Shtukas in mixed characteristic

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VB = vector bundle

LS = pro-étale local system

tor = torsors.

I. Prelim. on $\mathcal{Y}$ and $\mathcal{X}$

II. pro-étale torsors $/S$ $\longleftrightarrow$ φ^{-1}-equiv. torsors $/\mathcal{Y}(S)$
(LS) (VB)

III. Sht_{S} (one leg) $= [\mathrm{Gr}_G / \mathcal{S}(\mathbb{Z}_p)] \; / \mathrm{Spd}\, \mathbb{Q}_p$

$\boxed{\text{I.}}$ Relative $\mathcal{Y}$ and $\mathcal{X}$

$\mathrm{Perf} \ni S \overset{\text{ex}}{=} \mathrm{Spa}(R, R^+)$, $\varpi \in R^+$ p.u.

The following is functorial in S:

$\mathrm{Spa}\, W(R^+)$

$\cup$

$\mathcal{Y}_{[0,\infty]}(S)$

$\cup$

$\mathcal{Y}_{[0,\infty)}(S) = S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p$

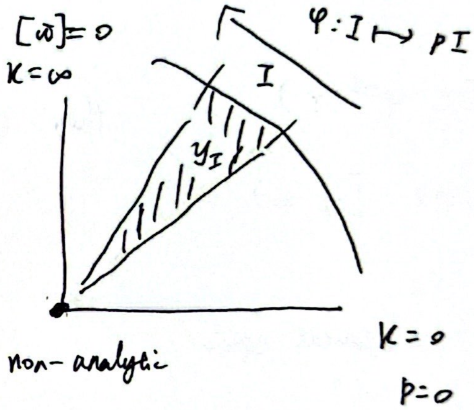
$\cup$

$\mathcal{Y}_{(0,\infty)}(S) = S \dot{\times} \mathrm{Spa}\, \mathbb{Q}_p$

] analytic adic spaces

$\downarrow$ mod $\varphi = \varphi_S$

$\mathcal{X}_{FF}$



[**Slogan**: "p-integral" $\Longleftrightarrow$ extend to $\mathcal{Y}_{[0,\infty)}$]

($p=0$)
$K=0$
$\downarrow$

• $\mathcal{G}$-torsors over $\mathcal{Y}$ and $\mathcal{X}$

<u>Def.</u> Assume $I \subset \varphi(I)$, then
$$\begin{array}{c} \mathcal{Y}_{\varphi(I)} \xrightarrow[\sim]{\varphi^{-1}} \mathcal{Y}_I \\ \cup \\ \mathcal{Y}_I \end{array}$$

<u>φ^{-1}-mod</u> on $\mathcal{Y}_I$ means $\left[\begin{array}{l} \bullet\ \ \mathcal{V}\ \ VB/\mathcal{Y}_I \\ \bullet\ \ \text{isom. } (\varphi^{-1})^* \mathcal{V} \xrightarrow{\sim} \mathcal{V} \end{array}\right.$

$\mathcal{G}/\mathbb{Z}_p$
smooth
(reduction)

<u>φ^{-1}-equiv. $\mathcal{G}$-torsor</u> on $\mathcal{Y}_I$ means.

$$\left[\begin{array}{l} \bullet\ \ P: \mathcal{G}\text{-torsor } /\mathcal{Y}_I \\ \bullet\ \ \text{isom. } (\varphi^{-1})^* \ P|_{\mathcal{Y}_I} \xrightarrow{\sim} P \end{array}\right.$$

If $I \subset \varphi^{-1}(I)$, then can define φ-mod, φ-eq. $\mathcal{G}$-torsor on $\mathcal{Y}_I$.

$[2, \infty)$ $[\frac{2}{p}, \infty)$

<u>Prop.</u> $\exists$ categorical equiv.

$$\left(VB/\mathcal{X}_{FF,S}\right) \simeq \left(\underset{(\varphi^{-1})}{\varphi\text{-mod}}/\mathcal{Y}_{(0,\infty)}\right) \simeq \left(\varphi\text{-mod}/\mathcal{Y}_{[2,\infty)}\right)$$
$$\simeq \left(\varphi^{-1}\text{-mod}/\mathcal{Y}_{(0,2]}\right)$$

Likewise $\left(\mathcal{G}\text{-tor}/\mathcal{X}_{FF,S}\right) \simeq$ -- $\simeq$ -- $\simeq \left(\varphi^{-1}\text{-eq. } \mathcal{G}\text{-tor}/\mathcal{Y}_{(0,2]}\right)$

<u>idea of pf</u>... Spread out via φ or φ^{-1}.

$\mathcal{G}/\mathbb{Z}_p$

$G = \mathcal{G}_{\mathbb{Q}_p}$
$\hat{\ }$ reductive

- Newton / Kottwitz maps

P : G-tor / $\mathcal{X}_{FF,S}$

obtain $|S| \longrightarrow |\mathrm{Bun}_G| = B(G) \xrightarrow{\nu_P} (X_*(T)^+_{\mathbb{Q}})^{\Gamma} \ni 0$ $\quad \Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$

$s \longmapsto P_{\bar{s}}$ $\qquad \xrightarrow{\kappa_P} \pi_1(G)_\Gamma \ni 0$

Fact $\quad (\nu_P, \kappa_P)$ is injective on $B(G)$

ν_P is injective if $G = GL_n$ (or if $\pi_1(G)_\Gamma$ is torsion free)

Thm 1 $\quad$ • ν_P is upper semicont. (Kedlaya-Liu '15)

• κ_P is locally const. (Fargues - Scholze)

Cor $\quad \overset{\text{"adm. locus"}}{S^a} = \{\nu_P = 0 = \kappa_P\} \subset S$ is open.

$= \{\nu_P = 0\}$ if $G = GL_n$ $\quad$ locus where P is trivial G-torsor $\forall$ geom. pt.

$\boxed{\text{II}}$ $\quad S \in \mathrm{Perf}$ $\qquad \mathbb{L} \longmapsto \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{X}_{FF,S}} \qquad \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{Y}_{[0,2]}(S)}$

Thm 2 (KL15) $\quad (\mathbb{Q}_p\text{-LS}/S) \simeq \left(\begin{array}{c}\mathrm{VB}/\mathcal{X}_{FF,S} \\ \text{trivial}, \forall \text{ geom. pt} \\ (\nu = 0, \text{"slope 0"})\end{array}\right) \simeq \left(\begin{array}{c}\varphi^{-1}\text{-mod}/\mathcal{Y}_{[0,2]} \\ \text{"slope 0"}\end{array}\right)$

$\uparrow$ $\qquad\qquad$ $\nearrow$ restrict

$(\mathbb{Z}_p\text{-LS}/S) \simeq (\varphi^{-1}\text{-mod}/\mathcal{Y}_{[0,2]}(S))$

$\mathbb{L} \longmapsto \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{Y}_{[0,2]}(S)}$

Note $\exists$ ext. from $(0, r]$ to $[0, r] \Rightarrow$ "slope 0"

Def For φ^{-1}-mod, $\mathcal{E}_\eta / \mathcal{Y}_{(0,r]}(S)$, define

$$\mathrm{Latt}(\mathcal{E}_\eta) : \mathrm{Perf}/S \longrightarrow \mathrm{Set}$$

$$S' \longmapsto \left\{ \begin{array}{l} \mathcal{E}' : \varphi^{-1}\text{-mod}/ \mathcal{Y}_{[0,r]}(S') \\ \alpha : \mathcal{E}_\eta \simeq \mathcal{E}' / \mathcal{Y}_{(0,r]}(S') \end{array} \right\}$$

"ext. of $\mathcal{E}_\eta$ to $\mathcal{Y}_{[0,r]}$"

Observe

$$\begin{array}{c} \mathrm{Latt}(\mathcal{E}_\eta) \\ \swarrow \quad \downarrow \\ S^a \subset S \\ \text{open} \end{array}$$

ext. to $\mathcal{Y}_{[0,r]} \Rightarrow$ "slope 0" $\forall$ geom. pt.

$$\mathcal{E}_\eta \big|_{\mathcal{Y}_{(0,r]}(S^a)} \xleftrightarrow{\text{Thm 2}} \mathbb{L}_\eta / S^a \quad \underline{\mathbb{Q}_p}\text{-LS.}$$

Moreover, $\mathrm{Latt}(\mathcal{E}_\eta)(S') = \{ \underline{\mathbb{Z}_p}\text{-lattices} \subset \mathbb{L}_\eta / S' \}$

$\uparrow$ Perf/S^a

$\uparrow$ pro-étale locally

If $n = \mathrm{rank}\, \mathcal{E}_\eta = \mathrm{rank}\, \mathbb{L}_\eta$,

$\uparrow$ $\underline{GL_n(\mathbb{Q}_p)/GL_n(\mathbb{Z}_p)}$ étale over base

$\rightarrow$ $\mathrm{Latt}(\mathcal{E}_\eta)$ is perf'd space

$$\begin{array}{c} \text{étale} \downarrow \\ S^a \subset S \\ \text{open} \end{array}$$

Rem ...

Analog for G-torsors

$\mathcal{G}/\mathbb{Z}_p$, $\quad G = \mathcal{G}_{\mathbb{Q}_p}$

$S \in \mathrm{Perf}$

Tannakian formalism gives:

Thm 3

$$\left(\underline{G(\mathbb{Q}_p)}\text{-tor}\right) \simeq \begin{pmatrix} G\text{-tor}/X_{FF,S} \\ \text{trivial } \forall \text{ geom. pt. of } S \\ (\nu = \kappa = 0) \end{pmatrix} \simeq \begin{pmatrix} \varphi^{-1}\text{-eq. } G\text{-torsor} \\ \text{on } \mathcal{Y}_{(0,\infty]}(S) \\ \nu = \kappa = 0 \end{pmatrix}$$

$$\left(\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-torsor}\right) \simeq \left(\varphi^{-1}\text{-eq. } \mathcal{G}\text{-torsor}/\mathcal{Y}_{[0,\infty]}(S)\right)$$

$$\left(\mathbb{P} = \rho \in \mathrm{Rep}_{\mathbb{Q}_p} G \mapsto \rho_* \mathbb{P}\right) \xmapsto{\text{Thm 2}} \left(\mathcal{E} : \rho \mapsto \mathcal{E}_\rho\right)$$

Idea (top row) To construct quasi-inverse.

key claim $\quad G\text{-tor}/X_{FF,S}$ is constructed pro-étale locally on S

$$\left(\Leftarrow \mathcal{E} \mapsto \underline{\mathrm{Isom}}_{G\text{-tor}}(G, \mathcal{E})\right)$$

pro-étale torsor for $\underline{\mathrm{Aut}}(G) = \underline{G(\mathbb{Q}_p)}$.

(reduce to S, strictly totally disconn'd)

conn. comps $=$ $\mathrm{Spa}(C, C^+)$

P_η: φ^{-1}-eq. $\mathcal{G}$-tor / $\mathcal{Y}_{(0,r]}(S)$

$\rightsquigarrow$ can define

$$\begin{tikzcd} \mathrm{Latt}(P_\eta) \arrow[d, "\text{étale}"'] \\ S^{a} \underset{\text{open}}{\subset} S \end{tikzcd} \quad \leftarrow \text{perf'd space}$$

$\boxed{\text{III}}$ $\mathcal{G}/\mathbb{Z}_p$ reduction $\quad K_p := \mathcal{G}(\mathbb{Z}_p)$

__Def__ $\quad \mathrm{Sht}_{\mathcal{G}}$: $\mathrm{Perf}/\mathrm{Spd}\,\mathbb{Z}_p \to \mathcal{G}\text{-}\mathrm{Sht}$ (groupoid)

$$S \longmapsto (S^\sharp, P, \varphi_P)$$

where $\cdot$ $S^\sharp/\mathrm{Spa}\,\mathbb{Z}_p \xrightarrow{\text{div}} \mathcal{Y}_{[0,\infty)}(S)$ is from $S \to \mathrm{Spd}\,\mathbb{Z}_p$

$\left(\begin{smallmatrix}\text{com.}\\ \text{omit}\end{smallmatrix}\right)$

$\cdot$ P : $\mathcal{G}$-tor / $\mathcal{Y}_{[0,\infty)}(S)$

$\cdot$ φ_P : $\varphi_S^* P \underset{S^\sharp}{\dashrightarrow} P$

means: $\left[\begin{array}{l} \text{isom. away from } S^\sharp \\ \text{mero. along } S^\sharp \end{array}\right.$

Gr_G: $\mathrm{Perf}/\mathrm{Spd}\,\mathbb{Q}_p \xrightarrow{\text{v-sheaf}}$ Set

trivial $\mathcal{G}$-tor

$$S \longmapsto (S^\sharp/\mathbb{Q}_p,\ \mathcal{E},\ \alpha : \mathcal{E}_1 \underset{S^\sharp}{\dashrightarrow} \mathcal{E})$$

com. omit

$\mathcal{G}$-tor. / $X_{FF,S}$

Lemma K_p-quot. of Gr_G is identified as

$$\mathrm{Gr}_{G,K_p} : \mathrm{Perf} / \mathrm{Spd}\, \mathbb{Q}_p \longrightarrow \text{Groupoid}$$

$$S \longmapsto (\underbrace{\mathcal{E}_0, \mathcal{E}}_{G\text{-tor} / X_{FF,S}}, \overset{\underline{G(\mathbb{Z}_p)}\text{-tor}}{\mathbb{P}}, \alpha : \mathcal{E}_0 \underset{S^\sharp}{\dashrightarrow} \mathcal{E})$$

s.t. • $\mathcal{E}_0$ is trivial, $\forall$ geom. pt

• $\mathcal{E}_0 \simeq \mathbb{P} \times^{\underline{G(\mathbb{Z}_p)}} (G \times X_{FF,S})$

pf $\mathrm{Gr}_G \simeq \underline{\mathrm{Isom}}_{\mathrm{Gr}_{G,K_p}} (\mathbb{P}, \underline{G(\mathbb{Z}_p)})$

Prop $\mathrm{Sht}_{G,\mathbb{Q}_p} \simeq \mathrm{Gr}_{G,K_p}$ as v-stacks / $\mathrm{Spd}\, \mathbb{Q}_p$

$\uparrow$ base change

pf $\boxed{\Rightarrow}$ Given $(\mathcal{P}/\mathcal{Y}_{[0,\infty)}, \varphi_{\mathcal{P}} : \varphi_S^* \mathcal{P} \underset{S^\sharp}{\dashrightarrow} \mathcal{P})$

ε small restrict to (avoid $S^\sharp$) $\rightsquigarrow \mathcal{Y}_{(0,\varepsilon)}$ $\qquad$ $\rightsquigarrow \mathcal{Y}_{(r,\infty)}$ $\quad r$ large

$\mathcal{E}_0 / X_{FF,S}$ $\qquad$ $\mathcal{E} / X_{FF,S}$

$\rightsquigarrow \mathcal{E}_0 \xrightarrow[S^\sharp]{\alpha := \varphi_{\mathcal{P}}^{n}} \mathcal{E}$

$\mathcal{P} \overset{\text{Thm 3}}{\rightsquigarrow} \mathbb{P}$ $\quad \underline{G(\mathbb{Z}_p)}$-tor.

$\boxed{\Leftarrow}$ ....