Higher Algebra

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Lecture 1. Introduction to ∞-categories

Assume Grothendieck universe $\mathcal{U}$.

elements of $\mathcal{U}$: <u>small</u> sets.

For us, categories have <u>large</u> object sets, <u>large</u> morphism sets.

Set: cat. of <u>small</u> sets.

<u>Def</u>. An ∞-cat. is a (large) simplicial set $\mathcal{C}$, s.t.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad 0<i<n$$

Functor $\mathcal{C} \to \mathcal{D}$ is a map of s.sets.

<u>Example</u>. C ordinary cat., define

$$(NC)_n = \{\text{Functors } [n] \to C\}$$

$$[n] = 0 \leq 1 \leq \cdots \leq n$$

$\mathcal{C}$ ∞-cat., $\mathcal{C}_0$: objects

$\mathcal{C}_1$: morphisms $f: a \to b$, $a = \partial_1 f$, $b = \partial_0 f$

Definition: $f, g : a \to b$ **equivalent** if $\exists\, \sigma \in \mathcal{C}_2$

$$\begin{tikzcd} & b \arrow[dr, "\mathrm{id}_b"] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "\sigma"] & & b \end{tikzcd}$$

, write $f \simeq g$.

Example. $\mathcal{C}$ Kan complex, then $\mathcal{C}$ is an ∞-cat.

In particular, $\mathrm{Sing}(X)$ is an ∞-cat. (X top. space)

Objects: points of X

Morphisms: paths in X

equivalence: homotopy of paths (rel. endpoints)

Def. $f : a \to b$, $g : b \to c$,

A composition of f, g is a 2-simplex σ

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h := \partial_1 \sigma"'] \arrow[rr, phantom, shift left=3, "\sigma"] & & c \end{tikzcd}$$

Write $h \simeq g \circ f$.

Def. $f : a \to b$ is a morphism. Call f equivalence if $\exists\, g : b \to a$, $\mathrm{id}_a \simeq g \circ f$.
$\mathrm{id}_b \simeq f \circ g$.

Def. $\mathcal{C}, D$ simplicial sets, $\mathrm{Fun}(\mathcal{C}, D)$ w/ $\mathrm{Fun}(\mathcal{C}, D)_n$ = set of maps $\mathcal{C} \times \Delta^n \to D$.

Proposition. $\mathcal{C}$ simplicial set, D ∞-cat., then $\mathrm{Fun}(\mathcal{C}, D)$ is ∞-cat.

Morphism $\eta: f \to g$ is $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ restricting to f, g.

"natural transformation"

η equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \iff \eta_c$ equiv. in $\mathcal{D}$, $\forall c \in \mathcal{C}_0$.

__Def.__ $f: \mathcal{C} \to \mathcal{D}$ is called __equivalence__ if $\exists\ g: \mathcal{D} \to \mathcal{C}$, natural equivalences

$f \circ g \xrightarrow{\sim} id_{\mathcal{D}}$, $g \circ f \xrightarrow{\sim} id_{\mathcal{C}}$.

Write $\mathcal{C} \simeq \mathcal{D}$.

__Def.__ $a, b \in \mathcal{C}$, define

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "(a,b)"] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

($\pi_0 \mathrm{Map}_{\mathcal{C}}(a,b)$ set of equiv. classes of morphisms $a \to b$)

__Def.__ $f: \mathcal{C} \to \mathcal{D}$ __fully faithful__ if $\mathrm{Map}_{\mathcal{C}}(a,b) \to \mathrm{Map}_{\mathcal{D}}(f(a), f(b))$

are homotopy equiv. for each $a, b \in \mathcal{C}$.

__Def__ $f: \mathcal{C} \to \mathcal{D}$ __ess. surj.__ if $\forall d \in \mathcal{D}$, have $c \in \mathcal{C}$ and an equiv. $d \simeq f(c)$.

__Prop.__ $f: \mathcal{C} \to \mathcal{D}$ is an __equiv.__ if and only if __f is fully faithful & essentially surj.__

Def. $S \subset \mathcal{C}_0$, define $\mathcal{C}_S \subset \mathcal{C}$ consists of all simplices w/ vertices in S.

If we first saturate S under equiv. of objects $S \subset \bar{S}$, then get

$\mathcal{C}_S \to \mathcal{C}_{\bar{S}}$ equiv.

Fix $a, b, c \in \mathcal{C}$, set

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(a,b,c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(a,b,c)"] & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Lemma. $\mathrm{Map}_{\mathcal{C}}(a,b,c)$ is Kan, and have

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(a,b,c) \arrow[r, "\partial_1"] \arrow[d, "(\partial_0, \partial_2)"', "\simeq"] & \mathrm{Map}_{\mathcal{C}}(a,c) \\ \mathrm{Map}_{\mathcal{C}}(b,c) \times \mathrm{Map}_{\mathcal{C}}(a,b) & \end{tikzcd}$$

induced by

$$\begin{tikzcd} \Delta^2 & \Delta^1 \arrow[l] \\ \Lambda^2_1 \arrow[u] & \end{tikzcd}$$

Choice of homotopy inverse s

$\circ_s$: $\mathrm{Map}_{\mathcal{C}}(b,c) \times \mathrm{Map}_{\mathcal{C}}(a,b) \xrightarrow{s} \mathrm{Map}_{\mathcal{C}}(a,b,c) \xrightarrow{\partial_1} \mathrm{Map}_{\mathcal{C}}(a,c)$

Similarly, can define $\mathrm{Map}_{\mathcal{C}}(a,b,c,d)$, parametrizes associativity of $\circ$.

Recall. Kan enriched in sSets

Construction. For J finite, nonempty, totally ordered set,

$\mathfrak{C}[\Delta^J]$
- objects: J
- $\mathrm{Hom}(i,j) = N\left\{ K \subset J : \begin{matrix} \min K = i \\ \max K = j \end{matrix} \right\}$
- composition: union of K's.

Def. Simplicial set S:

$$S_n = \mathrm{Fun}^{sSet}(\mathfrak{C}[\Delta^n], \mathrm{Kan})$$

0-simplices: Kan complexes

1-simplices: maps of Kan complexes

2-simplices:

$$\begin{tikzcd} & X_1 \arrow[dr, "f_0"] & \\ X_0 \arrow[ur, "f_2"] \arrow[rr, "f_1"'] & & X_2 \end{tikzcd}$$

+ a homotopy $f_1 \simeq f_0 \circ f_2$.

Thm. S is an ∞-cat, and for X, Y, have an equivalence

$$\mathrm{Map}_S(X,Y) \simeq \mathrm{Hom}_{\mathrm{Kan}}(X,Y).$$

Def. For a simplicially enriched cat. $\mathcal{C}$, define $N^{\Delta}(\mathcal{C})_n = \mathrm{Fun}^{sSet}(\mathfrak{C}[\Delta^n], \mathcal{C})$

Prop. If $\mathcal{C}$ is enriched in Kan complexes, $N^{\Delta}(\mathcal{C})$ is ∞-cat, and have

$$\mathrm{Map}_{N^{\Delta}(\mathcal{C})}(a,b) \simeq \mathrm{Hom}_{\mathcal{C}}(a,b).$$

Lecture 2 Limits in ∞-categories

$\mathcal{C}$ ∞-cat. $a, b \in \mathcal{C}$ $\longrightarrow$ $\mathrm{Map}_{\mathcal{C}}(a,b)$

I simplicial set, $\mathrm{Fun}(I, \mathcal{C}) =: \mathcal{C}^I$

Today: I small simplicial set, $\mathcal{C}$ an ∞-cat., $F : I \longrightarrow \mathcal{C}$

Def. A cone over F is a pair consisting of

an obj. $y \in \mathcal{C}$, together w/ a natural transformation

$\eta : c_y \longrightarrow F$

$c_y : I \longrightarrow \Delta^0 \xrightarrow{y} \mathcal{C}$ constant functor.

Construction. (y, η) cone over F, and $x \in \mathcal{C}$,

$\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$

as composite $\mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{c} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, c_y) \xrightarrow[\eta_*]{} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$

obtained from $\mathcal{C} \xrightarrow{c} \mathcal{C}^I$, pullback along $I \longrightarrow \Delta^0$

Def. A cone (y, η) over F is a limit cone in $\mathcal{C}$ if for each $x \in \mathcal{C}$, the map $\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{C}^I}(c_x, F)$ is a homotopy equivalence.

$y = \lim_I F = \lim_{i \in I} F(i)$

<u>Example</u>. $I = \phi$, $\mathrm{Fun}(I, \mathcal{C}) \simeq pt$

$y \in \mathcal{C}$ is limit if $\mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{\sim} pt$ is a homotopy equiv. for any $x \in \mathcal{C}$.

$y = * = pt$. terminal object

<u>Example</u>. I discrete set , $\mathrm{Fun}(I, \mathcal{C}) \simeq \prod_I \mathcal{C}$

A functor $I \to \mathcal{C}$ thus consists of a sequence of objects $\{y_i\}_{i \in I}$.

A cone is given by an object $y \in \mathcal{C}$ together w/ maps $\{\pi_i : y \to y_i\}_{i \in I}$.

This is a limit cone, if for each object $x \in \mathcal{C}$, we have

$\mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{(\pi_{i*})} \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, y_i)$ is an equivalence.

For example in S, products are given by products of Kan cpx's.

<u>Lemma</u>. Any two limits ob. y, y' are equivalent.

<u>Proof</u>. $\{y \to y'\} \simeq \{c_y \to F\}$ (y,η). (y', η')

$f \longleftarrow\!\mapsto \eta$ $(\eta' \circ f \simeq \eta)$

$g : y' \to y$ similarly constructed $(\eta \circ g \simeq \eta')$

$\eta \circ g \circ f \simeq \eta' \circ f \simeq \eta$

$\Rightarrow g \circ f \simeq id$ □

<u>Remark</u>. One can show that the ∞-cat. of limit cones over $F : I \to \mathcal{C}$

is <u>trivial</u> or <u>empty</u>.

Remark. I is a simplicial set,

$I \subset I'$ an ∞-cat. obtained by gluing in fillers for those that do not have fillers.

$$\mathrm{Fun}(I', \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(I, \mathcal{C})$$

$$I = \left\{ \begin{tikzcd} & 0 \arrow[d] \\ 0' \arrow[r] & 1 \end{tikzcd} \right\} = \Delta^1 \amalg_{\Delta^0} \Delta^1$$

$$F : I \to \mathcal{C} \qquad \left\{ \begin{tikzcd} & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd} \text{ in } \mathcal{C} \right\}$$

Lemma. A natural transformation $ca \to F$ is up to equivalence given by maps

$i : a \to b$, $j : a \to c$ together w/ an equiv. $h \circ i \simeq k \circ j$ of maps $a \to d$ in $\mathcal{C}$

Proof. Since $I = \Delta^1 \amalg_{\Delta^0} \Delta^1$, we have $\mathrm{Fun}(I, \mathcal{C}) \simeq \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1}$. Thus a transf.

$ca \to F$ is given by

$$\begin{tikzcd} a \arrow[r, "id"] \arrow[d, "i"'] \arrow[dr] \arrow[dr, phantom, "\simeq"] & a \arrow[d, "f"] \\ b \arrow[r, "h"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[d, "f"'] & a \arrow[l] \arrow[d, "j"] \arrow[dl] \arrow[dl, phantom, "\simeq"] \\ d & c \arrow[l, "k"] \end{tikzcd}$$

Lemma. $\mathcal{C}$ Kan-enriched category, $F : I \to N_\Delta(\mathcal{C})$, $z \in S$, $x \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(z, \mathrm{Map}_{\mathcal{C}^I}(cx, F)) \simeq \mathrm{Map}_{S^I}(c_z, \underline{\mathrm{Hom}}(x, F(-)))$$

If $\mathcal{C} = S$, this space is hom. equiv. to $\simeq \mathrm{Map}_{S^I}(c_{z \times x}, F)$

Prop. A cone (y, η) over F in $\underline{S}$ is a limit precisely if for each $x \in S$, the map $[x, y]_S = \pi_0(\mathrm{Maps}_S(x, y)) \longrightarrow \pi_0(\mathrm{Maps}_{S^I}(c_x, F)) = [c_x, F]_{S^I}$ is an iso.

Proof We have

$$\begin{tikzcd}
{[\mathfrak{z}, \mathrm{Maps}_S(x, y)]_S} \arrow[r, phantom, "="] \arrow[d, "S"] & {[\mathfrak{z} \times x, y]_S} \arrow[d, "S"] \\
{[\mathfrak{z}, \mathrm{Maps}_{S^I}(c_x, F)]_S} \arrow[r, phantom, "\simeq"] & {[c_{\mathfrak{z} \times x}, F]_{S^I}}
\end{tikzcd}$$

□

$$\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr, phantom, "///"] & b \arrow[d] \\
d \arrow[r] & c
\end{tikzcd}$$

in S is a pullback if maps from x to a are up to homotopy the same as triples $x \to b$, $x \to d$, and a homotopy of the composite $x \to b \to c$ & $x \to d \to c$.

Prop. For any functor $F : I \to S$, the space $\mathrm{Map}_{S^I}(c_{pt}, F)$ is the limit of F. In particular, S has all (small) limits.

Proof. $\mathrm{Maps}_S(x, \mathrm{Maps}_{S^I}(c_{pt}, F)) \simeq \mathrm{Map}_{S^I}(c_x, F)$ □

Example. I arbitrary sSet, $F = c_x$ constant on space $x \in S$

$$\lim_I F = \mathrm{Map}_{S^I}(c_{pt}, c_x) \simeq \underline{\mathrm{Hom}}(I, \mathrm{Maps}(pt, x)) \cong \underline{\mathrm{Hom}}(I, x)$$

Example (sequential limits)

$I' = N(\mathbb{N}, <)^{op}$

$I = (\Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \cdots)^{op}$ $\quad 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to$.

$I \subset I'$

$\mathcal{C}$ Kan enriched cat. $\quad I \xrightarrow{F} N_\Delta(\mathcal{C})$, $x \in \mathcal{C}$

$\underline{Hom}(x,-) : \mathcal{C} \to sSet \quad \Rightarrow N_\Delta \underline{Hom}(x,-) : N_\Delta(\mathcal{C}) \to \mathcal{S}$

Thm. A cone (x,η) over F is a limit cone iff for any x, the induced cone $(\underline{Hom}_{\mathcal{C}}(x,y), N_\Delta \underline{Hom}_{\mathcal{C}}(x,\eta))$ is a limit cone in $\mathcal{S}$.

(Informally, $Map_{\mathcal{C}}(x, \lim_I F) \simeq \lim_{i \in I} Map_{\mathcal{S}}(x, F(i))$)

Proof.

limit cone in $\mathcal{C}$

$\Updownarrow$

$Map_{\mathcal{C}}(x,y) \xrightarrow{\sim} Map_{\mathcal{C}^I}(cx, F)$

$\Updownarrow$

$\underline{Hom}_{\mathcal{S}}(3, Map_{\mathcal{C}}(x,y)) \xrightarrow{\sim} \underline{Hom}_{\mathcal{S}}(3, Map_{\mathcal{C}^I}(cx, F))$

$\quad\quad \wr\wr \quad\quad\quad\quad\quad\quad \wr\wr$

$Map_{\mathcal{S}}(3, \underline{Hom}(x,y)) \Rightarrow Map_{\mathcal{S}^I}(c_3, \underline{Hom}(x, F(-)))$ $\square$

Remark:
- A cone over F is a functor $\dfrac{I \times \Delta^1}{I \times \{0\}} \to \mathcal{C}$
- There is a "smaller" model for this quotient called $I^\triangleleft = \Delta^0 * I$

$\cdot\ \mathrm{Map}_{\mathcal{C}^I}(cx, F) \simeq$
$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{C}^{I^{\triangleleft}} \arrow[d] \\ \Delta^0 \arrow[r, "{(x,F)}"] & \mathcal{C} \times \mathcal{C}^I \end{tikzcd} \simeq \begin{tikzcd} P'' \arrow[r] \arrow[d] & \mathcal{C}_{/F} \arrow[d] \\ \Delta^0 \arrow[r] & \mathcal{C} \end{tikzcd}$$

Lecture 3. Colimits and descriptions

Last time $F: I \to \mathcal{C}$

$$\mathrm{Map}_{\mathcal{C}}(x, \lim_{i \in I} F(i)) \simeq \mathrm{Map}_{\mathcal{C}^I}(cx, F) \simeq \lim_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, F(i))$$

Today colimits, $\mathcal{C}$ ∞-cat. $\rightsquigarrow$ $\mathcal{C}^{op}: \Delta \xrightarrow{(\ )^{op}} \Delta^{op} \xrightarrow{\mathcal{C}} \mathrm{Set}$.

$$\{\text{colimit of } F: I \to \mathcal{C}\} = \{\text{limit of } F^{op}: I^{op} \to \mathcal{C}^{op}\}$$

Explicitly, a cone under F is an object $y \in \mathcal{C}$ together w/ a natural transform.

$$\eta: F \to c_y.$$

This is a colimit cone if for any $z \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(y, z) \xrightarrow{\eta^*} \mathrm{Map}_{\mathcal{C}^I}(F, c_z) \text{ is an equiv.}$$

Proposition. Extending an object $y \in \mathcal{S}$ to a colimit cone over some fixed $F: I \to \mathcal{S}$ is equiv. to providing a natural isom.

$$[y, z]_{\mathcal{S}} \simeq [F, c_z]_{\mathcal{S}^I} \quad \text{natural in } z.$$

Examples • $I = \emptyset$, colimit over I is called <u>initial object</u>. write $\emptyset \in \mathcal{C}$

$$\mathrm{Map}_{\mathcal{C}}(\emptyset, z) \simeq * \qquad (\text{in } S.\ \emptyset = \emptyset)$$

• I discrete, $F = \{y_i\}_{i\in I}$. colimit is called coproduct, $\coprod_{i\in I} y_i = \operatorname{colim}_I F$

$$\mathrm{Map}_{\mathcal{C}}\Big(\coprod_{i\in I} y_i, z\Big) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}}(y_i, z)$$

In S, $\coprod_{i\in I} y_i$ is the disjoint union of Kan cpxes.

• $I = \left\{\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \\ 1' & & \end{array}\right\} = \Delta^1 \coprod_{\Delta^0} \Delta^1$, $F = \left\{\begin{array}{ccc} a & \rightarrow & b \\ \downarrow & & \\ c & & \end{array}\right\}$

$\operatorname{colim} F =: b \coprod_a c$

<u>Universal property</u> for any $z \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(b \sqcup_a c, z) \xrightarrow{\sim} \mathrm{Map}(b,z) \underset{\mathrm{Map}_{\mathcal{C}}(a,z)}{\times} \mathrm{Map}_{\mathcal{C}}(c,z)$$

pullback in S

<u>Pushouts in S</u>: Given $\begin{array}{ccc} a & \rightarrow & b \\ \downarrow & & \\ c & & \end{array}$ of spaces, we form

$$b \coprod_{a\times\{0\}} a \times \Delta^1 \coprod_{a\times\{1\}} c \quad \text{" Double mapping cylinder "}$$

Replace by a Kan cpx d.

Then
$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & b \arrow[d] \\ c \arrow[r] & d_{/K} \end{tikzcd}$$
in S is a pushout.

$$\left[I \to S, (I \times_S S_*)_{/K} \right]$$

Examples. (1) A coequalizer is a colimit over

$$I = (0 \rightrightarrows 1) = \frac{\Delta^1 \amalg \Delta^1}{\sim}$$

$$a \underset{g}{\overset{f}{\rightrightarrows}} b \to c$$

We claim the coequalizer is the pushout

$$\begin{tikzcd} a \amalg a \arrow[r, "(f,g)"] \arrow[d, "\nabla"'] & b \arrow[d] \\ a \arrow[r] \arrow[ur, phantom, "\lrcorner"] & c \end{tikzcd}$$

(2) Pushout
$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$
can be written as coequalizer

$$a \rightrightarrows b \amalg c \to d$$

(3) Sequential colimit: $I = \Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \cdots \to N(\mathbb{N}, <)$

$$(y_0 \to y_1 \to \cdots) \longrightarrow y_\infty = \operatorname{colim}_{\to} y$$

$$\coprod_{i \geq 0} y_i \underset{\text{shift}}{\overset{id}{\rightrightarrows}} \coprod_{i \geq 0} y_i \longrightarrow y_\infty$$ is a coequalizer.

Def. A simplicial set I is called *finite* if it has only finitely many non-degenerate simplices.

Thm. For an ∞-cat. $\mathcal{C}$, TFAE

(1) $\mathcal{C}$ admits an initial object and pushouts

(2) $\mathcal{C}$ admits all finite coproducts and all coequalizers

(3) $\mathcal{C}$ admits all finite colimits.

Proof. (1) $\Rightarrow$ (2) Coproduct $a \amalg b$ is the pushout
$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & a \arrow[d] \\ b \arrow[r] & a \amalg b \end{tikzcd}$$

(2) $\Rightarrow$ (1) ϕ coproduct.

(3) $\Rightarrow$ (1), (2) I finite simplicial set, $I_0 \subset I_1 \subset \cdots \subset I_n = I$, $n = \dim I$

$$\begin{tikzcd} \coprod_J \partial\Delta^n \arrow[r] \arrow[d] & I_{n-1} \arrow[d] \\ \coprod_J \Delta^n \arrow[r] \arrow[ur, phantom, "\lrcorner"] & I_n \end{tikzcd} \qquad F : I \to \mathcal{C}$$

$$\operatorname{colim}_I F \simeq \operatorname{cl}_{I_{n-1}} F \coprod_{\operatorname{cl}_{\amalg_J \partial\Delta^n} F|_{\amalg_J \partial\Delta^n}} \operatorname{cl}_{\amalg_J \Delta^n} F|_{\amalg_J \Delta^n}$$

$$\operatorname{colim}_{\amalg_J \Delta^n} G \cong \coprod_J G|_{\Delta^n} \simeq \coprod_J (G|_{\Delta^n})[n]$$

Thm. If an ∞-cat. $\mathcal{C}$ admits coproducts, then TFAE

(1) $\mathcal{C}$ admits all colimits ; (2) $\mathcal{C}$ admits pushouts ; (3) $\mathcal{C}$ admits coequalizers

(4) $\mathcal{C}$ admits geometric realizations, that is colimits indexed over $N(\Delta^{op})$

Proof. (2) $\Leftrightarrow$ (3). (1) $\Rightarrow$ (2), (3), (4).

(2,3) $\Rightarrow$ (1) : $I = \bigcup (I_0 \subset I_1 \subset \cdots)$

$$\operatorname*{colim}_{I} (F) \simeq \operatorname*{colim}_{n \in \mathbb{N}} \left(\operatorname*{colim}_{I_n} F|_{I_n} \right)$$

(4) $\Rightarrow$ (2)
$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$
is the colimit of the simplicial diagram

$$\substack{\to\\\to\\\to\\\to} b \amalg a \amalg a \amalg c \substack{\to\\\to\\\to} b \amalg a \amalg c \rightrightarrows b \amalg c$$

$\square$

Cor. The ∞-cat. $\mathcal{S}$ has all colimits.

Def. • The ∞-cat. $\mathcal{S}_*$ of pointed spaces is the $N_\Delta(\mathrm{Kan}_*)$, where Kan_* is the simplicially enriched cat. of pointed Kan cpxes.

• The ∞-cat. Cat_∞ of ∞-cat. is given by $N_\Delta(\mathrm{Cat}_\infty^\Delta)$, where $\mathrm{Cat}_\infty^\Delta$ is the simplicially enriched cat. whose objects are ∞-cats and whose morphisms

$$\underline{\mathrm{Hom}}_{\mathrm{Cat}_\infty^\Delta}(x,y) \subset \mathrm{Fun}(x,y) \quad \text{max'l sub Kan cpx.}$$

Thm. Cat_∞ and $\mathcal{S}_*$ have all limits and colimits.

Lecture 4 Derived categories as ∞-categories

For an abelian cat. $\mathcal{A}$, construct an ∞-cat. $\mathcal{D}(\mathcal{A})$

Remark

- objects: (unbounded) chain complexes in $\mathcal{A}$

- equiv. classes of morphisms usual RHom

Def. Write $Ch(\mathcal{A})$ for the cat. of chain cpxes, enriched over $Ch(\mathbb{Z})$

Constructions. Dold - Kan

$$\Gamma : Ch(\mathbb{Z})_{\geq 0} \xrightarrow{\sim} Fun(\Delta^{op}, Ab)$$

Lax monoidal

$$K: Ch(\mathbb{Z}) \longrightarrow Ch(\mathbb{Z})_{\geq 0} \xrightarrow{\Gamma} Fun(\Delta^{op}, Ab) \longrightarrow Fun(\Delta^{op}, Set)$$

dg cat. $\mapsto$ simplicially enriched cat.

$$\pi_n K(C) = H_n(C), \ n \geq 0$$

Def. For a dg cat. C, let $N_{dg}(C) = N_\Delta(C_\Delta)$

where C_Δ is the simplicially enriched cat. obtained from C by applying K to its Hom's

Def. For abelian cat. $\mathcal{A}$, define $K(\mathcal{A}) := N_{dg}(Ch(\mathcal{A}))$

Proposition. $K(\mathcal{A})$ has all finite limits & colimits.

- $0 \in K(\mathcal{A})$ is initial & terminal obj.

$$Map_{K(\mathcal{A})}(0, E) = K(Hom(0, E)) = \Delta^0$$
$$Map_{K(\mathcal{A})}(E, 0) = K(Hom(E, 0)) = \Delta^0$$

- $C, D \in K(\mathcal{A})$, then $C \oplus D$ is coproduct.

  Check that $\mathrm{Map}_{K(A)}(C \oplus D, E) \simeq \mathrm{Map}_{K(A)}(C, E) \times \mathrm{Map}_{K(A)}(D, E)$

  Similarly, $C \oplus D$ is also product.

- $C \xrightarrow{f} D$, $C \xrightarrow{g} D'$,

  $C(f,g)_n = D_n \oplus D_n' \oplus C_{n-1}$

  $\partial(d, d', c) = (\partial d + f(c), \partial d' - g(c), \partial c)$

  Square
$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & D \arrow[d] \\ D' \arrow[r] & C(f,g) \end{tikzcd}$$
  in $K(A)$. is pushout.

  Similarly, have a description for pullbacks.

In $K(\mathbb{Z})$, $(\cdots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \cdots) \to (\cdots 0 \to \mathbb{Z}/2 \cdots)$

Def. Let $\mathcal{C}$ be an ∞-cat., and $W \subseteq \mathcal{C}_1$ a subset of morphisms.

A functor $f: \mathcal{C} \to \mathcal{C}'$ is a Dwyer - Kan localization at W if

- f takes W to equivalences in $\mathcal{C}'$
- For every $\mathcal{D}$, $\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful, w/

essential image the full subcat. $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, on those functors $\mathcal{C} \to \mathcal{D}$ which take W to equivalences. Write $\mathcal{C}[W^{-1}] = \mathcal{C}'$.

Example. $\mathcal{C} = \Delta^1$, W consists of the nontrivial morphism $0 \to 1$,

then $\mathcal{C}[W^{-1}] \simeq \Delta^0$. Check that $\mathrm{Fun}(\Delta^0, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^W(\Delta^1, \mathcal{D})$

is an equiv.

ess. surj. :
$$\begin{tikzcd} d \arrow[d, "\simeq"] \\ d' \end{tikzcd} \sim \begin{tikzcd} d \arrow[d, "id"] \\ d \end{tikzcd}$$

fully faithful :
$$\mathrm{Map}\left(\begin{tikzcd} a \arrow[d, "f"] \\ a' \end{tikzcd}, \begin{tikzcd} b \arrow[d, "g"] \\ b' \end{tikzcd} \right) \simeq \text{Pullback} \left(\begin{tikzcd} & \mathrm{Map}_{\mathcal{D}}(a', b') \arrow[d, "f^*"] \\ \mathrm{Map}_{\mathcal{D}}(a,b) \arrow[r, "g_*"] & \mathrm{Map}_{\mathcal{D}}(a, b') \end{tikzcd} \right)$$

Proposition. $\mathcal{C}[W^{-1}]$ always exists. ; $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is essentially surj.

Proof.
$$\begin{tikzcd} \coprod_W \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \coprod_W \Delta^0 \arrow[r] & \mathcal{C}' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$
in Cat_∞.

Claim.
$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(\coprod_W \Delta^0, \mathcal{D}) \arrow[d] \\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[r] & \mathrm{Fun}(\coprod_W \Delta^1, \mathcal{D}) \end{tikzcd}$$
in Cat_∞.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \text{ in } \mathrm{Cat}_\infty, \qquad \begin{tikzcd} \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(B, \mathcal{E}) \arrow[d] \\ \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \arrow[r] & \mathrm{Fun}(A, \mathcal{E}) \end{tikzcd}$$

Check after $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{F}, -)$

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{F}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cap"] & \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathrm{Fun}(\mathcal{F}, \mathcal{E})) \arrow[d, phantom, "\cap"] \\
\mathrm{Fun}(\mathcal{F}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{F}, \mathcal{E}))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathrm{Fun}(\mathcal{F}, \mathcal{E})) \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Fun}(\mathcal{F}, \mathcal{E})) \arrow[r] & \cdots
\end{tikzcd}$$

(Cat_∞ is an $(\infty,2)$-category)

Def Let $W \subset K(A)_1$ be the set of quasi-isomorphisms, i.e. isomorphisms on homology

Let $D(A) = K(A)[W^{-1}]$.

Lecture 5. Slices and filtered colimits.

Recall $D(A) = K(A)[W^{-1}]$

Def. Let $\mathcal{C}$ be an ∞-cat, $x \in \mathcal{C}$

$$\begin{tikzcd}
\mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d, "d_0"] \\
\Delta^0 \arrow[r, "x"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_{x/} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d, "d_1"] \\
\Delta^0 \arrow[r, "x"] & \mathcal{C}
\end{tikzcd}$$

(objects $c \to x$) (objects $x \to c$)

Def. For $C \in K(A)$, $K(A)^{qi}_{/C} \subset K(A)_{/C}$

full subcat. on all $C' \to C$ where the map is a quasi-isom.

Dually, $K(A)^{qi}_{C/} \subset K(A)_{C/}$

Proposition. $$\mathrm{Map}_{D(A)}(C,D) \simeq \operatorname*{colim}_{\left(K(A)^{qi}_{/C}\right)^{op}} \mathrm{Map}_{K(A)}(-,D)$$

$$\simeq \operatorname*{colim}_{K(A)^{qi}_{D/}} \mathrm{Map}_{K(A)}(C,-)$$

Note. Map (as we defined it) is not a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$

Since quasi-isomorphisms become equivalences in $D(A)$, have

$$\operatorname*{colim}_{\left(K(A)^{qi}_{/C}\right)^{op}} \mathrm{Map}_{K(A)}(-,D) \longrightarrow \operatorname*{colim}_{\left(K(A)^{qi}_{/C}\right)^{op}} \mathrm{Map}_{D(A)}(-,D)$$

$$\operatorname*{colim}_{\left(K(A)^{qi}_{/C}\right)^{op}} \mathrm{Map}_{D(A)}(-,D) \simeq \mathrm{Map}_{D(A)}(C,D)$$

Def. $C \in K(A)$ K-projective, if $\mathrm{Map}_{K(A)}(C,-)$ sends quasi-isomorphisms to homotopy equivalences. Similarly, K-injective if $\mathrm{Map}_{K(A)}(-,C)$ sends quasi-isoms to homotopy equivalences.

Cor If C is K-projective or D is K-injective, $\mathrm{Map}_{D(A)}(C,D) \simeq \mathrm{Map}_{K(A)}(C,D)$

Recall Fundamental lemma of homological algebra

bounded below, levelwise proj. $\Rightarrow$ K-proj.

bounded above, levelwise inj. $\Rightarrow$ K-inj.

Given $C \in K(A)$ and a quasi-isom. $C' \to C$ w/ C' K-projective, then

$$\mathrm{Map}_{D(A)}(C, D) \simeq \mathrm{Map}_{K(A)}(C', D)$$

Def. An ∞-cat. $\mathcal{C}$ is called <u>filtered</u> if for any finite simplicial set K, every map $K \to \mathcal{C}$ extends over $(K \times \Delta^1) \amalg_{(K \times 1)} \Delta^0$

Dually, $\mathcal{C}$ is <u>cofiltered</u> if $\mathcal{C}^{op}$ is filtered.

<u>filtered colimits</u> are colimits indexed over filtered diagrams.

(dually cofiltered limits)

Lemma. If I is a filtered ∞-cat, J a finite simplicial set,

$$\begin{tikzcd} \mathrm{Fun}(I \times J, \mathcal{S}) \arrow[r, "\lim_J"] \arrow[d, "\mathrm{colim}_I"] & \mathrm{Fun}(I, \mathcal{S}) \arrow[d, "\mathrm{colim}_I"] \\ \mathrm{Fun}(J, \mathcal{S}) \arrow[r, "\lim_J"] & \mathcal{S} \end{tikzcd}$$

commutes.

Lemma If $K \in \mathcal{S}$ comes from a <u>finite</u> sSet,

$\mathrm{Map}(K, -) : \mathcal{S} \to \mathcal{S}$ commutes w/ filtered colimits.

Cor. $[K,-] = \pi_0 \operatorname{Map}(K,-)$

$\mathcal{S} \longrightarrow N(\mathrm{Set})$

commutes w/ filtered colimits.

Proof. $\pi_0 : \mathcal{S} \longrightarrow N(\mathrm{Set})$ commutes w/ <u>all</u> colimits.

Lemma. $K(A)^{qi}_{c/}$, $\left(K(A)^{qi}_{/c}\right)^{op}$ are both filtered. (if A has arbitrary colimits)

Proof. Check that $K(A)^{qi}_{c/} \subset K(A)_{c/}$ is closed under colimits.

Have $K(A)_{c/} \xrightarrow{\text{cofib};} K(A)$

$$(c \mapsto c') \longmapsto \operatorname{cofib}(c \to c') \simeq \operatorname{pushout}\left(\begin{tikzcd} c \arrow[r] \arrow[d] & c' \\ 0 & \end{tikzcd}\right)$$

$(c \to c')$ quasi-iso. $\Leftrightarrow$ $\operatorname{cofib}(c \to c')$ is acyclic.

$\square$

Prop. The derived ∞-cat. $D(A)$ has finite colimits & limits, and

$K(A) \longrightarrow D(A)$ preserves them.

Proof 0 is still initial and terminal in $D(A)$

Given a pushout
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
in $K(A)$,
$$\begin{tikzcd} \operatorname{Map}_{K(A)}(D,E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circledast"] & \operatorname{Map}_{K(A)}(C,E) \arrow[d] \\ \operatorname{Map}_{K(A)}(B,E) \arrow[r] & \operatorname{Map}_{K(A)}(A,E) \end{tikzcd}$$
in $\mathcal{S}$

$$\begin{tikzcd} \mathrm{Map}_{D(A)}(D, E) \arrow[r] \arrow[d] & \cdots \arrow[d] \\ \mathrm{Map}_{D(A)}(B, E) \arrow[r] & \cdots \end{tikzcd}$$

is a fibered colimit of ⊛ for E' for $K(A)^{qi}_{E/}$, i.e. it is a pullback.

Given $\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ C & & B \end{tikzcd}$ in $D(A)$

$$\pi_0 \mathrm{Map}_{D(A)}(A, C) \simeq \operatorname*{colim}_{K(A)^{qi}_{C/}} \pi_0 \mathrm{Map}_{K(A)}(A, -)$$

So there exists a quasi-isom.. $C \to C'$ s.t. $A \to C \to C'$ comes from $K(A)$.

$$\begin{tikzcd} C \arrow[d, "\simeq"] & A \arrow[l] \arrow[r] \arrow[d, "id"] & B \arrow[d, "\simeq"] \\ C' & A \arrow[l] \arrow[r] & B' \end{tikzcd}$$

← this row comes from $K(A)$. So its colimit exists in $D(A)$. upper row is equivalent !

<u>Prop</u>. $N(sSet)[W^{-1}] \simeq S$, where W is weak equivalences.

<u>Prop</u>. $N(Ch(A))[W^{-1}] \simeq D(A)$
W quasi-isoms

$N(Ch(A))[W^{-1}] \simeq K(A)$
W chain homotopy equiv.

Lecture 6 — Derived functors

Recall $\mathcal{A}$ abelian cat. $\Rightarrow D(\mathcal{A}) := K(\mathcal{A})[q.\text{iso}^{-1}]$

Assume that $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, i.e.

$$F(0) \simeq 0, \qquad F(A \oplus B) \cong FA \oplus FB$$

Construction. We get an induced functor

$$Ch(\mathcal{A}) \xrightarrow{Ch(F)} Ch(\mathcal{B})$$

$$(\cdots \to C_1 \to C_0 \to \cdots) \mapsto (\cdots F(C_1) \to F(C_0) \to F(C_{-1}) \to \cdots)$$

which is an enriched functor over $Ch(\mathbb{Z})$

$$\Rightarrow \underbrace{K(\mathcal{A}) = N_{dg}(Ch(\mathcal{A})) \xrightarrow{N_{dg}(F)} N_{dg}(Ch(\mathcal{B})) = K(\mathcal{B})}_{K(F)}$$

which commutes w/ finite limits & colimits.

We would like to get an induced functor $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$ s.t.

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "p_{\mathcal{A}}"'] & K(\mathcal{B}) \arrow[d, "p_{\mathcal{B}}"] \\ D(\mathcal{A}) \arrow[r, "D(F)"'] & D(\mathcal{B}) \end{tikzcd}$$

is a comm. square in Cat_∞

TFAE:

1. $\exists$ a functor $D(A) \to D(B)$ making the square commutative.

2. The functor $K(A) \to K(B)$ preserves quasi-isomorphisms

3. The functor $A \to B$ is exact.

$$\begin{tikzcd} K(A) \arrow[r, "K(F)"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\Nearrow\eta"] & K(B) \arrow[d, "p_B"] \\ D(A) \arrow[r, "LF"'] & D(B) \end{tikzcd}$$

<u>Def.</u> The <u>left derived functor</u> LF of F is given by a functor

$LF : D(A) \to D(B)$ together w/ a nat. transf. $\eta : LF \circ p_A \to p_B \circ K(F)$.

s.t. for any other functor $H : D(A) \to D(B)$ the map

$$\mathrm{Map}_{\mathrm{Fun}(D(A), D(B))}(H, LF) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(K(A), D(B))}(H \circ p_A, p_B \circ K(F))$$

is an equiv.

<u>Warning</u>: The left derived functor might not exist in all generality.

<u>Prop.</u> If LF exists, then it preserves finite limits & colimits.

Def. Given $(\mathcal{C}, W)$, and a functor $G: \mathcal{C} \to D$, then the left derived functor LG is given by

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d] \arrow[dr, phantom, "\Uparrow\eta"] & D \\ \mathcal{C}[W^{-1}] \arrow[ur, "LG"'] & \end{tikzcd}$$

$\eta: LG \to G$ s.t. the induced map

$$\mathrm{Map}(H, LG) \xrightarrow{\eta^*} \mathrm{Map}(H, G)$$ is an equiv. for each $H: \mathcal{C}[W^{-1}] \to D$.

- We say that LG is the <u>absolute left derived functor</u> if for any $T: D \to \mathcal{E}$, the induced triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d] \arrow[dr, phantom, "\Uparrow T\circ\eta"] & D \arrow[r, "T"] & \mathcal{E} \\ \mathcal{C}[W^{-1}] \arrow[urr, "T\circ LG"'] & & \end{tikzcd}$$

exhibits $T \circ LG$ as the left derived functor of $T \circ G$.

Rmk. $\mathrm{Fun}(\mathcal{C}[W^{-1}], D) \cong \mathrm{Fun}^W(\mathcal{C}, D) \subset \mathrm{Fun}(\mathcal{C}, D)$

For given $G \in \mathrm{Fun}(\mathcal{C}, D)$, we have $LG \in \mathrm{Fun}^W(\mathcal{C}, D)$, and

$$\mathrm{Map}(H, LG) \simeq \mathrm{Map}(H, G) \quad \text{for } H \in \mathrm{Fun}^W(\mathcal{C}, D)$$

<u>Theorem</u>. For any functor $G: K(A) \to D$, we have that

$$(LG)(C) = \varprojlim_{\hat{C} \to C} G(\hat{C}) \quad \text{provided that this limit exists for each } C \in K(A).$$

$$\left(\text{limit over } K(A)^{q\text{isom}}_{/C}\right)$$

<u>Example</u>. ① For $C = 0$, we have

$$(LG)(0) = \varprojlim_{\hat{C} \to 0} G(\hat{C}) = G(0) \quad \text{since } 0 \text{ initial in } K(A)^{q\text{isom}}_{/0}$$

② For any C, assume that there is a K-proj. obj. $\hat{C} \xrightarrow{\sim} C$, then we claim that

$p: \hat{C} \to C$ is initial in $K(A)^{qi}_{/C}$.

To see this, we note that for any other obj. $\hat{\hat{C}} \xrightarrow{\sim} C$

$$\begin{tikzcd}
\mathrm{Map}_{K(A)_{/C}}(\hat{C} \to C, \hat{\hat{C}} \to C) \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{K(A)}(\hat{C}, \hat{\hat{C}}) \arrow[d, "\wr"] \\
\Delta^0 \arrow[r, "p"'] & \mathrm{Map}_{K(A)}(\hat{C}, C)
\end{tikzcd}$$

$\Rightarrow \quad (LG)(C) = G(\hat{C})$.

<u>Note</u>. If for any object C there exists such a K-proj. resolution $\hat{C}$, then

$(LG)(C) = G(\hat{C})$ for any functor G.

In particular, this also is compatible w/ post composition w/ functors $T: D \to \mathcal{E}$, i.e. in this case every left derived functor is <u>absolute</u>!

<u>Example.</u> Assume that $F : \mathcal{A} \to \mathcal{B}$ additive, $\mathcal{A}$ is small, and $\mathcal{B}$ has all infinite products, and they are exact (AB4*)

Then
- $D(\mathcal{B})$ has all limits.
- The derived functor $LF : D(\mathcal{A}) \to D(\mathcal{B})$ exists
- LF preserves finite colimits & limits.

There is a dual notion of right derived functor.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d] \arrow[dr, phantom, "\overset{\eta}{\Nearrow}"] & \mathcal{D} \\ \mathcal{C}[W^{-1}] \arrow[ur, "RG"'] & \end{tikzcd}$$

i.e. $\eta : G \to RG$

Then the dual of our formula $K(\mathcal{A})^{op} \simeq K(\mathcal{A}^{op})$

reads as $RG(c) = \operatorname{colim}_{c \Rightarrow \hat{c}} G(\hat{c})$

<u>Example.</u> We have functors

$$\mathrm{Map}_{K(\mathcal{A})}(A,-) : K(\mathcal{A}) \to \mathcal{S}$$

$$\mathrm{Map}_{K(\mathcal{A})}(-,B) : K(\mathcal{A})^{op} \simeq K(\mathcal{A}^{op}) \to \mathcal{S}$$

Then the formulas for $\mathrm{Map}_{D(\mathcal{A})}(A,B)$ show that $\mathrm{Map}_{D(\mathcal{A})}(A,-) \simeq R\mathrm{Map}_{K(\mathcal{A})}(A,-)$,
$\mathrm{Map}_{D(\mathcal{A})}(-,B) \simeq R\mathrm{Map}_{D(\mathcal{A})}(-,B)$, $[\mathrm{Map}_{D(\mathcal{A})} = R\mathrm{Map}_{K(\mathcal{A})}]$

Proof sketch. $LG(c) \cong \varprojlim_{\hat{c} \xrightarrow{\sim} c} G(\hat{c})$

(1) Show that $LG(c) := \varprojlim_{\hat{c} \to c} G(\hat{c})$ defines a functor $K(A) \longrightarrow D$.

(2) It preserves weak equivalences.

(3) There is a nat'l transf. $\eta : LG \longrightarrow G$

(4) If G preserves q.isoms, then η is an equiv

(5) $\mathrm{Map}(H, LG) \xrightarrow{\eta_*} \mathrm{Map}(H, G) \xrightarrow{L} \mathrm{Map}(LH, LG) \xrightarrow[\eta_*]{} \mathrm{Map}(H, LG)$

$$LLG \underset{\eta L}{\overset{L\eta}{\rightrightarrows}} LG$$

Def. Let $\mathcal{C}, \mathcal{D}$ be ∞-cats w/ functors $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$

A natural transf. $\varepsilon : LR \longrightarrow \mathrm{id}_{\mathcal{D}}$ is called a counit of an adjunction if for a given pair of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$, the induced map

$$\mathrm{Map}_{\mathcal{C}}(c, Rd) \xrightarrow{L} \mathrm{Map}_{\mathcal{D}}(Lc, LRd) \xrightarrow{\varepsilon_d} \mathrm{Map}_{\mathcal{D}}(Lc, d)$$ is a homotopy equiv.

Dually, a transf. $\eta : \mathrm{id}_{\mathcal{C}} \to RL$ is called unit of an adjunction if the induced map $\mathrm{Map}_{\mathcal{D}}(Lc; d) \xrightarrow{R} \mathrm{Map}_{\mathcal{C}}(RLc, Rd) \xrightarrow{\eta_c^*} \mathrm{Map}_{\mathcal{C}}(c, Rd)$ is an equiv.

In either case, we say that L is left adjoint to R $(L \dashv R)$

Given L, R as above, and $\varepsilon : LR \to id$, $\eta : id \to RL$ nat'l transf.

We say that the <u>zigzag identities</u> hold if the composites

$$L = L \circ id \xrightarrow{\eta} L \circ R \circ L \xrightarrow{\varepsilon} id \circ L = L$$

$$R = id \circ R \xrightarrow{\eta} R \circ L \circ R \xrightarrow{\varepsilon} R \circ id = R$$

are equiv. to id_L resp. id_R (in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$)

<u>Prop</u>. If transformations ε and η satisfy the zigzag identities, then those are unit and counit of an adjunction. Conversely, given a unit of an adjunction η, there is a unique counit ε s.t. the zigzag identities are satisfied.

<u>Proof</u>. Assume ε, η satisfy the ZZ-identities, then

$$\mathrm{Map}_{\mathcal{C}}(c, Rd) \xrightarrow{L} \mathrm{Map}_{\mathcal{D}}(Lc, RLd) \xrightarrow{\varepsilon} \mathrm{Map}_{\mathcal{D}}(Lc, d)$$

$$\mathrm{Map}_{\mathcal{D}}(Lc, d) \xrightarrow{R} \mathrm{Map}_{\mathcal{D}}(RLc, Rd) \xrightarrow{\eta} \mathrm{Map}_{\mathcal{D}}(c, Rd)$$

are inverse to each other. "$\square$"

<u>Facts</u>: • Given $L : \mathcal{C} \to \mathcal{D}$, then the pair (R, η) is unique if it exists.

• The composite of left-adjoints is again left-adjoint, w/ right-adjoint given by the composite of right adjoints.

$$\mathrm{Cat}_\infty^L \simeq (\mathrm{Cat}_\infty^R)^{op}$$

I small ∞-cat, consider the functor

$$c_- = \Delta : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$$

Assume that Δ has a left-adjt L, then for any functor $F: I \to \mathcal{C}$, we get from the unit a map $F \to \Delta(LF)$ s.t. the induced map

$$\mathrm{Map}_{\mathcal{C}}(LF, y) \xrightarrow{\sim} \mathrm{Map}(F, \Delta y) \quad \text{is an equiv.}$$

$$\Rightarrow LF = \operatorname{colim}_I F.$$

Prop. • If $\mathcal{C}$ has all I-indexed colimits, then Δ admits a left-adjoint.

• Assume that $L: \mathcal{C} \to \mathcal{D}$ has for every d an object Rd w/ a map $LRd \to d$ s.t. $\mathrm{Map}_{\mathcal{C}}(c, Rd) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Lc, d)$ equiv., then L admits a right adjoint R given pointwise by R.

Examples • $S \xrightarrow{\pi_0} \mathrm{Set}$ a left adjoint

• $\mathcal{C}_{/x} \xrightarrow{\text{forget}} \mathcal{C}$ is a left adjoint if $\mathcal{C}$ has products.

Prop. Left adjoint functors preserve colimits, right adjoint functors prove limits.

Def. Given a functor $p: \mathcal{C} \to \mathcal{C}'$ and any functor $F: \mathcal{C} \to \mathcal{D}$, then a triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "p"'] \arrow[dr, phantom, "\Uparrow"] & \mathcal{D} \\ \mathcal{C}' \arrow[ur, "\mathrm{RKan}\, F"'] & \end{tikzcd}$$

is said to exhibit $\mathrm{RKan}\, F$ as the right Kan ext'n of F along p (short: right Kan ext'n) if it is terminal in the sense given for derived functors.

$$\mathrm{Map}(H, \mathrm{RKan}\, F) \xrightarrow{\sim} \mathrm{Map}(H \circ p, F)$$

Cor. If the right Kan extn RKan F exists for every $F: \mathcal{C} \to \mathcal{D}$, then the restriction functor $p^*: \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a right adjoint RKan.

$$\boxed{\mathcal{C}' = \mathcal{C}[W^{-1}]}$$

Lecture 7 non-abelian derived functors

$F: A \to B$ additive

(1) $K(F): K(A) \to K(B)$

(2)
$$\begin{tikzcd} K(A) \arrow[r, "K(F)"] \arrow[d] & K(B) \arrow[d] \\ D(A) \arrow[r, "LF"'] \arrow[ur, phantom, "\Nearrow"] & D(B) \end{tikzcd}$$

left derived functor (= right Kan ext'n)

Assume that $K(A)$ has enough K-projectives, then

① $K(A)^{K\text{-proj}} \xhookrightarrow{i} K(A) \xrightarrow{p} D(A)$ is an equiv. and i is left adjoint to the projection p.

② For any $F: K(A) \to \mathcal{D}$, the left derived functor $LF: D(A) \to \mathcal{D}$ is under this identification given by the restriction along i.

③ In particular, $LF: D(A) \to D(B)$ preserves connective objs

$$LF(D(A)_{\geq 0}) \subset D(B)_{\geq 0}$$

Questions. - How can we universally characterize $K(F): K(A)_{\geq 0} \to D(B)$?

- What if F is not additive ?

Yoneda Lemma.

We fix three Grothendieck universes.

{small sets} ⊂ {large sets} ⊂ {very large sets}

⇒ notion of categories / ∞-categories in all of these:

small ∞-cat: objs, morphisms, etc. are small.

large ∞-cat: —— " —— are large

very large ∞-cat: —— " —— are very large

S large ∞-cat. of small spaces (a.k.a. Kan cpxes)

$\hat{S}$ very large ∞-cat. of large spaces. $\qquad S \subset \hat{S}$

$\mathcal{C}$ any large ∞-cat. $\Rightarrow \mathrm{Map}_{\mathcal{C}}(a,b) \in \hat{S}$

$\mathrm{Cat}_\infty^{\mathrm{small}}$: large ∞-cat. of small ∞-cats

Cat_∞ : very large ∞-cat. of large ∞-categories

$\mathcal{C} \in \mathrm{Cat}_\infty \to \mathcal{C}[W^{-1}] \in \mathrm{Cat}_\infty$

$$\mathrm{Map}_{D(A)}(x,y) \simeq \underset{\tilde{x}\to x}{\mathrm{colim}}\ \mathrm{Map}_{K(A)}(\tilde{x},y) \in \hat{S}$$ A large abelian cat.

Def. We say that a large ∞-cat. $\mathcal{C}$ is <u>locally small</u> if for any pair $a,b \in \mathcal{C}$, the space $\mathrm{Map}_{\mathcal{C}}(a,b) \in \hat{S}$ is equiv. to an obj. in $S \subset \hat{S}$, i.e. is essentially small.

In this case,

$$\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \hat{S}$$ factors through $S \subset \hat{S}$.

<u>Construction</u>: For any large ∞-cat. $\mathcal{C}$, we have a functor

$$\hat{j} : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \hat{S}) = \hat{P}(\mathcal{C})$$
$$c \longmapsto \underline{c} = \mathrm{Map}_{\mathcal{C}}(-,c)$$

called the <u>Yoneda embedding</u>.

If $\mathcal{C}$ is locally small, then this factors as

$$j : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, S) = P(\mathcal{C}).$$

<u>Theorem</u> (Yoneda lemma): (1) The functor j is fully faithful.

(2) For any $F : \mathcal{C}^{op} \longrightarrow \hat{S}$ and any $x \in \mathcal{C}$, there is an equiv.

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{op}, S)}(\underline{x}, F) \simeq F(x)$$

(3) Every object $F \in \mathrm{Fun}(\mathcal{C}^{op}, \hat{S})$ is a (large) colimit of objects of the form $\underline{x}$ for $x \in \mathcal{C}$

Proof. $\mathrm{Map}_{\hat{P}(\mathcal{C})}(\underline{x},\underline{y}) \simeq \underline{y}(x) \simeq \mathrm{Map}_{\mathcal{C}}(x,y)$

For (3). For a given $F: \mathcal{C}^{op} \to \hat{\mathcal{S}}$ the pullback:

$$\begin{tikzcd}
\mathcal{C}_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(\mathcal{C}^{op}, \hat{\mathcal{S}})_{/F} \arrow[d] \\
\mathcal{C} \arrow[r, "j"'] & \mathrm{Fun}(\mathcal{C}^{op}, \hat{\mathcal{S}})
\end{tikzcd}$$

of ∞-cats

$\mathcal{C}_{/F}$ is large.

obj: $\underline{x} \to F$

morphism: $\begin{tikzcd}[column sep=small] \underline{x} \arrow[rr] \arrow[dr] & & \underline{y} \arrow[dl] \\ & F & \end{tikzcd}$

Now we claim that

$$F \simeq \operatorname*{colim}_{\underline{x} \in \mathcal{C}_{/F}} \underline{x}$$

Addendum: If $\mathcal{C}$ is small, then any $F: \mathcal{C}^{op} \to \mathcal{S}$ is a small colimit of representables.

Now let $F: \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{C}$ is small and $\mathcal{D}$ possibly large.

Prop. Assume that $\mathcal{D}$ admits all small colimits, then

① There is an essentially unique colimit preserving functor

$P(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, S) \longrightarrow D$ extending F:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "j"'] \arrow[dr, phantom, "\cong"] & D \\ P(\mathcal{C}) \arrow[ur] & \end{tikzcd}$$

This is also a left Kan ext'n.

② If D is locally small, then this is left adjoint to the restricted Yoneda embedding

$$D \to \mathrm{Fun}(D^{op}, S) \xrightarrow{F^*} \mathrm{Fun}(\mathcal{C}^{op}, S)$$

Cor. $\mathrm{Fun}^{colim}(P\mathcal{C}, D) \simeq \mathrm{Fun}(\mathcal{C}, D)$

i.e. $P\mathcal{C}$ is the universal ∞-cat. obtained from $\mathcal{C}$ by freely adjoining colimits.

Construction. Let K be any class of small colimit shapes, e.g.

K = all colimits, K = finite colimits, K = filtered, K = geom. realizations,
i.e. Δ^{op}-indexed colimits

We form $P^K(\mathcal{C}) \subset P(\mathcal{C})$ as the smallest full subcat. which contains representables and which is closed under K-indexed colimits.

Prop. We have for any large ∞-cat. D which admits K-indexed colimits, that restriction along $j: \mathcal{C} \to P^K(\mathcal{C})$ is an equiv.

$$\mathrm{Fun}^{K\text{-colim}}(P^K(\mathcal{C}), D) \underset{\text{"LKE"}}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, D)$$

Example. $\mathcal{C}$ any ∞-cat.

$$\mathrm{Ind}(\mathcal{C}) = P^{\text{filtered}}(\mathcal{C}) \qquad \left(= \mathrm{Fun}^{\text{fin-lim}}(\mathcal{C}^{op}, \mathcal{S}) \text{ if } \mathcal{C} \text{ has finite limits} \right)$$

Claim. (1) objects in $\mathrm{Ind}(\mathcal{C})$ are given by functors $I \xrightarrow{F} \mathcal{C}$ where I is filtered $\underset{i \in I}{\text{"colim"}} F_i \in \mathrm{Ind}(\mathcal{C})$.

(2) For any pair F, G, we have

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}\left(\underset{i\in I}{\text{"colim"}} F_i, \underset{j \in J}{\text{"colim"}} G_j\right) \simeq \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Map}_{\mathcal{C}}(F_i, G_j)$$

In particular, if $\mathcal{C}$ is a 1-cat., then so is $\mathrm{Ind}(\mathcal{C})$.

Prop (Dold-Kan + ...) Assume that $\mathcal{A}$ has enough compact projective objects, ($A \in \mathcal{A}$ cpt if $\mathrm{Hom}_{\mathcal{A}}(A, -)$ preserves filtered colimits), then we have an equiv.

$$\begin{aligned} D(\mathcal{A})_{\geq 0} &\simeq K(\mathcal{A}^{proj})_{\geq 0} \\ &\simeq P^{\Delta^{op}, \text{filtered}}(\mathcal{A}^{cp}) \\ &\simeq \mathrm{Fun}^{\pi}((\mathcal{A}^{cp})^{op}, \mathcal{S}) \quad \text{finite product preserving functors} \end{aligned}$$

Cor. A functor $D(\mathcal{A})_{\geq 0} \to \mathcal{E}$ that preserves geometric realizations and filtered colimits is uniquely determined by its restriction to $\mathcal{A}^{cp}$.

$$\begin{tikzcd} \mathcal{A}^{cp} \arrow[r] \arrow[d] & \mathcal{E} \\ D(\mathcal{A})_{\geq 0} \arrow[ur] & \end{tikzcd}$$

Def Let $\mathcal{C}$ be an ordinary cat. which admits small colimits and is generated under small colimits by $\mathcal{C}^{cp}$ ($x \in \mathcal{C}$ is called projective if $\mathrm{Hom}_{\mathcal{C}}(x,-)$ commutes w/ split coequalizers)

Then <u>the animation</u> $\mathrm{Ani}(\mathcal{C})$ is defined as $\mathcal{P}^{\Delta,\mathrm{fin}}(\mathcal{C}^{cp}) \subset \mathrm{Fun}^{\pi}((\mathcal{C}^{cp})^{op}, \mathcal{S})$

Example $\mathrm{Ani}(\mathrm{Set}) \simeq \mathcal{S}$.

Lecture 8 Spectra

$\mathcal{S}_*$ pointed spaces $(\mathcal{S}_{\partial^0/})$

$\Sigma, \Omega : \mathcal{S}_* \to \mathcal{S}_*$ defined via
$$\begin{tikzcd} X \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} , \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & pt \arrow[d] \\ pt \arrow[r] & X \end{tikzcd}$$

Def. A <u>spectrum</u> is a sequence of spaces X_i, $i \geq 0$ w/ equivalences $X_i \xrightarrow{\sim} \Omega X_{i+1}$.

Def $\mathrm{Sp} = \mathcal{E}q(\mathcal{S}_\bullet) \times_{\mathcal{S}_\bullet} \mathcal{E}q(\mathcal{S}_\bullet) \times_{\mathcal{S}_\bullet} \cdots$

w/ • $\mathcal{E}q(\mathcal{S}_\bullet) \subset \mathcal{S}_\bullet^{\Delta^1}$ full on equivalences

• $\searrow$: target

• $\swarrow$: $\Omega \circ$ source

<u>Objects</u>: list of $X_i \xrightarrow{\sim} \Omega X_{i+1}$.

<u>Maps</u>: Maps $X_i \xrightarrow{f_i} Y_i$ & homotopies
$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & Y_i \arrow[d] \\ \Omega X_{i+1} \arrow[r, "\Omega f_{i+1}"'] & \Omega Y_{i+1} \end{tikzcd}$$

Remark. $Sp \simeq \lim (\cdots \xrightarrow{\Omega} S_* \xrightarrow{\Omega} S_*)$ (in Cat_∞)

$$\mathrm{Map}_{Sp}(X,Y) = \lim \mathrm{Map}_{S_*}(X_i, Y_i)$$

Example. • $(HA)_i = K(A,i)$, equivalences

$$K(A,i-1) \xrightarrow{\sim} \Omega K(A,i)$$

• $X \in S_*$, $(\Sigma^\infty X)_i = \operatorname{colim}_k \Omega^k \Sigma^{k+i} X$

$$\Sigma^\infty S^0 =: \mathbb{S}$$

Lemma. $\mathrm{Map}_{Sp}(\Sigma^\infty X, Y) \simeq \mathrm{Map}_{S_*}(X, Y_0)$

Proof sketch: $X \to Y_0 \simeq \Omega^{k+i} Y_{k+i}$ is adjoint to $\Sigma^{k+i} X \to Y_{k+i}$

$\xRightarrow{\Omega^k} \Omega^k \Sigma^{k+i} X \to Y_i$

gives map $\operatorname{colim}_k \Omega^k \Sigma^{k+i} X \to Y_i$ $\square$

Def. $\Omega^\infty Y = Y_0$. $S_* \underset{\Sigma^\infty}{\overset{\Omega^\infty}{\rightleftarrows}} Sp$ ($\perp$)

Example. For $K \in S_*$ finite, have

$$\mathrm{Map}_{Sp}(\Sigma^\infty K, \Sigma^\infty X) \simeq \mathrm{Maps}_{S_*}(K, \Omega^\infty \Sigma^\infty X)$$

$$= \mathrm{Map}_{S_*}(K, \operatorname{colim}_k \Omega^k \Sigma^k X)$$

$$\simeq \operatorname{colim}_k \mathrm{Map}_{S_*}(K, \Omega^k \Sigma^k X) \simeq \operatorname{colim}_k \mathrm{Map}_{S_*}(\Sigma^k K, \Sigma^k X)$$

For $K = S^0$, get $\pi_n \mathrm{Map}(\mathbb{S}, \Sigma^\infty X) = \pi_n^s X$

stable homotopy gp.

<u>Lemma</u>. Sp has all limits, filtered colimits, a zero object.

Proof Zero obj. is given by $X_i = pt$.

Limits and filtered colimits commute w/ Ω.

<u>Lemma</u>. $\Omega : Sp \longrightarrow Sp$ is an equiv.

Proof. Ω takes $(X_0, X_1, \cdots) \longmapsto (\Omega X_0, X_0, X_1, \cdots)$

Inverse is given by shifting in the other direction.

Prop Sp has all colimits, and pushout squares are pullback squares.

Proof An ∞-cat. w/ finite limits, zero objects, and Ω is an equiv. has pushouts, and they are determined by the fact that pushout squares = pullback squares.

Once we have pushouts, have all colimits.

Note: $\Sigma : Sp \longrightarrow Sp$ is inverse to Ω. (i.e. Σ is a shift)

Y_i is $\Omega^\infty \Sigma^i Y$

<u>Example</u>. $[\Sigma^\infty X, \Sigma^n HA]_{Sp} = [X, K(A,n)]_{S_*} = H^n(X, A)$

$$[\Sigma^n \mathbb{S}, Y]_{Sp} = [S^n, \Omega^\infty Y]_{S_*} = \pi_n(\Omega^\infty Y)$$

Lemma. For every $Y \in Sp$, have $Y \simeq \operatorname{colim}_i \Sigma^{-i} \Sigma^\infty Y_i$

Proof.
$$\begin{aligned} \operatorname{Map}_{Sp}(Y, Z) &= \lim_i \operatorname{Map}_{S_*}(Y_i, Z_i) \\ &= \lim_i \operatorname{Map}_{Sp}(\Sigma^\infty Y_i, \Sigma^i Z) \\ &= \lim_i \operatorname{Map}_{Sp}(\Sigma^{-i}\Sigma^\infty Y_i, Z) \\ &= \operatorname{Map}_{Sp}(\operatorname{colim}_i \Sigma^{-i}\Sigma^\infty Y_i, Z) \end{aligned}$$

$$\begin{aligned} &\Sigma^{-i}\Sigma^\infty Y_i \\ \simeq\ &\Sigma^{-(i+1)}\Sigma^\infty \Sigma Y_i \\ \to\ &\Sigma^{-(i+1)}\Sigma^\infty Y_{i+1} \end{aligned}$$

□

Construction $C_* : Sp \longrightarrow D(\mathbb{Z})$ "Homology / Chains"

$$Y \longmapsto \operatorname{colim}_i C_*(Y_i)[-i]$$

preserves colimits, takes $\Sigma^\infty X \mapsto C_*(X)$

(in particular, $\mathbb{S} \mapsto \mathbb{Z}[0]$)

Def. Let $Sp^{fin} \subset Sp$ full subcat. of spectra of the form $\Sigma^{-n}\Sigma^\infty K$, K a finite space

Prop. $Sp = \operatorname{Ind}(Sp^{fin})$

Proof. Have functor Sp^{fin} in Sp are compact.

$$\begin{aligned} \operatorname{Map}_{Sp}(\Sigma^{-n}\Sigma^\infty K, \operatorname{colim}_j Y^j) &\simeq \operatorname{Map}_{S_*}(K, \Omega^\infty \Sigma^n \operatorname{colim}_j Y^j) \\ &\simeq \operatorname{colim}_j \operatorname{Map}_{S_*}(K, \Omega^\infty \Sigma^n Y^j) \end{aligned}$$

Write $Y = \operatorname{colim}_i \Sigma^{-i}\Sigma^\infty Y_i$, and each Y_i as filtered colimit of finite spaces.

Analogy: In $D(R) \supseteq \mathrm{Perf}(R)$ (full subcat. of perfect cpxes)

$D(R) = \mathrm{Ind}(\mathrm{Perf}(R))$. Think of Sp as "$D(\mathbb{S})$"

Additional subtleties arise from $\mathrm{Map}_{Sp}(\mathbb{S}, \mathbb{S})$ is not discrete.

Think about $C_* : Sp \to D(\mathbb{Z})$ as "base change along $\mathbb{S} \to \mathbb{Z}$"

Def. $X \in Sp$ is <u>n-connective</u> if $\pi_i X = 0$, $i<n$

$$\begin{array}{c} \pi_i X \\ \| \\ [\Sigma^i \mathbb{S}, X]_{Sp} \end{array}$$

<u>n-coconnective</u> if $\pi_i X = 0$, $i>n$.

$Sp_{\geq n} \subset Sp$

$Sp_{\leq n} \subset Sp$

Prop. $Sp_{\leq n} \to Sp$ has a left adjoint $\tau_{\leq n}$

Proof $X^n = X$, build X^i as follows:

pick generators of $\pi_i(X^{i-1})$, map $\bigoplus \Sigma^i \mathbb{S} \to X^{i-1}$

$$\begin{tikzcd} \bigoplus \Sigma^i \mathbb{S} \arrow[r] \arrow[d] & X^{i-1} \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X^i \end{tikzcd}$$

Let $\tau_{\leq n} X = \operatorname{colim}_i X^i$

- $\tau_{\leq n} Y \in Sp_{\leq n}$, and $\pi_i \tau_{\leq n} X = \pi_i X$ for all $i \leq n$

(since $\pi_i(\mathbb{S}) = 0$ for $i<0$)

- every map $X \to Y \in Sp_{\leq n}$ factors through all of these pushouts. and $\tau_{\leq n} X \to Y$

Def Let $\tau_{\geq n} X$ be the fiber of $X \to \tau_{\leq n-1} X$.

Lemma - If $X \in Sp_{\geq n}$, $Y \in Sp_{\leq n-1}$, $Map(X, Y) = 0$

- $Sp_{\geq n} \cap Sp_{\leq n} \simeq Ab$

Proof - Any map $X \to Y$ factors through $\tau_{\leq n-1} X \to Y$

$$\begin{tikzcd} \tau_{\leq n-1} X \arrow[d, phantom, "\wr"] \arrow[r] & Y \\ 0 & \end{tikzcd}$$

- For the second statement, observe that π_n gives a functor and that mapping spaces are discrete.

Since $\pi_i Map(X, Y) = Map(\Sigma^i X, Y) \simeq 0 \quad (i > 0)$

If $\pi_n(X)$ is free, then can choose a map

$\bigoplus \Sigma^n S \to X$ iso. on π_n, and thus an iso under $\tau_{\leq n}$.

(get an iso $Map(X, Y) \xrightarrow{\sim} Map(\bigoplus \Sigma^n S, Y) \simeq \prod \pi_n Y$)

If it is not free, pick $\bigoplus \Sigma^n S \to X$ surj. on π_n.

Fiber has free π_n. Use exact sequence for $\pi_* Map(-, Y)$.

Postnikov tower for any X, have a tower

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & \tau_{\leq n+1} X \arrow[d] & H(\pi_{n+1}(X))[n+1] \arrow[l] \\ X \arrow[ur] \arrow[r] \arrow[dr] & \tau_{\leq n} X \arrow[d] & \\ & \vdots & \end{tikzcd}$$

Lecture 9 Symmetric monoidal ∞-categories

Recall. An (ordinary) symm. monoidal cat. consists of

- $\mathcal{C}$ category
- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- $\mathbb{1}_\mathcal{C} \in \mathcal{C}$
- nat'l isoms
  - $(a \otimes b) \otimes c \xrightarrow{\sim} a \otimes (b \otimes c)$
  - $a \otimes \mathbb{1}_\mathcal{C} \xrightarrow{\sim} a \xleftarrow{\sim} \mathbb{1}_\mathcal{C} \otimes a$
  - $a \otimes b \xrightarrow{\sim} b \otimes a$

satisfies a number of coherence conditions.

$$
\begin{tikzcd}
((a\otimes b)\otimes c)\otimes d \arrow[d, "\wr"] \arrow[dr, "\sim"] & \\
(a\otimes (b\otimes c))\otimes d \arrow[d, "\wr"] \arrow[r, phantom, "|||"] & (a\otimes b)\otimes (c\otimes d) \arrow[d, "\wr"] \\
a\otimes ((b\otimes c)\otimes d) \arrow[r, "\sim"] & a\otimes (b\otimes (c\otimes d))
\end{tikzcd}
$$

commutes

- We abbreviate this as $(\mathcal{C}, \otimes)$ or $\mathcal{C}$.
- We will write $a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_n = (((a_1 \otimes a_2) \otimes a_3) \otimes \cdots \otimes a_n)$

- A <u>lax symmetric monoidal functor</u> between symm. monoidal cats $\mathcal{C}$ and $\mathcal{D}$

is given by
- $F : \mathcal{C} \to \mathcal{D}$
- nat'l morphisms $F(c) \otimes_{\mathcal{D}} F(c') \to F(c \otimes_{\mathcal{C}} c')$, $c, c' \in \mathcal{C}$

$$\mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$$

satisfying compatibility conditions w/ the assoc., unit., and sym. isomorphisms.

Such a lax sym. monoidal functor is called <u>(strongly) sym. monoidal</u> if the maps

$F(c) \otimes_{\mathcal{D}} F(c') \xrightarrow{\sim} F(c \otimes_{\mathcal{C}} c')$ and $\mathbb{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbb{1}_{\mathcal{C}})$ are isoms.

- A sym. monoidal cat. $(\mathcal{C}, \otimes)$ is called <u>closed</u> if for any obj. $c \in \mathcal{C}$, the functor

$-\otimes c : \mathcal{C} \to \mathcal{C}$ admits a right adjoint $(\underline{\mathrm{Hom}}(c, -))$

In particular. it preserves colimits in $\mathcal{C}$.

<u>Prop</u>. Assume that R is a comm. ring, then the cat. Mod_R admits an essentially unique sym. monoidal str. which is closed and w/ tensor unit R.

$$\otimes_R : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

<u>"Proof sketch"</u> - $R \otimes M \simeq M$

$$-(\bigoplus_I R) \otimes M \simeq \bigoplus_I (R \otimes M) \simeq \bigoplus_I M$$

N arb. R-module, $\bigoplus_J R \xrightarrow{f} \bigoplus_I R \twoheadrightarrow N$, i.e. $N = \mathrm{coker}(f)$

$$N \otimes M \cong \mathrm{coker}(f) \otimes M = \mathrm{coker}\left(\bigoplus_J M \xrightarrow{b} \bigoplus_I M\right) \qquad \square$$

The functor for any map $R \to S$ of comm. rings

$-\otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ is sym. monoidal, (i.e. inherits canonical refinement to a sym. monoidal functor)

$$(M \otimes_R N) \otimes_R S \cong (M \otimes_R S) \otimes_S (N \otimes_R S)$$

Goal today (1) Define sym. monoidal ∞-cat. s.t. $N\mathcal{C}$ is an example for an ord. sym. monoidal cat.

(2) Define lax / strong sym. monoidal functors

Thm ① The ∞-cat. Sp and $D(R)$ admit essentially unique closed sym monoidal strs. w/ units $\mathbb{S}$ resp. R.

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$-\otimes c : \mathcal{C} \to \mathcal{C}$

$$\otimes_{\mathbb{S}} : Sp \times Sp \to Sp. \qquad \overset{\otimes^L_R}{\underset{\otimes_R}{\shortparallel}} : D(R) \times D(R) \to D(R)$$

② The functors $C_* : Sp \to D(\mathbb{Z})$

$\Sigma^\infty_+ : \mathcal{S} \to Sp$

$\otimes^L_R S : D(R) \to D(S)$ for any map $R \to S$

inherit canonical strong sym. monoidal structures

"Proof sketch" $\quad X \in Sp$, - $\mathbb{S} \otimes X \simeq X$

$$- (\Sigma^\infty_+ Y) \otimes X$$
$$\simeq \Sigma^\infty_+ (\operatorname{colim}_Y (pt)) \otimes X$$
$$\simeq \operatorname{colim}_Y (\Sigma^\infty_+ pt \otimes X) \simeq \operatorname{colim}_Y X.$$

- $(\Sigma^{-n} y) \otimes X \simeq \Sigma^{-n}(y \otimes X)$

- general Spectrum y is $\operatorname{colim}_n (\Sigma^{\infty - n} y_n)$

$\Rightarrow$ This determines $y \otimes X$ up to equiv.

Let Fin_* be the cat. of finite pointed sets

$$\langle n \rangle = \{0, \dots, n\} \quad w/ \ 0 \text{ as base pt}$$

$N Fin_*$ is the nerve, i.e. the assoc. ∞-cat.

- $1 \leq i \leq n$, $\rho^i : \langle n \rangle \to \langle 1 \rangle$, $k \mapsto \begin{cases} 1, & k = i \\ 0, & \text{else} \end{cases}$

<u>Def</u> (first def) Segal $\quad$ A sym. monoidal ∞-cat. is a functor

$$\underline{e} : N Fin_* \to Cat_\infty$$

s.t. the induced maps $\quad \underline{e}(\langle n \rangle) \xrightarrow[\sim]{(\rho^i_*)} \prod_{i=1}^{n} e(\langle 1 \rangle)$

are equiv. for $n = 0, 1, \dots$

Notation. $\mathcal{C} = \underline{\mathcal{C}}(\langle 1 \rangle)$

$$\otimes : \mathcal{C} \times \mathcal{C} \xleftarrow{\sim} \underline{\mathcal{C}}(\langle 2 \rangle) \xrightarrow{m_*} \underline{\mathcal{C}}(\langle 1 \rangle) = \mathcal{C}$$

$$\begin{aligned} m : \langle 2 \rangle &\longrightarrow \langle 1 \rangle \\ 0 &\longmapsto 0 \\ 1 &\longmapsto 1 \\ 2 &\longmapsto 1 \end{aligned}$$

$$\mathbb{1} : \ pt = \underline{\mathcal{C}}(\langle 0 \rangle) \longrightarrow \underline{\mathcal{C}}(\langle 1 \rangle) = \mathcal{C}$$

Def A (strong) symm. monoidal functor is given by a nat'l transf. $\underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}$.

$$\mathrm{SymMonCat}_\infty \subset \mathrm{Fun}(N\mathrm{Fin}_*, \mathrm{Cat}_\infty)$$

Def (2nd, Lurie) A sym. monoidal ∞-cat. is a functor $\mathcal{C}^\otimes \longrightarrow N\mathrm{Fin}_*$ satisfying the following conditions:

(1) It is a coCartesian fibration

$$\begin{tikzcd} \mathcal{C}^\otimes_{\langle n \rangle} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{C}^\otimes \arrow[d] \\ pt \arrow[r, "\langle n \rangle"] & N\mathrm{Fin}_* \end{tikzcd}$$

$f : \langle n \rangle \longrightarrow \langle m \rangle$ in Fin_*

$$f_! : \mathcal{C}^\otimes_{\langle n \rangle} \longrightarrow \mathcal{C}^\otimes_{\langle m \rangle}$$

(2) The induced maps $\mathcal{C}^\otimes_{\langle n \rangle} \xrightarrow[\sim]{(\rho^i)_!} \prod \mathcal{C}^\otimes_{\langle 1 \rangle}$ are equivs for $n = 0, 1, \cdots$

Notation. $\mathcal{C} = \mathcal{C}^\otimes_{\langle 1 \rangle}$, $\quad \otimes : \mathcal{C} \times \mathcal{C} \xleftarrow{\sim} \mathcal{C}^\otimes_{\langle 2 \rangle} \xrightarrow{m_!} \mathcal{C}^\otimes_{\langle 1 \rangle} = \mathcal{C}$

<u>Thm</u> (Lurie) The two definitions "agree", that is for any such sym. monoidal cat.

$\mathcal{C}^{\otimes} \to N\mathrm{Fin}_*$, we get an induced functor $N\mathrm{Fin}_* \to \mathrm{Cat}_\infty$ and vice versa.
$$\langle n \rangle \mapsto \mathcal{C}^{\otimes}_{\langle n \rangle}$$

Let $(\mathcal{C}, \otimes)$ be an ord. sym. monoidal cat. We define $\mathcal{C}^{\otimes}$ as follows.

<u>Objects</u>. $c_1, \dots c_n$ objects in $\mathcal{C}$, $\langle n \rangle \in \mathrm{Fin}_*$

<u>morphisms</u> $(c_1, \dots, c_n)$ to $(d_1, \dots, d_m)$ are given by

- $\langle n \rangle \xrightarrow{f} \langle m \rangle$ a map in Fin_*
- for each $k \in \langle m \rangle \setminus 0$ a map

$$\bigotimes_{i \in f^{-1}(k)} c_i \longrightarrow d_k \quad \text{a map in } \mathcal{C}.$$

- There is a composition defined in the obvious way.

  and a functor $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$

A morphism $f : \langle n \rangle \to \langle m \rangle$ is called <u>inert</u> if the induced map

$$f|_{f^{-1}(\langle m \rangle \setminus 0)} : f^{-1}(\langle m \rangle \setminus 0) \to \langle m \rangle \setminus 0 \quad \text{is a bijection}$$

<u>Def</u>. A lax sym. monoidal functor between sym. monoidal ∞-cats $\begin{matrix}\mathcal{C}^{\otimes} \\ \downarrow \\ N\mathrm{Fin}_*\end{matrix}$ and $\begin{matrix}\mathcal{D}^{\otimes} \\ \downarrow \\ N\mathrm{Fin}_*\end{matrix}$

is a functor
$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr] & & \mathcal{D}^{\otimes} \arrow[dl] \\ & N\mathrm{Fin}_* & \end{tikzcd}$$
s.t. $F^{\otimes}$ sends coCartesian lifts of inert morphisms to coCartesian lifts.

Lecture 10. $\mathbb{E}_n$-algebras

$\mathcal{C}$ sym monoidal ∞-cat

(1) $\mathcal{C}$ sym. monoidal 1-cat. $\rightsquigarrow N\mathcal{C}$ sym. mon. ∞-cat

(2) $\mathcal{C}$ sym. monoidal top. enriched cat. $\rightsquigarrow N_\Delta(\mathcal{C})$ sym. mon. ∞-cat.

(3) $(\mathcal{S}, \times)$ sym. mon. ∞-cat.

(4) $(\mathcal{S}_*, \wedge)$, $X \times Y / X \vee Y$

Def An assoc. alg. in $\mathcal{C}$ is a sym. mon. functor

$$N(\mathrm{Ass}^{\otimes}_{act}) \longrightarrow \mathcal{C}$$

$$\mathrm{Alg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(N\mathrm{Ass}^{\otimes}_{act}, \mathcal{C})$$

- A comm. alg. in $\mathcal{C}$ is given by a sym. mon. functor

$$N(\mathrm{Comm}^{\otimes}_{act}) \longrightarrow \mathcal{C}$$

$$\mathrm{CAlg}(\mathcal{C}) = \mathrm{Fun}^{\otimes}(N(\mathrm{Comm}^{\otimes}_{act}), \mathcal{C})$$

Warning: In a 1-cat. $\mathcal{C}$, we have $\mathrm{CAlg}(\mathcal{C}) \overset{full}{\subset} \mathrm{Alg}(\mathcal{C})$

This is false in ∞-cats. $\mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{Alg}(\mathcal{C})$. NOT fully faithful

Def. Let $0 \leq n < \infty$. We define a sym. mon. ∞-cat. $(\mathbb{E}_n^{\otimes})_{act}$ as the homotopy coherent nerve of the top. enriched cat obj: $\underbrace{D^n \amalg \cdots \amalg D^n}_{k\text{-fold}}$, $D^n = (0,1)^n$

Dasero

Morphisms. $D^n \amalg \cdots \amalg D^n \longrightarrow D^n \amalg \cdots \amalg D^n$ are given by rectilinear embeddings,

i.e. embeddings of top. spaces which on each disk is rectilinear, i.e. given by an

affine linear map w/ matrix of the form $\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$, $\alpha_i \in \mathbb{R}_{>0}$

Composition is composition.

This cat. is top. enriched and sym. mon. by $\amalg$

Def. An $\mathbb{E}_n$-alg. in $\mathcal{C}$ is a sym. monoidal functor $(\mathbb{E}_n^{\otimes})_{act} \longrightarrow \mathcal{C}$.

$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) = \mathrm{Fun}^{\otimes}((\mathbb{E}_n)^{\otimes}_{act}, \mathcal{C})$

Mapping spaces in $(\mathbb{E}_n^{\otimes})_{act}$ are determined by

$$\begin{array}{rc} & \mathrm{Map}_{(\mathbb{E}_n^{\otimes})_{act}}(\overbrace{D^n \amalg \cdots \amalg D^n}^{k \text{ copies}}, D^n) \\ \text{homotopy equiv.} & \simeq \big\downarrow \text{ev at center points} \\ & \mathrm{Conf}_k(D^n) \subset (D^n)^k \end{array}$$

$\rightarrow \mathrm{Map}_{(\mathbb{E}_n^{\otimes})_{act}}(D^n, D^n) \simeq \mathrm{Conf}_1(D^n) \simeq pt$

$\mathrm{Map}_{(\mathbb{E}_n^{\otimes})_{act}}(D^n \amalg D^n, D^n) \simeq \mathrm{Conf}_2(D^n) \simeq S^{n-1}$

$\underline{A} : (\mathbb{E}_n^{\otimes})_{act} \longrightarrow \mathcal{C}$ underlying alg. is $\underline{A}(D^n) = A$

$S^{n-1} \longrightarrow \mathrm{Map}(A \otimes A, A)$ etc...

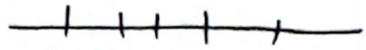
Prop There is an equiv.

$$N\,\mathrm{Ass}^{\otimes}_{act} \xrightarrow{\sim} (\mathbb{E}_1^{\otimes})_{act} \quad \text{of sym. mon. } \infty\text{-cats}$$

sending $S \longmapsto \coprod_S D^1$.

Proof (1) $(\mathbb{E}_1^{\otimes})_{act}$ is essentially an 1-cat.

$D^1 \amalg \cdots \amalg D^1 \longrightarrow D^1$



and equiv. to $\mathrm{Ass}^{\otimes}_{act}$

(2) This functor is sym monoidal

(3) Thus it is an equiv. of sym mon. ∞-cat. $\square$.

Cor. $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C})$.

[$\mathbb{E}_2$-alg. in the ∞-cat. of 1-cats : braided monoidal cats]

There are functors

$$(\mathbb{E}_0^{\otimes})_{act} \xrightarrow{\times D^1} (\mathbb{E}_1^{\otimes})_{act} \xrightarrow{\times D^1} (\mathbb{E}_2^{\otimes})_{act} \xrightarrow{\times D^1} \cdots$$

- Thus we get induced functor

$$\cdots \longrightarrow \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$$

Def. The ∞-cat. of $\mathbb{E}_\infty$-algs is defined as the inverse limit of this diagram in Cat_∞.

Thm We have an equiv. $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})$.

"Proof" $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \varprojlim \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \varprojlim \mathrm{Fun}^{\otimes}((\mathbb{E}_n^{\otimes})_{act}, \mathcal{C})$

$$\simeq \mathrm{Fun}^{\otimes}(\mathrm{colim}\,(\mathbb{E}_n^{\otimes})_{act}, \mathcal{C})$$
$$\simeq \mathrm{Fun}^{\otimes}(N\mathrm{Comm}^{\otimes}_{act}, \mathcal{C})$$

Need: $\mathrm{colim}\,(\mathbb{E}_n^{\otimes})_{act} \simeq N\mathrm{Comm}^{\otimes}_{act}$

Let $(\mathcal{C}, \otimes)$ be closed sym. mon., or weaker assume that

$- \otimes c : \mathcal{C} \to \mathcal{C}$ commutes w/ filtered colimits & geom. realizations (i.e. Δ^{op}-index colimits)

and $\mathcal{C}$ has all colimits & limits

Thm (1) The ∞-cat. $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$, $0 \le n \le \infty$ admits all limits and colimits.

the functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \xrightarrow[ev_{\langle 1 \rangle}]{\text{forget}} \mathcal{C}$ preserves filtered colimits, geom. realizations, and all limits.

Moreover, it detects equivalences

(2) For $n = \infty$, the coproduct is given by $A \otimes B$, $A, B \in \mathrm{CAlg}(\mathcal{C})$.

(3) The ∞-cat. $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ admits a sym. monoidal str. s.t.

$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$ is sym. monoidal

<u>Example</u> Consider $(\mathcal{S}, \times)$

$\mathrm{Alg}(\mathcal{S})$ the cat. of "monoids in $\mathcal{S}$"

$$X \times X \longrightarrow X$$

We have that $\pi_0(X)$ is an actual monoid in Set. We call X <u>group-like</u> if

$\pi_0 X$ is a gp. $\iff$ $X \times X \xrightarrow{\sim} X \times X$ homotopy equiv.

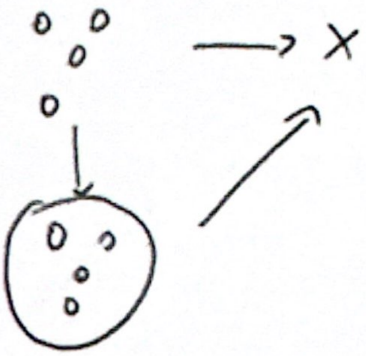
$$(a,b) \longmapsto (ab, b)$$

An $\mathbb{E}_n$-alg. X in $\mathcal{S}$ is called group-like if the underlying $\mathbb{E}_1$-alg. is.

For every pointed space $X \in \mathcal{S}_*$,

$$\Omega^n X \simeq \mathrm{Map}_*(S^n, X) \simeq \mathrm{Map}((D^n, \partial D^n), (X, *))$$

is canonically an $\mathbb{E}_n$-alg. in $(\mathcal{S}, \times)$



<u>Thm (Boardman–Vogt)</u> The functor $\Omega^n : \mathcal{S}_*^{n\text{-conn}} \longrightarrow \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$

is fully faithful w/ essential image the group-like $\mathbb{E}_n$-algs for any $0 \leq n < \infty$.

<u>Cor</u> For any $\mathbb{E}_n$-alg. A in $\mathcal{S}$ and any conn'd pointed space $X \in \mathcal{S}_*^{\mathrm{conn}}$, we have

that $\mathrm{Map}_{\mathbb{E}_n}(\Omega^n(\Sigma^n X), A) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, A)$

Proof. We can consider the subspace $A' \subset A$ consisting of the unit components

$\pi_0(A)' \subset \pi_0(A)$, then

$$\mathrm{Map}_{\mathbb{E}_n}(\Omega^n\Sigma^n X, A) \simeq \mathrm{Map}_{\mathbb{E}_n}(\Omega^n\Sigma^n X, A')$$

Similarly, $\mathrm{Map}_{S_*}(X, A) \simeq \mathrm{Map}_{S_*}(X, A')$

We can assume A is grouplike, i.e. $A = \Omega^n Y$.

$$\mathrm{Map}_{\mathbb{E}_n}(\Omega^n\Sigma^n X, \Omega^n Y) \simeq \mathrm{Map}_{S_*}(\Sigma^n X, Y) \simeq \mathrm{Map}_{S_*}(X, \Omega^n Y) \simeq \mathrm{Map}_{S_*}(X, A).$$

Sketch of proof. For $n=1$, $S_*^{\mathrm{conn}} \underset{\mathrm{Bar}}{\overset{\Omega}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{gp}}(S)$

Idea. Consider an inverse:

$$\mathrm{Bar}(G) = \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\left(\cdots \substack{\to\\\to\\\to\\\to} G\times G \substack{\to\\\to\\\to} G \rightrightarrows pt\right)$$

$$\Omega\,\mathrm{Bar}(G) \simeq G \qquad\qquad G \longrightarrow \underset{*}{\overset{EG}{\wr\!\wr}} \longrightarrow BG$$

- Second step $S_*^{2\text{-conn}} \rightleftarrows \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{gp}}(S_*^{\mathrm{conn}}) \underset{\mathrm{Bar}}{\overset{\Omega}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{gp}}(S))$

Thm (Dunn-Additivity) For any sym. monoidal ∞-cat. $\mathcal{C}$,

$$\mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})) \quad , n, m \geq 0.$$

$\mathcal{C}$ has geom. realization, and $\otimes$ commutes w/ them in both variables

Cor $\mathrm{Bar}: \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})_{/\mathbb{1}} \longrightarrow \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C})$, $A \longmapsto \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\left(\cdots \substack{\to\\\to\\\to} A\times A \substack{\to\\\to\\\to} A \rightrightarrows \mathbb{1}\right)$

Proof $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C}))$

Lecture 11 p-adic completion

A abelian grp <u>p-complete</u> if is complete (and separated) wrt. the p-adic topology on A (base $p^n A \subset A$)

$\Leftrightarrow$ $A \xrightarrow{\sim} \varprojlim A/p^n = A^\wedge_p$.

Def. For $X \in Sp$, $n \in \mathbb{Z}$, we set $X/n = \mathrm{cofib}(X \xrightarrow{n} X)$

(equi. $X/n \simeq X \otimes_{\mathbb{S}} \mathbb{S}/n$)

- If $n \mid m$, there is a canonical map $X/m \longrightarrow X/n$, obtained as the cofiber of

obtained as the cofiber of

$$\begin{tikzcd} X \arrow[r, "m/n"] \arrow[d, "m"'] & X \arrow[d, "n"] \\ X \arrow[r, "id"] \arrow[d] & X \arrow[d] \\ X/m \arrow[r] & X/n \end{tikzcd}$$

- we define for $X \in Sp$ a spectrum $X^\wedge_p = \varprojlim(\cdots \to X/p^3 \to X/p^2 \to X/p)$

There is a canonical map $X \longrightarrow X^\wedge_p$ induced from the maps $X \longrightarrow X/p^n$.

- We say a spectrum X is p-complete if $X \xrightarrow{\sim} X^\wedge_p$ is an equiv.

Let $Sp^\wedge_p$ be the full subcat. of Sp on p-complete spectra.

Thm. (1) For every spectrum X, the spectrum $X_p^\wedge$ is p-complete.

(2) The functor $(-)_p^\wedge : Sp \longrightarrow Sp_p^\wedge$ is left adjoint to the inclusion $Sp_p^\wedge \hookrightarrow Sp$.

$X \longrightarrow X_p^\wedge$

(3) The ∞-cat. $Sp_p^\wedge$ has all limits & colimits.

"Proof" ① - $Sp_p^\wedge$ is closed under limits and finite colimits.

- For any X, X/p is p-complete

- $X/p \xrightarrow{p^{n-1}} X/p^n \to X/p^{n-1}$ $\qquad X_p^\wedge = \varprojlim (X/p^n)$

② $(\ )_p^\wedge$ idempotent. i.e. $(X_p^\wedge)_p^\wedge \simeq X_p^\wedge$

$\mathrm{Map}(X, Y) \leftarrow \mathrm{Map}(X_p^\wedge, Y)$

③ limits ✓ finite colimits ✓ For an arb. diag $I \xrightarrow{X} Sp_p^\wedge$, the colimit is given by $\left(\operatorname{colim}_I^{Sp} X_i\right)_p^\wedge$.

Def. A map $f: X \to Y$ of spectra is called <u>p-adic equiv.</u> if the induced map $f_p^\wedge : X_p^\wedge \xrightarrow{\sim} Y_p^\wedge$ is an equiv. of spectra.

Note. If X & Y are p-complete, then f is a p-adic equiv. $\Leftrightarrow$ f is an equiv.

Prop. ① A map $f: X \to Y$ is a p-adic equiv. iff the map $f/p: X/p \to Y/p$ is an equiv.

② A map $f: X \to Y$ for X, Y connective ($\pi_i(X) = \pi_i(Y) = 0$, for $i<0$)

is a p-adic equiv. iff $X \underset{S}{\otimes} H\mathbb{F}_p \longrightarrow Y \underset{S}{\otimes} H\mathbb{F}_p$ is an equiv.

<u>Prop.</u> – A abelian gp, A is p-complete $\Rightarrow$ $HA \in Sp$ is p-complete

e.g. $H\mathbb{Z}_p^\wedge$ is p-complete

– Converse fails in general, but if A has bounded order of p^∞-torsion, then $\Leftarrow$ also holds.

<u>Def</u> An abelian gp A is called <u>derived p-complete</u> if HA is p-complete.

<u>Rmk</u> There is also a notion of p-completeness etc. in $D(\mathbb{Z})$

Similar to Sp. Then A is derived p-complete iff $A[0] \in D(\mathbb{Z})$ is p-complete

$\Leftrightarrow$ $A \xrightarrow{\sim} R\varprojlim \operatorname{cone}(A \xrightarrow{p^n} A) \in D(\mathbb{Z})$

<u>Thm</u> (1) A spectrum X is p-complete iff $\pi_n X$ is derived p-complete for any n.

(2) Assume $\pi_n X$ has bounded order of p^∞-torsion for each n, then

$\pi_*(X_p^\wedge) = \pi_*(X)_p^\wedge$

<u>Examples</u>.. $\pi_*(S_p^\wedge) = \begin{cases} \mathbb{Z}_p & * = 0 \\ p^\infty\text{-torsion in } \pi_*(S), & * > 0 \\ 0 & , * < 0 \end{cases}$

– $(H\mathbb{Q})_p^\wedge = 0$, $H\mathbb{Z}_p$ is p-complete

<u>Monoidal properties</u> – $Sp_p^\wedge$ admits a closed sym. monoidal str. w/ $X \hat{\otimes}_p Y := (X \underset{S}{\otimes} Y)_p^\wedge$

$Sp \xrightarrow{(\,)_p^\wedge} Sp_p^\wedge$ is strongly sym. monoidal.

- If R is an $\mathbb{E}_n$-algebra, so is $R_p^\wedge$, and $R \to R_p^\wedge$ is an $\mathbb{E}_n$-map exhibits

$R_p^\wedge$ as the initial p-complete $\mathbb{E}_n$-algebra under R.

$$\mathrm{Alg}_{\mathbb{E}_n}(Sp_p^\wedge) \underset{\subset}{\overset{(\cdot)_p^\wedge}{\longleftarrow}} \mathrm{Alg}_{\mathbb{E}_n}(Sp)$$

<u>Question</u>. Can we recover any spectrum X from $X_p^\wedge$, p primes ?

<u>Recall</u>. An abelian gp A is called <u>rat'l</u> if it is uniquely divisible.

$\Leftrightarrow$ it is a $\mathbb{Q}$-v.s. $\Leftrightarrow$ $A \xrightarrow{\sim} A \otimes_{\mathbb{Z}} \mathbb{Q} = A_{\mathbb{Q}}$ isom.

<u>Def</u>. A spectrum X is called rat'l if $\pi_n(X)$ is rat'l for each n.

We denote $Sp_{\mathbb{Q}} \subset Sp$ the full subcat. on all rat'l spectra.

<u>Thm</u> · X rat'l $\Leftrightarrow$ it admits the str of an $H\mathbb{Q}$-module.

· $X \otimes_{\mathbb{S}} H\mathbb{Q} =: X_{\mathbb{Q}}$ is rat'l, and $\pi_*(X_{\mathbb{Q}}) = \pi_*(X)_{\mathbb{Q}}$.

· The functor $- \otimes_{\mathbb{S}} H\mathbb{Q} : Sp \longrightarrow Sp_{\mathbb{Q}}$ is left adj't to the inclusion $Sp_{\mathbb{Q}} \subset Sp$.

$Sp_{\mathbb{Q}} \subset Sp$ is closed under limits & colimits.

· $Sp_{\mathbb{Q}} \simeq \mathrm{Mod}_{H\mathbb{Q}} \simeq D(\mathbb{Q})$

· $Sp_{\mathbb{Q}}$ is sym monoidal, $X \otimes Y := X \otimes_{\mathbb{S}} Y = X \otimes_{H\mathbb{Q}} Y$, $\mathbb{1} = H\mathbb{Q}$

Slogan. Any spectrum X can be "recovered" from $X_p^\wedge$, $\forall p$ & $X_{\mathbb{Q}}$

(and a certain "gluing map")

Maps $X \to X_p^\wedge$, $X \to X_{\mathbb{Q}}$, & $X_{\mathbb{Q}} \to (X_p^\wedge)_{\mathbb{Q}} =: X_{\mathbb{Q}_p}$

get a comm. square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & X_{\mathbb{Q}_p} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & (\prod_p X_p^\wedge)_{\mathbb{Q}} \end{tikzcd}$$

Hasse square / fracture square

Thm - This square is a pullback for any spectrum X

- We have a pullback of ∞-cats

$$\begin{tikzcd} Sp \arrow[r, "(\cdot)_p^\wedge"] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \prod_p Sp_p^\wedge \arrow[d] & (X_p)_{p \in P} \arrow[d, maps to] \\ Sp_{\mathbb{Q}}^{\Delta^1} \arrow[r, "\text{target}"] & Sp_{\mathbb{Q}} & (\prod X_p)_{\mathbb{Q}} \end{tikzcd}$$

- We have pullbacks

$$\begin{tikzcd} Alg_{\mathbb{E}_n}(Sp) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \prod_p Alg_{\mathbb{E}_n}(Sp_p^\wedge) \arrow[d] \\ (Alg_{\mathbb{E}_n}(Sp_{\mathbb{Q}}))^{\Delta^1} \arrow[r] & Alg_{\mathbb{E}_n}(Sp_{\mathbb{Q}}) \end{tikzcd}$$

$n = 0, 1, 2, 3, \dots, \infty$

Lecture 12. The Tate construction.

• Let G be a gp object in $(\mathcal{S}, \times)$, i.e. a group like $\mathbb{E}_1$-space.

We have $BG \in \mathcal{S}$, $\mathcal{C}$ any ∞-cat.

$$\mathcal{C}^{BG} = \mathrm{Fun}(BG, \mathcal{C}) \quad \text{object in } \mathcal{C} \text{ w/ } G\text{-action.}$$

Construction. For G as before, we consider G as an object in $\mathcal{S}^{B(G\times G)}$ by letting $G\times G$ act on G as $(g,h)\cdot x = gxh^{-1}$.

Def (Klein) We define a spectrum $D_G \in Sp^{BG}$ called the <u>dualizing spectrum of G</u> as follows: $D_G := (\Sigma^\infty_+ G)^{h(G\times 1)}$ w/ its remaining $G = (1\times G)$-action.

Example Assume G finite, then

$$\begin{tikzcd} (\Sigma^\infty_+ G)^{hG} = \left(\bigoplus_{g\in G} \mathbb{S}\right)^{hG} \arrow[dr] & & \simeq \mathbb{S} \\ & \bigoplus_{g\in G} \mathbb{S} \arrow[ur, "pr_e"'] & \end{tikzcd}$$

Proof $H^*(G, \bigoplus_{g\in G} \pi_*(\mathbb{S})) \Rightarrow \pi_*((\bigoplus_{g\in G} \mathbb{S})^{hG})$

It is a classical result that $H^*(G, \bigoplus_{g\in G} A) = \begin{cases} A, & * = 0 \\ 0, & * \neq 0 \end{cases}$

the HFPSS degenerates and gives the result.

$\Rightarrow D_G = \mathbb{S}^{\text{triv}}$

If G is a cpt Lie gp.

Theorem (..., Klein) We have $D_G = \mathbb{S}^{\mathfrak{g}}$

where $S^V = \Sigma^\infty(V^+)$ and G acts on $\mathfrak{g}$ by adjoint rep.

Example $\mathbb{T} = U(1) = S^1$, $D_{\mathbb{T}} = (\mathbb{S}^1)^{\text{triv}}$

Assume that $BG \simeq (M, m_0)$ where M is a closed smooth mfd, eg.

$G = \mathbb{Z}$, $BG \simeq S^1$; any closed mfd (M, m_0), $G = \Omega M$

Thm. We have $D_G = \mathbb{S}^{-T_{m_0}M}$ <— tangent space of M at m_0.

$\simeq \mathbb{S}^{-(\dim M)}$

as a functor $BG \simeq M \longrightarrow Sp$

that sends $m \longmapsto \mathbb{S}^{-T_mM}$

Rmk. One can define for any space X (in place of BG), a dualizing spectrum D_X

$D_X : X \to Sp$

for example, by define this functor on conn'd components, $X \simeq \coprod BG_i$

then on BG_i it is D_{G_i}

$\boxed{\text{Construction}}$ $\mathcal{C}$ stable ∞-cat., which has all limits & colimits.

For any $E \in Sp$ and $X \in \mathcal{C}$, there is an object $E \otimes X \in \mathcal{C}$ ($\mathcal{C}$ is a module over Sp)

defined s.t.

$$-\otimes X : Sp \to \mathcal{C}$$

- sends colimits of spectra to colimits in $\mathcal{C}$
- the composite $\quad S \xrightarrow{\Sigma^\infty_+} Sp \xrightarrow{-\otimes X} \mathcal{C}$

is given by the functor $\quad M \longmapsto M \otimes X = \operatorname{colim}_M c_X$

$$(\Sigma^\infty_+ M) \otimes X = M \otimes X$$

<u>Def</u>. $G \in \mathrm{Alg}^{gp}_{\mathbb{E}_1}(S)$ and $X \in \mathcal{C}^{BG}$. We define the <u>norm map</u>

$$Nm_G : (DG \underset{S}{\otimes} X)_{hG} \longrightarrow X^{hG}$$

defined as the composite

$$\left((\Sigma^\infty_+ G)^{h(G\times 1)} \underset{S}{\otimes} X \right)_{h(1\times G)}$$

$$\downarrow$$

$$\left((\Sigma^\infty_+ G \underset{S}{\otimes} X)^{h(G\times 1)} \right)_{h(1\times G)}$$

$$\downarrow$$

$$\left((\Sigma^\infty_+ G \underset{S}{\otimes} X)_{h(1\times G)} \right)^{h(G\times 1)} \simeq X^{hG}$$

X considered as $G\times G$ spectrum where $G\times 1$ acts trivially and $1\times G$ through the given G-action on X, i.e.

$$B(G\times G) \xrightarrow{pr_2} BG \xrightarrow{X} \mathcal{C}$$

<u>Ex</u>. (1) If G is finite, $X \in \mathcal{C}^{BG}$,

this depends on the choice $DG \simeq S$

$$(D_G \otimes X)_{hG} \simeq X_{hG} \longrightarrow X^{hG}$$

For $\mathcal{C} \in Sp$, $X = HM$, M abelian gp w/ G-action.

This map is a map

$$\begin{tikzcd} HM_{hG} \arrow[r, "Nm_G"] \arrow[d] & HM^{hG} \\ H(M_G) \arrow[r] & H(M^G) \arrow[u] \end{tikzcd}$$

this is induced from the classical norm map $M_G \longrightarrow M^G$

$$[m] \longmapsto \sum_{g \in G} gm$$

This follows from the fact that for general X, the composite

$$X \longrightarrow X_{hG} \xrightarrow{Nm_G} X^{hG} \longrightarrow X$$

is given by the sum of the maps $l_g : X \to X$

② For $G = \mathbb{T}$, we get a map

$$\Sigma X_{h\mathbb{T}} = (D_{\mathbb{T}} \otimes X)_{h\mathbb{T}} \longrightarrow X^{h\mathbb{T}}$$

Thm The norm map $(D_G \otimes X)_{hG} \to X^{hG}$ is an equiv provided that one of the following conditions hold.

(1) BG is a finite CW cpx

(2) X is induced, that is $X \simeq \Sigma^\infty_+ G \otimes Y$, where G acts on $\Sigma^\infty_+ G$

Proof (1) In this case, all limits in the interchange maps are finite.

(2) In the induced case, the colimits are still closed enough to finite. □

Example BG closed mfd, $\mathcal{C} = Sp$, $X = H\mathbb{Z}^{\text{triv}}$.
M

$$(DG \otimes X)_{hG} \xrightarrow{\sim} X^{hG} = H\mathbb{Z}^{BG} = \mathrm{map}(BG, X)$$

$$\pi_* (H\mathbb{Z}^{BG}) = H^{-*}(M, \mathbb{Z})$$

$$(DG \otimes X)_{hG} \simeq (\mathbb{S}^{-T.M} \otimes H\mathbb{Z})_{hG} \simeq (H\mathbb{Z}[-n])_{hG}$$

$$\pi_* = H_{*+n}(M, \mathbb{Z}^{\omega}) \xrightarrow{\sim} H^{-*}(M, \mathbb{Z})$$

orientation cov

This is Poincaré duality!

- Replacing $H\mathbb{Z}$ by any spectrum we get Poincaré duality in arbitrary homology-cohom. theory.

- If $X = BG$ is a finite CW cpx, the map is a generalized version of PD:

$$H_*(X, D_X) \longrightarrow H^{-*}(X)$$

D_X is a parametrized sphere ($D_G = \mathbb{S}(n)$ as underlying spectrum)

$\Leftrightarrow$ X is Poincaré duality space

Thm (Klein). The transformation $(DG \otimes -)_{hG} \to (-)^{hG}$ exhibits the functor $(DG \otimes -)_{hG}$ as the universal functor on $(-)^{hG}$ which preserves colimits i.e. the assembly map

(note that this uniquely determines DG)

$$\mathrm{Sp}^{BG} \longrightarrow \mathrm{Sp}$$

<u>Def</u> For $X \in \mathcal{C}^{BG}$, we define the Tate construction as

$$X^{tG} := \mathrm{cofib}\,(\mathrm{Nm}_G)$$

<u>Example</u> ① If BG is finite, then $X^{tG} = 0$ for all X

② For G finite and $X = HM$, M abelian gp w/ G-action, we have $(HM)^{tG} = \mathrm{cofib}\,(\mathrm{Nm}_G : HM_{hG} \to HM^{hG})$

The homotopy gps are given by $\pi_*((HM)^{tG}) = \hat{H}^{-*}(G, M)$

<u>Thm</u> Assume that $\mathcal{C}$ is a sym. monoidal ∞-cat. s.t. the tensor product commutes w/ colimits in both variables separately, then the functor

$(-)^{tG} : \mathcal{C}^{BG} \to \mathcal{C}$ admits a (unique) lax sym. mon. str s.t.

$(-)^{hG} \to (-)^{tG}$ admits a refinement to sym. mon. transformation.

<u>Cor.</u> If $A \in \mathrm{CAlg}(\mathcal{C}^{BG}) \simeq \mathrm{CAlg}(\mathcal{C})^{BG}$, then A^{tG} is a comm. alg

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<u>Lecture 13</u> <u>The Tate diagonal</u>

For abelian gps, $A \longrightarrow A \otimes A$ is NOT a homomorphism,
$x \longmapsto x \otimes x$ error terms $x \otimes y + y \otimes x$

Recall. $N : (A^{\otimes p})_{C_p} \longrightarrow (A^{\otimes p})^{C_p}$

$$x_1 \otimes \cdots \otimes x_p \longmapsto \sum_{\sigma \in C_p} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)}$$

Observation. The "diagonal" $\begin{array}{l} A \longrightarrow A^{\otimes p} \\ x \longmapsto x \otimes \cdots \otimes x \end{array}$ induces a homomorphism

$$A \longrightarrow (A^{\otimes p})^{C_p} / N(A^{\otimes p}{}_{C_p})$$

This homomorphism exhibits $(A^{\otimes p})^{C_p} / N(A^{\otimes p}{}_{C_p})$ as A/p

Thm $\exists!$ lax sym. monoidal nat'l transf $X \longrightarrow (X^{\otimes p})^{tC_p}$ (of functors $Sp \to Sp$)

Lemma The Singer construction $X \longmapsto (X^{\otimes p})^{tC_p}$ is exact.

Proof First look at $Y \simeq X \oplus Z$

$$(X \oplus Z)^{\otimes p} \simeq X^{\otimes p} \oplus Z^{\otimes p} \oplus \left[\bigoplus_{p} X^{\otimes p-1} \otimes Z \oplus \bigoplus_{\binom{p}{2}} X^{\otimes p-2} \otimes Z^{2} \oplus \cdots \right]$$

The "induced terms" vanish after $(-)^{tC_p}$.

Generally: think of $X \to Y \to Z$ as a "2-stage filtration"

$$\begin{array}{ccccccc} \cdots \to & 0 & \to & X & \to & Y \\ & \downarrow & & \downarrow & & \downarrow \\ & 0 & \to & X & \to & Z \end{array}$$

get filtration on $Y^{\otimes p}$: $\quad X^{\otimes p} \to \bigoplus_{p} X^{\otimes p-1} \otimes Y \to \cdots \to \bigoplus_{p} X \otimes Y^{p-1} \to Y^{\otimes p}$

$$X^{\otimes p} \qquad\qquad \bigoplus_{p} X \otimes Z^{p-1} \qquad\qquad Z^{\otimes p}$$

after Tate. $(X^{\otimes p})^{tC_p} \quad 0 \qquad 0 \quad (Z^{\otimes p})^{tC_p}$

get cofiber sequence $(X^{\otimes p})^{tC_p} \to (Y^{\otimes p})^{tC_p} \to (Z^{\otimes p})^{tC_p}$

Rmk. $((\Sigma X)^{\otimes p})^{tC_p} \simeq \Sigma (X^{\otimes p})^{tC_p}$

$\uparrow$

NOT Σ^p.

Indeed, $(\Sigma X)^{\otimes p} \simeq \mathbb{S}^p \otimes X^{\otimes p}$

but the C_p-action includes a nontrivial action on $\mathbb{S}^p$.

(regular representation sphere)

in particular, $(\mathbb{S}^p)^{tC_p} \simeq (\mathbb{S}^1)^{tC_p}$.

Lemma ("stable Yoneda") Let $\mathcal{C}$ be a stable ∞-cat.,

$$\mathrm{map}_{\mathrm{Fun}^{ex}(\mathcal{C}, Sp)}(\mathrm{map}(X,-), F) \simeq F(X)$$

Proof sketch. A nat'l transf $\mathrm{map}(X,-) \to F$ is a sequence of compatible nat'l

transformations $\mathrm{Map}(\Sigma^{-n}X,-) \to \Omega^n \Sigma^n F(-)$

i.e. a sequence of points in $\Omega^\infty \Sigma^n F(\Sigma^{-n}X) \simeq \Omega^\infty F(X)$

i.e. $\mathrm{Map}_{\mathrm{Fun}^{ex}(\mathcal{C}, Sp)}(\mathrm{map}(X,-), F) \simeq \Omega^\infty F(X)$ $\quad (\square)$

So nat'l transf. $X \to (X^{\otimes p})^{tC_p}$ corresponds to maps

$$\mathbb{S} \to (\mathbb{S}^{\otimes p})^{tC_p} \simeq \mathbb{S}^{tC_p}$$

Take
$$\begin{array}{ccccc} \mathbb{S} & \to & \mathbb{S}^{hC_p} & \xrightarrow{can} & \mathbb{S}^{tC_p} \\ \wr\!\! & & \wr\!\! & & \\ map(\Sigma^\infty pt, \mathbb{S}) & \to & map(\Sigma^\infty BC_p, \mathbb{S}) & & \end{array}$$

<u>Lemma</u>. $map(\mathbb{S}, -)$ is initial among lax sym. monoidal $\overset{exact}{\checkmark}$ functors $Sp \to Sp$.

<u>Thm</u>. For X bounded below spectrum, then

$$X \to (X^{\otimes p})^{tC_p}$$

exhibits $(X^{\otimes p})^{tC_p}$ as $X_p^\wedge$! In particular, $(X^{\otimes p})^{tC_p}$ is again bounded below!

In particular, $\mathbb{S}^{tC_p} \simeq \mathbb{S}_p^\wedge$ (Lin, Gunawardena)

Special form of Segal conjecture (Carlson)

<u>Def</u>. For an $\mathbb{E}_\infty$-ring R, have <u>Tate valued Frobenius</u>

$$R \xrightarrow{\Delta} (R^{\otimes p})^{tC_p} \xrightarrow{\mu^{tC_p}} R^{tC_p}$$

(using that mult. $R^{\otimes p} \to R$ is equiv.)

Same as map $(R^{\otimes p})_{hC_p} \to R$

Norm

(Spectral analogue of $R \xrightarrow{\varphi} R/p$ for ordinary comm. ring)
$x \longmapsto x^p$

Applied to $\mathrm{map}(\Sigma^\infty_+ X, E)$, get power operations