Deformations (b): representability and Schlessinger's criterion

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Lecture 1

Functors of Artin rings: representability & Schlessinger's criterion

We've seen that a scheme X can be completely recovered from its functor of points.

& (under mild hypotheses) the tangent space is recovered by looking at maps

$\mathrm{Spec}\, k[\varepsilon] \to X$.

We'll look at something in between: $\mathrm{Spec}\, A \to X$, A local Artin ring

(w/ image some $x \in X$)

From a moduli perspective, we're studying families over $\underbrace{\mathrm{Spec}\, A}_{\text{one point}}$, w/ a fixed restriction to

$\mathrm{Spec}\, k$. These are called "infinitesimal thickenings". and the data obtained is the

"complete local ring" of the moduli space at the chosen point.

Recovering complete local rings

(Temporary) notation. $\mathrm{Art}(k)$ is the cat. of local Artin rings w/ residue field k.

(morphism compatible w/ map to k).

Given X a locally noetherian scheme, and $x \in X$, $k = k(x)$. let

$$F_{X,x} : \mathrm{Art}(k) \to \mathrm{Set} \text{ given by } (A \twoheadrightarrow k) \mapsto \left\{ \mathrm{Spec}\, A \xrightarrow{f} X \text{ s.t. } f \circ (\mathrm{Spec}\, k \to \mathrm{Spec}\, A) = \mathrm{Spec}\, k \xrightarrow{x} X \right\}$$

Prop. Given X locally noetherian scheme, $x \in X$, the canonical map

$\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x} \to X$ induces a bijection $F_{\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x}} \xrightarrow{\sim} F_{X,x}$, and any complete local noeth. ring

R w/ residue field k $\mathrm{Spec}\, R \to X$ inducing such a bijection is canonically isom. to $\hat{\mathcal{O}}_{X,x}$.

Rmk. Last part anticipates prorepresentability.

Proof. The first statement is equiv. to saying any map $\mathrm{Spec}\, A \to X$ w/ image x factors uniquely through $\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x} \to X$.

It's an easy exercise that it factors through $\mathrm{Spec}\, \mathcal{O}_{X,x} \to X$ (indeed, this is true for any local ring A). So we need $\mathcal{O}_{X,x} \to A$ factors uniquely through $\mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x}$. But since A is artin., so some power m_x^n maps to 0 in A. i.e. $\mathcal{O}_{X,x} \to A$ factors through $\mathcal{O}_{X,x}/m_x^n$

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r] \arrow[dr] \arrow[dd] & \hat{\mathcal{O}}_{X,x} \arrow[d] \\ & \mathcal{O}_{X,x}/m_{\hat{x}}^n \arrow[dl] \\ A & \end{tikzcd}$$

, so get factorization through $\hat{\mathcal{O}}_{X,x}$.

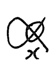
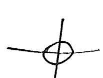
2nd part. point is that $\hat{\mathcal{O}}_{X,x}/m_{\hat{x}}^n$ and R/m_R^n both give Artin rings $\forall n$. Using a Yoneda-style trick, construct compatible maps $R \to \hat{\mathcal{O}}_{X,x}/m_{\hat{x}}^n$ & $\hat{\mathcal{O}}_{X,x} \to R/m_R^n$, use to construct $\hat{\mathcal{O}}_{X,x} \underset{\sim}{\overset{\sim}{\rightleftarrows}} R$.

Rmk. What data is in $\hat{\mathcal{O}}_{X,x}$?

(1) dim of X at x.

(2) "Singularity type" of X at x. something similar to a local ring of an analytic space.

(3) e.g. Cohen thm, X smooth $/k$ of dim n, $\hat{\mathcal{O}}_{X,x} \simeq k[[x_1, \dots, x_n]]$.

(4) e.g. $y^2 = t^3 - t^2$  , $\hat{\mathcal{O}}_{X,x} \simeq k[[u,v]]/(uv)$ 

(5) e.g. $y^2 = t^3$.  get $\hat{\mathcal{O}}_{X,x} \simeq k[[y,t]]/(y^2 - t^3) \not\simeq k[[s]]$, even though have a homeomorphism.

The functors of interest

We work in a relative setting: we'll fix Λ a complete local noetherian ring w/ res. field k. we'll consider $\mathrm{Art}(\Lambda, k)$ of Artin local Λ-algebras w/ residue field k.

Nonstandard terminology: A predeformation functor is a (covariant) functor

$F : \mathrm{Art}(\Lambda, k) \longrightarrow \mathrm{Set}$ s.t. $F(k)$ is the one point set.

Roughly, these arise by considering families over $\mathrm{Spec}\, A$ restricting to a fixed object over $\mathrm{Spec}\, k$. Starting w/ a global moduli functor, can obtain a pre-deformation functor by choosing an object $/k$ and restricting to Artin rings.

This doesn't always work well.

Lecture 2

Examples. For "nice" global moduli functor, it works well to simply restrict to $\mathrm{Art}(\Lambda, k)$ to obtain predeformation functors.

Ex. Deformations of a closed subscheme.

Let X_Λ be a scheme over $\mathrm{Spec}\, \Lambda$, write X for $X_\Lambda|_{\mathrm{Spec}\, k}$. Let $Z \subset X$ be a closed subscheme. $\mathrm{Def}_{Z,X} : \mathrm{Art}(\Lambda, k) \longrightarrow \mathrm{Set}$ is defined by

$$A \longmapsto \{ Z_A \subset X_\Lambda|_{\mathrm{Spec}\, A} \text{ closed subscheme, flat over } A. \text{ s.t. } Z_A|_{\mathrm{Spec}\, k} = Z \}.$$

Sometimes, simple restriction of functors isn't so good.

Ex. Deformations of a scheme. Fix X/k.

Def_X is defined by

$$A \mapsto \left\{ (X_A, \varphi) : X_A \text{ is flat over } \operatorname{Spec} A,\ \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & X_A \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd} \text{ induces an isom. } X \xrightarrow{\sim} X_A \times_A k \right\} \Big/ \simeq$$

Note: If we naively restricted functors, we still get a pre-deformation functor, but it's behavior will be worse.

Problem comes from auts of X not extending to X_A.

First hint that for moduli problems involving automorphisms, functors to Sets don't capture everything.

Ex. Deformations of a quasicoherent sheaf.

Fix X_Λ over $\operatorname{Spec} \Lambda$, set $X = X_\Lambda / k$. Fix $\mathcal{E}$ a q.coh. sheaf on X.

Define $\mathrm{Def}_{\mathcal{E}}$ by $A \mapsto \{ (\mathcal{E}_A, \varphi) : \mathcal{E}_A$ is a q.coh. sheaf on X_Λ / A, flat over A.

$\varphi : \mathcal{E}_A \to \mathcal{E}$ inducing $\mathcal{E}_A \otimes_A k \xrightarrow{\sim} \mathcal{E} \} / \simeq$.

Prorepresentability and hulls.

Def. Given $F : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$, let $\widehat{\mathrm{Art}}(\Lambda, k)$ be the cat. of complete local noetherian Λ-algebras, and $\hat{F} : \widehat{\mathrm{Art}}(\Lambda, k) \to \mathrm{Set}$ defined by

$\hat{F}(R) := \varprojlim_n F(R/m^n)$. We say F is prorepresentable if $\hat{F}$ is representable.

$\curvearrowright$: If we start w/ a global moduli problem, $\hat{F}$ is not necessarily obtained by considering families over R. This is the issue of effectivizability, cf. next week.

Def: Given $F, F' : Art(\Lambda, k) \longrightarrow Set$, then $f: F \to F'$ is (formally) <u>smooth</u> if

$\forall$ surjection $A \twoheadrightarrow B$ in $Art(\Lambda, k)$, the map $F(A) \twoheadrightarrow F(B) \times_{F'(B)} F'(A)$ is surjective.

Recall T_F, the tangent space of F, is $F(k[\varepsilon])$.

<u>Notation</u>. Given $R \in \hat{Art}(\Lambda, k)$, denote $h_R : \hat{Art}(\Lambda, k) \longrightarrow Set$ the functor of points of $\mathrm{Spec}\, R$, $h_R(R') = \mathrm{Mor}(R, R')$, and $\bar{h}_R$ is the restriction to $Art(\Lambda, k)$.

Def Let F be a predeformation functor. a pair (R, η), $\eta \in \hat{F}(R)$ is a <u>hull</u> for F,

if $\bar{h}_R \to F$ is smooth, and $T_{\bar{h}_R} \xrightarrow{\sim} T_F$ is an isom.

Prop. If (R, η) and (R', η') are hulls of F, then they are isomorphic.

Left as an exercise.

<u>Schlessinger's criterion</u>.

Def. A surjective map $f: A \to B$ in $Art(\Lambda, k)$ is a <u>small thickening</u> if $\ker f \simeq k$,

or equivalently, $m_A \cdot \ker f = 0$, and $\ker f$ is principal.

Rmk. It's easy to check that any surjection in $Art(\Lambda, k)$ can be written as a sequence of small thickenings.

Given $A' \to A$, $A'' \to A$

$$(*) \quad F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'')$$

Thm (Schlessinger) If F is a predeformation functor, consider:

(H1) $(*)$ is surjective when $A'' \twoheadrightarrow A$ (equiv., small thickening).

(H2) $(*)$ is bijective when $A'' = k[\varepsilon]$, $A = k$

(H3) T_F is finite dim'l

(H4) $(*)$ is bijective whenever $A' \overset{!}{=} A'' \twoheadrightarrow A$ (equiv., small thickening).

(H1)-(H3) $\Leftrightarrow$ F has a hull

(H1)-(H4) $\Leftrightarrow$ F is prorepresentable.

Lecture 3. Schlessinger. $A' \to A$, $A'' \to A$

$$(*) \quad F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

(H1) $(*)$ is surj. if $A'' \to A$ is a small thickening

(H2) $(*)$ is bijective if $A = k$, $A'' = k[\varepsilon]$

(H3) T_F is finite-dim'l

(H4) $(*)$ is bijective if $A' = A''$ and $A' \to A$ is a small thickening.

Rmk
- Fiber products may seem strange. We'll come back to this.
- (H1) & (H2) are essentially always satisfied.
- (H3) tends to be related to properness.
- (H4) is related to presence of automorphisms

Def A predeformation functor F is a deformation functor if it satisfies (H1) & (H2).

Note (H3) makes sense for any deformation functor.

$\boxed{\mathrm{Def}_X}$

Def: Given $(X_A, \varphi) \in \mathrm{Def}_X(A)$, an automorphism of (X_A, φ) (an infinitesimal aut. of X_A) is an aut. of X_A / A commuting w/ φ.

Thm. Let X be a scheme $/k$, and Def_X the functor of deformations of X. Then

(i) Def_X is a deformation functor.

(ii) Def_X satisfies (H3) if X is proper.

(iii) Def_X satisfies (H4) iff $\forall\, A' \to A$ a small thickening and $(X_{A'}, \varphi)$ over A', every automorphism of $(X_{A'}|_A, \varphi|_A)$ is the restriction of an aut. of $(X_{A'}, \varphi)$.

Proof

In particular, if $H^0(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)) = 0$, then (H4) is satisfied.

Cor If X is proper, then Def_X has a hull. and if further $H^0(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)) = 0$, then Def_X is prorepresentable.

Examples If X is a smooth proper curve. then Def_X has a hull, and Def_X is prorepresentable $\Leftarrow g \geq 2$.

Lemma Consider
$$\begin{tikzcd}[row sep=small, column sep=small]
N \arrow[rr, "p''"] \arrow[dr, "p'"] \arrow[dd] & & M'' \arrow[dr, "u''"'] \arrow[dd] & \\
& M' \arrow[rr, "u'"] \arrow[dd] & & M \arrow[dd] \\
B \arrow[rr] \arrow[dr] & & A'' \arrow[dr] & \\
& A' \arrow[rr] & & A
\end{tikzcd}$$
of compatible ring & module homo., and w/ $B = A' \times_A A''$, $N = M' \times_M M''$, and M' & M'' are flat over A' & A'', and:

(i) $A'' \twoheadrightarrow A$ w/ nilp kernel

(ii) u' induces an isom. $M' \otimes_{A'} A \xrightarrow{\sim} M$, and similarly for u''.

Then N is flat over B, and p' induces $N \otimes_B A' \xrightarrow{\sim} M'$, and similarly for p''.

Also, in the same situation, if we have L a B-module, and $q': L \to M'$ and $q'': L \to M''$ s.t. q' induces $L \otimes_B A' \xrightarrow{\sim} M'$, then $q' \times q'': L \xrightarrow{\sim} N$ is an isom.

Note: This is more general than is necessary for Schlessinger, since we don't restrict to Artin local rings (then all flat modules are free).

Prop. Given $A' \to A$, $A'' \to A$, where $A'' \to A$ is surjective w/ nilp. kernel, write $B = A' \times_A A''$. Then

(i) Given X' and X'' flat over A' and A'', and an isom. $\varphi: X'|_A \xrightarrow{\sim} X''|_A$, there exists Y flat over B, w/ maps
$$\begin{array}{ccc} X' & \xrightarrow{\varphi'} & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,A' & \to & \mathrm{Spec}\,B \end{array} \quad \text{and} \quad \begin{array}{ccc} X'' & \xrightarrow{\varphi''} & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,A'' & \to & \mathrm{Spec}\,B \end{array}$$
inducing isoms $X' \xrightarrow{\sim} Y|_{A'}$, $X'' \xrightarrow{\sim} Y|_{A''}$ and $\varphi = \varphi''|_A \circ \varphi'^{-1}|_A$.

(ii) Given Y_1, Y_2 flat over B, the nat'l map

$$\mathrm{Isom}_B(Y_1, Y_2) \xrightarrow{\sim} \mathrm{Isom}_{A'}(Y_1|_{A'}, Y_2|_{A'}) \underset{\mathrm{Isom}_A(Y_1|_A,\, Y_2|_A)}{\times} \mathrm{Isom}_{A''}(Y_2|_{A''}, Y_2|_{A''})$$

is a bijection.

<u>Proof</u> (i) We'll construct Y on the same topological space as X'.

We identify the spaces of X'' and $X''|_A$, and also $X'|_A$ using φ, and write $i : X'|_A \to X'$. Set

$$\mathcal{O}_Y(U) = \mathcal{O}_{X'}(U) \underset{\mathcal{O}_{X'|_A}(i^{-1}(U))}{\times} \mathcal{O}_{X''}(i^{-1}(U))$$

so

$$\mathcal{O}_Y = \mathcal{O}_{X'} \underset{i_* \mathcal{O}_{X'|_A}}{\times} i_* \mathcal{O}_{X''}.$$

The Lemma says that "$\mathcal{O}_Y$" is flat over B, and recovers $\mathcal{O}_{X'}$ & $\mathcal{O}_{X''}$ on restriction to A' & A''. Also we check that $\mathcal{O}_Y$ is in fact a sheaf, and defines a scheme str. which boils down to module fiber product commutes w/ localization.

(ii) is similar, using 2nd part of the Lemma.

<u>Proof of Thm</u>. (i) (H1) & (H2) satisfied.

(H1) follows from prop (i). (H2) uses (ii) of Prop.

uses $A = k$, so the φ in def'n of Def_X rigidify the isoms.

(iii) is similar. (ii) is true for smooth proper X from Martin's lecture.

See later lectures for general statements.

Lecture 4. The proof of Schlessinger's thm

$$(*) \quad F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'').$$

Thm Let F be a predeformation functor.

F has a hull $\iff$ (H1) – (H3) are satisfied.

F is prorep. $\iff$ (H1) – (H4) are satisfied.

Prop. Let F be a deformation functor. and $A' \to A$ a [small] thickening w/ kernel I,

For every $\eta \in F(A)$, when the set of $\eta' \in F(A')$ restricting to η is nonempty, it has a transitive action of $T_F \otimes_k I$. This action commutes w any morphism $F' \to F$ of deformation functors.

(H4) is satisfied $\iff$ $\forall A' \to A$ small thickenings and all $\eta \in F(A)$, this action is free.
(whenever the set is nonempty).

Def. A surjection $p: A' \twoheadrightarrow A$ in $\mathrm{Art}(\Lambda, k)$ is <u>essential</u> if $\forall q: A'' \to A'$ s.t. $p \circ q$ is surj., then q is surj.

Lemma. If p is a small thickening, p is <u>not</u> essential $\iff$ p has a section.

Ex. $k[\varepsilon] \rightarrow k$

$\mathbb{Z}/p^2 \to \mathbb{F}_p$ – essential.

Prop If (H1) – (H3) are satisfied, then F has a hull

Proof: 2 parts: construct the hull, then prove it is one.

We'll construct (R, ξ), $R \in \widehat{\mathrm{Art}}(\Lambda, k)$, $\xi \in \hat{F}(R)$, s.t. $\bar{h}_R \xrightarrow{\xi} F$ is smooth, and induces $T_R \xrightarrow{\sim} T_F$.

Let $\mathfrak{n}$ be the max'l ideal of Λ, $r = \dim T_F$ ($<\infty$ by (H3)).

Set $S = \Lambda[[t_1, \dots, t_r]]$, let $\mathfrak{m}$ be the max. ideal of S.

We'll construct R as S/J, where $J = \bigcap_{i \geq 2} J_i$. and the J_i are constructed inductively.

$J_2 = \mathfrak{m}^2 + \mathfrak{n} S$, $S/J_2 = k[T_S^*] \simeq k[T_F^*] \approx \underbrace{k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]}_{r \text{ times}}$

$R_2 = S/J_2$, and use (H2) to construct a $\xi_2 \in F(R_2)$ inducing a bijection $T_{R_2} \xrightarrow{\sim} T_F$.

Suppose we have $R_{i-1} = S/J_{i-1}$, and $\xi_{i-1} \in F(R_{i-1})$.

We'll choose J_i to be minimal among J satisfying

- $\mathfrak{m} J_{i-1} \subset J \subset J_{i-1}$.
- ξ_{i-1} can be lifted to an elt. of $F(R_i = S/J_i)$

First cond'n is preserved under arbitrary intersection. need to check that 2nd cond'n is too.

Note. J satisfying first cond. $\Leftrightarrow$ vector subspaces of $J_{i-1}/\mathfrak{m} J_{i-1}$, which is finite-dim'l.

This implies enough to check pairwise intersections.

Suppose J, K satisfy our conditions, claim $J \cap K$ does too.

Again using $J_{i-1}/\mathfrak{m} J_{i-1}$, we can replace K w/o changing $J \cap K$, so that $J + K = J_{i-1}$.

Then $S/J \times_{S/J_{i-1}} S/K \simeq S/(J \cap K)$. so by (H1), we have some elt of $F(S/J \cap K)$ restricting to ξ_{i-1}, which means $J \cap K$ satisfies our conditions.

So we can set J_i to be the minimal ideal satisfying our conditions & choose ξ_i lifting ξ_{i-1}.

Set $J = \bigcap_i J_i$, $R = S/J$.

If $R_i = S/J_i$, because $m^i \subset J_i$, we have $R = \varprojlim R/J_i$, and

$\xi = \varprojlim \xi_i$ makes sense. So (R, ξ) is our prospective hull.

$T_R \xrightarrow{\sim} T_F$ is immediate from choice of ξ_i. smoothness is harder.

Fix $p: A' \to A$ a small thickening, $\eta' \in F(A')$ s.t. $p(\eta') = \eta \in F(A)$.

and $u: R \to A$ s.t. $u(\xi) = \eta$. Want lift $u': R \to A'$ s.t. $u'(\xi) = \eta'$.

First construct any u' lifting u.

Since A is an Artin ring, u factors through $R \to R_i$, some i.

$$\begin{tikzcd}
 & R_{i+1} \arrow[rr, dashed] & & A' \arrow[d, "p"] \\
R \arrow[ur] \arrow[urrr, dashed, "u'"] \arrow[r] \arrow[rrr, bend right, "u"'] & R_i \arrow[rr] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{i+1} \arrow[r, dashed] \arrow[r, bend left, "w"] \arrow[dr] & R_i \times_A A' \arrow[d, "p_1"] \arrow[l, bend right, "s"'] \\
 & R_i
\end{tikzcd}$$

p_1 is a small thickening.

If we have a section, no problem.

If not, p_1 is essential, choose w as above, must be surjective.

Enough $\ker w \supset J_{i+1}$. This follows from (H1).

So we have some u', we want to have $u'(\xi) = \eta'$. But we have compatible transitive actions of $T_F \otimes I \simeq T_R \otimes I$ of $F(p)^{-1}(\eta)$ & $h_R(p)^{-1}(\eta)$.

$$h_R(p)^{-1}(\eta) = \{R \to A' \text{ s.t. } R \to A \text{ sends } \xi \text{ to } \eta\}.$$

So $\exists\ \tau \in T_F \otimes I$ sending $u'(\xi)$ to η'. Then we can modify u' by τ. and we'll have the desired u' lifting u, sending ξ to η'.

Lecture 5. $(*)\quad F(A'\times_A A'') \longrightarrow F(A')\times_{F(A)} F(A'')$

Proof of (rest of) Schlessinger's criterion. Already showed (H1)-(H3) $\Rightarrow$ have a hull.

Suppose F has a hull (R,ξ). (H3) follows from $T_R \approx T_F$, and R noetherian $\Rightarrow \dim T_R < \infty$.

Now suppose we have $p': A' \to A$, $p'': A'' \to A$ in $\mathrm{Art}(\Lambda, k)$, w/ p' surjection.

For (H1) want $(*)$ surjective.

Suppose have $\eta' \in F(A')$, $\eta'' \in F(A'')$, both restricting to $\eta \in F(A)$. Since $h_R \to F$ is smooth, (by exercise), it is surjective, so $\exists\ u': R \to A'$, s.t. $u'(\xi) = \eta'$. Also, using smoothness applied to p'', $\exists\ u'': R \to A''$ w/ $u''(\xi) = \eta''$. & $p''\circ u'' = p'\circ u'$.

Set $\zeta = u'\times_u u''(\xi) \in F(A'\times_A A'')$, this lifts (η', η'') and this proves (H1).

For (H2), we assume $A = k$, $A'' = k[\varepsilon]$, want $(*)$ injective.

Suppose $v \in F(A'\times_A A'')$ also restricts to η' and η'', want $v = \zeta$.

Keeping the same $u': R \to A'$, we apply smoothness to $A'\times_k k[\varepsilon] \to A'$ to obtain $q'': R \to k[\varepsilon]$ s.t. $u'\times q''(\xi) = v$. Because $T_R \approx T_F$, and had $u'\times u''(\xi) = \zeta$,

$u'', q'' \in T_R$. so since $u''(\xi) = \zeta|_{A''} = v|_{A''} = q''(\xi)$, so $u'' = q''$, so $\zeta = v$

This is (H2), so done. i.e. hull $\Rightarrow$ (H1)-(H3).

Now suppose (H1)-(H4) satisfied. Know have a hull (R,ξ), want that it pro rep. F.

i.e. $\forall A$, have $h_R(A) \xrightarrow{\sim} F(A)$. We prove this by induction on the length of A.

Let $p: A' \to A$ be a small thickening, w/ kernel I, and suppose $h_R(A) \xrightarrow{\sim} F(A)$,

want to conclude $h_R(A') \xrightarrow{\sim} F(A')$.

$\forall \eta \in F(A)$, have that $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are both pseudotorsors under (by (H4)).
$T_F \otimes I \simeq T_R \otimes I$, compatibly by functoriality. But have surjection, so they must be in bijection. Since this holds for all $\eta \in F(A)$, have bijection $h_R(A') \xrightarrow{\sim} F(A')$ so (R, ξ) prorep. F by induction.

If F is prorep., then $(*)$ is always bijective because $A' \times_A A''$ is a categorical fiber product in $\widehat{\mathrm{Art}}(\Lambda, k)$.

ξ

More examples

Deformations of a quotient sheaf. Let X_Λ be a scheme $/\Lambda$ w/ a q.coh. sheaf $\mathcal{E}_\Lambda$. Write $X, \mathcal{E}$ for restriction to k. Fix $\mathcal{E} \twoheadrightarrow \mathcal{F}$ a qcoh sheaf quotient.

$$\mathrm{Def}_{\mathcal{F},\mathcal{E}} \text{ sends } A \text{ to } \Big\{ \mathcal{E}_\Lambda|_A \twoheadrightarrow \underset{\substack{\uparrow \\ \text{flat over } A}}{\mathcal{F}_A},\ \text{restricting to } \mathcal{E} \twoheadrightarrow \mathcal{F} \text{ after } \otimes k \Big\}$$

Note: no auts to worry about, could even have notion of equality of quotients coming from equality of kernels.

Thm $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ is a deformation functor, and satisfies (H4).

If X_Λ is proper and $\mathcal{E}$ is coherent, then $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ satisfies (H3), so is prorep.

Note: For representability of global version (Quot scheme), need projective hypothesis. But we see that the local behavior is still scheme like under properness hypothesis.

This hints at algebraic spaces.

Sketch of proof of thm

Given $A' \to A$, $A'' \to A$ and $\mathcal{F}_{A'}$, $\mathcal{F}_{A''}$ both restricting to $\mathcal{F}_A$ on A.

Set $B = A' \times_A A''$, and set $\mathcal{F}_B = \mathcal{F}_{A'} \times_{\mathcal{F}_A} \mathcal{F}_{A''}$, get a [surjection]

$\mathcal{E}_B = \mathcal{E}_\Lambda|_B \to \mathcal{F}_B$.

$$\mathcal{E}_B \xrightarrow[\text{not nec. } \cong \text{, but OK.}]{} \mathcal{E}_{A'} \times_{\mathcal{E}_A} \mathcal{E}_{A''} \twoheadrightarrow \mathcal{F}_B$$

This gives (H1), but we actually constructed an inverse to (*), so get (H2), (H4) also.

The tangent space to $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ is $H^0(X, \mathcal{H}om(\mathcal{G}, \mathcal{F}))$, $\mathcal{G} = \ker(\mathcal{E} \twoheadrightarrow \mathcal{F})$. (exer.)

Under hypothesis, this is finite-dim'l, so (H3) is satisfied.

Cor. Given X_Λ/Λ, and $Z \subset X$, then $\mathrm{Def}_{Z,X}$ is a deformation functor, and satisfies (H4). If further X_Λ is proper $/\Lambda$, then (H3) is satisfied, so prorepresentable.

Proof. $\mathcal{E}_\Lambda = \mathcal{O}_{X_\Lambda}$

Example Given $X_\Lambda, Y_\Lambda/\Lambda$, $f: X \to Y$ over k.

Def_f sends A to $\{f_A : X_\Lambda|_A \to Y_\Lambda|_A \text{ over } A\}$ restricting to f on k

use graph immersion

Cor. If X_Λ & Y_Λ are loc. f. type $/\Lambda$, and X_Λ flat over Λ, Y_Λ separated, then Def_f satisfies (H1), (H2), (H4). If X_Λ & Y_Λ are proper, also (H3).

Lecture 6 Dimension of hulls

Mori used a lower bound on dimension of a space of morphisms (in terms of tangent and obstruction spaces) as a key technical tool to prove amazing thms about existence of rat'l curves on varieties.

<u>Background on obstruction theory</u>.

<u>Def</u>. $A' \xrightarrow{\pi} A$ in $\mathrm{Art}(\Lambda, k)$ is a <u>thickening</u> if it is surjective, w/

$\ker \pi \cdot m_{A'} = 0$. i.e. $\ker \pi$ has a k-vec. sp. str.

<u>Def</u>. Given a predeformation functor F, an <u>obstruction theory</u> for F is a vec. sp.

V/k, and $\forall\ A' \xrightarrow{\pi} A$ thickenings, and all $\eta \in F(A)$. an elt $ob(\eta, A') \in V \otimes_k \ker \pi$,

s.t. (i) $ob(\eta, A') = 0 \iff \exists\, \eta' \in F(A')$ s.t. $\eta'|_A = \eta$

(ii) If
$$\begin{tikzcd} A' \arrow[r] \arrow[dr] & A \\ & B \arrow[u] \end{tikzcd}$$
w/ $\ker(A' \to A) = I$, $\ker(A' \to B) = J$,

then $ob(\eta, B)$ is induced by $ob(\eta, A')$. $V \otimes I \to V \otimes I/J$.

<u>Thm</u>. Suppose F has a hull (R, ξ) and an obstruction theory taking values in V, then

$\dim \Lambda + \dim T_F - \dim V \le \dim R \le \dim \Lambda + \dim T_F$.

If Λ is regular, and the first inequality is an equality, then R is a complete intersection in $\Lambda[[t_1, \dots, t_r]]$.

<u>Lemma</u>. Suppose $F_1 \to F_2$ is a <u>smooth</u> morphism of predeformation functors, and we have an obstruction theory for F_2 taking values in V. Then we obtain an obstruction theory for F_1 taking values in V.

<u>Proof</u>. Given $A' \to A$, $\eta \in F_1(A)$, set $ob(\eta, A') = ob(f(\eta), A')$. By smoothness, this satisfies (i), and (ii) is a diagram chase.

<u>Proof of thm</u>. The Lemma reduces to the case $F = h_R$. since by def'n of a hull, $h_R \to F$ is smooth and induces an isom. $T_R \simeq T_F$.

Let $d = \dim T_R$, Schlessinger constructs R as S/J, where $S = \Lambda[[t_1, \dots, t_d]]$, so it's enough to prove that J can be gen. by $\leq \dim V$ elts.

By the Artin-Rees Lemma, we have $J \cap m_S^n \subset J \cdot m_S$ for some n. Set

$$A' = \Lambda[[t_1, \dots, t_d]] / (m_S J + m_S^n), \quad \text{and} \quad A = \Lambda[[t_1, \dots, t_d]] / (J + m_S^n),$$

this gives a thickening

$$0 \to \underset{\substack{\| \\ (J + m_S^n)/(m_S J + m_S^n) \; = \; J/m_S J}}{I} \to A' \to A \to 0.$$

From the quotient map $R = S/J \to A$, we have an obj. $\xi_A \in h_R(A)$ and an obstruction $\underset{\substack{\cap \\ V \otimes I}}{ob(\xi_A, A')}$ to lifting to a map $R \to A'$.

We can write $ob(\xi_A, A') = \sum_{j=1}^{\dim V} v_j \otimes \bar{x}_j$, where the v_j form a basis for V and $\bar{x}_j$ are images of some $x_j \in J$.

Want to show the x_j generate J. It's enough to see that the $\bar{x}_j$ gen. $I = J/m_S J$ by Nakayama. Consider $B = A'/(\bar{x}_j)$, this surj. onto A, w/ kernel I'.

We get $ob(\xi_A, B) \in V \otimes I'$. By functoriality, is zero, so we have a lift $R \to B$.

$$\begin{tikzcd}
S \arrow[r, "\ker J"] \arrow[d, dashed] & R \arrow[d] \arrow[dr] & \\
S \arrow[r] & B \arrow[r] & A
\end{tikzcd}$$

Want: $J \subset \underbrace{m_S J + (x_i) + m_S^n}_{= \ker(S \to B)}$

We can choose some $\varphi : S \to S$ making above commute by choosing $\varphi(t_i)$ appropriately. φ commutes w/ the two maps to A, is the identity modulo $J + m_S^n$. In particular, φ is the identity on m_S/m_S^2, so φ is an isom.

So $\varphi^{-1}(J) \subset J + m_S^n$. so $J \subset \varphi(J) + \underset{\overset{\|}{m_S^n}}{\varphi(m_S^n)}$

By commutativity of the square, $\varphi(J) \subset m_S J + (x_i) + m_S^n$, so

$$J \subset \varphi(J) + m_S^n \subset m_S J + (x_i) + m_S^n. \qquad \square$$

Example. Say X, Y smooth varieties, $f : X \to Y$, consider Def_f

Fact. tangent space is $H^0(X, f^*T_Y)$, and there is an obstruction theory in $H^1(X, f^*T_Y)$.

If X is a curve, then $H^0 - H^1$ of f^*T_Y is $\chi(f^*T_Y)$, and this is computed by Riemann-Roch.

Ex Deformations of a smooth surface X. Tangent space is $H^1(X, T_X)$, and there is an obstruction theory in $H^2(X, T_X)$. If we understand $H^0(X, T_X)$, then we can compute $H^1 - H^2$ of T_X by computing $\chi(T_X)$ via Riemann-Roch.

e.g. if X has finite automorphism gp in char. 0, $H^0(X, T_X) = 0$.

Lecture 7. Effectivity & Algebraization

Two remaining questions:

Q: (Effectivity) Suppose F is a deformation functor coming from a global problem, $R \in \widehat{\mathrm{Art}}(\Lambda, k)$ and $\eta \in \hat{F}(R)$, when does η come from a family over $\mathrm{Spec}\, R$ for the original problem?

Q: (Algebraization) In same situation, above answer is yes, so we have sth over $\mathrm{Spec}\, R$, when is this induced from an "algebraic object", e.g. from sth. over R' of f. type/base.

Effectivity

No general positive answer.

Main Tool for positive result is Grothendieck's Existence Theorem.

Thm. $f: X \to \operatorname{Spec} A$ proper, A a complete local noetherian ring.

Let $A_n = A/m_A^{n+1}$, and $X_n = X \otimes_A A_n$.

Given $\{\mathcal{F}_n\}$ a compatible system of coherent sheaves on X_n, $\exists$ $\mathcal{F}$ on X coherent w/

$\mathcal{F}|_{X_n} \simeq \mathcal{F}_n$, $\forall n$.

This gives a positive answer for effectivity, in the case of coherent sheaves on a proper scheme.

What about moduli of abstract schemes?

OK for curves, but fails for surfaces.

Specifically, fails for K3 surfaces ($K_X = 0$, $H^1(X, \mathcal{O}_X) = 0$).

In this case, if we look at Def_X, it looks like we have a 20 dim'l moduli space. only 19 of them are algebraic. In fact, have 20 dim'l space of analytic K3 surfaces, (algebraic locus is a countable union of 19-dim'l subspaces).

Patch: work w/ moduli of polarized varieties (i.e., w/ a choice of an ample line bundle).

It follows from GET (equiv. of cats version) that effectivity is satisfied for moduli of polarized (projective) varieties.

Algebraization

Artin consider (uni)versal families, proves a positive result quite generally, using earlier approximation theorems. This requires: base S finite type over a field or an excellent Dedekind domain (e.g. $\operatorname{Spec} \mathbb{Z}$)

Def: Let $F: Sch_S \to Set$ be a contravariant functor. We say F is <u>locally of finite presentation</u> over S if for all filtering projective systems of affine schemes $Z_\lambda \in Sch_S$, we have $\varinjlim F(Z_\lambda) = F(\varprojlim Z_\lambda)$

Why this? EGA: if $F = h_X$, some $X \in Sch_S$, then this is equiv. to $X \to S$ being locally of finite presentation.

Notation F is a deformation functor,

(R, ξ), $\xi \in \hat{F}(R)$ is <u>smooth</u> over R if the induced map $\bar{h}_R \to F$ is smooth.

$\uparrow$

$\widehat{Art}(\Lambda, k)$

<u>Thm</u> Suppose $F: Sch_S \to Set$ is locally of finite presentation, and $\eta_0 \in F(k)$, given some $\operatorname{Spec} k \to S$ of finite type, w/ image $s \in S$, Let R be a complete local noetherian $\mathcal{O}_{S,s}$-algebra, w/ residue field k, and suppose we have $\xi \in F(R)$, which induces η_0 over k, and w/ (R, ξ) smooth over the local deformation functor corresponding to η_0.
Then $\exists$ X of finite type $/S$, $x \in X$ closed, and $\eta \in F(X)$, w/ an isom. $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} R$.
s.t. η maps to $\xi_n \in F(R/m_R^{n+1})$, $\forall n$.

In general, this doesn't imply $\eta \mapsto \xi$ unless ξ is uniquely determined by the ξ_n.

<u>Thm</u> In situation "above", and if ξ is uniquely determined by the ξ_n, then (X, x, η) is unique up to étale morphisms.

$$\begin{tikzcd} & (X'', x'', \eta'') \arrow[dl, "\text{étale}"'] \arrow[dr, "\text{étale}"] & \\ (X, x, \eta) & & (X', x', \eta') \end{tikzcd}$$

Lecture 8 Groupoid perspective.

One nice property: when working w/ cats fibered, we can restrict naturally from global to local and get right result (e.g. we can specify pairs (X_A, φ): X_A flat over A, $\varphi: X \to X_A$ inducing $X \xrightarrow{\sim} X_A \otimes_A k$.)

Def. A cat. cofibered in groupoids over C is a cat. fibered in grpoids over C°.

Def. A grpoid is trivial if $\exists$ exactly one morphism from any object to any other.

"triv" trivial groupoid: any trivial grpoid is equiv. to $\circlearrowright\!\bullet$.

Rmk. Artin uses $(S1')$, Rim uses "homogeneous groupoids".

Def. A cat. cofibered in groupoids $\mathcal{S}$ over $Art(\Lambda, k)$ is a deformation stack if $\mathcal{S}_k$ is trivial, and $\forall$ $A' \to A$, $A'' \twoheadrightarrow A$, we have

(i) $\forall\ \eta_1, \eta_2 \in \mathcal{S}_{A' \times_A A''}$, the nat'l map

$$Mor_{A'\times_A A''}(\eta_1, \eta_2) \xrightarrow{\sim} Mor_{A'}(\eta_1|_{A'}, \eta_2|_{A'}) \underset{Mor_A(\eta_1|_A,\ \eta_2|_A)}{\times} Mor_{A''}(\eta_1|_{A''}, \eta_2|_{A''})$$

is a bijection

(ii) Given $\eta' \in \mathcal{S}_{A'}$, $\eta'' \in \mathcal{S}_{A''}$, and $\varphi: \eta'|_A \to \eta''|_A$, $\exists\ \varphi \in \mathcal{S}_{A'\times_A A''}$ inducing η', η'', φ on restriction.

Given $\mathcal{S}$, we write $F_{\mathcal{S}}: Art(\Lambda, k) \to Set$ for the functor of isom. classes.

Prop. Let $\mathcal{S}$ be a deformation stack, then $F_{\mathcal{S}}$ is a deformation functor.

Proof. $F_{\mathcal{S}}(k)$ is one pt set b/c $\mathcal{S}_k$ is trivial.

(H1) follows from (ii), (H2) follows from (i). in fact get injectivity of $(*)$ as long as $A = k$.

Rmk. Although being a deformation stack is formally stronger than satisfying (H1) & (H2), it seems in practice that any proof of (H1) & (H2) is really a proof of the deformation stack condition. E.g., Def_X. Earlier proposition actually proves the deformation stack conditions.

Lemma. If $\mathcal{S}$ is the local deformation problem at a point of an Artin stack, then $\mathcal{S}$ is a deformation stack.

Rmk The argument for Lemma involves the "asymmetry of only $A' \to A$ being surjective". b/c we have to use the formal criterion for smoothness applied to the smooth cover by a scheme.

[Lemma 1.4.4 of Olsson, Crystalline cohomology of stacks and Hyodo-Kato cohomology]

<u>More good properties of deformation stacks</u>

$A' \to A$, $\eta \in \mathcal{S}_A$, $\{\eta' \in \mathcal{S}_{A'} : \eta'|_A = \eta\}/\simeq$

ker I

action by $T \otimes I$

$\{(\eta', \varphi) : \eta' \in \mathcal{S}_{A'},\ \varphi : \eta'|_A \xrightarrow{\sim} \eta\}/\simeq$

is a pseudo-torsor over $T_{\mathcal{S}} \otimes I$

$$T_{\mathcal{S}} = T_{F_{\mathcal{S}}}.$$

- $A' \to A$, $\eta' \in \mathcal{S}_{A'}$, $\varphi \in \mathrm{Aut}(\eta'|_A)$, $\{\varphi' \in \mathrm{Aut}(\eta') : \varphi'|_A = \varphi\}$

is a torsor over $\mathrm{Aut}(\xi_\varepsilon) \otimes I$, ξ_ε is trivial def. over $k[\varepsilon]$.

Prop. If $\mathcal{S}$ is a deformation stack, then $F_{\mathcal{S}}$ satisfies (H4) iff for $A' \twoheadrightarrow A$ and all $\eta' \in \mathcal{S}_{A'}$, the map $\mathrm{Aut}(\eta') \to \mathrm{Aut}(\eta'|_A)$ is surjective.

In fancier language, in a global setting, (H4) $\Leftrightarrow$ the Isom functor is smooth at the identity.

Why deformation <u>stack</u>?

Why all these ring fiber products?

<u>Lemma</u>:

$$\begin{tikzcd} A' \times_A A'' \arrow[r] \arrow[d] & A'' \arrow[d] \\ A' \arrow[r] & A \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} B \arrow[r, "q'"] \arrow[d, "q''"'] & B' \arrow[d] \\ B'' \arrow[r] & B' \otimes_B B'' \end{tikzcd}$$

& $B \hookrightarrow B' \times B''$

& $q'(\ker q'')$ is an ideal

$$B \hookrightarrow B' \times B'' \Leftrightarrow \begin{array}{c} \mathrm{Spec}\, B' \sqcup \mathrm{Spec}\, B'' \\ \downarrow \text{ scheme-theoretically surj.} \\ \mathrm{Spec}\, B \end{array}$$

- $\otimes$ corresponds to fiber product of schemes, i.e. "intersections" from the point of view of descent theory.