Wild harmonic bundles and related topics

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Talk 1. X cplx mfld,

$(E, \bar\partial_E)$ holomorphic vector bundle

$\theta \in \underset{(\text{hol})}{\mathrm{End}(E) \otimes \Omega^1_X}$, $\theta \wedge \theta = 0$

h: metric of E

$\Rightarrow \nabla_h = \bar\partial_E + \partial_{E,h}$

$\theta^\dagger_h$ = adjoint of θ w.r.t. h

Def. h is pluri-harmonic if $\mathbb{D}^1 = \nabla_h + \theta + \theta^\dagger_h$ integrable

$\mathbb{D}^1 \circ \mathbb{D}^1 = 0$

$\Leftrightarrow \partial_E \theta = \bar\partial_E \theta^\dagger = 0$

$[\partial_E, \bar\partial_E] + [\theta, \theta^\dagger_h] = 0$

$(E, \bar\partial_E, \theta, h)$ harmonic bundle

Example $E = \mathcal{O}_X$, $\theta = df$ $(f \in \mathcal{O}_X)$

$h(1,1) = 1$

$\Rightarrow (E, \theta, h)$ harmonic bundle.

Lem

harmonic bundle of rank 1

$\Leftrightarrow$ E: holomorphic line bundle

θ: closed holomorphic 1-form

h. $\nabla_h \circ \nabla_h = 0$.

condition separated

Example. polarized cplx variation of Hodge str.

$V = \bigoplus_{p+q=w} V^{p,q}$ vector bundle.

tame: $\nabla = \theta^\dagger + \nabla^u + \theta$, Σ_θ: 0-sect.

nilpotent

$(V, \bar\partial, \theta, h)$ harmonic bundle

∇: integrable connection of V s.t. $\nabla : V^{p,q} \to V^{p+1,q-1} \otimes \Omega^{0,1}$

$\oplus V^{p,q} \otimes (\Omega^{0,1} \oplus \Omega^{1,0})$

$\oplus V^{p-1,q+1} \otimes \Omega^{1,0}$

(Griffith transversality)

$\langle\cdot,\cdot\rangle : V \times \bar{V} \to \mathbb{C}$ flat, $(-1)^w$- hermitian

$\bigoplus V^{p,q}$: orthogonal decomposition

$(\sqrt{-1})^{p-q}\langle\cdot,\cdot\rangle$ positive on $V^{p,q}$

$\left(\bigoplus (\sqrt{-1})^{p-q}\langle\cdot,\cdot\rangle\big|_{V^{p,q}}\right.$: pluriharmonic metric $\Big)$

$H \subset X$: normal crossing hypersurface

$(\underbrace{E, \bar{\partial}_E}, \theta, h)$ harmonic bundle on $X - H$.

$\Downarrow$

coherent sheaf on $T^*(X-H)$ $\longleftarrow$ E : $\mathrm{Sym}\,\Theta_{X \setminus H}$- module

Σ_θ : the support of

(spectral variety of $(E, \bar{\partial}_E, \theta, h)$)

(local)

$(E, \bar{\partial}_E, \theta, h)$ wild : $\overline{\Sigma_\theta} \subset T^*X(\log H) \otimes \mathcal{O}(NH)$

proper over X

$(E, \bar{\partial}_E, \theta, h)$ tame : $\overline{\Sigma_\theta} \subset T^*X(\log H)$ proper over X.

example. $\underline{E} = \mathcal{O}_X$, $\theta = df$, $\left(f \in \mathcal{O}_X(*H)\right)$.

sheaf of meromor function g on X s.t. pole of $g \subset H$

$h(1,1) = 1$

$\Rightarrow (E, \theta, h)$ wild harmonic bundle.

$\left(\Sigma_\theta = \mathrm{Im}(\theta) \subset T^*(X \setminus H)\right)$

Ex. of wild harmonic bundles / punctured disc

$Y = \{z \in \mathbb{C} : |z| < 1\}$, $\quad 0 \in Y$

$\mathfrak{a} \in \mathcal{O}_Y(*0)$.

(1) $L(\mathfrak{a}) = \left(\mathcal{O}_{Y\setminus 0},\ d\mathfrak{a},\ h(1,1) = 1\right)$

$\varphi : Y \to Y$, $\varphi(z) = z^m$

$\varphi_* L(\mathfrak{a})$ on $Y \setminus 0$.

(2) $(a, \alpha) \in \mathbb{R} \times \mathbb{C}$

$L(a, \alpha) := \left(\mathcal{O}_{Y\setminus 0},\ \alpha \frac{dz}{z},\ h(1,1) = |z|^{-2a}\right)$

(tame harmonic bundle)

monodromy of corresponding flat bundle.

$\exp(-2\pi\sqrt{-1}(\alpha - a - \bar{\alpha}))$

(3) $N \in M_r(\mathbb{C})$ nilpotent

$V(N)$ tame harmonic bundle underlying polarized variation of Hodge str.

w/ monodromy $e^{2\pi\sqrt{-1} N}$

(4) $\bigoplus_i \varphi_*(L(\mathfrak{a}_i)) \otimes L(a_i, \alpha_i) \otimes V(N_i)$ wild harmonic bundle

general wild harmonic bundles on $(Y, 0)$ are "close to" this type of wild harmonic bundles tame case (Simpson)

$(E, \bar\partial_E, \theta, h)$ tame on $(Y, 0)$

(shrink Y) $\quad (E, \bar\partial_E, \theta) = \bigoplus_{\alpha \in \mathbb{C}} (E_\alpha, \bar\partial_{E_\alpha}, \theta_\alpha)$

$$\left(\overline{\Sigma_{\theta_\alpha}} \text{ in } T^*Y(\log 0) \right) \cap \underbrace{T^*Y(\log 0)|_0}_{\mathbb{C}} = \{\alpha\}$$

$a \in \mathbb{R}$, $Y \overset{\text{open}}{\supset} U \ni 0$

$$P_a E(U) = \{ s \in E(U \setminus 0) : |s|_h = O(|z|^{-a-\varepsilon}),\ \forall \varepsilon > 0 \}$$

$\mathcal{O}_Y$-module $P_a E$.

$P_* E = (P_a E : a \in \mathbb{R})$

$PE = \bigcup_{a \in \mathbb{R}} P_a E$

Prop (Simpson) $\quad P_a E$: $\mathcal{O}_Y$-locally free modules

$$[\bar\partial_E, \partial_E] = -[\theta, \theta^\dagger] = O\left(|z|^{-2} (\log|z|)^{-2} dz\, d\bar z\right)$$

theory of acceptable bundles (Cornalba - Griffiths)

$$\theta(P_a E) \subset P_a E \otimes \Omega^1(\log 0)$$

$$\mathrm{Res}(\theta) \curvearrowright Gr_a^P(E) := P_a E \Big/ \bigcup_{b<a} P_b E,$$

$$\| $$

$$\bigoplus_{\alpha \in \mathbb{C}} Gr_{a,\alpha}^{P.\mathbb{E}}(E)$$

$N^{0}_{a,\alpha}$: nilpotent part of $\operatorname{Res}(\theta)$ on $\operatorname{Gr}^{P^{\mathbb{E}}}_{a,\alpha}(E)$

$$(E, \bar\partial_E, \theta, h) \sim \bigoplus L(a,\alpha) \otimes V(N^{0}_{a,\alpha})$$

$\lambda \in \mathbb{C}$

$\mathcal{E}^\lambda = (E, \bar\partial_E + \lambda\theta^+)$

$\mathbb{D}^\lambda = \bar\partial_E + \lambda\theta^+ + \lambda\partial_E + \theta$

$U \ni 0$, $P_a\mathcal{E}^\lambda(U) = \{ s \in \Gamma(U \setminus 0, \mathcal{E}^\lambda) : |s|_h = O(|z|^{-a-\varepsilon}), \forall \varepsilon > 0 \}$

$\mathcal{O}_Y$- mod $P_a\mathcal{E}^\lambda$. locally free

$\mathbb{D}^\lambda P_a\mathcal{E}^\lambda \subset P_a\mathcal{E}^\lambda \otimes \Omega^1_Y(\log 0)$

$\operatorname{Res}(\mathbb{D}^\lambda) \curvearrowright \operatorname{Gr}^P_b(\mathcal{E}^\lambda)$

$$\overset{\shortparallel}{\bigoplus_{\beta \in \mathbb{C}}} \operatorname{Gr}^{P^{\mathbb{E}}}_{(b,\beta)}(\mathcal{E}^\lambda)$$

$N^\lambda_{b\beta}$ nilpotent part

Prop (Simpson) $k(\lambda, \cdot) : \mathbb{R}\times\mathbb{C} \longrightarrow \mathbb{R}\times\mathbb{C}$

$$k(\lambda, (a,\alpha)) = (a + 2\operatorname{Re}(\lambda\bar\alpha),\ \alpha - a\lambda - \bar\alpha\lambda^2).$$

$$\dim \operatorname{Gr}^P_{a\alpha}(\mathcal{E}^\lambda) = \dim \operatorname{Gr}^P_{k(\lambda,a,\alpha)}(\mathcal{E}^\lambda)$$

$$N^{0}_{a,\alpha} \sim N^\lambda_{k(\lambda,a,\alpha)}$$

$\coprod_{\lambda\in\mathbb{C}} \operatorname{Gr}^{P^{\mathbb{E}}}_{k(\lambda(a,\alpha))}(\mathcal{E}^\lambda)$ $/\mathbb{C}_\lambda$ holomorphic vector bundle

$\coprod_{\lambda\in\mathbb{C}} N^\lambda_{k(\lambda.(a,\alpha))} \circlearrowleft$

W: weight filt. vector subbundles

$(E, \partial_E, \theta^{+}, h)$ tame harmonic bundle on $(Y^{+}, 0)$

$$\Rightarrow \coprod_{\mu} \mathrm{Gr}^{P\mathbb{E}}_{k(\mu,-a,\bar{\alpha})}(\mathcal{E}^{+\mu}) \text{ on } \mathbb{C}_{\mu}$$

$$\curvearrowright \coprod_{\mu} N^{+\mu}_{k(\mu,(-a,\bar{\alpha}))} \qquad W$$

$$\mathrm{Gr}^{P\mathbb{E}}_{k(\lambda,a,\alpha)}(\mathcal{E}^{\lambda}) \overset{\lambda=\mu^{-1}}{\simeq} \mathrm{Gr}^{P\mathbb{E}}_{k(\mu,-a,\bar{\alpha})}(\mathcal{E}^{+\mu})$$

$$\lambda^{-1} N^{\lambda}_{a,\alpha} = -\mu^{-1} N^{\mu}_{(-a,\bar{\alpha})}$$

$\Rightarrow$ $V_{a,\alpha}$ vector bundle on $\mathbb{P}^1$

$$N_{a,\alpha}: V_{a,\alpha} \to V_{a,\alpha} \otimes \mathcal{O}_{\mathbb{P}^1}(2)$$

$$S: V_{a\alpha} \otimes \sigma^{*} V_{a\alpha} \to \mathcal{O}_{\mathbb{P}^1}(0)$$

$\sigma \curvearrowright \mathbb{P}^1$
$\sigma(\lambda) = -\bar{\lambda}^{-1}$

$(\underset{W}{V_{a,\alpha}}, N_{a,\alpha}, S_{a,\alpha})$ polarized mixed twistor structure

$L \Leftrightarrow$ twistor nilpotent $V(N_{a,\alpha})$)

$$(E, \bar{\partial}_E, \theta, h) \sim \bigoplus_{\substack{-1<a\leq 0 \\ \alpha\in\mathbb{C}}} L(a\alpha) \otimes V(N_{a\alpha})$$

Talk 2

X cplx mfld, $H = \bigcup H_i$ normal crossing hypersurface

$$\left\{ \begin{array}{l} V : \mathcal{O}_X(*H)\text{- locally free module of finite rank,} \\ \nabla : V \longrightarrow \Omega^1_X \otimes V \quad \text{integrable connection} \end{array} \right.$$

meromorphic flat bundle

$\nabla(fs) = df \otimes s + f\nabla(s)$, $\quad \nabla \circ \nabla = 0$

$f \in \mathcal{O}_X(*H)$, $s \in V$

(V, ∇) regular singular $\Leftrightarrow$ $\exists$ L : lattice of V

(L : $\mathcal{O}_X$ locally free submodule,

$L \otimes \mathcal{O}_X(*H) = V$)

s.t. $\nabla(L) \subset L \otimes \Omega^1_X(\log H)$

(local systems on $X \setminus H$) $\simeq$ (regular singular mero. flat bundles on (X, H))

(Riemann - Hilbert correspondence)

Deligne

irreg singularity (1-dim case, classical)

$Y = \{ z \in \mathbb{C} : |z| < 1 \}$

(V, ∇) mero flat bundle $/(Y, 0)$

$\exists e \in \mathbb{Z}_{\geq 1}$

(1) $$(V, \nabla) \underset{\mathcal{O}_{\Delta,0}}{\otimes} \mathbb{C}[[z^{1/e}]] = \bigoplus_{\alpha \in z^{-1/e}\mathbb{C}[z^{-1/e}]} (\hat{V}_\alpha, \hat{\nabla}_\alpha)$$ (Hukuhara-Levelt - Turrittin decomposition)

$\hat{\nabla}_\alpha - d\alpha\text{-}\mathrm{id}_{V_\alpha}$: regular singular

(2) Suppose $(V, \nabla) \otimes \mathbb{C}[[z]] = \bigoplus_{\mathfrak{a} \in I\,(\subset z^{-1}\mathbb{C}[z^{-1}])} (\hat{V}_{\mathfrak{a}}, \hat{\nabla}_{\mathfrak{a}})$

$\varpi : \widetilde{Y}(0) \to Y$ oriented real blowup

$$\widetilde{Y}(0) = \left\{ (r, e^{\sqrt{-1}\theta}) : \begin{array}{l} 0 \le r < 1 \\ e^{\sqrt{-1}\theta} \in S^1 \end{array} \right\} \qquad \varpi(r, e^{\sqrt{-1}\theta}) = re^{\sqrt{-1}\theta}$$

$$\cong S^1 \times [0, 1)$$

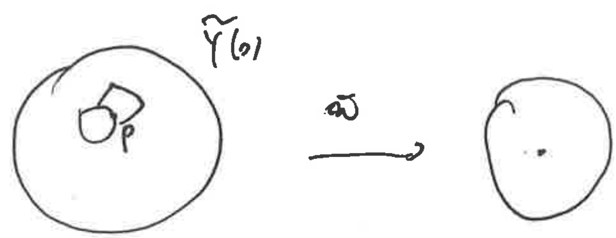
$$\mathcal{O}_{\widetilde{Y}(0)}(U) = \{ f \in C^\infty(U) : f|_{U \setminus \varpi^{-1}(0)} \text{ holomorphic} \}$$

$$(\widetilde{V}, \widetilde{\nabla}) := \varpi^{-1}(V, \nabla) \otimes_{\varpi^{-1}\mathcal{O}_Y} \mathcal{O}_{\widetilde{Y}(0)}$$

not unique $\downarrow$ $\exists$ decomp.

$p \in \varpi^{-1}(0)$, $\exists\ U_p$ nbhd of p in $\widetilde{Y}(0)$ s.t. $(\widetilde{V}, \widetilde{\nabla})|_{U_p} = \boxed{\bigoplus_{\mathfrak{a} \in I} (\widetilde{V}_{\mathfrak{a}p}, \widetilde{\nabla}_{\mathfrak{a}p})}$

$$(\widetilde{V}_{\mathfrak{a}p}, \widetilde{\nabla}_{\mathfrak{a}p})\Big|_{\widehat{\varpi^{-1}(0) \cap U_p}} = \varpi^{-1}(\hat{V}_{\mathfrak{a}}, \hat{\nabla}_{\mathfrak{a}})$$



$\le_p$ partial order on I

$$\mathfrak{a} \le_p \mathfrak{b} \iff -\mathrm{Re}(\mathfrak{a}) \le -\mathrm{Re}(\mathfrak{b}) \text{ on } U_p \setminus \varpi^{-1}(0)$$

(well defined for sufficiently small U_p)

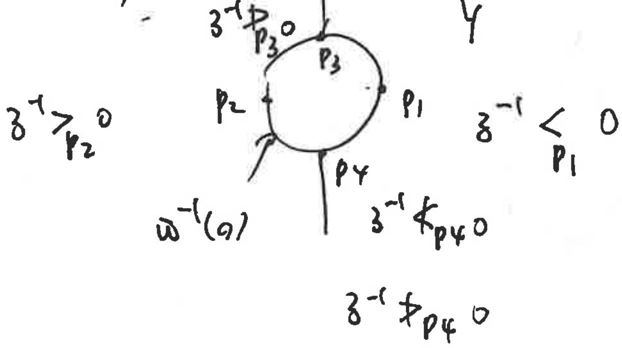
$$F_{\mathfrak{b}}^{p}(\widetilde{V}|_{U_p}) = \bigoplus_{\mathfrak{a} \le_p \mathfrak{b}} \widetilde{V}_{\mathfrak{a},p} \quad \leftarrow \text{well-defined}$$

Stokes filtration F^p indexed by $(I, \leq_p)$

p' close to p, $\quad a \leq_p b \Rightarrow a \leq_{p'} b$

($\Leftarrow$ not necessarily correct)

<u>ex</u> $I = \{z^{-1}, 0\}$



$$F^{p'}_a(V|_{U_{p'}}) = F^p_a(V|_{U_p})\big|_{U_{p'}} + \sum_{b <_{p'} a} F^{p'}_b(V|_{U_{p'}})$$

(3) $\mathcal{L}$ local system on $Y \setminus 0$ corresponding to $(V, \nabla)|_{Y\setminus 0}$

$\Rightarrow \widetilde{\mathcal{L}}$ local system on $\widetilde{Y}(0)$

$p \in \varpi^{-1}(0)$, $\quad F^p_a(\widetilde{\mathcal{L}}_p)$: induced by $F^p_a(\widetilde{V}|_{U_p})\big|_{U_p \setminus \varpi^{-1}(0)}$

F^p: filtration of $\widetilde{\mathcal{L}}_p$ indexed by $(I, \leq_p)$.

p' close to p, $\quad F^{p'}_a = F^p_a + \sum_{b <_{p'} a} F^{p'}_b \quad (*)$

$\mathbb{F} = \{F^p : p \in \varpi^{-1}(0)\}$ satisfies $(*)$ $\quad$ Stokes structure of $\mathcal{L}$ indexed by I $\quad$ (Deligne)

$$\begin{pmatrix} (V,\nabla) \text{ meromorphic flat bundle} \\ (V,\nabla)\otimes \mathbb{C}[[z]] = \bigoplus_{a\in I} (\hat{V}_a, \hat{\nabla}_a) \end{pmatrix} \simeq \begin{pmatrix} \text{local system w/} \\ \text{Stokes str. over } I \end{pmatrix}$$

(generalized R-H correspondence
Sibuya - Malgrange - Deligne)

$$\left(\mathrm{Gr}^{F^p}_a (\mathcal{L}_p)\big|_{U_{p'}} = \mathrm{Gr}^{F^{p'}}_a (\mathcal{L}_{p'}) \right)$$

$$\mathrm{Gr}^{F^p}_a (\tilde{V}|_{U_p})\big|_{U_{p'}} = \mathrm{Gr}^{F^{p'}}_a (\tilde{V}|_{U_{p'}})$$

$$\Rightarrow \mathrm{Gr}^{\mathbb{F}}_a (\tilde{V}) \text{ on } \tilde{Y}(0) \Rightarrow \mathrm{Gr}^{\mathbb{F}}_a (V) \text{ on } (Y,0)$$

meromorphic flat bundle

$$\mathrm{Gr}^{\mathbb{F}}_a (V) \otimes \mathbb{C}[[z]] \simeq (\hat{V}_a, \hat{\nabla}_a)$$

$\tilde{\mathcal{L}}$: local system on $\tilde{Y}(0)$, $\mathbb{F} = (F^p : p\in \varpi^{-1}(0))$ Stokes str. over I

$T>0$, $\mathbb{F}^{(T)} := (F^{(T)p} : p\in \varpi^{-1}(0))$ $F^{(T)p}_{Ta'} = F^p_a$

Stokes str. over $TI = \{Ta : a\in I\}$

$$(\tilde{\mathcal{L}}, \mathbb{F}) \rightsquigarrow (\tilde{\mathcal{L}}, \mathbb{F}^{(T)})$$

$$(V,\nabla) \Rightarrow (V^{(T)}, \nabla^{(T)})$$

$\xleftarrow{S}\!\!\!\rightarrow$ $s\in S$, $s.I = \{sa : a\in I\}$

$(\tilde{V}, \tilde{\nabla})$ mero. flat bundle over $S\times(\Delta,0)$

s.t. $(\tilde{V},\tilde{\nabla})|_{\{1\}\times(\Delta,0)} = (V,\nabla)$ index set of $(\tilde{V},\tilde{\nabla})|_{S\times(\Delta,0)}$
$= s.I$.

X (higher dim case)

$(V,\nabla)/(X,H)$ Naive hope: $p\in H$, $(X_p, z_1,\dots,z_n)$ coord. $X_p\cap H=\bigcup_{i=1}^{\ell}\{z_i=0\}$

if this is not satisfied, p is called <u>turning point</u>

$$\begin{cases} X'_p \xrightarrow{\varphi} X_p \quad (z_1^{1/e},\dots,z_\ell^{1/e}, z_{\ell+1},\dots,z_n)\mapsto(z_1,\dots,z_n) \\ \ \ p' \in X'_p, \quad p\in X_p \\ \varphi^*(V,\nabla)|_{\hat{p}'} = \bigoplus_{a\in I}(V_a,\nabla_a) \quad \nabla_a - da_i \ \text{reg. sing} \end{cases}$$

I: non degenerate

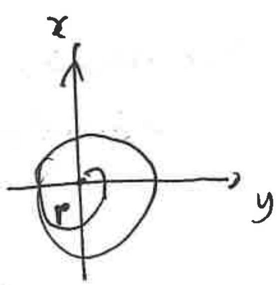
ex. (V_0,∇_0) mero. flat bundle / $\mathbb{P}^1$

index set ↗ $V_0=\mathcal{O}_{\mathbb{P}^1}(*0)^{\oplus 2}$, $\nabla(v_1,v_2)=(v_1,v_2)\begin{pmatrix}0&1\\z^{-1}&0\end{pmatrix}d(z^{-1})$

$\pm C z^{-\frac{3}{2}}$

$\Phi:\mathbb{C}^2 \dashrightarrow \mathbb{P}^1$ $\Phi(x,y)=[x:y]$

$\Phi^*(V,\nabla)$: $(0,0)$ turning pt



$\pm C(x/y)^{-\frac{3}{2}}$

(V,∇) good $\overset{\text{def}}{\iff}$ (V,∇) has no turning pt

(1) (2) (3) can be generalized for mer. flat bundles

<u>Th</u> (Kedlaya, M. algebraic case) $(V,\nabla)/(X,H)$, $\exists\varphi: X'\to X$ proj. birat'l

s.t. $\varphi^{-1}(H)=NC$, $\varphi^{-1}(V,\nabla)=$ good

$Y = \{|z|<1\}$

$(E, \bar\partial_E, \theta, h)$ wild $/ (Y, 0)$

$$(E, \bar\partial_E, \theta) = \bigoplus_{a\in I} (E_a, \bar\partial_{E_a}, \theta_a)$$

s.t. $\theta_a - da$ tame

$\mathcal{E}^\lambda = (E, \bar\partial_E + \lambda\theta^\dagger)$, $\mathbb{D}^\lambda$

$$P\mathcal{E}^\lambda(U) = \{ s \in \Gamma(U\setminus 0, \mathcal{E}^\lambda) : |s|_h = O(|z|^{-N}), \exists N\}$$

loc. free $\mathcal{O}_Y(*0)$-module $P\mathcal{E}^\lambda$

$\coprod_{\lambda\in\mathbb{C}_\lambda} P\mathcal{E}^\lambda / \mathbb{C}_\lambda \times X$ (not holomorphic)

$\coprod_{\lambda\in\mathbb{C}_\lambda} (P\mathcal{E}^\lambda, \mathbb{D}^\lambda)^{\left(\frac{1}{1+|\lambda|^2}\right)}$ holomorphic on $\mathbb{C}_\lambda\times Y$

$\lambda\neq 0$, $\mathrm{Gr}^{\mathbb{F}}_{(1+|\lambda|^2)a}(P\mathcal{E}^\lambda, \mathbb{D}^\lambda) := \mathrm{Gr}^{\mathbb{F}}_{\frac{1+|\lambda|^2}{\lambda}a}(P\mathcal{E}^\lambda, \mathbb{D}^{\lambda f})$ ← flat connection corresponding to $\mathbb{D}^\lambda$

$\lambda=0$ $\quad \mathrm{Gr}^{\mathbb{F}}_a(P\mathcal{E}^0, \mathbb{D}^0) := (P\mathcal{E}^0_a, \mathbb{D}^0)$

$\coprod_\lambda \mathrm{Gr}^{\mathbb{F}}_{(1+|\lambda|^2)a}(P\mathcal{E}^\lambda,\mathbb{D}^\lambda)\big|_{Y\setminus 0}$ on $\mathbb{C}_\lambda\times(Y\setminus 0)$

$(E, \partial_E, \theta^\dagger, h)$ on $(Y^\dagger, 0) \Rightarrow \coprod_\mu \mathrm{Gr}^{\mathbb{F}}_{(1+|\mu|^2)a}(P\mathcal{E}^{\dagger\mu}, \mathbb{D}^{\dagger\mu})$, $\mathbb{C}_\mu\times(Y^\dagger - 0)$

$\lambda \neq 0$

$\lambda = \mu^{-1}$ $\quad \mathrm{Gr}^{\mathbb{E}}_{(1+|\lambda|^2)a}(\mathcal{P}\Xi^{\lambda}) = \mathrm{Gr}^{\mathbb{E}}_{(1+|\mu|^2)\bar{a}}(\mathcal{P}\Xi^{\dagger\mu})$

$\rightsquigarrow$ variation of polarized pure twistor str.

$\Rightarrow (E_a, \bar{\partial}_E, \theta_a, h_a) \qquad h - \oplus h_a = O(\exp(-\varepsilon|z|^{-1}))$

harmonic metric

$(E, \bar{\partial}_E, \theta, h) \sim \oplus (E_a, \bar{\partial}_{E_a}, \theta_a, h_a)$