Geometric Langlands for projective curves

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Lecture 1. Setup X/k smooth projective curve (conn'd)

Fix a "sheaf theory"

$\mathrm{Lisse}(Y) \subset \mathrm{Shv}(Y)$, $\forall Y/k$

Examples 1) $\mathrm{char}\, k = 0$

$\mathrm{Shv}(Y) = D\text{-mod}(Y)$

$\mathrm{Lisse}(Y) = \{\text{v.b. w/ } \nabla\}$ ← really : Y sm.

$\{F : H^i(F) \text{ is a colim of v.b. w/ } \nabla\}$

2) $k = \mathbb{C}$ $\mathrm{Shv}(Y)$ = Sheaves on $Y(\mathbb{C})^{an}$

$\mathrm{Lisse}(Y) = \{\text{cohomologies are locally constant}\}$

3) ℓ prime $\neq \mathrm{char}(k)$, $\mathrm{Shv} = \overline{\mathbb{Q}}_\ell$ - étale sheaves

Lisse = as above

Lisse $\xleftrightarrow{\text{syn.}}$ local system

Geometric class field theory.

Construction: σ rank 1 loc. sys on X

$\rightsquigarrow \chi_\sigma \in \mathrm{Shv}(\mathrm{Bun}_{\mathbb{G}_m})$ w/ some properties

χ_σ ← prototypical "eigensheaf".

$\mathrm{Bun}_{\mathbb{G}_m}$ = moduli stack of line bundles on X $\simeq \mathrm{Jac}\,X \times B\mathbb{G}_m \times \mathbb{Z}$

$\mathrm{Bun}^n_{\mathbb{G}_m} = \{\mathcal{L} : \deg \mathcal{L} = n\}$

$\mathrm{Bun}_{\mathbb{G}_m} = \coprod_{n \in \mathbb{Z}} \mathrm{Bun}^n_{\mathbb{G}_m}$

What do we want?

- $\chi_\sigma|_{\text{triv}} \simeq e$ ← notation for field of coeffs of my sheaf theory.

  $\mathcal{O}_X \in \mathrm{Bun}_{\mathbb{G}_m}$ — fixed isom. "normalization"

- $\chi_\sigma|_X$ along $X \xrightarrow{AJ} \mathrm{Bun}_{\mathbb{G}_m}$, $x \longmapsto \mathcal{O}(x)$

  I obtain σ.

Example Betti setting $k = \mathbb{C}$

$$\mathrm{Jac}\, X \overset{\text{as cpx mfds}}{\simeq} H^1(X, \mathcal{O}) / H^1(X, \mathbb{Z}(1))$$

Choose $x_0 \in X$

$$AJ_{x_0} : X \longrightarrow \mathrm{Jac}\, X$$
$$x \longmapsto \mathcal{O}(x - x_0)$$

$$\rightsquigarrow \pi_1(X) \longrightarrow \pi_1(\mathrm{Jac}(X)) \simeq H^1(X; \mathbb{Z}(1))$$

factors as
$$\begin{array}{ccc} \pi_1(X)^{ab} & \longrightarrow & \pi_1(\mathrm{Jac}(X)) \simeq H^1(X; \mathbb{Z}(1)) \\ \| & & \\ H_1(X, \mathbb{Z}) & & \end{array}$$

Exer. $H_1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathbb{Z}(1))$ is the PD isomorphism.

$$\begin{array}{ccc} \sigma : \pi_1(X)^{ab} & \longrightarrow & e^\times \\ \wr\! \| & \nearrow & \\ \pi_1(\mathrm{Jac}) & & \end{array}$$

← defines χ_σ.

- $x \in X$

$$H_x : \mathrm{Bun}_{\mathbb{G}_m} \xrightarrow{\sim} \mathrm{Bun}_{\mathbb{G}_m}$$

$$\mathcal{L} \mapsto \mathcal{L}(x)$$

$$H_X : X \times \mathrm{Bun}_{\mathbb{G}_m} \longrightarrow \mathrm{Bun}_{\mathbb{G}_m}$$

$$(x, \mathcal{L}) \longmapsto \mathcal{L}(x)$$

$$H_X^*(\chi_\sigma) \simeq \sigma \boxtimes \chi_\sigma \in \mathrm{Shv}(X \times \mathrm{Bun}_{\mathbb{G}_m}) \quad + \text{compats.}$$

"Hecke property"

<u>Idea</u> $\exists \leq 1$ such χ_σ for trivial reasons

Hecke property says: $\chi_\sigma |_{\mathcal{L}(x)} \simeq \sigma_x \otimes \chi_\sigma|_{\mathcal{L}}$.

in particular, $\mathcal{L} \simeq \mathcal{O}(D)$, $D = \sum n_i x_i$ divisor

$$\chi_\sigma|_{\mathcal{L}} \simeq \bigotimes_i \sigma_{x_i}^{\otimes n_i}$$

"χ_σ is overdetermined".

<u>Souped up version</u>:

$$\mathrm{Sym}^n X = \{\text{eff. divisors on } X \text{ of deg } n\} = \{\underset{\deg \mathcal{L} = n}{\mathcal{L}} + s \in \Gamma(\mathcal{L}) \setminus \{0\}\}$$

$$\Big\downarrow p_n$$

$$\mathrm{Bun}^n_{\mathbb{G}_m}$$

fiber at $\mathcal{L}$ is $\Gamma(\mathcal{L}) \setminus 0$.

Define: $\sigma^{(n)} \in \mathrm{Shv}(\mathrm{Sym}^n X)$ as:

$$X^n \xrightarrow{\mathrm{add}_n} \mathrm{Sym}^n X$$

$\mathrm{add}_{n*}(\sigma \boxtimes \cdots \boxtimes \sigma) \circlearrowleft \mathfrak{S}_n$ and I take $\mathfrak{S}_n$-invts

$\sigma^{(n)}$ has fiber $\otimes \sigma_{x_i}^{\otimes n_i}$ at $D = \sum n_i x_i$

By desiderata: $p_n^*(\chi_\sigma) \simeq \sigma^{(n)}$

compatible w/ Hecke property in a nat'l sense.

Claim: $\exists \leq 1$ such χ_σ

Idea: χ_σ is determined by $\chi_\sigma|_{\mathrm{Bun}^{2n}_{\mathbb{G}_m}}$ for any n w/ Hecke property.

$n \gg 0$ $(n > 2g-2)$

$\sigma^{(n)}$ descends to $\mathrm{Bun}^n_{\mathbb{G}_m}$

Riemann-Roch $\Rightarrow \mathrm{Sym}^n X \longrightarrow \mathrm{Bun}^n_{\mathbb{G}_m}$ is smooth and surj. w/ conn'd (simply) fibers

$\Rightarrow p_n^*(\chi_\sigma)$ determines $\chi_\sigma|_{\mathrm{Bun}^n_{\mathbb{G}_m}}$ uniquely $\Rightarrow \leq 1$ $(\Rightarrow = 1)$

Baby GL for $G = \mathbb{G}_m$.

Goal: Discuss enhancements of this story.

For a while, want to discuss $G = PGL_2$.

Main players.

A GL_2-bundle on $X \Leftrightarrow \mathcal{E}$ v.b. of rk 2

A PGL_2-bundle on $X \Leftrightarrow$ " a v.b. of rk 2 well-defined up to $-\otimes \mathcal{L}$".

Bun_{PGL_2} = moduli stack of PGL_2-bundles on X.

This is our major player.

$\mathcal{E}$ v.b. rk 2 $\deg \mathcal{E} \in \mathbb{Z}$

$\deg(\mathcal{E} \otimes \mathcal{L}) = \deg \mathcal{E} + 2 \deg \mathcal{L} \Rightarrow \deg P \bmod 2$ well-defined for $P \in \mathrm{Bun}_{PGL_2}$

$\mathrm{Bun}_{PGL_2} = \mathrm{Bun}_{PGL_2}^{even} \amalg \mathrm{Bun}_{PGL_2}^{odd}$

<u>Idea</u>: motivated by GCFT & Langlands

given an SL_2-local system σ on X, irr., hope $\exists\, \mathcal{F}_\sigma \in \mathrm{Shv}(\mathrm{Bun}_{PGL_2})$ canonically.

Will discuss in detail.

$\wr$ $\mathcal{F}_\sigma$ is NOT a local system, but <u>is</u> an irred. (on each conn'd component) perverse shf.

B = Borel $= \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset PGL_2$

T = Cartan $= \mathbb{G}_m$

$\mathrm{Bun}_B = \{ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0 \}$

$\mathrm{Bun}_B^n = \{$ ext'ns w/ $\deg \mathcal{L} = n \}$

$$\begin{tikzcd} & \mathrm{Bun}_B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_{PGL_2} = G & & \mathrm{Bun}_{\mathbb{G}_m} = T \end{tikzcd}$$

<u>Def</u>. $CT_* : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_T)$ is $q_* p^!$ is the constant term functor.

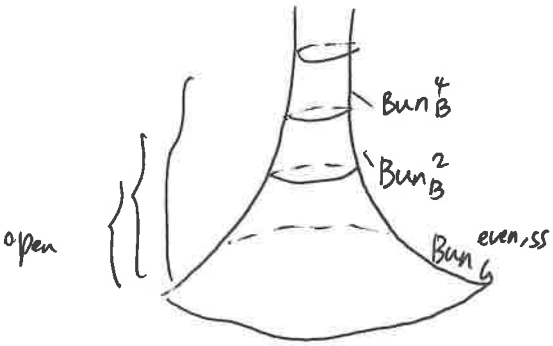
<u>Def</u>. $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ is <u>cuspidal</u> if $CT_*(\mathcal{F}) = 0$

<u>Idea</u>: CT_* is an analogue of Jacquet functor.

<u>Philosophy</u>: cuspidal $\longleftrightarrow$ irreducibility on spectral side

<u>So</u>: F_σ should be cuspidal.

HN picture of $\mathrm{Bun}_{PGL_2}^{(\text{even})}$



<u>Claim</u>: $n > 1$,

$\mathrm{Bun}_B^n \longrightarrow \mathrm{Bun}_{PGL_2}$ is a locally closed embedding.

w/ disjt images

$$\mathrm{Bun}_B^n = \{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0\}, \quad \deg \mathcal{L} = n$$

$\dim \mathrm{Bun}_B^n = C - n$

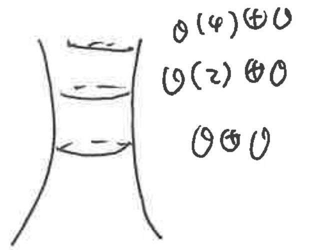
<u>Exer</u>. Check the injectivity of this map.

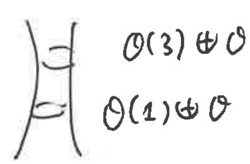
Bun_G^{ss} ← semi-stable

bundles not in the image of these maps

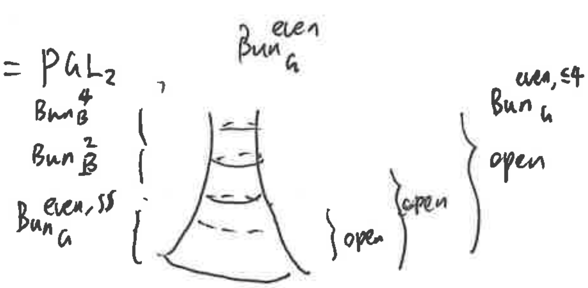
<u>Eg</u>. $\mathcal{O} \oplus \mathcal{O}$ = trivial bundle is semi-stable.

$\mathbb{P}^1$: $\mathcal{O}(n) \oplus \mathcal{O}$





<u>Lecture 2</u>. $G = PGL_2$,



σ irred. SL_2 loc. sys on X

<u>Want</u> $F_\sigma \in \mathrm{Shv}(\mathrm{Bun}_G)$

cuspidal $:= CT_*(F_\sigma) = 0$

$CT_* = \bigoplus CT_*^n, \quad n \in \mathbb{Z}$

$n \gg 0 \ (n > 2g-2)$

$$\begin{tikzcd} & Bun_B^n \arrow[dl, "p_n"'] \arrow[dr, "q_n"] & \\ Bun_G & & Bun_{G_m}^n \end{tikzcd}$$

$Bun_B^n \longrightarrow Bun_{G_m}^n = \{\deg \mathcal{L} = n\}$

$\parallel$

$\{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0\}$

this ext'n automatically splits $(H^1(\mathcal{L}) = 0)$

$CT_*^n = q_{n*}\, p_n^!$

fiber of q_n at $\mathcal{L}$: $\mathbb{B} H^0(\mathcal{L})$

Cor. $q_{n*} : Shv(Bun_B^n) \longrightarrow Shv(Bun_{G_m}^n)$ is an equiv.

Remind: $Shv(B G_a) = Shv(pt)$

Cor. F cuspidal $\Rightarrow$ $CT_*^n F = 0$, $\quad p_n^! F = 0$, $n \gg 0$.

$\Rightarrow$ F vanishes in $!$-sense at ∞.

$CT_! = \bigoplus CT_!^n$

$$\begin{tikzcd} & Bun_B^n \arrow[dl, "p_n"'] \arrow[dr, "q_n"] & \\ Bun_G & & Bun_{G_m}^n \end{tikzcd}$$

$CT_!^n = q_{n!} \cdot p_n^*$

Thm (Drinfeld-Gaitsgory) : $CT_*^n \simeq CT_!^{-n}$ $\quad$ (Braden's thm)

$CT_*^{-n}(F) = 0 \Rightarrow CT_!^n(F) = 0$

Cor. $n \gg 0$, F cuspidal $\Rightarrow p_n^* F = 0 \Rightarrow F$ vanishes in $*$-sense also

$\Rightarrow F$ is cleanly (both $*$ & $!$) extended from $Bun_G^{\leq 2g-2}$

How to construct $\mathcal{F}_\sigma$?

Digression. Inspiration from no. theory.

$\sigma : \underset{\text{Langlands - Weil - Deligne gp}}{\underline{W_{\mathbb{Q}}}} \longrightarrow SL_2(\mathbb{C})$ unramified at finite places, irred. (odd)

$\simeq Gal(\bar{\mathbb{Q}} | \mathbb{Q})$

Expectation: $\exists \quad f_\sigma(q) = \sum_{n \geq 0} a_n q^n$ modular form attached to σ.

$q = e^{2\pi i \tau}, \quad Im(\tau) > 0$

$f(\tau) = \underset{\text{factor}}{(\quad)} f\left(\frac{-1}{\tau}\right)$

How to construct f_σ :
- $a_0 = 0$ (cuspidality)
- $a_1 = 1$ (normalization)
- $a_{nm} = a_n a_m, \ (n, m) = 1$

p prime $\rightsquigarrow Fr_p \in W_{\mathbb{Q}}$

- $a_p = tr(\sigma(Fr_p))$ Frobenius
- $a_{p^{n+1}} = a_p a_{p^n} - (p^{\text{arch. factor}}) \cdot a_{p^{n-1}}, \ \forall n \geq 1$

Langlands says (here): f_σ is modular.

Goal: Write $\mathcal{F}_\sigma$ in similar terms.

Analogy.

$$\begin{tikzcd} & \{\text{modular forms}\} \arrow[dl, "a_0"'] \arrow[dr, "a_n"] & & \longleftrightarrow & & Shv(Bun_G) \arrow[dl, "CT_*"'] \arrow[dr, "coeff_D"] & \\ \mathbb{C} & & \mathbb{C} & & Shv(Bun_{G_m}) & & Vect \end{tikzcd}$$

$$\{n \geq 1\} \simeq \{p_1^{\nu_1} \cdots p_k^{\nu_k}\} = \mathrm{Div}^{\mathrm{eff}}(\mathrm{Spec}\,\mathbb{Z})$$

Our Fourier-Whittaker coeffs are indexed by effective divisors D on X.

<u>Rmk.</u> In general, $\check{\Lambda}^+$ - valued divisors.

- $\mathbb{G}_m$: $\chi_\sigma|_{\mathcal{O}(D)}$ any $\underbrace{D\text{-divisor}}_{\mathbb{Z}\text{-valued divisor}}$
- PGL_2 $\mathrm{coeff}_D(F_\sigma)$ important that it's $\underbrace{\mathbb{Z}^{\geq 0}\text{-div.}}_{\text{effective}}$
- G : $D = \sum \check{\lambda}_x [x]$, $\check{\lambda}_x \in \check{\Lambda}^+$.

$D = 0$, $\quad \mathrm{Coeff} = \mathrm{coeff}_0$

$$\begin{tikzcd} & \mathrm{Bun}^{\Omega}_{N=\mathbb{G}_a} = \{0 \to \Omega \to \mathcal{E} \to \mathcal{O} \to 0\} \arrow[dl, "p"'] \arrow[dr, "\psi"] & \\ \mathrm{Bun}_G & & \mathbb{A}^1 = H^1(X, \Omega) \end{tikzcd}$$

$$\{0 \to \Omega \to \mathcal{E} \to \mathcal{O} \to 0\} = R\Gamma(X, \Omega)[1]$$

$$\mathrm{Coeff}(F) = C^\cdot(\mathrm{Bun}^{\Omega}_{\mathbb{G}_a}, p^! F \overset{!}{\otimes} \psi^! \exp)$$

where $\exp \in \mathrm{Shv}(\mathbb{A}^1)$ is exp. D-mod

or AS sheaf

or "Kir" $\approx j_!$ const.

Analogous to: $f \mapsto \int_{\mathbb{R}/\mathbb{Z}} f(\tau) e^{-2\pi i \tau} d\tau$

More generally, for $D \geq 0$, $\mathrm{Bun}_N^{\Omega(-D)} = \{0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O} \to 0\}$

$$\begin{tikzcd} & \mathrm{Bun}_N^{\Omega(-D)} \arrow[dl, "p_D"'] \arrow[r, "\mathrm{Baer}"] \arrow[drr, bend right, "\psi_D"'] & \mathrm{Bun}_N^{\Omega} \arrow[dr] & \\ \mathrm{Bun}_G & & & \mathbb{A}^1 \end{tikzcd}$$

$$\Omega(-D) \to \Omega$$

$$\mathrm{Coeff}_D(F) = C^\cdot(p_D^! F \overset{!}{\otimes} \psi_D^! \exp)$$

Variant: $d \geq 0$,

$$\mathrm{coeff}_d : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Sym}^d X)$$

universal version of the above $\quad \mathrm{coeff}_d(\mathcal{F})|^!_D = \mathrm{coeff}_D(\mathcal{F})$

Expectation for $\mathcal{F}_\sigma$:

$$\mathrm{coeff}(\mathcal{F}_\sigma) \simeq e - \text{coeff. field}$$

more generally, $\quad \mathrm{coeff}_d(\mathcal{F}_\sigma) \simeq \sigma^{(d)}$

$\sigma^{(d)} \in \mathrm{Shv}(\mathrm{Sym}^d X)$ like last time

$\mathrm{add}_* (\underbrace{\sigma \otimes \cdots \otimes \sigma}_{d})^{S_d}$ $\quad$ fiber at $D = \sum n_i x_i$ is $\otimes \mathrm{Sym}^{n_i}(\sigma_{x_i})$

Rmks $\quad D = \sum_{\substack{\text{distinct}\\ \text{pts}}} x_i \quad , \quad \mathrm{coeff}_D(\mathcal{F}_\sigma) = \otimes \sigma_{x_i}$

$\mathrm{coeff}_x(\mathcal{F}_\sigma) = \sigma_x \quad \longleftarrow a_p = \mathrm{tr}(\sigma(\mathrm{Fr}_p))$.

2) This is the right analogue of the recursion from before

$$\left(\dim V = 2. \quad \underset{a_{p^n} \cdot a_p}{\mathrm{Sym}^n(V) \otimes V} \simeq \underset{a_{p^{n+1}}}{\mathrm{Sym}^{n+1}(V)} \oplus \underset{a_{p^{n-1}}}{\mathrm{Sym}^{n-1}(V)} \right)$$

3) $\sigma^{(d)}$ is a baby version of what's called a Laumon sheaf.

Q: Do the coeffs of $\mathcal{F}$ determine $\mathcal{F}$?

Example. $\mathcal{F} = e_{\mathrm{Bun}_G}$ = const sheaf, $\mathrm{coeff}_d(\mathcal{F}) = 0$, $\forall d$.

What about cuspidal sheaves?

i) Sort of yes: see: $\mathrm{coeff}_d(\mathcal{F})$ (and $\mathrm{coeff}_{d+1}(\mathcal{F})$) for some $d \gg 0$

uniquely determines cuspidal $\mathcal{F}$

ii) Sort of no: $\bigoplus_{d \gg 0} \mathrm{Coeff}_d : \mathrm{Shv}(\mathrm{Bun}_G)_{cusp} \longrightarrow \prod \mathrm{Shv}(\mathrm{Sym}^d X)$

is not fully-faithful for trivial reasons.

Rmk.

There's a fix for ii) for general reductive gps. due to Gaitsgory:

view Whittaker coeffs as $\mathrm{coeff}^{\mathrm{true}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Rep}\, \check{G}_{\mathrm{Ran}}$.

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ • F is f.f.

• F is conservative

Lecture 3.

Recap: σ irred. SL_2 locsys on X

Want: $\mathcal{F}_\sigma$ on Bun_G, $G = PGL_2$

cuspidal, irred. perverse sheaf

on each conn'd cpt

$\mathrm{Coeff}_d : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Sym}^d X)$

Analogous to pullback along Abel-Jacobi and to q-expansion.

Motivated by no. theory:

said: $\mathrm{coeff}_d(\mathcal{F}_\sigma) = \sigma^{(d)} \in \mathrm{Shv}(\mathrm{Sym}^d X)$

Now: Explain why this pins down F_σ.

Another description of coeff d (special to GL_n)

Express coeff d by FT.

Fourier–Deligne transform: V vs. $V^\vee$

$$\mathrm{Shv}(V) \simeq \mathrm{Shv}(V^\vee)$$

use $\mathrm{pair}^!(\exp)$ as a kernel

$$\begin{array}{c} V \times V^\vee \\ \downarrow \mathrm{pair} \\ \mathbb{A}^1 \end{array}$$

Ditto for v.b. over some base

$d \geq 0$ (maybe $d \gg 0$)

$$\begin{tikzcd} & S = \mathrm{Bun}^d_{\mathbb{G}_m} & \\ E = \left\{ \begin{array}{l} \mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m} \\ s \in R\Gamma(\mathcal{L}) \end{array} \right\} \arrow[ur] & & E^\vee = \left\{ \begin{array}{l} \mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m} \\ s \in R\Gamma(\mathcal{L})^\vee \end{array} \right\} \arrow[ul] \end{tikzcd}$$

$$= \left\{ \begin{array}{l} \mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m} \\ s \in R\Gamma(\mathcal{L}^\vee \otimes \Omega)[1] \end{array} \right\}$$

Dual bundle

$$E^\vee = \left\{ \begin{array}{l} \mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m} + \\ 0 \to \Omega \to \mathcal{E} \to \mathcal{L} \to 0 \end{array} \right\} \simeq \mathrm{Bun}_B^{-d+2g-2} = \{0 \to \mathcal{L}_0 \to \mathcal{E}_0 \to \mathcal{O} \to 0\}$$

$$\mathrm{Bun}^d_{\mathbb{G}_m} \overset{0}{\subset} E = \{ \mathcal{L}^{\deg d} + s \in \Gamma(\mathcal{L}) \}$$

$$\cup\ j \text{ open}$$

$$\mathrm{Sym}^d X \simeq \{ \mathcal{L} + \text{non-zero } s \in \Gamma(\mathcal{L}) \}$$

$$\begin{tikzcd}
 & & & \mathrm{Bun}^d_{\mathbb{G}_m} \arrow[d, "i"] & \\
 & \mathrm{Bun}_B^{-d+2g-2} = E^\vee \arrow[dl, "p_{-d+2g-2}"'] \arrow[dr, "q_{-d+2g-2}"] & & E \arrow[dl] & \mathrm{Sym}^d X \arrow[l, hook', "j"'] \\
\mathrm{Bun}_G & & \mathrm{Bun}^d_{\mathbb{G}_m} & &
\end{tikzcd}$$

$\nwarrow$ Smooth pullback for $d \gg 0$

$\uparrow$ complement to zero section

<u>Claim</u>: $\mathrm{Coeff}_d(\mathcal{F})$ is calculated via:

a) pullback $\mathcal{F}$ along $p_{\dots}$

b) Fourier transform

c) restrict along j

Geometry of Bun_G:

Said: $n > 1$ $\mathrm{Bun}_B^n \to \mathrm{Bun}_G$ is locally closed (HN-strata)

<u>Now</u>: $n << 0$, $\mathrm{Bun}_B^n \to \mathrm{Bun}_G$ is smooth (tangent space calculation + RR)

<u>Claim</u>, Can recover $\mathcal{F}_\sigma$ (up to even v.s. odd)

$\uparrow$ putative

from $\mathrm{Coeff}_d(\mathcal{F}_\sigma)$ $d \gg 0$.

<u>Idea</u> 1) Can recover $\mathcal{F}_\sigma$ from $p^!_{-d+2g-2}(\mathcal{F}_\sigma)$ $\leftarrow$ gen'l fact about irred. perv. sheaves & smooth maps w/ conn'd fibers

2) FT is an equiv., so I can recover $\mathcal{F}_\sigma$ from $FT(p^!_{-d+2g-2}\mathcal{F}_\sigma)$.

3) This FT is cleanly extended from open $\mathrm{Sym}^d X \Rightarrow$ I can recover $\mathcal{F}_\sigma$ from $j^* FT(p^!_{\dots}\mathcal{F}_\sigma)$.

Clean: Because $i^! FT$ = $*$-pushforward of $p^!_{-d+2g-2}(\mathcal{F}_\sigma)$

along q = $CT_{-d+2g-2}(\mathcal{F}_\sigma) = 0$

Thm (Drinfeld, Laumon, Gaitsgory): $\mathcal{F}_\sigma$ exists. (PGL_2/GL_2)

Thm (Frenkel – Gaitsgory – Vilonen) Extension of this story to PGL_n/GL_n

Generalized recently in work w/ Arinkin – Beraldo – Campbell – Chen – Faergeman – Gaitsgory – Lin – Rozenblyum

G (split) reductive, $\check{G}$ = dual gp char $k = 0$

σ irred. loc. sys for $\check{G}$ (σ does not factor any proper parabolic)

Thm: $\mathcal{F}_\sigma$ exists.

1) $\exists!\ \mathcal{F}_\sigma \in \mathrm{Shv}(\mathrm{Bun}_G)^{\heartsuit}_{cusp}$ that is a Hecke eigensheaf for σ equipped w/ an isom. $\mathrm{coeff}_0(\mathcal{F}_\sigma) \simeq e^{[\cdots]}$ ← sheaf coeff. field.

2) $\mathcal{F}_\sigma \in \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$, i.e. $SS(\mathcal{F}_\sigma) \subset Nilp$
singular support

3) $CC(\mathcal{F}_\sigma) = [Nilp]$ subject to stupid hypotheses.
characteristic cycle

3) $\Rightarrow$ generic rank of $\mathcal{F}_\sigma$ is $\prod_{\substack{d_i \\ \text{exp. of } G}} d_i^{(g-1)(2d_i-1)}$, g = genus

PGL_2 $Nilp^{reg, n}$ $n \geq 0$ mult. $= 2^n$.

0

2^{3g-3} 2^{3g-3}

4) $\mathcal{F}_\sigma$ is semisimple. $\mathcal{F}_\sigma \simeq \bigoplus_{V \text{ irrep of } S_\sigma} \mathcal{F}_{\sigma,V}^{\dim V}$, $S_\sigma = \mathrm{Aut}(\sigma) \supseteq Z_{\check{G}}$ finite index, $\mathcal{F}_{\sigma,V}$ irred, perverse, distinct for $V \neq W$.

Hardest part (biggest difficulties):

NO FT for general reductive gps.