Perverse sheaves and their categorification

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Lecture 1. Perverse sheaves as objects of mixed functoriality.

① What are they informally?

Goresky – MacPherson 1977 : Intersection homology

$IH_\bullet$ (Singular $\mathbb{C}$-vars) w/ Poincaré duality

Cohomology	midway $IH_\bullet$	Homology
sheaves $\mathcal{F}$	$\boxed{\text{1/2-sheaves}}$	Cosheaves
 $\mathcal{F}(U) \to \mathcal{F}(V)$	formalized as perverse sh.	$\mathcal{F}(V) \to \mathcal{F}(U)$

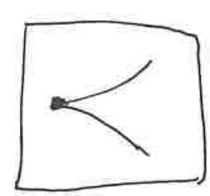
E.g. functions — $\boxed{\text{Half-densities}}$ — distributions, generalized volume forms

$\sqrt{Vol}$, $K^{1/2}$

Form a Hilbert space

② Formal def. [BBD] 1980.

(X $\mathbb{C}$-mfd, $S = \{X_\alpha\}$ $\mathbb{C}$-strat.) stratified $\mathbb{C}$-mfd.



X_α smooth $\mathbb{C}$-mfds, not necessarily closed

$Sh(X,S)$ $\quad$ S-constructible sheaves $\mathcal{F}$. $\mathcal{F}|_{X_\alpha}$ loc. const.

$D(X,S)$ = constructible complexes $\{ \cdots \to \mathcal{F}^{-1} \xrightarrow{d} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \to \cdots \}$

$\circlearrowleft$

* Verdier duality $\qquad$ $\underline{H}^i(\mathcal{F}^\bullet)$ constructible.

* $Sh(X,S) = Cosh(X,S)$

* $\circlearrowleft$ $Perv(X,S) \subset D(X,S)$

$Perv(X,S)$ defined by conditions on $\underline{H}^i(\mathcal{F})$.

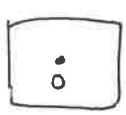
- $\underline{H}^0$ sits on $\dim_{\mathbb{C}} 0$
- $\underline{H}^{-1}$ on $\dim_{\mathbb{C}} \leq 1$
- $\underline{H}^{-2}$ on $\dim_{\mathbb{C}} \leq 2$

& another dual set of conditions

Old dream: give a definition w/o derived cats. & w/o analysis.

Riemann-Hilbert correspondence: $Perv(X,S) \simeq$ holonomic regular $\mathcal{D}$-modules w/ char. var. bounded by S.

$\left(\subset \bigcup_\alpha \overline{T^*_{X_\alpha} X} \right)$

Ex. $Perv(\mathbb{C}, 0)$ 

$\mathcal{F} \xleftrightarrow{1:1} \left\{ \Phi \underset{b}{\overset{a}{\rightleftarrows}} \Psi \text{ s.t. } T_\Phi = 1 - ba,\ T_\Psi = 1 - ab \text{ are isomorphisms} \right\}$

Φ = space of vanishing cycles, Ψ = space of nearby cycles

These make ϕ, ψ into loc. systems on S^1.

<u>Exercise</u>: Such data $\overset{1:1}{\longleftrightarrow}$ comm. diagrams

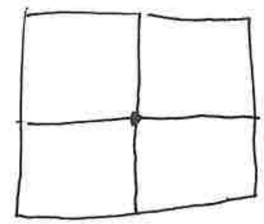
$$\begin{tikzcd}
E_{-+} \arrow[d, "\wr"] & E_{0+} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & E_{++} \arrow[d, "\wr"] \\
E_{-0} & E_{00} \arrow[l] \arrow[r] & E_{+0} \\
E_{--} \arrow[u, "\wr"] & E_{0-} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & E_{+-} \arrow[u, "\wr"]
\end{tikzcd}$$

ψ

$E_{00} = \phi \oplus \psi$

cosheaf

sheaf in this direction

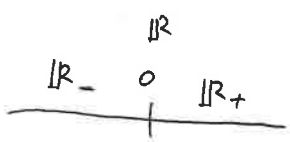


9 cells

<u>Thesis</u>: Perverse sheaves $\simeq$ sheaves in Re-direction, cosheaves in Im-direction.

④ <u>Cellular sheaves</u>

$X, \; S = \{U_\alpha\}_{\alpha \in A}$
cells $\simeq \overset{\circ}{B}{}^n$ some



$Sh(X, S) = \{ S\text{-constr. sheaves} \; . \; F|_{U_\alpha} \text{ constant} \}$

Assume S is <u>quasi-regular</u>. $(\overline{U_\alpha} \subset \overline{B}{}^n)$

$\Rightarrow F_\alpha = \Gamma(U_\alpha; F)$ stalk

OK NOT OK

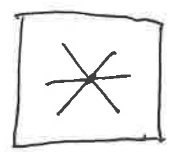
$U_\beta \subset \overline{U_\alpha} \rightsquigarrow \gamma_{\beta\alpha} : F_\beta \longrightarrow F_\alpha$ generalization

Classical: $Sh(X, S) = \mathrm{Rep}\left(\underset{\text{cell closure inclusions}}{(A, \leq)} \to Vect \right)$

$Cosh(X, S) = \mathrm{Rep}\left(\underset{\text{Specialization}}{(A, \geq)} \to Vect \right)$

⑤ Janus sheaves: hyperplane arrangements

$\mathbb{R}^n \supset \mathcal{H}$ arrangement of hyperplanes through 0



13 faces

$\mathbb{C}^n \supset \mathcal{H}_{\mathbb{C}}$ complexification

$S^{(0)}$ = Stratif. of $\mathbb{C}^n$ by open flats

$\Big(\underbrace{L = \bigcap_{H \in I \subset \mathcal{H}} H_{\mathbb{C}}}_{\text{flat}}\Big) \setminus$ all proper subflats

$Perv(\mathbb{C}^n, \mathcal{H}) = Perv(\mathbb{C}^n, S^{(0)}) = ?$

$\mathcal{C} = \{ \text{faces of } \mathcal{H} \text{ in } \mathbb{R}^n \}$ cell dec. of $\mathbb{R}^n$

$S^{(2)} = \mathcal{C} \times \mathcal{C}$ = product cell decomp. of $\mathbb{C}^n$

$\{A + iB\}_{A, B \in \mathcal{C}}$ refines $S^{(0)}$

$\leq$ splits into $\underset{\text{horiz.}}{\leq'}$ $\underset{\text{vert}}{\leq''}$ Anodyne ineq. When in same $S^{(0)}$-strat. ?

$$Sh(\mathbb{C}^n, S^0) = \underset{\text{ano - iso}}{Rep}\left((\mathcal{C}\times\mathcal{C}, \leq) \to Vect \right)$$

Def A Janus sheaf (diagram) is a datum of

- Vect. spaces $E_{A,B}$ $\forall A, B \in \mathcal{C}$
- $E_{A,B} \xrightarrow{\delta'} E_{C,B}$ for $A \leq C$ transitive
- $E_{A,B} \xleftarrow{\delta''} E_{A,D}$ for $B \leq D$ transitive
- compatibility : the whole thing commutative $A \leq C$, $B \leq D$

  comm. sq.
- Ano $\to$ Iso.

Th (K.-Schechtman) $Perv(\mathbb{C}^n, \mathcal{H}) \simeq$ such Janus sheaves.

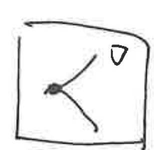
⑦ Janus sheaves : symmetric products

$$Sym^n(\mathbb{C}) = \mathbb{C}^n / \mathfrak{S}_n$$

- Space of monic polynomials

  discriminant

$$\{ f(z) = z^n + a_1 z^{n-1} + \dots + a_n \} = \mathbb{C}^n \supset \nabla = \{ f : \Delta(f) = 0 \}$$



- of effective divisors

$$D = \sum_{z \in \mathbb{C}} n_z \cdot z, \quad n_z \in \mathbb{Z}_{\geq 0}, \quad \sum n_z = n$$

stratif. by multiplicity

$\{$contingent matrices of content $n\}$ $= CM_n = \coprod CM_n(p,q)$ $p \times q$ mat.

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1q} \\ \vdots & & \vdots \\ m_{p1} & \cdots & m_{pq} \end{bmatrix}$$

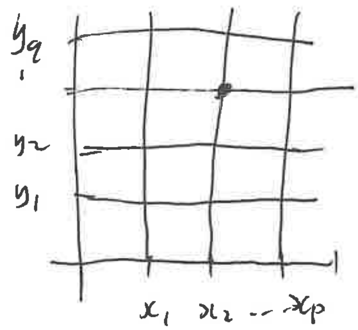
$m_{ij} \in \mathbb{Z}_{\geq 0}$,

$\sum m_{ij} = n$

no zero row or column

label a cell decomposition of $\mathrm{Sym}^n \mathbb{C}$.

$D = \sum n_z \cdot z \rightsquigarrow$ matrix $M(D)$



$x_\nu = \mathrm{Re}(z : n_z \neq 0)$

$y_\nu = \mathrm{Im}(z : n_z \neq 0)$

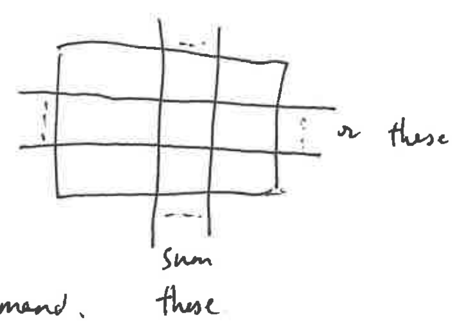
$$M(D)_{ij} = n_{x_i + \sqrt{-1}\, y_j}$$

$$U_M = \{D : M(D) = M\}.$$

$S^{(2)} = \{U_M\}$ cell decomp. refines $S^{(0)}$

Partial order splits into $\leq'$, $\leq''$.

by horiz. or vert. contractions



Anodyne: when in any sum, ≤ 1 nonzero summand.

Eg $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$||$

<u>Janus sheaf</u>: $M \longmapsto E_M$

$\begin{matrix} \cap \\ CM_n \end{matrix}$

δ' for $\leq'$, δ'' for $\leq''$

covariant contravariant

+ compatibility $M \geq'' N \leq' P$

$$\underset{\rightarrow}{\delta'_{NP}} \; \underset{\leftarrow}{\delta''_{MN}} = \sum_{M \leq' Q \geq'' P} \delta''_{P \leftarrow Q} \, \delta'_{M \to Q}$$

— Ano – iso.

<u>Thm</u> $\mathrm{Perv}(\mathrm{Sym}^n \mathbb{C}, S^{(0)}) = \{\text{such Janus sheaves}\}$.

⑧ <u>Example</u>: <u>Hopf alg.</u> / field k

$A = \bigoplus_{n \geq 0} A_n$, $A_0 = k$

$\mu_{mn}: A_m \otimes A_n \longrightarrow A_{m+n}$ $\mu_{0,n} = \mu_{n,0} = \mathrm{id}$

$\Delta_{mn}: A_{m+n} \longrightarrow A_m \otimes A_n$ $\Delta_{0,n} = \Delta_{n,0} = \mathrm{id}$

$\rightsquigarrow$ Janus sheaf on each $\mathrm{Sym}^n(\mathbb{C})$

$E_M = \bigotimes A_{m_{ij}}$ $\qquad \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \leq' \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$M \leq' N$ $\qquad \delta_{M,N}$: comultipl.

$$E_M \to E_N$$

$M \leq'' N$ $\qquad \delta_{NM} : E_N \to E_M$ mult.

Compatibility holds.

$A_p \otimes A_q \xrightarrow{\mu_{pq}} A_{p+q} \xrightarrow{\Delta_{rs}} A_r \otimes A_s$ $\qquad$ composition

$$= \sum_{\substack{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \to \begin{bmatrix} p \\ q \end{bmatrix} \\ \downarrow \\ [r \ s]}} (\mu_{m_{11},m_{12}} \otimes \mu_{m_{21},m_{22}}) \circ (\Delta_{m_{11},m_{21}} \otimes \Delta_{m_{12},m_{22}})$$

Get $\underbrace{\mathcal{F}_n}_{(E_M)_{M \in CM_n}}$ for $\forall n \in \mathrm{Perv}(\mathrm{Sym}^n(\mathbb{C}))$

<u>Thm</u> (1) $(\mathcal{F}_n)$ is <u>factorizable</u>. $\forall$ disjoint $U_1, U_2 \subset \mathbb{C}$

- $\mathrm{Sym}^{p+q}(U_1 \amalg U_2) \supset \mathrm{Sym}^p(U_1) \times \mathrm{Sym}^q(U_2)$

$$\mathcal{F}_{p+q}\big|_{\mathrm{Sym}^p(U_1) \times \mathrm{Sym}^q(U_2)} \simeq \mathcal{F}_p\big|_{\mathrm{Sym}^p(U_1)} \boxtimes \mathcal{F}_q\big|_{\mathrm{Sym}^q(U_2)}$$

(2) All factorizable perverse sheaves are obtained this way.

Lecture 2. Categorification of perverse sheaves

① Categorification in general

$$\begin{tikzcd} \text{Triangulated categories } /k \arrow[r, "K_0"', "\text{Grothendieck group}"'] & \text{Vector spaces } /k \arrow[l, dashed, bend right, "\text{lift data}"'] \end{tikzcd}$$

$\sim$ {complexes of some kind}

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, "+1"] & \end{tikzcd}$$

$$\mathcal{V} \longmapsto \bar{V} = K_0(\mathcal{V}) \otimes k$$

$[A]$ generators

$[A] - [B] + [C] = 0$

$Hom(A, B)$ – vector space

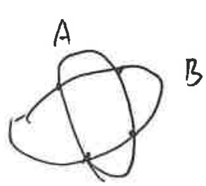
or a complex (dg enhanced cat.) $\qquad (a, b)$ bilinear form

② Ex. Fukaya category as categorification of (co)homology

$M \quad 2n$-dim C^∞-mfd, cpt

$V = H_n(M; \mathbb{R}) \simeq H^n(M; \mathbb{R}) \qquad (a, b)$ intersection form

n-dim. cycles $\qquad \# A \cap B$



$\overset{\text{lift}}{\rightsquigarrow}$ (M, ω) symplectic $\quad Fuk(M)$ category

Ob = Lagrangian varieties

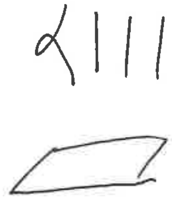
$$Hom^\bullet(A, B) = \Big(\bigoplus_{x \in A \cap B} k_x, \text{ grading}, d_{Floer} \Big)$$

categorifies part of H_n

③ <u>Coefficients</u> ?

For usual coh., $\exists\ H^\bullet(X, \mathcal{F})$ ← sheaf or complex of sh.

Typical use: in fibrations



$$\begin{array}{c} Y \\ \downarrow p \\ X \end{array}$$

$$H^\bullet(Y;\mathbb{R}) \overset{\text{Leray SS}}{\Leftarrow} \underbrace{H^i(X, R^j p_*(\underline{\mathbb{R}}_Y))}_{\text{direct images}}$$

<u>For Fukaya ?</u>

sheaves of categories ?

$U \longmapsto \mathcal{C}(U)$ cat.

$\mathcal{C}(U) \longrightarrow \mathcal{C}(V)$

concept known as <u>stack</u>.

<u>Standard</u>.

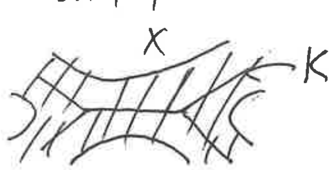
<u>Not quite</u>, perverse sheaves!

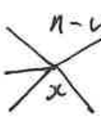
④ <u>Why perverse</u> ?

Kontsevich's localization on a Lagrangian skeleton $\rightsquigarrow$ $\mathcal{R}_K$ a sheaf / stack of categories on K.

Say $\dim_{\mathbb{R}} = 2$. X surface, $\partial X \neq \phi$

$X \supset K$ spanning graph



Stalk at  $= D\mathrm{Rep}(A_{n-1}\text{-quiver}) \leftarrow K_0 \simeq \mathbb{Z}^{n-1}$

Rot ↻ $\qquad 1 \to 2 \to \cdots \to n-1$ ↻ Coxeter functor

$$\mathrm{Fuk}(X) \overset{\text{Kontsevich}}{\simeq} R\Gamma(K, \mathcal{R}_K)$$

Cohom.

<u>Observe</u>: $\mathcal{R}_K$ is a categorification of $\underline{H}^1_K(\underline{k}_X)$ coh. w/ supp.

$\underline{H}^{\neq 1}_K = 0$

 rk at $x = n-1$

$\underline{k}_X[1]$ is perverse

has only $\underline{H}^0_K$.

<u>Prop</u>. For a constructible (w.r.t. $A \subset X$, some points) complex $\mathcal{F}$ on X, TFAE:

(i) $\forall$ graph $K \subset X$, $\underline{H}^{\neq 0}_K(\mathcal{F}) = 0$

(ii) $\mathcal{F}$ is perverse.

⑤ How to categorify perv. sheaves?

Term: perv. schobers

Def of perv. sheaves as complexes not good;

"Elementary" descriptions useful.

$\underline{Ex}$. $X = \underset{\text{disk}}{D} \subset \mathbb{C}$

$S = \{0\} \sqcup D - \{0\}$.

$\text{Perv} \leftrightarrow \left\{ \Phi \underset{b}{\overset{a}{\rightleftarrows}} \Psi \right\}$ $\begin{matrix} 1-ab \\ 1-ba \end{matrix}$ invert.

Categorifies to: <u>spherical functors</u>

$D_0 \underset{a^*}{\overset{a}{\rightleftarrows}} D_1$ enhanced triang. cat.

(R. Anno, T. Logvinenko)

unit
counit $aa^* \longrightarrow Id_{D_1}$, $a^*a \longleftarrow Id_{D_0}$

a called spherical $\iff$ cones of these are equiv. of cats.

⑥ <u>Schobers on surfaces</u>

$\underset{\text{top. surface}}{X} \supset \underset{\text{finite set}}{A}$ $S = \{X - A, \text{ points of } A\}$

Naively: a (schober on (X, A)) = datum

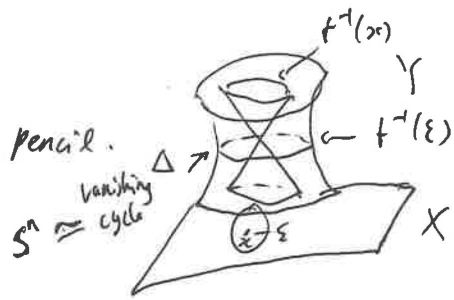
- A local system of triang. cat. on $X - A$
- A spherical functor datum near $\forall x \in A$
- glued compatibility

$T_{D_1} = \text{cone}(aa^* \to Id_{D_1})$

$\underline{Ex}$. $X = \mathbb{C}$, $f : \overset{\text{Kähler}}{Y^{n+1}} \longrightarrow X$ holomorphic Lefschetz pencil.

$\mathcal{L}_f$ Lefschetz schober on X

$A = \{\text{singular values}\}$



- stalk at $x \notin A$ $=$ Fuk$(f^{-1}(x))$

Φ at $x \in A$ $\simeq D(\mathrm{Vect}_k) \longrightarrow \underset{\mathrm{Fuk}(f^{-1}(\varepsilon))}{\psi}$

$k \longmapsto \Delta$

$\boxed{\text{categorifies certain perv. sheaf } L_f.}$

⑦ Corolla data for R_K(schober)?

$X \supset K$ $\quad \mathfrak{S}$ schober

Want: a sheaf of categories $R_K(\mathfrak{S})$ on K categorifying $\mathbb{H}^0_K$(perv. sheaf)

Answer: this is Waldhausen S-construction.

⑧ S-construction (used in alg. K-theory)

Originally: for abelian cat. $\mathcal{A}$ (e.g. R-Mod)

Naively: $\forall n$, $S_n^{\text{naive}}(\mathcal{A})$ = cat. of filtered obj. $A_1 \subset A_2 \subset \cdots \subset A_n$ isoms.

$\partial_i : S_n^{\text{naive}} \longrightarrow S_{n-1}^{\text{naive}}$, $i = 0, \ldots, n$

$\partial_{i \neq 0}$ = dropping A_i, $\partial_0(A_1 \subset \cdots \subset A_n) = A_2/A_1 \subset A_3/A_1 \subset \cdots \subset A_n/A_1$.

Subtlety: simplicial identities hold not strictly.

understand isom.

$$(A_3/A_1)/(A_2/A_1) \underset{\text{isom.}}{\overset{?}{=}} A_3/A_2$$

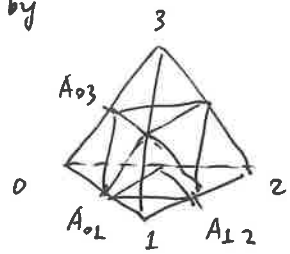
MK learned this from [Hinich-Schechtman]

G. Segal : use all $A_{ij} = A_j/A_i$, $0 \le i < j \le n$ ($A_0 = 0$) as independent as part of structure.

$S_n(\mathcal{A})$ = cat. of diagrams formed by

A_{ij} $0 \le i < j \le n$

+ morphisms + SES



$$0 \to A_{ij} \to A_{ik} \to A_{jk} \to 0$$

$0 \le i < j < k \le n$ octahedron.

For triangulated cat: similarly SES $\rightsquigarrow$ exact triang.

Important : $S_\cdot(\mathcal{V})$ is not just simplicial, but paracyclic.

On $S_n(\mathcal{V}) \circlearrowleft$ Rot τ_n $\quad \tau_n^{n+1}$ = shift by 2.

$\hookrightarrow$ $n+1$ -valent

$S_2(\mathcal{V})$ = {exact. triangles}

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, "+1"] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] & C \arrow[dl] \\ A[1] \arrow[u, "+1"] & \end{tikzcd}$$

Relative S-constr. Waldhausen.

$F : \mathcal{A} \to \mathcal{B}$ exact

$$\begin{tikzcd} S_n(F) \arrow[r] \arrow[d] & S_{n+1}(\mathcal{B}) \arrow[d, "\partial_{n+1}"] \\ S_n(\mathcal{A}) \arrow[r, "f_*"'] & S_n(\mathcal{B}) \end{tikzcd}$$

defined as fiber product

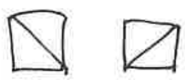
Properties [DKS, S, 2106.02873]

(1) If F spherical, then $S_\bullet(F)$ is paracyclic.

(2) $S_\bullet(F)$ is <u>2-Segal</u>.

Map ( $\to S_\bullet(F)$) is indep. of triang.

triang. polygon



This translates into:

(1) For a schober $\mathfrak{S}$, the sheaf $R_K(\mathfrak{S})$ on K w/ stalks $S_n(F)$ is well-defined

(2) $R\Gamma(K, R_K(\mathfrak{S}))$ is indep. on spanning $K \supset A$.

$\uparrow$

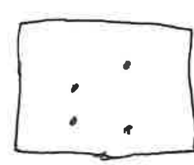
this can be called $\mathrm{Fuk}(X, \mathfrak{S})$

Lecture 3. Fourier transform for schobers and the Algebra of the Infrared.

① FT for functions, D-modules and perverse sheaves.

$f(w)$ holomorphic in $\mathbb{C}_w$, possibly multivalued e.g. $\sqrt{\frac{z-1}{z+i}}$

$\mathbb{C}_w$



$\check{f}(z) = \int_\gamma f(w)\, e^{-zw}\, dw$

$\gamma = \gamma_z$ contour s.t. $\int$ converges

(exp. decays

many choices

$\check{f}$ again multivalued

Formal FT: isomorphism of rings

$$D_w = \mathbb{C}\langle w, \partial_w \rangle \xrightarrow{FT} D_z = \mathbb{C}\langle z, \partial_z \rangle$$

$$w \longmapsto \partial_z$$

$$\partial_w \longmapsto -z$$

f satisfies $P(f) = 0 \underset{\text{if all is good}}{\Rightarrow} \check{P}(\check{f}) = 0.$

M left D_w-module.

$Sol(M) = \underline{RHom}_D(M, \mathcal{O}_{\mathbb{C}_w})$ solution complex of sheaves

$\underline{H}^0 = \underline{Hom}$ = sheaf of sol.

$M = D/D \cdot P$. $\underline{Hom}_D(M, \mathcal{O}) = \{f : P(f) = 0\}$

$\underline{H}^1 = \text{coker}(\underline{P})$

$\{\mathcal{O} \xrightarrow{P} \mathcal{O}\}$

Riemann-Hilbert correspondence:

(1) M is holonomic (such as D/DP) $\Rightarrow$ $Sol(M)$ is perverse

(2)
$$\begin{tikzcd} D\text{-}Mod^{hol} \arrow[r, "Sol"] & Perv(\mathbb{C}) \\ D\text{-}Mod^{h.reg} \arrow[u, hook] \arrow[ur, "\sim"'] & \end{tikzcd}$$

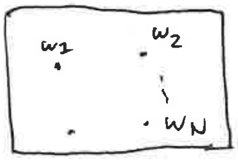
Regular means solution grows $\leq$ polynomially

e.g. $f' = f$, $f = e^w$ irregular at ∞.

FT for perverse sheaves

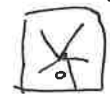
holonomic, irregular

$$\mathrm{Perv}(\mathbb{C}) \ni \mathcal{F} \xleftarrow[\text{Sol}]{(2)} \underset{\substack{\text{hol.}\\ \text{reg.}\\ \text{over } \mathcal{D}_w}}{\mathcal{M}} \xrightarrow{FT} \underset{\text{over } \mathcal{D}_z}{\check{\mathcal{M}} = FT^*(\mathcal{M})} \longrightarrow \underset{\substack{\shortparallel \\ \mathrm{Sol}(\check{\mathcal{M}})}}{\check{\mathcal{F}}}$$



$A \subset \mathbb{C}_w$ sing pts

<u>Malgrange</u> : (1) $\check{\mathcal{F}} \in \mathrm{Perv}(\mathbb{C}_z, 0)$



$\mathbb{C}_z$

Sing. of $\check{\mathcal{M}}$ at 0 regular, at ∞ irregular.

<u>"irregularity"</u> can be described.

$\zeta \in S^1$

<u>Stokes filtration</u> on space of sol. on each ray $\mathbb{R}_+ \cdot \zeta$

$\forall \lambda$, which grow $\leq e^{\lambda R}$ on $R\zeta$, $R \to \infty$.

<u>Not realized before</u> : a <u>Good</u> answer involves convex geometry of A. & schoberizes

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② What is "Algebra of IR" (Gaiotto- Moore- Witten)

Physical theory $\xrightarrow{\text{IR limit}}$ Vacua + tunnelling between them

2d SUSY

$\{\text{Vacua}\} = \mathbb{V}$ typically finite

$$\begin{array}{ccc} & \{v_1, \cdots, v_N\} & \\ \text{central charge of SUSY alg} \downarrow & \downarrow \text{typically embedding} & \\ \mathbb{C} & \supset & A \end{array}$$

$v_i \longmapsto$ local D-brane cat. $\mathcal{D}_i$

IR formalism

Global D-brane cat. corresponding to a half-plane

Tunnelling $\longmapsto$ functors $T_{ij} : \mathcal{D}_i \longrightarrow \mathcal{D}_j$

Our interpretation [K.– Soibelman – Soukhanov]

These data describe a schober $\mathfrak{S}$ w/ sing. at A

$D_i = \Phi$ cat. of vanishing cycles

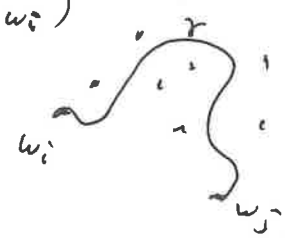
Ex. Landau–Ginzburg theory assoc. to $W : X \xrightarrow{\text{hol., proper}} \mathbb{C}$, $\mathfrak{S} = \mathcal{L}_W$ Lefschetz schober

Kähler
CY

③ Picard – Lefschetz theory for perverse sheaves

$\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A = \{w_1, \cdots w_N\})$

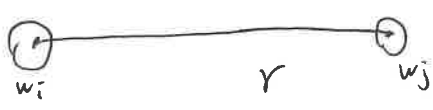
Φ_i space of vanishing cycles on w_i (loc. sys. on $S^1_{w_i}$)



γ path from w_i to w_j avoiding other w_k.

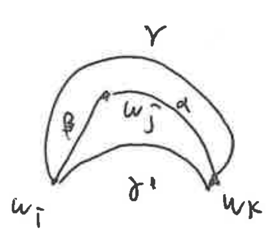
$t_{ij}(\gamma) : \Phi_i \to \Phi_j$ transport map

$$\Phi_i \xrightarrow{a_i} \Phi_i \xrightarrow[\text{par. transl.}]{} \Phi_j \xrightarrow{b_j} \Phi_j$$



depends on γ

Elementary move

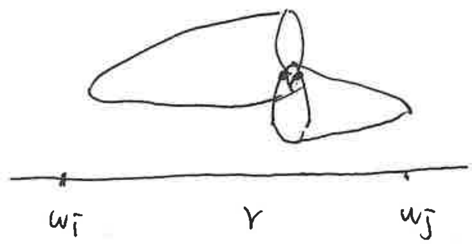


PL identity: $t_{ik}(\gamma') = t_{ik}(\gamma) - t_{jk}(\alpha)\, t_{ij}(\beta)$.

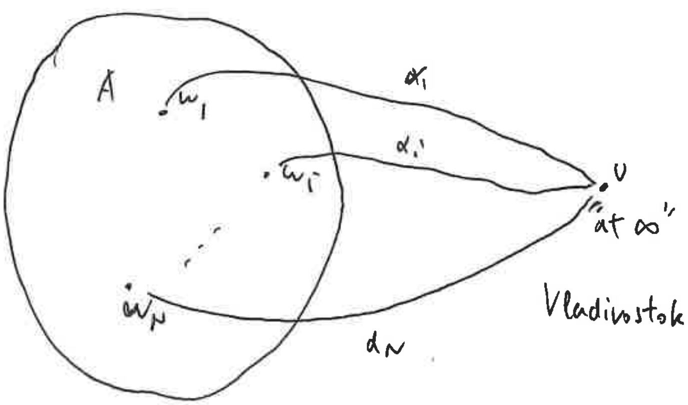
Usual PL theory for $\mathcal{F} = L_W$ Lefschetz perv. sh.

$\Phi_i \simeq k$

$t_{ij}(k) \in \mathbb{Z}$ int. # of thimbles



⊕ Perv $(\mathbb{C}, A)$ explicitly per S. Gelfand, MacPherson, Vilonen '96



choose "spider"

set of non intersecting paths

$$\mathcal{F} \xleftrightarrow{GMV} \text{datum of} \quad \begin{matrix} \Phi_1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} \\ \vdots \\ \Phi_N \underset{b_N}{\overset{a_N}{\rightleftarrows}} \end{matrix} \Psi = \mathcal{F}_v$$

Perv $(\mathbb{C}, A)$ / {const. sheaves}

$\{ \phi_i \to \phi_j ,\ t_{ij} \}$

gives $t_{ij}^{\text{Vlad}} : \phi_i \xrightarrow{a_i} \psi \xrightarrow{b_j} \phi_j$

Physically, t_{ij}^{straight} more immediate.

Our thesis: GMV = GMW.

⑤ PL theory for schobers

Φ_i : categories

$T_{ij}(\gamma): \Phi_i \to \Phi_j$ transport functors

PL triangle :

$$T_{jk}(\alpha)\, T_{ij}(\beta) \to T_{ik}(\gamma) \to T_{ik}(\gamma')$$

Again, can reduce to GMV data

$$\begin{tikzcd} \Phi_1 & & \Psi \arrow[ll, "b_1"'] \arrow[dll, "b_N"] \\ \Phi_N & & \end{tikzcd}$$

N spherical functors

Rectilinear $T_{ij}[w_i, w_j] \xleftrightarrow{\text{our interpretation}}$ Tunnelling functors of algebra of IR

⑥ Stokes data for $\check{\mathcal{F}}$: baby IR algebra
perv. sh.

Malgrange : $\check{\mathcal{F}} \in \mathrm{Perv}(\mathbb{C}, 0) \longleftrightarrow \{\check{\Phi} \rightleftarrows \check{\Psi}\}$

$\check{\Phi} = \Psi_{\mathcal{F}}$

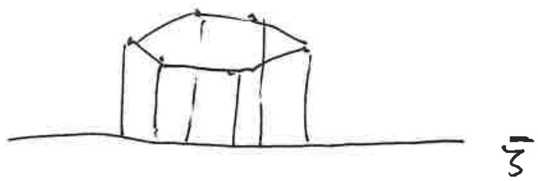
$\check{\Psi} = \Phi_1 \oplus \cdots \oplus \Phi_N$ not canonically

$\check{\Psi}_\zeta =$ stalk at $\zeta \in S^1_\infty = \mathbb{H}^0($  $, \mathcal{F})$

shifted ½ plane of decay

carries Stokes ~~filtration~~ filtration.

$\zeta \longmapsto$ order $\leq_\zeta$ on $\{1, \cdots N\}$ by decay of exp



Switches at $\zeta_{ij} = \mathrm{dir}(z_i \to z_j)$

$\check{\Psi}_\zeta$ filtered by $(\{1, \dots, N\}, \leq_\zeta)$

$\{$such data$\}$ = $H^1(S^1$, some sheaf of non ab. groups$)$

$$\Big\updownarrow$$

by one Stokes matrix

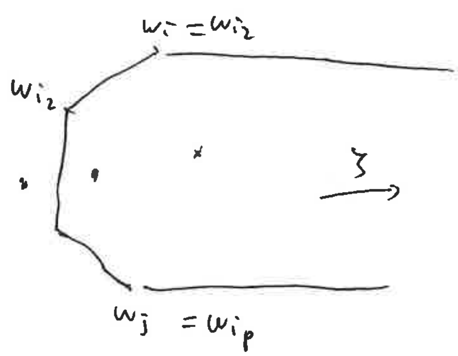
fix ζ $\qquad C : \bigoplus \phi_i \longrightarrow \bigoplus \phi_i$ upper triangular w.r.t. $\leq_\zeta$.

$$\begin{pmatrix} C_{ij} : \phi_i \longrightarrow \phi_j \\ i \leq_\zeta j \end{pmatrix} = ?$$

in Vladivostok picture

Studied by Malgrange, Mochizuki, O'Agnolo, Kashiwara.
1705.07610

<u>Our approach</u>:



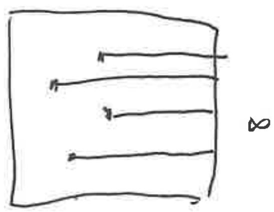
$C_{ij} = ?$

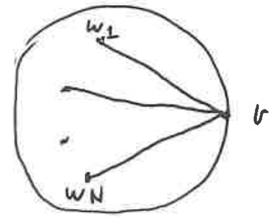
look at ζ-convex paths from w_i to w_j

Thm [K-S-S] $c_{ij} = \sum_{\substack{\text{such} \\ \text{paths}}} \overset{\text{straight}}{t_{i_{p-1} i_p}} \cdots \overset{\text{straight}}{t_{i_1 i_2}}$

⑦ Fukaya - Seidel cat. w/ coeff. in a schober $\mathfrak{S}$.

depends on 1/2 plane — dir. $\zeta \in S^1_\infty$





Vlad. spider

$$\begin{array}{l} \Phi_1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} \Psi \\ \vdots \\ \Phi_N \underset{b_N}{\overset{a_N}{\rightleftarrows}} \Psi \end{array}$$

$T_{ij}(\zeta)$: Vladivostok transport

They form a monad $T_{jk}(\zeta)\, T_{ij}(\zeta) \longrightarrow T_{ik}(\zeta)$ $(i<j<k)$

(assoc. alg. of functors) $a_k^* \underbrace{a_j a_j^*}_{\downarrow \, id} a_i \longrightarrow a_k^* a_i$

Def $FS(\mathfrak{S}, \zeta)$ = cat. of alg. over this monad

Triangulated cat w/ a semiorthogonal decomp. $\Phi_1, \dots, \Phi_N$

Direct analog of $\Psi(\check{F})_\zeta$.

The IR complex.

Rectilinear (free) monad $R(\zeta) = R_{ij}(\zeta)_{i<j}$

$$R_{ij}(\zeta) = \bigoplus_{\substack{\zeta\text{-conv.} \\ \gamma}} T_\gamma \qquad \text{Composition: } \underbrace{\text{concatenation / or } 0}_{\text{free}}$$

<u>Thm</u> [KSS] (1) $\exists$ differential in $R(\zeta)$ s.t. $d^2 = 0$ in derived cat. (from maps in PL triangle)

(2) $\exists$ Postnikov system ("filtration") in $T(\zeta)$ whose assoc. graded is $R(\zeta)$,

and d = connecting maps.

$\approx$ describing $\check{\mathfrak{S}}$ w/ Stokes structure

Further: extend [K.- Kontsevich- Soibelman] to schobers.