Cycles on special fiber of Shimura varieties and arithmetic applications

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Lecture 1. <u>Tate conjecture</u>. k finite field / no. field. X smooth proj. var. $/k$.

$$CH^i(X) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \xrightarrow{cl_X} H^{2i}_{et}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i))^{Gal_k} \quad \text{is } \underline{\text{surjective}}.$$

Tate classes

$CH^i(X)$: free $\mathbb{Z}$-mod. gen. by codim i cycles $/k$ up to rat'l equiv.

<u>Rmk</u> k'/k finite. then Tate conj. for k' $\Rightarrow$ Tate conj. for k.

If a finite group $H \curvearrowright X$, can study (for each irred. rep ρ of H),

$$H^{2i}_{et}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i))[\rho]^{Gal_k} = \mathrm{Hom}_H(\rho, H^2_{et}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i)))^{Gal_k}.$$

More generally, $\underbrace{\mathrm{Corr}(X,X)}_{\text{alg. of correspondences}} \curvearrowright H^{2i}_{et}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i))$.

<u>Small goal</u>: For an irrep ρ of $\mathrm{Corr}(X,X)$, when $H^{2i}_{et}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i))[\rho]^{Gal_k}$ is "computable" and "generic", show that $CH^i(X)[\rho] \to H^{2i}_{et}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i))[\rho]^{Gal_k}$ is surjective.

<u>Shimura varieties</u>. (G, X) Shimura datum

- G reductive group $/\mathbb{Q}$
- $X = G(\mathbb{R})$-conj. class of homom. $h: \underset{\substack{\| \\ \text{"}\mathbb{C}^\times\text{"}}}{\mathbb{S}} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}$. s.t.

(SV 1) For any $h \in X$

$$\begin{tikzcd}
 & h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \arrow[r] \arrow[d, equal] & G_{\mathbb{C}} \\
 & \text{"}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^{\times}\text{"} \arrow[d, equal] & \\
\mathbb{G}_{m,\mathbb{C}} \arrow[r, "i_1"] \arrow[rruu, bend right, "\mu = \mu_h"'] & \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \; \mathbb{C}^{\times} \times \mathbb{C}^{\times} &
\end{tikzcd}$$

minuscule cocharacter

(SV2) $\mathrm{Ad}_{h(i)}$ is the Cartan involution of $G^{ad}(\mathbb{R})$

(positivity of polarization)

Rmk $X \simeq G(\mathbb{R}) / \text{max'l cpt subgp of } G(\mathbb{R}) \text{ mod center}$

- $E = E(G, X)$ = reflex field = field of definition of the $G(\mathbb{C})$-conj. class of μ

= finite ext'n of $\mathbb{Q}$ inside $\mathbb{C}$

For a neat open cpt subgroup $K \subset G(\mathbb{A}_f)$

$\rightsquigarrow$ $Sh = Sh_K(G, X)$ quasi-proj. smooth var. $/E$

s.t. $Sh_K(G,X)(\mathbb{C}) = G(\mathbb{Q}) \backslash \left(X \times \left(G(\mathbb{A}_f)/K \right) \right)$

$= \coprod_i \Gamma_i \backslash X$ for some subgp $\Gamma_i \subset G(\mathbb{Q})$

commensurable to $G(\mathbb{Z})$

Assume that (G, X) is of Hodge type i.e. embedding of Shimura data

$(G,X) \hookrightarrow (GSp_{2g}, \mathfrak{h}_g^{\pm})$ $\qquad \mathfrak{h}_g^{\pm} = \left\{ Z \in \mathrm{Mat}_{g \times g}(\mathbb{C})^{sym} ; \ \mathrm{Im}\, Z \begin{smallmatrix} >0 \\ <0 \end{smallmatrix} \right\}$

$GSp_{2g} \overset{\shortparallel}{} (\mathbb{Q}^{2g}, \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix})$

$\mathcal{A}$ = universal AV

$$Sh_K(G,X)_{/E} \longrightarrow Sh_{K_{GSp}}\left(GSp_{2g}, h_g^{\pm}\right)_{\otimes E} = \text{Siegel moduli space of AVs}$$

moduli of AVs w/ additional structures

e.g. nontrivial endomorphisms, or Hodge cycles.

Fix a prime p s.t. $G_{\mathbb{Q}_p}$ is unramified $\rightsquigarrow$ G extends to a reductive gp $/\mathbb{Z}_{(p)}$.

($\Rightarrow E$ is unram. @ p)

$K = K^p K_p \subset G(\mathbb{A}_f)$ $\qquad\qquad$ $K^p = G(\mathbb{A}_f^p)$

Assume K_p is hyperspecial $= G(\mathbb{Z}_p)$

<u>Thm</u> (Kisin, Vasiu) $p>2$. The tower of Shimura varieties $\left(Sh_{K^pK_p}(G,X)\right)_{K^p}$ admits an $\underset{\text{smooth}}{\wedge}$ integral canonical model $/\mathcal{O}_{E,(p)}$.

Fix an isom. $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$ $\rightsquigarrow$ p-adic embedding $E \hookrightarrow \overline{\mathbb{Q}_p}$, place $\mathfrak{p}$

Put $\boxed{Sh} = Sh_K(G) = Sh_K(G,X) \underset{\mathcal{O}_{E,(p)}}{\otimes} \mathbb{F}_p$

The (tame) Hecke algebra (essentially the away from p part)

$$\mathcal{H}_K = C_c\left(K\backslash G(\mathbb{A}_f)/K, \overline{\mathbb{Q}_\ell}\right) \quad \text{acts on} \quad H^*_{\acute{e}t}\left(Sh_K(G)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}_\ell}\right)$$

by correspondences.

$$\wr\wr$$

$$H^*_{\acute{e}t}\left(Sh_{\overline{\mathbb{F}_p}}, \overline{\mathbb{Q}_\ell}\right)$$

For an irrep π_f^K of $\mathcal{H}k_K$ (typically, π automorphic rep of $G(\mathbb{A})$, then $(\pi_f)^K$)

Put $W^i(\pi_f^K) := \mathrm{Hom}_{\mathcal{H}k_K}\left(\pi_f^K, H^*_{et}\left(Sh_K(G)_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell\right)\right)$

$$\circlearrowleft \; \mathrm{Gal}_E$$

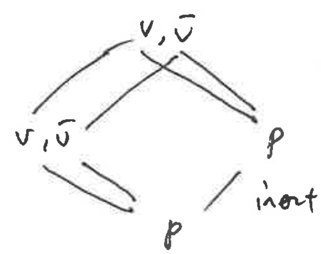
$$W^i(\pi_f^K)\Big|_{\mathrm{Gal}_{\mathbb{F}_p}} \simeq \mathrm{Hom}_{\mathcal{H}k_K}\left(\pi_f^K, H^*_{et}(Sh_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)\right)$$

$$\mathrm{Hom}_{\mathcal{H}k_{K^p}}\left(\pi_f^{K,p}, H^*_{et}(Sh_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)\right)$$

__Goal:__ Find cycles that generate $W^{mid}\left(\pi_f^K\left(\frac{mid}{2}\right)\right)^{\mathrm{Frob}_p=1}$

__Working example__. moduli of AVs w/ action by $\mathcal{O}_E$.

$$\begin{tikzcd}[column sep=small]
 & E = CM \arrow[dl, dash] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\sqrt{-D}) = E_0 \arrow[dr, dash, "2"', "\text{imag quad.}"'] & & F = \text{totally real} \arrow[dl, dash, "m"] \\
 & \mathbb{Q} &
\end{tikzcd}$$



$\mathbb{C} \simeq \bar{\mathbb{Q}}_p$. $\quad \mathrm{Hom}(E, \mathbb{C}) = \mathrm{Hom}(E, \bar{\mathbb{Q}}_p) = \mathrm{Hom}(\mathcal{O}_E, \overset{\sigma}{\circlearrowright}\mathbb{Z}_p^{ur})$

$$= \{\tau_1, \dots, \tau_m, \bar{\tau}_1, \dots, \bar{\tau}_m\}. \text{ so that } \tau_{i+1} = \sigma \circ \tau_i$$

Take V = Hermitian $\mathcal{O}_{E,(p)}$-module (perfect) free of rank n s.t.

$$\mathrm{sig}\left(V \otimes_{F,\tau_i} \mathbb{R}\right) = (a_i, n - a_i).$$

$$G \approx GU(n)$$

$$G \hookrightarrow GSp_{2mn}$$

Fact. $\left\{ \begin{array}{l} \text{Herm. perfect forms } (\ ,\ ) \text{ on } V \quad V\times V \to \mathcal{O}_{E,(p)} \\ \text{s.t. } (ax, by) = a\bar{b}(x,y) \\ \overline{(x,y)} = (y,x) \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{antisym. perfect forms } \langle\ ,\ \rangle : V\times V \to \mathbb{Z}_{(p)} \\ \text{s.t. } \langle ax, y\rangle = \langle x, \bar{a}y\rangle \\ \text{and } \langle x,y\rangle = -\langle y,x\rangle \end{array} \right\}$

$$(\ ,\ ) \longmapsto \langle x,y\rangle = \mathrm{Tr}_{E/\mathbb{Q}}\left(\sqrt{-D}\,(x,y)\right)$$

$$G = GU(V) = \{(g,c) \in GL_{\mathcal{O}_{E,(p)}}(V) \times \mathbb{Z}_{(p)}^{\times} : \langle gx, gy\rangle = c\langle x,y\rangle,\ \forall x,y \in V\}$$

similitude unitary gp

$$0 \to \mathrm{Res}_{E/\mathbb{Q}} U(V) \to GU(V) \xrightarrow{c} \mathbb{G}_m \to 1$$

$$GU(V) \hookrightarrow GSp_{2mn}$$

$$X \approx \prod_{i=1}^{m} U(a_i, n-a_i) / U(a_i) \times U(n-a_i)$$

$$\dim Sh = \dim_{\mathbb{C}} X = \sum_{i=1}^{m} a_i(n-a_i)$$

In this case, $Sh_K(G)$ is "almost" the moduli of AVs (S : $\mathbb{Z}_p$-scheme)

- A/S AV of dim mn w/ $\mathcal{O}_E \hookrightarrow \mathrm{End}_S(A)$, $1 \mapsto \mathrm{id}_A$ + <u>signature condition.</u>
- $\lambda : A \to A^\vee$ prime-to-p polarization s.t.

$$\begin{array}{ccc} A & \xrightarrow{\bar{a}} & A \\ \lambda\downarrow & & \downarrow\lambda \\ A^\vee & \xrightarrow{a^\vee} & A^\vee \end{array} \qquad a \in \mathcal{O}_E$$

- level str... $\quad E_p = F_p \oplus F_p$, $\underline{U(V) \cong GL}$

<u>Description of $W^{mid}(\pi_f^K)$.</u> Let $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ pinning be the Langlands dual gp of G.

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \phi_p \rangle, \quad \phi_p = \text{geom. Frobenius}$$

(with $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acting on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$)

Langlands dual gp ${}^L G_p = \hat{G} \rtimes \langle \phi_p \rangle$.

(In our examples, $\hat{G} = G_m \times \prod_{i=1}^{m} GL_n$ $\circlearrowleft$ ϕ_p permutes these factors

$\rightsquigarrow \; {}^L G_p^{ess} = \left(\prod_{i=1}^{m} GL_n \right) \rtimes \langle \phi_p \rangle$)

Input 1. Langlands parameter

$$\begin{tikzcd}
\mathrm{rec}(\pi_{f,p}) : W_{\mathbb{Q}_p} \arrow[r, two heads] & \langle \phi_p \rangle \arrow[r] \arrow[dr, equal] & {}^L G_p = \hat{G} \rtimes \langle \phi_p \rangle \arrow[d] \\
& & \langle \phi_p \rangle
\end{tikzcd}$$

So $\mathrm{rec}(\pi_{f,p})(\phi_p) = \hat{g}_p \phi_p$ for some $\hat{g}_p \in \hat{G}$

may conj. by $\hat{G}$ to ensure (assuming semisimplicity)

$$\mathrm{rec}(\pi_{f,p})(\phi_p) = \hat{t}_p \phi_p \quad \text{for } \hat{t}_p \in \hat{T}.$$

(In our example, $\mathrm{rec}(\pi_{f,p})^{ess}(\phi_p) = (t_1, \dots, t_m)\phi_p \in (\diagdown)^m \rtimes (\phi_p)$

$$\mathrm{rec}(\pi_{f,p})^{ess}(\phi_p^m) = (t_1, \dots, t_m)\phi_p (t_1, \dots, t_m)\phi_p \cdots$$

$$= (t_1, \dots, t_m)(t_2, \dots, t_m, t_1)(t_3 \cdots) \cdots (t_m, t_1, \dots t_{m-1}) \phi_p^m$$

$$= (t_1 \cdots t_m, t_1 \cdots t_m, \dots, t_1 \cdots t_m) \phi_p^m$$

Key
$$\left(\begin{tikzcd}
\mathrm{rec}_{\mathbb{Q}_p} \; \langle \phi_p \rangle \arrow[r] \arrow[dr, equal] & \prod_{i=1}^{m} GL_n \rtimes (\phi_p) \arrow[d] \\
& \langle \phi_p \rangle
\end{tikzcd} \right) \Big/ \mathrm{Ad}\left(\prod GL_n\right) \simeq \left\{ \mathrm{rec}_{\mathbb{Q}_{p^m}} \langle \phi_p^m \rangle \longrightarrow GL_n \right\} \Big/ \mathrm{Ad}(GL_n)$$

<u>LHS</u> LLC for $\left(\mathrm{Res}_{\mathbb{Q}_{p^m}|\mathbb{Q}_p} GL_n\right)(\mathbb{Q}_p)$ $=$ LLC for $(GL_{n,\mathbb{Q}_{p^m}})(\mathbb{Q}_{p^m})$ <u>RHS</u>

<u>Input 2</u>. rep. assoc. to μ (minuscule cochar.)

<u>Recall</u>. $h: \mathbb{S} \longrightarrow G_{\mathbb{R}}$ $\rightsquigarrow$ conj. class of $\mu: \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$

$\rightsquigarrow$ weight for $\hat{G}$. $\rightsquigarrow$ $\mu^* = -w_0(\mu)$. $V_{\mu^*} = V_\mu^* =$ h.w. rep. of $\hat{G}$ of h.w. μ^*.

<u>Want</u>: $W^*(\pi_f^k)\Big|_{\mathrm{Gal}\, \mathbb{F}_{p^m}}$ $\mathbb{F}_p \subset \mathbb{F}_{p^m}$

$$\begin{array}{ccccc}
\langle \phi_{p^m} \rangle & \xrightarrow{\mathrm{rec}(\pi_{f,p})} & {}^L G_p & \xrightarrow{r_{\mu^*}} & \mathrm{End}(V_{\mu^*}) \\
\phi_{p^m} & \longmapsto & t_{p,m}\phi_{p^m} & \longmapsto & r_{\mu^*}(t_{p,m}) \\
 & & \shortparallel & & \\
 & & (t_p \phi_p)^m & &
\end{array}$$

<u>Expectation</u>. $\underset{\substack{\uparrow \\ \text{in Grothendieck gp}}}{[W^*(\pi_f^k)]} = \underbrace{a_G(\pi_f^k)}_{\text{automorphic multiplicity}} V_{\mu^*}$

<u>Often</u> concentrated in deg $d = \dim Sh$.

$$\left(W^d(\pi_f^k)\left(\tfrac{d}{2}\right)\right)^{\phi_p = 1}$$

<u>Example</u> Assume π_f^k has trivial central char. $\Rightarrow$ $t_1 \cdots t_n = p^{n-1}$

$\mathrm{rec}(\pi_{f,p})(\phi_p^m) = \Delta(t) \cdot \phi_p^m$, $t = \mathrm{diag}(t_1, \ldots, t_n)$

Eg $m=2$, $a_1=1$, $a_2=n-1$. $\mu^* \in X^*(GL_n^2)$

$\shortparallel$

$(1,0,\cdots,0), (1,\cdots,1,0)$

$$V_{\mu^*} = \mathrm{Std} \boxtimes \wedge^{n-1}\mathrm{Std} \qquad \dim = 1\cdot(n-1)+(n-1)\cdot 1 = 2(n-1)$$

evals of (t,t) $\begin{matrix} t_1 & p^{n-1}/t_1 \\ \vdots & \vdots \\ t_n & p^{n-1}/t_n \end{matrix}$ $\rightsquigarrow$ evals $= p^{n-1}\dfrac{t_i}{t_j}$ for all $i,j \in \{1,\cdots,n\}$

$$\left(V_{\mu^*}(n-2)\right)^{\phi_{p^n}^2} \supset n\text{-dim'l subspaces}$$

$\uparrow$ in "generic" cases, this is an equality.

In general, define $\Lambda^{\mathrm{Tate}} = \left\{ \lambda \in X^*(\hat{T}) : \sum_{i=1}^{m} \phi_p^i(\lambda) \in X_*(Z_G) \subset X^*(\hat{T}) \right\}$

when G splits$/\mathbb{Q}_p$, $\Lambda^{\mathrm{Tate}} = X_*(Z_G)$. Moreover if $\underbrace{G_{ss}}_{\text{splits}/\mathbb{Q}_p,\ +} \Rightarrow \Lambda^{\mathrm{Tate}} = 0$.

$$V_{\mu^*}^{\mathrm{Tate}} = \bigoplus_{\lambda \in \Lambda^{\mathrm{Tate}}} V_{\mu^*}(\lambda) \qquad\qquad V_{\mu^*}^{\mathrm{Tate}} = V_{\mu^*}(0)$$

Hope when π_f^K is generic, construct cycles for $V_{\mu^*}^{\mathrm{Tate}}$.

Slogan Irred comp. of the basic locus of Sh generates "generic" Tate classes.

Theorem (X.-Zhu) Assume (G,X) is Hodge type, Z_G is conn'd. Assume $V_{\mu^*}^{\mathrm{Tate}} \neq 0$, $d = \dim \mathrm{Sh}$. Then (a) $\exists$ an inner form G' of G s.t. $G'(\mathbb{A}_f) = G(\mathbb{A}_f)$, $G'(\mathbb{R})$ is cpt mod center. ($\Rightarrow \mathrm{Sh}_{K'}(G')$ is discrete)

(b) $\mathrm{Sh}_K(G)^{\mathrm{basic}} \leftarrow$ basic locus has pure dim $\frac{d}{2}$ (d even)

Moreover, $\overline{\mathbb{Q}}_\ell\left[\mathrm{Irr}\left(Sh_K(G)^b\right)\right] \simeq H_*^{BM}\left(Sh_K(G)^b_{\overline{\mathbb{F}}_p}; \overline{\mathbb{Q}}_\ell\right)$

$$\simeq C^*\left(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell\right) \otimes V_{\mu^*}^{Tate}$$

Say $r = \dim V_{\mu^*}^{Tate}$,

$$\begin{tikzcd} & I_j \arrow[dl] \arrow[dr, "\text{locally closed}"] & \\ Sh_{K'}(G') & & Sh_K(G)^b \end{tikzcd}$$

irred. var. of dim $\frac{d}{2}$.

$Sh_K(G)^b = \bigcup_j \mathrm{image}(I_j)$

par. by $Sh_{K'}(G') = G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K$

(2) For an irrep π_f^K of $\mathcal{H}_K$ if the Satake par. of $\pi_{f,p}$ is <u>generic</u> wrt. V_{μ^*}

(in our case, $\mathrm{rec}(\pi_{f,p})(\phi_p^m)$ has distinct evals)

$$H_*^{BM}\left(Sh_K(G)^{basic}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\right)[\pi_f] \otimes \pi_f^K \longrightarrow H^*_{\acute{e}t,c}\left(Sh_K(G)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\left(\tfrac{d}{2}\right)\right)^{\phi_{p^m}=1} \qquad (*)$$

geom. realisation of Jacquet – Langlands corr.

is <u>injective</u>.

(3) When (G,X) is Kottwitz arithmetic Sh. var., (so that $\mathrm{rec}_G(\pi)$ can be computed)

and assume the Satake par. of $\pi_{f,p}$ is strongly regular wrt. V_{μ^*}

(in our case $\dim\left(V_{\mu^*}\left(\tfrac{d}{2}\right)\right)^{\phi_{p^m}=1} = \dim V_{\mu^*}^{Tate}$)

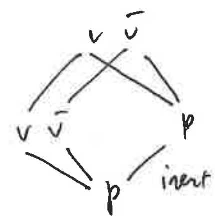
then $(*)$ surjective (for dimension reasons)

<u>Lecture 2</u>. <u>An example + geometric Satake theory</u>

<u>Example</u> (Tian – X.)

$$\begin{tikzcd} & E \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ E_0 \arrow[dr, dash, "2"'] & & F = \text{real quadratic} \arrow[dl, dash, "2"] \\ & \mathbb{Q} & \end{tikzcd}$$

imag quad



$\mathrm{Hom}(E, \mathbb{C}) = \{\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2\}$

$V \otimes_{F, \tau_i} \mathbb{R} = V_i$

V Herm. space over E of sig. $(1,1)$ at both τ_1, τ_2

V' ———— of sig $(0,2)$ at τ_1, $(2,0)$ at τ_2

s.t. $V \otimes_F \mathbb{A}_{F,f} \simeq V' \otimes_F \mathbb{A}_{F,f}$

$\rightsquigarrow \quad G = GU(V) = G\big(U(1,1) \times U(1,1)\big) \underset{\text{inner forms}}{\longleftrightarrow} G' = GU(V') = G\big(U(0,2) \times U(2,0)\big)$

Assume that V and V' are unramified @ p, i.e. $V_{\mathbb{Q}_p}$ & $V'_{\mathbb{Q}_p}$ admit self-dual lattices.

Fix an open cpt (neat) subgp $K \subset G(\mathbb{A}_f) = G'(\mathbb{A}_f)$

$\rightsquigarrow \quad \underset{\text{2-dim'l}}{\underset{\uparrow}{Sh_K(G)/\mathbb{Z}_{p^2}}}, \quad \underset{\text{0-dim'l}}{\underset{\uparrow}{Sh_K(G')/\mathbb{Z}_{p^2}}}$

$G = G\left(\prod U(a_i, n-a_i)\right)$ $\quad \dim_{\mathbb{C}} \dfrac{U(a_i, n-a_i)}{U(a_i) \times U(n-a_i)}$

$\dim Sh_K G = \sum a_i(n-a_i)$

$Sh_K(G) = Sh_K(G)_{\mathbb{F}_{p^2}}$

In this case, $V_{\mu^*}^{\mathrm{Tate}} = \left(\mathrm{std}_2^* \otimes \mathrm{std}_2^*\right)^{\text{central weights}}$ is 2-dim'l.

So we expect 2 types of cycles.

$\exists$ following moduli problem:

param. isogenies $\alpha: A \to B$ satisfying certain properties.

$$\begin{tikzcd} & I_1, I_2 \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\text{locally closed embedding}"] & \\ Sh_K(G') & & Sh_K(G) \end{tikzcd}$$

(B, λ_B, η_B^p) $\qquad$ (A, λ_A, η_A^p)

discrete Shimura varieties $\to Sh_K(G')$

At each point $x \in I_1(\overline{\mathbb{F}}_p)$, (related to $H(V)_{\mathbb{Q}_p} \simeq GL_{2,\mathbb{Q}_{p^2}}$

$\curvearrowright V = \mathrm{Std} \oplus \mathrm{Std}^{*\sigma}$

dual to each other

$$\underline{A_x} \rightsquigarrow \underset{\underset{\mathcal{O}_E \otimes \mathbb{Z}_p}{\circlearrowleft}}{A_x[p^\infty]} = A_x[v^\infty] \oplus A_x[\bar{v}^\infty]$$

} Dieudonné module

$$D(A_x) = \text{rank } 2 \ W(\overline{\mathbb{F}}_p) \otimes \mathcal{O}_{E_v}$$

$$\begin{array}{ccc} \shortparallel & & \wr\!\shortmid \\ D(A_x)_{\tau_1} \oplus D(A_x)_{\tau_2} & & W(\overline{\mathbb{F}}_p) \oplus W(\overline{\mathbb{F}}_p) \end{array}$$

(Same for B)

$$\begin{tikzcd}
W(\overline{\mathbb{F}}_p)^{\oplus 2} = D(A_x)_{\tau_2} \arrow[r, "V", "\textcircled{1}"'] \arrow[d, "\alpha_{\tau_2}"] & D(A_x)_{\tau_1} \arrow[r, "V", "\textcircled{1}"'] \arrow[d, "\alpha_{\tau_1}"] \arrow[l, bend right, "F"'] & D(A_x)_{\tau_2} \arrow[d, "\alpha_{\tau_2}"] \arrow[l, bend right, "F"'] \\
D(B_x)_{\tau_2} \arrow[r, "V", "\textcircled{2}"'] & D(B_x)_{\tau_1} \arrow[r, "V", "\textcircled{2}"'] \arrow[l, bend right, "F"'] & D(B_x)_{\tau_2} \arrow[l, bend right, "F"']
\end{tikzcd}$$

$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ "$\times p$"

$$\frac{D(A_x)_{\tau_1}}{V(D(A_x)_{\tau_2})} = 1\text{-dim'l over } \overline{\mathbb{F}}_p$$

Option 1 $\alpha_{\tau_2} \cong$, α_{τ_1} has coker $\overline{\mathbb{F}}_p$

(i.e. I_1 is the moduli space of $(A, \lambda_A, \eta_A^p, B, \lambda_B, \eta_B^p, \underbrace{\alpha: A \to B}_{\mathcal{O}_E\text{-linear isogeny}}) / S$

$(A, \lambda_A, \eta_A^p) \in Sh_K(G)(S)$

$(B, \lambda_B, \eta_B^p) \in Sh_K(G')(S)$

Compatibility of $\lambda_A, \lambda_B, \eta_A^p, \eta_B^p$.

s.t. – $\ker \alpha \subset A[p]$

– $\alpha_*: H_1^{dR}(A/S)_{\tau_1} \xrightarrow{\sim} H_1^{dR}(B/S)_{\tau_1}$

$\alpha_*: H_1^{dR}(A/S)_{\tau_2} \longrightarrow H_1^{dR}(B/S)_{\tau_2}$ has coker locally free of rk 1 over $\mathcal{O}_S$

Option 2. α_{τ_2} has cokernel $\simeq \overline{\mathbb{F}}_p$, α_{τ_1} "$\underline{\times p}$" $\rightsquigarrow I_2$ a similar moduli problem.

- $\underline{\text{Fibers of } I_1}$: Choice of $D(A_x)_{\tau_1}$ are given by $\mathbb{P}^1(D(B_x)_{\tau_1}/p) = \mathbb{P}^1/\overline{\mathbb{F}}_p$

knowing $(B_x, \lambda_{B_x}, \eta_{B_x}^p)$

By Serre-Tate-Grothendieck-Messing, gives the needed $A_x \to B_x$.

<u>Subtlety</u>: In fact, $I_1 = \mathbb{P}\left(H_1^{dR}(B/Sh_K(G'))_{\tau_2}\right)$

<u>Cohomological consequences</u>

irrep of $\mathcal{H}_{K^p}$.

$$H^0(Sh_K(G')_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)^{\oplus 2}[\pi_f^p]$$

$$\begin{tikzcd} & I_1, I_2 \arrow[dl, "\mathbb{P}^1"'] \arrow[dr] & \\ Sh_K(G') & & Sh_K(G) \end{tikzcd}$$

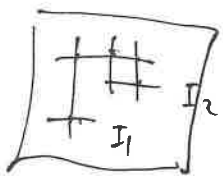
$$= \bigoplus_{i=1}^{2} H^0(I_i, \bar{\mathbb{F}}_p, \bar{\mathbb{Q}}_\ell)[\pi_f^p]$$

$$\xrightarrow{\text{Gysin}} H^2(Sh_K(G)_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell(1))[\pi_f^p] \quad \mathrm{Frob}_{p^2} = 1$$

Res

$$\bigoplus_{i=1}^{2} H^2(I_i, \bar{\mathbb{F}}_p, \bar{\mathbb{Q}}_\ell(1))[\pi_f^p]$$

$$= H^0(Sh_K(G')_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)[\pi_f^p]$$



<u>Fact</u>.

$$\underbrace{\begin{tikzcd} I_1 \arrow[d, "\mathbb{P}^1"'] & I_1 \cap I_2 \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\wr"] & I_2 \arrow[d, "\mathbb{P}^1"] \\ Sh_K(G') & Sh_{K^p \mathrm{Iw}_p}(G') \arrow[l, dashed] \arrow[r, dashed] & Sh_K(G') \end{tikzcd}}_{T_p\text{-operator}}$$

$I_i \hookrightarrow Sh_K(G)$ closed embedding if K^p small enough

The intersection matrix is
$$\begin{pmatrix} -2p & T_p \\ T_p S_p^{-1} & -2p \end{pmatrix}$$
central twist

Evaluate this at <u>π_f^p isotypical</u> part

$$\rho_\pi(\mathrm{Frob}_{p^2}) = \begin{pmatrix} \alpha_\pi & 0 \\ 0 & \beta_\pi \end{pmatrix}, \quad \alpha_\pi \beta_\pi = p^2. \quad \det\begin{pmatrix} -2p & \alpha_\pi + \beta_\pi \\ \alpha_\pi + \beta_\pi & -2p \end{pmatrix} = 4p^2 - (\alpha_\pi + \beta_\pi)^2 = 4\alpha\beta - (\alpha_\pi + \beta_\pi)^2$$
$$= -(\alpha_\pi - \beta_\pi)^2$$

If $\alpha_\pi \neq \beta_\pi$, then $\det \neq 0$ $\Rightarrow$ $H^0(Sh_K(G')_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)^{\oplus 2}[\pi_f^p] \hookrightarrow H^2(Sh_K(G)_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(1))[\pi_f^p]$

+ "dimension counting" (some further assumptions on π) $\Rightarrow$ $\cong$.

On the Galois side, we see

$\boxed{\rho_\pi^{\otimes 2}}(\mathrm{Frob}_{p^2})$ has evals $\alpha_\pi^2, \alpha_\pi\beta_\pi, \alpha_\pi\beta_\pi, \beta_\pi^2$

$$\Big\} \quad \underline{\mathrm{Frob}_p\text{-action}}$$

$$\left(\otimes\text{-}\mathrm{Ind}_{\mathrm{Gal}_{\mathbb{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbb{Q}_p}} \rho_\pi\right)\Big|_{\mathrm{Gal}_{\mathbb{Q}_{p^2}}}$$

<u>How to prove such a result in general?</u>

$$\begin{array}{ccc} Sh_K(G)^{\mathrm{perf}} & \ni & A \\ \downarrow & & \big\downarrow \\ \mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\mu} & & (D(A[p^\infty]), F, V) \end{array}$$

$FV = VF = p$

$$F: {}^{\sigma}D(A) \longrightarrow D(A)$$

$D(A_x)$ is a free $\mathbb{Z}_{p^2} \otimes W(\bar{\mathbb{F}}_p)$-mod of rk 2

$\Leftrightarrow$ a $\underbrace{\mathrm{Res}_{\mathbb{Z}_{p^2}/\mathbb{Z}_p} GL_2}_{= G_{\mathbb{Q}_p}}$ -torsor over $W(\bar{\mathbb{F}}_p)$

${}^{\sigma}D(A) \xrightarrow{F} D(A)$ a "modification" of $G_{\mathbb{Q}_p}$-torsors of type μ^*.

<u>Hope</u>: construct correspondences

$$\begin{tikzcd} & \mathrm{Sht}^{\mathrm{loc},\underline{a}}_{G_{\mathbb{Q}_p},\mu|\lambda} \arrow[dl] \arrow[dr, "\mathbb{P}^1"] & \\ \mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\mu} & & \mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\lambda} \\ Sh_K(G) \arrow[u] & I \arrow[l, dashed] \arrow[r, dashed, "\mathbb{P}^1"] & Sh_K(G') \arrow[u] \end{tikzcd}$$

Upshot: First develope the theory locally using GRT tools, then "pull" them back to Shimura varieties

$H^0(Sh_K(G')_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell) \longrightarrow H^2(Sh_K(G)_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell(1))[\pi_f^p]$ realizes Jacquet-Langlands corr. geometrically.

$\sim\!\!\sim$

Geometric Satake corr. (Fix prime p)

$\mathcal{O} = \begin{cases} \mathbb{F}_p[\![\varpi]\!] & F = \mathrm{Frac}(\mathcal{O}) \\ \mathbb{Z}_p, & \varpi = p \end{cases}$, Fix a reductive gp $G/\mathcal{O}$.

$L^n G(R) = G(R[\![\varpi]\!]/\varpi^n)$

For an $\mathbb{F}_p$-alg. R, define $\begin{cases} \mathcal{O} = \mathbb{F}_p[\![\varpi]\!], \quad L^+G(R) = G(R[\![\varpi]\!]),\ LG(R) = G(R(\!(\varpi)\!)). \\ \mathcal{O} = \mathbb{Z}_p, \quad \text{we require } R \text{ to be perfect}, \end{cases}$

$\rightsquigarrow L^+G(R) := G(W(R)),\ LG(R) = G(W(R)[\tfrac{1}{p}])$

$Gr = LG/L^+G$ affine grassmannian

Cartan decomposition When $G = GL_n$.

$$Gr = \coprod_{\underline{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_n)} \mathring{Gr}_{\underline{\lambda}} \longleftarrow x \in \mathring{Gr}_{\underline{\lambda}} \Leftrightarrow \frac{\mathcal{E}_x}{\varpi^N \bar{\mathbb{F}}_p[\![\varpi]\!]} \simeq \frac{\bar{\mathbb{F}}_p[\![\varpi]\!]}{\varpi^{\lambda_1+N}} \oplus \cdots \oplus \frac{\bar{\mathbb{F}}_p[\![\varpi]\!]}{\varpi^{\lambda_n+N}}$$

locally closed stratification

In general, $Gr = \coprod_{\lambda \in X_*(T)^+} \mathring{Gr}_\lambda$ ← Smooth of dim $\langle 2\rho, \lambda \rangle$, ρ = half sum of pos. roots

& $Gr_\lambda := \overline{\mathring{Gr}_\lambda} = \bigcup_{\lambda' \leq \lambda} \mathring{Gr}_{\lambda'}$.

In particular, $G = GL_n$, $\lambda = \omega_a = (\underbrace{1,\dots,1}_{a}, \underbrace{0,\dots,0}_{n-a})$, $Gr_{\omega_a} = Gr(n, n-a)$

$\hat{}$ $n-a$ dim'l sub

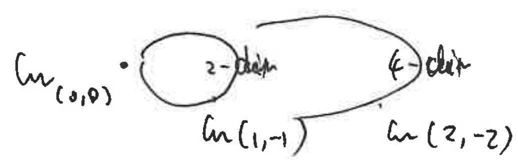
Thm ("Absolute" Geom Satake, Mirkovic - Vilonen, Ginzburg, Lusztig)

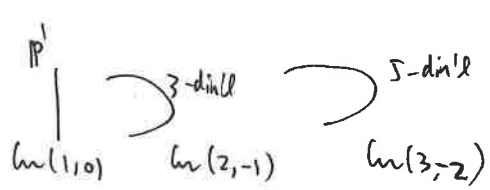
Let $\hat{G}$ be the Langlands dual gp. Then there exists an equiv. of tensor cats

$$Perv_{L^+G}(Gr) \xrightarrow{\oplus H^*(Gr,-)} Rep(\hat{G})$$

$$IC_{Gr_\mu} \longleftrightarrow V_\mu \qquad \mu \in X_*(T)^+ = X^*(\hat{T})^+$$

Ex $G = GL_2$ $\qquad \pi_0(Gr) = \mathbb{Z}$, $\qquad Gr_\lambda \mapsto \sum \lambda_i$





$\mathcal{F}_1 \boxtimes \mathcal{F}_2 \qquad p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$

$$Gr \times Gr \xleftarrow{p} LG \times Gr \xrightarrow{q} LG \overset{L^+G}{\times} Gr = Gr \tilde{\times} Gr \xrightarrow{m} Gr$$

$\mathcal{E}_2 \dashrightarrow \mathcal{E}_0$, $\mathcal{E}_1 \dashrightarrow \mathcal{E}_0$ $\qquad \mathcal{E}_2 \dashrightarrow \mathcal{E}_1 \dashrightarrow \mathcal{E}_0 \longmapsto \mathcal{E}_2 \dashrightarrow \mathcal{E}_0$

$$\mathcal{F}_1 \text{"}\tilde{\otimes}\text{"} \mathcal{F}_2 = m_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$$

"meaning"

$$\begin{tikzcd} Gr_\mu \tilde{\times} Gr_\lambda \arrow[d, "m"] \\ Gr_{\mu+\lambda} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_2 \arrow[r, dashed, "\le\lambda"] & \mathcal{E}_1 \arrow[r, dashed, "\le\mu"] & \mathcal{E}_0 \\ & \mapsto & \\ \mathcal{E}_2 \arrow[rr, dashed, "\le\mu+\lambda"] & & \mathcal{E}_0. \end{tikzcd}$$

$\hat{G}$ side. $V_\mu \otimes V_\lambda = \bigoplus_{\nu \in X^*(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(V_\nu, V_\mu \otimes V_\lambda) \otimes V_\nu$

$\Big\}$ Sat

$$m_!(IC_\mu \tilde{\boxtimes} IC_\lambda) \simeq \bigoplus_{\nu \in X^*(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(V_\nu, V_\mu \otimes V_\lambda) \otimes IC_\nu.$$

$G = GL_2$ $\quad \mu = (1,0), \quad \lambda = (0,-1)$

$$std_2 \otimes std_2^* = tr \oplus \mathrm{Sym}^2 \otimes \det^{-1}$$

$$\begin{array}{ccc} Gr_{(1,0)} \tilde{\times} Gr_{(0,-1)} & = \mathbb{P}^1 \tilde{\times} \mathbb{P}^1 \supset \mathbb{P}^1 \\ \downarrow & \downarrow f \ \text{blow up at the singular pt} \\ Gr_{(1,-1)} & \cdot\bigcirc \end{array}$$

$$Rf_*(\mathbb{Q}_\ell) = \mathbb{Q}_\ell \oplus \mathbb{Q}_{\ell, pt}[-2](-1), \qquad \mathbb{Q}_\ell = IC_{Gr_{(1,-1)}}[-2]$$

Lecture 3. Categorical trace construction

Recall $k = \mathbb{F}_p$, $\mathcal{O} = \begin{cases} \mathbb{F}_p[[\varpi]] \\ \mathbb{Z}_p \end{cases}$, G reductive gp $/\mathcal{O}$

$\rightsquigarrow Gr = LG/L^+G = \{\mathcal{E} \dashrightarrow \mathcal{E}_{triv}$ modification of the trivial G-torsor $/\mathrm{Spa}\,\mathcal{O}\}$, $\circlearrowleft L^+G$

$$= \coprod_{\lambda \in X_*(T)^+} \underbrace{L^+G \varpi^\lambda L^+G / L^+G}_{Gr^\circ_\lambda} = \{\mathcal{E} \overset{\lambda}{\dashrightarrow} \mathcal{E}_{triv}\}, \qquad \overline{Gr^\circ_\lambda} = Gr_\lambda = \coprod_{\lambda' \le \lambda} Gr^\circ_{\lambda'}$$

Rmk $Hk = [L^+G \backslash Gr] = \{\mathcal{E} \dashrightarrow \mathcal{E}'$ modification of G-torsors$\}$

Can define "$\mathrm{Perv}(Hk)$" $= \mathrm{Perv}_{L^+G}(Gr)$ by truncation.

⌈ Given $\mu \in X_*(T)^+$, L^+G-action on Gr_μ factors through $L^+G \to L^N G$

Define $\mathrm{Perv}_{L^+G}(Gr) := \varinjlim_{\mu \in X_*(T)^+} \varprojlim_N \mathrm{Perv}_{L^+G}(Gr_\mu)$ (assuming $\pi_1(G) = \{1\} = \pi_0(LG)$) ⌋

Theorem (Absolute geom. Satake) There's an equiv. of tensor categories

$$\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}) \xleftarrow[\sim]{H^*(Gr,-)} \mathrm{Perv}_{L^+G}(Gr_{\bar{k}}) = \mathrm{Perv}(Hk_{\bar{k}})$$

$$V_\mu \longleftarrow\!\shortmid \; IC_{Gr_\mu}$$

$$V \otimes W \longleftrightarrow \mathcal{F} * \mathcal{G}$$

Geom. Satake vs. classical

$P^0_{L^+G}(Gr) = \{L^+G\text{-equiv. perverse sheaves of pure weight } 0\}$

$\Downarrow$

Simple objects if $\mu \in X_*(T)^{+,\sigma=1}$.

IC^N_μ s.t. $IC^N_\mu|_{\mathring{Gr}_\mu} = \bar{\mathbb{Q}}_\ell[\langle 2\rho, \mu\rangle](\langle \rho, \mu\rangle)$ (Fix a $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$)

if μ is not σ-inv, $\left(\bigoplus_i IC^N_{\sigma^i(\mu)}\right)$

The geom Satake equiv. upgrades to

$$P^0_{L^+G}(Gr) \xrightarrow{\sim} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L G)$$

σ arithmetic Frob

ϕ. geom. Frob

Theorem The following diagram commutes (ϕ geom. Frob)

$$\begin{tikzcd}
{[\mathcal{F}]} \arrow[d, mapsto] & K_0(P^{\circ}_{L^+G}(\mathrm{Gr})) \arrow[r, "\mathrm{Sat}", "\sim"'] \arrow[d, "\substack{\text{sheaf func.}\\ \text{dictionary}}"] & K_0(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)) \arrow[r] & \overline{\mathbb{Q}}_\ell[\hat{G}\phi / \mathrm{Ad}\,\hat{G}] = \overline{\mathbb{Q}}_\ell[\hat{G}/\mathrm{Ad}_\phi \hat{G}] \arrow[d, "\cong"] \\
x \mapsto \mathrm{Tr}(\phi_x, \mathcal{H}^*(\mathcal{F})) & \mathcal{H}_G = C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}); \overline{\mathbb{Q}}_\ell) \arrow[r, "\mathrm{Sat}^{cl}"] & C_c^\infty(T(\mathcal{O})\backslash T(F), \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[X_*(T)^{\phi=1}]^{W_0} & \overline{\mathbb{Q}}_\ell[\hat{T}\phi/\hat{T}]^{W_0} = \overline{\mathbb{Q}}_\ell[X^*(\hat{T})^{\phi=1}]^{W_0}
\end{tikzcd}$$

$$[V] \longmapsto \chi_V|_{\hat{G}\phi}$$

Local shtukas

$$\mu_\bullet = (\mu_1, \dots, \mu_r)$$

Def Given dominant coweights $\mu_1, \dots, \mu_r \in X_*(T)^+$,

$$\mathrm{Hk}^{loc}_{\mu_\bullet} = \{ \mathcal{E}_r \overset{\leq \mu_r}{\dashrightarrow} \mathcal{E}_{r-1} \overset{\leq \mu_{r-1}}{\dashrightarrow} \cdots \overset{\leq \mu_2}{\dashrightarrow} \mathcal{E}_1 \overset{\leq \mu_1}{\dashrightarrow} \mathcal{E}_0 \}$$

For a perfect k-alg. R and $\mathcal{E}$ a G-torsor over $\mathrm{Spec}\, R[\![\varpi]\!]$ or $\mathrm{Spec}\, W(R)$,

define ${}^\sigma\mathcal{E} := (\sigma_R \times \mathrm{id}_\varpi)^*(\mathcal{E})$.

Define the prestack $\mathrm{Sht}^{loc}_{\mu_\bullet}$ to classify (for a perfect k-alg R)

- an R-pt of Hk_{μ}, i.e. $\mathcal{E}_r \xrightarrow{\leq \mu_r} \mathcal{E}_{r-1} \to \cdots \to \mathcal{E}_1 \xrightarrow{\leq \mu_1} \mathcal{E}_0$
- an isom. $\mathcal{E}_0 \xrightarrow{\sim} {}^\sigma\mathcal{E}_r$

We have

$$\begin{tikzcd}
\mathrm{Sht}^{loc}_{\mu_\bullet} \arrow[r, "\Phi^{loc}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hk}^{loc}_{\mu_\bullet} \arrow[d, "\pi_r,\, \pi_0"] \\
\mathbb{B}L^+G \arrow[r, "1 \times \sigma"] & \mathbb{B}L^+G \times \mathbb{B}L^+G
\end{tikzcd}$$

Fact Given a Sh. var of Hodge type $Sh_G(K)/\mathbb{F}_{p^N}$, K_p hyperspecial

$\mu: \mathbb{G}_m \to G$ minuscule cochar

$$\begin{array}{ccc} Sh_G(K)^{perf} & & A \\ \downarrow loc_p & & \downarrow \\ Sht_\mu^{loc} & & D' := D(A[p^\infty]) \end{array}$$

covariant Dieudonné mod

$, V: D' \xrightarrow{\sigma(\mu)} \sigma(D')$

Consider $\mathbb{D} = (\sigma^{-1} D', V, \text{additional str.}) \in Sht_\mu^{loc}$.

Hope.

$$\begin{tikzcd} \underset{(A,\lambda,\eta)}{Sh_G(K)} & \underset{\overset{p}{(\text{isog. } A\to B)}}{I} \arrow[l] \arrow[r] \arrow[d] & \underset{(B,\lambda',\eta')}{Sh_{G'}(K)} \arrow[d] \\ Sht_\mu^{loc} \arrow[from=u, "loc_p"] & Sht_{\mu\mid\lambda}^{loc} \arrow[l] \arrow[r] & Sht_\lambda^{loc} \end{tikzcd}$$

$G(\mathbb{A}_f) = G'(\mathbb{A}_f)$

$\mathcal{E} \xrightarrow{\mu} \sigma\mathcal{E}$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\leq\mu"] \arrow[d, "\beta"] & \sigma\mathcal{E} \arrow[d, "\sigma(\beta)"] \\ \mathcal{E}' \arrow[r, "\leq\lambda"'] & \sigma\mathcal{E}' \end{tikzcd}$$

$\mathcal{E}' \xrightarrow{\lambda} \sigma\mathcal{E}'$

Def. Suppose given a corr. of spaces

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

and $\mathcal{F}_1 \in D_b(X_1)$, $\mathcal{F}_2 \in D_b(X_2)$

A cohomological corresp. is a map $u: c_1^* \mathcal{F}_1 \to c_2^! \mathcal{F}_2$.

$$\mathrm{Corr}_C((X_1,\mathcal{F}_1),(X_2,\mathcal{F}_2)) = \mathrm{Hom}_C(c_1^*\mathcal{F}_1, c_2^!\mathcal{F}_2)$$

u gives $H_c^*(X_{1,\bar{k}}, \mathcal{F}_1) \xrightarrow{c_1^*} H_c^*(C_{\bar{k}}, c_1^*\mathcal{F}_1) \xrightarrow{H_c^*(u)} H_c^*(C_{\bar{k}}, c_2^!\mathcal{F}_2) \xrightarrow{Tr} H_c^*(X_{2,\bar{k}}, \mathcal{F}_2)$

* Can define composition, pullback, pushforward of coh. corresp. (under some conditions)

$\mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}})$ = cat. of perverse sheaves on $\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}$ + morphisms are cohom. corresp.

$$\begin{tikzcd} & \mathrm{Sht}^{\mathrm{loc}}_{\mu|\nu} \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \mathcal{F}_\mu \backslash \mathrm{Sht}^{\mathrm{loc}}_{\mu} & & \mathrm{Sht}^{\mathrm{loc}}_{\nu} - \mathcal{F}_\nu \end{tikzcd}$$

$\mathrm{Sht}^{\mathrm{loc}} = \bigcup \mathrm{Sht}^{\mathrm{loc}}_{\mu}$

$$c_1^* \mathcal{F}_\mu \longrightarrow c_2^! \mathcal{F}_\nu$$

$$\Phi^{\mathrm{loc}} : \mathrm{Sht}^{\mathrm{loc}} \longrightarrow \mathrm{Hk}^{\mathrm{loc}}_{IC_\mu}$$

$$\begin{tikzcd} \mathrm{Coh}([\cdot/\hat{G}]) = \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}) \arrow[r, "\mathrm{Sat}", "\sim"'] \arrow[d] & \mathrm{Perv}(\mathrm{Hk}^{\mathrm{loc}}_{\bar{k}}) \arrow[d, "\Phi^{\mathrm{loc},*}"] \\ \mathrm{Coh}_{fr}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]) \arrow[r, dashed, "\exists", "S"'] & \mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}) \end{tikzcd}$$

* Stack of unramified local Langlands parameters

$$\mathrm{Loc}_{{}^L G} = \left[\mathrm{Hom}_{gp}\left(\begin{tikzcd} W_{\mathbb{F}_p} = \langle \sigma \rangle \arrow[r] \arrow[dr, "id"'] & \hat{G} \rtimes \langle \sigma \rangle \arrow[d] \\ & \langle \sigma \rangle \end{tikzcd} \right) / \mathrm{Ad}\, \hat{G} \right] = [\hat{G}\sigma / \mathrm{Ad}\, \hat{G}] = [\hat{G} / \mathrm{Ad}_\sigma \hat{G}]$$

$$V_\mu \to \tilde{V}_\mu \qquad \to [\cdot/\hat{G}]$$

Expectation (Langlands) $\quad H^*_c(\mathrm{Sh}_K(G)_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell) \approx \bigoplus_\pi \pi_f^K \otimes (r_\mu \circ \mathrm{rec}_{\pi_p})$

(Drinfeld) $\quad H^*_c(\mathrm{Sh}_K(G)_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell) \approx R\Gamma(\overset{[\hat{G}/\mathrm{Ad}_\sigma \hat{G}]}{\overset{\shortparallel}{\mathrm{Loc}_{{}^L G}}}, \mathcal{A}^\bullet \otimes \tilde{V}_\mu)$

$$W_{\mathbb{F}_{p^N}} \xrightarrow{\mathrm{rec}\, \pi_p} {}^L G \xrightarrow{r_\mu} GL(V_\mu)$$

$\mathcal{A}^\bullet$: almost skyscraper sheaf $\bigoplus_\pi \pi_f^K \otimes \delta_{\mathrm{rec}\, \pi_p}$

$\mathcal{A}^\bullet$ depends only on $G(\mathbb{A}_f)$ & K

$$R\Gamma(\mathrm{Loc}_{{}^LG}, A \otimes \widetilde{V_\mu}) \otimes \mathrm{Hom}_{\mathrm{Loc}_{{}^LG}}(\widetilde{V_\mu}, \widetilde{V_\lambda}) \to R\Gamma(\mathrm{Loc}_{{}^LG}, A \otimes \widetilde{V_\lambda})$$

$$\mathrm{Coh}_{tr}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}])$$

Abstract proof (categorical trace)

- E - base comm. ring, $A = E$- alg, not nec. comm.

$Tr(A) = A/(ab - ba : a, b \in A)$ quot as E-mod

$= HH_0(A) = \mathrm{colim}(A \otimes A \rightrightarrows A)$

Universal property: universal for maps $f: A \to V$, V E-mod s.t. $f(ab) = f(ba)$

$(\mathcal{C}, \otimes, \mathbb{1})$ E-linear monoidal cat

$\sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ autom.

(e.g. $\mathcal{C} = \mathrm{Coh}([\cdot/\hat{G}]) = \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$)

Define the (twisted) categorical trace to be the 2-colimit of

$$\mathcal{C}^{\otimes 3} \substack{\to\\\to\\\to} \mathcal{C}^{\otimes 2} \rightrightarrows \mathcal{C}$$

$$(X, Y) \mapsto \begin{cases} X \otimes Y \\ \sigma Y \otimes X \end{cases}$$

$$(X, Y, Z) \mapsto \begin{cases} X, Y \otimes Z \\ \sigma Z \otimes X, Y \\ Z, X \otimes \sigma Y \end{cases}$$

Universal for functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \\ & \mathrm{Tr}_\sigma(\mathcal{C}) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

s.t. $\exists$ functorial isom

$$\begin{array}{ccc} \alpha_{X,Y}: \ F(X \otimes Y) & \Rightarrow & F(\sigma Y \otimes X) \\ \| & & \| \\ F(X) \otimes F(Y) & \simeq & F(\sigma Y) \otimes F(X) \end{array}$$

<u>Explicit construction of $\mathrm{Tr}_\sigma(\mathcal{C})$</u> [Assume that every obj. $V \in \mathcal{C}$ admits a left dual V^*

i.e. $\exists$ $\mathrm{coev}_V: \mathbb{1} \to V \otimes V^*$

$\mathrm{ev}_V: V^* \otimes V \to \mathbb{1}$

satisfying some condition]

<u>Object</u>: for each $V \in \mathrm{Obj}(\mathcal{C})$ $\rightsquigarrow$ obj $\tilde{V} \in \mathrm{Tr}_\sigma(\mathcal{C})$

<u>Define</u> $\mathrm{Hom}_{\mathrm{Tr}_\sigma(\mathcal{C})}(\tilde{X}, \tilde{Y})$

$$= \Big(\bigoplus_{V, W \in \mathcal{C} \times \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X, V \otimes W) \otimes \mathrm{Hom}_{\mathcal{C}}(\sigma W \otimes V, Y) \Big) / \sim$$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, bend right, "u'"'] & V \otimes W \arrow[d, "\sum \alpha_i \otimes \beta_i"] & ?? & \sigma W \otimes V \arrow[r, "v"] \arrow[d, "\sum \sigma\beta_i \otimes \alpha_i"] & Y \\ & V' \otimes W' & & \sigma W' \otimes V' \arrow[ur, "v'"'] & \end{tikzcd}$$

$\sum \alpha_i \otimes \beta_i \in \mathrm{Hom}_{\mathcal{C}}(V, V') \otimes \mathrm{Hom}_{\mathcal{C}}(W, W')$ $\qquad (u \otimes v) \sim (u' \otimes v')$

<u>Prop</u> For $\mathcal{C} = \mathrm{Coh}([\cdot/\hat{G}])$, $\mathrm{Tr}_\sigma(\mathcal{C}) = \mathrm{Coh}_{fr}\Big(X \underset{\Delta \to X \times X \leftarrow 1 \times \sigma}{\times} X \Big)$

$\phantom{For \mathcal{C} = \mathrm{Coh}(} \| $

$\phantom{For \mathcal{C} = \mathrm{Coh}(} X$

$$= \mathrm{Coh}_{fr}\Big(\underbrace{[\cdot/\hat{G}] \underset{[\cdot/\hat{G}] \times [\cdot/\hat{G}]}{\times} [\cdot/\hat{G}]}_{= [\hat{G} / \mathrm{Ad}_\sigma \hat{G}]} \Big)$$

Conclusion: $\mathrm{Coh}_{fr}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]) = \mathrm{Tr}_\sigma(\mathrm{Coh}[\cdot/\hat{G}])$

Need to show, for $V_\mu, V_\lambda \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$

← comes from partial Frob à la V. Lafforgue

$\exists$ isom. $\alpha_{\mu,\lambda} : IC_\mu * IC_\lambda \simeq IC_{\sigma(\lambda)} * IC_\mu$ on $Sht^{loc}_{\bar{k}}$

$$\mathcal{E} \xrightarrow{\lambda} \mathcal{E}' \xrightarrow{\mu} {}^\sigma\mathcal{E} \longmapsto \mathcal{E}' \xrightarrow{\mu} {}^\sigma\mathcal{E} \xrightarrow{\sigma(\lambda)} {}^\sigma\mathcal{E}'$$

LHS is the "IC" on

$$\begin{tikzcd} Sht^{loc}_{\mu,\lambda} \arrow[d, "m"] & IC_\mu \tilde{\otimes} IC_\lambda \arrow[d, mapsto] \\ Sht^{loc}_{\mu+\lambda} & m_*(IC_\mu \tilde{\otimes} IC_\lambda) \end{tikzcd}$$

$Sht^{loc}_{\sigma(\lambda),\mu}$ $\quad \mathcal{F} \xrightarrow{\mu} \mathcal{F}' \xrightarrow{\sigma(\lambda)} {}^\sigma\mathcal{F}$

$$\begin{tikzcd}[column sep=small]
 & & Sht^{Hk,\le\nu}_{\lambda|\mu} \arrow[dl] \arrow[drr] & & & \\
 & Sht^{loc,0}_{\lambda|(\sigma\nu^*,\eta)} \arrow[dl] \arrow[dd] \arrow[dr] & & & Sht^{loc,0}_{(\eta,\nu^*)|\mu} \arrow[dl] \arrow[dr] \arrow[dd] & \\
Sht^{loc}_{\lambda} \arrow[dd] & & Sht^{loc}_{\sigma(\nu)^*,\eta} \arrow[dd] & Sht^{loc}_{\eta,\nu^*} \arrow[l, "F", "\sim"'] \arrow[dd] & & Sht^{loc}_{\mu} \arrow[dd] \\
 & Hk^{loc,0}_{\lambda|(\sigma\nu^*,\eta)} \arrow[dl] \arrow[dr] & & & Hk^{loc,0}_{(\eta,\nu^*)|\mu} \arrow[dl] \arrow[dr] & \\
Hk^{loc}_{\lambda} & & Hk^{loc}_{\sigma(\nu)^*,\eta} & Hk^{loc}_{\eta,\nu^*} & & Hk^{loc}_{\mu}
\end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\lambda"] \arrow[d, "\nu"'] \arrow[dr, dashed, "\eta"] & {}^\sigma\mathcal{E}' \arrow[d, "\sigma(\nu)"] \\ \mathcal{E} \arrow[r, "\mu"'] & {}^\sigma\mathcal{E} \end{tikzcd}$$

$\mathcal{E}' \xrightarrow{\eta} {}^\sigma\mathcal{E} \xrightarrow{\sigma(\nu)} {}^\sigma\mathcal{E}'$ $\qquad \mathcal{E} \to \mathcal{E}' \to {}^\sigma\mathcal{E}$

$\mathcal{E}' \xrightarrow{\lambda} {}^\sigma\mathcal{E}' \qquad (\mathcal{E}' \to {}^\sigma\mathcal{E} \to {}^\sigma\mathcal{E}') \qquad \mathcal{E} \xrightarrow{\mu} {}^\sigma\mathcal{E}$

$\uparrow F$

$(\underset{\nu}{\mathcal{E} \to \mathcal{E}'} \underset{\eta}{\to {}^\sigma\mathcal{E}})$

$Hk^{loc,0}_{\mu_\cdot|\nu_\cdot}$ classifies

$$\begin{array}{ccccc} \mathcal{E}'_t & \xrightarrow{\nu_t} & \cdots & \xrightarrow{\nu_1} & \mathcal{E}'_0 \\ \| & & & & \| \\ \mathcal{E}_s & \xrightarrow{\mu_s} & \cdots & \xrightarrow{\mu_1} & \mathcal{E}_0 \end{array}$$

$$\mathrm{Cor}_{Sht^{Hk}_{\lambda|\mu}}\left((Sht^{loc}_{\mu}, IC_\mu), (Sht^{loc}_{\lambda}, IC_\lambda)\right)$$

$$\uparrow$$

$$\mathrm{Cor}\left((Sht^{loc}_{\mu}, IC_\mu), (Sht^{loc}_{\eta,\nu^*}, Sat(V)\tilde{\otimes}Sat(W))\right) \otimes \mathrm{Cor}\left((Sht_{\sigma(\nu)^*,\eta}, Sat(\sigma W)\tilde{\otimes}Sat(V)), (Sht_\lambda, IC_\lambda)\right)$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\mathrm{Cor}(Hk \qquad Hk) \otimes \mathrm{Cor}(Hk \qquad Hk)$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\mathrm{Hom}_{\hat{G}}(V_\mu, V\otimes W) \otimes \mathrm{Hom}_{\hat{G}}(\sigma W \otimes V, V_\lambda)$$

Lecture 4. Irreducible components of affine Deligne-Lusztig varieties

Recall. commutative diagram.

$$\begin{tikzcd}
V_\mu \in \underset{\overset{\Vert}{\operatorname{Rep}\hat{G}}}{\operatorname{Coh}([\cdot/\hat{G}])} \arrow[r, "\operatorname{Sat}", "\sim"'] \arrow[d, "\pi^*"] & \operatorname{Perv}(\operatorname{Hk}^{loc}_{\bar{k}}) = \operatorname{Perv}_{L^+G}(\operatorname{Gr}_{\bar{k}}) \ni IC_\mu \arrow[d, "\Phi^{loc*}"] \\
\pi^* V_\mu = \tilde{V}_\mu \in \operatorname{Coh}_{fr}([\hat{G}/\operatorname{Ad}_\sigma \hat{G}]) \arrow[r, dashed, "\exists S"] & \operatorname{Perv}^{corr}(\operatorname{Sht}^{loc}_{\bar{k}}) \ni \Phi^{loc*} IC_\mu
\end{tikzcd}$$

$\Phi^{loc} : \operatorname{Sht}^{loc}_{\bar{k}} \to \operatorname{Hk}^{loc}_{\bar{k}}$

Goal today

1. understand homs in $\operatorname{Coh}_{fr}[\hat{G}/\operatorname{Ad}_\sigma \hat{G}]$

2. understand the geom. of

$$\begin{tikzcd}
& \operatorname{Sht}^{loc}_{\mathbb{1}|\mu} \arrow[dl] \arrow[dr] & \\
{[\cdot/G(\mathbb{Z}_p)]} = \operatorname{Sht}^{loc}_{\mathbb{1}} & & \operatorname{Sht}^{loc}_{\mu}
\end{tikzcd}$$

$$X = [\hat{G}/\operatorname{Ad}_\sigma \hat{G}]$$

$$\operatorname{Hom}_X(\tilde{V}_\lambda, \tilde{V}_\mu) = \operatorname{Hom}_{\hat{G}}(\mathcal{O}_{\hat{G}} \otimes V_\lambda, \mathcal{O}_{\hat{G}} \otimes V_\mu)^{\operatorname{Ad}_\sigma \hat{G}}$$

$$= \Gamma(\hat{G}, \mathcal{O}_{\hat{G}} \otimes \underbrace{\boxed{V_\lambda^* \otimes V_\mu}}_{= V})^{\operatorname{Ad}_\sigma \hat{G}}$$

Def'n. For $\hat{G}$-rep'n V, define $J(V) = \Gamma(\hat{G}, \mathcal{O}_{\hat{G}} \otimes V)^{\operatorname{Ad}_\sigma \hat{G}}$

$$= \{ f : \hat{G} \to V : f(hg\sigma(h)^{-1}) = h \cdot f(g) \}$$

(For simplicity, will assume $\sigma = $ triv, i.e. G splits / $\mathbb{Q}_p$ or $\mathbb{F}_q((\varpi))$)

- Chevalley restriction map $\hat{T} < \hat{G}$, $J_{\hat{G}}(V) = \Gamma(\hat{G}, \mathcal{O}_{\hat{G}} \otimes V)^{\operatorname{Ad}\hat{G}} \overset{\text{injective}}{\hookrightarrow} \Gamma(\hat{T}, V)^{\operatorname{Ad}_{N_{\hat{G}}}(\hat{T})} = \Gamma(\hat{T}, V(0))^W$

$$f \longmapsto f|\hat{T}$$

injective b/c $\mathrm{Ad}_{\hat{G}}(\hat{T}) \supset \hat{G}^{\mathrm{reg,ss}}$ open dense in $\hat{G}$

When $V = \mathbb{1}$, $J_{\hat{G}}(V) \xrightarrow{\sim} J_{\hat{T}}(V(0))^W$ is an isom.

$$\mathcal{H}^{sph}_G = \bar{\mathbb{Q}}_\ell[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)]$$

Sat SS,

$$\begin{tikzcd} J_{\hat{G}}(V) \arrow[r, hook, "\mathrm{Res}_V"] & J_{\hat{T}}(V(0))^W \\ \Gamma(\hat{G}, \mathcal{O}_{\hat{G}})^{\mathrm{Ad}\hat{G}} = J_{\hat{G}}(\mathbb{1}) \arrow[u, phantom, "\circlearrowleft"] \arrow[r, "\mathrm{Res}_{\mathbb{1}}", "\sim"'] & J_{\hat{T}}(\mathbb{1})^W \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

injection of $J_{\hat{G}}(\mathbb{1}) \simeq J_{\hat{T}}(\mathbb{1})^W$ - modules

work of Balagovic describes the image.

<u>Fact.</u> Res_V is generically an isom. over $J_{\hat{T}}(\mathbb{1})^W$.

<u>Theorem</u> Assume that $\hat{G}$ is semisimple and simply-conn'd. Write $X = [\hat{G}/\mathrm{Ad}\,\hat{G}]$

(1) $J_{\hat{G}}(V)$ is a free module over $\mathcal{O}_{\hat{T}}^W = \mathcal{O}_{\hat{G}}^{\mathrm{Ad}\hat{G}}$ of rank $= \dim V(0)$

& Res_V is gen. isom.

$$J_{\hat{G}}(V_\mu) = \mathrm{Hom}_X(\tilde{\mathbb{1}}, \tilde{V_\mu}) \longrightarrow \mathrm{Mor}_{\mathrm{Perv}^{\mathrm{loc}}(\mathrm{Sht}^{\mathrm{loc}})}(\widetilde{IC}_{\mathbb{1}}, \widetilde{IC}_\mu)$$

$$\begin{tikzcd} & \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}|\mu} \arrow[dl] \arrow[dr] & & \mathrm{Sht}^{\mathrm{loc}}_{\mu|\mathbb{1}} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}} & & \mathrm{Sht}^{\mathrm{loc}}_{\mu} & & \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}} \end{tikzcd}$$

(2) Consider $\mathrm{Hom}_X(\tilde{\mathbb{1}}, \tilde{V_\mu}) \times \mathrm{Hom}_X(\tilde{V_\mu}, \tilde{\mathbb{1}}) \longrightarrow \mathrm{Hom}_X(\tilde{\mathbb{1}}, \tilde{\mathbb{1}})$

not perfect

$$\begin{tikzcd} J_G(V_\mu) \arrow[d, "\mathrm{Res}_{V_\mu}"] & \times & J_G(V_\mu^*) \arrow[d, "\mathrm{Res}_{V_\mu^*}"] \arrow[r, "(*)"] & J_{\hat{G}}(\mathbb{1}) = \mathcal{O}_{\hat{G}}^{\mathrm{Ad}\hat{G}} = \mathcal{H}_G^{sph} \arrow[d, equal] \\ J_{\hat{T}}(V_\mu(0))^W & \times & J_{\hat{T}}(V_\mu^*(0))^W \arrow[r] & J_{\hat{T}}(\mathbb{1})^W \end{tikzcd}$$

Fixing a $J_{\hat{G}}(\mathbb{1})$-basis of $J_{\hat{G}}(V_\mu)$ and $J_{\hat{G}}(V_\mu^*)$, the determinant of $(*)$ belongs to

$$\mathrm{disc}_{long}^{m_\ell} \cdot \mathrm{disc}_{short}^{m_s} \cdot \bar{\mathbb{Q}}_\ell^{\times} \quad , m_\ell, m_s \in \mathbb{Z}$$

where $\mathrm{disc}_{long} = \prod_{\substack{\alpha \in \Pi \\ \alpha \text{ long root}}} (e^{\alpha} - 1) \in \bar{\mathbb{Q}}_\ell[\hat{T}]^W$, same for disc_{short}.

Sketch of proof of (1): $J_{\hat{G}}(V) = (\mathcal{O}_{\hat{G}} \otimes V)^{\mathrm{Ad}\,\hat{G}}$

$$\underset{\text{Peter-Weyl}}{=} \Big(\bigoplus_{\lambda \in X^*(\hat{T})^+} V_\lambda \otimes V_\lambda^* \otimes V \Big)^{\mathrm{Ad}\,\hat{G}}$$

$k = \bar{\mathbb{Q}}_\ell$

$$= \bigoplus_{\lambda \in X^*(\hat{T})^+} \underset{\substack{\| \\ \mathrm{Ind}_{\hat{B}^-}^{\hat{G}} k_\lambda}}{\mathrm{Hom}_{\hat{G}}(V_\lambda, V_\lambda \otimes V)} = \bigoplus_{\lambda \in X^*(\hat{T})^+} \mathrm{Hom}_{\hat{B}^-}(k_\lambda, V_\lambda \otimes V) \ni \varphi : k_\lambda \to V_\lambda \otimes V$$

$\mathrm{Hom}_{\hat{B}^-}(k_\lambda, V_\lambda \otimes V) \to V(0)$

$v_\lambda \mapsto v_\lambda$ h.w. vector

$\varphi(v_\lambda) = v_\lambda \otimes \underset{c(\varphi)}{\boxed{?}}$ + other terms

If $\lambda' \geq \lambda$, there is a nat'l map

$$\begin{tikzcd} \mathrm{Hom}_{\hat{B}^-}(k_\lambda, V_\lambda \otimes V) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\hat{B}^-}(k_{\lambda'}, V_{\lambda'} \otimes V) \arrow[dl] \\ & V(0) & \end{tikzcd}$$

This defines an increasing filtration Fil_V on $V(0)$, given by the corresponding images.

Rmk $\mathcal{O}_{\hat{G}}^{\mathrm{Ad}\,\hat{G}} \curvearrowright J_{\hat{G}}(V)$

$\|$

$\bigoplus_{\delta \in X^*(\hat{T})^+} (V_\delta \otimes V_\delta^*)^{\hat{G}}$

$\|$

$\mathrm{Hom}_{\hat{G}}(V_\delta, V_\delta) \simeq k$

$\mathrm{Hom}_{\hat{G}}(V_\delta, V_\delta) \ni f_\delta$

f_δ will send $\mathrm{Hom}_{\hat{G}}(V_\lambda, V_\lambda \otimes V)$ to

$\mathrm{Hom}_{\hat{G}}(V_\delta \otimes V_\lambda, V_\lambda \otimes V_\delta \otimes V)$

$= \mathrm{Hom}_{\hat{G}}(V_{\lambda+\delta}, V_{\lambda+\delta} \otimes V) \oplus$ other terms

Thus, $J_{\hat{G}}(V) = \bigoplus_{\lambda \in X^*(\hat{T})^+} \mathrm{fil}_\lambda V(0)$

$\circlearrowleft$

$\mathcal{O}_{\hat{G}}^{\mathrm{Ad}\,\hat{G}} \ni f_\delta$ $\quad$ f_δ will send $\mathrm{fil}_\lambda V(0)$ to something in $\mathrm{fil}_{\lambda+\delta} V(0)$

+ " lower terms "

<u>Fact</u>. $\mathrm{fil}_\bullet V(0)$ is a " nice " filtration so that when $\lambda \gg 0$, $\mathrm{fil}_\lambda V(0) = V(0)$

$$\dim V(0) = \sum_{\lambda \in X^*(\hat{T})^+} \dim \mathrm{gr}_\lambda V(0)$$

<u>Fact</u>. $\exists$ " nice " basis of each $\mathrm{gr}_\lambda V(0)$ s.t. $\exists$ lift of these basis

$$a_{\lambda, i} \in \mathrm{Hom}_{\hat{G}}(V_\lambda, V_\lambda \otimes V) \subset J_{\hat{G}}(V)$$

$$J_{\hat{G}}(V) = \bigoplus_{\lambda, i} J_{\hat{G}}(\mathbb{1}) \, a_{\lambda, i}$$

<u>Example</u> $\hat{G} = SL_2$, $V = \mathrm{Std} \otimes \mathrm{Std} = \mathbb{1} \oplus \mathrm{Sym}^2$

$$J_{\hat{G}}(V) = \bigoplus_{\lambda \in X^*(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(V_\lambda, V_\lambda \otimes V) \qquad J_{\hat{G}}(\mathbb{1}) = \bigoplus_{\delta \in X^*(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(\underset{\substack{\cup \\ T_\delta}}{\mathrm{Sym}^\delta}, \mathrm{Sym}^\delta)$$

$\lambda = 0$, $\quad \mathrm{Hom}(\mathbb{1}, \mathbb{1} \otimes (\mathbb{1} \oplus \mathrm{Sym}^2)) = 1\text{-dim'l} \ni a_0$

$\lambda = 1$ $\quad \underbrace{\mathrm{Hom}(\mathrm{Std}, \mathrm{Std} \otimes (\mathbb{1} \oplus \mathrm{Sym}^2))}_{\substack{\| \\ 2\text{-dim'l}}} \ni T_1(a_0) - * \cdot a_0$

$\ni a_1$

$\updownarrow$

$\mathrm{Hom}(\mathrm{Std}, \mathrm{Std} \otimes \mathrm{Sym}^2)$

$\lambda > 1$, $\quad \mathrm{Hom}(\mathrm{Sym}^\lambda, \mathrm{Sym}^\lambda \otimes (\mathbb{1} \oplus \mathrm{Sym}^2))$

$\mathrm{Sym}^\lambda \oplus \mathrm{Sym}^{\lambda-2} \oplus \mathrm{Sym}^\lambda \oplus \mathrm{Sym}^{\lambda+2}$

See $J_{\hat{G}}(\mathbb{1}) \curvearrowright J_{\hat{G}}(V_\mu)$ geometrically.

- Case of $J_{\hat{G}}(\mathbb{1}) = \mathcal{O}_{\hat{G}}^{\mathrm{Ad}\,\hat{G}} = C_c^\infty\left(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})\right)$

$$\left[G(\mathcal{O})\backslash G(F)/G(\mathcal{O})\right] = \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}|\mathbb{1}} \quad \begin{tikzcd} \mathcal{E} \arrow[r, "\sim"] \arrow[d, dash] & \sigma\mathcal{E} \arrow[d, dash] \\ \mathcal{F} \arrow[r, "\sim"] & \sigma\mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_{triv} \arrow[r, "id"] \arrow[d, "g"'] \arrow[dr, phantom, "\curvearrowright"] & \mathcal{E}_{triv} \arrow[d, "\sigma(g)"] \\ \mathcal{E}_{triv} \arrow[r, "id"] & \sigma\mathcal{E}_{triv} \end{tikzcd} \qquad \sigma(g) = g$$

$\mathcal{O} = \mathbb{Z}_p$ or $\mathbb{F}_p[[\varpi]]$

$$\begin{tikzcd} & \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}|\mathbb{1}} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}} & & \mathrm{Sht}_{\mathbb{1}} \end{tikzcd}$$

$[\cdot/G(\mathcal{O})] = \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}}$

$$\mathrm{Sht}_{\mathbb{1}} = \{\mathcal{E} \xrightarrow{\sim} \sigma\mathcal{E}\} = \left[L^+G/\mathrm{Ad}_\sigma L^+G\right] \overset{\text{Lang Thm}}{=\!=} \left[\cdot/G(\mathcal{O})\right]$$

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Perv}^{\mathrm{con}}}\left(\mathbb{1}_{\mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}}}, \mathbb{1}_{\mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}}}\right) = \mathrm{Hom}_{\mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}|\mathbb{1}}}(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell) = C_c^\infty\left(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})\right) \\ \mathrm{Hom}_X(\tilde{\mathbb{1}}, \tilde{\mathbb{1}}) = J(\hat{G}) = \bigoplus_{\delta \in X^*(T)^+} \mathrm{Hom}_{\hat{G}}(V_\delta, V_\delta) \ni T_\delta \arrow[u] \end{tikzcd}$$

T_δ is supported on $\mathrm{Sht}^{\mathrm{loc}, \leq\delta}_{\mathbb{1}|\mathbb{1}}$ $\quad \begin{tikzcd} \mathcal{E} \arrow[r, "\sim"] \arrow[d, dash, "\leq\delta"'] & \sigma\mathcal{E} \arrow[d, "\leq\sigma(\delta)"] \\ \mathcal{F} \arrow[r, "\sim"] & \sigma\mathcal{F} \end{tikzcd}$

- Action $J_{\hat{G}}(V_\mu)$

$$\begin{tikzcd} & \mathrm{Sht}^{\mathrm{loc}}_{\mu|\mathbb{1}} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{\mathrm{loc}}_{\mu} & & \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}} \end{tikzcd}$$

$\sim\!\sim$ classify

$$\begin{tikzcd} & \mathrm{Sht}^{\mathrm{loc}, \leq\delta}_{\mathbb{1}|\mathbb{1}} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{\mathrm{loc}}_{\mathbb{1}} & & \mathrm{Sht}^{\mathrm{loc}}_{\delta} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\leq\mu"] \arrow[d, dash, "\leq\nu"'] & \sigma\mathcal{E}' \arrow[d, "\leq\sigma(\nu)"] \\ \mathcal{E} \arrow[r, "\sim"] \arrow[d, dashed, "\delta"] & \sigma\mathcal{E} \arrow[d, dashed, "\sigma(\delta)"] \\ \mathcal{E} \arrow[r, "\sim"] & \sigma\mathcal{E} \end{tikzcd}$$

$$\begin{aligned} & J_G(V_\mu) \\ & \quad \| \\ \hookrightarrow & \bigoplus_{\nu \in X^*(T^+)} \mathrm{Hom}_{\hat{G}}(V_\nu, V_\nu \otimes V) \\ & \quad \cup \\ J_G(\mathbb{1}) = & \bigoplus_\delta \mathrm{Hom}_{\hat{G}}(V_\delta, V_\delta) \end{aligned}$$

Pullback this picture to Shimura varieties

$$\begin{tikzcd}[column sep=small] & & I^{\le \nu+\delta} \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & I^{\le \nu} \arrow[dl] \arrow[dr] & & T_{p,\delta} \arrow[dl] \arrow[dr] & \\ Sh_G(K) & & Sh_{G'}(K) & & Sh_{G'}(K) \end{tikzcd}$$

$T_{p,\delta}$ ← usual Hecke corresp

$Sh_{G'}(K)$ discrete

<u>Upshot</u> $\bigcup_\nu I^{\le\nu}$ = union of many irred components

"generated" by those corresponding to basic element $a_{\lambda,i}$'s.

Fibers of the correspondence

$$\begin{tikzcd} & X_{\mu^*}(1)^{\le\nu} \arrow[r] \arrow[d] & pt \arrow[d] \\ Sht^{loc}_{\mu} & Sht^{loc\,\le\nu}_{M|1} \arrow[l] \arrow[r] & Sht^{loc}_{1} \\ Sh_G(K) \arrow[u] & I^{\le\nu} \arrow[l] \arrow[r] \arrow[u] & Sh_{G'}(K) \arrow[u] \end{tikzcd}$$

$Sht^{loc}_1 = [pt/G(\mathcal{O})]$

classifies

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\le\mu"] \arrow[d, "g"] & {}^\sigma\mathcal{E} \arrow[d, "\sigma(g)"] \\ \mathcal{E}_{triv} & = {}^\sigma\mathcal{E}_{triv} \end{tikzcd}$$

belongs to Gr

Condition: $g^{-1}\sigma(g) \in Hk_{\mu^*}$.

<u>Affine Deligne-Lusztig variety</u>: $\mu \in X_*(T)^+$, $b \in G(\breve{F})$

$$X_{\mu^*}(b) = \{ h \in Gr : h^{-1} b\, \sigma(h) \in \overline{Gr_{\mu^*}} \}$$

$\circlearrowleft$

$$J_b(F) = \{ g \in G(\breve{F}) : g^{-1} b\, \sigma(g) = b \}$$

$\shortparallel$

$G_p^{(?)}$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\le\mu"] \arrow[d, "h"] & {}^\sigma\mathcal{E} \arrow[d, "\sigma(h)"] \\ \mathcal{E}_{triv} & {}^\sigma\mathcal{E}_{triv} \arrow[l, "b"'] \\ \mathcal{E}_{triv} \arrow[u, leftarrow, "g"] \arrow[ur, phantom, "\wr\wr\wr"] & \mathcal{E}_{triv} \arrow[l, "b"'] \arrow[u, leftarrow, "\sigma(g)"'] \end{tikzcd}$$

Fact. J_b is always an inner form of a Levi of G

When $b = 1$ (or central), $J_b(F) = G(F)$

When $b = \varpi^{\tau^*}$ for some $\tau \in X_*(T)^+$, $J_b = M_\tau$ ← Levi defined by τ (max'l Levi for which τ is central)

When $b = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$, $J_b = D_{\mathbb{Q}_p}^\times$ division quaternion alg. $D_{\mathbb{Q}_p}$

$\updownarrow$

$GL_2(\mathbb{Q}_p)$

Thm (X.-Zhu) When G is semisimple and adjoint, split $/\mathbb{Q}_p$, $\mathbb{F}_p((\varpi))$

Assume $\tau \in X_*(T)^+$ s.t. $V_\mu(\tau) \neq 0$, then $X_{\mu^*}(\varpi^{\tau^*})$ is equi-dim'l of dim $\langle \rho, \mu - \tau \rangle$. Moreover, there's a bijection $\mathrm{Irr}(X_{\mu^*}(\varpi^{\tau^*})) = \coprod_{\text{basis of } V_\mu(\tau)} \underbrace{M_\tau(F)}_{\overset{\|}{J_b(F)}} / M_\tau(\mathcal{O})$

$\|$

Same as set of hyperspecial subgps of $M_\tau(F)$.

Explicitly, $X_{\mu^*}(\varpi^{\tau^*}) = \bigcup_{\text{basis } a \text{ of } V_\mu(\tau)} M_\tau(F) \overset{M_\tau(\mathcal{O})}{\times} \boxed{X_{\mu^*}(\varpi^{\tau^*})^a}$

$\circlearrowleft$ $\qquad\qquad\qquad\qquad$ ↑ irred

$J_b = M_\tau(F)$

Rmk $M_\tau(\mathcal{O}) \circlearrowright X_{\mu^*}(\varpi^{\tau^*})^a$ similar to "parabolic" Deligne-Lusztig var.

$\downarrow$

$M_\tau(\mathcal{O}/\varpi^2) \circlearrowleft$ $\qquad$ In general, the action doesn't factor through $M_\tau(\mathbb{F}_q)$.

$2 \sim$ linear in μ/a

Rmk Chen-Zhu's conj. (proved by Sian Nie)

$\mathrm{Irr}(X_{\mu^*}(b)) = \coprod_{\text{basis } a \text{ of } V_\mu(\nu_b)} J_b(\mathbb{Q}_p) / \text{Stab group}$ ← max'l parahoric describe explicitly

↖ some weight close to Newton polygon of b

Semi-infinite orbits

locally closed

$U(\breve{F}) \curvearrowright Gr$

$\uparrow$

unipotent of Borel

$$Gr = \coprod_{\lambda \in X_*(T)} \underbrace{U(\breve{F}) \cdot \varpi^{\lambda} G(\breve{\mathcal{O}}) / G(\breve{\mathcal{O}})}_{S_\lambda}$$

semi-infinite orbits

<u>Important</u>: $H_c^{i}(S_\lambda \cap Gr_\mu, IC_\mu) = \begin{cases} V_\mu(\lambda) & \text{if } i = \langle \rho, \mu + \lambda \rangle \\ 0 & \text{o/w} \end{cases}$

So $\mathrm{Irr}(S_\lambda \cap Gr_\mu)$ form a basis of $V_\mu(\lambda)$.

Take $\nu \in X_*(T)$, $S_\nu \cap X_{\mu^*}(\varpi^{\tau^*})$ $\longleftarrow$ <u>Goal</u>: This has pure dim $\langle \rho, \mu - \tau \rangle$

$\uparrow$

classifies

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\leq \mu"] \arrow[d, "\alpha"'] & {}^{\sigma}\mathcal{E} \arrow[d, "\sigma(\alpha)"] \arrow[dl, dashed] \\ \mathcal{E}_{triv} \arrow[r, "\varpi^{\tau}"] & {}^{\sigma}\mathcal{E}_{triv} \end{tikzcd}$$

$$\begin{tikzcd} S_\nu \cap X_{\mu^*}(\varpi^{\tau^*}) \arrow[rr] \arrow[d, "\text{maps to } \alpha"'] & & (S_{\sigma(\nu)-\tau} \tilde{\times} S_\tau) \cap (Gr \tilde{\times} Gr_\mu) \arrow[dl, "m"'] \arrow[d, "pr_2"] \\ \alpha \in S_\nu \arrow[r, "{(id,\ \varpi^{-\tau} \cdot \sigma)}"] & S_\nu \times S_{\sigma(\nu)-\tau} & {} \end{tikzcd}$$

$\eta \downarrow$ $\quad \leq \mu \downarrow$

$$\begin{tikzcd} S_\nu \cap X_{\mu^*}(\varpi^{\tau^*}) \arrow[r] \arrow[d] & S_{\sigma(\nu)-\tau} \tilde{\times} (S_\tau \cap Gr_\mu) \arrow[r] \arrow[d] & S_\tau \cap Gr_\mu \\ S_\nu \arrow[r] & S_\nu \times S_{\sigma(\nu)-\tau} & \end{tikzcd}$$

$S_\tau \cap Gr_\mu = \bigcup_{\text{basis } a \text{ of } V_\mu(\tau)} (S_\tau \cap Gr_\mu)^{a}$

Define $\left(S_\nu \cap X_{\mu^*}(\varpi\tau^*)\right)^a$ to be the preimage of $(S_\tau \cap \mathrm{Gr}_\mu)^a$.

Upshot. $S_\nu \cap X_{\mu^*}(\varpi\tau^*)$ up to an affine bundle is an étale $U_\tau(\mathbb{Q}_p)$-torsor over $(S_\tau \cap \mathrm{Gr}_\mu)^a$

$\exists$ "minimal" irred comp $\longleftrightarrow$ $U_\tau(\mathbb{Z}_p)$-torsor $/ (S_\tau \cap \mathrm{Gr}_\mu)^a$.

Lecture 5 Coherent sheaves on moduli stack of Langlands parameters

§1. moduli stack of Galois repns

Coeffs $\Lambda = \mathbb{F}_\ell, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, $\ell \neq p$

F_x = local field $\supset \mathcal{O}_x \twoheadrightarrow \mathcal{O}_x/\varpi_x = \mathbb{F}_q$, $q = p^r$

X proj sm. geom conn curve$/\mathbb{F}_q$, $F = \mathbb{F}_q(X)$ f. field

$D \subset X$ finite set of pts of X

$\hat{G}$ affine alg gp $/\Lambda$

$$\begin{array}{ccccccccc} 1 & \to & I_{F_x} & \to & \mathrm{Gal}_{F_x} & \to & \sigma^{\hat{\mathbb{Z}}} & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & I_{F_x} & \to & W_{F_x} & \to & \sigma^{\mathbb{Z}} & \to & 1 \end{array}$$

$W_{F_x} \ni \sigma, \tau$, $\sigma\tau\sigma^{-1} = \tau^q$

pro-p $\searrow$ $I_{F_x} \supset P_{F_x}$

$\dfrac{I_{F_x}}{P_{F_x}} \simeq \hat{\mathbb{Z}}^{(p)}(1) \ni \tau$

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(\overline{X-D}) & \to & \pi_1(X-D) & \to & \sigma^{\hat{\mathbb{Z}}} & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & \pi_1(\overline{X-D}) & \to & W_{F,D} & \to & \sigma^{\mathbb{Z}} & \to & 1 \end{array}$$

$\mathrm{Loc}_{F_x} = \mathrm{Loc}_{W_{F_x}, \hat{G}} = \Big[\underbrace{\mathrm{Hom}_{(ct)}(W_{F_x}, \hat{G})}_{R_{W_{F_x}, \hat{G}}} \Big/ \hat{G} \Big]$; $\mathrm{Loc}_{F,D} = \mathrm{Loc}_{W_{F,D}, \hat{G}}$

Rmk ① Local case as "$\# P_{F_x}$" is invertible in Λ,

$$\mathrm{Loc}_{W_{F_x},\hat{G}} = \coprod_{\substack{\rho_0 : P_{F_x} \to \hat{G}(\Lambda) \\ \text{up to conj.}}} \mathrm{Loc}_{\Gamma_q, \boxed{\mathrm{Cent}_{\hat{G}}(\rho_0)}}^{\hat{H}} = \langle \sigma, \tau \in \hat{H}^2 : \sigma\tau\sigma^{-1} = \tau^q \rangle$$

block

∞ disj.t union

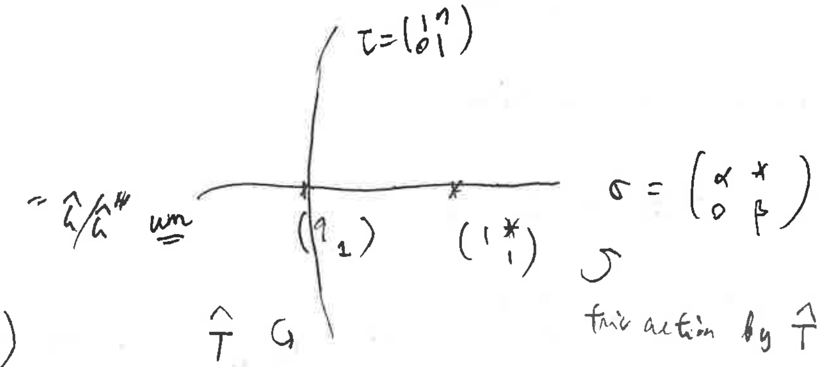
$\Gamma_q = \langle \sigma, \tau : \sigma\tau\sigma^{-1} = \tau^q \rangle$ ← image of τ is forced to be topologically unipotent

Eg. $\hat{G} = GL_2$, $\rho_0 = \mathbb{1}$. $R^{\mathrm{tame}}_{W_{\mathbb{Q}_p}, GL_2} = \{ (\sigma, \tau) \in GL_2^2 : \sigma\tau\sigma^{-1} = \tau^q \} / \Lambda$

$q \not\equiv 1$ (mod ℓ)

"Up to conj." $\sigma = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}$ ← usually 0, $\tau = \begin{pmatrix} \zeta & * \\ 0 & \zeta' \end{pmatrix}$, $\zeta^{q-1} = \zeta'^{\,q-1} = 1$

$R^{\mathrm{unip}}_{W_{\mathbb{Q}_p}, GL_2} := \{ \tau \text{ unipotent} \}$



$$\begin{array}{ccc} \mathrm{Loc} = [R/\hat{G}] & & \sigma \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \hat{T} \quad G \\ \downarrow & \swarrow & \\ \hat{G}/\!/\hat{G} & \mathrm{pt} & \end{array}$$

$$\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -* \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & qn \\ & 1 \end{pmatrix} \Rightarrow n = 0$$

$$\sigma = \begin{pmatrix} q & \\ & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} q & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \begin{pmatrix} q & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & qn \\ 0 & 1 \end{pmatrix}$$

GL_3: $\begin{pmatrix} q^2 & & \\ & q & \\ & & 1 \end{pmatrix}$ $\quad \tau = \begin{pmatrix} 1 & * & \\ & 1 & * \\ & & 1 \end{pmatrix}$

$$\mathrm{Loc}^{\mathrm{unip}}_{W_{\mathbb{Q}_p},\, GL_2} = \underset{\substack{\| \\ \mathrm{Loc}^{\mathrm{unr}}_{W_{\mathbb{Q}_p},\, GL_2} \\ \| \\ [\mathbb{A}^1/\mathbb{G}_m^{\mathrm{triv}}]}}{\left[GL_2 / \mathrm{Ad}\, GL_2 \right]} \cup \underset{\simeq [\mathbb{A}^1/\mathbb{G}_m^{\mathrm{mult}}]}{\mathrm{Loc}^{\mathrm{st}}_{W_{\mathbb{Q}_p},\, GL_2}}$$

Global case. $\mathrm{Loc}_{W_{F,D},\hat{G}} = \coprod_{\substack{\theta:\ \text{pseudorep'n} \\ \text{of } \pi_1^{\mathrm{geom}}(X-D)}} \mathrm{Loc}^{\theta}_{W_{F,D},\hat{G}}$ nice algebraic stack / $\mathbb{Z}_\ell$

This is not trivial. Use de Jong's conjecture (proved by Gaitsgory).

If $\rho: W_{F,D} \longrightarrow GL_n(\mathbb{F}_\ell((t)))$ is a cts rep'n, then

$\rho(\pi_1^{\mathrm{geom}}(X-D))$ is finite.

② In general, $R_{W_{F_x},\hat{G}}$ has derived structures. but when $\underline{\underline{F_x}}$ local + $\hat{G}$ reductive,

$\Rightarrow$ $R_{W_{F_x},\hat{G}}$ is classical.

③ (Assume $\Lambda = \overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$)

Say $(\rho: W_{F_x} \longrightarrow \hat{G}) \in \mathrm{Loc}_{W_{F_x},\hat{G}}$ is an elliptic point if $\underset{\substack{\| \\ S_\rho}}{\mathrm{Cent}_{\hat{G}}(\rho)} / Z_{\hat{G}}$
or $W_{F,D}$ $\quad$ $W_{F,D}$ $\quad$ is finite.

In this case, $[*/S_\rho] \xhookrightarrow{\text{clopen}} \mathrm{Loc}_{W_{F_x},\hat{G}}$
$\quad$ $W_{F,D}$

This S_ρ is related to the L-packets.

④ Should have used ${}^cG = \hat{G} \rtimes (\mathbb{G}_m \times \mathrm{Gal}_{F(x)})$ instead of $\hat{G}$.

but over $\mathbb{Z}[q^{1/2}]$ this has the same story as ${}^LG$.

§2 Local and global conjectures

- $X \supset D / \mathbb{F}_q$

$I = \{1, \dots, m\}$ finite set, K level, $K = \prod_{x \notin |D|} G(\mathcal{O}_x) \times \prod_{x \in |D|} K_x$

$W = \boxtimes_i V_{\mu_i} \in \mathrm{Rep}(\hat{G}^I)$

$IC_{\mu_\bullet}$

$\mathrm{Sht}_{(X-D)^I, K}$ classifies $\{\underline{x} \in (X-D)^I, \mathcal{E} \dashrightarrow {}^\tau\mathcal{E}, K\text{-level str.}\}$

isom. away from $x_1, \dots, x_m$

$$\begin{array}{c} \mathrm{Sht}_{(X-D)^I, K} \\ \downarrow p \\ (X-D)^I \ni \Delta(\bar{\eta}) \end{array}$$

Define $H(I, W) = p_!(IC_{\mu_\bullet})\big|_{\Delta(\bar{\eta})}$ $\curvearrowleft$ $\pi_1((X-D)^I, \Delta(\bar{\eta}))$

$\circlearrowleft$ Hk_K

$\pi_1((X-D)^I, \Delta(\bar{\eta})) \xrightarrow{\text{Drinfeld Lemma}} \pi_1(X-D, \bar{\eta})^I$

__Basic question__: what is $H(I, W)$?

- G/k split reductive group, equipped w/ pinning (B, T, e)

Fix $\psi_0 : (F \backslash \mathbb{A}_F, +) \to \Lambda^\times$ additive char. (Whittaker datum)

Local conj.

There's a natural fully faithful functor

$A_x : \mathrm{Rep}_{f.g.}(G(F_x)) \to \mathrm{Coh}(\mathrm{Loc}_{W_{F_x}, \hat{G}})$

everything is derived

generated by $c\text{-ind}_{K_x}^{G(F_x)} \Lambda$ for some K_x open compact by colimits.

Global conj.

$$\begin{tikzcd} \mathrm{Loc}_{W_{F,D}, \hat{G}} \arrow[r, "r"] & \prod_{x \in D} \mathrm{Loc}_{W_{F_x}, \hat{G}} \arrow[d] \\ & {[\cdot/\hat{G}]} \arrow[ul, "\widetilde{W}"] \end{tikzcd}$$

$W = \otimes V_{\mu_\bullet}$

__Conj__: $K \rightsquigarrow \boxtimes_{x \in D} A_{K_x}$ over $\prod_{x \in D} \mathrm{Loc}_{W_{F_x}, \hat{G}}$

(1) $G = \mathbb{G}_m$

$$\Gamma(\mathrm{Loc}_{W_{F_x}, \mathbb{G}_m}) = C_c(F_x^\times, \Lambda)$$

A_x is just LCFT $\rightsquigarrow$ tori

(2) For $K_x \subset G(F_x)$ open cpt, put

$$A_{K_x} = A_x(\text{c-ind}_{K_x}^{G(F_x)} \Lambda) \in \mathrm{Coh}(\mathrm{Loc}_{W_{F_x}, \hat{G}})$$

When K_x is hyperspecial,

$$A_{K_x} = \mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{F_x}} = \mathcal{O}_{[\hat{G}/\mathrm{Ad}\,\hat{G}]}$$

<u>Fully faithful means</u>

$$\mathrm{End}_{G(F_x)}(\text{c-ind}_{K_x}^{G(F_x)} \Lambda) = \mathrm{End}_{\mathrm{Loc}_{F_x}}(A_{K_x})$$

$$\begin{array}{c} \| \\ \mathcal{H}k_{K_x} \end{array} \qquad \text{i.e. } \mathcal{H}k_{K_x} \curvearrowright A_{K_x}.$$

<u>Rmk.</u> When $K_x =$ hyperspherical,

$$^{cl}\mathrm{End}_{\mathrm{Loc}_{F_x}}(\mathcal{O}_{\mathrm{Loc}^{\mathrm{ur}}_{F_x}}) = \Gamma(\mathrm{Loc}^{\mathrm{ur}}_{F_x}, \mathcal{O})$$

$$= \Gamma(\hat{G}/\mathrm{Ad}\,\hat{G}, \mathcal{O})$$

$$= \Lambda[\hat{G}]^{\mathrm{Ad}\,\hat{G}} \underset{\#}{\cong} \mathcal{H}k_{K_x}$$

<u>derived version</u> (Tony Feng) $q \equiv 1$ model

$$R\mathrm{End}_{G(F_x)}(\text{c-ind}_{K_x}^{G(F_x)} \Lambda) = R\mathrm{End}_{\mathrm{Loc}_{F_x}}(A_{K_x})$$

$$\begin{array}{ccc} H_K(I, W) = R\Gamma(\mathrm{Loc}_{W_{F,D}, \hat{G}}, & f^!(\boxtimes_x A_{K_x}) & \otimes \widetilde{W}) \\ \circlearrowleft \quad \circlearrowleft & \circlearrowleft & \circlearrowleft \\ \mathcal{H}k_K \quad W_{F,D} & \mathcal{H}k_{K_x} & W_{F,D} \end{array}$$

This explains the action of $\mathcal{H}k_{K_x}$, $x \in D$

For $y \notin D$,

$$\begin{tikzcd} \mathrm{Loc}_{F,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Loc}_{F, D\cup\{y\}} \arrow[d] \\ \mathrm{Loc}^{\mathrm{unr}}_{W_{F_y}} \arrow[r] & \mathrm{Loc}_{W_{F_y}} \end{tikzcd}$$

$$\Rightarrow R\Gamma(\mathrm{Loc}_{F,D}, f^!(\boxtimes_x A_{K_x}) \otimes \widetilde{W})$$

$$= R\Gamma(\mathrm{Loc}_{F, D\cup\{y\}}, f_y^!(\boxtimes_x A_{K_x} \otimes \underset{\substack{\circlearrowleft \\ \mathcal{H}k_y}}{\mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{W_{F_y}}}}) \otimes \widetilde{W})$$

There's a tautological $W_{F,D}$-action on $\widetilde{W}$

$$\mathrm{Loc}_{F,D} = [R_{W_{F,D}, \hat{G}}/\hat{G}]$$
$$\begin{array}{c} \Psi \\ \rho \end{array}$$

$\gamma \in W_{F,D}$ acts on the fiber $\widetilde{W}_\rho$

by ${}^{\text{"}w\text{"}}\rho(\gamma) \in \hat{G}$

Special fiber cycles interpreted. Fix $x_0 \in D$

$K_{x_0} =$ hyperspecial

"combine $A_{K_{x_0}}$ w/ $\widetilde{W}$"

<u>Upshot</u>: $\mathrm{Hom}_{\mathrm{Loc}^{\mathrm{unr}}_{F_{x_0}}}(\mathcal{O}^{\mathrm{unr}} \otimes \widetilde{W}, \mathcal{O}^{\mathrm{unr}} \otimes \widetilde{W}')$

$\otimes\, R\Gamma(\mathrm{Loc}_{F,D}, f^!(\boxtimes_{x \neq x_0} A_x \otimes (A_{x_0} \otimes \widetilde{W})))$

$\to R\Gamma(\mathrm{Loc}_{F,D}, f^!(\boxtimes_{x \neq x_0} A_x \otimes (A_{x_0} \otimes \widetilde{W}')))$

(3) compatibility w/ parabolic induction

$$M \twoheadleftarrow{q} P \xhookrightarrow{r} G$$

$$\hat{M} \xleftarrow{\hat{q}} \hat{P} \xrightarrow{\hat{r}} \hat{G}$$

Expect to realize this by

$$\begin{tikzcd} & ? \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{I}_{x_0, W} & & \mathrm{Sht}^{I}_{x_0, W'} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Rep}_{f.g.}(M(F)) \arrow[rr, "\mathbb{A}_M"] \arrow[d, "\text{infl}"] & & \mathrm{Coh}(\mathrm{Loc}_{F_x, \hat{M}}) \arrow[d, "\hat{q}^!"] \\ \mathrm{Rep}_{f.g.}(P(F)) \arrow[d, "\text{c-ind}_{P(F)}^{G(F)}(-)"] & \curvearrowright & \mathrm{Coh}(\mathrm{Loc}_{F_x, \hat{P}}) \arrow[d, "\hat{r}_*"] \\ \mathrm{Rep}_{f.g.}(G(F)) \arrow[rr, "\mathbb{A}_G"] & & \mathrm{Coh}(\mathrm{Loc}_{F_x, \hat{G}}) \end{tikzcd}$$

E.g. $M = T$

$$\text{c-ind}_{T(\mathcal{O})}^{T(F)} \Lambda \longleftrightarrow \mathcal{O}_{\mathrm{Loc}^{un}_{F_x}, \hat{T}}$$

$$\wr$$

$$\text{c-ind}_{B(F)}^{G(F)}\left(\text{c-ind}_{T(\mathcal{O})}^{T(F)} \Lambda\right)$$

$$\cong$$

$$\text{c-ind}_{Iw}^{G(F)} \Lambda$$

$\hat{q}^! \mathcal{O} = \mathcal{O}$

$$\begin{tikzcd} & \mathrm{Loc}^{unip}_{\hat{B}} \arrow[r] \arrow[dl] & \mathrm{Loc}^{tame}_{\hat{B}} \arrow[d, "\hat{q}"] \arrow[dr, "\hat{r}"] & \\ \mathrm{Loc}^{un}_{\hat{T}} \arrow[rr] & & \mathrm{Loc}_{\hat{T}} & \mathrm{Loc}^{tame}_{\hat{G}} \end{tikzcd}$$

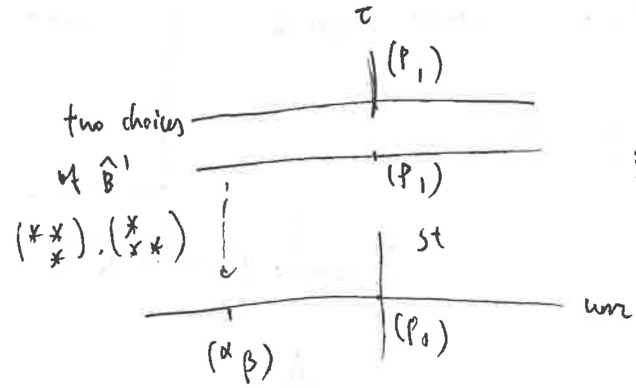
$\mathcal{O}$

$\hat{r}_* \mathcal{O}_{\mathrm{Loc}^{unip}_{\hat{B}}}$

"False example" $\hat{G} = GL_2$

some Borel

$$\begin{tikzcd} \mathrm{Loc}^{unip}_{\hat{B}} = \left[Rw^{unip}_{F_x, \hat{B}} / \hat{B} \right] \arrow[d] & Rw^{unip}_{F_x, \hat{B}} \overset{\hat{B}}{\times} \hat{G} = \{ (\sigma, \tau, \hat{B}') : \sigma, \tau \in \hat{B}', \ \sigma\tau\sigma^{-1} = \tau^p \} \arrow[l] \arrow[d] \\ \left[Rw_{F_x, \hat{G}} / \hat{G} \right] & Rw_{F_x, \hat{G}} \arrow[l] \end{tikzcd}$$



$\hat{B}^{+} = \begin{pmatrix} * & * \\ & * \end{pmatrix}$

$\hat{B}^{-} = \begin{pmatrix} * & \\ * & * \end{pmatrix}$

$$f_* \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{\hat{B}}} = \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{\hat{G}}} \oplus \mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{\hat{G}}} =: \mathrm{Spr}$$

Springer sheaf.

Theorem (Hemo-Zhu) When $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathrm{Spr}_{\hat{G}}$ is concentrated in deg 0, and

$$\mathcal{H}k_{Iw} \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{tame}}_{\hat{G}}}(\mathrm{Spr})$$

Rmk In terms of global conjecture.

$$H(I,W) = R\Gamma\left(\mathrm{Loc}_{W_{F,D},\hat{G}},\ f^! \mathrm{Spr}_{Iw_x} \otimes \widetilde{W}\right) \qquad (\mathrm{Spr}_{Iw_x})_{\rho_x} \otimes W$$

$\mathrm{Loc}_{W_{F,D},\hat{G}} \ni \rho$ elliptic, $S_\rho = \{1\}$ $\quad H_K(I,W)_\rho = \left(f^! \mathrm{Spr}_{Iw_x} \otimes \widetilde{W}\right)_\rho$ $\;\|\;$ Galois rep'n $\otimes\, \pi_\rho^K$ $\;\|\;$ $(\pi_\rho^{(x)})^{K^{(x)}} \otimes \pi_{\rho,x}^{Iw_x}$

ρ : isolated point

$$\begin{array}{ccc} \mathrm{Loc}_{W_{F,D},\hat{G}} & & \rho \\ \downarrow & & \downarrow \\ \mathrm{Loc}_{W_{F_x},\hat{G}} & & \rho_x \end{array}$$

$\rho_x \to$ unr: π_{ρ_x} unr PS, $\pi_{\rho_x}^{Iw_x} = $ 2-dim'l

$\rho_x \to$ St: π_{ρ_x} St, $\pi_{\rho_x}^{Iw_x} = $ 1-dim'l

Related.

$$\left(\text{c-ind}_K^{PGL_2} \Lambda\right)^{\oplus 2} \longrightarrow \text{c-ind}_{Iw}^{PGL_2} \Lambda \longrightarrow \left(\text{c-ind}_K^{PGL_2} \Lambda\right)^{\oplus 2}$$

Composition is $\begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix}$

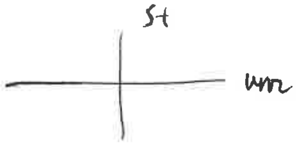
$$\begin{array}{ccc} & X_0(p) & \\ \swarrow & & \searrow \\ X_{hyp} & & X_{hyp} \end{array}$$

Coh. side

$$\mathcal{O}_{\mathrm{unr}}^{\oplus 2} \longrightarrow \mathrm{Spr} \longrightarrow \mathcal{O}_{\mathrm{unr}}^{\oplus 2}$$

$$\shortparallel$$

$$\mathcal{O}_{\mathrm{unip}} \oplus \mathcal{O}_{\mathrm{unr}} \neq \mathcal{O}_{\mathrm{St}} \oplus \mathcal{O}_{\mathrm{unr}}^{\oplus 2}$$



(4) Bernstein–Zelevinsky duality $\qquad F = F_x$

$$\mathbb{D}^{\mathrm{coh}} : \mathrm{Rep}(G(F), \Lambda) \longrightarrow \mathrm{Rep}(G(F), \Lambda)$$

contravariant

$$\pi \longmapsto R\mathrm{Hom}_{G(F)}(\pi, C_c(G(F), \Lambda))$$

$$c\text{-ind}_K^{G(F)} \Lambda \longmapsto c\text{-ind}_K^{G(F)} \Lambda$$

$$\mathbb{A} \circ \mathbb{D}^{\mathrm{coh}} = c^* \circ \mathbb{D}^{GS} \circ \mathbb{A}$$

Grothendieck–Serre duality on $\mathrm{Loc}_{F, \hat{G}}$

c: Chevalley involution $\curvearrowright (\hat{G}, \hat{B}, \hat{T})$, $\lambda \mapsto -w_0(\lambda)$
(Cartan ?)

(5) $$\mathbb{A}(c\text{-ind}_{U(F)}^{G(F)} \psi_0) = \mathcal{O}_{\mathrm{Loc}_{F_x, \hat{G}}}$$

not f.g., need ind-complete

Structure sheaf is NOT coherent
b/c ∞-ly many components.

Geometrization of $\mathrm{Hom}_{\mathrm{Loc}_{F_x, \hat{G}}}(\mathbb{A}_{Iw} \otimes \tilde{W}, A_{Iw} \otimes \tilde{W}')$

(G, X) Shimura datum unram. @ p (typically $V^{\mathrm{Tate}} = 0$)

K_p = Iwahori, G' = inner form of G, $G'(\mathbb{A}_f^p) \simeq G(\mathbb{A}_f^p) \supset K^p$ $G'(\mathbb{R})$

compact mod center

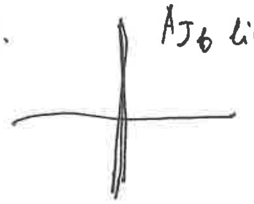
$G'(\mathbb{Q}_p) = J_b$

e.g. $G = GL_2$, $G' = D^\times/\mathbb{Q}$ ram. at p, ∞

Thm (Hemo-Zhu) There's a global Jacquet-Langlands transfer map

$$\mathrm{Hom}_{\mathrm{Loc}^{\mathrm{unip}}_{\hat{G}}}\left(A_{J_b}, \tilde{V}_\mu \otimes \mathrm{Spr}\right) \longrightarrow \mathrm{Hom}_{Hk_{K^p}}\left(C_c\left(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K', \bar{\mathbb{Q}}_\ell\right), H^*(Sh_\mu, \Psi_\mu)\right)$$

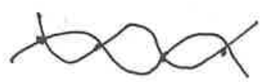
Eg. A_{J_b} lives over St



H^0

H^1

$X_0(p)$



— $X^{ss} \simeq Sh_{D^\times}$

$R^1\Psi = \mathbb{Q}_{\ell, X^{ss}}$

$R^0\Psi = \mathbb{Q}_{\ell, X_0(p)_{\mathbb{F}_p}}$

Lecture 6. Euler system and application to Beilinson-Bloch-Kato Conjecture

Joint work w/ Y. Liu, Y. Tian, W. Zhang, X. Zhu

§1. Introduction to Beilinson-Bloch-Kato conj.

Let X be a proj. smooth variety $/\mathbb{Q}$, $\dim X = d$.

$$CH^n(X) \otimes \bar{\mathbb{Q}}_\ell \xrightarrow{cl} \underbrace{H^{2n}_{et}(X, \bar{\mathbb{Q}}_\ell(n))}_{\text{absolute étale cohomology}}$$

Note $\exists$ Hochschild-Serre spectral seq

$$E_2^{ij} = H^i(\mathrm{Gal}_{\mathbb{Q}}, H^j_{et}(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))) \Rightarrow H^{i+j}_{et}(X, \bar{\mathbb{Q}}_\ell(n))$$

$\longrightarrow$ sub $\qquad\qquad\qquad\qquad$ $\longrightarrow$ sub

$$\begin{array}{ll}
H^2(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))^{\mathrm{Gal}_{\mathbb{Q}}} & \\
H^1(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))^{\mathrm{Gal}_{\mathbb{Q}}} & H^1(\mathrm{Gal}_{\mathbb{Q}}, H^1(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))) \\
H^0(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))^{\mathrm{Gal}_{\mathbb{Q}}} & H^1(\mathrm{Gal}_{\mathbb{Q}}, H^0(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n)))
\end{array}$$

<u>1st graded piece</u>:

$$\begin{tikzcd}
CH^n(X) \otimes \bar{\mathbb{Q}}_\ell \arrow[r, "cl"] \arrow[dr, "cl^0"'] & H^{2n}_{\acute{e}t}(X, \bar{\mathbb{Q}}_\ell(n)) \arrow[d] \\
& H^{2n}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))^{\mathrm{Gal}_{\mathbb{Q}}}
\end{tikzcd}$$

<u>Tate conjecture</u>: cl^0 is surjective.

Conj.

<u>Relation to L-function</u>: $\dim_{\bar{\mathbb{Q}}_\ell}\left(H^{2n}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))^{\mathrm{Gal}_{\mathbb{Q}}}\right) = -\mathrm{ord}_{s=1} L(H^{2n}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n)), s)$ $\quad \mathrm{Gal}_{\mathbb{Q}} \curvearrowright$

(<u>Rmk</u>. nice $\mathrm{Gal}_{\mathbb{Q}} \curvearrowright V$ irred, $L(V,s)$ has a simple pole at $s=1$ $\Leftrightarrow$ V is trivial)

$\Leftrightarrow L(V,s) = \zeta(s)$

& if V irred nontrivial, $L(V,1) \neq 0$

<u>2nd graded piece</u>: Define $CH^n(X)^0_{\bar{\mathbb{Q}}_\ell} := \ker(cl^0)$

$$\begin{array}{ccc}
CH^n(X)^0_{\bar{\mathbb{Q}}_\ell} & \xrightarrow{cl} & H^1(\mathrm{Gal}_{\mathbb{Q}}, H^{2n-1}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))) \\
& & \cup \\
& \overset{\text{Abel-Jacobi map}}{\xrightarrow{AJ}} & H^1_f(\mathrm{Gal}_{\mathbb{Q}}, H^{2n-1}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell(n))) \leftarrow \underline{\text{Selmer gp}}
\end{array}$$

This is a conj. to factor through H^1_f — Known for some unitary Sh. var., a cor. of wt-monodromy conj.

$$\begin{tikzcd} X \arrow[d, "\text{proper smooth}"'] & \mathfrak{X} \arrow[l] \arrow[d, "\text{proper sm.}"] \\ \operatorname{Spec}\mathbb{Q} & \operatorname{Spec}\mathbb{Z}[\tfrac{1}{N}] \arrow[l] \end{tikzcd}$$

$\ell \mid N$

$\mathrm{Gal}_{\mathbb{Q},N}$ = Gal gp of max'l ext'n of $\mathbb{Q}$ unram. outside N

$$\begin{tikzcd} \mathrm{Gal}_{\mathbb{Q}} \arrow[d] \arrow[r, phantom, "\curvearrowright"] & H^{2n-1}_{et}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell(n)) =: V \\ \mathrm{Gal}_{\mathbb{Q},N} \arrow[ur, phantom, "\curvearrowright"] & \end{tikzcd}$$

$$H^1_f(\mathrm{Gal}_{\mathbb{Q}}, V) = \{x \in H^1(\mathrm{Gal}_{\mathbb{Q},N}, V) : \mathrm{loc}_p(x) \in \boxed{H^1_f(\mathrm{Gal}_{\mathbb{Q}_p}, V)}\}$$

$$\begin{tikzcd} H^1(\mathrm{Gal}_{\mathbb{Q},N}, V) \arrow[d, "\mathrm{loc}_p \quad p \mid N"] \\ H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V) \end{tikzcd}$$

where – for $\ell \neq p$,

$$1 \to I_{\mathbb{Q}_p} \to \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Gal}_{\mathbb{F}_p} \to 1$$

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathrm{Gal}_{\mathbb{F}_p}, V^{I_{\mathbb{Q}_p}}) \arrow[r] \arrow[d, equal] & H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V) \arrow[r] & H^1(I_{\mathbb{Q}_p}, V)^{\mathrm{Gal}_{\mathbb{F}_p}} \arrow[r] \arrow[d, equal] & 0 \\ & H^1_f(\mathrm{Gal}_{\mathbb{Q}_p}, V) & & H^1_{sing}(\mathrm{Gal}_{\mathbb{Q}_p}, V) & \end{tikzcd}$$

H^1_{unr}

exact annihilator

Interesting fact: Tate duality

$$\begin{array}{ccc} H^1_f & \longleftrightarrow & H^1_f \\ \cap & & \cap \end{array}$$

$$H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V) \times H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V^*(1))$$

$$\to H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \overline{\mathbb{Q}}_\ell(1)) = \overline{\mathbb{Q}}_\ell$$

$$\Rightarrow H^1_f = (H^1_{sing})^*.$$

Important fact: (if V is unram. @ p), then $x \in H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V)$

$$\longleftrightarrow \quad 0 \to V \to E_x \to \mathbb{Q}_\ell \to 0$$

$x \in H^1_f \iff E_x$ is unramified.

- for $\ell = p$, $H^1_f(\mathrm{Gal}_{\mathbb{Q}_p}, V) := \ker\left(H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V) \to H^1(\mathrm{Gal}_{\mathbb{Q}_p}, V \otimes \mathbb{B}_{cris})\right)$

V unram. $\ell \neq p$

$\Updownarrow$

V crystalline $\ell = p$

When V is crystalline, $x \in H^1_f(\mathrm{Gal}_{\mathbb{Q}_p}, V)$

$\iff$ for $0 \to V \to E_x \to \mathbb{Q}_p \to 0$, E_x is crystalline.

"proof" V crystalline $\iff \dim V = \dim (V \otimes \mathbb{B}_{cris})^{\mathrm{Gal}_{\mathbb{Q}_p}}$

$$0 \to V \otimes \mathbb{B}_{cris} \to E_x \otimes \mathbb{B}_{cris} \to \mathbb{B}_{cris} \to 0$$

$$0 \to (V \otimes \mathbb{B}_{cris})^{\mathrm{Gal}_{\mathbb{Q}_p}} \to (E_x \otimes \mathbb{B}_{cris})^{\mathrm{Gal}_{\mathbb{Q}_p}} \to \overset{1}{\underset{\cup}{\mathbb{Q}_p}}$$

$$\xrightarrow{\delta} H^1(\mathbb{Q}_p, V \otimes \mathbb{B}_{cris})$$

Note. $\dim (E_x \otimes \mathbb{B}_{cris})^{\mathrm{Gal}_{\mathbb{Q}_p}} = \dim E_x \iff \delta(1) = 0$

Note $\delta(1) =$ image of x in $H^1(\mathbb{Q}_p, V \otimes \mathbb{B}_{cris})$

Beilinson-Bloch-Kato conj.

① The Abel-Jacobi map $CH^n(X)^0_{\mathbb{Q}_\ell} \twoheadrightarrow H^1_f(\mathrm{Gal}_{\mathbb{Q}}, H^{2n-1}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)))$ is ~~surj.~~ an isom

② $\dim H^1_f(\mathrm{Gal}_{\mathbb{Q}}, \underbrace{H^{2n-1}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))}_{=V,\ V \simeq V^*(1)}) = \mathrm{ord}_{s=n} L(H^{2n-1}_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell), s)$

center of func. eqn

Eg. $X = E$ elliptic curve, $n = 1$

$$E(\mathbb{Q}) \xrightarrow{AJ_\ell} H^1_f(\mathrm{Gal}_{\mathbb{Q}}, \underbrace{H^1_{\acute{e}t}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(1))}_{\mathrm{Tate}_\ell(E) \otimes \mathbb{Q}_\ell}), \quad P \mapsto cl(P - \text{"}0\text{"})$$

In this case, B-SD conj. $\Rightarrow$ rank $E(\mathbb{Q}) = \dim E(\mathbb{Q}) \otimes \mathbb{Q}_\ell = \dim H^1_f(\mathrm{Gal}_{\mathbb{Q}}, \mathrm{Tate}_\ell(E)) \otimes \mathbb{Q}_\ell$

Goal: In very special cases, provide some $\underset{\sim}{\text{evidence}}$ when rank is 0 or 1.

§2. Statement of main theorem (Rankin-Selberg case)

$$\begin{array}{l} F = \mathrm{CM} \\ \mid 2 \quad c \\ F^+ = \text{totally real} \\ \mid \\ \mathbb{Q} \end{array}$$

Fix $n \geq 2$. Π_n = cuspidal automorphic rep'n of $GL_n(\mathbb{A}_F)$

s.t. * Π_n is conjugate self-dual:

$$\Pi_n \circ c \simeq \Pi_n^\vee$$

* $\Pi_{n,\infty}$ has infinitesimal character as the trivial rep'n

($\longleftrightarrow$ wt 2 for modular forms, $H^*(Sh, \mathbb{Q}_\ell)$)

Π_{n+1} ...... $GL_{n+1}(\mathbb{A}_F)$

$\rightsquigarrow$ associated Galois rep'n $\rho_n: \mathrm{Gal}_F \to GL_n(\bar{\mathbb{Q}}_\ell)$ s.t. $\rho_n^c = \rho_n^*(1-n)$.

ρ_{n+1}.

Thus, putting $\rho = \rho_n \otimes \rho_{n+1}(n)$, so that $\rho^c \simeq \rho^*(1)$

(ρ is conjugate self-dual for "arithmetic dual". $\{n, n+1\} = \{n_0, n_1\}$, n_0 even, n_1 odd

Thm (LTXZZ) Assume $F^+ \neq \mathbb{Q}$, let E be the coeff. field of Π_n & Π_{n+1}, it's a no. field. Suppose — $\exists$ a very special inert prime $\mathfrak{p}$ of F^+ s.t. $\Pi_{n_0,\mathfrak{p}}$ is Steinberg, $\Pi_{n_1,\mathfrak{p}}$ is unram.

$$\begin{array}{ll} F & \mathfrak{P} \\ \mid & \mid \text{ inert} \\ F^+ & \mathfrak{p} \\ \mid & \\ \mathbb{Q} & p \end{array}$$

& $F^+_{\mathfrak{p}} \simeq \mathbb{Q}_p$, p unram. in F

page 44.

+ Satake parameter contains 1 exactly once.

- $\exists$ w_0, w_1 places of F s.t. $\Pi_{n_0, w_0}, \Pi_{n_1, w_1}$ are supercuspidal

$\varepsilon(\Pi_n \times \Pi_{n+1}, \text{center}) = 1$

<u>Then</u> (1) $L(\Pi_n \times \Pi_{n+1}, \text{center}) \neq 0 \Rightarrow$ for all but finitely many places λ of E,

<u>rk 0</u>

$$H^1_f(\mathrm{Gal}_F, \rho_\lambda(n)) = 0$$

$\llcorner$ rep's / Coeff in E_λ

<u>rk 1</u> (2) + Stronger technical assumptions + $\varepsilon(\Pi_n \times \Pi_{n+1}, \text{center}) = -1$ $(\Rightarrow L(\text{center}) = 0)$

If certain class $[\Delta(Sh)] \in H^1_f(\mathrm{Gal}_F, \rho_\lambda(n)) \Rightarrow \dim H^1_f(\mathrm{Gal}_F, \rho_\lambda(n)) = 1$

$\not\parallel$
0

$\Downarrow$ expected

$L'(\text{center}) \neq 0$

Kolyvagin, Bertolini - Darmon

$\downarrow$

Idea of the proof (Euler system argument + <u>Geometric construction</u>)

Make stronger hypo. ① " L^{alg}" $(\Pi_n \times \Pi_{n+1}, \text{center}) \not\equiv 0 \bmod \ell \quad \ell \gg 0$

$$\Rightarrow H^1_f(\mathrm{Gal}_F, \overline{\rho_\lambda}(n)) = 0$$

(weaker hypothesis $\Rightarrow H^1_f(\mathrm{Gal}_F, \rho_\lambda/\ell^N(n))$ bounded indep. of N)

Suppose not, $0 \neq x \in H^1_f(\mathrm{Gal}_F, \overline{\rho_\lambda}(n))$

$H^1_f(\mathrm{Gal}_{F_\mathfrak{p}}, \overline{\rho_\lambda}(n))$

Chebotarev density $\Rightarrow \exists$ an adm. prime $\mathfrak{p}$ s.t. $\mathrm{loc}_\mathfrak{p}(x) \neq 0$

adm. means $\mathfrak{p}$ very special inert

$$\begin{array}{cc} F & \mathfrak{p} \\ | & |\ \text{inert} \\ F^+ & \mathfrak{p} \\ | & | \\ \mathbb{Q} & p \end{array}$$

$F^+_\mathfrak{p} = \mathbb{Q}_p$

puman ρ_{n_0}, ρ_{n_1} unram @ $\mathfrak{p}$

Π_n, Π_{n+1}

even $\bar\rho_{n_0}\left(\frac{n_0}{2}\right)(\mathrm{Frob}_{p^2}) = \begin{pmatrix} \ddots & & & \\ & p^2/\alpha & & \\ & & \boxed{\begin{matrix} p^{-2} & \\ & 1 \end{matrix}} & \\ & & & \alpha \ddots \end{pmatrix}$ mod ℓ ~ level raising part!

odd $\bar\rho_{n_1}\left(\frac{n_1-1}{2}\right)(\mathrm{Frob}_{p^2}) = \begin{pmatrix} \ddots & & & & \\ & \beta & & & \\ & & \alpha & & \\ & & & 1 & \\ & & & & \alpha^{-1} \\ & & & & & \beta^{-1} \ddots \end{pmatrix}$ mod ℓ $\qquad \ell \nmid \# U_{n_0}(\mathbb{F}_p)$

"genericity condition" $\bar\rho = \bar\rho_{n_0} \otimes \bar\rho_{n_1}(n)\ (\mathrm{Frob}_{p^2})$ eigenvalues mod ℓ

has exactly one pair of $\{p^{-2}, 1\}$

What's special about these adm. primes?

$\bar\rho = \mathbb{F}_\ell \oplus \mathbb{F}_\ell(1) \oplus$ sth else

unram.

nontrivial H^0, $H^2 = 0$ $\qquad$ nontriv H^2, $H^0 = 0$

Fact. $H^1(\mathrm{Gal}_{\mathbb{Q}_{p^2}}; \bar\rho) = H^1(\mathrm{Gal}_{\mathbb{Q}_{p^2}}; \mathbb{F}_\ell) \oplus H^1(\mathrm{Gal}_{\mathbb{Q}_{p^2}}; \mathbb{F}_\ell(1))$

$\bar\rho^c \simeq \bar\rho^*(1)$ $\qquad H^1_{\mathrm{ur}}(\mathrm{Gal}_{\mathbb{Q}_{p^2}}, \mathbb{F}_\ell) \oplus H^1_{\mathrm{sing}}(\mathrm{Gal}_{\mathbb{Q}_{p^2}}, \mathbb{F}_\ell(1))$

$\bar\rho|_{\mathrm{Gal}_{\mathbb{Q}_{p^2}}}$ is self-dual. $\qquad \mathbb{F}_\ell \longleftrightarrow \mathbb{F}_\ell$

duality gives a

perfect pairing here

Recall. $H^1_f(\mathrm{Gal}_F, \bar\rho(\#)) = \{ x \in H^1(\mathrm{Gal}_{F,N}, \bar\rho(\#)) : \mathrm{loc}_v(x) \in H^1_f(\mathrm{Gal}_{F_v}, \bar\rho(\#)) \}$

$\forall v | N$ $\qquad$ x "finite" everywhere

Key input For every adm. prime $\mathfrak{p}$, if we can find $Z_{\mathfrak{p}} \in H^1(\mathrm{Gal}_{F,N}, \bar\rho^c(\#))$

construction uses geom of Sh. var. $\qquad \bar\rho^c(\#) = \bar\rho^*(1)$

which is - "finite" at all $v \mid N$, i.e. $\mathrm{loc}_v(Z_p) \in H^1_f(\mathrm{Gal}_{F_v}, \bar\rho^c)$

- "singular" at p, i.e.

$$\begin{array}{ccc} H^1(\mathrm{Gal}_{F_v}, \bar\rho^c) & \longrightarrow & H^1_{sing}(\mathrm{Gal}_{F_v}, \bar\rho^c) \\ Z_p & \longmapsto & \neq 0 \end{array}$$

Then we are done! b/c

$$\underset{\substack{\uparrow \\ \mathrm{CFT}}}{0} = \sum_v \langle \mathrm{loc}_v(x), \mathrm{loc}_v(Z_p) \rangle = \underbrace{\sum_{v \mid N} \langle \mathrm{loc}_v(x), \mathrm{loc}_v(Z_p) \rangle}_{=0} + \langle \overset{\overset{0}{\not\shortparallel}}{\mathrm{loc}_p(x)}, \overset{\overset{0}{\not\shortparallel}}{\mathrm{loc}_p(Z_p)} \rangle$$

$$\begin{array}{ccccccccc} 0 & \to & H^2(\mathrm{Gal}_F, \mathbb{F}_\ell(1)) & \to & \bigoplus_v H^2(\mathrm{Gal}_{F_v}, \mathbb{F}_\ell(1)) & \xrightarrow{\sum \mathrm{inv}} & \mathbb{F}_\ell & \to & 0 \\ & & \shortparallel & & \shortparallel\ \mathbb{F}_\ell & & & & \\ 0 & \to & \mathrm{Br}_F[\ell] & \to & \bigoplus_v \mathrm{Br}_{F_v}[\ell] & \xrightarrow{\sum \mathrm{inv}} & \mathbb{F}_\ell & \to & 0 \end{array}$$

__Reflection__: just need to construct such a Z_p.

__Rmk__. construction of $Z_p \in H^1(\mathrm{Gal}_{F,N}, \bar\rho^c)$

__Essential difficulty__: any time we have a motivic obj.

$$CH^?(X) \longrightarrow H^1_f(\mathrm{Gal}_{F,N}, H^\bullet(X))$$

$\uparrow$ by wt-mon. conj.

such class is automatically "finite" everywhere!

- Have to manually manipulate the data to "create" some ramification! through modl congruence

① (Kolyvagin) make a tamely ram. ext'n of F_p, and then take some "clever" trace

② (Bertolini–Darmon) pass to a ramified inner form of G.

③ (Kato) varying weights p-adically. $H^1_f(\mathbb{Q}_p, V)$ is sensitive to weights.

<u>Construction of Z_p</u>:

Π_n, Π_{n+1}, $\varepsilon(\Pi_n \times \Pi_{n+1}, \text{center}) = 1$.

- $\exists$ V Herm. space for F/F^+, dim n, sig. $(n, 0)$ $\rightsquigarrow U(V)$

$W = V \oplus \mathbb{1}$ $\rightsquigarrow U(W)$

<u>Fact</u> Π_n transfers to π_n on $U(V)$, $\Pi_{n+1} \rightsquigarrow \pi_{n+1}$ on $U(W)$.

* "<u>nearby Herm. space</u>" V' Herm space for F/F^+, dim n, sig $(n-1, 1)$

at some $\tau: F^\times \to \mathbb{C}$, $(n, 0)$ other

$$V' \otimes_F \mathbb{A}_{F,f}^{(p)} \simeq V \otimes_F \mathbb{A}_{F,f}^{(p)}$$

But $V' \otimes F_p$ is a ram. Herm. space for $\mathbb{Q}_{p^2}/\mathbb{Q}_p = F_p/F_p^+$.

$W' = V' \oplus \mathbb{1}$

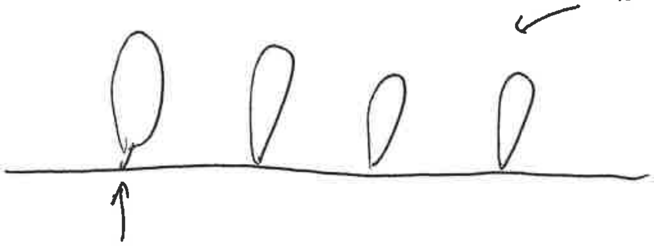
<u>Geom. obj.</u>: $K^p \subset U(V')(\mathbb{A}_{F^+,f}^{(p)}) \times U(W')(\mathbb{A}_{F^+,f}^{(p)}) = U(V) \times .. \times U(W)$

$$\underbrace{Sh_{U(V')} \xhookrightarrow{\Delta} Sh_{U(V')} \times Sh_{U(W')}}_{\text{Geometric obj.}} \Big/ \mathcal{O}_{F,(p)}$$

<u>Geometry</u>. $Sh_{U(V')}$ has semistable reduction at p.

$Sh_{U(V'), \mathbb{F}_p}$

lots of other $\mathbb{P}^{n-1}$'s, parametrized by $Sh_{U(V)}$



$\supset M^{\circ, ss}$ $\quad$ $\dim \approx \lfloor \frac{n-1}{2} \rfloor$ described by $Sh_{U(V)}(K')$

M° $\dim n-1$

intersection = Fermat hypersurface in $\mathbb{P}^{n-1}$

$$x_0^{p+1} + x_1^{p+1} + \cdots + x_{n-1}^{p+1} = 0$$

$$H^0(Sh_{U(V')}, \mathbb{F}_\ell) \xrightarrow{cl} H^{2n}_{\text{ét}}(Sh_{U(V')} \times Sh_{U(W')}, \mathbb{F}_\ell^{(n)})_{m_{\bar\rho}} \qquad \bar\rho \text{ cusp}$$

$$1 \longmapsto [Z_p]$$

$$\downarrow$$

$$H^1(Gal_F, H^{2n-1}_{\text{ét}}(Sh_{U(W')} \times Sh_{U(W')}, \bar F, \mathbb{F}_\ell(n))_{m_{\bar\rho}})$$

$$\downarrow loc_p \quad \text{singular part}$$

$$H^1_{sing}(Gal_{F_p}, H^{2n-1}_{\text{ét}}(Sh_{U(V')} \times Sh_{U(W')}, \bar F, \mathbb{F}_\ell(n))_{m_{\bar\rho}})$$

$$\| \qquad \text{uses Rapoport–Zink spectral seq.}$$

const func.

$$H^0(Sh_{U(V)}, \mathbb{F}_\ell) \longrightarrow H^0(Sh_{U(V)} \times Sh_{U(W)}, \mathbb{F}_\ell)_{m_{\bar\rho}}$$

sing part of $loc_p(Z_p)$

const func $\longmapsto$ $\displaystyle\int_{\Delta(Sh_{U(V)})} \phi_{U(V)} \otimes \phi_{U(W)}$ for $\phi_{U(V)}, \phi_{U(W)}$ in $\pi_n \otimes \pi_{n+1}, 6$

$$\underline{\underline{GGP}} \sqrt{(*)\, L^{alg}(\pi_n \times \pi_{n+1}, \text{center})}$$