The Weil Conjectures

Вадим Вологодский

Lecture 1. Introduction.

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_k(x_1, \dots, x_n) = 0 \end{cases} \quad (*) \qquad f_i \in \mathbb{Z}[x_1, \dots, x_n]$$

Fix a prime p.

Let N_m be the # of solutions to $(*)$ in $\mathbb{F}_{p^m}$.

Weil's conjecture (Dwork's thm) $Z(t) := \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} t^m\right) \in \mathbb{Q}[[t]]$

is rational, $Z(t) \in \mathbb{Q}(t)$.

Remark. $t \frac{Z'(t)}{Z(t)} = t \frac{\partial}{\partial t} \log Z(t) = \sum_{m=1}^{\infty} N_m \cdot t^m$

Cor: $\sum_{m=1}^{\infty} N_m \cdot t^m$ is rational.

Rmk Let $a_1, a_2, \dots,$ $a_i \in \mathbb{C}$ a sequence.

$\exp\left(\sum_{m \geq 1} \frac{a_m}{m} \cdot t^m\right)$ is rational

$\Leftrightarrow \exists\, \alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in \mathbb{C}$ s.t. $\forall m$, $a_m = \sum_{i=1}^{r} \alpha_i^m - \sum_{j=1}^{s} \beta_j^m$

Proof $(\Leftarrow)$ $\exp\left(\sum_{m \geq 1} \frac{\alpha^m t^m}{m}\right) = \frac{1}{1 - \alpha t}$

Ex. (Gauß) $p \neq 2, 3$

$$y^2 = x^3 - 1 \quad , \quad Z(t) = \frac{1 - a_p t + p t^2}{1 - pt}$$

1) $a_p = p - N_1$,

$$a_p = 0 \qquad p \equiv 2 \pmod 3$$

$$a_p = tr(\pi) = \pi + \bar{\pi} \qquad p \equiv 1 \pmod 3$$

$$\pi \in \mathbb{Z}[\omega] \ , \ \omega = e^{\frac{2\pi i}{3}} \ , \quad \pi\bar{\pi} = p$$
$$= \mathbb{Z}[X]/(X^2 + X + 1)$$

Rmk. Consider the elliptic curve $E \subset \mathbb{P}^2_{\mathbb{C}}$, $y^2 z = x^3 - z^3$,

$$End(E) = \mathbb{Z}[\omega]$$

Weil's idea

$$X = Spec\ \mathbb{F}_p[X_1, \cdots, X_n]/(f_1, \cdots, f_k).$$

$$N_m = |X(\mathbb{F}_{p^m})|.$$

$$X_{\bar{\mathbb{F}}_p} = X \times Spec\ \bar{\mathbb{F}}_p$$

$$F : X_{\bar{\mathbb{F}}_p} \longrightarrow X_{\bar{\mathbb{F}}_p} , \quad F^*(x_i) = x_i^p.$$

$$X(\mathbb{F}_{p^m}) = X_{\bar{\mathbb{F}}_p}^{F^m}$$

$$\text{Vect}_k, \ \text{char } k = 0$$
$$\rotatebox{-90}{$\in$}$$

There should be cohomology theory, $X_{\bar{\mathbb{F}}_p} \rightsquigarrow \underset{\overset{\circlearrowleft}{F^*}}{H^*(X_{\bar{\mathbb{F}}_p})}$, $\underset{\overset{\circlearrowleft}{F^*}}{H^*_c(X_{\bar{\mathbb{F}}_p})}$, s.t. $\left| X_{\overline{\mathbb{F}}_p}^{F^m} \right| = \sum_{d\geq 0} (-1)^d \operatorname{tr}\left(F^{m*} : H_c^d (X_{\overline{\mathbb{F}}_p}) \ell \right)$

$= \sum_{i=1}^{r} \alpha_i^m - \sum_{j=1}^{s} \beta_j^m$, where α_i, β_j are eigenvalues of F^*.

Zeta function

X scheme of finite type over $\mathbb{Z}$,

$|X|$ set of closed points of X

$x \in |X|$, $\mathcal{O}_{X,x} \supset m_x$, $\mathcal{O}_{X,x} / m_x = k(x)$ residue field

$N(x) := |k(x)|$.

<u>Lemma</u>. If F is a field which is f.g. as a ring, then F is finite.

<u>Cor</u>. $N(x) < \infty$.

<u>Def</u>. Zeta function: $\zeta(s) = \prod_{x\in|X|} \frac{1}{1-N(x)^{-s}}$, $s \in \mathbb{C}$

<u>Lemma</u>. The product converges absolutely and uniformly on any compact in $\{ s : \operatorname{Re} s > \dim X \}$

<u>Cor</u>. $\zeta(s)$ is a holomorphic function on $\{ s : \operatorname{Re} s > \dim X \}$

<u>Rmk</u> $X = U_1 \cup U_2$, $\zeta(s, X) = \frac{\zeta(s, U_1) \cdot \zeta(s, U_2)}{\zeta(s, U_1 \cap U_2)}$

<u>Ex.</u> $\zeta(s, \operatorname{Spec} \mathbb{Z}) = \prod_p \frac{1}{1-p^{-s}} = \sum_{n\geq 1} \frac{1}{n^s}$

Ex. $X / \mathbb{F}_q$, $x \in |X|$, $k(x) \supset \mathbb{F}_q$, $|k(x)| = q^n$ for some n. $\deg x := n$

$$\zeta(s, X) = \prod_{x \in |X|} \frac{1}{1 - q^{-s \deg x}} \overset{t=q^{-s}}{=} \prod_{x \in |X|} \frac{1}{1 - t^{\deg x}} = Z(\cdot X, t)$$

Claim. $Z(X,t) = \exp\left(\sum \frac{N_m}{m} t^m \right)$, $N_m = X(\mathbb{F}_{q^m})$.

Lecture 2 Enough to prove, $\forall x \in |X|$,

$$\log \frac{1}{1 - t^{\deg x}} = \sum_{m \geq 1} \frac{|\{\alpha : \mathrm{Spec}(\mathbb{F}_{q^m}) \to X : \mathrm{Im}(\alpha) = x\}|}{m} t^m$$

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{F}_{q^m}) \arrow[r, dashed] \arrow[dr, "\alpha"'] & \mathrm{Spec}(k(x)) \arrow[d] \\ & X \end{tikzcd} \qquad \begin{tikzcd} k(x) \arrow[r, "\alpha"] & \mathbb{F}_{q^m} \\ \cup & \cup \\ \mathbb{F}_q \arrow[r, phantom, "="] & \mathbb{F}_q \end{tikzcd}$$

$$|\{\alpha : \mathrm{Spec}(\mathbb{F}_{q^m}) \to X : \mathrm{Im}(\alpha) = x\}| = \begin{cases} 0, & \deg x \nmid m \\ \deg x, & \deg x \mid m \end{cases}$$

So previous sum $= \sum_{l \geq 1} \frac{\deg x}{l \deg x} t^{l \deg x}$

Lemma. X scheme of f.type $/\mathbb{Z}$, $\zeta(X, s) = \prod_{x \in |X|} \frac{1}{1 - N(x)^{-s}}$ converges for $\mathrm{Re}\, s > \dim X$.

Pf. $(X/\mathbb{F}_q,)$ $t = q^{-s}$, $\zeta(X, s) = Z(t)$.

$$t \frac{\partial}{\partial t}\left(\log Z(t)\right) = \sum_{m\geq 1} N_m \cdot t^m$$

Want: converges for $0 < |t| < q^{-\dim X}$.

$$\Uparrow$$

$$N_m = O\left(q^{m \dim X}\right)$$

May assume X integral and affine. Then by Noether normalization,

$$\exists \quad \begin{tikzcd} X \arrow[d, "- \text{finite}"] \\ \mathbb{A}^{\dim X}_{\mathbb{F}_q} \end{tikzcd} \qquad \Rightarrow \left|X(\mathbb{F}_{q^m})\right| \leq C \left|\mathbb{A}^{\dim X}_{\mathbb{F}_q}(\mathbb{F}_{q^m})\right| = C q^{m \dim X}.$$

Weil's conjectures

1. For any $X/\mathbb{F}_q$, $Z(x,t)$ is rational.

- X smooth, projective, geometrically irreducible. $d = \dim X$

2. $$Z\left(X, \frac{1}{q^d t}\right) = \pm q^{\frac{dE}{2}} t^E Z(x,t)$$

$$E = \chi(X) = (\Delta_X, \Delta_X) \in \mathbb{Z}$$

$$\Delta_X \hookrightarrow X \times X$$

$$\zeta(X, d-s) \leftrightsquigarrow \zeta(X, s)$$

3. $$Z(X,t) = \frac{P_1(t) \cdots P_{2d-1}(t)}{P_0(t) \cdots P_{2d}(t)}, \quad P_i(t) \in \mathbb{Z}[t],$$

$$P_i(t) = \prod_j (1 - \alpha_{ij} t), \quad |\alpha_{ij}| = q^{i/2}$$

Proof for curves

Set up: X_0 smooth projective geometrically connected curve $/\mathbb{F}_q$.

$g := h^0(\Omega^1_{X_0}) = \dim H^0(X_0, \Omega^1_{X_0})$.

Riemann – Roch. $\mathcal{L} \in \mathrm{Pic}(X_0)$,

$$h^0(\mathcal{L}) - h^0(\mathcal{L}^* \otimes \Omega^1_{X_0}) = 1-g+\deg \mathcal{L}.$$

Cor: $\deg \Omega^1_X = 2g-2$,

If $\deg \mathcal{L} > 2g-2$, then $h^0(\mathcal{L}) = 1-g+\deg\mathcal{L}$.

Thm. $X_0/\mathbb{F}_q$, then $|\mathrm{Pic}^0(X_0)| < \infty$.

Pf. $0 \longrightarrow \mathrm{Pic}^0(X_0) \longrightarrow \mathrm{Pic}(X_0) \longrightarrow \mathbb{Z}$

Pick any $\mathcal{L}$ w/ $\deg(\mathcal{L}) = d > 2g$, then $h^0(\mathcal{L}) \neq 0$.

$\mathcal{L} \simeq \mathcal{O}(D)$, where D is an effective divisor of degree d.

$\mathrm{Div}\, X_0 = \mathbb{Z}[|X_0|]$.

$\#$ effective divisors of degree $d < \infty$

$$\Uparrow \quad |\{x \in |X| : \deg x \le d\}| < \infty.$$

$$\Rightarrow \quad |\mathrm{Pic}^d(X_0)| < \infty.$$
$$\shortparallel$$
$$|\mathrm{Pic}^0(X_0)|$$

Thm ⓐ $Z(X_0, t)$ is rational.

ⓑ $$Z(X_0,t) = \frac{f(t)}{(1-t)(1-qt)}$$

$f(t) \in \mathbb{Z}[t]$, $\deg f \leq 2g$. $f(0) = 1$, $f(1) = |\mathrm{Pic}^0(X_0)| =: h$.

Pf ⓐ $$Z(X_0,t) = \prod_{x\in|X_0|} \frac{1}{1-t^{\deg x}} = \sum_{\substack{D\geq 0 \\ D\in \mathrm{Div}(X_0)}} t^{\deg(D)}$$

$$= \sum_{\mathcal{L}\in \mathrm{Pic}(X_0)} \boxed{\ ?\ }\ t^{\deg(\mathcal{L})}$$

$$\underset{\mathbb{P}(H^0(X_0,\mathcal{L}))}{\underset{\parallel}{\left\{\begin{matrix} D\geq 0 \\ \mathcal{O}(D)\simeq\mathcal{L}\end{matrix}\right\}}} \neq \phi \iff H^0(X_0,\mathcal{L})\neq 0.$$

$$= \sum_{\mathcal{L}\in \mathrm{Pic}(X_0)} \left|\mathbb{P}(H^0(X_0;\mathcal{L}))\right| t^{\deg(\mathcal{L})}$$

$$= \underbrace{\sum_{0\leq \deg\mathcal{L}\leq 2g-2} \frac{q^{h^0(\mathcal{L})}-1}{q-1} t^{\deg(\mathcal{L})}}_{\underset{g_1(t)}{\shortparallel}} + \underbrace{\sum_{\deg\mathcal{L}>2g-2} \frac{q^{1-g+\deg\mathcal{L}}-1}{q-1} t^{\deg\mathcal{L}}}_{\underset{g_2(t)}{\shortparallel}}$$

$g_1(t) \in \mathbb{Z}[t]$.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0(X_0) \arrow[r] & \mathrm{Pic}(X_0) \arrow[r] \arrow[dr] & \mathbb{Z} \\ & & & e\mathbb{Z} \arrow[u, phantom, sloped, "\subset"] \arrow[r] & 0 \end{tikzcd}$$

$$g_2(t) = |\mathrm{Pic}^0(X_0)| \sum_{ed > 2g-2} \frac{q^{1-g+de}-1}{q-1} t^{de}$$

$$= \frac{|\mathrm{Pic}^0(X_0)|}{q-1}\left(q^{1-g} \frac{(qt)^{d_0 e}}{1-(qt)^e} - \frac{t^{d_0 e}}{1-t^e} \right)$$

where d_0 = smallest integer w/ $d_0 e > 2g-2$.

Cor. $\lim_{t\to 1} (t-1) Z(X_0, t) = \frac{|\mathrm{Pic}^0(X_0)|}{e(q-1)}$

Consider $X_0' = X_0 \otimes \mathbb{F}_{q^e} / \mathbb{F}_{q^e}$.

Claim $Z(X_0', t^e) = \prod_{i=1}^{e} Z(X_0, \varepsilon^i t)$, $\varepsilon = \exp\left(\frac{2\pi\sqrt{-1}}{e}\right)$.

$$= \left(Z(X_0, t)\right)^e$$

LHS has simple pole at $t=1$, RHS has pole of order e.

$\Rightarrow e = 1$.

Lecture 3 Last time, X_0 smooth projective curve $/\mathbb{F}_q$,

$\Gamma(X_0, \mathcal{O}_{X_0}) = \mathbb{F}_q$.

$Z(X_0, t) = \frac{f(t)}{(1-t)(1-qt)}$, $f(t) \in \mathbb{Z}[t]$, $f(0) = 1$, $f(1) = |\mathrm{Pic}^0(X_0)|$

Idea. $$Z(X_0, t) = \prod_{x \in |X_0|} \frac{1}{1 - t^{\deg x}}$$

$$= \sum_{d \geq 0} c_d t^d,$$ where c_d = # effective divisors of deg. d.

For $d > 2g-2$, $$c_d = |\mathrm{Pic}^d(X_0)| \cdot |\mathbb{P}^{d-g}(\mathbb{F}_q)|$$

$$= |\mathrm{Pic}^0(X_0)| \cdot |\mathbb{P}^{d-g}(\mathbb{F}_q)|$$

Kapranov's motivic zeta function.

X_0 quasi-projective $/\mathbb{F}_q$

$$S^d X_0 = \overbrace{X_0 \times \cdots \times X_0}^{d} / \sigma_d$$

$$\mathrm{Mor}(X_0^d/\sigma_d, Z) = \mathrm{Mor}(X_0^d, Z)^{\sigma_d}$$

$$(S^d X_0)(\mathbb{F}_q) = \left((S^d X_0)(\overline{\mathbb{F}}_q)\right)^{\mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_q)}$$

$$= \left(S^d(X_0(\overline{\mathbb{F}}_q))\right)^{\mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_q)}$$

= effective 0-cycles on X_0 of degree d

$$= \left\{ \sum a_x \cdot x \in \mathbb{Z}[|X_0|] ;\ a_x \geq 0,\ \sum a_x \deg x = d \right\}$$

$$|X_0| = X_0(\overline{\mathbb{F}}_q) / \mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_q)$$

$$Z(X_\circ, t) = \prod_{x \in |X_\circ|} \frac{1}{1 - t^{\deg x}}$$

$$= \sum_{d \geq 0} c_d \, t^d, \qquad c_d = \# \text{ of effective } 0\text{-cycles of degree } d$$

$$= |(S^d X_\circ)(\mathbb{F}_q)|$$

$$= \sum_{d \geq 0} |(S^d X_\circ)(\mathbb{F}_q)| \, t^d$$

k any field.

<u>Def</u>. $$K(\mathrm{Var}_k) = \mathbb{Z}[\mathrm{Var}_k] \Big/ \left(\begin{array}{c} U \overset{\text{open}}{\subset} X \\ [X] - [U] - [X \setminus U] \end{array} \right)$$

$$[X_1][X_2] = [(X_1 \times X_2)_{\mathrm{red}}]$$

$X_\circ$ quasi-projective $/k$,

$$Z_M(X_\circ, t) = \sum_{d \geq 0} [S^d X_\circ] \, t^d \in K(\mathrm{Var}_k)[[t]]$$

<u>Ex</u>. ⓐ $$K(\mathrm{Var}_{\mathbb{F}_q}) \longrightarrow \mathbb{Z}$$

$$[X_\circ] \longmapsto |X_\circ(\mathbb{F}_q)|$$

$$Z_M(X_\circ, t) \rightsquigarrow Z(X_\circ, t)$$

ⓑ $$K(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}$$

$$[X] \longmapsto \chi(X(\mathbb{C}))$$

Notation: $\mathbb{L} = [\mathbb{A}^1]$

Thm. X_0 smooth proper curve $/k$. $\Gamma(X_0, \mathcal{O}_{X_0}) = k$.

Assume $X_0(k) \neq \emptyset$, then $Z_M(X_0,t) = \dfrac{f(t)}{(1-t)(1-\mathbb{L}t)}$, $f(t) \in K(\mathrm{Var}_k)[t]$.

Pf.

$$\begin{tikzcd} S^d X_0 \arrow[d, "AJ"] \\ \underline{\mathrm{Pic}}^d(X_0) \end{tikzcd}$$

$x_0 \in X_0(k)$ Picard scheme

$\mathrm{Mor}(S, \underline{\mathrm{Pic}}(X_0))$

$= \{$ line bundles over $X_0 \times S$ w/ a trivialization over $x_0 \times S \}$

$$\begin{tikzcd} (y_1, y_2, \cdots, y_d) \arrow[r, mapsto] & \mathcal{O}_{X_0}(\Sigma y_i) \\ S^d X_0 \arrow[u, phantom, "\in" rotate=90] & \end{tikzcd}$$

$AJ^{-1}(\mathcal{L}) = \mathbb{P}(H^0(X_0, \mathcal{L}))$

If $d > 2g-2$, $AJ^{-1}(\mathcal{L}) = \mathbb{P}^{d-g}$

Moreover, $S^d X_0 = \mathbb{P}(E_d)$, where E_d is a vector bundle over $\underline{\mathrm{Pic}}^d(X_0)$.

$$\begin{tikzcd} \mathcal{L}_{univ} \arrow[d] \\ \underline{\mathrm{Pic}}^d(X_0) \times X_0 \arrow[d, "pr"] \\ \underline{\mathrm{Pic}}^d(X_0) \end{tikzcd}$$

$E_d = pr_* \mathcal{L}_{univ}$

$$Z_M(X_0, t) = \sum_{0 \le d \le 2g-2} [S^d X_0] t^d + \sum_{d > 2g-2} [\mathbb{P}(E_d)] t^d$$

$$\begin{tikzcd} E \arrow[d] \\ Y \end{tikzcd} \qquad [\mathbb{P}(E)] = [Y][\mathbb{P}_k^{\mathrm{rk}E-1}].$$

$$\mathrm{Pic}^d(X_0) = \mathrm{Pic}^0(X_0) : X_0(k) \neq \emptyset$$

$$= \text{polynomial} + \sum_{d>2g-2} [\mathrm{Pic}^d(X_0)][\mathbb{P}^{d-g}]\, t^d$$

Problem: Is the statement true if $X_0(k) = \emptyset$?

Lecture 4.

X_0 smooth, proper curve $/\mathbb{F}_q$, genus $(X_0) = g$

Thm $$Z\left(X_0, \frac{1}{qt}\right) = q^{1-g} t^{2-2g} Z(X_0, t).$$

Pf. $$Z(X_0, t) = \sum_{\mathcal{L}} \frac{q^{h^0(\mathcal{L})}-1}{q-1} t^{\deg \mathcal{L}}$$

$$= \underbrace{\sum_{0\le \deg \mathcal{L} \le 2g-2}}_{A} + \underbrace{\sum_{\deg \mathcal{L} > 2g-2}}_{B}$$

Serre duality $h^0(\mathcal{L}) - h^0(\mathcal{L}^* \otimes \Omega^1) = 1 - g + \deg \mathcal{L}.$

$$\frac{q^{h^0(\mathcal{L}^*\otimes\Omega^1)}-1}{q-1} t^{\deg(\mathcal{L}^*\otimes\Omega)} = q^{g-1} t^{2g-2} \frac{q^{h^0(\mathcal{L})} - q^{\deg \mathcal{L}+1-g}}{q-1} (q^{-1}t^{-1})^{\deg \mathcal{L}}$$

$$\sum_{0\le \deg \mathcal{L} < g-1} \frac{q^{h^0(\mathcal{L})}-1}{q-1} t^{\deg \mathcal{L}} + \sum_{\deg \mathcal{L} = g-1} \frac{q^{h^0(\mathcal{L})}-1}{q-1} t^{g-1}$$
$$+ \sum_{g-1 < \deg \mathcal{L} \le 2g-2} \frac{q^{h^0(\mathcal{L})} - q^{\deg \mathcal{L}+1-g}}{q-1} t^{\deg \mathcal{L}} \Big] u(t)$$

The sum $u(t) + q^{g-1} t^{2g-2} u\left(\frac{1}{qt}\right) +$ middle term

$Z(X_0,t)$ is the sum above and $\displaystyle\sum_{\deg L \geqslant g} \frac{q^{\deg L + 1 - g} - 1}{q-1} t^{\deg L}$

$$\parallel$$

$$\frac{|Pic^0(X_0)|\, t^g}{(1-t)(1-qt)}$$

Ex. Motivic functional equation for $g=2$.

$$Z_{Mot}(X_0, t) = \sum_d [S^d X_0] t^d$$

If $d > 2g-2 = 2$, $\quad [S^d X_0] = [\underline{Pic}^0(X_0)][\mathbb{P}^{d-g}]$

$$Z_{Mot}(X_0,t) = 1 + [X_0]t + [S^2 X_0]t^2 + \cdots$$

$$S^2 X_0 \xrightarrow{AJ} \underline{Pic}^2(X_0) = \underline{Pic}^0(X_0)$$

birational equivalence

$\chi(S^2 X_0) = ?$

$$\begin{tikzcd} X_0^2 \supset \Delta_{X_0} \arrow[d] \\ S^2 X_0 \supset \Delta X_0 \end{tikzcd}$$

$$\chi(S^2 X_0) = \chi(S^2 X_0 \setminus \Delta_{X_0}) + \chi(X_0)$$

$$= \frac{1}{2}\chi(X_0^2 \setminus \Delta_{X_0}) + \chi(X_0)$$

$$= \frac{1}{2}\chi(X_0)^2 + \frac{1}{2}\chi(X_0) = 1$$

$[S^2X_0] = [\underline{\mathrm{Pic}}^0(X_0)] + \mathbb{L}$ ($S^2X_0 = \underline{\mathrm{Pic}}^2(X_0)$ blow up at 1 pt)

$$Z_{\mathrm{Mot}}(X_0,t) = 1 + [X_0]t + \mathbb{L}t^2 + [\underline{\mathrm{Pic}}^0(X_0)]\left(t^2 + [\mathbb{P}^1]t^3 + [\mathbb{P}^2]t^4 + \cdots\right)$$

$$Z(X_0,t) = \frac{\prod_{i=1}^{2g}(1-\varpi_i t)}{(1-t)(1-qt)}$$

Thm $|\varpi_i| = q^{\frac{1}{2}}$. (Rmk: $Z(X_0, q^{-s}) = \zeta(s)$, thm $\Rightarrow$ all zeroes of $\zeta(s)$ are on the line $\mathrm{Re}\, s = \frac{1}{2}$).

Claim. Thm holds $\Leftrightarrow$ $\underset{\substack{\shortparallel \\ N_n}}{|X_0(\mathbb{F}_{q^n})|} = q^n + O(q^{\frac{n}{2}})$, $n \to \infty$

Pf. $t\frac{\partial}{\partial t}\log Z(X_0,t) = \sum N_n t^n$

$$\shortparallel$$

$$\sum \left(1 + q^n - \sum_{i=1}^{2g}\varpi_i^n\right)t^n$$

Thm $\Rightarrow$ $|N_n - 1 - q^n| \le 2g\sqrt{q^n}$

Lemma. $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$, $\sum_{i=1}^{k}\lambda_i^n$ is bounded as $n \to \infty$, then $|\lambda_i| \le 1$.

$|\varpi_i| \le \sqrt{q}$.

Using functional equation, if ϖ is a zero of $Z(X_0,t)$, then $\frac{q}{\varpi}$ is also a zero

$\Rightarrow |\varpi_i| = \sqrt{q}$.

PF. $X = X_0 \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q}$, $Y = X \times X$.

Thm. $\exists !$

$$\begin{tikzcd} \mathrm{Div}(Y) \otimes \mathrm{Div}(Y) \arrow[r, "{(\cdot,\cdot)}"] \arrow[d] & \mathbb{Z} \\ \mathrm{Pic}(Y) \otimes \mathrm{Pic}(Y) \arrow[ur] & \end{tikzcd}$$

s.t. if C is a smooth curve, $(C, D) = \deg \mathcal{O}(D)\big|_C$.

Ex. $\Delta_X \hookrightarrow X \times X = Y$, $(\Delta_X, \Delta_X) = \deg \mathcal{O}(\Delta_X)\big|_{\Delta_X} = \deg T_X = 2-2g$.

$NS(Y) := \mathrm{Div}(Y) \big/$ divisors numerically equiv. to 0 $\{D : \forall C, (C,D)=0\}$

$NS(Y) \otimes NS(Y) \xrightarrow{(\cdot,\cdot)} \mathbb{Z}$

Thm (Hodge Index) If H is ample, then $(\cdot,\cdot)$ is negative definite on $H^\perp \subset NS_{\mathbb{Q}} = NS \otimes \mathbb{Q}$.

Cor. $C, D \in H^\perp$, $(C,D)^2 \leq (C,C)(D,D)$.

$X = X_0 \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q} \xrightarrow{F} X_0 \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q} = X$, $F = Fr^n \otimes Id$ $q = p^n$

$\deg F = q$, $\underset{\text{tangent lines are horizontal}}{dF = 0}$. Let $\Gamma_F \hookrightarrow Y$ be the graph of F,

$X \xhookrightarrow[(Id,F)]{} X \times X = Y$, $(\Gamma_F, \Delta_X) = |X_0(\mathbb{F}_q)|$

$$H = \underset{\overset{\shortparallel}{V_1}}{[x_0 \times X]} + \underset{\overset{\shortparallel}{V_2}}{[X \times x_0]} \quad \text{ample.}$$

$$(V_1, \Gamma_F) = 1, \quad (V_2, \Gamma_F) = \deg F = q$$

$$(\Gamma_F, \Gamma_F) = \deg(F^* T_X) = q(2-2g)$$

$$(\Gamma_F - V_2 - qV_1, \ \Delta_X - V_1 - V_2)^2$$

$$\leq (\Gamma_F - V_2 - qV_1, \ \Gamma_F - V_2 - qV_1)(\Delta_X - V_1 - V_2, \ \Delta_X - V_1 - V_2)$$

$$\Rightarrow \quad |X_0(\mathbb{F}_q) - (q+1)| \leq 2g\sqrt{q}. \qquad \square$$

<u>Lecture 5</u> $\quad X_0/\mathbb{F}_q, \quad F \curvearrowright X = X_0 \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q}$

$$|X_0(\mathbb{F}_{q^n})| = \sum_{i \geq 0} (-1)^i \operatorname{tr}(F^n \curvearrowright H^i_c(X))$$

$$= \alpha_1^n + \cdots + \alpha_s^n - \beta_1^n - \cdots - \beta_r^n$$

$$\Rightarrow Z(X_0, t) = \frac{\prod_{i=1}^{r}(1-\beta_i t)}{\prod_{j=1}^{s}(1-\alpha_j t)}$$

Riemann Hypothesis $\Leftrightarrow$ For smooth projective X, eigenvalues of $F \curvearrowright H^i_c(X)$ have norm $q^{i/2}$.

Thm $X/\mathbb{C}$ curve (smooth, proper). $F : X \to X$, $\deg F = q$,

then eigenvalues of $F^* \curvearrowright H^i(X, \mathbb{C})$ have absolute value $q^{i/2}$.

⌈ $H^i(X, \mathbb{C}) = H^i(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \Rightarrow$ eigenvalues are algebraic. ⌋

$E = \mathbb{C}/\mathbb{Z}^2 \overset{top.}{=} S^1 \times S^1$

$F : S^1 \times S^1 \longrightarrow S^1 \times S^1$

$(z_1, z_2) \mapsto (z_1^n, z_2^m)$.

$\deg F = nm$, eigenvalues on H^1 are n and m.

Pf. $i = 1$. $F^* \curvearrowright H^1(X, \mathbb{C})$

$$(\,,\,) : H^1(X, \mathbb{C}) \otimes H^1(X, \mathbb{C}) \longrightarrow \mathbb{C}$$

$$\alpha \otimes \beta \longmapsto i \int_X \alpha \wedge \bar{\beta}$$

$$(\alpha, \beta) = \overline{(\beta, \alpha)}$$

$$\int_X F^*(\alpha) \wedge F^*(\bar{\beta}) = \int_X F^*(\alpha \wedge \bar{\beta})$$

$$= q \int_X \alpha \wedge \bar{\beta}.$$

$(F^*\alpha, F^*\beta) = q(\alpha, \beta)$ $\qquad$ $(\,,\,)$ is indefinite!

$$\begin{tikzcd} H^1_{DR}(X) \arrow[r, phantom, "="] & H^1(X, \mathbb{C}) \\ \Gamma(X, \Omega^1) = F^1 \arrow[u, hook] & \end{tikzcd}$$

Hodge decomposition. $F^1 \oplus \overline{F^1} \xrightarrow{\sim} H^1(X, \mathbb{C})$

Claim F^1 and $\bar{F}^1$ are orthogonal. $(\,,\,)|_{F^1}$ is positive definite,

$(\,,\,)|_{\overline{F^1}}$ is negative definite.

Pf. $0 \neq \alpha \in \Gamma(X, \Omega^1)$, want $i \int_X \alpha \wedge \bar{\alpha} > 0$.

Indeed, locally, $\alpha = f(z)\,dz$, $\alpha \wedge \bar{\alpha} = |f(z)|^2 dz \wedge d\bar{z}$

$$= -i\,|f(z)|^2 dx \wedge dy$$

$(\bar{\alpha}, \bar{\alpha}) = -(\alpha, \alpha)$.

$\alpha, \beta \in F^1$, then $\int \alpha \wedge \beta = 0$ $\Leftrightarrow$ $(F^1, \bar{F^1}) = 0$

$$\underset{\overset{\circlearrowleft}{F^*/\sqrt{q}}}{F^1} \oplus \underset{\overset{\circlearrowleft}{F^*/\sqrt{q}}}{\bar{F^1}}$$

unitary.
$\Downarrow$
has eig. values
of norm 1.

$\square$

$X/\mathbb{C}$ smooth projective, $d = \dim_{\mathbb{C}} X$,

$F : X \circlearrowleft$, $\deg F = q$, $F^* \circlearrowleft H^i(X, \mathbb{C})$, eigenvalues have norm $q^{\frac{i}{2d}}$?

Ex. $X = \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{F = (f, g)} \mathbb{P}^1 \times \mathbb{P}^1$

$\deg f = n$, $\deg g = m$, $\deg F = nm$.

$H^2(X) = H^2(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1)$, $F^* = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$. ☹

Modification (Serre) $X \hookrightarrow \mathbb{P}^N$. $F : X \circlearrowleft$, $F^*(\mathcal{O}(1)) = \mathcal{O}(q) \Rightarrow \deg F = q^{\dim X}$

eigenvalues have norm $q^{\frac{i}{2}}$.

Pf. $(\cdot, \cdot) : H^r(X, \mathbb{C}) \otimes \overline{H^r(X, \mathbb{C})} \longrightarrow \mathbb{C}$

$\omega = c_1(\mathcal{O}(1))$, $(\alpha, \beta) = i^r \int_X \alpha \wedge \bar{\beta} \wedge \omega^{d-r}$

$(\alpha, \beta) = \overline{(\beta, \alpha)}$.

Ex. $X = \mathbb{P}^1 \times \mathbb{P}^1$

$$(\cdot,\cdot) : H^2(X) \otimes \overline{H^2(X)} \longrightarrow \mathbb{C}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{, signature } (1,1)$$

Thm (Hard Lefschetz)

$$H^{d-r}(X) \xrightarrow[\sim]{\wedge \omega^r} H^{d+r}(X)$$

Cor. $(\cdot,\cdot)$ is nondegenerate.

Cor. If $k < d$, $H^k(X) \overset{\wedge \omega}{\hookrightarrow} H^{k+2}(X)$

Def. $r \geq 0$, $PH^{d-r}(X) := \ker\left(H^{d-r}(X) \xrightarrow{\wedge \omega^{r+1}} H^{d+r+2}(X)\right)$

Ex. $H^{d-r}(X) = PH^{d-r}(X) \oplus \omega PH^{d-r-2}(X) \oplus \omega^2 PH^{d-r-4}(X) \oplus \cdots$

d=3. $H^1(X) \hookrightarrow H^3(X) \xrightarrow{\omega} H^5(X)$

$$H^3(X) = \ker \omega \oplus \omega H^1(X).$$

$$H^r_{DR}(X) = F^0 \supset \cdots \supset F^{r-1} \supset F^r$$

$$\bigoplus_{p+q=r} \underset{\displaystyle H^{p,q}}{\underset{\parallel}{F^p \cap \overline{F^q}}} \xrightarrow{\sim} H^r(X) \qquad \underset{\displaystyle PH^{p,q}}{\underset{\parallel}{PH^r(X) \cap H^{p,q}}}$$

$$H^r_{DR}(X) = \bigoplus_{p+q=r} PH^{p,q}(X) \oplus \omega \cdot \bigoplus_{p+q=r-2} PH^{p,q}(X) \cdots$$

Thm. $(\cdot,\cdot)\big|_{{}_w PH^{pq}}$ is definite.

$$\overset{\uparrow}{\circlearrowleft}_F$$

Lecture 6. Last: $X/\mathbb{C}$ smooth projective, $(X, \omega = c_1(\mathcal{O}(1)) \in H^2(X,\mathbb{C}))$.

$H^i(X;\mathbb{C})$ has a canonical positive definite Hermitian form.

Ex. $\dim X = 1$, $H^1(X,\mathbb{C}) = F^1 \oplus \bar{F}^1$

$i\int \alpha\wedge\bar{\beta}$, $\alpha,\beta \in F^1$, $-\int \alpha\wedge\bar{\beta}$, $\alpha,\beta\in \overline{F^1}$.

* * *

Poincaré duality. X smooth oriented cpt manifold, $\dim_{\mathbb{R}} X = n$, $\left[\begin{array}{l} H^*(A) \\ := H^*(A;\mathbb{C}) \end{array}\right]$

$H^n(X\times X) \ni [\Gamma_f]$ $\quad X \xrightarrow{f} X$

$\Gamma_f \subset X\times X$

Want: algebra structure on $H^n(X\times X)$ s.t. $[\Gamma_f]\cdot[\Gamma_g] = [\Gamma_{gf}]$.

First construction:

$$\begin{tikzcd}
(x_1,x_2,x_3) \arrow[r, mapsto] \arrow[dd, mapsto, bend right] & (x_1,x_2,x_2,x_3) \\
X\times X\times X \arrow[r, "i"] \arrow[d, "\pi_{13}"'] & (X\times X)\times(X\times X) \\
X\times X & \\
(x_1,x_3) &
\end{tikzcd}$$

$$\alpha,\beta \in H^n(X\times X),\quad \alpha\cdot\beta := \pi_{13*}\, i^*\big(\underset{\substack{\uparrow \\ H^{2n}(X\times X\times X\times X)}}{\alpha\boxtimes\beta}\big)$$

$$i^*\big([\Gamma_f]\boxtimes[\Gamma_g]\big) = \big[\{(x, f(x), gf(x)) \in X\times X\times X\}\big]$$

$$\pi_{13*}\big(\qquad\qquad\big) = \pi_{13*}\big(\qquad\qquad\big) = [\Gamma_{gf}]$$

Second construction.

$$H^n(X\times X) = \bigoplus_{p+q=n} H^p(X)\otimes H^q(X)$$

$$= \bigoplus_{p+q=n} \operatorname{End}(H^p(X))$$

Exercise. Show that it gives the same algebra structure.

$$f : X \longrightarrow X, \quad [\Gamma_f] \in H^n(X\times X) = \bigoplus_{0\leq p\leq n} \operatorname{End}(H^p(X)) \xrightarrow{tr} \mathbb{C}$$

$$(f^* : H^p(X) \circlearrowleft)$$

$$tr(\{f_p \in \operatorname{End}(H^p(X))\})$$
$$= \sum (-1)^p \, tr(f_p \curvearrowright H^p(X))$$

$$H^n(X\times X) \xrightarrow{tr} \mathbb{C}$$

$$tr(\alpha) = \int_{X\times X} [\alpha]\wedge[\Delta_X]$$

$$tr([C]) = |C\cap\Delta_X|$$

$$tr(\Gamma_f) = \#\text{ fixed points.}$$

$X\times X \circlearrowleft \sigma$ permutation

$\alpha\in H^n(X\times X) \circlearrowleft {}^t$ – the map induced by σ.

Ex. $$tr(\alpha\circ\beta^t) = \int_{X\times X} \alpha\wedge\beta.$$

Algebraic: X smooth proj. curve over $k=\bar{k}$.

$$\operatorname{Cor}(X,X) := NS(X\times X).$$

$$f: X\longrightarrow X \rightsquigarrow [\Gamma_f] \in \operatorname{Cor}(X,X)$$

1st Construction.

$$\begin{tikzcd}
X\times X\times X \arrow[r, hook, "i"] \arrow[d, "\pi_{13}"'] & X\times X\times X\times X \\
X\times X &
\end{tikzcd}$$

$\alpha, \beta \in \mathrm{Cor}(X,X)$.

$$\alpha\circ\beta = \pi_{13*}\left(i^*(\beta\boxtimes\alpha)\right).$$

$$\begin{tikzcd}
1 \arrow[rrrr, mapsto] & & & & {[X\times pt]} \\
\mathrm{End}(\mathbb{Z}) \arrow[u] \arrow[d, equal] & \times\, \heartsuit \,\times \arrow[dd, dashed, "?"] & \mathrm{End}(\mathbb{Z}) \arrow[rr] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \mathrm{Cor}(X,X) \\
\mathbb{Z} & & 1 \arrow[rr, mapsto] & & {[pt\times X]} \\
& \mathrm{End}(\underline{\mathrm{Pic}}^0(X)) & & &
\end{tikzcd}$$

$\underline{\mathrm{Pic}}^0(X)$ is an abelian variety,

Fix $x_0\in X$,

$$\begin{tikzcd}
\mathcal{O}(x-x_0) & f: \underline{\mathrm{Pic}}^0(X) \arrow[r] & \underline{\mathrm{Pic}}^0(X) \\
x \arrow[u, mapsto] & X \arrow[u, hook, "AJ"'] &
\end{tikzcd}$$

$f\circ AJ: X \to \underline{\mathrm{Pic}}^0(X) \rightsquigarrow$ line bundle on $X\times X$.

$\rightsquigarrow [f] \in \mathrm{Cor}(X,X)$.

Thm. $\mathrm{End}(\mathbb{Z}) \times \mathrm{End}(\underline{\mathrm{Pic}}^0(X)) \times \mathrm{End}(\mathbb{Z}) \xrightarrow{\sim} \mathrm{Cor}(X,X)$.

Construction of the Inverse map:

$$\mathrm{Cor}(X,X) \longrightarrow \mathrm{End}(\underline{\mathrm{Pic}}^0(X)).$$

$$\mathrm{Pic}(X\times X)\Big/\mathrm{Pic}(X)\times\mathrm{Pic}(X) \simeq \mathrm{Mor}^0(X, \underline{\mathrm{Pic}}^0(X))$$

$$\begin{tikzcd} \mathrm{Mor}^0(X, \underline{\mathrm{Pic}}^0(X)) \arrow[d, phantom, "\shortparallel"] \\ \{f : X \to \underline{\mathrm{Pic}}^0(X) : f(x_0)=0\} \end{tikzcd}$$

$$\mathrm{Mor}(\overset{x_0}{\overset{\uparrow}{X}}, \underline{\mathrm{Pic}}(X)) = \mathrm{Pic}(X\times X)\big/\mathrm{Pic}(X)$$

$$\shortparallel$$

$$\mathrm{Mor}^0(X, \underline{\mathrm{Pic}}^0(X)) \times \underline{\mathrm{Pic}}(X)$$

<u>Key claim</u>: $\mathrm{Mor}^0(X, \underline{\mathrm{Pic}}^0(X)) \overset{\simeq}{\leftarrow} \mathrm{End}(\underline{\mathrm{Pic}}^0(X))$

$$X \xrightarrow{AJ} \underline{\mathrm{Pic}}^0(X) \longrightarrow \underline{\mathrm{Pic}}^0(X)$$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "AJ"'] & & \underline{\mathrm{Pic}}^0(X) \\ \underline{\mathrm{Pic}}^0(X) \arrow[rru, dashed, "\exists"] & & \\ & S^d X \arrow[lu, "\iota"] \arrow[ruu] & X^d \arrow[l] \arrow[uu, "\Sigma\circ f"'] \end{tikzcd}$$

$d>2g$

fibers are projective spaces

Claim. $\mathrm{Mor}(\mathbb{P}^N, \underline{\mathrm{Pic}}^0(X))$ all constant.

<u>Summary</u>: $\mathrm{Pic}(X\times X)\big/\mathrm{Pic}(X)\times\mathrm{Pic}(X) \simeq \mathrm{Hom}(\underline{\mathrm{Pic}}^0(X), \underline{\mathrm{Pic}}^0(X)).$

$$\begin{tikzcd}
\mathrm{Pic}(X\times X) \arrow[r] \arrow[d] \arrow[dr] & \mathrm{Pic}(X\times X)/\mathrm{Pic}(X)\times\mathrm{Pic}(X) \arrow[d, "\wr"] \\
\mathrm{Cor}(X,X) \arrow[r, dashed] & \mathrm{End}(\underline{\mathrm{Pic}}^0(X))
\end{tikzcd}$$

<u>Lecture 7</u>. X smooth proj. curve $/ k = \bar{k}$

$NS(X\times X) \circlearrowright$ permutation, $\alpha \rightsquigarrow \alpha^t$.

<u>Thm</u> (Weil)

$$NS(X\times X) \xrightarrow{\sim} \mathbb{Z}\times \underset{\substack{\uparrow \\ \text{in the cat. of} \\ \text{group schemes}}}{\mathrm{End}}(\underline{\mathrm{Pic}}^0(X)) \times \mathbb{Z}$$

$$\alpha \longmapsto (\alpha^0, \alpha^1, \alpha^2)$$

$$\alpha^0 = (\alpha, x_0\times X) \in \mathbb{Z}$$

$$\alpha^2 = (\alpha_0, X\times x_0)$$

$$\begin{tikzcd}
\mathrm{Pic}(X\times X) \arrow[r, two heads] \arrow[d] & NS(X\times X) \\
\mathrm{Pic}(X\times X)/\mathrm{Pic}^0(X)\times\mathrm{Pic}^0(X) \arrow[ur] \arrow[r, phantom, "\simeq"] & \mathbb{Z}\times\mathrm{End}_{gr}(\underline{\mathrm{Pic}}^0(X))\times\mathbb{Z} \arrow[u]
\end{tikzcd}$$

This gives a ring structure on $NS(X\times X)$.

$$[C]\circ[D] = \pi_{13*}\, i^*[D\times C]$$

$$\begin{tikzcd} X\times X\times X \arrow[r, "i", "2=3"'] \arrow[d, "\pi_{13}"'] & X\times X\times X\times X \\ X\times X & \end{tikzcd}$$

Ex. $X \overset{f}{\underset{g}{\rightrightarrows}} X$ $\qquad [\Gamma_f],[\Gamma_g]\in NS(X\times X)$

$$[\Gamma_f]\circ[\Gamma_g] = [\Gamma_{f\circ g}]$$

$$\operatorname{tr}: NS(X\times X)\longrightarrow \mathbb{Z}$$
$$\operatorname{tr}(\alpha) \doteq (\alpha, \Delta_X)$$

- $(\alpha\circ\beta)^t = \beta^t\circ\alpha^t.$
- $\operatorname{tr}(\alpha^t\circ\beta) = (\alpha,\beta).$

$$\begin{array}{ccc} \operatorname{tr}(\alpha^t\circ\beta) & = & (\alpha,\beta). \\ \| & & \| \\ (\alpha^t\times\beta,\ \Delta_{1=4,\,2=3}) & = & (\alpha\times\beta,\ \Delta_{1=3,\,2=4}) \end{array}$$

$\forall \underset{0\neq}{\alpha}\in \operatorname{End}_{gr}(\underline{\operatorname{Pic}}^0(X))\subset NS(X\times X),$

$$\begin{array}{c} \operatorname{tr}(\alpha^t\circ\alpha) < 0. \\ \| \\ (\alpha,\alpha) \end{array}$$

The Hodge index theorem.

Rmk. $k = \mathbb{C}$.

Thm.

$$\begin{tikzcd}
\mathrm{Pic}(X \times X) \arrow[r, "c_1"] \arrow[d] & H^2(X\times X; \mathbb{Z}) \arrow[r, "\sim"] & \overset{\mathbb{Z}}{\overset{\shortparallel}{\mathrm{End}(H^0(X,\mathbb{Z}))}} \times \mathrm{End}(H^1(X;\mathbb{Z})) \times \overset{\mathbb{Z}}{\overset{\shortparallel}{\mathrm{End}(H^2(X;\mathbb{Z}))}} \\
 & & H^1(X,\mathbb{Z}) \otimes \mathbb{C} = H^1_{DR}(X) \supset F^1 = \Gamma(X, \Omega^1) \\
NS(X \times X) \arrow[rr, "\sim"'] & & \mathbb{Z} \times \mathrm{End}_{HS}(H^1(X)) \times \mathbb{Z} \arrow[d, equal, "\text{in the cat. of Hodge str}"] \\
 & & \mathbb{Z} \times \mathrm{End}_{gr}(\underline{\mathrm{Pic}}^0(X)) \times \mathbb{Z}
\end{tikzcd}$$

$$tr(\alpha) = \sum_{i=0}^{2} (-1)^i \, tr\left(\alpha^i \curvearrowright H^i(X; \mathbb{C})\right)$$

Pf. $X \times X = Y$.

Hodge (1,1) - theorem

$$\mathrm{Pic}(Y) \xrightarrow{c_1} H^2(Y;\mathbb{Z}) \cap \overset{F^1 \cap \bar{F}^1}{\overset{\shortparallel}{H^{1,1}(Y)}}$$

Indeed,

$$0 \to \mathbb{Z} \to \mathcal{O}^* \to \mathcal{O} \to 0$$

$$\begin{tikzcd}
H^1(Y, \mathcal{O}^*_{an}) \arrow[r] & H^2(Y, \mathbb{Z}) \arrow[r] \arrow[d] & H^2(Y, \mathcal{O}_{an}) = H^2_{DR}(Y)/F^1 \\
 & H^2_{DR}(Y) = R^2\Gamma(Y, \mathcal{O}_{an} \to \Omega^1_{an} \to \dots) \arrow[ur] &
\end{tikzcd}$$

$$H^1(X, \mathbb{C}) \otimes H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C}) = H^{1,1}$$

$$\overset{\shortparallel}{H^{10} \oplus H^{01}} \qquad \overset{\shortparallel}{H^{10} \oplus H^{01}}$$

$$\Rightarrow \mathrm{Pic}(X \times X) \xrightarrow{c_1} \mathrm{End}_{HS}(H^*(X)) \subset \mathrm{End}(H^*(X;\mathbb{Z}))$$

$$\begin{tikzcd}
\mathrm{Pic}(X\times X) \arrow[r, two heads, "c_1"] \arrow[d] & \mathrm{End}_{HS}(H^*(X)) \\
NS(X\times X) \arrow[ur, dashed, "\sim"'] &
\end{tikzcd}$$

Hodge index Thm $\Rightarrow$ $0 \neq \alpha \in H^2(X;\mathbb{Z}) \cap H^{1,1}$

$(\alpha, \omega) = 0 \quad \rightarrow \quad (\alpha, \alpha) < 0$

class of an ample line bundle

$\mathrm{Pic}^0(X) = H^1(X;\mathcal{O})/H^1(X,\mathbb{Z})$

$$\begin{tikzcd}
H^0(X,\mathcal{O}) \arrow[r] \arrow[d, equal] & H^0(X,\mathcal{O}^*) \arrow[r] \arrow[d, equal] & H^1(X,\mathbb{Z}) \arrow[r, hook] & H^1(X,\mathcal{O}) \arrow[r] & H^1(X,\mathcal{O}^*) \arrow[r, "c_1"] & H^2(X;\mathbb{Z}) \\
\mathbb{C} \arrow[r, two heads] & \mathbb{C}^* & & & &
\end{tikzcd}$$

<u>Fact.</u> $\underline{\mathrm{Pic}}^0(X)_{an} \simeq H^1(X,\mathcal{O}_{an})/H^1(X,\mathbb{Z})$

<u>Sketch:</u>

$$\begin{tikzcd}
\mathrm{Lie}\,\underline{\mathrm{Pic}}^0(X)_{an} \arrow[r, "\exp"] \arrow[d, equal] & \underline{\mathrm{Pic}}^0(X) \\
H^1(X,\mathcal{O}_{an}) \arrow[ur] &
\end{tikzcd}$$

<u>Lecture 8.</u> $X/\mathbb{C}$

$NS(X\times X) \simeq \mathrm{End}_{HS}(H^*(X)) \simeq \mathbb{Z} \times \mathrm{End}_{gr}(\underline{\mathrm{Pic}}^0(X)) \times \mathbb{Z}$

Want $\mathrm{End}_{HS}(H^1(X)) = \mathrm{End}_{gr}(\underline{\mathrm{Pic}}^0(X))$

$$\begin{tikzcd}
\psi = d\varphi \arrow[d, phantom, "\curvearrowright"] & \varphi \arrow[d, phantom, "\curvearrowright"] \\
\mathrm{Lie}\, \underline{\mathrm{Pic}}^0(X)_{an} \arrow[r, "\exp"] \arrow[d, equal] & \underline{\mathrm{Pic}}^0(X)_{an} \\
H^1(X, \mathcal{O}) \arrow[ur] &
\end{tikzcd}$$

$$H^1(X,\mathcal{O}) / H^1(X, \mathbb{Z}) \xrightarrow{\sim} \underline{\mathrm{Pic}}^0(X)_{an}$$

Why $\mathrm{Lie}\, \underline{\mathrm{Pic}}^0(X)$ is $H^1(X, \mathcal{O})$?

$$\begin{aligned}
&\mathrm{Lie}\, \underline{\mathrm{Pic}}^0(X) \\
&= \left\{ \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2 \xrightarrow{\varphi} \underline{\mathrm{Pic}}^0(X) \;:\; \varphi|_{\mathrm{Spec}\, k} = 0 \right\} \\
&= \ker\left(\mathrm{Pic}\left(X \times \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2\right) \longrightarrow \mathrm{Pic}(X) \right) \\
&= \ker\left(H^1\left(X \times \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2, \mathcal{O}^*\right) \longrightarrow H^1(X, \mathcal{O}^*) \right)
\end{aligned}$$

$$\overset{\substack{\mathcal{O}_X \\ \wr\wr}}{1 + \mathcal{O}_X \varepsilon} \longrightarrow \mathcal{O}^*_{X \times \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2} \underset{\longleftarrow}{\longrightarrow} \mathcal{O}^*_X \longrightarrow 0$$

$$\mathrm{End}_{\mathbb{Z}}\left(H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z}) \right) = \left\{ \psi \curvearrowright H^1(X, \mathcal{O}) \;:\; \psi(H^1(X, \mathbb{Z})) \subset H^1(X, \mathbb{Z}) \right\}$$

$$0 \longrightarrow \underset{\substack{\| \\ F^1}}{\Gamma(X, \Omega^1)} \longrightarrow H^1(X, \mathbb{Z}) \otimes \mathbb{C} \longrightarrow H^1(X, \mathcal{O}) \longrightarrow 0$$

$$= \mathrm{End}_{HS}(H^1(X)) \subset \mathrm{End}\left(H^1(X, \mathbb{Z})\right)$$

Ex. Show for any $0 \neq \alpha \in \mathrm{End}_{HS}(H^1(X))$, $\underset{\substack{\| \\ -(\alpha, \alpha)}}{\mathrm{tr}(\alpha^t . \alpha)} > 0$.

α^t is the adjoint to α w.r.t. Poincaré form on H^1

Hint: Let α^* be the adjoint to α w.r.t. the positive definite Hermitian form on $\underset{F^1 \oplus \overline{F^1}}{\overset{\shortparallel}{H^1(X, \mathbb{C})}}$. $\left[\begin{array}{ll} r', r \in F^1, & (r, r') = i \int r \overline{r'} \\ r, r' \in \overline{F^1}, & (r, r') = -i \int r \overline{r'} \end{array} \right]$

then $\operatorname{tr}(\alpha^* \alpha) > 0$. Check that if $\alpha \in \operatorname{End}_{HS}(H^1(X))$, then $\alpha^t = \alpha^*$.

Observation: $\underline{\operatorname{Pic}}^0(X)_{an} = H^1(X, \mathcal{O}) / H^1(X, \mathbb{Z})$

$$\Rightarrow \underset{\ker\left(\operatorname{Pic}^0(X) \xrightarrow{n} \operatorname{Pic}^0(X)\right)}{\overset{\shortparallel}{\operatorname{Pic}^0(X)[n]}} = \frac{\frac{1}{n} H^1(X, \mathbb{Z})}{H^1(X, \mathbb{Z})} = H^1(X, \mathbb{Z}/n\mathbb{Z})$$

X smooth projective curve $/ k = \overline{k}$

Def. $H^1_{\acute{e}t}(X, \mathbb{Z}/n(1)) := \operatorname{Pic}^0(X)[n]$.

$$H^1_{\acute{e}t}(X; \mathbb{Z}_\ell(1)) := \varprojlim_n H^1_{\acute{e}t}(X, \mathbb{Z}/\ell^n(1))$$

Rmk. $k = \mathbb{C}$, $H^1_{\acute{e}t}(X; \mathbb{Z}_\ell(1)) = H^1_{Betti}(X, \mathbb{Z}) \otimes \mathbb{Z}_\ell$.

$$\begin{tikzcd} NS(X \times X) \arrow[r, phantom, "\simeq"] & \mathbb{Z} \times \arrow[d] & \operatorname{End}_{gr}(\underline{\operatorname{Pic}}^0(X)) \arrow[d] & \times \mathbb{Z} \arrow[d] \\ & \mathbb{Z}_\ell \times & \operatorname{End}(H^1_{\acute{e}t}(X, \mathbb{Z}_\ell(1))) & \times \mathbb{Z}_\ell \end{tikzcd}$$

Thm (Weil). $\ell \neq \operatorname{char} k$, then $H^1_{\acute{e}t}(X, \mathbb{Z}_\ell(1))$ is a free $\mathbb{Z}_\ell$-module of rank $2g$.

and $\forall\ \alpha \in NS(X\times X)$, $\quad tr(\alpha) = \sum_{i=0}^{2} (-1)^i\, tr\left(\alpha^* \curvearrowright H^i_{\acute{e}t}(X, \mathbb{Z}_\ell(1))\right)$

$$= (\alpha, x_0 \times X) - tr(\alpha^* \curvearrowright H^1_{\acute{e}t}) + (\alpha, X\times x_0)$$

Cor. $\quad F: X \longrightarrow X$, $\deg F = q$, then char poly. of $F^* \curvearrowright H^1_{\acute{e}t}(X, \mathbb{Z}_\ell(1))$ has integral coefficients independent of $\ell \neq \mathrm{char}\, k$, and its roots have absolute value $q^{\frac{1}{2}}$.

Pf $\quad$ Let χ_F be the char. poly

$$(\Gamma_F, \Delta_X) = \underset{\overset{\shortparallel}{1}}{(\Gamma_F, x_0\times X)} - tr(F^* \curvearrowright H^1_{\acute{e}t}) + \underset{\overset{\shortparallel}{q}}{(\Gamma_F, X\times x_0)}$$

$tr(F^* \curvearrowright H^1_{\acute{e}t}) \in \mathbb{Z}$. $\quad$ Do the same for powers of F,

$\quad tr(F^{n*} \curvearrowright H^1_{\acute{e}t}) \in \mathbb{Z}$, $\forall n$.

$\Rightarrow \chi_F(t) \in \mathbb{Q}[t]$.

$NS(X\times X)$ has finite rank $\Rightarrow \Gamma_F \in NS(X\times X)$ is integral. i.e.

$\exists\ \underset{\overset{\cup}{\text{monic}}}{m(t) \in \mathbb{Z}[t]}$ s.t. $m(\Gamma_F) = 0$. $\Rightarrow m(F^* \curvearrowright H^1_{\acute{e}t}(X, \mathbb{Z}_\ell(1)) = 1$.

roots of $F^* \curvearrowright H^1_{\acute{e}t}(X, \mathbb{Z}_\ell(1))$ are alg. integers.

$\Rightarrow \chi_F(t) \in \mathbb{Z}[t]$.

$\Gamma_F^t \circ \Gamma_F = q\, \Gamma_{\Delta X}$. : $\quad tr(\Gamma_F^t \circ F \circ \alpha) = (\Gamma_F, \Gamma_F \circ \alpha) = F^*(\Gamma_{\Delta X}, \alpha)$

$$= q(\Gamma_{\Delta X}, \alpha) = q\, tr\, \alpha$$

$\Rightarrow \mathrm{tr}\left((\Gamma_F \circ \alpha)^t \circ (\Gamma_F \circ \beta) \right) = q\, \mathrm{tr}(\alpha^t \circ \beta)$

But $\mathrm{tr}(\alpha^t \circ \alpha) > 0$ $\Rightarrow$ multiplication by Γ_F on $NS(X \times X)$ rescales a positive form by q $\Rightarrow$ roots of $m(t)$ have absolute value $q^{\frac{1}{2}}$.

Lecture 9. Étale fundamental group

X scheme, $\bar{x} \in X(\bar{k})$ $\rightsquigarrow$ profinite group $\pi_1^{\text{ét}}(X, \bar{x})$

§1. G - discrete group.

G-sets = the cat. of sets w/ an action of G.

$$\begin{tikzcd} F: G\text{-sets} \arrow[r] & \text{Sets} \\ X \arrow[u, phantom, "\in"] \arrow[r, mapsto] & X \arrow[u, phantom, "\in"] \end{tikzcd}$$

Lemma. $G \xrightarrow{\sim} \mathrm{Aut}(F)$. $(*)$

Pf. $g \in G$, $X \in G$-sets,

$$\begin{tikzcd} F(X) \arrow[r, "g"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & F(X) \arrow[d, "\pi"] \\ F(X') \arrow[r, "g"] & F(X') \end{tikzcd}$$

, $\pi: X \to X'$ in G-sets

This defines $(*)$.

Consider G as a G-set, $F(X) = \mathrm{Mor}_{G\text{-sets}}(G, X)$,

$$\mathrm{Aut}(F) \underset{\text{Yoneda}}{\xrightarrow{\sim}} \mathrm{Aut}_{G\text{-sets}}(G) = G.$$

Ex. Let T be a (nice) connected top. space, $a \in T$,

$$\begin{tikzcd}
\text{Covers}(T) \arrow[rr, "\simeq"] \arrow[dr, "F_a"'] & & \pi_1(T,a)\text{-sets} \arrow[dl, "\text{Forget}"] \\
& \text{Sets} &
\end{tikzcd}$$

$F_a(U \xrightarrow{p} T) = p^{-1}(a) \in \text{Sets}$

Functor in the other direction: Fix a universal cover
$$\begin{tikzcd}
U_{\text{univ}} \arrow[d, "p"] \\
T
\end{tikzcd}$$
and $\tilde{a} \in p^{-1}(a)$.

$\mathrm{Aut}_{\text{covers}}(U_{\text{univ}}) = \pi_1(T,a)$.

$X \in \pi_1(T,a)$-sets, $\rightsquigarrow$ $U_{\text{univ}} \overset{\pi_1(T,a)}{\times} X$

Cor. $\pi_1(T,a) \simeq \mathrm{Aut}(F_a)$

§2. G – topological group

G-sets = cat. of sets w/ a conti. action of G

$\cup$ $\qquad\qquad$ $G \times X \longrightarrow \underset{\substack{\uparrow \\ \text{discrete.}}}{X}$ is continuous

Finite G-sets.

$\downarrow F$

Finite sets

Profinite completion of G: $\quad G^\vee := \varprojlim_{\substack{U \triangleleft G \\ \text{open of} \\ \text{fin. index}}} G/U.$

$G^\vee$ - top. group, basis of topology on $G^\vee$ is

$$\ker\left(G^\vee \to G/U\right).$$

$$G \xrightarrow{\text{conti.}} G^\vee.$$

Universal property:

$$\begin{tikzcd} G \arrow[r, "\text{conti}'"] \arrow[d] & H^{\text{profinite}} \\ G^\vee \arrow[ur, dashed, "\exists!"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

Lemma. $G^\vee \xrightarrow{\sim} \operatorname{Aut}(F)$.

Rmk. Basis of topology on $\operatorname{Aut}(F)$ is

$$U_X = \ker\left(\operatorname{Aut}(F) \to \operatorname{Aut}(F(X))\right)$$

$X \in$ Finite-G-sets.

$\operatorname{Aut}(F)$ is a profinite group $\operatorname{Aut}(F) = \varprojlim_{X \in \text{Finite-G-sets}} \operatorname{Aut}(F)/U_X$.

Pf.

$$\begin{tikzcd} G \arrow[r, "\text{conti}'\text{-hom.}"] \arrow[d] & \operatorname{Aut}(F) \\ G^\vee \arrow[ur, "\varphi"] & \end{tikzcd}$$

φ is injective: $G/U \in$ Finite-G-set.

$$\begin{tikzcd} G^\vee \arrow[r] \arrow[dr, bend right] & \operatorname{Aut}_{G\text{-sets}}(G/U) \arrow[d, phantom, "\text{IS}"] \\ & G/U \end{tikzcd}$$

Surjectivity. φ is surjective:

$G^\vee$ is compact $\Rightarrow$ enough to check that $\operatorname{Im}\varphi$ is dense.

$\Updownarrow$

$\forall\, X \in$ Finite-G-sets,

$\gamma \in \mathrm{Aut}(F)$, $\exists\, g \in G$ s.t.

$$\begin{tikzcd} F(X) \arrow[r, bend left, "\gamma_*"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\parallel"] & F(X) \end{tikzcd}$$

$U = \ker(G \longrightarrow \mathrm{Aut}(F(X)))$.

$$\begin{tikzcd} \text{Finite-}G\text{-sets} \arrow[r, "F"] & \text{Sets} \\ \text{Finite-}G/U\text{-sets} \arrow[u, hook] \arrow[ur, "\bar{F}"] & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, dashed, Rightarrow, "g"{pos=0.1}, "\bar{\gamma}"'{pos=0.9}] & G/U \arrow[d, "\wr"] \\ \mathrm{Aut}(F) \arrow[r] & \mathrm{Aut}(\bar{F}) \\ \gamma \arrow[r, mapsto] & \bar{\gamma} \end{tikzcd}$$

§3. Goal: $F: \mathcal{C} \longrightarrow$ Finite sets.

$$\begin{tikzcd} & & \text{Finite sets} \\ \mathcal{C} \arrow[r] & \text{Finite-}\mathrm{Aut}(F)\text{-sets} \arrow[ur] & (*) \\ X \arrow[r, mapsto] & F(X) & \end{tikzcd}$$

Goal: give conditions for $(*)$ to be an equivalence.

Def. $F: \mathcal{A} \to \mathcal{B}$ a functor.

1. If $\mathcal{A}$ has finite limits,

(⟺ $\mathcal{A}$ has final object $*_{\mathcal{A}}$ and fiber products).

and F commutes w/ them. (⟺ $F(*_A) = *_B$ final obj. in $\mathcal{B}$,

$F(X) \times_{F(Z)} F(Y) \xrightarrow{\sim} F(X \times_Z Y)$).

then F is left exact.

2. If $\mathcal{A}$ has finite colimits, (ϕ_A, $X \cup_Z Y$),

and F commutes w/ them, then F is right exact.

3. F is exact ⟺ F is right & left exact.

Def. $F: \mathcal{C} \to$ Finite Sets.

$(\mathcal{C}, F)$ is a Galois category, if

① $\mathcal{C}$ has finite limits & colimits, F is exact.

② $\forall X \in \mathcal{C}$ is a finite coproduct of connected objects

($X \in \mathcal{C}$ is connected, if $X \neq \phi_{\mathcal{C}}$ and $\forall X' \hookrightarrow X$, either

$X' = \phi_{\mathcal{C}}$ or $X' = X$).

③ F reflects isomorphisms: ($X \xrightarrow{a} Y$, $F(a)$ is iso. ⟺ a is iso.)

Thm (Grothendieck) If $(\mathcal{C}, F)$ is a Galois category, then

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\sim"] \arrow[dr, "F"'] & \text{Finite-Aut}(F)\text{-sets} \arrow[d] & (*) \\
& \text{Finite-Sets} &
\end{tikzcd}$$

Beginning of proof:

Lemma: $a, b : X \rightrightarrows Y$, X is connected. If $\exists \cdot x \in F(X)$

s.t. $F(a)(x) = F(b)(x)$, then $a = b$.

Proof. Equilizer $Eq(a,b)$

$$F(Eq(a,b)) = Eq(F(a), F(b)) \ni x$$

$$\Rightarrow \underset{\substack{\nparallel \\ \phi_{\mathcal{C}}}}{Eq(a,b)} \hookrightarrow X \quad \Rightarrow \quad Eq(a,b) = X \quad \Rightarrow a = b.$$

Cor. $(*)$ is faithful.

Lecture 10 §1. Étale morphisms.

Def - Prop: A morphism $f : X \longrightarrow Y$ of schemes is étale if TFEC hold:

1. f is locally of finite presentation (lfp)

$$\Gamma@ \ f : \underset{\substack{\| \\ \mathrm{Spec} A}}{X} \to \underset{\substack{\| \\ \mathrm{Spec} B}}{Y} \text{ is lfp if } \begin{tikzcd} B \arrow[r, "\sim"'] \arrow[rr, bend right, "f^*"'] & A[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \arrow[r] & A \end{tikzcd}$$

finitely many relations

① $f: X \to Y$ is lfp if $\forall$ affine open

$$\begin{tikzcd} X \arrow[r] & Y \\ \operatorname{Spec} A \arrow[u, phantom, "\cup"] \arrow[r, "f"] & \operatorname{Spec} B \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$f|_A$ is lfp.

Fact: lfp is Zariski local.

Fact: If $f: X \to Y$ a flat, lfp morphism, then f is open.

Ex. $\operatorname{Spec} k(t) \to \operatorname{Spec} k[t]$ is flat, but not open. ⌟

and has the "infinitesimal lifting property":

$$\begin{tikzcd} X \arrow[d, "f"'] & T^0 \arrow[l] \arrow[d, hook, "\substack{\text{square 0}\\\text{thickening}}"] & = \operatorname{Spec} A/I \\ Y & T_{\text{affine}} \arrow[l] \arrow[ul, dashed, "\exists !"'] & = \operatorname{Spec} A \end{tikzcd}$$

$I^2 = 0$

Ex.

$$\begin{tikzcd} X \arrow[d, "f"'] & \operatorname{Spec} k \arrow[l, hook'] \arrow[d] \\ Y & \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \arrow[l, hook'] \arrow[ul, dashed, "\exists !"'] \end{tikzcd}$$

f is iso. on Zariski tangent spaces.

2. f is lfp, flat, and $\Omega_{X/Y} = 0$.

3. f is lfp, flat, and $\forall y \in Y$,

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k(y) \arrow[r, hook] & Y \end{tikzcd}$$

the fiber $X_y = \coprod_{i \in I} \operatorname{Spec} k_i$

w/ $k_i \supset k(y)$ finite separable ext'n.

4. $\forall x \in X$, $\exists$ $x \in U \overset{\text{open}}{\subset} X$, $y = f(x) \in W \overset{\text{open}}{\subset} Y$

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l, phantom, "\supset"] \arrow[d] \arrow[r, phantom, "="] & \operatorname{Spec} A \arrow[d] \\
Y & W \arrow[l, phantom, "\supset"] \arrow[r, phantom, "="] & \operatorname{Spec} B
\end{tikzcd}$$

$A = B[x_1, \cdots, x_n] \big/ (f_1, \cdots, f_n)$

w/ $\det\left(\frac{\partial f_i}{\partial x_j}\right)$ invertible in A.

<u>Lemma</u>.

$$\begin{tikzcd}
X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

f, g étale $\Rightarrow$ h étale.

<u>Pf</u>. Check I.L.P for h

$$\begin{tikzcd}
T^\circ \arrow[r, hook] \arrow[d, "p"'] & T_{\text{affine}} \arrow[dl, dashed, "\tilde{q}"'] \arrow[d, "q"] \arrow[dr] & \\
X \arrow[r, "h"] & X' \arrow[r, "g"'] & Y
\end{tikzcd}$$

$f = g \circ h$ is étale,

$\exists !\ \tilde{q}$ s.t. $g \circ h \circ \tilde{q} = g \circ q$ $(*)$

$\tilde{q}|_{T^\circ} = p$.

Want $h \circ \tilde{q} = q$.

know $h \circ \tilde{q}|_{T^\circ} = q|_{T^\circ}$ and $(*)$. Since g is étale $\Rightarrow h \circ \tilde{q} = q$.

<u>Cor</u>. $X \xrightarrow{f} Y$ is étale, then $X \xrightarrow{\Delta_X} X \times_Y X$ is open.

<u>Pf</u>.

$$\begin{tikzcd}
X \arrow[rr, "\Delta_X"] \arrow[dr] & & X \times_Y X \arrow[dl, "pr"] \\
& Y &
\end{tikzcd}$$

$\Rightarrow \Delta_X$ is étale $\Rightarrow$ open.

FEt_X = object: finite étale $Y \to X$

morphisms

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

($\Rightarrow f_* \mathcal{O}_Y$ is vector bundle over X)

Lemma. Let $f: Y \to X$ be a finite flat $\underset{\text{lfp}}{\wedge}$ morphism.

f is étale $\Leftrightarrow$ $\forall\ x \in \operatorname{Spec} A \subset X$, s.t. $B = \mathcal{O}_Y(f^{-1}(\operatorname{Spec} A))$

B is a free A-module, the trace form

$$Q: B \times B \longrightarrow A \qquad \text{is nondegenerate.}$$
$$(b_1, b_2) \longmapsto tr_{B/A}(b_1 b_2)$$

Pf. Reduce to $Y = \operatorname{Spec} k$.

$Q: K \times K \longrightarrow k$ is nondegenerate $\Leftrightarrow$ $K|k$ is separable.

Def. A geometric point of X is $\bar{x}: \operatorname{Spec} K \longrightarrow X$, $K = \bar{K}$.

Given $\bar{x}$, define $F_{\bar{x}}: \mathrm{FEt}_X \longrightarrow$ Finite Sets

$$F_{\bar{x}}(Y) = Y_{\bar{x}}(K).$$

$$\begin{tikzcd} \bigsqcup \operatorname{Spec} K = Y_{\bar{x}} \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} K \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Thm. If X is connected, then $(\mathrm{FEt}_X, F_{\bar{x}})$ is a Galois category.

Def. $\pi_1^{\text{ét}}(X, \bar{x}) := \operatorname{Aut}(F_{\bar{x}})$.

Cor. $\mathrm{FEt}_X \simeq \text{Finite-}\pi_1^{\acute{e}t}(X, \bar{x})\text{-sets}.$

Pf. Limits in FEt_X

$$* = X \xrightarrow{\mathrm{Id}} X$$

fiber products:

$$\begin{tikzcd} Y_1 \arrow[dr] & & Y_3 \arrow[dl] \\ & Y_2 \arrow[d] & \\ & X & \end{tikzcd} \Rightarrow Y_1 \times_{Y_2} Y_3 \in \mathrm{FEt}_X.$$

Colimits. initial object $\emptyset$

coproducts. $Y_1 \longrightarrow X$, $Y_2 \longrightarrow X \rightsquigarrow Y_1 \sqcup Y_2 \longrightarrow X.$

Coequilizers:

$$\begin{tikzcd} Y_1 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

[X, $\mathcal{A}$ sheaf of algebra

$\underline{\mathrm{Spec}}_X \mathcal{A} \longrightarrow X$ s.t.

$\forall\, T \xrightarrow{\pi} X$,

$\mathrm{Mor}(T, \underline{\mathrm{Spec}}_X \mathcal{A}) = \mathrm{Mor}(\mathcal{A}, \pi_* \mathcal{O}_T)$]

$\underline{\mathrm{Spec}}_X f_{i*} \mathcal{O}_{Y_i}$

Take $\mathcal{A} = eq\left(f_{2*}\mathcal{O}_{Y_2} \overset{a^*}{\underset{b^*}{\rightrightarrows}} f_{1*}\mathcal{O}_{Y_1} \right)$

Claim. $\underline{\mathrm{Spec}}_X \mathcal{A} = \mathrm{coeq}(a, b) \in \mathrm{FEt}_X.$

Pf. Ex.

Connected components $Y \longrightarrow X$, $Y = \bigsqcup_i Y_i$.

$$\begin{tikzcd} Y_i \arrow[dr, "\text{Connected}"'] & & Y' \arrow[ll, hook', "i"'] \arrow[dl] \\ & X & \end{tikzcd} \Rightarrow i \text{ is iso.}$$

$F_{\bar{x}}$ reflects iso.

$$\begin{tikzcd} Y_1 \arrow[rr, "g"] \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Want: if $g : F_{\bar{x}}(Y_1) \xrightarrow{\sim} F_{\bar{x}}(Y_2)$, then g is iso.

Pf. May assume that Y_2 is connected. $g_* \mathcal{O}_{Y_1}$ is vector bundle of some rank d.

Since Y_2 is connected, d is constant.

Looking at $Y_{1,\bar{x}}$ and $Y_{2,\bar{x}}$, we see that $d = 1 \Rightarrow g$ is iso.

Lecture 11. X connected scheme, $\bar{x} : \operatorname{Spec} \bar{k} \to X$ geometric point.

$\rightsquigarrow \pi_1^{ét}(X, \bar{x})$.

Finite-$\pi_1^{ét}(X, \bar{x})$-sets $\simeq FEt_X$

$$\begin{tikzcd} X \arrow[rr, "f"] & & X' \\ & \operatorname{Spec} \bar{k} \arrow[ul, "\bar{x}"] \arrow[ur, "f(\bar{x})"'] & \end{tikzcd}, \quad \begin{tikzcd} FEt_{X'} \arrow[rr] \arrow[dr, "F_{f(\bar{x})}"'] & & FEt_X \arrow[dl, "F_{\bar{x}}"] \\ & \text{Finite Sets} & \end{tikzcd}, \quad \begin{tikzcd} Y \arrow[d] & \rightsquigarrow & Y \times_{X'} X \arrow[d] \\ X' & & X \end{tikzcd}$$

$$\rightsquigarrow \pi_1^{ét}(X, \bar{x}) \longrightarrow \pi_1^{ét}(X', f(\bar{x})).$$

Dependence of the base point.

Def. $\mathcal{C}$ cat., $F : \mathcal{C} \to$ Finite Sets is a fiber functor

$\Leftrightarrow$ $(\mathcal{C}, F)$ is a Galois category.

Lemma. Any two fiber functors $F_1, F_2 : \mathcal{C} \to$ Finite Sets are isomorphic.

Pf. $(\mathcal{C}, F_1) \simeq (\text{Finite}-G\text{-sets}, \underset{\overset{\shortparallel}{F_1}}{\text{Forgetful functor}})$

$$F_1(X) = \underset{\substack{H \subset G \\ \text{open, normal}}}{\underrightarrow{\text{Colim}}} \underbrace{X^H}_{\overset{\shortparallel}{\mathrm{Mor}_{\mathcal{C}}(G/H, X)}}$$

$$\mathrm{Mor}(F_1, F_2) = \underset{H \subset G}{\underleftarrow{\lim}} \underbrace{F_2(G/H)}_{\text{finite}} \quad (\text{Yoneda Lemma})$$

$\overset{\text{claim}}{\neq} \phi$

$H \subset H'$, $F_2(G/H) \twoheadrightarrow F_2(G/H')$.

Similarly, $\mathrm{Mor}(F_2, F_1) \neq \phi$.

$$F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_1$$

$\mathrm{Mor}(F_1, F_1) = \underset{H \subset G}{\underleftarrow{\lim}} G/H = G = \mathrm{Aut}(F_1)$ $\Rightarrow$ $\beta \circ \alpha$ is iso.

Cor. $\pi_1^{\acute{e}t}(X, \bar{x}) \simeq \pi_1^{\acute{e}t}(X, \bar{x}')$. canonical up to composition w/ an inner auto.

Ex $X = \mathrm{Spec}\, k$. $\bar{x} : \mathrm{Spec}\, \bar{k} \to \mathrm{Spec}\, k$.

$$\mathrm{FEt}_X : \begin{tikzcd} \mathrm{Spec}\, A \arrow[d] \\ \mathrm{Spec}\, k \end{tikzcd}$$

$A = \prod_{i \in I} k_i$

$k_i | k$ finite separable

$F_{\bar{x}}(A) = \mathrm{Mor}(A, \bar{k})$

$= \mathrm{Mor}(A, k^{sep}) \curvearrowleft \mathrm{Gal}(k^{sep}|k)$

$$\mathrm{Gal}(k^{sep}|k) \xrightarrow[\sim]{\text{claim}} \mathrm{Aut}(F_{\bar{x}}) = \pi_1^{\acute{e}t}(X, \bar{x})$$

$F_{\bar{x}}(A) = \varinjlim_{\substack{k \subset k' \subset k^{sep} \\ \text{Galois, finite}}} \mathrm{Mor}(A, k')$, then use Yoneda Lemma.

Ex. X

$$\begin{tikzcd}
\bar{x} : \mathrm{Spec}\, \mathbb{F}_q \arrow[r] \arrow[d, squiggly] & X \\
\hat{\mathbb{Z}} = \pi_1^{\acute{e}t}(\mathrm{Spec}\, \mathbb{F}_q) \arrow[r] & \pi_1^{\acute{e}t}(X)
\end{tikzcd}$$

$\hat{\mathbb{Z}} = \langle Fr_q \rangle$

$\rightsquigarrow$ get a conj. class Fr_x in $\pi_1^{\acute{e}t}(X)$.

Ex. $\pi_1^{\acute{e}t}(\mathbb{P}^1_k) = 0$.

$$\begin{tikzcd}
Y \arrow[d, "f"] \\
\mathbb{P}^1
\end{tikzcd}$$

Y - connected curve.

$\deg f = d$.

$d \cdot 2 = 2 - 2\,\mathrm{genus}(Y)$

$\Rightarrow d = 1$.

* * *

Complex varieties

$X/\mathbb{C}$ finite type.

$\rightsquigarrow$ $X^{an} = X(\mathbb{C})$ w/ usual topology

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{C}} & U \arrow[l, hook'] \arrow[r, hook, "open"'] & X
\end{tikzcd}$$

$U^{an} \underset{open}{\subset} X^{an}$

$U^{an} \subset \mathbb{C}^n$

$\uparrow$ is equipped w/ induced topology

Lemma. If $f: X \to Y$ is proper, then

$X^{an} \to Y^{an}$ is proper ($\Leftrightarrow$ closed and fibers are compact).

Idea of proof.

1. reduce to projective f.

   Use Chow's Lemma: $\exists$ a projective morphism $X' \to X$ s.t. $X' \to Y$ is also proj.

2. $$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^n \times Y \arrow[d] \\ & Y \end{tikzcd}$$ Use $\mathbb{P}^{n,an}$ is compact.

Lemma. If $f: X \to Y$ is étale, then f^{an} is a local homeomorphism.

Pf. Reduce to $Y = \mathbb{A}^n$. May assume X, Y affine.

$$\begin{tikzcd} \operatorname{Spec} \bar{B} = X \arrow[r, hook] \arrow[d] & W_{affine} \arrow[d, dashed, "\text{étale}"] \\ \operatorname{Spec} A/I = Y \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$

$\bar{B} = A/I[x_1, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_n)$,

$\det\left(\frac{\partial \bar{f}_i}{\partial x_j}\right)$ is invertible.

$$W_{affine} = \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_n)\left[\det\left(\frac{\partial f_i}{\partial x_j}\right)^{-1}\right]$$

then use inverse function theorem.

Cor. We get $FEt_X \to \{\text{finite covering spaces of } X^{an}\}$

$$\begin{tikzcd} Y \arrow[d] & \rightsquigarrow & Y^{an} \arrow[d] \\ X & & X^{an} \end{tikzcd}$$

Thm. This functor is an equivalence.

Cor. If X is connected, then so is X^{an}, and $\forall\ \bar{x} \in X(\mathbb{C})$,

$$\underset{\substack{| \\ \text{profinite completion}}}{\pi_1^{top}(X^{an}, \bar{x})^{\wedge}} \xrightarrow{\sim} \pi_1^{\acute{e}t}(X, \bar{x})$$

Ex. $\pi_1^{\acute{e}t}(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}) = \underset{\llcorner \text{free group over 2 generators}}{\hat{F_2}}$

Rmk. X^{an} depends on $X \longrightarrow \operatorname{Spec} \mathbb{C}$.

$$\begin{tikzcd}
X \arrow[d] \arrow[dr] & \\
\operatorname{Spec} \mathbb{C} \arrow[r, "\sigma", "\sim"'] & \operatorname{Spec} \mathbb{C}
\end{tikzcd}$$

$X^{an} \neq X_{\sigma}^{an}$

though $X(\mathbb{C}) = X_\sigma(\mathbb{C})$.

Serre constructed an example of smooth projective $X/\underset{\substack{| \\ \text{number field,}}}{k}$

and $k \underset{i_2}{\overset{i_1}{\rightrightarrows}} \mathbb{C}$ s.t. $\pi_1^{top}((X \otimes_{k, i_1} \mathbb{C})^{an}) \ncong \pi_1^{top}((X \otimes_{k, i_2} \mathbb{C})^{an})$

But their profinite completions must be isomorphic

Lecture 12. **Thm.** $X/\mathbb{C}$ $\quad FEt_X \xrightarrow{\sim} \{\text{finite covers of } X^{an}\}$

Proof for smooth, projective X.

$$\begin{tikzcd} Z \arrow[d, "f"'] & f^{-1}(U^{an}) \arrow[l, phantom, "\supset"] \arrow[d] \\ X^{an} & U^{an} \arrow[l, phantom, "\supset"] \end{tikzcd}$$

$X \supset U$

$$\begin{tikzcd} \mathcal{O}(f^{-1}(U^{an})) & \subset & \mathcal{O}^{an}(f^{-1}(U^{an})) \\ \mathcal{O}(U) \arrow[u, phantom, "\cup \text{ integral closure}"] & & \end{tikzcd}$$

Def. Let Z be a (smooth) cpt analytic mfd, $\mathcal{L}$ a holomorphic line bundle over Z. We say that $\mathcal{L}$ is positive if $\mathcal{L}$ has a Hermitian metric whose curvature form $\omega \in \Omega^{1,1}(Z)$ is positive: $\underset{0 \neq}{\theta} \in T_{Z,z}$, $i\,\omega(\theta, \bar{\theta}) > 0$.

Connection: $$\begin{tikzcd} \mathcal{L} \arrow[r, "\nabla"] \arrow[dr, "\nabla^{0,1}"'] & (\Omega^{1,0}_Z \oplus \Omega^{0,1}_Z) \otimes \mathcal{L} \arrow[d] \\ & \Omega^{0,1}_Z \otimes \mathcal{L}, \end{tikzcd}$$

For any holomorphic section s of $\mathcal{L}$, $\nabla^{0,1}(s) = 0$.

Claim: for any metric on $\mathcal{L}$, $\exists!\ \nabla$ s.t.

① $\nabla^{0,1}$ is the one above.

② ∇ preserves the metric.

Ex. $Z = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(1)$ is positive

Thm (Kodaira) If $\mathcal{L}$ is positive, and E any holomorphic bundle, then for $N >> 0$, $E \otimes \mathcal{L}^N$ is generated by global sections. and $H^q(Z, E \otimes \mathcal{L}^N) = 0$, $q > 0$.

<u>Thm</u> (Kodaira) If $\mathcal{L}$ is positive, then for $N >> 0$,

$$Z \hookrightarrow \mathbb{P}\left(H^0(Z, \mathcal{L}^{\otimes N})^*\right)$$. In particular, Z is projective.

$$\left\{\begin{array}{c}\text{Smooth projective} \\ \text{varieties } /\mathbb{C}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{cpt} \\ \text{holomorphic manifolds} \\ \text{that admit a positive lb.}\end{array}\right\}$$

$$X \longmapsto X^{an}$$

<u>Pf.</u> X smooth, projective.

$$\begin{tikzcd} Z \arrow[d, "f"'] & \\ X^{an} \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

$f^*(\mathcal{O}_{X^{an}}(1))$ is positive.

$\Rightarrow Z$ algebraic, $Z = Y^{an}$ and f is algebraic.

$$(f_* \mathcal{O}_Y)^{an} \underset{\uparrow}{=} f_* \mathcal{O}_Z$$

Chow's thm

<u>Cor.</u> $X, Y / \mathbb{C}$, $\bar{x} \in X(\mathbb{C})$, $\bar{y} \in Y(\mathbb{C})$,

$$\pi_1^{\acute{e}t}(X \times Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\acute{e}t}(X, \bar{x}) \times \pi_1^{\acute{e}t}(Y, \bar{y})$$

<u>Rmk.</u> FALSE for schemes over $\overline{\mathbb{F}_p}$.

$\mathbb{A}^1_{\overline{\mathbb{F}_p}}$: $\pi_1^{\acute{e}t}(\mathbb{A}^1_{\overline{\mathbb{F}_p}}) \neq 0$.

$$\begin{tikzcd} \operatorname{Spec} \overline{\mathbb{F}_p}[t] \arrow[d, "\text{finite étale}"] \\ \operatorname{Spec} \overline{\mathbb{F}_p}[X] \end{tikzcd}$$

$X \mapsto t^p - t$

$\operatorname{spec} k(a)[t] / t^p - t - a$

$$\pi_1^{\acute{e}t}(\mathbb{A}^1_{\overline{\mathbb{F}_p}} \times \mathbb{A}^1_{\overline{\mathbb{F}_p}}) \longrightarrow \pi_1^{\acute{e}t}(\mathbb{A}^1_{\overline{\mathbb{F}_p}}) \times \pi_1^{\acute{e}t}(\mathbb{A}^1_{\overline{\mathbb{F}_p}})$$

NOT injective $\Leftrightarrow \exists$ finite étale cover $\begin{array}{c} X \\ \downarrow \\ \mathbb{A}^2_{\overline{\mathbb{F}_p}}\end{array}$ – connected which splits over $\mathbb{A}^1_{\overline{\mathbb{F}_p}} \times 0$ & $0 \times \mathbb{A}^1_{\overline{\mathbb{F}_p}}$.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} \overline{\mathbb{F}}_p[t] \arrow[d] \\
\operatorname{Spec} \overline{\mathbb{F}}_p[v,u] \arrow[r] & \operatorname{Spec} \overline{\mathbb{F}}_p[x] \\
x \arrow[r, mapsto] & uv
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, hook] & \mathbb{A}^3, \; t^p - t = uv \arrow[d] \\
 & \mathbb{A}^2
\end{tikzcd}$$

Claim. $k = \bar{k} \subset \mathbb{C}$, $\quad X/k$. $\quad X_{\mathbb{C}} = X \otimes_k \mathbb{C} \longrightarrow X$,

$$\boxed{\pi_1^{\text{ét}}(X_{\mathbb{C}}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X)}$$

Observation. X/k , $X_{k^{sep}} = X \otimes_k k^{sep}$

$$\circlearrowleft$$
$$\operatorname{Gal}(k^{sep}|k)$$

$\rightsquigarrow \operatorname{Gal}(k^{sep}|k) \to \operatorname{Out} \operatorname{Aut}\left(\pi_1^{\text{ét}}(X_{k^{sep}})\right)$.

Ex. $X = \mathbb{G}_m = \operatorname{Spec} \mathbb{Q}[x, x^{-1}]$, $\quad k = \mathbb{Q}$.

$\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}) = \pi_1^{\text{ét}}(X_{\mathbb{C}}) = \hat{\mathbb{Z}}$

$$\begin{tikzcd}
X_{n,\overline{\mathbb{Q}}} \arrow[r] \arrow[d, equal] & X_{\overline{\mathbb{Q}}} \arrow[d, equal] \\
\operatorname{Spec} \overline{\mathbb{Q}}[y, y^{-1}] \arrow[r] & \operatorname{Spec} \overline{\mathbb{Q}}[x, x^{-1}] \\
x \arrow[r, mapsto] & y^n
\end{tikzcd}$$

$$\begin{aligned}
\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}) &= \operatorname{Aut}(F) \\
&\simeq \varprojlim \operatorname{Aut}\begin{pmatrix} X_{n,\overline{\mathbb{Q}}} \\ \downarrow \\ X_{\overline{\mathbb{Q}}} \end{pmatrix} \\
&= \varprojlim \mu_n =: \hat{\mathbb{Z}}(1)
\end{aligned}$$

Ex. $X = \mathbb{P}^1 - \{0, 1, \infty\}$, $k = \mathbb{Q}$,

$$\mathrm{Gal}(\bar{\mathbb{Q}} \mid \mathbb{Q}) \longrightarrow \mathrm{Out\,Aut}\left(\pi_1^{\acute{e}t}(X_{\bar{\mathbb{Q}}})\right) = \mathrm{Out\,Aut}\left(\hat{F_2}\right)$$

free on 2 generators.

Thm. This is an injection.

Idea of proof: Thm (Belyi) For any smooth projective curve $Y/\bar{\mathbb{Q}}$,

$\exists$ $Y_{\bar{\mathbb{Q}}} \longrightarrow \mathbb{P}^1_{\bar{\mathbb{Q}}}$ étale over $X \subset \mathbb{P}^1_{\bar{\mathbb{Q}}}$.

Assume $\mathrm{Id} \neq \sigma \in \ker\varphi$,

Pick $j \in \bar{\mathbb{Q}}$ s.t. $\sigma(j) \neq j$.

Pick an elliptic curve $E/\bar{\mathbb{Q}}$ w $j(E) = j$.

$$\begin{tikzcd}
E \arrow[d, "f"'] \arrow[drr] & & \\
\mathbb{P}^1_{\bar{\mathbb{Q}}} \arrow[rr, "\sigma"] \arrow[d] & & \mathbb{P}^1_{\bar{\mathbb{Q}}} \arrow[d] \\
\mathrm{Spec}\,\bar{\mathbb{Q}} \arrow[rr, "\sigma"] & & \mathrm{Spec}\,\bar{\mathbb{Q}}
\end{tikzcd}$$

Lecture 13. Topological invariance of étale topology

Def. $f: X \to Y$ is universal homeomorphism, if $\forall$ $Y' \to Y$,

$X \times_Y Y' \to Y'$ is a homeomorphism.

Ex. ⓐ $\mathrm{Spec}\, A/I \hookrightarrow \mathrm{Spec}\, A$, $I^N = 0$.

ⓑ $X/\mathbb{F}_p$, $\mathrm{Fr}: X \to X$

Thm. If $f: X \to Y$ is a universal homeomorphism, then

$$\begin{tikzcd} FEt_Y \arrow[r, "\sim"] & FEt_X \end{tikzcd}$$

$$\begin{tikzcd} Y' \arrow[d] & & Y'\times_Y X \arrow[d] \\ Y \arrow[rr, mapsto, shorten=8mm] & & X \end{tikzcd}$$

Schemes étale over $Y \xrightarrow{\sim}$ Schemes étale over X

Cor. If Y is connected, then $\pi_1^{ét}(X) \xrightarrow{\sim} \pi_1^{ét}(Y)$.

Pf. For $f: X \to Y$ closed embedding w/ $I^N = 0$.

Base change is fully faithful:

$$\begin{tikzcd} Y_1 \arrow[dr, "\text{étale}"'] & & Y_2 \arrow[dl, "\text{étale}"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} Y_1 \times_Y X \arrow[r, "h"] & Y_2 \times_Y X \end{tikzcd} \overset{\text{want}}{\rightsquigarrow} \begin{tikzcd} Y_1 \arrow[rr, "\exists!\,\tilde{h}"] \arrow[dr] & & Y_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

$$\begin{tikzcd} Y_1 \times_Y X \arrow[r] \arrow[dr] \arrow[d, "\text{infinitesimal thickening}"'] & Y_2 \times_Y X \arrow[d] \\ Y_1 \arrow[r, dashed, "\exists!\,\tilde{h}"] \arrow[dr] & Y_2 \arrow[d, "\text{étale}"] \\ & Y \end{tikzcd}$$

Base change is essentially surjective:

Given $X' \to X$, want $\forall x' \in X'$, $\exists x' \in U' \subset X'$, and étale $W' \to Y$ s.t. $W' \times_Y X \simeq U'$.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, dashed] & Y \end{tikzcd}$$

$X = \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A = Y$, $I^N = 0$.

$X' = \operatorname{Spec} B$, $B = A/I[x_1, \dots, x_n] \big/ (f_1, \dots, f_n)$

$\det(\partial f_i / \partial x_j) \in B^*$.

Take $Y' = \operatorname{Spec} B'$, where $B' = A[x_1, \cdots, x_n]/(\tilde{f}_1, \cdots, \tilde{f}_n)$,

$\tilde{f}_i$ arbitrary lift of f_i, $Y' \to Y$ étale.

Thm ① Let A be a complete local ring, ($m \overset{\max}{\subset} A$, $A \cong \varprojlim A/m^n$)

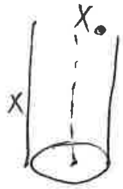
$f : X \to \operatorname{Spec} A$ proper morphism, $X_0 = X \otimes_A \overset{k}{\overset{\|}{A/m}}$, then

$FEt_X \xrightarrow{\sim} FEt_{X_0}$.

Cor. $\pi_1^{et}(X_0) \xrightarrow{\sim} \pi_1^{et}(X)$.

Rmk.

① Picture. A small nbhd of X_0 contracts to X_0.



The thm is false w/out properness assumption.

$X = \mathbb{P}^1 \times \operatorname{Spec} \mathbb{C}[[t]] - \{0, \infty\}$. $X_0 = \mathbb{G}_m \xleftarrow{x^2 \leftarrow\!\shortmid x} \mathbb{G}_m = X_0'$

$$\begin{tikzcd}
X_0' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d] \arrow[dr, phantom, "?"] & X' \otimes \mathbb{C}((t)) \arrow[l, hook'] \arrow[d, "splits"] \\
X_0 \arrow[r, hook] & X & \mathbb{P}^1 \otimes \mathbb{C}((t)) \arrow[l, hook']
\end{tikzcd}$$

② How we use the thm: $\eta \in \operatorname{Spec} A$
generic pt

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] & X_\eta \arrow[l] \arrow[d] & & X_{\bar{\eta}} \arrow[ll, "A/m = k"', "\mathrm{Frac}(A) = K"] \arrow[d] \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A & \operatorname{Spec} K \arrow[l] & & \operatorname{Spec} \bar{K} \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1^{et}(X_{\bar{\eta}}) \arrow[r] \arrow[rrr, bend right, "sp"'] & \pi_1^{et}(X_\eta) \arrow[r] & \pi_1^{et}(X) & \pi_1^{et}(X_0) \arrow[l, "\sim"']
\end{tikzcd}$$

Next thm. If the fibers of $X \to \operatorname{Spec} A$ are geometrically connected, then $sp: \pi_1^{ét}(X_{\bar\eta}) \to \pi_1^{ét}(X_0 \otimes_k \bar k)$ is surjective, and iso. on prime-to-char k quotients.

Pf. for noetherian A.

$X_n = X \times \operatorname{Spec} A/m^{n+1}$.

$$\begin{tikzcd} \cdots & X_2 \arrow[l] & X_1 \arrow[l, hook'] & X_0 \arrow[l, hook'] \end{tikzcd}$$

$\mathfrak{X} := \varinjlim X_n$

Grothendieck algebraization thm.

$X \to \operatorname{Spec} A$ proper,

$$\begin{tikzcd} {} & Z_2 \arrow[l] \arrow[d, "h_2"] \arrow[dr, phantom, "\urcorner"] & Z_1 \arrow[l, hook'] \arrow[d, "h_1"] \arrow[dr, phantom, "\urcorner"] & Z_0 \arrow[l] \arrow[d, "h_0"] \\ {} & X_2 \arrow[l] & X_1 \arrow[l] & X_0 \arrow[l] \end{tikzcd}$$

h_i finite,

$Z_i \times_{X_i} X_{i-1} = Z_{i-1}$

then $\exists$ a finite $h: Z \to X$ s.t. $Z_n = X_n \times_X Z$.

Pf of Thm ①. Focus on the essential surjectivity

$$\begin{tikzcd} \cdots & X_2' \arrow[d] & X_1' \arrow[l] \arrow[d] & X_0' \arrow[l, "\text{finite étale}"'] \arrow[d, "\text{finite étale}"] \\ & X_2 & X_1 \arrow[l] & X_0 \arrow[l] \end{tikzcd}$$

Apply algebraization, get a finite
$$\begin{tikzcd} X' \arrow[rr, "h"] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$
. Claim. $h \in FEt_X$.

$h_* \mathcal{O}_{X'}$ is vector bundle; sheaf of relative Kähler differential is 0.

Lecture 14

A complete local ring

$f: X \to \operatorname{Spec} A$ proper, $A/m = k$

$X_0 = X \otimes_A k$

Thm. $\mathrm{FEt}_X \xrightarrow{\sim} \mathrm{FEt}_{X_0}$

$Y \to X \mapsto Y \times_X X_0 \to X_0.$

Cor. $\pi_1^{\acute{e}t}(X_0) \xrightarrow{\sim} \pi_1^{\acute{e}t}(X)$, X, X_0 connected.

$\pi_1^{\acute{e}t}(\operatorname{Spec} A) \xleftarrow{\sim} \pi_1^{\acute{e}t}(\operatorname{Spec} k) = \mathrm{Gal}(\bar{k}|k)$

Pf. $X_n = X \otimes_A A/m^{n+1}$

$$\begin{tikzcd}
Z \arrow[d, "h"] & & Z_2 \arrow[d, "h_2"] & Z_1 \arrow[l] \arrow[d, "h_1"] & Z_0 \arrow[l] \arrow[d, "h_0"] & \in \mathrm{FEt}_{X_0} \\
X & \cdots & X_2 \arrow[l] & X_1 \arrow[l] & X_0 \arrow[l] &
\end{tikzcd}$$

Claim. $\exists!\ Z \to X \in \mathrm{FEt}_X$ s.t. $Z \times_X X_n = Z_n$

$\cdots \leftarrow h_{n*}\mathcal{O}_{Z_n} \leftarrow h_{n+1*}\mathcal{O}_{Z_{n+1}} \leftarrow \cdots \quad \rightsquigarrow h_*\mathcal{O}_Z$

Def. $\mathrm{Coh}(\mathfrak{X}) := \varprojlim \mathrm{Coh}(X_n)$

$\Phi: \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(\mathfrak{X})$

$\Phi(\mathcal{F}) = (\mathcal{F}|_{X_n}, n = 1, 2, \cdots)$

Example.

$$\begin{tikzcd} X = \mathbb{A}^1 \times \operatorname{Spec} k[[t]] \arrow[d] \\ \operatorname{Spec} k[[t]] \end{tikzcd}$$

$$\operatorname{Coh}(X) = \operatorname{Mod}^{fg}\left(k[[t]][x]\right) \xrightarrow{\Phi} \operatorname{Coh}(\mathfrak{X}) = \varprojlim \operatorname{Mod}^{fg}\left(k[[t]][x]/t^{n+1}\right)$$

$$= \operatorname{Mod}^{fg}\left(k[x][[t]]\right)$$

Theorem. $X \longrightarrow \operatorname{Spec} A$ proper, then Φ equivalence.

Rmk. $\mathcal{F} \in \operatorname{Coh}(X)$, $\mathcal{F}|_{X_n} = \mathcal{F}_n$,

$$H^q(X, \mathcal{F}) \xrightarrow{\sim} \varprojlim H^q(X_n, \mathcal{F}_n) \quad \text{(theorem of formality)}$$

X smooth curve $/ \mathbb{Z}_p$,

$$\pi_1^{\acute{e}t}(X_{\bar{\mathbb{Q}}_p}) \overset{?}{\rightsquigarrow} \pi_1^{\acute{e}t}(X_{\mathbb{F}_p}).$$

A noe. complete local ring

$X \longrightarrow \operatorname{Spec} A$ proper $\qquad \eta = \operatorname{Spec} K \hookrightarrow \operatorname{Spec} A \hookleftarrow \operatorname{Spec} k$.

$$\begin{tikzcd} \pi_1^{\acute{e}t}(X_{\bar\eta}) \arrow[r] \arrow[d, phantom, "\parallel" description] \arrow[rrr, bend right, "sp"'] & \pi_1^{\acute{e}t}(X_\eta) \arrow[r] & \pi_1^{\acute{e}t}(X) & \pi_1^{\acute{e}t}(X_k) \arrow[l, "\sim"'] \\ X_K \otimes_K \bar{K} & & & \end{tikzcd}$$

Thm. $X \longrightarrow \operatorname{Spec} A$ smooth proper, X_0 connected, $\bar{k} = k = A/m$,

then $sp: \pi_1^{\acute{e}t}(X_{\bar\eta}) \longrightarrow \pi_1^{\acute{e}t}(X_k)$ is surjective.

(1) If $\operatorname{char} k = 0$, sp is isom. (2) $\operatorname{char} k = p > 0$, $\forall$ finite G, $(|G|, p) = 1$, $\pi_1^{\acute{e}t}(X_{\bar\eta}) \to G$ factors through $\pi_1^{\acute{e}t}(X_0)$.

__Cor__ $k = \bar{k}$, $X \subset \mathbb{P}^n_k$ complete intersection, $\dim X \geqslant 2$, then

$\pi_1^{\text{ét}}(X) = 0$.

<u>Lecture 15</u>. A complete DVR, $A/m = k = \bar{k}$,

$X \xrightarrow{f} \operatorname{Spec} A$ proper

$$\begin{tikzcd}
X_{\bar{K}} \arrow[r] \arrow[d] & X_K \arrow[r] \arrow[d] & X \arrow[d] & X_0 \arrow[l] \arrow[d] \\
\operatorname{Spec} \bar{K} \arrow[r, hook] & \operatorname{Spec} K \arrow[r, hook] & \operatorname{Spec} A & \operatorname{Spec} k \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1^{\text{ét}}(X_{\bar{k}}) \arrow[r] \arrow[rrr, bend right, "sp"'] & \pi_1^{\text{ét}}(X_K) \arrow[r] & \pi_1^{\text{ét}}(X) & \pi_1^{\text{ét}}(X_0) \arrow[l, "\sim"']
\end{tikzcd}$$

__Thm.__ If f is also smooth, then sp is surjective, and for any cts hom.

$\pi_1^{\text{ét}}(X_{\bar{K}}) \xrightarrow{\varphi} \underset{\substack{\uparrow \\ \text{finite gp}}}{G}$ w/ $(|G|, \operatorname{char} k) = 1$, φ factors through sp.

__Ex.__ $k = \bar{k}$, $e \in E$ elliptic curve $/k$,

$$\pi_1^{\text{ét}}(E, e) = \lim_{\substack{\text{Galois covers} \\ E' \xrightarrow{f} E \\ \text{equipped w/ } e' \in E', \\ f(e') = e.}} \operatorname{Aut}_{\mathrm{FEt}_E}\begin{pmatrix} E' \\ \downarrow \\ E \end{pmatrix}$$

$$E \xrightarrow{f^t} \underset{\substack{| \\ \text{elliptic curve}}}{E'} \xrightarrow{f} E \qquad f \circ f^t = \deg f =: d$$

$$\overset{\rotatebox{90}{$\in$}}{e'} \longmapsto e.$$

char $k=0$; then $E \xrightarrow{d} E$ is finite étale.

$$\pi_1^{\acute{e}t}(E,e) = \lim_{d} \mathrm{Aut}\begin{pmatrix} E \\ \downarrow d \\ E \end{pmatrix} = \varprojlim_{d} E[d](k),$$

$$\begin{tikzcd} E[d] \arrow[r] \arrow[d] & E \arrow[d, "d"] \\ \mathrm{Spec}\, k \arrow[r, hook, "e"] & E \end{tikzcd}$$

$$= \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, E(k))$$

char $k = p > 0$

$$\begin{tikzcd} E \arrow[r] \arrow[rr, bend left, "d"] \arrow[dr, "Fr^n"'] & E' \arrow[r, "f"] & E \\ & E \arrow[u] \arrow[ur, "h_d"] & \end{tikzcd}$$

finite étale

$$\pi_1^{\acute{e}t}(E) = \varprojlim_{d} \mathrm{Aut}\begin{pmatrix} E \\ \downarrow h_d \\ E \end{pmatrix} = \varprojlim_{d} E[d](k) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, E(k))$$

$$\pi^{\acute{e}t}(k) = \prod_{\ell} T_\ell(E), \qquad T_\ell(E) = \lim_{n} E[\ell^n](k).$$

If char $k=0$, then for every ℓ, $T_\ell(E) \simeq \mathbb{Z}_\ell^2$,

$$\pi_1^{\acute{e}t}(E) \simeq \hat{\mathbb{Z}}^2.$$

char $k = p > 0$, for $\ell \neq p$, $T_\ell(E) \simeq \mathbb{Z}_\ell^2$,

$$T_p(E) = \begin{cases} \mathbb{Z}_p & \text{ordinary} \\ 0 & \text{supersingular} \end{cases}$$

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "Fr"'] & & E \\ & E \arrow[ur, "V = Fr^t"'] & \end{tikzcd}$$

Ex. Let $E \to \mathrm{Spec}\, \mathbb{Z}_p$ be an elliptic curve $/ \mathbb{Z}_p$.

$A = W(\bar{\mathbb{F}}_p) \supset \mathbb{Z}_p$, $\mathrm{Frac}(A) = k$.

$$\begin{tikzcd} \pi_1^{\acute{e}t}(E_{\bar{k}}) \arrow[r, "sp"] \arrow[d, phantom, "\wr"] & \pi_1^{\acute{e}t}(E_{\bar{\mathbb{F}}_p}) \arrow[d, phantom, "\wr"] \\ \prod_\ell T_\ell(E_{\bar{k}}) \arrow[r] & \prod_\ell T_\ell(E_{\bar{\mathbb{F}}_p}) \end{tikzcd}$$

$$\begin{tikzcd} E[d](\bar{k}) \arrow[dd, bend right] \\ E[d](\mathcal{O}_{\bar{k}}) \arrow[u, "\wr"'] \arrow[d] \\ E[d](\bar{\mathbb{F}}_p) \end{tikzcd}$$

valuation criterion

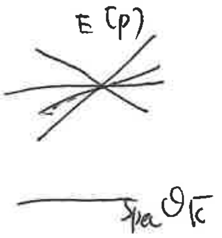
$$\begin{tikzcd} E[d] \arrow[r] \arrow[d] & E \arrow[d, "d"] \\ \operatorname{Spec} \mathcal{O}_{\bar{K}} \arrow[r] & E \end{tikzcd}$$

finite étale. $(d,p)=1$

disjoint union of copies of $\operatorname{Spec} \mathcal{O}_{\bar{K}}$

$l \neq p$, $T_l(E_{\bar{K}}) \xrightarrow[\sim]{sp} T_l(E_{\bar{\mathbb{F}}_p})$

$l = p$, $\underset{\underset{\mathbb{Z}_p^2}{\text{is}}}{T_p(E_{\bar{K}})} \overset{sp}{\twoheadrightarrow} \underset{\underset{\mathbb{Z}_p \text{ or } 0.}{\text{is}}}{T_p(E_{\bar{\mathbb{F}}_p})}$



$$\begin{tikzcd} & E(\bar{\mathbb{F}}_p) \arrow[dr, "V"] & \\ E(\bar{\mathbb{F}}_p) \arrow[ur, "Fr", "\sim"'] \arrow[rr, "p"] & & E(\bar{\mathbb{F}}_p) \end{tikzcd}$$

Th. curves of genus > 1 over $\bar{\mathbb{F}}_p$ $\longrightarrow$ profinite groups

$C \rightsquigarrow \pi_1^{\text{ét}}(C)$ is finite-to-one.

What do we have to prove?

$$\begin{tikzcd} \tilde{Y} \arrow[d, dashed, "\exists"'] & Y \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner"] \\ X_{\bar{A}} & X_{\bar{K}} \arrow[l] \end{tikzcd}$$

– a Galois cover w/ Galois group G.

$(|G|, \text{char } k) = 1$.

$\bar{A} \subset \bar{K}$

integral closure of A in $\bar{K}$

Fundamental group of normal schemes

Lemma A: a noetherian normal domain, $K = \mathrm{Frac}(A)$, and $K \subset L$ a finite separable ext. then the integral closure B of A in L is finite over A.

Pf. $Q: L \times L \to K$, $Q(x,y) = \mathrm{Tr}_{L|K}(xy)$

Pick a K-basis $\beta_1, \dots, \beta_n$ for L w/ $\beta_i \in B$.

Let $\beta_1^*, \dots, \beta_n^*$ be the dual basis.

Since $\mathrm{tr}_{L|K}(B) \subset A$, $B \subset \underbrace{A\beta_1^* \oplus A\beta_2^* \oplus \dots \oplus A\beta_n^*}_{\text{finite } A\text{-mod}} \subset L$. $\square$

Construction: X noetherian, normal, integral scheme, function field of $X =: K$, and let $L|K$ finite separable ext'n,

Normalization of X in L is a normal integral Y w/ finite morphism $Y \xrightarrow{f} X$ and $Y \times_X \mathrm{Spec}\, K = \mathrm{Spec}\, L$.

Lemma. Normalization exists and unique up to unique isom.

Fact: If $Y \to X$ is étale, X is normal, then Y is normal.

Cor. X noetherian, normal, integral. $K =$ func. field (X), then $\mathrm{FEt}_X \hookrightarrow \mathrm{FEt}_{\mathrm{Spec}\, K}$ is fully faithful, $\begin{matrix} Y \\ \downarrow \\ X \end{matrix} \mapsto \begin{matrix} Y \times_X \mathrm{Spec}\, K \\ \downarrow \\ \mathrm{Spec}\, K \end{matrix}$

Cor. $\pi_1^{ét}(\mathrm{Spec}\, k) \twoheadrightarrow \pi_1^{ét}(X)$.

Lecture 16.

$$\begin{tikzcd} & X \arrow[d] & \\ \mathrm{Spec}\, K \arrow[r, hook] & \mathrm{Spec}\, A & \mathrm{Spec}\, k \arrow[l, hook'] \end{tikzcd}$$

$$\begin{tikzcd} Y_K \arrow[d, "\text{Galois cover}"] \\ X_K \end{tikzcd}$$

$(|G|, \mathrm{char}\, k) = 1$

$\exists\, K \subset L$ s.t. $Y_L \to X_L$ extends to finite étale cover of $X \underset{A}{\otimes} B$, where B is the integral closure of A in B.

Last time, X integral, Noetherian, <u>normal</u>, $K =$ func. field (X)

$$\begin{tikzcd} \mathrm{FEt}_X \arrow[r, hook] & \mathrm{FEt}_{\mathrm{Spec}\, K} \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[d] & \rightsquigarrow & Y \times_X \mathrm{Spec}\, K = \mathrm{Spec}\, L \arrow[d] \\ X & & \mathrm{Spec}\, K \end{tikzcd}$$

normalization of X in L

Cor. $\pi_1^{ét}(\mathrm{Spec}\, k) \twoheadrightarrow \pi_1^{ét}(X)$

$$\begin{tikzcd} \pi_1^{ét}(\mathrm{Spec}\, k) \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Gal}(\bar{k}|k) \end{tikzcd}$$

Thm (Nagata-Zariski) Let X be a regular connected scheme, $Y \xrightarrow{f} X$ a normalization of X in a finite separable extension of $K = ff(X) \subset L$.

Assume that $\exists\, U \subset X$ w/ $\operatorname{codim}_X (X \setminus U) \geq 2$, and that $f : f^{-1}(U) \to U$ is étale, then f is étale.

Cor. If $\operatorname{codim}_X (X \setminus U) \geq 2$, then $\pi_1^{\text{ét}}(U) \xrightarrow{\sim} \pi_1^{\text{ét}}(X)$.

Ex X/k smooth surface, $a \in X(k)$,

$\pi_1^{\text{ét}}(X - a) \xrightarrow{\sim} \pi_1^{\text{ét}}(X)$.

Ex. $\pi_1(\mathbb{C}^n - 0) = \{1\}$, $n \geq 2$.

$X = U_1 \cup U_2$, $\quad \pi_1(X) = \operatorname{colim} \left(\begin{tikzcd} \pi_1(U_1 \cap U_2) \arrow[r] \arrow[d] & \pi_1(U_2) \\ \pi_1(U_1) & \end{tikzcd} \right)$.

Pf. Step 1. Assume in addition that f is flat. We show that f is étale.

Set $f_* \mathcal{O}_Y = A$ - a vec. bdle

$Q : A \otimes_{\mathcal{O}_X} A \to \mathcal{O}_X$, $\quad Q(x,y) = tr(xy)$

Want Q is nondegenerate: $A \xrightarrow{\sim} A^*$.

May assume $A = \mathcal{O}_X^{\oplus n}$. Let h be the determinant of the Gram matrix of Q, $h \in \mathcal{O}_X(X)$. Want $h \in \mathcal{O}(X)^*$. But $h|_U \in \mathcal{O}(U)^*$, and $\operatorname{codim}_X (X \setminus U) \geq 2$.
$\Rightarrow h \in \mathcal{O}(X)^*$.

Step 2. $\dim X = 2$. Want f is flat.

$U \xhookrightarrow{j} X \hookleftarrow X \setminus U \ni a$. Replace X by $\operatorname{Spec} \mathcal{O}_{X,a}$.

$$\begin{tikzcd} m \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\shortparallel"] & \mathcal{O}_{X,a} \\ (x_1, x_2) & \end{tikzcd} \qquad \mathcal{A} = j_* A|_U$$

Check: A_a is a flat $\mathcal{O}_{X,a}$-module.

$$\operatorname{Tor}_i^{\mathcal{O}_{X,a}}(A_a, k) = 0,\ i > 0 \iff \overset{-2}{A_a} \xrightarrow{(x_1, -x_2)} \overset{-1}{A_a \oplus A_a} \xrightarrow{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \overset{0}{A_a}$$

acyclic in negative degrees

$$0 \to \mathcal{O}_{X,a} x_1 x_2 \to \mathcal{O}_{X,a} x_1 \oplus \mathcal{O}_{X,a} x_2 \to \mathcal{O}_{X,a} \to k \to 0$$

$$\begin{tikzcd} A_a \arrow[r, "x_1"] \arrow[d, "x_2"'] & A_a \arrow[r, dashed] \arrow[d, "x_2"] & A_a/x_1 \arrow[d, dashed, "x_2"] \\ A_a \arrow[r, "x_1"] & A_a \arrow[r, dashed] & A_a/x_1 \end{tikzcd}$$

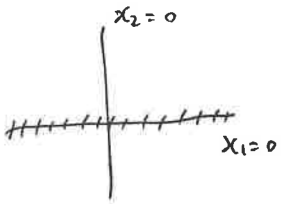
Enough to check:

① $A_a \xrightarrow{x_1} A_a$ is injective

② $A_a/x_1 \xrightarrow{x_2} A_a/x_1$ is injective.

① is OK.

② $s \in A_a$. $x_2 s = x_1 s'$, then $\frac{s}{x_1}$ is regular on U.



$\Rightarrow s \in A_a$. $\mathcal{A} = j_* A|_U$.

Step 3. $\dim X \geq 3$, induction on $\dim X$. Replace X by $\operatorname{Spec} \hat{\mathcal{O}}_{X,a}$ (some work)

Pick $h \in m - m^2$.

$$\begin{tikzcd} X & \operatorname{Spec} \hat{\mathcal{O}}_{X,a}/h = X_0 \arrow[l] \\ U \arrow[u, hook, "j"] & \operatorname{Spec} \hat{\mathcal{O}}_{X,a}/h - \{m\} = U_0 \arrow[l] \arrow[u, hook] \end{tikzcd}$$

$$\begin{tikzcd}
\pi_1^{\acute{e}t}(X) \arrow[r, equal] & \pi_1^{\acute{e}t}(\mathrm{Spec}\, k) \\
\pi_1^{\acute{e}t}(X_0) \arrow[u, "\wr"] \arrow[r, equal] & \pi_1^{\acute{e}t}(\mathrm{Spec}\, k) \arrow[u, equal]
\end{tikzcd}$$

Want: finite étale cover of U extends to X

Restrict it to U_0. Extend to X_0, and then to X.

Need: $\pi_1(U_0) \twoheadrightarrow \pi_1(U)$.

$$\Updownarrow$$

$$\begin{tikzcd}
Y \arrow[d, "\in \mathrm{FEt}_U"] & & V \times_U U_0 = V_0 \arrow[d] \\
U & \Longrightarrow & U_0
\end{tikzcd}$$

Y - conn'd ... is connected.

$$U \longleftarrow U_n = \mathrm{Spec}\, \hat{\mathcal{O}}_{X,a}/h^{n+1} - \{m\} \hookleftarrow U_0$$

$$V_n = V \times_U U_n. \qquad \left(\hat{\mathcal{O}}_{X,a} = \varprojlim \mathcal{O}_{X,a}/h^{n+1} \right)$$

Enough to show $\Gamma(V, \mathcal{O}_V) \xrightarrow{\sim} \varprojlim \Gamma(V_n, \mathcal{O}_{V_n})$

$\Uparrow$ some dévissage

$$\Gamma(U, \mathcal{O}_U) \xrightarrow{\sim} \varprojlim \Gamma(U_n, \mathcal{O}_{U_n})$$

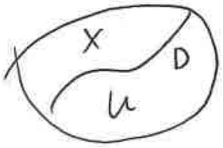
$$\Uparrow$$

$$\Gamma(U, \mathcal{O}_U) = \Gamma(X, \mathcal{O}_X), \quad H^1(U, \mathcal{O}_U) = 0.$$

Lecture 17. Last time: X regular connected, $U \subset X$, $\mathrm{codim}_X(X-U) > 1$, then $\pi_1^{\acute{e}t}(U) \xrightarrow{\sim} \pi_1^{\acute{e}t}(X)$.

Ramification theory

Set up:



A DVR,

$K = \mathrm{Frac}(A)$, $L \supset K$ finite separable ext., $B =$ integral closure of A in L.

$(\pi) = m \subset A$, $k = A/m$.

Prop. B is a PID w/ finitely many max'l ideals $m_1, \dots, m_n$.

Pf. B is normal, Krull dim $(B) = 1 \Rightarrow B$ is a Dedekind domain.

(B_{m_i} is DVR). B is a free A-module of rank $[L:K]$. ($B \otimes_A K = L$)

$\underset{B \otimes_A A/m}{\overset{B/mB}{\shortparallel}}$ is a k-alg. of dim $[L:K]$ $\Rightarrow$ B/mB has finitely many max. ideals

$\Rightarrow$ B has finitely many max. ideals.

Consider

$$B \twoheadrightarrow B/m_1^2 \times B/m_2^2 \times \dots \times B/m_n^2 \quad (\text{surj. by CRT})$$

Take $\pi_i \in m_i - m_i^2$ s.t. its image in B/m_j is not 0 for every $j \neq i$,

then $m_i = (\pi_i)$. □

Set $k(m_i) = B/m_i$, $[k(m_i):k] =: f_i$

$\pi = u\,\pi_1^{e_1} \cdots \pi_n^{e_n}$, $u \in B^\times$, i.e. $\pi B = m_1^{e_1} \cdots m_n^{e_n}$.

e_i ramification indices.

Lemma. $[L:K] = \sum_{i=1}^{n} e_i f_i$.

Pf. $[L:K] = \dim_k B/\mathfrak{m}B = \sum_i \dim_k B/\mathfrak{m}_i^{e_i} = \sum_i e_i \dim_k B/\mathfrak{m}_i$ □.

Rmk. If A is complete, then $n=1$.

Pf. $B = \varprojlim B/\mathfrak{m}^l B = \prod_{i=1}^{n} \varprojlim B/\mathfrak{m}_i^{e_i l}$

B is domain $\Rightarrow$ $n=1$.

Prop. Assume $L|K$ Galois, then $G = \mathrm{Gal}(L|K)$ acts transitively on $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\}$.

In particular, $f_1 = f_2 = \cdots = f_n = f$, $e_1 = e_2 = \cdots = e_n = e$. $[L:K] = nef$.

Pf. $\prod_{\sigma\in G} \sigma(\pi_1) \in K \cap B = A$.
$\in \mathfrak{m}$

$\Rightarrow \prod_{\sigma\in G} \sigma(\pi_1)$ is divisible by π_i. □

Def. $D_{\mathfrak{m}_i} := \{\sigma\in G : \sigma(\mathfrak{m}_i) = \mathfrak{m}_i\}$

decomposition group.

Prop. ⓐ $D_{\mathfrak{m}_i} \twoheadrightarrow \mathrm{Aut}(k(\mathfrak{m}_i)|k) = \mathrm{Aut}(k(\mathfrak{m}_i)^{sep}|k)$

ⓑ $k(\mathfrak{m}_i)|k$ is normal.

Pf. $k \subset k(m_i)^{sep} \subset k(m_i)$

$$\begin{tikzcd} k(m_i)^{sep} \arrow[d, phantom, "\parallel"] \\ k(\bar{b}) \end{tikzcd}$$

$\leftarrow$ primitive element thm

Pick b
$$\begin{tikzcd} B \arrow[r] & B/m_i \\ b \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \bar{b} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$
s.t. $b \in m_j$ for all $j \neq i$.

$\prod_{\sigma \in G} (x - \sigma(b)) \in A[X]$ monic.

$\Rightarrow$ all conjugates of $\bar{b}$ are of the form $\overline{\sigma(b)}$ for $\sigma \in G$.

Observe that $\sigma(b) \equiv 0 \ (m_i)$ for $\sigma \notin D_{m_i}$.

$\Rightarrow$ all conjugates of $\bar{b}$ have the form $\overline{\sigma(b)}$ for $\sigma \in D_{m_i}$.

$$1 \longrightarrow \underset{\substack{| \\ \text{inertia subgp.}}}{I_{m_i}} \longrightarrow D_{m_i} \longrightarrow \operatorname{Aut}(k(m_i)|k) \longrightarrow 1$$

Construction: $\varphi : I_{m_i} \longrightarrow \mu_{e_i}(k(m_i)) = \{ \varepsilon \in k(m_i) : \varepsilon^{e_i} = 1 \}$. $\quad \underset{\cup}{k(m_i)^*}$

$$\varphi_i(\sigma) = \frac{\sigma(\pi_i)}{\pi_i} \mod m_i$$

$$\varphi_i(\sigma_1 \sigma_2) = \frac{\sigma_1 \sigma_2(\pi_i)}{\sigma_2(\pi_i)} \frac{\sigma_2(\pi_i)}{\pi_i} \quad (m_i)$$

$$= \varphi_i(\sigma_1) \, \varphi_i(\sigma_2)$$

$$|D_{m_i}| = \frac{[L:K]}{n}$$

$$|I_{m_i}| = \frac{[L:K]}{n\,[k(m_i)^{sep}:k]} = e\,[k(m_i):\underbrace{k(m_i)^{sep}]}_{\text{power of char. } k}$$

$$\mu_{e_i}(k(m_i)) = \mu_{|I_{m_i}|}(k(m_i)).$$

Prop. $\ker \varphi_i$ is $\{1\}$ if char $k=0$, and $\ker\varphi_i$ is a p-group if char $k=p>0$.

Pf. $\ker\varphi_i$ acts trivially on m_i^r/m_i^{r+1}. ☐

Lecture 18 Last time: ramification.

A DVR $m \subset A$, $k = A/m$. $K = \mathrm{Frac}(A)$.

$L \supset K$ Galois ext'n (finite), $\mathrm{Gal}(L|K) =: G$, $A \subset B$ integral closure.
$\hat{K} \subset \hat{L}$

$m_1, \dots, m_n$ max. ideals in B. $[k(m_i):k] = f$ $mB = m_1^e m_2^e \dots m_n^e$

$fen = [L:K]$.

Decomposition groups $D_{m_i} < G$, stabilizer of m_i

$$1 \to I_{m_i} \to D_{m_i} \twoheadrightarrow \mathrm{Aut}(k(m_i)|k) \to 1$$

$$|I_{m_i}| = e[k(m_i):k(m_i)^{sep}], \quad |D_{m_i}| = \frac{|G|}{n} = f\cdot e$$

$$I_{m_i} \xrightarrow{\varphi_i} \mu_e(k(m_i)), \quad \underbrace{I_{m_i} \longrightarrow \mathrm{Aut}_{k(m_i)}(m_i/m_i^2) = k(m_i)^*}_{\varphi_i}$$

Prop. $\ker \varphi_i = P_i$ is trivial if $\mathrm{char}\, k = 0$, and a p-gp if $\mathrm{char}\, k = p > 0$.

Cor. Write $e = e' p^N$, $(e', p) = 1$, $\mathrm{char}\, k = p$
($e = e'$ if $\mathrm{char}\, k = 0$),

then $I_{m_i} \twoheadrightarrow \mu_e(k(m_i)) = \mu_{e'}(k(m_i)) \simeq \mathbb{Z}/e'\mathbb{Z}$

Pf. $|\mathrm{Im}\, \varphi_i| = e'$. $\square$

Cor. I_{m_i} is solvable.

Cor. $\mathrm{Gal}(\overline{\mathbb{Q}_p} | \mathbb{Q}_p)$ is solvable.

Pf. $\underset{K}{\underset{\parallel}{\mathbb{Q}_p}} \subset L$, $\mathrm{Gal}(L | \mathbb{Q}_p) = G$ $n = 1$.

$A = \mathbb{Z}_p$, $m_B \subset B \subset L$

$$1 \to \underbrace{I}_{\text{solvable.}} \to \underset{G}{\underset{\parallel}{D_{m_B}}} \to \underbrace{\mathrm{Gal}(k(m_B) | \mathbb{F}_p)}_{\text{abelian}} \to 1$$

Def. $L|K$ is tame iff $|I_{m_i}|$ is not divisible by $\mathrm{char}\, k$. & $k(m_i) = k(m_i)^{sep}$

$\Leftrightarrow$ φ_i is injective.

Ex. A complete (DVR), k perfect. $L|K$ finite Galois.

$n = 1$, $D = G$, $P \subset I \subset G$, $P = \ker \varphi$. $I/P \xrightarrow[\sim]{\varphi} \mu_{e'}(k(m_B))$.

$$K \subset \overset{L^I}{\overset{\parallel}{K_{nr}}} \subset \overset{L^P}{\overset{\parallel}{K_{tame}}} \subset L.$$

$$\mathrm{Gal}(K_{nr}|K) \xrightarrow{\sim} \mathrm{Gal}(k(m_B)|k)$$

Prop. $\overset{I/p = \langle \sigma \rangle \curvearrowright}{K_{tame}} = K_{nr}\left(a^{\frac{1}{e'}}\right)$, where a has valuation 1 in K_{nr}.

Pf Observe that by Hensel's lemma K_{nr} has all e'-th roots of unity.

Thus, by Kummer, $K_{tame} = K_{nr}\left(a^{\frac{1}{e'}}\right)$ for some $a \in K_{nr}$.

$$\begin{tikzcd}
\sigma \arrow[d, mapsto] & I/p \arrow[r, "\varphi", "\sim"'] \arrow[dr, "\sim"] & \mu_{e'}(k(m_B)) & \xi^r \\
\frac{\sigma(a^{\frac{1}{e'}})}{a^{\frac{1}{e'}}} & & \mu_{e'}(k_{nr}) \arrow[u, "\sim"'] & \xi \arrow[u, mapsto]
\end{tikzcd}$$

$(r, e') = 1$.

$\mathrm{val}(a) \equiv r \ (e')$.

$\Rightarrow$ replaces a by r^{-1} mod e'

$$K \subset K_{nr} \subset K_{tame} \subset \bar{K}$$

$$\begin{tikzcd}
\mathrm{Gal}(K_{nr}|K) \arrow[r, phantom, "="] & \mathrm{Gal}(\bar{k}|k) \arrow[d, phantom, "\cong" rotate=90] \\
\pi_1^{\text{ét}}(\mathrm{Spec}\, k) \arrow[u] \arrow[r] & \pi_1^{\text{ét}}(\mathrm{Spec}\, A)
\end{tikzcd}$$

$$K_{tame} = \bigcup_{(e, \mathrm{char}\, k) = 1} K_{nr}\left(\pi^{\frac{1}{e}}\right) \subset \bar{K}$$

$$\mathrm{Gal}(K_{tame}|K_{nr}) = \varprojlim_{\substack{e \\ (e, \mathrm{char}\, k) = 1}} \mu_e(k_{nr}) \approx \prod_{\ell \neq \mathrm{char}\, k} \mathbb{Z}_\ell .$$

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(K_{tame}|K_{nr}) \arrow[r] & \mathrm{Gal}(K_{tame}|K) \arrow[r] & \mathrm{Aut}(\bar{k}|k) \arrow[r] \arrow[l, bend right] & 1 .
\end{tikzcd}$$

$K \subset K_{nr} \subset K_{tame}$

$$\mathrm{Gal}(K_{tame} | K) = \mathrm{Aut}(\bar{k}|k) \ltimes \prod_{\ell \neq \mathrm{char}\, k} \mathbb{Z}_\ell(1)$$

$$\ker\left(\mathrm{Gal}(\bar{K}|K) \to \underbrace{\mathrm{Gal}(K_{tame}(K))}_{}\right) \text{ is a } p\text{-group.}$$

Back to specialization thm:

$X \to \mathrm{Spec}\, A$ smooth, proper. A complete DVR.

$X_0 = X \otimes_A k$ connected, $k = \bar{k}$.

$$\begin{tikzcd} \pi_1(X_{\bar{K}}) \arrow[r] \arrow[rrr, bend right, "sp"'] & \pi_1(X_K) \arrow[r] & \pi_1(X) & \pi_1(X_0) \arrow[l, hook', "\simeq"'] \end{tikzcd}$$

<u>Thm.</u> sp is surjective, and iso. on prime to char k quotients.

<u>Need</u>: For a Galois cover $Y_K \to X_K$ w/ G of order coprime to char k, $\exists$ a finite $L|K$ s.t. $Y_L \to X_L$ extends to a finite étale cover $Y_B \to X_B$.

Take normalization $Y \to X$ of X in $ff(Y_K) \supset ff(X_K)$.

$$\begin{matrix} X \\ \cup \\ U \end{matrix}$$

locus where f is étale

If f is not étale, then look at the generic pt $\eta \in X_0$.

$\mathcal{O}_{X,\eta} \subset$ integral closure in $ff(Y)$. is ramified. $L = K(\pi^{\frac{1}{e}})$.

<u>Lecture 19</u> What we want to prove.

A DVR,
$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "\text{smooth}"] & X_K \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Spec}\, A & \mathrm{Spec}\, K \arrow[l, hook'] \end{tikzcd}$$
, $X_0 \otimes_k \bar{k}$ connected.

$A \supset (\pi)$

Let $Y_K \longrightarrow X_K$ be a Galois cover w/ $\mathrm{Gal}(Y_K | X_K) = G$, $(|G|, \mathrm{char}\, k) = 1$,

then $\exists$ $\begin{matrix} L & \supset & K \\ \cup & & \cup \\ B & \supset & A \end{matrix}$ finite separable s.t. $Y_L \longrightarrow X_L$ extends to finite étale cover $Y_B \longrightarrow X_B$.

Step 1. May assume $Y_{\overline{K}}$ is connected.

Step 2. Let Y be the normalization of X in $\mathcal{K}(Y_K)$

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "f", "\text{finite}"'] & Y_K \arrow[l] \arrow[d, "\text{étale}"] \\ \eta \in X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] & X_K \arrow[l] \arrow[d] \\ \mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Spec}\, A & \mathrm{Spec}\, K \arrow[l] \end{tikzcd}$$

normal

Let η be the generic pt of X_0.

$$\begin{tikzcd} \mathrm{Spec}\, \mathcal{O}_{Y, f^{-1}(\eta)} = \mathrm{Spec}\, \mathcal{O}_{X,\eta} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[loop right, "G"] \\ \mathrm{Spec}\, \mathcal{O}_{X,\eta} \arrow[r] & X \end{tikzcd}$$

$$\begin{matrix} \mathcal{O}_{X,\eta} & \subset & \mathcal{K}(X_K) \\ \cap & & \cap \\ \mathcal{O}_{Y, f^{-1}(\eta)} & \subset & \mathcal{K}(Y_K) \end{matrix}$$

Let $m_1, \cdots, m_n$ be max'l ideals of $\mathcal{O}_{Y, f^{-1}(\eta)}$.

- G acts transitively on $\{m_1, \cdots, m_n\}$. (Galois cover)

$f^{-1}(\eta) = \{\eta_1, \cdots, \eta_n\}$

$(\pi) = \underset{\substack{\shortparallel \\ (\pi')}}{m} \subset \mathcal{O}_{X,\eta}$. $\quad \pi = \pi'^{E} u$, $\quad \mathcal{O}_{X,\eta}/(\pi) =$ local ring of $\overset{\text{smooth}}{X_0}$ at η, reduced.

$\Rightarrow (\pi) = (\pi')$.

$(\pi) = m = m_1^e \cdots m_n^e$ $\quad \mathcal{O}_{Y, f^{-1}(\eta)} \subset \mathcal{O}_{Y,\eta_i} \ni (\pi_i)$

$m_i = (\pi_i)$

$\pi = u \pi_i^e$, $u \in \mathcal{O}_{Y,\eta_i}^{\times}$.

$efn = |G|$, $f = [k(\eta_i) : k(\eta)]$.

$\Rightarrow (e, \mathrm{char}\, k) = 1$, $(f, \mathrm{char}\, k) = 1$.

$\Rightarrow \quad k(\eta_i) \mid k(\eta)$ is separable.

Set $L = K(\pi^{\frac{1}{e}})$,

$$\begin{tikzcd} L \arrow[d, phantom, "\cup"] \\ B \end{tikzcd}$$

Set Y_B to be the normalization of X in $ff(Y_L) \overset{Y_{\bar{K}} \text{ connected}}{=} ff(Y_K) \otimes_K L$.

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{Y,\eta_i}[u^{\frac{1}{e}}] \arrow[r] \arrow[d, "\text{fin. étale}"'] & \operatorname{Spec} \mathcal{O}_{Y_B, f_B^{-1}(\eta)} \arrow[r] \arrow[d] & Y_B \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{Y,\eta_i} \arrow[r, phantom, "\subset"] & \operatorname{Spec} \mathcal{O}_{Y, f^{-1}(\eta)} \arrow[r] \arrow[d] & Y \arrow[d] \\
& \operatorname{Spec} \mathcal{O}_{X,\eta} \arrow[r] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{X,\eta} \arrow[rr, phantom, "\subset"] \arrow[d, phantom, "\cap"] & & \mathcal{O}_{Y_B, f_B^{-1}(\eta)} \arrow[d, phantom, "\cap"] \\
ff(X) \subset ff(Y) & \subset & ff(Y_B)
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{O}_{X_B,\eta} \arrow[dr, hook, "\text{étale}"] & \\
\mathcal{O}_{X,\eta} \arrow[ur, hook, "\text{ramification is } e"] \arrow[dr, hook, "\text{ramification is } e"'] & & \mathcal{O}_{Y,\eta_i}[u^{\frac{1}{e}}] \\
& \mathcal{O}_{Y,\eta_i} \arrow[ur, hook, "\text{étale}"'] &
\end{tikzcd}$$

$$\begin{tikzcd} Z_1 \arrow[d, "\text{étale}"] \\ Z_2 \end{tikzcd}$$

Z_1 is normal $\Rightarrow$ Z_2 is normal

Step 3

$$\begin{tikzcd}
Y_B \times_{X_B} \operatorname{Spec} \mathcal{O}_{B,\eta} \arrow[r] \arrow[d, "\text{étale}"'] & Y_B \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{B,\eta} \arrow[r] & X_B
\end{tikzcd}$$

- finite, Y_B is normal.

étale in codim. 1.

X_B is regular

By purity, $Y_B \to X_B$ is étale. ☐

$K = \mathbb{C}((t))$, $A = \mathbb{C}[[t]]$.

$$\begin{tikzcd}
Y \arrow[d] & Y_K = X_K \otimes_K \mathbb{C}((t^{\frac{1}{e}})) \arrow[l] \arrow[d] \\
X & X_K \arrow[l]
\end{tikzcd}$$

$Y = X \otimes_A \mathbb{C}[[t^{\frac{1}{e}}]]$

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}[[t^{\frac{1}{e}}]] \arrow[d] \\ \operatorname{Spec} \mathbb{C}[[t]] \end{tikzcd}$$

$$\begin{tikzcd}
Y_k \arrow[r, "u \in \mathcal{O}_X^\times"] \arrow[d] & \mathbb{G}_{m,k} \arrow[d] & \zeta \arrow[d, mapsto] \\
X_k \arrow[r, "u \cdot t"] & \mathbb{G}_{m,K} & \zeta^e
\end{tikzcd}$$

Specialization Thm:

$$\begin{tikzcd}
X \arrow[d] \\
\operatorname{Spec} A
\end{tikzcd}$$
smooth, proper, A complete DVR, $k = \bar{k}$, X_0 connected.

$$\begin{tikzcd}
\pi_1(X_{\bar{K}}) \arrow[r] \arrow[rrr, bend right, "sp"'] & \pi_1(X_K) \arrow[r] & \pi_1(X) & \pi_1(X_0) \arrow[l, "\sim"']
\end{tikzcd}$$

isom. on coprime to char k quotients.

Rmk. $D = \{ z \in \mathbb{C} : |z| < 1 \}$

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d, "- \text{smooth, proper,}"] & X_{D^*} \arrow[l] \arrow[d] & X_{\mathcal{H}} \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & D & D^* \arrow[l] & \mathcal{H} \arrow[l]
\end{tikzcd}$$

$\exp(2\pi i \tau) \leftarrow\!\shortmid \tau$ $\qquad \mathcal{H} = \{ \tau : \operatorname{Im} \tau > c \}$

$$\begin{tikzcd}
\pi_1(X_{\mathcal{H}}) \arrow[r] \arrow[rrr, bend right, "\sim", "sp"'] & \pi_1(X_{D^*}) \arrow[r] & \pi_1(X) & \pi_1(X_0) \arrow[l, "\sim"']
\end{tikzcd}$$

Ex. X_0/k $\quad k = \bar{k}$, char $k = p > 0$,
smooth proper curve

then $\exists$ smooth proper curve $X \to \operatorname{Spec} W(k)$ s.t. $X \otimes_{W(k)} k = X_0$.

$$\begin{tikzcd}
 & & & \mathcal{O}(1) & \mathcal{O}(1) \\
X \arrow[d] & \cdots & X_2 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_0 \arrow[l] \arrow[d] \\
\operatorname{Spec} W(k) & & \operatorname{Spec} W(k)/p^3 \arrow[ll] & \operatorname{Spec} W(k)/p^2 \arrow[l] & \operatorname{Spec} k \arrow[l]
\end{tikzcd}$$

$$\boxed{\begin{array}{l} \text{Obstruction:} \\ \quad H^2(X_0, T_{X_0}) \end{array}}$$

Lecture 20.

How to use specialization thm.

$K = \operatorname{Frac}(A)$

$X_0 / \overline{\mathbb{F}}_q = k$ smooth projective curve, then there $\exists$ smooth proj. $X \to \operatorname{Spec} \overset{\overset{A}{\|}}{W(k)}$

s.t. $X \otimes_A k = X_0$. $\quad g = \text{genus}(X_0)$

$$\pi_1^{et}(X_{\overline{K}}) \overset{sp}{\twoheadrightarrow} \pi_1^{et}(X_0)$$

Thm. $\bar{k} = k \subset k' = \overline{k'}$, $\quad Y/k$ connected, finite type.

$$\pi_1^{et}(Y \otimes_k k') \xrightarrow{\varphi} \pi_1^{et}(Y)$$

ⓐ φ is surjective

ⓑ φ is an isom. if Y is proper or char $k = 0$.

$\overline{K} \hookrightarrow \mathbb{C}$, $\quad \pi_1^{et}(X_{\overline{K}}) \xleftarrow{\sim} \pi_1^{et}(X_{\mathbb{C}})$

$$\pi_1^{et}(X_{\mathbb{C}}) = \left(\text{free on } a_1, b_1, \cdots, a_g, b_g \Big/ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1\right)^\wedge$$

Rmk. $\operatorname{Gal}(\overline{K}|K) \to \operatorname{Out}(\pi_1^{et}(X_{\overline{K}}))$.

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{et}(X_{\overline{K}}) \arrow[r] & \pi_1^{et}(X_K) \arrow[r] & \pi_1^{et}(\operatorname{Spec} K) = \operatorname{Gal}(\overline{K}|K) \arrow[r] \arrow[l, bend right, dashed] & 1. \end{tikzcd}$$

Def. Let G be a profinite group, denote by G' the max'l coprime to char k quotient of G.

$$G' = \varprojlim_{\substack{H \subset G \\ \text{open, normal} \\ ([G:H], \operatorname{char} k) = 1}} G/H$$

$$\pi_1^{\text{ét}}(X_{\bar{k}})' \underset{\sim}{\xrightarrow{sp}} \pi_1^{\text{ét}}(X_0)'.$$

Ex 2.

$$\begin{tikzcd} X \arrow[d] \\ C \end{tikzcd}$$

Smooth, proper
w/ conn'd fibers
C is connected.

$X, C / k = \bar{k}$,

$c_1, c_2 \in C(k)$, $\pi_1^{\text{ét}}(X_{c_1}) \overset{?}{\sim} \pi_1^{\text{ét}}(X_{c_2})$

Thm. $\pi_1^{\text{ét}}(X_{c_1})' \simeq \pi_1^{\text{ét}}(X_{c_2})'$.

Pf. May assume C is a smooth curve.

$K = ff(C)$, $\hat{K}_{c_i} = \operatorname{Frac}(\hat{\mathcal{O}}_{C,c_i}) \supset K$

$\overline{\hat{K}_{c_i}} \supset \bar{K}$

$X_{\bar{K}} = X \times_C \operatorname{Spec} \bar{K}$,

$$\underbrace{\pi_1^{\text{ét}}(X_{\bar{K}})}_{\text{proper}} \xleftarrow{\sim} \pi_1^{\text{ét}}(X_{\overline{\hat{K}_{c_i}}}) \xrightarrow{sp} \pi_1^{\text{ét}}(X_{c_i})$$, sp is isom. on prime to char k quotient.

HW

1. ⓐ $\bar{k} = k \subset k' = \overline{k'}$, Y/k scheme of finite type.

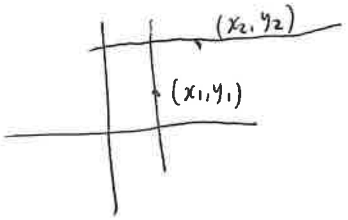
If Y is connected, then $Y_{k'}$ is connected.

Pf. $\Gamma(X_{k'}, \mathcal{O}_{Y_{k'}}) = \Gamma(Y, \mathcal{O}_Y) \otimes_k k'$

$$= \varinjlim_{\substack{k \subset A \subset k' \\ \text{finite type}/k}} \Gamma(Y, \mathcal{O}_Y) \otimes_k A$$

Enough to show that for conn'd $X, Y/k$, $X \times_k Y$ is connected.

$\boxed{\overline{X(k)} = X}$ $\qquad X \times Y = \underset{(x_1,y_1)}{Z_1} \sqcup \underset{(x_2,y_2)}{Z_2}$,



$(X \times \{y_1\}) \cup (\{x_2\} \times Y) \subset X \times Y$.

ⓑ X/k connected. $\pi_1^{et}(X_{k'}) \twoheadrightarrow \pi_1^{et}(X)$ surj.

$\Leftrightarrow$ for any connected $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$, $Y_{k'}$ is conn'd.

ⓒ char $k = 0$, $k = \bar{k}$,

$$\pi_1^{et}(X \times Y) \xrightarrow{\sim} \pi_1^{et}(X) \times \pi_1^{et}(Y)$$

Step 1: Reduction to $k \subset \mathbb{C}$. $\qquad Sch_k^{ft} = \varinjlim_{\substack{k' \subset k \\ \text{f.g. subfields}}} Sch_{k'}^{ft}$

Step 2. OK for $k = \mathbb{C}$

Step 3. $k \subset \mathbb{C}$.

$(x,y) \in X \times Y(k)$, $g_X \in \pi_1^{et}(X) \longrightarrow \pi_1^{et}(X \times Y)$

$g_Y \in \pi_1^{et}(Y) \longrightarrow \pi_1^{et}(X \times Y)$

Claim: Images commute $g_X g_Y = g_Y g_X$.

$$\begin{tikzcd} \tilde{g}_X \in \pi_1^{et}(X_{\mathbb{C}}) \arrow[r, two heads] \arrow[rr, bend left, mapsto] & \pi_1^{et}(X) \ni g_X & {} \\ \tilde{g}_Y \in \pi_1^{et}(Y_{\mathbb{C}}) \arrow[r, two heads] \arrow[rr, bend right, mapsto] & \pi_1^{et}(Y) \ni g_Y & {} \end{tikzcd}$$

$\tilde{g}_X \tilde{g}_Y \tilde{g}_X^{-1} \tilde{g}_Y^{-1} = 1$ in $\pi_1^{et}(X_{\mathbb{C}} \times Y_{\mathbb{C}})$, project to $\pi_1^{et}(X \times Y)$.

$$\begin{tikzcd} \pi_1^{et}(X \times Y) \arrow[r, shift left] & \pi_1^{et}(X) \times \pi_1^{et}(Y) \arrow[l, bend right, "\varphi"'] \end{tikzcd}$$

Claim: φ is surjective.

$$\begin{tikzcd} \pi_1^{et}(X \times Y) & \pi_1^{et}(X) \times \pi_1^{et}(Y) \arrow[l, "\varphi"'] \\ \pi_1^{et}(X_{\mathbb{C}} \times Y_{\mathbb{C}}) \arrow[u] & \pi_1^{et}(X_{\mathbb{C}}) \times \pi_1^{et}(Y_{\mathbb{C}}) \arrow[l, "\sim"'] \arrow[u, two heads] \end{tikzcd}$$

$\square$

ⓓ char $k = 0$. $k \subset k'$, $\pi_1^{et}(X_{k'}) \xrightarrow{\sim} \pi_1^{et}(X)$.

Need any cover of $X_{k'}$ has the form $Y \times_k k'$ for some $Y \longrightarrow X$.

Choose some f.g. algebra A/k, $k \subset A \subset k'$ s.t.

$$\begin{tikzcd} \text{our cover} \arrow[r, dash] & X_{k'} \times_{X_A} Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ & X_{k'} \arrow[r] & X_A \end{tikzcd}$$

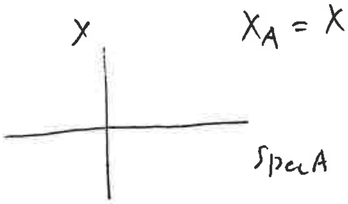
Pick any homom. $A \longrightarrow k$

$$\begin{tikzcd}
X_{k'} \times_{X_A} Y' \arrow[r] \arrow[d] & Y' \arrow[d] & Y' \times_{X_A} X \arrow[l] \arrow[d] \\
X_{k'} \arrow[r] & X_A & X \arrow[l]
\end{tikzcd}$$

May assume that $Y' \times_{X_A} \operatorname{Spec} A$ is trivial

Then $Y' = \left(Y' \times_{X_A} X\right) \times \operatorname{Spec} A$.

$X_A = X \times \operatorname{Spec} A$



ⓔ $\pi_1^{\acute{e}t}\left(\mathbb{A}^1_{\overline{\mathbb{F}_p(t)}}\right) \not\twoheadrightarrow \pi_1\left(\mathbb{A}^1_{\overline{\mathbb{F}_p}}\right)$

$\overline{\mathbb{F}_p(t)} = k'$, $\overline{\mathbb{F}_p} = k$

Fix $c \in k'[x]$, consider

$$\begin{tikzcd}
Y_c = \operatorname{Spec} k'[x,y]/y^p - y - c \arrow[d] \\
\operatorname{Spec} k'[x]
\end{tikzcd}$$

$Y_c = Y_{c'} \iff c - c' = a^p - a$ for some $a \in k'[x]$.

Take $c = tx$.

K, char $K = p$.
$\operatorname{Hom}(\operatorname{Gal}(\bar{K}|K), \mathbb{Z}/p)$
$= \operatorname{Coker}\left(K \xrightarrow{a \mapsto a^p - a} K\right)$

ⓕ Show that $\pi_1^{\acute{e}t}\left(\mathbb{A}^1_{\overline{\mathbb{F}_q}}\right)$ is not abelian.

Drinfeld's curve:

$$\begin{tikzcd}
C = \operatorname{Spec} \overline{\mathbb{F}_q}[x,y]/(x^q y - x y^q = 1) \arrow[d] \\
\mathbb{A}^1 = \operatorname{Spec} \overline{\mathbb{F}_q}[f],
\end{tikzcd}$$

$\circlearrowleft SL_2(\mathbb{F}_q)$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$f \mapsto x^{q^2} y - x y^{q^2}$

$$\begin{array}{l} H^1_{\acute{e}t}(C, \mathbb{Q}_\ell) = \mathrm{Hom}_{Cont}(\pi_1^{\acute{e}t}(C), \mathbb{Q}_\ell) \\ \quad \circlearrowleft \\ SL_2(\mathbb{F}_q) \end{array}$$

Thm. All cuspidal representations of $SL_2(\mathbb{F}_q)$ appears in $H^1_{\acute{e}t}(C, \mathbb{Q}_\ell) \otimes \overline{\mathbb{Q}_\ell}$ w/ mult. one.

$$U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

V irred. is called cuspidal if $V^U = 0$.

Lecture 21 Grothendieck topology

$Sh_{\acute{e}t}(X)$

Idea: X scheme.

Et_X category of étale schemes over X

$Ob(Et_X) = \{ U \xrightarrow{\text{étale}} X \}$

Morphisms.
$$\begin{tikzcd} U \arrow[r] \arrow[dr] & U' \arrow[d] \\ & X \end{tikzcd}$$

$PSh_{\acute{e}t}(X) = \text{Functors}(Et_X^{op}, \text{Sets (or Ab Groups)})$

$PSh_{\acute{e}t}(X) \supset Sh_{\acute{e}t}(X)$

Rmk. ⓐ If $U, U' \subset X$, then $\mathrm{Mor}\left(\begin{smallmatrix} U \\ \downarrow \\ X \end{smallmatrix}, \begin{smallmatrix} U' \\ \downarrow \\ X \end{smallmatrix} \right) = \begin{cases} \phi, & U \not\subset U' \\ X, & U \subset U'. \end{cases}$

ⓑ. If $\begin{smallmatrix} U \\ \downarrow \\ X \end{smallmatrix}$ a Galois cover, then $\mathrm{Aut}\left(\begin{smallmatrix} U \\ \downarrow \\ X \end{smallmatrix} \right) = \mathrm{Gal}(U/X)$

1. Let $\underline{\mathcal{C}}$ (e.g. $\underline{\mathcal{C}} = Et_X$) be a category.

$U \in ob(\underline{\mathcal{C}})$

Def. A sieve $\underline{U}$ on U is a full sub cat. of

$\underline{\mathcal{C}}/U = \{ V \in \underline{\mathcal{C}}, V \to U \}$ s.t.

if $V \xrightarrow{\varphi} U \in ob(\underline{U})$ and $W \xrightarrow{\psi} V$ a morphism in $\underline{\mathcal{C}}$,

then $\varphi \circ \psi : W \to U \in ob(\underline{U})$

Construction. For $\{ \varphi_i : U_i \to U \}$ a sieve generated by $\{\varphi_i\}$ consists of all $V \to U$ that factor through one of φ_i.

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[dr] & U_i \arrow[d, "\varphi_i"] \\ & U \end{tikzcd}$$

Ex. X top, $\underline{\mathcal{C}}$ cat. of open sets

$U_1, U_2 \overset{\varphi_i}{\subset} X = U$. Sieve generated by $\{ \varphi_i, i = 1,2 \}$ consists of $V \subset X$

s.t. either $V \subset U_1$ or $V \subset U_2$.

Construction. Let $\underline{U}$ be a sieve on U. and $V \xrightarrow{\varphi} U$ a morphism in $\underline{\mathcal{C}}$.

the restriction $\overset{\text{sieve on } V}{\underline{U}_V}$ of $\underline{U}$ to V is

$$\{ W \xrightarrow{\psi} V : \varphi \circ \psi : W \to U \in \mathrm{Ob}(\underline{U}) \}$$

Def. A topology on $\underline{\mathcal{C}}$ consists of

$\forall$ $U \in \mathrm{ob}(\underline{\mathcal{C}})$ a collection of sieves on U ("covering sieves") s.t.

ⓐ $\mathcal{C}/U$ is a covering sieve

ⓑ If $\underline{U}$ is a covering sieve on U, $V \to U$, then $\underline{U}_V$ is a covering sieve on V.

ⓒ If $\underline{U}$ is a covering sieve, $\underline{U}'$ any sieve on U s.t. $\forall$ $V \to U \in \mathrm{ob}(\underline{U})$, U'_V is a covering sieve, then U' is a covering sieve.

Ex. ⓐ X, $\underline{\mathcal{C}}$ = open sets. $\underline{U}$ is a covering sieve on U iff

$$\bigcup_{V \to U \in \mathrm{ob}(\underline{U})} V = U.$$

ⓑ $\underline{\mathcal{C}} = \mathrm{Et}_X$, $\underline{U}$ is a covering sieve if $\displaystyle\bigcup_{V \xrightarrow{\varphi} U \in \mathrm{ob}(\underline{U})} \varphi(V) = U$.

$$\begin{tikzcd} U_i \times_U V \arrow[r] \arrow[d] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r] & U \end{tikzcd}$$

$\underline{U}_V$ is gen. by $\{U_i \times_U V\}$.

Def. PSh on $\mathcal{C}$ are functors $F: \mathcal{C}^{op} \to Sets$ (Ab, Rings)

For a sieve $\underline{U}$ on U, a $\underline{U}$-locally given section of F consists of

$\forall\ V \to U \in Ob(\underline{U})$, $s_V \in F(V)$ s.t. $\forall\ W \to V$, $s_V|_W = s_W$.

Def A presheaf F is a sheaf, if for any covering sieve $\underline{U}$,

$F(U) \xrightarrow{\sim}$ $\underline{U}$-locally given sections.

$\begin{matrix} \psi \\ s \end{matrix} \mapsto \{s_V\}$.

Ex X top. $\mathcal{C}$ = open sets

F $\underline{U}$ a covering sieve generated by $U_i \hookrightarrow U$, $\bigcup U_i = U$.

$\underline{U}$ - locally given sections $F(U) \to \prod_i F(U_i) \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$

$F(U) \xrightarrow{\sim} \text{coeq}(\alpha_1, \alpha_2)$

⑥ $\underline{\mathcal{C}} = Et_X$, $\underline{U}$ is gen. by $U_i \xrightarrow{\varphi_i} U$, $U = \bigcup Im\, \varphi_i$

$F(U) \to \prod_i F(U_i) \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$

$F(U) \xrightarrow{\sim} \text{coeq.} (\alpha_1, \alpha_2)$.

Ex. $\mathcal{C} = G$-sets.

A covering sieve consists of $U_i \xrightarrow{\varphi_i} U$ if $\bigcup_i \mathrm{Im}\,\varphi_i = U$.

$Sh_{\underline{\mathcal{C}}} \xrightarrow{\sim} G\text{-sets}$, $\mathcal{F} \longmapsto \mathcal{F}(G) \in G\text{-sets}$

$\mathcal{F}(U) = \mathrm{Mor}_G(U,S) \longleftarrow S \in G\text{-sets}$

Lecture 22 Faithfully flat descent.

Question

$$\begin{tikzcd} Y\times_X Y \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \arrow[rd, shift left] \arrow[rd, shift right] & \\ Y \arrow[r, "\varphi"] \arrow[d, "\pi"] & Z \\ X \arrow[ru, dashed, "\psi"'] & \end{tikzcd}$$

$\varphi \circ p_1 = \varphi \circ p_2$

Want a unique ψ s.t. $\psi \circ \pi = \varphi$.

$$\begin{tikzcd} Y = Y\times_X Y \arrow[d, shift right] \arrow[d, shift left] \arrow[rd, shift left, "\parallel"] \arrow[rd, shift right] & \\ \text{Ex. } Y = \mathbb{G}_m \sqcup \operatorname{Spec} k \arrow[r, "Id"] \arrow[d] & Z = Y \\ X = \mathbb{A}^1_k \arrow[ru, dashed, "\text{does NOT exist}"'] & \end{tikzcd}$$

Site = a category w/ Grothendieck topology.

Faithfully flat topology: (fidelment plat qc)

Def. A family $\{\varphi_i : U_i \to X\}$ of flat morphisms is a fpqc covering iff $\forall$ affine open $T \subset X$, $\exists$ affine open $T_1 \subset U_{i_1}, \dots, T_n \subset U_{i_n}$ s.t. $\bigcup_{j=1}^{n} \varphi_{i_j}(T_j) = T$.

(In particular, $\coprod_i U_i \longrightarrow X$ is faithfully flat.)

Ex ⓐ $X = \mathbb{A}^1_k$, $\mathcal{U} = \bigsqcup_{x \in \mathbb{A}^1_k} \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1, x} \longrightarrow \mathbb{A}^1_k = X$

faithfully flat, but not a fpqc covering.

ⓑ $\mathbb{G}_m \sqcup \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1, 0} \longrightarrow \mathbb{A}^1$ is fpqc cover.

fpqc site: $\underline{\mathcal{C}} = Sch_X$

Covering sieves = sieves generated by fpqc coverings $\{U_i \to Y\}$, $Y \in Sch_X$.

Lemma. A presheaf $\mathcal{F}$ on $Sch_{X, fpqc}$ is sheaf iff

1.) $\forall\, Y \in Sch_X$, $\mathcal{F}$ is a Zariski sheaf on Y

2.) For every flat and surjective $\operatorname{Spec} B \to \operatorname{Spec} A$ over X,

$$\mathcal{F}(\operatorname{Spec} A) \longrightarrow \mathcal{F}(\operatorname{Spec} B) \rightrightarrows \mathcal{F}\left(\operatorname{Spec}\left(B \otimes_A B\right)\right)$$

is an equilizer diagram.

Proof. Omitted.

Thm.
$$\begin{tikzcd}
Y \times_X Y \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
Y \arrow[r, "\varphi"] \arrow[d, "\pi"] & Z \\
X \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

$\varphi \circ p_1 = \varphi \circ p_2$

Assume that π is a fpqc covering, then $\exists !\, \psi$ s.t. $\psi \circ \pi = \varphi$

Let $\underline{\mathcal{U}}$ be a sieve on X. A $\underline{\mathcal{U}}$-locally given qcoh. sheaf on X is

ⓐ $\forall$ $U \to X \in Ob(\underline{\mathcal{U}})$, a qcoh. sheaf $\mathcal{F}_U$

ⓑ $\forall$ $V \xrightarrow{\varphi} U$ over X, $\rho_\varphi : \mathcal{F}_V \xrightarrow{\sim} \varphi^* \mathcal{F}_U$

ⓒ For
$$\begin{tikzcd} W \arrow[r, "\psi"] & V \arrow[d, "\varphi"] \\ & U \end{tikzcd}$$
over X,

$$\begin{tikzcd} \mathcal{F}_W \arrow[rr, "\rho_{\varphi\circ\psi}", "\sim"'] \arrow[dr, "\rho_\psi", "\sim"'] & \circlearrowright & \psi^*\varphi^*\mathcal{F}_U \\ & \psi^*\mathcal{F}_V \arrow[ur, "\psi^*\rho_\varphi", "\sim"'] & \end{tikzcd}$$
is comm.

$\rightsquigarrow$ Category $\mathrm{QCoh}(\underline{U})$.

$$\mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(\underline{U})$$
$$\mathcal{F} \longmapsto \{ \underset{\underset{\underline{U}}{\cap}}{U} \xrightarrow{\varphi} X \longmapsto \mathcal{F}_U = \varphi^*\mathcal{F} \}.$$

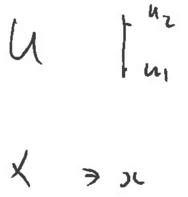
Thm. If $\underline{U}$ is a fpqc covering sieve, then $\mathrm{QCoh}(X) \xrightarrow{\sim} \mathrm{QCoh}(\underline{U})$.

Ex. $X = U_1 \cup U_2$, $\mathrm{QCoh}(\underline{U}) = \{ (\mathcal{F}_{U_i}, \mathcal{F}_{U_1}|_{U_1\cap U_2} \simeq \mathcal{F}_{U_2}|_{U_1\cap U_2}) \}$.

Ex. Let $\overset{U}{\overset{\shortparallel}{\mathrm{Spec}\, B}} \longrightarrow \mathrm{Spec}\, A = X$ be a finite Galois cover,

$G = \mathrm{Gal}(U/X)$, $\underline{U}$ is generated by $U \longrightarrow X$

$$p_1^*\mathcal{F}_U \simeq \mathcal{F}_{U\times_X U} \simeq p_2^*\mathcal{F}_U \qquad \mathcal{F}_U$$
$$U\times_X U\times_X U \rightrightarrows U\times_X U \underset{p_2}{\overset{p_1}{\rightrightarrows}} U \longrightarrow X$$



$\mathcal{F}_U$, $p_1^*\mathcal{F}_U \underset{\sim}{\xrightarrow{\rho}} p_2^*\mathcal{F}_V$

$$\mathrm{QCoh}(\underline{U}) = \left\{ \left(\mathcal{F}_U,\ p_1^*\mathcal{F}_U \xrightarrow[\sim]{\rho} p_2^*\mathcal{F}_U\right) : p_{13}^*\rho = (p_{23}^*\rho)\circ(p_{12}^*\rho) \right\}$$

$$\begin{tikzcd}
U \times_X U \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & X \\
& W \arrow[ul, "{(\varphi_1,\varphi_2)}"] \arrow[u, shift left] \arrow[u, shift right, "\varphi_1"'] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{F}_W = \varphi_1^*\mathcal{F}_U \arrow[dl, phantom, "\wr"] \\
\varphi_3^*\mathcal{F}_U \arrow[r, phantom, "\wr"] & \varphi_2^*\mathcal{F}_U
\end{tikzcd}$$

is $(\varphi_1,\varphi_2)^*\rho$

$$\begin{tikzcd}
(u,gu) & (u,g) \arrow[l, mapsto] & \\
U \times_X U \arrow[d, shift left] \arrow[d, shift right] & U \times_{\mathrm{Spec}\,\mathbb{Z}} G \arrow[l, "\simeq"'] \arrow[dl, shift right, "p_1"'] \arrow[dl, shift left, "a = \text{action}"] & \\
U \arrow[rr] \arrow[d, equal] & & X \arrow[d, equal] \\
\mathrm{Spec}\,B & & \mathrm{Spec}\,A
\end{tikzcd}$$

$$p_1^*\mathcal{F}_U \overset{\rho}{\simeq} a^*\mathcal{F}_U$$

$$U \times_X U \times_X U = U \times G \times G$$

$\mathrm{QCoh}(\underline{U})$ = the cat. of G-equiv. sheaves on U.

= B-modules M w/ an action of G

s.t. $g.(b.v) = g(b).v$

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\sim"'] \arrow[d, equal] & \mathrm{QCoh}(\underline{U}) \\
A\text{-modules} & \\
N \arrow[r, mapsto] & N \otimes_A B \arrow[loop below, "G"] \\
M^G & M \arrow[l, mapsto]
\end{tikzcd}$$

<u>Lecture 23</u>. Thm. Let $\underline{U}$ be a covering sieve on X for fpqc topology.

Then $\mathrm{QCoh}(X) \xrightarrow[\sim]{\pi^*} \mathrm{QCoh}(\underline{U})$

Pf Step 1. Reduce to the case when $X = \operatorname{Spec} A$ and $\underline{\mathcal{U}}$ is generated by flat surjective $\operatorname{Spec} B \to \operatorname{Spec} A$.

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \\
\mathcal{U} \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}$$

Omitted.

Step 2. Theorem is true if $\operatorname{Spec} B \overset{\leftarrow}{\longrightarrow} \operatorname{Spec} A$ admits a section.

Indeed, $\underline{\mathcal{U}} = \mathrm{Sch}_{\operatorname{Spec} A}$:

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \arrow[l, bend right, hook] \\
\mathcal{U} \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}$$

$\operatorname{Spec} A \xrightarrow{Id} \operatorname{Spec} A \in \operatorname{Ob}(\underline{\mathcal{U}})$

Step 3. Lemma. Assume that $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat and surjective, then

① A sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact

$\Leftrightarrow$ $0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$ is exact.

② M is a finite (resp. finitely presented ; locally free of finite rank), A-module

$\Leftrightarrow$ $M \otimes_A B$ is finite (resp. f.p.; loc. free of f. rk) B-module.

Pf. ① $\Rightarrow$ OK, since $- \otimes_A B$ is exact.

$\Leftarrow$ Need if $M \otimes_A B = 0$, then $M = 0$. If $M \neq 0$, then $\exists\ \overset{\overset{0}{\nparallel}}{A/a} \hookrightarrow M$

$M \otimes_A B = 0 \Rightarrow A/a \otimes_A B = 0 \Rightarrow B/aB = 0$.

$$\begin{tikzcd} V(aB) \neq \phi \arrow[d, phantom, "\shortparallel"] & \\ \pi^{-1}(V(a)) \arrow[r, phantom, "\subset"] \arrow[d] & \operatorname{Spec} B \arrow[d, "\pi"] \\ \phi \neq V(a) \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

a contradiction.

② ⓐ If $x_1, \cdots, x_n \in M \otimes_A B$, there exists a finite $M' \subset M$ s.t.

$$x_1, \cdots, x_n \in M' \otimes_A B \subset M \otimes_A B.$$

If $x_1, \cdots, x_n$ generate $M \otimes_A B$, then $M' \otimes_A B = M \otimes_A B \Rightarrow M' = M$.

ⓑ f.p.:

Choose $0 \to N \to A^n \to M \to 0$, want N is finite.

$0 \to N \otimes_A B \to B^n \to M \otimes B \to 0 \Rightarrow N \otimes_A B$ is finite $\Rightarrow N$ is finite

ⓒ locally free of finite rank modules = flat and finite presentation.

Want if $M \otimes_A B$ is flat, then M is flat.

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & M \otimes_A N_1 \arrow[r] & M \otimes_A N_2 \arrow[r] & M \otimes_A N_3 \arrow[r] & 0 \quad \text{exact} \\ & & \arrow[u, Rightarrow] & & \end{tikzcd}$$

$(\Leftarrow)$ $0 \to B \otimes_A M \otimes_A N_1 \to B \otimes_A M \otimes_A N_2 \to B \otimes_A M \otimes_A N_3 \to 0$ exact

Step 4. $$\begin{tikzcd} U \times_X U \times_X U \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U \times_X U \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & U \arrow[r] \arrow[d, phantom, "\shortparallel"] & X \arrow[d, phantom, "\shortparallel"] \\ & & \operatorname{Spec} B & \operatorname{Spec} A \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}_U, & p_1^* \mathcal{F}_U \arrow[r, "\varphi", "\simeq"'] \arrow[d, phantom, "\wr"] & p_2^* \mathcal{F}_U \arrow[dl, phantom, "\wr"] \\ & \mathcal{F}_{U \times_X U} & \end{tikzcd}$$

$$\mathrm{QCoh}(\underline{U}) = \left\{ \mathcal{F}_U,\ p_1^* \mathcal{F}_U \overset{\varphi}{\simeq} p_2^* \mathcal{F}_U, \right.$$

$$\begin{tikzcd}
p_1^* \mathcal{F}_U \arrow[rr, "p_{13}^* \varphi", "\sim"'] \arrow[dr, "p_{12}^* \varphi"'] & \circlearrowright & p_3^* \mathcal{F}_U \\
& p_2^* \mathcal{F}_U \arrow[ur, "\sim", "p_{23}^* \varphi"'] &
\end{tikzcd} \Bigg\}$$

$\mathcal{F}_U = \widetilde{M'}$, M' B-module

$\mathcal{F}_{U \times_X U} = \widetilde{M''}$, M'' $B \otimes_A B$-module

$$\begin{tikzcd}
U \arrow[d, "\pi"] \\
X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\pi^*"] & \mathrm{QCoh}(\underline{U}) \arrow[l, bend left, "\pi_* \text{ right adjoint}"]
\end{tikzcd}$$

$$\begin{tikzcd}
M' \arrow[r, bend left, "\partial_1"] \arrow[r, bend right, "\partial_2"'] \arrow[d, equal] & M'' \arrow[d, equal] \\
\Gamma(U, \mathcal{F}_U) & \Gamma(U \times_X U, \mathcal{F}_{U \times_X U})
\end{tikzcd}$$

$$\Gamma(U, \mathcal{F}_U) \xrightarrow{\partial_i} \Gamma(U \times_X U, p_i^* \mathcal{F}_U)$$

$$\begin{tikzcd}
A \subset B \arrow[r, bend left, "\partial_1"] \arrow[r, bend right, "\partial_2"'] & B \otimes_A B
\end{tikzcd}$$

$$\pi_*(\mathcal{F}_{\underline{U}}) := \mathrm{Eq}(M' \rightrightarrows M'')$$

Need $\pi^* \pi_* \to \mathrm{Id}$, $\mathrm{Id} \to \pi_* \pi^*$ s.t.

$$\begin{tikzcd}
\pi_* \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \pi_* \pi^* \pi_* \arrow[r] & \pi_*
\end{tikzcd}$$

$$\begin{tikzcd}
\pi^* \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \pi^* \pi_* \pi^* \arrow[r] & \pi^*
\end{tikzcd}$$

$$B \underset{A}{\otimes} \mathrm{Eq}(M' \rightrightarrows M'') \longrightarrow M'$$

$$M \longrightarrow \mathrm{Eq}(M \underset{A}{\otimes} B \rightrightarrows M \underset{A}{\otimes} B \underset{A}{\otimes} B)$$

Step 5. Want $\pi^* \pi_* \xrightarrow[\sim]{} id$, $id \xrightarrow[\sim]{} \pi_* \pi^*$

By Lemma, it suffices to prove this after a faithfully flat base change $A \to C$.

$Y = \mathrm{Spec}\, C$

$$\begin{tikzcd} U \times_X Y \arrow[r] \arrow[d] & U \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, shift left, "\pi^*"] \arrow[d] & \mathrm{QCoh}(\underline{U}) \arrow[l, shift left, "\pi_*"] \arrow[d] \\ \mathrm{QCoh}(Y) \arrow[r, shift left] & \mathrm{QCoh}(\underline{U} \times_X Y) \arrow[l, shift left] \end{tikzcd}$$

Take $C = B$.

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d, "\text{has a section}"'] & U \arrow[d] \\ U \arrow[r] \arrow[u, dashed, bend right] & X \end{tikzcd}$$

Cor. $F \in \mathrm{QCoh}(X)$,

$\mathcal{F} : (\underset{\substack{\uparrow \\ \mathrm{Sch}_X}}{U \xrightarrow{f} X}) \longmapsto \Gamma(U, f^* F)$ is a sheaf on $\mathrm{Sh}_{X, fpqc}$.

Pf. Let $(Y \to X) \in Sch_X$, $\underline{U}$ a covering sieve on Y,

need $\mathcal{F}(Y) \xrightarrow{\sim} \mathcal{F}(\underline{U})$.

May assume $Y = X$.

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[rr, "\sim"'] \arrow[d, equal] & & \mathcal{F}(\underline{U}) \arrow[d, equal] \\
Mor_{QCoh(X)}(\mathcal{O}_X, \mathcal{F}) \arrow[rr, "\sim"] & & Mor_{QCoh(\underline{U})}(\mathcal{O}_X, \mathcal{F})
\end{tikzcd}$$

$$M \xrightarrow{\sim} Eq\left(M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B\right).$$

Thm. (Faithfully flat descent for morphisms). For any $X \in Sch_S$,

$\underset{Sch_S}{\underset{\rotatebox{90}{$\in$}}{Y}} \longmapsto Mor_S(Y, X)$ is a sheaf on $Sch_{S, fpqc}$.

Cor.

$$\begin{tikzcd}
U \times_X U \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, shift left] \arrow[drr, shift right, "="] & & \\
U \arrow[rr] \arrow[d, "\text{fpqc cover}"'] & & X \\
Y \arrow[urr, dashed] & &
\end{tikzcd}$$

Pf. For X, S affine. $X = Spec\, B$, $S = Spec\, A$.

$$Mor_S(Y, X) = Mor_{alg}(B, \Gamma(Y, \mathcal{O}_Y))$$

Then follows from $\mathcal{O} \in Sch_{S, fpqc}$.

Lecture 24. Notation. ⓐ Big fpqc site $Sch_{X, fpqc}$

$\mathcal{C} = Sch_X$, $Y \in Sch_X$

Covering sieves on Y are generated by fpqc covers $\{U_i \to Y\}$.

ⓑ small étale site of X, $X_{ét}$

$\mathcal{C}$ = schemes étale over X.

topology defined by étale covers.

ⓒ Big étale site over X, $Sch_{X, ét}$

$\mathcal{C} = Sch_X$, topology is given by étale covers of $Y \in Sch_X$.

ⓓ X_{Zar}, $Sch_{X, Zar}$.

§1. Torsors

$\mathcal{C}$ site, $\underline{G}$ a sheaf of groups on $\mathcal{C}$.

$\underline{G}$ - torsor is a sheaf T of sets w/ an action of $\underline{G}$: $\underline{G} \times T \to T$ which is <u>locally</u> isomorphic to $\underline{G}$ w/ canonical action. $\underline{G} \times \underline{G} \to \underline{G}$.

locally = $\forall X \in ob(\mathcal{C})$, $\exists$ a covering sieve $\underline{\mathcal{U}}$ on X s.t.

$\forall \begin{matrix} U \\ \downarrow \\ X \end{matrix} \in ob(\underline{\mathcal{U}})$,

$(T, \underline{G} \times T \to T)\big|_{\mathcal{C}/U} \simeq (\underline{G}, \underline{G} \times \underline{G} \to \underline{G})\big|_{\mathcal{C}/U}$.

Ex. Consider the group scheme GL_n over $\operatorname{Spec} \mathbb{Z}$.

$$GL_n = \operatorname{Spec} \mathbb{Z}[a_{ij}, \det(a_{ij})^{-1}].$$

$$\underline{GL}_n(X) = \operatorname{Mor}(X, GL_n) = GL_n(\Gamma(X, \mathcal{O}_X))$$

$\underline{GL}_n$ is a sheaf on Sch_{fpqc}.

Thm. $\underline{GL}_n$ - torsors on $Sch_{X, fpqc}$ = vector bundles over X of rank n.

= $\underline{GL}_n$ - torsors on X_{Zar} = $\underline{GL}_n$ -torsors on $X_{ét}$.

Pf. For any comm. ring R,

$$\underset{\text{groupoid}}{\underbrace{GL_n(R) - \text{torsors}}} \simeq \underset{\text{groupoid}}{\underbrace{\text{free } R\text{- modules of rank } n}}$$

$$T \longmapsto E = \underset{\substack{\Psi \\ a}}{T} \overset{GL_n(R)}{\times} R^n = T \times R^n / GL_n(R)$$

$$(a, \underset{\substack{\Psi \\ R^n}}{x})$$

$$(a, x) + (a, y) = (a, x+y).$$

$$\underset{\substack{\circlearrowleft \\ GL_n(R)}}{\operatorname{Isom}_R(R^n, E)} \longleftarrow\!\!\shortmid E$$

$\Leftarrow$ E vec. bdle over X,

$$\begin{tikzcd} T(Y \xrightarrow{f} X) = \mathrm{Iso}_{\mathcal{O}_Y}(\mathcal{O}_Y^n, f^*E) \arrow[d, phantom, "\circlearrowleft"] \\ \underline{GL}_n(Y) \end{tikzcd}$$

T is a sheaf on $Sch_{X, fpqc}$: for any $E \in QCoh(X)$,

$$\begin{tikzcd} Y \arrow[d, "f"] \\ X \end{tikzcd} \rightsquigarrow \Gamma(Y, f^*E) \text{ is a fpqc sheaf.}$$

$$\begin{tikzcd} \mathcal{O}_Y^n \arrow[r] \arrow[dr, "id"'] & f^*E \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathcal{O}_Y^n \end{tikzcd}$$

$\Rightarrow$ Choosing a covering sieve $\underline{U}$ on X s.t. $\forall\, U \in \underline{U}$,

$T|_{Sch_U} \simeq \underline{GL}_n|_{Sch_U}$.

For $U \in ob(\underline{U})$, define $E_U = T(U) \overset{\underline{GL}_n(U)}{\times} \mathcal{O}_U^n$

\
vector bundle over U

$U \in Ob(\underline{U}) \rightsquigarrow$ vec. bdle E_U over U.

$\{E_U,\ U \in Ob(\underline{U})\} \in QCoh(\underline{U}) \simeq QCoh(X)$.

$W \xrightarrow{f} U$, $E_W \Leftarrow f^*E_U$..
$T(U) \longrightarrow T(W)$ This gives E.

§2. Geometric torsors

G a flat group scheme finitely presentable over S.

Def. A G-torsor over S is a scheme $X \xrightarrow{f} S$ w/ an action of G

$$G \times_S X \longrightarrow X \quad \text{s.t.}$$

1. f is fpqc cover

2. $$G \times_S X \longrightarrow X \times_S X \quad \text{is an isom}$$
$$(g,s) \longmapsto (x, gx)$$

$$G \rightsquigarrow \underline{G}\begin{pmatrix} X \\ \downarrow \\ S \end{pmatrix} = \mathrm{Mor}_S(X, G)$$

$\underline{G}$ is a sheaf of groups on $\mathrm{Sch}_{S,\,\mathrm{fpqc}}$.

A G-torsor $X \longrightarrow S$, define $\underline{G}$-torsor $\underline{X}$: (*)

$$\underline{X}\begin{pmatrix} Y \\ \downarrow \\ S \end{pmatrix} = \mathrm{Mor}_S(Y, x)$$

$\underline{X}$ is a sheaf on $\mathrm{Sch}_{S,\,\mathrm{fpqc}}$.

Prop. Assume that G is affine, then

(geometric) G-torsors $=$ $\underline{G}$-torsors.

Pf $\Leftarrow$: T, $\underline{G} \times T \longrightarrow T$

(*) fully faithful (Yoneda)

T, $G\times T \longrightarrow T$, pick a covering sieve $\underline{\mathcal{U}}$ s.t.

$T|_U$ is trivial for any $U\in ob(\underline{\mathcal{U}})$.

$T|_U$ determines a geometric G-torsor $T^{geom}_U \approx U\times_S G$

$T^{geom}_U \xrightarrow{f_U} U$, Set $A_U = f_{U*}\,\mathcal{O}_{T^{geom}_U}$

$\{A_U,\ U\in ob(\underline{\mathcal{U}})\}$ is an algebra object in $QCoh(\underline{\mathcal{U}})$. Thus it defines an algebra $\mathcal{A}$ in $QCoh(S)$.

Take $X = \underline{Spec}\,\mathcal{A}$.

Rmk. Along the way, we see that we can fpqc descend affine schemes:

$$\begin{tikzcd} Y\times_X Y \arrow[d, shift left] \arrow[d, shift right] & \\ Y \arrow[d] & Z \arrow[l] \\ X & \end{tikzcd}$$

$$Z\times_{Y,p_1}(Y\times_X Y) \simeq Z\times_{Y,p_2}(Y\times_X Y) \quad + \text{ cocycle condition}$$

Lecture 25. Cohomology.

General theory: $\mathcal{C}$ - site

$PSh(\mathcal{C}) = PSh(\mathcal{C}, Ab) \hookleftarrow Sh(\mathcal{C})$

Lemma. This has a left adjoint, $\mathcal{F} \mapsto \mathcal{F}^{\#}$ (sheafification)

Construction of $\mathcal{F}^{\#}$.

Define $\varepsilon : PSh(\mathcal{C}) \to PSh(\mathcal{C})$

$$\varepsilon(\mathcal{F})(X) = \underset{\substack{\text{covering} \\ \text{sieves on } X}}{\operatorname{colim}} \mathcal{F}(\underline{U})$$

Lemma ⓐ For any $\mathcal{F} \in PSh(\mathcal{C})$, $\varepsilon(\mathcal{F})$ is separated:

$$(\varepsilon\mathcal{F})(X) \hookrightarrow \varepsilon\mathcal{F}(\underline{U}) \quad \text{for any covering sieve.}$$

ⓑ If $\mathcal{F}$ is separated, then $\varepsilon\mathcal{F} \in Sh(\mathcal{C})$.

$$\mathcal{F}^{\#} := \varepsilon(\varepsilon\mathcal{F}).$$

Proof is omitted.

More generally, let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$.

$$\begin{tikzcd} PSh(\mathcal{C}, Mod(\mathcal{O})) \arrow[r, shift right, "\# - \text{left adjoint}"'] & Sh(\mathcal{C}, Mod(\mathcal{O})) \arrow[l, shift right] \end{tikzcd}$$

Lemma $Sh(\mathcal{C}, Mod(\mathcal{O}))$ is an abelian cat.

Pf. $\operatorname{coker}(\mathcal{F}_1 \to \mathcal{F}_2) = \left[X \mapsto \operatorname{coker}(\mathcal{F}_1(X) \to \mathcal{F}_2(X)) \right]^{\#}$.

Lemma. $Sh(\mathcal{C}, Mod(\mathcal{O}))$ has enough injectives.

Pf. Suppose $\mathcal{A}$ is an abelian cat.

① $\mathcal{A}$ has all direct sums

② Direct sums are exact

$0 \to A_i \to B_i \to C_i \to 0$ exact $\Rightarrow$ $0 \to \oplus A_i \to \oplus B_i \to \oplus C_i \to 0$ exact

③ Filtered colimits are exact.

④ $\mathcal{A}$ has a generator, $\exists\, U \in ob(\mathcal{A})$ s.t.

$$\begin{aligned} \mathcal{A} &\to Ab \\ X &\mapsto Mor_{\mathcal{A}}(U, X) \end{aligned}$$

reflects iso.

Fact: Then $\mathcal{A}$ has enough injectives.

Functor of global sections

$Sh(\mathcal{C}, Mod(\mathcal{O})) \xrightarrow{\Gamma} Ab$, $\Gamma(\mathcal{F}) = Mor(\mathcal{O}, \mathcal{F})$

If X is a final object in $\mathcal{C}$, then $\Gamma(\mathcal{F}) = \mathcal{F}(X)$.

Def. $H^i(\mathcal{C}, \mathcal{F}) = R^i\Gamma\mathcal{F} = Ext^i(\mathcal{O}, \mathcal{F})$

Ex. Interpretation of H^1.

Let G be a sheaf of abelian groups.

G-torsors in $\mathcal{C}$ $\simeq$ Extensions

$$0 \to G \to \tilde{T} \xrightarrow{\pi} \underline{\mathbb{Z}} \to 0$$

$$\begin{tikzcd} T(U) = \pi^{-1}(1) \subset \tilde{T}(U) & \tilde{T} \arrow[l, mapsto] \\ G(U) \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$\Rightarrow$ $H^1(X, G) \cong$ iso. classes of G-torsors

Corollary of fpqc descent:

$G, \mathcal{F} \in \mathrm{QCoh}(X)$

$$\begin{tikzcd}
\text{Ext'ns } 0 \to G \to \tilde{\mathcal{F}} \to \mathcal{F} \to 0 \text{ in } \mathrm{Sh}(X_{Zar}, \mathrm{Mod}(\mathcal{O}_X)) \arrow[r, "\sim"'] & \text{Ext'ns } 0 \to G \to \tilde{\mathcal{F}} \to \mathcal{F} \to 0 \text{ in } \mathrm{Sh}(\mathrm{Sch}_{X, fpqc}, \mathrm{Mod}(\mathcal{O}_X))
\end{tikzcd}$$

<u>Cor</u>. $H^1_{fpqc}(X, G) \xleftarrow{\sim} H^1_{Zar}(X, G)$, for $G \in \mathrm{QCoh}(X)$.

$$\begin{tikzcd}
H^1_{fpqc}(X,G) \arrow[d, equal] \\
H^1_{ét}(X,G)
\end{tikzcd}$$

<u>Fact</u>. This is also true for H^q, $q \in \mathbb{N}$.

<u>Ex</u>. Let C be a finite abelian gp.

$H^1_{ét}(X, \underline{C}) = \underline{C}$-torsors $=$ C-torsors

$$\begin{tikzcd}
Y \arrow[d] & C \arrow[l, phantom, "\circlearrowleft"] \\
X
\end{tikzcd}$$

$C \times Y \xrightarrow{\sim} Y \times_X Y$

$= \mathrm{Hom}(\pi_1(X, x), C)$

if X is connected

<u>Kummer's Theory</u>.

$$1 \to \mu_n \to G_m \xrightarrow{z \mapsto z^n} G_m \to 1$$

$$(*) \quad 1 \to \underline{\mu_n} \to \underline{G_m} \to \underline{G_m} \to 1 \qquad \mathrm{Sh}(\mathrm{Sch}_{fpqc})$$

$\mathbb{G}_m(X) = \mathcal{O}(X)^*$, $\underline{\mu_n}(X) = \{ f \in \mathcal{O}(X)^* : f^n = 1 \}$.

Thm ⓐ $(*)$ is exact in $Sh(Sch_{fpqc})$.

ⓑ $(*)$ is exact in $Sh(Sch_{\mathbb{Z}[\frac{1}{n}], \acute{e}t})$

Pf ⓑ

$$\begin{tikzcd}
\mathbb{G}_m \times \operatorname{Spec} \mathbb{Z}[n^{-1}] \arrow[r, "z \mapsto z^n"] & \mathbb{G}_m \times \operatorname{Spec} \mathbb{Z}[n^{-1}] \\
U \arrow[u] \arrow[ur] \arrow[r] & X \arrow[u]
\end{tikzcd}$$

$\mathbb{G}_m \times \operatorname{Spec} \mathbb{Z}[n^{-1}] \to \mathbb{G}_m \times \operatorname{Spec} \mathbb{Z}[n^{-1}]$ is an étale cover,

$U \to \mathbb{G}_m \times \operatorname{Spec} \mathbb{Z}[n^{-1}]$ is an étale cover

---

Lecture 26 Last time:

$$1 \to \underline{\mu_n} \to \underline{\mathbb{G}_m} \xrightarrow{z \mapsto z^n} \underline{\mathbb{G}_m} \to 1$$

is exact in $Sh(Sch_{\mathbb{Z}[\frac{1}{n}], \acute{e}t})$.

Cor. $X / \operatorname{Spec} \mathbb{Z}[n^{-1}]$,

$$1 \to \underline{\mu_n} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 1$$ is exact in $Sh(X_{\acute{e}t})$

Notation: $\underline{\mu_n} = \mathbb{Z}/n(1)$

If $\varepsilon \in \Gamma(X, \mathcal{O}_X)$ is a primitive n-th root of 1, then $\mathbb{Z}/n \xrightarrow{\sim} \mathbb{Z}/n(1)$, $1 \longmapsto \varepsilon$

Cor

$$1 \longrightarrow H^0(X, \mathbb{Z}/n(1)) \longrightarrow \Gamma(X, \mathcal{O}_X^*) \xrightarrow{\wedge n} \Gamma(X, \mathcal{O}_X^*)$$

$$\longrightarrow H^1_{\acute{e}t}(X, \mathbb{Z}/n(1)) \longrightarrow \underset{\substack{\shortparallel \\ \mathrm{Pic}(X)}}{H^1_{\acute{e}t}(X, \mathcal{O}_X^*)} \xrightarrow{n} \underset{\substack{\shortparallel \\ \mathrm{Pic}(X)}}{H^1_{\acute{e}t}(X, \mathcal{O}_X^*)}$$

1st Chern class

$$\xrightarrow{c_1} H^2_{\acute{e}t}(X, \mathbb{Z}/n(1)) \longrightarrow \cdots$$

Ex. X smooth proper curve / $k = \bar{k}$, $(\mathrm{char}\, k, n) = 1$

$$1 \to \mu_n(k) \to k^* \xrightarrow{n} k^* \to H^1_{\acute{e}t}(X, \mathbb{Z}/n(1)) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X)$$

$$\to H^2_{\acute{e}t}(X, \mathbb{Z}/n(1)) \to \cdots$$

$$H^1_{\acute{e}t}(X, \mathbb{Z}/n(1)) = \mathrm{Pic}\, X \underset{\substack{\uparrow \\ n\text{-torsion}}}{[n]}$$

$$0 \to \underline{\mathrm{Pic}}^0(X) \to \underline{\mathrm{Pic}}(X) \to \mathbb{Z} \to 0$$

$\underline{\mathrm{Pic}}^0(X)$ is an abelian variety

$$\begin{tikzcd} \underline{\mathrm{Pic}}^0(x) \arrow[d, "n", "\text{of degree } n^{2g}"'] \\ \underline{\mathrm{Pic}}^0(x) \end{tikzcd}$$

$$\Rightarrow \mathrm{Pic}(X)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$$

$$H^1_{\acute{e}t}(X, \mathbb{Z}/n) = \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), \mathbb{Z}/n)$$

$\mathrm{char}\, k = p$, $\pi_1^{top}(\text{surface of genus } g)^{\wedge} \longrightarrow \pi_1^{\acute{e}t}(X)$ iso on coprime to p quotients

Functoriality

Def. Let $f: X \longrightarrow Y$ be a morphism. $\mathcal{F} \in PSh(X_{\bar{e}t})$

$$f_*\mathcal{F}\left(\begin{array}{c} U \\ \downarrow \\ Y \end{array}\right) = \mathcal{F}\left(\begin{array}{c} X \times_Y U \\ \downarrow \\ X \end{array}\right)$$

$$\begin{tikzcd} X \times_Y U \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

If $\{U_i \to U\}$ is a cover, then $\{U_i \times_Y X \to U \times_Y X\}$ is a cover.

$\Rightarrow$ $\mathcal{F} \in Sh(X_{\bar{e}t})$, then $f_*\mathcal{F}$ is also a sheaf.

Lemma $f_*: Sh(X_{\bar{e}t}) \longrightarrow Sh(Y_{\bar{e}t})$ has a left adjoint f^{-1} which is exact.

Pf.
$$f^{-1}\mathcal{F} = \left(\begin{array}{c} U \\ \downarrow \\ X \end{array} \rightsquigarrow \varinjlim_{\begin{tikzcd}[ampersand replacement=\&] U \arrow[r] \arrow[d] \& V \arrow[d] \\ X \arrow[r] \& Y \end{tikzcd}} \mathcal{F}\left(\begin{array}{c} V \\ \downarrow \\ Y \end{array}\right) \right.$$

Ex complete the proof.

Digression. $\varepsilon: PSh(\mathcal{C}) \circlearrowleft$

$$\varepsilon\mathcal{F}(U) = \varprojlim_{\substack{\underline{U} \text{ covering} \\ \text{sieves}}} \mathcal{F}(\underline{U})$$

$$\varepsilon(\varepsilon\mathcal{F}) = \mathcal{F}^{\#}.$$

Ex. $\varepsilon\mathcal{F}$ need not be a sheaf. Take space X (e.g. $X = \mathbb{R}$), $\mathcal{F}(U) = \mathbb{Z}$ for all U.
$(\varepsilon\mathcal{F})(U) = \mathbb{Z}$ for all $U \neq \emptyset$ and $(\varepsilon\mathcal{F})(\emptyset) = 0$.

Stalks

Def A geometric point of X is $\bar{x}\colon \operatorname{Spec} K \longrightarrow \underset{\substack{\Downarrow \\ x\text{ - image}}}{X}$, where K is separably closed.

$\bar{x}$ is algebraic if $k(x) \subset K$ is an alg. ext'n.

Def An étale nbhd of $\bar{x}$ is

$$\begin{tikzcd}
 & U \arrow[d, "\text{étale}"] \\
\operatorname{Spec} K \arrow[r, "\bar{x}"] \arrow[ur, "\tilde{x}"] & X
\end{tikzcd}$$

Def $$\mathcal{F}_{\bar{x}} = \underset{\substack{\uparrow \\ Ab}}{\bar{x}^{-1}\mathcal{F}} = \lim_{\substack{\text{nbhds} \\ \text{of } \bar{x}}} \mathcal{F}(U).$$

Assume that $\bar{x}$ is algebraic, $k(x) \subset K$. Then $\mathcal{F}_{\bar{x}}$ carries $\operatorname{Gal}(K|k(x))$-action

$$\sigma\Big((U,\tilde{x}) \mapsto \underset{\substack{\uparrow \\ \mathcal{F}(U)}}{s_{U,\tilde{x}}}\Big) = s_{U,\tilde{x}\circ\sigma}$$

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d, "\pi"'] \arrow[dr, "\bar{x}"] & \\
\operatorname{Spec} k(x) \arrow[r, "x"] & X
\end{tikzcd}$$

$$\operatorname{Sh}(\operatorname{Spec} k(x)_{\text{ét}}) \xrightarrow[\sim]{\pi^{-1}} \operatorname{Rep}_{\text{cts}} \operatorname{Gal}(K|k(x))$$

$$\mathcal{F}_{\bar{x}} = \bar{x}^{-1}\mathcal{F} = \pi^{-1}(x^{-1}\mathcal{F})$$

Def. Let $\bar{x}$ be an algebraic pt of X, w/ $K = k(x)^{sep}$

$$\begin{tikzcd} & U \arrow[d] \\ \bar{x} : \operatorname{Spec} k \arrow[r] \arrow[ur] & X \end{tikzcd}$$

Set $\mathcal{O}^{sh}_{X,\bar{x}} := (\mathcal{O}_X)_{\bar{x}}$.

Strict henselisation

Def A local ring (A, m) is henselian if $\forall$ monic $f(T) \in A[T]$ and every $\alpha \in A/m = k$ s.t. $\bar{f}(\alpha) = 0$, $\bar{f}'(\alpha) \neq 0$, $\exists\, \tilde{\alpha} \in A$ s.t. $f(\tilde{\alpha}) = 0$, $\tilde{\alpha} \bmod m = \alpha$.

A is strict henselian if $A/m = (A/m)^{sep}$.

Ex. $\mathbb{Z}_p$ henselian

$\mathcal{O}^{sh}_{X,\bar{x}}$ strictly henselian

$$X = \operatorname{Spec} \mathbb{Z} \xleftarrow{\bar{x}} \operatorname{Spec} \overline{\mathbb{F}}_p$$

$$\mathcal{O}^{sh}_{X,x} = \overline{\mathbb{Q}} \cap W(\overline{\mathbb{F}}_p) \subset \widehat{\overline{\mathbb{Q}}}_p$$

Lecture 27

Def. ⓐ A local ring $A \supset m$, $k = A/m$ is henselian, if $\forall\, f(T) \in A[T]$ monic, and every $\alpha \in k$ w/ $\bar{f}(\alpha) = 0$, $\bar{f}'(\alpha) \neq 0$, $\exists\, \tilde{\alpha} \in A$ w/ $f(\tilde{\alpha}) = 0$ and $\tilde{\alpha} \bmod m = \alpha$.

ⓑ Furthermore, A is strictly henselian if in addition $k = \bar{k}$.

Th TFAE

① A is henselian

② $\forall\ f(T) \in A[T]$ monic, and any $\bar{f} = g_0 h_0 \in k[T]$ w/ g_0, h_0 monic and $(g_0, h_0) = 1$, $\exists\ f = gh$ w/ monic g, h s.t. $g \equiv g_0 \bmod m$, $h \equiv h_0 \bmod m$.

③ For any diagram

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[d, "\pi"', "\text{– étale}"] \\ \operatorname{Spec} k \arrow[ur, "\hat{i}"] \arrow[r, "i"] & \operatorname{Spec} A \arrow[u, dashed, bend left, "s"] \end{tikzcd}$$

$\exists$ a section s, $\pi \circ s = id$,

$s \circ i = \hat{i}$.

④ Every finite A-alg. B is a product of local rings.

$B = \prod_i B_i \xrightarrow{\sim} \prod_i B_{m_i}$, m_i max'l ideals of B

Ex. ④ fails for local rings.

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} k[z^{\frac{1}{2}}] \arrow[d] & = \mathbb{A}^1 \arrow[d] \\ \operatorname{Spec} k[z]_{(z-1)} \arrow[r] \arrow[d, phantom, "\parallel"] & \operatorname{Spec} k[z] & = \mathbb{A}^1 \\ A & & \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec} B \otimes_A k \arrow[r, "\text{induces iso. on } \pi_0"] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

B is not a product of local rings.

Ex. Assume $A = \hat{A}$, B – finite A-algebra, $B = \varprojlim_n B/m_A^n B$

$$B/m_A^n B = B \otimes_A A/m_A^n$$

$$\underbrace{B/m_A^n B \longrightarrow B/m_A B}_{\text{same set of idempotents}}$$

idempotents in B

$\hat{\updownarrow}$

idempotents in $B/m_A B$.

Def. For $A \supset m$, henselisation of A at m is

$$A^h := \operatorname{colim} B \qquad , \quad \text{strict henselisation}$$

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[d, "\text{étale}"] \\ \operatorname{Spec} k \arrow[r, hook] \arrow[ur] & \operatorname{Spec} A \end{tikzcd}$$

$$A^{sh} := \operatorname{colim} B = i^{-1} \mathcal{O}^{ét}_{\operatorname{Spec} A}$$

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[d, "-\text{étale}"] \\ \operatorname{Spec} \bar{k} \arrow[r, hook] \arrow[ur] & \operatorname{Spec} A \end{tikzcd}$$

Ex. A^h is henselian.

$$\operatorname{Spec} \bar{k} \xrightarrow{i} \operatorname{Spec} A$$

Ex. A - DVR, $K = \operatorname{Frac}(A)$, then

$$A^h = \hat{A} \cap K^{sep}$$

Pf. $A^h \subset \hat{A}$ because $\hat{A}$ is henselian.

$A^h \subset K^{sep}$

Claim. $\forall$ finite Galois ext'n $\overset{K^{sep}}{\overset{\cup}{L}}|K$, $B = L \cap \hat{A} \subset A^h$.

Pf.

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[d, "\text{étale}"] \\ \operatorname{Spec} k \arrow[r, hook] \arrow[ur] & \operatorname{Spec} A \end{tikzcd} \qquad B/m_A B = k$$

Pf. ② $\Rightarrow$ ④ Assume first $B = A[T]/f$, f monic.

If $\bar{f}$ is a power of irreducibles, then B is local.

If not, $\bar{f} = g_0 h_0$, g_0, h_0 monic, coprime, ② $\Rightarrow f = gh$

$\rightarrow B \xrightarrow{\sim} B/g \times B/h$. since $(g, h) = 1$.

General case: If B is not local, pick a nontrivial idempotent $\bar{b}$ in $B/m_A B$.

$$\begin{tikzcd} C = A[T]/(f(T)) \arrow[r] & B \\ T \arrow[r, mapsto] & b \equiv \bar{b} \text{ mod } m_A \end{tikzcd} \qquad f \text{ monic}$$

$$\begin{tikzcd} C/m_A C \arrow[r, two heads] & k[\bar{b}] \subset B/m_A B \\ \text{idempotent - } \bar{c} \arrow[u, phantom, "\in"] & \bar{b} \arrow[u, phantom, "\in"] \end{tikzcd}$$

Lift $\bar{c}$ to an idempotent $c \in C$. Let e be its image in B,

$$B = eB \times (1-e)B$$

④ ⇒ ③ Zariski's Main Theorem.

Let $X \to Y$ be a quasi-finite, separated morphism, Y quasi compact, then

∃ factorization
$$\begin{tikzcd} X \arrow[r, "\text{open}"] \arrow[d] & X' \arrow[dl, "\text{finite}"] \\ Y & \end{tikzcd}$$

If $X = \text{Spec } B$, $Y = \text{Spec } A$, then we can take $X' = \text{Spec } B'$,

$B' = \{ b \in B : b \text{ is integral over } A \}$.

$$\begin{tikzcd} & \text{Spec } B \arrow[r, "\text{open}"] \arrow[d, "\text{étale}"] & \text{Spec } B' \arrow[dl, "\text{finite}"] \arrow[r, phantom, "\overset{④}{\cong}"] & \bigsqcup_i \text{Spec } B'_i \\ \text{Spec } k \arrow[ur] \arrow[r] & \text{Spec } A & & \end{tikzcd}$$

Assume $\text{Spec } k \subset \text{Spec } B'_1$, $\Rightarrow \text{Spec } B'_1 \subset \text{Spec } B$

$$\begin{tikzcd}
 & \operatorname{Spec} B_i' \arrow[d, "\substack{\text{finite}\\ \text{étale}}"'] \arrow[r, phantom, "\subset"] & \operatorname{Spec} B \arrow[dl, "\text{étale}"] \\
\operatorname{Spec} k \arrow[ur] \arrow[r] & \operatorname{Spec} A \arrow[ur, dashed, bend left] &
\end{tikzcd}$$

$B_i' / m_A B_i' = k$

④ ⇒ ②: $B = A[T]/(f(T))$,

$$\begin{tikzcd}
B/m_A B \arrow[r, phantom, "\simeq"] & {} & k[T]/g_0 \times k[T]/h_0 & \\
B \arrow[u] \arrow[r, phantom, "\simeq"] & B_1 \arrow[d, equal] & \times \arrow[u] & B_2 \arrow[d, equal] \\
 & A[T]/g & \times & A[T]/h
\end{tikzcd}$$

g, h monic,

$\deg g = \deg g_0$, $\deg h = \deg h_0$, $g \equiv g_0 \mod m_A$, $h \equiv h_0 \mod m_A$

$f \mid g \cdot h \Rightarrow f = g \cdot h$.

① ⇒ ③:

Def $A \longrightarrow B$ is standard étale if $B = \left(A[T]/(f(T))\right)_{g(T)}$ w/

$f'(T)$ invertible in B.

Thm. $A \longrightarrow B$ is étale if $\forall\, \mathfrak{p} \in \operatorname{Spec} B$, $\exists\, g \in B$ s.t. $g \notin \mathfrak{p}$ and $A \to B_g$ is standard étale.

③ ⇒ ① clear.

Lecture 28

Lemma. A strictly henselian;

For any sheaf $\mathcal{F} \in Sh((\operatorname{Spec} A)_{\acute{e}t})$, $H^q(\operatorname{Spec} A, \mathcal{F}) = \begin{cases} 0, & q>0 \\ i^{-1}\mathcal{F}, & q=0 \end{cases}$

$i : \operatorname{Spec} k \to \operatorname{Spec} A$.

Pf. $i^{-1}\mathcal{F} = \operatorname{colim} \mathcal{F}(U) = \operatorname{colim} \mathcal{F}(\operatorname{Spec} B) = \mathcal{F}(\operatorname{Spec} A)$

$\quad \|$

$(i^{-1}\mathcal{F})(\operatorname{Spec} k)$

$$\begin{tikzcd} & U \arrow[d, "\text{étale}"] \\ \operatorname{Spec} k \arrow[r] \arrow[ru] & \operatorname{Spec} A \end{tikzcd} \qquad \begin{tikzcd} & \operatorname{Spec} B \arrow[d, "\text{étale}"] \\ \operatorname{Spec} k \arrow[r] \arrow[ru] & \operatorname{Spec} A \arrow[u, bend left, "\exists !"] \end{tikzcd}$$

As i^{-1} is exact, Γ is also exact $\Rightarrow H^q = 0$ for $q > 0$.

Thm. Let $\pi : X \to Y$ be a finite morphism, then for any $\mathcal{F} \in Sh(X_{\acute{e}t})$,

$R^q \pi_* \mathcal{F} = 0 \quad , q > 0$.

Pf. Want: $\forall$ geometric pt $y : \operatorname{Spec} k \to Y$, $(R^q\pi_*\mathcal{F})_y = 0$.

$$(R^q\pi_*\mathcal{F})_y = \operatorname{colim} H^q(X \times_Y U, \mathcal{F})$$

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k \arrow[r] \arrow[ru] & Y \end{tikzcd}$$

Lemma. For any qc and separated $\pi : X \to Y$,

$$(R^q\pi_*\mathcal{F})_y = H^q(X \times_Y \operatorname{Spec} \mathcal{O}^{sh}_{Y,y}, \mathcal{F})$$

$X \times_Y \mathcal{O}^{sh}_{Y,y} = \operatorname{Spec} B$, w/ B a finite $\mathcal{O}^{sh}_{Y,y}$ algebra.

$B = B_1 \times B_2 \times \cdots \times B_n$, where B_i are strictly henselian.

$\Rightarrow H^q(\mathrm{Spec}\, B, \mathcal{F}) = 0$, $q > 0$.

<u>Cor</u>. $f: X \longrightarrow Y$ finite, $H^q(X, \mathcal{F}) = H^q(Y, \pi_* \mathcal{F})$.

Cohomology of a point.

$$\begin{tikzcd} Sh_{\acute{e}t}(\mathrm{Spec}\, k) \arrow[rr, equal] \arrow[rd, "\Gamma"'] & & \mathrm{Mod}^{cont}(G) \arrow[ld, "M \mapsto M^G"] \\ & Ab & \end{tikzcd}$$

$G = \mathrm{Gal}(k^{sep}/k)$.

$\mathcal{F} \mapsto \operatorname{colim}_{k \subset k' \subset k^{sep}} \mathcal{F}(\mathrm{Spec}\, k') = \mathcal{F}_{k^{sep}}$

$M = \bigcup_{\substack{H \subset G \\ \text{open}}} M^H.$

<u>Def</u>. For a profinite G and $M \in \mathrm{Mod}^{cont}(G)$, set

$$H^q(G, M) = \mathrm{Ext}^q_{\mathrm{Mod}^{cont}(G)}(\mathbb{Z}, M).$$

<u>Ex</u>. $H^q(G, M) = \operatorname{colim}_{\substack{H \subset G \\ \text{open}}} H^q(G/H, M^H)$

$$H^q_{\acute{e}t}(\mathrm{Spec}\, k, \mathcal{F}) = H^q(G, \mathcal{F}_{k^{sep}})$$

<u>Ex</u>. $k = \mathbb{F}_p$, $G = \hat{\mathbb{Z}} \ni Fr$

ⓐ For every torsion G-module $M \in \mathrm{Mod}^{cont}(G)$,

$$H^q(G, M) = H^q(\underset{0}{M} \xrightarrow{Fr - Id} \underset{1}{M})$$

⑥ $H^2(G; \mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$.

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

$$H^2(G, \mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{cts}(G, \mathbb{Q}/\mathbb{Z})$$

* * *

Cohomology of curves

Thm. For any curve $X / k = \bar{k}$, and any $\mathcal{F} \in Sh_{\acute{e}t}(X)$,

$H^q_{\acute{e}t}(X, \mathcal{F}) = 0$, $q > 2$.

Brauer group.

K field. A/K associative algebra, $\dim_K A < \infty$.

Prop. TFAE (central simple algebra)

① Center $A = K$ and A has no nontrivial 2-sided ideals

② $A \otimes_K \bar{K} \simeq \mathrm{Mat}_n(\bar{K})$

③ $A \otimes_K K^{sep} \simeq \mathrm{Mat}_n(K^{sep})$

④ $A \otimes A^{op} \xrightarrow{\sim} \mathrm{End}_K(A) = \mathrm{Mat}(K)$

$a \otimes a' \mapsto (x \mapsto a x a')$

⑤ $A \simeq \mathrm{Mat}_m(D)$, where D is a division algebra w/ center$(D) = K$.

If A and B are c.s., then $A \otimes_K B$ is.

Det $A \sim B \iff A \simeq \mathrm{Mat}_m(D),\ B \simeq \mathrm{Mat}_{m'}(D)$

$$\iff A \underset{k}{\otimes} B^{op} \simeq \mathrm{Mat}_\ell(k)$$

$\mathrm{Br}(k) =$ monoid (w.r.t. $\otimes$) of iso. classes of c.s. algebras

$$\left\{ \begin{matrix} \text{iso. classes of c.s. algebras} \\ \text{of } \dim A = n^2 \end{matrix} \right\} \xleftrightarrow[\sim]{1:1} H^1(\overset{\mathrm{Gal}(k^{sep}|k)}{\overset{\parallel}{G}}, PGL_n(k^{sep}))$$

$$\text{ii}$$

$$\{\text{set of } PGL_n\text{-torsors on } (\mathrm{Spec}\, k)_{\acute{e}t}\}$$

$$A \longmapsto T = \{\varphi : \mathrm{Mat}_n(k^{sep}) \xrightarrow{\sim} A \otimes k^{sep}\}$$

$$g \in PGL_n(k^{sep}) = \mathrm{Aut}(\mathrm{Mat}_n(k^{sep}))$$

$$g.\varphi = g \circ \varphi.$$

$$\sigma \in G, \quad \varphi^\sigma = \sigma \circ \varphi \circ \sigma^{-1}$$

$$\left(T \overset{PGL_n(k^{sep})}{\times} \mathrm{Mat}_n(k^{sep})\right)^G \longleftarrow\!\shortmid\ T$$

$$1 \to G_m \to GL_n \to PGL_n \to 1$$

$$\rightsquigarrow \quad H^1(G, PGL_n(k^{sep})) \to H^2(G, (k^{sep})^\times)$$

Thm. $\mathrm{Br}(k) \xrightarrow{\sim} H^2(G, k^{sep,*})$

Pf. $H^1(G; GL_n(k^{sep})) = \{*\} \Rightarrow$ injectivity

Surjectivity: take $\alpha \in H^2(G, k^{sep*})$.

Pick finite Galois $k^{sep} \supset L \supset k$ s.t. image of α in $H^2(\mathrm{Gal}(k^{sep}|L), k^{sep*})$ is 0. Set $n = [L:k]$.

$$\begin{tikzcd}
k^{sep,*} \arrow[r] \arrow[d, equal] & (L \otimes k^{sep})^* \arrow[r] \arrow[d, hook] & (L\otimes k^{sep})^*/k^* \arrow[r] \arrow[d] & 1 \\
k^{sep,*} \arrow[r] & GL_n(k^{sep}) \arrow[r] & PGL_n(k^{sep}) \arrow[r] & 1
\end{tikzcd}$$

$$\begin{tikzcd}
H^1(G, PGL_n(k^{sep})) \arrow[r] & H^2(G, k^{sep,*}) & \\
\square & & \\
H^1(G, (L\otimes k^{sep})^*/k^{sep,*}) \arrow[uu] \arrow[r] & H^2(G, k^{sep,*}) \arrow[uu, equal] \arrow[r] \arrow[ddr, "Res"] & H^2(G, (L\otimes k^{sep})^*) \arrow[dd, equal, "\text{Shapiro Lemma}"] \\
\tilde{\alpha} \arrow[r, mapsto] \arrow[uu, bend left] & \alpha \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[dr, mapsto] & \\
& & 0 \in H^2(\mathrm{Gal}(k^{sep}|L), k^{sep,*})
\end{tikzcd}$$

$\square$

Lecture 29 Last time: $\mathrm{Br}(k) = H^2(\mathrm{Gal}(k^{sep}|k), k^{sep,*})$

Ex. $\mathrm{Br}(\mathbb{F}_q) = H^2(\hat{\mathbb{Z}}, \bar{\mathbb{F}}_q^*) = 0$.

Def. K is a C_r-field if $\forall$ homogeneous $f(T_1, \dots, T_n) \in k[T_1, \dots, T_n]$, $\deg f = d$, $0 < d^r < n$, $\exists$ $0 \neq (a_1, \dots, a_n) = \alpha \in k^n$ s.t. $f(\alpha) = 0$.

Prop. If K is a C_1-field, then $Br(K)=0$.

Pf. Let A be a central simple K-alg.

$$\begin{tikzcd}
A \arrow[r, hook, "Nr"] \arrow[d] & K \arrow[r, phantom, "\subset"] & K^{sep} \\
A \otimes_k K^{sep} \arrow[rr, phantom, "\cong"] \arrow[rr, bend left, "i"] \arrow[rr, bend left=50, "\sigma\circ i"'] & & Mat_n(K^{sep}) \arrow[u, "det"']
\end{tikzcd}$$

conjugate

$\sigma \in Gal(K^{sep}|K)$,

$\det\circ\sigma\circ i = \det\circ i$

Nr is a polynomial function in $\dim_K A$ variables of deg $\sqrt{\dim_K A}$.

Let $A = D$ be a division algebra, $\dim D = n^2$.

By def'n of C_1-field; applied to Nr, $\exists\ \alpha\in D-\{0\}$ s.t. $Nr(\alpha)=0$. (if $n>1$) □

Prop. Let X be a smooth proj. conn'd curve $/k=\bar{k}$, $K=k(X)$, then K is a C_1-field.

Pf. $f(T_1,\cdots,T_n)\in K[T_1,\cdots,T_n]$, $f=\sum a_{i_1\cdots i_n}T_1^{i_1}\cdots T_n^{i_n}$

pick $H>0$ s.t. $a_{i_1\cdots i_n}\in\Gamma(X,\mathcal{O}(H))$

For $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\alpha_i\in\Gamma(X,\mathcal{O}(eH))$, $e>0$

$f(\alpha)\in\Gamma(X,\mathcal{O}((1+de)H))$, $d=\deg f$.

$f(\alpha)=0$ is a system of $\dim\Gamma(X,\mathcal{O}((1+de)H))$ homogeneous equations in $n\dim\Gamma(X,\mathcal{O}(eH))$ variables.

$$\dim\Gamma(X,\mathcal{O}((1+de)H)) = (1+de)\overset{\deg H}{\overset{\parallel}{c}}+1-g \quad (e\gg 0)$$

$$n\dim\Gamma(X,\mathcal{O}(eH)) = n(ec+1-g) \quad (e\gg 0) \quad \wedge \quad (d<n)$$

$\Rightarrow f(\alpha)=0$ has a non-zero sol'n.

Cor (Tsen) For every algebraic ext'n $L \supset K$, $Br(L) = 0$.

$(K = k(X),\ \dim X = 1,\ k = \bar{k})$

Prop. Let K be a field, Assume that for any algebraic ext'n $L \supset K$, $Br(L) = 0$, then

① For any torsion $G = Gal(k^{sep}|k)$ - module M, $H^q(G, M) = 0$, $q > 2$.

② $H^q(G, k^{sep,*}) = 0$, $q > 0$.

Digression: "Methode de la trace"

$f : Y \longrightarrow X$ finite étale map.

Define $tr : f_* f^{-1} \longrightarrow Id$ (by adjunction, have $Id \to f_* f^{-1}$. This is the nonobvious direction)

$$Sh(X_{\acute{e}t}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} Sh(Y_{\acute{e}t})$$

for étale $U \longrightarrow X$ s.t. $Y \underset{X}{\times} U \underset{U}{\simeq} U \times S$ ← finite set , $(*)$

then $(f_* f^{-1}(F))(U) = F(U)^{|S|} \xrightarrow{\Sigma} F(U)$.

$U \to X$ w/ $(*)$ form a covering sieve on X

$U_1 = Y$

$$\begin{tikzcd} Y \underset{X}{\times} U_1 \arrow[d, shift right] & = U_1 \sqcup W \arrow[dl] \\ U_1 \arrow[u, bend right, "\Delta"'] & \end{tikzcd}$$

$W \to U_1$ has smaller degree.

$U_2 \to U_1$ s.t. $W \underset{U_1}{\times} U_2 = U_2 \sqcup W' \ \ldots$

$$\begin{tikzcd}
\mathrm{Id} \arrow[r] \arrow[rr, bend right, "\deg f \cdot \mathrm{Id}"'] & f_* f^{-1} \arrow[r, "tr"] & \mathrm{Id}
\end{tikzcd}$$

$(\exists\ \mathrm{Id} \longrightarrow f^{-1} f_*)$

$$\begin{tikzcd}
H^q(X, F) \arrow[r] \arrow[rr, bend left, "\deg f \cdot \mathrm{Id}"] \arrow[dr] & H^q(X, f_* f^{-1} F) \arrow[r] \arrow[d, equal] & H^q(X, F) \\
& H^q(Y, f^{-1} F) \arrow[ur] &
\end{tikzcd}$$

<u>Prop</u>. K, $\forall\ K \subset L$ algebraic, $Br(L) = 0$

① For torsion M, $H^q(G, M) = 0$, $q > 2$.

② $H^q(G, K^{sep,*}) = 0$, $q > 0$.

<u>Pf</u>. $0 \to \mu_\ell \to K^{sep,*} \xrightarrow{\wedge \ell} K^{sep,*} \to 0$, $\ell \nmid \mathrm{char}\, K$.

$\Rightarrow H^2(H, \mu_\ell) = 0$ for all open $H \subset G$.

$0 \to \mathbb{Z}/\ell\mathbb{Z} \to K^{sep} \xrightarrow{x \mapsto x^\ell - x} K^{sep} \to 0$ $\quad \ell = \mathrm{char}\, K$

$H^q(G, K^{sep}) = 0$, $q > 0$ $\Rightarrow$ $H^q(G, \mathbb{Z}/p) = 0$, $q > 1$.

$\Big(|\Gamma| < \infty$, $\underset{\underset{H^q(\{e\}, k)}{\shortparallel}}{H^q(\Gamma, k[\Gamma])} = 0$, for example, by Shapiro lemma $\Big)$.

$$\begin{tikzcd}
Y = \mathrm{Spec}\, L \arrow[d] \\
X = \mathrm{Spec}\, K
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{\mathrm{Gal}(K^{sep}|L)}{\overset{\shortparallel}{\mathrm{Mod}(G)}} \arrow[r, shift left, "f^{-1} = \mathrm{Res}^G_H"] & \mathrm{Mod}(H) \arrow[l, shift left, "f_* = \mathrm{Ind}^G_H"] \arrow[r, phantom, "\text{—}\ \mathrm{Gal}(K^{sep}|K)"] & {}
\end{tikzcd}$$

$$M \mapsto \begin{array}{c} \mathbb{Z}[G] \underset{\mathbb{Z}[H]}{\otimes} M \\ \shortparallel \xleftarrow{\quad} H \text{ has finite index in } G. \\ \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \end{array}$$

Lecture 30

Prop. K field, Assume $\forall$ algebraic $L|K$, $Br(L) = 0$.

Then ① For any torsion G-module M,

$$\begin{tikzcd} G \arrow[d, equal] \\ Gal(K^{sep}|K) \end{tikzcd}$$

$$H^q(G, M) = 0, \quad q \geq 2.$$

② $$H^q(G, K^{sep,*}) = 0, \quad q \geq 1.$$

Pf. $H^2(G, \mu_\ell) = 0$, $\ell \neq \text{char } K$.

$H^2(G, \mathbb{Z}/p\mathbb{Z}) = 0$, $\text{char } K = p$.

① M torsion module

$$H^q(G, \varinjlim M_i) = \varinjlim H^q(G, M_i)$$

$\Rightarrow$ may assume M is finite $\rightsquigarrow$ reduce to $\ell M = 0$, ℓ prime.

$$M = (\mathbb{Z}/\ell)^n.$$

Case 1

$q = 2$, $\ell \neq \text{char } K$.

Pick open normal $H \subset G$ s.t. $M^H = M$, $\mu_\ell \subset (K^{sep})^H$.

$G \supset H' \supset H$. s.t. $H'/H \subset G/H$ is a Sylow ℓ-subgp.

Then
$$\begin{tikzcd} H^2(G, M) \arrow[r, hook, "i"] & H^2(H', M) \arrow[d, equal] \\ & H^2(G, \mathrm{Ind}_{H'}^{G} M) \arrow[ul, "tr"] \end{tikzcd}$$

$tr \circ i = [G : H']$ prime to ℓ, invertible on M

$\Rightarrow i$ injective

$M^{H'} \neq 0$. $\underbrace{M}_{\text{vector space over } \mathbb{F}_\ell} \circlearrowleft \text{\textcircled{H'/H}}$ - ℓ-group

prove by induction on $|H'/H|$

$\mathbb{F}_\ell[\mathbb{Z}/\ell] = \mathbb{F}_\ell[x]/(x^\ell - 1) = \mathbb{F}_\ell[x]/(x-1)^\ell$

$$\begin{tikzcd}
\mathbb{Z}/\ell \arrow[r, hook, "H'\text{-modules}"] \arrow[d, equal] & M \\
\mu_\ell &
\end{tikzcd}$$

$1 \to \mu_\ell \to M \to M' \to 0$

$H^2(H', \mu_\ell) = 0$.

Do induction on order of M ... $H^2(H', M') = 0$

$\Rightarrow H^2(H', M) = 0$

$q > 2$. For any class $\alpha \in H^q(G, M)$, $\exists\ M \hookrightarrow M'$ s.t.

$$\begin{tikzcd}
H^q(G,M) \arrow[r] & H^q(G,M') \\
\alpha \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & 0 \\
H^q(G/H,M) \arrow[u, mapsto] \arrow[uu, bend left] \arrow[r] & H^q(1, \mathrm{Ind}_1^{G/H} M) \arrow[uu]
\end{tikzcd}$$

$M' := \mathrm{Ind}_H^G M$.

$0 \to M \to M' \to M'' \to 0$

$$\underset{\substack{\| \text{ induction on } q \\ 0}}{H^{q-1}(G, M')} \to H^q(G, M) \to H^q(G, M')$$

① $\Rightarrow$ For any G-module M, $H^q(G, M) = 0$, $q > 2$

$0 \to M_{\text{tor}} \to M \to M \otimes \mathbb{Q} \to \underset{\text{torsion}}{M'} \to 0$

$$H^q(G, M_{tor}) = H^q(G, M') = H^q(G, M \otimes \mathbb{Q}) = 0, \quad q > 1.$$

Cor. X curve $/ k = \bar{k}$, $K = k(X)$.

$$H^q(\mathrm{Gal}(K^{sep}|k), K^{sep,*}) = 0, \quad q > 0.$$

Let X be a smooth curve $/ k = \bar{k}$.

$j : \eta \to X$ generic pt

$i_x : x \hookrightarrow X$ closed points

$$1 \to \mathcal{O}_X^* \to j_* \mathcal{O}_\eta^* \xrightarrow{div} \bigoplus_{x \in |X|} i_{x*} \mathbb{Z}_x \to 0$$

Lemma. $R^q j_* \mathcal{O}_\eta^* = 0$, $q > 0$.

Pf. $$(R^q j_* \mathcal{O}_\eta^*)_{\underset{\underset{x}{\shortparallel}}{\bar{x}}} = H^q(\underset{\underset{\mathrm{Spec}\, L}{\shortparallel}}{\eta \times_X \mathrm{Spec}\, \mathcal{O}^{sh}_{X,x}}, \mathcal{O}_\eta^*) = H^q(\mathrm{Gal}(L^{sep}|L), L^{sep,*}) = 0$$

where L is an alg. ext'n of K

$$\begin{array}{ccc} R^q\Gamma(X, Rj_* \mathcal{O}_\eta^*) & = & H^q(\eta, \mathcal{O}_\eta^*) \\ \shortparallel & & \shortparallel \end{array}$$

Cor. $H^q(X, j_* \mathcal{O}_\eta^*) = H^q(\mathrm{Gal}(k^{sep}|k), k^{sep,*}) = 0$, $q > 0$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y, & Sh(X) \arrow[r, "f_*"] \arrow[rr, bend right, "\Gamma_X"'] & Sh(Y) \arrow[r, "\Gamma_Y"] & Ab & D^+(Sh(X)) \arrow[r, "Rf_*"] \arrow[rr, bend right, "R\Gamma_X"'] & D^+(Sh(Y)) \arrow[r, "R\Gamma_Y"] & D^+(Ab) \end{tikzcd}$$

Cor. $H^q(X, \mathcal{O}_X^*) = 0, \ q > 1$.

Pf $H^q(X, i_{x,*}\mathbb{Z}_x) = H^q(\mathrm{Spec}\, k, \mathbb{Z}_x) = 0, \ q \geq 1$.

Cor. $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ [already seen by fpqc descent]

Cor. $(n, \mathrm{char}\, k) = 1$, X smooth proj,

$$\underset{\substack{\| \\ \mathbb{Z}/n(1)}}{H^q(X, \mu_n)} = \begin{cases} \mu_n(k), & q = 0 \\ \mathrm{Pic}(X)[n], & q = 1 \\ \mathbb{Z}/n, & q = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Pf. $1 \to \mu_n \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 1$

Need $\mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \cong \mathbb{Z}/n\mathbb{Z}$.

$$0 \to \underset{\substack{(\\ \text{abelian} \\ \text{variety}}}{\underline{\mathrm{Pic}}^0(X)} \longrightarrow \underset{\substack{\uparrow \\ \text{Picard scheme}}}{\underline{\mathrm{Pic}}(X)} \xrightarrow{\deg} \mathbb{Z} \to 0$$

$$\underline{\mathrm{Pic}}^0(X) \overset{\text{degree } n^{2g}}{\xrightarrow{\times n}\!\!\!\!\rightarrow} \underline{\mathrm{Pic}}^0(X)$$

$$\mathrm{Pic}(X)[n] \approx (\mathbb{Z}/n)^{2g}$$

Now X smooth curve, not proper. $X \overset{j}{\hookrightarrow} \bar{X}$, $\bar{X} - X = \{x_1, \cdots, x_r\}$.

$$0 \to \mu_n \to \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0$$

$$\begin{tikzcd}
 & & & & \bigoplus_{m=1}^{r} i_{x_m *}\mathbb{Z}_{x_m}/n \arrow[d, equal] & \\
0 \arrow[r] & j_*\mu_n \arrow[r] & j_*\mathcal{O}_X^\times \arrow[r, "n"] & j_*\mathcal{O}_X^\times \arrow[r] & R^1 j_*\mu_n \arrow[r] & R^1 j_*\mathcal{O}_X^\times = 0
\end{tikzcd}$$

$$\left(R^q j_* \mathcal{O}_X^\times \right)_x = H^q\left(X \times_{\bar{X}} \operatorname{Spec} \mathcal{O}_{X,x}^{sh}, \mathcal{O}_X^\times\right)$$

$q>0$

$= 0$

$$R\Gamma(X, \mu_n) = R\Gamma(\bar{X}, Rj_*\mu_n)$$

$$\mu_n \to Rj_*\mu_n \to \bigoplus_{m=1}^{r} i_{x_m,*}\mathbb{Z}_{x_m}/n[-1] \xrightarrow{+1}$$

$$R\Gamma(\bar{X}, \mu_n) \to R\Gamma(\bar{X}, Rj_*\mu_n) \to R\Gamma\left(\bar{X}, \bigoplus_{m=1}^{r} i_{x_m,*}\mathbb{Z}_{x_m}/n[-1]\right) \xrightarrow{+1}$$

$$0 \to H^1(\bar{X}, \mu_n) \to H^1(X, \mu_n) \to \bigoplus_{m=1}^{r} \mathbb{Z}/n \to \mathbb{Z}/n \to H^2(X, \mu_n)$$

$$\begin{tikzcd}
\mathbb{Z}^r \arrow[r] & \operatorname{Pic}(\bar{X}) \arrow[r, two heads] & \operatorname{Pic}(X) \arrow[r] & 0 \\
 & \operatorname{Pic}^0(\bar{X}) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

$$H^2(X, \mu_n) = \operatorname{Pic}(X)/n\operatorname{Pic}(X) = 0.$$

Lecture 31. X smooth proj. curve $/ k = \bar{k}$.

char $k \nmid n$, $U = X - \{x_1, \dots, x_r\}$.

$$\begin{tikzcd}
0 \arrow[r] & H^1_{et}(X, \mu_n) \arrow[r] \arrow[d, phantom, "\wr\|"] & H^1_{et}(U, \mu_n) \arrow[r] & \bigoplus_{i=1}^{r} \mathbb{Z}/n\mathbb{Z} \arrow[r, "\Sigma"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\
 & \operatorname{Pic}(X)[n] \approx (\mathbb{Z}/n\mathbb{Z})^{2g} & & & &
\end{tikzcd}$$

$$\Rightarrow \quad \left| H^1_{\text{ét}}(U, \mu_n) \right| < \infty .$$
$$\wr\!\wr$$
$$(\mathbb{Z}/n\mathbb{Z})^{2g+r-1}$$

<u>Rmk</u>. If char $k = p$, $U \neq X$,

$$\left| H^1_{\text{ét}}(U, \mathbb{Z}/p) \right| = \infty \; ; \qquad 0 \to \mathbb{Z}/p \to \mathcal{O}_U \xrightarrow{f \mapsto f^p - f} \mathcal{O}_U \to 0$$

$$\parallel \qquad\qquad\qquad H^1_{\text{ét}}(\underset{\substack{\uparrow \\ \text{affine}}}{U}, \mathcal{O}_U) = 0$$

$$\operatorname{coker}\left(\mathcal{O}(U) \xrightarrow{f \mapsto f^p - f} \mathcal{O}(U) \right)$$

$$\dim H^1_{\text{ét}}(\mathbb{A}^1_{\overline{\mathbb{F}}_p}, \mathbb{Z}/p) \neq 0.$$

<u>Constructible sheaves</u>:

<u>Def</u>. $\mathcal{F}$ is finite locally constant if $\exists$ étale cover $[U_i \to X]$ s.t.

$\mathcal{F}|_{U_i} \simeq \underline{A_i}$, where A_i are finite abelian gps.

$\mathcal{F}$ is locally constant if $\mathcal{F}$ is representable by a finite étale commutative gp scheme $G \xrightarrow{\pi} X$: $\rightsquigarrow$ $\mathcal{F}(U \to X)$ = group of U-pts of G.

$$\begin{tikzcd} G \times_X U \arrow[d] & G \arrow[d, "\pi"] \\ U \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

If X is connected, x_0 geometric pt,

$$\left\{\begin{matrix}\text{locally constant} \\ \text{sheaves}\end{matrix}\right\} \simeq \left\{\text{finite } \pi_1^{\acute{e}t}(X, x_0)\text{-modules}\right\}$$

$$\pi_1^{\acute{e}t}(X, x_0) \longrightarrow \operatorname{Aut}(A) \qquad A = \mathcal{F}_{x_0}.$$

$$|A| < \infty.$$

$$\begin{tikzcd} X \arrow[d] \\ \operatorname{Spec} k, \end{tikzcd} \qquad \operatorname{Gal}(k^{sep}/k) \curvearrowright \mu_n(k^{sep})$$

Def. Let X be a qcqs scheme. A sheaf $\mathcal{F}$ is finite constructible if $\exists \underset{\text{finite stratification}}{\wedge} X = \bigsqcup X_i$, where X_i are locally closed, s.t. $\mathcal{F}|_{X_i}$ is locally constant.

Ex. If $f : Y \to \overset{\text{noetherian}}{X}$ is finite, then $f_*(\mathbb{Z}/n\mathbb{Z})$ is constructible.

e.g. $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$
$x \longmapsto x^2$

$$\begin{tikzcd} Y \times_X U \arrow[r, hook] \arrow[d, "f_U"', "\text{étale}" {swap, xshift=-12pt}] & Y \arrow[d, "f"] & Y \times_X Z \arrow[l] \arrow[d, "f_Z"] \\ U \arrow[r, phantom, "\subset"] & X \arrow[r, phantom, "\supset"] & Z \end{tikzcd}$$

$$(f_* \mathbb{Z}/n\mathbb{Z})_U = f_{U*}(\mathbb{Z}/n\mathbb{Z})$$

$$(f_* \mathbb{Z}/n\mathbb{Z})_Z = f_{Z*}(\mathbb{Z}/n\mathbb{Z})$$

Lemma. ⓘ Constructible sheaves form an abelian cat., ⓘⓘ constr. sheaves $\subset$ sheaves is exact. ⓘⓘⓘ $\forall\ \mathcal{F} \xrightarrow{\alpha} \mathcal{F}'$, if $\mathcal{F}$ is constr., then so is $\mathrm{Im}\,\alpha$.

Extension by 0

Let $j : U \to X$ be an étale morphism. then j^{-1} has a left adjoint $j_!$.

$$j_!^{Psh}\mathcal{F}\left(\begin{smallmatrix} W \\ \downarrow \\ X \end{smallmatrix}\right) = \bigoplus_{\substack{W \to U \\ \text{over } X}} \mathcal{F}\left(\begin{smallmatrix} W \\ \downarrow \\ U \end{smallmatrix}\right), \qquad j_!\mathcal{F} = \left(j_!^{Psh}\mathcal{F}\right)^{\#}.$$

$$j^{-1}\mathcal{G}\left(\begin{smallmatrix} U' \\ \downarrow \\ U \end{smallmatrix}\right) = \mathcal{G}\left(\begin{smallmatrix} U' \\ \downarrow \\ X \end{smallmatrix}\right)$$

$$\mathrm{Mor}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}(\mathcal{F}, j^{-1}\mathcal{G}).$$

Properties: $j_!$ is exact.

$$(j_!\mathcal{F})_{\bar{x}} = \bigoplus \mathcal{F}_{\bar{u}}$$

$$\begin{tikzcd} & U \arrow[d] \\ \mathrm{Spec}\,k \arrow[r, "\bar{x}"'] \arrow[ur, dashed, "\bar{u}"] & X \end{tikzcd}$$

$\bar{x} : \mathrm{Spec}\,\overset{k^{sep}}{\overset{\shortparallel}{k}} \to X$

Base change

$$\begin{tikzcd} U \times_X Y \arrow[r, "j'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

$\mathcal{F} \in Sh(U)$. $j_! f'^{-1}\mathcal{F} \xrightarrow{\sim} f^{-1}j_!\mathcal{F}$.

$$\Uparrow$$

$f'^{-1}\mathcal{F} \to j'^{-1}f^{-1}j_!\mathcal{F} \Leftarrow \mathcal{F} \to j^{-1}j_!\mathcal{F}$

Check isom. on stalks

If j is <u>finite</u> étale, then $j_! = j_*$.

Recall $j_* j^{-1} \mathcal{F} \xrightarrow{tr} \mathcal{F}$

$$\begin{tikzcd}
j_* j^{-1}\left(\boxed{j_! \mathcal{G}}\right) \arrow[r] & j_! \mathcal{G} \\
j_* \mathcal{G} \arrow[u, hook] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

<u>Claim</u>. If $j : U \to X$ étale and qc, then $j_! \mathbb{Z}/n\mathbb{Z}$ is constr.

<u>Pf</u>. $\exists\ X = \coprod_i X_i$ s.t. $U \times_X X_i \to X_i$ is finite

<u>Thm</u>. X a curve $/ k = \bar{k}$, $\mathcal{F}$ finite constructible, $n\mathcal{F} = 0$, char $k \nmid n$.

then $\left| H^q_{et}(X, \mathcal{F}) \right| < \infty$, and $H^q = 0$ for $q > 2$.

(if X is affine, then $H^q = 0$, $q > 1$)

<u>Pf</u>. $\mathcal{F}|_U$ a local system, $U \hookrightarrow X$.

$0 \to j_! \mathcal{F}|_U \to \mathcal{F} \to \mathcal{F}' \to 0$.

$\mathcal{F}'$ has finite support.

Key step: X smooth, $U \overset{j}{\hookrightarrow} X$, $\mathcal{F} = j_! \mathcal{L}$ ← local system of $\mathbb{Z}/\ell$-modules

char $k \nmid \ell$, ℓ prime.

Pick a finite Galois cover

$$\begin{tikzcd}
& G \curvearrowright \tilde{U} \arrow[d, "h"] \\
U' \arrow[ur] \arrow[dr, "f"'] & \\
& U
\end{tikzcd}$$

prime to ℓ cover

s.t. $h^* \mathcal{L} \simeq \underline{\mathbb{F}_\ell}^m$.

$H \subset G$ Sylow's ℓ-subgp

$$\begin{tikzcd}
U' \arrow[r, hook, "j'"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
U \arrow[r, hook, "j"'] & X
\end{tikzcd}$$

X' = normalization of X in U'

$$\begin{tikzcd}
H^q_{\acute{e}t}(X, j_! \mathcal{L}) \arrow[r, hook] \arrow[d, shift right] & H^q(X', j'_! f^{-1} \mathcal{L}) \arrow[d, equal] \\
H^q(X, j_! f_* f^{-1} \mathcal{L}) \arrow[u, shift right, "tr"'] \arrow[r, phantom, "="] & H^q_{\acute{e}t}(X, f'_* j'_! f^{-1} \mathcal{L})
\end{tikzcd}$$

$f^{-1}\mathcal{L}$ has a filtration w/ $Gr \simeq \underline{\mathbb{Z}/\ell\mathbb{Z}}$.

Lecture 32. Proper base change.

$$\begin{tikzcd}
X = X_n \arrow[d, "f_n"] \\
X_{n-1} \arrow[d, "f_{n-1}"] \\
\vdots \arrow[d] \\
X_0 = \operatorname{Spec} k
\end{tikzcd}$$

$$R\Gamma(X, \mathcal{F}) = Rf_{1*} \circ \cdots \circ Rf_{n*} \mathcal{F}$$

$$\begin{tikzcd}
X \times_S S' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"'] & S
\end{tikzcd}$$

$\mathcal{F} \in Sh_{\acute{e}t}(X)$,

$$g^{-1} \circ f_* \mathcal{F} \longrightarrow f'_* \circ g'^{-1} \mathcal{F}$$

$$(\Leftrightarrow) \quad f_* \mathcal{F} \longrightarrow \underbrace{g_* f'_*}_{f_* g'_*} g'^{-1} \mathcal{F}$$

$$\begin{tikzcd}
D(Sh_{\acute{e}t}(X)) \arrow[r, shift left, "Rf_*"] & D(Sh_{\acute{e}t}(S)) \arrow[l, shift left, "f^{-1}"]
\end{tikzcd}$$

$$(*) \quad g^{-1} Rf_* \mathcal{F} \xrightarrow{\text{base change morphism}} Rf'_* \circ g'^{-1} \mathcal{F}$$

Thm. If f is proper and $\mathcal{F}$ is a torsion sheaf. then $(*)$ is an isom

Cor. $\bar{s} \to S$ a geometric pt,

$$(R^q f_* \mathcal{F})_{\bar{s}} \xrightarrow{\sim} H^q(X_{\bar{s}}, \mathcal{F}|_{X_s})$$

Cor $\Rightarrow$ Thm :

$$\begin{array}{c} (R^q f_* \mathcal{F})_{\bar{s}} = H^q(X_{\mathcal{O}^{sh}_{S,\bar{s}}}, \mathcal{F}) \\ \downarrow\wr \\ H^q(X_{\bar{s}}, \mathcal{F}) \end{array}$$

$$\begin{tikzcd} X_{\bar{s}} \arrow[r] \arrow[d] & X_{\mathcal{O}^{sh}_{S,\bar{s}}} \arrow[r] \arrow[d] & X \arrow[d] \\ \bar{s} \arrow[r] & \operatorname{Spec} \mathcal{O}^{sh}_{S,\bar{s}} \arrow[r] & S \end{tikzcd}$$

Thm' :

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ s \arrow[r, hook] & S \end{tikzcd}$$

$S = \operatorname{Spec} A$, A henselian,

$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X_0, \mathcal{F})$, f proper.

Ex. $X = S - s$. fails

Prop. Let $f : X \to S$ be a cts proper separated map of topological spaces, $s \in S$

$\forall \mathcal{F}$, $(R^q f_* \mathcal{F})_s \xrightarrow{\sim} H^q(X_s, \mathcal{F})$.

Pf. $X \hookleftarrow X_s$, $H^q(X_s, \mathcal{F}) = \lim\limits_{X_s \subset U \subset X} H^q(U, \mathcal{F})$

Ex.

$$\begin{tikzcd} S^1 \times \mathbb{C} - \{(a,0),(b,0)\} \arrow[d] \\ \mathbb{C} \end{tikzcd}$$

Künneth formula

$X, Y \ / k = \bar{k}$, X is proper

$$R\Gamma(X \times Y, \mathbb{Z}/n) \simeq R\Gamma(X, \mathbb{Z}/n) \overset{\mathbb{L}}{\underset{\mathbb{Z}/n}{\otimes}} R\Gamma(Y, \mathbb{Z}/n).$$

Pf.

$$\begin{tikzcd} X \times Y \arrow[r, "g"] \arrow[d, "f"] & X \\ Y \arrow[d, "\pi"] & \\ \operatorname{Spec} k & \end{tikzcd}$$

check $\Rightarrow$ on stalks

$$\pi^{-1} R\Gamma(X, \mathbb{Z}/n) \xrightarrow{\sim} Rf_* \mathbb{Z}/n$$

$$R\Gamma(X, \mathbb{Z}/n) \xrightarrow{g^*} R\Gamma(X \times Y, \mathbb{Z}/n)$$

$$R\Gamma(Y, C) \simeq C \overset{\mathbb{L}}{\underset{\mathbb{Z}/n}{\otimes}} R\Gamma(Y, \mathbb{Z}/n)$$

Proof of proper base change:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ s \arrow[r] & S \end{tikzcd}$$

f proper , $S = \operatorname{Spec} A$, A henselian

$$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X_0, \mathcal{F}).$$

Step 1. $q = 0$, $\mathcal{F} = \mathbb{Z}/n$

Prop. A noetherian, henselian. $S = \operatorname{Spec} A$, $f : X \to S$ proper.

$$\pi_0(X_0) \xrightarrow{\sim} \pi_0(X).$$

Pf.

$$\begin{tikzcd} \operatorname{Idem}(\Gamma(X, \mathcal{O}_X)) \arrow[r, "\text{Want}", "\sim"'] \arrow[rd, phantom, "\wr"] & \operatorname{Idem}(\Gamma(X_0, \mathcal{O}_{X_0})) \\ & \operatorname{Idem}(\Gamma(Y, \mathcal{O}_X) \otimes_A \hat{A}) \\ \operatorname{Idem}(\Gamma(X, \mathcal{O}_X)/m) \arrow[ru, phantom, "="] & \end{tikzcd}$$

$m \subset A$ max'l ideal,

$\Gamma(X, \mathcal{O}_X)$ is a finite A-alg.,
$\rightarrow$ product of local algebras

Formal function thm:

$$\varprojlim_n \Gamma(X, \mathcal{O}_X)/m^n = \varprojlim_n \Gamma(X_n, \mathcal{O}_{X_n}) , \quad X_0 \hookrightarrow X_n \hookrightarrow X$$

Step 2 $q = 1$, $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$, A noetherian

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}\text{ - torsors over } X$$

$$\begin{tikzcd}
FEt_X \arrow[r, "\sim"'] & FEt_{X_0}
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[d] & & X_0 \times_X Y \arrow[d] \\
X \arrow[rr, phantom, "\rightsquigarrow"] & & X_0
\end{tikzcd}$$

OK for $A = \hat{A}$.

General case: Artin's approximation

$$\begin{tikzcd}
A \arrow[r, hook] & \hat{A} \arrow[r] & A/m \\
 & A' \arrow[u] \arrow[ul, dashed] & \text{- f.g ring} \\
 & A[x_1, \dots x_n]/(f_1, \cdots f_m) \arrow[u, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes_A A' \arrow[d] \\
X
\end{tikzcd}$$

Step 3. Thm is true for finite f.

$$\begin{tikzcd}
X \arrow[d] \\
S
\end{tikzcd}
\qquad S \text{ is strictly henselian.}$$

Step 4. $X \xrightarrow{f} Y \xrightarrow{g} S$. If Thm holds for f and g, then it is true for $g\circ f$.

Step 5. $X \xrightarrow{f} Y \xrightarrow{g} S$ f is surjective. Thm for f & $g\circ f$ $\Rightarrow$ Thm for g.

Step 6. Enough to prove Thm for
$$\begin{tikzcd} X = \mathbb{P}^1 \times S \arrow[d] \\ S \end{tikzcd}$$

Chow's lemma:
$$\begin{tikzcd} \mathbb{P}^m \times X \arrow[d] & & \\ X \arrow[d] & X' \arrow[l, "\text{proper birat'l}"'] \arrow[ul] \arrow[r, hook] & \mathbb{P}^n \times S \arrow[dll] \\ S & & \end{tikzcd}$$

$$\begin{tikzcd} S^n \mathbb{P}^1 \simeq \mathbb{P}^n \\ (\mathbb{P}^1)^n \arrow[u, "\text{finite surj.}"'] \end{tikzcd}$$

Step 7.

Prop $X_0 \hookrightarrow X$. Assume that for all $n \geq 0$, and finite $X' \to X$,

$H^q(X', \mathbb{Z}/n\mathbb{Z}) \to H^q(X_0', \mathbb{Z}/n)$, $X_0' = X_0 \times_X X'$ is iso. for $q=0$

and surj. for $q>0$, then for all torsion $\mathcal{F}$, $H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X_0, \mathcal{F})$.

Step 8. Key Lemma:

$f: X \to \operatorname{Spec} A$, A strictly henselian, noetherian, f is proper, and $\dim X_0 \leq 1$,

Then $H^q(X, \mathbb{Z}/n\mathbb{Z}) \to H^q(X_0, \mathbb{Z}/n\mathbb{Z})$ is iso. for $q=0$, surj. for $q>0$.

Pf OK, $q=0,1$ and empty for $q>2$.

Case $\mathrm{char}\, k \nmid n$:

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r] \arrow[d] & H^2(X, \mu_n) \arrow[d] \\
\mathrm{Pic}(X) \arrow[r, two heads] & H^2(X_0, \mu_n)
\end{tikzcd}$$

Claim. $\mathrm{Pic}(X) \twoheadrightarrow \mathrm{Pic}(X_0)$

Want $\mathrm{Div}^{eff}(X) \twoheadrightarrow \mathrm{Div}^{eff}(X_0)$ effective Cartier divisors

$$\begin{array}{c} \mathrm{Div}^{eff}(X) \twoheadrightarrow \mathrm{Div}^{eff}(X_0) \\ \qquad\qquad\qquad \rotatebox{90}{$\in$} \\ \qquad\qquad\qquad D \end{array}$$

May assume that $\mathrm{supp}\, D = x \in X_0$, given locally by $t_0 = 0$, $t_0 \in \mathcal{O}_{X_0, x}$.

Pick $x \in U \subset X$ and a lift $t \in \mathcal{O}(U)$ of t_0.

$U \supset Y = \{t=0\}$, $Y \cap X_0 = x$, $Y \to S$ is quasi-finite.

$Y = Y_1 \sqcup Y_2$, $Y_1 \to S$ finite, $Y_2 \cap X_0 = \phi$

Shrinking U, we may assume that $Y = Y_1$.

Set $D|_{X-Y} = 0$, $D|_U = \mathrm{div}\, t$.

Lecture 33. Last time: $X \xrightarrow{f} S$ proper

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r] \arrow[d] & X \times_S \mathrm{Spec}\, \mathcal{O}^{sh}_{S,\bar{s}} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}\, k(\bar{s}) \arrow[r] & \mathrm{Spec}\, \mathcal{O}^{sh}_{S,\bar{s}} \arrow[r] & S
\end{tikzcd}$$

F torsion sheaf, $\bar{s} \to S$ geometric pt,

$$(R^q f_* F)_{\bar{s}} = H^q\left(X \times_S \mathrm{Spec}\, \mathcal{O}^{sh}_{S,\bar{s}}, F\right) \xrightarrow{\sim} H^q(X_{\bar{s}}, F)$$

$$\begin{tikzcd} \Leftrightarrow X\times_S S' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f \text{ proper}"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

$$g^{-1} Rf_* \mathcal{F} \xrightarrow{\sim} Rf'_* \cdot g'^{-1} \mathcal{F}$$

<u>Applications</u>:

1. cohomology w/ compact support.

X/k separated finite type, $\mathcal{F}$ - torsion sheaf.

$$\begin{tikzcd} \exists\ X \arrow[rr, hook, "\text{open}"] \arrow[dr] & & \bar{X} \arrow[dl, "\text{proper}"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

(Nagata)

$$H^q_c(X, \mathcal{F}) := H^q(\bar{X}, j_!\mathcal{F})$$

<u>Claim</u>. H^q_c is well-defined.

closure of X in $\bar{X}_1 \times \bar{X}_2$

$$\begin{tikzcd} & \bar{X}_1 & \\ X \arrow[ur, hook, "j_1"] \arrow[r, "\text{open}"] \arrow[dr, "j_2"'] & \bar{X}_3 \arrow[u] \arrow[d] \arrow[r, "\text{closed}"] & \bar{X}_1 \times \bar{X}_2 \\ & \bar{X}_2 & \end{tikzcd}$$

$\Rightarrow$ May assume

$$\begin{tikzcd} & \bar{X}_1 \arrow[d, "p \text{ proper}"] \\ X \arrow[ur, "j_1"] \arrow[r, "j_2"'] & \bar{X}_2 \end{tikzcd}$$

<u>Lemma</u> $Rp_*(j_1!\mathcal{F}) = j_2!\mathcal{F}$.

⌈ If so, $H^q(\bar{X}_1, j_1!\mathcal{F}) = R^q\Gamma(\bar{X}_2, Rp_* j_1!\mathcal{F})$

$= H^q(\bar{X}_2, j_2!\mathcal{F})$ ⌋

<u>Pf</u>. Use proper base change.

$$j_!\mathcal{F} \longrightarrow Rp_*(j_1!\mathcal{F}) \qquad \bar{s} \in \bar{X}_2 - X, \quad (j_2!\mathcal{F})_{\bar{s}} = 0$$

$$(Rp_*(j_1!\mathcal{F}))_{\bar{s}} = R\Gamma(p^{-1}(\bar{s}), \overbrace{j_1!\mathcal{F}|_{p^{-1}(\bar{s})}}^{\overset{0}{\shortparallel}}) = 0$$

More generally, let $X \xrightarrow{f} S$ be a separated morphism of finite type, S noetherian. $\exists$

$$\begin{tikzcd} X \arrow[r, hook, "j\ \text{open}"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\text{proper}\ \bar{f}"] \\ & S \end{tikzcd}$$

$\mathcal{F}$ torsion.

$$Rf_! \mathcal{F} := R\bar{f}_* (j_! \mathcal{F}).$$

Claim: $Rf_! \mathcal{F}$ well-defined.

Claim. $(R^q f_! \mathcal{F})_{\bar{s}} = H^q_c(X_{\bar{s}}, \mathcal{F})$.

$$\begin{tikzcd} (R^q f_! \mathcal{F})_{\bar{s}} \arrow[dr, phantom, "\shortparallel"] & & H^q_c(X_{\bar{s}}, \mathcal{F}) \arrow[dl, phantom, "/\!/"] \\ & H^q(\bar{X}_{\bar{s}}, j_! \mathcal{F}) & \end{tikzcd}$$

$$\begin{tikzcd} X \times_S S' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd} \qquad g^{-1} Rf_! \mathcal{F} \xrightarrow{\sim} Rf'_! \, g'^{-1} \mathcal{F}.$$

Thm. Assume that dimensions of fibers of $X \xrightarrow{f} S$ $\le n$, then $\forall$ torsion $\mathcal{F}$,

$R^q f_! \mathcal{F} = 0$ for $q > 2n$.

Cor. $X / k = \bar{k}$ proper, then $H^q_{\text{ét}}(X, \mathcal{F}) = 0$, $q > 2n$.

Pf. WLOG, $S = \operatorname{Spec} \bar{k}$.

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

$$U \overset{\substack{\text{dense} \\ \text{affine} \\ \text{open}}}{\underset{j}{\hookrightarrow}} X \overset{i}{\hookleftarrow} X - U = Z$$

$H^q_c(X, i_* F|_Z)$

$\shortparallel$

$H^q_c(Z, F|_Z)$ $\qquad$ $H^q_c(X, j_! F|_U) = H^q_c(U, F|_U)$.

$$\begin{tikzcd} \exists\, U \arrow[r, "\substack{\text{finite}\\ \text{surj.}}", "g"'] \arrow[rrrrr, bend right, "f"'] & \mathbb{A}^n \arrow[r, "\pi_n"] & \mathbb{A}^{n-1} \arrow[r, "\pi_{n-1}"] & \cdots \arrow[r] & \mathbb{A}^1 \arrow[r, "\pi_1"] & \operatorname{Spec} \bar{k} \end{tikzcd}$$

$Rf_! F = R\pi_{1!} \circ \cdots \circ R\pi_{n!} \cdot Rg_! F$

$Rg_! F = Rg_* F = g_* F$

$R\pi_{n!}(g_* F)$ is supported in degrees $0, 1, 2$.

$R\pi_{n-1!}(R\pi_{n!}(g_* F))$ is supported in degrees $0, 1, 2, 3, 4$.

$\cdots\cdots$

Thm. If F is finite constructible, $[X = \sqcup X_i,\ F|_{X_i}$ is local system of finite gps$]$ $X \xrightarrow{f} S$, finite type, separated, S noetherian, then $R^q f_! F$ is constr.

Cor. $X/k = \bar{k}$, $|H^q_c(X, F)| < \infty$.

$\mathbb{Z}_\ell$ - sheaves.

Ex. X normal, $H^1_{ét}(X, \underset{\substack{|\\ \text{constant sheaf}}}{\underline{\Lambda}}) = \operatorname{Hom}_{cts}(\pi_1^{ét}(X), \Lambda)$

In particular, if Λ is torsion free, then $H^1_{ét}(X, \underline{\Lambda}) = 0$.

Ex.

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d] \\ {} \end{tikzcd}$$

$= X / \bar{k}$

$H^1_{ét}(X, \mathbb{Z}) = \mathbb{Z}$.

Lecture 34 ℓ-adic sheaves

Def. X noetherian scheme. A $\mathbb{Z}_\ell$-sheaf $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 1}$ where $\mathcal{F}_n$ is a constr. sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, together w/ $\cdots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$

s.t. $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isom. $\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \xrightarrow{\sim} \mathcal{F}_n$.

$$\mathrm{Mor}(\mathcal{F}, \mathcal{F}') = \varprojlim \mathrm{Mor}(\mathcal{F}_n, \mathcal{F}'_n).$$

A $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ is lisse (smooth) if each $\mathcal{F}_n$ is locally constant.

If X is connected, $x_0 \in X$,

lisse sheaves $\simeq$ finite $\mathbb{Z}_\ell$-modules w/ cts action of $\pi_1^{ét}(X, x_0)$.

$\uparrow$ for ℓ-adic top.

$$\pi_1^{ét}(X) \to \mathrm{Aut}(V) \to \mathrm{Aut}(V/\ell^n) \quad \text{kernel is open, } \forall n.$$

Ex. char $k \neq \ell$, $\varprojlim \mu_{\ell^n}(\bar{k}) = \mathbb{Z}_\ell(1)$

Lemma 1. Let $\cdots \to G_2 \to G_1$ be an inverse system of constr. $\mathbb{Z}/\ell$-modules, Assume that $\forall k$,

$$G_{n+1}/\ell^k G_{n+1} \xrightarrow{\sim} G_n/\ell^k G_n \quad \text{for } n \gg 0,$$

That is $\{G_n/\ell^k G_n\}$ becomes eventually constant.

Call $\mathcal{F}_k$ the corresponding sheaf, then $\{\mathcal{F}_k\}_{k\geq 1}$ is a $\mathbb{Z}_\ell$-sheaf.

Lemma $\mathbb{Z}_\ell$-sheaves form an abelian cat.

Pf. Let $\Phi : \{\mathcal{F}_n\} \to \{\mathcal{G}_n\}$ be a morphism,

$$\operatorname{Coker} \Phi = \left\{\operatorname{coker}(\mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{G}_n)\right\}_{n\geq 1}.$$

$\ker \Phi$ is defined using Lemma 1 applied to

$$\left\{ \bigcap_{m\geq n} \operatorname{Im}\left(\ker \varphi_m \longrightarrow \ker \varphi_n\right)\right\}.$$

Ex. $\Phi : \mathbb{Z}_\ell \xrightarrow{\times \ell} \mathbb{Z}_\ell$

$$\begin{tikzcd} \mathbb{Z}_\ell \arrow[r, "\times\ell"] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z}_\ell \arrow[d, phantom, "\shortparallel"] \\ \mathcal{F} & \mathcal{G} \end{tikzcd}$$

$$\begin{tikzcd} \ker \varphi_m = \ker(\mathbb{Z}/\ell^m \xrightarrow{\times \ell} \mathbb{Z}/\ell^m) \arrow[d, "0"'] \\ \ker \varphi_n = \ker(\mathbb{Z}/\ell^n \xrightarrow{\times \ell} \mathbb{Z}/\ell^n) = \ell^{n-1}\mathbb{Z}/\ell^n\mathbb{Z} \end{tikzcd}$$

$\ker \Phi = 0$.

Def. $\mathbb{Z}_\ell$- sheaf $\mathcal{F}$ is a torsion sheaf if $\ell^n \mathcal{F} = 0$ for some n.

$\mathbb{Q}_\ell$- sheaves $=$ $\mathbb{Z}_\ell$- sheaves / torsion sheaves

- $\mathrm{Ob}(\mathbb{Q}_\ell\text{-sheaves}) = \mathrm{Ob}(\mathbb{Z}_\ell\text{- sheaves})$

$$\mathrm{Mor}_{\mathbb{Q}_\ell\text{-sheaves}}(\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{\mathbb{Z}_\ell\text{-sheaves}}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell .$$

Def. $\mathcal{F}$ - $\mathbb{Z}_\ell$- sheaf, X separated of finite type over $k = \bar{k}$, char $k \neq \ell$

$$H^q(X, \mathcal{F}) := \varprojlim H^q(X, \mathcal{F}_n) \underset{\text{true, but difficult!}}{\overset{?}{=}} \mathrm{Ext}^q_{\mathbb{Z}_\ell\text{- sheaves}}(\mathbb{Z}_\ell, \mathcal{F})$$

$$H^q_c(X, \mathcal{F}) = \varprojlim H^q_c(X, \mathcal{F}_n)$$

For $\mathbb{Q}_\ell$ - sheaves, $\mathcal{F} = \mathcal{F}' \otimes \mathbb{Q}_\ell$, $H^q(X, \mathcal{F}) := H^q(X, \mathcal{F}') \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

Prop. ⓐ $H^q_c(X, \mathcal{F})$, $H^q(X, \mathcal{F})$ are finite $\mathbb{Z}_\ell$- modules.

ⓑ $H^*(X, ?)$, $H^*_c(X, ?)$ are δ- functors.

Pf. **Thm.** $H^q_c(X, \mathcal{F}_n)$ is finite.

$H^q(X, \mathcal{F}_n)$ is finite if $\ell \neq$ char k.

$X \xrightarrow{f} S$, $\mathcal{F}$ finite constructible, then $Rf_! \mathcal{F}$ is constructible.

If S/k, $n\mathcal{F} = 0$ for some n invertible in k, then $Rf_* \mathcal{F}$ is constructible.

Observe that if $\cdots C_3 \to C_2 \to C_1$ is an inverse system of finite ab. gps, then $R^1 \varprojlim C_\cdot = 0$ if $\forall n, \exists m$, $\mathrm{Im}(C_m \to C_n) = \mathrm{Im}(C_{m+1} \to C_n) = \cdots$

@ Assume that $\mathcal{F}$ is torsion free.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & \mathcal{F}_n \arrow[r, "\times \ell"] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_{n-1} \arrow[r, "\ell"] \arrow[d] & \mathcal{F}_{n-1} \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

$\cdots \to H^q(X, \mathcal{F}) \xrightarrow{\times \ell} H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}_1) \to H^{q+1}(X, \mathcal{F}) \to \cdots$ is exact

$\Rightarrow \left| H^q(X, \mathcal{F})/\ell \right| < \infty$

$H^q(X, \mathcal{F}) = \varprojlim$ finite $\mathbb{Z}/\ell^n$-modules

$\Rightarrow \underset{\overset{\shortparallel}{M}}{H^q(X, \mathcal{F})}$ is ℓ-adically complete

$\Rightarrow \quad M \xrightarrow{\sim} \varprojlim M/\ell^n, \ |M/\ell| < \infty \Rightarrow M$ is f.g. over $\mathbb{Z}_\ell$.

For any $\mathcal{F}$, $0 \to \mathcal{F}_{tor} \to \mathcal{F} \to \underset{}{\overset{\text{torsion free}}{\mathcal{F}'}} \to 0$

$0 \to \mathcal{F}_{tor, n} \to \mathcal{F}_n \to \mathcal{F}_n' \to 0$

$\cdots \to H^q(X, \mathcal{F}_{tor}) \to H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}') \to \cdots$ exact

$X / k = \bar{k} \rightsquigarrow \quad H^q(X; \mathbb{Q}_\ell)$

char $k \neq \ell \qquad H^q_c(X; \mathbb{Q}_\ell)$

$k = \mathbb{C}, \qquad H^q_c(X, \mathbb{Q}_\ell) \simeq H^q_c(X(\mathbb{C})_{an}) \otimes \mathbb{Q}_\ell$

$\Rightarrow \dim_{\mathbb{Q}_\ell} H^q_c(X, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_{\ell'}} H^q_c(X, \mathbb{Q}_{\ell'})$ for different ℓ, ℓ'.

$X \circlearrowleft \varphi \rightsquigarrow H^q(X, \mathbb{Q}_\ell) \circlearrowleft \varphi^*$

$$\det(t - \varphi^*) = \chi_\varphi(t).$$

Does $\chi_\varphi(t)$ have integral coefficients?

Are they depend on ℓ?

Lecture 35. char $k \neq \ell$, $k = \bar{k}$.

X smooth proj. curve $/k$. $\quad U \underset{open}{\overset{j}{\hookrightarrow}} X$, $\quad U = X - \{x_1, \dots, x_r\}$.

$$H^q_c(U, \mathbb{Q}_\ell) = H^q(X, j_! \mathbb{Q}_\ell)$$

$$0 \to j_! \mathbb{Q}_\ell \to \mathbb{Q}_\ell \to \bigoplus_{m=1}^{r} i_{m,*} \mathbb{Q}_\ell \to 0$$

$$H^q(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z}) \qquad 0 \to \overset{\mathbb{Z}/\ell^n}{\overset{\wr}{\mu_{\ell^n}}} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

$$= \begin{cases} \mathbb{Q}_\ell, & q = 0 \\ \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \underset{\overset{\|}{T_\ell(\mathrm{Pic}^0(X))}}{\varprojlim \mathrm{Pic}(X)[\ell^n]}, & q = 1 \end{cases}$$

$\lfloor \mathbb{Q}_\ell, \quad q=2.$

Make it canonical: replace $\mathbb{Q}_\ell$ by $\mathbb{Q}_\ell(1)$,

$$H^q(X, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell \otimes \varprojlim_n H^q(X, \mu_{\ell^n}).$$

$r>0$, $$H_c^q(U, \mathbb{Q}_\ell(1)) = \begin{cases} 0, & q=0 \\ \text{a vec sp. of dim} = 2g+r-1, & q=1 \\ \mathbb{Q}_\ell, & q=2 \end{cases}$$

Poincaré duality:

ⓐ For any $\mathbb{Q}_{\bar\ell}$- local system $\mathcal{F}$ on U,

$$H^q(U, \mathcal{F}^*) \otimes H_c^{2-q}(U, \mathcal{F}(1)) \to H^2(U, \mathbb{Q}_\ell(1)) \xrightarrow[\sim]{tr} \mathbb{Q}_\ell$$

is nondegenerate.

ⓑ Finite coefficients. Let $\mathcal{F}$ be a constr. sheaf of $\mathbb{Z}/n$-modules,

$$\operatorname{Ext}^q_{Sh(U_{\acute{e}t}, \mathbb{Z}/n)}(\mathcal{F}, \mathbb{Z}/n\mathbb{Z}) \otimes H_c^{2-q}(U, \mathcal{F}(1)) \to H^2(X, \mathbb{Z}/n(1)) = \mathbb{Z}/n\mathbb{Z}.$$

is nondegenerate.

$$\begin{tikzcd} \operatorname{Ext}^q_{Sh(X_{\acute{e}t}, \mathbb{Z}/n)}(j_!\mathcal{F}, \mathbb{Z}/n) \otimes \operatorname{Ext}^{2-q}_{Sh(X_{\acute{e}t}, \mathbb{Z}/n)}(\mathbb{Z}/n\mathbb{Z}, j_!\mathcal{F}(1)) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^2_{Sh(X_{\acute{e}t}, \mathbb{Z}/n)}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}(1)) \arrow[d, equal] \\ \operatorname{Ext}^{2-q}_{Sh(X_{\acute{e}t})}(\mathbb{Z}, j_!\mathcal{F}(1)) & H^2(X, \mathbb{Z}/n(1)) = \mathbb{Z}/n \end{tikzcd}$$

Cor. For a $\mathbb{Q}_\ell$- local system $\mathcal{F}$,

$$\begin{tikzcd} H^2_c(U,\mathcal{F}) \arrow[r, phantom, "\simeq"] & H^0(U, \mathcal{F}^*(1))^* \arrow[d, equal] \\ & (\mathcal{F}_{\bar{s}})_{\pi_1^{et}(X,\bar{s})}(1) \end{tikzcd}$$

$G \curvearrowright V$, $((V^*)^G)^* = V_G$.

Lefschetz formula.

$k = \mathbb{F}_q$, X/k separated scheme of finite type.

$$X_{\bar{k}} = X \times_k \bar{k} \xrightarrow{F = Fr_q \times Id} X \times_k \bar{k}$$

$$\boxed{X_{\bar{k}}^F(\bar{k}) = X(k)} \qquad X_{\bar{k}} \circlearrowleft F$$

Thm. Let $\mathcal{F}$ be a $\mathbb{Q}_\ell$-sheaf on X, then

$$\sum_{x \in X(k)} Tr(F_{\bar{x}} \mid \mathcal{F}_{\bar{x}}) = \sum_i (-1)^i Tr(F^* \mid H^i_c(X_{\bar{k}}; \mathcal{F})).$$

$$F^{-1}\mathcal{F} \to \mathcal{F} \quad {}^{''}\, \mathcal{F} \times \bar{k}$$

$$\mathcal{F} \to F_* \mathcal{F}$$

$$\begin{tikzcd} H^q_c(X_{\bar{k}}, \mathcal{F}) \arrow[r] \arrow[rr, bend right, "F^*"'] & H^q_c(X_{\bar{k}}, F_*\mathcal{F}) \arrow[r, phantom, "="] & H^q_c(X_{\bar{k}}, \mathcal{F}) \end{tikzcd}$$

Remarks. $\mathrm{Fr}_q = F \circ \pi : X_{\bar k} \longrightarrow X_{\bar k}$

$\pi = \mathrm{Id} \times \mathrm{Fr}_q$

Lemma. For any scheme $Y / \mathbb{F}_q$, (e.g. $X_{\bar k}$), Fr_q acts trivially on $Y_{\text{ét}}$:

$$\begin{tikzcd}
 & U' = U \times_{(Y,\mathrm{Fr})} Y \arrow[r] \arrow[d, "\text{ét}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "f\ \text{ét}"] \\
U \arrow[ur, "F_{U/Y}", "\sim"'] \arrow[urr, bend left=40, "\mathrm{Fr}"] \arrow[r, "f\ \text{ét}"'] & Y \arrow[r, "\mathrm{Fr}"] & Y
\end{tikzcd}$$

$F_{U/Y} = (\mathrm{Fr}, f)$

Pf. $F_{U/Y}$ is finite étale

$\Rightarrow$ enough to check that $F_{U/Y}$ is iso on geometric fibers.

$Y = \operatorname{Spec} \bar k$. $U = \operatorname{Spec} \bar k$ $\square$

Cor. $\mathrm{Fr}_q^{-1} \mathcal{F} = \mathcal{F} = \mathrm{Fr}_{q*} \mathcal{F}$.

Cor. Fr_q acts trivially on $H^q(Y, \mathcal{F})$. $\square$

$$\begin{tikzcd}
X_{\bar k} \arrow[d, "pr"] \\
X
\end{tikzcd}$$

$pr = pr \circ \pi$

$$\begin{tikzcd}
pr^{-1}\mathcal{F} \arrow[d, equal] & = \pi^{-1} pr^{-1} \mathcal{F} \\
\mathcal{F}_{X_{\bar k}} &
\end{tikzcd}$$

$$\begin{tikzcd}
F^{-1}\mathcal{F}_{X_{\bar k}} \arrow[rrr, bend right=20] & = F^{-1}\pi^{-1}\mathcal{F}_{X_{\bar k}} & = \mathrm{Fr}_q^{-1}\mathcal{F}_{X_{\bar k}} \arrow[r, "\sim"] & \mathcal{F}_{X_{\bar k}}
\end{tikzcd}$$

Rmk. For any k and any X/k, $\mathrm{Gal}(\bar k | k) \curvearrowright H^q(X_{\bar k}, \mathcal{F})$
$\mathcal{F}$ on X,

$k = \mathbb{F}_q$, $\quad \langle Fr_q \rangle = \hat{\mathbb{Z}} = Gal(\bar{\mathbb{F}}_q \mid \mathbb{F}_q)$

$$Fr_q^{-1*} = F^* .$$

Ex. $X = \mathbb{P}^1_{\mathbb{F}_q}$, $\quad \mathcal{F} = \mathbb{Q}_\ell$.

$$\sum_{x \in X(k)} \underset{\substack{\| \\ q+1}}{\mathrm{Tr}(F_x \mid \mathcal{F}_{\bar{x}})} = 1 + tr\Big(F^* \mid \underset{\substack{\| \\ q}}{\overset{\substack{\mathbb{Q}_\ell(-1) \\ \|}}{H^2(\mathbb{P}^1_{\bar{k}}, \mathbb{Q}_\ell)}}\Big)$$

Lecture 36. $\quad k = \mathbb{F}_q$, $\quad X/k$ separated of finite type.

$X_{\bar{k}} = X \times_k \bar{k}$. $\qquad Fr = F \circ \pi$

$$\begin{tikzcd} X_{\bar{k}} \arrow[d, "pr"'] \\ X \end{tikzcd}$$

$\mathcal{F}$ on $X \rightsquigarrow \mathcal{F}_{X_{\bar{k}}} := pr^{-1}\mathcal{F}$.

$\pi^{-1}\mathcal{F}_{X_{\bar{k}}} \xrightarrow{\sim} \mathcal{F}_{X_{\bar{k}}}$: $pr \circ \pi = pr$.

$$\mathcal{F}_{X_{\bar{k}}} \simeq Fr^{-1}\mathcal{F}_{X_{\bar{k}}} = F^{-1}\pi^{-1}\mathcal{F}_{X_{\bar{k}}} = F^{-1}\mathcal{F}_{X_{\bar{k}}}$$

$$\Rightarrow \quad F^{-1}\mathcal{F}_{X_{\bar{k}}} \overset{\text{Weil structure}}{\xrightarrow{\sim}} \mathcal{F}_{X_{\bar{k}}} .$$

Sheaves on $X \longrightarrow$ Weil sheaves on $X_{\bar{k}}$

that is ($\mathcal{F}$ on $X_{\bar{k}}$, w/ isom. $F^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$).

$\mathbb{Q}_\ell$- local systems on a normal conn'd $X \simeq Rep_{cts}(\pi_1^{ét}(X))$

$$\begin{tikzcd}
 & & & \mathbb{Z} \arrow[d, hook] & \\
1 \arrow[r] & \pi_1^{et}(X_{\bar{k}}) \arrow[r] & \pi_1^{et}(X) \arrow[r, "\psi"] & Gal(\bar{k}|k) \arrow[r] \arrow[d, phantom, "\parallel"] & 1 \\
 & & & \hat{\mathbb{Z}} &
\end{tikzcd}$$

Weil's local systems $\simeq \mathrm{Rep}_{cts}(\psi^{-1}(\mathbb{Z}))$.

Thm For any Weil sheaf on $X_{\bar{k}}$,

$$(\mathcal{F},\ F^{-1}\mathcal{F} \overset{F}{\underset{\sim}{\to}} \mathcal{F}) \rightsquigarrow \mathcal{F}_{F(x)} = (F^{-1}\mathcal{F})_x \overset{F_x}{\underset{\sim}{\to}} \mathcal{F}_{\bar{x}}$$

$$\sum_{x \in X(k) = X_{\bar{k}}^{F}(\bar{k})} Tr(F_x \mid \mathcal{F}_{\bar{x}}) = \sum_i (-1)^i Tr(F^* \mid H_c^i(X_{\bar{k}}; \mathcal{F})).$$

Rmk. The same is true if we replace $\mathbb{Q}_\ell$ by any finite ext'n $k|\mathbb{Q}_\ell$.

Def Let $\mathcal{F}$ be a $\mathbb{Q}_\ell$-sheaf on $X/\mathbb{F}_q = k$.

$$L(X, \mathcal{F}, t) = \prod_{x \in |X|} \frac{1}{\det(1 - Fr_x^{-1} t^{\deg x} \mid \mathcal{F}_{\bar{x}})} \in \mathbb{Q}_\ell[[t]].$$

$$\begin{tikzcd}
\mathrm{Spec}\, \overline{k(x)} \arrow[r] \arrow[rr, bend right, "\bar{x}"'] & \mathrm{Spec}\, k(x) \arrow[r, "x"] & X
\end{tikzcd}$$

$$Fr_x \in Gal(\overline{k(x)} \mid k(x))$$

Eg.

$\rho : \pi_1^{et}(X) \to GL(n, \mathbb{Q}_\ell) \rightsquigarrow \mathcal{F}$.

$x \in |X| \quad Fr_x \in \pi_1^{et}(\mathrm{Spec}\, k(x)) \to \pi_1^{et}(X)$

Thm 2. $L(x, \mathcal{F}, t) = \prod_i \det\left(1 - F^* t \mid H^i_c(X_{\bar{k}}, \mathcal{F}_{X_{\bar{k}}})\right)^{(-1)^{i+1}}$

Pf. Thm 1 $\Rightarrow$ Thm 2.

Apply $t \frac{\partial}{\partial t} \log$

Use $t \frac{\partial}{\partial t}\left(\log \det\left(1 - f t \mid V\right)\right) = \sum_{n \geq 1} \operatorname{tr}(f^n \mid V)\, t^n$

$$t \frac{\partial}{\partial t} \log L(X, \mathcal{F}, t) = \sum_{x \in |X|} \deg x \sum_{n \geq 1} \operatorname{Tr}\left(\operatorname{Fr}_{\bar{x}}^{-n} \mid \mathcal{F}_{\bar{x}}\right) t^{n \deg x}$$

$$= \sum_{d \geq 1} \sum_{\substack{\bar{x} \in X(\bar{\mathbb{F}}_q) \\ \deg x = d}} \sum_n \operatorname{Tr}\left(F_{\bar{x}}^{nd} \mid \mathcal{F}_{\bar{x}}\right) t^{nd}$$

$$= \sum_{m \geq 0} \sum_{x \in X(\mathbb{F}_{q^m})} \operatorname{Tr}\left(F_x^m \mid \mathcal{F}_{\bar{x}}\right) t^m$$

$$= \sum_i (-1)^i \sum_m \operatorname{Tr}\left(F^{*m} \mid H^i_c(X_{\bar{k}}, \mathcal{F}_{X_{\bar{k}}})\right) t^m \qquad \square$$

Ex X smooth curve $/k$, $\mathcal{F}$ irred. local system, $\operatorname{rk} \mathcal{F} > 1$.

$$\pi_1^{\text{ét}}(X) \to GL_n(\mathbb{Q}_\ell) = \operatorname{Aut}(V)$$

Then $L(X, \mathcal{F}, t)$ is a polynomial.

Pf. $$L(X, \mathcal{F}, t) = \frac{\det(1 - F^* t \mid H^1_c)}{\det(1 - F^* t \mid H^0_c)\det(1 - F^* t \mid H^2_c)}$$

$$H^0_c(X_{\bar{k}}, \mathcal{F}) = \begin{cases} V^{\pi_1^{\text{ét}}(X_{\bar{k}})}, & X \text{ is proper} \\ 0, & X \text{ is not proper} \end{cases}$$

$$H^2_c(X_{\bar k}, \mathcal{F}) = \left(H^0(X_{\bar k}, \mathcal{F}^*)\right)^* = V_{\pi_1^{\acute{e}t}(X_{\bar k})}$$

$$1 \longrightarrow \pi_1^{\acute{e}t}(X_{\bar k}) \longrightarrow \pi_1^{\acute{e}t}(X) \longrightarrow \hat{\mathbb{Z}} \longrightarrow 1$$

$$V^{\pi_1^{\acute{e}t}(X_{\bar k})} = 0 = V_{\pi_1^{\acute{e}t}(X_{\bar k})}$$

Sheaves $\rightsquigarrow$ functions

Weil sheaves on $X_{\bar k}$ $\rightsquigarrow$ $\mathrm{Fun}(X(k), \mathbb{Q}_\ell)$

$$\left(\mathcal{F},\ F^{-1}\mathcal{F} \xrightarrow[\sim]{F} \mathcal{F}\right) \rightsquigarrow \chi_{\mathcal{F}}(x) = \mathrm{Tr}(F_x \mid \mathcal{F}_{\bar x})$$

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} X_{\bar k} \arrow[r, "F"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & X_{\bar k} \arrow[d, "f"] \\ Y_{\bar k} \arrow[r, "F"'] & Y_{\bar k} \end{tikzcd}$$

$$\begin{tikzcd} \text{Weil sheaves on } X_{\bar k} \arrow[r, bend left, "R^q f_!"] \arrow[d] & \text{Weil sheaves on } Y_{\bar k} \arrow[l, bend left, "f^{-1}"] \arrow[d] \\ \mathrm{Fun}(X(k), \mathbb{Q}_\ell) \arrow[r, bend left, "\Sigma"] & \mathrm{Fun}(Y(k), \mathbb{Q}_\ell) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

$$X(k) \longrightarrow Y(k) \qquad (\Sigma \chi)(y) = \sum_{f(x)=y} \chi(y)$$

$$\chi_{f^{-1}\mathcal{F}} = f^*(\chi_{\mathcal{F}})$$

Th3. $\Sigma(\chi_{\mathcal{F}}) = \sum_i (-1)^i \chi_{R^i f_! \mathcal{F}}$

Thm 3 $\Leftrightarrow$ Thm 1   Use base change.

Pf of Thm 1.

Step 1 Reduction to curves

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact

If Thm 1 holds for 2 of $\mathcal{F}_i$'s, then it holds for the other.

$$\begin{tikzcd} U \arrow[r, hook, "j"] & X & Z = X - U \arrow[l, hook', "i"'] \end{tikzcd}$$

dense affine open

$0 \to j_! \mathcal{F}|_U \to \mathcal{F} \to i_* \mathcal{F}|_Z \to 0$

$\Rightarrow$ May assume that X is affine.

$$\begin{tikzcd} X \arrow[d, "f_0\ \text{finite}"] \\ \mathbb{A}^n \arrow[d, "f_1"] \\ \mathbb{A}^{n-1} \arrow[d] \\ \vdots \\ pt \end{tikzcd}$$

$$\begin{tikzcd} \text{Thm 1 for } X \\ \text{Thm 3 for each } f_i \arrow[u, Rightarrow] \\ \text{Thm 1 for fibers of } f_i \arrow[u, Rightarrow] \end{tikzcd}$$

Enough to prove Thm 1 for $\mathbb{A}^1$.

Lecture 37.

Goal: C_0 a smooth curve $/ k = \mathbb{F}_q$,

$C = C_0 \otimes_k \bar{k}$, $F : C \to C$ geometric Frobenius

$(\mathcal{F}, \; F^{-1}\mathcal{F} \xrightarrow[\sim]{F} \mathcal{F})$ Weil sheaf on C

Thm.
$$\sum_{x \in C_0(k) = C^F(k)} \mathrm{Tr}(F_{\bar{x}} \mid \mathcal{F}_{\bar{x}}) = \sum_i (-1)^i \, \mathrm{Tr}(F^* \mid H^i_c(C, \mathcal{F}))$$

Tool: Thm (Weil)

C smooth proj. curve $/ k = \bar{k}$, $f : C \to C$, $\Gamma_f, \Delta \subset C \times C$

$$(\Gamma_f, \Delta) = \sum_i (-1)^i \, \mathrm{Tr}(f^* \mid H^i(C, \mathbb{Q}_\ell^{(1)})) = 1 - \mathrm{Tr}(f^* \mid H^1(C; \mathbb{Q}_\ell)) + \deg f$$

$$H^1(C; \mathbb{Q}_\ell^{(1)}) = \mathbb{Q}_\ell \otimes \varprojlim_n \mathrm{Pic}(C)[\ell^n]$$

Ex. Let $\mathcal{F}_0$ be a local system on C_0 that corresponds to an Artin rep'n of $\pi_1^{\text{ét}}(C_0)$.

$$\begin{tikzcd} \pi_1^{\text{ét}}(C_0) \arrow[r] \arrow[d] & GL(V), \\ G \arrow[ur] & \end{tikzcd} \qquad V / \mathbb{Q}_\ell$$

finite quotient

Pf of Thm for such $\mathcal{F}_0$:

$$\begin{tikzcd} G \circlearrowright \tilde{C_0} \arrow[d, "h_0"] & \tilde{C} \arrow[l] \arrow[d, "h"] \arrow[dl, phantom, "\urcorner"] \circlearrowleft G \\ C_0 & C \arrow[l] \end{tikzcd}$$

finite étale corresponding to $\pi_1^{\text{ét}}(C_0) \to G$.

Lemma. $H^q(C, \mathcal{F}) = \Big(H^q(\underset{\underset{G}{\circlearrowleft}}{\tilde{C}}, \mathbb{Q}_\ell) \otimes \underset{\underset{G}{\circlearrowleft}}{V}\Big)^G$

Pf. $\big(H^q(\tilde{C}, \mathbb{Q}_\ell) \otimes V\big)^G = H^q(C, h_* \mathbb{Q}_\ell \otimes V)^G$

$= H^q(C, (h_* \mathbb{Q}_\ell \otimes V)^G)$ $\quad ((-)^G$ is exact$)$

$(h_* \mathbb{Q}_\ell \otimes V)^G = \mathcal{F}$: $\quad (\mathbb{Q}_\ell[G] \otimes V)^G \simeq \mathrm{Hom}_G(\mathbb{Q}_\ell[G], V) \simeq V$.

$$\sum_{q=0}^{2} (-1)^q \, \mathrm{Tr}\left(F^* \,\middle|\, (H^q(\tilde{C}, \mathbb{Q}_\ell) \otimes V)^G\right)$$

$$= \frac{1}{|G|} \sum_q (-1)^q \sum_{g \in G} \mathrm{Tr}(F^* g^{-1} \mid H^q(\tilde{C}, \mathbb{Q}_\ell)) \, \mathrm{Tr}(g \mid V)$$

$$W \circlearrowleft \overset{\overset{(F^*)}{\shortparallel}}{\mathbb{Z}} \times G \qquad \mathrm{Tr}(F^* \mid W^G) = \frac{1}{|G|} \sum_g \mathrm{Tr}(F^* g^{-1} \mid W)$$

$$= \frac{1}{|G|} \sum_{x \in C_0(k)} \sum_{g \in G} \left| \{ y \in \tilde{C} \text{ over } x \text{ s.t. } F(y) = g(y) \} \right| \cdot \mathrm{Tr}(g \mid V)$$

$$= \frac{1}{|G|} \sum_{x \in C_0(k)} \sum_{g \in G} \left| \{ g' \in G : g' \mathrm{Fr}_x^{-1} = g g' \} \right| \cdot \mathrm{Tr}(g \mid V) \qquad \Updownarrow \; g = g' \mathrm{Fr}_x^{-1} g'^{-1}$$

$$\begin{tikzcd} G \circlearrowright \tilde{C}_{0,x} \arrow[r] \arrow[d] & \tilde{C}_0 \arrow[d] \\ \mathrm{Spec}\, k \arrow[r, "x"] & C_0 \end{tikzcd}$$

$$= \frac{1}{|G|} \sum_{x \in C_0(k)} \sum_{g' \in G} \mathrm{Tr}(g' \mathrm{Fr}_x^{-1} g'^{-1} \mid V)$$

$$(\tilde{C}_{0,x}(k), F) \simeq (G, \mathrm{Fr}_x)$$

$$= \sum_{x \in C_0(k)} \mathrm{Tr}(\mathrm{Fr}_x^{-1} \mid V) = \sum_{x \in C_0(k)} \mathrm{Tr}(F_x \mid \mathcal{F}_{\bar{x}})$$

Want.

Thm. Let $\mathcal{F}_0$ be a constructible flat $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf on C_0, $(\mathcal{F}, F^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{F})$ the corresponding sheaf on C, then

$$\sum_q (-1)^q \, \mathrm{Tr}(F^* \mid H_c^q(C, \mathcal{F})) = \sum_{x \in C_0(k)} \mathrm{Tr}(F_x \mid \mathcal{F}_{\bar{x}}) \,. \quad ?$$

Doesn't make sense.

How to make sense of $\mathrm{Tr}(F^* \mid H^q_c(C, F))$?

Idea: Represent $H^q_c(C, F)$ as cohomology of a finite $\overset{R\Gamma(C,F) = P^\bullet}{\text{Complex}}$ of finite free $\mathbb{Z}/\ell^n\mathbb{Z}$-modules w/ F^* action, and define $\sum_q (-1)^q \mathrm{Tr}(F^* \mid H^q_c) = \sum_i (-1)^i \mathrm{Tr}(F^* \mid P^i)$

Traces.

Let Λ be a finite ring (eg. $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}[G]$)

M - left Λ-module, $f \circlearrowright M$.

$\mathrm{Tr}(f \mid M) = ?$

Def. $\Lambda^b = \Lambda / [\Lambda, \Lambda]$ ab gp generated by $ab - ba$, $a, b \in \Lambda$

For a free Λ-module $M = \Lambda^{\oplus m}$, $\mathrm{End}_\Lambda(\Lambda^{\oplus m}) = \mathrm{Mat}_m(\Lambda)$

$$\begin{bmatrix} a : \Lambda^m \to \Lambda^m \\ a(v) = vA \end{bmatrix} \longleftarrow\!\shortmid A$$

$$\mathrm{Tr} : \mathrm{End}_\Lambda(\Lambda^{\oplus m}) \simeq \mathrm{Mat}_m(\Lambda) \xrightarrow{tr} \Lambda \longrightarrow \Lambda^b$$

Ex. $\Lambda^{\oplus n} \xrightarrow{a} \Lambda^{\oplus m} \xrightarrow{b} \Lambda^{\oplus n}$, then $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

Let P be a finite projective Λ-module,

$\varphi : P \circlearrowleft$, choose $P \overset{a}{\hookrightarrow} \Lambda^m \xrightarrow{b} P$ s.t. $\Lambda^m = \mathrm{Im}(a) \oplus \ker(b)$, $b \circ a = Id_P$

$\mathrm{Tr}(\varphi \mid P) := \mathrm{Tr}(a \circ \varphi \circ b) \in \Lambda^b$

__Ex__. $\mathrm{Tr}(\varphi | P)$ is well-defined: $\mathrm{Tr}(\varphi \circ \psi) = \mathrm{Tr}(\psi \circ \varphi)$.

$D(\Lambda) = D(\mathrm{Mod}(\Lambda))$

$K(\Lambda)$

homotopy cat. of
chain cpxes of Λ-modules

$$AC(\Lambda) \longrightarrow K(\Lambda) \longrightarrow D(\Lambda)$$

__Fact__. If $P^\bullet$ is bounded from above complex of projective modules, then $\forall M$,

$$\mathrm{Mor}_{K(\Lambda)}(P^\bullet, M) = \mathrm{Mor}_{D(\Lambda)}(P^\bullet, M)$$

__Def__. $K_{perf}(\Lambda) \subset K(\Lambda)$ is the full subcat. formed by finite cpxes of finite proj. Λ-modules.

$$\begin{tikzcd}
K_{perf}(\Lambda) \arrow[r, hook] \arrow[dr, "\sim"'] & D(\Lambda) \arrow[d, phantom, "\cup"] \\
 & D_{perf}(\Lambda)
\end{tikzcd}$$

__Main def'n__. $M \in D_{perf}(\Lambda)$,

$\varphi \curvearrowright M$. Write $\underset{\underset{\varphi}{\circlearrowleft}}{M} = \underset{\underset{\varphi^\bullet}{\circlearrowleft}}{P^\bullet} \in K_{perf}(\Lambda)$

$$\mathrm{Tr}(\varphi | M) := \sum_i (-1)^i \,\mathrm{Tr}(\varphi^i | P^i) \in \Lambda^b .$$

Lecture 38

$$\begin{tikzcd} D(\Lambda) \\ \cup \end{tikzcd}$$

$K_{perf}(\Lambda) \simeq D_{perf}(\Lambda)$ – cpxes q.isom. to bdd cpxes of finite projective Λ-modules

Def. $P^\bullet \in D_{perf}(\Lambda)$, $f \in \mathrm{End}_{D(\Lambda)}(P^\bullet)$

where P^i's are finite projective.

Define $\mathrm{Tr}(f|P^\bullet) = \sum_i (-1)^i \,\mathrm{Tr}(f^i|P^i) \in \Lambda^b = \Lambda/[\Lambda,\Lambda]$

Lemma $\mathrm{Tr}(f)$ is well-defined.

Pf $f \circlearrowright P \circlearrowleft f'$, $f - f' = dh + hd$

Need $\sum_i (-1)^i \,\mathrm{Tr}(f^i|P^i) = \sum_i (-1)^i \,\mathrm{Tr}(f'^i|P^i)$

$$\mathrm{Tr}(dh) = \sum_i (-1)^i \,\mathrm{Tr}(P^i \xrightarrow{dh} P^i)$$

$$= \sum_i (-1)^{i-1}\,\mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) = -\mathrm{Tr}(hd)$$

$\Rightarrow \mathrm{Tr}(hd + dh) = 0.$

$$\begin{tikzcd} P^\bullet \arrow[loop below, "f"] \arrow[r, bend left, "\varphi"] & Q^\bullet \arrow[l, bend left, "\psi"] \end{tikzcd}$$

, $g = \varphi f \psi$

$\varphi \circ \psi \sim \mathrm{Id}$

$\psi \circ \varphi \sim \mathrm{Id}$

Need $\mathrm{Tr}(f) = \mathrm{Tr}(g)$

$$\mathrm{Tr}(g) = \mathrm{Tr}(\varphi \circ f \circ \psi) = \mathrm{Tr}(f \circ \psi \circ \varphi) = \mathrm{Tr}(f)$$

Def. $M \in D^-(\Lambda) \subset D(\Lambda)$

bounded from above cpx

M has a finite Tor-dimension iff $\exists n$ s.t. $\forall K \in \mathrm{Mod}^r(\Lambda)$,

$H^i\left(K \overset{L}{\underset{\Lambda}{\otimes}} M^\cdot\right) = 0$, $i < n$.

Ex. $\Lambda = \mathbb{Z}/\ell^2\mathbb{Z}$, $M = \mathbb{Z}/\ell$ has infinite Tor-dimension.

$$\cdots \xrightarrow{\ell} \mathbb{Z}/\ell^2 \xrightarrow{\ell} \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell$$

Lemma. Assume that Λ is noetherian. Then, for $M \in D^-(\Lambda)$, M is perfect

$\Leftrightarrow$ M has finite Tor-dimension and $H^i(M)$ are finite Λ-modules.

Pf $\Rightarrow$ obvious

$\Leftarrow$.

$$\cdots M^{-m} \to \cdots \to M^{m-1} \to M^m \cdots = M$$

Replace M by $\cdots \to P^0 \to \cdots \to P^{m-1} \to P^m$

where P^i are finite projective.

$$P^\cdot \overset{\text{q isom}}{\simeq} \tau_{\geq n} P = P^n/\mathrm{Im}\, d \to P^{n+1} \to \cdots \to P^m$$

Claim. $P^n/\mathrm{Im}\, d$ is flat.

$(P^{n+1} \to \cdots \to P^m) \to \tau_{\geq n} P^\cdot \to P^n/\mathrm{Im}\, d\, [-n]$ dist. tr.

$$K \overset{L}{\otimes} (P^{n+1} \to \cdots \to P^m) \to K \overset{L}{\otimes} \tau_{\geq n} P^\cdot \to K \overset{L}{\otimes} P^n/\mathrm{Im}\, d\, [n] \xrightarrow{+1} \cdots$$

<u>Claim</u>: $\Rightarrow \mathrm{Tor}_i^\Lambda (k, P^n/\mathrm{Im}\, d) = 0, \quad i > 0.$

$$\begin{tikzcd}
H^{n-1}(k \overset{L}{\otimes} \tau_{\geq n} P) \arrow[r] \arrow[d, phantom, "\parallel"] & H^{n-1}(K \overset{L}{\otimes} P^n/\mathrm{Im}\, d\, [-n]) \arrow[r] \arrow[d, phantom, "\parallel"] & H^n(k \otimes P^{n+1} \to \cdots \to k \otimes P^m) \arrow[d, phantom, "\parallel"] \\
0 & \mathrm{Tor}_1(k, P^n/\mathrm{Im}\, d) & 0
\end{tikzcd}$$

<u>Fact</u>. If Λ is Noetherian, V is finite Λ-module, V is projective $\Leftrightarrow$ V is flat.

$\Rightarrow P^n/\mathrm{Im}\, d$ is projective.

<u>Thm</u>. $\mathcal{F}$ constructible flat $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf on $C/k = \bar{k}$, then $\overset{\mathrm{char}\, k \neq \ell}{}$

$$R\Gamma(C, \mathcal{F}),\ \underset{\substack{\parallel \\ R\Gamma(\bar{C}, j_! \mathcal{F}),\ C \overset{j}{\hookrightarrow} \bar{C}}}{R\Gamma_c(C, \mathcal{F})} \in D_{\mathrm{perf}}(\mathbb{Z}/\ell^n\mathbb{Z}).$$

<u>Pf</u>. $R\Gamma(C, \mathcal{F}) \overset{L}{\underset{\Lambda}{\otimes}} k \simeq R\Gamma(C, \mathcal{F} \underset{\Lambda}{\otimes} k)$ universal coefficient theorem.

$H^i(R\Gamma(C, \mathcal{F} \underset{\Lambda}{\otimes} k)) = 0, \quad i < 0.$

<u>Thm</u> $C_0/k = \mathbb{F}_q$, $C \simeq C_0 \otimes \bar{k}$, $\mathcal{F}_0$ constr. flat $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf on C_0.

$(\mathcal{F}, F^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{F})$ on C,

$$\sum_{x \in C_0(k)} \mathrm{Tr}(F_x \mid \mathcal{F}_{\bar{x}}) = \mathrm{Tr}(F^* \mid R\Gamma_c(C, \mathcal{F})).$$

Prop. $0 \to \mathcal{F}_0^0 \to \mathcal{F}_0^1 \to \mathcal{F}_0^2 \to 0$ SES, $\mathcal{F}_0^i$ flat, then

$$\sum_{i=0}^{2} (-1)^i \operatorname{Tr}\left(F^* \mid R\Gamma_c(C, \mathcal{F}^i)\right) = 0.$$

Pf.

$$\begin{tikzcd}
R\Gamma_c(C, \mathcal{F}^0) \arrow[r] \arrow[loop below, "F^*"] & R\Gamma_c(C, \mathcal{F}^1) \arrow[r] \arrow[loop below, "F^*"] & R\Gamma_c(C, \mathcal{F}^2) \arrow[r, "+1"] \arrow[loop below, "F^*"] & {}
\end{tikzcd}$$

dist. Δ

Lemma

$$\begin{tikzcd}
M_0 \arrow[r, "\alpha"] \arrow[loop below, "\varphi_0"] & M_1 \arrow[r, "\beta"] \arrow[loop below, "\varphi_1"] & M_2 \arrow[loop below, "\varphi_2"]
\end{tikzcd}$$

dist. triangle in $D_{perf}(\Lambda)$

$$\alpha\varphi_0 = \varphi_1\alpha, \quad \beta\varphi_1 = \varphi_2\beta$$

$$\boxed{\text{FALSE!}}$$

$$\operatorname{Tr}(\varphi_1) = \operatorname{Tr}(\varphi_0) + \operatorname{Tr}(\varphi_2)$$

$\Lambda = \mathbb{Z}/\ell^2\mathbb{Z}$,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}/\ell^2 \arrow[loop above, "\varphi_0=0"] \arrow[d, "\alpha"', "\mathrm{Id}"] & M_0 \\
\mathbb{Z}/\ell^2 \arrow[loop left, "\ell"] \arrow[r, "\ell"] \arrow[d, "\mathrm{Id}"'] & \mathbb{Z}/\ell^2 \arrow[loop right, "0"] \arrow[d, "\beta"'] \arrow[dl, "h=\mathrm{Id}"'] & M_1 \\
\mathbb{Z}/\ell^2 \arrow[r] \arrow[loop below, "\varphi_2=0"] & 0 & M_2
\end{tikzcd}$$

$$\beta\varphi_1 - \varphi_2\beta = dh + hd$$

Def. $(M, F) \in \operatorname{Fil}^f(\operatorname{Mod}(\Lambda))$ is finite-proj, if $\operatorname{gr}^p M$ is finite proj. $\forall p$

$DF_{perf}(\Lambda) \subset DF(\Lambda)$.

Lemma. $M^\cdot \in DF_{perf}(\Lambda)$, $f : M^\cdot \circlearrowleft$, then $\operatorname{Tr} f = \sum_p \operatorname{Tr}(gr^p f)$.

$$\mathcal{F}^0 \subset \mathcal{F}^1 \in D^+F(Sh_{ét}(C))$$

$$F^* \sim R\Gamma_c(\mathcal{F}^0 \subset \mathcal{F}^1) \in DF_{perf}(\Lambda)$$

$$gr^0 R\Gamma_c(\mathcal{F}^0 \subset \mathcal{F}^1) = R\Gamma_c(\mathcal{F}^0)$$

$$gr^1 R\Gamma_c(\mathcal{F}^0 \subset \mathcal{F}^1) = R\Gamma_c(\mathcal{F}^2).$$

Lecture 39. Properties of traces

Claim. $\forall\, G < \infty$, Λ - comm. ring, (e.g. $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$)

P finite projective $\Lambda[G]$-module, $f \in \operatorname{End}_{\Lambda[G]}(P)$.

$\operatorname{Tr}_\Lambda(f|P)$ is divisible by $|G|$.

Λ comm. ring, $|G| < \infty$.

$$\begin{tikzcd}
\Lambda[G] \arrow[d] & \\
(\Lambda[G])^b = \Lambda[G]/[\Lambda[G], \Lambda[G]] \arrow[r, "\varphi"] & M \arrow[d, dash] \\
 & \Lambda\text{-module}
\end{tikzcd}$$

$$\{\varphi\} \leftrightarrow \text{class functions } G \to M$$

$$\varepsilon : \Lambda[G] \longrightarrow \Lambda, \quad \varepsilon\left(\sum \lambda_g g\right) = \lambda_e, \quad \underset{\text{unit}}{e \in G}$$

Lemma P finite proj. $\Lambda[G]$-module, $f \in \mathrm{End}_{\Lambda[G]}(P)$,

Then $\varepsilon\left(\mathrm{Tr}_{\Lambda[G]}(f|P)\right)|G| = \mathrm{Tr}_\Lambda(f|P)$.

Pf. $$\underset{\substack{\circlearrowleft \\ f}}{P} \oplus \underset{\substack{\circlearrowleft \\ 0}}{P'} = \Lambda[G]^{\oplus m},$$

$$\mathrm{Tr}_{\Lambda[G]}(f|P) = \mathrm{Tr}_{\Lambda[G]}\left((f,0)\,|\,\Lambda[G]^{\oplus m}\right)$$

This reduces the proof to $P = \Lambda[G]$.

f right multiplication by $f = \sum \lambda_g g$,

$$\mathrm{Tr}_\Lambda(f|\Lambda[G]) = \sum_g \lambda_g \,\mathrm{tr}_\Lambda(g|\Lambda[G]) = |G|\lambda_e = \varepsilon(f)|G|.$$

Def. $\mathrm{Tr}^G_\Lambda(f|P) := \varepsilon\left(\mathrm{Tr}_{\Lambda[G]}(f|P)\right)$.

Lemma. Let P be a finite projective $\Lambda[G]$-module, M - $\Lambda[G]$-module, finite proj./Λ.

Then $P \otimes_\Lambda M$ is a finite proj. $\Lambda[G]$-module.

Pf 1. $P \otimes_\Lambda M \simeq P \otimes_\Lambda M_{\text{triv}}$

Pf 2. $\mathrm{Hom}_{\Lambda[G]}(P \otimes_\Lambda M, N) = \mathrm{Hom}_{\Lambda[G]}(P, \mathrm{Hom}_\Lambda(M,N))$.

Lemma. $u \in \mathrm{End}_{\Lambda[G]}(P)$, $v \in \mathrm{End}_{\Lambda[G]}(M)$,

$$\mathrm{Tr}^G_\Lambda(u \otimes v \mid P \otimes_\Lambda M) = \mathrm{Tr}^G_\Lambda(u \mid P)\, \mathrm{Tr}_\Lambda(v \mid M).$$

Def A monoid extension Γ of $\mathbb{N}$ by G is

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] & \tilde{\Gamma} \arrow[r, "\pi"] & \mathbb{Z} \arrow[r] & 1 \\
 & & \Gamma \arrow[u, phantom, sloped, "\subset"] \arrow[r] & \mathbb{N} \arrow[u, phantom, sloped, "\subset"] &
\end{tikzcd}$$

group ext'n

s.t. $\Gamma = \pi^{-1}(\mathbb{N})$

$\gamma \in \Gamma$, $Z_\gamma := \{ g \in G : g\gamma = \gamma g \} \subset G$

Lemma P - $\Lambda[\Gamma]$-module, finite projective over $\Lambda[G]$,

$$\mathrm{Tr}_\Lambda(\gamma \mid P) = \# Z_\gamma \; \mathrm{Tr}^{Z_\gamma}_\Lambda(\gamma \mid P).$$

Lemma P - $\Lambda[\Gamma]$-module, finite projective over $\Lambda[G]$.

M - $\Lambda[\Gamma]$-module, finite projective $/\Lambda$

$$\mathrm{Tr}^{Z_\gamma}_\Lambda(\gamma \mid P \otimes_\Lambda M) = \mathrm{Tr}^{Z_\gamma}_\Lambda(\gamma \mid P)\, \mathrm{Tr}_\Lambda(\gamma \mid M).$$

Lemma. P - $\Lambda[\Gamma]$-module, finite projective $/\Lambda[G]$,

$$P_G = \Lambda \otimes_{\Lambda[G]} P. \qquad \mathrm{Tr}_\Lambda(\underset{\mathbb{N} = \Gamma/G}{\underset{\uparrow}{1}} \mid P_G) = \sum_{\Gamma \ni \gamma \mapsto 1 \in \mathbb{N}}{}' \; \mathrm{Tr}^{Z_\gamma}_\Lambda(\gamma \mid P)$$

sum over G-conj. classes

Pf. after multiplication by $|G|$.

$$|G| \, \mathrm{Tr}_\Lambda (1 | P_G) = \sum_{r \mapsto 1} \mathrm{Tr}_\Lambda (r | P_G)$$

$$= \mathrm{Tr}_\Lambda \Big(\sum_{r \mapsto 1} r \,\Big|\, P_G \Big) = \mathrm{Tr} \Big(\sum_{r \mapsto 1} r \,\Big|\, P \Big)$$

$$\begin{tikzcd} P^G \arrow[r, hook, "a"] & P \arrow[r, two heads, "b"] & P_G \arrow[ll, bend left, "c"] \end{tikzcd} \quad , \; c = \sum_{r \mapsto 1} r$$

$$\mathrm{Tr}(a \circ c \circ b) = \mathrm{Tr}(b \circ a \circ c)$$

$$\mathrm{Tr}_\Lambda \Big(\sum_{r \mapsto 1} r, \, P \Big) = \sum_{r \mapsto 1}{}' \frac{|G|}{|Z_r|} \mathrm{Tr}(r, P) = |G| \sum_{r \mapsto 1}{}' \mathrm{tr}_\Lambda^{Z_r} (r, P)$$

Ex. A finite étale group scheme $\underline{G} / \mathbb{F}_q$, $G = \underline{G}(\overline{\mathbb{F}}_q)$

$\circlearrowleft$

$\mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_q) \ni \mathrm{Fr}$.

$\rightsquigarrow \tilde{\Gamma} = G \rtimes \underset{\substack{\rotatebox{90}{$\in$} \\ 1}}{\mathbb{Z}}$

The conjugation action of 1 is Fr.

Conjugacy classes of $r \mapsto 1$:

$$g' \, g \, 1 \, g'^{-1} = g' g \underbrace{1 \, g'^{-1} \, 1^{-1}}_{\mathrm{Fr}(g'^{-1})} 1 = g' g \, \mathrm{Fr}(g'^{-1}) \, 1$$

$$g \sim g' g \, \mathrm{Fr}(g')^{-1}$$

$\wr\wr$

G - torsors over $\mathrm{Spec}\, \mathbb{F}_q$. $\quad H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_q), G)$

Assume that Λ admits a finite resolution by $\Lambda[\Gamma]$-modules

$$0 \to P_r \to \cdots \to P_1 \to P_0 \to \Lambda$$ where P_i are finite projective $/\Lambda[G]$.

$$H_i(P._G) = H_i(\Lambda \overset{L}{\underset{\Lambda[G]}{\otimes}} \Lambda) = H_i(G, \Lambda).$$

$$\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P.) = \frac{1}{|Z_\gamma|} \mathrm{Tr}(\gamma, P.)$$

$$= \frac{1}{|Z_\gamma|} \mathrm{Tr}(\gamma, \Lambda) = \frac{1}{|Z_\gamma|}$$

$$\mathrm{Tr}_\Lambda(1, P._G) = \left| B\underline{G}(\mathbb{F}_q) \right| = \sum_{\gamma \mapsto 1}' \frac{1}{|Z_\gamma|}$$

$$\|$$

$$\mathrm{Tr}_\Lambda(1, \Lambda \overset{L}{\underset{\Lambda[G]}{\otimes}} \Lambda) = \mathrm{Tr}(F^* | H.(B\underline{G}, \Lambda))$$

Lecture 40. G finite gp

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] \arrow[d, phantom, "\|"] & \tilde{\Gamma} \arrow[r] \arrow[d, phantom, "\cup"] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cup"] & 1 \\
1 \arrow[r] & G \arrow[r] & \Gamma \arrow[r] & \mathbb{N} = \mathbb{Z}_{\geq 0} \arrow[r] & 1
\end{tikzcd}$$

$\Lambda[\Gamma]$-module P, P finite proj. $/\Lambda[G]$

$$\mathrm{Tr}_\Lambda(1_{\mathbb{N}} | P_G) = \sum_{\substack{\gamma \mapsto 1_{\mathbb{N}} \\ \gamma \in \Gamma}}' \text{“}\frac{\mathrm{Tr}_\Lambda(\gamma | P)}{|Z_\gamma|}\text{”} = \sum_{\substack{\gamma \mapsto 1_{\mathbb{N}} \\ \gamma \in \Gamma}}' \mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma | P)$$

Thm. $C_0 / k = \mathbb{F}_q$, $\ell \neq \operatorname{char} k = p$ smooth curve. $C = C \otimes \bar{k}$. $\mathcal{F}_0$ constr. $\mathbb{Z}/\ell^n\mathbb{Z}$-flat sheaf on C_0, $\mathcal{F}$ its pullback to C, $F^{-1}\mathcal{F} \xrightarrow[\sim]{F} \mathcal{F}$, then

$$\begin{tikzcd} \sum_{x \in C_0(k)} \mathrm{Tr}(F_x \mid \mathcal{F}_{\bar{x}}) \arrow[d, equal] \arrow[r, phantom, "="] & \mathrm{Tr}(F^* \mid R\Gamma_c(C, \mathcal{F})) \arrow[d, equal] \\ T'(C_0, \mathcal{F}_0) & T''(C_0, \mathcal{F}_0). \end{tikzcd}$$

Pf. $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$.

Step 1. $Z_0 = C_0 - U_0 \xhookrightarrow{i} C_0 \xhookleftarrow{j} U_0$

$$T'(C_0, \mathcal{F}_0) = T'(U_0, \mathcal{F}_0|_{U_0}) + T'(C_0, i_* \mathcal{F}_0|_{Z_0})$$

$$T''(C_0, \mathcal{F}_0) = T''(U_0, \mathcal{F}_0|_{U_0}) + T''(C_0, i_* \mathcal{F}_0|_{Z_0})$$

$$0 \to j_! \mathcal{F}_0|_{U_0} \to \mathcal{F}_0 \to i_* \mathcal{F}_0|_{Z_0} \to 0$$

$$\begin{tikzcd} D(\mathrm{Mod}(\Lambda)) \ni & R\Gamma_c(U, \mathcal{F}|_U) \arrow[r] \arrow[loop below, "F^*"] & R\Gamma_c(C, \mathcal{F}) \arrow[r] \arrow[loop below, "F^*"] & R\Gamma_c(C, i_* \mathcal{F}|_Z) \arrow[r, "+1"] \arrow[loop below, "F^*"] & {} \\ D(\mathrm{Mod}(\Lambda[F^*])) \arrow[u] \arrow[ur, phantom, sloped, "\in"] & & & & \end{tikzcd}$$

Use additivity of traces.

Step 2. $\dim \operatorname{supp} \mathcal{F}_0 = 0$.

$$\operatorname{supp} \mathcal{F}_0 = \{x\} \subset |C_0|.$$

$$\mathcal{F}_{0\,\bar{x}} \circlearrowleft \mathrm{Gal}(\bar{k} \mid k(x))$$

$$\begin{tikzcd} \operatorname{Spec}(\bar{k} \otimes_k k(x)) \arrow[r] \arrow[d] & C \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r, hook] \arrow[d] & C_0 \\ \operatorname{Spec} k & \end{tikzcd}$$

$$R\Gamma_c(C, \mathcal{F}) = \mathrm{Ind}_{\mathrm{Gal}(\bar{k}|k(x))}^{\mathrm{Gal}(\bar{k}|k) \circlearrowleft F^{-1}} \mathcal{F}_{0,\bar{x}}$$

$$\circlearrowleft$$

$$\mathrm{Gal}(\bar{k}|k) \ni F^{-1}$$

$$\mathrm{Tr}(F^* \mid R\Gamma_c(\mathcal{F})) = \begin{cases} 0 & , k(x) \neq k \\ \mathrm{Tr}(F_x \mid \mathcal{F}_{\bar{x}}) & , k(x) = k . \end{cases}$$

<u>Step 3</u>. Situation: C_0 affine curve, $\mathcal{F}_0$ a local system, $C_0(k) = \phi$.

WTS $T'' = 0$

conn'd Galois cover ↗

$$\begin{tikzcd} S_0 \circlearrowleft G \arrow[d, "f"] \\ C_0 \end{tikzcd}$$

$$f^{-1}\mathcal{F}_0 = \underline{M} \quad , \overset{G}{\circlearrowleft} M = \Gamma(S_0, f^{-1}\mathcal{F}_0)$$

$$(f_* f^{-1} \mathcal{F}_0)_G \xrightarrow[\sim]{tr} \mathcal{F}_0$$

$$\|$$

$$(f_* \underline{\Lambda} \otimes_{\Lambda} M)_G$$

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, "f"] & \overset{G \circlearrowleft}{\bar{S}_0} \arrow[d] \\ C_0 \arrow[r, hook] & \bar{C}_0 \end{tikzcd}$$

$\bar{C}_0$ smooth proj.

$$R\Gamma_c(C, \mathcal{F}) = \Lambda \overset{L}{\underset{\Lambda[G]}{\otimes}} \left(R\Gamma_c(C, f_*\Lambda) \overset{L}{\underset{\Lambda}{\otimes}} M \right)$$

$$= \Lambda \overset{L}{\underset{\Lambda[G]}{\otimes}} \left(\underbrace{R\Gamma_c(S, \Lambda) \overset{L}{\underset{\Lambda}{\otimes}} M}_{\overset{!!}{P^\cdot}} \right)$$

$$\overset{F^*}{\circlearrowleft}$$

$$P^\cdot \in D_{perf}(\mathrm{Mod}(\Lambda[G]))$$

$P^\bullet$ has finite Tor-dimension.

$$H^i(P^\bullet \overset{L}{\underset{\Lambda(G)}{\otimes}} N) = 0, \quad i<0.$$

$\Gamma = G \times \mathbb{N}$. $\quad$ Γ acts on $P^\bullet$

$1_{\mathbb{N}}$ acts by F^*.

$$\mathrm{Tr}_\Lambda(F^* \mid R\Gamma_c(C,\mathcal{F})) = \sum_{\gamma \mapsto 1}{}' \, \mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma \mid R\Gamma_c(S,\Lambda) \overset{L}{\underset{\Lambda}{\otimes}} M)$$

$$= \sum_{g \in G/\mathrm{Ad}} \mathrm{Tr}_\Lambda^{Z_g}((g^{-1}F)^* \mid R\Gamma_c(S,\Lambda) \underset{\Lambda}{\otimes} M)$$

$$= \sum_{g \in G/\mathrm{Ad}} \mathrm{Tr}_\Lambda^{Z_g}((g^{-1}F)^* \mid R\Gamma_c(S,\Lambda))\, \mathrm{Tr}_\Lambda(g^{-1} \mid M)$$

ETS that $\forall g \in G$, $\mathrm{Tr}_\Lambda^{Z_g}((g^{-1}F)^* \mid R\Gamma_c(S,\Lambda)) = 0$.

Replacing $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ by $\Lambda' = \mathbb{Z}/\ell^N\mathbb{Z}$, $N >> n$.

We reduce $|Z_g|\, \mathrm{Tr}^{Z_g}((g^{-1}F)^* \mid R\Gamma_c(S,\Lambda)) = 0$.

$$\mathrm{Tr}_\Lambda((g^{-1}F)^* \mid R\Gamma_c(S,\Lambda)) = \mathrm{Tr}_\Lambda((g^{-1}F)^* \mid R\Gamma_c(\bar{S},\Lambda)) - \mathrm{Tr}_\Lambda((g^{-1}F)^* \mid R\Gamma_c(\bar{S}, i_*\Lambda_{\bar{S}-S}))$$

$$\begin{tikzcd} S \arrow[loop above, "g^{-1}F"] \arrow[r, hook] \arrow[d] & \bar{S} \arrow[d] & \bar{S}-S \arrow[l, hook'] \arrow[d] \\ C \arrow[loop below, "F"] \arrow[r] & \bar{C} & \bar{C}-C \arrow[l] \end{tikzcd}$$

$$= |\bar{S}^{g^{-1}F}(\bar{k})| - |(\bar{S}-S)^{g^{-1}F}(\bar{k})| = 0$$

$X_0 / k = \mathbb{F}_q$

$$Z(X_0, t) = \prod_{x \in |X_0|} \frac{1}{1 - t^{\deg x}} \in \mathbb{Z}[[t]]$$

$$Z(X_0, t) = \prod_{i=0}^{2\dim X_0} \overset{P_i(t)}{\overset{\shortparallel}{\det\left(1 - F^* t \mid H^i_c(X, \mathbb{Q}_\ell)\right)}}^{(-1)^{i+1}}$$

$$= \frac{P_1(t)\; P_3(t) \cdots}{P_0(t)\; P_2(t) \cdots}$$

Want: If X_0 is smooth projective,

$P_i(t) \in \mathbb{Z}[t]$.

$P_i(\alpha) = 0 \Rightarrow |c(\alpha)| = q^{-i/2}$, $\forall\, c : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

$$\begin{tikzcd} \mathfrak{X}_0 \arrow[d, "f"] \\ S \end{tikzcd}$$

$$(R^i f_! \mathbb{Q}_\ell)_{\bar{s}} = H^i_c(\mathfrak{X}_s, \mathbb{Q}_\ell)$$

Lecture 41 Smooth base change Thm.

R strictly henselian ring (e.g. complete local ring, $R/m = k = \bar{k}$)

$\bar{k} = k = R/m$, $K = \mathrm{Frac}(R)$

$$\begin{tikzcd} X_k \arrow[r, hook] \arrow[d] & X \arrow[d, "\pi"] & X_{\bar{K}} \arrow[l] \arrow[d] \\ \mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}\, R & \mathrm{Spec}\, \bar{K} \arrow[l] \end{tikzcd}$$

Thm If π is smooth, $\mathrm{char}\, k \nmid n$, $H^i_{\acute{e}t}(X, \mathbb{Z}/n) \xrightarrow{\sim} H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}/n)$

Cor. Assume in addition, π is proper,

$$H^i(X_k, \mathbb{Z}/n) \xleftarrow{\sim} H^i_{et}(X, \mathbb{Z}/n) \xrightarrow{\sim} H^i_{et}(X_{\bar{k}}, \mathbb{Z}/n)$$

Cor. If $X \xrightarrow{\pi} S$ smooth, proper, $n \in \mathcal{O}(S)^*$, then

$R^i \pi_* (\mathbb{Z}/n)$ are local systems.

Pf. $\dim S = 1$. $s \in S$,

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{S,s} \arrow[r] & S \\
\operatorname{Spec} \mathcal{O}^{sh}_{S,s} \arrow[u] \arrow[ur, "v"'] \arrow[d, equal] & \\
R &
\end{tikzcd}$$

Want $v^* R^i \pi_* \mathbb{Z}/n$ is constant.

$$\begin{tikzcd}
X_R \arrow[r] \arrow[d, "\pi_R"'] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] & S
\end{tikzcd}$$

$\quad\quad\quad\ \ \|$

$R^i \pi_{R*} \mathbb{Z}/n$

$\operatorname{Spec} R = S$, $\pi_R = \pi$, $\mathcal{F} = R^i \pi_* \mathbb{Z}/n$.

WTS $\underline{\mathcal{F}_k} = \underline{H^0(S, \mathcal{F})} \xrightarrow{\sim} \mathcal{F}$.

Enough to check that this is $\simeq$ on fibers at $\operatorname{Spec} \bar{k} \to \operatorname{Spec} R$,

$$\mathcal{F}_k \xleftarrow{\sim} H^i(X, \mathbb{Z}/n) \underset{\text{smooth base change}}{\xrightarrow{\sim}} \mathcal{F}_{\bar{k}} = H^i(X_{\bar{k}}, \mathbb{Z}/n)$$

k field, char $k \neq \ell$,

X/k

$$\begin{tikzcd}
 & H^2_{\acute{e}t}(X, \mathbb{Z}_\ell(1)) \arrow[d] \\
c_1 : \mathrm{Pic}(X) \arrow[ur] \arrow[r] & H^2_{\acute{e}t}(X, \mathbb{Q}_\ell(1)).
\end{tikzcd}$$

$$1 \to \mu_{\ell^n} \to \mathcal{O}_X^* \xrightarrow{\ell^n} \mathcal{O}_X^* \to 1$$

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \to H^2(X, \mu_{\ell^n})$$

$$x = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n_{\bar{k}}, \mathbb{Q}_\ell(1)) = H^2(\mathbb{P}^n_{\bar{k}}, \mathbb{Q}_\ell)(1)$$

$$\rightsquigarrow \mathbb{Q}_\ell(-1) \to H^*(\mathbb{P}_{\bar{k}}, \mathbb{Q}_\ell) \rightsquigarrow \oplus \mathbb{Q}_\ell(-i) \to H^*(\mathbb{P}_{\bar{k}}, \mathbb{Q}_\ell)$$

$$H^*(\mathbb{P}^n_{\bar{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(-1) \oplus \cdots \oplus \mathbb{Q}_\ell(-n).$$

OK if char $k = 0$.

Char $k = p$, may assume $k = \mathbb{F}_p$.

$R = W(\bar{\mathbb{F}}_p)$.

$$\begin{tikzcd}
\mathbb{P}^n_R \arrow[d, "\pi"'] \\
\mathrm{Spec}\, R
\end{tikzcd}
\qquad \mathbb{Q}_\ell(-1) \to R^2\pi_* \mathbb{Q}_\ell$$

$$\mathbb{Q}_\ell(-1)[-2] \to R\pi_* \mathbb{Q}_\ell \quad \text{on } \mathrm{Spec}\, R$$

$$\Updownarrow \text{ adjunction}$$

$$\mathbb{Q}_\ell(-1)[-2] \to \mathbb{Q}_\ell \quad \text{on } \mathbb{P}^n_R$$

$$\Uparrow$$

$$c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n_R, \mathbb{Q}_\ell(1))$$

b.c. true over $\underline{\overline{W(\bar{\mathbb{F}}_p)[\tfrac{1}{p}]}}$

$$\underset{\text{local system}}{\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell(-n)} \xrightarrow[\sim]{\perp} \oplus \underbrace{R^* \pi_* \mathbb{Q}_\ell}_{\text{local system}}$$

Cor

Weak Lefschetz.

Smooth hypersurface. $\overset{\{f=0\}}{\overset{\shortparallel}{X}} \hookrightarrow \mathbb{P}^{d+1}_k$

$$H^i(\mathbb{P}^{d+1}_{\bar k}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(X_{\bar k}, \mathbb{Q}_\ell), \quad i < \dim X.$$

Use that X is liftable over $W(\bar k)$ and the result over $\mathbb{C}$.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}^{d+1}_{W(\bar k)} \arrow[dl, "\pi'"] \\ \operatorname{Spec} W(\bar k) & \end{tikzcd} \qquad R^i\pi'_* \mathbb{Q}_\ell \xrightarrow{\sim} R^i\pi_* \mathbb{Q}_\ell$$

Cor $X \hookrightarrow \mathbb{P}^{d+1}$ hypersurface

$$H^i(X_{\bar k}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-\frac{i}{2}), & i \text{ even } \neq d. \\ 0 & , i \text{ odd}, i \neq d. \end{cases}$$

Pf $H^{2d}(X_{\bar k}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-d)$

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\text{finite morphism}"] \arrow[d, "\pi"'] & \mathbb{P}^d_{W(\bar k)} \arrow[dl, "\pi'"] \\ \operatorname{Spec} W(\bar k) & \end{tikzcd}$$

induces an iso.

$$\begin{matrix} R^{2d}\pi'_* \mathbb{Q}_\ell \xrightarrow{\sim} R^{2d}\pi_* \mathbb{Q}_\ell \\ \shortparallel \\ \mathbb{Q}_\ell(-d) \end{matrix}$$

Poincaré duality.

$$H^i(X_{\bar k}, \mathbb{Q}_\ell) \otimes H^{2d-i}(X_{\bar k}, \mathbb{Q}_\ell) \underset{\text{perfect pairing}}{\longrightarrow} H^{2d}(X_{\bar k}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-d).$$

Thm. $X/\mathbb{F}_q$ smooth, projective, then

ⓐ $\det\left(\mathrm{Id} - tF^* \mid H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right) \in \mathbb{Q}[t]$.

ⓑ Eigenvalues of $F^* \curvearrowright H^i$ have absolute value $q^{i/2}$.

(for all $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$)

Proof for hypersurfaces:

Key Lemma $\mathcal{U} \subset \mathbb{P}^1_{\mathbb{F}_q}$ open

$\mathcal{F}$ - $\bar{\mathbb{Q}}_\ell$ - local system $(\rho : \pi_1^{et}(\mathcal{U}) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell))$

Assume that $\forall x \in |\mathcal{U}|$,

Coeff. of $P_x = \det(1 - tF_x \mid \mathcal{F}_{\bar{x}})$ are real $(\forall \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C})$.

Assume that $\exists x_0 \in |\mathcal{U}|$ s.t. the eigenvalues have $|\cdot| = 1$, then the same is true for all x.

$X \hookrightarrow \mathbb{P}^{d+1}_{\mathbb{F}_q}$ $\{f = 0\} = X$

$\deg f = n$

One checks the thm for $\underbrace{x_0^n + x_1^n + \cdots + x_{d+1}^n}_{g} = 0$

$$Z(X,t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg x}} = \frac{P_1(t)\, P_3(t) \cdots}{P_0(t)\, P_2(t) \cdots}, \quad P_i(t) = \det(1 - tF^* \mid H^i_c(X, \mathbb{Q}_\ell))$$

Consider $\mathcal{X} \subset \mathbb{P}^{d+1} \times \mathbb{A}^1$

$$\begin{tikzcd} \mathcal{X}_U \arrow[r, phantom, "\subset"] \arrow[d, "\pi", "\text{smooth}"'] & \mathcal{X} \arrow[d] \\ U \arrow[r, phantom, "\subset"] & \mathbb{A}^1 \end{tikzcd}$$

$\|$

$\{tf+(1-t)g=0\}$

Apply the key Lemma to $R^d \pi_* \overline{\mathbb{Q}}_\ell \left(\frac{d}{2}\right) = \mathcal{F}$.

- - - - - - - - - - - - - - - - - - - -

Lecture 42. **Key Lemma** $U \subsetneq \mathbb{P}^1_{\mathbb{F}_q}$, $\mathcal{F}$ - $\overline{\mathbb{Q}}_\ell$ - local system

Assume that

① $\forall\, x \in |U|$, $P_x = \det(1 - tF_x | \mathcal{F}_x)$ has real coefficients.

② $\exists\, x_0 \in |U|$, s.t. roots of P_{x_0} have $|\cdot| = 1$. Then $\forall\, x \in |U|$, roots of P_x have $|\cdot| = 1$.

$\boxed{1}$ $\overline{\mathbb{Q}}_\ell$ - local sys. is
$$\begin{tikzcd} & GL_n(K) \arrow[d] \\ \rho : \pi_1^{et}(U) \arrow[r] \arrow[ru] & GL(V) \end{tikzcd}$$
$\mathbb{Q}_\ell \overset{\text{finite}}{\subset} K \subset \overline{\mathbb{Q}}_\ell$

$\boxed{2}$
$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{et}(U_{\overline{\mathbb{F}}_q}) \arrow[r] & \pi_1^{et}(U) \arrow[r] & \pi_1^{et}(\operatorname{Spec} \mathbb{F}_q) \arrow[r] \arrow[d, equal] & 1 \\ & \langle Fr_{k(x)} \rangle = \pi_1^{et}(\operatorname{Spec} k(x)) \arrow[ru] \arrow[rr] \arrow[rrd, bend right, mapsto] & & \langle Fr_q \rangle & \\ & & & Fr_q^{\deg x} \arrow[u, phantom, "\in" rotate=90] & \end{tikzcd}$$

$F_q = Fr_q^{-1}$

$x \in U$, $\operatorname{Spec} k(x) \longrightarrow U$

$F_x = Fr_{k(x)}^{-1}$

$\boxed{3}$ We apply Lemma to

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \\ U \end{tikzcd}$$
- smooth proper family of hypersurfaces of dim d

$\mathcal{F} = R^d \pi_* \overline{\mathbb{Q}}_\ell \left(\frac{d}{2}\right)$

Pf. <u>Step 1</u> (Rankin trick) $\rho : \pi_1^{ét}(U) \to GL(V)$

$\forall\ k > 0$, Eigenvalues of F_q on $\left(V^{\otimes k}\right)_{\pi_1^{ét}(U_{\bar{\mathbb{F}}_q})}$ have $|\cdot| = 1$.

$$\begin{tikzcd}
V^{\otimes 2k} \arrow[r, two heads] & \left(V^{\otimes 2k}\right)_{\pi_1(U_{\bar{\mathbb{F}}_q})} \\
\circlearrowleft & \circlearrowleft \\
F_{x_0} \arrow[r, squiggly] & F_q^{\deg x}
\end{tikzcd}$$

$\Rightarrow$ eigenvalues of $F_q^{\deg x}$ have $|\cdot| = 1$.

<u>Step 2</u>.

$$Z\left(U, \mathcal{F}^{\otimes 2k}, t\right) = \prod_{x \in |U|} \det\left(1 - F_x\, t^{\deg x} \,\middle|\, \mathcal{F}_{\bar{x}}^{\otimes 2k}\right)^{-1}$$

$$\shortparallel$$

$$\frac{\det\left(1 - F^* t \,\middle|\, H^1_c(U_{\bar{\mathbb{F}}_q}, \mathcal{F}^{\otimes 2k})\right)}{\det\left(1 - F^* t \,\middle|\, H^2_c(U_{\bar{\mathbb{F}}_q}, \mathcal{F}^{\otimes 2k})\right)}$$

(H^0_c trivial)

Poincaré duality. $H^2_c\left(U_{\bar{\mathbb{F}}_q}, \mathcal{F}^{\otimes 2k}\right) = \left(H^0\left(U_{\bar{\mathbb{F}}_q}, (\mathcal{F}^{\otimes 2k})^\vee\right)\right)^\vee(-1) = \left(V^{\otimes 2k}\right)_{\pi_1^{ét}(U_{\bar{\mathbb{F}}_q})}(-1)$

$\circlearrowleft$

F_q eigenvalues have $|\cdot| = q$

$\Rightarrow$ poles of $Z(U, \mathcal{F}^{\otimes 2k}, t)$ are all on $|t| = \frac{1}{q}$.

$\det\left(1 - F_x t^{\deg x} \,\middle|\, \mathcal{F}_{\bar{x}}^{\otimes 2k}\right)$ is a power series w/ positive coeff.s.

As $Z(U, \mathcal{F}, t)$ converges on $|t| < \frac{1}{q}$,

$\det\left(1 - F_x t^{\deg x} \,\middle|\, \mathcal{F}_{\bar{x}}^{\otimes 2k}\right)^{-1}$ converges on the same disk.

$\rightarrow$ Eigenvalues of $F_x \circlearrowleft \mathcal{F}_{\bar{x}}$ have $|\cdot| \leq \sqrt[2k]{q^{\deg x}}$.

Take $k \to \infty$ $\Rightarrow |\cdot| \leq 1$

Step 3. Consider $\mathcal{L} = \det \mathcal{F} = \Lambda^{\mathrm{rk}\,\mathcal{F}} \mathcal{F}$. Want $F_x \curvearrowright \mathcal{L}$ is multiplication by a number of $|\cdot| = 1$.

Lemma. Let $\mathcal{L}$ be a local system on $\mathcal{U}$ of rank 1, then $\exists\, m > 0$ s.t. the action of $\pi_1^{\text{ét}}(\mathcal{U}_{\overline{\mathbb{F}}_q})$ on $\mathcal{L}_x^{\otimes m}$ is trivial for all x.

Pf.
$$\begin{tikzcd} & & \pi_1^{\text{ét}}(\mathcal{U}_{\overline{\mathbb{F}}_q}) \arrow[d] & \mu_\infty(K) - \text{finite gp} \arrow[d] & \\ \mathcal{L} \arrow[r, leftrightsquigarrow] & \rho : & \pi_1^{\text{ét}}(\mathcal{U}) \arrow[r] \arrow[d] & \mathcal{O}_K^* \arrow[r, "x \mapsto \log x^N"] & \mathcal{O}_K \\ & & \hat{\mathbb{Z}} & \mathbb{Q}_\ell \subset K \subset \overline{\mathbb{Q}_\ell} & \end{tikzcd}$$

We'll prove that $\mathrm{Hom}(\pi_1^{\text{ét}}(\mathcal{U}_{\overline{\mathbb{F}}_q}), \mathcal{O}_K)^{\mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q)} = 0$.

$$\Leftrightarrow \quad H^1_{\text{ét}}(\mathcal{U}_{\overline{\mathbb{F}}_q}, \mathcal{O}_K)^{\hat{\mathbb{Z}}} = 0.$$

$\mathcal{U} = \mathbb{P}^1 - \{x_0, \cdots, x_n\}$. may assume $x_i \in \mathbb{P}^1(\mathbb{F}_{q'})$

$H^1_{\text{ét}}(\mathcal{U}_{\overline{\mathbb{F}}_q}, \mathcal{O}_K) = \mathcal{O}_K(-1)^{\oplus n}$ as a module over $\mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_{q'})$

$$\Rightarrow \quad H^1_{\text{ét}}(\mathcal{U}_{\overline{\mathbb{F}}_q}, \mathcal{O}_K)^{\hat{\mathbb{Z}}} = 0.$$

Poincaré duality. $\mathcal{F}$ - $\mathbb{Z}/n\mathbb{Z}$- local system on smooth curve $\mathcal{U}/k$

$(n, \mathrm{char}\, k) = 1$. $\qquad \mathcal{U} \overset{j}{\hookrightarrow} C$ - smooth proper

$$\begin{array}{ccccc} H^2_c(\mathcal{U}_{\bar{k}}, \mathcal{F}) \otimes H^0(\mathcal{U}_{\bar{k}}, \mathcal{F}^*) & \longrightarrow & \mathbb{Z}/n\mathbb{Z}(-1). & & \\ \parallel \qquad\qquad \parallel & & & & \\ \mathrm{Ext}^2_{Sh(C)}(\mathbb{Z}/n\mathbb{Z}, j_!\mathcal{F}) \otimes \mathrm{Ext}^0_{Sh(C)}(j_!\mathcal{F}, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^2(C, \mathbb{Z}/n\mathbb{Z}) & = & \mathbb{Z}/n\mathbb{Z}(-1) \end{array}$$

OK if $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$

Pick an étale cover $U' \xrightarrow{\pi} U$ s.t. $\pi^*\mathcal{F}$ is constant.

$$0 \to \mathcal{F}' \to \pi_*\pi^*\mathcal{F} \xrightarrow{tr} \mathcal{F} \to 0$$

$$H_c^*(U_{\bar{k}}, \pi_*\pi^*\mathcal{F}) = H_c^*(U', \pi^*\mathcal{F})$$

$$\begin{tikzcd}
0 \arrow[r] & H_c^2(U_{\bar{k}}, \mathcal{F})^\vee \arrow[r] \arrow[d, "\textcircled{a}"', "\wr"] & H_c^2(U_{\bar{k}}, \pi_*\pi^*\mathcal{F})^\vee \arrow[r] \arrow[d, "\wr"', "\textcircled{b}"] & H_c^2(U_{\bar{k}}, \mathcal{F}')^\vee \arrow[d, "\textcircled{c}"'] \\
0 \arrow[r] & H^0(U_{\bar{k}}, \mathcal{F}) \arrow[r] & H^0(U_{\bar{k}}, \pi_*\pi^*\mathcal{F}) \arrow[r] & H^0(U_{\bar{k}}, \mathcal{F}')
\end{tikzcd}$$

By Five Lemma, ⓐ is injective. Now apply the same argument to $\mathcal{F}'$, we see that ⓒ is injective $\Rightarrow$ ⓐ is surjective by five Lemma.

Lecture 43 **Thm 1.** $f: X \to S$ smooth proper, $n \in \mathbb{Z}$, $n \in \mathcal{O}(S)^*$,

then $R^q f_* \mathbb{Z}/n$: local system.

Application: $U \subset \operatorname{Spec}\mathbb{Z}$, open, $(\ell) \notin U$,

$f: X \to U$ smooth proper, then $\forall (p) \in U$,

$$\dim_{\mathbb{Q}_\ell} H^q(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell} H^q(X_{\mathbb{C}}, \mathbb{Q}_\ell) \simeq \dim_{\mathbb{Q}_\ell} H^q(X(\mathbb{C}), \mathbb{Q}_\ell).$$

Thm 2. $S = \operatorname{Spec} R$, R strictly henselian domain, (e.g. $R = \bar{k}[[t]]$, $R = W(\bar{k})$).

$$\bar{\eta} \longrightarrow S \longleftarrow s$$

$f : X \to S$ smooth

then
$$\begin{tikzcd} H^q(X, \mathbb{Z}/n) \arrow[r, "\sim"] \arrow[d] & H^q(X_{\bar\eta}, \mathbb{Z}/n). \\ H^q(X_s, \mathbb{Z}/n) & \end{tikzcd}$$

$X_{\bar\eta} \to X_\eta \to X$

Picture over $\mathbb{C}$

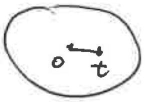
$$\begin{tikzcd} & D^\circ \arrow[d, phantom, "\cap" rotate=90] & \\ f : X \arrow[r] & D \subset \mathbb{C} \\ X_0 \arrow[u, hook] \arrow[r] & 0 \arrow[u, phantom, "\in" rotate=90] \\ a \arrow[u, phantom, "\in" rotate=45] & \end{tikzcd}$$

$D = \{|z| < 1\}$

$f^{-1}(D^\circ) \to D^\circ$

fiber bundle

$$\begin{tikzcd} f^{-1}((0,t]) \arrow[r, "j"] \arrow[d, phantom, "\simeq" rotate=90] & f^{-1}([0,t]) & X_0 \arrow[l] \\ X_t & & \end{tikzcd}$$



Fact. If f is smooth, j is homotopy equiv.

$\rightsquigarrow H^*(X_t) \xleftarrow{\sim} H^*(f^{-1}([0,t])) \to H^*(X_0)$

In general,
$$\begin{tikzcd} H^q(f^{-1}([0,t]), Rj_*\mathbb{Z}) \arrow[d, "\wr"', "f \text{ proper}"] & = & H^q(f^{-1}((0,t]), \mathbb{Z}) \arrow[d, phantom, "\simeq" rotate=90] \\ H^q(X_0, \underset{\text{complex of vanishing cycles}}{i^{-1}Rj_*\mathbb{Z}}) & & H^q(X_\varepsilon) \end{tikzcd}$$

$\mathbb{Z} \xrightarrow{can} i^{-1}Rj_*\mathbb{Z} \longrightarrow \text{cone}(can) \xrightarrow{+1}$

How to compute $\left(i^{-1} R j_* \mathbb{Z}\right)_a$?

$$= \left(R j_* \mathbb{Z}\right)_a$$

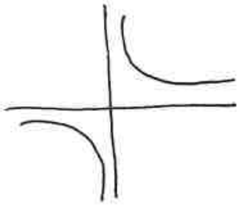
$$\left(R^q j_* \mathbb{Z}\right)_a = H^q\left(X \cap B_\varepsilon \cap f^{-1}((0, \delta t]) \,;\, \mathbb{Z}\right) \qquad \delta \ll \varepsilon$$

Ex.
$$\begin{tikzcd} D \arrow[d, "f"] \\ D \end{tikzcd} \quad f(z) = z^2$$

$$\left(i^{-1} R^q j_* \mathbb{Z}\right)_0 = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & q = 0 \\ 0, & q > 0 \end{cases}$$

Ex.
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathbb{C}^2 \arrow[d] \\ D \arrow[r, hook] & \mathbb{C} \end{tikzcd}$$

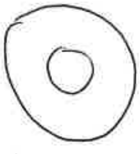
$$f(x_1, x_2) = x_1 x_2$$



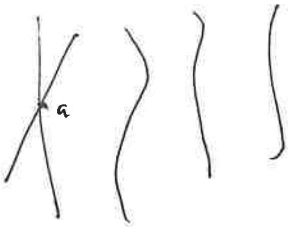
$$E = X \cap B_\varepsilon \cap f^{-1}((0, \delta t]), \mathbb{Z})$$

$$H^q(E, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{cases} |x_1| < \varepsilon, \ |x_2| < \varepsilon \\ 0 < x_1 x_2 < \delta \end{cases}$$



$x_1 x_2 = t$



$$\mathbb{Z} \to i^{-1} R j_* \mathbb{Z} \to \delta_a[-1] \to$$

Ex.
$$\begin{tikzcd} X \arrow[d] \\ \operatorname{Spec} k[\![t]\!] \end{tikzcd} \quad k = \bar{k}$$

What is vanishing cycle ?

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X_\eta \arrow[l] \arrow[d] & X_{\bar\eta} \arrow[l] \arrow[d] \arrow[lll, bend right, "j"'] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[[t]] & \operatorname{Spec} k((t)) \arrow[l] & \operatorname{Spec} \overline{k((t))} \arrow[l]
\end{tikzcd}$$

$\circlearrowleft$ $\mathrm{Gal}(\overline{k((t))} \mid k((t)))$

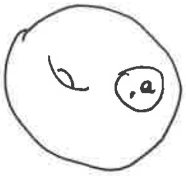
$$i^{-1} R j_* \mathbb{Z}/n\mathbb{Z} \circlearrowleft^{\mathrm{Gal}(\overline{k((t))}/k((t)))}$$

<u>Goal</u>: if f is smooth, $\mathbb{Z}/n \xrightarrow{\sim} i^{-1} R j_* \mathbb{Z}/n$

<u>Thm</u>. C smooth curve over $\mathbb{C}$

$a \in C$, $C^0 = C - a$

$$\begin{tikzcd}
X \arrow[d, "f"'] \\
C^0
\end{tikzcd}$$
smooth, proper, $R^q f_* \mathbb{C}$ - local sys. over C^0



$\boxed{\text{The monodromy around } a \text{ is quasi-unipotent.}}$

(eigenvalues = roots of unity)

<u>Pf (Sketch)</u>

$$\begin{tikzcd}
\bar{X} \times_C \mathcal{H} \arrow[r] \arrow[d] & \bar{X} \times_C D \arrow[r, hook] \arrow[d, "\text{Smooth}"'] & \bar{X} \arrow[d, "\bar{f}"] & X \arrow[l] \arrow[d, "f"] \\
\mathcal{H} \arrow[r, "\exp"'] & D \arrow[r, hook] & C & C^0 \arrow[l]
\end{tikzcd}$$

locally $f(x_1, \cdots, x_n) = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$

$$\underset{\mathbb{Z}}{\underset{\circlearrowleft}{H^*(X_0, i^{-1} R j_* \mathbb{C})}} = H^*(X_t, \mathbb{C})$$

Lecture 44

$\bar{x} : \operatorname{Spec} k(\bar{x}) \longrightarrow X$

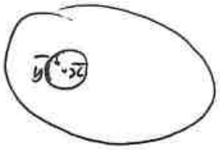
$x = \operatorname{Im} \bar{x} , \quad k(\bar{x}) = \overline{k(x)}$

$$\tilde{X}^{\bar{x}} = \operatorname{Spec} \mathcal{O}^{sh}_{X,\bar{x}} = \varprojlim \operatorname{Spec} \mathcal{O}(U)$$

$$\begin{tikzcd}
 & U \arrow[d, "\text{étale}"] \\
\operatorname{Spec} k(\bar{x}) \arrow[ur] \arrow[r] & X
\end{tikzcd}$$

$\bar{x}$ specialization of $\bar{y}$ if

$\bar{y}$ geom. pt of $\tilde{X}^{\bar{x}}$

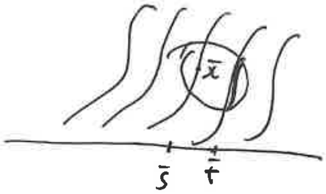


Def. $f : X \longrightarrow S$, $\bar{s}, \bar{t}$ geom. pt of S, $\bar{t} \rightsquigarrow \bar{s}$

$\bar{x}$ pt of X over $\bar{s}$, then $\tilde{X}^{\bar{x}}_{\bar{t}} := \tilde{X}^{\bar{x}} \times_{\tilde{S}^{\bar{s}}} \bar{t}$

Def f locally acyclic (la)

$$\Leftrightarrow \quad H^q(\tilde{X}^{\bar{x}}_{\bar{t}}, \mathbb{Z}/n) = \begin{cases} 0, & q > 0 \\ \mathbb{Z}/n, & q = 0. \end{cases}$$



$$\Updownarrow$$

$X_{\bar{t}} \xrightarrow{j} X \xhookleftarrow{i} X_{\bar{s}}$ $\quad i^{-1} R j_* \mathbb{Z}/n = \mathbb{Z}/n$

Lemma Composition of la morphisms is la.

Thm Smooth morphism is la.

Cor. $X \xrightarrow{f} S = \tilde{S}^{\bar{s}}$ smooth, $\bar{\eta}$ generic pt of S, $X_{\bar{\eta}} \xrightarrow{j} X$.
then $R j_* \mathbb{Z}/n \xleftarrow{\sim} \mathbb{Z}/n$, $H^*(X, \mathbb{Z}/n) \xrightarrow{\sim} H^*(X_{\bar{\eta}}, \mathbb{Z}/n)$.

$$H^*(X_{\bar{s}}, \mathbb{Z}/n) \longleftarrow H^*(X, \mathbb{Z}/n) \xrightarrow{\sim} H^*(X_{\bar{\eta}}, \mathbb{Z}/n)$$

② Local w.r.t. S and X, $S = \tilde{S}^{\bar{s}} = \operatorname{Spec} A$,

$$\begin{tikzcd} X = \mathbb{A}^{1,0}_A = \operatorname{Spec} A[T]^{sh} = \operatorname{Spec} A\{T\} \arrow[d] & \\ \operatorname{Spec} A & \operatorname{Spec} k(\bar{t}) \arrow[l] \end{tikzcd} \quad , \bar{t}$$

$$\begin{tikzcd} X \arrow[r, "\text{étale}"] \arrow[d, "f"'] & \mathbb{A}^n_S \arrow[dl] \\ S & \end{tikzcd}$$

Want $H^q(X_{\bar{t}}, \mathbb{Z}/n) = \begin{cases} 0, & q > 0 \\ \mathbb{Z}/n, & q = 0 \end{cases}$

Since $X_{\bar{t}}$ projective limit of affine curves, $H^q(X_{\bar{t}}, \mathbb{Z}/n) = 0$ for $q \geq 2$.

Enough to consider $\bar{t} = \bar{\eta}$ generic geom. pt of A

Observation: $A' \supset A$ finite extension

$$A'\{T\} \xleftarrow{\sim} A\{T\} \otimes_A A'$$

<u>$q=0$</u>. Want $X_{\bar{t} = \bar{\eta}}$ connected $\qquad X_{\bar{\eta}} = X_{\eta} \otimes_{k(\eta)} k(\bar{\eta})$

If $X_{\bar{\eta}}$ is disconnected, $\exists\ k(\eta) \overset{\text{finite}}{\subset} k \subset k(\bar{\eta})$ s.t. X_k is disconn'd.

$$\begin{tikzcd} & X \otimes A' \arrow[dl, "\text{smooth}"'] \arrow[dr] & \\ \overset{\text{normal}}{\operatorname{Spec} A'} \arrow[d, dash] \arrow[dr] & & X \arrow[dl, "\text{smooth}"] \\ \text{normalization of } A \text{ in } k & \operatorname{Spec} A & \end{tikzcd}$$

$\Rightarrow X \otimes_A A'$ is normal

On the other hand, $X \otimes_A A'$, X_k disconn'd $\Rightarrow X \otimes_A A' = \operatorname{Spec} A'\{T\}$ disconn'd.

Contradiction.