Classical and geometric Langlands over function fields

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Lecture 1. X smooth proper curve $/k$

Bun_n - the moduli space of rank n alg. vec. bdles on X

$\mathrm{Bun}_n(k)$ = vec. bdles on X of rank n

$\mathrm{Hom}(S, \mathrm{Bun}_n)$ = vec. bdle of rank n on $X \times S$

$\uparrow$

regarded as a groupoid

Lemma: Bun_n is an algebraic stack, locally of the form

Y/G.
- Y - scheme,
- G - alg. grp

$x \in X$

$$\begin{tikzcd} & H_x \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

$H_x = (M, M', M \overset{\alpha}{\hookrightarrow} M')$

- M, M' are rank n vec. bdles on X
- M'/M is a skyscraper at x

$\mathrm{Hom}(S, H_x) = (\mathcal{M}, \mathcal{M}', \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$

- $\mathcal{M}, \mathcal{M}'$ are vec. bdles on $X \times S$
- $\mathcal{M}'/\mathcal{M}$ is supported on $\{x\} \times S$, and is a line bundle on it.

$\overleftarrow{h}(\mathcal{M}, \mathcal{M}', \alpha) = \mathcal{M}$

$\overrightarrow{h}(\mathcal{M}, \mathcal{M}', \alpha) = \mathcal{M}'$

Fix $\mathcal{M}$ and let's describe $\overleftarrow{h}^{-1}(\mathcal{M})$.

<u>Lemma</u>. $\overleftarrow{h}^{-1}(\mathcal{M}) \simeq \mathbb{P}(\mathcal{M}_x)$

Similarly, $\overrightarrow{h}^{-1}(\mathcal{M}') \simeq \mathbb{P}((\mathcal{M}'_x)^*)$

Our group today is GL_n.

$k = \mathbb{F}_q$

$\mathrm{Bun}_n(\mathbb{F}_q) \simeq GL_n(K) \backslash GL_n(\mathbb{A}) / GL_n(\mathbb{O})$

$\mathbb{A}$ ring of adeles of the global field $K = \mathbb{F}(X)$

$\cup$

$\mathbb{O}$ integral adeles

$\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ - the space of automorphic functions

$x \in |X|$

$T_x \in \mathrm{End}(\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q)))$

$$\begin{tikzcd}
 & H_x(\mathbb{F}_q) \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_n(\mathbb{F}_q) & & \mathrm{Bun}_n(\mathbb{F}_q)
\end{tikzcd}$$

$$T_x(f) = \overrightarrow{h}_!(\overleftarrow{h}^*(f))$$

$$(T_x(f))(\mathcal{M}') = \sum_{\substack{(\mathcal{M}, \mathcal{M}', \alpha) \\ \uparrow \\ H_x(\mathbb{F}_q)}} f(\mathcal{M})$$

<u>Lemma</u>. T_x pairwise commute.

Seeking common eigenvectors, i.e. $f \in \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ s.t.

$$T_x(f) = \lambda_x \cdot f$$

$$|X| \xrightarrow{\lambda} \overline{\mathbb{Q}}_\ell$$

What are the λ's that can appear?

Langlands: $\pi_1^{\text{ét}}(X) \xrightarrow{\sigma} GL_n(\overline{\mathbb{Q}}_\ell)$

Set $\lambda_x = \mathrm{Tr}(\sigma(\mathrm{Frob}_x))$, $\mathrm{Frob}_x \in \pi_1^{\text{ét}}(X)/\text{conjugation}$

__Langlands' conjecture__. The system of eigenvalues of the action of $\{T_x\}$ on $\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ are all of the above form.

__Thm__ (L. Lafforgue, 2000) Conj. is true.

$(\underset{\text{vec. sp.}}{V}, \; T_x \in \mathrm{End}(V), \;\; x \in |X|)$

1) A comm. alg.

2) $A \curvearrowright V$

3) $\forall\, x \in |X|$, $\exists\, a_x \in A$ s.t. its action on V equals T_x

$\mathrm{Spec}(A)$

And we'll say that V "spectrally decomposes over $S = \mathrm{Spec}\, A$".

If $\{a_x\}$ generate A, and $v \in V$ is s.t. $T_x \cdot v = \lambda_x \cdot v$,

$\{\lambda_x\} \in S$.

Can think of V as a q.coh. sheaf over S, and eigenvectors correspond to subsheaves that are torsion.

__Question__: Does there exist an affine scheme over $\bar{\mathbb{Q}}_\ell$ $(\mathrm{LocSys}_n^{\text{arith}})$ s.t.

- Its $\bar{\mathbb{Q}}_\ell$-points are n-dim'l Galois rep'ns up to semisimplification
- $\Gamma(\mathrm{LocSys}_n^{\text{arith}}, \mathcal{O}) \curvearrowright \mathrm{Func}_c(\mathrm{Bun}_n)$

Recall

• $\forall x \longrightarrow \underset{\uparrow}{a_x}$ whose action equals T_x

$\Gamma(\mathrm{LocSys}_n^{\mathrm{arith}}, \mathcal{O})$

σ_1 and σ_2 that have isom. semisimplifications

$$\mathrm{Tr}(\sigma_1(\mathrm{Frob}_x)) = \mathrm{Tr}(\sigma_2(\mathrm{Frob}_x))$$

Thm (P. Scholze, X.-W. Zhu, AGKRRV) (a) $\exists$ an algebraic stack / $\bar{\mathbb{Q}}_\ell$ whose $\bar{\mathbb{Q}}_\ell$-pts are exactly n-dim'l Galois rep'ns. $\mathrm{LocSys}_n^{\mathrm{arith}}$

$$\mathrm{LocSys}_n^{\mathrm{arith,\,coarse}} = \mathrm{Spec}\,\Gamma(\mathrm{LocSys}_n^{\mathrm{arith}}, \mathcal{O})$$

(b) $\bar{\mathbb{Q}}_\ell$-pts of $\mathrm{LocSys}_n^{\mathrm{arith,\,coarse}}$ are isom. classes of n-dim'l Galois rep'ns up to semisimplification.

Thm (V. Lafforgue, X. Gong, AGKRRV) • $\exists\ \Gamma(\mathrm{LocSys}_n^{\mathrm{arith,\,coarse}}, \mathcal{O}) \curvearrowright \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$

• $\forall x$, $\exists\ a_x \in \Gamma(\mathrm{LocSys}_n^{\mathrm{arith,\,coarse}}, \mathcal{O})$ s.t. a_x acts as T_x.

Upshot: $\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ spectrally decomposes over $\mathrm{LocSys}_n^{\mathrm{arith,\,coarse}}$.

Question: Can we describe $\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ explicitly in terms of $\mathrm{LocSys}_n^{\mathrm{arith}}$?

Lecture 2. X curve $/k$

Bun_n - moduli stack of rank n vec. bdles on X

$x \in |X|$

$$\begin{tikzcd} & H_x \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

Classical story: $k = \mathbb{F}_q$

$$\begin{tikzcd} & H_x(\mathbb{F}_q) \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_n(\mathbb{F}_q) & & \mathrm{Bun}_n(\mathbb{F}_q) \end{tikzcd}$$

$$T_x : \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q)) \longrightarrow \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$$

$$T_x(f) = \overleftarrow{h}_! \, \overrightarrow{h}^*(f)$$

Thm. $\exists$ an algebraic stack over $\bar{\mathbb{Q}}_\ell$ $\mathrm{LocSys}_n^{\mathrm{arith}}$ whose $\bar{\mathbb{Q}}_\ell$-pts are

$$\sigma : \pi_1^{\text{ét}}(X) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell) \;/ \mathrm{Ad}(GL_n(\bar{\mathbb{Q}}_\ell))$$

Thm $A = \Gamma(\mathrm{LocSys}_n^{\mathrm{arith}}, \mathcal{O})$, $A \curvearrowright \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$

Moreover, $\forall\, x \in |X| \rightsquigarrow a_x \in A$, $a_x(\sigma) = \mathrm{Tr}(\sigma(\mathrm{Frob}_x))$

a_x acts as T_x.

NB. For GL_n, the elements as generate A.

Cor. If $f \in \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ is a joint eigenvector, i.e.

$$T_x(f) = \lambda_x \cdot f, \quad \text{then } \exists\, \sigma : \pi_1^{\acute{e}t}(X) \to GL_n(\overline{\mathbb{Q}_\ell})$$

s.t. $\lambda_x = \mathrm{Tr}(\sigma(\mathrm{Frob}_x))$.

$\mathrm{Spec}(A) =: \mathrm{LocSys}_n^{\mathrm{arith,\,coarse}}$

Conclusion: $\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ admits a spectral decomposition over $\mathrm{LocSys}_n^{\mathrm{arith,\,coarse}}$

Question: can we describe $\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$ explicitly in terms of $\mathrm{LocSys}_n^{\mathrm{arith}}$?

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k - algebraically closed

$\mathcal{Y}$ algebraic stack over k $\rightsquigarrow \mathrm{Shv}(\mathcal{Y})$

Examples.

- $k = \mathbb{C}$, $\mathrm{Shv} := \mathrm{Shv}^{\mathrm{Betti}}$

  classical sheaves of e- vec. sps where e is an arbitrary alg. closed field of char. 0

  - $\mathrm{Shv}^{\mathrm{Betti}}(\mathcal{Y}) \supset \mathrm{Shv}^{\mathrm{Betti,\,const}}(\mathcal{Y})$

- k - arbitrary alg. closed of char 0, $\mathrm{Shv}(\mathcal{Y}) := \mathrm{Dmod}(\mathcal{Y})$

In this case, the field of coefficients $e = k$.

$$\mathrm{Dmod}(\mathcal{Y}) \supset \mathrm{Dmod}^{hol}(\mathcal{Y}) \supset \mathrm{Dmod}^{RS}(\mathcal{Y})$$

<u>NB</u>. If $k = \mathbb{C}$, $e = \mathbb{C}$, Riemann - Hilbert:

$$\mathrm{Shv}^{\text{Betti, const}}(\mathcal{Y}) \simeq \mathrm{Dmod}^{RS}(\mathcal{Y}).$$

- k - arbitrary, $\ell \neq \mathrm{char}(k)$

$$\mathrm{Shv}^{\text{ét}}(\mathcal{Y}) \supset \mathrm{Shv}^{\text{ét, const}}(\mathcal{Y})$$

$\boxed{\text{All categories are derived}}$

$$\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$$

$f^! : \mathrm{Shv}(\mathcal{Y}_2) \longrightarrow \mathrm{Shv}(\mathcal{Y}_1)$

$f_* : \mathrm{Shv}(\mathcal{Y}_1) \longrightarrow \mathrm{Shv}(\mathcal{Y}_2)$

$(f_!, f^!)$ - adjoint pair

(f^*, f_*) - adjoint pair

$\mathcal{F}_1 \in \mathrm{Shv}(\mathcal{Y}_1)$, $\mathcal{F}_2 \in \mathrm{Shv}(\mathcal{Y}_2)$,

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

$\mathrm{Shv}(\mathrm{Bun}_n)$ - category of automorphic sheaves

$$\begin{tikzcd}
 & H_x \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_n & & \mathrm{Bun}_n
\end{tikzcd}$$

$$T_x(\mathcal{F}) = \overrightarrow{h}_* \circ \overleftarrow{h}^!(\mathcal{F})[-(n-1)]$$

Observe: T_x and T_y commute for $x \neq y$.

$$\begin{tikzcd}
 & H \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "s"] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_n & X & \mathrm{Bun}_n
\end{tikzcd}$$

$$T : \mathrm{Shv}(\mathrm{Bun}_n) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_n \times X)$$

$$T(\mathcal{F}) = (\overrightarrow{h} \times s)_* \circ \overleftarrow{h}^!(\mathcal{F})[-(n-1)]$$

<u>Def.</u> An $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_n)$ is a ptwise Hecke eigensheaf if $\forall x$,

$\exists\, \Lambda_x \in \mathrm{Vect}_e$ s.t. $T_x(\mathcal{F}) \overset{\alpha_x}{\simeq} \mathcal{F} \otimes \Lambda_x$ s.t. the following diagram commutes

$\forall x \neq y$:

$$\begin{tikzcd}
T_x \circ T_y(\mathcal{F}) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & T_y \circ T_x(\mathcal{F}) \arrow[d, phantom, "\wr"] \\
T_x(\mathcal{F} \otimes \Lambda_y) \arrow[d, phantom, "\wr"] & T_y(\mathcal{F} \otimes \Lambda_x) \arrow[d, phantom, "\wr"] \\
T_x(\mathcal{F}) \otimes \Lambda_y \arrow[d, phantom, "\wr"] & T_y(\mathcal{F}) \otimes \Lambda_x \arrow[d, phantom, "\wr"] \\
\mathcal{F} \otimes \Lambda_x \otimes \Lambda_y \arrow[r, phantom, "\simeq"] & \mathcal{F} \otimes \Lambda_y \otimes \Lambda_x
\end{tikzcd}$$

Def. $\mathcal{F}$ is a Hecke eigensheaf if

$$T(\mathcal{F}) \overset{\alpha}{\simeq} \mathcal{F} \boxtimes \Lambda$$

$\Lambda \in \mathrm{Shv}(X)$.

Exercise: Formulate the commutativity requirement on this isom.

Theorem: If $\mathcal{F}$ is a non-zero eigensheaf, then this $\Lambda \in \mathrm{Shv}(X)$ is a rank n $\boxed{\text{local system}}$ on X.

local system = locally constant sheaf = lisse = " $\pi_1(X) \to GL_n(e)$ "

Λ is " the same " as σ.

Before: $\Gamma(\mathrm{LocSys}_n^{\mathrm{arith}}, \mathcal{O}) \sim \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q))$

Now: $\mathrm{QCoh}(\mathrm{LocSys}_n) \overset{\text{Want}}{\sim} \mathrm{Shv}(\mathrm{Bun}_n)$

alg. stack over e

Before: $a_x \in \Gamma(\mathrm{LocSys}_n^{\mathrm{arith}}, \mathcal{O})$

$a_x(\sigma) = \mathrm{Tr}(\sigma(\mathrm{Frob}_x))$, a_x acts as T_x.

Now: $\mathcal{A}_x \in \mathrm{QCoh}(\mathrm{LocSys}_n)$, $\mathcal{A}_x|_\Lambda = \Lambda_x$, $\mathcal{A}_x$ acts as T_x

In this case, we'll say that $\mathrm{Shv}(\mathrm{Bun}_n)$ admits a spectral decomp.

over LocSys_n.

$$T : \mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n \times X)$$

Instead of $\mathcal{A}_x \in \mathrm{QCoh}(\mathrm{LocSys}_n)$, we want

$$\mathcal{A} \in \mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}(X)$$

B_1, B_2 - assoc. algs, $(B_1\text{-mod}) \otimes (B_2\text{-mod}) \simeq (B_1 \underset{e}{\otimes} B_2)\text{-mod}$

DG categories over e.

Want:

$T : \mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n \times X)$ acts as $\mathcal{A}$.

$$\mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n)$$

$$\mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}(X) \otimes \mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n) \otimes \mathrm{Shv}(X)$$

$$\mathcal{A} : \underbrace{\mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n) \otimes \mathrm{Shv}(X) \xrightarrow{\boxtimes} \mathrm{Shv}(\mathrm{Bun}_n \times X)}_{T}$$

Lecture 3.

- D-modules: $\mathrm{Dmod}(\mathcal{Y}) \supset \mathrm{Dmod}^{hol}(\mathcal{Y}) \supset \mathrm{Dmod}^{RS}(\mathcal{Y})$
- sheaves in classical topology $\mathrm{Shv}_{Betti}(\mathcal{Y}) \supset \mathrm{Shv}^{const.}_{Betti}(\mathcal{Y})$
- ℓ-adic sheaves $\mathrm{Shv}^{const.}_{\acute{e}t}(\mathcal{Y})$

$$\begin{tikzcd} & H \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}\times s"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times X \end{tikzcd}$$

$$T(\mathcal{F}) = (\overrightarrow{h}\times s)_* \cdot \overleftarrow{h}^!(\mathcal{F})[-(n-1)]$$

- LocSys_n - stack over the field of coefficients e whose e-points are locally constant sheaves of rank n on X.

- $\mathcal{A} \in \mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}(X)$

$\forall\ \Lambda \in \mathrm{LocSys}_n, \qquad \mathcal{A}_\Lambda \in \mathrm{Shv}(X)$
$\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad \Lambda$

- $\mathrm{QCoh}(\mathrm{LocSys}_n) \curvearrowright \mathrm{Shv}(\mathrm{Bun}_n)$ s.t.

$$\mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}(\mathrm{Bun}_n) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_n)$$

$$\rightsquigarrow \underbrace{\mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}(X)}_{\mathcal{A}} \otimes \mathrm{Shv}(\mathrm{Bun}_n) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_n) \otimes \mathrm{Shv}(X)$$

$$\begin{tikzcd} Shv(Bun_n) \arrow[r, "A"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & Shv(Bun_n) \otimes Shv(X) \arrow[d, "\boxtimes"] \\ Shv(Bun_n) \arrow[r, "T"'] & Shv(Bun_n \times X) \end{tikzcd}$$

Shv = Dmod

Dmod on X equipped w/ an action of R

//

$Hom(Spec\, R, LocSys_n) :=$ vec. bdles of rk n on $Spec(R) \times X$ equipped w/ a connection along X

$Hom(Spec\, R, Bun_n) =$ vec. bdles of rk n on $Spec\, R \times X$

$\check{G} = GL_n$

$Hom(Spec\, R, LocSys_{\check{G}}) =$ symmetric monoidal functors from $Rep(\check{G}) \to \{$Dmods on X w/ action of $R\}$

$\qquad \cong$

$\qquad R\text{-mod} \otimes Dmod(X)$

$\qquad =$

$\qquad QCoh(Spec\, R) \otimes Dmod(X)$

$Hom(Spec\, R, Bun_n) =$ symmetric monoidal functors $Rep(G) \to QCoh(Spec\, R \times X)$

Tannaka: Rank n bdles on $Y \Leftrightarrow Rep(GL_n) \to QCoh(Y)$

right t-exact

$S \to LocSys_n \rightsquigarrow A_S \in QCoh(S) \otimes Dmod(X)$

$\rightsquigarrow A \in QCoh(LocSys_n) \otimes Dmod(X)$

$\mathcal{Y}$ alg. stack

To specify $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$

$$\Updownarrow$$

To specify $\mathcal{F}_S \in \mathrm{QCoh}(S)$

$\forall S \to \mathcal{Y}$ s.t.

$$\begin{tikzcd} S_1 \arrow[r] & \mathcal{Y} \\ S_2 \arrow[u, "f"] \arrow[ur] & \end{tikzcd}$$

$$f^*(\mathcal{F}_{S_1}) \simeq \mathcal{F}_{S_2}$$

<u>Thm</u> (Drinfeld – G.) $\exists!$ $\mathrm{QCoh}(\mathrm{LocSys}_n) \curvearrowright \mathrm{Dmod}(\mathrm{Bun}_n)$

s.t. $\mathcal{A}$ acts as T.

Can we describe $\mathrm{Dmod}(\mathrm{Bun}_n)$ completely in terms of LocSys_n?

<u>Conj.</u> (GLC) $\mathrm{Dmod}(\mathrm{Bun}_n) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_n^{dR})$

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}) & = & \mathrm{Ind}(\mathrm{Perf}(\mathcal{Y})) \\ \cap & & \cap \\ \mathrm{IndCoh}(\mathcal{Y}) & := & \mathrm{Ind}(\mathrm{Coh}(\mathcal{Y})) \end{array}$$

$\mathrm{Shv} = \mathrm{Shv}_{\mathrm{Betti}}$

$\mathrm{LocSys}_n^{\mathrm{Betti}}$ $\quad \mathrm{Hom}(\overset{\mathrm{Spec}\,R}{\overset{\shortparallel}{S}}, \mathrm{LocSys}_n^{\mathrm{Betti}})$ = symmetric monoidal functors

$\mathrm{Rep}(GL_n) \to$ R-modules in $\mathrm{Shv}_{\mathrm{Betti}}(X)$

$\simeq \mathrm{QCoh}(S) \otimes \mathrm{Shv}_{\mathrm{Betti}}(X) \supset \mathrm{QCoh}(S) \otimes \mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{loc.const}}(X)$

$S \to \mathrm{LocSys}_n^{\mathrm{Betti}}$

$\rightsquigarrow \ \mathcal{A}_S \in \mathrm{QCoh}(S) \otimes \mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{loc.\,const}}(X)$

$\rightsquigarrow \ \mathcal{A} \in \mathrm{QCoh}(\mathrm{LocSys}_n) \otimes \mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{loc.\,const}}(X)$

$$\begin{tikzcd}
\mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n) \arrow[r, "\mathcal{A}"] \arrow[d, equal] & \mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n) \otimes \mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{loc\text{-}const}}(X) \arrow[d] \\
\mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n) \arrow[r, "T"'] & \mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n \times X)
\end{tikzcd}$$

Impossible to have an action of $\mathrm{QCoh}(\mathrm{LocSys}_n^{\mathrm{Betti}})$ on $\mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n)$ w/ this property.

Can we describe $\mathcal{F} \in \mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n)$ s.t. $T(\mathcal{F}) \in \mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n) \otimes \mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{loc.\,const}}(X)$.

<u>Thm</u> (Nadler – Yun, AGKRRV) This happens iff $\mathcal{F}$ has nilp. singular support.

$$\mathcal{Y} \rightsquigarrow T^*\mathcal{Y} \supset \mathcal{N}$$

$\mathcal{Y} = \mathrm{Bun}_n$

$T^*\mathcal{Y} = \mathrm{Higgs}$

$\mathrm{Shv}^{\mathrm{Betti}}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}^{\mathrm{Betti}}(\mathcal{Y})$

points of Bun_n are vec. bdles M of rk n.

pts of Higgs $M \xrightarrow{A} M \otimes \omega_X$

$\mathrm{Nilp} \subset \mathrm{Higgs}$

↳ A is nilpotent.

$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \subset \mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_n)$

<u>Thm</u> (Nadler - Yun, AGKRRV) $\exists!$ action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_n)$ on

$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_n)$ s.t. $\mathcal{A}$ acts as T.

<u>Conj</u> (GLC) Ben-Zvi - Nadler.

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{Betti}}_n)$$

$\mathrm{Shv} := \mathrm{Shv}^{\mathrm{const}}_{\text{ét}}$

$\mathrm{Dmod}^{\mathrm{hol}}$

$\mathrm{Shv}^{\mathrm{const.}}_{\mathrm{Betti}}$

$n = 1$

$\mathrm{Bun}_n = \mathrm{Pic}$

$$\mathrm{Dmod}(\mathrm{Pic}) \overset{\text{Fourier-Mukai}}{=\!=} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_1)$$

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{loc.const}}(\mathrm{Pic}) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_1)$$

$\mathrm{LocSys}^{\mathrm{dR}}_n \rightsquigarrow$ spectral decomp. thm

$\mathrm{LocSys}^{\mathrm{Betti}}_n \rightsquigarrow$ Spectral decomp. thm.

$\boxed{\mathrm{LocSys}^{\mathrm{restr.}}_n \rightsquigarrow \text{spectral decomp. thm.}} \longleftrightarrow$ classical theory.

Lecture 4. $\mathrm{LocSys}^{\mathrm{Betti}}_n$

$$\mathrm{Hom}(\mathrm{Spec}\, R, \mathrm{LocSys}^{\mathrm{Betti}}_n) = \mathrm{Rep}(GL_n) \longrightarrow \mathrm{Shv}_{\mathrm{loc.const}}(X, R)$$

$$\simeq R\text{-mod}(\mathrm{Shv}_{\mathrm{loc.const}}(X))$$

$$\simeq R\text{-mod} \otimes \mathrm{Shv}_{\mathrm{loc.const}}(X)$$

$$F(\text{standard}) = \mathcal{A}_R \in R\text{-mod} \otimes \mathrm{Shv}_{\mathrm{loc.const}}(X)$$

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

$$\|$$

$$\{\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G) \text{ s.t. } T(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const}}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X)\}$$

Thm. $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_n) \sim \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)$ so that

$$\begin{tikzcd}
\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[r, "A"] \arrow[d, equal] & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const}}(X) \arrow[d, equal] \\
\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[r, "T"] & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const}}(X)
\end{tikzcd}$$

Conj (GLC Betti Version)

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{Betti}}_n)$$

Shv will be $\begin{cases} \mathrm{Dmod}^{hol} & e = k \\ \mathrm{Shv}^{Betti}_{const} & e = \mathbb{C} \\ \mathrm{Shv}^{\acute{e}t}_{const} & e = \overline{\mathbb{Q}}_\ell \end{cases}$

$\mathrm{LocSys}^{restr.}_n$

$$\mathrm{Hom}(\mathrm{Spec}\, R, \mathrm{LocSys}^{restr}_n) = \mathrm{Rep}(GL_n) \xrightarrow{F} \mathrm{Spec}(R) \otimes \mathrm{Shv}_{loc.\,const}(X).$$

$$\begin{array}{l} \mathcal{A}_R \in \mathrm{Spec}(R) \otimes \mathrm{Shv}_{loc\,const}(X) \\ \parallel \\ F(\mathrm{standard}) \end{array}$$

$$\{ \mathcal{A}_R : \mathrm{Spec}(R) \to \mathrm{LocSys}^{restr}_n \}$$
$$\Updownarrow$$
$$\mathcal{A} \in \mathrm{QCoh}(\mathrm{LocSys}^{restr}_n) \otimes \mathrm{Shv}_{loc.\,const}(X)$$

$$\mathrm{LocSys}^{restr.}_n \longrightarrow \mathrm{LocSys}^{Betti}_n$$

can have "infinite dim'l local systems"

$$\mathrm{Shv}^{Betti,\,constr.}_{loc.\,const} \subset \mathrm{Shv}^{Betti}_{loc.\,const}$$

How does $\mathrm{LocSys}^{restr}_n$ look like?

It equals the disjoint union of formal completions of closed subsets in $\mathrm{LocSys}^{Betti}_n$ consisting of local system w/ a given semisimplification

$$\mathrm{LocSys}_n^{\text{restr.}} = \bigsqcup_{\sigma} \mathrm{LocSys}_{n,\sigma}^{\text{restr.}}$$

$$I \subset R \longrightarrow R'$$

$$\begin{tikzcd}
R\text{-mod} \otimes \mathrm{Shv}^{\text{constr.}}_{\text{loc.const}}(X) \arrow[d, "\Gamma"] \arrow[r, phantom, "\subset"] & R\text{-mod} \otimes \mathrm{Shv}_{\text{loc-const}}(X) \arrow[d] \\
R'\text{-mod} \otimes \mathrm{Shv}^{\text{constr.}}_{\text{loc.const}}(X) \arrow[r, phantom, "\subset"] & R'\text{-mod} \otimes \mathrm{Shv}_{\text{loc.const}}(X)
\end{tikzcd}$$

Thm

$$\{ \mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G) \text{ s.t. } T(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\text{loc.const}}(X) \}$$

$$= \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

Thm $\exists!$ $\mathrm{QCoh}(\mathrm{LocSys}_n^{\text{restr.}}) \curvearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$

$$\begin{tikzcd}
\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \arrow[r, "A"] \arrow[d, equal] & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \otimes \mathrm{Shv}_{\text{loc.const}}(X) \arrow[d, equal] \\
\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \arrow[r, "T"] & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \otimes \mathrm{Shv}_{\text{loc.const}}(X)
\end{tikzcd}$$

Cor. $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \simeq \bigoplus_{\sigma} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)_\sigma$

runs over semisimple local systems

Thm (GLC, nilpotent version)

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_n^{\mathrm{restr.}})$$

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}^{\mathrm{Betti,constr.}}(\mathrm{Bun}_n) & \simeq & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_n^{\mathrm{restr.}}) \\ \cap & & \cap \\ \mathrm{Shv}_{\mathrm{Nilp}}^{\mathrm{Betti}}(\mathrm{Bun}_n) & \simeq & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_n^{\mathrm{Betti}}) \end{array}$$

$k = \overline{\mathbb{F}}_q$, assume X is defined over $\mathbb{F}_q$

<u>Slogan</u>: $\mathrm{Tr}(\mathrm{Frob}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)) = ?$

$$\begin{tikzcd} e \arrow[r] \arrow[rrr, bend right, "\mathrm{Tr}(F,V)"'] & V\otimes V^* \arrow[r, "F\otimes id"] & V\otimes V^* \arrow[r] & e \end{tikzcd}$$

$\mathcal{O}$ symmetric monoidal cat.

$V \in \mathcal{O}$ dualisable object

$$1_{\mathcal{O}} \longrightarrow V \otimes V^*$$

$$V \otimes V^* \longrightarrow 1_{\mathcal{O}}$$

$$V \xrightarrow{F} V$$

$\mathrm{Tr}(F,V) \in \mathrm{End}(1_{\mathcal{O}})$, $1_{\mathcal{O}} \longrightarrow V\otimes V^* \xrightarrow{F\otimes id} V\otimes V^* \longrightarrow 1_{\mathcal{O}}$

$\mathcal{O} = \mathrm{DGCat}_{/e}$

Fact. Any compactly generated cat. is dualisable.

$$C \xrightarrow{F} C$$

$$\begin{gathered} \mathrm{Tr}(F, C) \in \mathrm{Vect}_e \\ \| \\ HH(F, C) \end{gathered}$$

$1_{\mathcal{O}} = \mathrm{Vect}_e, \quad \underset{\mathrm{DGCat}_e}{\mathrm{End}}(\mathrm{Vect}_e, \mathrm{Vect}_e) = \mathrm{Vect}_e$

$C = \mathrm{QCoh}(Y)$

$Y \xrightarrow{\phi} Y$

$F = \phi_*$

$\mathrm{Tr}(\phi_*, \mathrm{QCoh}(Y)) \overset{\text{claim}}{=\!=} \Gamma(Y^\phi, \mathcal{O}_{Y^\phi})$

$$\mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \xrightarrow{\sim} \mathrm{QCoh}(Y \times Y) \xrightarrow{\Delta_Y^*} \mathrm{QCoh}(Y) \xrightarrow{\Gamma(Y,-)} \mathrm{Vect} \qquad \text{counit}$$

$$\begin{array}{ccccccc} e & \longmapsto & \mathcal{O}_Y & & & & \\ \mathrm{Vect} & \longrightarrow & \mathrm{QCoh}(Y) & \xrightarrow{\Delta_*} & \mathrm{QCoh}(Y \times Y) & \xleftarrow{\sim} & \mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \end{array} \qquad \text{unit}$$

$$\begin{tikzcd}
Y^\phi \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner" very near start] & & Y \arrow[r] \arrow[d, "\Delta"] & pt \\
Y \arrow[r, "\Delta"] \arrow[d] & Y \times Y \arrow[r, "\phi \times id"] & Y \times Y & \\
pt & & &
\end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[r, "F_1"] \arrow[d, "\phi"'] & C_1 \arrow[d, "\phi"] \arrow[dl, Rightarrow, "\alpha"'] \\ C_2 \arrow[r, "F_2"'] & C_2 \end{tikzcd}$$

ϕ admits a continuous right adjoint

$$\mathrm{Tr}(F_1, C_1) \xrightarrow{\alpha} \mathrm{Tr}(F_2, C_2)$$

(c, F) $\quad c \in C$ compact object , $c \xrightarrow{\alpha} F(c)$

$\rightsquigarrow$ Chern class $\quad d(c, \alpha) \in \mathrm{Tr}(F, C)$

$C = \mathrm{Shv}(Y)$, $\quad Y$ over k, $\quad C$ linear over e

$Y \xrightarrow{\phi} Y$

$\mathrm{Tr}(\phi_*, Y) \underset{\longrightarrow}{\overset{?}{=}} C^{\cdot}(Y^{\phi}, \omega_{Y^{\phi}})$

far from being an isom.

$\mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \xrightarrow{\sim} \mathrm{QCoh}(Y \times Y)$

$\mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y) \xrightarrow[\not\sim]{\boxtimes} \mathrm{Shv}(Y \times Y)$

Y over $\bar{\mathbb{F}}_q$, defined over $\mathbb{F}_q$, ϕ geometric Frobenius

$Y^{\phi} = Y(\mathbb{F}_q)$

local term

$\mathrm{Tr}(\mathrm{Frob}, \mathrm{Shv}(Y)) \xrightarrow{LT} \mathrm{Funct}(Y(\mathbb{F}_q))$

$\mathcal{F} \in \mathrm{Shv}(Y)$, $\mathcal{F} \xrightarrow{\alpha} \mathrm{Frob}_*(\mathcal{F})$

$d(\mathcal{F}, \alpha) \in \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(Y))$,

Thm. $\quad LT(d(\mathcal{F}, \alpha)) = \mathrm{Funct}(\mathcal{F}, \alpha)$

$$
\begin{tikzcd}
\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathrm{Bun}_n)) \arrow[d, "LT"] & \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)) \arrow[l] \arrow[dl, "\sim"] \\
\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q)) &
\end{tikzcd}
$$

<u>Thm</u> (AGKRRV) isom.

$\mathrm{Frob} \curvearrowright \mathrm{LocSys}_n^{\mathrm{restr.}}$

$\mathrm{LocSys}_n^{\mathrm{arith.}} := (\mathrm{LocSys}_n^{\mathrm{restr.}})^{\mathrm{Frob}}$

$$\mathrm{QCoh}(\mathrm{LocSys}_n^{\mathrm{restr.}}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \longrightarrow \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)$$

$$
\begin{tikzcd}
\mathrm{Tr}(\mathrm{Frob}, \mathrm{QCoh}(\mathrm{LocSys}_n^{\mathrm{restr.}})) \arrow[d, phantom, "\wr\!\wr"] \arrow[r, phantom, "\otimes"] & \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)) \arrow[d, phantom, "\wr\!\wr"] \arrow[r] & \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)) \arrow[d, phantom, "\wr\!\wr"] \\
\Gamma(\mathrm{LocSys}^{\mathrm{arith}}, \mathcal{O}) \arrow[r, phantom, "\otimes"] & \mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q)) \arrow[r] & \mathrm{Funct}_c(\mathrm{Bun}(\mathbb{F}_q))
\end{tikzcd}
$$

$\uparrow$

V. Lafforgue's action.

GLC: $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_n^{\mathrm{restr.}})$

$$
\begin{array}{ccc}
\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_n)) & = & \mathrm{Tr}(\mathrm{Frob}, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_n^{\mathrm{restr}})) \\
\wr\!\wr & & \| \\
\mathrm{Funct}_c(\mathrm{Bun}_n(\mathbb{F}_q)) & & ?
\end{array}
$$

$$\mathrm{Tr}\left(\mathrm{Frob},\ \mathrm{QCoh}\left(\mathrm{LocSys}_n^{\mathrm{restr.}}\right)\right)$$

$$\shortparallel$$

$$\Gamma\left(\mathrm{LocSys}_n^{\mathrm{arithm}},\ \mathcal{O}\right)$$

$$\mathrm{Tr}\left(\mathrm{Frob},\ \mathrm{IndCoh}_{\mathrm{Nilp}}\left(\mathrm{LocSys}_n^{\mathrm{restr.}}\right)\right) \simeq \Gamma\left(\mathrm{LocSys}_n^{\mathrm{arith}},\ \omega\right)$$

<u>Conj</u> $\mathrm{Funct}_c\left(\mathrm{Bun}_n(\mathbb{F}_q)\right) = \Gamma\left(\mathrm{LocSys}_n^{\mathrm{arith}},\ \omega\right)$