Root datum

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July 31, 2023

For Russian mathematicians, the most important Lie algebra is $\mathfrak{sl}_2$, while for American mathematicians, the most important one is E_8.

A member of the Russian mafia

1 D_ℓ ($\ell \geq 3$)

1.1 Root datum

Let $\varepsilon_1, \cdots, \varepsilon_\ell$ be the standard basis in $\mathbb{R}^\ell$.

We have

- the set of roots $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq \ell\}$;
- the set of simple roots $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \cdots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell\}$;
- the highest root $\theta = \varepsilon_1 + \varepsilon_2$;
- Dynkin–Langlands dual $D_\ell^\vee = D_\ell$;
- Coxeter number $h = 2\ell - 2$;
- dual Coxeter number $h^\vee = 2\ell - 2$;
- fundamental weights

$$\begin{aligned} \omega_i &= \varepsilon_1 + \cdots + \varepsilon_i \text{ for } i < \ell - 1 \\ \omega_{\ell-1} &= \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{\ell-1} - \varepsilon_\ell) \\ \omega_\ell &= \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{\ell-1} + \varepsilon_\ell) \end{aligned}$$

- the half sum of positive roots $\rho = (\ell - 1)\varepsilon_1 + (\ell - 2)\varepsilon_2 + \cdots + \varepsilon_{\ell-1}$.

1.2 Matrix realization

In terms of classical Lie algebras, D_ℓ is realized by $\mathfrak{so}(2\ell, \mathbb{C})$. But it is more convenient to use the following description:

$$D_\ell = \mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{gl}(2\ell, \mathbb{C}) : B, C \text{ skew-symmetric} \right\}.$$

- The Cartan algebra $\mathfrak{h}$ can be chosen as diagonal matrices in $\mathfrak{g}$;

- $\varepsilon_i \in \mathfrak{h}^*$ is the linear map sending $\mathrm{diag}(x_1, \cdots, x_\ell, -x_1, \cdots, -x_\ell)$ to x_i;
- The normalized Killing form $\varkappa(X, Y) = \mathrm{tr}(XY)$;
- We can fix
$$\begin{aligned} e_{\varepsilon_i - \varepsilon_j} &= f_{\varepsilon_j - \varepsilon_i} = E_{ij} - E_{j+\ell, i+\ell}, \, 1 \le i, j \le \ell, i \neq j \\ e_{\varepsilon_i + \varepsilon_j} &= f_{-\varepsilon_i - \varepsilon_j} = E_{i, j+\ell} - E_{j, i+\ell}, \, 1 \le i < j \le \ell \\ e_{-\varepsilon_i - \varepsilon_j} &= f_{\varepsilon_i + \varepsilon_j} = E_{i+\ell, j} - E_{j+\ell, i}, \, 1 \le i < j \le \ell \end{aligned}$$
The root decomposition is given by $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$. The coroots are computed by $h_\alpha = [e_\alpha, f_\alpha]$. Explicitly, we have
$$\begin{aligned} h_{\varepsilon_i - \varepsilon_j} &= H_i - H_j, \, 1 \le i, j \le \ell, i \neq j \\ h_{\varepsilon_i + \varepsilon_j} &= H_i + H_j, \, 1 \le i < j \le \ell \\ h_{-\varepsilon_i - \varepsilon_j} &= -H_i - H_j, \, 1 \le i < j \le \ell \end{aligned}$$
Here $H_i = E_{ii} - E_{i+\ell, i+\ell}$.

2 E_6

2.1 Root datum

Let $\varepsilon_1, \cdots, \varepsilon_8$ be the standard basis in $\mathbb{R}^8$.

We have

- the set of roots
$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 5\} \cup \left\{ \pm\frac{1}{2} \left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5} (-1)^{p_i} \varepsilon_i \right) : \sum_{i=1}^{5} p_i \text{ is even} \right\};$$
- the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_6\}$, where
$$\begin{aligned} &\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7), \\ &\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1, \alpha_4 = \varepsilon_3 - \varepsilon_2, \\ &\alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4; \end{aligned}$$
- the highest root $\theta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$;
- Dynkin–Langlands dual $E_6^\vee = E_6$;
- Coxeter number $h = 12$;
- dual Coxeter number $h^\vee = 12$;

- fundamental weights

$$\begin{aligned}
\omega_1 &= \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\
\omega_2 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8) \\
\omega_3 &= \frac{5}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \\
\omega_4 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8 \\
\omega_5 &= \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_4 + \varepsilon_5 \\
\omega_6 &= \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5
\end{aligned}$$

- the half sum of positive roots $\rho = \varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4\varepsilon_5 + 4(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)$

3 E_7

3.1 Root datum

Let $\varepsilon_1, \cdots, \varepsilon_8$ be the standard basis in $\mathbb{R}^8$.

We have

- the set of roots

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 6\} \cup \{\pm(\varepsilon_7 - \varepsilon_8)\} \cup \left\{ \pm\frac{1}{2}\left(\varepsilon_7 - \varepsilon_8 + \sum_{i=1}^{6}(-1)^{p_i}\varepsilon_i\right) : \sum_{i=1}^{6} p_i \text{ is odd} \right\};$$

- the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_7\}$, where

$$\begin{aligned}
&\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7), \\
&\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1, \alpha_4 = \varepsilon_3 - \varepsilon_2, \\
&\alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5;
\end{aligned}$$

- the highest root $\theta = \varepsilon_8 - \varepsilon_7$;
- Dynkin–Langlands dual $E_7^\vee = E_7$;
- Coxeter number $h = 18$;
- dual Coxeter number $h^\vee = 18$;

- fundamental weights

$$
\begin{aligned}
\omega_1 &= \varepsilon_8 - \varepsilon_7 \\
\omega_2 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - 2\varepsilon_7 + 2\varepsilon_8) \\
\omega_3 &= \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - 3\varepsilon_7 + 3\varepsilon_8) \\
\omega_4 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + 2(\varepsilon_8 - \varepsilon_7) \\
\omega_5 &= \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \frac{3}{2}(\varepsilon_8 - \varepsilon_7) \\
\omega_6 &= \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8 \\
\omega_7 &= \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7)
\end{aligned}
$$

- the half sum of positive roots $\rho = \varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4\varepsilon_5 + 5\varepsilon_6 - \frac{17}{2}\varepsilon_7 + \frac{17}{2}\varepsilon_8$.

4 E_8

4.1 Root datum

Let $\varepsilon_1, \cdots, \varepsilon_8$ be the standard basis in $\mathbb{R}^8$.

We have

- the set of roots $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 8\} \cup \left\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{p_i}\varepsilon_i : \sum_{i=1}^{8} p_i \text{ is even}\right\}$;
- the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_8\}$, where

$$
\begin{aligned}
&\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7), \\
&\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1, \alpha_4 = \varepsilon_3 - \varepsilon_2, \alpha_5 = \varepsilon_4 - \varepsilon_3, \\
&\alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5, \alpha_8 = \varepsilon_7 - \varepsilon_6;
\end{aligned}
$$

- the highest root $\theta = \varepsilon_7 + \varepsilon_8$;
- Dynkin–Langlands dual $E_8^\vee = E_8$;
- Coxeter number $h = 30$;
- dual Coxeter number $h^\vee = 30$;

- fundamental weights

$$
\begin{aligned}
\omega_1 &= 2\varepsilon_8 \\
\omega_2 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_5 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8) \\
\omega_3 &= \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 7\varepsilon_8) \\
\omega_4 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8 \\
\omega_5 &= \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 4\varepsilon_8 \\
\omega_6 &= \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 3\varepsilon_8 \\
\omega_7 &= \varepsilon_6 + \varepsilon_7 + 2\varepsilon_8 \\
\omega_8 &= \varepsilon_7 + \varepsilon_8
\end{aligned}
$$

- the half sum of positive roots $\rho = \varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4\varepsilon_5 + 5\varepsilon_6 + 6\varepsilon_7 + 23\varepsilon_8$.

References

[BHJF+21] Richard Borcherds, Mark Haiman, Theo Johnson-Freyd, Nicolai Reshetikhin, and Vera Serganova. Berkeley Lectures on Lie Groups and Quantum Groups. 2021.

[Bou] N Bourbaki. *Groupes et algèbres de Lie Chapitres 4, 5 et 6*. Springer.