The geometric Satake correspondence

Thomas Haines

Lecture 1. Satake isom. for p-adic groups.

F na. local field.

$\cup$

$\mathcal{O}$ ring of integers

$\in$

ϖ unif.

$k = \mathbb{F}_q$ res. field. Fix prime ℓ, $(\ell, q) = 1$.

$G \supset B \supset T$ all split $/F$.

$/\mathcal{O}$

$W_0 = N_G T(F)/T(F)$

finite Weyl gp

$K = G(\mathcal{O}) \subset G(F)$

$\mathcal{H}_K(G) := C_c(K \backslash G(F)/K ; \bar{\mathbb{Q}}_\ell)$ assoc. $\bar{\mathbb{Q}}_\ell$-alg.

$\mathrm{vol}_{dg}(K) = 1$.

$$f_1 * f_2 (g) = \int_{G(F)} f_1(gx^{-1}) f_2(x)\, dx$$

Thm (Satake for split gp)

$$\mathcal{H}_K(G) \cong \underbrace{\bar{\mathbb{Q}}_\ell[X_*(T)]^{W_0}}_{\text{comm. f.d. } \bar{\mathbb{Q}}_\ell\text{-alg.}}$$

RHS $= \mathcal{H}_{T(\mathcal{O})}(T(F))$

du on $U(F)$

$\mathrm{vol}_{du}\, U(\mathcal{O}) = 1$

$$S(f)(t) = \delta_B^{\frac{1}{2}}(t) \int_{U(F)} f(tu)\, du$$

$\delta_B : T(F) \to \bar{\mathbb{Q}}_\ell^{\times} \ni q^{\frac{1}{2}}$

$$\delta_B(t) = \left|\det\left(\mathrm{Ad}(t)\right)\middle|\,\mathrm{Lie}\,U(F)\right)\Big|_F \in q^{\mathbb{Z}} \subset \overline{\mathbb{Q}}_\ell^\times$$

General groups:

$G \supset P = MN$ min'l F-parabolic.

A = max'l F-split torus, $M = C_G(A)$ = min'l F-Levi

Some Bruhat - Tits theory

Kottwitz hom. : $\breve{F} = \widehat{F^{un}}$

$$\kappa_G : G(\breve{F}) \twoheadrightarrow \pi_1(G)_I$$

$I \subset \Gamma = \mathrm{Gal}(\overline{F}|F)$
inertia

$\pi_1(G) = X_*(T)/Q^\vee$, $Q^\vee$ = abs. coroot lattice.

$G = GL_n$: $\kappa_G = |\det| : GL_n(\breve{F}) \to \mathbb{Z}$

$$G(\breve{F})_1 := \ker \kappa_G,$$

If G/F, set $\kappa_G : G(F) \twoheadrightarrow \pi_1(G)_I^\sigma$, σ = any Frob. $\in \Gamma$

$$G(F)_1 := G(F) \cap G(\breve{F})_1.$$

Let $\mathcal{A}$ = apt. in BT bld corr to A. Let $o \in \mathcal{A}$ be a special vertex.

$\rightsquigarrow$ BT group scheme $\mathcal{G} = \mathcal{G}_o / \mathcal{O}$.

- $\mathcal{G}(\mathcal{O}) = \mathrm{Fix}_{G(F)^\circ} \cap G(F)_1$

- $\mathcal{G}$ smooth, affine, geom. conn. $/\mathcal{O}$.

$K := \mathcal{G}(\mathcal{O})$ special max'l parahoric subgp. of $G(F)$.

.. $\Lambda_M = M(F)/M(F)_1$ f.g. abel. gp.

$$W_0 \circlearrowright \qquad M(F)_1 = K \cap M(F) = !\ \text{parahoric subgp in } M(F)$$

$\shortparallel$

$N_G A(F)/M(F)$ rel. Weyl gp of (G, A)

Thm (H. – Rostami, 2009)

$$\mathcal{H}_K(G) \cong \bar{\mathbb{Q}}_\ell [\Lambda_M]^{W_0}.$$

Rmk. Cartier (Casselman) proved

$$\mathcal{H}_{\tilde{K}}(G) = \bar{\mathbb{Q}}_\ell [\tilde{\Lambda}_M]^{W_0}, \quad \tilde{K} \supset K \quad \underline{\text{special max'l compact}}.$$

$$\tilde{\Lambda}_M = M(F)/M(F)^1, \quad M(F)^1 \subset M(F) \quad !\text{-max'l cpt in } M(F).$$

Special case: G qs $/F$, $G \supset B \supset T = C_G(A)$

$$\begin{array}{c} B \\ \shortparallel \\ TU \end{array}$$

K again $\mathcal{G}(\mathcal{O})$, special max'l parahoric.

$$\Lambda_M = \Lambda_T = X_*(T)_I^\sigma = X^*(\hat{T}^I)^\sigma$$

Get $\mathcal{H}_K(G) \cong \bar{\mathbb{Q}}_\ell[X^*(\overset{\hat{G}^I \downarrow}{\hat{T}^I})_\sigma]^{W_0}$

conn'd comp.

Caution $\Rightarrow$ $\mathcal{H}_{\bar{K}}(G) \cong \bar{\mathbb{Q}}_\ell[\underbrace{X^*(\hat{T}^{I,\circ})_\sigma}_{\left(X_*(T)_I/\text{torsion}\right)^\sigma}]^{W_0}$

$\hat{G}$ = Langlands dual gp + I-action

fixes splitting $\hat{G} \supset \hat{B} \supset \hat{T}$

roots: $\left(X^*(\hat{T}) \supset \Phi^\vee,\ X_*(\hat{T}) \supset \Phi,\ \Delta^\vee\right) \circlearrowleft I$

Φ = abs. roots for (G, B, T)

$\Phi^\vee$ coroots

<u>Thm</u> The $\bar{\mathbb{Q}}_\ell$-gp $\hat{G}^I$ is reductive, $\hat{G}^{I,\circ} \supset \hat{T}^{I,\circ}$ max'l torus, w/ root system

$\left[(\Phi^\vee)^\diamond\right]_{red} \leftarrow$ if $(a, 2a) \in R^\diamond$, discard $2a$

R any root system w/ I-action, $R^\diamond$ = set of I-averages

$\cap$ $\qquad \cap$

$V \qquad V^I$

Change notation: $k = \bar{k}$ any field alg. closed, $\ell \neq \operatorname{char} k$,

$F = k((t)) \supset \mathcal{O} = k[[t]]$, G/F conn. red. (Steinberg $\Rightarrow$ q.s.)

$G \supset B \supset T = C_G(A)$, $\mathcal{G}$ = special max'l parahoric $/\mathcal{O}$

$B = TU$

Construct ind-sch. $/_k$ $\mathrm{Gr}_{\mathcal{G}}$ affine grassmannian.

$$\begin{matrix} \mathrm{Gr}_{\mathcal{G}} \\ \circlearrowleft \\ L^+\mathcal{G} \end{matrix}$$

Main Thm:

$\left(P_{L^+\mathcal{G}} \left(\mathrm{Gr}_{\mathcal{G}} ; \bar{\mathbb{Q}}_\ell \right), \star \right)$ is Tannakian cat. w/ fiber functor

$$\mathcal{F} \longmapsto R^*\Gamma(\mathrm{Gr}_{\mathcal{G}}, \mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} R^i\Gamma(\mathrm{Gr}_{\mathcal{G}}, \mathcal{F}).$$

which is isom. to $\left(\mathrm{Rep}\left(\hat{G}^I\right), \otimes \right)$.

Two major steps:

① construct $\left(P_{L^+\mathcal{G}} \left(\mathrm{Gr}_{\mathcal{G}} ; \bar{\mathbb{Q}}_\ell \right), \star \right)$ + show it is Tannakian

② Identify Tannakian grp as $\hat{G}^I$.

Xinwen Zhu: Main Thm G/F tamely ramified.

Timo Richarz: removed tameness assumption

Earlier, Timo proved split case in a novel way, using Larsen-Kazhdan-Varshavsky char of H in terms of Grothendieck semiring of rep'ns.

Idea: know set of irred. objects $\{V_\mu\}$ in Rep H and $V_\mu \otimes V_\lambda = \bigoplus_\nu V_{\mu,\lambda}^{\nu} \otimes V_\nu$ mult.

allows one to recover H.

Group Theoretic preliminaries

[HR08] Det. Iwahori - Weyl gp is

(G, A)

$$W = N_G T(F) / T(F)_1 . \quad W_0 = N_G T(F) / T(F)$$

$$T(F) / T(F)_1 \xrightarrow[\sim]{k_T} X_*(T)_I .$$

$$W_0 = (N_G T(F) \cap K) / T(F)_1$$

$$\boxed{W \cong X_*(T)_I \rtimes W_0}$$

$$V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \circlearrowleft I .$$

$$\cup$$

$$V^I = V_I$$

BT theory provides set of affine roots for (G, A)

$$\Phi_{aff} = \{ \alpha + r : \alpha \text{ rel. root } \in X^*(A), r \text{ certain } \in \mathbb{R} \}$$

Φ_{aff} affine linear functionals on V^I

$\rightsquigarrow$ affine hyperplanes $H_{\alpha+r} = \{ v \in I^r : \alpha(v) + r = 0 \}$

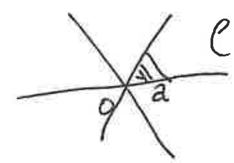
V^I is underlying apt $\mathcal{A}$ for (G, A) in $\mathcal{B}(G, F)$

Φ_{aff}, $\{H_{\alpha+r}\} \longrightarrow$ Coxeter complex, alcoves, facets.

Fix dominant Weyl chamber $\mathcal{C} \subset \mathcal{A}$, $\mathcal{C} = \{ v \in V^I : \langle \alpha, v \rangle > 0, \forall \alpha \in \operatorname{Lie} B \}$.

Choose origin in $\mathcal{A}$ to corr. to choice of $\mathcal{G}_0 = \mathcal{G}$.

require base alcove $\mathbf{a} \subset \mathcal{C}$, $\mathcal{C}$ has apex 0.



$0 \rightsquigarrow \mathcal{G}_0$

$\mathbf{a} \rightsquigarrow \mathcal{G}_{\mathbf{a}}$ BT Iwahori grp sch.

(W_{aff}, S_{aff}) = Coxeter system given by simple affine reflections in V^I through walls of $\mathbf{a}$.

W acts on V^I

$\shortparallel$

$X_*(T)_I \rtimes W_0$, W_0 = grp generated by reflections through walls of $\mathbf{a}$ containing 0.

$\lambda \in X_*(T)_I$ acts by translation by $\underline{\underline{-\lambda}}$.

W permutes Φ_{aff}, $\{H_{\alpha+r}\}$, $\{\text{alcoves}\}$

$\cup$

W_{aff} acts simply transitively on set of alcoves.

$\therefore$ having fixed $\mathbf{a}$, get canonical decomp. $W = W_{aff} \rtimes \Omega_{\mathbf{a}}$,

$\Omega_{\mathbf{a}} \subset W$ is Stab. of $\mathbf{a}$.

W_{aff} has length func. $\ell: W_{aff} \to \mathbb{Z}_{\geq 0}$, Bruhat order $\leq$

Extend these to W:
- require Ω = length 0 elts in W
- $w_1 \omega_1 \leq w_2 \omega_2 \iff w_1 \leq w_2$ in W_{aff}, $\omega_1 = \omega_2$ in Ω.

$\therefore$ W has structure of a quasi-Coxeter group.

Prop. $\exists!$ reduced root system $\Sigma \subset V^I$ s.t. Φ_{aff} consists of $\alpha+n$, $\alpha \in \Sigma$, $n \in \mathbb{Z}$.

Thus $W_{aff} = W_{aff}(\Sigma \times \mathbb{Z})$

Prop. The following statements on Σ hold

(1) Σ described in terms of I-action on abs. roots $(\Phi \supset \underset{\uparrow\text{simple}}{\Delta})$ as follows:

$\forall \alpha \in \Delta$, set $N_I' \alpha = \begin{cases} \sum_{\beta \in I\cdot\alpha} \beta, & \text{if } \{\beta \in I\cdot\alpha\} \text{ pairwise } \perp \\ 2\sum_{\beta \in I\cdot\alpha} \beta, & \text{otherwise} \end{cases}$

$N_I'\Delta =$ simple roots in a root system in V^I

which is $\cong$ set of simple roots in Σ.

(2) $\exists$ identification of (based) root systems $\Sigma^\vee = (\Phi^\vee, \Delta)_{red}$

i.e. Σ is of type dual to $\Phi(\hat{G}^{I,\circ}, \hat{T}^{I,\circ})$.

<u>Three decompositions of $G(F)$</u> $\supset K$

① Bruhat-Tits. $K_a = \mathcal{G}_a(\mathcal{O})$

$\forall w \in W$, fix any lift $\dot{w} \in N_G T(F)$.

Then $G(F) = \coprod_{w \in W} K_a \dot{w} K_a$.

Rmk. $G = GL_n$, can prove by hand using row & column operations.

② Cartan: $\forall \lambda \in X_*(T)_I$,

choose any Kottwitz lift $t^\lambda \in T(F)$, $k_T(t^\lambda) = \lambda \in X_*(T)_I$.

Then $$G(F) = \coprod_{\lambda \in \underbrace{X_*(T)_I^+}_{\text{dominant}}} K t^\lambda K \qquad [K = \mathcal{G}(\mathcal{O})]$$

(for Σ roots)

Rmk. Can be deduced formally from ①, using BN-pair relations, etc.

③ Iwasawa decomposition. $$G(F) = \coprod_{\lambda \in X_*(T)_I} U(F) t^\lambda K$$

$\left(\forall \lambda \in X_*(T)_I,\ \mu \in X_*(T)_I^+,\ U(F) t^\lambda K \cap K t^\mu K \neq \phi \Leftrightarrow \mu - \lambda = \text{sum of pos. coroots for } \Sigma^\vee \right)$

Prop (see notes)

$$\underset{\substack{\text{dom. for} \\ \text{abs. simple} \\ \text{roots}}}{X_*(T)^+} \twoheadrightarrow \underset{\substack{\text{dom. for} \\ \Delta(\Sigma) = N_I' \Delta}}{X_*(T)_I^+}$$

Lecture 2

2 corrections

① $\Phi_{aff} \neq \Sigma \times \mathbb{Z}$, they just define the same hyperplanes in $\mathcal{A}$.

② $X_*(T)^+ \longrightarrow X_*(T)_I^+$

can only prove surjective when $Z(G)$ is conn'd.

Announcement: Focus on split case

- final version of notes will cover split case
- remarks at end of lecture about general case.

I-action on $(X^*(T) \supset \Phi,\ X_*(T) \supset \Phi^\vee,\ \Delta)$

$\rightsquigarrow\ X_*(T)_I,\ \Sigma,\ \Sigma^\vee,\ (\check{\Phi}^0)_{red}$ (split: $= \check{\Phi}$)

In split case, I acts trivially, $\Sigma = \Phi$.

Examples of K, K_a in split case:

- $\mathcal{G} = G/\mathcal{O}$, $K = \mathcal{G}(\mathcal{O})$,
  $\cup$
- $\mathcal{G}_a$, $K_a = \mathcal{G}_a(\mathcal{O}) = \{g \in \mathcal{G}(\mathcal{O}) : g \bmod \varpi \in B(k)\}$

e.g. $G = GL_n$, our convention give

$$K_a = \begin{bmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \mathcal{O}^\times & \\ \varpi\mathcal{O} & & \ddots \\ & & & \mathcal{O}^\times \end{bmatrix}$$

Ind-scheme

k - field.

Def. An ind-scheme is a $\operatorname{colim}_{i \in I} F_i$, $F_i : \mathrm{Aff}_k \to \mathrm{Sets}$

I directed set, and each $F_i \cong$ k-sch X_i, where colim_i is taken

in $\mathrm{Presheaf}(\mathrm{Aff}_k)$.

Fact: filtered colim commute w/ finite limits, such as equilizers,

so if F_i are $\mathcal{C}$-sheaves for some Grothendieck topology $\mathcal{C}$ on Aff_k,

then so is $\operatorname{colim}_i F_i$, and is also colim in $\mathrm{Sh}_\mathcal{C}(\mathrm{Aff}_k)$.

Recall. presheaf $F : \mathrm{Aff}_k \to \begin{matrix}\mathrm{Sets} \\ \mathrm{Groups}\end{matrix}$ is a $\mathcal{C}$-sheaf if $\forall$ $\mathcal{C}$-cover

$\operatorname{Spec} R' \to \operatorname{Spec} R$, $\quad F(R) \to F(R') \rightrightarrows F(R' \otimes_R R')$ is equalizer

in target cat.

$\mathcal{C}$ Zar, étale, fppf, fpqc.

X sch $\Rightarrow$ X is fpqc sheaf $\quad \therefore$ any ind-scheme is also an fpqc sheaf.

<u>Strict</u> ind-scheme : $X_i \to X_j$ closed immersions.

<u>Sheafification</u>. $F \in \text{Presheaf}(\text{Aff}_k, \text{Sets})$, $\mathcal{C}$ Grothendieck top.

$\exists$ sheafification F^{++} and canonical $F \to F^{++}$, w/ universal property $\forall$ sheaf F',

$\text{Hom}_{\text{Presh}}(F, F') = \text{Hom}_{Sh}(F^{++}, F')$.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \\ F^{++} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

<u>Loop groups + positive loop groups.</u>

$G / k((t))$ - affine

$\mathcal{G} / k[[t]]$, $\quad \mathcal{G} \times_{\text{Spec}\, k[[t]]} \text{Spec}\, k((t)) = G$.

Define presheaves

$LG : R \mapsto G(R((t)))$

$L^+\mathcal{G} : R \mapsto \mathcal{G}(R[[t]])$.

<u>Exercise</u>. Prove

(1) LG is rep'd by ind-affine group ind-scheme.

(2) $L^+\mathcal{G}$ is a group k-scheme (of infinite type $/k$).

<u>Def</u>. $\text{Gr}_{\mathcal{G}}$ is the étale sheafification of the presheaf $R \mapsto LG(R) / L^+\mathcal{G}(R)$.

<u>Rmk</u> NOT always the case that $\text{Gr}_{\mathcal{G}}(R) = LG(R) / L^+\mathcal{G}(R)$.

However, it <u>is</u> true when G <u>split</u>, R <u>local</u>.

<u>Goal</u>. When G reductive $/ k((t))$, $\mathcal{G} = G_{\mathcal{O}}$

$Gr_{\mathcal{G}}$ is represented by an ind-projective ind-scheme $/k$.

Start w/ $G = GL_n$, $\mathcal{G} = GL_{n,\mathcal{O}}$.

Starting point: $LG(R) = GL_n(R((t)))$ acts on set

$$\mathrm{Latt}_n(R) = \{ R[[t]] \text{ lattices } L \subset R((t))^n \}$$

<u>Def</u>. R any ring, $\Lambda = R[[t]]^n$. An $R[[t]]$-lattice is an $R[[t]]$-submod $L \subset R((t))^n$ s.t. $\exists N$ w/

- $t^N \Lambda \subset L \subset t^{-N} \Lambda$
- L is $R[[t]]$-projective

Will turn out that $Gr_{\mathcal{G}}(R) = \mathrm{Latt}_n(R) = \underset{N}{\mathrm{colim}} \underbrace{\mathrm{Latt}_{n,N}(R)}_{\text{proj. } k\text{-sch.}}$

<u>Main prop</u>. Let R be any ring, $\Lambda = R[[t]]^n$, TFAE as conditions on $R[[t]]$-module L w/ $t^N \Lambda \subset L \subset \Lambda$ for $N \in \mathbb{N}$:

(1) L is $R[[t]]$-proj.

(2) Λ / L (hence $L / t^N \Lambda$) are R-proj.

Lemma R any ring, M f.p. R-mod, then M is R-flat $\Leftrightarrow$ R-proj.

In particular, if R is Noetherian, then for M finite $/R$, flat $\Leftrightarrow$ proj.

$(1) \Rightarrow (2)$: $t^N \Lambda \subset L \subset \Lambda$.

Claim. L f.g. $/R_t = R[[t]]$.

This is a local prop: ETS for $L_\mathfrak{p}$ f.g. $/ R_{t,\mathfrak{p}}$, $\forall$ prime $\mathfrak{p}$.

$t^N \Lambda_\mathfrak{p} \subset L_\mathfrak{p} \subset \Lambda_\mathfrak{p}$

$R_t \to R((t))$ flat $\Rightarrow$ $L_\mathfrak{p} \otimes_{R_{t,\mathfrak{p}}} R((t))_\mathfrak{p} = R((t))_\mathfrak{p}^n$ ⊛

OTOH, $L_\mathfrak{p}$ is proj $/R_{t,\mathfrak{p}}$ ← local, so $L_\mathfrak{p}$ is $R_{t,\mathfrak{p}}$-free (Kaplansky's Thm)

⊛ $\Rightarrow$ finite type.

L $R[[t]]$-proj. $t^{-1}L/L$ $R = R[[t]]/tR[[t]]$ - proj.

$t^{-m}L/L$ R-proj, $\forall m \geq 1$ $\Rightarrow$ R-flat

$R((t))^n/L = \bigcup_m t^{-m}L/L$ R-flat
$(t^n \Lambda)$

$0 \to L/t^N\Lambda \to R((t))^n/t^n\Lambda \to R((t))^n/L \to 0$

$R((t))^n/t^n\Lambda$ R-flat, $R((t))^n/L$ R-flat

Tor-vanishing $\Rightarrow$ $L/t^N\Lambda$ R-flat. Similarly Λ/L R-flat

$$0 \to \underbrace{L/t^N\Lambda}_{\text{finite}} \to \Lambda/t^N\Lambda \to \underbrace{\Lambda/L}_{\substack{\text{f.p. as } R\text{-mod} \\ R\text{-flat} \\ \text{hence } R\text{-proj.}}} \to 0$$

(2) $\Rightarrow$ (1) $\quad t^N\Lambda \subset L \subset \Lambda, \quad \Lambda/L,\ L/t^N\Lambda \quad R$-proj.

Want: $\quad L \quad R_t$-proj.

First, claim ETS for R-Noetherian.

$$R = \operatorname{colim}_i R_i, \qquad R_i \text{ noeth.}$$

$Gr_N^f(R)$: $\quad \Lambda/L \quad R$-proj.

$Gr_N(R)$: $\quad L \quad R_t$-proj.

$$Gr_N \hookrightarrow Gr_N^f$$

$$Gr_N(R_i) \xrightarrow{\sim} Gr_N^f(R_i)$$

$$Gr_N^f(\operatorname{colim}_i R_i) = \operatorname{colim}_i Gr_N^f(R_i)$$

So assume R Noeth, $R_t = R[\![t]\!]$ Noeth. $\quad L$ finite R_t-module.

$$\begin{array}{ccc} \Lambda_0 = R[t]^n, & & \Lambda = R[\![t]\!]^n \\ \cup & & \cup \\ L_0 & \longleftrightarrow & L \\ \cup & & \cup \\ t^N\Lambda_0 & & t^N\Lambda \end{array}$$

$L = L_0 \otimes_{R[t]} R[\![t]\!]$

(use $R[t] \to R[\![t]\!]$ is flat)

ETS: L_0 is $R[t]$-proj.

Since $R[t]$ Noeth, L_0 fin. $/R[t]$,

ETS L_0 $R[t]$-flat.

ETS $\forall$ max'l $q \subset R[t]$, $L_{0,q}$ $R[t]_q$-flat.

q lies over $p \subset R$.

Lemma (see notes) $\exists a \in R - p$ s.t. R_a / p_a is a field, and

$R_a[t]_{q_a} = R[t]_q$.

Further, since $R_a/p_a \hookrightarrow R_a[t]/q_a$, see q_a lies over p_a max'l

$\therefore$ replace R w/ R_a, assume $\underset{\substack{\parallel \\ m}}{p}$ max'l in R, $k = R/m$.

Want: $L_{0,q}$ is a free $R[t]_q$-mod.

Apply following to $R_m \to R[t]_q$, $M = L_{0,q}$

Lemma $\underset{\substack{\cup \\ m}}{R} \to S$ map of Noeth. local rings

M finite S-module, s.t.

(a) M/mM free S/mS-mod

(b) M flat $/R$.

Then M free $/S$. (and S flat $/R$)

(b) $R[t]_q = \underset{\substack{\uparrow \\ \text{some } S}}{S^{-1}} R[t]_m$

$$L_{o,q} = L_{o,m} \otimes_{R[t]_m} R[t]_q$$

L_o is R-flat (recall Λ_o/L_o, Λ_o R-proj.)

Want $L_{o,q}$ to be R_m-flat.

$N \hookrightarrow P$ $\quad R_m$-mod.

$L_{o,m} \otimes_{R_m} N \hookrightarrow L_{o,m} \otimes_{R_m} P$ also $R[t]_m$-linear

$$R[t]_q \otimes_{R[t]_m} L_{o,m} \otimes_{R_m} N \hookrightarrow R[t]_q \otimes_{R[t]_m} L_{o,m} \otimes_{R_m} P \quad \Rightarrow \text{(b) holds.}$$

(a) This is statement $L_{o,q}/m L_{o,q} = (L_o/mL_o)_q$ is free

$R[t]_q / m R[t]_q = (R[t]/m R[t])_q$ - mod

ETS L_o/mL_o is free $\underbrace{k[t]}_{\text{PID}}$-mod

$$0 \to \underbrace{L_o \otimes_R R/m}_{\text{has no } k[t]\text{-torsion}} \to \Lambda_o \otimes_R R/m \to \overbrace{(\Lambda_o/L_o)}^{R\text{-proj.}} \otimes_R R/m \to 0$$

$\therefore$ (a) holds. $\square$

We used: If $X = \mathrm{Gr}_N^f$, then $\operatorname{colim}_i R_i = R \Rightarrow X(\operatorname{colim}_i R_i) = \operatorname{colim}_i X(R_i)$

$t^N \Lambda \subset L \subset t^{-N}\Lambda$, $t^{-N}\Lambda / t^N \Lambda \cong R^{2nN}$ $\circlearrowleft t$

$\mathrm{Gr}_N^f \overset{\text{closed}}{\subset} \coprod_r \mathrm{Gr}(2nN, r)$

being t-stable is a closed condition.

$\therefore$ Gr_N^f is a proj. k-scheme.

<u>Lemma</u> A morphism of schemes $X \to S$ is locally of finite presentation, iff $\forall$ directed set I and inverse system of affine schemes $\{T_i\}$ over S,

$\mathrm{Hom}_S(\varprojlim_i T_i, X) = \varinjlim_i \mathrm{Hom}_S(T_i, X)$.

<u>Lemma</u> $L \in \mathrm{Gr}_N(R)$, then $\exists$ Zariski cover $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ s.t.

$L \otimes_{R[[t]]} R'[[t]]$ is $R'[[t]]$-free.

<u>Pf.</u> $R_t = R[[t]]$. We know L finite $/R_t$.

$\exists\, g_1, \dots, g_r \in R_t$, $(g_1, \dots, g_r)_{R_t} = (1)$, s.t. L_{g_i} are $(R_t)_{g_i}$-free, $\forall i$.

Set $f_i = g_i(0) \in R$, $(f_1, \dots, f_r)_R = (1)$, $f_i \in R_{t,g_i}^{\times}$ if $f_i \neq 0$.

$L \otimes_{R_t} R_{f_i}[[t]] = \left(L \otimes_{R_t} R_{t,g_i}\right) \otimes_{R_{t,g_i}} R_{f_i}[[t]]$, $R_{f_i}[[t]]$-free. $\square$

Define $\mathrm{Latt}_n = \operatorname{colim}_N \mathrm{Gr}_N^f$ is an ind-proj. ind-scheme $/k$, hence fpqc sheaf on Aff_k.

Note $LG(R)/L^+G(R)$ identifies w/ $R[[t]]$ - free L in $R((t))^n$.

$$\begin{tikzcd}
(LG/L^+G)_{\text{presheaf}} \arrow[r, "a", "\text{mono, Zar. locally epi}"'] \arrow[d, "b"] & \mathrm{Latt}_n \\
LG/L^+G \arrow[ur, dashed, "\exists! c"'] &
\end{tikzcd}$$

mono, epi, hence isom.

$LG/L^+G = (LG/L^+G)^{++}_{\text{presh}}$

Used:

Lemma The following hold:

(1) $F \mapsto F^{++}$ preserves finite limits

(2) In any category w/ fiber products, $F \to F'$ is mono $\iff$ $F \xrightarrow{\Delta} F \times_{F'} F \rightrightarrows F$ is an equalizer.

- Thus, $F \mapsto F^{++}$ preserves monos.

- Morphism of sheaves is isom $\iff$ mono + locally epi

Torsor description of Gr_G.

Lecture 3. Summary of last time.

$\mathrm{Gr}_{GL_n,\mathcal{O}} = LGL_n / L^+GL_{n,\mathcal{O}}$ is isomorphic as étale sheaf to

$\mathrm{Latt}_n = \operatorname{colim}_N \mathrm{Latt}_{n,N}$ $\therefore$ $\mathrm{Gr}_{GL_n,\mathcal{O}}$ is rep'd by an ind-proj. ind-sch. $/k$.

Goal: Prove $\mathrm{Gr}_{\mathcal{G}}$ ind-proj. ind-scheme $/k$ $\forall$ (special max'l) parahoric $\mathcal{G}/\mathcal{O}$.

Assuming $\mathcal{G}/\mathcal{O}$ split, for simplicity.

<u>Torsor description of $\mathrm{Gr}_{\mathcal{G}}$</u>

Let $G \to X$ any affine group scheme / scheme X.

Let $\mathcal{C}$ be a Grothendieck topology on Aff_X.

<u>Def</u>. A (right) $\mathcal{C}$-torsor $\mathcal{E}$ on X is a $\mathcal{C}$-sheaf on Aff_X w/ right action $\mathcal{E} \times_X G \to \mathcal{E}$ s.t. $\forall R$, $\mathcal{E}(R) \circlearrowleft G(R)$ is simply transitive if it's not empty, and s.t. $\forall R$, $\exists$ $\mathcal{C}$-cover $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ w/ $\mathcal{E}(R') \neq \phi$.

k any field, $\quad \mathbb{D}_R = \mathrm{Spec}\, R[[t]]$

$R \in \mathrm{Aff}_k \quad \quad \mathbb{D}_R^* = \mathrm{Spec}\, R((t))$. ($\mathbb{D}_R^* \subset \mathbb{D}_R$)

Assume $\mathcal{G} \to \mathbb{D}_k$ affine gp sch. of f.t.

<u>Def</u>. $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} : \mathrm{Aff}_k \longrightarrow (\mathrm{Sets})$

$R \longmapsto \{(\mathcal{E}, \alpha)\}/\cong$

- $\mathcal{E} \to \mathbb{D}_R$ right $\mathcal{G} \times_{\mathbb{D}_k} \mathbb{D}_R$-torsor ("$\mathcal{G}$-torsor")
- $\alpha \in \mathcal{E}(\mathbb{D}_R^\times)$, i.e. isom. of $\mathcal{G}$-torsor $\mathcal{E}|_{\mathbb{D}_R^\times} \xrightarrow{\sim} \mathcal{E}_0|_{\mathbb{D}_R^\times}$

Here $\mathcal{E}_0 =$ trivial $\mathcal{G}$-torsor.

$(\mathcal{E}, \alpha) \xrightarrow{\sim} (\mathcal{E}', \alpha')$ is a map $\mathcal{E} \xrightarrow{\pi} \mathcal{E}'$ of $\mathcal{G}$-torsors ($\Rightarrow \cong$) s.t. $\alpha = \alpha' \circ \pi$.

Lemma. Properties

(0) $\mathcal{E}$ is rep'd by an affine sch $+ \mathcal{E} \to \mathbb{D}_k$

(effectivity of étale descent of affine schemes)

(1) $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ has base pt $(\mathcal{E}_0, \mathrm{id})$

(2) $L\mathcal{G}$ acts on the left on $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$: $(g, (\mathcal{E}, \alpha)) \mapsto (\mathcal{E}, g \circ \alpha)$.

(3) $\mathcal{G} \mapsto \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ is functorial in $\mathcal{G}$ (can "pushout").

(4) $\mathrm{Gr}_{GL_n, \mathcal{O}}^{\mathrm{tor}} \cong \mathrm{Latt}_n$

$$\mathrm{Latt}_n = R \mapsto [\text{vector bundles on } \mathbb{D}_R]$$

Given GL_n-torsor $\mathcal{E}$ on $\mathbb{D}_R$, get $\mathcal{E} \times^{GL_n, \mathcal{O}} \mathcal{O}^n$ a vector bdle

Given vector bdle V, $\mathcal{E} = \underline{\mathrm{Isom}}(\mathcal{O}^n, V)$.

$\therefore$ $\mathrm{Gr}_{GL_n, \mathcal{O}}^{\mathrm{tor}}$ is étale sheaf (fpqc)

$$\mathrm{Gr}_{GL_n, \mathcal{O}} \xrightarrow{\sim} \mathrm{Gr}_{GL_n, \mathcal{O}}^{\mathrm{tor}}$$

$$g \in LGL_{n, \mathcal{O}} \mapsto (\mathcal{E}_0, g)$$

Thm. $\mathcal{G}$ smooth affine grp sch. / $\mathbb{D}_k$.

(I) $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} \to \mathrm{Spec}\, k$ is rep'd by a separated ind-scheme of ind-finite type/k

(II) If $\mathcal{G}$ reductive, then $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ is ind-proj. /k.

<u>Prop</u> (key, Xinwen Zhu) Let $\mathcal{G} \hookrightarrow \mathcal{H}$ be a closed immersion of f.t. aff gp schemes/ $\mathbb{D}_k$ s.t. the fppf quotient $\mathcal{H}/\mathcal{G}$ is rep'd by quasi-affine (affine) scheme $/\mathbb{D}_k$, then $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} \to \mathrm{Gr}_{\mathcal{H}}^{\mathrm{tor}}$ is rep'd by a qc immersion (closed immersion)

<u>Pf</u>:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathrm{Spec}\, R \\ \downarrow & \ulcorner & \downarrow (\mathcal{E},\alpha) \\ \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} & \longrightarrow & \mathrm{Gr}_{\mathcal{H}}^{\mathrm{tor}} \end{array}$$

need this is a locally closed qc immersion (closed immersion)

$$\mathcal{E} \underset{\alpha}{\overset{\pi}{\rightleftarrows}} \mathbb{D}_R \quad \text{str.}$$

$$\mathcal{E}/\mathcal{G} \underset{\tilde{\alpha}}{\overset{\tilde{\pi}}{\rightleftarrows}} \mathbb{D}_R$$

By effectivity of descent for quasi-affine schemes, $\exists$ W aff of f.p. $/\mathbb{D}_R$ and qc open embedding

$$\begin{array}{ccc} \mathcal{E}/\mathcal{G} & \hookrightarrow & W \\ {\scriptstyle \tilde{\alpha}} \nwarrow \searrow {\scriptstyle \tilde{\pi}} & & \swarrow {\scriptstyle p} \\ & \mathbb{D}_R & \end{array} \qquad (*)$$

Def of fiber ${}^{\mathbb{D}_R}$ of qt $[\mathcal{E}/\mathcal{G}]$ yields identification

$$\mathcal{F}\Big([R \to R']\Big) = \Big\{ \text{sections } \beta \text{ of } \tilde{\pi} \text{ over } \mathbb{D}_{R'} : \beta\big|_{\mathbb{D}^*_{R'}} = \tilde{\alpha}\big|_{\mathbb{D}^*_{R'}} \Big\}$$

Lemma. $V \underset{s}{\overset{p}{\rightleftarrows}} \mathbb{D}_R$ aff. sch. of fin. pres. and s sections over $\mathbb{D}_R^*$,

then $[R \to R'] \mapsto \{ \text{sections } s' \text{ of } p \text{ over } \mathbb{D}_{R'} : s'|_{\mathbb{D}^*_{R'}} = s|_{\mathbb{D}^*_R} \}$

is rep'd by a closed subscheme of Spec R.

Pf. $V \hookrightarrow \mathbb{A}^N_{\mathbb{D}_R}$, $N \gg 0$.

$s = (s_1(t), \ldots, s_N(t))$, $s_i = \sum s_{ij} t^j \in R((t))$.

Presheaf is rep'd by Spec A, $A = R/\langle s_{ij} : j < 0 \rangle$

Apply Lemma to $(*)$, $\{ \beta \text{ sections of } \tilde{\pi} \quad \beta|_{\mathbb{D}^*_{R'}} = \tilde{\alpha}|_{\mathbb{D}^*_{R'}} \}$

$= \{ \text{Sections of } p \text{ landing in open } \mathcal{E}/\mathcal{G} \subset W \}$.

So we get open in Spec A

Lemma. $\mathcal{G}$ flat affine gp scheme / $\mathbb{D}_k$, then

Pappas- Rapoport

(a) $\exists$ closed immersion $\mathcal{G} \hookrightarrow GL_{n,\mathcal{O}} \times GL_{1,\mathcal{O}}$ s.t. $GL_{n,\mathcal{O}} \times GL_{1,\mathcal{O}} / \mathcal{G}$ is quasi-affine

(b) If $\mathcal{G}$ reductive (e.g. $\mathcal{G} = G_{\mathcal{O}}$ split), then $GL_{n,\mathcal{O}} \times GL_{1,\mathcal{O}} / \mathcal{G}$ is affine.

J. Alper

Cor. $\forall$ flat affine $\mathcal{G}/\mathbb{D}_k$, $Gr^{tor}_{\mathcal{G}}$ is representable by ...

If $\mathcal{G}$ is reductive, ... ind-proper hence ind-proj.

Cor. If $\mathcal{G} \to \mathbb{D}_k$ smooth and affine, then $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ is an isom.

Pf. (X. Zhu) $L\mathcal{G}(R)/L^+\mathcal{G}(R) \longrightarrow \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}(R)$

$$g \longmapsto (\mathcal{E}, g)$$

Sheafifies to $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ which is a mono. So ETS étale-locally an epi.

Given $(\mathcal{E}, \alpha) \in \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}(R)$, need to know $\exists$ $\underset{\text{cover}}{\text{étale}}$ $R \to R'$ s.t.

$\mathcal{E}_{\mathbb{D}_{R'}} \cong \mathcal{E}_{0, \mathbb{D}_{R'}}$.

$R[[t]] \longrightarrow R$

$\mathbb{D}_R \longleftarrow \operatorname{Spec} R$

$$\mathcal{E} \times_{\mathbb{D}_R} \operatorname{Spec} R \qquad \overbrace{\mathcal{G} \times_{\mathbb{D}_R} \operatorname{Spec} R}^{\mathcal{G}_R \text{ smooth } / \operatorname{Spec} R} \text{ - torsor}$$

$$\begin{array}{ccc} & & \mathcal{E} \times_{\mathbb{D}_R} \operatorname{Spec} R \\ & \nearrow & \downarrow \\ \operatorname{Spec} R' & \to & \operatorname{Spec} R \end{array}$$

Then inf. lifting, this section lifts to

$$\operatorname{Spf}(R'[[t]]) \longrightarrow \mathcal{E} \times_{\mathbb{D}_R} \mathbb{D}_{R'}$$

i.e. to $\operatorname{Spec} R'[[t]] \longrightarrow \underbrace{\mathcal{E} \times_{\mathbb{D}_R} \mathbb{D}_{R'}}_{\text{affine scheme}}$. $\square$

These ingredients prove Thm. (G reductive)

$$\mathrm{Gr}_{\mathcal{G}} \cong \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} \overset{\text{closed}}{\hookrightarrow} \mathrm{Gr}^{\mathrm{tor}}_{GL_n \times GL_1} \cong \mathrm{Latt}_n \times \mathrm{Latt}_1$$

$\therefore$ $\mathrm{Gr}_{\mathcal{G}}$ ind-proj. ind-scheme $/k$. $\square$

$\left(P_{L+g}\left(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell\right), *\right)$

k field ($k = \bar{k}$, or $k = \mathbb{F}_q$), ℓ prime, $\ell \nmid \operatorname{char} k$.

X/k f.t. separated k-scheme.

$\rightsquigarrow D^b_c(X, \Lambda)$, Λ ℓ-torsion abelian gp
BBD eg. $\mathbb{Z}/\ell^n\mathbb{Z}$

$$D^b_c(X, \mathbb{Z}_\ell) \otimes \mathbb{Q} = \varprojlim_n D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \mathbb{Q}$$
$\shortparallel$
$D^b_c(X, \mathbb{Q}_\ell)$

$\rightsquigarrow D^b_c(X, \overline{\mathbb{Q}}_\ell)$ + 6-functors Rf_*, $Rf_!$, f^*, $f^!$,
$\cup$ $R\mathcal{H}om(-,-)$, $-\overset{L}{\otimes}-$, $\mathbb{D}_X$
$P(X, \overline{\mathbb{Q}}_\ell)$ perverse sheaves

$${}^pD^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) = \{ \mathcal{F} \in D^b_c : \dim \operatorname{supp} \mathcal{H}^i \mathcal{F} \leq -i, \ \forall i \in \mathbb{Z} \}$$
$${}^pD^{\geq 0}(X, \overline{\mathbb{Q}}_\ell) = \{ \mathbb{D}_X \mathcal{F} : \mathcal{F} \in {}^pD^{\leq 0} \}$$

Def / Thm.

${}^pD^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) \cap {}^pD^{\geq 0}(X, \overline{\mathbb{Q}}_\ell) = P(X, \overline{\mathbb{Q}}_\ell)$ is abelian cat, all of whose objects have finite length.

- $j: U \hookrightarrow X$ open, $j_{!*}: P(U) \longrightarrow P(X)$
- description of simple objects in $P(X)$.

$\mathcal{L}$ lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $U \overset{\text{open}}{\subset} X$, $U \xrightarrow{j} \bar{U} \xrightarrow{i} X$

$IC(\bar{U}, \mathcal{L}) = i_* j_{!*} \mathcal{L}[\dim U]$, simple if $\mathcal{L}$ is irred.

Lecture 4

Recall. $\mathcal{G} \to \mathcal{H}$, $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{H}}$

$\mathcal{H}/\mathcal{G}$ rep'd by q-affine scheme $/\mathbb{D}_k$

$(\mathcal{E}, \alpha): \mathrm{Spec}\, R \to \mathrm{Gr}_{\mathcal{H}}$

$$\mathcal{F} = \mathrm{Spec}\, R \times_{\mathrm{Gr}_{\mathcal{H}}} \mathrm{Gr}_{\mathcal{G}}$$

(*) $\mathcal{F}(R \to R') = \{ \text{sections } \beta \text{ of } \tilde{\pi} \text{ over } \mathbb{D}_{R'} : \beta|_{\mathbb{D}^*_{R'}} = \tilde{\alpha}|_{\mathbb{D}_{R'}} \}$

Here $\mathcal{E} \xrightarrow{\pi} \mathbb{D}_R$

$\mathcal{E}/\mathcal{G} \xrightarrow{\tilde{\pi}} \mathbb{D}_R$

given $\alpha : \mathbb{D}^\times_R \to \mathcal{E}$

$\tilde{\alpha} : \mathbb{D}^\times_R \to \mathcal{E}/\mathcal{G}$

Key point: recall why $[\mathcal{E}/\mathcal{G}]$ is defined the way it is.

fiber of $[\mathcal{E}/\mathcal{G}]$ over X:

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{E} \\ \downarrow & & \\ X & & \end{array}$$

$\mathcal{G}$-equiv. morphism

$\mathcal{G}$-torsor

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\ulcorner"] & P \arrow[r] \arrow[d] & \mathcal{E} \\ X' \arrow[r] & X & \end{tikzcd}$$

Lemma. Assume $\mathcal{H}/\mathcal{G}$ exists as q-aff. sch. $/X$,

$\mathcal{E} \to X$ given $\mathcal{H}$-torsor

$\mathcal{E}/\mathcal{G} \to X$

(a) sections $X \to \mathcal{E}/\mathcal{G}$ correspond bijectively to isom. classes of $\mathcal{G}$-torsors $\mathcal{E}_{\mathcal{G}} \to X$ + $\mathcal{G}$-equiv. morphism $\mathcal{E}_{\mathcal{G}} \to \mathcal{E}$ over X

($\mathcal{E}_\mathcal{G} \to X$ gives $\mathcal{G}$-str. to $\mathcal{E}$, so induced map $\mathcal{E}_\mathcal{G} \overset{\mathcal{G}}{\times} \mathcal{H} \xrightarrow{\sim} \mathcal{E}$ of $\mathcal{H}$-torsors)

(b) Suppose $\alpha : X \to \mathcal{E}$ is a section lifting the section $X \to \mathcal{E}/\mathcal{G}$

then it induces a ! section $X \to \mathcal{E}_\mathcal{G}$ comp. w/ $X \to \mathcal{E}$.

$$\begin{tikzcd} \mathcal{E}_\mathcal{G} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X \arrow[r] \arrow[ur, "\alpha"] \arrow[u, bend left, dashed] & \mathcal{E}/\mathcal{G} \end{tikzcd}$$

Apply Lemma to $(*)$ w/ $X = \mathbb{D}_{R'}$, $X^* = \mathbb{D}^*_{R'}$, $\mathcal{E}^* = \mathcal{E}|_{\mathbb{D}^*_R}$,

$\alpha = \alpha|_{\mathbb{D}^*_{R'}}$, $\beta = \beta|_{\mathbb{D}^*_{R'}}$.

$$\begin{tikzcd} \mathcal{F}^*_\mathcal{G} \arrow[r] \arrow[d] & \mathcal{E}^* \arrow[d] \\ X^* \arrow[r, "\alpha"'] \arrow[ur, "\alpha"] & \mathcal{E}^*/\mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_\mathcal{G} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X \arrow[r, "\beta"] & \mathcal{E}/\mathcal{G} \end{tikzcd}$$

Lemma $\Rightarrow$ $\mathcal{E}_\mathcal{G}|_{X^*} \cong \mathcal{F}^*_\mathcal{G}$. Then given α exists, get from (b),

get section of $\mathcal{E}_\mathcal{G} \to X$ over $\mathbb{D}^*_{R'}$.

$P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G})$

$L^+\mathcal{G} \curvearrowright \mathrm{Gr}_\mathcal{G} = \{(\mathcal{E}, \alpha)\}$

$L\mathcal{G} \curvearrowright \quad \overset{\shortparallel}{(L\mathcal{G}/L^+\mathcal{G})^{\text{ét}}}$

$g \cdot (\mathcal{E}, \alpha) = (\mathcal{E}, g \circ \alpha)$

Equivariant perverse sheaves

X f.t. separated k-scheme. (k as above)

G f.t. smooth connected k-grp scheme, acting on X on the left

$m: G \times G \to G$

$a: G \times X \to X$ $\qquad D(X) := D^b_c(X, \bar{\mathbb{Q}}_\ell)$

$e: X \to G \times X$ $\qquad P(X) := P(X, \bar{\mathbb{Q}}_\ell)$

$pr_2: G \times X \to X$

Def. $K \in P(X)$ is <u>G-equivariant</u> if $\exists$ isom. $\varphi: a^* K \xrightarrow{\sim} pr_2^* K$ in $D(G \times X)$.

Rmk ("Naive version") Can define G-equiv. object $K \in D(X)$, but require

<u>rigidity</u> $e^* \varphi: K \xrightarrow{id} K$ + <u>cocycle condition</u>: $(m \times id_X)^* \varphi = pr_{23}^* \varphi \circ (id_G \times a)^* \varphi$.

Point: for <u>perverse</u> K G-equiv. for <u>smooth</u> + <u>conn</u> gp G, these 2 properties are automatic. Apply following to $p_3: G \times G \times X \to X$.

BBD 4.2.5: $f: X \to Y$ smooth of rel. dim d w/ geometrically conn. $\overset{\text{dim } d}{\vee}$ fibers

then $f^*[d]: P(Y) \to P(X)$. <u>fully faithful</u>.

Exercise $G \curvearrowright X$ as above, $K \in P_G(X)$, Q is a subqt of K in $P(X)$, then $Q \in P_G(X)$.

Perverse sheaves on orbit spaces

G as above, $H \subset G$ $\overset{\text{smooth}}{\text{connected}}$ closed subgp.

$X = G/H$ (assume exists as scheme).

$G \curvearrowright X$.

Prop. Any $K \in P_G(X)$ is of the form $K \simeq \bar{\mathbb{Q}}_\ell^r[\dim X]$, some $r \geq 0$.

Pf. $G \to G/H$ smooth w/ geom. conn. fibers

So BBD 4.2.5, can assume $H = \bullet$, $G = X$.

$K \in P_G(G)$, $i: G \xrightarrow{\sim} G \times e \hookrightarrow G \times G$

$s: G \to \operatorname{Spec} k$, $e: \operatorname{Spec} k \to G$

Use equivariance and $pr_2 \circ i = e \circ s$

$K = id_G^* K = i^* a^* \cong i^* pr_2^* K = s^* K_0$, $K_0 = e^* K$ = stalk of K at e

$\Rightarrow K$ = constant complex of f.d. $\bar{\mathbb{Q}}_\ell$ - v.s.

OTOH, $\exists$ open smooth $U \overset{j}{\subset} G$ and lisse $\bar{\mathbb{Q}}_\ell$ - sheaf $\mathcal{L}$ on U,

$j^* K = \mathcal{L}[d]$, $d = \dim U = \dim X$. $\Rightarrow K \cong \bar{\mathbb{Q}}_\ell^r[\dim X]$. $\square$

Rmk. H not connected, $H^\circ \subset H$, assume H/H° étale, $\pi: G/H^\circ \to G/H$ étale.

Cartan decomposition (in split case)

$$G(F) = \coprod_{\mu \in X_*(T)^+} \mathcal{G}(\mathcal{O}) t^\mu \mathcal{G}(\mathcal{O}).$$

Consequence: as (in the split case)

$$\mathrm{Gr}_{\mathcal{G}}(k) = LG(k)/L^+\mathcal{G}(k) = G(F)/G(\mathcal{O}),$$

get $\mathrm{Gr}(k) = \coprod_{\mu} L^+\mathcal{G}\, t^{\mu} e_0 (k)$, $\quad e_0 = $ base pt.

$$= \coprod_{\mu} \mathrm{Gr}_{\mu}(k).$$

<u>Def</u> $\mathrm{Gr}_{\mu} = \mathrm{Gr}_{\mathcal{G},\mu} = L^+\mathcal{G}$ - orbit of $t^{\mu}e_0$ in $\mathrm{Gr}_{\mathcal{G}}$.

$\mathrm{Gr}_{\leq\mu} = \mathrm{Gr}_{\mathcal{G},\leq\mu}$: scheme theoretic image of

$$L^+\mathcal{G} \longrightarrow \mathrm{Gr}_{\mathcal{G}}$$
$$g \longmapsto g t^{\mu} e_0$$

<u>Notation</u>: $\mathcal{O}_{\mu} = \mathrm{Gr}_{\mathcal{G},\mu}$, $\quad \overline{\mathcal{O}_{\mu}} = \mathrm{Gr}_{\mathcal{G},\leq\mu}$

<u>Lemma</u>. $\mu, \lambda \in X_*(T)^+$,

(a) $\mathrm{Gr}_{\leq\mu} = \coprod_{\lambda\leq\mu} \mathrm{Gr}_{\lambda}$. $\quad \lambda \leq \mu : \mu - \lambda = $ sum of pos. coroots

(b) $\mathrm{Gr}_{\leq\mu}$ is irred. proj. of dim $d_{\mu} := \langle 2\rho, \mu\rangle$

$$\downarrow$$
$$\sum_{\alpha>0} \alpha.$$

(c) $\mathrm{Gr}_{\mathcal{G},\underline{red}} = \operatorname{colim}_{\mu} \mathrm{Gr}_{\mathcal{G},\leq\mu}$

Need to understand $P_{L^+G}(\mathrm{Gr}_{\leq\mu})$.

<u>Prop</u>. Fix $\mu \in X_*(T)^+$, then stabilizer of $t^{\mu}e_0$ of L^+G acting on Gr_{μ} is a <u>smooth</u> <u>conn'd</u> gp scheme, isom. to $P_{\mu} \ltimes \left(L^{++}\mathcal{G} \cap t^{\mu} L^+\mathcal{G}\, t^{-\mu}\right)$.

P_μ = parabolic subgp $\longleftrightarrow \mu$

$L^{++}G = \ker\left[L^+G \to G,\ t \mapsto 0\right]$

Pf. In notes. Recently proved that stabilizer is smooth + conn'd.

[Richarz - Scholbach, Motivic geom. Satake]

Irred. objects in $P_{L^+G}(\mathrm{Gr}_{\le\mu}) = P_{\underset{\text{quotient}}{\text{finite type}}}(\mathrm{Gr}_{\le\mu})$

Rmk Action of L^+G factors through a finite type quotient of L^+G: look at

GL_n case.

$$\begin{array}{c} t^{-N}\Lambda \\ \cup \\ L \\ \cup \\ t^N\Lambda \end{array} \qquad \Gamma_{2N} \subset GL_n(\mathcal{O})$$

$\therefore$ irred. equiv. lisse $\bar{\mathbb{Q}}_\ell$-sh. on Gr_μ are $\bar{\mathbb{Q}}_\ell[d_\mu]$,

hence

Cor. Irred. objects in $P_{L^+G}(\mathrm{Gr}_G)$ are the

$IC_\mu = i^\mu_* j^\mu_{!*}\ \bar{\mathbb{Q}}_\ell[d_\mu]$, $i^\mu : \mathrm{Gr}_{\le\mu} \hookrightarrow \mathrm{Gr}_G$, $j^\mu : \mathrm{Gr}_\mu \overset{\text{open}}{\hookrightarrow} \mathrm{Gr}_{\le\mu}$.

$\mu \in X_*(T)^+$.

Want: $(\mathrm{Rep}(\hat{G}), \otimes) \cong (P_{L^+G}(\mathrm{Gr}_G), *)$

$$\begin{array}{cc} V_\mu & IC_\mu \\ \mu \in X^*(\hat{T})^+ = X_*(T)_+ & \end{array}$$

Goal. $P_{L^+G}(\mathrm{Gr}_G)$ is semisimple, w/ simple objects IC_μ.

ETS: given dominant λ, μ. $\mathrm{Ext}^1_{P(\mathrm{Gr}_G)}(IC_\lambda, IC_\mu) = 0$.

$$\mathrm{Ext}^1_{P(\mathrm{Gr}_G)}(IC_\lambda, IC_\mu) = \mathrm{Hom}_{D(\mathrm{Gr}_G)}(IC_\lambda, IC_\mu[1])$$

Case (i) $\lambda = \mu$. $\mathcal{O}_\mu \overset{j}{\hookrightarrow} \overline{\mathcal{O}}_\mu \overset{i}{\hookleftarrow} \overline{\mathcal{O}}_\mu \setminus \mathcal{O}_\mu$.

Distinguished Δ: $i_! i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+1}$

$\rightsquigarrow$ exact sequence

$$\underset{(B)}{\mathrm{Hom}(IC_\mu, i_! i^! IC_\mu[1])} \to \mathrm{Hom}(IC_\mu, IC_\mu[1]) \to \underset{(A)}{\mathrm{Hom}(IC_\mu, j_* j^* IC_\mu[1])}$$

(A) $= 0$: (A) $= \mathrm{Hom}(j^* IC_\mu, j^* IC_\mu[1])$

$= \mathrm{Hom}_{D(\mathcal{O}_\mu)}(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell[1])$

$= \mathrm{Ext}^1_{\mathrm{Sh}\,\mathcal{O}_\mu}(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell[1]) \overset{(+)}{=} H^1_{\text{ét}}(\mathcal{O}_\mu, \overline{\mathbb{Q}}_\ell)$

(+): both are derived fcts of $\mathrm{Sh}_{\text{ét}}(\mathcal{O}_\mu, \overline{\mathbb{Q}}_\ell) \xrightarrow{\Gamma} \underline{Ab}$

So ETS $H^1_{\text{ét}}(\mathcal{O}_\mu, \overline{\mathbb{Q}}_\ell) = 0$.

$$\underbrace{L^{++}G / L^{++}G \cap L^{\geq\mu}G}_{\text{pro-unip}} \to \mathcal{O}_\mu = L^+G / (L^{++}G \cap L^{\geq\mu}G) \rtimes P_\mu$$

$$\downarrow$$

$$L^+G / L^{++}G \rtimes P_\mu = G/P_\mu$$

So ETS $H^1(G/P_\mu, \bar{\mathbb{Q}}_\ell) = 0$.

This is classical, we'll show later that $H^{odd}(\text{Sch. var. } \bar{\mathbb{Q}}_\ell) = 0$.

(B) $= 0$: we'll use $i^* IC_\mu$ lies in $p \le -1$

(standard fact: KW, III 5.1)

$\therefore$ $i^! IC_\mu$ lies in $p \geq 1$

$$\mathrm{Hom}(IC_\mu, i_! i^! IC_\mu[1]) = \mathrm{Hom}(\underset{p\le -1}{i^* IC_\mu}, \underset{p \geq 0}{i^! IC_\mu[1]})$$

<u>Case (ii)</u> $\lambda \neq \mu$, and $\lambda \le \mu$ or $\mu \le \lambda$.

If $\lambda < \mu$. $i : \bar{O}_\lambda \hookrightarrow \bar{O}_\mu$

$$\mathrm{Hom}(i_* IC_\lambda, IC_\mu[1]) = \mathrm{Hom}(\underset{p\le 0}{IC_\lambda}, \underbrace{i^! IC_\mu[1]}_{\text{Claim: } p\geq 1}) \overset{\therefore}{=} 0$$

Equiv.: $i^* IC_\mu$ in $p \le -2$.

If $\mu < \lambda$, $i : \bar{O}_\mu \hookrightarrow \bar{O}_\lambda$

$\mathrm{Hom}(IC_\lambda, i_* IC_\mu[1])$

$$= \mathrm{Hom}(\underset{p\le -2}{i^* IC_\lambda}, \underset{p \geq -1}{IC_\mu[1]}) = 0$$

<u>Case (iii)</u> $\lambda \not\le \mu$, $\mu \not\le \lambda$. Still need claim.

Lecture 5. Assume for notation simplicity. G split, $\mathfrak{g} = G/\mathcal{O}$.

$P_{L^+G}(Gr_G)$

simple objects : IC_μ , $\mu \in X_*(T)^+$.

Semisimplicity follows from

$\forall \lambda, \mu \in X_*(T)^+$, $\mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\mu[1]) = 0$:

given ext'n in $P(Gr_G)$

$$0 \to IC_\mu \to F \to IC_\lambda \to 0$$

gives dist. Δ in $D(Gr_G)$, hence exact seq.

$$\mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\mu) \to \mathrm{Hom}_{D(Gr_G)}(IC_\lambda, F) \to \mathrm{Hom}_{D(Gr_G)}(IC_\lambda, \overset{\overset{id}{\uparrow}}{IC_\lambda})$$

$$\to \underset{\underset{0.}{\|}}{\mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\mu[1])} \to \cdots$$

then similar argument shows any $F \in P_{L^+G}(Gr_G)$ is SS by induction on length.

Technical Lemma. If $i: \overline{\mathcal{O}_\lambda} \hookrightarrow \overline{\mathcal{O}_\mu}$, $\lambda < \mu$ $\Rightarrow$ $i^* IC_\mu$ in ${}^pD^{\leq -2}$.

Case 1 $\lambda = \mu$ ✓

Case 2 $\lambda \neq \mu$, $\lambda < \mu$ or $\mu < \lambda$

$\lambda < \mu$, ✓

$\mu < \lambda$, $i: \overline{\mathcal{O}_\mu} \hookrightarrow \overline{\mathcal{O}_\lambda}$

$$\mathrm{Hom}(IC_\lambda, i_* IC_\mu[1]) = \mathrm{Hom}(\underset{p\leq -2}{i^* IC_\lambda}, \underset{p \geq -1}{IC_\mu[1]}) = 0$$

<u>Case 3</u> $\lambda \nleq \mu$, $\mu \nleq \lambda$

$\pi_0(\mathrm{Gr}_G) = \pi_1(G) := X_*(T)/Q^\vee$

WLOG, λ, μ "in same conn. cpt"

$\mu - \lambda \in Q^\vee$

$\exists\ \nu \in X_*(T)_+$, $\lambda < \nu$, $\mu < \nu$.

$$\begin{tikzcd}
\overline{O}_\lambda \times_{\overline{O}_\nu} \overline{O}_\mu \arrow[r, hook, "\iota_1"] \arrow[d, hook, "\iota_2"'] & \overline{O}_\mu \arrow[d, hook, "i_2"] \\
\overline{O}_\lambda \arrow[r, hook, "i_1"] & \overline{O}_\nu
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{Hom}(i_{1*} IC_\lambda, i_{2*} IC_\mu[1]) &= \mathrm{Hom}(i_2^* i_{1*} IC_\lambda, IC_\mu[1]) \\
&= \mathrm{Hom}(\iota_{1*} \iota_2^* IC_\lambda, IC_\mu[1]) \\
&= \mathrm{Hom}(\underbrace{\iota_2^* IC_\lambda}_{p \le -1}, \underbrace{\iota_1^! IC_\mu[1]}_{p \ge 0}) = 0
\end{aligned}$$

<u>Pf of Technical Lemma</u>

(use facts about Schubert var.)

Know $i^* IC_\mu \in {}^pD^{\le -1}$.

Need to show $i^* IC_\mu \in {}^pD^{\le -2}$.

Will show: $\forall$ odd j, $i^* IC_\mu \in {}^pD^{\le j} \Rightarrow i^* IC_\mu \in {}^pD^{\le j-1}$. (*)

Use 2 facts

$d_\mu = \dim \mathrm{Gr}_{\le\mu} = \langle 2\rho, \mu\rangle$

(I) $\mathcal{H}^i IC_\mu$ vanish unless $i \equiv d_\mu \pmod 2$

(parity vanishing)

(II) $\dim \mathrm{Gr}_{\le\lambda} \equiv \dim \mathrm{Gr}_{\le\mu} \pmod 2$, $\forall \lambda \le \mu$.

(II) : clear. $\mu - \lambda = \sum_{\text{some } \alpha} \alpha^\vee$, $\langle 2\rho, \alpha^\vee\rangle =$ even.

(I) Use $Fl_G := LG/\underbrace{L^+\mathcal{G}_a}_{\text{Iwahori}} \xrightarrow{p} LG/L^+G = \mathrm{Gr}_G$

smooth rel. dim d

w/ fiber $\cong G/B$

$$p^{-1}(\mathrm{Gr}_{\le\mu}) = Fl_{\le \underbrace{t_\mu w_0}_{=: w_\mu}} \qquad y \mapsto p(y) = x$$

BBD 4.2.5 $\quad p^*[d]\, IC_\mu = IC_{w_\mu}$

$$\mathcal{H}^{i-d}_y IC_{w_\mu} = \mathcal{H}^0_x IC_\mu$$

$IC_{w_\mu} = j_{!*} \overline{\mathbb{Q}}_\ell [d+d_\mu]$

Demazure resolution

$\widetilde{Fl}_{w_\mu} \xrightarrow{\pi} Fl_{w_\mu}$, fibers are paved by affine spaces.

smooth $\quad$ birational

$$IC_{w_\mu} \overset{\oplus}{\hookrightarrow} \pi_* \overline{\mathbb{Q}}_\ell [d+d_\mu]$$

$\Rightarrow \mathcal{H}^{i-d}_y IC_{w_\mu} = 0$ unless $i-d-(d+d_\mu)$ even, i.e. unless $i - d_\mu$ even. $\Rightarrow$ (I)

Claim: (I), (II) $\Rightarrow$ ⊛. Apply following to $K = i^* IC_\mu$

<u>Lemma</u>. $K \in D(Gr_{\leq \mu})$ s.t.

(1) $\mathcal{H}^i K = 0$ unless $i \equiv d_\mu \pmod 2$

(2) $\forall i$, $\dim \operatorname{supp} \underbrace{\mathcal{H}^i K}_{\text{equivariant}} \equiv d_\mu \pmod 2$

Then $K \in {}^pD^{\leq j} \Rightarrow K \in {}^pD^{\leq j-1}$ if j <u>odd</u>.

<u>Pf</u>. $K \in {}^pD^{\leq j} \Leftrightarrow \dim \operatorname{supp} \mathcal{H}^i K[j] \leq -i$, $\forall i$

$\Leftrightarrow \dim \operatorname{supp} \mathcal{H}^i K \leq -i+j$, $\forall i$

WLOG, $i = 2m + d_\mu$

$\Leftrightarrow \dim \operatorname{supp} \mathcal{H}^i K \leq -2m - d_\mu + j$

LHS $\equiv d_\mu\ (2)$, RHS $\not\equiv d_\mu\ (2)$ (j odd)

$\Leftrightarrow \dim \operatorname{supp} \mathcal{H}^i K \leq -2m - d_\mu + j - 1$. □

Have proved $P_{L^+G}(Gr_G)$ is semisimple, w/ simple objects IC_μ.

Construct $\star$ convolution product on $P_{L^+G}(Gr_G)$.

$(\mathcal{H}_K(G), *)$ $\quad f_1, f_2$, $f_1 * f_2 (g) = \int_G f_1(x) f_2(x^{-1}g)\, dx$

$f_1 = \mathbb{1}_{Kt^{\mu_1}K}$, $f_2 = \mathbb{1}_{Kt^{\mu_2}K}$

$\# \{xK : xK \in Kt^{\mu_1}K,\ x^{-1}gK \in Kt^{\mu_2}K\}$

$K \xrightarrow{\mu_1} xK \xrightarrow{\mu_2} gK$ $\quad (g^{-1}x \in Kt^{-\mu_2}K)$

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G = LG \overset{L^+G}{\times} LG/L^+G$$

$$= LG \times LG \Big/_{a_2} L^+G \times L^+G$$

$a_2 : (x,y)\cdot(h_1,h_2) = (xh_1, h_1^{-1} y h_2)$

$a_1 : (x,y)(h_1,h_2) = (xh_1, yh_2)$

$$(LG \times LG)\Big/_{a_1} L^+G \times L^+G = \mathrm{Gr}_G \times \mathrm{Gr}_G$$

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$
$$(x,y) \longmapsto xy$$

Convolution Diagram:

$$\begin{array}{c} \mathcal{F}_1 \quad \mathcal{F}_2 \\ \mathrm{Gr}_G \times \mathrm{Gr}_G \end{array} \xleftarrow[a_1\ qt]{p} LG \times LG \xrightarrow[a_2\ qt]{q} \mathrm{Gr} \tilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$

Thm. The following hold

(a) p, q are "smooth of same rel. dim".

(b) m locally trivial (in stratified sense) and semismall.

Def of convolution

$\mathcal{F}_1, \mathcal{F}_2 \in P_{L^+G}(\mathrm{Gr}_G)$, $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in P_{equiv}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$

$p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ perverse "up to shift", a_1- equiv

autom. a_2- equiv. (uses $\mathcal{F}_1, \mathcal{F}_2$ L^+G - equiv.)

<u>Descent Lemma</u> $\Rightarrow$ $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ descends, i.e. $\exists!$ perverse sheaf $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$

on $Gr_G \tilde{\times} Gr_G$ s.t. $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$

<u>Def</u>. $\mathcal{F}_1 * \mathcal{F}_2 = Rm_*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$.

This is perverse since m is semismall and L^+G-equiv.

$$Gr_G \times Gr_G \xleftarrow[a_1 \, q_+]{p} LG \times LG \xrightarrow[a_2 - q_+]{q} Gr \tilde{\times} Gr \xrightarrow{m} Gr_G$$

$$\cup$$

$$\overline{\mathcal{O}}_\lambda \times \overline{\mathcal{O}}_\mu \longleftarrow p_1^{-1}(\overline{\mathcal{O}}_\lambda) \times p_1^{-1}(\overline{\mathcal{O}}_\mu) / L^{\geq n}G \; / L^{\geq n}G \longrightarrow \overline{\mathcal{O}}_\lambda \tilde{\times} \overline{\mathcal{O}}_\mu \xrightarrow{m} \overline{\mathcal{O}}_{\lambda+\mu}$$

Finite-dim'l version:

$Gr_{\leq \mu} =: \overline{\mathcal{O}}_\mu$

$Gr_\mu = \mathcal{O}_\mu$

$n \gg 0$, action on $\overline{\mathcal{O}}_\lambda, \overline{\mathcal{O}}_\mu, \overline{\mathcal{O}}_\lambda \tilde{\times} \overline{\mathcal{O}}_\mu, \overline{\mathcal{O}}_{\lambda+\mu}$

factors through $G_n = LG / L^{\geq n}G$.

$$L^{\geq n}G = \ker\left(G(R[\![t]\!]) \to G(R[t]/t^n)\right)$$
$$\cap$$
$$L^+G$$

Why p, q (resp. p_n, q_n) smooth of same rel. dim.?

p_n smooth

q is Zariski-locally isom. to p.

$$\begin{array}{l} Gr_G \tilde{\times} Gr_G \\ \downarrow pr_2 \\ Gr_G \end{array}$$

Zar. locally trivial on base (big cell)

m locally trivial: $\overline{\mathcal{O}}_\mu \tilde{\times} \overline{\mathcal{O}}_\lambda = \coprod_{\substack{\mu' \leq \mu \\ \lambda' \leq \lambda}} \mathcal{O}_{\mu'} \tilde{\times} \mathcal{O}_{\lambda'} \xrightarrow{m} \overline{\mathcal{O}}_{\lambda+\mu} = \coprod_{\nu \leq \lambda+\mu} \mathcal{O}_\nu$

Property: given $y \in \mathcal{O}_\nu \subset \operatorname{im}(m|_{(-)})$, $\exists$ open V, $y \in V \subset \mathcal{O}_\nu$

and $m|_{(-)}^{-1}(V) \cong V \times m|_{(-)}^{-1}(y)$.

Def. of semismallness here is:

r-fold convolution morphism

$(\mu_1, \ldots, \mu_r) = \mu_\bullet$, $|\mu_\bullet| = \sum \mu_i$

$\lambda < |\mu_\bullet|$, $y = t^\lambda e_0 \in \mathcal{O}_\lambda$

$$m_\mu : \overline{\mathcal{O}_{\mu_1}} \tilde{\times} \cdots \tilde{\times} \overline{\mathcal{O}_{\mu_r}} \longrightarrow \overline{\mathcal{O}_{|\mu_\bullet|}}$$
$$\cup$$
$$\mathcal{O}_\lambda \ni y$$

$\dim m_\mu^{-1}(y) \le \langle \rho, |\mu_\bullet| - \lambda \rangle$

By locally triviality, equiv. to

$\dim m_\mu^{-1}(\mathcal{O}_\lambda) \le \langle \rho, |\mu_\bullet| + \lambda \rangle$

Strategy: $\dim(S_\lambda \cap \mathcal{O}_\mu) \le \langle \rho, \mu + \lambda \rangle$, $\mu \in X_*(T)^+$, $\lambda \in \Omega(\mu)$.

$\Omega(\mu)$ = wt space of $V_\mu \in \operatorname{Rep}(\hat{G})$

$= \{ \lambda \in X_*(T) : w\lambda \le \mu, \forall w \in W \}$.

$S_\lambda := LU t^\lambda e_0 \subset Gr_G$ (locally closed sub-ind sch.)

<u>Sketch of reduction</u>: $m_{\mu_\bullet}^{-1}(S_\lambda \cap \overline{\mathcal{O}_\mu}) = \bigcup_{\nu_\bullet = (\nu_1, \ldots, \nu_r)} (S_{\nu_1} \cap \overline{\mathcal{O}_{\mu_1}}) \tilde{\times} \cdots \tilde{\times} (S_{\nu_r} \cap \overline{\mathcal{O}_{\mu_r}})$, $\mu = |\mu_\bullet|$

$\dim \mathrm{LHS} \le \max\limits_{\nu_\bullet} \sum\limits_i \langle \rho, \nu_i + \mu_i \rangle = \langle \rho, \lambda + \mu \rangle.$

Need to show $\dim(S_\mu \cap \bar{O}_\mu) \le \langle \rho, \lambda + \mu \rangle$

[MV07], [NP01]

ETS for $k = \bar{\mathbb{F}}_q$ (generic flatness), Fix $\mathbb{F}_q$ (vary q)

$K = G(\mathbb{F}_q[[t]])$, $U = U(\mathbb{F}_q((t)))$.

By Weil conj, ETS

$$\lim_{q \to \infty} \frac{\# (Ut^\lambda K/K \cap Kt^\mu K/K)}{q^{\langle \rho, \mu + \lambda \rangle}} = m_\mu(\lambda)$$

mult. of λ in V_μ of $\hat{G}$.

$\mathrm{LHS} = (\mathbb{1}_{Kt^\mu K})^\vee (t^\lambda) \, q^{\langle \rho, \lambda \rangle}$

By Macdonald's formula, this is the coeff. of $t^{-w_0\lambda}$ in

$$\frac{q^{\langle \rho, \lambda + \mu \rangle}}{W_\mu(q^{-1})} \sum_{w \in W_0} w \left(\prod_{\alpha > 0} \frac{1 - q^{-1} t^{-\alpha^\vee}}{1 - t^{-\alpha^\vee}} \right) \cdot t^{w\mu}$$

Devide this by $q^{\langle \rho, \mu + \lambda \rangle}$ and take the limit as $q \to \infty$. The Weyl character formula implies we get $m_\mu(\lambda)$. This completes the proof. $\square$

Lecture 6. Goal: $(P_{L^+G}(Gr_G), *)$ is a neutral Tannakian cat.

Sym. monoidal cat. $(\mathcal{C}, \otimes, I)$. $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, identity object I

right unit: $r_A : A \otimes I \xrightarrow{\sim} A$, left unit: $l_A : I \otimes A \xrightarrow{\sim} A$.