Higher Rankin-Selberg integrals over function fields

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§1. Rankin-Selberg integrals

§2. Higher Gross-Zagier formula

§3. Main result

C curve $/\mathbb{F}_q$

§1. $G = GL_n \times GL_{n+1}$ $\qquad f = f_n \otimes f_{n+1} \in \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))_{cusp}$

$\cup \qquad (g, \begin{pmatrix} g & \\ & 1 \end{pmatrix})$

$H = GL_n \qquad \hat{\uparrow}\, g$

Hecke eigenform

$\sigma_f = \sigma_n \otimes \sigma_{n+1}$

rk n $\qquad$ rk $n+1$

irred. local system on C

$$\sigma_f \rightsquigarrow L(s, \sigma_f) = \det\left(1 - q^{-s}\,\mathrm{Frob}^* \big|_{H^1(\sigma_f)}\right)$$

$$\| $$

$$\mathrm{tr}\left(\mathrm{Frob}^*, \wedge^{\bullet}\left(H^1(\sigma_f)(s)\right)\right)$$

$\mathrm{Whit} : \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)) \longrightarrow \overline{\mathbb{Q}_\ell}$

$$I(\tfrac{1}{2}, f) = \int_{\mathrm{Bun}_H(\mathbb{F}_q)} f$$

Thm (Jacquet - Piatetski-Shapiro - Shalika)

$$\frac{I(\frac{1}{2}, f)}{\mathrm{Whit}(f)} \overset{(\text{const})}{=} L(\tfrac{1}{2}, \sigma_f)$$

$\tilde{C} \xrightarrow{\nu} C$ unramified 2 fold cover

A variant for $n=1$. $\qquad G = PGL_2$

$\cup$

$H = (\mathrm{Res}_{\tilde{C}/C}\, \mathbb{G}_m)/\mathbb{G}_m \qquad J(f) = \int_{\mathrm{Bun}_H(\mathbb{F}_q)} f$

<u>Thm</u> (Waldspurger)

$$\left(\frac{J(f)}{Whit(f)}\right)^2 = \text{const.}\ L\left(\tfrac{1}{2}, \nu^* \sigma_f\right)$$

$$\frac{J(f)^2}{\langle f, f\rangle} = \frac{1}{2}\cdot q^{g-1}\ \frac{\tilde{L}\left(\frac{1}{2}, \nu^* \sigma_f\right)}{\tilde{L}\left(1, Ad\,\sigma_f\right)}$$

§2. classical $\rightsquigarrow$ arithmetic

$$Bun_G(\mathbb{F}_q) \underset{r=0}{\overset{r\geq 0}{\rightleftarrows}} Sht_{G,\underline{\mu}}/C^r \quad , \underline{\mu} = (\mu_1, \dots, \mu_r) \in X_*(T)^r$$

$$= \left\{ \left(\underset{\underset{Frob^*\mathcal{E}_r}{\wr|}}{\mathcal{E}_0} \overset{\mu_1}{\underset{c_1}{\dashrightarrow}} \mathcal{E}_1 \dashrightarrow \cdots \overset{\mu_r}{\underset{c_r}{\dashrightarrow}} \mathcal{E}_r \right) : \begin{array}{l}\mathcal{E}_i : G\text{-bundles}/C \\ c_i \in C\end{array} \right\}$$

$G = PGL_2$

$\cup$

$H = (Res_{\tilde{C}/C}\, \mathbb{G}_m)/\mathbb{G}_m$

take $\mu = \begin{pmatrix}1 & \\ & 0\end{pmatrix}$, $\underline{\mu} = (\mu, \dots, \mu)$

$$\begin{array}{ccccc} Bun_H(\mathbb{F}_q) & & \overset{\dim r}{\searrow}\ Sht_{H,\underline{\mu}}/\tilde{C}^r & = \left\{ Frob^*\mathcal{L}_r \simeq \mathcal{L}_0 \dashrightarrow \cdots \dashrightarrow \mathcal{L}_r \right\}/Pic(C)(\mathbb{F}_q) & \mathcal{L}_i \in Pic(\tilde{C}) \\ \downarrow & \rightsquigarrow & \downarrow \pi_r & \downarrow & \\ Bun_G(\mathbb{F}_q) & & \dim 2r \rightarrow Sht_{G,\underline{\mu}} \underset{C^r}{\times} \tilde{C}^r & \mathcal{E}_i = \nu_* \mathcal{L}_i & \end{array}$$

$$\mathbb{1}_{Bun_H(\mathbb{F}_q)} \in Fun_c(Bun_G(\mathbb{F}_q)) \rightsquigarrow [Z_{\underline{\mu}}] := \pi_{\underline{\mu},!}[Sht_{H,\underline{\mu}}] \in H_c^{mid}(Sht_{G,\underline{\mu}} \underset{C^r}{\times} \tilde{C}^r)$$

Heegner–Drinfeld cycle

<u>Thm</u> (Yun–Zhang)

$$\langle [Z_{\underline{\mu}}]_{\pi_f}, [Z_{\underline{\mu}}]_{\pi_f} \rangle = q^{\dim Bun_H} \frac{1}{2(\log q)^r} \frac{\tilde{L}^{(r)}\left(\frac{1}{2}, \nu^*\sigma_f\right)}{\tilde{L}\left(1, Ad\,\sigma_f\right)}$$

§3. $G = GL_n \times GL_{n+1}$ $\qquad \sigma = \sigma_n \otimes \sigma_{n+1}$

$\cup$ $\qquad\qquad \uparrow \quad \nearrow$

$H = GL_n$ $\qquad$ local sys. on C

irr. $/ C_{\overline{\mathbb{F}}_q}$

$$\mu_+ = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad \mu_- = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 & \\ & & & -1 \end{pmatrix} \in X_*(T_H) \subset X_*(T_G)$$

$\underline{\varepsilon} \in \{\pm 1\}^r \rightsquigarrow \mu_{\underline{\varepsilon}} = (\mu_{\varepsilon_1}, \dots, \mu_{\varepsilon_r})$

$$\pi^d_{\underline{\varepsilon}} : \underset{\dim nr}{Sht^d_{H, \mu_{\underline{\varepsilon}}}} \longrightarrow \underset{\dim 2nr}{Sht^d_{G, \mu_{\underline{\varepsilon}}}}$$

$d \leftarrow$ degree

$\longleftarrow \neq \phi \iff \sum \varepsilon_i = 0$

$$\rightsquigarrow [Z^d_{\mu_{\underline{\varepsilon}}}] = \pi_{\underline{\varepsilon}, !}[Sht^d_{H, \mu_{\underline{\varepsilon}}}] \in H^{BM}_{mid}(Sht^d_{G, \mu_{\underline{\varepsilon}}})$$

$$\cong$$

$$H^{mid}_c(Sht^d_{G, \mu_{\underline{\varepsilon}}})^*$$

$$\langle , \rangle^\circ : H^{mid}_c(Sht^\circ_{G, \mu_{\underline{\varepsilon}}})^{\otimes 2} \longrightarrow \overline{\mathbb{Q}}_\ell$$

$M = H^1(\sigma) \oplus H^1(\sigma^*)$

e.g. $r=2$ $\quad M^{\otimes 2}_{\Sigma=0} = H^1(\sigma) \otimes H^1(\sigma^*) \oplus H^1(\sigma^*) \otimes H^1(\sigma) \subset M^{\otimes 2}$

<u>Thm.</u> (in progress) $\exists$ nat'l map

$$M^{\otimes r}_{\Sigma=0} \xrightarrow{d_\sigma} \prod_{d \in \mathbb{Z}} \bigoplus_{\underline{\varepsilon} \in \{\pm 1\}^r_{\Sigma=0}} H^{mid}_c(Sht^d_{G, \mu_{\underline{\varepsilon}}})$$

$$\text{let } Z_\sigma := \sum_{\substack{d \in \mathbb{Z} \\ \underline{\varepsilon} \in \{\pm 1\}, \Sigma=0}} [Z^d_{\mu_{\underline{\varepsilon}}}] \Big|_{M^{\otimes r}_{\Sigma=0}, d_\sigma}$$

finite sum.

$$\langle , \rangle^\circ_\sigma = \langle , \rangle^\circ \Big|_{M^{\otimes r}_{\Sigma=0} \otimes M^{\otimes r}_{\Sigma=0}, d_\sigma \otimes d_{\sigma^*}} \quad \text{is non-degenerate}$$

$$\rightsquigarrow \langle , \rangle^{\circ, *}_\sigma : ((M^*)^{\otimes r})^{\otimes 2} \longrightarrow \overline{\mathbb{Q}}_\ell$$

$$\langle Z_{r,\sigma}, Z_{r,\sigma} \rangle_{\sigma}^{0,*} = q^{\dim \mathrm{Bun}_H} \frac{1}{(\log q)^{r+2}} \frac{\tilde{L}^{(r)}\left(\frac{1}{2}, \sigma \otimes \sigma^*\right)}{\operatorname{Res}_{s=1} \tilde{L}(s, \mathrm{Ad}\,\sigma_n) \cdot \operatorname{Res}_{s=1} \tilde{L}(s, \mathrm{Ad}\,\sigma_{n+1})}$$

Rmk (conj). $\alpha_\sigma\left(M^{\otimes r}_{\Sigma=0}\right) = \pi_\sigma$ - isotypic part of $\prod_{d \in \mathbb{Z}} \bigoplus_{\Sigma} H_c^*(\mathrm{Sht}_{G,\mu})$

Ingredients in the proof

GLC for GL_n (Frenkel - Gaitsgory - Vilonen)

Categorical trace (AGKRRV)

$\Rightarrow \alpha_\sigma$

Geometric relative Langlands (BZSV) $\to$ global $\to$ Kolyvagin system (Yun-Zhang)

$\to$ local $\to$ Braverman- Finkelberg- Ginzburg - Travkin