Arithmetic Quantum Field Theory

Shurui Liu

TQFT

4d TQFT

dim 4	X^4	$\longmapsto$	$F(X) \in \mathbb{C}$
dim 3	X^3	$\longmapsto$	$F(X) \in \mathrm{Vect}_{\mathbb{C}}$
(w/o boundary) dim 2	X^2	$\longmapsto$	$F(X) \in \mathrm{LinCat}_{\mathbb{C}}$
dim 1	X^1	$\longmapsto$	$F(X) \in 2\text{-Cat}$

(1) $\partial X^4 = Y^3$

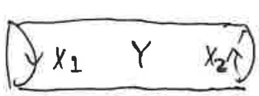
$F(X) \in F(Y)$ a vector in $F(Y)$

(2) $\partial X^3 = Y^2$

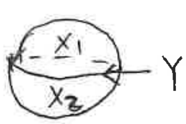
$F(X) \in F(Y)$ obj. in a cat

(3) ......

Ref Lurie, cobordism hypothesis.

e.g. 

$F(Y) \in \mathrm{Hom}(F(X_1), F(X_2))$

e.g. $X^4 =$ 

$F(X) \in \mathbb{C}$

$F(X_1) \in F(Y)$ a vector in v.s.

$F(X_2) \in F(Y)^*$ a covector

$F(X) = \langle F(X_1), F(X_2) \rangle$

Slogan: Langlands program , G split reductive gp , $\check{G}$

Geom / Autom. side   Spectral / Galois side

$$\mathcal{A}_G \xrightarrow{\sim} \mathcal{B}_{\check{G}}$$

e.g. X Riemann surface / $\mathbb{C}$ or curve / $\overline{\mathbb{F}_q}$

$\dim X = 2$, $\mathcal{A}_G(X) = \mathrm{Shv}(\mathrm{Bun}_G(X))$.

$\mathcal{B}_{\check{G}}(X) = \mathrm{IndCoh}(\mathrm{Loc}_{\check{G}}(X))$.

e.g. C curve / $\mathbb{F}_q$

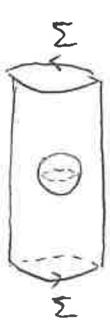
C $\dim = 3$

$$C_{\overline{\mathbb{F}_p}} \xrightarrow{Fr} C_{\overline{\mathbb{F}_p}}$$

$$\mathcal{A}_G \simeq \mathcal{B}_{\check{G}}$$

Operators

$M = \Sigma^2 \times [0,1] - B$, $\partial M = \Sigma \amalg (-\Sigma) \amalg S^2$



$\mathcal{F}(M) : \mathcal{F}(\Sigma) \otimes \mathcal{F}(S^2) \to \mathcal{F}(\Sigma)$ for TQFT $\mathcal{F}$.

$\mathcal{F} = \mathcal{A}_G$

① $\mathcal{A}_G(S^2)$ is monoidal

② $\mathcal{A}_G(\Sigma) \in \mathrm{Mod}_{\mathcal{A}_G(S^2)}(\mathrm{LinCat})$ $\rightsquigarrow$ Hecke operators

$$S^2 = D \coprod_{\substack{S^1 \\ \wr\wr \\ D^\times}} D$$

<u>AG</u>. Ravioli $= D \coprod_{D^\times} D$.

$$\mathcal{A}_G(S^2) = \mathrm{Shv}(\mathrm{Bun}_G(S^2))$$

$$= \mathrm{Shv}(L^+G \backslash LG / L^+G).$$

$\mathcal{A}_G \simeq \mathcal{B}_{\check{G}}$ $\quad S^2$

derived Satake equiv.

functoriality $\longleftrightarrow$ interface of TQFT

H, G are groups, $M \circlearrowleft G \times H$

$$\rightsquigarrow \quad \begin{tikzcd} \mathcal{A}_H \arrow[r, "\theta_M"] \arrow[d, phantom, "\wr\wr"] \arrow[dr, phantom, "///"] & \mathcal{A}_G \arrow[d, phantom, "\wr\wr"] \\ \mathcal{B}_{\check{H}} \arrow[r, "\mathcal{L}_{\check{M}}"] & \mathcal{B}_{\check{G}} \end{tikzcd}$$

e.g. $\quad H = e, \; M = T^*X$

$$\theta_M(S^2) \in \mathrm{Shv}(\mathrm{Bun}_G(S^2)) = \mathrm{Shv}(L^+G \backslash LG / L^+G)$$

$$\theta_M(S^2) = p_* \omega$$

$$p: \mathrm{Map}(S^2, X/G) = \mathrm{Bun}_G^X(S^2)$$
$$\downarrow$$
$$\mathrm{Map}(S^2, pt/G) = \mathrm{Bun}_G(S^2)$$

$$\check{M} \simeq \mathrm{Spec}(H^*(\mathrm{Sat}_G(p_* \omega)))$$

$$\mathcal{L}_{\check{M}}(S^2) = m_* \mathcal{O}_{\check{M}/\check{G}}$$

$$\check{M}/\check{G} \xrightarrow{m} \check{\mathfrak{g}}^*/\check{G}$$

§ Local conj.

$\mathbb{F} = \mathbb{C}, \ \overline{\mathbb{F}_q}$ resp.

$k = \mathbb{C}, \ \overline{\mathbb{Q}_\ell}$ resp.

$\mathrm{Sat}^{\text{naive}}_G : D(\mathrm{Perv}(L^+G \backslash LG / L^+G)) \xrightarrow{\sim} \mathrm{Rep}(\check{G})$

$\mathrm{Sat}_G : \mathrm{Shv}(L^+G \backslash LG / L^+G) \xrightarrow{\sim} \mathrm{QCoh}(\check{\mathfrak{g}}^{*\,\unicode{x25B1}} / \check{G}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{*}/\check{G})^{\unicode{x25B1}}$

$xy - yx = \hbar[x,y]$

$\mathrm{Sat}_{G,\hbar} : \mathrm{Shv}(L^+G \backslash LG / L^+G \rtimes \mathrm{Aut}(D)) \simeq \mathrm{Mod}^{\check{G}}(U_\hbar(\check{\mathfrak{g}}^{\unicode{x25B1}}))$

$\mathrm{Aut}(D)$ = auto. gp of D preserving origin $(= \mathrm{Aut}(k[[t]]))$; $\hbar$: deg 2

<u>Shearing $\unicode{x25B1}$</u> : $\check{\mathfrak{g}}^{*\,\unicode{x25B1}} = \check{\mathfrak{g}}^*[-2] \circlearrowleft \mathbb{G}_m^{gr}$ wt $= -2$

(1) $V \in \mathrm{Rep}(\mathbb{G}_m^{gr}) \iff V = \bigoplus_{i \in \mathbb{Z}} V_i$

$V^{\unicode{x25B1}} = \bigoplus_{i \in \mathbb{Z}} V_i \langle i \rangle$ shift homological $i + (\cdots)$

$\unicode{x25B1} : \mathrm{Rep}(\mathbb{G}_m^{gr}) \longrightarrow \mathrm{Rep}(\mathbb{G}_m^{gr})$

(2) $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Rep}(\mathbb{G}_m^{gr})}(\mathrm{LinCat}_k)$, $\mathcal{C}^{\unicode{x25B1}} := \mathcal{C} \otimes_{\mathrm{Rep}(\mathbb{G}_m^{gr})} \mathrm{Rep}(\mathbb{G}_m^{gr})^{\unicode{x25B1}}$

$\mathrm{Rep}(\mathbb{G}_m^{gr}) \overset{(-)^{\unicode{x25B1}} \otimes (-)}{\curvearrowright} \mathrm{Rep}(\mathbb{G}_m^{gr})^{\unicode{x25B1}} \overset{\otimes}{\curvearrowleft} \mathrm{Rep}(\mathbb{G}_m^{gr})$

e.g. $\mathcal{C} = \mathrm{Mod}(A)$, $\mathcal{C}^{\unicode{x25B1}} \simeq \mathrm{Mod}(A^{\unicode{x25B1}})$

$A \in \mathrm{Alg}(\mathrm{Rep}(\mathbb{G}_m^{gr}))$

- $\mathrm{Rep}^{\text{super}}_{(-1)^{2\check\rho}}(\check G) \xleftarrow[\sim]{\otimes \text{ sym. monoidal}} D(\mathrm{Perv}(L^+G\backslash LG/L^+G))$

(not modifying commutativity constraints)

- $\langle 1\rangle := [1] \circ \Pi$ ← parity shift $(\frac{1}{2})$

$[1]$: coh. shift -1

<u>(unramified) local conjecture</u> (G, M) polarized hyperspherical $M = T^*_\Psi X$

$\exists$ equiv $\mathrm{Shv}^!(LX/L^+G) \simeq \mathrm{QCoh}^{\shortparallel}(\check M/\check G)$ ← neutral G_m or trivial

$\check M = \check G \times^{\check G_X} V_X$

$\mathrm{Shv}(L^+G\backslash LG/L^+G) \circlearrowright \mathrm{Shv}^!(LX/L^+G)$, $\quad \mathrm{QCoh}^{\shortparallel}(\check M/\check G) \simeq \mathrm{QCoh}^{\shortparallel}(V_X^{\vee}/\check G_X) \circlearrowleft \mathrm{QCoh}^{\shortparallel}(\check{\mathfrak g}^{\vee *}/\check G)$ $\quad V_X = S_X \oplus (\check{\mathfrak g}_e \oplus \mathfrak g^{\perp})$

(1) Hecke action

$$\begin{array}{ccc} & L^+G\backslash LG \times^{L^+G} LX & \\ \swarrow & & \searrow \\ L^+G\backslash LG/L^+G & & L^+G\backslash LX \end{array}$$

$\leftrightarrow \quad \check M/\check G \xrightarrow{m} \check{\mathfrak g}^*/\check G$

$(-) \otimes m^*(-)$.

$\mathrm{QCoh}^{\shortparallel}(\check M/\check G) = \mathrm{Mod}_{m_* \mathcal O^{\shortparallel}_{\check M/\check G}}(\mathrm{QCoh}^{\shortparallel}(\check{\mathfrak g}/\check G))$

$\Rightarrow \{\underline V \otimes \mathcal O^{\shortparallel}_{\check M/\check G}\}$ generates $\mathrm{QCoh}^{\shortparallel}(\check M/\check G)$

(2) basic objects $L^+X/L^+G \xhookrightarrow{i} LX/L^+G$, $\delta_1 = i_* \omega_{L^+X/LG} \xleftrightarrow{\mathbb L} \mathcal O^{\shortparallel}_{\check M/\check G}$

$$PL_X := \mathcal{E}nd_{Sat_G}(\delta_1) \in Shv(L^+G \backslash LG / L^+G)$$
$$\wr\wr$$
$$QCoh(\check{\mathfrak{g}}^{\square}/\check{G})$$

$$PL_{X,\hbar} := \mathcal{E}nd_{Sat_{G,\hbar}}(\delta_1) \in Shv(L^+G \backslash LG / L^+G \rtimes Aut(D))$$

$\Rightarrow$ $PL_X \longleftrightarrow \mathcal{O}^{\square}_{\check{M}/\check{G}}$

$$PL_{X,\hbar} \longleftrightarrow \mathcal{O}_{\hbar}(\check{M}/\check{G})^{\square}$$
$\downarrow$ flat (conj.)
$k[\hbar]$

(3) loop rotation $Aut(D) \longleftrightarrow$ Poisson str. on $\check{M}/\check{G}$.

<u>Applications</u>: (0) $Shv^!(LX/L^+G) \xrightarrow{\sim} Shv^*(LX/L^+G)$

(1) $X = H$ gp $\qquad \check{M} = T^*\check{H} = \check{H} \times \check{\mathfrak{h}}^*$

$G = H \times H$

$V_X = \mathfrak{h}^{\vee *}$, $\check{G}_X = \check{H}$, G_m^{gr} wt $= -2$

$\Rightarrow$ ① $Shv(L^+H \backslash LH / L^+H) \simeq QCoh^{\square}(\check{\mathfrak{h}}/\check{H})$

not diagonal

$$T^*\check{H}/\check{H}\times\check{H} = \mathfrak{h}^*/\check{H} \xrightarrow{m} \mathfrak{h}^*/\check{H} \times \mathfrak{h}^*/\check{H}$$

② $(-) * i^{-1}(-) \longleftrightarrow m^*(- \boxtimes -)$

$$Shv(L^+H \backslash LH / L^+H) \simeq Shv(L^+H \backslash LH / L^+H) \circlearrowleft Shv(L^+H \backslash LH / L^+H)$$

$$\mathcal{F}_1 * \delta_1 * \text{inverse}(\mathcal{G}) \longleftrightarrow m^*(\mathcal{F} \boxtimes \mathcal{G})$$

③ loop $\longleftrightarrow$ Poisson ④ $- \overset{!}{\otimes} - \longleftrightarrow ?$

(2) $X = pt$

$$\mathrm{Shv}(*/L^+G) \simeq \mathrm{Mod}(H^*(pt/G)) \xrightarrow{\sim} \mathrm{QCoh}^{\square}(\check{\mathfrak{g}}/\!/\check{G})$$

$$\mathrm{QCoh}^{\square}(\check{M}/\check{G}) \simeq \mathrm{QCoh}^{\square}(\Sigma) \xrightarrow[\text{Kostant thm}]{\sim}$$

Σ: Kostant slice

(3) $\check{M} = pt \circlearrowleft \check{G}$

$M = T^*G /\!/_{\psi} U$, $\mathrm{Shv}(LG/L^+G)^{U,\psi} \simeq \mathrm{Rep}(\check{G})^{\square}$

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