Perfect complexes v.s. coherent sheaves

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Goal: When $D^b_{coh}(X) \xrightarrow{\text{fully faithful}} \mathrm{IndPerf}(X)$?

Recall. D_{qc} :
D^b_{coh} : Prestack $\longrightarrow$ Ani right Kan ext'n of
D_{perf} :

$$\begin{array}{ccc} \text{Prestack} & \longrightarrow & \text{Ani} \\ \cup & \nearrow & \\ \text{(fpqc) Algebraic Stack} & & \end{array}$$

$R \mapsto D(R)$

$R \mapsto D^b_{coh}(R)$ R noetherian

$R \mapsto D_{perf}(R)$

$$D_{qc}(\mathcal{X}) = \lim_{\mathrm{Spec} R \to \mathcal{X}} D(R)$$

$$D_{perf}(\mathcal{X}) = \lim_{\mathrm{Spec} R \to \mathcal{X}} D_{perf}(R) \quad , \quad D_{perf}(R) = D(R)^{dual} = D(R)^{\omega}$$

$$\begin{array}{c} D_{perf}(\mathcal{X}) \\ \wr\| \\ D_{qc}(\mathcal{X})^{dual} \end{array}$$

But in general, $D_{qc}(\mathcal{X})^{\omega} \neq D_{perf}(\mathcal{X})$.

and $D_{qc}(\mathcal{X})$ may not be compactly genrated.

[$D_{qc}(\mathcal{X})^{\omega} \subset D_{perf}(\mathcal{X})$ if $\mathcal{X}$ has representable diagonal. qcqs alg. sp.]

(e.g. (Neeman) $D_{qc}(B\mathbb{G}_a, \mathbb{F}_q)$ has no non-zero compact obj.)

$D_{qc}(BG, \mathbb{F}_p)$ $G \supset \mathbb{G}_a$

Fact. If $\mathcal{O}_{\mathcal{X}}$ is compact, then $D_{perf}(\mathcal{X}) = D_{qc}(\mathcal{X})^{\omega}$.

$\mathrm{Hom}(P, -) \simeq \mathrm{Hom}(\mathcal{O}, P^* \otimes -)$

$\mathcal{O}_{\mathcal{X}}$ compact : call $\mathcal{X}$ concentrated

$\Leftrightarrow$ $\mathrm{Hom}(\mathcal{O}_{\mathcal{X}}, -) = R\Gamma(\mathcal{X}, -)$ has finite cohomological dim.

(counterexample , BG , G reductive / $\mathbb{F}_p$
not linearly reductive).

- $\mathcal{X}$ classic, $D^{b,(+)}\mathrm{Coh}(\mathcal{X})$, $\mathrm{Coh}(\mathcal{X}) = D_{\mathrm{Coh}}(\mathcal{X})^{\heartsuit}$

$$\begin{array}{c} D^{b,(+)}\mathrm{Coh}(\mathcal{X}) \\ \wr\mid \\ D^{b,(+)}_{\mathrm{Coh}}(\mathcal{X}) \end{array}$$

- In char. 0, every qcqs alg. stack w/ affine stabilizer and finitely presented inertia is concentrated, and $D_{qc}(\mathcal{X})$ is dualizable.

If furthermore $X \underset{\text{locally}}{\overset{\text{étale}}{\simeq}} U/GL_n$, U quasi-affine.

$$D_{qc}(\mathcal{X}) \simeq \mathrm{Ind}_{(\omega_0)} \mathrm{Perf}(\mathcal{X}).$$

$D\mathrm{Perf}(R) \simeq D^b_{\mathrm{Coh}}(R)$ if R noetherian classic and regular.

$$\begin{array}{ccc} D\mathrm{Perf}(\mathcal{X}) & \simeq & D^b_{\mathrm{Coh}}(\mathcal{X}) \quad \text{if } \mathcal{X} \text{ is smooth} \\ \downarrow & & \\ \mathrm{Ind}\,\mathrm{Perf}(\mathcal{X}) & & \end{array}$$

Def We say an alg stack $\mathcal{X}$ satisfies resolution property, if for any $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, we can find locally free sheaf $\mathcal{E}$ s.t. $\mathcal{E} \twoheadrightarrow \mathcal{F}$.

$$\left(\Rightarrow \mathcal{E}^\bullet : \cdots \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to 0 \right)$$
$$\underset{\mathcal{F}}{\overset{\text{q.isom.}}{\wr\wr}}$$

For any $\mathcal{F} \in D^b_{Coh}(\mathcal{X})$, if $\mathcal{X}$ has resolution property, can find $\mathcal{E}^\bullet$ complex of locally free sheaves, $\mathcal{E}^\bullet \underset{\text{q.isom.}}{\simeq} \mathcal{F}$.

$$D_{perf}(\mathcal{X}) \hookrightarrow D_{qc}(\mathcal{X}) \rightsquigarrow$$

$$\begin{array}{ccc} \text{"colim"}\, P_i & \longmapsto & \operatorname{colim} P_i \\ \mathrm{Ind}\, D_{perf}(\mathcal{X}) & \underset{y}{\overset{\perp}{\rightleftarrows}} & D_{qc}(\mathcal{X}) \\ \wr\wr & & \\ \mathrm{Fun}^{ex}(\mathrm{Perf}(\mathcal{X})^{op}, D(\mathbb{Z})) & & \Psi \\ \mathrm{Hom}(-, \mathcal{F}) & \longleftarrow\!\!\shortmid & \mathcal{F} \\ \| \;\; =: h_{\mathcal{F}} & & \\ \text{"colim"}\, P_i & & \end{array}$$

$\mathcal{F}, \mathcal{G} \in D^b_{Coh}(\mathcal{X})$.

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \overset{?}{=} \mathrm{Hom}(h_{\mathcal{F}}, h_{\mathcal{G}})$$
$$\simeq \mathrm{Hom}(\underbrace{y^L(h_{\mathcal{F}})}_{\substack{\| \\ y^L y(\mathcal{F}) \overset{\text{want}}{=\!=} \mathcal{F}}}, \mathcal{G})$$

$$\mathrm{Hom}(\operatorname{colim}_i P_i, \mathcal{G})$$
$$= \lim_i \mathrm{Hom}_{D_{qc}}(P_i, \mathcal{G})$$
$$= \lim_i \mathrm{Hom}_{\mathrm{Ind\,Perf}}(P_i, h_{\mathcal{G}})$$
$$= \mathrm{Hom}(\text{"colim"}\, P_i, h_{\mathcal{G}})$$

<u>resolution property</u>: $\mathrm{Ind\,Perf}(\mathcal{X}) \xrightarrow{y^L} D_{qc}(\mathcal{X})$

image generate the target $\Leftrightarrow$ y is conservative

$\mathcal{X}$ has resolution property + concentrated

$F \in D^b_{coh}(\mathcal{X})$, $y^L h_F = \text{colim}\, P_i$

$y^L(y(F)) = \text{colim}\, P_i \longrightarrow F$.

$P \in D_{perf}(\mathcal{X})$

$h_F = \text{"colim"}\, P_i$

$\text{Hom}(P, F) = \text{Hom}_{\text{Ind Perf}}(P, \text{colim}\, P_i)$

$= \text{colim}\, \text{Hom}(P, P_i)$

$\text{Hom}(P, \text{colim}\, P_i) \longrightarrow \text{Hom}(P, F) \simeq \text{colim}\, \text{Hom}(P, P_i)$

P compact $\Rightarrow$ $\sim$

$D_{qc}(\mathcal{X}) \xrightarrow{\sim} \text{IndPerf}(\mathcal{X})$.

Thm (Totaro - Gross) $\mathcal{X}$ algebraic stack, $\mathcal{X}$ has resolution property

$\Leftrightarrow$ $\mathcal{X} \simeq [U/GL_n, \mathbb{Z}]$ U quasi-affine

e.g. $k = \bar{k}$ field. X/G. X quasi-projective w/ G-equiv. ample line bundle

(in char. 0, always exists such line bundle)

Summary. Focus on $\mathcal{X} = X/G$, $\text{IndPerf}(X/G) \simeq D_{qc}(X/G)$.

Describe essential image of $y|_{D^b_{coh}(\mathcal{X})}$.

Prop. $F \in \text{IndPerf}(\mathcal{X})$ lies in $\text{Im}(y|_{D^b_{coh}(\mathcal{X})})$

$\Leftrightarrow$ (i) $F|_{D^{\geq n}_{perf}(\mathcal{X})}$ is representable by some obj. in $D_{perf}(\mathcal{X})$ for any n

(ii) $F|_{D^{\leq m}_{perf}(\mathcal{X})} = 0$, $\exists\, m$

$$D_{perf}^{\leq m} = D_{coh}^{\leq m} \cap D_{perf}, \qquad D_{perf}^{\geq n} = \text{complexes of locally free sheaves of deg} \geq n.$$

Important facts: $Ext^i(\mathcal{E}, K) = 0, \quad i > \dim X$ for $\mathcal{E}$ locally free, $K \in Coh(\mathcal{X})$

$$\| $$

$$R^i\Gamma(\mathcal{X}, \mathcal{E}^\vee \otimes K) = 0 \text{ when } i > \dim X.$$

$\Rightarrow$ (i) $\pi_0 Hom(\mathcal{E}, \mathcal{F}) = 0$ if $\mathcal{E} \in D_{perf}^{\geq m}$, $\mathcal{F} \in D^{<n}$, $m-n \geq \dim X$.

$$\| $$

$$Hom(\mathcal{O}, \mathcal{E}^* \otimes \mathcal{F})$$

$$\uparrow$$

$$D^{<-m+n}, \quad -m+n \leq -\dim X$$

(ii) for any $P \in D_{perf}^{\geq n + \dim X + 1}$, $P \simeq$ summand of complex of the form consisting of $\mathcal{L}^{\otimes m} \otimes V$ in degree $\geq n$ $\subset D_{perf}^{\geq n}$

$V \in Rep(G)$

$\mathcal{L}$ fixed ample line bundle on X/G

$P \overset{q.isom}{\simeq} P^\bullet$ complex consisting of $\mathcal{L}^{\otimes m} \otimes V$'s

$$P^{\geq n} \to P^\bullet \to P^{<n} \to P^{\geq n}[1]$$

hope split

$\pi_0 Hom(P^\bullet, P^{<n}[1]) = 0$? ✓

[in fact, $D_{perf}^{\geq n} \supset D_{coh}^{\geq n - \dim X} \cap D_{perf}$ but won't need this]

<u>Necessity</u>. $\mathcal{F} \in D_{coh}^b(X/G)$, $h_{\mathcal{F}}|_{D_{perf}^{\leq m}} = 0$ for $m \ll 0$.

$\mathcal{F} \simeq \mathcal{F}^\bullet$ complex of locally free sheaves.

$$\mathcal{F}^{\geq N} \to \mathcal{F}^\bullet \to \text{cofib}$$

want

$$\pi_0 \operatorname{Hom}(-, \mathcal{F}^{\geq N}) \simeq \pi_0 \operatorname{Hom}(-, \mathcal{F}^\bullet) \quad \text{for } (-) \in D_{perf}^{\geq n}$$

want $\pi_0 \operatorname{Hom}(-, \text{cofib}) = 0$

define $N < n - \dim X$

Fix $n < m - \dim X$, by (1), $F|_{D_{perf}^{\geq m}}$ should be rep. by $\mathcal{F}^\bullet \in D_{perf}(X)$

$$F|_{D_{perf}^{\leq m}} = 0.$$

for any $p \in \operatorname{Perf}(X/G)$, $F(p) \simeq F(p^{\geq m}) = \operatorname{Hom}(p^{\geq m}, \mathcal{F}^\bullet)$

$$\simeq \operatorname{Hom}(p^{\geq m}, \overbrace{\tau^{\geq n}\mathcal{F}^\bullet}^{\text{actually } \in D_{coh}^{\geq m}})$$

$(\operatorname{Hom} = \pi_0 \operatorname{Map})$

$$\simeq \operatorname{Hom}(p, \underbrace{\tau^{\geq n}\mathcal{F}^\bullet}_{\text{coh. cpx rep. } F})$$

$i : \operatorname{Perf}(X/G) \xhookrightarrow{\text{fully faithful.}} \mathcal{C}$ stable ∞-cat.

$\mathcal{C} \xrightarrow{\Phi} \operatorname{Ind}\operatorname{Perf}(X/G) : M \mapsto \operatorname{Hom}(i(-), M)$

<u>Assume</u>. this factors through $\Phi : \mathcal{C} \to D^b_{coh}(X/G)$

<u>Assume</u>: $\mathcal{C}$ admits a bounded t-str, there are 3-conditions

A) Φ is of bounded amplitude: $\Phi : \mathcal{C}^{\tau, \leq 0} \to D_{coh}^{\leq d}$, $\mathcal{C}^{\tau, \geq 0} \to D_{coh}^{\geq -d}$ $\exists\, d$

B) $\exists\, k > 0$ s.t. $F \in \mathcal{C}$, $\Phi(F) \in D_{coh}^{\leq 0} \Rightarrow F \in \mathcal{C}^{\tau \leq k}$.

C) $\exists\, \delta > 0$ s.t. $F \in \mathcal{C}$ we have $H^i(\Phi(F)) = 0$, $i \in [-\delta, \delta] \Rightarrow H^{0,\tau}(F) = 0$

Prop (i) B) $\Rightarrow$ Φ is fully faithful.

(ii) A), C) $\Rightarrow$ Φ is an equiv.

Proof WLOG, $k=0$. "$i \dashv \Phi$": $\mathrm{Hom}_{\mathcal{C}}(i(\underset{\mathrm{Perf}(X/G)}{\mathcal{F}}), M) = \mathrm{Hom}(\mathcal{F}, \Phi(M))$

$$\simeq \mathrm{Hom}(\Phi(i(\mathcal{F})), \Phi(M)) \quad (\Phi \circ i = \mathrm{id})$$

$\mathrm{Hom}(M_1, M_2) \simeq \mathrm{Hom}(\Phi(M_1), \Phi(M_2))$ if $M_1 \in \mathrm{Im}(i)$

Fix M_1, M_2, find n s.t. $M_2 \in \mathcal{C}^{\tau \geq n}$, $\Phi(M_2) \in D^{\geq n}_{\mathrm{coh}}$

$N < n$, $\mathcal{F}^\bullet \simeq \Phi(M_1)$ by locally frees, $\mathcal{F}_N[N] \to \mathcal{F}^{\geq N} \to \mathcal{F}^\bullet$, $\mathcal{F}_N \in \mathrm{Coh}(X/G)$

$$\underset{\mathcal{F}^\bullet}{\mathrm{Hom}(\Phi(M_1), \Phi(M_2))} \simeq \mathrm{Hom}(\mathcal{F}^{\geq N}, \Phi(M_2)) \simeq \mathrm{Hom}(i(\mathcal{F}^{\geq N}), M_2)$$

$i(\mathcal{F}^{\geq N}) \to M_1 \to \mathrm{cofib}$, $\Phi(\mathrm{cofib}) = \mathrm{cofib}(\mathcal{F}^{\geq N} \to M_1) = \mathcal{F}_N[-N+1] \in D^{\leq N-1}$

$\to \mathrm{cofib} \in \mathcal{C}^{\tau, \leq N-1}$

$\Rightarrow \mathrm{Hom}(i(\mathcal{F}^{\geq N}), M_2) \simeq \mathrm{Hom}(M_1, M_2)$

(ii). C) $\Rightarrow$ B) $\underline{\Phi(\mathcal{F}) \in D^{<-\delta} \Rightarrow \mathcal{F} \in \mathcal{C}^{<0}}$

How to check $F|_{D^{\geq n}_{\mathrm{perf}}}$ is representable

$X/G \xrightarrow[\mathcal{L}]{\text{proj.}} \mathrm{Spec}\, A/G$, then homogeneous coord. ring $\hat{\mathcal{O}}(X) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ noetherian

$\tilde{\Phi}: \underset{\mathcal{C}}{M} \mapsto \bigoplus_{n \geq 0} \mathrm{Hom}^G_{\mathrm{deeq}}(i(\mathcal{L}^{\otimes n}), M) = \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(i(\mathcal{L}^{\otimes n}), \mathcal{O}_G * M)$

$\widetilde{\Phi}_m \quad \bigoplus_{n \geq m} \cdots$

Prop TFAE

(1) $F \longmapsto \mathrm{Hom}(i(F), M)$ is representable by obj. in D_{perf}

when restricted to $D_{perf}^{\geq m}$

(2) $\widetilde{\Phi}(M[n])$ is finitely gen. for all n, and $\widetilde{\Phi}(M[n]) = 0$, $n >> 0$

(3) $\widetilde{\Phi}(M[n]) = 0$, $n >> 0$, $\exists\, m > 0$, $\widetilde{\Phi}_m(M[n])$ is finitely gen. for all M.

$$\begin{array}{ccc} \mathrm{Perf}(X/G) & \longrightarrow & \mathcal{C} \\ \downarrow & & \vdots \\ \mathrm{Perf}(X) & & \mathcal{C} \otimes_{\mathrm{Perf}(X/G)} \mathrm{Perf}(X) \end{array}$$