Hodge-Tate period map

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<u>Outline</u>

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1) SVs

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m$ be the Deligne torus,

<u>Def</u>. A <u>Shimura datum</u> is a pair (G, X) where $G/\mathbb{Q}$ is a reductive gp (conn'd).

and X is a $G(\mathbb{R})$-conj. class of homomorphisms $h: \mathbb{S} \longrightarrow G_{\mathbb{R}}$ s.t.

- For all $h \in X$, $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \longrightarrow GL(\mathfrak{g}_{\mathbb{R}})$ has weights in $\{(-1,1), (0,0), (1,-1)\}$.

 (Convention: $z \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ acts on $V^{p,q} \subset V_{\mathbb{C}}$ as $z^p \bar{z}^q$.)

- For all $h \in X$, adjoint action of $h(i)$ on $G^{ad}_{\mathbb{R}}$ (adjoint gp) is a <u>Cartan involution</u>.

Recall: a Cartan involution on $G'/\mathbb{R}$ is an involution θ s.t.

$$\{ g \in G'(\mathbb{C}) : g = \theta(\bar{g}) \} \text{ is compact.}$$

[Ex. $G' = SL_2/\mathbb{R}$, $\theta = \mathrm{ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$]

- $G^{ad}_{\mathbb{Q}} = \prod_i H_i / \mathbb{Q}$, H_i simple, then for all $h \in X$,

$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \longrightarrow G^{ad}_{\mathbb{R}} \longrightarrow H_i \text{ is nontrivial.}$$

For any cpt open $K \subset G(\mathbb{A}_f)$, ("level")

$$X_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

Fact (Baily-Borel, Borel). For K sufficiently small, X_K is the $\mathbb{C}$-pts of a (uniquely determined) alg var. $Sh_K(G, X)_{\mathbb{C}} / \mathbb{C}$.

Hecke action: right action of $G(\mathbb{A}_f)$ on $\varprojlim_K Sh_K(G, X)_{\mathbb{C}}$

for $g \in G(\mathbb{A}_f)$, $Sh_K(G, X)_{\mathbb{C}} \xrightarrow{g} Sh_{g^{-1}Kg}(G, X)_{\mathbb{C}}$

$$(x, h) \longmapsto (x, hg)$$

Fact: Can use (G, X) to define $E(G, X) \subset \mathbb{C}$, finite $/\mathbb{Q}$ s.t.

$\exists \varprojlim_K Sh_K(G, X)/E$ w/ Hecke action $/E$.

(defined in terms of Hodge cocharacter)

2) PEL type Shimura varieties

PEL data.

Def. A PEL datum is a tuple $(B, *, V, (\cdot,\cdot), h)$,

- $B/\mathbb{Q}$ is finite dim. semisimple alg.
- $*: B \to B$ involution that is positive, i.e. $\mathrm{tr}_{B/\mathbb{Q}}(bb^*) > 0$ for nonzero $b \in B$.
- V is a finite left B-module
- $(\cdot,\cdot): V \times V \to \mathbb{Q}$ bilinear, nondegenerate, alternating, s.t.

$$(bv, w) = (v, b^* w)$$

Induces an involution $* : \mathrm{End}(V) \to \mathrm{End}(V)$
$$A \mapsto \text{adjoint } A^*$$

extends $*$ on B, along $B \to \mathrm{End}(V)$.

Let $G/\mathbb{Q}$ be alg. gp w/ functor of pts

$$R \mapsto \{x \in \mathrm{Aut}_{B\otimes R}(V \otimes R) : xx^* \in R^\times\}$$

- $h: \mathbb{S} \to G_{\mathbb{R}}$ s.t.
  - $h(\bar{z}) = h(z)^*$ for $z \in \mathbb{C}^\times$.
  - $(v, h(i)w)$ is symmetric bilinear positive definite on $V_{\mathbb{R}}$.
  - Induced Hodge structure on $V_{\mathbb{R}}$ has weights $(1,0), (0,1)$

Let X be $G(\mathbb{R})$-conj. class of h.
Fact. (G, X) is a Shimura datum (if G is conn'd)

Let $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ Hodge decomposition

$$E_0 = \mathbb{Q}[\{ tr(b|V^{1,0}) : b \in B \}] \subset \mathbb{C}$$

<u>Fact</u> $E(G, X) = E_0$.

How to get a feeling for G?

Say B is simple algebra, F = center of B (no. field)

$F^+ = F^{*=id}$, F^+ totally real.

Let $G_1/\mathbb{Q}$

$$G_1(R) = \{ x \in Aut_{B \otimes R}(V \otimes R) : xx^* = id \}$$

$G_1 = Res_{F^+/\mathbb{Q}} G_0$, for some G_0 / F^+.

$$n = \frac{1}{2}[F:F^+] \sqrt{\dim_F End_B(V)} \quad \in \mathbb{Z} \quad (\text{by } \exists \text{ of } h)$$

(A) F/F^+ is cpx quadratic ext, G_0 is inner form of quasi-split unitary gp over F^+ of A_{n-1}

(C) $F = F^+$ is totally real, G_0 is symplectic gp in $2n$ variables

(D) ———— " ————, G_0 is orthogonal gp in $2n$ variables.

Ex. Siegel case Fix $g \in \mathbb{Z}_{\geq 1}$

- $B = \mathbb{Q}$
- $* = id$
- $V = \mathbb{Q}^{2g}$
- $(\cdot,\cdot)$ coming from $\begin{bmatrix} & I_g \\ -I_g & \end{bmatrix} = J$,

  $*$ on $End(V)$: $A \longmapsto J^{-1} A^T J$,
- $G = GSp_{2g} / \mathbb{Q}$
- $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ is $z = a+bi \longmapsto a I_{2g} + bJ$
- $X = \{ A \in Sym_{g \times g}(\mathbb{C}) : Im(A) > 0 \text{ or } Im(A) < 0 \}$
- $F = F^+ = \mathbb{Q}$, $G_0 = G_1 = Sp_{2g} / \mathbb{Q}$, $n = g$

  Type (C).

In general, $B = \prod_i B_i$ for simple $B_i / \mathbb{Q}$.

V also decomposes. $(\cdot,\cdot)$ also compatible,

"up to similitude"; G is "product" of $Res_{F^+|\mathbb{Q}} G_0$ as above.

We assume.

- Type A, C : no factors of type D

  $\Rightarrow$ G conn'd

  $\Rightarrow$ Hasse principle "doesn't fail too badly".

Dage r

- Smooth integral structure at p

  - $B_{\mathbb{Q}_p} = \prod_i \mathrm{Mat}_{n_i \times n_i}(F_i)$ for $F_i / \mathbb{Q}_p$ unram.

  - $\exists$ $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B \subset B$, $*$-inv

    $\mathcal{O}_B \otimes \mathbb{Z}_p \subset B_{\mathbb{Q}_p}$ is max'l $\mathbb{Z}_p$-order.

  - $\exists$ $\mathbb{Z}_{(p)}$-lattice $\Lambda_0 \subset V$, stable under $\mathcal{O}_B$, self-dual under $(\cdot,\cdot)$.

Fix $\mathcal{O}_B, \Lambda_0$: obtain $G_{\mathbb{Z}_{(p)}} / \mathbb{Z}_{(p)}$.

$$R \longmapsto \{ x \in \mathrm{Aut}_{\mathcal{O}_B \otimes R}(\Lambda_0 \otimes R) : xx^* \in R^\times \}.$$

$G_{\mathbb{Z}_{(p)}} \times \mathbb{Q} = G.$

<u>Moduli interpretation</u>.

We define moduli problem of ab. var. $/ \mathcal{O}_{E_0} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, w/

P - polarization

E - endomorphisms by $\mathcal{O}_B$

L - level $K = K^p K_p$
  - $K^p \subset G(\mathbb{A}_f^p)$ cpt open
  - $K_p := G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ hyperspecial.

<u>Warning</u>: $\mathbb{C}$-pts of moduli problem is finite disjt union of copies of X_K.

Def Let S_K^{pre} be the presheaf of groupoids over $\mathcal{O}_{E_0} \otimes \mathbb{Z}_{(p)}$ whose value

S is as follows:

objects: quadruples $(A, \iota, \lambda, \bar{\eta})$ where

- A/S is ab. sch. of rel. dim $\frac{1}{2}\dim_{\mathbb{Q}} V$
- $\iota : \mathcal{O}_B \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ s.t. for $b \in \mathcal{O}_B$,

$$\det_{\mathcal{O}_S}(\iota(b) \mid \mathrm{Lie}\, A) = \det(b \mid V^{1,0})$$

$$\underset{\mathcal{O}_S(S)}{\uparrow} \qquad\qquad \underset{\mathcal{O}_{E_0} \otimes \mathbb{Z}_{(p)}.}{\uparrow}$$

- $\lambda : A \to A^\vee$ is prime-to-p quasi-isogeny s.t.

  - λ symmetric w.r.t. $A^{\vee\vee} \cong A$
  - $\exists\, n \in \mathbb{Z}_{>0}$, (loc. constant) s.t. $n\lambda$ is a polarization at geom. pts of S. (ample line bundle)
  - $\mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$
    $\circlearrowleft * \qquad \circlearrowleft$ Rosati involution (from λ)

- $\bar{\eta}$ is K^p-orbit of a "trivialization" η of the locally constant pro-étale sheaf $\underline{H_1}(A, \mathbb{A}_f^p)$ on S, under action of $\underline{G(\mathbb{A}_f^p)}$.

$\bar{\eta}$ is "$\tilde{\pi}_1$-invt".

"Trivialization": $\underline{H_1}(A, \mathbb{A}_f^p) \simeq \underline{V_{\mathbb{A}_f^p}}$ (over some cover)

Compatible w/ B-actions, $\langle \cdot, \cdot \rangle$ up to a scalar in $\underline{\mathbb{A}_f^{p\times}}$

LHS: Weil pairing

Isomorphism $(A, \iota, \lambda, \bar{\eta}) \xrightarrow{f} (A', \iota', \lambda', \bar{\eta}')$ is prime-to-p quasi-isogeny

$A \xrightarrow{f} A'$ s.t.

- $f^\vee \circ \lambda' \circ f = c\lambda$ for some $c \in \underline{\mathbb{Z}_{(p)}^*}(S)$

- f commutes w/ actions of $\mathcal{O}_B$

- $\bar{\eta} = \bar{\eta}' \circ f_*$

Fact, S_K^{pre} is a DM stack, which if K' is sufficiently is representable by a smooth quasi-proj. scheme $/ \mathcal{O}_{E_0} \otimes \mathbb{Z}_{(p)}$.

We assume K^p is sufficiently small.

3) PEL SVs as v-sheaves

Chose $\mathfrak{p} | p$ of E_0, $E = E_{0,\mathfrak{p}}$, $\mathcal{O}_E$ ring of integers

$$S_K := S_K^{pre} \times_{\mathcal{O}_{E_0} \otimes \mathbb{Z}_{(p)}} \mathcal{O}_E.$$

$$S_{K,E} := S_K \times_{\mathcal{O}_E} E$$

Work over $\mathrm{Spa}\, E := \mathrm{Spa}(E, \mathcal{O}_E)$

Def. The <u>adic Shimura variety</u> at level K is

$$S_{K,\diamond} := (S_{K,E}^{ad})^\diamond \quad / \mathrm{Spd}\, E$$

Have universal abelian sch. $A \to S_{K,E}$ (upto prime-to-p quasi-isog.)

Proof

$A^{ad} := \mathcal{A} \longrightarrow S^{ad}_{K,E}$

$A^{\diamond} \xrightarrow{\pi} S_{K,\diamond}$ proper map of diamonds.

Let $T_p A := \underline{\mathrm{Hom}}_{\mathbb{Z}_p}(R^1\pi_* \underline{\mathbb{Z}_p}, \underline{\mathbb{Z}_p})$

"Tate module of universal object"

Def. <u>Shimura variety at infinity level</u> is

$$S_{K^p,\diamond} = \underline{\mathrm{Isom}}[T_p A, \underline{\Lambda_0 \otimes \mathbb{Z}_p}]$$

$$\downarrow$$

$$S_{K,\diamond}$$

O_B-linear, preserve $\langle\cdot,\cdot\rangle$ up to a constant in $\mathbb{Z}_p^\times$.

Rmk. Usually defined as

$$\varprojlim_{K_p \subset G(\mathbb{Q}_p)} S_{K^p K_p,\diamond}$$ in cat. of diamonds.

known to be representable by perfectoid space.

Next "good reduction locus".

Let $\mathcal{S}_K /_{\mathrm{Spf}\, O_E} :=$ p-adic completion of S_K / O_E

Def. • At level K:

$$S^{\circ}_{K,\diamond} := \left(\mathcal{S}^{ad}_K \times_{\mathrm{Spa}\, O_E} \mathrm{Spa}\, E\right)^{\diamond}$$

Fact: $S^{\circ}_{K,\diamond} \hookrightarrow S_{K,\diamond}$ "open subdiamond".

Lemma. $S^{\circ}_{K,\diamond}$ is (analytic) sheafification of

$$\mathrm{Perf} \to \mathrm{Set}, \quad S = \mathrm{Spa}(R, R^+) \mapsto \{(S^{\#}, \mathrm{Spf}\, R^{\#+} \to \mathcal{S}_K)\}$$

where $S^{\#} = \mathrm{Spa}(R^{\#}, R^{\#+})$ untilt of S over E.

· Infinite level $S^{\circ}_{K^p,\diamond}$ is pullback of $S^{\circ}_{K,\diamond}$ from $S_{K,\diamond}$ to $S_{K^p,\diamond}$.

(For any $X/\mathcal{O}_E$ "nice" w/ p-adic completion $\mathfrak{X}/\mathrm{Spf}\, \mathcal{O}_E$,

$$\mathfrak{X}^{ad} \times_{\mathrm{Spa}\, \mathcal{O}_E} \mathrm{Spa}\, E \overset{\text{open}}{\hookrightarrow} (X_E)^{ad}.$$
)

(if X proper, isom.)

(Baby example: for $X = \mathbb{A}^1_{\mathcal{O}_E}$, $\mathrm{Spa}(E\langle t\rangle, \mathcal{O}_E\langle t\rangle) \hookrightarrow (\mathbb{A}^1_E)^{ad}$)

Rmk. Have "integral model" of good reduction locus.

$$(\mathcal{S}_K^{ad})^{\diamond} / \mathrm{Spd}\, \mathcal{O}_E \quad (v\text{-sheaf})$$

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4) Hodge-Tate period map

Choose · $\overline{\mathbb{Q}_p} \,|\, E \,|\, \mathbb{Q}_p$, $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$

· $G_{\overline{\mathbb{Q}_p}} \supset B \supset T$ · μ be dominant cocharacter corresponding to $G(\overline{\mathbb{Q}_p})$-conj. class of ν_h^{-1}. where

Def. 1.

$\nu_h : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p}$ is Hodge cocharacter.

<u>Recall</u>: Gr_G B^+_{dR}-affine grassmannian lying over $\mathrm{Spd}\, E$.

μ determines Schubert cell $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_G$.

μ is minuscule, so $\mathrm{Gr}_{G,\mu} = \mathrm{Gr}_{G,\leq\mu}$ is proper.

$$\begin{array}{c} \| \text{ Bialynicki-Bi} \\ \mathcal{F}\ell^{\diamond}_{G,\mu} \end{array}$$

Abbreviate $G_{\mathbb{Q}_p}$ to G.

Fix $K^p \subset G(\mathbb{A}_f^p)$

<u>Thm (Scholze, Caraiani-Scholze)</u> There exists a $G(\mathbb{Q}_p)$-equiv. "Hodge-Tate period map"

$$S^{\circ}_{K^p,\diamond} \xrightarrow{\pi^{\circ}_{HT}} \mathrm{Gr}_{G,\mu} = \mathcal{F}\ell^{\diamond}_{G,\mu}$$

also $G(\mathbb{A}_f^p)$-equiv: for $g \in G(\mathbb{A}_f^p)$.

$$\begin{tikzcd}
S^{\circ}_{K^p,\diamond} \arrow[r, "\pi^{\circ}_{HT}"] \arrow[d, "g"'] & \mathrm{Gr}_{G,\mu} \\
S^{\circ}_{g^{-1}K^p g,\diamond} \arrow[ur, "\pi^{\circ}_{HT}"'] &
\end{tikzcd}$$

Vague idea of construction.

$S^{\circ}_{K^p,\diamond} : \mathrm{Perf} \to \mathrm{Set}$ is (analytic) sheafification of

$$S = \mathrm{Spa}(R,R^+) \mapsto \{(S^\sharp, \mathcal{A}, \beta)\}$$

- $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp +})$ untilt of S over $\mathrm{Spa}\, E$
- $\mathfrak{A} / \mathrm{Spf}\, R^{\sharp +}$ formal ab. sch. w/ " G-str."
- $\beta \in \underline{\mathrm{Isom}}(T_p A, \underline{\Lambda_0 \otimes \mathbb{Z}_p})$, A is " generic fiber of $\mathfrak{A}$ "

Want $(S^\sharp, \mathfrak{A}, \beta) \rightsquigarrow$ elt of Gr_G

(then show in $\mathrm{Gr}_{G,\mu}$)

Consider " prismatic Dieudonné module " of $\mathfrak{A}[p^\infty]$, modulo comparison isomorphisms.

trivializations form a $G_{\mathbb{Q}_p}$-torsor over $\mathrm{Spec}\, B_{dR}(R^\sharp)$, β provides a trivialization over $\mathrm{Spec}\, B_{dR}(R^\sharp)$. $\rightsquigarrow$ elt of Gr_G.

Rank one geometric pt $\mathrm{Spa}(C, \mathcal{O}_C)$, C alg. closed perf. field, char $C = p$

Have untilt $(C^\sharp, \mathcal{O}_{C^\sharp}) / (E, \mathcal{O}_E)$

We give an elt. of $\mathcal{F}\ell^\diamond_{G,\mu}(C, \mathcal{O}_C) = \mathcal{F}\ell_{G,\mu}(C^\sharp, \mathcal{O}_{C^\sharp})$

$$\| $$

$$\left\{ W \subset V \otimes C^\sharp : \begin{array}{c} \text{max. isotropic,} \\ B\text{-invt} \end{array} \right\}$$

$\mathcal{G} = \mathfrak{A}[p^\infty]$,

$$0 \to \mathrm{Lie}\, \mathcal{G} \otimes_{\mathcal{O}_{C^\sharp}} C^\sharp(1) \to T_p \mathcal{G}(\mathcal{O}_{C^\sharp}) \otimes C^\sharp \to \cdots \to 0$$

$$T_p \mathcal{G}(\mathcal{O}_{C^\sharp}) \otimes C^\sharp \overset{\beta}{\cong} V \otimes C^\sharp$$

Pick $W =$