Parity sheaves and modular representations

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Geometry of linkage (w/ S. Riche)

Beilinson - Bernstein localisation

$\mathfrak{g}/\mathbb{C}$ semisimple

$$D_{G/B}\text{-mod} \xrightarrow[\sim]{\Gamma} U(\mathfrak{g})/(Z_+)\text{-mod}$$

: direct connection

: block by block

: Kazhdan - Lusztig conjecture

Geometry Satake equivalence

G/k reductive group $\rightsquigarrow$ $G^\vee/\mathbb{C}$ dual group

$Gr := G^\vee((t))/G^\vee[\![t]\!]$ ind-scheme

Eg: $G^\vee = SL_2$, $Gr = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^3 \sqcup \dots$

gluing maps are complicated : $\neq \mathbb{P}^\infty$.

$$(\mathrm{Rep}\, G, \otimes) \xrightarrow{\sim} (\mathrm{Perv}_{G^\vee\mathbb{C}[\![t]\!]}(Gr, k), *)$$

: indirect connection (Tannakian formalism)

: block decomposition opaque

: Lusztig character formula ??

Equivariant localization : $S^1 \curvearrowright X$. Philosophy : $X \leftrightsquigarrow X^{S^1}$

Eg. $\chi(X) = \chi(X^{S^1})$ "because" $\chi\left(\begin{smallmatrix}\text{nontrivial}\\ \text{orbit}\end{smallmatrix}\right) = 0$

<u>Smith theory</u> $S^1 \rightsquigarrow \mu_p$ when taking mod p coefficients.

<u>Eg.</u> $\mu_p \curvearrowright X$, $\chi(X) \equiv \chi(X^{\mu_p}) \mod p$

"because" $\chi\begin{pmatrix}\text{non trivial}\\ \Downarrow \\ \text{free orbit}\end{pmatrix} \equiv 0 \mod p.$

<u>Treumann</u> $\mu_p \curvearrowright X$, $\subset \mathbb{C}^\times \curvearrowright$, char $k = p$

$$Sm_{\mathbb{C}^\times}(X^{\mu_p}) = \frac{D^b_{\mathbb{C}^\times \mu_p}(X^{\mu_p}, k)}{\begin{pmatrix}\mu_p\text{- perfect}\\ \text{complexes}\end{pmatrix}} \simeq D^b_{\mathbb{C}^\times}(X^{\mu_p}, k[\mu_p])$$

restr. to $\mu_p \subset \mathbb{C}^\times$ yields a $\uparrow$

$\leftarrow$ $\mathcal{F}$ s.t. $res^{\mathbb{C}^\times}_{\mu_p}(\mathcal{F})$ is μ_p- perfect

2- periodic: $[2] \simeq [0]$.

<u>Lemma (T)</u>: $i : X^{\mu_p} \hookrightarrow X$

$\mathcal{F} \in D^b_{\mu_p \, \mathbb{C}^\times}(X, k)$, then $i^! \mathcal{F} \longrightarrow i^* \mathcal{F}$ is an isom. in $Sm_{\mathbb{C}^\times}(X^{\mu_p})$

<u>Ex.</u> $Sm_{\mathbb{C}^\times}(pt) \simeq k[x^{\pm 1}]$- dgmod

$\uparrow$ deg 2

<u>Key observation</u>: $\mathbb{G}_m \curvearrowright Gr$ "loop rotation"

$Gr = G^\vee(\mathbb{C}(\!(t)\!))/G^\vee(\mathbb{C}[\![t]\!])$ $\lambda \cdot t \mapsto \lambda t$

$$(Gr)^{\mu_p} = \coprod_{\gamma \in X_*(\check{T})/(W_p)} G^\vee((t^p)) \cdot t^\gamma =: Gr_\gamma$$

$\leftarrow$ NO DOT!

Eg. $\quad Gr_0 = \check{G}((t^p)) / \check{G}[[t^p]]$

$Gr_{\text{reg. wt}}$ = affine flag variety for $\check{G}((t^p))$

Compare: $\quad \mathrm{Rep}\, G = \bigoplus_{\gamma \in X/(W_p \cdot)} \mathrm{Rep}_\gamma$

Theorem (Bezrukavnikov - Gaitsgory - Mirković - Riche - Rider)

$$\mathrm{Perv}_{\check{G}[[t]]}(Gr, k) \xrightarrow{\sim} \mathrm{Perv}_{IW}(Gr, k)$$

$$IC_\lambda \rightsquigarrow IC_{\lambda+\rho}$$

Iwahori-Whittaker sheaves, need $\check{G} \rightsquigarrow \check{G}/\overline{\mathbb{F}}_\ell$, $\ell \neq p$

$$\begin{tikzcd}
D^b_{IW, \mathbb{G}_m}(Gr) \arrow[r, "i^{!*}"] & Sm_{IW, \mathbb{G}_m}((Gr)^{\mu_p}) & D^b_{IW, \mathbb{G}_m}((Gr)^{\mu_p}) \arrow[l, "q"'] \\
\cup & \cup & \cup \\
\mathrm{Perv} & & \\
\cup & & \\
\mathrm{Tilt} \arrow[r, "i^{!*}", "\sim"'] & \mathrm{Parity}^{\text{even}} & \mathrm{Parity}^{\text{even}} \arrow[l]
\end{tikzcd}$$

$\mathbb{G}_m$: loop rotation

Th. (Riche - W.): $\mathrm{Tilt} \xrightarrow[\sim]{i^{!*}} \mathrm{Parity}^{\text{even}}$

$\mathrm{Parity}^{\text{even}}$ ↪ stalks and costalks vanish outside even degrees

Consequences: ① Linkage principle: indecomposable objects may only live on one component!

② q arrow $\Rightarrow$ character formula for tilting modules for all p.

Philosophy:

(Borel - de Siebenthal)

$G^\vee((t^\ell))$ are like pseudo-Levi subgroups

w/ Weyl groups are W_ℓ's.