Construction of the functor

Tannakian formalism [The constructions in this section is 1-categorical].

$\Lambda = \overline{\mathbb{Q}}_\ell$ $\quad$ H Λ-alg gp. (reductive) $\quad$ A, B are Λ-algebras (comm.) equipped w/ H-action.

$$\begin{tikzcd}
\mathrm{Rep}(H) \arrow[r, "\text{monoidal}"] \arrow[d, "\text{monoidal}"'] & B\text{-mod}^H = \mathrm{Coh}^H(\mathrm{Spec}\, B) \\
A\text{-mod}^H = \mathrm{Coh}^H(\mathrm{Spec}\, A) \arrow[ur, dashed] &
\end{tikzcd}$$

$$\left\{ \begin{array}{l} \text{the data of monoidal ext'ns} \\ A\text{-mod}^H_{fr} = \mathrm{Coh}^H_{fr}(\mathrm{Spec} A) \to B\text{-mod}^H \end{array} \right\} \simeq \left\{ H\text{-equiv. morphism } A \to B \right\}$$

In general, given

$$\begin{tikzcd}
\mathrm{Rep}(H) \arrow[r, "F \text{ monoidal}"] \arrow[d] & \mathcal{C} \\
A\text{-mod}^H_{fr} \arrow[ur, dashed] &
\end{tikzcd}$$

$\mathcal{C}$ ← general Λ-linear monoidal cat.

$\mathcal{C}$ is symm. monoidal cat.

① WLOG, can assume F is symmetric monoidal:

define $\mathcal{C}^{sym}$: $\quad Ob(\mathcal{C}^{sym}) = Ob(\mathrm{Rep}(H))$

$\mathrm{Hom}_{\mathcal{C}^{sym}}(V, W) = \{ \phi \in \mathrm{Hom}_{\mathcal{C}}(F(V), F(W)) : \forall\, U \in \mathrm{Rep}(H)$, the diagram

$$\begin{tikzcd}
& F(U)\otimes F(V) \arrow[rr, "id\otimes\phi"] \arrow[dl] & & F(U)\otimes F(W) \arrow[dr, "\sim"] & \\
F(U\otimes V) \arrow[d, "\wr"] & & & & F(U\otimes W) \arrow[d, "\wr"] \\
F(V\otimes U) \arrow[dr, "\sim"] & & & & F(W\otimes U) \\
& F(V)\otimes F(U) \arrow[rr] & & F(W)\otimes F(U) \arrow[ur, "\sim"] &
\end{tikzcd}$$

is comm. $\}$

The pentagon identity of $\mathcal{C}$ forces $\mathcal{C}^{sym}$ to be a sym. monoidal cat., and

$F: \mathrm{Rep}(H) \to \mathcal{C}$ factors through $\mathcal{C}^{sym}$. Moreover, the ext'n

$A\text{-mod}^{H}_{fr} \to \mathcal{C}$ that we want also must factor through $\mathcal{C}^{sym}$.

② Reduce to the affine case: $B = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathbb{1}_{\mathcal{C}}, F(\mathcal{O}(H))) \circlearrowleft H$

$$\left.\begin{array}{l} \mathcal{O}(H) \text{ comm. algebra obj. in } \mathrm{Ind}(\mathrm{Rep}(H)) \\ F \text{ sym. monoidal} \end{array}\right] \quad F(\mathcal{O}(H)) \text{ comm. alg. obj. in } \mathrm{Ind}(\mathcal{C})$$

$\rightsquigarrow$ B is a commutative algebra.

$$\Phi : \mathrm{Ess\,Im}(F) \longrightarrow B\text{-mod}^{H}_{fr}$$
$$X \longmapsto \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, X \otimes F(\mathcal{O}(H)))$$

By def'n, Φ induces an isom. $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, F(\mathcal{O}(H))) = B \xrightarrow{\sim} \mathrm{Hom}_{B\text{-mod}^{H}_{fr}}(\Phi(\mathbb{1}_{\mathcal{C}}), \Phi(F(\mathcal{O}(H))))$.

But $\mathrm{Rep}(H)$ is semisimple, and every irred. $V \in \mathrm{Rep}(H)$ is a direct summand of $\mathcal{O}(H)$,

so $\forall\, V \in \mathrm{Rep}(H)$, we have an isom.

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, F(V)) \xrightarrow{\sim} \mathrm{Hom}_{B\text{-mod}^{H}_{fr}}(\Phi(\mathbb{1}_{\mathcal{C}}), \Phi(F(V)))$$

But $\mathrm{Rep}(H)$ is rigid, so Φ is fully faithful. $\rightsquigarrow$ reduce to the affine case.

Localization of perfect complexes. Let X be an Artin stack over Λ, $U \subset X$ open immersion

w/ closed complement Z $\rightsquigarrow$ $D_{perf}(U)$ = idempotent completion of $D_{perf}(X) / D_{perf}(X)_Z$

$D_{perf}(X)_Z$: full subcat. spanned by cplxes supp. on Z.

G conn'd reductive gp / $\overline{\mathbb{F}}_q$, fix pinning (G, B, T, e)

Langlands dual $\check{G} \supset \check{B} \supset \check{T}$ / Λ,

Grothendieck altration $\tilde{\check{\mathfrak{g}}} \longrightarrow \check{\mathfrak{g}}$

Springer resolution $\tilde{\check{\mathcal{N}}} \longrightarrow \check{\mathcal{N}}$

$St := \tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}$, $St' := \tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \check{\mathcal{N}}$

$$\begin{tikzcd} C\tilde{\mathfrak{g}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \check{\mathfrak{g}} \times \check{\mathfrak{t}} \times \check{G}/\check{U} \arrow[d] \\ \tilde{\check{\mathfrak{g}}} \arrow[r] & \check{\mathfrak{g}} \times \check{\mathfrak{t}} \times \check{G}/\check{B} \end{tikzcd}$$

$C\tilde{\mathfrak{g}} \longrightarrow \check{\mathfrak{g}} \times \check{\mathfrak{t}} \times \check{G}/\check{U}$ is the "universal infinitesimal stabilizer" for the

$\check{G} \times \check{T}$ -action on $\check{G}/\check{U}$.

<u>Idea</u>. H Λ-alg gp $\curvearrowright$ smooth Λ-scheme X, every $v \in \mathfrak{h} \overset{= \mathrm{Lie}(H)}{\rightsquigarrow}$ vector field ∂_v on X

the universal infinitesimal stabilizer is a closed subscheme $Z \longrightarrow \mathfrak{h} \times X$

st the fiber at $v \times X$

$$\begin{tikzcd} Z_v \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & v \times X \arrow[d] \\ Z \arrow[r] & \mathfrak{h} \times X \end{tikzcd}$$

Z_v = infinitesimal stabilizer of v.

i.e. vanishing locus of ∂_v.

Upshot:

- There's a purely algebro-geometric way to construct Z (we can describe the ideal sheaf explicitly)
- The same construction works for affine but not necessarily smooth X.

Apply the construction to the $\check{G}\times\check{T}$-action on $\overline{\check{G}/\check{N}}$ (affine closure), get

$\overline{C\tilde{\check{\mathfrak{g}}}} \longrightarrow \overline{\check{G}/\check{N}}$ together w/ a Cartesian diagram

$$\begin{tikzcd} C\tilde{\check{\mathfrak{g}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \check{G}/\check{N} \arrow[d] \\ \overline{C\tilde{\check{\mathfrak{g}}}} \arrow[r] & \overline{\check{G}/\check{N}} \end{tikzcd}$$

$C\tilde{\check{\mathfrak{g}}} \hookrightarrow \overline{C\tilde{\check{\mathfrak{g}}}}$ is an open subscheme

(but not dense in general).

We have a very explicit description of $\mathcal{O}(\overline{C\tilde{\check{\mathfrak{g}}}})$ (Drinfeld-Plücker relation)

<u>Prop.</u> Let $\mathcal{C}$ be a Λ-linear monoidal cat. Suppose we are given

1) A monoidal functor $F: \mathrm{Rep}(\check{G}\times\check{T}) \longrightarrow \mathcal{C}$

2) A tensor endomorphism $E.$ of $F|_{\mathrm{Rep}(\check{G})}$: $E_{V_1\otimes V_2} = E_{V_1}\otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1}\otimes E_{V_2}$.

3) An action of $\mathcal{O}(\check{\mathfrak{t}})$ on F by endomorphisms: for $f\in\mathcal{O}(\check{\mathfrak{t}})$, we have

$$f_{V_1\otimes V_2} = f_{V_1}\otimes \mathrm{id}_{F(V_2)} = \mathrm{id}_{F(V_1)}\otimes f_{V_2}$$

4) A "highest wt arrow" $b_\lambda : F(V^\lambda) \longrightarrow F(\Lambda_\lambda)$ for every $\lambda\in X_*(T)^+$.

simple $\check{G}$-module of h.w. λ ; $\check{T}$-module

s.t. the following diagrams are comm.

$$\begin{tikzcd} F(V^\lambda\otimes V^\mu) \arrow[r] \arrow[d, "b_\lambda\otimes b_\mu"'] \arrow[dr, phantom, "\text{///}"] & F(V^{\lambda+\mu}) \arrow[d, "b_{\lambda+\mu}"] \\ F(\Lambda_\lambda)\otimes F(\Lambda_\mu) \arrow[r, "\sim"'] & F(\Lambda_{\lambda+\mu}) \end{tikzcd} \qquad \begin{tikzcd} F(V^\lambda) \arrow[r, "b_\lambda"] \arrow[d, "E_{V^\lambda}"'] \arrow[dr, phantom, "\text{///}"] & F(\Lambda_\lambda) \arrow[d, "\lambda"] \\ F(V^\lambda) \arrow[r, "b_\lambda"] & F(\Lambda_\lambda) \end{tikzcd}$$

then F extends uniquely to a monoidal functor $\mathrm{Coh}^{\check{G}\times\check{T}}_{fr}(\overline{C\tilde{\check{\mathfrak{g}}}}) \longrightarrow \mathcal{C}$

s.t. E goes to the tautological endomorphism on $\mathrm{Coh}^{\check{G}}_{fr}(\tilde{\check{\mathfrak{g}}})$, $\mathcal{O}(\check{\mathfrak{t}})$-action comes from

[illegible]

the projection $\overline{C\check{\mathfrak{g}}} \to \check{\mathfrak{t}}$, and the h.w. arrows come from $\mathcal{O}(\check{G}/\check{U})$.

Using this, we were able to construct a monoidal functor $\mathrm{Coh}^{\check{G}\times\check{T}}_{fr}(\overline{C\check{\mathfrak{g}}}) \to P_{I^+I^+}$

extending the monodromic central sheaves & the monodromic Wakimoto modules.

$$\begin{tikzcd}
\overline{C_{st}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \check{\mathfrak{g}} \arrow[d, "\Delta"] \\
\overline{C\check{\mathfrak{g}}} \times \overline{C\check{\mathfrak{g}}} \arrow[r] & \check{\mathfrak{g}} \times \check{\mathfrak{g}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{C_{st'}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \check{\mathfrak{t}} \times \{0\} \arrow[d] \\
\overline{C_{st}} \arrow[r] & \check{\mathfrak{t}} \times \check{\mathfrak{t}}
\end{tikzcd}$$

$C_{st} \subset \overline{C_{st}}$, $C_{st'} \subset \overline{C_{st'}}$,

$St \simeq C_{st} / \check{T}\times\check{T}$, $St' \simeq C_{st'} / \check{T}\times\check{T}$.

Now we apply the previous prop. in the following setting: $\check{G}$ is replaced by $\check{G}\times\check{T}$
$\check{T}$ $\check{T}\times\check{T}$,

$\mathcal{C} = \mathrm{Fun}(D_{I^+I^+}, D_{I^+I^+})$

Two actions of $\mathrm{Rep}(\check{T})$: left & right convolution w/ Wakimoto sheaves

Action of $\mathrm{Rep}(\check{G}^2)$: restrict to diagonal $\check{G}$, then apply central functor equipped w/ nearby cycle monodromy

$\mathcal{O}(\check{\mathfrak{t}}^2)$-action: torus monodromy.

Highest weight arrows.

$\rightsquigarrow \mathrm{Coh}^{\check{G}^2\times\check{T}^2}_{fr}(\overline{C\check{\mathfrak{g}}}^2) \to \mathrm{Fun}(D_{I^+I^+}, D_{I^+I^+})$

the action of $\check{h}^2$ factors through restriction to diagonal ⇝ factors through

$$\mathrm{Coh}^{\check{h}\times\check{t}^2}_{fr}(\overline{C_{\check{\mathfrak{g}}}^2}) \longrightarrow \mathrm{Fun}(D_{I^+I^+}, D_{I^+I^+})$$

The tensor endomorphism factors through diagonal ⇝ factors through

$$\mathrm{Coh}^{\check{h}\times\check{T}^2}_{fr}(\overline{C_{St}}) \longrightarrow \mathrm{Fun}(D_{I^+I^+}, D_{I^+I^+}).$$

Similarly, get $\mathrm{Coh}^{\check{h}\times\check{T}}_{fr}(\overline{C_{St'}}) \longrightarrow \mathrm{Fun}(D_{I^+I}, D_{I^+I})$

Evaluation at $T_{w_0}^{mon}$, T_{w_0} gives

$$\Phi_{fr} : \mathrm{Coh}^{\check{h}\times\check{t}^2}_{fr}(\overline{C_{St}}) \longrightarrow P^{Wak}_{I^+I^+} \longrightarrow D_{I^+I^+}$$

$$\Phi_{fr'} : \mathrm{Coh}^{\check{h}\times\check{t}^2}_{fr}(\overline{C_{St'}}) \longrightarrow P^{Wak}_{I^+I} \longrightarrow D_{I^+I}.$$

<u>Perfect complexes</u>. • Let X be an Artin stack $/\Lambda$, $U\subset X$ open immersion, Z closed complement ⇝ $D_{perf}(U)$ = idempotent completion of $D_{perf}(X)/D_{perf}(X)_Z$

↑ full subcat. spanned by cpxes supp. on Z.

• $X = Y/H$ H reductive gp $/\Lambda$, Y affine, of finite type

then $\mathrm{Coh}(X) = \mathrm{Coh}^H(Y)$ has enough projectives,

$D^-\mathrm{Coh}(X) \xrightarrow{\sim} D^-_{coh}(X)$.

$\Phi_{fr}: \mathrm{Coh}_{fr}^{\check{G}\times\check{T}^2}(\overline{C_{st}}) \longrightarrow D_{I^+ I^+}$.

Taking left derived functor (right Kan ext'n) ($\mathrm{Coh}_{fr}^{\check{G}\times\check{T}}(\overline{C_{st}})$ consists of compact, projective generators of $\mathrm{Coh}^{\check{G}\times\check{T}}(\overline{C_{st}})$)

$$\begin{tikzcd} L\Phi_{fr}: D^- \mathrm{Coh}^{\check{G}\times\check{T}^2}(\overline{C_{st}}) \arrow[r] \arrow[d, "\wr"] & D_{I^+I^+} \\ D^-_{\mathrm{coh}}(\check{G}\backslash \overline{C_{st}}/\check{T}^2) \arrow[ur] & \end{tikzcd}$$

restricting to perfect cpxes $\rightsquigarrow$ $D_{perf}(\check{G}\backslash \overline{C_{st}}/\check{T}^2) \longrightarrow D_{I^+I^+}$.

Moreover, it factors through $D_{perf}(\check{G}\backslash \overline{C_{st}}/\check{T}^2) \longrightarrow \underbrace{D^{Wak}_{I^+I^+}} \longrightarrow D_{I^+I^+}$

full subcat. gen. by Wakimoto sheaves under fibers & cofibers:

$$\begin{tikzcd} D_{perf}(\check{G}\backslash \overline{C_{st}}/\check{T}^2) \arrow[r] \arrow[dr] & D^{Wak}_{I^+I^+} \arrow[d, "gr"] \\ & D_{perf}(\check{T}\backslash \cdot) \end{tikzcd}$$

Corresponding to pullback along

$$\check{T}\backslash \cdot \longrightarrow \check{G}\backslash \tilde{\mathfrak{g}} \xrightarrow{\Delta} \check{G}\backslash St \simeq \check{G}\backslash C_{st}/\check{T}^2 \subset \check{G}\backslash \overline{C_{st}}/\check{T}^2$$

$\cdot \longrightarrow \tilde{\mathfrak{g}} = \check{G}\overset{\check{B}}{\times}\check{b}$ denotes the point $[1:0]$.

reduce to the fact:
- monodromy acts trivially on the Wakimoto grading
- the h.w. arrow is projection of V^λ to the h.w. space $V^\lambda(\lambda)$.

$D_{perf}(\check{G}\backslash \overline{C_{st}}/\check{T}^2)_{\dots} \longrightarrow 0$. But $D_{I^+I^+}$ is Karoubian, so

get
$$\begin{tikzcd}
D_{perf}(\check{H}\backslash C_{St}/\check{T}^2) \arrow[r] \arrow[d, phantom, "\wr"] & D_{I^+I^+} \\
D_{perf}(\check{H}\backslash St) \arrow[ur] &
\end{tikzcd}$$

the log monodromy endomorphism is pro-nilpotent

$\rightsquigarrow$ factors through $\Phi_{perf} : D_{perf}(\check{H}\backslash \widehat{St}) \longrightarrow D_{I^+I^+}$.

completion of St along the inverse image of $\check{\mathcal{N}} \subset \check{\mathfrak{g}}$. NOT a formal scheme.

Similarly, have a functor $\Phi_{perf'} : D_{perf}(\check{H}\backslash St') \longrightarrow D_{I^+I}$.

<u>Prop</u>. We have the following commutative diagrams

$$\begin{tikzcd}
D_{perf}(\check{H}\backslash \widehat{St}) \arrow[r, "i^*"] \arrow[d, "\Phi_{perf}"'] \arrow[dr, phantom, "\wr\wr\wr"] & D_{perf}(\check{H}\backslash St') \arrow[d, "\Phi_{perf'}"] \\
D_{I^+I^+} \arrow[r, "\pi_*"'] & D_{I^+I}
\end{tikzcd} \qquad ①$$

$$\begin{tikzcd}
D^b Coh^{\check{H}}(\widehat{\tilde{\mathfrak{g}}}) \arrow[r, "pr^*_{Spr,1}"] \arrow[d, "\Phi_{IW}"'] \arrow[dr, phantom, "\wr\wr\wr"] & D_{perf}(\check{H}\backslash \widehat{St}) \arrow[d, "\Phi_{perf}"] \\
D^{I^+}_{IW} \arrow[r, "Av_{IW}"'] & D_{I^+I^+}
\end{tikzcd} \qquad ②$$

$$\begin{tikzcd}
D^b Coh^{\check{H}}(\widehat{\tilde{\mathfrak{g}}}) \arrow[r, "(pr'_{Spr,1})^*"] \arrow[d, "\Phi_{IW}"'] \arrow[dr, phantom, "\wr\wr\wr"] & D_{perf}(\check{H}\backslash St') \arrow[d, "\Phi_{perf'}"] \\
D^{I^+}_{IW} \arrow[r, "\pi_* \circ Av_{IW}"'] & D_{I^+I}
\end{tikzcd} \qquad ③$$

Pf. Commutativity of ① basically follows from the construction and $\pi_*(T_{w_0}^{mon}) \simeq T_{w_0}$

↑

the tannakian construction.

Commutativity of ② also follows from the tannakian construction and the fact

that $Av_{IW}(\Delta_e^{IW}) \simeq T_{w_0}^{mon}$ (proved by Daniel last time).

The third diagram is a concatenation of the first two.

Prop. $\Phi_{perf} : D_{perf}(\check{G}\backslash \hat{St}) \longrightarrow D_{I^+I^+}$ is fully faithful.

Pf. It suffices to show that for $V, V' \in Rep(\check{G})$, $\lambda, \mu, \lambda', \mu' \in X_*(T)$,

Φ_{perf} induces a homotopy equivalence between mapping spaces

$$Map_{D_{perf}(\check{G}\backslash \hat{St})}\left(V \otimes \mathcal{O}_{\hat{St}}(\lambda,\mu),\ V' \otimes \mathcal{O}_{\hat{St}}(\lambda',\mu') \right)$$

$$\xrightarrow{\sim} Map_{D_{I^+I^+}}\left(Z^{mon}(V) * J_\lambda^{mon} * T_{w_0}^{mon} * J_\mu^{mon},\ Z^{mon}(V') * J_{\lambda'}^{mon} * T_{w_0}^{mon} * J_{\mu'}^{mon} \right)$$

By adjunction, reduce to the case $\lambda = \mu' = 0$, $V = triv$.

$$RHS = Map_{D_{I^+I^+}}\left(T_{w_0}^{mon} * J_\mu^{mon},\ Z^{mon}(V') * \underbrace{J_{\lambda'}^{mon} * T_{w_0}^{mon}}_{\simeq Av_{IW} \circ Av^{IW}} \right)$$

$$\simeq Map_{D_{IW}^{I^+}}\left(Av^{IW}(\underbrace{T_{w_0}^{mon} * J_\mu^{mon}}),\ Av^{IW}(J_{\lambda'}^{mon} * Z^{mon}(V')) \right)$$

$$\underset{\text{monodromic AB}}{\simeq} Map_{D^bCoh^{\check{G}}(\hat{\tilde{\mathfrak{g}}})}\left(p_{Spr}^* \, p_{Spr*}(\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}}(\mu)),\ \mathcal{O}_{\hat{\tilde{\mathfrak{g}}}}(\lambda') \otimes V' \right)$$

flat base change

$$\simeq \mathrm{Map}_{D^b\mathrm{Coh}^{\check{G}}(\hat{\mathfrak{g}})}\left(pr_{\mathrm{Spr},2*}\, pr_{\mathrm{Spr},1}^{*}\left(\mathcal{O}_{\hat{\mathfrak{g}}}(\mu)\right), \mathcal{O}_{\hat{\mathfrak{g}}}(\lambda')\otimes V'\right)$$

$pr_{\mathrm{Spr},2}^{*} = pr_{\mathrm{Spr},2}^{!}$

$$\simeq \mathrm{Map}_{D_{\mathrm{perf}}(\check{G}\backslash \hat{St})}\left(pr_{\mathrm{Spr},1}^{*}(\mathcal{O}(\hat{\mathfrak{g}}(\mu))), pr_{\mathrm{Spr},2}^{*}\left(\mathcal{O}_{\hat{\mathfrak{g}}}(\lambda')\otimes V'\right)\right)$$

everything is Calabi-Yau

$\simeq$ LHS.

We need to check that this homotopy equivalence is the same as the map induced by Φ_{perf}. This amounts to the content of previous proposition.