r \right)$

A point of $\ker(\hat{G}(T') \longrightarrow \hat{G}(T))$ has coord. $(t_1,\dots,t_r)$ w/ $t_i \in IT'$

$\left([p^n]).(t_1,\dots,t_r) \right)_i \in (IT')^2$

Induction $\rightsquigarrow$ $\left([p^{m\nu}][t_1,\dots,t_r) \right)_i \in (IT')^{2^\nu} \subset (IT')^{\nu+1}$

Faithful:

If $f \in \mathrm{Hom}(G_{1,S'}, G_{2,S'})$ reduces to 0 on S,

then $\forall\, T'$, the image of $f(T')$ lies in $\ker(G_{2,S'}(T') \to G_{2,S}(T))$

which is killed by $p^{m\nu}$ $\Rightarrow p^{m\nu} f = 0 \Rightarrow f = 0$.

Full: Let $f \in \mathrm{Hom}(G_{1,S}, G_{2,S})$. Let $x \in G_{1,S'}(T')$ w/ $\bar{x} \in G_{1,S}(T)$.

Since $G_{2,S}$ is formally sm. $/S'$, $\exists$ a lift $\tilde{f}(\bar{x})$ of $G_{2,S'}(T')$ of $f(\bar{x})$.

Since $p^{m\nu}$ kills $\ker(G_{2,S'}(T') \to G_{2,S}(T))$, the elt

$\tilde{g}(x) := p^{m\nu}\tilde{f}(\bar{x})$ is well-defined $\rightsquigarrow \tilde{g} \in \mathrm{Hom}(G_{1,S'}, G_{2,S'})$ lifting $p^{m\nu} f$.

Essential surj. [Illusie, Déformations de groupes de Barsotti-Tate (dáprès A. Grothendieck) Théorème 4.4]

(2) Full faithfulness proved in the same way. Essential surj.: [Mumford GIT]

Thm (Serre - Tate) Equiv. of cats

$$(\text{ab. sch.}/S') \longrightarrow (\text{ab. sch. } A_S/S,\ p\text{-div. gp } G_{S'}/S',$$
$$\rho : A_S[p^\infty] \xrightarrow{\sim} G_S \cdot \times_{S'} S\ )$$

$$A_{S'} \longmapsto (A_S,\ A_{S'}[p^\infty],\ id).$$

Pf. Faithfulness from injectivity of $\mathrm{Hom}(A_{1,S'}, A_{2,S'}) \to \mathrm{Hom}(A_{1,S}, A_{2,S})$

Full: Let $f : A_{1,S} \to A_{2,S}$ and $f^\infty : A_{1,S'}[p^\infty] \longrightarrow A_{2,S'}[p^\infty]$.

We can lift $p^{m\nu} f$ to $\tilde{g} : A_{1,S'} \longrightarrow A_{2,S'}$.

$\tilde{g}[p^\infty] = p^{m\nu} f^\infty$

$A_{1,S'}[p^{m\nu}] \subset \ker \tilde{g}$, $\tilde{g}$ is divisible by $p^{m\nu}$.

Essential surj. : take $(A_S, G_{S'}, \rho)$ and pick any ab. sch. $A'_{S'}$ over S'

w/ isog. $f : A'_S \to A_S$. Get an isog. $\rho \cdot f[p^\infty] : A'_S[p^\infty] \to G_S$.

Replacing f by $p^{m\nu} f$, $\rho \cdot f[p^\infty]$ lifts uniquely to an isog. $\tilde{\rho} : A'_{S'}[p^\infty] \to G_{S'}$.

Replace $A'_{S'}$ by $A'_{S'}/\ker(\tilde{\rho})$.

<u>Igusa variety</u> $(G = GSp, X)$ Shimura datum.

level $K = K_p K^p$, $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. $K^p \subset G(\mathbb{A}_f^p)$ small enough
open cpt

$\mathcal{S} = \mathcal{S}_{K_pK^p}$ integral model of the Shimura variety.

moduli space of $(A, \lambda: A \to A^\vee$ prime to p q-isog. $\bar{\eta})$

$(A, \lambda, \bar{\eta}) \xrightarrow{\sim} (A', \lambda', \bar{\eta}')$ is a prime-to-p q-isog. $f: A \to A'$

s.t. $f^\vee \circ \lambda' \circ f = c\lambda$ for $c \in \mathbb{Z}_{(p)}^*$ and $\bar{\eta} = \bar{\eta}' \circ f_*$.

$x \in \mathcal{S} \times_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \rightsquigarrow \mathbb{D}(A_x[p^\infty])[\frac{1}{p}]$

$b \in B(G) = G(L)/\sim$, $L = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$

$x \sim y \Leftrightarrow x = g y \sigma(g)^{-1}, \exists g \in G(L)$

σ Frob. of $L/\mathbb{Q}_p$

Fix a completely slope divisible p-divisible gp $\mathbb{X}_b$ over $\overline{\mathbb{F}}_p$ representing the corresponding isog. class.

$\mathbb{X}_b = \bigoplus_{i=1}^{r} \mathbb{X}_i$, $\mathbb{X}_i$ isoclinic p-div gps of strictly decreasing slopes $\lambda_i \in [0,1]$.

<u>Def</u>. $Ig^b / \operatorname{Spec} \overline{\mathbb{F}}_p : (\overline{\mathbb{F}}_p\text{-alg.}) \to (Set)$

$R \mapsto \{(A, \rho) : A \in \mathcal{S}(R), \rho: A[p^\infty] \xrightarrow{\sim} \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R\}$

preserving polarization up to $\mathbb{Z}_p^\times(R)$.

<u>Prop</u>. The functor Ig^b is representable by a scheme.

Pf. Enough to prove $Ig^b \longrightarrow \mathcal{S} \times_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$ is relatively representable.

Let $\mathcal{A}$ be the universal abelian var. $/ \mathcal{S}$

Each functor $T \mapsto \{ \rho : \mathcal{A}_T[p^n] \xrightarrow{\sim} \mathbb{X}_b[p^n] \times_{\overline{\mathbb{F}}_p} T \}$ representable by

a finite type affine sch. $/ \mathcal{S} \times_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$.

An inverse limit of schemes along affine transition maps is representable.

Lemma $Ig^b(R) = \{ (A, \rho) : A \in \mathcal{S}(R)$ up to p-power isog. respecting extra str.

$\rho : A[p^\infty] \dashrightarrow \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R$ q-isog. preserving pol. up to

$\underline{\mathbb{Q}_p}^\times (R) \} / \sim$

Pf. Given $A \in \mathcal{S}(R)$ w/ a q-isog. $\rho : A[p^\infty] \dashrightarrow \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R$, define

$A' = A / \ker (p^N \rho)$, $p^N \rho$ actual isog.

$\pi : A \to A'$ qt map, $A[p^\infty] \overset{p^{-N}}{\dashrightarrow} A[p^\infty] \xrightarrow{\pi(p^N)} A'[p^\infty] \overset{\rho'}{\xrightarrow{\sim}} \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R$

Given $\lambda_A : A \to A^\vee$ prime to p q-isog. and $c \lambda_A[p^\infty] = \rho^\vee \circ \lambda_{\mathbb{X}_b} \circ \rho$ for some

$c \in \underline{\mathbb{Q}_p}^\times (R)$

π isog. preserving pol. $\Rightarrow d \lambda_A = \pi^\vee \circ \lambda_{A'} \circ \pi$ for some $d \in \underline{\mathbb{Q}}^\times (R)$

$p^n \rho = \rho' \circ \pi$, where ρ' isom. of p-div. gp w/ pol.

$\Rightarrow p^{-2N} d c^{-1} \in \underline{\mathbb{Z}_p}^\times (R)$

d fixed up to a unit in $\mathbb{Z}_{(p)}^{\times}(R)$ uniqueness of $\lambda_{A'}$ also ensures

$\lambda_{A'}$ is prime to p.

Cor (1) The formal gp sch. $\mathrm{Aut}_G(\hat{\mathbb{X}}_b)$ where

$$\hat{\mathbb{X}}_b(S) = \varprojlim \left(\cdots \xrightarrow{p} \mathbb{X}_b(S) \xrightarrow{p} \mathbb{X}_b(S) \right)$$

acts continuously on Ig^b.

(2) Ig^b is perfect, i.e. Frobenius map is an automorphism.

Pf (1) follows from the Lemma by acting on ρ

(2) pulling back under Frob induces an equiv. on the cat. of abelian vars

(resp. p-div. gps) up to p-power isog. (resp. q-isog.)

Fix a lift $(\mathbb{X}_b)_{\breve{\mathbb{Z}}_p}$ of $\mathbb{X}_b$ up to q-isog. w/ polarization

$\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$

For $R \in \mathrm{Nilp}^{op}_{\breve{\mathbb{Z}}_p}$, $Ig^b_{\breve{\mathbb{Z}}_p}(R) = \{ (A, \rho) : A \in S(R), \rho : A[p^\infty] \xrightarrow{\sim} (\mathbb{X}_b)_{\breve{\mathbb{Z}}_p} \times_{\breve{\mathbb{Z}}_p} R$

respecting extra str. $\}$

Def. $\mathfrak{X}^b : \mathrm{Nilp}^{op}_{\breve{\mathbb{Z}}_p} \to \mathrm{Set}$

$$R \mapsto \{ (A, \rho) : A \in S(R),\ \rho : A[p^\infty] \times_R R/p \dashrightarrow \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R/p$$

q-isog. respecting extra str. $\}$

Rapoport - Zink space

$\mathcal{M}^b : \mathrm{Nilp}^{op}_{\breve{\mathbb{Z}}_p} \longrightarrow \mathrm{Set}$

$$R \longmapsto \{ (G, \rho) : G\ p\text{-div. gp} / R,\ \rho : \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R/p \dashrightarrow G \times_R R/p$$

q-isog. respecting extra str. $\} / \sim$

Fact $\mathcal{M}^b$ is representable by a formal scheme, formally smooth.

$$\begin{array}{ccccc} Ig^b_{\breve{\mathbb{Z}}_p} \times_{\breve{\mathbb{Z}}_p} \mathcal{M}^b & \longrightarrow & \mathfrak{X}^b & & (A, \rho) \\ \downarrow p_2 & & \downarrow & & \downarrow \\ \mathcal{M}^b & = & \mathcal{M}^b & & (A[p^\infty], \rho) \end{array}$$

$Ig^b_{\breve{\mathbb{Z}}_p} \times_{\breve{\mathbb{Z}}_p} \mathcal{M}^b \longrightarrow \mathfrak{X}^b$

Let $(A, \rho), (G, \rho') \in \left(Ig^b_{\breve{\mathbb{Z}}_p} \times_{\breve{\mathbb{Z}}_p} \mathcal{M}^b \right)(R)$

$A \in S(R)$, $\rho : A[p^\infty] \xrightarrow{\sim} (\mathbb{X}_b)_{\breve{\mathbb{Z}}_p} \times_{\breve{\mathbb{Z}}_p} R$

G p-div gp, $\rho' : G \times_R R/p \dashrightarrow (\mathbb{X}_b) \times_{\overline{\mathbb{F}}_p} R/p$

lifts uniquely to a q-isog. $\rho' : G \dashrightarrow (\mathbb{X}_b)_{\breve{\mathbb{Z}}_p} \times_{\breve{\mathbb{Z}}_p} R$ by Serre-Tate

$A \in S(R)$, $A[p^\infty] \dashrightarrow G$ composite

By Lemma, $\exists !$ q-isog. of p-power order $A \dashrightarrow A'$ s.t. $A[p^\infty] \dashrightarrow A'[p^\infty]$

gets identified w/ $A[p^\infty] \dashrightarrow G$

$(A', A'[p^\infty] \underset{R}{\times} R/p = G \underset{R}{\times} R/p \xrightarrow{\rho'} (\mathbb{X}_b) \underset{\mathbb{F}_p}{\times} R/p) \in \mathfrak{X}^b(R)$

$\mathfrak{X}^b \longrightarrow Ig^b_{\breve{\mathbb{Z}}_p} \underset{\breve{\mathbb{Z}}_p}{\times} \mathcal{M}^b$

Let $(A', \rho') \in \mathfrak{X}^b(R)$. then $(A'[p^\infty], \rho') \in \mathcal{M}^b(R)$.

$A' \in S(R)$. $\rho' : A'[p^\infty] \underset{R}{\times} R/p \dashrightarrow \mathbb{X}_b \underset{\mathbb{F}_p}{\times} R/p$

$A'[p^\infty] \dashrightarrow (\mathbb{X}_b)_{\breve{\mathbb{Z}}_p} \underset{\breve{\mathbb{Z}}_p}{\times} R$ by Serre-Tate

$\exists!$ q-isog. of p power order $A' \dashrightarrow A$ s.t. the induced q-isog. $\rho : A[p^\infty] \longrightarrow (\mathbb{X}_b)_{\breve{\mathbb{Z}}_p} \underset{\breve{\mathbb{Z}}_p}{\times} R$ isom

$(A, \rho) \in Ig^b_{\breve{\mathbb{Z}}_p}(R)$

(G, X) PEL type $\quad \mathcal{S} \quad \iota : \mathcal{O}_B \longrightarrow End(A) \otimes \mathbb{Z}_{(p)}$ satisfying some cond'n