(Co) standard modules

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$$T_w \mapsto (-1)^{\ell(w)}$$

$$\begin{tikzcd}
Z(\mathbb{H}) \subset R = \mathbb{C}[v,v^{-1}]\{e^\lambda\} \arrow[d, phantom, "\cong"] & \subset \mathbb{H} = \mathbb{C}[v,v^{-1}]\{e^\lambda T_w\} \arrow[d, phantom, "\cong"] & \curvearrowright M_{asp} = \mathbb{H} \otimes_{H_f} \mathbb{C}[v,v^{-1}] \arrow[d, phantom, "\cong"] \\
K^{\hat{G}\times\mathbb{G}_m}(pt) \subset K^{\hat{G}\times\mathbb{G}_m}(\tilde{\mathcal{N}}) & \subset K^{\hat{G}\times\mathbb{G}_m}(Z) & \curvearrowright K^{\hat{G}\times\mathbb{G}_m}(\hat{\tilde{\mathcal{N}}})
\end{tikzcd}$$

$$\begin{tikzcd}
\hat{\tilde{\mathcal{N}}} \arrow[r, hook] \arrow[d] & \hat{\tilde{\mathcal{N}}} \times_{\hat{\mathfrak{g}}} \hat{\tilde{\mathcal{N}}} \\
pt &
\end{tikzcd}
\qquad \curvearrowright \quad K^{\hat{G}\times\mathbb{G}_m}(\hat{\tilde{\mathcal{N}}} \times_{\hat{\mathfrak{g}}} W)$$

Take W to be a nilp. orbit $\mathcal{O}$,

$a \in \hat{G}/\!/\hat{G}\times\mathbb{G}_m = \operatorname{Spec} K^{\hat{G}\times\mathbb{G}_m}(pt)$

$= (s,t)$

Choose $X \in \mathcal{O}^a$

i.e. $\mathrm{Ad}_s X = tX$.

$$K^{\hat{G}\times\mathbb{G}_m}(\hat{\tilde{\mathcal{N}}}_{\mathcal{O}}) \otimes_{K^{\hat{G}\times\mathbb{G}_m}(pt)} \mathbb{C}_a \overset{\text{localization}}{\simeq} \bigoplus_{X \in \mathcal{O}^a / C_{\hat{G}}(s)} K(\mathcal{B}^s_X)$$

$$\begin{tikzcd}
\tilde{\mathcal{N}}^a \arrow[r, "\pi^a"] & \hat{\mathcal{N}}^a \arrow[d, phantom, "\cup"] \\
 & \mathcal{O}^a \arrow[d, phantom, "\ni"] \\
\mathcal{B}^s_X \arrow[uu] \arrow[r] & X
\end{tikzcd}$$

$\mathcal{B}^s_X = \{$Borels of $\hat{G}$ containing s, $u = \exp(X)$ $\}$

$X \rightsquigarrow u = \exp(X)$.

Let $C(s,u) = \pi_0(C_{\hat{G}}(s,u))$

<u>Def</u>. Let $(a=(s,t), \cancel{X}, \rho)$, u, $X \in \hat{\mathcal{N}}^a$, $\rho \in \mathrm{Irr}(C(s,u))$

The standard module of $\mathbb{H}$ $\mathcal{M}_{(a,u,\rho)} = K(\mathcal{B}^s_X)_\rho$

$= \mathrm{Hom}_{C(s,u)}(\rho, K(\mathcal{B}^s_X))$.

$a = (s,t) \in \hat{G}/\!\!/\hat{G} \times \mathbb{G}_m$.

$R_a \mathcal{H}_a = \mathcal{H} \otimes_{Z(\mathcal{H})} \mathbb{C}_a$

$\pi^a : \tilde{\mathcal{N}}^a \longrightarrow \mathcal{N}^a$

$\cong$

${}_c H^{BM}(Z_a) \curvearrowright H^{BM}(\tilde{\mathcal{N}}^a)$

<u>Lemma.</u> $\tilde{\mathcal{N}}^a$ is smooth.

$H^{BM}(\tilde{\mathcal{N}}^a)$ $\cong$ $\cong$

$\omega_{\tilde{\mathcal{N}}^a} = \mathbb{C}[2d_a]$,

$\mathrm{REnd}(\pi^a_* \omega_{\tilde{\mathcal{N}}^a}) \curvearrowright \mathrm{RHom}(\mathbb{C}, \pi^a_* \omega_{\tilde{\mathcal{N}}^a})$

$d_a = \dim \tilde{\mathcal{N}}^a$.

$X \xrightarrow{f} Y$

X smooth, f proper, $\mathrm{REnd}(Rf_* \omega_X[-d_X]) \curvearrowright \mathrm{RHom}(\mathcal{B}, Rf_* \omega_X[-d_X])$

<u>Decomposition thm.</u> $\mathcal{A} = Rf_* \omega_X[-d_X] \overset{\text{non-canonical}}{\simeq} \bigoplus V_{(Z,\mathcal{L})} \otimes IC(Z, \mathcal{L})$

$Z \overset{\text{closed}}{\hookrightarrow} Y$

$\cup$ open

Z°, $\mathcal{L} \in \mathrm{Loc}(Z^\circ)$

Irr. rep. of $\mathrm{Ext}^\bullet(\mathcal{A}, \mathcal{A}) \longleftrightarrow V_{(Z,\mathcal{L})}$

<u>Prop.</u> (1) π^a is surjective

(2) $\mathcal{N}^a$ has only finitely many $C_{\hat{G}}(s)$-orbits.

<u>Cor.</u> Z appearing in the decomposition must be of the form $Z = \bar{\mathcal{O}}$ for some orbit $\mathcal{O} \subset \mathcal{N}^a$. $\mathcal{L}$ is a $C_{\hat{G}}(s)$-equiv. local sys. on $C_{\hat{G}}(s)/C_{\hat{G}}(s,u) = \mathcal{O}$

$\mathcal{L} \longleftrightarrow$ Irr rep. of $\pi_0(C_{\hat{G}}(s,u)) = C(s,u)$.

Sketch of proof of the prop:

(1) π^a surjective $\Leftrightarrow$ $\mathcal{B}_u^s \neq \phi$.

$\Leftrightarrow$ $\exists$ Borel of $\hat{G}$ containing $(s, u = \exp(X))$

$sus^{-1} = u^t \Rightarrow \langle s, u \rangle$ generate a solvable subgp of $\hat{G}$

$\Rightarrow$ belongs to some Borel.

(2)
$$\begin{tikzcd}[column sep=small]
& \{(s,X) \in \hat{G} \times \hat{\mathfrak{g}} : \mathrm{Ad}_s X = tX\}/\hat{G} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\hat{G}/\hat{G} \supset \underset{\text{conj. class of } s}{C} & & \hat{\mathcal{N}}/\hat{G}
\end{tikzcd}$$

$s \qquad\qquad X$

$$p_1^{-1}(s) = \hat{\mathcal{N}}^a$$

We need to show there are only finitely many $C_{\hat{G}}(s)$-orbits.

$$\hat{\mathcal{N}}^a / C_{\hat{G}}(s) = p_1^{-1}(C)/\hat{G}$$

Enough: $p_1^{-1}(C)/\hat{G} \xrightarrow{p_2} \hat{\mathcal{N}}/\hat{G}$ has finite fibers.

$$\begin{tikzcd}
T/W \arrow[d, equal] & T_X \text{ max. torus} \arrow[l, "\text{finite}"'] \arrow[d, phantom, "\subset"] & \hat{\mathcal{N}}/\hat{G} \arrow[d, phantom, "\ni"] \\
\hat{G}/\hat{G} & C_{\hat{G}}(X)/\mathrm{Ad}_s C_{\hat{G}}(X) \arrow[l] \arrow[r] & X
\end{tikzcd}$$

$a = (s, t)$ $\qquad \hat{\mathfrak{g}} = \bigoplus_{\lambda} \hat{\mathfrak{g}}_{\lambda}$, λ eigenvalue of Ad_s

$$\hat{\mathcal{N}}^a = \hat{\mathfrak{g}}_t \cap \hat{\mathcal{N}}$$

Observation.. if t is NOT a root of 1, then $\boxed{\hat{\mathcal{N}}^a = \hat{\mathfrak{g}}_t}$

$\ni$
X'

$ad_{X'} : \hat{\mathfrak{g}}_{\lambda} \to \hat{\mathfrak{g}}_{t\lambda} \to \hat{\mathfrak{g}}_{t^2\lambda} \to \cdots$

$\pi^a : \tilde{\hat{\mathcal{N}}}^a \longrightarrow \hat{\mathcal{N}}^a$

$\mathcal{B}_u^s = (\pi^a)^{-1}(X)$.

Thm. $H^{BM}_{odd}(\mathcal{B}_u^s ; \mathbb{C}) = 0$.

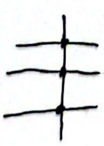
Usual Springer fiber: $H^{BM}_{odd}(\mathcal{B}_u, \mathbb{C}) = 0$

Fact. $\mathcal{B}_u$ is connected equidim, but $\mathcal{B}_u^s$ may not be connected, nor equidimensional.

Ex. $X \in \hat{\mathcal{N}}$ is called subregular if $\hat{G} \cdot X \subset \hat{\mathcal{N}}$ is of codim 2.

$\mathcal{B}_u$ is always a chain of $\mathbb{P}^1$'s.

$\hat{G} = G_2$, $\qquad \mathcal{B}_u$



D_4

$\mathcal{B}_u^s \leftarrow$ particular s

$C(s, u) = \mathfrak{S}_3$

$\pi^a : \tilde{\hat{\mathcal{N}}}^a \longrightarrow \hat{\mathcal{N}}^a$

$$A = R(\pi^a)_* \, \omega_{\tilde{\hat{\mathcal{N}}}^a}[-d_a] \simeq \bigoplus_{(\mathcal{O},\mathcal{L})} V_{(\mathcal{O},\mathcal{L})} \otimes IC(\mathcal{O},\mathcal{L})$$

$\mathcal{O}$ $\quad C_{\hat{\mathfrak{g}}}(s)$-orbit in $\hat{\mathcal{N}}^a$

$\mathcal{L} \longleftrightarrow \chi \in \mathrm{Irr}(C(s,u))$

Lemma. Let $i_{\mathcal{O}} : \mathcal{O} \hookrightarrow \hat{\mathcal{N}}^a$ be the locally closed embedding,

$$\epsilon_{\mathcal{O}'} : (i_{\mathcal{O}'})^! IC(\mathcal{O},\mathcal{L}) \to (i_{\mathcal{O}'})^* IC(\mathcal{O},\mathcal{L})$$

$\epsilon_{\mathcal{O}'} = 0$ if $\mathcal{O}' \neq \mathcal{O}$. $\quad \epsilon_{\mathcal{O}'}$ is an isom. if $\mathcal{O}' = \mathcal{O}$.

Pf. $\mathcal{O}' \notin \bar{\mathcal{O}}$, obvious $\epsilon_{\mathcal{O}'} = 0$.

$\mathcal{O}' \in \bar{\mathcal{O}}$, $\mathcal{O}' \neq \mathcal{O}$.

$$\mathcal{F} \in \mathrm{Perv} \iff \begin{array}{l} i_{\mathcal{O}'}^! \mathcal{F} \in D(\mathrm{Loc}(\mathcal{O}'))^{\geq -\dim \mathcal{O}'} \\ i_{\mathcal{O}'}^* \mathcal{F} \in D(\mathrm{Loc}(\mathcal{O}'))^{\leq -\dim \mathcal{O}'} \end{array}$$

In addition, if $\mathcal{F} = IC$ $\quad$ "$\leq$" $\Rightarrow$ "$<$" unless $\mathcal{O}'$ is the largest stratum in the supp. of $\mathcal{F}$. $\square$

Cor. $x \in \mathcal{O}$, $i_x : \{x\} \hookrightarrow \mathcal{O}$

$$\mathrm{Im}\left((i_x^! \, i_{\mathcal{O}}^! A)_{\mathcal{L}} \to (i_x^! \, i_{\mathcal{O}}^* A)_{\mathcal{L}} \right) = V_{(\mathcal{O},\mathcal{L})}$$

$$\begin{tikzcd} \{x\} \arrow[r, hook, "i_x"] \arrow[rr, bend right, "i_x"'] & \mathcal{O} \arrow[r, hook, "i_{\mathcal{O}}"] & \hat{\mathcal{N}}^a \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B}_u^s \arrow[r] \arrow[d] & \tilde{\hat{\mathcal{N}}}^a \arrow[d] \\ x \arrow[r, phantom, "\in"] & \hat{\mathcal{N}}^a \end{tikzcd}$$

Def. $M_{(a,x,\mathcal{L})} := (i_x^! A)_{\mathcal{L}} = (R\mathrm{Hom}((i_x)_! \mathbb{C}, A))_{\mathcal{L}}$

$\cong (H^{BM}(\mathcal{B}_u^s))_{\mathcal{L}}$ $\quad \boxed{\text{Standard module}}$

Cor. Every irr. rep. of $\mathbb{H}$ appears as a quotient of a standard module.

Cor. Suppose $t \neq$ root of 1. Let $u \in \mathcal{O} \overset{\text{open}}{\subset} \hat{\mathcal{N}}^a = \hat{\mathfrak{g}}_t$, then

$M_{(a,u,\mathcal{L})}$ is irred.

Thm. Suppose $t \neq$ root of unit, then $V_{(\mathcal{O},\mathcal{L})} \neq 0$ as long as

$M_{(a,u,\mathcal{L})} \neq 0$, i.e. $\mathcal{L} \leftrightarrow \chi \in \mathrm{Irr}(C(s,u))$ appears in $H^{BM}(\mathcal{B}_u^s)$.

Ex. $\hat{G} = G_2$, $a = (s,q)$, u subregular, $C(s,u) = \mathfrak{S}_3$

$\curvearrowright H^{BM}(\mathcal{B}_u^s)$

$\mathrm{Irr}(\mathfrak{S}_3) = \begin{Bmatrix} \text{trivial, sgn} \\ \text{2-dim} \end{Bmatrix}$

$\mathbb{P}^1$

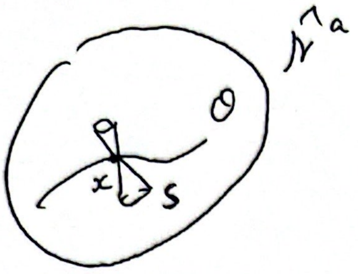
Only trivial, 2-dim appear in $H^{BM}(\mathcal{B}_u^s)$.

(s, u, sgn) will correspond to a <u>super cuspidal</u> rep in the LLC.

$x \in \mathcal{O} \overset{i_\mathcal{O}}{\hookrightarrow} \hat{\mathcal{N}}^a$

$i_x^! i_\mathcal{O}^! A \to i_x^! i_\mathcal{O}^* A$ $\qquad M_{(a,u,\mathcal{L})} \twoheadrightarrow L_{(a,u,\mathcal{L})} \twoheadrightarrow M^\vee_{(a,u,\mathcal{L})}$

Def $(i_x^! i_\mathcal{O}^* A)_\mathcal{L} = M^\vee_{(a,u,\mathcal{L})}$ costandard rep.



S transversal slice. $\tilde{S} = (\pi_a^{-1})(S)$

$\circlearrowleft$ $\qquad$ $\uparrow$ smooth, homotopy retract

$C_{\hat{G}}(s,u)^{red}$. $\quad \pi_0(-) = C(s,u)$ $\quad$ to B_u^s.

Lemma:

$$RHom((i_S)_! \mathbb{C}, \mathcal{A})_{\mathcal{L}} \cong M^\vee_{(a,u,\mathcal{L})}$$

$$\parallel$$

$$H^{BM}_\bullet(\tilde{S})_{\mathcal{L}}$$

$$\begin{tikzcd}
M_{(a,u,\mathcal{L})} \arrow[r, two heads] \arrow[d, equal] & L_{(a,u,\mathcal{L})} \arrow[r, hook] & M^\vee_{(a,u,\mathcal{L})} \arrow[d, equal] \\
RHom((i_x)_! \mathbb{C}, \mathcal{A}) \arrow[rr] & & RHom((i_S)_! \mathbb{C}, \mathcal{A})
\end{tikzcd}$$