The anti-spherical projector

Daniel Kim

$G \supset B \supset T / \mathbb{C}$, $\quad \Bbbk = \mathbb{Q}$

$LG \supset L^+G \supset I \supset I^+ \xrightarrow{\psi} \mathbb{G}_a$

$$\begin{tikzcd}
\mathbb{Z}[X_*(T)]^W[q^{\pm 1}] \arrow[r] & \mathbb{Z}[X_*(T)][q^{\pm 1}] \arrow[d, hook] & \\
& H_I \arrow[d, dash, dashed] & \curvearrowright \ H_I \otimes_{H_{fin}, \varepsilon} \mathbb{Z}[q^{\pm 1}] \arrow[d, dash, dashed] \\
& D_{I^+ I^+} & D_{IW}
\end{tikzcd}$$

$D_{II} = D(I \backslash LG / I) \supset P_{II}$

$D_{I^+I} = D(I^+ \backslash LG / I)$

$\quad = D((I, \hat{u}) \backslash LG / I) \supset P_{I^+I}$

$D_{I^+I^+} = D((I, \hat{u}) \backslash LG / (I, \hat{u}))$

$\quad = D(I^+ \backslash LG / (I, \hat{u})) \supset P_{I^+I^+}$

$\varepsilon : H_{fin} \longrightarrow \mathbb{Z}[q^{\pm 1}]$

$\quad T_w \mapsto q^{l(w)}$

$$\zeta = \sum_{w \in W_{fin}} T_w \in H_{fin}$$

$\mathbb{Z}[q^{\pm 1}] \xrightarrow{\zeta} H_{fin}$ is a left H_{fin}-module homomorphism

$\mathbb{Z}[q^{\pm 1}] \simeq H_{fin} \cdot \zeta$.

<u>Goal</u>: categorify $H_I \longrightarrow H_I \cdot \zeta$ right multiplication by ζ.

§. Tilting objects

$$\begin{array}{c} I^+\backslash LG/I^+ \\ \uparrow j_w \end{array}$$

For $w \in \widetilde{W}$, $\quad pr_w : I^+\backslash (LG)_w / I^+ \longrightarrow T$

$$\Delta_w^{mon} = j_{w!}\, pr_w^*(\widetilde{ch}) \in D_{I^+I^+}$$

$$\nabla_w^{mon} = j_{w*}\, pr_w^*(\widetilde{ch}) \in D_{I^+I^+}$$

$$\Delta_w = j_{w!}\, \Bbbk[\cdots] \in D_{I^+I}$$

$$\nabla_w = j_{w*}\, \Bbbk[\cdots] \in D_{I^+I}$$

Def For an object $\mathcal{F} \in D_{I^+I^+}$, it is <u>tilting</u> if it has two finite filtrations whose $\begin{cases} \text{graded are } \Delta_w^{mon} \\ \text{graded are } \nabla_w^{mon} \end{cases}$

Similarly for $\mathcal{F} \in D_{I^+I}$.

$$\rightsquigarrow \mathcal{T} \subset P_{I^+I} \subset D_{I^+I}$$

tilting sheaves

$$\mathcal{T}^{mon} \subset P_{I^+I^+} \subset D_{I^+I^+}$$

Prop. $\mathcal{T}^{mon}$ is closed under $\star^{I^+}$. An object $\mathcal{F} \in P_{I^+I^+}$ is tilting

$\Leftrightarrow \pi_* \mathcal{F}$ is tilting

$$\pi : I^+\backslash LG/I^+ \longrightarrow I^+\backslash LG/I$$

$$\boxed{\begin{array}{l} \pi_* \Delta_w^{mon} \simeq \Delta_w \\ \pi_* \nabla_w^{mon} \simeq \nabla_w \end{array}}$$

Prop. For $w \in \widetilde{W}$, there exists uniquely (up to isom.) an indecomposable object

$T_w^{mon} \in \mathcal{T}^{mon}$, $T_w \in \mathcal{T}$ s.t. $\quad \mathrm{supp}(T_w^{mon}) = I^+\backslash (LG)_{\leq w}/I^+$

$\qquad \mathrm{supp}(T_w) = I^+\backslash (LG)_{\leq w}/I$

Thm. There exists a monoidal functor $\mathbb{V} : \mathcal{T}^{mon} \longrightarrow \mathrm{IndCoh}(\hat{\mathfrak{t}} \times \hat{\mathfrak{t}})$

which is fully faithful on $\mathcal{T}^{mon}_{\leq w_0}$ s.t. $\mathbb{V}(\mathcal{T}^{mon}_s) = \omega_{\hat{\mathfrak{t}} \times_{\hat{\mathfrak{t}}/\!/\{1,s\}} \hat{\mathfrak{t}}}$ for $s \in W_{fin}$

w_0 largest elt in W_{fin} — simple reflection

§. Averaging functors

In general, $H \subset G$ closed subgp w/ smooth quotient, $G \curvearrowright X$.

$$\rightsquigarrow \quad \mathrm{Res}^G_H : D(G\backslash X) \longrightarrow D(H\backslash X)$$

left adjoint $\mathrm{Av}^G_{H,!} : D(H\backslash X) \longrightarrow D(G\backslash X)$

right adjoint $\mathrm{Av}^G_{H,*} : D(H\backslash X) \longrightarrow D(G\backslash X)$

(descending G-equiv. $G \overset{H}{\times} X \longrightarrow X$)

$$D^{I}_{IW} = D(I\backslash LG/(I^{op,+}, \psi))$$

$$D^{I^+}_{IW} = D((I,\hat{u})\backslash LG/(I^{op,+}, \psi))$$

$$\mathrm{Av}_{IW} : \quad \begin{tikzcd} D^{I^+}_{IW} \arrow[r, shift left, "\mathrm{Res}^{I^{op,+}}_{I^+\cap I^{op,+}}"] & D((I,\hat{u})\backslash LG/(I^{op,+}\cap I^+)) \arrow[l, bend right, "\mathrm{Av}^\psi_!"'] \arrow[l, bend left, "\mathrm{Av}^\psi_*"] \arrow[r, shift left, "\mathrm{Av}^{I^+}_{(I^{op,+}\cap I^+),!}"] & D_{I^+I^+} \arrow[l, bend left, "\mathrm{Res}"] \end{tikzcd}$$

$$\mathrm{Av}_{IW} = \mathrm{Av}^{I^+}_{(I^{op,+}\cap I^+),!} \circ \mathrm{Res}^{I^{op,+}}_{I^+\cap I^{op,+}}$$

<u>Fact</u>: $\mathrm{Av}^\psi_! = \mathrm{Av}^\psi_*$

"clean" extension

(uses ψ is non-degenerate)

$$\mathrm{Av}^\psi_{!/*} = \pi_{!/*}(\psi \overset{H}{\boxtimes} \sim)$$

Prop. $\mathrm{Av}^{IW} : D_{I^+ I^+} \longrightarrow D_{IW}^{I^+}$ can be identified w/ $(-) \overset{I^+}{\star} \Delta_e^{IW, mon}$ $:= \mathrm{Av}^{IW}(\Delta_e^{mon})$

$\Rightarrow$ $\mathrm{Av}_{IW} \circ \mathrm{Av}^{IW} : D_{I^+I^+} \longrightarrow D_{I^+I^+}$

$$\shortparallel$$

$$(-) \overset{I^+}{\star} \mathrm{Av}_{IW}(\Delta_e^{IW, mon})$$

Thm $\mathrm{Av}_{IW}(\Delta_e^{IW, mon}) \simeq T_{w_0}^{mon}$

Prop $\mathrm{Av}_{IW} \circ \mathrm{Av}^{IW} = (-) \overset{I^+}{\star} T_{w_0}^{mon}$ is self-adjoint

Right adjoint $= \mathrm{Av}_{IW, *} \circ \mathrm{Av}^{IW} = \mathrm{Av}_{IW} \circ \mathrm{Av}^{IW}$

Prop $\mathrm{End}(T_{w_0}^{mon}) \simeq \mathcal{O}(\hat{t} \underset{t /\!/ W_{fin}}{\times} \hat{t})$

$\mathbb{V}(T_{w_0}^{mon}) \simeq \omega_{\hat{t}} \underset{\check{t} /\!/ W_{fin}}{\times} \hat{\check{t}}$

$\uparrow$

Cofree of rank 1

$\Rightarrow \mathrm{End} = \mathcal{O}(\cdots)$

$P_{I^+I^+}^{fin} \subset P_{I^+I^+}$ subcat. supported on $_{I^+}\backslash L^+G / _{I^+}$

Consider Serre quotient $P_{I^+I^+}^{fin} \longrightarrow \overline{P_{I^+I^+}^{fin}}$ by the Serre subcat. gen. by simple objects not Δ_e^{mon}.

$$\begin{tikzcd} P_{I^+I^+}^{fin} \arrow[rr, "(-) \overset{I^+}{\star} T_{w_0}^{mon}"] \arrow[dr] & & P_{I^+I^+}^{fin} \\ & \overline{P_{I^+I^+}^{fin}} \arrow[ur, dashed] & \end{tikzcd}$$

because Av^{IW} kills IC_w for $w \notin {}^f\widetilde{W}$.

Prop. The endo functor

$$(-) \overset{I^+}{\star} T_{w_0}^{mon} : \overline{P^{fin}_{I^+I^+}} \longrightarrow \overline{P^{fin}_{I^+I^+}} \quad \text{is isom. to}$$

$$\mathcal{O}(\check{t}) \underset{\mathcal{O}(\check{t}/\!/W_f)}{\otimes} (-) \qquad \text{where } \mathcal{O}(\check{t}/\!/W_f) \subset \mathcal{O}(\check{t})\text{-action on } \overline{P^{fin}_{I^+I^+}}$$

is given by monodromy.

Check what happens to Δ_e^{mon}

$$\Delta_e^{mon} \overset{I^+}{\star} T_{w_0}^{mon} \simeq T_{w_0}^{mon}$$

$$\begin{array}{c} \hat{\check{t}} \underset{\check{t}/\!/W_{fin}}{\times} \hat{\check{t}} \\ \downarrow \quad \text{proj. to 2nd factor} \\ \hat{\check{t}} \end{array}$$

Fact: $T_{w_0}^{mon}$ in its standard / costandard filtration each $\Delta_w^{mon} / \nabla_w^{mon}$ $(w \in W_{fin})$

appears w/ mult. 1.

§. Projection of central functors

$$Z : Rep(\check{G}) \longrightarrow D_{II} \longrightarrow D_{I^+I}$$

$$Z^{mon} : Rep(\check{G}) \longrightarrow D_{I^+I^+}$$

are compatible: $\pi_* \circ Z^{mon} \simeq Z$.

Prop. For $V \in Rep(\check{G})$, $\quad T_{w_0}^{mon} \overset{I^+}{\star} Z^{mon}(V) \in J^{mon}$, $T_{w_0} \overset{I}{\star} Z(V) \in J$.

Serre quotient $P_{II} \longrightarrow {}^f P_{II}$ ignoring $w \notin {}^f W$.

$$\begin{array}{c} {}^f P_{II} \\ \wr\wr \\ P^I_{IW} \end{array}$$

$$\begin{tikzcd} T_{w_0} \overset{I}{\star} (-) : P_{II} \arrow[r] \arrow[dr] & P_{I^+I} \\ & {}^f P_{II} \arrow[u] \end{tikzcd}$$

$Av^{IW}(Z(V))$ admits a filtration by $Av^{IW}(\Delta_w)$, $Av^{IW}(\nabla_w)$ for $w \in \tilde{W}$.

$\rightarrow$ $T_{w_0} \overset{I}{\star} Z(V)$ admits filtration by $T_{w_0} \overset{I}{\star} \Delta_w$ / $T_{w_0} \overset{I}{\star} \nabla_w$ ---

?

This has filtration by Δ_w ($w \in W_{fin}$)

$\downarrow$

$T_{w_0} \overset{I}{\star} \Delta_w$ has a filtration by Δ_{vw}

$\Rightarrow$ tilting.

§. Compatibility w/ Springer

$P_{Spr} : \check{B} \backslash \hat{\check{n}} \longrightarrow \check{G} \backslash \check{\mathfrak{g}}$ (inside $\check{B}$)

$\hat{\check{N}} \hookrightarrow \hat{\check{\mathfrak{g}}} \searrow \check{\mathfrak{g}}$

$\Phi^{mon}_{IW} : \mathrm{IndCoh}(\check{B} \backslash \hat{\check{n}}) \xrightarrow{\sim} D^{I^+}_{IW}$

<u>Thm.</u> $P_{Spr}^* P_{Spr*}$ on the left corresponds to $T^{mon}_{w_0} \overset{I^+}{\star} (-)$ on the right.

Prop. $Av^{IW}(T_{w_0}^{mon})$ on the right corresponds to $\mathcal{O}(\check{t}) \underset{\mathcal{O}(\check{t}/\!/W_{fin})}{\otimes} \omega$ on the left.

Av^{IW} factors through $\overline{P^{fin}_{I^+I^+}}$ $\Rightarrow$ $Av^{IW}(T_{w_0}^{mon}) = \mathcal{O}(\check{t}) \underset{\mathcal{O}(\check{t}/\!/W_{fin})}{\otimes} \underbrace{Av^{IW}(\Delta_e^{mon})}_{\omega}$

Prop. For $\mathcal{F} \in \mathrm{IndCoh}(\check{\mathfrak{n}}\backslash\check{\mathfrak{g}})$, we have

$$T_{w_0}^{mon} \overset{I^+}{\star} \Phi_{IW}^{mon}(p_{Spr}^* \mathcal{F}) \cong \Phi_{IW}^{mon}\left(\mathcal{O}(\check{t}) \underset{\mathcal{O}(\check{t}/\!/W_{fin})}{\otimes} p_{Spr}^*(\mathcal{F})\right)$$

Prop. $p_{Spr *}\mathcal{F} = 0$ on the left hand side corresponds to $T_{w_0}^{mon} \overset{I^+}{\star} \mathcal{F} = 0$ on the right.

Let $F_\Xi = T_{w_0}^{mon} \overset{I^+}{\star} \mathcal{F}$ — self-dual

$F_{Spr} = p_{Spr}^* p_{Spr *}$ — self-dual

$F_\Xi \cdot p_{Spr}^* = p_{Spr}^* \cdot p_{Spr *} \cdot p_{Spr}^* = F_{Spr} \cdot p_{Spr}^*$

$\check{B}\backslash\check{\mathfrak{b}} \xrightarrow{p_{as}} \check{\mathfrak{n}}\backslash\check{\mathfrak{g}}$, $p_{as *}\mathcal{O} \cong \mathcal{O}(\check{t}) \underset{\mathcal{O}(\check{t}/\!/W_{fin})}{\otimes} \mathcal{O}$

$\rightsquigarrow F_\Xi \cdot F_{Spr} \cong F_{Spr} \cdot F_{Spr} \longrightarrow \mathrm{Id}$

; self duality

$\rightsquigarrow F_{Spr} \xrightarrow{c} F_\Xi$

Want: $\forall \mathcal{F}$, $c_\mathcal{F} : F_{Spr}(\mathcal{F}) \longrightarrow F_\Xi(\mathcal{F})$ is an isom

$$\mathrm{Hom}(\mathcal{G}, \mathrm{cofib}(c_{\mathcal{F}})) = \mathrm{cofib}\big(\mathrm{Hom}(\mathcal{G}, F_{Spr}(\mathcal{F})) \to \mathrm{Hom}(\mathcal{G}, F_{\Xi}(\mathcal{F}))\big)$$

$\mathrm{cofib}(c_{\mathcal{F}})$ is zero if either $\mathcal{F} \in \mathrm{im}\, F_{Spr}$, or $\mathcal{F} \in \ker F_{Spr}$.

$$= \mathrm{cofib}\big(\mathrm{Hom}(F_{Spr}(\mathcal{G}), \mathcal{F}) \to \mathrm{Hom}(F_{\Xi}(\mathcal{G}), \mathcal{F})\big)$$

is also 0 when $\mathcal{G} \in \ker F_{Spr}$ or $\mathcal{G} \in \mathrm{im}\, F_{Spr}$.

$\Rightarrow \mathrm{cofib}(c_{\mathcal{F}})$ lies in the left orthogonal of both $\ker F_{Spr}$ & $\mathrm{im}\, F_{Spr}$

Self-adjoint $\Rightarrow$ $\mathrm{im}\, F_{Spr}$ & $\ker F_{Spr}$ are each other's orthogonal

$\Rightarrow \mathrm{cone}(c_{\mathcal{F}}) = 0$