Alterations

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Lecture 1

Def Let X be reduced (locally) noetherian scheme. A resolution of singularities of X is a proper map

$$\begin{tikzcd} X' \arrow[d, "f"] \\ X \end{tikzcd}$$

s.t.
① proper
② X' is regular
③ f birat'l $:=$ $\exists$ dense open $U \subset X$, $U' \subset X'$ s.t. $f : U' \xrightarrow{\sim} U$.

f proper $\Downarrow$

③' $\exists$ dense open $U \subset X$ s.t. $f^{-1}(U) \subset X'$ dense and $f^{-1}(U) \xrightarrow{\sim} U$.

Could be greedy: any such $U \subset \mathrm{Reg}(X) = \{x \in X : \mathcal{O}_{X,x} \text{ is regular}\}$ (open in X when X is excellent)

so could ask if we can achieve resolution using $U = \mathrm{Reg}(X)$?

Also, could ask $(X' - f^{-1}(U))_{\mathrm{red}} \subset X'$ is a "strict normal crossings divisor" (sncd) in X'. [for ③']

Def. Let S be a regular scheme. Let $D \subset S$ be an effective Cartier divisor : $I_D \subset \mathcal{O}_S$ invtble.

Say D is sncd if
① D reduced
② All irred. components $D_i \subset D$ $(i \in I)$ (w/ reduced str.) are regular
③ For $J \subset I$, $D_J := \bigcap_{i \in J} D_i$ $\boxed{\text{regular}}$ of pure codim $\#J$ in S:

$$\dim \mathcal{O}_{D_J,\xi} = \dim \mathcal{O}_{S,\xi} - \#J, \quad \forall\, \xi \in D_J.$$

Exer. For $\xi \in D$, $(I_D)_\xi \subset \mathcal{O}_{S,\xi}$ = regular local ($\Rightarrow$ UFD)

$$(I_D)_\xi = (f) = \prod_{i \in J \subset I} t_i,$$ where $(t_i) = (I_{D_i})_\xi$, for D_i irred. comp. of D through ξ.

__Exer.__ Reduced Cartier $D \subset S$ is an sncd

$\Leftrightarrow$ irred. factors of local generator of $(I_D)_\xi$ constitute part of a regular system of parameters in $\mathcal{O}_{S,\xi}$ - regular.

__Non-ex.__ $C = \{y^2 = x^2(x+1)\}$ over k, $\mathrm{char}(k) \neq 2$

In $\hat{\mathcal{O}}_{C,(0,0)}$ have $\sqrt{1+x}$, so $y = \pm x\sqrt{1+x}$.

$k[[x,y]]/(y^2-x^2(x+1))$ is __not__ domain.

$$\begin{tikzcd} & k[[x,y]]/(y-x\sqrt{1+x}) \\ & \cong k[[x]] \\ k[[x,y]]/(y^2-x^2(x+1)) = k[[x,y]]/(y-x\sqrt{1+x})(y+x\sqrt{1+x}) \arrow[uur] \arrow[r] & k[[x,y]]/(y+x\sqrt{1+x}) \\ & \cong k[[x]] \end{tikzcd}$$

__not__ sncd, but "formal-locally" looks like one.

This C becomes reducible after pullback C' along étale nbhd

$\mathrm{Spec}(k[x,y,\sqrt{1+x}]_{(1+x)}) \longrightarrow \mathrm{Spec}(k[x,y]_{(1+x)})$ is __sncd__.

__Def.__ Say reduced effective Cartier divisor $D \subset S \leftarrow$ regular is a __normal crossing divisor__ (ncd) if $\forall \xi \in D$, $\exists$ étale map $(U',\xi') \xrightarrow{f} (S,\xi)$ s.t. $f^{-1}(D) \subset U'$ is sncd.

__Later__ (use Artin approx) For regular S that is excellent, Cartier $D \subset S$ is ncd

$\Leftrightarrow$ $\mathrm{Spec}(\hat{\mathcal{O}}_{D,\xi}) \subset \mathrm{Spec}(\hat{\mathcal{O}}_{S,\xi})$ is sncd $\forall \xi \in D$.

__Rmk__ If $D \subset S$ = regular is ncd, then $D \subset S$ is sncd

$\Leftrightarrow$ all irred comp. $D_i \subset D$ (w/ reduced str.) are regular.

<u>Thm</u> (Hironaka) For X = reduced, sep'td of f.-type over field k of char. 0,

$\exists$ resol'n of sing. $X' \xrightarrow{f} X$ s.t.

① f isom. over $\mathrm{Reg}(X) = X^{sm} \subset X$

$\boxed{\text{Kollár has 30-page pf}}$

② $f^{-1}(X - \mathrm{Reg}(X)) \subset X'$ is ncd.

<u>Def</u>. X = integral noetherian scheme. An <u>alteration</u> of X is ~~proper~~ $f : X' \longrightarrow X$

where
- X' is integral
- f dominant proper ($\Rightarrow$ surjective)
- $[k(X') : k(X)] < \infty$

($\underset{\text{Spreading out}}{\Longleftrightarrow}$ $\exists$ dense open $U \subset X$ s.t. $\begin{matrix} f^{-1}(U) \\ \downarrow \\ U \end{matrix}$ is finite flat (even finite étale when $k(X') \mid k(X)$ is sep'ble))

<u>Thm</u> (de Jong) X = integral, sep'td f.-type $/k$ = field. ("variety")

Pick <u>any</u> closed $Z \subsetneq X$. Then $\exists$

$$\begin{tikzcd} Z' \arrow[r, phantom, "\subset"] \arrow[d] & X' \arrow[r, "\text{open}", "\text{dense}"'] \arrow[d, "f"] & \bar{X}' = \text{regular projective var.}/k \\ Z \arrow[r, hook] & X & \end{tikzcd}$$

where f is alteration, $f^{-1}(Z) \cup (\bar{X}' - X')$ w/

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

reduced str. is sncd in $\bar{X}'$. $\bar{X}' - f^{-1}(X - Z)$

<u>Rmk</u> ① For k perfect, can arrange $k(X') \mid k(X)$ is sep'ble)

② Pf gives <u>no</u> control on how $X - Z$ relates to $\mathrm{Reg}(X)$.

Lecture 2. Applications and non-applications of de Jong's thm

① **Grauert-Remmert thm** X = normal lft. $\mathbb{C}$-scheme

$$\begin{tikzcd} F_{\acute{e}t}(X) = \left\{ \begin{array}{c} E \\ \downarrow \\ X \end{array} \text{ finite étale} \right\} \arrow[d] & E \to X \arrow[d, mapsto] \\ F_{\acute{e}t}(X^{an}) = \left\{ \begin{array}{c} E' \\ \downarrow \\ X(\mathbb{C}) \end{array} \begin{array}{c} \text{finite-degree} \\ \text{"covering maps"} \end{array} \right\} & E^{an} \to X^{an} \end{tikzcd}$$

is an equiv.

For proper X, use GAGA.

In SGA 1, Exp XII gives general cases, via Hironaka "applied" to affine $X \hookrightarrow \bar{X}$ = proj. closure.

Can get by w/ de Jong's thm.

② **Artin comparison thm.** X sep'td, f. type $/\mathbb{C}$. $\mathcal{F}$ constructible abelian sheaf on $X_{\acute{e}t}$ (or ℓ-adic). The nat'l map $H^i(X_{\acute{e}t}; \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an})$ is an isom.

(also for H^*_c: much easier)
(reason: has excision seq.)

Pf in SGA4 uses Hironaka. — can adapt Berkovich's non-arch. pf to $\mathbb{C}$-case; to bootstrap to ℓ-adic case, can use alterations (Deligne).

③ **Deligne's theory of mixed Hodge structure**

- can replace birat'l w/ alteration.

Non-application X = smooth sep'd f. type $/\mathbb{C}$.

$$\Omega^\bullet_{X/\mathbb{C}} = (\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \xrightarrow{d} \Omega^2_{X/\mathbb{C}} \to \cdots)$$

$$H^i_{dR}(X/\mathbb{C}) := \mathbb{H}^i(X, \Omega^\bullet_{X/\mathbb{C}}) \xrightarrow{\sim} \mathbb{H}^i(X^{an}, \Omega^\bullet_{X^{an}/\mathbb{C}}) \overset{\text{Poincaré}}{\simeq} H^i(X(\mathbb{C}); \mathbb{C})$$

Thm (G.) isom. [IHES 29]

For X proper, use GAGA.

Čech-type reduction to q-proj. X

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, equal] & \bar{X}' \arrow[d, "\text{Hironaka}"] & \bar{X}' - X \arrow[l, hook'] & = \text{ncd in } \bar{X}' \\ X \arrow[r, hook] & \bar{X} & = \text{projective} \end{tikzcd}$$

$H^i_{dR}(X/\mathbb{C})$ behaves poorly w.r.t. finite flat covers,

and de Jong's thm doesn't control q-finite $\cap$ étale locus in base.

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Regular v.s. Smooth. Let S be scheme lft $/k$ = field

Say S is *regular* if all $\mathcal{O}_{S,s}$ are regular (suffices to check at closed pts)

Say S is *k-smooth* if $S_{\bar{k}}$ is regular $\Leftrightarrow$ $S_{k'}$ is regular $\forall$ finite k'/k

$\Leftrightarrow$ $S_{k'}$ reg. $\forall$ k'/k

$\Leftrightarrow$ $S_{k'}$ reg. for any perfect k'/k

$\Leftrightarrow$ $S \to \operatorname{Spec}(k)$ satisfies inftesimal smoothness criterion.

k-perfect $\Rightarrow$ k-smooth $\Leftrightarrow$ regular

o/w usually false.

Ex. Let k be imperfect, char $p > 0$. $a \in k - k^p$ (e.g. $k = \mathbb{F}_p(t)$, $a = t$)

Pick $m > 1$, $p \nmid m$. Let $C = \{y^m = x^p - a\} \subset \mathbb{A}^2_k$.

$$\begin{tikzcd} x^p - a \in k[x] \text{ irred.} \Rightarrow & k[x,y]/(y^m - (x^p - a)) \text{ is Dedekind} \\ & k[x] \arrow[u, hook] \end{tikzcd}$$

(int. closure of $k[x]$ in $k(x)(\sqrt[m]{x^p - a})$)

So C is regular.

<u>Over $\bar{k}$</u>: $y^m = (x - \alpha)^p$, so "$y^m = u^p$" $(m > 1)$

$\alpha^p = a$

so $C_{\bar{k}}$ reduced $(p \nmid m)$, but singularity at $(\alpha, 0)$.

$C - \underset{\substack{\shortparallel \\ (x^p - a,\, y)}}{\xi}$ is k-smooth.

<u>Ex2</u> (MacLane) Let $K | k$ f.gen'd of tr. deg 1

$$\begin{tikzcd} K \arrow[d, dash, "\text{finite}"'] & & C \arrow[d, "\text{finite flat}"] & = \text{normalization } w/\ k(C) = K \\ k(x) & \rightsquigarrow & \mathbb{P}^1_k & \end{tikzcd}$$

If C has dense open U = k-smooth, then $\exists$ étale $U \longrightarrow \mathbb{A}^1_k$

sep'ting $\rightsquigarrow x$
tr. basis $/k$
for $k(U) = k(C) = K$

$K | k$ having "separating" tr. basis

$\Leftrightarrow$ $K | k$ "sep'ble" (§ 26 of CRT)

$\Leftrightarrow$ $\exists$ k-smooth dense open $U \subset C$.

$k = \mathbb{F}_p(s,t)$, $\nearrow$ $C = \{sx^p + ty^p = 1\} \subset \mathbb{A}^2_k$

$C_{\bar{k}} = \{(\sigma x + \tau y)^p = 1\}$, $\sigma^p = s$, $\tau^p = t$

Exer. - Dedekind. everywhere non-reduced

- $k \subset k(C)$ is alg. closed.

Upshot: Let X be "variety" $/k$. If $K = k(X)|k$ is NOT separable (no septing tr. basis) then $\nexists$ generically k-smooth alteration $X' \to X$: $\underbrace{k \subset k(X) \subset k(X')}_{\text{Sep ble}} \Rightarrow k(x)|k$ sepble

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Lecture 3. Semi-stability and Excellence

[Correction: Grauert-Remmert Thm ("Riemann Existence Thm") does NOT need normality, any lft $\mathbb{C}$-scheme OK]

Basic idea of de Jong Consider $Z \subsetneq \overset{\text{variety}}{X}$ over k.

Step 0. Use Chow's lemma + normalization to reduce $X = $ (q-)proj, normal, $k = \bar{k}$.

This settles $d = 1$. Induct on $d > 1$.

Step 1 Use blow-up to pass to $\begin{matrix} X \\ \downarrow \\ \mathbb{P}^{d-1} \end{matrix}$ w/ fibers smooth geom. conn'd curve of genus g over dense open.

Step 2. Use properness of $\overline{M}_{g,n}$ $(n \geq 2g-2)$ to find alteration $Y' \to \mathbb{P}^{d-1}_k$ s.t.

$$\begin{tikzcd} \exists & X' \arrow[r, two heads, "\text{alteration}"] \arrow[d] & X \arrow[d] \\ & Y' \arrow[r] & \mathbb{P}^{d-1} \end{tikzcd}$$

semistable

may be awful

Here Z is useful.

Step3 Apply induction to $Y' (\supsetneq Z')$ to get to

└ locus of non-smooth fibers

$$\begin{tikzcd} X \arrow[d, "\text{sst, smooth over } Y-\tilde{Z}"] & \\ Y \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\text{||}"] & \tilde{Z} \arrow[l, phantom, "\text{Sncd}"', shift right=3] \\ \text{Smooth} & \end{tikzcd}$$

Concrete blowup conclude.

Semistable curve over a field:

Thm. Let X be a pure 1-dim'l f.type scheme $/k$. Pick $x \in X$ closed.

TFAE:

① For some $\bar{x} \in X_{\bar{k}}$ over x, $\widehat{\mathcal{O}}_{X_{\bar{k}}, x} \simeq \bar{k}[\![t]\!]$ or $\bar{k}[\![u,v]\!]/(uv)$

② Either $x \in X^{sm}$ or $\exists$ common étale nbhd $/k$

$$\begin{tikzcd} & (U,u) \arrow[dl, "\text{étale}"'] \arrow[dr, "\text{étale}"] & \\ (X,x) & & (\{uv=0\}, (0,0)) \end{tikzcd}$$

] X is semistable at x.

$\Rightarrow$ X smooth on $V - \{x\}$ for open $V \ni x$.

Pf ① ⇒ ② uses Artin approximation. ("ordinary double pt" singularities)

[§ 2, Ch III] of Freitag - Kiehl. ☒

$*$: $k(X)/k$ is separable! $(\underbrace{k \subset k(X) \subset k(U)}_{\text{separable}})$

Def Say X is semistable if so at all closed $x \in X$; then $X^{sm} \subset X$ is dense open.

We'll have a "sst reduction thm" for smooth proper geom. conn'd curves over

$K = \mathrm{Frac}(R)$ for DVR R.

Start w/

Thm. (sst reduction for ab. var.) Let A be ab. var. $/K = \mathrm{Frac}(R)$ for dvr R (res. field k)

$\exists$ finite sep'ble ext'n $K'|K$ so for semi-local int. closure $R' \subset K'$ of R.

have $\mathcal{A}' = \text{Néron}(A_{K'})$ has "semistable reduction": $(\mathcal{A}'_{\bar{k}})^0 \not\supset \mathbb{G}_a$

($\Leftrightarrow$ has "affine part" a torus)

Enough to take $K'|K$ split $A[\ell]$ for $\ell \neq \mathrm{char}(k)$
ℓ odd or $\ell = 4$.

Take X = smooth proper geom. conn'd curve $/K = \mathrm{Frac}(R)$

$A = \mathrm{Pic}^0_{X/K}$ ("Jacobian" of X/K), $\dim A = g := \mathrm{genus}(X)$.

$\exists$ "minimal regular proper model" $\mathcal{X} \xrightarrow[\text{flat}]{\text{proper}} \mathrm{Spec}(R)$ of X:

- $\mathcal{X}$ = regular
- "no contractions" [minimality]

Unique if $g > 0$.

Thm (Deligne-Mumford for $g \geq 2$, Deligne-Rapoport for $g = 1$)

A has semistable reduction $\Leftarrow\!\!\Rightarrow$ $\mathcal{X} \longrightarrow \mathrm{Spec}(R)$ has sst special fiber.

$\uparrow$ when A has sst reduction, $\mathrm{Pic}^0_{\mathcal{X}/R} \simeq (\mathcal{A})^0$

Combine Thms (+ε for $g=0,1$) $\Rightarrow$ $\exists$ finite separable $K'|K$ so $X_{K'} = \mathcal{X}' \otimes_{R'} K'$

for $\mathcal{X}' \longrightarrow \mathrm{Spec}(R')$ proper flat w/ sst special fibers.

Excellence. Ref Ch. 13 of Matsumura C.A

EGA IV_2, §5 - §7 esp. §7.8 ff

Def. Say (noeth) A is catenary if $\forall\ \mathfrak{p} \subsetneq \mathfrak{p}'$ in A, all max'l chains

$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}'$ have $n = \dim A_{\mathfrak{p}'} - \dim A_{\mathfrak{p}}$.

Say A is universally catenary if all f.gen. A-algs are catenary.

Ex (17.4 in [CRT]) Every (qt of) CM ring is catenary

If CM ring $B \twoheadrightarrow A$, then $\underbrace{B[T_1, \cdots, T_n]}_{CM} \twoheadrightarrow A[T_1, \cdots, T_n]$

so such A are univ. catenary.

Ex. Every complete local noeth. $A \twoheadleftarrow B$ = reg!

Lecture 4 Excellence II

Def. A noeth ring A is excellent if

① univ. catenary

② (G-ring) $\forall\ \mathfrak{p} \in \mathrm{Spec}\, A$, fiber algebras of $\mathrm{Spec}(\hat{A}_{\mathfrak{p}}) \longrightarrow \mathrm{Spec}(A_{\mathfrak{p}})$ are geometrically regular:

$$\begin{tikzcd} \hat{A}_{\mathfrak{p}} \\ A_{\mathfrak{p}} \arrow[u] \end{tikzcd}$$

$\forall$ prime $\mathfrak{q} \subset \mathfrak{p}$, $\hat{A}_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{q})$ is regular & remains so after all finite ext'ns on $k(\mathfrak{q})$

③ $\mathrm{Reg}(A') \subset \mathrm{Spec}(A')$ is open $\forall$ f.g. A-alg A'.

② + ③ = quasi-excellent

Rmk. To verify ③, one can use more robust equiv. formulations: §32 B Thm 73 in [CA] or [EGA IV_2, 6.12.4]

Examples ① Consider $A = R = \mathrm{DVR}$, $K = \mathrm{Frac}(R)$: content of excellence is $\hat{K} \underset{K}{\otimes} K'$ is reduced for all finite $K'|K$ $\Leftrightarrow$ $\hat{K}|K$ is separable in sense of fields ([CRT, §26])

See [BLR, §3.6, Ex11] gives DVR in char p not a G-ring.

② If C is regular curve over field k, then $R = \mathcal{O}_{C,c}$, $\forall$ closed $c \in C$ is G-ring:

$\hat{\mathcal{O}_{C,c}} \underset{\mathcal{O}_{C,c}}{\otimes} K'$ is reduced $\forall$ finite ext'n K' of $k(C)$.

(think about normalization of C in $K'|k(C)$).

③ Any Dedekind A in generic char 0 (e.g. $\mathbb{Z}$) is excellent.

Thm. (Grothendieck - Nagata, EGA IV_2, §7.8)

① Every complete local noeth ring & Dedekind domain of gen. char 0 is excellent.

* ② Excellence inherited by f.gen. algs and localization at mult. set.

③ A = excellent & reduced, $\Rightarrow$ normalization $A \to \tilde{A}$ is modulo-finite

④ Let P be any of long list of "homological" properties of noeth. local rings (e.g. Cohen-Macaulay, normal, Gorenstein, ...)

$\mathbb{P}(B) = \{ \mathfrak{p} \in \operatorname{Spec}(B) : B_{\mathfrak{p}} \text{ satisfies } \mathbb{P} \}$

For excellent A,
- $\mathbb{P}(A)$ is open
- for any ideal $I \subset A$ and $f: \operatorname{Spec} \hat{A} \to \operatorname{Spec} A$, $f^{-1}(\mathbb{P}(A)) = \mathbb{P}(\hat{A})$

(For A local, $I =$ max'l, $[A$ has $\mathbb{P} \iff \hat{A}$ does$]$)

Def A locally noeth scheme X is excellent if every affine open $\operatorname{Spec} A \subset X$ has A excellent ($\iff$ for one affine open cover $\{\operatorname{Spec} A_i\}$, all A_i are excellent)

Rmk. $\exists$ "Jacobian criterion" for excellence of regular $\mathbb{Q}$-algs ([CA, Thm 101]), and applies to $\mathcal{O}^{an}_{\mathbb{C}^n,0}$, so $\mathcal{O}_{X,x}$ for $\mathbb{C}$-analytic spaces are excellent. Thus, if $X = Y^{an}$ for lft Y over $\mathbb{C}$, then for $y \in Y(\mathbb{C})$:

$$\begin{tikzcd} \mathcal{O}_{Y,y} \arrow[r] \arrow[d] & \mathcal{O}_{X,y} \arrow[d] \\ \hat{\mathcal{O}}_{Y,y} \arrow[r, phantom, "\simeq"] & \hat{\mathcal{O}}_{X,y} \end{tikzcd}$$

, so "$\mathbb{P}(\mathcal{O}_{Y,y}) \iff \mathbb{P}(\mathcal{O}_{Y^{an},y})$"

true but harder for rigid analytic spaces.

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Relations of G-ring to resolution of singularities

Thm [EGA IV_2, 7.9.5] Let X be loc. noeth. Assume $\forall$ open $U \subset X$, all finite $Y \to U$ w/ Y integral ($Y \twoheadrightarrow$ irred comp. of U) have resolution of singularities, then all $\mathcal{O}_{X,x}$ are quasi-excellent (esp. G-rings)

$\wr$ EGA version omits U, seems error.

Pf. Key case. $X = \operatorname{Spec} A$, want $\operatorname{Spec} \hat{A}_{\mathfrak p} \to \operatorname{Spec} A_{\mathfrak p}$ to have geom. regular fiber algebras. For $\mathfrak q \subset \mathfrak p$, pass to $A/\mathfrak q$ to reduce A = local domain w/ fr. field K and want $\hat{A} \otimes_A K'$ regular $\forall$ finite $K'|K$. $K' = \operatorname{Frac}(A')$ for A-finite A' (semi-local)

$$\hat{A} \otimes_A K' \simeq (\hat{A} \otimes_A A') \otimes_{A'} K' = \Big(\prod_{m'} (A'_{m'})^{\wedge} \Big) \otimes_{A'} K'$$

Rename $A'_{m'}$ as A, and key issue: $\hat{A} \otimes_A K$ = regular when A admits resolution?

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y = \text{regular, integral} \arrow[d, "f = \text{resolution}"] \\ X' = \operatorname{Spec}\hat{A} \arrow[r, "h"'] & X = \operatorname{Spec} A \end{tikzcd}$$

f isom. over dense open $U \subset X$

so f' is isom. over $h^{-1}(U) \subset X'$

$\cup$

X'_η

Would suffice to show Y' = regular.

Note f' is proper onto local X', and $\hat{A}$ = exc. $\Rightarrow$ $\operatorname{Reg}(Y') \subset Y'$ is open.

f' = proper $\Rightarrow$ open in Y' is $|Y'|$ if contains Y'_0.

So enough $\hat{\mathcal{O}}_{Y',y'}$ reg. for $y' \in Y'_0$. $Y'_n \simeq Y_n$, $\forall n \geq 0$.

$\Rightarrow \hat{\mathcal{O}}_{Y',y'} \xleftarrow{\sim} \hat{\mathcal{O}}_{Y,y}$ = regular $\because$ Y is reg!

Lecture 5. Preliminary Reduction Steps

One more general def'n: **Def.** A modification of an integral noeth scheme S is a proper birat'l map $\psi: S' \to S$, $\eta' \mapsto \eta$ for S' integral.

Def. For $\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$ finite type, separated for integral noeth X, S. the <u>strict transform</u> of f w.r.t. a modification $\psi : S' \to S$ is

$$\begin{tikzcd}
X' \arrow[r] \arrow[dr, bend right] & \overbrace{X \times_S S'}^{\text{often disconn'd}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
 & S' \arrow[r, "\psi"'] & S
\end{tikzcd}$$

For surjective f
(This <u>is</u> integral)

and $X' \to X$ is modification too.

schematic closure of $X_\eta \otimes_\eta \eta' \simeq X_\eta$ in $X \times_S S'$.

Rmk. If closed subscheme $\mathfrak{X}' \subset X \times_S S'$ that is <u>S'-flat</u> w/ $\mathfrak{X}'_{\eta'} = X_\eta \otimes_\eta \eta'$ $(\neq \phi)$

then $\mathfrak{X}'$ <u>is</u> the strict transform: for integral $\mathfrak{X}'$ flat over S',

$\mathfrak{X}'_{\eta'} \longrightarrow \mathfrak{X}'$ has sch. closure $\mathfrak{X}'$.

Thm 4.1. $X =$ variety $/ k =$ field $(:=$ integral, sep'td, fin. type$)$, $Z \subsetneq X$ proper closed

$\exists$ alteration
$$\begin{tikzcd}
X_1 \arrow[r, hook, "\text{open}", "j_1"'] \arrow[d, "\varphi_1"] \arrow[dr] & \overline{X_1} = \text{regular proj.} / k \\
X \arrow[dr] & \mathrm{Spec}(k_1) \arrow[d, dash] \\
 & \mathrm{Spec}(k)
\end{tikzcd}$$
s.t. for $Z_1 = \varphi_1^{-1}(Z) \subsetneq X_1$,

$Z_1 \cup (\overline{X}_1 - X_1) =$ "$\partial_{\overline{X}_1}(X_1 - Z_1)$"

is sncd in $\overline{X_1}$.

If k perfect, can arrange φ_1 is generically étale $\left(\Leftrightarrow k(X_1) | k(X) \text{ is } \underline{\text{sep'ble}} \text{ finite} \right)$
and geom. integral

Rmk By construction, X_1 is gen. smooth over finite ext'n $k_1 | k$.

Even if k alg. closed in $k(X)$, we cannot rule out $k_1 \neq k$.

So X_1 may not be geom. integral over k. (even if X is geom. integral)

$\underline{(4.2-4.10)}$: Induct on $d = \dim(X) \geq 0$.

If $d=0$, then $X = \mathrm{Spec}(k_1)$ for finite $k_1|k$ and $Z = \emptyset$, so done, $\overline{X_1} = X_1 = X$.

We'll grant full result in dim $d > 0$ over $\bar{k}$.

Aim to deduce in dim d over k.

Pick

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "\pi"', "\text{integral surj.}"] & (X_{\bar{k}})_{\mathrm{red}} \\ X & \end{tikzcd}$$

an irred. comp. (w/ reduced), so variety $/\bar{k}$ of dim d.

($X_{\bar{k}} \to X$ flat, so all generic pts of $X_{\bar{k}}$ lie over $\eta \in X$)

Let $Z' = \pi^{-1}(Z) \subsetneq X'$. Apply full result $/\bar{k}$ to $Z' \subsetneq X'$.

$\boxed{\text{not relative}}$

$$\begin{tikzcd} X_1' \arrow[r, hook, "\text{open}", "j_1'"'] \arrow[d, "\begin{smallmatrix}\text{gen.}\\ \text{étale} = \varphi_1' \\ \text{alteration}\end{smallmatrix}"'] & \overline{X_1'} = \text{smooth proj. var.}/\bar{k} \\ X' & \end{tikzcd}$$

$\Updownarrow$ regular

, for $Z_1' = \varphi_1'^{-1}(Z') \subsetneq X_1'$

we have $\underbrace{Z_1' \cup (\overline{X_1'} - X_1')}_{\text{"}\partial(\overline{X_1'} - Z_1')\text{"}} \subset \overline{X_1'}$ is $\underset{\sim}{\text{sncd}}$

union of smooth irred comps w/ smooth $\cap$'s of expected dim.

$\bar{k} = \varinjlim_{\alpha} k_\alpha$ for $k \subset \underset{\text{finite}}{k_\alpha} \subset \bar{k}$

Use general $\varinjlim$ formalism in [EGA IV$_3$, §8, §9, §11, ...]

(any "finitely presented alg. geom. situation" over $A = \varinjlim_\alpha A_\alpha$ descend to some A_{α_0} along w/ all the properties) to get above setup over $\bar{k}$ to arise over some finite ext'n $K|k$:

① $(X_{\bar{k}})_{\mathrm{red}} = (X_K)_{\mathrm{red}} \otimes_K \bar{k}$

② reduced irred. comp. $X'' \subset (X_K)_{\mathrm{red}}$ s.t. $X'' \otimes_K \bar{k} = X' \subset (X_{\bar{k}})_{\mathrm{red}}$, $Z'' \subset X''$ preimage of Z w/ $Z''_{\bar{k}} = Z'$.

③
$$\begin{tikzcd} X_1'' \arrow[r, hook, "\text{open}", "j_1''"'] \arrow[d, "\varphi_1''", "\substack{\text{gen. étale}\\ \text{alterations}}"'] & \overline{X_1}'' \\ X'' & \end{tikzcd}$$

$\overline{X_1}''$ = K-smooth (⇒ reg) proj. var / K

(⇒ proj. /k)

s.t. $\partial_{\overline{X_1}''}(X_1'' - Z_1'') =$ sncd in $\overline{X_1}''$

Aim

this works as φ_1

$$\begin{tikzcd} X_1'' \arrow[r] \arrow[d] & \overline{X_1}'' \\ X'' \subset (X_K)_{red} \arrow[d] \arrow[dr, phantom, ""] & \\ X & \end{tikzcd}$$

X'' has dense open also open in $(X_K)_{red}$

gen. étale → ↓ } finite flat over dense open

when $K|k$ is sep'ble (eg. k perfect)

X = integral

$\boxed{\text{Now } k = \bar{k}}$ Chow's Lemma ([EGA II, §5]) ∃ q-proj. var. $X' \xrightarrow{\pi} X$ modification

Pass to $(X', \pi^{-1}(Z))$, so $\boxed{X = \text{q-proj}}$

Pick $X \overset{\text{open}}{\hookrightarrow} \overline{X}$ = proj. var.

Pass to $(\overline{X}, Z \cup (\overline{X} - X))$ to arrange $X = \underline{\text{proj}}$.

$$\begin{tikzcd} X' \arrow[r, hook, "\text{open}"] \arrow[d] & \overline{X}' \arrow[d, "\text{reg. alt.}"] \\ X \arrow[r, hook] & \overline{X} \end{tikzcd}$$

Pass to $X' = Bl_Z(X) \underset{\text{mod.}}{\twoheadrightarrow} X$ (if $Z \neq \phi$) and $Z' = \pi^{-1}(Z)$

so $|Z'| = |\text{Cartier}|$.

Last : pass to $\widetilde{X} \xrightarrow{\pi} X$ normalization and $\widetilde{Z} = \pi^{-1}(Z)$ so $\boxed{X = \text{normal}}$

Settles $d=1$: $Z = \underbrace{\{\text{finite}\} \subset X}_{\text{sncd}}$ = smooth proj. curve

Now $\boxed{d \geq 2}$.

Lecture 6   Curve fibration I.

Controlling the finite fiber locus and enlarging Z.

We have reduced the task to constructing a generically étale alteration $\varphi: X' \to X$ of a normal proj'. var. X of dim $d \geq 2$ over $k = \bar{k}$. Although it is generally hopeless to construct φ to be étale or even flat over a specific pt $x \in X(k)$, we can at least exert some mild control over the locus of $x \in X$ w/ finite fibers: its complement has codim ≥ 2.

Prop. Any altration $\varphi: X' \to X$ between proj. vars over $k = \bar{k}$ w/ normal X is finite over a dense open $U \subset X$ w/ complement of codim ≥ 2.

Pf Let $R = \mathcal{O}_{X,x}$ for a point $x \in X$ w/ codim 1, i.e. $\dim R = 1$. By normality, any such R is a discrete val. ring. We will show that the proper map $X'_R = X \times_X \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ obtained by localization to R is a finite morphism. Once this is shown, by "spreading out" principles, we obtain an open nbhd $V \subset X$ around x s.t. $\varphi^{-1}(V) \to V$ is finite. The non-empty union U of all such V's is then an open subset of X for which $\varphi^{-1}(U) \to U$ is finite (as may be verified over each member V of an open cover of U), and the proper closed set $X - U$ contains no points in X w/ a 1-dim'l local ring on X, so $X - U$ has codim ≥ 2 at all of its points.

Noting that $X'_R \to \mathrm{Spec}(R)$ has generic fiber $\eta' \to \eta$ that is finite, we are reduced to proving finiteness of any proper map $f: Y \to \mathrm{Spec}(R)$ between integral schemes for which R is a DVR w/ module-finite integral closure in all finite – not necessarily separable! – ext'n of its fraction field (e.g. $\mathcal{O}_{X,x}$ as above, by spreading-out from module-finiteness for integral closures w/ affine varieties over a field), and the generic fiber Y_η is η-finite. Finiteness for f is the same as quasi-finiteness for f since f is proper. So our task concerns only the closed fiber Y_0 of Y: we just need to check that it has only finitely many points.

Let R' be the R-finite normalization of R in the finite ext'n $k(\eta')$ of the fraction field $k(\eta)$ of R. In particular, R' is a semi-local Dedekind domain.

By the valuative criterion for properness, adapted to the case of Dedekind domains (rather than just DVRs), the given map $\eta' \to Y$ over $\operatorname{Spec}(R)$ extends uniquely to an R-map $h: \operatorname{Spec}(R') \to Y$. This latter map between integral schemes is an isom. between generic fibers over R, so it is dominant and it is also proper because it is a map between proper R-schemes. $\Rightarrow$ surjective. Hence, Y_0 is the image of the special fiber of $\operatorname{Spec}(R')$ over $\operatorname{Spec}(R)$. The R-finiteness of R' then ensures that Y_0 consists of only finitely many pts, as desired.

Why enlarging Z is harmless.

Suppose that $Z \subset \tilde{Z} \subsetneq X$ is a containment of proper closed subsets w/ Z the supp. of a Cartier divisor in X and there exists a regular alteration $\varphi: X' \to X$ s.t. $\varphi^{-1}(\tilde{Z})_{\text{red}} \subset X'$ is an sncd. Since $\varphi^{-1}(\tilde{Z})_{\text{red}}$ is then a union of regular regular closed subschemes w/ pure codim. 1 whose successive intersections are regular w/ the "expected" dimension, any union among its reduced irred. components is also of that type and hence is also an sncd. Thus, for dimension reasons we get $\varphi^{-1}(Z)_{\mathrm{red}}$ is an sncd. This freedom to increase Z later in the argument will be used a lot (w/o comment).

* * *

A model case of curve fibration via blow-up

Let $X = \mathbb{P}^d$. Pick $p \in \mathbb{P}^d(k)$. Taking a hyperplane $H = \mathbb{P}^{d-1}$ in $\mathbb{P}^d$ not containing p, each line ℓ in $\mathbb{P}^d$ passing through p corresponds to exactly one point in H ($\ell \cap H$). This makes H represent the functor of "lines in $\mathbb{P}^d$ passing through p".

There is also the classical description $Bl_p(\mathbb{P}^d) = \{(q,\ell) \in \mathbb{P}^d \times \mathbb{P}^{d-1} : q \in \ell\}$ (as a closed subscheme of $\mathbb{P}^d \times \mathbb{P}^{d-1}$) via a *unique* isom. over $\mathbb{P}^d$, identifying the blow-up map w/ the first projection $\varphi : Bl_p(\mathbb{P}^d) \to \mathbb{P}^d$ that is an isom. over $\mathbb{P}^d - \{p\}$.

The second proj. $\pi : Bl_p(\mathbb{P}^d) \to \mathbb{P}^{d-1}$ is a Zariski $\mathbb{P}^1$-bundle, as one can check over open affine spaces, most concretely described pointwise by the observation

$$\pi^{-1}(\ell) = \ell \times \{\ell\} \subset \mathbb{P}^d \times \{\ell\} \quad \text{as a scheme-theoretic fiber.}$$

Now consider a reduced (possibly reducible) proper closed $Z \subsetneq \mathbb{P}^d$ and pick $p \notin Z$, so φ restricts to an isom. $\varphi^{-1}(Z) \xrightarrow{\sim} Z$. Let π_Z be the restriction of π to $\varphi^{-1}(Z)$ (which is henceforth identified w/ Z). Note $\pi_Z^{-1}(\{\ell\}) = (\ell \cap Z) \times \{\ell\}$ as schemes, and this is k-finite since $p \in \ell$ but $p \notin Z$. This shows that π_Z is proper & quasi-finite hence finite.

We expect that if p has been chosen at random, then for "most" lines ℓ in $\mathbb{P}^d$ containing p, the finite scheme $\ell \cap Z$ should be geometrically reduced (and thus étale, as it is k-finite). To show this, we will use Bertini theorems. More broadly,

HOPE. For a "random" choice of p, the finite $\pi_Z : Z \longrightarrow \pi(Z)$ is gen. étale, and even birat'l if $\dim Z < d-1$.

The above ideas will be upgraded to fiber a general X of dim. d in curves over $\mathbb{P}^{d-1}$.

Lemma (4.11) Consider X a proj. var. of dim $d \geq 2$ over an alg closed field k and $Z \subsetneq X$ the support of a Cartier divisor. There exists a finite subset $S \subset X^{sm}(k)$ outside Z and $f : X' := Bl_S(X) \longrightarrow \mathbb{P}^{d-1}_k$ over k s.t.

1) all fibers of f are pure dim. 1 and the open relative smooth locus $Sm(X'/\mathbb{P}^{d-1}) \subset X'$ of f is fiberwise dense;

2) for the structure map $\varphi : X' \longrightarrow X$ that is an isom. over $X - S \supset Z$, the map
$$Z \xleftarrow{\sim} \varphi^{-1}(Z) \xrightarrow{f} \mathbb{P}^{d-1}_k$$
is generically étale & finite.

Also, if X is normal then we can arrange there to exist a dense open $U \subset \mathbb{P}^{d-1}_k$ s.t. $f^{-1}(U) \longrightarrow U$ has smooth geom. conn'd fibers.

To prove this we will find a suitable finite and generically étale $X \xrightarrow{h} \mathbb{P}^d$ and take $S = h^{-1}(p)$ for a "good" $p \in \mathbb{P}^d_k$ well-positioned outside $h(Z) \subset \mathbb{P}^d_k$.

Idea of the proof: We shall use a projective Noether normalization expressing X as finite over a projective d-space. The finite map will be built as a composition of successive projections away from pts into successive hyperplanes. To get started, since X is proj. we can pick a closed immersion of it into $\mathbb{P}^N_k$ for some N. If $N = d$ then $X = \mathbb{P}^d_k$ and so we can use the "warm-up" example.

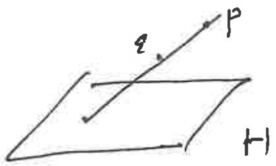
Suppose $N > d$. Pick a k-point $p \in \mathbb{P}^N_k - X$. As in the "warm-up" example, identify $\mathbb{P}^{N-1}_k$ as the scheme of lines in $\mathbb{P}^N_k$ passing through p. Consider the k-map $\pi_p : \mathbb{P}^N_k - \{p\} \to \mathbb{P}^{N-1}_k$ taking q to the unique line through q & p. Let $\pi_{p,X}$ be the restriction of π_p to $X \subset \mathbb{P}^N_k - \{p\}$. Note $\pi_p^{-1}(\{\ell\}) = \ell - \{p\}$ as schemes, so $\pi_{p,X}^{-1}(\{\ell\}) = (\ell - \{p\}) \cap X$
$= \ell \cap X$.

Note that $\pi_{p,X} : X \longrightarrow \mathbb{P}^{N-1}$ is quasi-finite (as $\ell \cap X \subsetneq \ell$ since $p \notin X$) and proper, hence finite. If $d = N-1$, and (as we hope to happen for a random p relative to X), "most" lines ℓ containing p have $\ell \cap X$ geom. reduced (and hence étale since it is of dim. 0), the finite map $\pi_{p,X}$ is generically étale onto $\mathbb{P}^{N-1}_k$. If instead $d \le N-2$, then we hope for "most" p that $\pi_{p,X} : X \longrightarrow \pi(X) \subset \mathbb{P}^{N-1}_k$ is birat'l onto its image, as is $Z \longrightarrow \pi(Z)$ (because Z may be reducible, so birat'lity requires being attentive to different irred. components not landing on top of each other under π_p). When this is the case then by using such a p we can reduce to $\pi(Z) \subset \pi(X) \subset \mathbb{P}^{N-1}_k$ and proceed by downward induction on N.

Lecture 7. Curve Fibration II

Coord. free version of $\pi_p : \mathbb{P}^N - \{p\} \longrightarrow H \simeq \mathbb{P}^{N-1}$

$$\begin{array}{c} H \\ \| \\ \text{“}\{\ell \ni p\}\text{”} \end{array}$$



$\pi_p(q) = \overline{pq} \cap H$

(Explicit: $p = [1:0:\cdots:0]$, $H = \{x_0 = 0\}$)

$\dim V = N+1 \geq 2$, $W \subset V$ hyperplane, so V/W is 1-dim'l.

$\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V)) \longleftarrow \mathbb{P}(V/W) = pt = \{p\}$ representing $\lambda_0 : V \twoheadrightarrow k$ killing W

$= $ scheme representing $\left[R \longmapsto \{V_R \twoheadrightarrow \mathcal{L}\}/\simeq \right]$

$$\begin{array}{ccc} \mathbb{P}(V) - \mathbb{P}(V\backslash W) & \longrightarrow & \mathbb{P}(W) \\ \lambda & \longmapsto & \lambda|_W \end{array}$$

Prop 2.1.1. Let k be a field, $X \subset \mathbb{P}^N_k$ closed gen. smooth subscheme of pure dim. $d < N$.

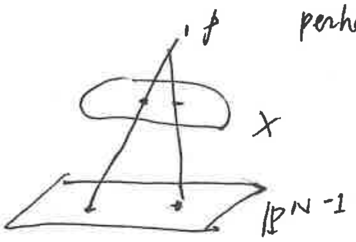
$\exists$ dense open $U \subset \mathbb{P}^N_k - X$ s.t. $\forall$ finite separable $k'|k$ and $p \in U(k')$ (so $p \notin X_{k'}$)

the finite map $\pi_p : X_{k'} \longrightarrow \mathbb{P}^{N-1}_{k'}$ satisfies

(Rem: Apply to original irred. X and perhaps reducible Z)

(α) birat'l onto image if $d \leq N-2$

(β) gen. étale onto $\mathbb{P}^{N-1}_{k'}$ if $d = N-1$.



Pf. Grant $\exists$ such U' over k_s: let's build U over k.

By coefficient-chasing, $\exists$ finite Galois ext'n $K|k$ inside k_s and open $\mathcal{U} \subset \mathbb{P}^N_k$ s.t. $\mathcal{U} \otimes_k k_s = U'$.

Then $\mathcal{U}$ solves problem over K (exercise: $\mathcal{U}(K') \subset \mathcal{U}'(k_s)$ for $K \subset K' \subset k_s$)

Let $\bigwedge_{\gamma \in \mathrm{Gal}(K|k)} \gamma^*(\mathcal{U}) \subset \mathbb{P}^N_K$ is Gal$(K|k)$-stable, non-empty open.

so it is $\mathcal{U}_K$ for some open $\mathcal{U} \subset \mathbb{P}^N_k$ which works.

Now $\underline{k = k_s}$, so any smooth k-scheme Y has $Y(k) \subset Y$ Zariski-dense.

Rmk. For (β) it suffices to check on dense open in $X^{sm} \subset X$, hence to treat all irred. comp. $X_i \subset X$ separately.

For (α), must treat X as a single entity. Too global to permit such reduction step.
(minor glitch in [de J]).

for $d = N-1$

Idea. For $V = \coprod V_i$ to be X^{sm}, then for line $\ell \subset \mathbb{P}^N_k$ through $p \notin X$, if ℓ misses $X - X^{sm} =$ (codim ≥ 2 in $\mathbb{P}^N$), then $\ell \cap X = \ell \cap V$ as schemes.

We'd like $\ell \cap V$ to be etale and meet $\underline{\text{every}}$ V_i for "most" ℓ through "most" $p \notin X$.

To do this, we need Bertini Thms (Jouanolou's book)

Setup. F field, $Z \subset \mathbb{P}^N_F$ is loc. closed subscheme of pure dim d.

Let $G = Gr(r, N)$ be Grassmannian of codim-r linear subspaces of $\mathbb{P}^N_F$ w/ fixed $1 \leq r \leq d$.

Eg. $d = N-1$, $r = N-1$, $G = \{\ell \subset \mathbb{P}^N\}$.

$$\begin{tikzcd} V \cap Z_G \arrow[r, phantom, "\subset"] & Z_G \\ V \arrow[u, phantom, "\cap"] \arrow[dr] \arrow[r, phantom, "\subset"] & \mathbb{P}^N \times G \arrow[u, phantom, "\cap"] \arrow[d] \\ & G \end{tikzcd}$$

universal "codim r" linear subspace.

$$\begin{tikzcd} V \cap Z_G \arrow[rr, hook] \arrow[dr] & & Z_G \arrow[dl] \\ (\dagger) & G & \end{tikzcd}$$

For field $F'|F$, $L \in G(F')$ a codim. r linear subspace $L \subset \mathbb{P}^N_F$

the diagram $(\dagger)$ pulls back to

$$\begin{tikzcd} \underbrace{L \cap Z_{F'}}_{\text{inside } \mathbb{P}^N_{F'}.} \arrow[r, phantom, "\subset"] \arrow[dr] & Z_{F'} \arrow[d] \\ & \mathrm{Spec}(F') \end{tikzcd}$$

Part I of [Jou] gives

① [Thm 6.10(2), Cor 6.11(1b)] If Z is smooth, $\exists$ dense open $\Omega_1 \subset G$ s.t. $Z_{F'} \cap L$ is smooth * of pure dim r * $\forall$ $F'|F$, $L \in \Omega_1(F')$.

use

② [Thm 6.10(2)] If Z is geom. reduced $/F$, then ... Ω_2 ... * *

③ [Thm 6.10(3)] If Z is geom. irred $/F$ and $r \leq d-1$, then ... Ω_3 ... * *

Rmk (i) [Thm 6.10(1)] If $r > d$, then $\exists$ dense open $\Omega \subset G$ so $Z_{F'} \cap L = \emptyset$ for $L \in \Omega(F')$ for $F'|F$.

(ii) Pf of ③ is "global", unlike ①, ②.

Let's return to Prop 2.11 over $F = k = k_s$, $d = N-1$.

Apply ② to conn'd (= irred.) comp. of X^{sm} w/ $q = d-1$: for "most" lines $\ell \subset \mathbb{P}^N_k$

and Rem (i) to $X - X^{sm}$: $\ell \cap X = \ell \cap X^{sm}$ = geom. reduced, hence étale

(∵ pure dim. 0)

Pick such ℓ_0, and $p \in \ell_0 - (X \cap \ell_0)$, Look at $\pi_p : X \longrightarrow \mathbb{P}^{N-1}_k$:

$\pi_p^{-1}(\{\ell\}) = \ell_0 \cap X$ étale, meets every irred. comp. of X^{sm}!

$x \longmapsto \{\ell_0\}$

$$\left(\Omega^1_{X/\mathbb{P}^{N-1}}\right)(x) = \underbrace{\left(\Omega^1_{(\ell_0 \cap X)/k}\right)}_{=0}(x) = 0$$

étale

So $\Omega^1_{X/\mathbb{P}^{N-1}}$ is zero near such x, so over dense open in X!

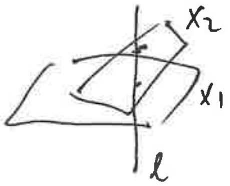
Lecture 8. Bertini & Birat'l projections

$X \subset \mathbb{P}^N$ of pure dim $d = N-1$: found dense open $\Omega \subset G$ = Grass of lines in $\mathbb{P}^N$ $(Gr(N-1, N))$

so for $\ell \in \Omega(k)$, and $p \in \ell - (\ell \cap X)$,

$$\pi_p : X \xrightarrow{\text{finite}} \mathbb{P}^{N-1}$$ is gen. étale (a priori gen. flat) (miracle flatness) on X^{sm}

~~flat away from dim ≤ N-2~~



So étaleness is property of fibers.

Why ~~do~~ such $p \in \mathbb{P}^N - X$ (indep. of mention of ℓ) sweep out at least some dense open?

Answer: incidence correspondence

$G = Gr(N-1, N) = Gr$ of lines in $\mathbb{P}^N$

$\mathbb{P}^N \times G \supset \{(p,\ell) : p \in \ell\}$ = "universal line"

$$\begin{tikzcd} & \{(p,\ell) : p \in \ell\} \arrow[dl, "pr_1"'] \arrow[d, "pr_2"] \\ \mathbb{P}^N & G \supset \Omega \end{tikzcd}$$

$\mathbb{P}^{N-1}$- bundle $\longrightarrow pr_1$

pr_2 (Zariski $\mathbb{P}^1$- bundle) $\Rightarrow$ (integral!)

Ω dense open

$$\underbrace{pr_1(\underbrace{pr_2^{-1}(\Omega)}_{\text{open}})}_{\Downarrow \text{ open in } \mathbb{P}^N} \cap (\mathbb{P}^N - X) \text{ is open.}$$

Now consider $\boxed{d \leq N-2}$. We'll use higher codim. Bertini : find "good"

$W \subset \mathbb{P}^N$ of $\underline{\text{codim } d}$ s.t. $W \cap X$ is "nice" and search for lines in W

$\dim W = N-d \geq 2$

By Bertini for $\underbrace{X^{sm}}_{\text{pure dim } d} = \coprod V_i$, $V_i = X_i^{sm} - \left(\bigcup_{j\neq i} (X_i \cap X_j)\right)$.

$G = Gr(d, N)$. $\exists\, W \in G(k)$ s.t. $W \cap X = W \cap X^{sm}$ and each $W \cap X_i = W \cap V_i$ is $\underline{\text{nonempty étale}}$.

($\Rightarrow k$-pts)

Pick $q_j \in W \cap V_j$ and line $\ell_j \subset W$ through q_j

$\underline{\text{not}}$ equal to $\overline{q_j p}$ for $p \in W \cap X = \{\text{finite}\}$

so $\ell_j \cap X = \{q_j\}$ as schemes.

Pick $p_j \in \ell_j - \underbrace{(\ell_j \cap X)}_{\text{finite } (\subset W \cap X)}$

Look at
$$\begin{tikzcd} X \arrow[r, "\text{finite}"] & \mathbb{P}^{N-1} & \arrow[l] \{\ell \ni p_j\} \\ Y_j \arrow[u, phantom, "\cup"] \arrow[ru] & & \end{tikzcd}$$
π_{p_j} :

Claim. $\underbrace{\pi_{p_j} : Y_j \longrightarrow \mathbb{P}^{N-1}}^{\text{does not "use" } \ell_j}$ is birat'l onto image.

Pf $\pi_{p_j}^{-1}(\{\ell_j\}) = \ell_j \cap X_j = \{q_j\}$ as scheme

Since $\ell_j \cap X = \{q_j\}$

$$\begin{tikzcd} X_j \arrow[d, "\text{finite}"] \\ \pi_{p_j}(X_j) \end{tikzcd}$$
w/ degree 1 fiber over $\{\ell_j\} = k\text{-pt}$

Want to deduce <u>generic fiber</u> has degree 1.

Lemma For finite surj. map of noeth schemes $\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd}$

and $y \in \overline{\{\eta\}}$, then $\deg_\eta(X_\eta) \leq \deg_y(X_y)$.

Pf WLOG $y \neq \eta$. (Better: use Nakayama)

By Krull-Akizuki: $\exists$ DVR R and $\mathrm{Spec}(R) \xrightarrow{\alpha} Y$

$\bullet \longmapsto y$

$O \longmapsto \eta$.

$\rightsquigarrow$ massive ext'n on $k(y)$.

base change α so $Y = \mathrm{Spec}(R)$.

$X = \mathrm{Spec}(A)$ finite R-alg A, so $A \overset{R\text{-mod}}{\simeq} R^m \oplus (\text{torsion})$

so $\dim_\eta A_\eta = m$. $A/\mathfrak{m}$ has $\dim \geq m$.

<u>Upshot</u>: $\forall j$, have $p_j \in \mathbb{P}^N - X$ so $\pi_{p_j}: X_j \to \mathbb{P}^{N-1} = \{\ell \ni p_j\}$ is birat'l onto image.

<u>Exer.</u> use incidence relation idea to show $\exists$ dense open $U_j \subset \mathbb{P}^N - X$ consisting of such p_j's. For $U = \bigcap_j U_j$ = dense open in $\mathbb{P}^N - X$, so for $p \in U(k)$.

$\pi_p: X \longrightarrow \mathbb{P}^{N-1} = \{\ell \ni p\}$ takes each X_j birat'l onto image.

Need to ensure $\pi_p(X_i) \neq \pi_p(X_j)$ for $i \neq j$ ($\Rightarrow \pi_p: X \to \pi_p(X)$ birat'l)

For each i, seek $\ell \subset \mathbb{P}^N$ through p w/ ℓ meets X_i but <u>NOT</u> $\underbrace{X_j}_{\text{codim} \geq 2 \text{ in } \mathbb{P}^N}$.

<u>Summarize</u> (for Prop 4.11) $\underset{\substack{\text{pure} \\ \dim \\ d-1 \\ |\text{Cartier}|}}{Z} \subsetneq \underset{\substack{\text{irred.} \\ \dim d < N}}{X} \subset \mathbb{P}^N_k$

Run 2.11 $N-d$ times to get

$$\begin{tikzcd} X \arrow[r, "\text{finite}", "\text{gen. étale}"'] & \mathbb{P}^d_k \\ Z \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\text{birat'l}"'] & \pi(Z) \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

$\pi:$

Seek $\xi \in \mathbb{P}^d_k$ "good" w.r.t. $\pi(Z)$ (in sense of toy case) $\left(\pi_\xi : \pi(Z) \xrightarrow[\text{gen. étale}]{\text{finite}} \mathbb{P}^{d-1}\right)$ and take $S = \pi^{-1}(\xi)$

Lecture 9. Curve Fibration III

Recap Over $k = \bar{k}$,

$$\begin{tikzcd} \mathbb{P}^N \supset X \arrow[r, "\pi", "\text{finite, gen. étale}"'] & \mathbb{P}^d \\ |\text{Cartier}| = Z \arrow[u, phantom, "\cup"] \arrow[r, "\text{birat'l}"'] & \pi(Z) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

] composition of $N-d$ "generic" point projections.

Let $\Omega \subset \mathbb{P}^d$ be a dense open s.t. $\pi^{-1}(\Omega) \xrightarrow{\text{finite}} \Omega$ is étale. $(\Rightarrow \pi^{-1}(\Omega) \subset X^{sm})$

Pick $\xi \in \Omega$ in "good position" w.r.t. $\pi(Z) \underset{f}{\subsetneq} \mathbb{P}^d$:

$$\begin{tikzcd} pr_\xi : \mathbb{P}^d - \{\xi\} \arrow[r] & \mathbb{P}^{d-1} = \{\ell \ni \xi\} \\ \pi(Z) \arrow[u, phantom, "\cup"] \arrow[ur, "\text{finite, gen. étale}"'] & \end{tikzcd}$$

Let $S = \pi^{-1}(\xi) \subset X^{sm}$ be a nonempty finite set; consider

$$\begin{tikzcd} & X' = \{(x,\ell) \in X \times \mathbb{P}^{n-1} : \pi(x) \in \ell\} \arrow[dl, "f = pr_2"'] \arrow[d, "\varphi = pr_1"] \arrow[r] \arrow[dr, phantom, "\lrcorner"] & Bl_\xi(\mathbb{P}^d) \arrow[d] \\ \mathbb{P}^{d-1} & X \arrow[r, "\pi", "\text{étale over } \Omega"'] & \mathbb{P}^d \end{tikzcd}$$

$\left(= \{(p,\ell) \in \mathbb{P}^d \times \overbrace{\mathbb{P}^{d-1}}^{\{\ell \ni \xi\}} : p \in \ell\}\right)$

Observe π is <u>flat</u> over $\Omega \ni \xi$, so

$$\begin{tikzcd} X' \arrow[rr, "\exists!", "\simeq"'] \arrow[dr] & & Bl_S(X) \arrow[dl] \\ & X & \end{tikzcd}$$

(" blow-up commutes w/ flat base change)

also note φ isom over $X - S$

Want to study

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"'] & Bl_\xi(\mathbb{P}^d) \arrow[dl, "pr_2'"] \\ \mathbb{P}^{d-1} & \end{tikzcd}$$

$f^{-1}(\{\ell\}) \simeq \pi^{-1}(\ell)$ pure dim 1.

$$\begin{tikzcd} \pi : X \arrow[r] & \mathbb{P}^d \\ \cup & \cup \\ \pi^{-1}(\Omega) \arrow[r, "\text{étale}"] & \Omega \ni \xi \\ & \cup \\ \underbrace{\pi^{-1}(\Omega)\cap\pi^{-1}(\ell)}_{\text{Smooth}} \arrow[r, "\text{finite}"', "\text{étale}"' near end] & \underbrace{\ell\cap\Omega}_{\text{dense open in } \ell} \neq \phi \ (\text{contains } \xi) \end{tikzcd}$$

and omits only
finite subset of pure curve $\pi^{-1}(\ell)$

So $\pi^{-1}(\ell)$ is gen. smooth
$$\begin{array}{c} \pi^{-1}(\ell) \\ \shortparallel \\ f^{-1}(\{\ell\}) \end{array}$$

<u>Q</u>. How does $f^{-1}(\{\ell\})^{sm}$ relate to $sm(X'/\mathbb{P}^{d-1})$?

These are <u>same</u> by

<u>Lemma</u>. (weak 2.8) Let $A \to B$ be a local map of CNL rings w/

① A $\boxed{\text{domain}}$ of dim δ

② $B/m_A B \simeq k[\![t_1, \cdots, t_r]\!]$ for $k = A/m_A$, w/ dim $B = \delta + r$.

Then $B \underset{A}{\simeq} A[\![T_1, \cdots, T_r]\!]$

We'll apply to $A = \hat{\mathcal{O}}_{\mathbb{P}^{d-1}, f(x')}$, $B = \hat{\mathcal{O}}_{X', x'}$ for $x' \in f^{-1}(\{\ell\})^{sm}$ a k-pt $(\{\ell\} \in \mathbb{P}^{d-1}(k))$

$\delta = d-1$, $r = 1$

So $B/m_A B = \hat{\mathcal{O}}_{f^{-1}(\{\ell\}), x'}$ $(\simeq k[\![t]\!])$

Then conclusion $\Rightarrow$ $X' \to \mathbb{P}^{d-1}_k$ is smooth at x'.

Pf lift $t_1, \dots, t_r$ to $T_1, \dots, T_r \in m_B$, so have (by completeness), a!

local A-alg. map $\alpha: A[\![X_1, \dots, X_r]\!] \longrightarrow B$

$X_i \longmapsto T_i$

α mod m_A is an isom. by hypothesis.

so α mod m_A^2 is surjective by

"$m_A \cdot (B/m_A B)$"

$$\begin{tikzcd} & m_A B/m_A^2 B \arrow[d, phantom, "\cap"] \\ A/m_A^2[\![X_1, \dots X_r]\!] \arrow[r] \arrow[d] & B/m_A^2 B \arrow[d] \\ k[\![X_1, \dots X_r]\!] \arrow[r] & B/m_A B \end{tikzcd}$$

Repeat by successive approx. over A.

By completeness of B, A w.r.t. m_A-adic top.

we get surjectivity of $\alpha: \underbrace{A[\![X_1, \dots X_r]\!]}_{\dim \delta + r} \twoheadrightarrow \underset{\dim \delta + r}{B}$.

$\boxed{\text{domain}}$

If $\exists\, g \neq 0$ in $\ker(\alpha)$, then $B \simeq A[\![X_1, \dots X_r]\!]/\ker(\alpha)$ would have $\dim < \delta + r$. contradiction.

$\therefore \ker(\alpha) = 0$.

Remaining. Assume X normal, want $\exists$ dense open $U \subset \mathbb{P}^{d-1}$ s.t.

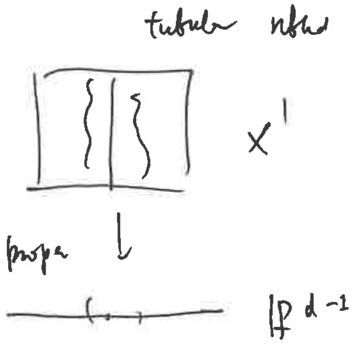
$f^{-1}(U) \longrightarrow U$ is smooth ($\Leftrightarrow$ all fibers of X' over U are smooth)

and $\boxed{\text{geom. conn'd}}$ —— (next time show f is own Stein factorization, so geom. conn'd fibers : EGA III_1, 4.3.3)

For this we'll need a bit more "genericity" on ξ.

$$\begin{tikzcd} X' \arrow[d, "f"'] \arrow[d, phantom, shift left=4ex, "\text{proper}"] \\ \mathbb{P}^d \end{tikzcd}$$

If $\exists$ one smooth fiber X'_y, then open $V = \mathrm{Sm}(X'/\mathbb{P}^{d-1}) \supset X'_y$



$\Rightarrow U := \mathbb{P}^{d-1} - f(X'-V)$ works for smoothness.

Need to find some $\ell \ni \xi$ s.t. $\pi^{-1}(\ell) \subset X$ is smooth.

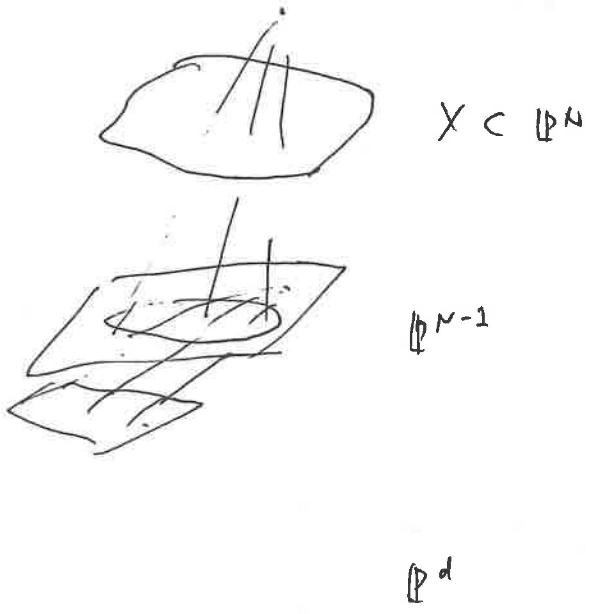
Note X normal $\Rightarrow$ $X - X^{sm} \subset X$ has dim $\leq d-2$

Want to interpret $\pi^{-1}(\ell) = X \cap L$ for $L \subset \mathbb{P}^d$ linear through ξ of codim $d-1$

(generic such L <u>miss</u> $X - X^{sm}$)

So run Bertini for $X^{sm} \subset \mathbb{P}^N$

<u>Difficulty</u>:



Make L as span of $N-d$ pts in original $\mathbb{P}^N$.

Lecture 10 Stein factorization and 3-pt Lemma

<u>Recap</u> The fibers of

$$\begin{tikzcd} X' = Bl_\xi(X) \arrow[d, "f"] \\ \mathbb{P}^{d-1} = (\{\ell \ni \xi\}) \end{tikzcd}$$

over k-pts are $X \cap L$ for certain $L \in Gr(d-1, N)(k)$

Varying all choices underlying constr. of G, the L's attained sweep out at least a dense open in $G = Gr(d-1, N)$ (see App B) Had X normal proj. var. of dim $d \geq 2$ in $\mathbb{P}^N_k$, so $X - X^{sm}$ has dim $\leq d-2$

$\therefore$ for "most" L, have $L \cap (X - X^{sm}) = \emptyset$, so $L \cap X = L \cap X^{sm} =$ smooth + irred

Upshot: if choose ξ (and pt proj's preceding it) generic enough, (Bertini for $X^{sm} \to \mathbb{P}^N$)

then $\exists\ y = \{\ell\} \in \mathbb{P}^{d-1}(k)$ s.t. $X'_y = \pi^{-1}(\ell)$ is smooth + conn'd

By "weak Z-8" from last time, $X' \xrightarrow{f} Y = \mathbb{P}^{d-1}$ is smooth at all pts of X'_y, so by properness get $sm(X'/Y) \supset f^{-1}(U)$ for some dense open $U \subset Y$.

$$\begin{tikzcd} X' \arrow[r, phantom, "\supset"] \arrow[d, "f"'] & f^{-1}(U) \arrow[d, "\text{smooth}"] \\ Y = \mathbb{P}^{d-1} \arrow[r, phantom, "\supset"] & U \end{tikzcd}$$

(and has conn'd fiber X'_y for some (even "most") $y \in U(k)$)

We'll show $f: X' \to Y$ is own Stein factorization $(\mathcal{O}_Y \simeq f_* \mathcal{O}_{X'})$, so all fibers X'_y are geom. conn'd $(y \in Y)$

$$\begin{tikzcd} X' \arrow[dr] \arrow[ddr, bend right, "f"'] & \\ & \underline{\mathrm{Spec}}_Y(f_* \mathcal{O}_{X'}) \;]\ \text{finite!} \arrow[d, "h \text{ finite}"] \\ & Y = \mathbb{P}^{d-1} = \text{normal} \end{tikzcd}$$

$\Rightarrow h$ is isom. if birat'l.

Let's localize at $y_0 \in Y(k)$ s.t. X'_{y_0} smooth conn'd.

Let $R = \mathcal{O}_{Y, y_0} =$ normal noeth. domain, $Z = X \times_Y \mathrm{Spec}(\mathcal{O}_{Y, y_0})$, then

$$\begin{tikzcd} Z \arrow[d] \\ \mathrm{Spec}(R) \end{tikzcd}$$

proper smooth

$\left(Sm(Z/R) \supset Z_{y_0} \Rightarrow Sm(Z/R) = Z \right)$

w/ special fiber Z_0 geom. conn'd. Want $R \to \mathcal{O}(Z)$ is isom.

$\left(\Rightarrow \mathrm{Frac}(R) \xrightarrow{\sim} \mathcal{O}(Z_\eta) \right)$.

so get birat'lity of h.

__Lemma__. If $Z \to \mathrm{Spec}\, R$ is proper flat w/ R local noeth., and Z_0 is geom. conn'd and geom reduced, then $R \xrightarrow{\sim} \mathcal{O}(Z)$.

Pf Let $k = R/m$, so $k \xrightarrow{\sim} H^0(Z_0, \mathcal{O})$

Z_0 is proper, geom. conn'd, geom. reduced

$$\begin{tikzcd} H^0(Z, \mathcal{O}_Z) \arrow[r] & H^0(Z_0, \mathcal{O}_{Z_0}) \\ R \arrow[u] \arrow[r, two heads] & k \arrow[u, "\wr"'] \end{tikzcd}$$

] surjective!

i.e. $\varphi^0_m : H^0(Z, \mathcal{O}_Z)/m \twoheadrightarrow H^0(Z_0, \mathcal{O}_{Z_0}) = k$ and $\mathcal{O}_Z$ is __R-flat__.

By cohomology and base change, φ^0_m is an isom.

$\therefore R \to H^0(Z, \mathcal{O}_Z)]$ is an __isom.__ mod m. So Nakayama Lemma $\Rightarrow$

finite R-mods

$R \twoheadrightarrow H^0(Z, \mathcal{O}_Z)$

$$\begin{tikzcd} Z & \mathrm{Spec}\,(\mathcal{O}_{Z,z_0}) \arrow[l] \arrow[dl] \\ \mathrm{Spec}\,(R) \arrow[from=u] & \end{tikzcd}$$

For $z_0 \in Z_0$, so

$\underbrace{R \twoheadrightarrow \mathcal{O}(Z) \to \mathcal{O}_{Z,z_0}}_{\text{local, flat } (\because Z \text{ is } R\text{-flat}) \Rightarrow \text{f. flat} \Rightarrow \text{injective.}}$

$\therefore R \twoheadrightarrow \mathcal{O}(Z)$ has kernel 0. □

Rename $Bl_S(X)$ as X to get to

Current Situation,

"(i) – (iv)" X proj. var. / $k = \bar{k}$ of dim $d \geq 2$

$Z = |\text{Cartier}| \subsetneq X$, $Z \neq \emptyset$

(v) X normal

(vi) $\exists\, f : X \longrightarrow Y$ = proj. var. of dim $d-1$ (not assumed normal) s.t.

a) All fibers X_y are geom. conn'd of pure dim 1

→ b) $sm(X/Y) \subset X$ meets each X_y is dense open.

c) $\exists$ dense open $U \subset Y$ s.t. $f^{-1}(U) \longrightarrow U$ smooth.

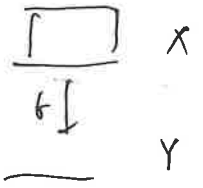
d)

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\text{finite}"', "\text{gen. étale}"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

Rmk. 1) If original k were alg closed, no "actual" alterations

2) Can replace Z w/ $Z \cup D$ for Cartier $D \subset X$ also finite gen. étale / Y

Lemma 4.13 (3-pt Lemma).



Let $X \xrightarrow{f} Y$ map of proj. var. / $k = \bar{k}$ w/ dim $X = d \geq 1$ satisfying (vi) a), b)

$\exists$ Cartier $D \subset X$ s.t. (i) $D \longrightarrow Y$ finite, gen. étale

(ii) $\forall\, y \in Y(k)$, $sm(X/Y) \cap D$ meets X_y^{sm} in ≥ 3 pts per irred. comp.

Rmk. Once Lemma proved, replace Z w/ $Z \cup D$, we'll have $sm(X/Y) \cap Z$ meets each X_y in ≥ 3 pts per irred. comp. (all in X_y^{sm}).

Start of pf. Build D in stages. ((ii) will entail noeth. induction on Y)

Made as hyperplane sections $X \cap H$ for well-chosen proj. embedding $X \hookrightarrow \mathbb{P}^N$.

("Relative Bertini")

For very ample $\mathcal{L}$ on X, consider $X \hookrightarrow \mathbb{P}(\overbrace{\Gamma(X, \mathcal{L})}^{V}) = \{\text{hyperplanes in } V\}$

k-pts: $x \longmapsto \ker(ev_x : \Gamma(X, \mathcal{L}) \twoheadrightarrow \mathcal{L}(x) \simeq k)$

$\mathbb{P}^\vee = \mathbb{P}(V^*)$ whose pts are lines in $V \supset \ell$

$= \{\text{hyperplanes in } \mathbb{P}\}$

$$\begin{tikzcd} \ell \arrow[d, mapsto] \\ \mathbb{P}(V/\ell) \subset \mathbb{P}(V) = \mathbb{P} \end{tikzcd}$$

Step 0:

Seek $H \subset \mathbb{P}^N$ s.t. $H \cap X_y$ is 0-dim'l, $\forall y \in Y(k)$ i.e. H not contain irred. comp. of X_y.

Then at least $D = X \cap H \longrightarrow Y$ is q-finite, hence finite.

Idea: Show locus of "bad" H in $\mathbb{P}^\vee$ is closed set of dim $< \dim \mathbb{P}^\vee$.

Lecture 11. Finiteness aspect of 3-pt lemma

$$\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd}$$

Lemma. Let $f: X \longrightarrow Y$ be k-map between proj. var./$k = \bar{k}$,

w/ $d = \dim X \geq 1$, and

a) All X_y $\underset{\text{geom. conn'd}}{\wedge}$ have pure dim 1 $(\Rightarrow \dim Y = d-1)$

b) $sm(X/Y) \underset{\text{open}}{\subset} X$ is fibrewise dense

∃ Cartier $D \subset X$ s.t. $D \longrightarrow Y$ finite and gen. étale s.t. $D \cap \mathrm{Sm}(X/Y)$ meets each $X_{\bar{y}}$ in ≥ 3 pts per $\underbrace{\text{irred. comp.}}_{\text{automatically in } X_{\bar{y}}^{sm}}$.

Pick $\mathcal{L}$ very ample lb on X, $\mathbb{P} = \mathbb{P}(\Gamma(X,\mathcal{L}))$ $\left[\underbrace{\mathbb{P}(V)}_{\substack{\| \\ \mathrm{Proj}(\mathrm{Sym}(V))}}(k) = \{\text{hyperplanes in } V\} \right]$

$$X \hookrightarrow \mathbb{P}$$

on k-pts, $x \mapsto \ker\left(ev_x : \Gamma(X,\mathcal{L}) \to \mathcal{L}(x) \simeq k\right)$.

Rmk. Hyperplanes in $\mathbb{P}$ are $H_s = \mathbb{P}(\Gamma(X,\mathcal{L})/ks)$ for nonzero $s \in \Gamma(X,\mathcal{L})$

and $X \cap H_s = Z(s) \subsetneq X$ since $s \neq 0$.

So this is Cartier in X, and "usually" a variety if $d \geq 2$.

Let $\mathbb{P}^\vee = \mathbb{P}(\Gamma(X,\mathcal{L})^*)$, so $\mathbb{P}^\vee(k) = \{\text{hyperplanes in } \Gamma(X,\mathcal{L})^*\}$

$$= \{(\Gamma(X,\mathcal{L})/\ell)^* = H_\ell\}$$

Seek "good" $H = H_\ell$: $X \cap H$ is q-finite $(\Rightarrow \text{finite})$ over Y)

$$\begin{tikzcd} \mathcal{Z} = \{(H,x) \in \mathbb{P}^\vee \times X : x \in H\} \arrow[r, phantom, "\underset{\text{closed}}{\subset}"] \arrow[dr, bend right, "\varphi"'] & \mathbb{P}^\vee \times X \arrow[d, "1 \times f"] \\ & \mathbb{P}^\vee \times Y \supset T \end{tikzcd}$$

Rmk. For irred. comp. C of any X_y, $C \cap H \neq \emptyset$ ("Bezout")

$$\varphi^{-1}(\underbrace{(H,y)}_{k\text{-pt}}) = \{(H,x) \in \mathbb{P}^\vee \times X : x \in H \cap X_y\} \simeq H \cap \underbrace{X_y}_{\text{pure dim } 1}$$

and "bad" H are those w/ some (H, y) having fiber dim 1.

Have closed $T = \{(H, y) : H \cap X_y \text{ is 1-dim'l}\}$ (semi-cont. of fiber dim.)

$pr_1(T) \subset \mathbb{P}^\vee$ is locus of "bad" H w.r.t. $X \xrightarrow{f} Y$. so to find (many) "good"

$H \in \mathbb{P}^\vee(k)$ suffices $\dim T \overset{?}{<} \dim \mathbb{P}^\vee$.

Let's look at $pr_2 : T \longrightarrow Y$, look at fibers over $Y(k)$

$$\dim T \leq \dim Y + \max_{y \in Y(k)} \underbrace{\dim \left(pr_2^{-1}(y)\right)}_{}.$$

controlled by images of

$\Gamma(X, \mathcal{L}) \longrightarrow \Gamma(C, \mathcal{L}|_C)$ for irred. comp. C of X_y

$$pr_2^{-1}(y) = \bigcup_{\substack{C \text{ irred.} \\ \text{comp. of } X_y}} \{H \in \mathbb{P}^\vee : H \supset i(C)\} \qquad i : C \hookrightarrow X_y \longrightarrow X \longrightarrow \mathbb{P}$$

$$H \supset i(C) \iff H \supset \underbrace{\text{span}(i(C))}_{\Lambda_C}$$

$H \supset \Lambda_C$ amounts to $1 + \dim \Lambda_C$ indep. linear cond. on H

So $\{H \in \mathbb{P}^\vee : \underline{H \supset i(C)}\} = W_C$ is a linear subspace of $\mathbb{P}^\vee$ w/ codim $1 + \dim \Lambda_C$

$\Updownarrow$

$H \supset \Lambda_C$

$$\dim T \leq \dim Y + \max_{\substack{C \text{ irred} \\ \text{comp. of} \\ \text{some } X_y}} \left(\dim \mathbb{P}^\vee - (1 + \dim \Lambda_C)\right) = \dim Y + \dim \mathbb{P}^\vee - \min_{\substack{C \text{ irred} \\ \text{comp of} \\ \text{some } X_y}} (1 + \dim \Lambda_C) \overset{?}{<} \dim \mathbb{P}^\vee$$

$\Lambda_C = \text{span}(i(C))$ in $\mathbb{P}$

So want all $1 + \dim \Lambda_C > \dim Y$ (over all C)

$$C \subset X \subset \mathbb{P}(\Gamma(X, \mathcal{L}))$$

$$\Lambda_C = \mathbb{P}(\Gamma(X,\mathcal{L})/V_C), \qquad V_C = \ker(\Gamma(X,\mathcal{L}) \to \Gamma(C, \mathcal{L}|_C)).$$

$$(C \subset H_s \iff s|_C = 0 \text{ in } \Gamma(C, \mathcal{L}|_C))$$

$$1 + \dim \Lambda_C = \dim\Big(\underbrace{\Gamma(X,\mathcal{L})/V_C}_{}\Big)$$

$$= \mathrm{im}\,(\Gamma(X,\mathcal{L}) \to \Gamma(C, \mathcal{L}|_C))$$

Want for <u>all</u> irred comp. C of all X_y have $\mathrm{im}\,(\Gamma(X,\mathcal{L}) \to \Gamma(C, \mathcal{L}|_C))$ has dim $\geq 1 + \dim Y$.

<u>Lemma</u>. For $Y \hookrightarrow \mathbb{P}^M_k$ and $\mathcal{L} = \mathcal{O}(1)$ and (reduced) irred. closed $C \subset X$, the image of $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(C, \mathcal{L}^{\otimes n}|_C)$ has dim $\geq n+1$. (use $n = \dim Y$)

<u>Pf</u>.

Pick $H_1, H_2 \subset \mathbb{P}^M$ s.t. the nonempty $H_1 \cap C$, $H_2 \cap C$ are finite <u>disjoint</u>. i.e. $s_1, s_2 \in \Gamma(\mathbb{P}^M, \mathcal{O}(1))$ s.t. $Z(s_i) \cap C$ are finite nonempty disjoint.

$s_1|_C, s_2|_C \in \Gamma(C, \mathcal{L}|_C)$ are lin. <u>indep</u>. Let $V = ks_1|_C + ks_2|_C \subset \Gamma(C, \mathcal{L}|_C)$ is 2-dim'l

suffices that

$$\underset{(n+1)\text{-dim'l}}{\mathrm{Sym}^n(V)} \longrightarrow \Gamma(C, \mathcal{L}^{\otimes n}|_C) \quad \text{is injective.}$$

Suppose $\sum_{i=0}^{n} a_i \, (s_1|_c)^{\otimes i} \cdot (s_2|_c)^{\otimes(n-i)} \overset{?}{=} 0$ in $\Gamma(C, \mathcal{L}^{\otimes n}|_C)$

"Divide" by $s_2|_C^n \in \Gamma(C, \mathcal{L}|_C^{\otimes n})$. $f := \frac{s_1|_C}{s_2|_C} \in k(C) - k$

$\sum_{i=0}^{n} a_i f^i = 0$ in $k(C)$. impossible ∵ $f \notin k = \bar{k}$.

all $a_i = 0$

Lecture 12. Finer structure of D and Z: $X \hookrightarrow \mathbb{P}(\Gamma(X, \underset{\substack{\uparrow \\ \text{very ample}}}{\mathcal{L}}))$

for $n \geq \dim Y$

Pass to $\mathcal{L}^{\otimes n}$ $\Rightarrow$ $\exists$ dense open locus $\Omega \subset \mathbb{P}^N = \{H \subset \mathbb{P}\}$ w/

$$\begin{tikzcd} H \in \Omega(k) \Rightarrow & D : X \cap H \arrow[r, hook] \arrow[dr, "\underline{\text{finite}}"'] & X \arrow[d, "f"] \\ & & Y \end{tikzcd}$$

For dim reason, each irred. comp. D_i of D is finite onto Y.

Recall: $\mathcal{U} = sm(X/Y) \subset X$ is fibrewise dense: $\mathcal{U}_{\bar{y}} \subset X_{\bar{y}}$ dense $(\subset X_{\bar{y}}^{sm})$

Want ① $D \to Y$ gen. étale (two possible def'ns equivalent)

② $\forall y \in Y(k)$, $D \cap sm(X/Y)$ meets each irred. comp. of X_y in ≥ 3 pts.

$(\underset{k=\bar{k}}{\Rightarrow}$ same $\forall X_{\bar{y}}$ $\forall y \in Y)$

Pick $y_1 \in Y(k)$, seek H_1 so for $D_1 = X \cap H_1$, have $\text{Ét}(D_1/Y) \supset (D_1)_{y_1}$

and $D_1 \cap sm(X/Y)$ meets each irred. comp. of X_{y_1} in ≥ 3 pts.

(then we'll see same automatic $\forall\, y \in U_1(k)$ for some open $U_1 \ni y_1$ in Y)

$\mathrm{sm}(X/Y)_{y_1} \subset X_{y_1}$ is dense open, so consider H_1 misses finite set $X_{y_1} - \mathrm{sm}(X/Y)_{y_1}$.

for such "good" H_1, have $(D_1)_{y_1} \subset X_{y_1}$ is contained in $\mathrm{sm}(X/Y)_{y_1} \subset X_{y_1}^{\mathrm{sm}}$.

Every irred. comp C of X_{y_1} meets H_1, and dense open locus of H_1's have

$H_1 \cap \underbrace{\mathrm{sm}(X/Y)_{y_1}}_{\substack{\text{smooth} \\ \text{q-proj.} \\ \text{curve}}}$ étale $\uparrow$ geom. reduced.

As long as $H_1 \cap C$ has degree ≥ 3, we'll have ② for D_1 at $y = y_1$. — make sure when passed to $\mathcal{L}^{\otimes n}$ we take $n \geq 3$.

Let's show $D_1 \longrightarrow Y$ is étale at all pts over y_1. We have arranged $(D_1)_{y_1}$ is étale ($(D_1)_{y_1} \subset \mathrm{sm}(X/Y)_{y_1}$)

Lemma.

$$\begin{tikzcd} X \arrow[d, "f"', "\substack{\text{rel.}\\ \text{curve}}"] & D \arrow[l, hook'] \arrow[dl] & \ni x \arrow[d, mapsto] & \text{k-pts.} \arrow[l] \arrow[dl] \\ Y & & y & \end{tikzcd}$$

Assume f smooth at x, D_y is étale at x.

Then $D \to Y$ is étale at x.
$x \mapsto y$

? $\widehat{\mathcal{O}}_{Y,y}$ may not be a domain

Pf. $\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{Y,y}[[t]]$ as $\widehat{\mathcal{O}}_{Y,y}$-alg

Want: $\widehat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \widehat{\mathcal{O}}_{D,x}$. $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/(h)$ some $h \in m_x$.

So $\widehat{\mathcal{O}}_{D,x} \simeq \widehat{\mathcal{O}}_{X,x}/(h) \simeq \widehat{\mathcal{O}}_{Y,y}[[t]]/(h) \overset{?}{\xleftarrow{\sim}} \widehat{\mathcal{O}}_{Y,y}$.

$k = \mathcal{O}_{D_y,x} = \widehat{\mathcal{O}}_{D_y,x} \simeq \widehat{\mathcal{O}}_{D,x}/m_y \simeq \bar{k}[[t]]/(\bar{h})$ $\bar{h} = h \bmod m_y \subset \mathcal{O}_{X_y,x}$.

$\bar{h} = t \cdot (\text{unit})$ in $k[\![t]\!]$

$\therefore h \in (\text{unit}) \cdot t + \hat{m_y}[\![t]\!]$ in $\hat{\mathcal{O}_{Y,y}}[\![t]\!]$

$\xrightarrow[\text{exer}]{}$ $\hat{\mathcal{O}_{Y,y}}[\![x]\!] \to \hat{\mathcal{O}_{Y,y}}[\![t]\!]$ is isom so $\hat{\mathcal{O}_{Y,y}} \xrightarrow{\sim} \hat{\mathcal{O}_{Y,y}}[\![t]\!]/(h)$.

$x \longmapsto h$

$$\begin{tikzcd} D_1 \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

$\text{Ét}(D_1/Y) \overset{\text{open}}{\subset} D_1$ contains $(D_1)_{y_1}$

$D_1 \cap \text{Sm}(X/Y) \overset{\text{open}}{\subset} D_1$ contains $(D_1)_{y_1}$

$$\begin{tikzcd} Z \arrow[d, "\varphi"', "\text{closed}"] \\ Y \end{tikzcd}$$

$Z \overset{\text{open}}{\supset} V \supset Z_{y_1} \implies V \supset \varphi^{-1}(U_1)$ for some open $U_1 \ni y_1$

$\therefore \exists$ open $U_1 \subset Y$ around y_1, so $(D_1)_{U_1}$ is U_1-étale.

and $(D_1)_{U_1} \subset \text{Sm}(X_{U_1}/U_1)$

$\Rightarrow$ $\forall y \in U_1(k)$ $(D_1)_y \cap \text{Sm}(X/Y)_y$ meets each irred. comp of X_y in ≥ 3 pts

$\left(1^{\text{on}} \text{ for } n \geq 3\right)$

If $U_1 = Y$, done.

If not, pick $y_2 \in (Y - U_2)(k)$ to get $D_2 = H_2 \cap X$ where ensure

$H_2 \not\ni$ gen. pts of D_1.

Now $D_1 + D_2$ $(\mathcal{I}_{D_1} \mathcal{I}_{D_2})$ has dense open as scheme that is dense open in $D_1 \amalg D_2$

So $D_1 + D_2 \to Y$ is still gen. étale, so solves problem over $U_1 \cup U_2$

$\cdots$ finish by q-compactness of Y (or noeth. induction)

Using D from above, replace Z by $(Z \cup D)_{\text{red}}$ to arrange:

(vi) (e) $\text{Sm}(X/Y) \cap Z_{\bar{y}}$ has ≥ 3 pts in each irred. comp. of $X_{\bar{y}}$, $\forall y \in Y$.

Rmk

$$\begin{tikzcd}
X \arrow[d, "f"'] & (X \times_Y Y')_{red} \arrow[l] \arrow[d] & (Z \times_Y Y')_{red} \arrow[l, hook'] \\
Y & Y' \arrow[l, "\text{alteration (gen-étale)}"] &
\end{tikzcd}$$

This (X', Z') satisfies all running properties

except maybe normality of X.

(details in notes)

4.16 Want to make alteration of Y so Z becomes $\bigcup_i \sigma_i(Y)$ for sections

$$\begin{tikzcd}
X \arrow[d, "f"'] \\
Y \arrow[u, bend right, "\sigma_1, \dots \sigma_m"']
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[r, phantom, "\subset"] \arrow[dr] & X \arrow[d] \arrow[r, phantom, "\supset"] & X_U \arrow[d] & Z_U \arrow[l, hook'] \arrow[dl, "\text{finite étale}"] \\
& Y \arrow[r, phantom, "\supset"] & U &
\end{tikzcd}$$

dense open

$$\begin{tikzcd}
Z_\eta \arrow[d] \arrow[r, phantom, "= \coprod \eta_i"] & \\
\ni \eta &
\end{tikzcd}$$

$k(\eta_i) | k(\eta)$ finite sep'ble

Pick $K | k(Y)$ big Galois finite splits all η_i/η

Use alteration Y' = normalization of Y in $\mathrm{Spec}(K) \longrightarrow \eta$

This gives $Z_\eta = \coprod_{i=1}^{m} \eta$

Z_i = closure in Z of $\eta_i = \eta$

$$\begin{tikzcd}
Z_i \arrow[d, "\text{finite birat'l}"] \\
Y = \text{normal}
\end{tikzcd}
\implies
\begin{tikzcd}
Z_i \arrow[r, hook] \arrow[d, "\wr"'] & X \\
Y \arrow[u, bend right, "\sigma_i"'] &
\end{tikzcd}$$

inverse

Lecture 13. Stable marked curves

- $\# \mathrm{Aut} < \infty$
- fibral ampleness (flat descent)

Def. Fix integers $g, n \geq 0$, s.t. $2g-2+n > 0$.

$$\begin{pmatrix} g \geq 2, \text{ any } n! \\ g = 1, \ n \geq 1 \\ g = 0, \ n \geq 3 \end{pmatrix}$$

An <u>n-pointed stable genus g curve</u> $/S$ is

$$\begin{tikzcd} \mathcal{C} \arrow[d, "f"'] \\ S \arrow[u, bend right, "\underbrace{\sigma_1, \dots, \sigma_n}_{\text{ordered } n\text{-tuple}}"'] \end{tikzcd}$$

s.t.

1) f proper flat finitely presented w/ all $\mathcal{C}_{\bar{s}}$ are conn'd <u>semistable</u> curves

2) All σ_i are pointwise disjoint and $\sigma_i \in \mathcal{C}^{sm}(S)$. ($\Rightarrow$ reduced)

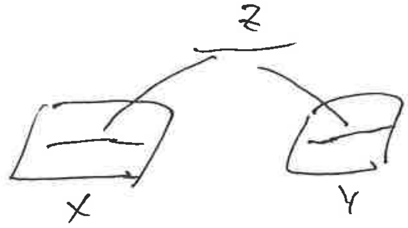
3) $h^1(\mathcal{O}_{\bar{s}}, \mathcal{O}) = g$

4) Any irred. comp. Z of $\mathcal{C}_{\bar{s}}$ w/ $Z \simeq \mathbb{P}^1$ has ≥ 3 special pts ($\sigma_i(\bar{s})$'s and/or meet other irred. components).

Ex.

To make interesting examples, digress to discuss <u>gluing along closed sets</u> and <u>gluing pts together</u>.

I) Gluing along closed set.



$$X \coprod_Z Y = \left(|X| \coprod_{|Z|} |Y|, \ \mathcal{O}_X \times_{\mathcal{O}_Z} \mathcal{O}_Y \right)$$ ← has pushout univ. property

When schemes, <u>can</u> cover X, Y, Z by compatible affine opens

and $\mathrm{Spec}(A) \coprod_{\mathrm{Spec}(C)} \mathrm{Spec}(B) \simeq \mathrm{Spec}\left(A \times_C B\right)$.

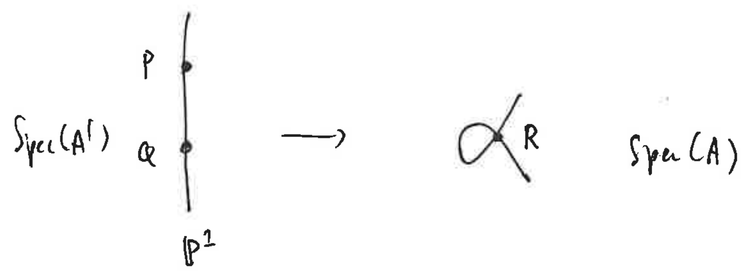
$$\begin{tikzcd} A \arrow[r, two heads, "f"] & C & B \arrow[l, two heads, "g"'] \end{tikzcd}$$

$$A \times_C B = \{(a,b) : f(a) = g(b)\}$$

Rmk. If $A \twoheadrightarrow C \twoheadleftarrow B$ as R-algs for noeth. R w/ A, B f. type $/R$, then

$A \times_C B \subset A \times B$ also f. type $/R$.

II) Self-gluing / k = field



$$\begin{tikzcd} \{f \in A' : f(P) = f(Q)\} \arrow[d, equal] & \\ A \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A' \arrow[d, two heads] \\ k \arrow[r, "\Delta"] & k \times k \end{tikzcd}$$

$C' \longrightarrow C$

$\hat{A}_R = k[\![u]\!] \times_k k[\![v]\!] = k[\![u,v]\!]/(uv)$

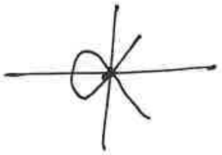
$\subset \hat{A}'_P \times \hat{A}'_Q$

Ex $A' = k[t]$, $\{P, Q\} = \{0, 1\}$

$$\underset{}{A} = k[\underset{X}{t(t-1)}, \underset{Y}{t^2(t-1)}] \subset k[t]$$

$$\simeq k[X,Y]/(Y^2 = X^3 + XY)$$

char$(k) \neq 2$, $\{P, Q\} = \{1, -1\}$, then $X = t^2 - 1$, $Y = t(t^2-1)$, $Y^2 = X^2(X+1)$



This allows "gluing" any finite set of k-pts in a f. type k-scheme. Has expected top. space.

More generally, $A'/I \simeq A'/J$, $\overset{\text{(likely not needed)}}{\overbrace{V(I) \cap V(J) = \phi}}$ inside Spec A'.

What's g?

Examples of stable marked curves

① $\mathbb{P}^1$ w/ n glued pairs of distinct k-pts

$C \xleftarrow{\text{normalization}} \mathbb{P}^1$

For C $\overset{\text{geom.}}{\checkmark}$integral $\overset{\text{proper}}{\checkmark}$ curve $/k$ w/ normalization $\tilde{C} \xrightarrow{\pi} C$.

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\tilde{C}} \to \underbrace{\mathcal{K}}_{} \to 0$$

skyscraper supp.
on non-reg. pts

$$0 \to k \xrightarrow{\sim} k \to H^0(C, \mathcal{K}) \to H^1(C, \mathcal{O}_C) \to H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \to H^1(C, \mathcal{K}) = 0.$$

$$g_C = g_{\tilde{C}} + h^0(C, \mathcal{K})$$

If C is semistable w/ k-rat'l singularities, then $\mathcal{K} = \bigoplus_{\text{sing}} (k \times k)/k$

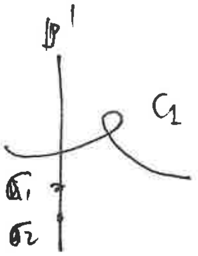
$\nearrow$

(hint: $K_{\hat{p}}$).

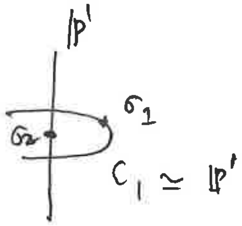
$H^0(C, \mathcal{K}) = k^{\# \text{ sing}}$.

Upshot, "$\mathbb{P}^1$ w/ n-gluings" has $g = n$, so stable $\Leftrightarrow$ $n \geq 2$

ⓔ



or



stable

$g = 1 + g(\tilde{C_1})$; now $h^0(\mathcal{O}_{\tilde{C}}) = 2$.

Exercises ($k = \bar{k}$) (i) sst w/ $g = 0$ $\Leftrightarrow$ tree of $\mathbb{P}^1$'s.

(ii) Stable w/ $g = 0$, $n = 3$ $\Leftrightarrow$ $(\mathbb{P}^1, \{0, 1, \infty\})$

(iii) Stable w/ $g = 1$ and $n = 1$:  $\int$ elliptic curve

$n>1$

"loop of marked $\mathbb{P}^1$'s"

and append tree of (marked) $\mathbb{P}^1$'s onto this.

(iv): For stable marked curve, any irred. comp. C w/ $h^1(\mathcal{O}_C)=1$ has at least one special pt. (Companion to $\mathbb{P}^1$ condition in def'n of stable curve)

Two main refs [DM] $g\geq 2,\ n=0$] lots of use of coherent duality.
(for rel. theory) [Knudsen] general]

To build moduli "spaces" of such data, need

Lemma. For proper $f: X\to S$ that's flat, surjective, f. presented, then

$\{s\in S: X_{\bar{s}}$ sst conn'd curve$\}\subset S$ is open.

Lecture 14. Stability and ampleness

Let's see where "$2g-2+n>0$" comes from.

Lemma. Let X be a proper conn'd sst curve over $k=\bar{k}$, w/ distinct points $\sigma_1,\dots,\sigma_n\in X^{sm}(k)$.

TFAE ① $(X;\sigma_1,\dots,\sigma_n)_{k[\varepsilon]}$ has no nontrivial aut $/k[\varepsilon]$ lifting id$/k$

② The lft k-group $\underline{\mathrm{Aut}}_{(X,\underline{\sigma})/k}$ is étale (automatically f. type)

②' $\mathrm{Aut}(X;\sigma_1,\dots,\sigma_n)$ is finite.

③ (stability) (i) Every irred. comp. Z of X w/ $Z\simeq\mathbb{P}^1$ has ≥ 3 special pts
(ii) Every irred. comp. C of X w/ arithmetic genus $h^1(C,\mathcal{O})=1$ has ≥ 1 special pt

③ (ii) automatic when $2g-2+n>0$ ((ii) not mentioned in [DM])

Pf: Sketch ③ $\Rightarrow$ ① (rest is easier)

Aut of $X_{k[\varepsilon]}$ lifting id/k is $\varphi : \mathcal{O}_X[\varepsilon] \simeq \mathcal{O}_X[\varepsilon]$

$$f + g\varepsilon \mapsto f + (g + D(f))\varepsilon$$

for $D : \mathcal{O}_X \to \mathcal{O}_X$ a k-linear derivation.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d, "d"'] & \mathcal{O}_X \\ \Omega^1_{X/k} \arrow[ur, "\delta \in \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)"'] & \end{tikzcd}$$

so $\delta|_{X^{sm}}$ "is" a vector field $\vec{v}_\delta$ on X^{sm}.

Check $\varphi \circ \sigma_i = \sigma_i$ over $k[\varepsilon]$ $\Leftrightarrow$ $\vec{v}_\delta(\sigma_i) = 0$ in $T_{\sigma_i}(X^{sm})/k$.

For $n=0$, $g \geq 2$: in [§1.4, DM] uses coherent (for sst curves) duality on X to show if ③ holds and $\delta \in \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$ w/ $\vec{v}_\delta(\sigma_i) = 0$, $\forall i$, then $\delta = 0$ (so φ is id).

Adapt this to general (g,n) w/ full force of ③.

* * *

Consider $(X; \sigma_1, \cdots \sigma_n)$ as in Lemma

For $\sigma_i \in X^{sm}(k)$, so $\overset{\mathcal{O}_X(-\sigma_i)}{\overset{\shortparallel}{I_{\sigma_i}}} \subset \mathcal{O}_X$ is <u>invertible</u>.

The scheme X is Gorenstein $\left(\hat{\mathcal{O}}_x \simeq \begin{cases} k[[t]] & x \in X^{sm}(k) \\ k[[u,v]]/(uv), & x \notin X^{sm}(k) \end{cases} \right)$

so "dualizing complex" is just a coherent sheaf $\underbrace{\omega_{X/k}}_{\text{invertible}}[1]$

and naturally $\omega_{X/k}|_{X^{sm}} \simeq \Omega^1_{X^{sm}/k}$.

Moreover, formation of $\omega_{X/k}$ commutes w/ "étale localization":

$$\begin{tikzcd} f: U \arrow[rr, "\text{étale}"] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd} \Rightarrow \quad f^* \omega_{X/k} \simeq \omega_{U/k}.$$

So $\omega_{X/k}$ can be "computed".

$$\tilde{\omega} = \omega_{X/k} \otimes I_{\sigma_1}^{\vee} \otimes \cdots \otimes I_{\sigma_n}^{\vee}$$

$$= \omega_{X/k}\left(\Sigma \sigma_i\right)$$

For invertible $\mathcal{L}$ on X, $\chi(\mathcal{L}^{\otimes m}) = \underbrace{(\deg \mathcal{L})}_{\text{def'n}} \cdot m + (\text{const})$

as $\deg(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg \mathcal{L}_1 + \deg \mathcal{L}_2$

(see [BLR; §9.1] for discussion of $\deg \mathcal{L}$ on singular curves)

Coherent duality $\underset{\substack{(X \text{ sst}) \\ \text{conn'd}}}{\Rightarrow}$ $\deg(\omega_{X/k}) = 2g-2$ for $g = h^1(X, \mathcal{O})$

$\deg(I_{\sigma_i}) = \deg(-\sigma_i) = -1$

$\Rightarrow \deg(\tilde{\omega}) = 2g-2+n$

Lemma. In above setup, stability ③ $\Leftrightarrow$ $\tilde{\omega}$ is ample.

($\Leftrightarrow$ $\tilde{\omega}|_{X_i}$ ample, $\forall$ irred. comp. X_i of X)

$\Rightarrow 2g-2+n > 0$

Pf Case $n=0$, $g \geq 2$: Thm 1.2 in [DM], Pf uses coherent duality

Adapts to $g \geq 2$, any n, and $g \leq 1$ and $n \geq 4$.

For $g \leq 1$, $n \leq 3$, see Cor 1.10 in [Knudsen].

Cor. Fix g, n w/ $2g-2+n>0$. Consider

$$\begin{tikzcd} X \arrow[d, "f"', "\text{proper} \\ \text{fppf}"] \\ S \arrow[u, bend right=60, "\sigma_1, \dots, \sigma_n"'] \end{tikzcd}$$

Then $S^{st} = \{ s \in S : (X_{\bar{s}}; \sigma_1(\bar{s}), \dots, \sigma_n(\bar{s}))$ is stable n-ptd genus g-curve $\}$

is <u>open</u> in S.

Pf. Last time $\Rightarrow$ can pass to open $\Omega \subset S$ via sst geom. conn'd curve fibers

and $\sigma_i \in Sm(X/S)$.

For $i \neq j$: $\sigma_i \times \sigma_j : S \longrightarrow X \times X \Rightarrow (\sigma_i \times \sigma_j)^{-1}(\Delta) \subset S$ is "bad" closed set

so <u>open</u> condition to have σ_i's disjoint.

By relative coherent duality, have <u>invertible</u> $\omega_{X/S}$ on X whose formation commutes w/ any base change on S and étale localization on X.

$I_{\sigma_i} \subset \mathcal{O}_X$ are invertible $\because \sigma_i \in X^{sm}(S)$

$\uparrow$

Formation of I_σ commutes w/ any base change

so $\tilde{\omega} = \omega_{X/S} \otimes I_{\sigma_1}^{\vee} \otimes \cdots \otimes I_{\sigma_n}^{\vee}$ commutes w/ base change.

$\boxed{\text{EGA IV}_3,\ 9.9.6\ (\text{no flatness})}$

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\text{inv. on}"] & X \arrow[d, "\text{proper} \\ \text{f. presented}"] \\ & S \end{tikzcd}$$

- $\Omega = \{ s \in S : \mathcal{L}|_{X_s}$ is ample on $X_s \}$ is <u>open</u> in S
- $\forall$ affine open $\operatorname{Spec} A \subset \Omega$, $\mathcal{L}|_{X_A}$ is A-ample. $\left(X_A \overset{j}{\hookrightarrow} \mathbb{P}^N_A \text{ so } \mathcal{L}_A^{\otimes(-)} \simeq j^* \mathcal{O}(1) \right)$

Lecture 15. Moduli of stable curves

$$\begin{tikzcd} Z = \bigcup \sigma_i(Y) \arrow[rrd, bend right] & \subset X \arrow[d, "f"'] & \supset X_U \arrow[d, "\substack{\text{smooth} \\ \text{geom. conn'd} \\ \text{fibers,} \\ \text{genus } g \geq 0}"] & \supset Z_U = \coprod_{i=1}^{m} U \quad (m \geq 3) \\ & Y & \underset{\substack{\text{dense} \\ \text{open}}}{\supset} U & \end{tikzcd}$$

Eventually, "alter" $(X, Z) \to Y$ to a stable marked curve (over Y).

$$\begin{tikzcd} (X, Z) \arrow[d] \\ Y \end{tikzcd}$$

then alter Y (by induction)

This requires using "moduli stack" $\overline{M}_{g,n}$

$$\begin{tikzcd} \mathcal{C}_{g,n} \arrow[d] & X_U \arrow[l] \arrow[d] \\ \overline{M}_{g,n} \arrow[u, bend right, "\underline{\sigma}"'] & U \subset Y \arrow[l] \end{tikzcd}$$

$\overline{M}_{g,n}$ $\shortparallel$ (smooth) proper $/k$

Consider functor: $\overline{M}_{g,n} : S \longmapsto \left\{ \begin{array}{c} (\mathcal{C}, \underline{\sigma}) \\ \downarrow \\ S \end{array} \begin{array}{l} \text{stable } n\text{-ptd} \\ \text{genus } g \text{ curve} \end{array} \right\} / \simeq$

$(2g - 2 + n > 0)$

via base change $(S' \to S) \mapsto \left(\overline{M}_{g,n}(S) \longrightarrow \overline{M}_{g,n}(S') \right)$

Usually not an étale sheaf on Sch (or Sch/k), so cannot be rep'ble.

Ex. k field, $\operatorname{char}(k) \neq 2$. fix elliptic curve E_0/k,
$(g, n) = (1, 1)$

$$\begin{tikzcd} X' = E_0 \times \mathbb{G}_m \arrow[d] \arrow[r] & C \arrow[d] \\ S' = \mathbb{G}_m \arrow[u, bend left, "\substack{\sigma'(t) \\ = (0,t)}"] \arrow[r, "t^2"'] & \mathbb{G}_m \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

$$\begin{tikzcd}
X' \arrow[r, "\simeq", "(x,t)\mapsto(-x,-t)"'] \arrow[d, bend right] & X' \arrow[d, bend left] \\
\mathbb{G}_m \arrow[r, "-t", "\simeq"'] & \mathbb{G}_m
\end{tikzcd}$$

$\underset{q+}{\rightsquigarrow}$

<u>Check</u>. $(C,\sigma) \not\simeq (E_0 \times \mathbb{G}_m, 0)$ $/S$

but become isom over $S' \to S$.

$\overline{M_{1,1}}(S) \to \overline{M_{1,1}}(S')$ <u>not</u> injective

<u>Rmk</u>. Append elliptic curve leaf, get singular examples w/ any (g,n) for $g \geq 1$.

To remove nontrivial auts, one idea is to equip data w/ "enough" extra structure.

If attached to each object ξ is a "well-behaved" fiberwise ample L_ξ, then can try

$$\begin{tikzcd}
\xi \arrow[r, hook] \arrow[dr] & \mathbb{P}(f_*(L_\xi^{\otimes 100})) \simeq \mathbb{P}^N_S \arrow[d] \arrow[dl, shorten <=2em] \\
& S
\end{tikzcd}$$

<u>App C.2, C.3</u>: For any $\begin{matrix}(X, \underline{\sigma}) \\ f \downarrow \\ S\end{matrix}$, $\begin{matrix}L_{X/S,\underline{\sigma}} \\ \shortparallel \\ L_{X/S}\end{matrix} = \begin{matrix}\omega_{X/S}(\Sigma\sigma_i)^{\otimes 4} \\ \\ \underline{\underline{\text{fiberwise ample}}}.\end{matrix}$

Satisfies 1) $L_{X/S}$ is canonical w.r.t. isom. in $(X, \underline{\sigma})$

and formation commutes w/ all base change on S

2) (using cohom. + base change) $f_* L_{X/S}$ is a vector bundle on S,

whose formation naturally commutes w/ any base change on S, and

rk is $N = N(g,n,4) = 4(2g-2+n) + (1-g)$.

3) $f^*(f_* L_{X/S}) \longrightarrow L_{X/S}$ is <u>surjective</u> defining closed immersion.

$$\begin{tikzcd}
L_{X/S} & \mathcal{O}(1) \arrow[l, squiggly] & \\
X \arrow[r, hook] \arrow[dr] & \mathbb{P}(f_* L_{X/S}) \arrow[u, phantom, "\wr"] \arrow[d] & \mathbb{P}^{N-1} \text{ Zar. locally on } S \\
& S &
\end{tikzcd}$$

has fiberal Hilb poly w.r.t. $\mathcal{L}_{X/B}$ given by $\overset{\Phi\,\shortparallel}{\Phi_{g,n}(t)} = (2g-2+n)t + (1-g) \in \mathbb{Z}[t]$

Look at $\mathrm{Hilb}^{\Phi}_{\mathbb{P}^{N-1}_{\mathbb{Z}}/\mathbb{Z}} = H$

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[dr] & \mathbb{P}^{N-1}_H \arrow[d] \\ & H = \text{q-proj.}/\mathbb{Z} \end{tikzcd}$$

univ. flat family w/ Hilb poly Φ

$\mathcal{Z}^n = \underbrace{\mathcal{Z} \times_H \cdots \times_H \mathcal{Z}}_{n}$ represents flat families

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr] & \mathbb{P}^{N-1}_T \arrow[d] \\ & T \end{tikzcd}$$

equipped w/ ordered n-tuple $\sigma_1, \dots, \sigma_n \in Y(T)$.

Let $\exists$ open $\Omega \subset \mathcal{Z}^n_{/H}$ represents such $(Y, \underline{\sigma})$ that are stable n-ptd genus g curves.

$\rightsquigarrow$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "j"'] & \mathbb{P}^{N-1}_\Omega \arrow[d] \\ & \Omega \end{tikzcd}$$

has no relation of $j^*\mathcal{O}(1)$ w/ $\mathcal{L}_{C/\Omega,\underline{\sigma}}$.

Imposing further closed condition on Ω and passing to PGL_N-torsor over <u>that</u> forces $j^*\mathcal{O}(1)$ to relate well to $\mathcal{L}_{C/\Omega,\underline{\sigma}}$ (see App C.2, C.3)

<u>Upshot</u>: can build quasi-projective $\mathbb{Z}$-scheme $\overline{Y}_{g,n}$ that represents functor

$$S \rightsquigarrow \left((C,\underline{\sigma})/_S,\ \mathbb{P}(f_*\mathcal{L}_{C/S}) \overset{PGL_N}{\overset{\circlearrowright}{\simeq}} \mathbb{P}^{N-1}_S \right)$$

Morally, $\overline{Y}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ is PGL_N-torsor.

<u>Def</u>. $\overline{\mathcal{M}}_{g,n}(S)$ = cat. of $(C, \underline{\sigma})$ over S, w/ $\overset{\text{(gpoid)}}{\simeq}$ as only maps

$S \rightsquigarrow \overline{\mathcal{M}}_{g,n}(S)$

"presheaf" in gpoids via base change

$$\begin{tikzcd} S' \arrow[d] & & \overline{\mathcal{M}}_{g,n}(S') \\ S \arrow[rr, squiggly, phantom, "\rightsquigarrow"] & & \overline{\mathcal{M}}_{g,n}(S) \arrow[u, "b_{S'/S}"'] \end{tikzcd}$$

base change w/ "nice" isom

$$b_{S'/S} \simeq b_{S''/S'} \circ b_{S'/S}$$

Since $\mathcal{L}_{X/S}$ is rel. ample (EGA miracle ...)

$$\begin{tikzcd} S' \arrow[d, "\text{fpqc}"'] \\ S \end{tikzcd} \quad \overline{\mathcal{M}}_{g,n}(S) \underset{\substack{\text{equiv} \\ \text{of cats}}}{\simeq} \left\{ (X', \theta) : X' \in \overline{\mathcal{M}}_{g,n}(S'),\ \theta : p_1^*(X') \simeq p_2^*(X') \text{ in } \overline{\mathcal{M}}_{g,n}(S' \times_S S') \right\}$$

$$X \longmapsto (X_{S'}, \theta_{can})$$

(2 descent is ! up to ! isom) "Stack in gpoids for fpqc + Zar top"

$\overline{\mathcal{M}}_{g,n}$ is Artin stack of f. type /$\mathbb{Z}$ due to $Y_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$.

Even a DM stack (notes ...)

$\uparrow$ $\underline{\mathrm{Aut}}_{(X, \underline{s})/k}$ étale

---

Lecture 16. Smoothness and properness of $\overline{\mathcal{M}}_{g,n}$

What is an Artin or DM stack over a scheme S?

(for fppf top - equiv in DM case to use étale top, and coverings that have "descent" for fpqc)

Fibered groupoid

"presheaf in groupoids"

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ Sch \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{X}(S) \arrow[r, "f^*"] & \mathcal{X}(S') \\ S \arrow[u, squiggly] & S' \arrow[l, "f"] \arrow[u, squiggly] \end{tikzcd}$$

w/ some "assoc" axiom on f^*'s.

Sit. 1) Descent for fppf topology ("stack"): for covering $S' \to S$, have equivalence

$\boxed{\text{Stack}}$

$$\mathcal{X}(S) \xrightarrow{\sim} \Big\{ (\xi, \theta) : \xi \in \mathcal{X}(S'),\ \theta : p_1^*(\xi) \simeq p_2^*(\xi) \text{ in } \mathcal{X}(S' \times_S S') \overline{\text{ satisfying cocycle in }} \underbrace{\mathcal{X}(S' \times_S S' \times_S S')}_{\text{descent datum}} \Big\}$$

$$\xi \longmapsto (\xi_{S'}, \theta_{can})$$

$+\ \mathcal{X}(\amalg S_i) \simeq \prod \mathcal{X}(S_i)$

$+$ Zariski $(S' = \amalg U_i')$

(algebraicity condition)

2) $\mathcal{X} \xrightarrow{\Delta_{\mathcal{X}}} \mathcal{X} \times \mathcal{X}$

rel. rep'ble in alg. spaces.

$$\begin{tikzcd} & \mathcal{Y}' \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

define $(\mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}')(S)$

$$= \{ (\xi, \xi', \varphi) : \xi \in \mathcal{Y}(S),\ \xi' \in \mathcal{Y}'(S),\ \varphi : p(\xi) \simeq p'(\xi') \text{ in } \mathcal{Z}(S) \}$$

$Hom(-,T) \to \mathcal{X}$ as fibered cats $\overset{\text{Yoneda}}{\Longleftrightarrow}$ $\xi \in \mathcal{X}(T)$

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times \mathcal{X} \\ \underbrace{Isom_T(x,x')}_{\text{alg. space}} \arrow[u] \arrow[r] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[u] \end{tikzcd}$$

$\hookleftarrow (x, x'),\ x, x' \in \mathcal{X}(T)$

Guarantees any

$$\begin{tikzcd} T \arrow[d] & \boxed{\text{alg. space}} \arrow[l] \arrow[d] \\ \mathcal{X} & T' \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \boxed{\text{alg-sp}} \arrow[r, "\text{smooth surjection}"] \arrow[d] & T \arrow[d] \\ X \arrow[r] & \mathcal{X} \end{tikzcd}$$

(Artin stack) Require $\exists$ "smooth cover" by a scheme: $X \longrightarrow \mathcal{X}$ rel. rep'ble in smooth surjections.

Ex. $\mathcal{X} = \overline{\mathcal{M}_{g,n}}$

$X = \underbrace{\overline{Y_{g,n}} \longrightarrow \overline{\mathcal{M}_{g,n}}}_{PGL_N\text{-torsor}}$

If $\exists$ $X \longrightarrow \mathcal{X}$ rel. reptble in étale surjections

say $\mathcal{X}$ is DM stack

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ S \end{tikzcd}$$

map of fibered cats on Sch $\Leftrightarrow$ $\widetilde{\mathcal{X}}$ as fibered cat on Sch_S.

$\mathcal{X}(T) = \{(T \xrightarrow{h} S), \xi \in \widetilde{\mathcal{X}}(T \to S)\} \leftsquigarrow \widetilde{\mathcal{X}}$

What does smoothness or properness <u>mean</u> for $\mathcal{X} \to \mathcal{Y}$ between Artin stacks?

Focus on $\mathcal{X} \xrightarrow{f} S$ = scheme.

<u>Def.</u> $\mathcal{X}$ is smooth over S if some ($\Rightarrow$ any) smooth scheme over $X \longrightarrow \mathcal{X}$ has X smooth over S. [For $Z' \underset{\text{smooth}}{\twoheadrightarrow} Z \to S$, Z' is S-smooth $\Leftrightarrow$ Z is].

Similarly define $\mathcal{X}$ being l.f.type$/S$ or ltp using $X \xrightarrow{sm} \mathcal{X}$.

Say $\mathcal{X}$ is <u>q-sept'd</u> over S if $\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X}$ is rept'd in qc maps.

Say $\mathcal{X}$ is <u>fp</u> $/S$ if lfp and qs and $\boxed{qc}/_S$

↑ $\exists$ qc scheme cover over open affines of S.

<u>Thm.</u> If $\mathcal{X} \to S$ (= Noeth) is lfp, then smoothness can be checked using infinitesimal deformation theory with Artin local rings having alg. closed res. field.

$\mathcal{X}$
$\downarrow$
S is <u>sept'd</u> if $\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X}$ is rept'd by closed immersions

is <u>proper</u> if sept'd, f. type (qc + lft), and <u>univ. closed</u>:

$$\begin{tikzcd} \forall\, T \to S, & {|\mathcal{X}_T|} \arrow[d, "\text{is closed}"] \\ & {|T|} \end{tikzcd}$$

$|\mathcal{X}_T| \sim$ assoc. top. space (qt of X_T for smooth scheme cover $X \to \mathcal{X}$).

$\exists$ Chow's Lemma for DM/Artin stacks (notes)

$\Rightarrow$ $\exists$ val. criterion for properness (inferred from scheme case).

<u>Val. criterion</u>.

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\text{f. type}"'] & \mathrm{Spec}(K) \arrow[l, "\xi"'] \arrow[d] \\ S \arrow[d, phantom, "\shortparallel"] & \mathrm{Spec}(R) \arrow[l] \arrow[ul, dashed] \arrow[d, phantom, "\shortparallel"] \\ \text{(noeth)} & \text{DVR} \end{tikzcd}$$

$\xi \in \mathcal{X}(K)$

For schemes, this is <u>equiv.</u> to usual criterion [$\exists$ finite ext'n $K'|K$ and ext'n of val. to K' w/ val. ring $R' = \mathrm{DVR} \supset R$. s.t. $\xi_{K'} \in \mathcal{X}(K')$ comes from $\underbrace{\mathcal{X}(R')}_{!\,\tilde{\xi} \in}$.

<u>Rmk</u>. Suffices to use R complete w/ alg. closed res. field.

<u>Goal</u>: Want $\overline{\mathcal{M}_{g,n}} \longrightarrow \mathrm{Spec}(\mathbb{Z})$ to be smooth + proper.

These will be proved by induction on n w/ g fixed:

$g \geq 2$: $n \geq 0$
$g = 1$; $n \geq 1$
$g = 0$: $n \geq 3$

<u>Base case</u>: $\boxed{g \geq 2}$ $\overline{\mathcal{M}_{g,0}}$ smooth + proper $/\mathbb{Z}$

in [DM] via
- deformation theory for smoothness
- val. criterion for properness ("stable reduction thm for curves")

$\uparrow$
requires refinement of val. criterion

$\boxed{g=1}$ $\overline{\mathcal{M}}_{1,1}$ studied via def. theory + val. criterion in [Deligne-Rapoport].

$\boxed{g=0}$ $\overline{\mathcal{M}}_{0,3} \simeq \operatorname{Spec} \mathbb{Z}$ $(\mathbb{P}^1, \{0,1,\infty\})$ is ! object/s.

Induction (next time):

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1} \arrow[r, phantom, "\overset{\exists}{\simeq}"] & \mathcal{Z}_{g,n} \arrow[d] & \text{univ. curve} \\ & \overline{\mathcal{M}}_{g,n} & \text{] grant smooth proper } /\mathbb{Z} \end{tikzcd}$$

Lecture 17 Smoothness + properness of $\overline{\mathcal{M}}_{g,n}/\mathbb{Z}$

We know: $\overline{\mathcal{M}}_{g,n}$ = f. presented DM stack $/\mathbb{Z}$ $(2g-2+n>0)$

$\Delta_{\overline{\mathcal{M}}_{g,n}}$ is sep't'd : Isom-schemes are septd (q.-proj. loc. on base)

Stages:
- smoothness, yield information above $\underbrace{\mathcal{M}}_{\text{open substack for smooth curves}} \subset \overline{\mathcal{M}}$
- septdness via (refined) val. criterion (use !ness of stable reduction for curves)

properness of $\Delta_{\mathcal{X}/S}: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ ↗

- properness via refined val criterion

Induct on n, fixed $g \geq 0$. so base case "$\mathcal{M}$" $\left[\begin{array}{l} \overline{\mathcal{M}}_{g,0} \quad (g \geq 2) \\ \overline{\mathcal{M}}_{1,1}, \ \overline{\mathcal{M}}_{0,3} = \operatorname{Spec} \mathbb{Z} \end{array}\right.$ no nontrivial aut.

$[\ \overline{\mathcal{M}}_{0,3}(S) = \{(\mathbb{P}^1_S, \{0,1,\infty\})\}\ ]$

Once base cases done: use Knudsen's contraction + stabilization constructions to build

$$\overline{\mathcal{M}}_{g,n+1} \overset{*}{\simeq} \mathcal{Z}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

proper, not smooth

handle $\mathbb{Z}$-smoothness here via def. theory + Artin approx.

Smoothness $/\mathbb{Z}$ for $\overline{\mathcal{M}} = \overline{\mathcal{M}_{g,0}}, \overline{\mathcal{M}_{1,1}}$

inf'l smoothness criterion: $\overline{\mathcal{M}}(A) \to \overline{\mathcal{M}}(A/I)$ is ess. surj. for Artin local A w/ alg. closed res. field, $I \subset A$ w/ $I^2 = 0$ (even $\mathfrak{m} I = 0$).

(deduced from scheme case via smooth scheme cover for f. pres. $\mathcal{X} \to S$ = noeth)

[DM, §1] for $g \geq 2$. [DR, II, Prop 2.7 ; III Thm 1.2] for $g = 1$

This shows (via inspection of def ring) that $\exists$ invertible ideal $I_\infty \subset \mathcal{O}_{\overline{\mathcal{M}}}$ so

$$\overline{\mathcal{M}} - \underbrace{Z(I_\infty)}_{\mathbb{Z}\text{-flat}} = \mathcal{M} \qquad \left[\mathcal{O}_{\overline{\mathcal{M}}}(T \to \overline{\mathcal{M}}) = \mathcal{O}(T)\right]$$

so $\mathcal{M} \subset \overline{\mathcal{M}}$ is a rel. sch. dense: (dense and remains so after any base change).

[EGA IV_3, 11.10.8-10]

So $\mathcal{M} \times \mathcal{M} \subset \overline{\mathcal{M}} \times \overline{\mathcal{M}}$ also dense open.

Septdness of $\overline{\mathcal{M}} \to \operatorname{Spec} \mathbb{Z}$: properness of the (septd!) $\Delta_{\overline{\mathcal{M}}/\mathbb{Z}}$:

$$\begin{tikzcd} \overline{\mathcal{M}} \arrow[r] & \overline{\mathcal{M}} \times \overline{\mathcal{M}} \\ \mathcal{M} \arrow[u, phantom, sloped, "\subset"] \arrow[r] \arrow[ur, phantom, "\urcorner"] & \mathcal{M} \times \mathcal{M} \arrow[u, phantom, sloped, "\subset"] \arrow[u, phantom, "\text{dense open}"'] \end{tikzcd}$$

Apply refined val. criterion (pf in notes):

Given: $\mathcal{U} \overset{\text{dense open}}{\subset} \mathcal{X}$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\text{septd, f-type}"] \\ S = \text{noeth} \end{tikzcd}$$

for properness, suffices to check val. criterion using complete DVRs R having alg. closed res. field w/ K-pts in $\mathcal{U}$!

For us, it ensures only need bijectivity of $\mathrm{Isom}_R(x, y) \xrightarrow{\sim} \mathrm{Isom}_K(x_K, y_K)$ when $x, y \in \overline{\mathcal{M}}(R)$ s.t. $x_K, y_K \in \mathcal{M}(K)$

This is !ness part of stable reduction for smooth curves $(g \geq 1)$

stable R'-model over some $K'|K$ is ! up to ! isom (ell. curve for $g=1$)

$$\left(\text{e.g. } \mathrm{Aut}_R(\mathcal{C},\underline{\sigma}) \simeq \mathrm{Aut}_K(\mathcal{C}_K,\underline{\sigma}_K)\right)$$

Properness (given know $\overline{\mathcal{M}}$ is sep'td /$\mathbb{Z}$!)

Apply refined val. criterion to

$$\begin{tikzcd}
\mathcal{M} \arrow[r, phantom, "\subset"] \arrow[r, phantom, "\text{dense open}"'] & \overline{\mathcal{M}} \arrow[d, "\text{sep't'd, f. type}"] \\
 & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}$$

exactly existence in stable reduction thm for smooth curves /K !! [DM, §2], [DR, III, Prop 1.6 (i) – (iii)]

$\boxed{R=\hat{R},\ k=\bar{k}}$

Induction,

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+1} \arrow[r, phantom, "\overset{?}{\simeq}"] & (\mathcal{Z}_{g,n}, \sigma_1, \dots \sigma_n) \arrow[d] \\
 & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

As for univ. family $Z \longrightarrow \mathrm{Hilb}_{X/S}$

may not be stable as $(n+1)$-ptd curve

$$\mathcal{Z}_{g,n}(S) = \{(\underbrace{\mathcal{C},\sigma_1,\dots,\sigma_n}_{\text{stable}};\ \sigma_{n+1})\}$$

extra $\sigma_{n+1} \in \mathcal{C}(S)$

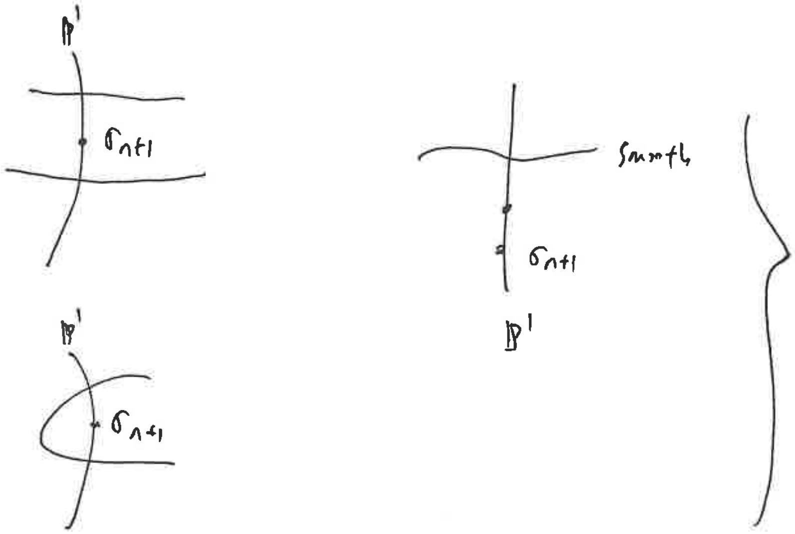
[maybe in $\mathcal{C}^{\mathrm{sing}}(\bar{s})$

$\sigma_{n+1}(\bar{s}) \overset{?}{=} \sigma_3(\bar{s})$ . . .]

$$\overline{\mathcal{M}}_{g,n+1} \xrightarrow{?} \mathcal{Z}_{g,n} \quad \sim \text{forget } \sigma'_{n+1}$$

$$(\mathcal{C}',\sigma'_1,\dots,\sigma'_{n+1}) \mapsto (\underbrace{\mathcal{C}',\sigma'_1,\dots,\sigma'_n}_{\text{may not be stable as } n\text{-ptd curve}};\ \sigma'_{n+1})$$

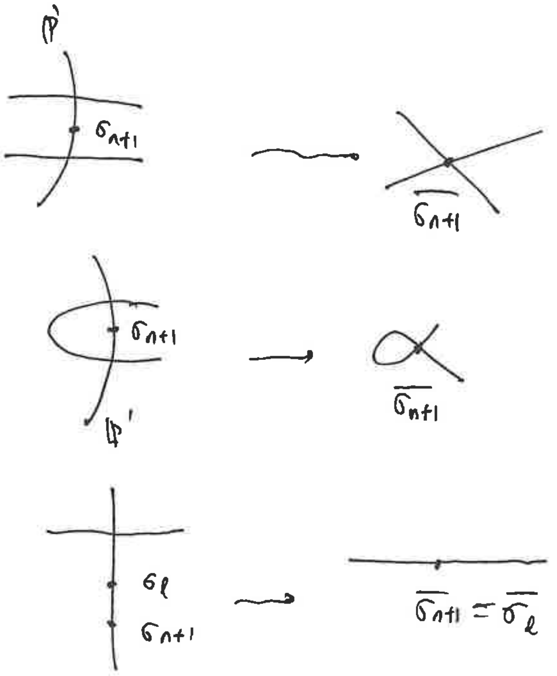
Ex.



only problems / $k = \bar{k}$.

Fix.

Contract!



$$\mathcal{C}' \xrightarrow{f} S$$

$$\mathcal{C}' \simeq \mathrm{Proj}_S\Big(\bigoplus_{m\geq 0} f_*\big(\underbrace{\tilde{\omega}^{\otimes m}_{\mathcal{C}'/S}}_{\omega_{\mathcal{C}'/S}(\sigma_1' + \dots + \sigma_{n+1}')}\big)\Big)$$

$$c(\mathcal{C}') = \mathrm{Proj}_S\Big(\bigoplus_{m\geq 0} f_*\big(\mathcal{L}^{\otimes m}\big)\Big)$$

$$\mathcal{L} = \omega_{\mathcal{C}'/S}(\sigma_1 + \dots + \sigma_n)$$

Knudsen shows this "works", using $\overline{\sigma_1'}, \dots \overline{\sigma_n'}$; $\overline{\sigma_{n+1}'}$

Build inverse process called stabilization.

Lecture 18. Alteration to dominate by stable curve I

Finish pf of $\mathbb{Z}$-smoothness of $\overline{\mathcal{M}_{g,n}}$ via induction on n.

$$\begin{tikzcd} \overline{\mathcal{M}_{g,n+1}} \arrow[r, phantom, "\overset{\text{Knudsen}}{\simeq}"] & \mathcal{Z}_{g,n} \arrow[r, phantom, "\overset{\text{open}}{\supset}"] \arrow[d, "\text{proper f. pres.}\ f"'] & \mathcal{U} \arrow[d, phantom, "\parallel"] \\ & \overline{\mathcal{M}_{g,n}} = \mathbb{Z}\text{-smooth} & f\text{-smooth locus} \end{tikzcd}$$

$\mathcal{U}$ is certainly $\mathbb{Z}$-smooth.

Issue. $\underbrace{\mathbb{Z}\text{-smoothness at pts } x \in |\mathcal{Z}_{g,n}| \text{ over } y \in |\overline{\mathcal{M}}_{g,n}| \text{ outside } \mathcal{U}}_{\text{Want to describe "local str." of } \mathcal{Z}_{g,n} \text{ near such } x.}$

Use description of $\hat{\mathcal{O}}_{\mathcal{Z},x} = ???$

For DM stack $\mathcal{X}$ and étale scheme cover

$$\begin{tikzcd} X \arrow[r, two heads, "\text{étale}"] & \mathcal{X} & X' \arrow[l, two heads] \\ {|X|} \arrow[r, two heads] & {|\mathcal{X}|} & {|X'|} \arrow[l, two heads] \\ \tilde{x} \arrow[r, mapsto] & x & \tilde{x}' \arrow[l, mapsto] \end{tikzcd}$$

$$\begin{tikzcd} \xi \arrow[rr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & \tilde{x}' \arrow[ddd, mapsto, bend left] \\ X \times_{\mathcal{X}} X' \arrow[r, two heads, "\text{ét}"] \arrow[d, "\text{ét}"'] & X' \arrow[d] & \\ X \arrow[r] & \mathcal{X} & \\ \tilde{x} \arrow[r, mapsto] & x & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}^{sh}_{X \times_{\mathcal{X}} X', \xi} & \mathcal{O}^{sh}_{X', \tilde{x}'} \arrow[l, "\sim"'] \\ \mathcal{O}^{sh}_{X, \tilde{x}} \arrow[u, "\wr"] & \end{tikzcd}$$

$$\mathcal{O}^{sh}_{\mathcal{X},x} := \mathcal{O}^{sh}_{X,\tilde{x}}, \text{ get } \widehat{\mathcal{O}^{sh}_{\mathcal{X},x}} \text{ or } \hat{\mathcal{O}}_{\mathcal{X},x}$$

Knudsen described $\widehat{\mathcal{O}^{sh}_{\mathcal{Z}_{g,n},x}}$ as algebra over $\widehat{\mathcal{O}^{sh}_{\overline{\mathcal{M}}_{g,n},y}}$

[Kn, Thm 2.7]

$\Downarrow$ Artin approx.

$(\mathcal{Z}_{g,n}, x) \to (\overline{\mathcal{M}}_{g,n}, y)$ has an étale nbhd in common w/

$$\begin{tikzcd} (0,0,0) \arrow[d, mapsto] & \in & \operatorname{Spec}\left(R[t,x,y]/(xy-t)\right) \arrow[d] \arrow[r, phantom, "\underset{(*)}{\simeq}"] & \operatorname{Spec}\left(R[x,y]\right) \\ \underbrace{0}_{\text{over some pt of Spec}(R).} & \in & \operatorname{Spec}(R[t]) = \mathbb{Z}\text{-smooth (induction)} & \\ & & \text{f.g. } \mathbb{Z}\text{-alg.} \arrow[u] & \end{tikzcd}$$

$\Rightarrow R$ is $\mathbb{Z}$-smooth

$\Rightarrow (*)$ is $\mathbb{Z}$-smooth!

$$\begin{tikzcd} \text{Back to} & X \arrow[d, "f"] \arrow[r, phantom, "\supset"] & Z = \bigcup_{i=1}^{n} \sigma_i(Y) \\ & Y \arrow[r, phantom, "\supset"] \arrow[d, equal] & U \\ & \text{q. proj. var.}/k=\bar{k} & \end{tikzcd}$$

$, n \geq 3$ $(\text{so } 2g-2+n \geq 1 > 0)$

proper flat w/ geom. conn'd gen. smooth fibers of dim 1

dense open

s.t.

$$\begin{tikzcd} X_U \arrow[d] \\ U \end{tikzcd}$$

<u>smooth</u>, genus $g \geq 0$.

and $\sigma_i|_U$ are pairwise disjoint

($\because$ true over η_Y).

$\Rightarrow$ stable n-pt'd genus g $/U$

$$\begin{tikzcd} & Y & & \\ \Rightarrow & U \arrow[u, hook] \arrow[r, "(X_U, \Sigma_U)"] \arrow[dr] & \mathcal{M}_{g,n}/k \arrow[d] \arrow[r, phantom, "\subset"] & \overline{\mathcal{M}_{g,n}}/k = \text{proper}/k \\ & & \mathrm{Spec}(k) & \end{tikzcd}$$

$$\begin{tikzcd} & & S' \arrow[d, "\underline{U\text{-mod}}"'] \arrow[ddr, bend left, "\exists \varphi'"] & \\ \underline{\text{Recall}}: & U \arrow[r, "\text{open}"] \arrow[drr, "\varphi"] \arrow[dd, "\text{f. type}"'] & S = \text{noeth. integral} & \\ & & & M \arrow[dll, "\text{sep'd}"] \\ & B = \text{noeth} & & \end{tikzcd}$$

$$\left(\begin{tikzcd} U' \arrow[d, "\wr"'] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\lrcorner"] & S' \arrow[d] \\ U \arrow[r, phantom, "\subset"] & S \end{tikzcd} \quad \text{s.t. } \varphi'|_{U'} = \varphi \right)$$

$$\begin{tikzcd} S' := \overline{\Gamma_\varphi} \subset S \times_B M \arrow[r] \arrow[r, bend left=40, "\varphi'"] & M \end{tikzcd}$$

$$\Gamma_\varphi = U \underset{\substack{\uparrow \\ \text{closed}}}{\subset} \overset{\cup}{U \times_B M}$$

closed $\because \Delta_{M/B}$ = closed immersion

For M a f. pres. sep't'd DM stack, $\overline{\Gamma_\varphi}$ only stack. but have Chow's Lemma for DM-stacks (notes) $\Longrightarrow$ can fix to get gen. étale alteration $\begin{matrix} Y' \\ \downarrow \\ Y \end{matrix}$ so if pass to Y', strict transform

$\underbrace{X' \text{ of } X}_{}$, $\exists\ Y \longrightarrow \overline{\mathcal{M}_{g,n}}$ s.t. $\mathcal{U}$ recovers $(X_{\mathcal{U}}, \sigma_{\mathcal{U}})$

rename as Y, X

Upshot: $\exists$ stable n-pt'd genus g $(\mathcal{C}, \Sigma)$ over Y and

$$\begin{tikzcd} (\mathcal{C}_{\mathcal{U}}, \Sigma_{\mathcal{U}}) \arrow[rr, "\psi", "\simeq"'] \arrow[dr] & & (X_{\mathcal{U}}, \sigma_{\mathcal{U}}) \arrow[dl] \\ & \mathcal{U} & \end{tikzcd}$$

Intuition: global stable reduction thm for $(X_{\mathcal{U}}, \sigma_{\mathcal{U}}) \to \mathcal{U}$

$$\begin{tikzcd} \Gamma_\psi \arrow[r, phantom, "\underset{\text{closed}}{\subset}"] \arrow[d, "\simeq"'] & \mathcal{C}_{\mathcal{U}} \times_{\mathcal{U}} X_{\mathcal{U}} \arrow[dl] \\ \mathcal{C}_{\mathcal{U}} & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} T = \overline{\Gamma_\psi} \arrow[r, phantom, "\subset"] & \mathcal{C} \times_Y X \arrow[d, phantom, "\cap\ \text{closed}"] \\ & \mathbb{P}^N_Y \end{tikzcd}$$

Dream

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "??"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

extending ψ (automatic $\Sigma_i \mapsto \sigma_i$)

Aim. Find modification $Y' \longrightarrow Y$ s.t. after passing to Y', X', get $T' \longrightarrow \mathcal{C}'$ isom
(use 3-pt Lemma)

Step 0 Find modif. so T' is Y'-flat (w/ geom. conn'd fiber of dim 1)

Facts Using graph closure trick w/ Hilbert schemes $\left(\begin{matrix} T_{\mathcal{U}} \simeq \mathcal{C}_{\mathcal{U}} \\ \downarrow \quad \text{is flat} \\ \mathcal{U} \end{matrix} \right)$

$\left(T_{\mathcal{U}} = \mathcal{U}\text{-flat} \Rightarrow \mathcal{U} \longrightarrow \mathrm{Hilb}_{\mathbb{P}^N_{\mathcal{U}}/\mathbb{Z}} \right)$

extend to Y, or modif. of Y ... (notes)

After U-modif. of Y, can ensure T is Y-flat $(\Rightarrow T = \overline{T_U} \subset \mathcal{C} \times_Y X)$.

Pass to $\widetilde{Y} \to Y$, so Y normal.

Step 1 (next time) $T \to Y$ is its own Stein factorization $(\Rightarrow$ geom. conn'd fibers$)$

Thm [EGA IV_3, 12.2.1(ii)] $Z \to S$ proper flat f.pres. $\Rightarrow \dim(Z_s)$ is loc. const. in s.

Lecture 19. Normality and applications

$$\begin{tikzcd} \mathcal{C} \arrow[dr] & & X \arrow[dl] \\ & Y \arrow[ul, bend left, "\underline{\sigma}"] \arrow[ur, bend right, "\underline{\sigma}"'] & \end{tikzcd}$$

and $= $ proj. normal var.

$$\begin{tikzcd} \mathcal{C}_U \arrow[rr, "\simeq", "\varphi"'] \arrow[dr] & & X_U \arrow[dl] \\ & U & \end{tikzcd}$$

$$\begin{tikzcd} T = \overset{= \text{ proj. var.}}{\overline{\Gamma_\varphi}} \subset \mathcal{C} \times_Y X \arrow[d] \arrow[d, bend right, "\text{flat}"'] \\ Y \end{tikzcd}$$

$(\Gamma_\varphi \xrightarrow{\sim} \mathcal{C}_U, X_U)$

The map $T \xrightarrow{pr_1} \mathcal{C}$ is proper birat'l ($\simeq$ over U)

and want it to be an isom. [so

$$\begin{tikzcd} \mathcal{C} & T \arrow[l, "pr_1"', "\sim"] \arrow[r, "pr_2"] & X \end{tikzcd}$$

over Y, $\mathcal{C} \xrightarrow{\widetilde{\varphi}} X$ agrees w/ φ over U

and $\Gamma_{\widetilde{\varphi}} \simeq \mathcal{C} = Y$-flat, so

$\Gamma_{\widetilde{\varphi}} = \overline{\Gamma_{\widetilde{\varphi}}|_U} = \overline{\Gamma_\varphi} = T$

Step 1 Prove sst curve w/ smooth gen. fiber over conn'd normal noeth base $(\mathcal{C} \to Y)$ is normal.

Step 2 Study $T_{\bar{y}} \subset \mathcal{C}_{\bar{y}} \times X_{\bar{y}}$ to infer $\underset{\text{conn'd curve}}{\boxed{T_{\bar{y}}}} \longrightarrow \mathcal{C}_{\bar{y}}$ are q. finite, so

$pr_1 : \underset{\text{proj. var.}}{T} \longrightarrow \underset{\text{normal}}{\mathcal{C}}$ is proper + q. finite, so finite, hence $\simeq$.

$\ast$ field

Ex $A =$ normal noeth domain, $K = \mathrm{Frac}(A)$, $a \in A - \{0\}$

$$B = A[x,y]/(xy-a)$$

$$= A\text{-free on } \{1, x, x^2, x^3, \ldots, y, y^2, y^3, \ldots\}$$

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[d] \arrow[r, phantom, "="] & X \arrow[d] \\ \mathrm{Spec}\, A \arrow[r, phantom, "="] & S \end{tikzcd}$$

$$B \hookrightarrow B_K = k[x, \tfrac{1}{x}]$$

Check by hand (working inside normal B_K) that B is normal

Hint. $A = \bigcap_{ht=1} A_{\mathfrak{p}} = \mathrm{DVR}$

Pf of Step 1

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$ sst curve ($\Rightarrow$ flat)

$S =$ conn'd normal noeth $\Rightarrow$ irred., $\exists!$ gen. pt $\eta \in S$

Want to verify all $\mathcal{O}_{X,x}$ are normal: Serre's homological criteria "$R_1 + S_2$"

(R_1) $\mathcal{O}_{X,x}$ field or DVR when $\dim \mathcal{O}_{X,x} \le 1$

(S_2) [given R_1] $\underbrace{\operatorname{depth} \mathcal{O}_{X,x} \ge 2}_{\exists \text{ reg. seq. of length 2 in } \mathfrak{m}_x}$ when $\dim \mathcal{O}_{X,x} \ge 2$

We know all $\mathcal{O}_{S,s}$ satisfy "$R_1 + S_2$".

f flat $\underset{\text{dim. formula}}{\Rightarrow}$ $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{S,f(x)} + \dim \mathcal{O}_{X_{f(x)},x}$

When $\dim \mathcal{O}_{S,f(x)} \ge 2$, then $\dim \mathcal{O}_{X,x} \ge 2$ and has depth ≥ 2:

use length-2 reg. seq. from $\mathcal{O}_{S,f(x)}$ (reg. seq. in $\mathcal{O}_{X,x}$ by flatness!)

Can now focus on dim $\mathcal{O}_{S,f(x)} \leq 1$ – then $\mathcal{O}_{S,f(x)}$ = field ($f(x) = \eta$)

or DVR (∵ S is normal)

By hypothesis, X_η is $\underline{\text{smooth}}$ (curve), so if $f(x) = \eta$, then $\mathcal{O}_{X,x} = \mathcal{O}_{X_\eta, x}$ = field or DVR

(X_η = smooth curve)

$\underline{\text{Remains}}$. $\mathcal{O}_{S,f(x)} =: R$ is DVR, so can localize S at $f(x)$, so $S = \text{Spec}(R)$ (care about $\mathcal{O}_{X,x}$).

Let $\pi \in R$ uniformizer.

Now $x \in X_0$ = special fiber and

$$\dim \mathcal{O}_{X,x} = 1 + \dim \mathcal{O}_{X_0,x} = \left[\begin{array}{ll} 1, & x \in X_0 \text{ generic pt} \\ 2, & x \in X_0 \text{ closed pt} \end{array}\right.$$

$\underline{\dim \mathcal{O}_{X,x} = 1}$: $\pi \in \mathfrak{m}_x$ is not zero divisor ($\mathcal{O}_{X,x}$ = R-flat)

$$\mathcal{O}_{X,x}/(\pi) = \underbrace{\mathcal{O}_{X_0,x}}_{\text{reduced curve}} = \text{field} \quad \text{gen. pt}$$

[Ch I, §2, Prop 2] in Serre's Local Fields ⇒ $\mathcal{O}_{X,x}$ = DVR.

$\underline{\dim \mathcal{O}_{X,x} = 2}$: Want reg. seq. of length 2.

$$\mathcal{O}_{X,x}/\underbrace{(\pi)}_{\text{not zero div.}} = \underbrace{\mathcal{O}_{X_0,x}}_{\text{reduced curve (pure dim 1)}} \leftarrow \text{closed} = 1\text{-dim'l}$$

Serre's "$R_0 + S_1$" criterion for reducedness: $\exists\, \bar{t} \in$ max. ideal of $\mathcal{O}_{X_0,x}$ not a zero-divisor, so $\{\pi, t\} \subset \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is reg. seq.

Upshot: $\mathcal{C}$ is normal!

To show $T \xrightarrow{pr_1} \mathcal{C}$ is isom, remains to check it is q-finite (by properness)

want $\underbrace{T_{\bar{y}}}_{\text{conn'd curve}} \longrightarrow \underbrace{\mathcal{C}_{\bar{y}}}_{\text{conn'd sst curve}}$ to be q. finite. $\forall$ geom. pts $\bar{y} \in Y$

$T_{\bar{y}} \subset \mathcal{C}_{\bar{y}} \times X_{\bar{y}}$ $\quad (T \subset \mathcal{C} \times_Y X)$

Suffices to rule out some irred. comp. of $T_{\bar{y}}$ crushed to pt in $\mathcal{C}_{\bar{y}}$.

Lemma 4.20 Consider irred. comp. decompositions of conn'd curves over $\bar{y}$:

$$\begin{tikzcd}
 & T_{\bar{y}} = T_1 \cup \cdots \cup T_t \subset \mathcal{C}_{\bar{y}} \times X_{\bar{y}} \arrow[dl, "(pr_1)_{\bar{y}}"'] \arrow[d, "(pr_2)_{\bar{y}}"] & \\
\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s = \mathcal{C}_{\bar{y}} & X_{\bar{y}} = X_1 \cup \cdots \cup X_r &
\end{tikzcd}$$

(i) $\forall X_i$, $\exists!\ j = j(i)$ s.t. $T_j \twoheadrightarrow X_i$ and

(*) $T_{j(i)} \longrightarrow \mathcal{C}_{\bar{y}}$ is not constant.

and $\exists$ open $V \subset X$ w $V \cap X_{\bar{y}}$ dense and $pr_2^{-1}(V) \xrightarrow{\sim} V$.

(ii) $\forall \mathcal{C}_\alpha$, $\exists!\ \gamma = \gamma(\alpha)$ s.t. $T_\gamma \twoheadrightarrow \mathcal{C}_\alpha$ and $\exists$ open $W \subset \mathcal{C}$ s.t.

$W \cap \mathcal{C}_{\bar{y}}$ dense, $\forall \bar{y}$ w $pr_1^{-1}(W) \xrightarrow{\sim} W$.

Rmk (*) is exactly due to $(\mathcal{C}, \underline{\Sigma})$ being stable and $(X, \underline{\sigma}) \to Y$ satisfying 3-pt Lemma.

Role of (ii) will be to help in pf of (*).

Lemma $\Rightarrow$ pr_1 is q-finite. Suppose not, so $\exists\, \bar{y}$ and $T_j \subset T_{\bar{y}}$ sent to pt in $\mathcal{C}_{\bar{y}}$.

By (i), such $j \neq j(i)$, $\forall i$, so $T_j \longrightarrow X_{\bar{y}}$ is pt. But $T_j \subset \mathcal{C}_{\bar{y}} \times X_{\bar{y}}$. contradiction.

Pf of (i) $\quad T \xrightarrow{\mathrm{pr}_1} X \quad$ isom. over $\mathcal{U}$, $T_{\mathcal{U}} = \Gamma_\psi$, $\psi : \mathcal{C}_{\mathcal{U}} \simeq X_{\mathcal{U}}$ so dominant and proper, so surjective.

$$\begin{tikzcd} T_{\bar{y}} \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & X_{\bar{y}} \arrow[d, phantom, "\shortparallel"] \\ T_1 \cup \cdots \cup T_t & X_1 \cup \cdots \cup X_2 \end{tikzcd}$$

so $\forall X_i$, some $T_j \twoheadrightarrow X_i$. Next time: !ness of j, and existence of V.

Lecture 20 Altering to a stable curve II

Situation.

$$T = \Gamma_\psi \subset \mathcal{C} \times_Y X$$

$$\text{stable}\left[\begin{tikzcd} & T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\text{flat}"] & \\ \mathcal{C} \arrow[dr, "h"] & & X \arrow[dl, "f"'] \\ & Y \arrow[ul, bend left, "\underline{\tau}"] \arrow[ur, bend right, "\underline{\sigma}"'] & \end{tikzcd}\right.$$

Y $\shortparallel$ normal ($\Rightarrow \mathcal{C}$ normal)

Goal: $p_1 : T \longrightarrow \mathcal{C}$ (proper birat'l) is q-finite, i.e. $T_{\bar{y}} \longrightarrow \mathcal{C}_{\bar{y}}$ does not send any irred. comp. of $T_{\bar{y}}$ onto a pt.

- All $T_{\bar{y}}$ are conn'd curves
- $(\tau_i, \sigma_i) : Y \longrightarrow \underset{\text{closed}}{T} \subset \mathcal{C} \times_Y X$ (check on $\mathcal{U}$)

$$\left.\begin{tikzcd}[column sep=tiny] & T_{\bar{y}} = \bigcup_{j=1}^{t} T_j \arrow[dl] \arrow[dr] & \\ \bigcup_{\alpha=1}^{\ell} \mathcal{C}_\alpha = \mathcal{C}_{\bar{y}} & & X_{\bar{y}} = \bigcup_{i=1}^{s} X_i \end{tikzcd}\right] \begin{array}{l} \text{irred compts} \\ \text{(all 1-dim'l)} \end{array}$$

Last time we showed

$= Y\text{-smooth} \Rightarrow \text{normal!}$

build inside $\mathrm{sm}(X/Y) \subset X$

ⓟ① $\forall i, \exists!\, T_j = T_{j(i)} \overset{p_2}{\twoheadrightarrow} X_i$ and $\exists$ open $V \subset X$ fiberwise dense $/Y$

and $p_2^{-1}(V) \xrightarrow{\sim} V$.

ⓟ② $\forall \alpha, \exists!\, T_\gamma = T_{\gamma(\alpha)} \overset{p_1}{\twoheadrightarrow} \mathcal{C}_\alpha$ and $\exists$ open $W \subset \mathcal{C}$ fiberwise dense $/Y$ s.t.

$p_1^{-1}(W) \xrightarrow{\sim} W$.

<u>We need</u> (to prove) $(*)$: each $T_{j(i)}$ in ⓟ① is <u>not</u> sent to a pt in $\mathcal{C}_{\bar y}$ (asymmetric assertion)

This will use ⓟ② and stability of $(\mathcal{C}, \underline{\tau})$ and "3-pt lemma" for $(X, \underline{\sigma})$.

Assume some $(p_1)_{\bar y}(T_{j(i)}) = \{c\} \in \mathcal{C}_{\bar y}$. Seek $\Rightarrow\Leftarrow$.

Note $(\tau_i(\bar y), \sigma_i(\bar y)) \in T_{\bar y}$, w/ $\tau_i(\bar y) \in \mathcal{C}^{sm}_{\bar y}$.

$T_{j(i)} \overset{p_2}{\twoheadrightarrow} X_i \supset X_i^{sm} \ni \overset{= x_\alpha}{\sigma_\alpha(\bar y)}, \overset{= x_\beta}{\sigma_\beta(\bar y)}, \overset{= x_\gamma}{\sigma_\gamma(\bar y)}$ 3 distinct pts

We're going to argue separately depending on whether or not $c \in \{\tau_\alpha(\bar y), \tau_\beta(\bar y), \tau_\gamma(\bar y)\}$

<u>Aim</u>
$$\left[\begin{array}{l} \bullet\ c \notin \{\dots\} \Rightarrow \exists\ 3 \text{ irred. comp of } \mathcal{C}_{\bar y} \text{ through } c. \\ \bullet\ c \in \{\dots\} \subset \mathcal{C}^{sm}_{\bar y} \Rightarrow \exists\ 2 \text{ irred comp. of } \mathcal{C}_{\bar y} \text{ through } c \end{array}\right] \Rightarrow\Leftarrow$$

$$\begin{tikzcd}
T \arrow[r, "p_2"] & X \\
\cup & \cup \\
p_2^{-1}(\mathrm{sm}(X/Y)) \arrow[r] & \mathrm{sm}(X/Y) \ni X_\alpha, X_\beta, X_\gamma \\
& \text{normal } (\because Y\text{-smooth, } Y \text{ normal}) \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

proper birat'l $(/Y)$ so is own Stein factorization.

So p_2 has geom. conn'd fibers $/ \mathrm{sm}(X/Y)$

$\longrightarrow$ $p_2^{-1}(X_\alpha)$, $p_2^{-1}(X_\beta)$, $p_2^{-1}(X_\gamma)$ all <u>conn'd</u>, so either pts ~~or~~ conn'd chain of irred. comps T_j. All three meet $T_{j(i)} \twoheadrightarrow X_i$.

So $p_1(p_2^{-1}(X_\alpha))$, $p_1(p_2^{-1}(X_\beta))$, $p_1(p_2^{-1}(X_\gamma))$ all pt or conn'd chain of irred comps in $\ell_{\bar{y}}$, all $\ni c$.

<u>Case 1</u>. $c \notin \{\tau_\alpha(\bar{y}), \tau_\beta(\bar{y}), \tau_\gamma(\bar{y})\}$.

We have $t_\alpha = (\tau_\alpha(\bar{y}), \underset{X_\alpha}{\overset{\shortparallel}{\sigma_\alpha(\bar{y})}}) \in p_2^{-1}(X_\alpha)$, so $\underset{\tau_\alpha(\bar{y}) \neq c}{\overset{\shortparallel}{p_1(t_\alpha)}} \in p_1(p_2^{-1}(X_\alpha))$

$(\in T_{\bar{y}}\,!!)$

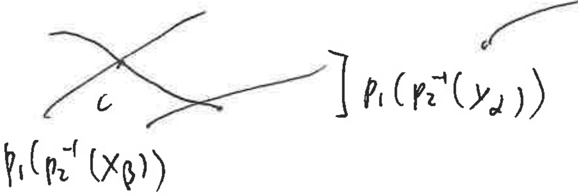
$\therefore$ $p_1(p_2^{-1}(X_\alpha))$ is conn'd chain of irred comps, containing c.

In particular, $p_2^{-1}(X_\alpha) \subset T_{\bar{y}}$ is conn'd chain of irred. comp.

Likewise, $p_2^{-1}(X_\beta)$, $p_2^{-1}(Y_\gamma) \subset T_{\bar{y}}$ are conn'd chains of irred comp.

and same for $p_1(p_2^{-1}(X_\beta))$, $p_1(p_2^{-1}(Y_\gamma)) \subset \ell_{\bar{y}}$.

$X_\alpha, X_\beta, X_\gamma \in X_i$ distinct, so $p_2^{-1}(X_\alpha)$, $p_2^{-1}(X_\beta)$, $p_2^{-1}(X_\gamma) \subset T_{\bar{y}}$ are <u>disjoint</u> conn'd unions of irred comp.



ⓟ2 gives each irred comp of $p_2^{-1}(X_\alpha)$ <u>not</u> sent to a pt is ! one going onto its irred comp image in $\ell_{\bar{y}}$.

provides a <u>different</u> irred comp of $\ell_{\bar{y}}$ through c $(p_2^{-1}(X_\beta), p_2^{-1}(X_\alpha)$ disjoint$)$.

Same for γ, get 3 irred comp of $\mathcal{C}_{\bar{y}}$ through c. $\Rightarrow\Leftarrow$.

Case 2 $\quad c \in \{\tau_\alpha(\bar{y}), \tau_\beta(\bar{y}), \tau_\gamma(\bar{y})\} \subset \mathcal{C}_{\bar{y}}^{sm}$,

The same reasoning gives 2 irred comp of $\mathcal{C}_{\bar{y}}$ through c ($\because$ $c \neq 2$ of these 3 markings) $\square$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\beta"] \arrow[dr, "h"'] & & X \arrow[dl, "\sigma"] \\ & Y \arrow[ul, bend left, "\underline{\tau}"] \arrow[ur, bend right, "\underline{\sigma}"'] & \end{tikzcd}$$

$X \supset Z = |\text{Cartier}| = \bigcup_{i=1}^{n} \sigma_i(Y)$, $\quad \beta \circ \tau_i = \sigma_i$

β_U is isom.

(so β is modification)

$$\beta^{-1}(Z) \subset \underbrace{\left(\bigcup_{i=1}^{n} \tau_i(Y)\right) \cup h^{-1}(D)}_{} = \mathfrak{z}, \quad \text{for } D = (Y - U)_{red} \subsetneq Y$$

Saw can replace "Z" w/ bigger proper closed subset. So can replace (X, Z) w/ $(\mathcal{C}, \mathfrak{z})$

By induction, $\exists$ gen. étale alteration $\varphi : \overset{\text{smooth}}{Y'} \to Y$ s.t. $\varphi^{-1}(D) = D'$ is sncd in Y'.

$$\begin{tikzcd} \mathcal{C}_{Y'} \arrow[r, "\tilde{\varphi}"] \arrow[d, "h'", "\substack{\text{sst}\\\text{curve}}"'] & \mathcal{C} \\ Y' & \end{tikzcd}$$

is gen. étale alteration

smooth over $U' = \varphi^{-1}(U)$ $\quad Y' = \underline{\text{smooth}}$

$$\tilde{\varphi}^{-1}(\mathfrak{z}) = \left(\bigcup_{i=1}^{n} \tau_i'(Y')\right) \cup h'^{-1}(\underline{D'})$$

$\uparrow$ $\in Sm(\mathcal{C}'_{Y'}/Y')$ $\qquad$ sncd in Y'

Lecture 21 Artin approximation

Motivation via semistable curves Eg. Take R a DVR w/ unif. π, residue field k, frac. field K.

"standard" semistable curves over R: $\quad C_n := \operatorname{Spec}\left(R[u,v]/(uv - \pi^n)\right)$, $n \geq 1$.

generic fiber $\mathrm{Spec}(K[u,u^{-1}])$ — K-smooth, special fiber $\mathrm{Spec}(k[u,v]/(uv))$

union of coord axes in $\mathbb{A}^2_k$.

Special fiber has 2 irred components w/ a (k-rat'l) singularity $\xi = (u,v)$ that is the unique point at which C_n is <u>not</u> R-smooth. Thus, $C_n - \{\xi\}$ is regular for all n, and the completion of the local ring of C_n at this point is $\hat{\mathcal{O}}_{C_n,\xi} = \hat{R}[[u,v]]/(uv-\pi^n)$

For $n=1$, $uv - \pi \in \mathfrak{m} - \mathfrak{m}^2$, so this quotient ring is regular (and hence C_1 is regular)

But for $n \geq 2$, $uv - \pi^n \in \mathfrak{m}^2$ and so this is not regular since $\dim(\mathfrak{m}/\mathfrak{m}^2) = 3$ (w/ basis $\{\bar{\pi}, \bar{u}, \bar{v}\}$). In fact,

$$\mathrm{Bl}_\xi(C_n) = \begin{cases} \text{regular}, & n=2 \\ \text{covered by copies of } C_{n-2}, & n \geq 3 \end{cases}$$

This gives us an inductive process of (coord-free!) singularity resolution for C_n: blow up at the (finitely-many) non-regular points, then repeat. In particular, such blow-up yields a scheme that is again a semistable R-curve

For us, "singularity" = "non-regular point", it's a property of a scheme at a pt, not a relative property (as for smoothness over a base)

* * *

We want to generalize the above example for general (possibly non-proper) semistable curves $C \to \mathrm{Spec}\, R$ w/ smooth generic fiber. In particular, such a C may not be covered by C_n's for the Zariski topology, and the non-regular locus of the special fiber may not consist of k-rat'l pts (and such sing. might involve "self-crossing" of a single irred. comp. as for the nodal plane cubic). Artin approx. will rescue the situation by showing that C_n's provide an adequate model for making calculations for a general semistable curves over a DVR (w/ smooth gen. fiber) if we work "étale-locally".

We will also generalize to a higher-dim'l regular base, i.e. when we have a semistable curve (w/ smooth generic fiber) $X \longrightarrow Y$ w/ $\dim(Y) > 1$ and we will be able to push $\mathrm{Sing}(X) = X - \mathrm{Reg}(X)$ into codim ≥ 3 via an intrinsic blow-up process. This will be done while keeping track of the specified proper closed subset $Z \subsetneq X$.

§

A local map of local rings $A \to A'$ is called local-étale if $A' = B_{\mathfrak{p}}$ for an étale A-alg. B and a prime ideal $\mathfrak{p}$ of B.

If $A \to A' = B_{\mathfrak{p}}$ is local-étale (w/ B étale $/A$ and $k'|k$ finite separable ext'n of residue fields) and X a connected A-scheme equipped w/ a pt x_0 over the closed pt then an A-map $f: X \to \mathrm{Spec}(A')$ carrying x_0 to the closed pt (if one exists!) is uniquely determined by the induced map on residue fields $k' \to k(x_0)$. In particular, if $A \to A'$ is residually trivial, then there is at most one $X \to \mathrm{Spec}(A')$ over A carrying x_0 to the closed pt.

Reason f determined by its effect on residue fields: $f, g: X \rightrightarrows \mathrm{Spec}(A')$ A-maps carrying x_0 to the closed pt, and inducing the same map $k' \to k(x_0)$ on res. fields

$$F = (f,g) : X \longrightarrow \mathrm{Spec}(A') \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(A') = \mathrm{Spec}(A' \otimes_A A') = \Delta \amalg S'$$

w/ diagonal $\Delta = \mathrm{Spec}(A')$ splitting off as a clopen subscheme (as $A' = B_{\mathfrak{p}}$ w/ A-étale B)

$F(x_0) \in \Delta$ by the equality of maps $h' \rightrightarrows k(x_0)$. $\underset{\neq \phi}{F^{-1}(\Delta)} \overset{\text{clopen}}{\subset} X \overset{\text{conn'd}}{\Rightarrow} F^{-1}(\Delta) = X$

So $F(X) \subset \Delta$, $f = g$.

Cor. the collection of all residually trivial local-étale A-algs has a *unique* str. of directed system via local A-alg maps: uniqueness is clear by the preceding discussion.

existence: A, A'' local-étale / A w/ res. field k, then the local ring $A' \otimes_A A''$ at the evident k-pt is local-étale over A and receives local A-alg. maps from A' and A''.

Def. A local ring A is *henselian* if for any local-étale $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ w/ trivial res. field ext'n has a section.

[Equiv. in terms of Hensel's lemma: if $F \in A[T]$ is monic, then any monic *coprime* factorization $F_0 = G_0 H_0$ over the res. field lifts to a monic fact. in $A[T]$]

Prop For A a local ring, its *henselisation* is the local ring $A^h := \varinjlim_{A \to A'} A'$. Colimit over the directed system of local-étale A-algs w/ trivial res. field ext'n.

This ring is henselian, and has the universal property that any local map from A to a henselian local ring extends !-ly to A^h. Moreover, henselisation preserves *noetherian* excellence, reducedness, regularity & normality, and commutes w/ reduction modulo an ideal, so it also preserves completion in the noetherian case (i.e. $\widehat{A^h} = \widehat{A}$).

Eg. R DVR w/ frac field K, $D \subset \mathrm{Gal}(K_s|K)$ a choice of decomposition group.

$R' =$ integral closure of R in $K_s^D \subset K$, $m' = R' \cap m_s$, $R'_{m'}$ is a henselisation of R.

Thm (Artin-Popescu approximation) Let (A, m) be an excellent noetherian local ring w/ max'l ideal m. $B = A[x_1, \dots, x_n]/(f_1, \dots, f_r)$ a finite-type A-alg. The canonical map $\mathrm{Hom}_A(B, A^h) \longrightarrow \mathrm{Hom}_A(B, \hat{A})$ has dense image in the $\hat{m}$-adic topology. More precisely, given an A-alg hom. $\phi: B \to \hat{A}$ and $n \geq 1$ as large as we please, there exists an A-alg. map $\psi: B \to A^h$ s.t. $\psi \equiv \phi \pmod{\hat{m}^n}$.

Prop A excellent local ring, C_1, C_2 local A-algs essentially of finite type, w/ local structure maps $A \rightrightarrows C_1, C_2$. If $\hat{C_1} \simeq \hat{C_2}$ as A-algs, then there exists a common residually trivial local-étale alg. C over C_1 and C_2.

Lecture 22 Ordinary double pt singularities

Ex. $\mathbb{Z}_{(p)}^h$ is a local ring for $\overline{\mathbb{Q}}^{D_p} \subset \mathbb{Q}_p$

Last time: A = excellent local

$$\begin{tikzcd} C_1 & & C_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$
] local, ess. f. type: $C_i = \underbrace{(B_i)}_{\text{f. type }/A}{}_{p_i}$.

If $\hat{C_1} \simeq \hat{C_2}$ over A, then $\exists$
$$\begin{tikzcd} & \mathrm{Spec}(C) \arrow[dl] \arrow[dr] & \\ \mathrm{Spec}(C_1) & & \mathrm{Spec}(C_2) \end{tikzcd}$$
] local-étale, res. trivial

Spreading out:

$$\begin{tikzcd}
(X_1, x_1) \arrow[dr, "\text{étale}"'] & & (X_2, x_2) \arrow[dl] \\
& (\operatorname{Spec} A, \mathfrak{m}) &
\end{tikzcd}$$

If $\widehat{\mathcal{O}}_{X_1,x_1} \overset{f}{\simeq} \widehat{\mathcal{O}}_{X_2,x_2}$ over $\hat{A}$, then $\exists$ common res. trivial étale nbhd

$$\begin{tikzcd}
& (X, x) \arrow[dl] \arrow[dr] & \\
(X_1, x_1) & & (X_2, x_2)
\end{tikzcd}$$

, recovers f mod $\max^{1000}$ ...

Ex. Let X be sst curve $/k = \bar{k}$, $x \in X(k)$ non-smooth.

Then $\widehat{\mathcal{O}}_{X,x} \simeq k[[u,v]]/(uv) \simeq \widehat{\mathcal{O}}_{Y,y}$ for $Y = \{uv = 0\} \subset \mathbb{A}^2_k$, $y = (0,0)$

$$\begin{tikzcd}
& (Z, z) \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\
X & & Y
\end{tikzcd}$$

Def. $X \xrightarrow{} S$ flat, fp, $\ni s$. Say a closed pt $x \in X_s$ is an ordinary double pt singularity

if $\widehat{\mathcal{O}}_{X_{\bar{s}},\bar{x}} \simeq k(\bar{s})[[t_1, \ldots, t_n]]/(q)$

for non-deg. quad. form q on $k(\bar{s})^n$.

Def For f-dim'l nonzero vector space V over a field k

a non-zero quad. form $q: V \to k$ is non-degenerate if

$(q = 0) \subset \mathbb{P}(V^*) \simeq \mathbb{P}^{n-1}$ is smooth

$(n = \dim V)$

Even For symmetric bilinear $B_q : V\times V \to k$, $B_q(v,v') = q(v+v') - q(v) - q(v')$

non-deg $\Leftrightarrow$ B_q perfect provided char $\neq 2$ or dim V = even.

Ex $n=2$, $q = t_1 t_2$.

Basic ex. of ord. double pt sing.

$$\begin{tikzcd}
& \vec{0}_s \ni x \in X = \mathrm{Spec}\left(A[t_1,\dots,t_n]/(Q-a)\right) \arrow[d] \\
s \arrow[d, equal] & \in S = \mathrm{Spec}(A) \arrow[d, equal] \\
\text{closed} & \text{local}
\end{tikzcd}$$

for - $Q = \sum_{i\le j} a_{ij} t_i t_j$ residually non-deg $\Rightarrow$ some $a_{ij} \in A^\times$.

So $Q - a \in k(\xi)[t]$ is not zero-div ($\xi \in S$) $\xrightarrow[\text{criterion (passing to noeth } A)]{\text{local flatness}}$ A flatness

- $a \in m$.

Lemma $\forall\, \wp \in \mathrm{Spec}(A)$, $Q_\wp \in k(\wp)[t_1,\dots,t_n]$ is non-deg.

Pf.

$$\begin{tikzcd}
H = (Q=0) \arrow[r, phantom, "\subset"] \arrow[dr, "{\text{proper, flat, f.p.}}"'] & \mathbb{P}^{n-1}_A \arrow[d] \\
& \mathrm{Spec}\, A
\end{tikzcd}$$

Given H_0 smooth (residually non-deg)

$\mathrm{Sm}(H/\mathrm{Spec}\,A) \overset{\text{open}}{\subset} H$ contains H_0

$\Rightarrow \mathrm{Sm}(H/\mathrm{Spec}\,A) = H$.

To analyze non-smooth locus in Basic Ex, want to analyze non-smooth locus in $(q=c) \subset \mathbb{A}^n_k$ for non-deg q.

(see Lemma, $q = Q_\wp$)

Lemma Let k = field, q = non-deg quad form on k^n. Pick $c \in k$. Let $X = \mathrm{Spec}\left(k[t_1,\dots,t_n]/(q-c)\right)$

$\underline{c=0}$ X smooth away from $\vec{0}$

$\underline{c\neq 0}$ X smooth $\underline{\text{except}}$ if char$(k)=2$ and n odd; then non-smooth at 1 geom pt.
(think $n=1$).

$\underline{\text{Pf}}$ $n=1$ easy. $\underline{\text{General } n\geq 2}$: WLOG $k=\bar{k}$.

See Exer 4 in HW 2:
$$q\simeq\begin{cases} x_1x_2+x_3x_4+\cdots+x_{n-1}x_n, & n \text{ even}\\ x_1x_2+\cdots+x_{n-2}x_{n-1}+x_n^2, & n \text{ odd}\end{cases}$$
Do explicit Jacobian calculation.

Back to Basic Ex:
$$\begin{tikzcd} X = \operatorname{Spec}\left(A[t_1,\dots,t_n]/(Q-a)\right) \arrow[d] \\ S=\operatorname{Spec} A \end{tikzcd}$$

or res. char $\neq 2$
$\downarrow$
For n even, (eg. $n=2$), non-smooth locus is $\underline{\text{exactly}}$ zero section over $\operatorname{Spec}(A/a)\subset S$.

[FK, Ch III, §2] use Artin to prove:

$\underline{\text{Str. Thm}}$. Consider $\begin{matrix} X\\ \downarrow\\ S\end{matrix}$ flat. f. presented. $x\in X_s$ ord. double pt sing.

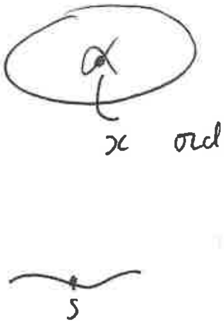
$n=1+\dim X_s$ even if char$(k(s))=2$. Pick $Q\in\mathcal{O}^{sh}_{S,s}[t_1,\dots,t_n]$ any res. non-deg. quad. form. ① $\exists$ $a\in m_{\mathcal{O}^{sh}_S}$ s.t. $\mathcal{O}^{sh}_{X,x}\simeq\left(\mathcal{O}^{sh}_{S,s}[t_1,\dots,t_n]/(Q-a)\right)^{sh}_{(s,0)}$

$\Leftrightarrow$ $\exists$ étale nbhd (S',s') of (S,s) and $\simeq$ Basic Ex's over (S',s') w/ common étale nbhd w/ (X,x).

② $a\mathcal{O}^{sh}_{S,s}$ is unique. ⊛ a unique up to unit if ideal invertible.

Cor $k(x)/k(s)$ separable!

Lecture 23 Refined str. of sst sing.



$$\begin{tikzcd}
X \arrow[d] & (X',x') \arrow[l] \arrow[r] & \mathrm{Spec}\left(A'[t]/(q_n-a')\right) \arrow[d, "\text{``basic example''}"] & (\vec{0},s') \arrow[d, mapsto] \\
S & (S',s') \arrow[l] \arrow[r] & \mathrm{Spec}(A') & s'
\end{tikzcd}$$

$a'(s')=0$, a' is ! up to units if not a zero divisor

Recall: provided require n even if $2 \notin A'^{\times}$, non-smooth locus upstairs in Basic Ex is zero section over $\mathrm{Spec}(A'/a')$.

For $n=2$, want to get "$a' \in \mathcal{O}(S)$".

Rmk If $k(x)=k(s)$, then above can be done w/ res. trivial étale nbhds at cost of unknown Q is place of q_n ($\bar{Q}$ over $k(s)$ unknown).

Focus on case $n=2$: sst curve $\begin{matrix} X \\ f\downarrow \\ \mathrm{Spec}(A)\end{matrix}$ no properness or connectedness conditions.

Setup: (A,m) = reduced local noeth. ring

Gen. fibers are smooth.

Prop. For $x \in X_0 - X_0^{sm}$, $\exists$ $a \in m$ not zero divisor s.t. (X,x) has étale nbhd in common w/ $\left(\mathrm{Spec}\left(A[u,v]/(uv-a)\right), \bar{0}\right)$. Such a is unique up to $A^{\times}$.

Toy Ex $A = R = \mathrm{DVR}$, "$uv = \pi^n$" for $n \to \infty$.

Pf. $\exists$ local-étale ext'n $A \to \overset{\text{(reduced)}}{A'}$ and $a \in A'$ giving étale-local str., want this a' to be <u>not</u> zero-divisor ($\Rightarrow$! up to unit) and to come from A" (after $A'^{\times}$-scaling)

ideal $a'A'$ is invertible

Step 1. Check a' nonzero at gen pts of A' (hence of A)

Step 2. Descend ideal $a'A'$ to $I \subset A$ via intrinsic construction.

$I \otimes_A A' = IA' = a'A'$ = invertible (step 1) $\Rightarrow$ I invertible $/A$ (descent)

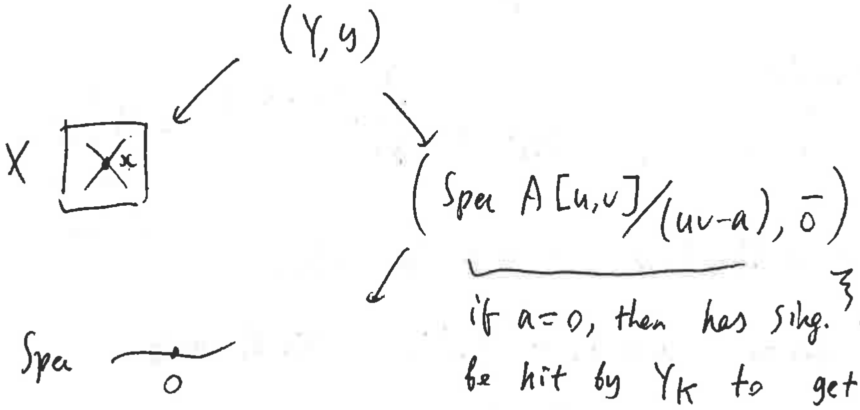
but A local, so $I = aA$ for $a \in \mathfrak{m}$. (so $a \in a'(A')^{\times}$)

Step 1. Rename A' as A (pass to $X_{A'}$ and $x' \in X_{0'}$ over x) to reduce to checking if $\exists\, a \in \mathfrak{m}$, then nonzero at gen pts.

Spec A 

$A \hookrightarrow \prod_{\min \mathfrak{p}} A/\mathfrak{p}$

Pass to $A/\mathfrak{p}$ for min $\mathfrak{p}$ so A <u>domain</u>, $K = \mathrm{Frac}(A)$. Want to check $a \neq 0$ when X_K smooth.



if $a = 0$, then has sing. in K-fiber, need it to be hit by Y_K to get sing. in X_K ($\Rightarrow\Leftarrow$)

$$\begin{tikzcd} \varphi(y) \in \varphi(Y) \underset{\text{open}}{\subset} \operatorname{Spec} A[u,v]/(uv) \\ \bar{o} \in \operatorname{Spec}(A) \arrow[u, "(u=v=0)"'] \end{tikzcd}$$

<u>local</u>

$\varphi(Y) \cap \operatorname{Spec}(A)$ is open, hence full, so contain ξ.

<u>Step 2</u> Need construction in A to descend $a'A' \subset A'$.

$I = \operatorname{ann}_A(\Omega^2_{X/A,x})$ $\left(\Omega^1_{X/A} = \text{invertible on } X^{sm}, \text{ so } \Omega^2_{X/A}\big|_{X^{sm}} = 0\right)$

Let's see this works.

$$\begin{tikzcd} & (Y,y) \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\ (X,x) & & (\operatorname{Spec} A'[u,v]/(uv-a'), \bar{o}) \end{tikzcd}$$

$$A' \underset{A}{\otimes} I \overset{\substack{\text{see notes} \\ \downarrow}}{=} \operatorname{ann}_{A'}(\Omega^2_{X_{A'}/A', x'}) = \operatorname{ann}_A(\Omega^2_{\underset{\overline{Y/A}}{Y/A'}, y})$$

$$\begin{tikzcd} X & X_{A'} \arrow[l, "\text{f-flat}"'] \\ x & x' \arrow[l, mapsto] \end{tikzcd}$$

(since $A \to A'$ étale)

$$= \operatorname{ann}_{A'}\left(\Omega^2_{\underbrace{A'[u,v]/(uv-a')}_{B'}}, \bar{o}\right) \overset{*}{=} a'A'$$

$$\Omega^1_{B'/A'} = \frac{B'du \oplus B'dv}{(udv + vdu)}, \quad \Omega^2_{B'/A'} = \frac{B'(du \wedge dv)}{(u,v)} = \underbrace{(A'/a')}_{\text{already local}}(du \wedge dv) \quad \square$$

<u>Def</u>. Let R be DVR, $K = \operatorname{Frac}(R)$, $X \longrightarrow \operatorname{Spec}(R)$, sst curve w/ X_K smooth, $x \in X_0 - X_0^{sm}$. $\pi \in R$ unif.

(X, x) has étale nbhd in common w/ $\left(\operatorname{Spec} R[u,v]/(uv-\pi^n), \bar{o}\right)$

for $\underbrace{!\ n = n_{x_0}}_{\text{measure of irregularity at } x_0} \geq 1.$

$n_{x_0} = 1 \iff x_0 \in \operatorname{Reg}(X).$

__Prop__ Assume $n_{x_0} \geq 2$, let $X' = Bl_{x_0}(X)$. Assume $X_0 - X_0^{sm} = \{x_0\}$.
$(X'_K = X_K)$

Then X' is <u>sst R-curve</u> and

1). $n_{x_0} = 2$ or 3, then X' is regular

2) $n_{x_0} \geq 4 \Rightarrow X'$ has ! non-reg. pt $x_0' \in X_0'$ w/ $k(x_0') = k(x_0)$ and $n_{x_0'} = n_{x_0} - 2$.

__Part I__ Reduce to case $X = C_{n_{x_0}}$. see notes.

__Part II__ next time review of blow-ups and compute for C_n.

Lecture 24. Blow-up of sst curve over DVR I

Discuss generalities on blow-up from A.1 − A.2.

A = ring, $I = (f_1, \cdots, f_n)$, $Z = \operatorname{Spec}(A/I) \hookrightarrow X = \operatorname{Spec}(A)$

Want to construct / describe $Y = Bl_Z(X) = Bl_I(A) = \operatorname{Proj}\left(\bigoplus_{m \geq 0} I^m\right) \subset \mathbb{P}_A^{n-1}$

$\bigoplus I^m \twoheadleftarrow A[T_1, \cdots, T_n]$

f_j in deg 1 $\longleftarrow$ T_j

Y has univ. property of being <u>final</u> among all

$$\begin{tikzcd} & Y = Bl_Z(X) \arrow[d] \\ Y' \arrow[ur, dashed, "\exists!"] \arrow[r, "\varphi"] & X = \operatorname{Spec}(A) \end{tikzcd}$$

s.t. $I\mathcal{O}_{Y'} \subset \mathcal{O}_{Y'}$ is invertible ($\iff \varphi^{-1}(Z) \subset Y'$ has invertible ideal sheaf)

$Bl_\phi(X) = X$

$Bl_X(X) = \phi$

For any such Y', have open cover by

$Y_i' = \{ f_i$ is local basis of $I\mathcal{O}_{Y'} \}$ $\rightsquigarrow$ " open subfunctor "

(if have invertible $J \subset R$ = local and generating set $\{f_1, \dots f_n\}$, then some f_j is a basis of J)

Want to describe $Y_i \subset Y = Bl_Z(X)$

$$A\left[\tfrac{f_j}{f_i}\right]_{j\neq i} \subset A\left[\tfrac{1}{f_i}\right]$$

$$Y_i' \longrightarrow \mathrm{Spec}\left(A[T_{ij} : j\neq i] \big/ (f_j - T_{ij} f_i)_{j\neq i}\right) \longrightarrow \{\text{kill } f_i^\infty\text{-torsion}\} = \mathrm{Spec}\left(A[T_{ij}: j\neq i]\big/(f_j - T_{ij} f_i)_{j\neq i} \big/ f_i^\infty\text{-tor}\right)$$

$\updownarrow$ f_i^∞-torsion

$$\mathrm{Spec}\left(A[T_{ij}: j\neq i]\big/(f_j - T_{ij} f_i)_{j\neq i} \big/ f_i^\infty\text{-tor}\right) \simeq Y_i$$

$(f_1, \dots f_n)\mathcal{O}_{Y_i'} = I\mathcal{O}_{Y_i'} = f_i \mathcal{O}_{Y'}$ and even freely gen'td.

Locally, $f_j = t_{ij} f_i$

$t_{ij} \in \mathcal{O}_{Y_i'}$

Ex. [Ch IV, Thm 2.2, Cor 2.5] : if $\{f_i\}$ reg seq. in $A \Rightarrow f_i^\infty$-torsion $= 0$.
Fulton - Lang

$Y_i = \mathrm{Spec}\left(A\left[\frac{f_j}{f_i}\right]_{j\neq i}\right)$ ~~represents~~ final among " f_i is basis of $I\mathcal{O}_Y$ " on cat. of A-schemes

Rmk. For noeth X, normalization $\tilde{X} \longrightarrow X$ is final among normal X-schemes $T \to X$ w/ dominant str. map

$$\begin{tikzcd} \text{normal} = T \arrow[r, "\exists !"] \arrow[dr] & \tilde{X} \arrow[d] \\ & X \end{tikzcd}
\qquad
\begin{tikzcd} \text{random} = T \arrow[r, "\text{random}"] \arrow[dr] & \tilde{X} \arrow[d] \\ & X \end{tikzcd}$$

$$Y_i \underset{\text{open}}{\supset} \{ f_i, f_{i'} \text{ both basis} \} \xleftarrow{} T_{ii'} \text{ unit}$$

$$\parallel$$

$$Y_{i'} \underset{\text{open}}{\supset} \{ f_i, f_{i'} \text{ both basis} \} \xleftarrow{} T_{i'i} \text{ unit}$$

$$\mathbb{A}^{n-1}_A \underset{\text{closed}}{\supset} Y_i = \operatorname{Spec}\left(A[T_{ji} : j \neq i] \big/ (f_j - T_{ji} f_i) \big/ (f_i^{\infty}\text{-torsion}) \right)$$

$$\cup \qquad = \operatorname{Spec}\left(A\left[\tfrac{f_j}{f_i}\right]_{j \neq i} \right)$$

$$\{T_{ii'} \neq 0\} \qquad T_{ii'} \neq 0 \ \cup$$

$$\wr\!\mid \ \text{glue as for } \mathbb{P}^{n-1}_A \qquad \operatorname{Spec}\left(A\left[\tfrac{f_j}{f_i}\right]_{j \neq i} \right)_{f_{i'}/f_i} \qquad \frac{f_j}{f_i} = \frac{f_{i'}}{f_i} \cdot \frac{f_j}{f_{i'}}$$

$$\{T_{i'i} \neq 0\} \qquad \wr\!\mid$$

$$\cap$$

$$\mathbb{A}^{n-1}_A \underset{\text{closed}}{\supset} Y_{i'} \underset{\text{open}}{\supset} \operatorname{Spec}\left(A\left[\tfrac{f_j}{f_{i'}}\right]_{j \neq i'} \right)_{f_i/f_{i'}}$$

This glues Y_i's along open to make $Bl_I(A) \underset{\text{closed}}{\subset} \mathbb{P}^{n-1}_A = \operatorname{Proj}(A[T_1, \dots, T_n])$

and $Y_i = Y \cap D_+(T_i)$

Blow up and base change

$$\begin{tikzcd}
Y' = Bl_{I'}(A') \arrow[r, "\exists !"] \arrow[d] & Bl_I(A) = Y \arrow[d] \\
X' = \operatorname{Spec}(A') \arrow[r] & \operatorname{Spec}(A) = X \\
Z' \arrow[u, phantom, "\cup"] \arrow[r, "\urcorner"] & Z \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$(I' = IA' = (f_1, \dots, f_n)A')$

Since $I\mathcal{O}_{Y'} = (\underbrace{IA'}_{I'})\mathcal{O}_{Y'}$ = invertible

but common square usually not cartesian!

$$\begin{tikzcd}
Y' \arrow[r, "?"] \arrow[d] & Y_{A'} \arrow[dl] \\
\operatorname{Spec}(A') &
\end{tikzcd}$$

— no reason for $I\mathcal{O}_{Y_{A'}} = I(A' \otimes_A \mathcal{O}_Y)$ to be invt.

Problem. $I \otimes_A A' \twoheadrightarrow IA' = IA'$ usually not inj.

OK for $A \to A'$ flat, so for $A \to A'$ flat, $Y' \xrightarrow{\sim} Y_{A'}$.

In general,

$$\begin{tikzcd} \widetilde{X} = Bl_Z(X) \arrow[d, "q"] \\ X \end{tikzcd}$$

call $\widetilde{Z} = q^{-1}(Z) \subset \widetilde{X}$ the exceptional divisor

(def'd by $I\mathcal{O}_{\widetilde{X}}$ = invertible)

Ex. X = f. type $/k$. $x \in |X|$ closed pt, $k' = k(x) =$ k-finite

$$\begin{tikzcd} \widetilde{X} = Bl_{\{x\}}(X) \arrow[d, "q"] \\ X \end{tikzcd}$$

What is $q^{-1}(x)$? By flat base change,

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$$

so exc. divisor is same as for $Bl_{m_x}(\mathcal{O}_x)$

$$\begin{tikzcd} = \mathrm{Proj}\left(\bigoplus_{n \geq 0} m_x^n\right) \arrow[d] & \xleftarrow[\text{fiber}]{\text{special}} \mathrm{Proj}\left(\left(\bigoplus_{n\geq 0} m_x^n\right) \otimes_{\mathcal{O}_x} k'\right) \\ \mathrm{Spec}(\mathcal{O}_x) & = \mathrm{Proj}\left(\underbrace{\bigoplus_{n \geq 0} m_x^n / m_x^{n+1}}_{k'\text{-alg}}\right) \end{tikzcd}$$

Also use $\widehat{\mathcal{O}_x}$.

Ex. $\mathcal{O}_x$ = regular $\Rightarrow$ $\underbrace{k'[t_1,\dots, t_n]}_{k'\text{-basis of } m_x/m_x^2} \xrightarrow{\sim} \bigoplus m_x^n / m_x^{n+1}$ $\left(= \mathrm{Sym}_{k'}(m_x/m_x^2)\right)$

$\therefore q^{-1}(x) \simeq \mathbb{P}^{n-1}_{k'}$ ($n = \dim \mathcal{O}_x$)

$R = \mathrm{DVR}$, $\pi =$ unif, $k = R/\mathfrak{m}$, $A = R[u,v]/(uv - \pi^n)$ $(n \geq 2)$

$$\begin{tikzcd} C_n = X = \mathrm{Spec}(A) \arrow[d, "\text{sst}"] \\ \mathrm{Spec}(R) \end{tikzcd}$$

, smooth away from $\xi = (u,v,\pi) \in X_0$.

Want to "calculate" $Bl_{(u,v,\pi)}(A)$

This will be covered by 3 affine spaces: $D_+(u)$, $D_+(v)$, $D_+(\pi)$.

where resp. u, v, π is free basis of pullback ideal.

$D_+(u)$ $v = v'u$

$\pi = \pi' u$

new variables

("T_{ij}")

$$\Big(\underbrace{A[v',\pi'] / (v - v'u, \pi - \pi'u)}_{R[u,v,v',\pi'] / (uv = \pi^n,\ v - v'u,\ \pi - \pi'u)} \Big) / (u^\infty\text{- torsion})$$

$$\left.\begin{array}{l} uv = \pi^n \\ v = v'u \\ \pi = \pi' u \end{array}\right] v'u^2 = \pi^n = (\pi')^n u^n \quad \boxed{n \geq 3}$$

$v' = (\pi')^n u^{n-2}$ mod u^∞-tor.

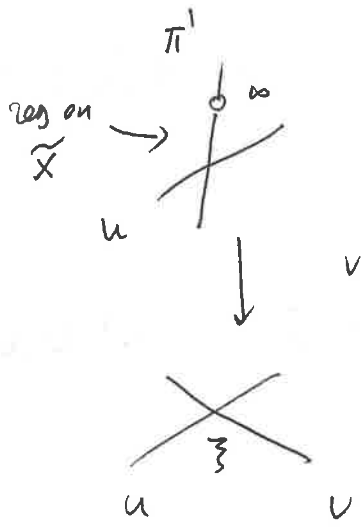
$D_+(u) = \mathrm{Spec}\left(R[u,\pi'] / (\pi - \pi' u) \right)$

= regular

$v' = (\pi')^n u^{n-2}$

$v = v'u = (\pi')^n u^{n-1}$

$$\begin{tikzcd} D_+(\pi) \bmod \pi \arrow[d] \\ \operatorname{Spec}(A/\pi) = X_0 \end{tikzcd}$$



$v = (\pi')^n u^{n-1}$

where $u=0$, nec. $v=0$ on $D_+(u)$

$D_+(v)$: similar w/ $v, \tilde{\pi}'$.

Lecture 25 . Blow-up for sst curves over dvr II

$$\xi = (u = v \overset{=\pi}{=} 0) \in \operatorname{Spec}\Big(\underbrace{R[u,v]/(uv - \pi^n)}_{A} \Big) = C_n = X, \quad n \geq 2$$

$$Bl_\xi(X) \supset D_+(u),\, D_+(v),\, D_+(\pi)$$

$$D_+(v) = \operatorname{Spec}\left(\frac{R[v, \tilde{\pi}']}{(v\tilde{\pi}' - \pi)} \right)$$

$$D_+(u) = \operatorname{Spec}\left(\frac{R[u, \pi']}{u\pi' - \pi} \right)$$ w/ $v = v'u$ for $v' = \pi'^n u^{n-2}$, $\pi = \pi' u$

$D_+(u)_k \cap D_+(v)_k$ is open locus in $D_+(u)_k$, $D_+(v)_k$ where :

v' is unit on $D_+(u)_k$, u' is unit on $D_+(v)_k$.

where $v' = \pi'^n u^{n-2}$ $\qquad u' = \tilde{\pi}'^n v^{n-2}$

$\boxed{n=2}$ π'-axis away from 0 in $D_+(u)_k$, $\tilde{\pi}'$-axis away from 0 in $D_+(v)_k$.

On here, $\pi' \tilde{\pi}' = 1$. get $\mathbb{P}^1_k$.

$\boxed{n\geq 3}$ On $D_+(u)_k$, $v' = \pi'^n u^{n-2}$ divisible by $\pi' u = 0$ on $D_+(u)_k$

So overlap $D_+(u)_k \cap D_+(v)_k = \phi$.

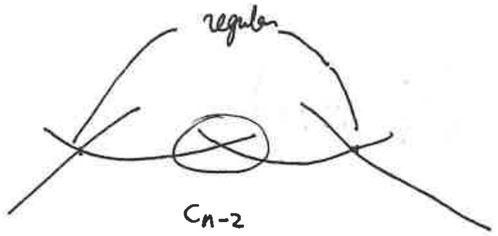
Need to compute $D_+(\pi)$. $u = u''\pi$, $v = v''\pi$, $\pi^n = uv = \pi^2 u'' v''$

$\Rightarrow u''v'' = \pi^{n-2}$

$$D_+(\pi) = \left(\mathrm{Spec}\ A[u'', v'']/(u''v''-\pi^{n-2})\right) = C_{n-2}$$

<u>$n=2$</u> $u''v''=1$, so u'', v'' are units, $D_+(\pi) \subset D_+(u)\cap D_+(v)$

Study where $D_+(\pi)_k$ meets $D_+(u)_k$, $D_+(v)_k$ by finding unit locus for multipliers $\pi', u'', \tilde{\pi}', u''$



As keep blowing-up, ! non-reg pt eventually reach regular scheme

$X' \to X = C_n$ $(n\geq 2)$, w/ X'_ξ is "chain" of $n-1$ copies of $\mathbb{P}^1_k$.

Def An <u>open sst curve</u> $X \xrightarrow{f} S$ is flat f.pres $X \to S$ w/ all geom. fibers sst curves. $\mathrm{Sm}(X/S) = \{x\in X : f \text{ is smooth}\} = \{x \in X: x \text{ smooth in } X_{f(x)}\} \overset{\text{open}}{\subset} X$

$\mathrm{Sing}(X/S) = X - \mathrm{Sm}(X/S)$ closed

Rmk If S regular, $\mathrm{Sing}(X) = X - \mathrm{Reg}(X) \overset{\text{usually not equality.}}{\subset} \mathrm{Sing}(X/S)$

__Claim__ $\mathrm{Ann}_{\mathcal{O}_X}(\Omega^2_{X/S}) \subset \mathcal{O}_X$ is finitely $\overset{\text{qcoh}}{\text{gen.}}$ ideal, formation commutes w/ base change on S. and its vanishing locus is $\mathrm{sing}(X/S)$.

On $\mathrm{sm}(X/S)$, $\Omega^2_{X/S} = 0$ so all clean

__Pf.__ Assertions are étale local on X, S, so reduce to $S = \mathrm{Spec}(A)$,

$X = \mathrm{Spec}\left(\overbrace{A[u,v]}^{B}/(uv-a)\right)$ for $a \in A$.

Direct calculation as before: $\Omega^2_{B/A} = \left(B/(u,v)\right) du \wedge dv$

$= (A/a)\; du \wedge dv$

$\left(\mathrm{Fitt}_2(\Omega^1_{X/S})\right)$

$$\begin{tikzcd} & X \arrow[d, "f"'] \\ D \arrow[r, phantom, "\subset"] & Y \end{tikzcd}$$

proper sst curve

w/ geom conn'd fibers

smooth over $U = Y - D$

D sncd

Y = smooth proj. of dim $d-1 \geq 1$

$Z = f^{-1}(D) \cup \left(\bigcup_{i=1}^{n} \sigma_i(Y)\right)$

$\boxed{d=2}$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y = \text{curve} \end{tikzcd}$$

$\mathrm{sm}(X/Y)$ complement of singularities in finitely many fibers

Pass to $\underset{\text{DVR}}{\underset{\parallel}{\mathcal{O}_{Y,y_j}}}$ for $D = \{y_1, \dots, y_m\}$ to see that blowing up X at f. many non-reg pts eventually reaches smooth surface.

For $d = 2$, still need to analyze $\tilde{Z}$ in restriction.

$\boxed{d \geq 3}$ $\quad \mathrm{Sing}(X) \subset \mathrm{Sing}(X/Y) \subset f^{-1}(D)$

has codim ≥ 2 in X = normal (sst curve $/Y$ = smooth and smooth over U)

Consider $T \subset \mathrm{Sing}(X)$ an irred comp w/ codim 2 (if any exist)

$$\begin{tikzcd} \overset{\dim d-2}{T} \arrow[r, phantom, "\subset"] \arrow[d, "{\substack{\text{q.finite} \\ \Downarrow \\ \text{finite}}}"'] & X \arrow[d, "f"] \\ \underset{\text{pure dim } d-2}{D} \arrow[r, phantom, "\subset"] & Y \end{tikzcd}$$

so
$$\begin{tikzcd} T \arrow[d, "\text{finite}"] \\ D_i \end{tikzcd}$$
D_i = irred comp of $(D = \text{sncd})$

(smooth)

Let η_i be gen pts of D_i, so $\mathcal{O}_{Y,\eta_i} = \text{DVR}$.
$\overset{\shortparallel}{R}$

$$\begin{tikzcd} Bl_T(X) \arrow[rd] \arrow[d] & \\ Y & X \arrow[l] \end{tikzcd}$$

over $\mathcal{O}_{Y,\eta_i}$

$Bl_T(X)$ is "less non-reg" generically along T: measure of irreg at η_T

for $\begin{matrix} X_R \\ \downarrow \\ \mathrm{Spec}(R) \end{matrix} \supset X_0 \ni \eta_T$ goes down.

<u>Puzzle</u>: what happens along <u>rest</u> of T?

Lecture 26. Local structure along codim 2 singularities

Put resolution task (ignoring Z) for $\begin{matrix} X \\ \downarrow \\ Y \end{matrix}$ in a broader setting:

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \arrow[r, phantom, "\overset{\text{sncd}}{\supset}"] & D \end{tikzcd}$$

proper sst curve

w/ geom. conditions

regular
excellent
conn'd
(noeth)

Assume f smooth over $S-D$

$D = \bigcup_i D_i$, D_i irred (reduced) hence regular

codim 1

Note $I_{D_i} \subset \mathcal{O}_S$ are <u>invertible</u>.

($\mathcal{O}_{S,s}$ = regular local $\Rightarrow$ UFD $\Rightarrow$ ht-1 primes are principal)

Also, $J \subset I \Rightarrow D_J := \bigcap_{i \in J} D_i$ has codim $|J|$.

Excellence will be convenient for passing to $\hat{\mathcal{O}}_{S,s}$ as the base (not f-type $/k$)

<u>Recall</u> $\operatorname{sing}(X) = X - \operatorname{Reg}(X) = \text{closed} \subset \operatorname{sing}(X/S)$ $(:= X - \operatorname{sm}(X/S))$

For irred comp T of $\operatorname{sing}(X)$, $\operatorname{codim}_X(T) \geq 2$ $\because$ X normal.

Last time:
$$\begin{tikzcd} T \arrow[d, "\text{finite}"] \\ D_{i_T} \end{tikzcd}$$
if $\operatorname{codim}_X(T) = 2$.

Want to "remove" all such T of codim 2 in X.

$$\begin{tikzcd} \operatorname{Bl}_T(X) \arrow[r] \arrow[d] & X \arrow[dl] \\ S & \end{tikzcd}$$

For generic pt η_{i_T} of D_{i_T}, passing to $\operatorname{Spec}(\underset{\text{DVR}}{\overline{\mathcal{O}_{S,\eta_{i_T}}}})$

non-reg near η_T gets "better" in $\operatorname{Bl}_T(X)$.

General setup: X also conn'd, hence integral.

Is
$$\begin{tikzcd} \operatorname{Bl}_T(X) \arrow[d] \\ S \end{tikzcd}$$
a sst curve? (still smooth over $S-D$. just $X|_{S-D}$)

Thm. $\exists$ modification $\phi: X_1 \to X$ (i.e. proper birat'l) w/ "center" $\subset \mathrm{Sing}(X)$

($\subset \mathrm{Sing}(X/S)$)

i.e. ϕ isom. over $\mathrm{Reg}(X)$

s.t. ① $X_1 \to S$ is sst curve

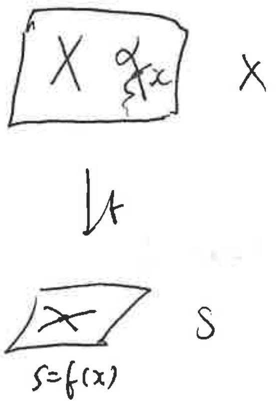
② $\mathrm{codim}_{X_1}(\mathrm{sing}(X_1)) \geq 3$.

Pf. Just have to make sure $Bl_T(X)$ is sst curve for irred comp $T \subset \mathrm{Sing}(X)$ w/ $\mathrm{codim}_X(T) \geq 2$ (this doesn't change $X - T$)

(over DVR, $Bl_\xi(X)$ has ! non-reg pt over ξ)

Need to understand étale-local str of (X, T) for $T \subset \mathrm{sing}(X)$ of codim 2 in X.

Rmk We'll see $T \underset{\text{finite}}{\twoheadrightarrow} D_{i,T}$ is étale, so such irred T are even regular



Pick $x \in X_s^{\mathrm{sing}}$ (e.g. $x \in \mathrm{Sing}(X)$, $s = f(x)$)

Want to describe $\hat{\mathcal{O}}_{X,x}$ as $\hat{\mathcal{O}}_{S,s}$-algebra. Note $k(x)/k(s)$ is finite sep'ble.

Let $D_1, \dots, D_m$ be irred comp. of D pass through s

$(I_{D_i})_s = (t_i)$ for $t_i \in m_s$. Want description in terms of t_i's.

Beware that $k(x) \neq k(s)$ may happen.

If $k(x) = k(s)$, then refined str. theorem for ord. double pt singularities gives

$\hat{\mathcal{O}}_x \simeq \hat{\mathcal{O}}_s[[u,v]]/(Q-a)$ for some residually non-deg $Q(u,v)$ and $a \in \hat{m}_s$

Issues: – Want to relate a to t_i's

– Have to allow $k(x) \neq k(s)$.

Can write $k(x) = k(s)[z]/(F_0)$ for sep'ble monic irred $F_0 \in k(s)[z]$.

$R = \underbrace{\hat{\mathcal{O}_s}[z]/(F)}_{\text{finite free } \hat{\mathcal{O}_s}\text{-module}}$ for $\underline{\underline{\text{monic}}}$ lift $F \in \hat{\mathcal{O}_s}[z]$ of F_0

w/ $R/\hat{m_s}R = k(s)[z]/(F_0) = k(x) =$ field

So $\hat{\mathcal{O}_s} \to R$ is finite $\underline{\text{étale}}$ (check discriminant) w/ R $\underline{\text{local}}$. w/ res field $k(x)/k(s)$

Hensel's lemma (for F) ↗ ∃!

$$\begin{tikzcd} & \hat{\mathcal{O}_x} \\ R \arrow[ur, "\exists !"] & \\ & \hat{\mathcal{O}_s} \arrow[uu] \arrow[ul, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec}(k(x)\otimes_{k(s)} k(x)) \arrow[r, phantom, "\subset"] \arrow[d] & X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] & x \arrow[d, mapsto] \\ \bullet \arrow[r, phantom, "\in"] & \operatorname{Spec}(R) \arrow[r] & \operatorname{Spec}(\hat{\mathcal{O}_s}) \arrow[r] & S & s \end{tikzcd}$$

Let $x'' = \Delta(x) \in x \times_S x$

$\underline{\text{Exer}}$. $\hat{\mathcal{O}_{X'',x''}} \simeq \hat{\mathcal{O}_{X,x}}$ as R-alg.

Focus on case $k(x) = k(s)$, $S = \operatorname{Spec}(A)$ for complete regular local A

$D \subset S$ w/ irred comp $\operatorname{Spec}(A/t_i) =$ regular

$s =$ closed pt

$\underline{\text{Claim}}$ $\hat{\mathcal{O}_x} \simeq A[[u,v]]/(Q - t_1^{n_1}\cdots t_m^{n_m})$ w/ res. non-deg $Q(u,v)$ and some $n_j \geq 1$

(non-reg $\Leftrightarrow$ $\sum n_j \geq 2$)

$\underline{\text{Key ideas}}$: $|\operatorname{Spec}(A/(a))| \overset{?}{\subset} |\operatorname{Spec}(A/\prod t_i)| = D$ $\quad (\prod t_i)^N = a(-)$

$\operatorname{rad}(a) \supset \operatorname{rad}(\prod t_i)$ in A = regular $\Rightarrow$ UFD $\quad$ ↑ irred in UFD A

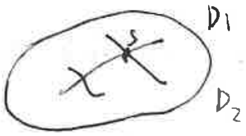
Lecture 27 Geometry for codim 2 singularities

Rmk.

$$\begin{tikzcd}
x'' = \Delta(x) & x' \arrow[rr, mapsto, bend left] & & \\
X'' \arrow[u, Rightarrow] \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] & \ni x \in X_s^{sing} \arrow[d, mapsto] \\
\mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(\hat{\mathcal{O}_s}) \arrow[r] & S & s
\end{tikzcd}$$

$\underbrace{\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(\hat{\mathcal{O}_s})}_{\text{f. ét lifting}}$

finite separable $k(x) | k(s)$



$D \subset S$
sncd

$I_{D_i, s} = (t_i)$ in $\mathcal{O}_s$ = reg local

$\{t_1, t_2, \dots\}$ part of regular system of parameters

$\overset{\nearrow}{\hat{\mathcal{O}_s}} / (t_i) = \hat{\mathcal{O}}_{D_i, s}$ = regular $\Rightarrow$ domain

So t_i irred in $\hat{\mathcal{O}_s}$

* $\{t_1, t_2, \dots\}$ also part of reg system of parameters of R = reg

(so each t_i irred. in R = UFD)

Pf (*) $(\hat{\mathcal{O}_s}, \hat{m_s}) \longrightarrow (R, n)$ étale, $\Rightarrow n/n^2 = k(x) \otimes_{k(s)} \hat{m_s}/\hat{m_s}^2$

$\{t_1, t_2, \dots\}$ lin. indep. $/k(x)$ $\Leftarrow$ $\{t_1, t_2, \dots\}$ lin indep $/k(s)$

Saw: when $k(x) = k(s)$, have $\hat{\mathcal{O}_x} \simeq \overbrace{\hat{\mathcal{O}_s}}^{A} [\![u, v]\!] / (Q - a)$ for $Q(u,v)$ = reg. non-deg quad form $/\hat{\mathcal{O}_s}$, $a \in \hat{m_s}$ to be described.

$\nearrow$ can replace w/ any lift of same reduction $/k(s)$, so can arrange $Q \in \mathcal{O}_s[u, v]$

__Claim.__ $|\mathrm{Spec}(A/a)| \subset \underset{\mathrm{Spec}(A/\prod t_i)}{\overset{}{D}}$ inside $\mathrm{Spec}\, A$ (viewed as base S)

$$D = \mathrm{Spec}(A/\pi t_i)$$

Grant claim, then $\mathrm{rad}(\prod t_i) \subset \mathrm{rad}(a)$, so $(\prod t_i)^N \in (a) = aA$

so $\prod t_r^N = ab$ in A = reg $\Rightarrow$ UFD

$\uparrow$

non-associate ireds in A

$\Rightarrow a \in A^\times \prod_i t_i^{n_i}$ some $n_i > 0$

$\Rightarrow$ replacing Q, a by common $A^\times$-multiple would give $\hat{\mathcal{O}}_x \simeq A[\![u,v]\!]/(Q \cdot \prod t_r^{n_i})$

— could then make linear change of var so $Q \in \mathcal{O}_S[u,v]$.

__Pf of claim.__

$$\begin{tikzcd}
 & (Y,y) \arrow[ld, "\acute{e}t"'] \arrow[d, "\acute{e}t"] \\
(X,x) \arrow[d, "f"'] & (\mathrm{Spec}\, A[u,v]/(Q-a), (m_S,0,0)) \arrow[ld] \\
(S=\mathrm{Spec}\, A, s) &
\end{tikzcd}$$

Artin approx.

Since f smooth over $S-D$, enough to show all fibers of f over $\mathrm{Spec}(A/a)$ are __not__ smooth.

$$\begin{tikzcd}
 & (\mathrm{Spec}\, A[u,v]/(Q-a), \arrow[ld, "h"'] & (s,0,0)) \\
\mathrm{Spec}(A) & & \\
\mathrm{Spec}(A/a) \arrow[u, phantom, "\cup"] \arrow[uur, "\sigma=(u=v=0)"] & & \\
 & s \arrow[uuur, mapsto, bend right] &
\end{tikzcd}$$

local closed subscheme inside $\mathrm{Sing}(h)$ X

$q(Y)$ open, meets $\sigma(Z(a))$, so $q(Y) \supset \sigma(Z(a))$

So $q^{-1}(\sigma(Z(a))) \twoheadrightarrow \sigma(Z(a))$

all non-sm/S since q is étale $\Rightarrow q'(q^{-1}(\sigma(Z(a)))) \subset X$

all non-smooth/S, maps onto $Z(a)$!

Back to original setup (over general conn'd, noeth excellent regular S)

$$\begin{tikzcd} T \arrow[d, "q.fin."'] & \subset & X \arrow[d, "f\ \text{proper}"] \\ D & \subset & S \end{tikzcd}$$

Assume $\exists\, T \subset \mathrm{Sing}(X)$ $(\subset \mathrm{Sing}(X/S))$

ired comp of codim 2 in X

$$\Big(\dim \mathcal{O}_{X,\eta_T} = 2\Big)$$

$$\begin{array}{c} \| \\ 1 + \underbrace{\dim \mathcal{O}_{Y,f(\eta_T)}}_{=1} \end{array}$$

, so $f(\eta_T) \in D$ is <u>generic</u>.

$$\begin{tikzcd} T \arrow[d, "\text{finite}"'] \\ D_{i_T} \end{tikzcd}$$

D_{i_T} = regular

<u>Claim</u>. $T \twoheadrightarrow D_{i_T}$ is étale (so T regular, hence $\underset{\text{domain},\ \forall x \in T}{\underset{\|}{\hat{\mathcal{O}}_{T,x}}}$)

<u>Pf</u>. Pass to affine open $\mathrm{Spec}\, B \subset D$, so situation is

$$\begin{tikzcd} \mathrm{Spec}(C) \arrow[d] \\ \mathrm{Spec}(B) \end{tikzcd}$$

B = regular ($\Rightarrow$ normal) domain

C = domain

$B \underset{\text{finite}}{\hookrightarrow} C$ w/ $\Omega^1_{C/B} = 0$

(= "unram" $\Leftrightarrow$ étale fibers)

[FK, lemma 1.5]

$\Rightarrow B \to C$ étale!

because $T \subset \mathrm{Sing}(X/S) \underset{\text{étale fibers}}{\longrightarrow} S$

For $x \in T \subset \text{sing}(X)$ w/ $\text{codim}_X(T) = 2$

$$\begin{tikzcd}
 & & (Y,y) \arrow[dl] \arrow[dr] & \\
x \in T \arrow[d] & (X,x) \arrow[dr] & & \left(\text{Spec } A[u,v]/(Q - \prod t_i^{n_i}),\ (s,0,0)\right) \arrow[dl] \\
D_{i_T} & & (S,s) \arrow[d, phantom, "\shortparallel"] & \\
 & & \text{Spec}(A) \arrow[d, phantom, "\shortparallel"] & \\
 & & \text{regular local} &
\end{tikzcd}$$

Claim $n_{i_T}(x) = n_{i_T}(\eta_T)$

$= \text{measure of irreg at } \eta_T$

Pf. localize at $f(\eta_T) = \eta_{D_{i_T}}$!

Turns in dvr base, all t_i for $i \neq i_T$ become units and have étale nbhd in common between (X, η_T) and $\left(\text{Spec } R[u,v]/(Q - t_{i_T}^{n_{i_T}}(\text{unit}))\right)$

w/ uniformizer t_{i_T}.

$\therefore$ η_T has measure of irreg $n_{i_T}(x)$.

Lecture 28 Semistability of a blow-up

$$\begin{tikzcd}
T \arrow[r, phantom, "\subset"] \arrow[d] & X \arrow[d, "f"] \\
D_{i_T} \arrow[r, phantom, "\subset"] & Y
\end{tikzcd}$$

$T \subset \text{sing}(X)$, $D_{i_T} \subset D \subset Y$, D sncd

f sst proper w/ geom conn'd fibers

Y = k-smooth proj. var $/k = \bar{k}$

f smooth over $Y - D$.

irred. $\overbrace{T \subset X}^{\text{codim } 2}$, $T \subset \text{sing}(X)$

$\left(\Rightarrow X \text{ normal, so } \text{codim}_X(\text{sing}(X)) \geq 2\right)$

We saw 1) $T \twoheadrightarrow D_{i_T}$ is finite étale, so T also regular ($\Leftrightarrow$ k-smooth)

2) $R = \mathcal{O}_{Y, \eta_{i_T}}$ = DVR and η_T is non-reg pt in special fiber of $X_R \to \mathrm{Spec}(R)$.

If n_T = measure of irreg of X_R at η_T ≥ 2, then this is exponent of t_{i_T} is local-étale description of (X, x):

$$\hat{\mathcal{O}_x} \simeq (\hat{\mathcal{O}}_{f(x)})' [[u, v]] / (Q - \prod t_i^{n_i(x)})$$

as algebras over $(\hat{\mathcal{O}}_{f(x)})'$ = f. étale $\hat{\mathcal{O}}_{f(x)}$-alg w/ res. field $k(x) \mid k(f(x))$.

where Q is res. non-deg quad form, and t_i's generate $I_{D_i, f(x)}$ for $D_i \ni f(x)$

$n_i(x) \geq 0$.

We proved (via étale-nbhd zigzag) that $n_{i_T}(x) = n_T$ for $x \in T$
by Artin approx ≥ 2

Goal. (i) $X' = Bl_T(X)$ is again sst $/Y$, smooth over $Y - D$ since $X' \to X$ is isom. over $X - T$.

(ii) $X' \to Y$ is "better" than $X \to Y$ (e.g. fewer codim 2 irred comp of sing. locus upstairs, or lower measure irreg on one such).

We'll see: $\exists$ at most one codim 2 irred comp $T' \subset \mathrm{sing}(X')$ over T, and if so ($\Leftrightarrow n_T \geq 4$) then $n_{T'} = n_T - 2$. When $n_T = 2$ or 3, no such T'.

Rmk $\exists$ only finitely many such T (all irred comp of $\mathrm{Sing}(X)$)

Rmk Once we pass to $\mathrm{codim}_X(\mathrm{Sing}(X)) \geq 3$, still need to grapple w/ $Z \subset X$.

For sst /Y of $Bl_T(X)$, the task is Zariski-local on X near each $x \in T$.
(i.e., "open sst")

$$\begin{tikzcd} Bl_T(X) \arrow[d] \arrow[dr, "\simeq \text{ over } X-T"] & \\ Y & X \arrow[l] \end{tikzcd}$$

By Artin approx, (w/o residual triviality!)

$$\begin{tikzcd} & & (W,w) \arrow[dll, "q_1"', "\text{étale}"] \arrow[d, "q_2"] \\ T \subset (X,x) \arrow[d] \arrow[dr] & & \left(\mathrm{Spec}\left(R[u,v]/(uv - \prod t_i^{n_i(x)})\right), (y,0,0)\right) \arrow[d] \\ D_{i_T} \subset & (Y,y) \supset & (\mathrm{Spec}(R), y) \end{tikzcd}$$

- use str. thm w/ strict henselizations for ord double pt singularities

affine open meeting just $D_i \ni y$, $I_{D_i}\big|_{\mathrm{Spec}(R)} = (t_i)$

Since q_2 open, can replace X w/ $q_1(W) \ni q_1(w) = x$, so $q_1 : W \twoheadrightarrow X$, and $\mathrm{Sing}(W) = q_1^{-1}(\mathrm{Sing}\, X)$ since q_1 is étale.

$$\begin{tikzcd} \supset q_1^{-1}(T) = \text{pure codim } 2 \text{ in } W \quad (q_1 \text{ is flat q-finite}) \arrow[d, "q_1", "\text{étale}"'] \\ T = \text{regular} \Rightarrow q_1^{-1}(T) \text{ is } \underline{\text{regular}}, \text{ hence conn'd comp} \end{tikzcd}$$

= irred comp!

Thus, $\exists !$ irred comp. $\mathcal{Y}$ of $q_1^{-1}(T)$ through w, and shrink W some more around w, then get $\boxed{q_1^{-1}(T) = \mathcal{Y}}$

$$\begin{tikzcd}
 & Bl_{\mathcal{Y}}(W) \arrow[dl, "\text{étale}"'] \arrow[dr] & & \\
Bl_T(X) \arrow[dr] & \Uparrow \ \underline{\text{Cartesian}} & W \arrow[dl] & \\
 & X & \square & \mathcal{Y} \arrow[ul, hook'] \arrow[dl, "q_1"] \\
 & & T \arrow[ul, hook] & \\
 & & \ni x &
\end{tikzcd}$$

So $Bl_T(X)$ being open sst $/Y$ is reduced to same for $Bl_{\mathcal{Y}}(W)$.

~~In Sing(W), can arrange~~

$\Rightarrow \mathrm{sing}(W) = q_2^{-1}(\mathrm{sing}\, B)$

$$\begin{tikzcd}
w \in \mathcal{Y} \arrow[d] \arrow[dr, "\text{dominant}"] \arrow[dd, mapsto, bend right] & \subset W \arrow[dr, "q_2 \ \text{étale}"] & \\
q_2(\mathcal{Y}) & T' \overset{\text{irred}}{\subset} \mathrm{Sing}(B) \ \text{closed} & (\mathrm{Spec}\, B, (y,0,0)) \\
(y,0,0) & (\dim = d-2) \quad (\text{Codim} \geq 2) &
\end{tikzcd}$$

for $B = \dfrac{R[u,v]}{(uv - \prod t_i^{n_i(x)})}$

$\mathcal{Y} \subset q_2^{-1}(T') \subset \mathrm{Sing}(W)$

$\shortparallel$

regular, w/ $\mathcal{Y}$ as one of its irred comp (since T' also regular, all codim 2 irred comp of open sst over (Y, D) are regular)

$\mathcal{Y}$ is conn'd comp of $q_2^{-1}(T')$, so shrink W around $w \in \mathcal{Y}$ to get $q_2^{-1}(T') = \mathcal{Y}$.

Now

$$\begin{tikzcd}
 & Bl_{\mathcal{Y}}(W) \arrow[dl] \arrow[dr, "\text{étale}"] \arrow[d, phantom, "\checkmark"] & \\
(W, w) \arrow[dr] & & Bl_{T'}(\mathrm{Spec}\, B) \arrow[dl] \\
 & (\mathrm{Spec}\, B, (y,0,0)) &
\end{tikzcd}$$

sst $/Y$ is enough

$\underbrace{}_{}$ dim $d-2$ $\quad B = R[u,v] / (uv - \prod t_i^{n_i(x)})$

$T' \subset \mathrm{Sing}(B)$ is exactly (u, v, t_{i_T}) (see notes)

$\uparrow$

$n_{i_T}(x) = n_T \geq 2$.

Lecture 29 Irred compts of Sing(X)

Still ignoring $Z \left(= f^{-1}(D) \cup \left(\bigcup_j \tau_j(Y) \right) \right)$ inside X

and focusing on case $\mathrm{codim}_X(\mathrm{Sing}\, X) \geq 3$.

Assume $\mathrm{Sing}(X) \neq \emptyset$, pick $x \in \mathrm{Sing}(X)$

$$\begin{tikzcd}
 & (W, w) \arrow[dl, "\text{ét}", "q_1"'] \arrow[dr, "\text{ét}"', "q_2"] & \\
(X, x) \arrow[dr, "f"'] & & (\mathrm{Spec}(B), \xi) \arrow[d] \\
 & (Y, f(x)) \arrow[r, phantom, "\overset{\text{open}}{\supset}"] & (\mathrm{Spec}\, R, f(x))
\end{tikzcd}$$

$B = R[u,v] / (uv - \prod t_i^{n_i(x)})$

$]$ Suff. small so for $D_i \ni f(x)$, has $\text{ét}_{D_i}\big|_{\mathrm{Spec}(R)} = t_i R$

w/ $\{t_i\}$ part of reg system of parameters in $R_{f(x)}$.

Prop All nonzero $n_i(x)$ are equal to 1

Pf. Suppose some $n_{i_0}(x) \geq 2$. Then $T' = Z(u, v, t_{i_0}) \subset \mathrm{Spec}\, B$

$= \mathrm{Spec}(R/t_{i_0})$ ~~R = domain~~

w/ codim 2 in $\mathrm{Spec}(B)$ near ξ $\left(\begin{array}{l} \dim B_\xi \\ = 1 + \dim R_{f(x)} \\ = 2 + \dim(R_{f(x)}/t_{i_0}) \end{array} \right)$

by shrinking $\mathrm{Spec}(R)$ around ξ can arrange noeth R/t_{i_0} is domain ($\because R_{\cancel{\xi} f(x)}/t_{i_0}$ = reg local $\to$ domain)

and $T' \subset \mathrm{Sing}(B)$.

For η_{i_0} = gen pt of D_{i_0}, $A = R_{\eta_{i_0}}$ = DVR

has
$$\begin{tikzcd} \mathrm{Spec}(B_{\eta_{i_0}}) \arrow[d] \\ \mathrm{Spec}(A) \end{tikzcd}$$
is open sst curve over DVR, w/ $\eta_{T'}$ is non-smooth pt in special fiber w/ measure of irreg $= n_{i_0}(x) \geq 2$.

$\therefore \xi \in T' \subset \mathrm{Sing}(B) \subset \mathrm{Spec}(B)$ has T' as irred comp of $\mathrm{Sing}(B)$,

so $\mathrm{codim}_B (\mathrm{Sing}\, B) = 2$.

But given $q_2(W) \subset \mathrm{Spec}(B)$ hits $q_2(w) = \xi \in T'$, so $w \in q_2^{-1}(T') \subset W$

is nonempty inside $\mathrm{Sing}(W)$ $\left(= q_2^{-1}(\mathrm{Spec}\, B)\right)$

yet $q_2 =$ q-finite flat $\Rightarrow$ $\mathrm{codim}_{q_2^{-1}(T')}(W) = 2$

$\therefore \mathrm{Sing}(W) \subset W$ has codim 2

$$\begin{tikzcd} W \arrow[d, "q_1"', "\text{ét}"] & \supset \mathrm{Sing}(W) \arrow[d] & \supset T \arrow[d] & = \text{irred comp of codim 2 in } W \\ X & \supset \underset{\text{closed}}{\mathrm{Sing}(X)} & \supset \overline{q_1(T)} & \end{tikzcd}$$

has same codim in X as T does in W : 2

$\Rightarrow\Leftarrow$. $\Box$

Relabel D_i's so $f(x) \in D_i$ for $1 \leq i \leq \mu \leq n$.

so (X, x) has common $\overset{\text{local-}}{\text{étale}}$ nbhd w/ $\left(\mathrm{Spec}\, B, \xi = (f(x), 0, 0)\right)$

for $B = R[u,v] \Big/ \left(uv - \prod_{i=1}^{\mu} t_i\right)$. Note $\mu \geq 2$ since B_ξ inherits

non-reg from $\mathcal{O}_x$:
$$\begin{tikzcd} & \mathcal{O}_w & \\ \mathcal{O}_x \arrow[ur] & & \mathcal{O}_\xi \arrow[ul] \end{tikzcd}$$

$$B_\xi = \underbrace{R_{f(x)}[u,v]}_{\text{reg local}}{}_\xi \Big/ \left(\overset{\in \max^2}{\overbrace{uv - \prod t_i}}\right)$$

Since $t_i \in R_{f(x)}$ not in $\max^2$.

$\Rightarrow$ near $f(x)$ in Y

~~Prop ①~~ Near x, $\operatorname{Sing}(X)$ is covered by $\operatorname{Sing}(X) \cap f^{-1}(D_i \cap D_j)$ for $1 \le i < j \le \mu$.

because all pts in $\operatorname{Sing}(X)$ lie over at least two D_i's.

Prop ① Each (reduced) $\operatorname{Sing}(X) \cap f^{-1}(D_i \cap D_j)$ is regular

w/ irred-conn'd comp E_{ij} through x of codim 3 in $\mathcal{O}_x$

w/ $E_{ij} \xrightarrow[f]{} \underbrace{D_i \cap D_j}_{\text{regular } (\because D \underset{\text{sncd}}{\subset} Y)}$ étale

② If $E_\alpha, E_\beta \subset \operatorname{Sing}(X)$ are distinct irred comp w/ $E_\alpha \cap E_\beta \neq \emptyset$.

Then $E_\alpha \cap E_\beta$ is regular w/ codim 4 or 5 in X near each of its pts.

Pf. For ①, can zigzag through étale nbhd to pass to model case:

$(\operatorname{Spec}(B), \xi = (f(x), 0, 0))$ for $B = R[u,v]/(uv - \prod t_i)$

$$\operatorname{Sing}(B) \underset{\subset}{\supset} \bigcup_{i<j} Z(u,v,t_i,t_j)$$

Then

So $= \bigcup_{i<j}$ $Z(u,v,t_i,t_j) = \operatorname{Spec}(R/(t_i,t_j)) = D_i \cap D_j$

For ②, pick $x \in E_\alpha \cap E_\beta$: $\alpha = (i,j) \neq (i',j') = \beta$

Can zigzag to pass to model case :

$$R_{f(x)}\,[u,v]_{\mathfrak{z}} \big/ (u, v, t_i t_j, t_{i'}, t_{j'})$$

$$= \underbrace{R_{f(x)}}_{} \big/ \underbrace{(t_i, t_j, t_{i'}, t_{j'})}_{3 \text{ or } 4 \text{ here}}$$

$\dim = \dim B_{\mathfrak{z}} - 1$ $\Rightarrow$ codim 4 or 5. □

In original setup, $Z = f^{-1}(D) \cup \left(\bigcup_i \tau_i(Y) \right)$ for disjoint sections

$$\tau_i : Y \longrightarrow \mathrm{sm}(X/Y) \subset X^{\mathrm{sm}}$$

These τ_i's are "unaffected" by blow-up's at T_j'

$\operatorname{codim}_{X_1}(\operatorname{Sing} X_1) \geq 3$

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[dr, "f_1"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

] modification centred over $\operatorname{Sing}(X)$

so $Z_1 = \phi^{-1}(Z) = f_1^{-1}(D) \cup \left(\bigcup_i \tau_i'(Y) \right)$

for $\tau_i' = \phi^{-1} \circ \tau_i$.

Pass to (X_1, Z_1)

Lecture 30. Analyzing Z and resolving X

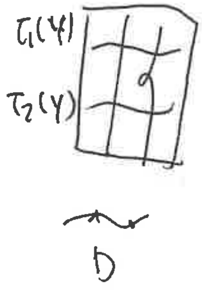
We reduced to case $\operatorname{codim}_X(\operatorname{Sing} X) \geq 3$. We also saw $\operatorname{Sing}(X)$ (if $\neq \emptyset$) has all irred comp of codim = 3 in X - blow-ups to remove these will ruin sst/Y.

So we need new axiomatic setup that omits mention of Y.

$\boxed{d=2}$ Here X is smooth, but $Z \subset X$ may be not sncd …

$Z := \left(f^{-1}(D) \cup \left(\bigcup_i \tau_i(Y)\right)\right)_{red}$ for pairwise disjoint $\tau_i : Y \to sm(X/Y) \subset X^{sm}$.

and $X \xrightarrow[sst]{f} Y$ smooth over $Y - D$ for $D \underset{sncd}{\subset} Y =$ smooth$/k$



$$\begin{tikzcd} X^{sm} \arrow[d] \\ Y \end{tikzcd}$$

<u>Prop</u> : $Z \cap X^{sm} \subset X^{sm}$ is a ncd (étale cover of $\mathcal{U} \supset Z \cap X^{sm}$, $\mathcal{U}$ open in X^{sm}, has preimage of $Z \cap X^{sm}$ as sncd)

<u>Rmk</u> App E gives for ncd $D \subset S$ = regular, $\exists$ modification (isom over $S - D$)

$\phi : S' \to S$ where S' = regular, $\phi^{-1}(D) \subset S'$ is <u>sncd</u>.

Thus Prop would settle case $d = 2$, or when $X = X^{sm}$ (any $d \geq 2$)

We could then assume $d \geq 3$.

<u>Pf</u> We'll analyse étale nbhd of (X^{sm}, x) for $x \in Z \cap X^{sm}$ in several cases.
— focus on k-pts ($k = \bar{k}$)

① $x \notin f^{-1}(D)$ (i.e. $x \in f^{-1}(Y - D)$) $\subset sm(X/Y)$, then $x \in \coprod \tau_i(Y)$ for

$\tau_i \in (sm(X/Y))(Y)$.

But section to smooth (open) curve over regular base Y is sncd.

$$\begin{tikzcd} sm(X/Y) \arrow[d, "\text{smooth (open) curve}"] \\ Y \arrow[u, bend left, "\tau_i"] \end{tikzcd}$$

② $x \in f^{-1}(D)$, $x \notin \bigcup t_i(Y)$. Distinguish $x \in Sm(X/Y)$ or not.

<u>$x \in Sm(X/Y)$</u>: $f: X \to Y$ <u>smooth</u> near x, yet $D \subset Y$ is sncd.

so easy to check $f^{-1}(D) \subset X$ near x is sncd.

<u>$x \notin Sm(X/Y)$</u>: near x, have "$Z = f^{-1}(D)$".

and have an étale nbhd of x:

$$\begin{tikzcd}
 & (W,w) \arrow[dl, "\acute{e}t"'] \arrow[d, "\acute{e}t"] \\
(X,x) \arrow[d] & (\mathrm{Spec}\, B, \xi) \arrow[d, "h"] \\
(Y, f(x)) \arrow[r, phantom, "\supset"] & \underbrace{(\mathrm{Spec}(R), f(x))}
\end{tikzcd}$$

"$\mu = 1$": $x \in X^{sm} = Reg(X)$

$$B = R[u,v]/(uv - t_{i_0})$$

$$\xi = (f(x), 0, 0)$$

small enough that $D_i \cap \mathrm{Spec}(R) = \mathrm{Spec}\,(\underset{\text{domain}}{\underbrace{R/(t_i)}})$

$$(I_{D_i}|_{\mathrm{Spec}(R)} = t_i R)$$

$\{t_i\}$ part of reg system of parameters of $R_{f(x)}$: What is $h^{-1}(D) \overset{?}{\subset} \mathrm{Spec}(B)$ near ξ.

Note $h(\xi) = f(x) \in D_{i_0}$

Want this to be sncd (near ξ): question about $B_\xi = \dfrac{(R_{f(x)}[u,v])_\xi}{(uv - t_{i_0})}$

Shrink $\mathrm{Spec}(R) \ni f(x)$ so only meets $D_j \ni f(x)$.

<u>$j \neq i_0$</u>: $h^{-1}(D_j)_\xi \longleftrightarrow \boxed{R_{f(x)}[u,v]_\xi} / (uv - t_{i_0}, t_j) = \overset{\text{regular local}}{\left(\underbrace{R_{f(x)}/(t_j)}_{\text{regular local}}[u,v]_\xi\right)} / (uv - t_{i_0})$

In here, $\{t_j\}_{j \neq i_0} \cup \{uv - t_{i_0}, u, v\}$ part of reg syst of parameters

$\boxed{j = i_0}$: $\quad h^{-1}(D_{i_0})_\xi \longleftrightarrow \left(R_{f(x)}/t_{i_0}\right)[u,v]_\xi/(uv)$

$= \text{union of } Z(t_{i_0}, u),\ Z(t_{i_0}, v)$

This is sncd

③ $x \in f^{-1}(D) \cap \underbrace{\tau_{i_0}(Y)}_{\subset \operatorname{Sm}(X/Y)}$ for some (unique) i_0.

Need to show $\left(f^{-1}(D) \cup \underbrace{\tau_{i_0}(Y)}_{\subset \operatorname{Sm}(X/Y)}\right)_{\text{red}}$ near $\overset{\in X^{sm}}{x}$ is ncd.

$\subset \operatorname{Sm}(X/Y)$, so $f: X \to Y$ is smooth near x.

For suff. small open $U \subset X$ around x:

$$\begin{tikzcd} U \arrow[r, "\text{étale}", "q"'] \arrow[dr] & \mathbb{A}^1_Y \arrow[d] \\ & Y \arrow[ul, bend left, "\tau_{i_0}"] \end{tikzcd}$$

(shrink)

where $q \circ \tau_{i_0} = 0\text{-section}$

So $\tau_{i_0}(Y) \underset{\text{clopen}}{\subset} q^{-1}(0\text{-section})$

$x \longmapsto (0, f(x))$

$$\begin{tikzcd} f^{-1}(D) \cup \tau_{i_0}(Y) \arrow[r, phantom, "\underset{\text{clopen}}{\subset}"] \arrow[d, phantom, "\cap"] & q^{-1}\left(\{0\}\times Y \cup \mathbb{A}^1_D\right) \arrow[r, phantom, "\subset"] \arrow[d, "\text{étale}"] & \mathbb{A}^1_Y & \\ U \arrow[rrrd, bend right, "\text{étale}"] & (\{0\}\times Y) \cup \mathbb{A}^1_D \arrow[r, phantom, "\underset{\text{sncd}}{\subset}"] & \mathbb{A}^1_Y & \\ & & & \left(D \underset{\text{sncd}}{\subset} Y\right) \end{tikzcd}$$

So gives ncd property near x.
(even sncd)

$\square$

<u>New axioms (w/o Y)</u>. X proj. var. $/k = \bar{k}$ of dim $d \geq 3$, $Z \subset X$ reduced closed

(i) $Z \cap X^{sm} \subset X^{sm}$ is ncd

(ii) All irred comp E of $\mathrm{Sing}(X)$ are smooth of codim $= 3$

and any $E \cap E'$ that's nonempty also smooth (maybe not transverse)

(iii) $\forall\ x \in (\mathrm{Sing}(X) \cap Z)(k)$

encoding Z near such x

$$\begin{tikzcd} \hat{\mathcal{O}_{X,x}} \simeq k[[u,v,t_1,\dots,t_{d-1}]]/(uv - t_1 \cdots t_s) \arrow[d] \\ \hat{\mathcal{O}_{Z,x}} \simeq \hat{\mathcal{O}_{X,x}}/(t_1 \cdots t_r) \end{tikzcd}$$

$$2 \leq s \leq r \leq d-1$$

(outside $\tau_i(Y)$'s) is $f^{-1}(D)$

By Artin approx, (iii) provides Basic Σ_x (if $\mathrm{Sing}(X) \neq \emptyset$) and meets Z

$k[u,v,t_1,\dots,t_{d-1}]/(uv - t_1 \cdots t_s)$, $Z \longleftrightarrow (t_1 \cdots t_r = 0)$

<u>Note</u> $I_Z \subset \mathcal{O}_X$ invertible

App F gives miracle: for $E \subset \mathrm{Sing}(X)$ irred comp, $(\mathrm{Bl}_E X$, preimage of $Z)$ satisfies (i) - (iii) w/ one less irred comp in $\mathrm{Sing}(X)$.