Hecke algebras with unequal parameters

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W (affine / finite) Weyl group $\quad s_i \ (i \in I)$

$s_i^2 = 1$

$\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \quad i \neq j$

$L : I \to \{1, 2, 3, \dots\}$

$L(i) = L(j)$ if s_i & s_j are conj. in W.

$\mathbb{Z}[W]$ $\qquad \mathcal{H}$ alg $/\mathbb{Z}[v, v^{-1}]$, $\quad T_i$, $i \in I$

Iwahori-Hecke alg. $\qquad (T_i + v^{-L(i)})(T_i - v^{L(i)}) = 0$

$w \in W$, $T_w = T_{i_1} \cdots T_{i_n}$ $\qquad \underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$

$w = s_{i_1} \cdots s_{i_n}$ reduced expr.

(T_w) basis of $\mathcal{H}$ $/\mathbb{Z}[v, v^{-1}]$.

G reductive conn. gp $/\overline{\mathbb{F}_q}$, $\quad \mathbb{F}_q$-rat'l str.

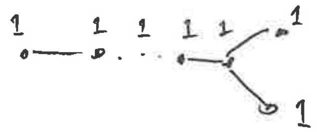
$B \subset G$ Borel, $/\mathbb{F}_q$.

$\mathrm{End}\left(\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\right) \underset{\text{Iwahori 1965}}{=} \mathcal{H}\Big|_{v = \sqrt{q}}$ for some W

L constant if G split $/\mathbb{F}_q$

L almost constant in general.

$SO(2n)^+$



$SO(2n)^-$

$F : G \to G$ Frobenius

Deligne – L. 1976. $w \in W$.

$$X_w = \{ B \text{ Borel subgp in } G : \operatorname{pos}(B, FB) = w \}$$

G^F acts on X_w, $G^F \curvearrowright H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$

irrep. ρ of G^F is <u>unipotent</u> if appears in $H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ for some w, i.

irrep ρ of G^F is <u>cuspidal</u> if $\forall \underset{\text{parabolic}/\mathbb{F}_q}{P \subsetneq G}$, $\rho^{U_P(\mathbb{F}_q)} = 0$.

L. 1977/1978: classification of unipotent reps.

Classification for $G = Sp(2n)$:

1) $n = k^2 + k$ $(k \geq 0)$, $\exists !$ unip $\overset{\text{cusp.}}{\text{rep.}}$ ρ_k

2) $n \neq k^2 + k$, $\nexists$ unip. cusp. rep.

P parabolic in $Sp_{2n} = G$, $L \simeq Sp_{2r} \times \underbrace{G_m \times \cdots \times G_m}_{n-r}$ $/\mathbb{F}_q$

$$\operatorname{End}_{G(\mathbb{F}_q)}\left(\operatorname{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)} (\rho_k \otimes 1) \right) \overset{\text{canonically}}{=} \mathcal{H}\Big(\quad \Big) \quad v = \sqrt{q}$$

$$\left\{ \text{unip. reps of } Sp_{2n}(\mathbb{F}_q)/\sim \right\} \xleftrightarrow[\text{bijection}]{} \coprod_{\substack{k\geq 0 \\ k^2+k \leq n}} \mathrm{Irr}\left(\mathcal{H}(\cdots\cdots)_{v=\sqrt{q}}\right)$$

3 models for affine Hecke algebra (equal parameter)

1) taking End(Ind(unip. cusp.))

(1)

2) perverse sheaves w/ weights on flag manifold $\times$ flag manifold

v indeterminant.

3) in terms of Langlands dual $/\mathbb{C}$, equiv. K-theory & $\mathbb{C}^\times$-action.

$$\bar{\ }: \mathcal{H} \longrightarrow \mathcal{H}, \quad T_i \mapsto T_i^{-1}, \quad v \mapsto v^{-1}.$$

KL (1978) $\quad w \in W$, $\exists !$ $C_w \in \mathcal{H}$ s.t. $\quad C_w = T_w \mod \sum_{y<w} \mathbb{Z}[v^{-1}] T_y$

$$\overline{C_w} = C_w$$

$\{C_w : w \in W\}$ basis of $\mathcal{H}$

$$C_x C_y = \sum h_{x,y,z} C_z \quad , \quad h_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$$

$$T_x T_y = \sum t_{x,y,z} T_z \quad . \quad t_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$$

$h_{x,y,z} \in v^N \mathbb{Z}[v^{-1}]$ $\quad$ (N some constant, indep. on x,y,z)

$t_{x,y,z} \in v^N \mathbb{Z}[v^{-1}]$

$z \longmapsto a(z)$

$W \longrightarrow \mathbb{N}$ $\quad$, $h_{x,y,z} \in v^{a(z)} \mathbb{Z}[v^{-1}]$, $\forall x,y$

$h_{x,y,z} \notin v^{a(z)-1} \mathbb{Z}[v^{-1}]$, some x,y

$$h_{x,y,z} = \underbrace{\gamma_{x,y,z}}_{\substack{\uparrow \\ \mathbb{Z}}} v^{a(z)} \quad + \text{ lower terms}$$

New algebra $J/\mathbb{Z}$ $\quad \{t_w : w \in W\}$

$$t_x t_y = \sum \gamma_{x,y,z} t_z$$

- associative
- unital

] NON TRIVIAL! Using geometry, Weil conj.

$\mathcal{H} \longrightarrow J \otimes \mathbb{Z}[v,v^{-1}]$ canonical alg. hom., isom. after ext. to $\mathbb{Q}(v)$.

W_J subgp of W gen. by $\{s_i : i \in J\}$

G red. conn. $/\overline{\mathbb{F}_q}$, $\quad W, I \supset J \quad \rightsquigarrow \mathcal{P}_J$ parabolic subgroups of type J.

$$Z_J = \{(P, gU_P) : P \in \mathcal{P}_J, g \in G/U_P\}, \quad Z_J = \bigcup_{w \in W_J \backslash W / W_J} {}^w Z_J$$

${}^w Z_J$: $\mathrm{pos}(P, gPg^{-1}) = w$

$J \subset J'$

$$Z_J \xleftarrow{c} Z_{JJ'} \xrightarrow{d} Z_{J'}$$

$$Z_{JJ'} = \{(P, gU_Q) : P \in \mathcal{P}_J, Q \in \mathcal{P}_{J'}, P \subset Q, gU_Q \in G/U_Q\}$$

$$c(P, gU_Q) = (P, gU_P)$$

$$d(P, gU_Q) = (Q, gU_Q)$$

$$f_{JJ'} : D(Z_J) \longrightarrow D(Z_{J'})$$
$$A \longmapsto d_! c^* A$$

Convolution : $D(Z_J) \times D(Z_J) \longrightarrow D(Z_J)$

$$Z_J \times Z_J \xleftarrow{b_1} \tilde{Z} \xrightarrow{b_2} Z_J$$

$$(P, gU_P), (P', g'U_{P'}) \leftarrow\!\shortmid \{ (\underset{P_J}{P}, \underset{P_J}{P'}, \underset{G/U_P}{gU_P}, \underset{G/U_{P'}}{g'U_{P'}}) : gPg'^{-1} = P' \} \mapsto (P, g'g U_P)$$

$$A, B \longmapsto A * B = b_{2!}(b_1^*(A \boxtimes B))$$ associative

<u>unipotent character sheaves on Z_J</u>

1) $J = \phi$, $Z_\phi \longrightarrow \mathcal{P}_\phi \times \mathcal{P}_\phi$

$(B, g U_B) \longmapsto (B, gBg^{-1})$

pull back of IC sheaves

for G-orbits.

2) general J, $\underbrace{{}^pH^i(f_{\phi J}(\text{unip. char.}))}$

take any simple constituent.

I_J set of iso. classes of unip. char. sheaves on Z_J.

K_J $\mathbb{Q}(v)$ - vec. sps w/ basis I_J.

$f_{JJ'} : K_J \to K_{J'}$, $J \subset J'$. (need weight filtration of perverse sheaves)

$$A \longmapsto \sum_{A_i \in I_{J'}} \sum_{j, h} (-1)^j v^h (\text{mult. of } A_i \text{ in } {}^pH^i(f_{JJ'}A)_h) A_i$$

weight h.

$A * B : K_J \times K_J \longrightarrow K_J$ assoc. alg.

not unital

$$\overline{K_J} = \frac{K_J}{\sum_{L \subsetneq J} t_{LJ} K_L} , \quad K_J = \bigoplus_{\substack{w \in W_J \backslash W / W_J \\ \cup \\ N(W_J)/W_J}} {}^{w} K_J$$

<u>Thm</u>. $\overline{K_J} = \bigoplus_{w \in N(W_J)/W_J} {}^{w} \overline{K_J}$

$\overline{K_J}$ assoc. alg , unital $/ \mathbb{Q}(v)$ (no unit over $\mathbb{Z}[v, v^{-1}]$)

Sometimes $\overline{K_J}$ gives Hecke algebra with unequal parameters.