

# Level rank dualities from d-Harish-Chandra series

Ting Xue

## §1. d-HC series.

$G$  conn. red /  $\overline{\mathbb{F}}_q$ , split /  $\mathbb{F}_q$ ,  $F: G \rightarrow G$  Frobenius

Deligne - Lusztig induction / restriction  $\mathbb{Z} \text{ In}(L^F) \xrightleftharpoons[\ast R_L^G]{R_L^G} \mathbb{Z} \text{ In}(G^F)$   
 $L: F\text{-stable Levi}$

Unip irreps  $U_{\text{ch}}(G^F) = \{p: (p: R_T^G(1)) \neq 0, T: F\text{-stable max. torus}\}$

Lusztig:  $U_{\text{ch}}(G^F)$  is indep of  $q$ , only depends on  $W_G$ . "generic"

Broué - Malle - Michel: define "generic" finite reductive gps

Write  $U_{\text{ch}}(G) = \text{set of unip chars of } G$ .

Thm (BMM'93)  $\forall d \in \mathbb{Z}_{>0}, \exists$  a partition

$$U_{\text{ch}}(G) = \bigsqcup_{(\mathbb{L}, \lambda)} U_{\text{ch}}(\mathbb{L}, \lambda)_d \leftarrow \{p: (p: R_{\mathbb{L}}^G(\lambda)) \neq 0\}$$

$\mathbb{L}$ -d-split Levi

$T: d\text{-split} \Leftrightarrow |T^F| = \Phi_d$

$\parallel$

$\mathbb{Z}_G(\mathbb{T})$

$\lambda \in U_{\text{ch}}(\mathbb{L})$  cuspidal  $(\ast R_M^{\mathbb{L}}(\lambda) = 0, \forall d\text{-split } M \subset \mathbb{L})$

If  $d=1$ , HC series

Moreover,  $\exists \varphi_{(\mathbb{L}, \lambda)_d}^{\text{BMM}}: U_{\text{ch}}(\mathbb{L}, \lambda)_d \xrightarrow{\sim} \text{In}(W_{(\mathbb{L}, \lambda)_d})$

$\hookrightarrow$  complex reflection group

$$\exists \varepsilon_{(\mathbb{L}, \lambda)_d}: U_{\text{ch}}(\mathbb{L}, \lambda)_d \rightarrow \{\pm 1\}$$

Ranks 1) d-HC series nontrivial only if  $\Phi_d(a) \mid |G^F|$

2)  $\exists$  d-split max. torus  $\Leftrightarrow$  d regular number of G.  $f(q)$

Brue - Malle '93. (conj.)  $W_{(L,\lambda)_d}$   $\rightsquigarrow$   $H_{(L,\lambda)_d}(x)$  generic cyclotomic Hecke alg.

1)  $H_{(L,\lambda)_d}(x) \mid_{x \mapsto z_d} =$  gp alg of  $W_{(L,\lambda)_d}$

2)  $H_{(L,\lambda)_d}(x) \mid_{x \mapsto q} = \text{End}_{G^F} \left( H_c^*(Y_{L \subset P}^G) [\lambda_q] \right)$

$\rightarrow$

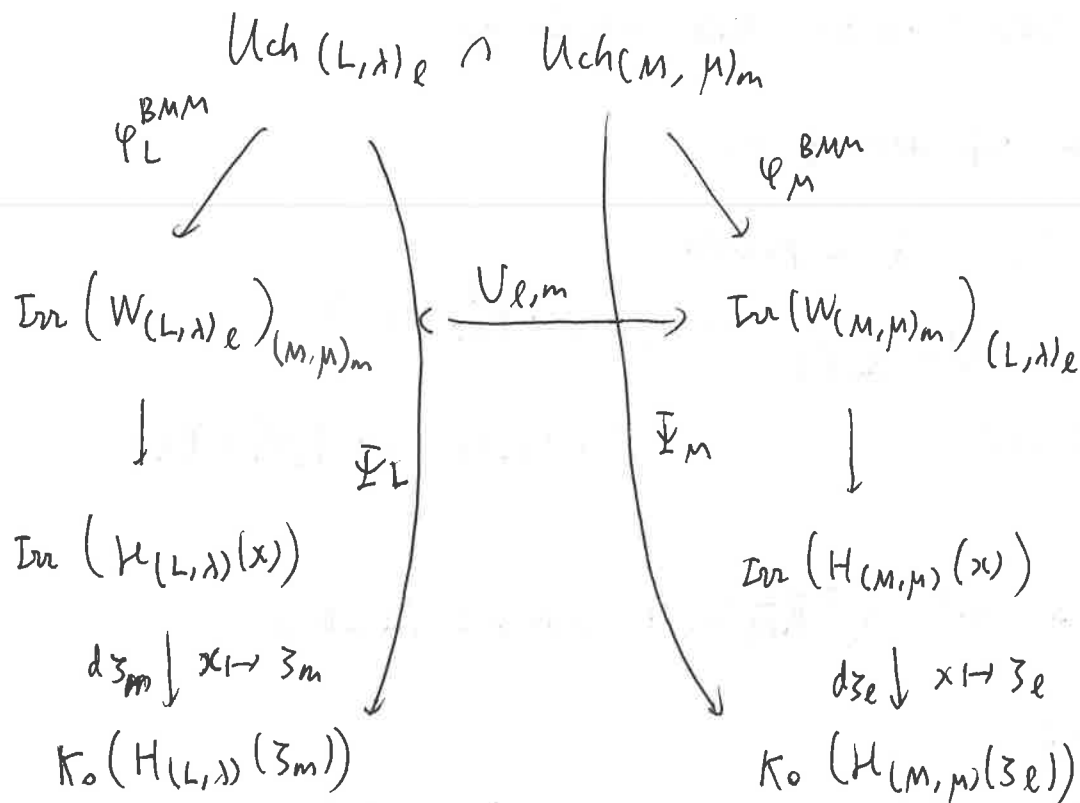
§2. Conjectural level-rank duality from d-HC series

fix  $l \neq m$ ,  $l, m \in \mathbb{Z}_{>0}$ ,

$$G = GL_N$$

$$W_{(L,\lambda)_d} \cong G_{d,1,2}$$

$$\cong \tilde{S}_2 \times (\mathbb{Z}/d)^2$$



Conj. (Trinh - X.)

1) The image of  $\mathbb{F}_\ell$  (resp.  $\mathbb{F}_m$ ) is a union of blocks of  $H_{(L, \lambda)_\ell}(\mathbb{F}_m)$   
(resp.  $H_{(M, \mu)_m}(\mathbb{F}_\ell)$ )

2) The bijection  $V_{\ell, m}$  categorifies to a derived equiv between highest wt covers  
of blocks of  $\text{Rep } H_{(L, \lambda)_\ell}(\mathbb{F}_m)$  (resp.  $\text{Rep } H_{(M, \mu)_m}(\mathbb{F}_\ell)$ ).

Thm (TX) For  $G = GL_n / GL_n$ ,  $(\ell, m)$  coprime, conjecture holds.

$V_{\ell, m} \longleftrightarrow$  Uglov's level-rank duality  
 $\longrightarrow$

§3 Connections w character sheaves on graded Lie algebras / affine Springer fibers

$m$  regular elliptic  $\ast$  of  $G \xleftrightarrow{RLY_G} \mathcal{O} := G \rightarrow G$  order  $m$

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$$

$G^\theta := G_0 \curvearrowright \mathfrak{g}_1$  has GIT-stable vectors

$$\text{char}_{G_0}(\mathfrak{g}_1) := \text{Four}(\text{SPerv}_{G_0}(\mathfrak{g}_{-1}^{\text{nil}}))$$

Vinberg:  $\mathfrak{g}_1 \xrightarrow{t} \mathfrak{g}_1 // G_0 \simeq e/c$ ,  $c = N_{G_0}(e)/Z_{G_0}(e)$  cplx redef. gp.

$$\Upsilon_t: \text{Perv}_{G_0}(t^{-1}(\bar{a})) \rightarrow \text{Perv}_{G_0}(t^{-1}(o)), \bar{a} \in e^{\text{rs}}/c$$

Thm (Vilonen - X., Grinberg - Vilonen - X.)

$$\text{Four}(P) \approx \text{IC}(G_{\pm}^{\text{rs}}, H_c)$$

$$\uparrow$$

$$\psi_0 \subseteq$$

↳ cyclotomic Hecke alg.

principal 1- HC series

principal m- HC series

$$W_{(A,1)} = W_A$$

$$W_{(T,1)m} = C$$

$$\mathcal{O}_{\frac{1}{m}}^{\text{rat}}(W_A)$$

↓

$$\text{Rep } H_{(A,1)}(\Sigma_m)$$

$$H_{(T,1)m}(1)$$

SI

$H_c$

$$\text{char}_{h_0}^{\text{cusp}}(g_{\pm})_{0, \text{st}}$$

Four

$$\text{spec}_{h_0}^{\text{cusp}}(g_{\pm}^{\text{nr}})_{0, \text{st}}$$

$$\downarrow \begin{matrix} G \vee X \\ \uparrow \\ V_X \end{matrix}$$

$$\downarrow \begin{matrix} \text{Lusztig-Yun} \\ + \text{Wille Liu, Ettinger} \end{matrix}$$

$$\text{Irr}_0(H_c)$$

↔

$$\text{Irr}_0^{\text{f-d.}}(\mathcal{O}_{\frac{1}{m}}^{\text{rat}}(W_A))$$

↑

conjecturally

induced by (1,m)- duality

Homogeneous affine Springer fibers

$$\gamma \in G(\mathbb{A}(t))^{\text{rs}}$$

homogeneous of slope  $\nu = \frac{1}{m}$

$$\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

Let  $\text{Sp}_{\gamma} \subset \text{Fl}$  affine Springer fiber

Thm (Oblozinski - Lus, 16)  $Brc \simeq \bigwedge_{j=1}^p H_{\varepsilon=1}^*(SP_r)_{st} \hookrightarrow H_{\frac{1}{m}}^{rat}(W_n)$

Conj (Tink - X.)  
1) The action of  $Brc$  factors through  $H_c \simeq H_{(T,1)_m}(1)$

$$2) \text{ Let } [\varepsilon_{2,\nu}] = \sum_{i,j} (-1)^i t^j [\bigwedge_j^p H_{\varepsilon=1}^i (SP_r)_{st}]$$

$$\text{Then } [\varepsilon_{2,\nu}] = \sum_{\rho \in Uch_{1,p} \cap Uch_{m,p}} \varepsilon(\rho) [\Delta_\nu(\varphi_1(\rho)) \otimes S(\varphi_m(\rho))]$$

$\uparrow$   
 std of  
 $H_{\frac{1}{m}}^{rat}(W_n)$

$\uparrow$   
 Spect of  $H_c$   
 mod

Rank Also expect  $[\varepsilon_{2,\nu}] = \sum_{\tau \rightarrow \sigma} \varepsilon_{\tau,\sigma} [L_\nu(\tau) \otimes D_\sigma]$

$\uparrow$   
 $(1,m)$  duality

$\uparrow$   
 $\text{In}^{tot}(H_{\frac{1}{m}}^{rat}(W))$

$\uparrow$   
 $\text{In}(H_c)$

1. The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that proper record-keeping is essential for the integrity of the financial system and for the ability to detect and prevent fraud. The document also notes that records should be kept for a sufficient period of time to allow for a thorough review if necessary.

2. The second part of the document outlines the procedures for the collection and distribution of funds. It states that all funds should be collected in a timely and accurate manner, and that they should be distributed to the appropriate parties as soon as possible. The document also provides guidance on how to handle any discrepancies or errors that may arise during the process.

3. The third part of the document discusses the role of the auditor in ensuring the accuracy and integrity of the financial records. It states that the auditor should conduct a thorough review of all records and should report any findings to the appropriate authorities. The document also provides guidance on how to respond to any allegations of fraud or other misconduct.

4. The fourth part of the document discusses the importance of maintaining the confidentiality of financial information. It states that all financial records should be kept secure and that access should be restricted to only those individuals who have a legitimate need to know. The document also provides guidance on how to handle any requests for information that may be made by third parties.

5. The fifth part of the document discusses the importance of maintaining the accuracy and integrity of the financial system. It states that all transactions should be recorded accurately and that the system should be subject to regular audits. The document also provides guidance on how to respond to any allegations of fraud or other misconduct.