Topics on Drinfeld modules and T-motives

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Lecture 1 Contents.

I. Basic theory of Drinfeld modules

Let F be a field w/ $\mathrm{char}(F) = p > 0$.

$$\tau_p : F \longrightarrow F, \text{ Frobenius endomorphism}$$
$$\alpha \longmapsto \alpha^p$$

§I.1 Additive polynomials

Def. A poly. $f(x) \in F[X]$ is additive, if $f(X+Y) = f(X) + f(Y) \in F[X,Y]$.

Prop. A poly. $f(x) \in F[X]$ is additive, iff $f(x) = \sum_{i=0}^{n} a_i X^{p^i}$, $a_i \in F$.

Pf. Exercise.

Rmk (1) If $\mathrm{char}\, F = 0$, then every additive polynomial in $F[X]$ is of the form ax, $a \in F$

(2) Let V be a fin. dim'l $\mathbb{F}_p$- vec. sp.,
$\wedge$
F (char $F = p > 0$)

$$f_V(x) = x \cdot \prod_{0 \neq v \in V} \left(1 - \frac{x}{v}\right) \in F[x] \text{ is additive.}$$

Let $F\{x\}$ be the set of all additive poly. in $F[x]$,

Observe: for $f_1, f_2 \in F\{x\}$, $f_1 \circ f_2 (x) := f_1(f_2(x)) \in F\{x\}$.

$\Rightarrow (F\{x\}, +, \circ)$ is a ring.

Question. When is it commutative ?

Rmk. (1) $F\{x\}$ is the endomorphism ring of $\mathbb{G}_a$ over F.

(2) Let $F[\tau_p]$ be the twisted polynomial ring over F w/ the multiplication law $\tau_p \cdot a = a^p \tau_p$, $\forall a \in F$.

Then we have the isom. $F[\tau_p] \xrightarrow{\sim} F\{x\}$

$$\sum_{n=0}^{N} a_n \tau_p^n \mapsto \sum_{n=0}^{N} a_n x^{p^n}$$

Recall. several needed properties:

Prop (Right division algorithm) Given $f, g \in F[\tau_p]$ w/ $g \neq 0$,

$\exists!\ h, r \in F[\tau_p]$ s.t. $\deg_{\tau_p} r < \deg_{\tau_p} g$ and $f = hg + r$.

Consequently, every left ideal of $F[\tau_p]$ is principal.

__Exercise__. Suppose F is perfect (i.e. τ_p is an isom), show that we have left division algorithm on $F[\tau_p]$.

__Exercise__. Suppose F is perfect, given a matrix $M \in Mat_{m\times n}(F[\tau_p])$, show that $\exists$ $U \in GL_m(F[\tau_p])$ and $V \in GL_n(F[\tau_p])$ s.t.

$$UAV = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \in Mat_{m\times n}(F[\tau_p])$$

Consequently, every f.g. left $F[\tau_p]$-module is isom. to

$$\bigoplus_{i=1}^{m} \frac{F[\tau_p]}{F[\tau_p]\cdot f_i} \quad \text{for } m \geq 0 \text{ and } f_i \in F[\tau_p].$$

__Rmk__. Given $l \in \mathbb{Z}_{>0}$, put $q = p^l$, $\tau_q = \tau_p^l$. Suppose the finite field $\mathbb{F}_q$ is contained in F, then the subring $F[\tau_q] \subset F[\tau_p]$ satisfies

$$\tau_q \cdot a = a^q \tau_q, \ \forall a \in F, \text{ and its image in } F\{x\}$$

$$\begin{array}{ccc} F[\tau_p] & \xrightarrow{\sim} & F\{x\} \\ \cup & & \cup \\ F[\tau_q] & \xrightarrow{\sim} & \{f(x) \in F\{x\} : f(x) \text{ is } \mathbb{F}_q\text{-linear}\} \end{array}$$

Suppose $V \subset F$ is an $\mathbb{F}_q$-vec. subspace of fin. dim., then $f_V(x) = x \cdot \prod_{0 \neq v \in V} \left(1 - \frac{x}{v}\right)$ is $\mathbb{F}_q$-linear.

<u>Def</u>. Given $f = a_0 + \cdots + a_n \tau_p^n \in F[\tau_p]$,

we set $\partial f := a_0$, called the derivative of f.

$\left(f(X) = a_0 X + a_1 X^p + \cdots + a_n X^{p^n}, \quad \frac{d}{dX} f(X) = a_0 = \partial f \right)$

Moreover, $\partial : F[\tau_p] \longrightarrow F$ is a ring hom.

<u>I.2 Definition of Drinfeld modules</u>

$\mathbb{F}_q$ finite field w/ q elts

$A = \mathbb{F}_q[t]$

An <u>A-field</u> is a pair (F, ι) where $\iota : A \longrightarrow F$ is a ring hom.

The kernel of ι is called the A-characteristic of F, denoted by $\mathrm{char}_A(F)$.

Put $\tau = \tau_q$.

<u>Def</u>. Given an A-field (F, ι), a Drinfeld A-module over F is a ring hom. $\rho : A \longrightarrow F[\tau]$ satisfying $\rho(A) \not\subset F$, and $\partial \circ \rho = \iota$.

$\left(\begin{array}{c} F[\tau] \\ \cong \\ \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a) \end{array} \right)$

Lecture 2

Rmk $\tau = \tau_q : \bar{F} \to \bar{F}$
$a \mapsto a^q$

$\rightsquigarrow$ $\bar{F}$ has an $F[\tau]$-module str.

ρ induces an A-module str on $\bar{F}$. diff. from the one induced by τ.

(2) ρ is actually uniquely determined by

$$\rho_t = \iota(t) + a_1\tau + \cdots + a_r\tau^r \quad \text{for some } r > 0$$

We call r the rank of ρ.

Moreover, for $a \in A$, $\deg_\tau \rho_a = r \deg a$.

(3) There is a notion of Drinfeld A-modules when A is the "ring of integers" in a global function field (w.r.t. a chosen place).

[DH87] Deligne – Husemöller, Survey on Drinfeld modules

[Go96] Basic structures of function field arithmetic.

[Ro02] Rosen, Number theory in function fields.

Example (Carlitz module) Let (F, ι) be an A-field.

Define $C : A \to F[\tau]$

$\overset{\parallel}{\mathbb{F}_q[t]}$ $t \mapsto C_t := \iota(t) + \tau$

Then $C_a = \sum_{i=0}^{\deg a} \iota(c_a^{(i)}) \tau^i \in F[\tau]$, where $c_a^{(i)} = \frac{(c_a^{(i-1)})^q - c_a^{(i-1)}}{t^{q^i} - t} \in A$ $\quad (c_a^{(0)} := a)$

Given an A-field F, suppose $\mathrm{char}_A(F) = (p) \neq 0$ for a monic $p \in A$

<u>Lemma</u>. For a Drinfeld A-module ρ of rank r over F, $\exists\ 0 < h \leq r$

s.t. $\rho_p = \underset{\neq 0}{c_{h \deg p}} \cdot \tau^{h \deg p} + \cdots + \underset{\neq 0}{c_{r \deg p}} \tau^{r \deg p}$

(We call h the <u>height</u> of ρ)

<u>Pf</u> Consider $\rho[p] := \{\alpha \in \bar{F} : \rho_p(\alpha) = 0\}$ $\quad (\rho_p(x) \in F\{x\})$

Then for $a \in A$ and $\alpha \in \rho(p)$, one has

$\rho_p(\rho_a(\alpha)) = \rho_a(\rho_p(\alpha)) = \rho_a(0) = 0$

$\Rightarrow \rho_a(\alpha) \in \rho[p]$.

$\therefore$ $\rho[p]$ is an A-submodule of $\bar{F}$ (via ρ)

and $\rho_p(\alpha) = 0$, $\forall \alpha \in \rho[p]$

$\Rightarrow$ $\rho[p]$ is a finite A/p-module (via ρ)

Write $\dim_{A/p} \rho[p] =: s \Rightarrow \#(\rho[p]) = q^{s \cdot \deg p}$.

On the other hand, $\rho_p(x) = \iota(\cancel{p}) x + c_1 x^q + \cdots + c_{r \deg p} \cdot x^{q^{r \deg p}}$

$\Rightarrow s < r$ and $c_i = 0$ if $i < (r-s) \deg p$ and $c_{(r-s) \deg p} \neq 0$.

Let $h = r - s$ $(0 < h \leq r)$ $\Rightarrow$ done! $\square$

Def. Let ρ and ρ' be two Drinfeld A-modules over a given A-field F.

A homomorphism $f: \rho \to \rho'$ over F is a twisted poly. $f \in F[\tau]$ s.t.

$$f \rho_a = \rho'_a f \quad , \forall a \in A.$$

We call f an isogeny, if $f \neq 0$.

Rmk. Identifying $F[\tau]$ w/ $\mathrm{End}_{F\text{-gp}}(\mathbb{G}_a)$, a homomorphism between ρ and ρ' is a group endo. on $\mathbb{G}_a$ preserving the A-module str. via ρ and ρ'.

Exercise: Given two Drinfeld modules ρ and ρ' over an A-field F, we have

$$\begin{array}{ll} \mathrm{Hom}_F(\rho, \rho') = 0 & \text{unless} \quad \mathrm{rank}(\rho) = \mathrm{rank}(\rho') \\ \quad \shortparallel & \qquad\quad\;\; \mathrm{height}(\rho) = \mathrm{height}(\rho'). \\ \{f: \rho \to \rho' \text{ hom}\} & \end{array}$$

If $\mathrm{Hom}_F(\rho, \rho') \neq 0$, we call ρ and ρ' are isogenious.

Lemma: Let ρ and ρ' be two Drinfeld A-modules of rank r over an A-field F.

Given an isogeny $f: \rho \to \rho'$, $\exists\ \hat{f}: \rho' \to \rho$ and $0 \neq a \in A$ s.t.

$$\hat{f} f = \rho_a \quad \text{and} \quad f \hat{f} = \rho'_a.$$

Pf. Given $0 \neq f \in \mathrm{Hom}_F(\rho, \rho')$, consider $\ker f = \{\alpha \in \bar{F} : f(\alpha) = 0\}$.

$\rightsquigarrow \ker f \subset \bar{F}$ is an A-submodule (via ρ). finite.

① $\boxed{f(x) \text{ is separable}}$ $\Rightarrow \exists\, 0 \neq a \in A$ s.t. $\rho_a(\alpha) = 0$, $\forall\, \alpha \in \ker f$.

Take $\hat{f}, r \in F[\tau]$ w/ $\deg r < \deg_\tau f$ s.t. $\rho_a = \hat{f} \cdot f + r$.

Then $r(\alpha) = 0$, $\forall\, \alpha \in \ker f$.

$\Rightarrow r = 0$ and $\rho_a = \hat{f} \cdot f$.

Moreover, $f \cdot \hat{f} \cdot f = f \rho_a = \rho'_a f \Rightarrow (f \cdot \hat{f} - \rho'_a)\, f = 0$ in $F[\tau]$

$\Rightarrow f \cdot \hat{f} = \rho'_a$. $\square$

Lecture 3. Recall $A = \mathbb{F}_q[t]$. fix an A-field F, i.e. $\iota : A \to F$.

ρ, ρ' : Drinfeld A-modules defined over F.

A homomorphism (an isogeny) from ρ to ρ' over F is a twisted poly.

$f \in F[\tau]$ satisfying $f \cdot \rho_a = \rho'_a \cdot f$, $\forall a \in A$.

i.e. $f : (\mathbb{G}_{a\,\bar{F}}, \rho) \twoheadrightarrow (\mathbb{G}_{a\,\bar{F}}, \rho')$ w/ finite kernel.

The following Lemma implies that being isogenous is an equiv. relation among Drinfeld modules.

Lemma ρ, ρ' Drinfeld A-modules defined over F.

$\forall$ isogeny $f: \rho \to \rho'$ over F, $\exists$ an isogeny $\hat{f}: \rho' \to \rho$ and nonzero $a \in A$

s.t. $\hat{f} \cdot f = \rho_a$ & $f \cdot \hat{f} = \rho'_a$ (in $F[\tau]$)

Pf Given an isogeny $f: \rho \to \rho'$, we write

$$f = (c_0 + c_1\tau + \cdots + c_\ell \tau^\ell)\tau^m \quad \text{w/ } c_0 \neq 0,\ c_\ell \neq 0,\ m \geq 0$$

Case 1. Suppose $m = 0$

$\Rightarrow f(X) = c_0 X + c_1 X^q + \cdots + c_\ell X^{q^\ell}$

$\Rightarrow f(X) \in F[X]$ is a separable poly.

Put $\ker f = \{\alpha \in \bar{F} : f(\alpha) = 0\}$.

Note $\forall \alpha \in \ker f$, $f(\rho_a(\alpha)) = \rho'_a(f(\alpha)) = 0$

$\Rightarrow \rho_a(\alpha) \in \ker f$ $\therefore$ $\ker f \subset (\bar{F}, \rho)$ is a finite A-submod
$\rho \circlearrowright A$

$\Rightarrow \exists\ 0 \neq a \in A$ s.t. $\rho_a(\alpha) = 0$, $\forall \alpha \in \ker f$.

Apply right division algorithm for ρ_a & f

$\Rightarrow$ have $\hat{f}, r \in F[\tau]$ s.t.
- $\deg_\tau r < \deg_\tau f$
- $\rho_a = \hat{f} \cdot f + r$.

$\Rightarrow \forall \alpha \in \ker f,\ r(\alpha) = 0$

$\Rightarrow$ # of zeroes of $r(X) \geq \# \ker f = \deg_X f(X)$

$$\underset{\deg_X r(X)}{\shortparallel}$$

$\Rightarrow r = 0$, $\rho_a = \hat{f} \cdot f$.

Moreover, $f \cdot \hat{f} \cdot f = f \cdot \rho_a = \rho'_a f$

$\Rightarrow \left(f \cdot \hat{f} - \rho'_a\right) \underset{\neq 0}{f} = 0 \in F[\tau] \Rightarrow f \cdot \hat{f} - \rho'_a = 0$.

<u>Claim</u>, $\hat{f} : \rho' \to \rho$ is an isogeny.

<u>Pf</u> $\forall b \in A$, to show $\forall b \in A$, $\hat{f} \cdot \rho'_b = \rho_b \cdot \hat{f}$.

Consider $f \cdot \hat{f} \cdot \rho'_b = \rho'_a \rho'_b = \rho'_{ab} = \rho'_{ba} = \rho'_b \rho'_a = \rho'_b f \hat{f} = f \rho_b \hat{f}$

$\Rightarrow \hat{f} \cdot \rho'_b = \rho_b \hat{f}$.

<u>Case 2</u> Suppose $m > 0$.

Consider $f \cdot \rho_a = \rho'_a \cdot f$, $\forall a \in A$

$$\left(c_0 + c_1\tau + \dots + c_\ell \tau^\ell\right)\tau^m \left(\iota(a) + \dots + * \tau^{2\deg a}\right)$$

the coeff. of τ^m LHS $c_0 \iota(a)^{q^m}$

RHS $\iota(a) \cdot c_0$

$\rightarrow c_0\left(\iota(a)^{q^m} - \iota(a)\right) = 0,\ \forall a \in A \Rightarrow \iota(a)^{q^m} = \iota(a),\ \forall a \in A$

$\Rightarrow \iota(A) \subset$ finite subfield of F.

$\because A/\ker \iota \hookrightarrow F$, $\ker \iota = (p)$ for $p \in A$ monic irreducible

Write $\rho_t = \iota(t) + a_1 \tau + \cdots + a_r \tau^r$

Define the Drinfeld A-module $\tilde{\rho}$ by

$$\tilde{\rho}_t := \iota(t) + a_1^{q^m} \tau + \cdots + a_r^{q^m} \tau^r$$

$$\Rightarrow \tau^m \rho_t = \tilde{\rho}_t \tau^m \Rightarrow \tau^m \rho_a = \tilde{\rho}_a \tau^m, \ \forall a \in A.$$

Recall $f = \underbrace{(c_0 + c_1 \tau + \cdots + c_\ell \tau^\ell)}_{\substack{\shortparallel \\ f^s}} \tau^m$

Ex. $f^s \tilde{\rho}_a = \rho'_a f^s$, $\forall a \in A$

i.e. $f^s : \tilde{\rho} \to \rho'$ is an isogeny.

By ①, for f^s & $\tilde{\rho}$ & ρ' $\Rightarrow \exists\, 0 \neq a_0 \in A$ s.t.

$$\hat{f^s} \cdot f^s = \tilde{\rho}_{a_0} \ \& \ f^s \cdot \hat{f^s} = \rho'_{a_0}$$

Note $\iota : A/(p) \hookrightarrow F$

$$\rho_p = \underset{\substack{\shortparallel \\ 0}}{\iota(p)} + g_1 \tau + \cdots + g_{r \deg p} \tau^{r \deg p}$$

$$\rho_{p^m} = \underbrace{\rho_p \cdots \rho_p}_{m} = g \cdot \tau^m \text{ for some } g \in F[\tau]$$

Define $a_1 = p^m a_0$ & $\hat{f} = g\, \widehat{f^s}$

$$\Rightarrow \hat{f}\cdot f = g\,\widehat{f^s}\, f^s \tau^m \xlongequal{\widehat{f^s}\cdot f^s = \tilde{\rho}_{a_0}} g\cdot \tilde{\rho}_{a_0}\tau^m \xlongequal{\tau^m \rho_a = \tilde{\rho}_a \tau^m} g\tau^m \rho_{a_0}$$

$$= \rho_{p^m}\rho_{a_0} = \rho_{p^m a_0} = \rho_{a_1}$$

Following the arguments of ①, we can derive $f\cdot\hat{f} = \rho'_{a_1}$

& $\hat{f}\cdot\rho'_b = \rho_b \hat{f}$, $\forall b \in A$. □

<u>Def</u> An isogeny $f: \rho \to \rho'$ of two Drinfeld A-modules ρ & ρ' over F is called an isom. over F if $\exists\, g: \rho' \to \rho$ isogeny over F s.t.

$f\cdot g = 1$ & $g\cdot f = 1$.

(i.e. $(\mathbb{G}_{a_{\bar{F}}}, \rho) \xrightarrow[f]{\simeq} (\mathbb{G}_{a_{\bar{F}}}, \rho')$)

<u>Exercise</u>. Such an isom. f must be in $F^\times$.

Fix ρ: Drinfeld A-module defined over F.

Define $\mathrm{End}_F(\rho) := \{ \underset{(\text{isogeny})}{\text{hom.}}\ f: \rho \to \rho \text{ over } F \} \subset F[\tau]$

called the endomorphism ring of the Drinfeld A-module ρ.

$\because \rho_a \rho_b = \rho_{ab} = \rho_{ba} = \rho_b \rho_a$, $\forall a, b \in A$ $\Rightarrow \underset{\text{comm. subring}}{\rho(A)} \subset \mathrm{End}_F(\rho)$.

Note We have an A-module str. on $\mathrm{End}_F(\rho)$ by $A \times \mathrm{End}_F(\rho) \to \mathrm{End}_F(\rho)$

$$a * f := \rho_a \cdot f \in F[\tau]$$

Thm ρ : Drinfeld A-module of rank r over F

$\Rightarrow$ ① $\mathrm{End}_F(\rho)$ is a free A-module of rank $\leq r^2$

② $k := \mathbb{F}_q(t)$ fraction field A, then

$\mathrm{End}_F(\rho) \otimes_A k$ is a division algebra over k.

Lecture 4. Pf. ② Note that every elt in $\mathrm{End}_F(\rho) \otimes_A k$ can be written as $f \otimes \frac{1}{c}$ for some $f \in \mathrm{End}_F(\rho)$, $c \in k^\times$

If $f \neq 0$, take $\hat{f}$ and a in the previous Lemma

$$(\hat{f} \cdot f = \rho_a = f \cdot \hat{f})$$

$$\Rightarrow (f \otimes \tfrac{1}{c})(\hat{f} \otimes \tfrac{c}{a}) = f\hat{f} \otimes \tfrac{1}{a} = \rho_a \otimes \tfrac{1}{a} = 1 \otimes \tfrac{a}{a} = 1 \text{ in } \mathrm{End}_F(\rho) \otimes_A k$$

Also $(\hat{f} \otimes \frac{c}{a})(f \otimes \frac{1}{c}) = 1$ $\therefore$ Every non zero elt in $\mathrm{End}_F(\rho) \otimes_A k$ has an inverse $\Rightarrow \mathrm{End}_F(\rho) \otimes_A k$ is a div. alg.

① Given a left $F[t]$ - module str. on $F[\tau]$:

$$F[t] \cong F \otimes_{\mathbb{F}_q} \mathbb{F}_q[t] \qquad (\text{via } \rho)$$

$$F \otimes_{\mathbb{F}_q} \mathbb{F}_q[t] \times F[\tau] \longrightarrow F[\tau]$$

$$(\alpha \otimes a) * f \longmapsto \alpha \cdot f \cdot \rho_a$$

Exercise: ① Show that $F[\tau]$ is a free $\mathbb{F}[t]$-module of rank r

(hint: $\{1, \tau, \dots, \tau^{r-1}\}$ is an $F[t]$-basis of $F[\tau]$).

② $\phi : F \otimes_{\mathbb{F}_q} \mathrm{End}_F(\rho) \longrightarrow \mathrm{End}_{F[t]}(F[\tau])$

$$\alpha \otimes f \longmapsto \phi_{\alpha \otimes f},$$

where $\phi_{\alpha\otimes f}(g) := \alpha g f, \ \forall g \in F[\tau]$.

Check that ϕ is a ring anti-homomorphism

Claim: ϕ is injective.

$\Rightarrow$ $F \otimes_{\mathbb{F}_q} \mathrm{End}_F(\rho)$ is actually a free $F[t]$-module of rank $\overset{n}{\leq} r^2$.

Take an $F[t]$-base $\{ \sum_{j=1}^{n_i} \alpha_{ij} \otimes f_{ij} =: \tilde{f_i} : i = 1, \dots, n \}$ of $F \otimes_{\mathbb{F}_q} \mathrm{End}_F(\rho)$.

Then, we can find n elts $\{ \underset{g_1}{\underset{\parallel}{f_{i_1 j_1}}}, \dots, \underset{g_n}{\underset{\parallel}{f_{i_n j_n}}} \}$ $\in \mathrm{End}_F(\rho)$ which form an

$F[t]$-base of $F \otimes_{\mathbb{F}_q} \mathrm{End}_F(\rho)$. ((Check!))

Then $F \otimes_{\mathbb{F}_q} \left(\dfrac{\mathrm{End}_F(\rho)}{\bigoplus_{i=1}^{n} A \cdot g_i} \right) = 0 \Rightarrow \mathrm{End}_F(\rho) = \bigoplus_{i=1}^{n} A \cdot g_i.$

Pf of claim: Suppose $\phi_{\sum_{i=1}^{m} \alpha_i \otimes f_i} = 0$ in $\mathrm{End}_{F[t]}(F[\tau])$

$$\begin{array}{c} \phi_{\sum_{i=1}^{m} \alpha_i \otimes f_i} \\ \| \\ \sum_{i=1}^{m} \phi_{\alpha_i \otimes f_i} = \alpha_i \sum_{i=1}^{m} \phi_{1 \otimes f_i} \quad (\alpha_i \in F) \end{array}$$

May assume ① m is minimal

② $\alpha_1 = 1$

Want: $\alpha_i \in \mathbb{F}_q \Rightarrow \overset{0}{\underset{\|}{}}\sum_{i=1}^{m} \phi_{\alpha_i \otimes f_i} = \phi_{1 \otimes \sum_{i=1}^{m} \alpha_i f_i}$

$$\Rightarrow \phi_{1 \otimes \sum_{i=1}^{m} \alpha_i f_i}(1) = 1 \cdot 1 \cdot \sum_{i=1}^{m} \alpha_i f_i = \sum_{i=1}^{m} \alpha_i f_i$$

$$\begin{array}{c} \| \\ 0 \end{array}$$

$\Rightarrow \phi_{1 \otimes \sum_{i=1}^{m} \alpha_i f_i} = 0.$ $\therefore$ ϕ is injective.

$\because \sum_{i=1}^{m} \phi_{\alpha_i \otimes f_i}(f) = 0, \quad \forall f \in F[\tau]$

$$\Rightarrow 0 = \tau \cdot \sum_{i=1}^{m} \phi_{\alpha_i \otimes f_i}(f) = \tau \cdot \sum_{i=1}^{m} \alpha_i f f_i$$

$$= \sum_{i=1}^{m} \alpha_i^q \tau f f_i = \sum_{i=1}^{m} \phi_{\alpha_i^q \otimes f_i}(\tau f) \quad , \forall f \in F[\tau]$$

$\therefore \sum_{i=1}^{m} \phi_{\alpha_i^q \otimes f_i}$ vanishes on $\tau F[\tau]$.

Note that for every f, $(\underset{\underset{F[t]}{\uparrow}}{t} - 2(t)) * f \in \tau F[\tau]$.

$\therefore \sum_{i=1}^{m} \phi_{\alpha_i^q \otimes f_i}\left((t-\iota(t)) * f\right) = 0$

$\parallel$

$(t-\iota(t)) * \left(\sum_{i=1}^{m} \phi_{\alpha_i^q \otimes f_i}(f)\right)$

$\because F[\tau]$ is free over $F[t]$, $\therefore \sum_{i=1}^{m} \phi_{\alpha_i^q \otimes f_i}(f) = 0, \ \forall f \in F[\tau]$.

Recall $\sum_{i=1}^{m} \phi_{\alpha_i \otimes f_i} = 0 = \sum_{i=1}^{m} \phi_{\alpha_i^q \otimes f_i}$

$\Rightarrow \sum_{i=1}^{m} \phi_{(\alpha_i - \alpha_i^q) \otimes f_i} = 0$

$\therefore \alpha_i = \alpha_i^q, \quad i = 1, 2, \cdots, m.$ $\square$

Rmk. ① The approach of the above thm comes from Anderson's idea.

② Let $\| f \otimes \frac{1}{c} \| = q^{(\deg_\tau f)/r - \deg_t c}$, $\forall f \otimes \frac{1}{c} \in \mathrm{End}_F(\rho) \otimes_A k =: \mathrm{End}_F^\circ(\rho)$

Exercise. $\|\cdot\| : \mathrm{End}_F^\circ(\rho) \longrightarrow \mathbb{R}_{\geq 0}$ is an absolute value satisfying

$\| \rho_a \| = q^{\deg a}, \quad \forall a \in A.$

Let $k_\infty := \mathbb{F}_q((t^{-1}))$, completion of k at ∞ $\left(\left|\frac{a}{b}\right|_\infty = q^{\deg a - \deg b}, \forall a, b \in A\right)$

Then $\|\cdot\|$ can be extended to an abs. val. on

$\boxed{\mathrm{End}_F^\circ(\rho) \otimes_k k_\infty}$ (i.e. completion of $\mathrm{End}_F^\circ(\rho)$ w.r.t. $\|\cdot\|$)

"ramified at ∞"

$\Rightarrow \operatorname{End}^\circ_F(\rho) \otimes_k k_\infty$ is a division alg. over k_∞.

$$\operatorname{End}^\circ_F(\rho) \otimes_k k_\infty = \operatorname{End}_F(\rho) \otimes_A k_\infty$$

③ If $\operatorname{char}_A(F) = 0$, then $\operatorname{rank}_A(\operatorname{End}_F(\rho)) \leq 2$, and $\operatorname{End}_F(\rho)$ is comm. (Analytic theory of Drinfeld modules)

Lecture 5. Rmk. Let F be an A-field w/ $\operatorname{char}_A(F) = 0$. Then $\operatorname{rank}_A(\operatorname{End}_F(\rho)) \leq 2$ for every Drinfeld A-module of rank 2 over F. ($\operatorname{char}_A(F)=0$)

A Drinfeld A-module ρ over F is called "CM" if $\operatorname{End}^\circ_{F^s}(\rho)$ has deg 2 over k.

F^s: separable closure of F

Ex. $\rho^{(2)} : A \to F[\tau]$

$t \mapsto \iota(t) + \tau^2$ $(\operatorname{char}_A(F)=0)$

Then $\operatorname{End}_{F^s}(\rho^{(2)}) \simeq \mathbb{F}_{q^2}[t]$ (Exercise)

§2.1. Torsions of Drinfeld modules

ρ: Drinfeld A-module of rank 2 over F (F: A-field)

$\rho[a] := \{\alpha \in \bar{F} : \rho_a(\alpha) = 0\}$, $\forall a \in A$

$\rho[a]$: the set of a-torsions of ρ.

Viewing $\bar{F}$ as an A-module via ρ, then $\rho[a]$ is a finite A-submod. of $\bar{F}$.

Prop Let ρ be a Drinfeld A-module of rank r over F

(1) For $a, b \in A$ w/ $(a,b) = 1$,

$$\rho[ab] = \rho[a] \oplus \rho[b]$$

(2) Given a monic irr. $\mathfrak{p} \in A$ w/ $\mathrm{char}_A(F) \neq \mathfrak{p}$, we have

$$\rho[\mathfrak{p}^n] \simeq (A/\mathfrak{p}^n)^r, \quad \forall n \in \mathbb{Z}_{\geq 0}$$

(3) Suppose $\mathrm{char}_A(F) = (\mathfrak{p}) \neq 0$, then

$$\rho[\mathfrak{p}^n] \simeq (A/\mathfrak{p}^n)^{r-h}, \quad \forall n \in \mathbb{Z}_{\geq 0}$$

(h = height of ρ).

Proof (1) is clear (exercise)

For (2) & (3), we first consider the case when $n = 1$.

$\Rightarrow$ $\rho[\mathfrak{p}]$ is an $A/\mathfrak{p}$-vec. sp. Put $d = \dim_{A/\mathfrak{p}} \rho[\mathfrak{p}]$

$\therefore$ $\#(\rho[\mathfrak{p}]) = q^{d \deg \mathfrak{p}}$

$$\| $$

$$\begin{cases} q^{r \deg \mathfrak{p}} & \text{if } \mathfrak{p} \nmid \mathrm{char}_A(F) \\ q^{(r-h) \deg \mathfrak{p}} & \text{if } \mathfrak{p} = \mathrm{char}_A(F). \end{cases}$$

For $n > 1$, consider the exact seq.

$$\begin{array}{ccccccccc} 0 \to & \rho[\mathfrak{p}] & \to & \rho[\mathfrak{p}^n] & \xrightarrow{\rho_{\mathfrak{p}}} & \rho[\mathfrak{p}^{n-1}] & \to 0 \\ & \alpha & \longmapsto & \alpha & & & \\ & & & \alpha & \longmapsto & \rho_{\mathfrak{p}}(\alpha) & \end{array}$$

The remaining argument is left as an exercise. $\square$

Let ρ be a Drinfeld A-module of rank r over F.

Given $m \in A$ coprime to $\mathrm{char}_A(F)$, let $F(\rho[m])$ be the field ext'n of F gen. by elts in $\rho[m]$.

Then ρ_m is separable $\Rightarrow$ $F(\rho[m]) \,|\, F$ is finite Galois.

$\rightsquigarrow$ an embedding

$$\mathrm{Gal}(F(\rho[m]) \,|\, F) \hookrightarrow \underset{\displaystyle \mathrm{Aut}_{A/m}(\rho[m])}{\mathrm{GL}_r(A/m) \atop \wr}$$

Rmk. ① Given a monic irr. $\mathfrak{p} \in A$,

$T_{\mathfrak{p}}(\rho) := \varprojlim_n \rho[\mathfrak{p}^n]$, which admits an action of $G_F = \mathrm{Gal}(\bar{F} | F)$

$\rightsquigarrow$ a rep'n of G_F on $V_{\mathfrak{p}}(\rho) = T_{\mathfrak{p}}(\rho) \otimes_{A_{\mathfrak{p}}} k_{\mathfrak{p}}$,

$\left(A_{\mathfrak{p}} = \varprojlim_n A/\mathfrak{p}^n, \quad k_{\mathfrak{p}} = \mathrm{Frac}(A_{\mathfrak{p}}) \right)$

② When ρ is rank 1

$\Rightarrow$ $\mathrm{Gal}(F(\rho[m])|F) \hookrightarrow (A/p)^\times$

$\Rightarrow$ $F(\rho[m])|F$ is abelian

3. Cyclotomic function fields:

Classical case: The cyclotomic field $\mathbb{Q}(\zeta_m)$, $\zeta_m = e^{\frac{2\pi i}{m}}$ has the following properties:

(1) $\mathrm{Gal}(\mathbb{Q}(\zeta_m)|\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$

$[\zeta_m \mapsto \zeta_m^a] \longleftarrow a \bmod m$

(2) The ring of integers in $\mathbb{Q}(\zeta_m)$ is $\mathbb{Z}[\zeta_m]$.

(3) A prime number $p \in \mathbb{Z}$ is ramified in $\mathbb{Q}(\zeta_m)$ iff $p \mid m$.

(4) For $p \nmid m$, one has $\mathrm{Frob}_p(\zeta_m) = \zeta_m^p$

$\Rightarrow$ p decomposes into $\varphi(m)/f_p$ primes in $\mathbb{Z}[\zeta_m]$,

where φ is the Euler function and f_p is the smallest positive integer f s.t. $p^f \equiv 1 \bmod m$.

Analogue in function fields

$F = k = \mathbb{F}_q(t) \overset{?}{\longleftrightarrow} \mathbb{F}_q[t] = A$ $(\mathrm{char}_A(k) = 0)$

Recall $e : A \to k[\tau]$

$t \longmapsto t + \tau$

For monic $m \in A$, $K_m := k(e[m])$

<u>Prop.</u> $Gal(K_m | k) \simeq (A/m)^\times$

$\left[\lambda \mapsto \rho_a(\lambda) \right] \longleftarrow a$

$\forall \lambda \in e[m]$

Suppose $m = p^n$, p monic irr. in A, $n \geq 1$

Put $e^*_{p^n}(x) = \dfrac{e_{p^n}(x)}{e_{p^{n-1}}(x)}$

<u>Lemma.</u> $e^*_{p^n}(x) \in A[x]$ is Eisenstein at p for every $n \geq 1$.

$(\Rightarrow$ $e^*_{p^n}(x)$ is irr. over k, $\deg e^*_{p^n} = q^{n \deg p} - q^{(n-1)\deg p}$

$= \# (A/p^n)^\times$

<u>Pf</u> Exercise.

(hint: ① $e_{p^n}(x) = e_p\left(e_{p^{n-1}}(x)\right)$

② Prove first the case when $n = 1$.

and use ① to show general n.)

<u>Lecture 6</u>. <u>Cor.</u> $e^*_{p^n}(x) \in k[x]$ is irr. and $K_{p^n} | k$ is totally ramified at p

Moreover, $Gal(K_{p^n} | k) \simeq (A/p^n)^\times$

Since $e[ab] = e[a] \oplus e[b]$, if $(a,b)=1$,

$\Rightarrow K_a \cap K_b = k$

Write $m = \prod_{i=1}^{s} p_i^{n_i}$, p_i monic irr. in A.

$\Rightarrow k_m = \prod_{i=1}^{s} k_{p_i^{n_i}}$ and $Gal(K_m|k) \simeq \prod_{i=1}^{s} Gal(k_{p_i^{n_i}}|k)$.

Moreover,

$$e[m] = \bigoplus_{i=1}^{s} e[p_i^{n_i}]$$

Prop. We have

$$\begin{tikzcd} Gal(k_m|k) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \prod_{i=1}^{s} Gal(k_{p_i^{n_i}}|k) \arrow[d, "\wr"] \\ (A/m)^\times \arrow[r, "\sim"] & \prod_{i=1}^{s} (A/p_i^{n_i})^\times \end{tikzcd} \Rightarrow Gal(k_m|k) \xrightarrow{\sim} (A/m)^\times$$

Exercise. Given $0 \neq \lambda \in e[m]$ and $\sigma \in Gal(K_m|k)$,

show that $\frac{\sigma(\lambda)}{\lambda}$ is a unit in the ring of integers of K_m $\widehat{\mathcal{O}_m}$

(the integral closure of A in K_m)

called cyclotomic unit.

Challenge: Let $C_m := \left\langle \frac{\sigma(\lambda)}{\lambda} : \begin{array}{l} \lambda \in e[m] \\ \sigma \in Gal(K_m|k) \end{array} \right\rangle$, then

$[\mathcal{O}_m^\times : C_m] < \infty$ (What is the index?)

Suppose $m = p^n$. Let λ_{p^n} be a root of $e^*_{p^n}(x)$

($\Leftrightarrow$ λ_{p^n} is a generator of $e[p^n]$)

$\Rightarrow$ discriminant of $A[\lambda_{p^n}] \subset \mathcal{O}_{p^n}$ is $p^\bullet$ $\Rightarrow$ $(A[\lambda_{p^n}])_q = (\mathcal{O}_p)_q$

for $q \neq p$

$e^*_{p^n}(x)$ is Eisenstein at p

Known result $\Rightarrow$ $(A[\lambda_{p^n}])_p = (\mathcal{O}_{p^n})_p$.

$\Rightarrow$ $A[\lambda_{p^n}] = \mathcal{O}_{p^n}$.

(checking directly, see [Ro 02, Prop 12.9])

For general $m = p_1^{n_1} \cdots p_s^{n_s}$,

$\because$ $K_{p_i^{n_i}}$ and $K_{p_j^{n_j}}$ are linearly disjoint w/ coprime discriminant.

$\Rightarrow$ $\mathcal{O}_m = \prod_{i=1}^{s} \mathcal{O}_{p_i^{n_i}}$. $\quad \mathcal{O}_{p_i^{n_i}} = A[\lambda_{p_i^{n_i}}]$

Take λ_m to be a generator of $e[m]$

$\Rightarrow$ $A[\lambda_m] \supset A[\lambda_{p_i^{n_i}}]$, $i = 1, \dots, s$.

$\Rightarrow$ $A[\lambda_m] = \mathcal{O}_m$

Prop. Given a monic m of A, we have $\mathcal{O}_m = A[\lambda_m]$ where λ_m is a generator of $e[m]$.

Moreover, a prime $p \in A$ is ramified in K_m iff $p \mid m$.

Let $m \in A$ be a monic poly.

Given $p \nmid m$, note that we have

$$e_p(x) \equiv x^{q^{\deg p}} \mod p.$$

Let β be a prime of $\mathcal{O}_m$ lying above p, then

$$e_p(\lambda_m) \equiv \lambda_m^{q^{\deg p}} \mod \beta$$
$$\equiv \mathrm{Frob}_p(\lambda_m) \mod \beta.$$

Let $\bar{e} : A \to A/p[\tau]$ reduction of e mod p

$t \mapsto \bar{t} + \tau$

$$\rightsquigarrow \begin{array}{ccc} e[m] & \longrightarrow & \bar{e}[m] \subset \mathcal{O}_m/\beta \\ \cap & \uparrow & \\ \mathcal{O}_m & \text{is a bijection } (\because p \nmid m) & \end{array}$$

$\Rightarrow e_p(\lambda_m) = \mathrm{Frob}_p(\lambda_m)$.

$\Rightarrow$ the order of $\mathrm{Frob}_p \in \mathrm{Gal}(K_m/k)$

the order of $(p \bmod m)$ in $(A/m)^\times$ = the smallest integer f_p s.t. $p^{f_p} \equiv 1 \mod m$

$\Rightarrow$ the decomposition group $D_\mathfrak{p}$ of $\mathfrak{p}$ in $\mathrm{Gal}(K_m|k)$ has order $f_\mathfrak{p}$

$\because$ $\#$ (primes of O_m lying above $\mathfrak{p}$) $= [\mathrm{Gal}(K_m|k) : D_\mathfrak{p}]$, we obtain:

<u>Prop</u>. Given a monic $m \in A$, let λ_m be a generator of $\mathcal{C}[m]$.

For $p \nmid m$, one has $\mathrm{Frob}_p(\lambda_m) = C_p(\lambda_m)$ (analogue of $\mathrm{Frob}_p(\zeta_m) = \zeta_m^p$)

In particular, $\mathfrak{p}$ decomposes into $\Phi(m)/f_\mathfrak{p}$ primes in O_m, where

$\Phi(m) = \# (A/m)^\times$ and f_p is the smallest primitive integer

s.t. $p^{f_p} \equiv 1 \bmod m$

In classical case, $\mathbb{Q}(\zeta_m)$ is a CM field w/ the totally real quadratic subfield

$\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ for $m > 2$.

In function field case, we have

<u>Prop</u>. Let J_m be the image of $\mathbb{F}_q^\times \hookrightarrow (A/m)^\times \simeq \mathrm{Gal}(K_m|k)$

$$\mathbb{F}_q^\times \longrightarrow J_m \subset \mathrm{Gal}(K_m|k)$$

Let $K_m^+ := k(\lambda_m^{q-1})$ (λ_m : generator of $\mathcal{C}[m]$)

Then (1) K_m^+ is the fixed field of J_m

(2) The infinite place of k splits completely in K_m^+

(3) Every place of K_m^+ lying above the infinite place of k is totally ramified in K_m.

$$\begin{tikzcd}[column sep=small] K_m \arrow[d, dash] & \tilde{\infty}_1 \arrow[d, dash, "q-1"] & \cdots & \tilde{\infty}_s \arrow[d, dash] \\ K_m^+ \arrow[d, dash] & \infty_1 \arrow[dr, dash] & \cdots & \infty_s \arrow[dl, dash] \\ k & & \infty & \end{tikzcd}$$

Proof. Observe that

① $K_m^+ = k(\lambda_m^{q-1}) \subset K_m^{J_m} \subset \underset{\substack{\|\\ k(\lambda_m)}}{K_m}$

② $[K_m : K_m^{J_m}] = \#(J_m) = q-1$

③ λ_m is a root of $X^{q-1} - \lambda_m^{q-1} \in K_m^+[X]$

$(\Rightarrow [K_m : K_m^+] \leq q-1)$

$\Rightarrow K_m^+ = K_m^{J_m}$ (1)

For (2) and (3), need "Newton polygons" of $e_m(X)$ to analyse the order of the roots of $e_m(X)$ at the infinite place.

(refer to [Ro 02, Prop 12.13 & Thm 12.14])

Classical Kronecker-Weber Thm:

$$\mathbb{Q}^{ab} = \mathbb{Q}(\mu_\infty), \quad \mu_\infty = \{\zeta_n : n \in \mathbb{Z}_{\geq 1}\}$$

However, the constant field of $K_m = \mathbb{F}_q$

(i.e. $K_m | k$ is "geometric")

Carlitz torsions are not enough to generate the max'l abelian ext'n of k.

In fact, put $t' = \frac{1}{t} \in k$, $A' = \mathbb{F}_q[t']$,

$C' : A' \to k[\tau]$ the Carlitz A'-module over k.

In fact, put $t' = \frac{1}{t} \in k$, $A' = \mathbb{F}_q[t']$,

$e' : A' \to k[\tau]$ the Carlitz A'-module over k.

Put $k'_m = k(e'[m'])$, $m' \in A'$.

Then $k^{ab} = \left(\prod_{m \in A} k_m\right) \cdot \left(\prod_{m' \in A'} k'_{m'}\right)$

[Hayes 74, Thm 7.2].

Lecture 7. §4. Uniformization of Drinfeld modules

Classical: ① $0 \to 2\pi\sqrt{-1}\,\mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \to 1$

$z \mapsto \exp(z)$

② Rank 2 case:

Let $E : y^2 = x^3 - ax - b$, $a, b \in \mathbb{C}$ w/ $\Delta(E) \neq 0$.

period lattice of E:

$$\Lambda_E := \left\{ \int_\gamma \frac{dx}{y} : \gamma \in H_1(E(\mathbb{C}); \mathbb{Z}) \right\}$$

$\cap$ $\mathbb{C}$ $\qquad$ $\uparrow$ homology classes of cycles of E.

$\rightsquigarrow$ Λ_E is a rank 2 $\mathbb{Z}$-lattice in $\mathbb{C}$

$\| \ \mathbb{Z}w_1 + \mathbb{Z}w_2$, w_1, w_2 fundamental periods of E.

Moreover, we have $0 \to \Lambda_E \to \mathbb{C} \xrightarrow{\exp_E} E(\mathbb{C}) \to 0$

where $\exp_E(z) := (\wp(z), \wp'(z))$, w/ $\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda_E}' \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$

4.1. Entire functions

$$k = \mathbb{F}_q(t) \supset A = \mathbb{F}_q[t],$$

$$\cap$$

$$k_\infty := \mathbb{F}_q((t^{-1})), \quad \mathbb{C}_\infty := \widehat{\overline{k_\infty}}$$

$$|\cdot|_\infty : k_\infty \longrightarrow \mathbb{R}_{\geq 0}$$

$$\wr$$

$$|\cdot|_\infty : \mathbb{C}_\infty \longrightarrow \mathbb{R}_{\geq 0} \quad \text{abs. value}$$

$$(A, k, k_\infty, \mathbb{C}_\infty) \longleftrightarrow (\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C})$$

Ideal property in nonarchimedean case:

$$\sum_{n=0}^{\infty} a_n \text{ converges} \Leftrightarrow \lim_{n\to\infty} a_n = 0 \qquad (a_n \in \mathbb{C}_\infty)$$

$$(\because |\cdot|_\infty \text{ is n.a.})$$

In particular, for a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}_\infty[\![x]\!]$, and

$\alpha \in \mathbb{C}_\infty$, one has $f(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n$ converges iff $\lim_{n\to\infty} a_n \alpha^n = 0$.

Def. An entire function on $\mathbb{C}_\infty$ is a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}_\infty[\![x]\!]$

s.t. $f(\alpha)$ converges for every $\alpha \in \mathbb{C}_\infty$.

$$\left(\underset{\text{exercise}}{\Longleftrightarrow} \lim_{n\to\infty} |a_n|_\infty \cdot q^{rn} = 0, \quad \forall r \in \mathbb{R} \right)$$

Example. (1) Every poly. in $\mathbb{C}_\infty[x]$ is entire.

(2) Given $c \in \mathbb{C}_\infty^\times$, $n \in \mathbb{Z}_{\geq 0}$ and a collection

$$\Lambda = \{\lambda_i : i \in I\} \quad w/ \ \lambda_i \in \mathbb{C}_\infty^\times \quad \text{which is}$$

"rigidly" discrete, i.e. $\# \{i \in I : |\lambda_i|_\infty \leq r\} < \infty$ for every $r \in \mathbb{R}_{\geq 0}$

Then $\#(I)$ is countable, and $cx^n \cdot \prod_{i \in I} \left(1 - \frac{x}{\lambda_i}\right)$ converges to an entire function on $\mathbb{C}_\infty$ $\Big($uniformly on $B(0,r) := \{\alpha \in \mathbb{C}_\infty : |\alpha| \leq r\}$ $\forall r \in \mathbb{R}_{\geq 0}\Big)$

Thm. Let $0 \not\equiv f$ be an entire function on $\mathbb{C}_\infty$

(1) Then f must be surj. if f is not a constant function

(2) For every $r \in \mathbb{R}_{>0}$, the cardinality of zeros λ of f (counting mult.) $w/$ $|\lambda|_\infty \leq r$ is <u>finite</u>.

(3) (Weierstraß factorization thm) Let $\{\lambda_i : i \in I\}$ be the zero set of f (counting multiplicity) excluding 0 if $f(0) = 0$. Then $\exists\, c \in \mathbb{C}_\infty^\times$ and $n \in \mathbb{Z}_{\geq 0}$ s.t. $f(x) = cx^n \prod_{i \in I} \left(1 - \frac{x}{\lambda_i}\right)$

Given $m \in \mathbb{Z}$, let $\mathbb{T}_{(m)} := \left\{ \sum_{n=0}^\infty a_n x^n \in \mathbb{C}_\infty[\![x]\!] : \lim_{n \to \infty} |a_n|_\infty q^{mn} = 0 \right\}$

$\left(\Leftrightarrow \sum_{n=0}^\infty a_n \alpha^n \text{ converges for every } \alpha \in B(0, q^m) \right)$

Observe: $\mathbb{T}_{(m)} \subset \mathbb{T}_{(m')}$ if $m \geq m'$.

Put $\mathbb{T} := \mathbb{T}_{(0)}$ (<u>Tate algebra</u>, one variable over $\mathbb{C}_\infty$).

<u>Lemma</u>. (1) There is a $\mathbb{C}_\infty$- alg. $\mathbb{T}_{(m)} \xrightarrow{\sim} \mathbb{T}$ $\quad (|t|_\infty = q)$

$$f(x) \longmapsto f(t^m x)$$

(2) Given $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}_\infty[[x]]$, put

$$\|f\| := \sup_n (|a_n|_\infty) \qquad (\text{Gauß norm})$$

Then $\|f_1 + f_2\| \leq \max(\|f_1\|, \|f_2\|)$

$\|f_1 \cdot f_2\| = \|f_1\| \cdot \|f_2\|$. $\qquad \forall\ f_1, f_2 \in \mathbb{T}$

(3) $\|f\| = \sup(|f(\alpha)|_\infty : \alpha \in B(0,1))$

$= \max(|f(\alpha)|_\infty : \alpha \in B(0,1))$.

(4) Let $\mathbb{T}^\circ := \{f \in \mathbb{T} : \|f\| \leq 1\}$ and

$\mathbb{T}^{\circ\circ} := \{f \in \mathbb{T} : \|f\| < 1\}$

Then $\mathbb{T}^\circ / \mathbb{T}^{\circ\circ} \simeq \bar{\mathbb{F}}_q[\mathbf{x}]$.

$f \bmod \mathbb{T}^{\circ\circ} \longmapsto \bar{f}$ $\quad , f \in \mathbb{T}^\circ$

<u>Pf</u>. Exercise.

$f \in \mathbb{C}_\infty[\![x]\!]$ is entire $\Leftrightarrow$ $f \in \mathbb{T}(m)$, $\forall m \in \mathbb{Z}_{\geq 0}$

Prop. (Weierstraß division thm) Given $f, g \in \mathbb{T}$ w/ $\underset{(WLOG)}{\underline{\|f\| = 1}}$, there exists a unique pair (q, r) where $q \in \mathbb{T}$ and $r \in \mathbb{C}_\infty[x]$ w/ $\deg r < \deg \bar{f}$

s.t. $g = q \cdot f + r$. (and $\|g\| = \max(\|q\|, \|r\|)$)

Pf. Exercise.

(hint: check first when $f \in \mathbb{C}_\infty[x]$ w/ $\deg f = \deg \bar{f}$

$$f = \sum_{n=0}^{\ell} a_n x^n, \ |a_n|_\infty \leq 1 \quad \text{and } |a_\ell|_\infty = 1$$

② For general f, $f = f_0 + f_1$, where $f_0 \in \mathbb{C}_\infty[x]$

and $\deg f_0 = \deg \bar{f_0}$ and $\|f_1\| < 1$)

Cor. (Weierstraß preparation thm) Given $f \in \mathbb{T}$, suppose $f(0) = c \neq 0$,

There exists unique subset $\underset{B(0,1) - \{0\}}{\{\lambda_1, \cdots, \lambda_d\} \cap}$ and $e_1, \cdots, e_d \in \mathbb{Z}_{\geq 1}$, $u(x) \in \mathbb{T}^{\circ\circ}$

s.t. $f(x) = c \cdot \prod_{i=1}^{d} \left(1 - \frac{x}{\lambda_i}\right)^{e_i} \left(1 + x u(x)\right)$

($\because$ $1 + x u(x)$ is non-vanishing on $B(0,1)$

$\therefore$ zero set of f is finite! (Counting multiplicity).)

Lecture 8. Pf. Take $c_0 \in \mathbb{C}_\infty^\times$ s.t. $\|\underbrace{c_0^{-1} f}_{f_0}\| = 1$

and $\bar{f_0} \in \bar{\mathbb{F}}_q[x]$ is monic.

Let $d_0 := \deg \bar{f_0}$. $\exists!\ (q_0, r_0)$ w/. $q_0 \in \mathbb{T}$ and $r_0 \in \mathbb{C}_\infty[x]$ w/

$\deg r_0 < \deg \bar{f_0}$ s.t. $x^{d_0} = q_0 f_0 + r_0$

$\because 1 = \|x^{d_0}\| = \max(\|q_0\|, \|r_0\|)$

$\Rightarrow \overline{x^{d_0} - r_0} = \bar{q_0} \cdot \bar{f_0}$

$\because \bar{f_0}$ is monic, and $\deg \bar{f_0} = d_0$

$\therefore \bar{q_0} = 1$ in $\bar{\mathbb{F}}_q[x]$.

$\Rightarrow q_0 = 1 + u_0(x)$, where $u_0(x) \in \mathbb{T}^{0,0}$ (i.e. $\|u_0\| < 1$)

$\Rightarrow q_0 \in (\mathbb{T}^\circ)^\times$ $\left(\because (1+u_0)^{-1} = 1 - u_0 + u_0^2 + \cdots + (-1)^n u_0^n + \cdots \right.$

$\shortparallel$ converges in $\mathbb{T}^\circ$ $\left.\right)$

$c_1 + c_2 x + \cdots + c_n x^n$

$\shortparallel$

$c_1(1 + x u(x))$

Write $q_0^{-1} = c_1(1 + x u(x))$, where $|c_1|_\infty = 1$ and $\|u\| < 1$.

On the other hand, we may uniquely express $x^{d_0} - r_0 = c_2 \prod_{i=1}^{d} \left(1 - \frac{x}{d_i}\right)^{e_i}$

(roots of $x^{d_0} - r_0 = q_0 \cdot f_0$ are all non-zero as $f_0(0) = c \neq 0$ and

$q_0(0) = 1 + u_0(0) \neq 0$)

$\therefore\ f = c_0 f_0$

$$= c_0 (x^{d_0} - z_0) q_0^{-1}$$

$$= c_0 c_1 c_2 \left(\prod_{i=1}^{d} \left(1 - \frac{x}{\alpha_i}\right)^{e_i} \right) (1 + x u(x))$$

$\because\ c = f(0) = c_0 c_1 c_2$ $\square$

Proof of Weierstraß factorization thm:

(1) follows immediately by (3). Indeed, for a nonconstant entire function

f and $c_0 \in \mathbb{C}_\infty$, $f - c_0$ is still non-constant, By (3), $f - c_0 = c \cdot x^n \prod_{i \in I} \left(1 - \frac{x}{\alpha_i}\right)$

and $I \neq \phi$. $\therefore\ \exists\ \alpha \in \mathbb{C}_\infty$ s.t. $f(\alpha) = c_0$.

(2) follows from the preparation thm.: given an entire function

$$0 \neq f = \sum_{n=0}^{\infty} a_n x^n = x^{n_0} \sum_{n=0}^{\infty} a_n' x^n \quad \text{w/ } a_0' \neq 0.$$

We may assume $f(0) = c \neq 0$ WLOG.

Put $f_m(x) = f(t^m x)$ for $m \in \mathbb{Z}_{\geq 0}$

$\because$ f is entire, $f_m \in \mathbb{T}$.

By preparation thm, $\exists!$ subset

$\{\alpha_{m,1}, \cdots, \alpha_{m,d_m}\} \subset B(0,1) - \{0\}$, $e_{m,1}, \cdots, e_{m,d_m} \in \mathbb{Z}_{\geq 1}$, $u_m \in \mathbb{T}^{00}$,

s.t. $f_m(x) = c \prod_{i=1}^{d_m} \left(1 - \frac{x}{\alpha_{m,i}}\right)^{e_{m,i}} (1 + x u_m(x))$

$\Rightarrow$ $f_m(x)$ has only finitely many zeroes in $B(0,1)$ (Counting mult.)

$\Rightarrow$ $f(x)$ has only finitely many zeroes in $B(0,q^m)$, $\forall m \in \mathbb{Z}_{\geq 0}$

(counting mult.)

For (3), we still assume $f(0) = c \neq 0$ WLOG.

Let $\Lambda_{(m)}$ be the zero set of f in $B(0,q^m)$ (counting mult.)

$$\Rightarrow f(x) = c \cdot \prod_{\lambda_m \in \Lambda_{(m)}} \left(1 - \frac{x}{\lambda_m}\right)\left(1 + \pi^m x\, u_m(\pi^m x)\right) \quad (\pi := t^{-1})$$

Let $g_m(x) = c \prod_{\lambda_m \in \Lambda_{(m)}} \left(1 - \frac{x}{\lambda_m}\right)$ and $u_m'(x) = \pi^m x \cdot u_m(\pi^m x)$

Then $\lim_{m \to \infty} u_m'(x) = 0$ uniformly on $B(0,r)$, $\forall r \in \mathbb{R}_{>0}$.

Then the identity $f = g_m \cdot (1 + u_m')$ and $\bigsqcup_m \Lambda_{(m)}$ is "rigidly" discrete

$\therefore$ $\lim_{m \to \infty} g_m$ exists and equals to f.

Rmk. Recall the <u>Newton polygon</u> of a polynomial tells the "order" of the zeros

This can be generalized to entire functions on $\mathbb{C}_\infty$.

§4.2 Exponential functions of Drinfeld modules

Let the A-field $F = \mathbb{C}_\infty$ $(\hookleftarrow A)$ $\quad (\mathrm{char}_A(\mathbb{C}_\infty) = 0)$

Let ρ be a Drinfeld A-module of rank r over $\mathbb{C}_\infty$.

Lemma. There exists a unique twisted power series

$$\exp_\rho = 1 + c_1\tau + \cdots + c_n\tau^n + \cdots \in \mathbb{C}_\infty[\![\tau]\!]$$

satisfying $\exp_\rho \cdot a = \rho_a \cdot \exp_\rho$, $\forall a \in A$.

Pf: Solving the functional equation:

$$\exp_\rho \cdot t = \rho_t \cdot \exp_\rho \qquad (*)$$

Write $\rho_t = t + g_1\tau + \cdots + g_r\tau^r \in \mathbb{C}_\infty[\tau]$.

$$(*) \iff c_m t^{q^m} = t\cdot c_m + \sum_{i=1}^{r} g_i \cdot c_{m-i}^{q^i}, \quad \forall m > 0$$

(m-th coeff of $\exp_\rho \cdot t$) $\qquad$ (put $c_0 = 1$ and $c_i = 0$ if $i<0$)

$$\therefore\ c_m = \frac{\sum_{i=1}^{r} g_i\, c_{m-i}^{q^i}}{t^{q^m} - t}$$

This recursive relation among c_m's gives us the unique twisted power series $\exp_\rho$.

Lecture 9. $\forall$ Drinfeld A-module ρ of rank r over $\mathbb{C}_\infty$, there exists a unique twisted power series

$$\exp_\rho = c_0 + c_1\tau + \cdots + c_n\tau^n + \cdots \in \mathbb{C}_\infty[[\tau]] \quad \text{w/ } c_0 = 1 \quad \text{satisfying}$$

$$\exp_\rho \cdot a = \rho_a \cdot \exp_\rho, \quad \forall a \in A.$$

$\ulcorner$ Only need to check $\rho_t = t + g_1\tau + \cdots + g_r\tau^r$

$$\exp_\rho \cdot t = \rho_t \cdot \exp_\rho$$

$$\rightsquigarrow \quad c_m = \frac{1}{t^{q^m} - t} \sum_{i=1}^{r} g_i\, c_{m-i}^{q^i} \quad (c_i = 0, \text{ if } i < 0) \qquad (*)$$

Exercise: F A-field $\mathrm{char}_A(F) \neq 0$, show that for a rank 1 Drinfeld A-mod ρ over A, the "exponential function" $\exp_\rho$ does not exist.

(Question: How about higher ranks?)

Exercise: The recursive relation $(*)$ implies $\lim_{m\to\infty} |c_m|_\infty^{q^{-m}} = 0$

$$\Rightarrow \lim_{m\to\infty} |c_m|_\infty \cdot |\alpha|_\infty^{q^m} = 0, \quad \forall \alpha \in \mathbb{C}_\infty$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m \alpha^{q^m} \text{ converges}, \quad \forall \alpha \in \mathbb{C}_\infty$$

i.e. $\exp_\rho(x) = \sum_{m=0}^{\infty} c_m x^{q^m} \in \mathbb{C}_\infty[[x]]$ is entire on $\mathbb{C}_\infty$.

Hint: $\exists M > 0$ s.t. $|c_{nr+l}|_\infty \le q^{-nq^{nr+l}} \cdot M$, $\forall n \in \mathbb{Z}_{\geq 0}$, $0 \le l \le r-1$

Lemma. The power series $\exp_\rho(x) = x + \sum_{m=1}^{\infty} c_m x^{q^m}$ is an entire function which is $\mathbb{F}_q$-linear.

Pf. Let $\underset{\substack{\uparrow \\ \mathbb{F}_q\text{-linear}}}{\exp_\rho^{(n)}(x)} = x + \sum_{m=1}^{n} c_m x^{q^m} \in \mathbb{C}_\infty[x]$

and $\exp_\rho(\alpha) = \lim_{n\to\infty} \exp_\rho^{(n)}(\alpha), \quad \forall \alpha \in \mathbb{C}_\infty$

$\therefore \exp_\rho(x)$ is $\mathbb{F}_q$-linear. $\square$

We call $\exp_\rho(x)$ the <u>exponential function</u> of ρ.

We have an exact seq. $0 \longrightarrow \Lambda_\rho \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_\rho} \mathbb{C}_\infty \longrightarrow 0$

$\Lambda_\rho := \{\alpha \in \mathbb{C}_\infty : \exp_\rho(\alpha) = 0\}$

Lemma. $\exp_\rho(x) = x \cdot \prod_{\lambda \in \Lambda_\rho}' \left(1 - \frac{x}{\lambda}\right)$

Moreover, Λ_ρ is an A-lattice of rank r in $\mathbb{C}_\infty$

($\Rightarrow$ Λ_ρ is a rigidly discrete free A-submod. of $\mathbb{C}_\infty$ of rank r).

Pf. ① Since $\exp_\rho(x-\lambda) = \exp_\rho(x), \ \forall \lambda \in \Lambda_\rho$, and $\mathrm{ord}_{x=0} \exp_\rho(x) = 1$

$\Rightarrow \mathrm{ord}_{x=\lambda} \exp_\rho(x) = 1, \quad \forall \lambda \in \Lambda_\rho$.

② From the F.E.: $\exp_\rho(ax) = \rho_a(\exp(x))$, $\forall a \in A$

$\Rightarrow \Lambda_\rho$ is an A-submod. of $\mathbb{C}_\infty$,

<u>Claim</u>. Λ_ρ is free of finite rank over A.

From the isom.

$$\begin{array}{ccc} \dfrac{a^{-1}\Lambda_\rho}{\Lambda_\rho} & \xrightarrow[\sim]{\exp_\rho(x)} & \rho[a] \qquad \forall a \in A \\ \wr\wr & & \wr\wr \\ (A/a)^{r'} & & (A/a)^{r} \end{array}$$

$$\Rightarrow \quad \mathrm{rank}_A \Lambda_\rho := r' = r$$

Consider $V_\rho = k_\infty \Lambda_\rho$: and claim that $\dim_{k_\infty}(V_\rho) < \infty$ $(\Lambda_\rho \subset V_\rho)$

The result follows from:

<u>Exercise</u>. Let V be a fin. dim'l vec. sp. over k_∞. A discrete A-submod of V is always free of rank $\leq \dim_{k_\infty} V$.

(hint: $A \subset k_\infty$ is discrete and cocompact)

Suppose $\dim_{k_\infty}(V_\rho) > r$. Take $\lambda_1, \dots, \lambda_{r+1} \in \Lambda_\rho$, which are linearly indep. over k_∞.

$$\begin{array}{ccc} \Lambda' := \Lambda_\rho \cap (k_\infty \lambda_1 + \dots + & k_\infty \lambda_{r+1}) & \subset V' \\ \cup & \parallel & \\ A\lambda_1 + \dots + A\lambda_{r+1} & V' & \end{array}$$

The above exercise implies Λ' is free of rank $r+1$ over A.

However, for $a \in A$, observe that $\frac{1}{a}\Lambda_\rho \cap V' = \frac{1}{a}\Lambda'$ (exercise)

$$\left(\frac{A}{a}\right)^{r+1} \simeq \frac{a^{-1}\Lambda'}{\Lambda'} \hookrightarrow \frac{a^{-1}\Lambda_\rho}{\Lambda_\rho} \simeq \rho[a] \simeq (A/a)^r, \quad \forall a \in A \qquad \#.$$

Prop Let ρ be a Drinfeld A-module of rank r over $\mathbb{C}_\infty$. There exists a unique A-lattice Λ_ρ of rank r in $\mathbb{C}_\infty$ satisfying that

$$\exp_\rho(x) = x \prod_{\lambda \in \Lambda_\rho}{}' \left(1 - \frac{x}{\lambda}\right)$$ and the diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_\rho \arrow[r] \arrow[d, "a(\cdot)"] & \mathbb{C}_\infty \arrow[r, "\exp_\rho"] \arrow[d, "a(\cdot)"] & \mathbb{C}_\infty \arrow[r] \arrow[d, "\rho_a"] & 0 \\
0 \arrow[r] & \Lambda_\rho \arrow[r] & \mathbb{C}_\infty \arrow[r, "\exp_\rho"] & \mathbb{C}_\infty \arrow[r] & 0.
\end{tikzcd}$$

Conversely, given an A-lattice Λ of rank r in $\mathbb{C}_\infty$, put

$$\exp_\Lambda(x) := x \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{x}{\lambda}\right)$$ entire. $\mathbb{F}_q$-linear.

$(\Lambda_{(m)} := \Lambda \cap B(0, q^m) \leftarrow$ finite $\mathbb{F}_q$-vec. sp.$)$

$$\exp_\Lambda^{(m)}(x) := x \prod_{\lambda \in \Lambda_{(m)}}{}' \left(1 - \frac{x}{\lambda}\right) \in \mathbb{C}_\infty[x]$$ $\mathbb{F}_q$-linear

and $\exp_\Lambda(\alpha) = \lim_{m \to \infty} \exp_\Lambda^{(m)}(\alpha)$, $\forall \alpha \in \mathbb{C}_\infty$.

For $a \in A$, set $\rho_a^\Lambda(x) := ax \prod_{w \in \frac{a^{-1}\Lambda}{\Lambda}}{}' \left(1 - \frac{x}{\exp_\Lambda(w)}\right) \in \mathbb{C}_\infty[x]$ $\mathbb{F}_q$-linear.

$\rightsquigarrow$ $\rho_a^\Lambda(x)$ corresponds to a twisted poly. $\rho_a^\Lambda \in \mathbb{C}_\infty[\tau]$.

<u>Lemma</u>, ① $\deg_\tau \rho_a^\Lambda = r \deg a$, $\forall a \in A$

② $\exp_\Lambda \cdot a = \rho_a^\Lambda \cdot \exp_\Lambda$, $\forall a \in A$

③ $\rho_a^\Lambda \rho_b^\Lambda = \rho_{ab}^\Lambda = \rho_b^\Lambda \rho_a^\Lambda$, $\forall a, b \in A$

($\rightsquigarrow$ $\rho^\Lambda : A \longrightarrow \mathbb{C}_\infty[\tau]$ is a Drinfeld A-module of rk r over $\mathbb{C}_\infty$)
$a \longmapsto \rho_a^\Lambda$

$(\partial \rho_a^\Lambda = a)$

<u>Lecture 10</u>. Pf: ① ② calculations.

③ From the F.E. in ②, we have $\rho_a^\Lambda(\rho_b^\Lambda(\exp_\Lambda(x))) = \rho_a^\Lambda(\exp_\Lambda(bx))$

$= \exp_\Lambda(abx) = \rho_{ab}^\Lambda \cdot \exp_\Lambda(x)$

$\exp_\Lambda$ surj. $\Rightarrow$ $\rho_a^\Lambda \rho_b^\Lambda = \rho_{ab}^\Lambda$.

<u>Prop</u>. Given an A-lattice Λ of rank r in $\mathbb{C}_\infty$, there exists a unique Drinfeld A-module ρ^Λ (of rank r) over $\mathbb{C}_\infty$ so that the period lattice is exactly Λ.

<u>Thm</u>. There is a bijection between the set of all Drinfeld A-modules of rank r over $\mathbb{C}_\infty$ and the set of A-lattices of rank r in $\mathbb{C}_\infty$. Moreover, put

$\mathrm{Hom}(\Lambda, \Lambda') = \{c \in \mathbb{C}_\infty : c \cdot \Lambda \subset \Lambda'\}$. Then $\mathrm{Hom}_{\mathbb{C}_\infty}(\rho, \rho') \xrightarrow{\sim} \mathrm{Hom}(\Lambda_\rho, \Lambda_{\rho'})$
$f \longmapsto \partial f$

Rmk. The above thm says in particular that the cat. $\mathcal{D}_A^2(\mathbb{C}_\infty)$ of Drinfeld A-modules of rank 2 over $\mathbb{C}_\infty$ is equiv. to the cat. $\mathcal{L}_A^2(\mathbb{C}_\infty)$ of A-lattices of rank 2 in $\mathbb{C}_\infty$.

Pf. It remains to show the bijection between Hom spaces.

Given $0 \neq f \in \mathrm{Hom}_{\mathbb{C}_\infty}(\rho, \rho')$, observe that f must be separable.

Put $c = \partial f \in \mathbb{C}_\infty^\times$.

Consider $c^{-1} f(\exp_\rho(ax)) = c^{-1} f \rho_a (\exp_\rho(x)) = c^{-1} \rho'_a f(\exp_\rho(x))$

$$= (c^{-1}\rho'_a c)(c^{-1} f(\exp_\rho(x)))$$

Let $\rho'' := c^{-1}\rho' c$, another Drinfeld A-mod. of rank 2 over $\mathbb{C}_\infty$.

From the uniqueness of the $\exp_{\rho''}$, we get

$$c^{-1} f(\exp_\rho(x)) = \exp_{\rho''}(x) = c^{-1}\exp_{\rho'}(cx).$$

$$\Rightarrow f(\exp_\rho(x)) = \exp_{\rho'}(cx).$$

$\therefore$ for $\lambda \in \Lambda$, we have $0 = f(\exp_\rho(\lambda)) = \exp_{\rho'}(c\lambda) \Rightarrow c\lambda \in \Lambda_{\rho'}$.

Conversely, given $c \in \mathbb{C}_\infty^\times$ w/ $c\Lambda_\rho \subset \Lambda_{\rho'}$, put $f_c(x) = cx \prod'_{w \in \frac{c^{-1}\Lambda_{\rho'}}{\Lambda_\rho}} \left(1 - \frac{x}{\exp_\rho(w)}\right)$

Moreover, we can check:

$$f_c(\exp_\rho(x)) = \exp_{\rho'}(cx) \qquad (\text{zero set} = c^{-1}\Lambda_{\rho'})$$

For $a \in A$, we have $f_c\,\rho_a(\exp_\rho(x)) = f_c(\exp_\rho(ax)) = \exp_{\rho'}(cax)$

$$= \rho'_a \exp_{\rho'}(cx) = \rho'_a f_c(\exp_\rho(x)).$$

$\Rightarrow f_c\rho_a = \rho'_a f_c$ ($\exp_\rho$ is surj.) (i.e. $f_c \in \mathrm{Hom}_{\mathbb{C}_\infty}(\rho,\rho')$)

Moreover, $\partial f_c = c$.

Finally, given $f \in \mathrm{Hom}_{\mathbb{C}_\infty}(\rho,\rho')$ w/ $\partial f = c \in \mathbb{C}_\infty^\times$, one has

$f - f_c \in \mathrm{Hom}_{\mathbb{C}_\infty}(\rho,\rho')$ w/ $\partial(f-f_c) = c-c = 0$.

$\Rightarrow f - f_c = 0.$ $\square$

<u>Cor</u> Let ρ and ρ' be two Drinfeld A-modules over $\mathbb{C}_\infty$, then

$\rho \underset{\mathbb{C}_\infty}{\simeq} \rho'$ iff $\exists\ c \in \mathbb{C}_\infty^\times$ s.t. $c\Lambda_\rho = \Lambda_{\rho'}$.

<u>Pf.</u> Let $f: \rho \xrightarrow{\sim} \rho'$ be an iso. $\overset{\text{exer.}}{\Longrightarrow} f \equiv c \in \mathbb{C}_\infty[\tau]$

$\rightsquigarrow c\Lambda_\rho \subset \Lambda_{\rho'}$ and $c = f(x) = c\cdot x \prod\limits'_{w \in \frac{c^{-1}\Lambda_{\rho'}}{\Lambda_\rho}} \left(1 - \frac{x}{\exp_\rho(w)}\right) \Rightarrow c^{-1}\Lambda_{\rho'} = \Lambda_\rho.$

(as k_∞-vec.sp.) $\square$.

Let $\Sigma(k_\infty^2, \mathbb{C}_\infty) = \{\text{embeddings } k_\infty^2 \hookrightarrow \mathbb{C}_\infty\}$

$$\varphi \longmapsto \varphi(A^2) \qquad \Downarrow \qquad \{\text{rank 2 } A\text{-lattices in } \mathbb{C}_\infty\}$$

Action $GL_2(k_\infty) \curvearrowright \mathcal{E}(k_\infty^2, \mathbb{C}_\infty)$

$$(\gamma * f)(a_1, \cdots, a_2) = f((a_1, \cdots, a_2)\cdot\gamma)$$

$$GL_2(A) \backslash \mathcal{E}(k_\infty^2, \mathbb{C}_\infty)$$
$$\downarrow \wr$$
$$\{\text{rank } 2 \ A \text{ lattices in } \mathbb{C}_\infty\}$$

Another action of $\mathbb{C}_\infty^\times$ on $\mathcal{E}(k_\infty^2, \mathbb{C}_\infty)$

$$(c * f)(\vec{a}) = c f(\vec{a}), \quad \forall \vec{a} \in k_\infty^2$$

$$\rightsquigarrow GL_2(A) \backslash \mathcal{E}(k_\infty^2, \mathbb{C}_\infty) / \mathbb{C}_\infty^\times$$
$$\updownarrow \wr$$
$$\{A\text{-lattices of rank } 2 \text{ in } \mathbb{C}_\infty\} / \simeq$$

$$\rightsquigarrow GL_2(A) \backslash \mathcal{E}(k_\infty^2, \mathbb{C}_\infty)(/\mathbb{C}_\infty^\times)$$
$$\downarrow \wr$$
$$\{\text{Drinfeld } A\text{-modules of rank } 2 \text{ over } \mathbb{C}_\infty\}(/\simeq)$$

<u>Exercise</u>. $\mathcal{E}(k_\infty^2, \mathbb{C}_\infty)/\mathbb{C}_\infty^\times \simeq \mathbb{P}^{2-1}(\mathbb{C}_\infty) - \coprod_{\vec{a} \in \mathbb{P}^{2-1}(k_\infty)} H_{\vec{a}} =: \Omega^2 \ (\text{or } \mathfrak{H}^2)$

called Drinfeld half space.

where $H_{\vec{a}} = \{(z_1 : \cdots : z_2) \in \mathbb{P}^{2-1}(\mathbb{C}_\infty) : a_1 z_1 + \cdots + a_2 z_2 = 0\}$
$(\vec{a} = (a_1, \cdots, a_2))$

Lecture 11 Fields of char $p \neq 0$.

Separable extensions

purely inseparable extensions

perfect fields

Goal of alg. number theory : understand Galois gp $Gal(\bar{\mathbb{Q}} | \mathbb{Q})$?

For us, $Gal\left(\overline{\mathbb{F}_q(\theta)} \mid \mathbb{F}_q(\theta)\right)$

function field char p, q a power of p

θ is a variable

max'l abelian ext'n of $\mathbb{F}_q(\theta)$

class field theory for $\mathbb{F}_q(\theta)$

Basic Problem: Hilbert's 12th problem.

Examples of works in char. $p \neq 0$ done by P. Deligne, S. Mori, V.G. Drinfeld.

More recently L. Lafforgue, Ngô Bau Chau.

Have Frobenius endomorphism $x \longmapsto x^p$, $x \longmapsto x^q$

on finite fields, Frobenius automorphisms

For those fin. dim. vec. sp., say over $\mathbb{F}_q$, we consider $\mathbb{F}_q$-linear maps.

Now K be a field . K char 0, prime subfield is $\mathbb{Q}$

K char p. prime subfield is $\mathbb{F}_p$.

If K has char. 0, K_0 = prime subfield, then K_0 - linear map are linear.

K has char. $p > 0$, then $\mathrm{End}(G_a/K)$ = set of Frobenius polynomials.

$$c_0 X + c_1 X^p + \cdots + c_n X^{p^n}$$

$$c_i \in K.$$

ring under composition.

If we restrict ourselves to $\mathbb{F}_q$ - linearity,

$\mathrm{End}_{\mathbb{F}_q}(G_a/K)$ = set of q-power $\mathbb{F}_q$ - linear map

$$\{c_0 X + c_1 X^q + \cdots + c_n X^{q^n}\}$$

Given any such polynomial, its zero set in K^a = alg. closure of K,

then zero set is $\mathbb{F}_q$- linear finite abelian gp.

In char. p, we have lots of finite abelian gps inside a field.

Twisted poly. ring $\mathbb{F}_q\{\tau\}$ non comm. $\tau : x \mapsto x^q$

Recall, valuation theory, absolute value.

$\alpha \in K$, $|\alpha| \geq 0$, $|\alpha| = 0 \iff \alpha = 0$

$|\alpha + \beta| \leq |\alpha| + |\beta|$, $|\alpha\beta| = |\alpha| \cdot |\beta|$.

archimedean absolute values in char. 0

In char. p, non-archimedean absolute value, strong $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$.

then call such absolute value a <u>valuation</u>.

Example. If field is $\mathbb{Q}$, all abs. values are equiv. to either

$|\cdot|_p$, p-adic valuation, p prime, n.a.

or $|\cdot|$ the usual abs. value. archimedean.

Example $k = \mathbb{F}_q(\theta)$, then have the degree valuation, degree in θ.

called also the ∞- valuation.

Also, for each monic irred. poly. $v \in \mathbb{F}_q[\theta]$,

$|\cdot|_v$, v-adic valuation.

This gives all valuations, mod equivalences

Example Completing $k = \mathbb{F}_q(\theta)$ under ∞, get Laurent series field

$$\sum_{n=-\infty}^{m} \alpha_n \theta^n$$

When $k = \mathbb{Q}$, $|\cdot|$ the usual abs. value, $\hat{k} = \mathbb{R}$.

$k = \mathbb{Q}(\sqrt{-1})$, $|\cdot|$ the usual abs. value, $\hat{k} = \mathbb{C}$.

$k = \mathbb{F}_q(\theta)$, $|\cdot|_\infty$ = degree valuation,

$\hat{k} = k_\infty$ notation

$= \mathbb{F}_q((\frac{1}{\theta}))$.

Note the alg. closure w/ extended valuation is not complete.

completion $\widehat{k^a} =: \mathbb{C}_\infty$, which is both complete & alg. closed.

Similarly, $\mathbb{C}_p = \widehat{\mathbb{Q}_p^a}$, alg. closure $\mathbb{Q}_p^a$ is not complete.

Note in fact, $\mathbb{Q}_p^a$ is inf. dim. over $\mathbb{Q}_p$,

not like $\mathbb{C}$ is quadratic over $\mathbb{R}$.

Similarly, here k_∞^a is inf. dim'l over k_∞

$\mathbb{C}_\infty$, $\mathbb{C}_p$ are inf. dim. space over k_∞, $\mathbb{Q}_p$, respectively.

Important for us!

We have Drinfeld modules over $\mathbb{C}_\infty$ of arbitrary rank

For elliptic curves over $\mathbb{C}$, only have dim 1 elliptic curves inside $\mathbb{P}^2_{\mathbb{C}}$.

Corresponds to lattices inside $\mathbb{C}$. rk 2 free discrete abelian gps

While you have discrete "$\mathbb{F}_q[\theta]$-lattices" of arbitrary rank inside $\mathbb{C}_\infty$.

We are interested in lattices; in function theory on complete valuated alg. closed fields

e.g. $\mathbb{C}_\infty$.

Classically, function theory on $\mathbb{C}$.

Entire function: classically, $f: \mathbb{C} \to \mathbb{C}$ holomorphic everywhere

For us, $f: \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$

Polynomial functions are entire functions.

Example. Classically, have the exponential function $z \longmapsto e^z$.

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Weierstraß factorization.

Picard theorem, e^z only omit 0

Valuation distribution study

Example $f: \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$. non constant

entire means it has Taylor expansion at 0 w/ infinite radius.

Given Drinfeld $\underset{\underset{\mathbb{F}_q[\theta]}{\shortparallel}}{A}$-module of rank $r > 0$,

Drinfeld exponential $\quad \exp_\rho(z) = \sum_{i=0}^{\infty} c_i z^{q^i}, \quad c_i \in \mathbb{C}_\infty$

surjective from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, also

$$\exp_\rho(z) = z \prod_{\substack{\lambda \neq 0 \\ \in \Lambda_\rho}} \left(1 - \frac{z}{\lambda}\right)$$

Λ_ρ is discrete free A-module rk r

For the Taylor coeff at 0, $\lim_{n \to \infty} |c_n|_\infty q^{rn} = 0, \ \forall r \in \mathbb{Q}$

Actually, we may write $\exp_\rho(z) = \lim_n f_n(z), \ n > 0$

$$f_n(z) = z \prod_{\substack{\lambda \in \Lambda_\rho \\ \deg \lambda \leq n}}' \left(1 - \frac{z}{\lambda}\right)$$

Recall ρ is only given by $\rho_\theta = \theta x + g_1 x^q + g_2 x^{q^2} + \dots + g_r x^{q^r}$, $g_i \in \mathbb{C}_\infty$, $g_r \neq 0$.

$\exp_\rho(z)$ are like infinite Frobenius poly. solving from

$$\exp_\rho(\theta z) = \rho_\theta(\exp_\rho(z))$$

<u>Ref</u> Y. 1985. Duke
1986 Inventiones Math

Growth of these entire functions.

$$M_{q^r}(f) = \sup_h (|c_h| q^{hr}) \quad \text{, on disc } |z| \le q^h$$

Classically, $f: \mathbb{C} \to \mathbb{C}$ entire of order ℓ if

$$|z| \le r \quad M_r(f) \le e^{\ell+\varepsilon} \quad \text{for } r \gg 0,\ \varepsilon > 0.$$

Then you get, order of $\exp_\rho(z)$ is $r \log q / \log \ell$.

On the other hand, these functions are $\mathbb{F}_q$-linear, Hence if $f(z_0) \in \bar{k}$,

then all its Taylor coeff at z_0 are algebraic.

$$f(z) = f(z_0) + \sum_h c_i (z-z_0)^{q^h}$$

$$= f(z_0 + (z-z_0))$$

Just as $f(z)$ satisfies some algebraic diff. eqn.

e.g. recall Weierstraß elliptic function

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$$

Schneider - Lang method. - C.L. Siegel

Review. very nice "uniformization"

Classically.

$$\begin{tikzcd}
0 \arrow[r] & 2\pi i\mathbb{Z} \arrow[r] \arrow[d, "n\cdot(\cdot)"] & \mathbb{C} \arrow[r, "\exp"] \arrow[d] & \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}) \arrow[r] \arrow[d, "(\cdot)^n"] & 0 \\
0 \arrow[r] & 2\pi i\mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}^\times \arrow[r] & 0
\end{tikzcd}$$

$\mathbb{Z}$-action.

E elliptic curve$/\mathbb{C}$, $\exp_E : z \mapsto (\wp_\Lambda(z), \wp'_\Lambda(z))$

Then

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_E \arrow[r] \arrow[d] & \mathbb{C} \arrow[r, "\exp_E"] \arrow[d] & E(\mathbb{C}) \arrow[r] \arrow[d, "{[n]}"] & 0 \\
0 \arrow[r] & \Lambda_E \arrow[r] & \mathbb{C} \arrow[r] & E(\mathbb{C}) \arrow[r] & 0
\end{tikzcd}, \quad \forall n \in \mathbb{Z}$$

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w\in\Lambda}' \left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right]$$

periodic meromorphic function, growth order 2.

alg. diff. eqn $\quad \wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - 60 G_2(\Lambda)\wp_\Lambda(z) - 140 G_3(\Lambda)$

$$G_w(\Lambda) := \sum_{w\in\Lambda}' \frac{1}{w^{2w}}, \quad w \geq 2$$

What we have:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_\rho \arrow[r] \arrow[d] & \mathbb{C}_\infty \arrow[r, "\exp_\rho"] \arrow[d, "a(\cdot)"] & \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \arrow[r] \arrow[d, "\rho_a"] & 0 \\
0 \arrow[r] & \Lambda_\rho \arrow[r] & \mathbb{C}_\infty \arrow[r] & \mathbb{G}_a(\mathbb{C}_\infty) \arrow[r] & 0
\end{tikzcd} \quad a \in A$$

Drinfeld correspondence.

$$\left\{\begin{matrix} \text{homothety} & A\text{-lattices} \\ \text{classes} & \cap \\ \text{rank } r & \mathbb{C}_\infty \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{Isom. classes} \\ \text{Drinfeld } A\text{-modules of rank } r \end{matrix}\right\}$$

In our talks, $A = \mathbb{F}_q[\theta]$.

But more generally, Drinfeld considered any proj. smooth curve over $\mathbb{F}_q$,

take a point ∞, fixed, $A :=$ all functions on that curve regular away from ∞.

Example. curve is $\mathbb{P}^1$, ∞ corresponds to degree valuation,

then $A \cong \mathbb{F}_q[\theta]$.

In general, A will be Dedekind domain in char. p, not UFD.

Now the moduli.

We study isom. classes e.g. moduli of elliptic curves.

On moduli you have modular forms.

More generally, in char $p \neq 0$, there is the moduli of ...

For rank one Drinfeld $\mathbb{F}_q[\theta]$-module,

i.e. $\rho_\theta = \theta\tau + g_1\tau^q$, $g_1 \in \mathbb{C}_\infty$

but up to isom., suffices to consider $\mathcal{C}_\theta = \theta\tau + \tau^q$ the Carlitz mod.

For rank 2, $\rho_\theta = \theta\tau + g_1\tau^q + \underset{\neq 0}{g_2}\tau^{q^2}$ $\rho = (g_1, g_2)$, $g_i \in \mathbb{C}_\infty$

<u>Lecture 12</u>. Start w/ Riemann, Riemann sphere

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C}) \cup \mathbb{A}'^1(\mathbb{C})$$

affine line $\mathbb{A}^1 \cap \mathbb{A}'^1 = \mathbb{G}_m$

$$\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(\tau), \quad \mathbb{C}(\mathbb{A}^1 \setminus \{0\}) = \mathbb{C}[\tau, \tfrac{1}{\tau}]$$

"Rigid" analytic geometry, non-archimedean

Why consider these?

GAGA principle : J-P. Serre

For our story :

Building block "Tate disc" $Sp(\mathbb{T}_1)$

$$\mathbb{T}_1 \text{ the Tate algebra } \mathbb{C}_\infty\langle\tau\rangle = \left\{\sum_{n=0}^{\infty} a_n\tau^n : \lim_n a_n = 0\right\}$$

Instead of $\mathbb{C}_\infty$, over any complete valuated field.

$Sp(\mathbb{T}_1)$ = max'l ideal space of $\mathbb{T}_1$

affinoid algebra.

$Sp(\mathbb{T}_1)(\mathbb{C}_\infty) \cong \{z : |z|_\infty \leq 1\}$

In elementary non-archimedean geometry, because of the strong Δ inequality,

$|x+y|_\infty \leq \max\{|x|_\infty, |y|_\infty\}$.

Strange phenomenon:

Facts: $|x|_\infty < |y|_\infty \Rightarrow |x-y|_\infty = |y|_\infty$

$\Rightarrow$ Every triangle is isoscile, every point inside an open disc is a center.

closed center

open means $|z-z_0|_\infty < c$

closed $\leq c$

c radius is important in that it has to satisfy $c \in |K^\times|_K$.

may be $q^{\mathbb{Z}}$ in our case

$q^{\mathbb{Q}}$

e.g. intersection of two open discs if not empty, should be again an disc.

<u>Back to $Sp(\mathbb{T}_1)$</u>

$Sp(\mathbb{T}_1)$ is the closed unit disc.

$\mathbb{P}^1(K) := Sp(\mathbb{T}_1) \cup Sp(\mathbb{T}_1')$

$Sp(\mathbb{T}_1) \cap Sp(\mathbb{T}') = Sp$ (Affinoid)

This affinoid $\left\{\sum_{n=-\infty}^{\infty} a_n z^n : \lim_{|n|\to\infty} a_n = 0\right\}$ "functions" on the "circle $|z|=1$" in $\mathbb{C}_\infty$.

Note: in nonarchimedean case, "circles" are also open.

Closed disc is disjoint union of two open sets ...

Because such K w/ the topology from the valuation is totally disconn'd.

Encounter there a "bad" case of classical analytic continuation.

Rigid analytic structures require more on "convergence", "open sets".

Can show that rigid meromorphic functions on $\mathbb{P}^1(\mathbb{C}_\infty)$ are the rat'l functions

So $Sp(T_1) \cap Sp(T_1') \cong Sp(K\langle z, \frac{1}{z}\rangle)$

Now affinoid domain inside $\mathbb{P}^1$.

analogue of closed affine subvarieties in algebraic geometry.

<u>Def</u> D Complement of finite unions of open discs are called connected affine domains in $\mathbb{P}^1(\mathbb{C}_\infty)$.

Then $\sigma(D)$ are affinoid, $\sigma \in PGL_2(\mathbb{C}_\infty)$.

$PGL_2(\mathbb{C}_\infty)$ acting on $\mathbb{P}^1(\mathbb{C}_\infty)$ as Möbius transform.

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad z \longmapsto \frac{az+b}{cz+d}.$$

<u>Recall</u> $PGL_2(\mathbb{C})$ is automorphism gp of $\mathbb{P}^1(\mathbb{C})$ in the sense of alg geometry or analytic geometry.

Now we have the same for $K = \mathbb{C}_\infty$, in alg. geom., in (rigid) analytic geom.

What is analytic continuation?

<u>Def</u> Grothendieck topology on a space X:

(1) Family $\mathcal{F}$ of subsets of X ; $\phi, X \in \mathcal{F}$

$U, V \in \mathcal{F}$, $U \cap V \in \mathcal{F}$

(2) For $U \in \mathcal{F}$, a collection $\mathrm{Cov}(U)$ of U by elts in $\mathcal{F}$, satisfies

$\{U\} \in \mathrm{Cov}(U)$, if $U, V \in \mathcal{F}$, $V \subset U$, $\mathcal{U} \in \mathrm{Cov}(V)$

$\Rightarrow \mathcal{U} \cap V \in \mathrm{Cov}(V)$

If $U \in \mathcal{F}$, $\{U_i\} \in \mathrm{Cov}(U)$, $\mathcal{V}_i \in \mathrm{Cov}(U_i)$, then $\bigcup_i \mathcal{V}_i \in \mathrm{Cov}(U)$

Call $\mathcal{F}$ collection of admissible open sets, $\mathcal{U} \in \mathrm{Cov}(V)$ admissible cover.

Sheaf of holomorphic functions on X

meromorphic functions

alg. of functions "regular" on each admissible domain given

then get $\mathcal{O}(X)$ sheaf.

<u>Example</u> $\mathcal{O}(\mathbb{P}^1(\mathbb{C}_\infty)) = \mathbb{C}_\infty$ Liouville.

Open discs in $\mathbb{P}^1(\mathbb{C}_\infty)$ are $\rho \in |\mathbb{C}_\infty^\times|$

$\{z : |z - z_0| < \rho\}$

"$\frac{1}{z}$" or $\{z : |z - z_0| > \rho\} \cup \{\infty\}$.

Then you have "Mittag-Leffler"

non-archimedean

Let $D \subset \mathbb{P}^1$ be connected affinoid domain, containing ∞.

$$D = \bigcap_i (\mathbb{P}^1 - D_i) \qquad \text{finite intersection}$$

$$D_i := \{ a \in \mathbb{C}_\infty : |a - a_i|_\infty < |\pi_i|_\infty \}, \quad \pi_i \in \mathbb{C}_\infty^\times$$

Let $\mathcal{O}(D)_+ := \{ f \in \mathcal{O}_D : f(\infty) = 0 \}$

Similarly $\mathcal{O}(\mathbb{P}^1 - D_i)_+$

$$\Rightarrow \mathcal{O}(D)_+ = \bigoplus_i \mathcal{O}(\mathbb{P}^1 - D_i)_+$$

$$\mathcal{O}(\mathbb{P}^1 - D_i)_+ = \left\{ \sum_{j=0}^{\infty} b_j \left(\frac{\pi_i}{z - a_i} \right)^j, \ \lim_j b_j = 0 \right\}$$

More interested in analytic structure on Drinfeld moduli

Ω_2 moduli of Drinfeld modules rank 2

or analytic structure on $\mathfrak{h}$ the upper half plane.

moduli of elliptic curves.

lattice $[w, 1] = \mathbb{Z}w + \mathbb{Z} \subset \mathbb{C}$, $z \in \mathfrak{h}$

Natural structure on $\mathfrak{h}$. in view of its compactification.

Consider "topology" on $\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}) = \mathfrak{h}^*$. compactification.

neighborhood of ∞

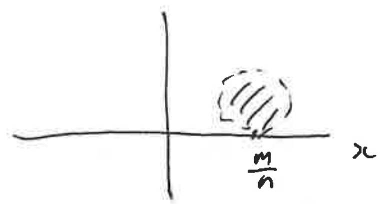
neighborhood of $\frac{m}{n} \in \mathbb{Q}$

$PGL_2(\mathbb{Q})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$

$\Downarrow$

σ

neighborhood at $\frac{m}{n}$



Quotient $\mathfrak{h}^*$ by $SL_2(\mathbb{Z})$, get the $\mathbb{P}^1$, the j-line.

Now for us, in char $p \neq 0$,

$\mathbb{F}_q[t]$, $t \neq \theta$ as another variable.

Rank 1 Drinfeld $\mathbb{F}_q[t]$-module

Carlitz module, action of $\mathbb{F}_q[t]$ by $t \mapsto \theta + \tau$, $\tau : x \mapsto x^q$

all rank one Drinfeld modules are isom. to Carlitz module.

Just like $\mathbb{C}/2\pi i \mathbb{Z} \xrightarrow{\exp} \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C})$ multi. gp

Write lattice $A \oplus Az$, $A = \mathbb{F}_q[\theta]$

$z \in \mathbb{C}_\infty - k_\infty = \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(k_\infty)$

Basic thm Drinfeld correspondence

$$\Lambda_\rho \overset{=[1,\omega]}{\longleftrightarrow} \rho$$

Drinfeld module $\quad \rho_t = \theta + g_1(\omega)\tau + g_2(\omega)\tau^2$

$$\neq 0$$

homothetic classes $\longleftrightarrow$ Isomorphism classes $/\mathbb{C}_\infty$

Notation $g_2 := \Delta$

$$GL_2(A)\backslash \mathfrak{h}^* \qquad \mathfrak{h} \text{ Drinfeld "plane".}$$

$$\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{F}_q(\theta))$$

<u>rigid analytic structure</u>.

Want $\quad GL_2(A)\backslash \mathfrak{h}^* \simeq \mathbb{P}^1(\mathbb{C}_\infty)$.

Introduce $\quad G_\ell(\Lambda_\rho) = G_\ell(\omega) := \sum'_{a,b \in \mathbb{F}_q[\theta]} \frac{1}{(a\omega+b)^\ell} \qquad \ell \geq 1$

weight ℓ

G_ℓ functions on Ω_2, weight ℓ.

$G_\ell \equiv 0$ if $\ell \not\equiv 0 \mod (q-1)$

In rank 2 case, we are interested in

$$(\theta^q - \theta)\, G_{q-1}(\Lambda) = g_1(\Lambda)$$

$$g_2(\Lambda) = (\theta^{q^2} - \theta)\, G_{q^2-1}(\Lambda) + G_{q-1}^{q+1}(\Lambda)(\theta^q - \theta)^q$$

$\Lambda \leftrightarrow \rho$, $\quad \rho_t = \theta + g_1(\Lambda)\tau + g_2(\Lambda)\tau^2$

g_i, G_i are " modular forms "

g_i wt $q^i - 1$

Recall $\mathfrak{h} \subset \mathbb{C}$

$SL_2(\mathbb{Z}) \backslash \mathfrak{h}$ moduli of elliptic curves

$$E: y^2 = 4x^3 - g_2 x - g_3$$

$$g_2 = 60\, G_2, \quad g_3 = 140\, G_3$$

Lecture 13. $\mathfrak{h} = \{z : \operatorname{Im} z > 0\} \subset \mathbb{C}$, classically, analytic structure

Nbd at cusps $\mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{R})$.

Now for rank 2 Drinfeld moduli,

$$\Omega_2(\mathbb{C}_\infty) = \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(k_\infty), \quad \text{cusps at } \mathbb{P}^1(k) \subset \mathbb{P}^1(k_\infty)$$
$$= \mathbb{C}_\infty - k_\infty$$

Consider rigid analytic structure.

Inside $\mathbb{P}^1(\mathbb{C}_\infty)$, complements of finite unions of open discs are called conn'd affinoid domains.

Any affinoid domain in $\mathbb{P}^1(\mathbb{C}_\infty)$ can be written as disjoint unions of conn'd affinoid domains uniquely. Here conn'd means that the constant functions $0, 1$ are the only idempotents of $\mathcal{O}(D)$, if D is the domain.

We are particularly interested in the following:

$$\mathbb{P}^1(\mathbb{C}_\infty) - \bigcup_{\eta \in \mathbb{P}^1(\mathbb{F}_q)} D^-(\eta, 1) =: D_0$$

where $D^-(\xi, 1) := \{ z \in \mathbb{C}_\infty : |z - \xi|_\infty < 1 \}$, $\xi \in \mathbb{F}_q$

$$D^-(\infty, 1) = \{\infty\} \cup \{ z \in \mathbb{C}_\infty : |z|_\infty > 1 \}$$

This is the $\mathbb{C}^\infty$-plane minus $q+1$ open discs.

Observe: these discs are disjoint.

$$\mathbb{P}^1(\mathbb{C}_\infty) \supset \Omega_2 \supset D_0$$

We know holomorphic functions on D_0.

Then, note $k_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ Laurent series, $\underset{\frac{1}{\theta}}{\overset{\pi}{\shortparallel}}$ uniformizer at place ∞.

Take, for $m \geq 0$ integer, enlarge a little bit, i.e. shrink open discs above

$$D_m := \mathbb{P}^1(\mathbb{C}_\infty) - \bigcup D^-(\xi_0 + \xi_1 \pi + \dots + \xi_m \pi^m, q^{-m})$$

$\xi = \xi_0 + \xi_1 \pi + \dots + \xi_m \pi^m$ inside k_∞. $\xi_0 \in \mathbb{P}^1(\mathbb{F}_q)$, $\xi_i \in \mathbb{F}_q$

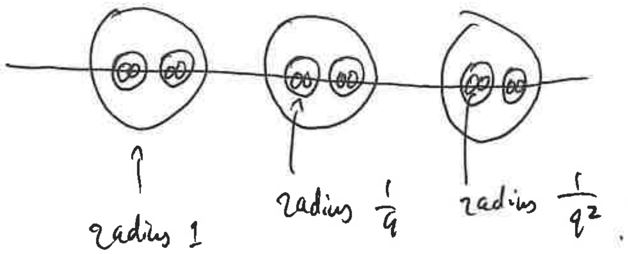
Here you have minus $(q+1) q^m$ open discs

$$D_0 \subset D_1 \subset D_2 \subset \dots \subset D_m \subset D_{m+1} \subset \dots$$

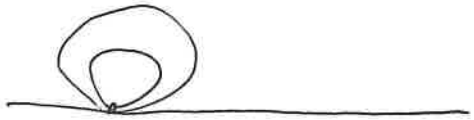
Union $\bigcup_m D_m = \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(k_\infty) = \Omega_2(\mathbb{C}_\infty)$.

This also gives $\mathcal{O}(\Omega_2(\mathbb{C}_\infty))$.

For $q=2$, $m=2$



Recall classically, open nbhds at cusps



Recall: $GL_2(A) \backslash \Omega_2(\mathbb{C}_\infty) / \mathbb{C}_\infty^\times$ parametrizes homothety classes of rk 2 A-lattices

will turn out to be algebraic.

$A = \mathbb{F}_q[\theta]$

On the other hand, Drinfeld correspondence

Isom. classes of Drinfeld rank 2 modules $/\mathbb{C}_\infty$.

$$Az + A = [z, 1] \longleftrightarrow (\rho_z)_t = \theta + g_1(z)\tau + g_2(z)\tau^2, \quad \tau: x \longmapsto x^2$$

The g_i, $i = 1, 2$ are "modular forms" on $\Omega_2(\mathbb{C}_\infty)$ satisfies

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$,

$$g_1\left(\frac{az+b}{cz+d}\right) = (cz+d)^{q-1} g_1(z)$$

$$g_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^{q^2-1} g_2(z)$$

Note $[z, 1] = [az+b, cz+d] \underset{\text{homothetic}}{\sim} \left[\frac{az+b}{cz+d}, 1\right]$

Exercise, express g_1, g_2 in terms of G_{q-1}, G_{q^2-1}, the Eisenstein series on $\Omega_2(\mathbb{C}_\infty)$.

We want to look closely at $\Omega_2(\mathbb{C}_\infty)$.

In fact, $\Omega_d(\mathbb{C}_\infty)$ for any rank d.

Def For $z \in \mathbb{C}_\infty$, introduce imaginary distance $|z|_i := \inf_{a \in k_\infty} |z-a|_\infty$

Facts: ① $z \in \mathbb{C}_\infty$, $|z|_i = 0 \iff z \in k_\infty$.

② $|z|_\infty \geq |z|_i$

③ For $z \in \mathbb{C}_\infty$, $a \in k_\infty$, $|az|_i = |a|_\infty |z|_i$

④ If $|z|_\infty \notin q^{\mathbb{Z}}$, $z \in \mathbb{C}_\infty$, then $|z|_\infty = |z|_i$

⑤ Consider residue reduction,

$\mathbb{C}_\infty^{\circ} := \{ z \in \mathbb{C}_\infty : |z|_\infty \leq 1 \}$

red: $\mathbb{C}_\infty^{\circ} \twoheadrightarrow \overline{\mathbb{F}_q}$, $z \longmapsto \bar{z}$

i.e. $\mathbb{C}_\infty^{\circ\circ} = \{ z \in \mathbb{C}_\infty : |z|_\infty < 1 \}$, $\mathbb{C}_\infty^{\circ} / \mathbb{C}_\infty^{\circ\circ} \cong \overline{\mathbb{F}_q}$ reduction.

For those z w/ $|z|_\infty = |z|_i$,

$|z|_\infty = |z|_i = 1 \iff \bar{z} \in \overline{\mathbb{F}_q} - \mathbb{F}_q$.

If $|z|_i < |z|_\infty = 1 \implies \bar{z} \in \mathbb{F}_q$.

Exercise: $GL_2(k_\infty)\left(\Omega_2(\mathbb{C}_\infty)\right) \subset \Omega_2(\mathbb{C}_\infty)$

For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k_\infty)$, $z \in \Omega_2(\mathbb{C}_\infty)$,

$$\left|\frac{az+b}{cz+d}\right|_i = |\sigma(z)|_i = |cz+d|_\infty^{-2} |\det \sigma|_\infty |z|_i$$

Hint: consider $z \mapsto \frac{1}{z}$, $z \mapsto z+a$.

This is exactly analogue of the classical story.

Now $z \in \Omega_2(\mathbb{C}_\infty)$, introduce a "norm" on $V := k_\infty^2 = k_\infty \times k_\infty$.

$$\alpha_z(a,b) \longmapsto |az+b|_\infty$$

$$(a,b) \in k_\infty^2$$

$$\alpha_z : k_\infty^2 \longrightarrow \mathbb{R}_{\geq 0}$$

Such function α satisfies

Def $\alpha : k_\infty^2 \longrightarrow \mathbb{R}_{\geq 0}$

$\alpha(x) \geq 0$, $\alpha(x) = 0 \iff x = (0,0)$

$\alpha(ax) = |a|_\infty \alpha(x)$ for $a \in k_\infty$.

$\alpha(x+y) \leq \max(\alpha(x), \alpha(y))$, $x, y \in k_\infty^2$.

Call norm α an integral norm if $\alpha(k_\infty^2) = |k_\infty|_\infty = q^{\mathbb{Z}} \cup \{0\}$.

α is said to be a rat'l norm if $\alpha(k_\infty^2) \subset q^{\mathbb{Q}} \cup \{0\}$.

$z \in \Omega_2(\mathbb{C}_\infty)$, $\alpha_z(a,b) = |az+b|_\infty$ is a rat'l norm.

<u>Question</u>. For what $z \in \Omega_2(\mathbb{C}_\infty)$, $|z|_\infty = |z|_i$ holds?

$\Omega_2(\mathbb{C}_\infty) \ni z \longmapsto \alpha_z$ called building map.

Call α_1, α_2 dilational equivalent, if $\alpha_1 = t\alpha_2$ w t real scalar.

For norm α, $[\alpha]$ dilation class of α.

Have a map $\Omega_2(\mathbb{C}_\infty) \ni z \longmapsto \alpha_z \longmapsto [\alpha_z] \in N(k_\infty^2)$

set of all dilation classes

Bruhat - Tits building map

or tree map

Key here: there is a natural tree structure on the set of all dilation classes.

<u>tree</u> is a 1-dim'l graph w/o circuits or back tracks

Now study this building map on $\Omega_2(\mathbb{C}_\infty)$.

Given a lattice (local) $M \subset k_\infty^2$ (free rank 2 over the valuation ring of k_∞)

$\mathcal{O}_\infty = \{z \in k_\infty : |z|_\infty \leq 1\}$

max'l cpt subring of k_∞

<u>Example</u>. Standard lattice $\mathcal{O}_\infty^2 \subset k_\infty^2$; $\pi\mathcal{O}_\infty \oplus \mathcal{O}_\infty, \cdots$

$M \otimes_{\mathcal{O}_\infty} k_\infty = k_\infty^2$ always holds

view M as fractional of standard lattices

More precisely, as $\sigma(\mathcal{O}_\infty^2)$, $\sigma \in GL_2(k_\infty)$.

The center of $GL_2(k_\infty)$ are scalar matrices

all dilation classes of lattices inside k_∞^2 parametrized by cosets

$$GL_2(k_\infty) / k_\infty^\times GL_2(\mathcal{O}_\infty)$$

Each lattice gives rise to a norm $M \longmapsto \alpha_M$.

$$x \in k_\infty^2, \quad \alpha_M(x) := \inf \left\{ \frac{1}{|a|_\infty} : ax \in M \right\}$$

Example This norm comes from "denominators"

If $M \subset M'$, then $\alpha_{M'} \leq \alpha_M$.

$$M = \mathcal{O}_\infty \times \frac{1}{\pi} \mathcal{O}_\infty,$$
$$\cup$$
$$* = \mathcal{O}_\infty \times \mathcal{O}_\infty \qquad \alpha_*(0, \tfrac{1}{\pi}) = q > \alpha_M(0, \tfrac{1}{\pi}) = 1.$$

The norms from lattice classes are always integral classes.

Conclusion: The integral norm classes parametrize the dilation classes of lattices.

The building map: $\Omega_2(\mathbb{C}_\infty) \longrightarrow N(k_\infty^2)$

$$\cup$$

$$GL_2(k_\infty) / k_\infty^\times GL_2(\mathcal{O}_\infty) \quad \text{integral classes}$$

Claim $\Omega_2(\mathbb{C}_\infty) \twoheadrightarrow N(k_\infty^2)$ surjective.

$\simeq T(\mathbb{Q})$ rat'l pts of Bruhat-Tits tree.

T has set vertices

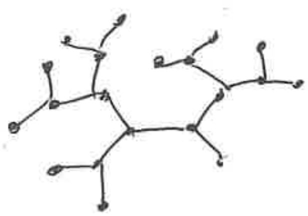
$$X(T) \cong GL_2(k_\infty)/k_\infty^\times GL_2(\mathcal{O}_\infty) \cong \left\{\begin{matrix}\text{integral}\\ \text{norm classes}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{set of}\\ \text{lattice classes}\end{matrix}\right\}$$

$$Y(T) \cong GL_2(k_\infty)/k_\infty^\times I(\mathcal{O}_\infty)$$

$$I(\mathcal{O}_\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\infty) : c \equiv 0 \mod \pi \right\}$$

called the Iwahori subgroup

$q=2$



$D_0 \subset \Omega_2(\mathbb{C}_\infty)$

$$\beta^{-1}(*) = D_0.$$

Building map $\qquad * =$ Class of standard lattice

<u>Lecture 14</u>. Back to Carlitz module

$C_t = \theta + \tau \in \mathbb{C}_\infty\{\tau\}$ endomorphism of $\mathbb{G}_a$

Its associated exponential function the Carlitz exponential

$$\exp_C(z) = \sum_{h=0}^{\infty} \frac{1}{D_h} z^{q^h}, \quad D_h \in \mathbb{F}_q[\theta] = A.$$

= like the factorial for A $\qquad$ explicitly work out!

$$= z \prod_{w \in \Lambda}{}' \left(1 - \frac{z}{w}\right)$$

Rank 1 says $\Lambda = A \cdot \tilde{\pi}$, $\tilde{\pi} \in \mathbb{C}_\infty$ trans. over k.

$$\tilde{\pi} = \underset{\substack{\uparrow \\ 2\sqrt{-1}}}{(-\theta)^{\frac{q}{q-1}}} \prod_{i=1}^{\infty} \underset{\substack{\uparrow \\ \pi}}{\left(1-\theta^{1-q^i}\right)^{-1}}$$, $(q-1)$-th root well-defined

analogue of $2\pi\sqrt{-1}$ classically.

The product expansion : analogue of Wallis's product formula for π.

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots$$

All rank one Drinfeld modules $\rho_t = \theta + g\tau$ deg 1 in τ

isom. to the Carlitz module

associated lattice of the form $\boxed{A\cdot w_0}$, $w_0 \in \mathbb{C}_\infty$.

$k\left(\sqrt[q-1]{-\theta}\right)$

$|$ Kummer

k base field $\supset \mathbb{F}_q^\times$, cyclic gp order $q-1$

In particular, we may take $w_0 = 1$.

Consider the lattice $\Lambda_\rho = A$, ρ the corresponding Drinfeld module.

Exponential $\exp_\rho(z)$ has zero set precisely A.

$$0 \longrightarrow A \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_\rho} \mathbb{C}_\infty \longrightarrow 0$$

$$\tilde{\pi}\cdot\exp_\rho(z) = \exp_c(\tilde{\pi} z) =: t^{-1}(z)$$

Claim This function $t(z)$ serves as a uniformizer at our cusp ∞ for $\Omega_2(\mathbb{C}_\infty)$.

Recall classically, nbhd at $i\infty$, or ∞,

$\mathrm{Im}\, z > c$, $z \in \mathbb{C}$, $c > 0$.

Use $z \longmapsto e^{2\pi i z}$ to map such nbhds onto balls inside the unit disc punctured at 0.

leads to q-expansions of functions modular on the upper half-plane.

Fourier expansion comes in.

E. Gekeler, D. Goss in 1980's, studied functions on $\Omega_2(\mathbb{C}_\infty)$.

Note, they introduced $t(z)$. $t(z)$ does not have zeros in $\Omega_2(\mathbb{C}_\infty)$ since A is excluded by $\mathbb{C}_\infty - k_\infty$.

We have introduced the imaginary distance $z \longmapsto |z|_i$

Now consider $\mathbb{C}_\infty / A \xrightarrow{\exp_\rho} \mathbb{C}_\infty$.

Introduce $\Omega_c := \{ z \in \Omega_2(\mathbb{C}_\infty) : |z|_i > c \}$. Take $c > 1$.

$A \backslash \Omega_c \hookrightarrow GL_2(A) \backslash \Omega_2(\mathbb{C}_\infty)$.

injective: If $\gamma \in GL_2(A)$, $\Omega_c \cap \gamma(\Omega_c) \neq \phi \Rightarrow \gamma \in B(A)$ Borel

consisting of $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ essentially "translation" by A.

$|z|_i > c$, $\left|\frac{az+b}{cz+d}\right|_i = \frac{1}{|cz+d|_\infty^2} |z|_i$ forces $c = 0$, then $a, d \in \mathbb{F}_q^\times$

Classically Borel gp of 2x2 matrices in $SL_2(\mathbb{Z})$ just translations in upper half plane.

Goal is to make $GL_2(A) \backslash \Omega_2(\mathbb{C}_\infty)$ into a rigid analytic space w/ nice analytic structure.

Consider such pieces $A \backslash \Omega_c$. Make these nbds at our cusp ∞.

$t(z)$ is a natural parameter here (just like $e^{2\pi i z}$ classically).

$A \backslash \Omega_c \underset{t}{\cong} D^+_{(r)} \setminus \{0\}$. , $r = r(c)$ conn'd affinoid domain w/ standard analytic structure.

Consequence: all our modular functions on Ω have t-expansions.

In non-archimedean geometry here.

$$|\exp_\rho(z)|_\infty = |z|_\infty \prod_{\substack{|a|_\infty \leq |z|_\infty \\ a \in A}}' \left|1 - \frac{z}{a}\right|_\infty$$

infinite product is actually a finite product

$$\left|1 - \frac{z}{a}\right|_\infty = 1 \quad \text{once } |z|_\infty > |a|_\infty.$$

$$= |z|_\infty \cdot \prod_{|a|_\infty < |z|_\infty} \left|\frac{z}{a}\right|_\infty \quad \text{since } \left|1 - \frac{z}{a}\right|_\infty = \left|\frac{z}{a}\right|_\infty$$

$$= |z|_\infty^{q^d} \Big/ \prod_{\deg < d} |a|_\infty \quad \text{if } |z|_\infty = q^{d-\varepsilon},\ 0 \leq \varepsilon < 1.\ d \in \mathbb{Z}$$

$\exp_\rho$ maps circles to circles

Prop $|z|_i \leq -\log_q |t(z)|_\infty \leq C_0 |z|_i$ for some $C_0 > 0$.

We are interested in the moduli for Drinfeld modules

$$GL_2(A) \backslash \Omega_2(\mathbb{C}_\infty)$$

Go back to the building map.

$$B: \Omega_2(\mathbb{C}_\infty) \longrightarrow N(k_\infty^2), \quad z \longmapsto [\alpha_z]$$

dilation class of norms on k_∞^2

$$\alpha_z(s,t) = |sz + t|_\infty.$$

$N(k_\infty^2)$ contains classes of integral norms α

$$\alpha(s,t) = \sup\{|s|_\infty, |t|_\infty\}, \quad (s,t) \in k_\infty^2$$

α corresponds to standard lattice $\mathcal{O}_\infty^2 \subset k_\infty^2$.

Take any basis $\{x_1, x_2\}$ of k_∞^2,

Norm $\alpha_{\{x_1,x_2\}}(sx_1 + tx_2) = \sup\{|s|_\infty, |t|_\infty\}$

$\mathcal{O}_\infty$-lattice $M_{\{x_1,x_2\}} = \mathcal{O}_\infty x_1 + \mathcal{O}_\infty x_2 \subset k_\infty^2$

norm $\alpha \longleftrightarrow$ lattice α_M, closed unit α-ball in k_∞^2

$$\alpha_M(x) = \inf\left\{\frac{1}{|a|_\infty} : a \in k_\infty, \ ax \in M\right\}$$

$GL_2(k_\infty)$ acts on $N(k_\infty^2)$ via changing basis of k_∞^2.

$GL_2(k_\infty)$ acts on set of integral norms

Action explicitly written

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha\right)(s,t) := \alpha\left((s,t)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \quad , (s,t) \in k_\infty^2$$

On the other hand, $GL_2(k_\infty)$ acts on $\Omega_2(\mathbb{C}_\infty)$ via Möbius transform.

Building map $\mathcal{B}: \Omega_2(\mathbb{C}_\infty) \longrightarrow N(k_\infty^2)$ is $GL_2(k_\infty)$-equivariant.

i.e. $[\alpha_{\gamma \cdot z}] = [\gamma \cdot \alpha_z] \qquad z \in \Omega_2(\mathbb{C}_\infty)$.

Dilation classes of integral norms parametrized by

$$GL_2(k_\infty) / k_\infty^* \, GL_2(\mathcal{O}_\infty) =: X(T)$$

this is integral points on $N(k_\infty^2)$

norms represented by norm w/ values

in $q^{\mathbb{Z}} \cup \{0\}$.

The building map is surjective, here $N(k_\infty^2)$ contains all rat'l norms.

First question: what is the inverse image of the standard norm class $*$?

$$\mathcal{B}^{-1}(*) = \mathbb{P}^1(\mathbb{C}_\infty) - \bigcup_{\eta \in \mathbb{P}^1(\mathbb{F}_q)} D^-(\eta, 1) = D_0 \subset \Omega_2(\mathbb{C}_\infty)$$

i.e. $\mathbb{P}^1$ minus $(q+1)$-open balls of radius 1, an affinoid domain.

Action by $\gamma \in GL_2(k_\infty)$, $\mathcal{B}^{-1}$ (any dilation class of lattice)

will be $\mathbb{P}^1(\mathbb{C}_\infty) - (q+1)$ open balls.

<u>Rmk</u> $GL_2(k_\infty)$ is infact the automorphism gp of $\Omega_2(\mathbb{C}_\infty)$ once the proper analytic structure is given on $\Omega_2(\mathbb{C}_\infty)$.

Classically, $GL_2^+(\mathbb{R})$ is the automorphism gp of the Riemann surface of the upper half plane.

There is a natural tree structure T from $GL_2(k_\infty)$

vertices the integral norm classes or lattice classes on k_∞^2

Vertices $X(T) \cong GL_2(k_\infty) / k_\infty^\times GL_2(\mathcal{O}_\infty)$

<u>Recall trees</u> graph w/ oriented edges

vertex set X, oriented edges Y

$$Y \longrightarrow X \times X \qquad Y \longrightarrow Y$$

$$y \longmapsto (\underset{\text{origin.}}{o(y)}, \underset{\text{terminal}}{t(y)}) \qquad y \longmapsto \bar{y} \qquad \bar{\bar{y}} = y,\ y \neq \bar{y}$$

Path, w/o backtracking

Groups acting on a graph X, If $G \backslash X$ the orbits

The quotient graph mod a gp action

Def. A tree is a conn'd graph w/o circuits.

A $(q+1)$-regular tree means every vertex has exactly $(q+1)$ edges connected.

Ref J.-P. Serre, Trees.

$GL_2(k_\infty)$ gives T w/ integral norms classes as vertices.

Two lattices M_1, M_2

$[M_1][M_2]$ is an edge if $\pi M_2 \subsetneq M_1 \subsetneq M_2$,

M_2/M_1 is $\mathcal{O}_\infty$-mod. of length 1, i.e. $M_2/M_1 \cong \mathbb{F}_q$.

Vertices given by $GL_2(k_\infty)/k_\infty^\times GL_2(\mathcal{O}_\infty)$

Oriented edges given by $GL_2(k_\infty)/I k_\infty^\times$

$$GL_2(\mathcal{O}_\infty) \supset I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \ (\mathrm{mod}\ \overset{\frac{1}{\theta}}{\overset{\shortparallel}{\pi}}) \right\}$$

$Y(T) \to X(T)$

$y \mapsto o(y)$ origin of edge is just the canonical map

Back to building map.

$$\Omega_2(\mathbb{C}_\infty) \longrightarrow N(k_\infty^2)$$
$$\cong$$
$$T(\mathbb{Q})$$

$GL_2(k_\infty)$ - action on both sides — analogue of $SL_2(\mathbb{Z})$

Note $GL_2(A) \subset GL_2(k_\infty)$, $GL_2(A)$-equiv. "arithmetic groups"

Intrested in $GL_2(A)\backslash \Omega_2(\mathbb{C}_\infty) \longrightarrow GL_2(A)\backslash N(k_\infty^2)$. $GL_2(A)$ NOT f.g.

Lecture 15

Recall. classically

$SL_2(\mathbb{Z})$ infinite "discrete" non-abelian gp

$PSL_2(\mathbb{Z})$ gen. by $-\frac{1}{z}$, $z+1$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Use Euclidean algorithm to derive this.

We want to understand $GL_2(\mathbb{F}_q[\theta])$ modular gp

More generally, if you have R arithmetic Dedekind domain, $GL_2(R)$ structure?

finite residue field

Find generators, relations.

In algebraic number theory, $GL_2(R)$, R ring of integers, can be complicated

Even for $SL_2(\mathbb{Z})$ or $GL_2(\mathbb{Z})$.

Subgroups may have more complicated structures.

Also generators are important, eg. $-\frac{1}{z}$ leads to functional eq'n for, say Riemann ζ-function.

Now $GL_2(\mathbb{F}_q[\theta])$, easiest description is

$$GL_2(\mathbb{F}_q[\theta]) \cong GL_2(\mathbb{F}_q) \underset{B(GL_2(\mathbb{F}_q))}{*} B(GL_2(\mathbb{F}_q[\theta]))$$

directions

B Borel subgp, given by $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : c=0, a, b, d \in A \right\}$

The $*$ here means <u>amalgam</u>. [Ref: J.-P. Serre, Trees]

$B(GL_2(\mathbb{F}_q)) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{F}_q\} \subset GL_2(\mathbb{F}_q)$

$$\begin{tikzcd} B(GL_2(\mathbb{F}_q)) \arrow[r, hook] \arrow[dr] & GL_2(\mathbb{F}_q) \\ & B(GL_2(\mathbb{F}_q[\theta])) \end{tikzcd}$$

amalgmate these two injections to get $GL_2(\mathbb{F}_q[\theta])$.

which is "universal" w.r.t. to above injections

If $\begin{tikzcd} \{e\} \arrow[r, hook] \arrow[d] & G_1 \\ G_2 \end{tikzcd}$, $G_1 \underset{\{e\}}{*} G_2$ is called the free products of the gps G_1 & G_2.

$GL_2(A)$ acts naturally on "symmetric" spaces (non-archimedean)

Introduced Bruhat-Tits tree T, also $\Omega_2(\mathbb{C}_\infty)$ by Drinfeld.

In genral, Bruhat-Tits buildings,

$\Omega_2(\mathbb{C}_\infty)$ moduli for rank 2 Drinfeld modules.

Have building map $B : \Omega_2(\mathbb{C}_\infty) \to T(\mathbb{Q}) \cong N(k_\infty^2)$.

equiv. $GL_2(k_\infty)$-action

$\cup$

$GL_2(\mathbb{F}_q[\theta])$-action

Analogue of $SL_2(\mathbb{Z})$ as Möbius transforms on the upper half plane.

We regard $\Omega_2(\mathbb{C}_\infty)$ as rigid analytic space.

Recall $\Omega_2(\mathbb{C}_\infty) \supset D_0$ conn'd affinoid domain.

D_0 consists of points z w/ $|z|_\infty = |z|_i = 1$.

$D_0 = \beta^{-1}(*)$, $*$ the standard vertex,

dilation class of $\mathcal{O}_\infty^2 \subset k_\infty^2$.

β^{-1} (any vertex) are of the same type.

$D_0 = \{ z \in \mathbb{C}_\infty : |z|_\infty = 1,\ |z - \zeta|_\infty = 1,\ \zeta \in \mathbb{F}_q^\times \}$

$\mathbb{P}^1(\mathbb{C}_\infty)$ minus $(q+1)$ open balls radius 1, centred at $\mathbb{P}^1(\mathbb{F}_q)$.

<u>Example</u> $\beta^{-1}([\pi\mathcal{O}_\infty \times \mathcal{O}_\infty]) = \{ z \in \mathbb{C}_\infty : |z|_\infty = |z|_i = \frac{1}{q} \}$

Vertices $[\pi \mathcal{O}_\infty \times \mathcal{O}_\infty]$ and $[\mathcal{O}_\infty^2] = *$ connected by an edge e (adjacent)

$\beta^{-1}(e - \text{end points}) = \{ z \in \mathbb{C}_\infty : \frac{1}{q} < |z|_\infty < 1 \}$ an "annulus".

Note $GL_2(k_\infty)$ also acts naturally on $\mathbb{P}^1(k_\infty)$.

We interprete $\mathbb{P}^1(k_\infty)$ on $T = T(\mathbb{G})$

$\mathbb{P}^1(k_\infty)$ regarded as parametrizing <u>ends</u> of T.

the set of ends of T is denoted by ∂T.

Let M, M' be lattices in k_∞^2.

Given integer $n > 0$, $\exists!$ lattice in k_∞^2

$M_n \subset M'$ s.t. $M'/M_n \cong \mathcal{O}_\infty / \pi^n \mathcal{O}_\infty$, i.e. of "distance" n to M

As n varies, $[M_n]$ corresponds bijectively to $\mathcal{O}_\infty$- submodule of $M'/\pi^n M'$ rank one, i.e. a point of $\mathbb{P}^1(M'/\pi^n M')$.

Note $\varprojlim_n \mathbb{P}^1(M'/\pi^n M') = \mathbb{P}^1(\mathcal{O}_\infty) \cong \mathbb{P}^1(k_\infty)$

$$\begin{array}{c} \| \\ \mathbb{P}^1(M') \end{array}$$

consists of lines in k_∞^2

Geometrically.

Two half-lines differ by a finite subgraph in $T(\mathcal{G})$ said to be equivalent, called ends on $T(\mathcal{G})$.

<u>Example</u> M_n as above

$L = \bigcap_n M_n$ is a unique line contained in M'.

Note any two vertices on T can be conn'd by a finite graph.

An end does not depend on its starting point.

Now a straight path in T is a subgraph isom. to

$$\cdots \underset{-2}{\circ} - \underset{-1}{\circ} - \underset{0}{\circ} - \underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} - \cdots$$

$\{$straight paths on $T\}$ $\longleftrightarrow$ $\{$pairs of distinct ends$\}$

$\longleftrightarrow$ $\{$direct sum decompositions of $k_\infty^2\}$

In particular, we have the standard

$$\mathcal{O}_\infty \times \mathcal{O}_\infty = \mathcal{O}_\infty^2 < k_\infty^2 = k_\infty \times k_\infty$$

$\underset{\text{standard}}{*}$ gives us a straight line on T.

a particular straight path from the end 0 to the end ∞ in $\mathbb{P}^1(k_\infty)$.

denoted by

$$\underset{0}{}\;(\cdots, v_{r-1}, v_r, v_{r+1}, \cdots) = (\cdots v_{-2}, v_{-1}, \underset{\underset{*}{\shortparallel}}{v_0}, v_1, v_2, \cdots)\;\underset{\infty}{}$$

Called the principal axis of the tree T.

let $v(r,w)$ represent $\begin{pmatrix} \pi^r, w \\ 0 & 1 \end{pmatrix}$, $r \in \mathbb{Z}$, $w \in \pi^r \mathcal{O}_\infty$

On T, $GL_2(k_\infty) / GL_2(\mathcal{O}_\infty) k_\infty^\times$ parametrizes

$S_X = \left\{ \begin{pmatrix} \pi^r & w \\ 0 & 1 \end{pmatrix} : r \in \mathbb{Z},\ w \in \pi^r \mathcal{O}_\infty \right\}$. system of coset representative.

More explicit description of vertices on T.

Since $GL_2(k_\infty)$ acts transitively on T, any vertex is obtained from $*$ by a "Möbius transform"

β^{-1} (any vertex) is $\mathbb{P}^1(\mathbb{C}_\infty)$ minus $(q+1)$ disjt open balls.

Picture of the rigid analytic space $\Omega_2(\mathbb{C}_\infty)$.

Start w/ $\beta^{-1}(*) = D_0$.

$\mathbb{P}^1(\mathbb{C}_\infty)$ minus $(q+1)$ - open balls disjoint.

each vertex has $(q+1)$ -adjacent edges.

Inverse image of each edge is $\mathbb{P}^1(\mathbb{C}_\infty)$ minus two open balls.

You "glue" two balls, one from the β^{-1} (edge), one from the β^{-1} (vertex) at the other other terminal vertex of the edge glue another vertex.

This is how you get the whole T on the right hand side of the building map.

On the other hand, you glue the inverse images under the building map

arrive at the rigid analytic structures on the left hand side, i.e. $\Omega_2(\mathbb{C}_\infty)$

You get infinitely many copies of $\mathbb{P}^1(\mathbb{C}_\infty)$ glued together.

Back to $\Omega_2(\mathbb{C}_\infty)$, $c>1$ real, $\Omega_c = \{ z \in \mathbb{C}_\infty : |z|_i > c \}$ imaginary distance

If $\Omega_c \cap \sigma(\Omega_c) \neq \phi$ w/ $\sigma \in GL_2(\mathbb{F}_q[\theta])$,

$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then must have $c=0$

recall transformation law $|\sigma(z)|_i = \dfrac{|\det \sigma|_\infty}{|cz+d|^2} |z|_i$

$\sigma \in B(GL_2(\mathbb{F}_q[\theta])) =: H$ $\quad a, d \in \mathbb{F}_q^\times$

$H \backslash \Omega_c \overset{t}{\hookrightarrow} GL_2(A) \backslash \Omega_2(\mathbb{C}_\infty)$

$H \backslash \Omega_c \underset{t^{q-1}}{\hookrightarrow} A \backslash \Omega_c \hookrightarrow A \backslash \mathbb{C}_\infty \cong \mathbb{C}_\infty = \mathbb{P}^1(\mathbb{C}_\infty) - \{0\}$.

Recall for $|z|_i > 1$. suppose z is of minimal value among $z \bmod A$

Computed $|\exp_\rho(z)|_\infty$ $\quad$ ρ is the rank one Drinfeld module corresponds to

$= |z|_\infty \prod_{\substack{0 \neq a \in A \\ |a|_\infty \leq |z|_\infty}} \left|1 - \frac{z}{a}\right|_\infty$ $\quad$ A-lattice in $\mathbb{C}_\infty$, namely A.

$= |z|_\infty \prod_{\substack{0 \neq a \in A \\ |a| < |z|}} \left|\frac{z}{a}\right|_\infty$.

Recall $\tilde{\pi}\exp_\rho(z) = t^{-1}(z)$, $\tilde{\pi}$ is the period of the Carlitz module

<u>Rmk</u> (1) If $|z|_i > 1$, then $|z|_i < |z|_\infty$ $\Rightarrow$ $|z|_\infty$ is not minimal among z mod A.

(2) If $|z|_i > 1$

$$A \backslash \Omega_C \hookrightarrow \mathbb{C}_\infty / A \underset{\exp_\rho}{\cong} \mathbb{C}_\infty \qquad \text{surjectivity}$$

consider z among z mod A of minimal abs. value is just consider the abs. val. on $\mathbb{C}_\infty$ as $\mathbb{C}_\infty / A$.

$$A \backslash \Omega_C \xrightarrow[\cong]{t^{q-1}} D^*(r) - \{0\} \text{ for some } r$$

$$\infty \longleftrightarrow 0$$

This is the one-point compactification of Ω_C.

i.e. adding cusp ∞ to $A \backslash \Omega_C$

<u>Claim</u>

$$GL_2(A) \backslash \Omega_2(\mathbb{C}_\infty) \cong \mathbb{C}_\infty$$

$$\mathbb{P}^1(\mathbb{C}_\infty)$$

$$GL_2(A) \backslash \mathbb{P}^1(\mathbb{C}_\infty) \cong \mathbb{P}^1(\mathbb{C}_\infty)$$

$$\mathbb{P}^1(k_\infty) \longleftrightarrow \infty$$

Compare to the classical case:

$\mathfrak{h}$ = upper half plane

$$SL_2(\mathbb{Z}) \backslash \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}) \cong \mathbb{P}^1(\mathbb{C})$$

(Classically parameters (uniformizer) $q = e^{2\pi i z}$.

In our world, we have uniformizer t from the Carlitz module