Motives of the Hitchin system

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Symm of $D^b_{coh}(-)$ $\underset{\text{algebraic cycles}}{\overset{\text{motives}}{\longleftarrow\!\!-\!-\!-\!-\!\longrightarrow}}$ Cohomology / Chow

abelian schemes

$f: A \longrightarrow B$ Mukai's transform $\xrightarrow[\text{Beauville, Deninger-Murre}]{}$ $N: A \to A$ mult. by N

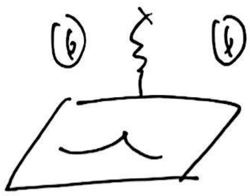
$[N]^* \curvearrowright H^*(A)$

$H^*(A) = \bigoplus_i H^*(A)_i$

$[N]^*\alpha = N^i\alpha$

Hitchin system

$f: M_{Hit} \longrightarrow B$ "D^b_{coh}"(M_{Hit}) $\xrightarrow{\text{Today}}$ $H^*(M_{Hit})$



Symmetries – geometric Langlands

– S-duality

$P=W$ conj.

Topological mirror conj.

BPS invariants

§1 Topology/$\mathbb{C}$ "GL_n" $(n, d) = 1$

rk degree

C curve of $g \geq 2$

$M_{n,d}$ = moduli of stable Higgs bundles $(\mathcal{E}, \theta)$

– $\mathcal{E}$ v.b. of rk n, deg d

– $\theta: \mathcal{E} \longrightarrow \mathcal{E} \otimes \omega_C$ Higgs field

Hitchin system

$$f_d : M_{n,d} \longrightarrow B_n$$

$$(\mathcal{E},\theta) \longmapsto \mathrm{char}(\theta) = \left(\mathrm{tr}(\theta), \mathrm{tr}(\Lambda^2\theta), \cdots, \det(\theta)\right)$$

$$\bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i})$$

$$B_n = \left\{ \begin{array}{c} C_a \subset T^*C \\ \quad {\scriptstyle n:1} \searrow \ \downarrow \\ \qquad C \end{array} \right\}$$

f_d is projective, (Lagrangian)

(i) Decomp. Thm (BBDG)

$$Rf_{d*}\underline{\mathbb{Q}} \simeq \bigoplus_i \underbrace{{}^p\mathcal{H}^i(Rf_{d*}\underline{\mathbb{Q}})}_{\text{semisimple perverse sheaves on } B_n}[-i]$$

(not canonical!)

Ngô: more precise description of the ss perverse sheaves that showed up.

(In general, still not completely understood yet)

$\rightarrow$ treats $G \rightsquigarrow$ fundamental Lemma

de Cataldo
Hausel
Migliorini

$$P_k H^*(M_{n,d}) = H^*\left(B_n, \bigoplus_{i\le k} {}^p\mathcal{H}^i(Rf_{d*}\underline{\mathbb{Q}})[-i]\right)$$

$$H^*(M_{n,d}) = H^*(B_n, Rf_{d*}\underline{\mathbb{Q}})$$

detects MHS of char. var. via nonabelian Hodge-theory

"$P=W$" : $M_{n,d} \overset{C^\infty}{\sim} M_{char}$ $\leftarrow$ character variety (affine)

$$P_k H^*(M_{n,d}) = W_{2k} H^*(M_{char})$$

('22 Maulik-S., Hausel-Mellit-Minets-Schiffmann)

(ii) Lefschetz symmetry $\quad R = $ rel. dim. f_d

η rel. ample on $M_{n,d} \to B_n$ $\qquad -\cup\eta^i : {}^p\mathcal{H}^{R-i} \xrightarrow{\sim} {}^p\mathcal{H}^{R+i}$ (Hard Lefschetz)

(iii) <u>d-indep</u>. $\quad (n,d) = (n,d') = 1$

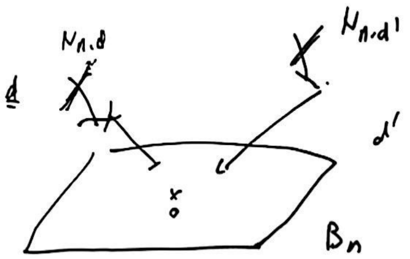
<u>Thm</u>. (Groechenig–Wyss–Ziegler, Maulik–S.)

$$\begin{tikzcd} M_{n,d} \arrow[dr, "f_d"'] & & M_{n,d'} \arrow[dl, "f_{d'}"] \\ & B_n & \end{tikzcd}$$

$$Rf_{d*}\underline{\mathbb{Q}} \simeq Rf_{d'*}\underline{\mathbb{Q}}$$

<u>Rmks</u> Thm $\Rightarrow$ $H^*(M_{n,d}) \simeq H^*(M_{n,d'})$

However, not true for moduli of stable vector bundles. $N_{n,d}$



§2. Algebraic cycles

(i), (ii), (iii) are all governed by alg. cycles. (MSY, 23, 24)

<u>Thm</u>: R = rel. dim. of $(f_d : M_{n,d} \to B_n)$

(a) $\exists\ \Delta_{M_{n,d}/B_n} = \pi_0 + \pi_1 + \cdots + \pi_{2R} \in CH_{\dim M}(M \times_B M)$

$$\pi_i \circ \pi_j = \delta_{ij}\pi_i$$

inducing $\displaystyle Rf_{d*}\underline{\mathbb{Q}} \simeq \bigoplus_{i=0}^{2R} {}^{P}\mathcal{H}^i(-)[-i]$

(b) The projectors $\{\pi_k\}$ are (essentially) given by Arinkin's Fourier–Mukai transform for compactified Jac

<u>Rmks</u> (1) (a) Confirms Corti–Hanamura's motivic decomp. conj. for Hitchin system.

(2) $\{\pi_k\} \xrightarrow{(a)} P$, $\{\pi_k\} \xrightarrow{(b)}$ tautological classes $\rightsquigarrow P = W$.

Thm ii   Lefschetz std conj. true for f_d:

$$\exists\ Z_{\eta,i} \in CH_*(M \times_B M) \quad \text{s.t.} \quad Z_{\eta,i} = (\eta^i)^{-1} \text{ on } Rf_* \underline{\mathbb{Q}}.$$

Thm iii   The motivic decomp obtained in Thm i is d-indep.