Relative perversity

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Starting point: Naive question

Is there some "relative" variant of perverse sheaves?

Setup: fix a prime ℓ, always work w/ $\mathbb{Z}[\frac{1}{\ell}]$-schemes. (All schemes qcqs)

Three scenarios we consider:

A) Λ an ℓ-torsion ring, $D_{\acute{e}t}(X, \Lambda)$ = left-completion of $D(X_{\acute{e}t}, \Lambda)$ $= \lim_{n\to\infty} D^{\geq -n}$

B) Λ as in A), but look at $D_{cons}(X, \Lambda) \subset D_{\acute{e}t}(X, \Lambda)$.

C) take $L \mid \mathbb{Q}_\ell$ an alg ext'n, look at $D_{cons}(X, L \text{ or } \mathcal{O}_L)$ = objects which become dualizable on a constructible stratification of X

$\subset D_{pro\acute{e}t}(X, \Lambda)$

Just write $D(X)$ for the relevant category in each scenario.

Theorem (H.- Scholze) Let $X \to S$ be a finitely presented map. Let $D(X)$ be as in scenarios A)- C). (In scenario C, assume that all constr. subsets of S have finitely many irred comp.) Then there is a t-structure ${}^{p/S}D^{\leq 0}(X)$, ${}^{p/S}D^{\geq 0}(X)$ on $D(X)$ characterized by the condition that $A \in D(X)$ lies in ${}^{p/S}D^{\leq 0}$ resp. ${}^{p/S}D^{\geq 0}$ iff $\forall\ \bar{s} \to S$ w/ fiber $X_{\bar{s}} \xrightarrow{h} X$, h^*A lies in ${}^{p}D^{\leq 0}(X_{\bar{s}})$ resp. ${}^{p}D^{\geq 0}(X_{\bar{s}})$.

(h^* for both !!!)

This t-structure interpolates between two extremes

i) If $X \to S = X$ is the identity, just get the standard t-str. on $D(X)$

ii) If $S = \mathrm{Spec}\, k$ is a pt, get the usual perverse t-structure on $D(X)$.

In general, this t-structure has no good "finite length" properties.

However, it does have good properties along these lines after imposing another condition namely the condition of being ULA.

Fix $X \xrightarrow{f} S$, $A \in D(X)$.

Intuition: A is universally locally acyclic (ULA) w.r.t. f if the coh. of $A|_{\text{slices of a small ball in } X}$ is constant as the slices vary.

Def'n A is ULA w.r.t. f if $\forall\, \bar{x} \to X$, $\bar{t} \rightsquigarrow f(\bar{x})$ specialization, the nat'l map $R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{t}, A)$ is an isom, and likewise after any base change.

Key things: 1) $X \to S$ smooth, Λ is ULA.

2) In reasonable situations, "any" sheaf is ULA over a dense open in the target. (Deligne)

Thm Fix $X \xrightarrow{f} S$ as before.

1) If $A \in D(X)$ is f-ULA, then all relative perverse truncations of A are f-ULA

$\Rightarrow$ $\mathrm{Perv}(X/S)$ = heart of rel. t-str. on $D(X)$ comes w/ full subcat $\mathrm{Perv}^{ULA}(X/S)$.

2) $\mathrm{Perv}^{ULA}(X/S)$ is stable under relative Verdier duality (i.e. $R\mathcal{H}om(-, f^! \Lambda)$)

$\rightsquigarrow$ Eg. in case C w/ $\Lambda = L$, $\mathrm{Perv}^{ULA}(X/S)$ is noetherian and artinian.

3) $\mathrm{Perv}^{ULA}(X/S)$ stable under subquotients.

Key special case of main thm:

valuation ring w/ unique non-zero prime ideal

$S = \mathrm{Spec}\, V$, V a rank one valuation ring w/ alg closed fraction field

("AIC" valuation ring)

absolutely integrally closed

$|S| = \underset{s}{\bullet} \leftsquigarrow \underset{\eta}{\bullet}$ $\quad X \longrightarrow S$ as before. $j : X_\eta \rightarrow X$, $i : X_s \rightarrow X$

[HA 1.4.4.11]

By a result of Lurie, always can define a t-structure whose connective part consists of sheaves which lie in ${}^pD^{\leq 0}$ (every fiber).

In the present case, get

$$ {}^{p/s}D^{\leq 0}(X) = \left\{ A \in D(X) \text{ s.t. } \begin{array}{l} j^*A \in {}^pD^{\leq 0}(X_\eta) \\ \text{and } i^*A \in {}^pD^{\leq 0}(X_s) \end{array} \right\} $$

$\Rightarrow$ By general nonsense, $A \in {}^{p/s}D^{\geq 0}$ iff $j^*A \in {}^pD^{\geq 0}(X_\eta)$
and $i^!A \in {}^pD^{\geq 0}(X_s)$.

<u>Claim</u>. The latter pair of conditions is equiv. to : j^*A, i^*A $\in {}^pD^{\geq 0}$.

Key idea: look at the exact triangle

$$i^! A \longrightarrow i^* A \longrightarrow \underbrace{i^* j_* j^* A}_{{}^pD^{\geq 0}} \longrightarrow$$

Assume $j^* A \in {}^pD^{\geq 0}$, then $i^* j_* : D(X_\eta) \longrightarrow D(X_s)$ is the nearby cycle functor, which in particular is perverse t-exact.

↖ Thm. of Gabber. (Illusie '94)

The idea in general case is to reduce to this (very!) special case by descent arguments. For this, we need very fine topologies.

Recall. v-topology $\subsetneq$ top. of universal submersions $\subsetneq$ arc-topology

$X \longrightarrow Y$ cover if $\forall$ Spec $V \to Y$, V val. ring can lift after replacing V by some V'/V faithfully flat

$X \longrightarrow Y$ univ. submersion if $|X| \to |Y|$ is a qt map after any base change on Y

Same as v-topology, but only test via rank $(\leq)1$ val. rings

$$\begin{tikzcd} \exists \quad X \arrow[d] & \operatorname{Spec} V' \arrow[l, dashed] \arrow[d] \\ Y & \operatorname{Spec} V \arrow[l] \end{tikzcd}$$

Thm (Bhatt - Mathew, Gabber) In each of scenarios A) - C), $X \mapsto D(X)$ is a $\overset{\text{hyper}}{\text{v-sheaf}}$ of stable ∞-cats. In scenarios B), C), it is a sheaf for arc-topology (Bhatt-Mathew). In scenario A), it is a universally submersive sheaf. (Gabber)

Idea of pf 1) $S = \operatorname{Spec} V$ rk 1 AIC val. ring, OK.

2) $S = \operatorname{Spec} V$ AIC val. ring, reduce to the previous case by approx. & descent.

3) S has each conn'd comp. $\simeq \operatorname{Spec} V$, V AIC val. ring.

reduce to previous case by pure topology +

<u>Lemma</u>. "the perverse coh. amplitude is a constructible func. on the base".

4) General S. Key pt: can pick a v-hypercover $S_\bullet \to S$ w/ all S_n as in 3).

Then already have t-str. you want on all $D(X \times_S S_n)$, and ~~trans~~; the pullbacks $D(X \times_S S_n) \to D(X \times_S S_m)$ are rel. perv t-exact.

$\Rightarrow$ Formal to get the desired t-str. on $D(X) = \lim\limits_{n \in \Delta} D(X \times_S S_n)$.

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S conn'd, $\bar{s} \to S$, S $\mathbb{Q}$-scheme or $\bar{s}$ dominates generic pt

$\mathrm{Perv}^{\mathrm{ULA}}(X/S) \longrightarrow \mathrm{Perv}(X_{\bar{s}})$ fully faithful.