The Hodge theory of the Decomposition Theorem

after M. A. de Cataldo & L. Migliorini

Geordie Williamson

X conn'd complex proj. var. $n = \dim X = \dim_{\mathbb{C}} X$

$H = \bigoplus H^i$, $H^i = H^i_{dR}(X; \mathbb{R})$

Remarkable theorems about H:

1) Poincaré duality

WL 2) Weak Lefschetz theorem

HL 3) Hard Lefschetz theorem : $\omega^i: H^{n-i} \xrightarrow{\sim} H^{n+i}$, ω = ample class = Chern_1 (ample lb.)

4) Hodge decomposition : $H_{\mathbb{C}} = \bigoplus H^{p,q}$

HR 5) Hodge-Riemann relations: if $n = 2m$ is even, $\langle -,- \rangle_{\text{Poincaré}}$ restricted to

$(\ker \omega) \cap H^{m,m} \cap H^n$ is $(-1)^m$-definite.

If X is singular, everything breaks down $\longrightarrow$ mixed Hodge theory (Deligne)

$\longrightarrow$ We can instead consider $IH^*(X; \mathbb{R})$ "intersection cohomology".

1) - 5) remain true : Goresky-MacPherson, Beilinson-Bernstein-Deligne-Gabber, Saito

1970's - 1988

How does Hodge theory extend to maps?

Throughout, $f: X \to Y$ proper morphism, X, Y proj., X smooth, $\dim X = n$

Deligne's degeneration theorem. Assume f is smooth (i.e. submersive, i.e. a C^∞-fibration).

Leray - Serre Spectral sequence:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathbb{R}_X) \overset{LS}{\Longrightarrow} H^{p+q}(X, \mathbb{R})$$

Deligne's thm: LS degenerates at E_2 $(E_2 = E_\infty)$

In fact

1) $\underset{\underset{D^b(Y)}{\cap}}{Rf_* \mathbb{R}_X} \simeq \bigoplus R^q f_* \mathbb{R}_X[-q]$ in $D^b(Y)$

HL on fibers

2) each $R^q f_* \mathbb{R}_X$ is semisimple.
loc. sys.

—— Σ

Everything breaks down for arbitrary map.

Perverse sheaves + DT

$D^b(Y)$ = bounded derived cat. of sheaves of $\mathbb{R}$-v.s. on Y

$\cup$

$D^b_c(Y)$, $\mathcal{H}^i$ constructible

$({}^pD^{\leq 0}, {}^pD^{\geq 0})$ perverse t-structure.

P_Y = perverse sheaves $= {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$.

${}^p\mathcal{H}^i : D^b_c(Y) \to P_Y$ perverse cohomology functors

Properties: → P_Y is abelian

→ given a conn'd smooth subvariety $Z \subset Y$ and a local system L on Z, $\exists\ IC(\bar{Z}; L) \in P_Y$ s.t.

1) $\mathrm{Supp}\ IC(\bar{Z}, L) = \bar{Z}$

2) $IC(\bar{Z}, L)|_Z \simeq L[\dim Z]$, 3) ...

$IC(\bar{Z}, \mathcal{L})$ are simple if $\mathcal{L}$ is, and all simple perverse sheaves are of this form.

Ex. if $\bar{Z}$ is smooth and If $\mathcal{L}$ extends to a local system $\bar{\mathcal{L}}$ on $\bar{Z}$, then $IC(\bar{Z}, \mathcal{L}) \simeq \bar{\mathcal{L}}[\dim Z]$.

$\mathcal{F} \in D^b_c(Y)$ is <u>Semisimple</u> if it is isomorphic to a direct sum of shifts of IC's.

Let $f : X \xrightarrow{\text{smooth}} Y$ be as above: (f proper)

<u>Decomposition Theorem</u>. $Rf_* \mathbb{R}_X$ is semisimple.

<u>Rmk</u> BBDG 1981, Saito 1988.

<u>Global structure of proof</u>.

Simultaneous induction on $\left(\dim Y, \text{"defect of semi-smallness"}\right)$

$$\text{"defect of semi-smallness"} = \max\{i : {}^p\mathcal{H}^i(Rf_* \mathbb{R}_X[n]) \neq 0\}$$

<u>Key case</u>: $Rf_* \mathbb{R}_X[n]$ is perverse, i.e. ${}^p\mathcal{H}^i(Rf_* \mathbb{R}_X[n]) = 0, \forall i \neq 0$.

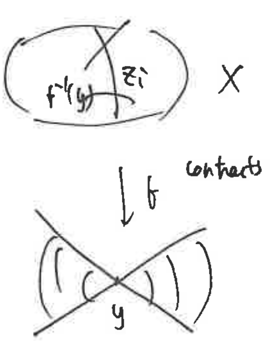
$\iff$ f is semi-small ("perverse finite")

Choose a stratification $Y = \coprod_{\lambda \in \Lambda} Y_\lambda$ s.t. f is a topological fibration over each Y_λ

Y_λ smooth conn'd w/ fiber F_λ (typically singular).

$|\Lambda| < \infty$

f is semi-small $\iff$ $\dim F_\lambda \leq \frac{1}{2} \operatorname{codim}(Y_\lambda \subset Y)$.



Z_i irred. comp. of $f^{-1}(y)$

Y singular surface

<u>DT for semismall f</u>:

$$Rf_* \mathbb{R}_X[n] = \bigoplus IC(\bar{Y}_\lambda, \mathcal{L}_\lambda),$$

where $\mathcal{L}_\lambda$ is the local system given by

$$y \mapsto H^{\operatorname{codim}(Y_\lambda \subset Y)}(f^{-1}(y)).$$

Rmk. 1) Decomposition is canonical in this case

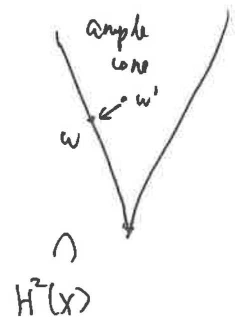
2) $Rf_* \mathbb{R}_X[n]$ is determined by $\{(Y_\lambda, \mathcal{L}_\lambda)\}_{\lambda \in \Lambda}$

Let $f: X \to Y$ be proper (not. nec. semi-small)

Thm (HL + HR for semi-small classes) HRSS $\quad \omega = f^*$(ample class on Y)

ω satisfies HL and HR on $H^*(X)$ $\iff$ f is semi-small.

Idea of proof of HRSS ($\Leftarrow$) Assume we know ω satisfies HL on $H^*(X)$



Using that the signature of a family of non-degenerate Hermitian forms cannot change, one deduces HR for ω.

Classical argument: weak Lefschetz + HR in dim $n-1$

$\Rightarrow$ HL in dim n

(Use that perverse sheaves satisfy weak Lefschetz).

HRSS $\Rightarrow$ DT for semismall f.

Assume $n = 2m$ is even

Semismall $\Rightarrow$ $\#\{y : \dim f^{-1}(y) = m\} < \infty$

Fix such a y and set $F = f^{-1}(y)$.

Set $i : \{y\} \hookrightarrow Y$.

Decomposition theorem predicts $Rf_* \mathbb{R}_X[n] \overset{(*)}{=} i_* V_y \oplus \mathcal{F}$ ← known by induction

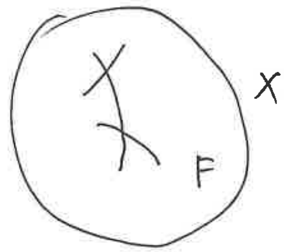
↑ skyscraper at y w/ fiber $H^n(F)$

Pagero

Establishing $(*)$ is key to DT in this case.

Consider: $$\begin{array}{ccccccc} \mathrm{Hom}(i_*\mathbb{R}_y,\ Rf_*\mathbb{R}_X[n]) & \times & \mathrm{Hom}(Rf_*\mathbb{R}_X[n],\ i_*\mathbb{R}_y) & \longrightarrow & \mathrm{End}(i_*\mathbb{R}_y) \\ \| \text{ can.} & & \| \text{ can.} & & \| \\ Q:\ H_n(F) & \times & H_n(F) & \longrightarrow & \mathbb{R} \end{array}$$

$F = \bigcup_{i\in I} Z_i \cup$ components of dim $< m$



$F \subset X$ is $\frac{1}{2}$ dim'l.

Lemma 1) Q is the intersection form

2) $(*)$ holds $\iff$ Q is non deg.

Claim. Q is $(-1)^m$-definite.

Why?

$$\begin{tikzcd} H_n(F) \arrow[rr, "cl\ \text{``class''}"] \arrow[dr] & & H^n(X) \\ & (H^{m,m}\cap H^n)\cap(\ker\beta) \arrow[ur] & \end{tikzcd}$$

$H_n(F) = \bigoplus_{i\in I}\mathbb{R}[Z_i]$

① cl is injective $\Leftarrow$ (mixed Hodge theory)

② cl is an isometry (tautology)

(Z_i are alg. cycles)

$\beta = f^*$ (ample class on Y) ($\{y\}\cap$ gen. hyperplane section is ϕ)

Now claim follows from HRSS + restriction of a non-deg. form is non-deg.