Non-vanishing of quantum geometric Whittaker coefficients

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$k = \mathbb{C}$, X sm proj. conn. curve $/k$

G simple, adjoint gp

$\check{G}$ Langlands dual

$\check{\Lambda}$ coweight lattice

I_G simple roots

Bun_G - stack of G-bundles on X

Notation. Let κ and $\check{\kappa}$ be $G(\check{G})$-inv. bilinear forms on $\mathfrak{g}(\check{\mathfrak{g}})$

$\kappa = c_\kappa \cdot \kappa_{Kill,\mathfrak{g}}$, $\check{\kappa} = c_{\check{\kappa}} \cdot \kappa_{Kill,\check{\mathfrak{g}}}$ s.t.

- $c_\kappa \neq -\frac{1}{2}$
- $(\kappa - \kappa_{crit})|_{\mathfrak{t}}$ and $(\check{\kappa} - \check{\kappa}_{crit})|_{\check{\mathfrak{t}} = \mathfrak{t}^*}$ are dual sym. bilinear form

$c_\kappa \in \mathbb{Q}$

$$D_\kappa(\mathrm{Bun}_G) := D_{\det_{\mathfrak{g}}^{c_\kappa}}(\mathrm{Bun}_G)$$

$\det_{\mathfrak{g}}$ - det-line bundle, fiber at $P_G \in \mathrm{Bun}_G$ is $\det(R\Gamma(X, \mathfrak{g}_{P_G}))$

Conj. (quantum GLC)

$$D_\kappa(\mathrm{Bun}_G) \overset{\mathbb{L}_\kappa}{\simeq} D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}})$$

For $K = K_{crit}$, $\check{K} = \infty$,

<u>Thm</u> (GLC) (ABCCFGLRR)
$$D_{K_{crit}}(Bun_G) \simeq QCoh\left(T^*_{def \check{g}} Bun_{\check{G}}\right)$$
$$\text{IS Thm. (S.)}$$
$$QCoh(LS_{\check{G}}(X))$$

What properties fix $\mathbb{L}_K$?

- Hecke eigenproperty : $\mathbb{L}_K$ intertwines this action w/ appropriate action on RHS.

$$D_K(Gr_G)^{L^+G} \curvearrowright D_K(Bun_G)$$

<u>Rmk</u>. $c_K = \frac{p_1}{p_2} \notin \mathbb{Z}$

$D_K(Gr_G)^{L^+G}$ is more degenerate.

For $\check{\lambda} \in \check{\Lambda}^+$, $D_K(L^+G \cdot t^{\check{\lambda}})^{L^+G} \neq 0$ iff $Kill(\check{\lambda}, -) \in p_2 \cdot \Lambda$

- Compatibility w/ Whittaker coeffs

<u>Rmk</u>. $k = \mathbb{F}_q$, $K = \underset{\text{field}}{\text{Fun}}(X)$

$$Bun_G(k) \simeq G(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$$

Coeff: $\text{Fun}^{G(K)}\left(G(\mathbb{A})/G(\mathbb{O})\right) \longrightarrow \text{Fun}^{N(\mathbb{A}),\chi}\left(G(\mathbb{A})/G(\mathbb{O})\right)$

$$f \longmapsto \int_{N(K)\backslash N(\mathbb{A})} f(n\cdot -)\, \chi^{-1}(n)\, dn$$

$$\begin{tikzcd}
 & D_\kappa(\mathrm{Gr})^{LN,\chi}_{\mathrm{Conf}} \arrow[r, phantom, "\simeq"] & \hat{\mathfrak{g}}_{-\check{\kappa}}\text{-mod}^{L^+\check{G}}_{\mathrm{Conf}} \\
 & D_\kappa(\mathrm{Bun}_G) \arrow[u, "\mathrm{Coeff}^{\mathrm{loc}}"] \arrow[r, "\mathbb{L}_\kappa"'] & D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}}) \arrow[u, "\Gamma"'] \\
D_\kappa(\mathrm{Bun}_G)^{\mathrm{cusp}} \arrow[ur, hook] \arrow[uur, bend left] & &
\end{tikzcd}$$

- Compatibility w/ CT.

<u>Whittaker coeffs (global)</u>

D : $\check{\Lambda}^+$-val. div. on X

define $\mathrm{Coeff}_D : D_\kappa(\mathrm{Bun}_G) \to \mathrm{Vect}$

<u>Def.</u> $\mathrm{Bun}_N^{\omega(-D)} = \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{\omega(-D)\}$, ω - dualizing sheaf

$$\begin{tikzcd}
 & \mathrm{Bun}_N^{\omega(-D)} \arrow[dl, "p"'] \arrow[dr, "\psi_D"] & \\
\mathrm{Bun}_G & & \mathbb{G}_a
\end{tikzcd}$$

$\omega(-D) \in \mathrm{Bun}_T$

$\forall \lambda : T \to \mathbb{G}_m$

$\lambda(\omega(-D)) = \omega(-\lambda(D))$

$\mathrm{Coeff}_D(\mathcal{F}) := C_{dR}(p^!(\mathcal{F}) \otimes \psi_D^!(\exp))\,[\cdots]$

<u>Thm 1</u> $\forall\, \overset{\overset{0}{\nparallel}}{\mathcal{F}} \in D_\kappa(\mathrm{Bun}_G)^{\mathrm{cusp}}$, $\exists\, \check{\Lambda}^+$-val. D s.t. $\mathrm{Coeff}_D(\mathcal{F}) \neq 0$.

Statement for $\mathrm{Shv}_{\kappa,\mathrm{Nilp}}(\mathrm{Bun}_G)$

Shv_κ = reg. hol. twisted D-modules

$$\underset{\mathrm{Map}(X, \mathcal{N}/G)^{\omega}_{\mathrm{red}}}{\overset{\shortparallel}{\mathrm{Nilp}}} \subset \underset{\mathrm{Map}(X, \mathfrak{g}/G)^{\omega}}{\overset{\shortparallel}{T^*\mathrm{Bun}_G}} = \{(P_G, \varphi) : P_G \text{ } G\text{-bundle}, \varphi \in \Gamma(X, \mathfrak{g}_{P_G} \otimes \omega)\}$$

Thm (Faltings, Ginzburg) $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$ is Lagrangian.

Thm 2 $\forall\, 0 \neq \mathcal{F} \in \mathrm{Shv}_{k,\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{cusp}}$, —"— .

$\Lambda \subset T^*\mathrm{Bun}_G$ closed conic Lagrangian.

Thm 3 ([NT], [FR]). Suppose $\Big(\overbrace{T^*_{\mathrm{Bun}_N^{\omega(-D)}}\mathrm{Bun}_G + d\phi_D}^{\mathrm{Kos}_D}\Big)$ intersects Λ $\pitchfork$ at a single ~~pt~~ pt $\{\lambda_D\}$,

then $\mathrm{Coeff}_D\big|_{\mathrm{Shv}_{k,\Lambda}(\mathrm{Bun}_G)}$ is t-exact and commutes w/ Verdier duality.

And $\underset{\underset{\mathrm{Shv}_{k,\Lambda}}{\uparrow}}{CC(\mathcal{F})} = \sum_{\beta \in \mathrm{Irr}(\Lambda)} c_{\beta,\mathcal{F}}(\beta)$, we have

$$\chi(\mathrm{Coeff}_{D,k}(\mathcal{F})) = c_{\beta_D,\mathcal{F}} \qquad \lambda_D \in \beta_D$$

$$\mathrm{Nilp}^{\mathrm{reg}} \hookrightarrow \mathrm{Nilp} \longleftarrow \mathrm{Nilp}^{\mathrm{irreg}}$$

$\cap$

$\mathrm{Map}_{\mathrm{gen}}(X, \mathcal{N}^{\mathrm{reg}}/G \subset \mathcal{N}/G)^{\omega}_{\mathrm{red}}$

Thm [BD] $$\mathrm{Nilp}^{\mathrm{reg}} = \bigcup_{\substack{\check\lambda \in \check\Lambda \\ \text{s.t. } \forall i, \\ (\alpha_i, \check\lambda) + (2g-2) \geq 0}} \mathrm{Nilp}^{\mathrm{reg},\check\lambda}$$

$\mathrm{Nilp}^{\mathrm{reg},\check\lambda}$ – smooth conn. of dim $=$ dim Bun_G.

Rmk $\mathrm{Kos}_D \cap \mathrm{Nilp}^{\mathrm{irreg}} = \emptyset$

Prop $(\mathrm{Nilp}^{\mathrm{reg}})^{\deg D - (2g-2)\check\rho} \cap \mathrm{Kos}_D = \{f_D^{\mathrm{glob}}\}$

Cor $\mathcal{F} \in Shv_{K, Nilp^{\nless \deg D - (2g-2)\check{\rho}}}(Bun_G)$,

$\mathrm{coeff}_{D,K}(\mathcal{F}) = \langle \mathcal{E}_{Nilp^{\deg D - (2g-2)}}, \mathcal{F} \rangle$

Claim $\forall \mathcal{F} \in Shv_{K, Nilp}(Bun_G)^{cusp}$, $SS(\mathcal{F}) \cap Nilp^{reg} \neq \phi$.

Proof of Thm 2 $\forall \mathcal{F} \in Shv_{K, Nilp}(Bun_G)^{cusp}, \heartsuit$

$Nilp^{\alpha} \subset SS(\mathcal{F})$ comp. w/ α minimal

Then $\mathrm{coeff}_{K, D_\alpha}(\mathcal{F}) \neq 0$ $\forall D_\alpha$ w/ $\deg D_\alpha - (2g-2)\check{\rho} = \alpha$. $\square$