Mixed characteristic shtukas

Jiahao Niu

Brief intro to global function field shtukas.

$X/\mathbb{F}_q$ smooth proj. curve, $k(X)$ function field

G split reductive gp $\rightsquigarrow$ $\mathrm{Bun}_{G,X} : S/\mathbb{F}_q \mapsto \{G\text{-torsors over } X \times_{\mathbb{F}_q} S\}$

Smooth Artin stack

The moduli space of shtukas: w/ legs labeled by $I = \{1, \dots, n\}$ (n legs)

$$\begin{array}{l} \mathrm{Sht}_{I,G} : S/\mathbb{F}_q \mapsto \{(x_i)_{i \in I} \in X^I(S),\ \mathcal{E} \in \mathrm{Bun}_{G,X}(S), \\ \downarrow \qquad\qquad\qquad \varphi : (\mathrm{Frob}_S \times \mathrm{id})^* \mathcal{E}|_{X \times_{\mathbb{F}_q} S \setminus \bigcup_i \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}|_{X \times_{\mathbb{F}_q} S \setminus \bigcup_i \Gamma_{x_i}} \} \\ X^I \end{array}$$

- this is an ind-Deligne-Mumford stack
- one can also add some "level structures" to make it an ind-scheme.
- By "geometric Satake equivalence": $\rightsquigarrow$ produce a functor

$$\mathrm{Rep}(\hat{G}^I) \longrightarrow D^b_c(\mathrm{Sht}_{I,G}) \xrightarrow[\text{middle degree}]{\text{cohomology}} \mathrm{Mod}_{\overline{\mathbb{Q}}_\ell} \circlearrowleft \begin{array}{c} \mathrm{Weil}(X^I) \\ \downarrow \text{Drinfeld Lemma} \\ \mathrm{Weil}(X)^I \end{array}$$

$\hat{G}$ dual gp

$$V \longmapsto S_V \longmapsto \mathcal{H}_{I,V}$$

Shtukas $\rightsquigarrow$ $V \mapsto \mathcal{H}_{I,V}$ for any I, V functorially.

e.g. $I = \phi$, $\mathbb{1}$ ← trivial rep., $H_{\phi,\mathbb{1}} = \{$ smooth unramified auto. forms $/ k(X) \}$

$\wr$

1 leg. $H_{\{1\},\mathbb{1}} \circlearrowleft \pi_1(X)$

$$\overset{\mathrm{Weil}(X)^2 \, \curvearrowright}{H_{\{1,2\}, V\boxtimes W}} \simeq H_{\{1\}, V\otimes W} \rightleftarrows H_{\{1\},\mathbb{1}}$$

$V\boxtimes W \in \mathrm{Rep}(\hat{G}^2)$

(chosing $\mathbb{1} \to V\otimes W$)
$V\otimes W \to \mathbb{1}$

find more symmetries to $H_{\{1\},\mathbb{1}}$ through $H_{I,V}$.

These will genrate an algebra $\mathcal{A} \subset \mathrm{End}(H_{\{1\},\mathbb{1}}^{\text{cuspidal}})$

$\mathcal{A} = \mathcal{O}($ moduli space of L-parameters for $\hat{G})$

cuspidal $= \bigoplus_{\mathcal{A} \xrightarrow{\sigma} \overline{\mathbb{Q}}_\ell} V_\sigma$ decomposition.

Local version $/ \mathbb{F}_q[[t]]$ X as above, $\hat{X}$ completion at one pt
$\hat{X} \simeq \mathrm{Spf}\, \mathbb{F}_q[[t]]$

$G = GL_n$

Def local shtukas a functor sending (perfectoid) adic space S

$$S \longmapsto \left\{ (x_i) \in \mathrm{Spa}\, \mathbb{F}_q[[t]](S),\ \mathcal{E} \text{ vec. bdle on } S\times \mathrm{Spa}\, \mathbb{F}_q[[t]],\ \varphi_{\mathcal{E}} : \varphi_S^* \mathcal{E}|_{S\times \mathrm{Spa}\, \mathbb{F}_q[[t]] \setminus \bigcup_i \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}|_{S\times \mathrm{Spa}\, \mathbb{F}_q[[t]] \setminus \bigcup_i \Gamma_{x_i}} \right\}$$

- $\varphi_{\mathcal{E}}$ meromorphic along $\bigcup_i \Gamma_{x_i}$

e.g. $S = \mathrm{Spa}(C, \mathcal{O}_C)$, $\quad S \times \mathrm{Spa}\, \mathbb{F}_q[[t]] = \mathbb{D}_C$ open unit disc.

$$\varphi_S : C \to C,\ x \mapsto x^p$$
$$t \mapsto t$$

Mixed version: $S \in \mathrm{Perf}$, "$S \times \mathrm{Spa}\, \mathbb{Z}_p$"

Def $\quad S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid space of char. p

$$S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p = \{([\varpi] \neq 0) \subset \underset{\mathrm{Spa}(W(R^+), W(R^+))}{\overset{\|}{\mathrm{Spa}\, W(R^+)}},\ \varpi \in R^+ \text{ pseudo uniformizer}\}$$

(as $S \subset \mathrm{Spa}(R^+, R^+)$, $\{\varpi \neq 0\}$)

- $S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p$ is an adic space.

$$\bigcup_{n \geq 1} \{ |p| \leq |[\varpi^{1/p^n}]| \neq 0 \} = \bigcup_{n \geq 1} \mathrm{Spa}(R_n, R_n^+)$$ ← these are sheafy

$\Uparrow$

stable uniform

$\Uparrow$

$\mathrm{Spa}(R_n, R_n^+)$ sous perfectoid

splitting as an R_n-module

$$R_n \hookrightarrow \underbrace{R_n \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p}_{\text{perfectoid}} \Longrightarrow$$

- $(S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p)^\diamond \simeq S \times (\mathrm{Spa}\, \mathbb{Z}_p)^\diamond$

$T \in \mathrm{Perf}^{\mathrm{aff}}$, $T = \mathrm{Spa}(A, A^+)$, $(S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p)^\diamond(T) \Leftrightarrow T^\sharp,\ \overset{\mathrm{Spa}(A^\sharp, A^{+\sharp})}{\overset{\|}{T^\sharp}} \to S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p$

$\Leftrightarrow \quad W(R^+) \to A^{+\sharp}$ sending $[\varpi] \mapsto$ unit in $A^\sharp$.

$\Leftrightarrow R^+ \longrightarrow (A^{+\sharp})^\flat \cong A^+ \quad , T^\sharp$

$R \to A$. $\varpi \mapsto$ unit in A.

$\rightsquigarrow (S \times (\mathrm{Spa}\, \mathbb{Z}_p)^\diamond)(T)$.

$\{$affinoid preadic space$\}$
$\cong$
$\{$complete Huber pair$\}^{op}$

Cor. $S^\sharp$ of $\overset{\mathrm{Perf}}{S} \longleftrightarrow$ sections of $(S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p)^\diamond \longrightarrow S$

$\mathcal{O}_{S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p} \twoheadrightarrow \mathcal{O}_{S^\sharp}$

this will give us a Cartier divisor (i.e. kernel will be locally free of rk 1)

Rmk. This is hard to check.

$C/\mathbb{Q}_p$ mixed char. $C^\flat/\mathbb{F}_p$

$A_{\inf} = W(\mathcal{O}_{C^\flat}) \overset{\theta}{\twoheadrightarrow} \mathcal{O}_C$ $\qquad$ Spa $A_{\inf}$

$\ker\theta = (\xi)$, $\xi = p - [p^\flat]$, $\quad p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots)$

x_k unique nonanalytic point , k residue field of $\mathcal{O}_{C^\flat}$.

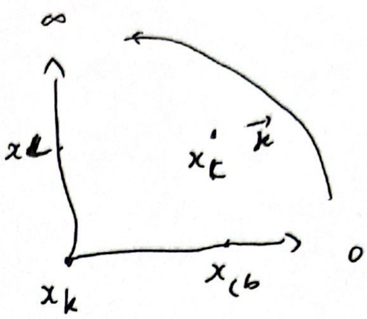
$x_{C^\flat} : A_{\inf} \twoheadrightarrow \mathcal{O}_{C^\flat} \hookrightarrow C^\flat$

$x_C : A_{\inf} \longrightarrow \mathcal{O}_C \to C$

$x_L : A_{\inf} \longrightarrow W(k) \to W(k)[\frac{1}{p}] =: L$

$\kappa : \mathcal{Y} = \mathrm{Spa}\, A_{inf} \setminus \{x_k\} \to [0,\infty]$

$$x \longmapsto \frac{\log |[p^\flat](\tilde{x})|}{\log |p(\tilde{x})|}$$



$\tilde{x}$ is the unique rank 1 generalization

$\kappa(x_C) = 1, \quad \kappa(x_{C^\flat}) = 0. \quad \kappa(x_L) = \infty, \quad \kappa \circ \varphi = p \circ \kappa.$

<u>Def</u>. $\mathrm{Sht}_I(S) = \{ (x_i)_{i \in I} \in (\mathrm{Spa}\, \mathbb{Z}_p)^\diamond(S)^I, \ \mathcal{E}$ vec. bdle on $\mathrm{Spa}\, \mathbb{Z}_p \dot{\times} S$.

$S \in \mathrm{Perf}_{\mathbb{F}_p}$

$\Updownarrow$

sections of $(S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p)^\diamond \to S$

$\Updownarrow$

untilt of S

$\varphi_\mathcal{E} : \varphi^* \mathcal{E}|_{\mathrm{Spa}\, \mathbb{Z}_p \dot{\times} S \setminus \bigcup_i \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}|_{\mathrm{Spa}\, \mathbb{Z}_p \dot{\times} S \setminus \bigcup_i \Gamma_{x_i}}$

meromorphic along $\bigcup_i \Gamma_{x_i}$ $\}$

$\downarrow$

$((\mathrm{Spa}\, \mathbb{Z}_p)^\diamond)^I$

<u>e.g.</u> Shtukas over $\mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat})$, $C/\mathbb{Q}_p$ complete alg. closed

$C^\flat / \mathbb{F}_p$

w/ legs at C

$\mathcal{E}$ is a vec. bdle / $\mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat}) \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p$

w/ $\varphi_\mathcal{E} : \varphi^* \mathcal{E}|_{\mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat}) \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p \setminus \Gamma_{x_C}} \xrightarrow{\sim} \mathcal{E}|_{\mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat}) \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p \setminus \Gamma_{x_C}}$

meromorphic along Γ_{x_C}.

(schematic version)

A Breuil–Kisin–Fargues module is a finite free $W(\mathcal{O}_{C^\flat})$-module M

$$\varphi_M : (\varphi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}] \quad \text{is an isom.}$$

$$(\xi = \ker(\theta : W(\mathcal{O}_{C^\flat}) \to C)).$$

Thm. (Fargues) $\{$Shtukas over $\mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat})$ w/ one leg at $\mathrm{Spa}(C, \mathcal{O}_C)\}$

$$\wr\wr$$

$\{$BKF-modules over $W(\mathcal{O}_{C^\flat})\}$.

$$\begin{tikzcd}
\{\text{p-divisible groups}/\mathcal{O}_C\} \arrow[r] \arrow[d] & \{\text{shtukas}/C^\flat \text{ w/ one leg at } C\} \simeq \{\text{BKF modules}\} \arrow[d, dashed, "?"] \arrow[d, bend left=60, "- \otimes_{W(\mathcal{O}_{C^\flat})} W(k)"] \\
\{\text{p-divisible gp}/k\} \arrow[r, "\simeq"] & \{\text{Dieudonne modules}\}
\end{tikzcd}$$

k residue field of $\mathcal{O}_C$.

Thm R is an integral perfectoid ring,

$$\{\text{p-divisible gps}/R\} \xhookrightarrow{\text{fully faithful}} \left\{ \begin{array}{l} \text{finite proj. } A_{\inf}(R)\text{-module } M \ \ (A_{\inf}(R) = W(R^\flat)) \\ \varphi_M : M[\tfrac{1}{\xi}] \underset{\varphi\text{-linear}}{\xrightarrow{\sim}} M[\tfrac{1}{\varphi(\xi)}] \end{array} \right\}$$

essential image : those M s.t. $M \subset \varphi_M(M) \subset \frac{1}{\varphi(\xi)} M$

$R = \mathcal{O}_C$: example

Pf. Both sides satisfy v-descent