Surfaces / $\mathbb{Z}$ + Enriques' classification

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Schemes $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ smooth, proper, geom. conn'd fibers of dim 2

Classification of surface / $k = \bar{k}$ arbitrary char.

Steps in classification:

(1) for any X = smooth proj. surface / $\bar{k}$, $\exists$ birat'l map $\underbrace{X \longrightarrow X^{min}}_{\text{is a blowup}}$

(2) X is minimal, then either X is ruled ($X \overset{\sim}{\dashrightarrow} \mathbb{P}^1 \times C$)

or K_X nef. ($\forall \underset{\text{curve}}{C} \subset X$, $K_X \cdot C \geq 0$)

(3) if K_X nef, $n \gg 0$, $|nK_X|$ base point free

$|nK_X| : X \longrightarrow \mathbb{P}^N$ image X^{can} canonical model.

ADE-singularities, but normal

$$X^{can} = \operatorname{Proj}\Big(\underbrace{\bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nK_X))}_{\text{finitely generated}}\Big)$$

convention: $\kappa(\text{ruled}) = -\infty$

(4) casework: $\kappa(X) := \dim X^{can}$ = "order of growth of $h^0(nK_X)$".

(a) $\kappa(X) = 2$ general type ??

(b) $\kappa(X) = 1$: "properly elliptic" $X \longrightarrow X^{can}$ = smooth curve.

$\mathcal{O}(nK_X) = \pi^* \mathcal{O}_{X^{can}}(1)$, so deg 0 on fibers thus fibers are curves w/ $P_g = 1$.

(c) $\kappa(X) = 0$ means $nK_X = 0$. [Mumford : $12 K_X \equiv 0$]

If $nK_X = 0$, then $\tilde{X} \to X$ take $z^n = \sigma$, $\sigma \in H^0(\mathcal{O}(nK_X))$

$K_{\tilde{X}} = 0$

if char $k \neq 2, 3$, know $p \nmid n$, then $\tilde{X} \to X$ étale μ_n- cover

Step 1. Def. a (-1) curve on X is an integral curve $C \subset X$ s.t.

(1) $C \simeq \mathbb{P}^1$

(2) $N_{C|X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, i.e. $C^2 = -1$

Thm Let X smooth surface, $C \subset X$ integral curve, $\exists\ f : X \to X'$ proper birat'l

$X \setminus C \xrightarrow{\sim} X' \setminus *$ and X' smooth $\iff$ C is a (-1)- curve and $f =$ Blowup at $*$.

Def. X is minimal if it contains no (-1) - curves.

Prop. any sequence $X \to X_1 \to X_2 \to \cdots$ of blowdowns terminates in a minimal surface

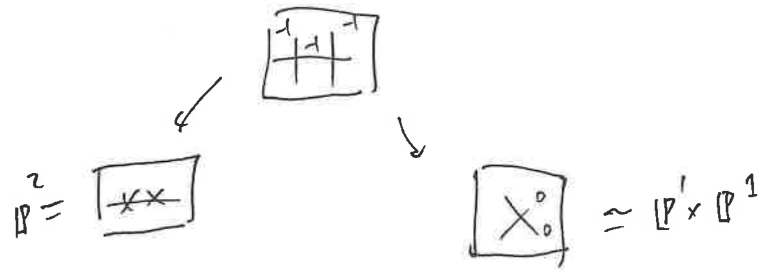
Proof. $\mathrm{Num}_X := \mathrm{Pic}(X)/\sim_{num}$ free $\mathbb{Z}$-module of finite rk $= \rho$

a blow down decreases ρ by 1.

2: ☆ X can have ∞-many (-1)-curves. (e.g. $Bl_{10\text{-pts}}\,\mathbb{P}^2$)

☆ minimal models are not unique

$$\begin{tikzcd} & F_1 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 & & \mathbb{P}^1\times\mathbb{P}^1 \end{tikzcd}$$



Thm. If $\exists$ min. model w/ K_X nef, then uniqueness holds.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X_1^{min} \arrow[rr, "\sim"'] & & X_2^{min} \end{tikzcd}$$

Step 2. ruledness.

Thm. if X minimal, either (1) X ruled; (2) K_X nef.

and K_X nef $\iff H^0(X, \mathcal{O}(nK_X)) \neq 0$ (i.e. $k(X) \geq 0$)

Proof first show if $|nK_X| \neq \phi$, then K_X is nef

if $C.K_X < 0$, $D \sim nK_X$ eff, $C.D < 0$, so $C \subset D$.

$D = aC + \sum D_j a_j$. $C.D = aC^2 + \sum a_j C.D_j < 0$. $C.D_j \geq 0$, $a_j \geq 0$, $a \geq 0$

$C^2 < 0$. Adjunction: $2p_g(C) - 2 = (C + K_X).C < 0$.

$\Rightarrow p_g = 0$, i.e. C smooth rat'l curve. $C^2 = -1$, $K_X.C = -1$ not possible.

WTS : X has a "pencil" of rat'l curves if K_X not nef.

$$\begin{array}{ccc} Bl\, X & \longrightarrow & X \\ \downarrow & & \\ C & & \end{array}$$

fibers are smooth rat'l curves

Tsen's theorem. any $\mathbb{P}^1$-bundle over a curve is ruled

$K = k(C)$ any conic over K has a point.

Step 3. Abundance.

Thm. If K_X nef, then $|nK_X|$ has no basepoints. $n \gg 0$

Thm (Mumford) Let X be a minimal surface, then

(a) $|12K_X| = \phi$, then X ruled

(b) $|12K_X| \neq \phi$ iff K_X nef and in this case either $12K_X \equiv 0$ or $|nK_X|$ basepoint free $n \gg 0$ and $\kappa(X) > 0$

"Proof": K_X nef $\Leftrightarrow$ X not ruled $\Leftrightarrow$ $|nK_X| \neq \phi$, $n \gg 0$.

Assume K_X nef $\Rightarrow$ $|12K_X| \neq \phi$.

Claim 1: $K_X^2 \geq 0$ (true for all nef classes, but uses N-M criterion)

Thm A case I. $K_X^2 = 0$, then either $2K_X = 0$ or $\exists$ a pencil of curves of arithmetic genus 1.

Thm C. case II. $K_X^2 > 0$, then $\kappa(X) = 2$ and $|2K_X| = \phi$ and $|nK_X|$ base pt free, $n \gg 0$.

Thm B. X minimal and $X \xrightarrow{f} C$ fibration in $P_g = 1$ curves.

(a) $mK_X = f^*(A)$, $A \subset C$ effective divisor

(b) if K_X nef and generic fiber smooth, $|12K_X| \neq \phi$

(c) if K_X nef and generic fiber singular, either $|2K_X| \neq \phi$ or X admits elliptic fibration.

Example $y^2 = x^p + t$ char $= p$ lies over $\mathbb{F}_p(t)$.

Claim curve is regular but not smooth.

$$X = \mathrm{Spec}\left(\mathbb{F}_p[x,y,t] / (y^2 = x^p + t)\right)$$

$$\downarrow$$

$p = 3$. $\mathrm{Spec}\,\mathbb{F}_p[t]$

Thm (Tate) If char $\neq 2, 3$, any $(p_g = 1)$ curve is smooth.

$\left[p_g < \frac{p-1}{2} \Rightarrow \text{smooth} \right]$