Depth zero affine Hecke category

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Lecture 1 $\quad G$ reductive group $/F$ na local field

(e.g. $G = GL_n$, $F = \mathbb{Q}_p$, $\mathbb{F}_p((t))$) $\quad GL_n(\mathbb{Q}_p) \supset GL_n(\mathbb{Z}_p) \supset \{1 + pM_n(\mathbb{Z}_p)\} \supset \{1 + p^2 M_n(\mathbb{Z}_p)\} \supset \cdots$

$G(F) \curvearrowleft (K_0 \supset K_1 \supset \cdots)$ locally cpt top. group, Haar measure dg

Recall a smooth repn of $G(F)$: $(V/\mathbb{C}, \rho)$

$$\rho: G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V) \quad \text{s.t.} \quad \forall v \in V,$$

$$\{g \in G(F) : gv = v\} \overset{\text{open cpt}}{\subset} G(F)$$

e.g. $K \subset G(F)$ open cpt, σ finite dim. rep. of $K = \varprojlim_i K_i$

K_i finite

$K \twoheadrightarrow K_i \xrightarrow{\sigma} GL(V_\sigma)$

$$\text{c-ind}_K^{G(F)} \sigma = \left\{ f : G(F) \longrightarrow V_\sigma : \begin{array}{l} f(kg) = \sigma(k) f(g) \\ \mathrm{supp}(f) \text{ is finite } \in K\backslash G(F) \end{array} \right\}$$

Lemma $\quad \text{c-ind}_K^{G(F)} \sigma$ is smooth.

e.g. $\sigma = \mathbb{1}$, $\quad \text{c-ind}_K^{G(F)} \sigma = C_c(K\backslash G(F)) = \bigoplus_{g \in K\backslash G(F)} \mathbb{1}_{Kg}$ infinite dim'l

Hecke algebra. K, σ as before

$$H_{(K,\sigma)} := \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} \sigma\right)$$

Lemma: $H_{(K,\sigma)} = \left\{ \Phi : G(F) \to \mathrm{End}(V) \ : \ \Phi(k_1 g k_2) = \sigma(k_1)\Phi(g)\sigma(k_2) \right\}$ $\forall k_1, k_2 \in K$

$(\Phi * f)(g)$ Supp$(\Phi) \overset{\text{finite}}{\subset} K\backslash G(F)/K$

$$\begin{tikzcd} V \arrow[r, "\sigma(k_1)"] \arrow[d, "\Phi(k_1 g k_2)"'] & V \arrow[d, "\Phi(g)"] \\ V \arrow[r, "\sigma(k_2)^{-1}"'] & V \end{tikzcd}$$

$$= \sum_{g' \in G/K} \Phi(gg')\left(f(g'^{-1})\right)$$

finite sum

$$= \sum_{g' \in G/K} \Phi(g')\left(f(g'^{-1}g)\right)$$

$\Phi_1, \Phi_2 \in H_{(K,\sigma)}$, $(\Phi_1 * \Phi_2)(g)$

$$= \sum_{g' \in G/K} \Phi_1(gg')\, \Phi_2(g'^{-1})$$

Lemma. $\mathrm{Hom}_{G(F)}\left(\text{c-ind}_K^{G(F)} \sigma, V\right) = \mathrm{Hom}_K\left(\sigma, V|_K\right)$

Rmk. Note if V is smooth, $\forall\, 0 \neq v \in V$, $K = \{g \in G(F) : gv = v\}$ open cpt, $V|_K \supset$ trivial rep. $\mathbb{C}\cdot v$ of K.

$\text{c-ind}_K^{G(F)}(\mathbb{1}) \longrightarrow V$. If V is irred. $\Rightarrow$ $\text{c-ind}_K^{G(F)} \mathbb{1} \twoheadrightarrow V$.

Rmk. The category of smooth reps of $G(F)$ $\mathrm{Rep}(G(F))$ is an abelian cat.

$\{\text{c-ind}_K^{G(F)} \sigma\}_{(K,\sigma)}$ are projective objects. (use Frobenius reciprocity)

Cor. $\exists$ fully faithful functor

$$D(H_{(K,\sigma)}^{op}\text{-mod}) \longrightarrow D(\mathrm{Rep}(G(F)))$$

$$M \longmapsto M \overset{L}{\underset{H_{(K,\sigma)}}{\otimes}} \text{c-ind}_K^{G(F)} \sigma$$

$$H_{(K,\sigma)} \longmapsto \text{c-ind}_K^{G(F)} \sigma$$

Rmk. $C_c^\infty(G) := \bigcup_K C_c^\infty(K \backslash G(F)/K)$

$\{\text{"non-degenerate" modules of } C_c^\infty(G)\} \simeq \mathrm{Rep}(G(F))$.

Important to study $H_{(K,\sigma)}$. When $\sigma = \mathbb{1}$, $H_K := H_{(K,\sigma)} = C_c(K \backslash G(F)/K)$

Ex. G split reductive group $/F$, defined over $\mathcal{O}$ $F \supset \mathcal{O}$

$K = G(\mathcal{O})$, $\sigma =$ trivial rep. of K. $\underset{\text{max. torus}}{T} \subset \underset{\text{Borel}}{B} \subset G$

$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$

$\cup$

$X_*(T)^+$

<u>Thm</u> (Satake isom.)

$$H_K \simeq \mathbb{C}[X_*(T)]^W \quad \text{as algebras.}$$

In particular, H_K is commutative.

<u>Hint</u>. $K \backslash G(F) / K \xleftrightarrow[\text{decomp.}]{\text{Cartan}} X_*(T)^+$

$K\lambda(\varpi)K \qquad \lambda : \mathbb{G}_m \to T$

ϖ uniformizer

e.g. $G = GL_n$

$$\left(GL_n(\mathbb{Q}_p) \right) = \left(GL_n(\mathbb{Z}_p) \right) \begin{pmatrix} \varpi^{d_1} & & \\ & \ddots & \\ & & \varpi^{d_n} \end{pmatrix} \left(GL_n(\mathbb{Z}_p) \right)$$

$d_1 \geq \cdots \geq d_n$

<u>Warning</u>. $\mathbb{1}_{K\varpi^\lambda K} \not\leftrightarrow \sum_{w \in W} e^{w(\lambda)}$

(under mild condition on G)

$c\text{-ind}_K^{G(F)} \mathbb{1}$ flat over H_K

$\{H_K\text{-mod}\} \longrightarrow \mathrm{Rep}(G(F))$

$M \longmapsto M \otimes_{H_K} \left(c\text{-ind}_K^{G(F)} \mathbb{1}\right)$

Rmk. This is not a block of $\mathrm{Rep}(G(F))$

Next $K = I$ Iwahori. H_I

G split $\supset B \supset T$ $\qquad F \supset \mathcal{O} \twoheadrightarrow k_F$

$$\begin{tikzcd}
G(\mathcal{O}) \arrow[r, "\bmod \varpi"] & G(k_F) \\
I \arrow[u, phantom, "\cup" rotate=-90] \arrow[r] & B(k_F) \arrow[u, phantom, "\cup" rotate=-90]
\end{tikzcd}$$

$$G(\mathcal{O})/I = G(k_F)/B(k_F) = (G/B)(k_F)$$

$$H_I \simeq C_c(I \backslash G(F)/I) \qquad \text{Iwahori - Hecke alg.}$$

$$I \backslash G(F)/I \xleftrightarrow{\sim} \widetilde{W} \quad \text{extended affine Weyl gp}$$

$$\begin{tikzcd}
1 \arrow[r] & T(F)/T(\mathcal{O}) \arrow[r] \arrow[d, phantom, "\wr" rotate=90] & \widetilde{W} := N_G(T)(F)/T(\mathcal{O}) \arrow[r] & N_G(T)(F)/T(F) = W \arrow[r] \arrow[l, bend left] & 1 \\
& X_*(T) & N_G(T)(\mathcal{O})/T(\mathcal{O}) = W \arrow[u] &
\end{tikzcd}$$

$$\widetilde{W} = X_*(T) \rtimes W \qquad\qquad W \backslash \widetilde{W}/W = X_*(T)/W = X_*(T)^+$$

$$G(F) = \bigsqcup_{\lambda} G(\mathcal{O})\, \varpi^\lambda\, G(\mathcal{O}) = \bigsqcup_{\substack{\lambda \\ w_1, w_2 \in W}} I w_1 I \varpi^\lambda I w_2 I.$$

Rmk $G = G_{sc}$ is simply conn'd

$\widetilde{W} = W_{aff}$ is a Coxeter gp

affine Weyl gp of $G(F)$

$(S \;, \ell : W_{aff} \to \mathbb{Z}_{\geq 0})$

simple reflections

Fact $IwI/I \simeq I/_{I \cap wIw^{-1}}$ ← finite set size $|k_F|^{\ell(w)}$

relies on the splitness

$S = \{ w \in \widetilde{W} : \ell(w) = 1 \}$

$$\widetilde{W} = W_{aff} = \left\langle s \in S : \begin{array}{l} s^2 = 1 \\ \underbrace{stst\cdots}_{n_{st}} = \underbrace{tstr\cdots}_{n_{st}} \end{array} \right\rangle$$

$n_{st} \in \{3, 4, 6, \infty\}$

In general (i.e. G may not be simply-connected)

$\ell : \widetilde{W} \to \mathbb{Z}_{\geq 0}$ defined by $\#(\overset{I/I \cap wIw^{-1}}{IwI/I}) = q^{\ell(w)}$

Let $\Omega = \{ w \in \widetilde{W} : \ell(w) = 0 \} = N_{G(F)}(I)/I$ subgp $\subset \widetilde{W}$

$$G_{sc} \longrightarrow G$$

$$T_{sc} \longrightarrow T$$

$$I_{sc} \longrightarrow I$$

$$W_{aff} = \widetilde{W}_{G_{sc}} \hookrightarrow \widetilde{W}$$

$$\begin{tikzcd}
1 \arrow[r] & W_{aff} \arrow[r] & \widetilde{W} \arrow[r] & \widetilde{W}/W_{aff} \arrow[r] \arrow[d, phantom, "\simeq"] & 1 \\
& & \Omega \arrow[u, hook] \arrow[ur, phantom, "\simeq"] & X_*(T)/X_*(T_{sc}) &
\end{tikzcd}$$

$\leftarrow$ coroot lattice

$$\begin{tikzcd}
1 \arrow[r] & X_*(T) \arrow[r] & \dfrac{N_G(T)(F)}{T(\mathcal{O})} \arrow[r] & W \arrow[r] & 1 \\
1 \arrow[r] & X_*(T_{sc}) \arrow[r] \arrow[u] & \dfrac{N_{G_{sc}}(T_{sc})(F)}{T_{sc}(\mathcal{O})} \arrow[r] \arrow[u] & W_{sc} \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

Let $T_w = \mathbb{1}_{IwI}$ $\quad w \in \widetilde{W}$

$\in C_c(I \backslash G(F)/I)$

$\{T_w\}$ from a basis of H_I

<u>Thm</u> (1) $G = G_{sc}$, $\widetilde{W} = W_{aff}$, H_I ($= H_{aff}$) has the following presentation

- $(T_s - q)(T_s + 1) = 0 \Leftrightarrow T_s^2 = (q-1)T_s + q$
  $s \in S$
- $\underbrace{T_s T_t T_s \cdots}_{n_{st}} = \underbrace{T_t T_s T_t \cdots}_{n_{st}}$ $\Leftarrow$ $T_{wv} = T_w T_v$ if $\ell(wv) = \ell(w) + \ell(v)$

2) In general (G not necessarily s.c.)

$$H_I = H_{aff} \rtimes \mathbb{C}[\Omega]$$

$$T_w \cdot T_\omega \quad \omega \in \Omega$$

$$= T_\omega \cdot T_{\omega^{-1} w \omega}$$

Lecture 2. G split reductive gp $/F \supset \mathcal{O} \overset{\varpi}{\ni} \twoheadrightarrow k_F$

$(G, B, T)/\mathcal{O}$

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & G(\mathcal{O}) \arrow[d, "\bmod \varpi"] \\ B(k_F) \arrow[r, hook] & G(k_F) \end{tikzcd} \qquad H_I = C_c(I \backslash G(F) / I, \mathbb{C})$$

$$\begin{tikzcd} 1 \arrow[r] & X_*(T) \arrow[r] \arrow[d, equal] & \widetilde{W} \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[l, bend right] \arrow[d, equal] & 1 \\ 1 \arrow[r] & X_*(T) \arrow[r] & \dfrac{N_G(T)(F)}{T(\mathcal{O})} \arrow[r] & \dfrac{N_G(T)(F)}{T(F)} \arrow[r] & 1 \end{tikzcd}$$

$$\lambda \longmapsto t_\lambda$$

$$G(F) = \coprod_{w \in \widetilde{W}} I w I$$

$$\varpi \longmapsto \lambda(\varpi) \longmapsto t_\lambda$$

$\lambda \in X_*(T)$: $G_m \to T \rightsquigarrow F^\times \ni \varpi \to T(F) \longrightarrow T(F)/T(\mathcal{O})$

Splitting $W = \dfrac{N_G(T)(\mathcal{O})}{T(\mathcal{O})} \hookrightarrow \dfrac{N_G(T)(F)}{T(\mathcal{O})}$

$$\widetilde{W} = X_*(T) \rtimes W$$

$$\widetilde{W} \cong X_*(T) \rtimes W$$

$$w = t_\lambda \cdot w_f$$

G simply-conn'd, $\left(W_{aff} = \widetilde{W} \text{ Coxeter gp}, \ S \right)$

$$\begin{tikzcd}
 & \mathbb{Q}^\vee \arrow[d, equal] & & & \\
1 \arrow[r] & X_*(T_{sc}) \arrow[r] \arrow[d] & W_{aff} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & X_*(T) \arrow[r] \arrow[d] & \widetilde{W} \arrow[r] \arrow[d, shift right] & W \arrow[r] & 1 \\
 & X_*(T)/\mathbb{Q}^\vee \arrow[r, "\sim"] & \pi_0(\widetilde{W}) \arrow[u, bend right, "\Omega"'] & &
\end{tikzcd}$$

$\mathbb{Q}^\vee$ coroot lattice

$$\Omega = \{ w \in \widetilde{W} : wIw^{-1} = I \} = N_{G(F)}(I)/I$$

$$l : \widetilde{W} \to \mathbb{Z}, \quad l(w) = \log_{\# k_F} \left(\# \, I/I \cap wIw^{-1} \right)$$

$$\Omega = \{ w : l(w) = 0 \}.$$

If $\widetilde{W} = W_{aff}$, $S = \{ w : l(w) = 1 \}$.

Let $\Phi(G,T) \subset X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ be the set of roots of (G,T)

Let $\Phi_{aff}(G,T) = \{ \underset{\alpha}{\underbrace{a+k}} : a \in \Phi(G,T), \ k \in \mathbb{Z} \}$

$\alpha, \beta, \cdots, \quad a = \dot{\alpha}, \ b = \dot{\beta}, \cdots$

$$A(G,T) = X_*(T_{ad}) \otimes \mathbb{R} \simeq \mathbb{Q}^\vee \otimes \mathbb{R}$$

$\Phi_{\mathrm{aff}}(G,T) \subset \mathrm{AffFun}(A(G,T))$

$$(a+k)(v) = a(v)+k$$

For $a \in \Phi(G,T) \rightsquigarrow U_a \xrightarrow{\sim} G_a$ up to multiplication by $\mathcal{O}^\times$.

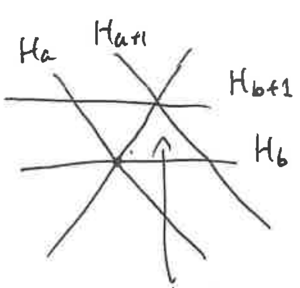
$\alpha \in \Phi_{\mathrm{aff}}(G,T)$, $\quad U_\alpha \subset U_a(F)$

$\alpha = a+k$, $\quad U_\alpha = \varpi^k \mathcal{O} \subset F$

$\boxed{U_\alpha \text{ indep. of the choice of } U_a \simeq G_a \text{ and } \varpi.}$

$$\begin{tikzcd} I \arrow[d] & \simeq & \prod\limits_{\substack{\alpha=a+1\\ a\in\Phi^-}} U_\alpha \times T(\mathcal{O}) \times \prod\limits_{\substack{\alpha=a\\ a\in\Phi^+}} U_\alpha \arrow[d] \\ G(F) & \hookleftarrow & U^-(F)\times T(F)\times U(F) \end{tikzcd}$$

$A(G,T)$



fund. alcove C

$C \subset \{v : a(v) \geq 0, \ \forall a \in \Phi^+\}$

$0 \in \bar{C}$

$\forall v \in A(G,T)$

$P_v = \langle T(\mathcal{O}), U_\alpha, \alpha(v) \geq 0 \rangle$

P_v parahoric

If $v \in C$, $P_v = I$.

If $v = 0$, $P_v = G(\mathcal{O})$

We can define an action of $\widetilde{W}$ on $A(G,T)$ by affine transformations.

$$\widetilde{W} = X_*(T) \rtimes W$$

$$\lambda \mapsto \bar{\lambda}, \quad W \curvearrowright X_*(T)$$

$$t_\lambda(v) = v - \bar{\lambda}$$

$$w(v) = w(v)$$

$$\widetilde{W} \curvearrowright \Phi_{aff}(G,T) \subset \mathrm{Aff\,Fun}(A(G,T))$$

$$(w.f)(v) = f(w^{-1}v)$$

<u>Lemma</u> $w \in \widetilde{W} \xrightarrow{\text{lifting}} \dot{w} \in G(F)$

$$\dot{w}\, U_\alpha\, \dot{w}^{-1} = U_{w(\alpha)}$$

$$\Omega = \{w \in \widetilde{W} : w(C) = C\}$$

$$\Phi^+_{aff} = \{\alpha : \alpha(C) > 0\}$$

$$l(w) = \#\{\alpha \in \Phi^+_{aff} : w(\alpha) \in \Phi^-_{aff}\}$$

$$I = P_v, \quad v \in A(G,T)$$

$$wIw^{-1} = P_{w(v)}$$

$S \subset \widetilde{W}$: $\forall \alpha \rightsquigarrow S_\alpha := t_{ka^\vee} s_a$, $\quad t_{ka^\vee} \in X_*(T)$, $s_a \in W$

$$\alpha = a + k$$

$$S_\alpha(v) = v - \alpha(v)a^\vee$$

$S_\alpha^2 = id$, S_α is an affine reflection w/ fixed pt

$$H_\alpha = \{\alpha = 0\}$$

$$W_{aff} = \langle S_\alpha \rangle_{\alpha \in \Phi_{aff}}$$

Fact: $\exists\, \Delta_{aff} \subset \Phi^+_{aff}$ affine simple roots s.t. every $\alpha \in \Phi^+_{aff}$ is a (unique, if G simple) linear combination $\{\alpha_i\}$ w/ non-negative integral coeff. ($\Delta_{aff} = \{\alpha_i\}$)

If G is simple, $\alpha_0 = 1 - \theta \leftarrow$ highest root

$\alpha_1 = a_1$

$\vdots \quad \nwarrow$ simple roots of G

$\alpha_\ell = a_\ell$

$S \in \tilde{W}$ are $\{s_i\}_{\alpha_i \in \Delta_{aff}}$.

Let $T_w = \mathbb{1}_{IwI}$

Thm. H_I has basis $\langle T_w \rangle_{w \in \tilde{W}}$ w/ relations

- $T_w T_v = T_{wv}$ if $\ell(w) + \ell(v) = \ell(wv)$
- $T_s^2 = (q-1) T_s + q$, $s \in S$, $q = \# k_F$

Pf

$$IwI \overset{I}{\times} IvI \longrightarrow G(F) \quad (\to IwvI \to)$$

$$g_1, g_2 \longmapsto g_1 g_2$$

$$(T_w * T_v)(g) = \sum_{g' \in G(F)/I} T_w(gg') \, T_v(g'^{-1})$$

$$IsI \overset{I}{\times} IsI \longrightarrow G(F)$$

preimage has $q-1$ elts $\longrightarrow s$

q elts $\longrightarrow 1$

Cor. $H_I = H_{aff} \rtimes \underset{\langle T_w : w \in \Omega\rangle}{\mathbb{C}[\Omega]}$ $\qquad \widetilde{W} = W_{aff} \rtimes \Omega$

$\simeq X_*(T) \rtimes W$

$\simeq \mathbb{C}[X_*(T)] \rtimes H_W$

$q^{\square}\, T_{t_\lambda} \longleftarrow\!\!| \ \lambda$ dominant

$$H_I\text{-mod} \overset{\sim}{\hookrightarrow} \operatorname{Rep}(G(F))^{[I]} \leftarrow \text{block of } \operatorname{Rep}(G(F))$$

$$M \longmapsto M \underset{H_I}{\otimes} \text{c-ind}_I^{G(F)} \mathbb{1}$$

Cor. $W \subset V$, V gen. by V^I

$\Rightarrow W$ gen. by W^I.

$$\begin{tikzcd}
I^+ \arrow[r] \arrow[d] & I \arrow[r, hook] \arrow[d] & G(\mathcal{O}) \arrow[d] \\
U(k_F) \arrow[r, hook] & B(k_F) \arrow[r, hook] & G(k_F)
\end{tikzcd}$$

$I/I^+ \overset{\sim}{\rightarrow} T(k_F)$

$$H_{I^+} = C_c(I^+ \backslash G(F) / I^+)$$

$$= \operatorname{End}_{G(F)}\left(\text{c-ind}_{I^+}^{G(F)} \mathbb{1}\right)$$

$$= \operatorname{End}_{G(F)}\left(\text{c-ind}_{I}^{G(F)} \operatorname{ind}_{I^+}^{I} \mathbb{1}\right)$$

$$\operatorname{ind}_{I^+}^{I} \mathbb{1} = \bigoplus_{\chi : T(k_F) \to \mathbb{C}^\times} \chi$$

$$\text{c-ind}_{I^+}^{G(F)} \mathbb{1} = \bigoplus_{\chi : T(k_F) \to \mathbb{C}^\times} \text{c-ind}_{I}^{G(F)} \chi$$

$$H_{(I,\chi)} = \operatorname{End}\left(\text{c-ind}_I^{G(F)} \chi\right) = \left\{ \underset{\text{cpt supp.}}{\Phi :} G \to \overset{\mathbb{C}}{\overset{\shortparallel}{\operatorname{End}(\chi)}} : \Phi(kgk') = \chi(k)\Phi(g)\chi(k') \right\}$$

(V,σ) rep. of K

$$H_{(K,\sigma)} = \{ \underset{\text{cpt supp.}}{\Phi : G(F) \longrightarrow \mathrm{End}(V)} : \Phi(k_1 g k_2) = \sigma(k_1)\Phi(g)\sigma(k_2) \}$$

$$= \bigoplus_{g \in K \backslash G / K} \underset{\{\Phi : \,\mathrm{supp}(\Phi) = K g K\}}{\overset{}{H_{K,\sigma,g}}}$$

$H_{K,\sigma,g} = \{\Phi : \mathrm{supp}(\Phi) = K g K\}$

Let $K_g = K \cap g K g^{-1}$

$K_g \subset K$

$K_{g^{-1}}$ ⊂

Lemma. $H_{K,\sigma,g} \neq 0 \iff \mathrm{Hom}_{K_g}(V|_{K_{g^{-1}}}, V|_{K_g}) \neq 0.$

$$\left(H_{K,\sigma,g} \xrightarrow{\sim} \mathrm{Hom}_{K_g}(V|_{K_{g^{-1}}}, V|_{K_g}) \right)$$

$K_g \xrightarrow{\sim} K_{g^{-1}}$

$k \longmapsto g^{-1} k g$

$k \in K_g$

$g = k g \, g^{-1} k^{-1} g$

$$\Phi(g) = \sigma(k)\, \Phi(g)\, \sigma(g^{-1}k^{-1}g)$$

$$\Updownarrow$$

$$\Phi(g)\,\sigma(g^{-1}kg) = \sigma(k)\,\Phi(g)$$

$H_{(I,\chi,w)} \neq 0$

$$\mathrm{Hom}_{I_w}(\chi|_{I_{w^{-1}}}, \chi|_{I_w}) \neq 0$$

$$\|$$

$$\mathrm{Hom}_{T(k_F)}({}^{w^{-1}}\chi, \chi) \neq 0$$

$$\begin{tikzcd}
I_w \arrow[r, hook] \arrow[d] \arrow[rr, bend left] & I \arrow[r] & T(k_F) \arrow[r, "\chi"] \arrow[d, "w^{-1}"] & \mathbb{C}^\times \\
I_{w^{-1}} \arrow[rr] & & T(k_F) &
\end{tikzcd}$$

$\underline{\text{or}}$. $H_{I,\chi} = \bigoplus_{\substack{w \in \widetilde{W} \\ {}^w\chi = \chi}} H_{I,\chi,w}$, $\quad \widetilde{W}_\chi := \{ w \in \widetilde{W} : {}^w\chi = \chi \}$

Lecture 3

$$H_{I^+} = \mathrm{End}\left(c\text{-ind}_{I^+}^{G(F)} \mathbb{1} \right)$$

$$= \mathrm{End}\left(\bigoplus_{\chi : T(k_F) \to \mathbb{C}^\times} c\text{-ind}_I^{G(F)} \chi \right)$$

$$= \bigoplus_{\chi, \chi'} \underbrace{\mathrm{Hom}\left(c\text{-ind}_I^{G(F)} \chi, \ c\text{-ind}_I^{G(F)} \chi' \right)}_{{}_\chi H_{\chi'}}$$

Last time, ${}_\chi H_{\chi'} = \{ f : G(F) \to \mathbb{C} \ : \ f(kgk') = \chi(k) f(g) \chi'(k'), \ k, k' \in I \}$

$$= \bigoplus_w {}_\chi H_{\chi'}^w, \longleftarrow \{ f : \mathrm{supp}(f) \subset IwI \}$$

$${}_\chi H_{\chi'}^w \neq 0 \iff \chi = w\chi' : T(k_F) \to \mathbb{C}^\times$$

$${}_\chi \widetilde{W}_{\chi'} := \{ w \in \widetilde{W} : \ \chi = w\chi' : \ T(k_F) \to \mathbb{C}^\times \}$$

$\chi = \chi'$, $\quad {}_\chi \widetilde{W}_\chi =: \widetilde{W}_\chi \leftarrow$ subgp of $\widetilde{W}$.

$$1 \to X_*(T) \to \widetilde{W}_\chi \twoheadrightarrow W_\chi \longrightarrow 1$$

$$W_\chi = \{ w \in W : \ w\chi = \chi \}$$

Let $\Phi_\chi^\vee = \{ \alpha^\vee \in \Phi^\vee : \ \chi \circ \alpha^\vee \text{ is trivial} \}$ ← subroot system

$\Phi_\chi^\vee \subset \Phi^\vee$ ← coroots of (G, T)

$\chi \circ \alpha^\vee$: $\quad \mathbb{G}_m \xrightarrow{\alpha^\vee} T$, $\quad k_F^\times \to T(k_F) \xrightarrow{\chi} \mathbb{C}^\times$

this is not a subroot system in general.

$\Phi_\chi \leftarrow$ dual of $\Phi_\chi^\vee$

$\Phi_\chi \subset \Phi$

$\Phi_{\chi, \mathrm{aff}} = \left\{ a + k : \ \begin{matrix} a \in \Phi_\chi \\ k \in \mathbb{Z} \end{matrix} \right\} \subset \Phi_{\mathrm{aff}}$, $\quad \Phi_{\chi,\mathrm{aff}} \subset \mathrm{Aff\ Fun}(X_*(T))$

$$W_x^\circ := \langle s_a : a \in \Phi_x \rangle \subset W_x$$

$$\widetilde{W}_x^\circ := \langle s_\alpha : \alpha \in \Phi_{x,\mathrm{aff}} \rangle$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & X_*(T) & \longrightarrow & \widetilde{W}_x & \longrightarrow & W_x & \longrightarrow & 1 \\
 & & \| & & \cup & & \cup & & \\
1 & \longrightarrow & X_*(T) & \longrightarrow & \widetilde{W}_x^1 & \longrightarrow & W^\circ_x & \longrightarrow & 1 \\
 & & \cup & & \cup & & \| & & \\
1 & \longrightarrow & \mathbb{Z}\Phi_x^\vee & \longrightarrow & \widetilde{W}_x^\circ & \longrightarrow & W_x^\circ & \longrightarrow & 1
\end{array}$$

$$G \rightsquigarrow \overset{\text{Langlands dual}}{(\hat{G}, \hat{B}, \hat{T}, \hat{e})}$$

$$\hat{B} \supset \hat{U} \qquad \hat{e} : \hat{U} \longrightarrow G_a$$

$$\hat{e}|_{\hat{U}_{a^\vee}} \xrightarrow{\sim} G_a$$

$$T \longrightarrow \hat{T}$$

$$\Phi^\vee(G,T) = \Phi(\hat{G},\hat{T})$$

$$X_*(T) = X^*(\hat{T})$$

$$\begin{array}{ccc}
X^*(T) & = & X_*(\hat{T}) \\
\cup & & \cup \\
\Phi(G,T) & = & \Phi^\vee(\hat{G},\hat{T})
\end{array}$$

$$\begin{array}{l}
\chi : T(k_F) \longrightarrow \mathbb{C}^\times \\
\qquad \| \\
X_*(T)\otimes k_F^\times \longrightarrow \mathbb{C}^\times
\end{array}$$

$$\longleftrightarrow \quad k_F^\times \longrightarrow X^*(T)\otimes \mathbb{C}^\times = X_*(\hat{T})\otimes\mathbb{C}^\times$$

$$k_F^\times \cong \mathbb{Z}/(q-1)\mathbb{Z} \qquad = \hat{T}(\mathbb{C})$$

$$\chi \in \hat{T}[q-1] \subset \hat{T}(\mathbb{C})$$

Now let $s \in \hat{T}$

$$1 \to \underset{\substack{\| \\ \hat{H}}}{Z_{\hat{G}}(s)^\circ} \to Z_{\hat{G}}(s) \longrightarrow \pi_0(s) := \pi_0(Z_{\hat{G}}(s)) \to 1$$

$\Phi_s^\vee = \{ a^\vee \in \Phi^\vee(G,T) = \Phi(\hat{G},\hat{T}) \ : \ a^\vee(s) = 1 \}$

$\hat{T} \xrightarrow{a^\vee} \mathbb{G}_m$

(If $s = x$, $\Phi_s^\vee = \Phi_x^\vee$)

$W_s = \{ w \in W(\hat{G},\hat{T}) \ : \ w(s) = s \}$

$\|$ if $s = x$

W_x

$W_s^\circ = \langle s_{a^\vee} : a^\vee \in \Phi_s^\vee \rangle$

<u>Lemma</u>. $\hat{T} \subset \hat{H}$ is a max. torus of $\hat{H}$, $\Phi(\hat{H},\hat{T}) = \Phi_s^\vee$, $W(\hat{H},\hat{T}) = W_s^\circ$

$B_{\hat{H}} = \hat{T} \cdot \prod_{a^\vee \in \Phi_+^\vee \cap \Phi_s^\vee} U_{a^\vee}$

<u>Rmk</u>. simple roots in $\Phi_s^\vee$ may not be simple in $\Phi^\vee$.

<u>Lemma</u>. $\pi_0(s) \cong \dfrac{W_s}{W_s^\circ}$

<u>Pf</u>. $N_{Z_{\hat{G}}(s)}(\hat{T}) \hookrightarrow N_{\hat{G}}(\hat{T})$

meets every component of $Z_{\hat{G}}(s)$.

$$1 \to \hat{T} \to N_{Z_{\hat{G}}(s)}(B_{\hat{A}}, \hat{T}) \twoheadrightarrow \pi_0(s) \to 1$$

$$W_s = \frac{N_{Z_{\hat{G}}(s)}(\hat{T})}{\hat{T}} \hookleftarrow \frac{N_{Z_{\hat{G}}(s)}(B_{\hat{A}}, \hat{T})}{\hat{T}} \overset{\sim}{\Rightarrow} \pi_0(s) = \frac{W_s}{W_s^\circ}$$

$$\begin{tikzcd} 1 \arrow[r] & W_s^\circ \arrow[r] & W_s \arrow[r] & \pi_0(s) \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}$$

canonical splitting

<u>Rmk.</u>

$$1 \to Z(\hat{A}) \to N_{Z_{\hat{G}}(s)}(B_{\hat{A}}, \hat{T}, e_{\hat{A}}) \to \pi_0(s) \to 1$$

This is usualy <u>non-split</u>. (cause a lot of problems)

<u>Rmk.</u> If $\hat{G}_{der} = [\hat{G}, \hat{G}]$ is simply-connected, then

$Z_{\hat{G}}(s)$ is conn'd (Steinberg)

$\left[\hat{G} = PGL_2, \quad s = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad Z_{\hat{G}}(s) = N_{\hat{G}}(\hat{T}) \right]$

<u>Lemma</u> $\pi_0(s)$ is abelian.

<u>Pf.</u>

$$\tilde{s} \longmapsto s$$

$$\hat{G}_{sc} \twoheadrightarrow \hat{G} \ni s \text{ ss}$$

$$1 \to \Gamma \to \hat{G}_{sc} \to \hat{G} \to 1$$

injective gp hom.

$$\pi_0(s) \to \Gamma \simeq \pi_1(G)$$

$$w \longmapsto \frac{w(\tilde{s})}{\tilde{s}}$$

<u>Ex.</u> $G = Sp(2n)$, $\hat{G} = SO(2n+1)$.

$X^*(T) = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i$, $\Phi = \{\pm\epsilon_i \pm \epsilon_j, \ \pm 2\epsilon_j\}$. simple roots $\{\epsilon_1 - \epsilon_2, \cdots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$

$X_*(T) = X^*(\hat{T}) = \oplus \mathbb{Z} \epsilon_i^\vee$

$\Phi^\vee = \{ \pm\epsilon_i^\vee \pm \epsilon_j^\vee , \pm \epsilon_j^\vee \}$, simple $\{ \epsilon_1^\vee - \epsilon_2^\vee, \ldots, \epsilon_{n-1}^\vee - \epsilon_n^\vee, \epsilon_n^\vee \}$

let $s \in \hat{T}$, $\epsilon_i^\vee(s) = -1$

$\Phi_s^\vee = \{ \pm \epsilon_i^\vee \pm \epsilon_j^\vee \}$ simple $\{ \epsilon_1^\vee - \epsilon_2^\vee, \ldots, \epsilon_{n-1}^\vee - \epsilon_n^\vee, \epsilon_{n-1}^\vee + \epsilon_n^\vee \}$

$$1 \longrightarrow \underset{\substack{\parallel \\ SO(2n)}}{\hat{H}} \longrightarrow Z_{\hat{G}}(s) = O(2n) \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

$\Phi_s = \{ \pm\epsilon_i \pm \epsilon_j \}$ simple $\{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n \}$

$s \in \hat{T} \rightsquigarrow (\hat{H}, B_{\hat{H}}, \hat{T}, e_{\hat{H}})$

$\downarrow$ dualize

(G, B, T, e) (H, B_H, T_H, e_H) ← endoscopic gp

$H = SO(2n)$

No map from $SO(2n) \to Sp(2n)$ if $n \geq 4$

In general, H is not a subgp of G.

$$\frac{\tilde{W}_x}{\tilde{W}_x^1} = \frac{W_x}{W_x^0} = \pi_0(x)$$

Come back to $\tilde{W}_x \supset \tilde{W}_x^1 \supset \tilde{W}_x^0$.

$$\begin{array}{ccccc} 1 \to & \mathbb{Z}\Phi_x^\vee & \to & \tilde{W}_x^0 & \\ & \cap & & \cap & \searrow W_x^0 \\ 1 \to & X_*(T) & \to & \tilde{W}_x^1 & \nearrow \end{array}$$

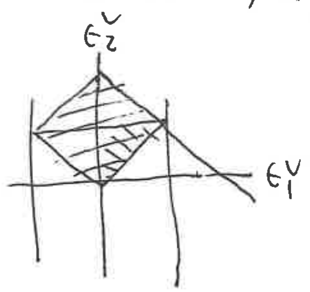
$\tilde{W}_x^0$ is the affine Weyl gp of H

$\cap$

$\tilde{W}_x^1$ - - - Iwahori-Weyl gp of H (extended affine)

$\pi_0(X) = \{1\}$ if Z_G is conn'd $\Leftrightarrow$ $\hat{G}_{der}$ is simply conn'd.

Ex. $G = Sp(4)$, $\hat{G} = SO(5)$



$\hat{H} = SO(4)$

$$\begin{tikzcd}
 & & & 1 & \\
 & & & \pi_0(X) \arrow[u] & \\
1 \arrow[r] & \tilde{W}_x^\circ \arrow[r] & \tilde{W}_x \arrow[r] & \Omega_x \arrow[r] \arrow[l, bend right] \arrow[u] & 1 \\
1 \arrow[r] & \tilde{W}_x^\circ \arrow[r] & \tilde{W}_x^1 \arrow[r] \arrow[u] & \Omega_x' = \pi_1(H) \arrow[u] & \\
 & & & 1 \arrow[u] &
\end{tikzcd}$$

$\Omega_X \simeq \{ w \in \tilde{W}_X :$ w preserves the fundamental alcove of $H \}$

$= w(\Phi^+_{X,aff}) = \Phi^+_{X,aff}$

<u>Lemma</u> (1) For every $w \in \Omega_X$, $\ell(wv) \geq \ell(w)$, $\forall v \in \tilde{W}_X^\circ$.

(2) If $v_1 \leq_X v_2$, then $wv_1 \leq wv_2$ in $\tilde{W}$

$\leq_X$: Bruhat order of $\tilde{W}_X^\circ$

Rmk. The converse is not always true.

(3) $\ell_X : \tilde{W}_X \to \mathbb{Z}_{\geq 0}$, (in general $\ell_X \neq \ell|_{\tilde{W}_X}$)

we have $\ell(v) - \ell(v') \geq \ell_X(v) - \ell_X(v')$ if $v' \leq_X v$.

$$\widetilde{W}_x \curvearrowright {}_x\widetilde{W}_{x'} \curvearrowleft \widetilde{W}_{x'}$$

Simply - transitive

$$\widetilde{W}^\circ_x \backslash {}_x\widetilde{W}_{x'} \simeq {}_x\widetilde{W}_{x'} / \widetilde{W}^\circ_{x'}$$

$$\widetilde{W}^\circ_x \backslash {}_x\widetilde{W}_{x'} \hookrightarrow {}_x\widetilde{W}_{x'}$$

minimal length lifting

Lecture 4 $\quad x, x' : T(k_F) \longrightarrow \mathbb{C}^\times$

$${}_x\widetilde{W}_{x'} = \{ w \in \widetilde{W} : wx' = x \} \quad , \quad \widetilde{W}_x := {}_x\widetilde{W}_x$$

$$\widetilde{W}_x \curvearrowright {}_x\widetilde{W}_{x'} \curvearrowleft \widetilde{W}_{x'}$$

$$\widetilde{W}_x \supset \widetilde{W}^\circ_x = \langle s_\alpha : \alpha \in \Phi_{x, \mathrm{aff}} \rangle$$

$$\begin{tikzcd} 1 \arrow[r] & W^\circ_x \arrow[r] & \widetilde{W}_x \arrow[r] & \Omega_x \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}$$

$$\Omega_x = \{ w \in \widetilde{W}_x : w(\Phi^+_{x,\mathrm{aff}}) \subset \Phi^+_{x,\mathrm{aff}} \}$$

$$\widetilde{W}^\circ_x \backslash {}_x\widetilde{W}_{x'} \xrightarrow{\sim} {}_x\widetilde{W}_{x'} / \widetilde{W}^\circ_{x'}$$

$$\| \qquad\qquad {}_x\Omega_{x'} \simeq \{ w \in {}_x\widetilde{W}_{x'} : w(\Phi^+_{x',\mathrm{aff}}) \subset \Phi^+_{x,\mathrm{aff}} \}$$

$$\Omega_x \xleftarrow{\sim} {}_x\Omega_{x'} \xrightarrow{\sim} \Omega_{x'}$$

Lemma (1) Let $w \in {}_x\Omega_{x'}$, then $\ell(w) < \ell(wv')$ for any $v' \in \tilde{W}^{\circ}_{x'} - \{1\}$

$\ell(w) < \ell(vw)$ for any $v \in \tilde{W}^{\circ}_{x} - \{1\}$

(2) Let $v_1 \leq_{x'} v_2$ Bruhat order in $\tilde{W}^{\circ}_{x'}$, then

$wv_1 \leq wv_2$ Bruhat order in $\tilde{W}$.

Rmk. $wv_1 \leq wv_2$ does not imply $v_1 \leq_{x'} v_2$ in general.

Pf. (1) We use (W, S) (quasi-) Coxeter gp,

Let $\alpha \in \Phi^+ \leftarrow$ (affine) root

$w(\alpha) > 0 \Rightarrow \ell(ws_\alpha) > \ell(w) \Rightarrow ws_\alpha \geq w$

$w(\alpha) < 0 \Rightarrow \ell(ws_\alpha) < \ell(w) \Rightarrow ws_\alpha < w$.

If $x \in w(\tilde{W}^{\circ}_{x'})$ is of minimal length, $\ell(xs_\alpha) > \ell(x)$, $\alpha \in \Phi^+_{x', aff}$

$\Rightarrow x(\alpha) > 0 \Rightarrow x = w$

On the other hand, wv $\quad v \neq 1 \in \tilde{W}^{\circ}_{x'}$

$\exists\ \alpha \in \Phi^+_{x', aff}$, $v(\alpha) < 0 \Rightarrow (wv)(\alpha) < 0$

$\Downarrow$

wv not of min. length

(2) $v_2 = v_1 s_\alpha$, α simple in $\Phi^+_{x', aff}$. $v_2 \geq_{x'} v_1 \Rightarrow v_1(\alpha) > 0$

$(wv_1)(\alpha) = w(v_1(\alpha)) > 0 \Rightarrow wv_1 s_\alpha > wv_1$.

Def. Let $w \in {}_x\Omega_{x'}$, $\begin{matrix} v \in \\ v' \in \end{matrix} w\left(\widetilde{W}^\circ_{x'}\right) = \left(\widetilde{W}^\circ_x\right) w$

$$\ell_w(v) = \ell_{x'}(w^{-1}v) = \ell_x(vw^{-1})$$

$v \underset{w}{\leq} v'$ if $w^{-1}v \underset{x'}{\leq} w^{-1}v'$

"groupoid" ${}_x\widetilde{W}_{x'} \cdot {}_{x'}\widetilde{W}_{x''} \to {}_x\widetilde{W}_{x''}$

${}_x\Omega_{x'} \cdot {}_{x'}\Omega_{x''} \to {}_x\Omega_{x''}$

Lemma $\ell_w(v) = \# \left\{ \alpha \in \Phi^+_{x', \mathrm{aff}} : v(\alpha) < 0 \right\}$

If we write $v = s_{i_1} \cdots s_{i_n} \tau$, s_{i_j} simple reflections in $\widetilde{W}$, $\tau \in \Omega$.
reduced expr.
$\ell(v) = n$

$$x \xrightarrow{s_{i_1}} \cdots x_{j-1} \xrightarrow{s_{i_j}} x_j \cdots s_{i_n}\tau x' \xrightarrow{s_{i_n}} \tau x' \xrightarrow{\tau} x'$$

$$\ell_w(v) = \# \{ j : s_{i_j} \in \widetilde{W}^\circ_{x_j} \}$$

$$\leq \ell(v)$$

$sx \widetilde{W} x$
$\underset{s}{\Psi}$ simple reflection in $\widetilde{W}$

If $s \notin \widetilde{W}^\circ_x$, then $s \in {}_{sx}\Omega_x$

Back to $H = \mathrm{End}\left(c\text{-}\mathrm{ind}_{I^+}^{G(F)} \mathbb{1}\right) = \bigoplus_{x,x'} {}_xH_{x'}$

$${}_xH_{x'} = \{ f : G(F) \to \mathbb{C} : f(k_1 g k_2) = x(k_1) f(g) x(k_2) \}$$

$\parallel$

$$\bigoplus_{w \in \widetilde{W}} {}_xH^w_{x'} = \{ f \in {}_xH_{x'} : \mathrm{supp}(f) \subset IwI \}$$

$${}_xH^w_{x'} \neq 0 \iff w \in {}_x\widetilde{W}_{x'}$$

$w \in {}_x\widetilde{W}_{x'}$, choose lifting $\dot{w} \in N_G(T)(F)$.

${}_xT^{\dot{w}}_{x'}(\dot{w}) = 1$, $\mathrm{supp}({}_xT^{\dot{w}}_{x'}) = IwI$.

$w \in {}_x\widetilde{W}_{x'}$, $v \in {}_{x'}\widetilde{W}_{x''}$,

$$\left({}_xT^{\dot{w}}_{x'}\ {}_{x'}T^{\dot{v}}_{x''}\right)(g) = \sum_{g' \in Iv^{-1}I/I} {}_xT^{\dot{w}}_{x'}(gg')\,{}_{x'}T^{\dot{v}}_{x''}(g'^{-1})$$

$$\overset{g' = g''\cdot\dot{v}^{-1}}{=\!=\!=} \sum_{g'' \in I^+/I^+ \cap v^{-1}I^+v} {}_xT^{\dot{w}}_{x'}(gg''\dot{v}^{-1})$$

Lemma. $\ell(wv) = \ell(w) + \ell(v) \Rightarrow {}_xT^{\dot{w}}_{x'}\ {}_{x'}T^{\dot{v}}_{x''} = {}_xT^{\dot{w}\dot{v}}_{x''}$

(If you don't choose $(\dot{w}\dot{v})$ carefully,

${}_xT^{\dot{w}}_{x'}\ {}_{x'}T^{\dot{v}}_{x''} = c\ {}_xT^{(\dot{w}\dot{v})}_{x''}$ for some constant c)

s simple reflection in $\widetilde{W}$, ${}_xT^{\dot{s}}_{sx}\ {}_{sx}T^{\dot{s}}_{x}$ $\quad \mathrm{supp} \subset I \cup IsI$

at $1 = q\,x(\dot{s}^{-2})$

$$\begin{pmatrix}1 & a\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -a^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & a\\ & 1\end{pmatrix} = \begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$$

$$\text{at } \dot{s} = \sum_{a \in k_F^\times} (x \circ \alpha_s^\vee)(a) = \begin{cases} (q-1), & s \in \widetilde{W}^\circ_x \\ 0, & s \notin \widetilde{W}^\circ_x,\ \alpha_s^\vee \circ x \neq 1 \end{cases}$$

Ex. $G = SL_2/\mathbb{Q}_p$ $\chi : k_F^\times \longrightarrow \{\pm 1\}$ Legendre symbol.

(unique nontrivial quadratic char.)

$\widetilde{W}^\circ_\chi \not\ni s$ $\dot{s} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $\dot{s}^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$

$\widetilde{W}_\chi \ni s$

$$\left({}_\chi T_\chi^{\dot{s}}\right)^2 = p\left(\frac{-1}{p}\right) {}_\chi T_\chi^{1} \qquad \boxed{\mathbb{Q}(\mu_p)}$$

Prop. (1) $w_1 \in {}_\chi \overset{\omega_1 \in {}_\chi \Omega_{\chi'}}{\widetilde{W}_{\chi'}}$, $w_2 \in {}_{\chi'} \overset{\omega_2}{\widetilde{W}_{\chi''}}$, $w = w_1 w_2 \in {}_\chi \overset{\omega = \omega_1 \omega_2}{\widetilde{W}_{\chi''}}$

then if $\ell_{\omega_1}(w_1) + \ell_{\omega_2}(w_2) = \ell_\omega(w)$

$$\Rightarrow {}_\chi T_{\chi'}^{\dot{w}_1} \, {}_{\chi'} T_{\chi''}^{\dot{w}_2} = c \, {}_\chi T_{\chi''}^{\dot{w}} , \quad c \neq 0$$

(2) Let $\alpha \in \Phi^+_{\chi, \mathrm{aff}}$ be a simple affine root

$$\left({}_\chi T_\chi^{\dot{s}_\alpha}\right)^2 = a \, {}_\chi T_\chi^{\dot{s}_\alpha} + b \, {}_\chi T_\chi^{1} , \quad a, b \neq 0 .$$

Hint: $\omega_1 = \omega_1$, $\omega_2 = \omega_2$, $\omega = \omega$, WTS

$${}_\chi T_{\chi'}^{\dot{\omega}_1} \, {}_{\chi'} T_{\chi''}^{\dot{\omega}_2} = c \, {}_\chi T_{\chi''}^{\dot{\omega}}$$

Induction on the length of w_2 , $w_2 = w_2' s_i$ ← simple reflection in $\widetilde{W}$

$\ell(w_2) = \ell(w_2') + 1$

$s_i \notin \widetilde{W}^\circ_{\chi''} \Rightarrow s_i \in {}_{s_i \chi''}\Omega_{\chi''} \Rightarrow w_2' \in {}_{\chi'}\Omega_{s\chi''}$

$${}_\chi T_{\chi'}^{\dot{w}_1} \, {}_{\chi'} T_{\chi''}^{\dot{w}_2} = c \, {}_\chi T_{\chi'}^{\dot{w}_1} \, {}_{\chi'} T_{s\chi''}^{\dot{w}_2'} \, {}_{s\chi''} T_{\chi''}^{\dot{s}_i}$$

induction $\quad c' \, {}_{x}T^{\dot{w}_1 \dot{w}_2'}_{s x''} \ {}_{s x''}T^{\dot{s}_i}_{x''}$

$$= c'' \, {}_{x}T^{\dot{w}_1 \dot{w}_2' \dot{s}_i}_{x''}$$

Lemma. Suppose $\exists\, M$ 1-dim repn of ${}_xH_x$, then

$${}_xH_x \cong \langle {}_x\tilde{T}^w_x \ ; \ w \in {}_x\tilde{W}_x \rangle$$

(1) ${}_x\tilde{T}^w_x \ {}_x\tilde{T}^v_x = {}_x\tilde{T}^{wv}_x$ if $\ell_{w_1}(w) + \ell_{w_2}(v) = \ell_w(wv)$

(2) $\left({}_x\tilde{T}^s_x\right)^2 = (q-1)\, {}_x\tilde{T}^s_x + q$, s simple reflection in $\tilde{W}^\circ_x$.

Let $m \neq 0 \in M$

$${}_x\tilde{T}^w_x \cdot m = q^{\ell_w(w)}\, m$$

Lecture 5. ${}_xH_x = \mathrm{End}\left(\text{c-ind}_I^{G(F)} x\right)$

${}_xH_{x'}$ supp ${}_x\tilde{W}_{x'} = \{ w \in \tilde{W} : wx' = x \}$

$$\downarrow$$

$$\tilde{W}^\circ_x \backslash {}_x\tilde{W}_{x'} = {}_x\Omega_{x'} = {}_x\tilde{W}_{x'} / \tilde{W}^\circ_{x'}$$

$\tilde{W}^\circ_x$ $\ \shortparallel$

$\langle s_\alpha$ reflections corresponding to α simple roots of $\Phi_{\mathrm{aff},x} \rangle$

may not be simple in Φ_{aff} $\to \alpha$

$\exists\, s_i, w,$

$s_\alpha = w s_i w^{-1}$, s_i simple reflection in W_{aff}, $w \in W_{aff}$

$\ell(s_\alpha) = 2\ell(w)+1$

$\ell_x(s_\alpha) = 1$

For a reduced word

$s_\alpha = t_1 \cdots t_{2r+1}$, t_i simple reflection

$w = s_{i_1} \cdots s_{i_r}$

$wx \;\cdots\; s_{i_{r-1}} s_{i_r} x \;\; s_{i_r} x \;\; x$

$\ell_x(s_\alpha) = \#\{t_i : t_i \in {}_{x_i}\widetilde{W}^{\circ}_{x_i}\}$

$\Rightarrow w \in {}_x \Omega_{w^{-1}x}$

$s_i \in {}_{w^{-1}x} \widetilde{W}^{\circ}_{w^{-1}x}$

For each simple reflection s_j, choose a lifting $\dot{s}_j$,

Let $\dot{s}_\alpha := \dot{s}_{i_1} \cdots \dot{s}_{i_r} \dot{s}_i \dot{s}_{i_r}^{-1} \cdots \dot{s}_{i_1}^{-1}$ $\qquad$ $\dot{s}_j = a^\vee(-1)$

$T^2_{\dot{s}_\alpha} = ?$ $\qquad$ Last time, $j = 1, \cdots, r,$

(1) $T_{\dot{s}_{i_j}} T_{\dot{s}_{i_j}^{-1}} = q T_1$

(2) $T_{\dot{s}_\alpha} = T_{\dot{s}_{i_1}} \cdots T_{\dot{s}_{i_r}} T_{\dot{s}_i} T_{\dot{s}_{i_r}^{-1}} \cdots T_{\dot{s}_{i_1}^{-1}}$

(3) $T^2_{\dot{s}_i} = (q-1) T_{\dot{s}_i} + q T_1$

(1) (2) (3) $\quad T^2_{\dot{s}_\alpha} = q^r \left((q-1) T_{\dot{s}_\alpha} + q \cdot q^r \right)$

$$\left(\frac{T_{\dot{s}_\alpha}}{q^r}\right)^2 = (q-1)\left(\frac{T_{\dot{s}_\alpha}}{q^r}\right) + q T_1$$

Goal: construct a 1-dim'l module of ${}_xH_x$ $\left(\Rightarrow {}_xH_x = {}_xH_x^\circ \rtimes \mathbb{C}[{}_x\Omega_x]\right)$

$F = \mathbb{F}_q((\varpi))$, $\mathbb{F}_q(\varpi) = K = ff(\mathbb{P}^1)$

$$C_c\left(G(K)\backslash G(\mathbb{A}_K) \Big/ \prod_{v\neq 0,\infty} G(\mathcal{O}_v) \times (I_0, x) \times (I_\infty^{opp,+}(1), \psi)\right)$$

$$\begin{tikzcd} I \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & B(k_F) \arrow[d, phantom, "\cap" rotate=90] \\ G(\mathcal{O}) \arrow[r] & G(k_F) \end{tikzcd} \qquad \begin{tikzcd} I^{op} \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & B^{op}(k_F) \arrow[d, phantom, "\cap" rotate=90] \\ G(\mathcal{O}) \arrow[r] & G(k_F) \end{tikzcd}$$

$$\underbrace{I \supset I^+}_{T(k_F)} \underbrace{\supset I^{++}}_{\prod U_\alpha(k_F)} = [I^+, I^+]$$

α affine simple root

for $G(K_0)$

$$\underbrace{I_\infty \supset I_\infty^+}_{T(k_F)} \underbrace{\supset I_\infty^{++}}_{\prod U_\alpha(k_F)}$$

for $G(K_\infty)$

α affine simple root

$$\psi : I_\infty^+ \longrightarrow I_\infty^+ / I_\infty^{++} \longrightarrow \mathbb{C}$$

$$I_\infty^+ / I_\infty^{++} = \prod_\alpha k_F$$

affine generic character

s.t. $\psi|_{k_F}$ is not trivial for each α.

$[U, U](\mathbb{F}_q)$

$\neq$

$[U(\mathbb{F}_q), U(\mathbb{F}_q)]$

$$C_c\left(G(K)\backslash G(\mathbb{A}_K) \Big/ \prod_{v\neq 0,\infty} G(\mathcal{O}_v) \times (I_0, \chi) \times (I_\infty^+, \psi)\right) =: {}_\chi M_\psi$$

$$= \left\{ f : \underset{\text{cptly supp}}{G(K)\backslash G(\mathbb{A}_K) \longrightarrow \mathbb{C}} \;:\; \begin{array}{l} f(g k_0) = f(g)\chi(k_0). \\ f(g k_\infty) = f(g)\psi(k_\infty) \end{array} \right\}$$

Prop (Gross, Heinloth-Ngô-Yun), G is simple simply conn'd, $\dim {}_\chi M_\psi = 1$.

In general, ${}_\chi M_\psi \supsetneq \mathbb{C}[\Omega]$.

$$G(k_F((\varpi))) = \{ \operatorname{Spec} k((\varpi)) \longrightarrow G \}$$

$$\bigcup$$

$$G(\mathbb{P}^1 - \{0\}) = \{ \mathbb{P}^1 - \{0\} \longrightarrow G \}$$

$$G(\mathbb{P}^1 - \{0\}) = G(K) \cap \prod_{v\neq 0} G(\mathcal{O}_v), \qquad \mathbb{P}^1 - \{0\} = \operatorname{Spec} k[\varpi^{-1}]$$

$$\bigcup$$

$$I_{\infty,\,pd} := \left\{ \underset{f(\infty)\in B(k_F)}{f : \mathbb{P}^1 - \{0\} \longrightarrow G} \right\}$$

$$G(K)\backslash G(\mathbb{A}_K) \Big/ \prod_{v\neq 0,\infty} G(\mathcal{O}_v) \times (I_0,\chi) \times (I_\infty^+, \psi)$$

$$\text{is}$$

$$(I, \chi)\backslash G(k((\varpi))) \big/ (I^+_{\infty,\,pd}, \psi)$$

Fact. $G(k((\varpi))) = \coprod_{w \in \tilde{W}} I \cdot w I^{op}_{\infty, pol}$ | $G = \coprod_{w \in W} BwB$ open cell Bw_0B

$= \coprod_{w \in W} B^- wB$ $\quad B^-B$

$$G(k((\varpi))) = \coprod_{w \in \tilde{W}} IwI \leftarrow \text{no open cell}$$

$$= \coprod_{w \in \tilde{W}} IwI^{op,+}_{\infty, pol} \quad , I \cdot I^{op,+}_{\infty, pol} \text{ open cell}$$

$$= I w_0 I^{+}_{\infty, pol} w_0^{-1}$$

$$I \cap I^{op,+}_{\infty, pol} = \{1\}$$

Lemma If $f \in C_c\left({}_{(I,\chi)} \backslash G(k((\varpi))) / (I^{op,+}_{\infty, pol}, \psi) \right)$

$\Rightarrow \operatorname{supp}(f) \subset \Omega$.

Conversely, $w \in \Omega$

$\exists f \neq 0$, supp. on $IwI^{op,+}_{\infty, pol}$ $\quad \Rightarrow$ Prop.

G/F equal char. local field $\rightsquigarrow$ $LG/_{k_F}$ loop group of G

$$LG : \mathrm{CAlg}_{k_F} \longrightarrow \mathrm{Grps}$$
$$R \longmapsto G(R((\varpi)))$$

$$LG(k_F) = G(F).$$

Fact, LG is represented by an ind-scheme.

$$LG = \varinjlim_i X_i \qquad X_i, \text{ affine } \overset{\text{inf. type}}{\text{scheme}} / k_F$$

$$X_i \hookrightarrow X_{i+1} \quad \text{fp. closed embedding}$$

$$G(\mathcal{O}) \rightsquigarrow L^+G : \mathrm{CAlg}_{k_F} \longrightarrow \mathrm{Grps}$$

$$R \longmapsto G(R[\![\varpi]\!])$$

__Fact__. L^+G is an affine gp scheme

$$\begin{array}{ccc} L^+G & \twoheadrightarrow & G \\ \cup & & \cup \\ I & \twoheadrightarrow & B \end{array}$$

affine gp scheme

$$I \supset \underset{\substack{\uparrow \\ \text{pro unipotent radical}}}{I^+} \supset I^{++} \supset \cdots$$

$I^{++} \leftarrow [I^+, I^+]$

Moy-Prasad filtration

$$I \twoheadrightarrow I/I^+ \simeq T \qquad I \cong T \ltimes I^+$$

__Def__. An affine pinning of LG is a ~~quadruple~~ triple (I, T, ψ).

where $I \subset LG$ is an Iwahori, $T \subset I$ is a max. torus.

$$\begin{tikzcd} I^+ \arrow[r, "\psi"] \arrow[d] & G_a \\ I^+/I^{++} = \prod\limits_{\substack{\alpha \text{ simple} \\ \text{affine root}}} G_a \arrow[ur] & \end{tikzcd}$$

$\psi|_{G_a} \longrightarrow G_a$ is an isom.

We'll consider $\underset{\substack{\| \\ M_\psi}}{N_{LG}(I, T, \psi)}\ (k_F) \curvearrowright \underset{\substack{\| \\ G(F)\backslash G(\mathbb{A}_K)/\cdots}}{X_{M_\psi}} \quad (I^+_\infty, \psi)$

Prop. We have a SES of comm. gps.

$$1 \to Z_G \to M_\psi \to \Omega \to 1$$

$${}_xH_x \curvearrowright {}_xM_\psi \circlearrowleft M_\psi(k)$$

$${}_xH_x \curvearrowright {}_xM_\psi \otimes_{\mathbb{C}[M_\psi]_x} \mathbb{C}$$
$$\mathbb{C}[M_\psi]_x \cong \Omega$$

<u>Lecture 6</u>. $G/F = k((t))$, LG/k, $k = \bar{k}$

<u>Def</u>. An affine pinning of LG is a triple (I, T, ϕ)

- I is an Iwahori
- $T \subset I$ a max'l torus
- $\phi : I^+ \longrightarrow \mathbb{G}_a$ s.t. $\phi|_{U_\alpha} : U_\alpha \xrightarrow{\sim} \mathbb{G}_a$ is an isom.

$$I^+/I^{++} = \prod_{\alpha \text{ simple affine root}} U_\alpha$$

$\underset{\text{ind-group}}{M_\phi} := N_{LG}(I, T, \phi)$

<u>Rmk</u>. $F = k((\varpi))$, $\mathbb{G}_m^{rot} \longrightarrow LG$. If G is almost simple, then all affine pinnings of LG are conjugate by $LG \rtimes \mathbb{G}_m^{rot}$, but they are not all conj. by LG.

Prop. We have SESs

$$1 \longrightarrow Z_G \longrightarrow M_\phi \longrightarrow \underset{\displaystyle \Omega}{\overset{\tilde{W}/W_{aff} \simeq \pi_0(LG)}{\|}} \longrightarrow 1$$

In addition, M_ϕ is comm.

Pf.

$$I \supset I^+ \supset I^{++} \supset \cdots$$

$$T = I/I^+, \quad \begin{array}{c} I^+/I^{++} =: V_I \circlearrowleft T \\ \wr\| \\ \mathrm{Lie}\, I^+/I^{++} = \mathrm{Lie}\, I^+/\mathrm{Lie}\, I^{++} \end{array}$$

P parahoric, Moy-Prasad fil.

$P = P(0) \supset P(1) \supset P(2) \supset \cdots$

$L_P = P/P(1)$,

$P(i)/P(i+1) \simeq \mathrm{Lie}\, P(i)/\mathrm{Lie}\, P(i+1)$

$(i \geq 1)$

$$1 \to T \to M = N_{LG}(I, T) \to \Omega \to 1$$

$$M_\phi = C_M(\phi).$$

Easy: $M_\phi \cap T = Z_G$

$$t \in T \cap M_\phi, \quad \begin{tikzcd} I^+ \arrow[r, "\mathrm{Ad}\, t"] \arrow[dr] & I^+ \arrow[d, "\phi"] \\ & G_a \end{tikzcd}$$

$\Rightarrow a(t) = 1, \forall a \in \Phi$

$w \in \Omega$, want $\dot{w} \in M$ a lifting of w

$\alpha \in \Phi_{aff}$ simple, $w(\alpha)$ another simple affine root. s.t. $\begin{tikzcd} U_\alpha \arrow[d, "\dot{w}"'] \arrow[dr, "\phi_\alpha"] & \\ U_{w(\alpha)} \arrow[r, "\phi_{w(\alpha)}"'] & G_a \end{tikzcd}$

For arbitrary lifting $\dot{w}'$,
$$\begin{tikzcd} U_\alpha \arrow[r, bend left, "\phi_\alpha"] \arrow[r, bend right, "\phi_{w(\alpha)} \cdot Ad_{\dot{w}'}"'] & G_a \end{tikzcd}$$

For any lifting $\dot{w}'$,

$$c_\alpha := (\phi_{w(\alpha)} \circ Ad_{\dot{w}'}) \circ \phi_\alpha^{-1} \in k^\times$$

If $\dot{w}'$ is replaced by $t\dot{w}'$,

$$c_\alpha \rightsquigarrow \dot{w}(\alpha)(t)\, c_\alpha$$

There is some relation between $\{c_\alpha\}$.

For simplicity, assume G almost simple.

$\{\alpha_0, \alpha_1, \cdots, \alpha_\ell\}$ affine simple roots

$\alpha_0 = 1-\theta$, $\alpha_1 = a_1$, $\alpha_\ell = a_\ell$

θ highest root $\quad \theta = \sum_{i=1}^{\ell} n_i a_i$

Let $n_0 = 1$, $\sum_{i=0}^{\ell} n_i \alpha_i = 1$

Let $c_i = c_{\alpha_i}$, $\prod_i c_i^{n_i}$ is indep. of the choice of $\dot{w}'$.

$$h : \tilde{V}_I^* \longrightarrow V_I^*/T \simeq \mathbb{A}^1$$
$$h(c_i) = \prod c_i^{n_i}$$

Rmk.

$$\begin{tikzcd} \phi \in \mathring{V}_I^* = \{c_i \neq 0, \forall i\} \arrow[d, "h"] \arrow[rr, phantom, ""] & & {} \arrow[d] \\ \mathbb{G}_m & & \prod c_i^{n_i} \end{tikzcd}$$

Lemma. The induced action of Ω on V_I^*/T is trivial.

$$\begin{tikzcd} M \curvearrowright V_I^* \arrow[r, "h"] & V_I^*/T \\ \phi \in \arrow[r] & \text{same pt} \\ \tilde{\omega}'(\phi) \arrow[ur] & \end{tikzcd}$$

Lemma $\Rightarrow$ $\exists$ lifting $\tilde{\omega}$ as desired.

Pf of. Lemma

$$\Omega \curvearrowright k[t]$$

$LG, I, T, \cdots$ are defined $/\mathbb{Z}$

$$\Omega \curvearrowright (V_I^*/T)/\mathbb{Z}$$

Enough to assume $k = \mathbb{C}$.

$$\mathfrak{g}^* \simeq \mathfrak{g}$$

$$(\phi_\alpha : \mathfrak{g}_\alpha \xrightarrow{\sim} \mathbb{C}) \in \mathfrak{g}_\alpha^* \rightsquigarrow \phi_\alpha \in L\mathfrak{g}$$

$T \curvearrowright V_I^* \simeq V_I \circlearrowleft T$ using some non-degenerate G-equiv pairing of $\mathfrak{g}$

$$
\begin{tikzcd}
LG & & m \arrow[ll] \\
Lg \arrow[d] & (\mathrm{Lie}\, I^+)^{\perp} \arrow[l, hook'] & V_I = \bigoplus \mathfrak{g}_\alpha \arrow[l, hook'] \arrow[d] \arrow[u, phantom, "\cup"] \\
L(\mathfrak{g}/G) & & V_I/T \circlearrowleft \Omega \arrow[ll] \arrow[d, phantom, "\parallel"] \\
& & \mathbb{A}^1
\end{tikzcd}
$$

if this is a closed embedding $\Rightarrow$ Ω acts trivially on V_I/T.

$$\mathfrak{g}/G = \mathrm{Spec}\, k[p_1, \dots, p_\ell]$$

$$\deg p_i = i+1$$

$$
\begin{tikzcd}
(\mathrm{Lie}\, I^+)^{\perp} \arrow[d] & \subset & \varpi^{-1}\mathfrak{g}(\mathcal{O}) \arrow[d] \\
\mathfrak{b} & \subset & \mathfrak{g} = \varpi^{-1}\mathfrak{g}(\mathcal{O})/\mathfrak{g}(\mathcal{O})
\end{tikzcd}
$$

<u>Lemma</u> $\Omega \curvearrowright V_I^*/T$ trivially.

Lg

$\varpi^{-1}\mathrm{Lie}\, I^+ \supset \varpi^{-1}\mathrm{Lie}\, I^{++} \cdots \supset \mathrm{Lie}\, I \supset \mathrm{Lie}\, I^+ \supset \mathrm{Lie}\, I^{++} \supset \cdots\ \varpi\, \mathrm{Lie}\, I$

$(Lg)^*$

$\varpi(\mathrm{Lie}\, I^+)^* \subset \varpi(\mathrm{Lie}\, I^{++})^{**} \subset \cdots \subset (\mathrm{Lie}\, I)^* \subset (\mathrm{Lie}\, I^+)^* \subset (\mathrm{Lie}\, I^{++})^* \subset \cdots \subset \varpi^{-1}(\mathrm{Lie}\, I)^*$

$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{B} F$. $\quad L\mathfrak{g} \otimes L\mathfrak{g} \to F \xrightarrow{\sum a_i \varpi^i \mapsto a_{-1}} k$

$F = k((\varpi))$

$\rightsquigarrow (L\mathfrak{g})^* \simeq L\mathfrak{g}$ as top. k-v.s.

$(L\mathfrak{g})^*$: top. dual

$$1 \to Z_G \to M_\phi \to \Omega \to 1$$

M_ϕ comm. : enough to show the commutator pairing

$$\Omega \times \Omega \to Z_G \text{ is trivial.}$$

Morphism defined / $\mathbb{Z}$

Suffices to show this when $k = \mathbb{C}$

Enough to show M_ϕ is comm. when $k = \mathbb{C}$

$$\phi \in L\mathfrak{g} = \mathfrak{g} \otimes k((\varpi))$$

$$\phi = \sum_{\alpha \in \text{affine simple roots}} X_\alpha = \varpi X_0 + \sum_{\alpha \text{ simple root}} X_\alpha \implies \text{mod } \varpi, \ \phi \text{ is regular}$$

nilp. in $\mathfrak{g}$

$G \curvearrowright \mathfrak{g}$ over $k[[\varpi]]$ $\implies$ ϕ as an elt in $\mathfrak{g}/F$ is regular.

Special fiber regular $\Rightarrow$ gen. fiber reg.

$$\Rightarrow \quad LC_G(\phi) \supset M_\phi$$

$$\nearrow \qquad \|$$

comm. $C_{L_G}(\phi)$

$\Rightarrow$ M_ϕ is commutative. $\square$

Rmk ϕ is regular semisimple.

Assume $Z(G)$ torus

$$1 \to Z(G)(k) \to M_\phi(k) \to \Omega \to 1$$

$$x H_x^{\Omega} \quad x A_\phi \curvearrowleft M_\phi(k)$$

$$\|$$

$$C_c\left(G(k)\backslash G(\mathbb{A}_k) \Big/ \prod_{v \neq 0,\infty} G(\mathcal{O}_v) \times (I_0, x) \times (I_\infty^{op,+}, \phi)\right)$$

$k = k(\varpi)$

$$\cong \quad k[\Omega].$$

$$M_\phi(k) \cdot I^{++} \xrightarrow{\Phi} k_F \to \mathbb{C}^\times$$

$$M_\phi(k) \ltimes I_\infty^{++} \xrightarrow{\tilde{x} \cdot \Phi} \mathbb{C}^\times$$

$$x A_{\tilde{x}\phi}$$

$$\|$$

$$C_c\left(G(k)\backslash G(\mathbb{A}_k) \Big/ \prod_{v \neq 0,\infty} G(\mathcal{O}_v) \times (I_0, x) \times (M_\phi(k) \ltimes I_\infty^{op,+}, \tilde{x}\phi)\right)$$

Lecture 7

(G, B, T, e)

$$LG \supset L^+G \supset I \supset I^+ \supset I^{++} \supset \cdots$$

$$\phi : I^+/I^{++} \longrightarrow \mathbb{G}_a \quad \text{non-deg.}$$

$$\begin{array}{ccc} & \wr\! & \\ d\phi : & \mathrm{Lie}\, I^+/\mathrm{Lie}\, I^{++} \longrightarrow k & \\ \Uparrow & \| & \\ V_I^* & V_I & \end{array}$$

$$M = N_{LG}(I, T), \qquad 1 \to T \to M \to \Omega \to 1$$

$$M \circlearrowright V_I^*$$

Lemma The action of Ω on $V_I^* /\!/ T$ is trivial

(can assume $k = \mathbb{C}$)

$$\left(\Rightarrow \quad 1 \to Z_G \to M_\phi \to \Omega \to 1 \right), \qquad M_\phi = C_M(\phi)$$

$$d\phi|_{L\mathfrak{g}_\alpha} =: d\phi_\alpha \in (L\mathfrak{g}_\alpha)^*, \quad d\phi_\alpha \neq 0$$

α affine simple root

$$d\phi = \sum d\phi_\alpha$$

Fix $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$ G-equiv. isom.

$$L\mathfrak{g}^* \xrightarrow{\sim} L\mathfrak{g}, \quad (L\mathfrak{g}_\alpha)^* \simeq L\mathfrak{g}_{-\alpha}$$

$$V_I^* \simeq \bigoplus_{\alpha \text{ simple}} L\mathfrak{g}_{-\alpha}$$

Kostant section

Fix principal sl_2-triple

principal nilp. $\{e, h, f\} \subset \mathfrak{g}$

e.g. sl_n, $f = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} = \sum_{\substack{\alpha \text{ finite} \\ \text{simple root}}} e_{-\alpha}$

$h = \begin{pmatrix} n-1 & & & \\ & n-3 & & \\ & & \ddots & \\ & & & -(n-1) \end{pmatrix}$

$$\begin{tikzcd} \mathcal{S} := f + \mathfrak{g}^e \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\!/G =: \mathfrak{c} \end{tikzcd}$$

$\mathbb{G}_m \circlearrowright \mathcal{S}$, $\mathbb{G}_m \circlearrowright \mathfrak{c}$

$\mathfrak{g}^e = \begin{pmatrix} 0 & c_1 & c_2 & \cdots & c_{n-1} \\ & \ddots & \ddots & & c_2 \\ & & \ddots & \ddots & c_1 \\ & & & & 0 \end{pmatrix}$

$\mathfrak{g}^e = \cdots \oplus \mathfrak{g}_\theta$ θ highest root

degree h $\leftarrow$ Coxeter number of $\mathfrak{g}$

$f + \mathfrak{g}^e = \begin{pmatrix} 0 & c_1 & c_2 & \cdots & c_{n-1} \\ 1 & \ddots & \ddots & & c_2 \\ & \ddots & \ddots & \ddots & c_1 \\ & & & 1 & 0 \end{pmatrix}$

$$\begin{tikzcd} & \{ f + t\lambda^{-1} e_\theta : \lambda \in k \} \arrow[dl] \arrow[d] \arrow[dr] & \\ \left(\bigoplus_{\alpha \text{ affine simple}} L\mathfrak{g}_{-\alpha} \right) = V_I^* \arrow[r] & L\mathcal{S} \arrow[r, "\sim"] & L\mathfrak{c} \circlearrowleft \Omega \end{tikzcd}$$

$V_I^* \longrightarrow V_I^*/\!/T$

everything is Ω-equiv.

Ω acts trivially on $L\mathfrak{c}$

$\rightarrow$ Ω acts trivially on $V_I^*/\!/T$.

$$x A_\phi := C_c\left(G(k) \backslash G(A_k) \Big/ (I_0, x) \times (I_\infty^{op,+}, \psi) \times \prod_{v \neq 0, \infty} G(O_v) \right) \quad K = k((t))$$

$H_x \curvearrowright x A_\phi \circlearrowleft M_\phi$

$$x A'_{\tilde{x}\phi} := C_c\left(\cdots \Big/ \cdots \times (M_\phi \ltimes I_\infty^{op,+}, \tilde{x}\phi) \times \cdots \right)$$

$\dim x A_\phi = \# \Omega$

$\dim x A'_{\tilde{x}\phi} = 1$

$$\begin{tikzcd} x & Z_G \arrow[d] & \\ & M_\phi \ltimes I_\infty^{op,+} \arrow[r, "(\tilde{x}, \phi)"] \arrow[d] & \mathbb{C}^\times \\ & \Omega & \end{tikzcd}$$

This is the 1-dim ${}_{x}\mathcal{H}_{x}$ -mod we want.

$\Rightarrow$ ${}_x\mathcal{H}_x = \left(\text{affine Hecke alg. for } H\right) \rtimes \mathbb{C}[\Omega_x]$

Z_G conn'd, $\Omega_x = \Omega$ for H ← endoscopic group assoc. to x

<u>Thm</u>. ${}_x\mathcal{H}_x \cong$ Iwahori - Hecke alg. of H.

${}_x A'_{\bar{x}\phi}$ is also a 1-dim module of

$$C_c\left(G(\mathcal{O}^{\{0,\infty\}}) \backslash G(\mathbb{A}^{\{0,\infty\}}) / G(\mathcal{O}^{\{0,\infty\}})\right) = \mathbb{T}^{\{0,\infty\}}$$

$$\mathbb{C}[\hat{G}]^{\hat{G}} \simeq \mathbb{T}_v = C_c\left(G(\mathcal{O}_v)\backslash G(k_v)/G(\mathcal{O}_v)\right)$$

$\downarrow$

$\{\sigma_v\}_{v \neq 0,\infty}$ conj. classes

$0 \neq f \in {}_x A'_{\bar{x}\phi}$ is cuspidal auto. rep'n.

<u>Thm</u> (Heinloth - Ngô - Yun, V. Lafforgue)

$\exists$ $\rho: \pi_1(\mathbb{P}^1_k - \{0,\infty\}) \longrightarrow \hat{G}$ $\qquad |\mathrm{tr}(\mathrm{Frob}_a)| \leq n q^{\frac{n-1}{2}}$ ← Weil bound

s.t. $\rho(\mathrm{Frob}_v) \sim \sigma_v$.

<u>Ex</u>. $G = GL_n$, $\hat{G} = GL_n$, $k = \mathbb{F}_q$, $a \in \mathbb{G}_m(\mathbb{F}_q)$

$$\mathrm{tr}(\mathrm{Frob}_a) = \sum_{\substack{(x_i) \in (\mathbb{F}_q^\times)^n \\ x_1 \cdots x_n = a}} \prod_i \chi_i(x_i)\,\phi\left(\sum x_i\right)$$

(Generalized) Kloosterman sum

$n=2$, $\chi = \mathrm{id}$

$$\sum_{x \in \mathbb{F}_q^\times} \phi\left(x + \frac{a}{x}\right)$$

$$x\mathcal{H}_x = \mathrm{End}\left(\text{c-ind}_{I}^{G(F)} x\right) \qquad x : I \to T(k) \to \mathbb{C}^\times$$

Two possible generalizations:

(1) $x : T(\mathcal{O}) \to \mathbb{C}^\times$ (Roche)

$$\begin{tikzcd} T(\mathcal{O}) \arrow[r] \arrow[d, two heads] & \mathbb{C}^\times \\ T(\mathcal{O}/\varpi^2) \arrow[ur] & \end{tikzcd}$$

$$I^+ \supset J_x \twoheadrightarrow T(\mathcal{O}/\varpi^2) \to \mathbb{C}^\times$$

$\exists$ a natural

$$\mathrm{End}\left(\text{c-ind}_{J_x}^{G(F)} x\right) = {}_x\mathcal{H}_x$$

$$\subset C_c(J_x \backslash G(F) / J_x)$$

(2) $P \subset G(F)$ parahoric (Morris)

$P \to L_P$ Levi quotient

$\quad\;\; \circlearrowright \sigma$

$\quad\;\; V$

σ irred. cuspidal rep'n of L_P
(not arise from parabolic induction)

$\mathcal{H}(P,\sigma) := \mathrm{End}\left(\text{c-ind}_P^G \sigma\right)$, then $\mathcal{H}(P,\sigma)^\circ$ is again an affine Hecke alg.

$\circlearrowleft$
$\Omega(P,\sigma)$ of an "endoscopic gp"

$\mathcal{H}(P,\sigma) \sim \mathcal{H}(P,\sigma)^\circ \rtimes \Omega(P,\sigma)$

Geometrization of $\chi : T(\mathbb{F}_q) \to \mathbb{C}^\times$

Let H be a conn'd alg gp $/k$. $(k = \bar{k})$

Let $\tilde{H} := \varprojlim_{H' \to H} H'$

where • H' conn'd alg. gp

• $H' \to H$ finite étale homomorphism

$\tilde{H}$ is a pro-alg. gp

$\ker(\tilde{H} \to H) =: \pi_1^c(H)$ pro-finite $/k$

Rmk (1) $H' \to H$ finite étale surj.

$\ker(H' \to H)$ is central in H'.

so $H' \to H$ is a central ext.

$\Rightarrow \pi_1^c(H)$ is abelian

(2) $\pi_1^{\text{ét}}(H) \twoheadrightarrow \pi_1^c(H)$

(3) H commutative, $\pi_1^c(H)$ was introduced by Serre.

(4) If H is a conn'd reductive gp

$\pi_1^{alg}(H) := \left(X_*(T) / \mathbb{Z}\Phi^\vee\right)(1)$

(5) If $H/k_1 \subset k$, $\pi_1^c(H) \circlearrowleft \mathrm{Gal}(k|k_1)$

Eg. char $k = p$, PGL_p

$$\pi_1^{alg}(H) \neq 1, \qquad \pi_1^c(H) = 1.$$

Lemma. H is commutative, $k = \overline{\mathbb{F}_p}$, (defined over $\mathbb{F}_q$)

then $\underset{\circlearrowright\, \mathrm{Frob}_q}{\pi_1^c(H)} = \varprojlim_n \underset{\circlearrowright\, \mathrm{Frob}_q}{H(\mathbb{F}_{q^n})}$

w/ transition maps $H(\mathbb{F}_{q^m}) \xrightarrow{N_m} H(\mathbb{F}_{q^n})$

Pf. $1 \to \Gamma \to H' \to H \to 1$

Choose q large enough s.t. $\Gamma \subset H'(\mathbb{F}_q)$

$$\begin{tikzcd} H'(\mathbb{F}_q) \arrow[dr] & \\ g \arrow[dr, mapsto] & H' \arrow[dr] & \\ & g^{-1}\mathrm{Frob}_q(g) & H' \end{tikzcd}$$

finite étale hom.

Gal w/ $H'(\mathbb{F}_q)$

$$\begin{tikzcd} 1 \arrow[r] & H'(\mathbb{F}_q) \arrow[r] & H' \arrow[r, "L", "g \mapsto g^{-1}\mathrm{Frob}_q(g)"'] & H' \arrow[r] & 1 \\ 1 \arrow[r] & \Gamma \arrow[r] \arrow[u, hook] & H' \arrow[r] \arrow[u, equal] & H \arrow[r] \arrow[u, hook] & 1 \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r] \arrow[dr, "L"'] & H' \arrow[d] \arrow[dr, "L"] & \\ & H \arrow[r] & H' \end{tikzcd}$$

$$\widetilde{H} = \varprojlim H' = \varprojlim H$$

p' char. exp. of k.

Rmk $H = T$, $\pi_1^c(T) = \varprojlim_{(n, \mathrm{char}\, k) = 1} T[n] \cong \hat{\mathbb{Z}}^{p'}(1)$

Lecture 8

H alg. group $/ k = \bar{k}$

$$1 \to \pi_1^c(H) \to \tilde{H} \to H \to 1$$

$$\tilde{H} = \varprojlim_{H' \to H} H', \qquad \begin{array}{l} H' \text{ conn'd} \\ H' \to H \text{ f. ét} \end{array}$$

Lemma. (1) Let $1 \to K \to H_1 \to H_2 \to 1$ w/ K finite étale

Then $1 \to \pi_1^c(H_1) \to \pi_1^c(H_2) \to K \to 1$.

(2) $1 \to H \to H_1 \to H_2 \to 1$, H, H_1, H_2 conn'd

Then $\pi_1^c(H) \to \pi_1^c(H_1) \to \pi_1^c(H_2) \to 1$.

Pf.

$$\begin{tikzcd} \tilde{H} \arrow[r] \arrow[d] & H \arrow[d] \\ \tilde{H}_1 \arrow[r] \arrow[d] & H_1 \arrow[d] \\ \tilde{H}_2 \arrow[r] & H_2 \end{tikzcd}$$

$\pi_1^c(H_1) \to \pi_1^c(H_2)$ surj.

Need to show

$$\pi_1^c(H) \twoheadrightarrow \ker(\pi_1^c(H_1) \to \pi_1^c(H_2))$$

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\ & \pi_1^c(H) \arrow[r] \arrow[d] & \pi_1^c(H_1) \arrow[r] \arrow[d] & \pi_1^c(H_2) \arrow[r] \arrow[d] & 1 \\ & \tilde{H} \arrow[r] \arrow[d] & \tilde{H}_1 \arrow[r] \arrow[d] & \tilde{H}_2 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & 1 \\ & 1 & 1 & 1 & \end{tikzcd}$$

$$\begin{tikzcd} & & \ker(\pi_1^c(H_1) \to \pi_1^c(H_2)) \arrow[d] \\ \tilde{H} \arrow[r, two heads] & (H \times_{H_1} \tilde{H}_1)^\circ \arrow[d] & \subset \ker(\tilde{H}_1 \to \tilde{H}_2) \\ & H & \end{tikzcd}$$

Λ is a finite $\mathbb{Z}_\ell$-alg. or alg. ext'n of $\mathbb{Q}_\ell$

$\pi_1^{\text{ét}}(H) \twoheadrightarrow \pi_1^c(H) \xrightarrow{\chi} \Lambda^\times$

$\rightsquigarrow$ Ch_χ local system on H

$m^* Ch_\chi \simeq Ch_\chi \boxtimes Ch_\chi$ on $H \times H$ $(m: H \times H \to H)$

satisfying a cocycle cond'n on $H \times H \times H$.

i.e. Ch_χ is a character sheaf (local system) on H.

Def A character sheaf on a (not necessarily conn'd) alg. group is a rk 1 Λ-local system $\mathcal{L}$ equipped w/ $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$ satisfying cocycle cond'n.

Let $CS(H, \Lambda)$ denote the Picard groupoid of character sheaves on H w/ coeff. Λ.

Rmk. (1) If H is conn'd, $CS(H, \Lambda)$ is a discrete groupoid.
It is an abelian gp.
(2) Being a character sheaf is a property rather than additional str. of $\mathcal{L}$.

Lemma (3) $\{\text{Cts hom. } \pi_1^c(H) \to \Lambda^\times\} \xrightarrow{\sim} CS(H, \Lambda)$

Let $R_{\pi_1^c(H), \mathbb{G}_m}$ be the moduli space $/\mathbb{Z}_\ell$ classifying (strongly) cts $\mathbb{G}_m$-rep^n of $\pi_1^c(H)$.

$$R_{\pi_1^c(H), \mathbb{G}_m}(\underset{\substack{\uparrow \\ \mathbb{Z}_\ell\text{-alg.}}}{A}) = \left\{ \rho : \pi_1^c(H) \longrightarrow A^\times \;:\; \begin{array}{l} A \text{ as } \pi_1^c(H)\text{-mod is a union} \\ A = \bigcup_i V_i \text{ of } \pi_1^c(H)\text{- submodules } V_i \\ \text{each } V_i \text{ is finite} / \mathbb{Z}_\ell \\ \& \ \pi_1^c(H) \longrightarrow \mathrm{Aut}(V_i) \text{ is cts} \end{array} \right\}$$

<u>Lemma</u> $R_{\pi_1^c(H), \mathbb{G}_m}$ is represented by an ind-scheme, ind-finite $/\mathbb{Z}_\ell$.

<u>Proof of the Lemma</u>

pro-ℓ-quotient of $\pi_1^c(H)$ is top. finitely generated.

$$1 \longrightarrow \pi_1^c(H)^{\ell'} \longrightarrow \pi_1^c(H) \longrightarrow \pi_1^c(H)_\ell \longrightarrow 1$$

$$\begin{array}{ccc} \circlearrowleft & & \\ \mathbb{G}_m & & \mathbb{Z}_\ell^2 \end{array}$$

<u>Ex</u> $H = \mathbb{G}_m^2$ torus, $\pi_1^c(H) = X_*(H) \otimes \hat{\mathbb{Z}}(1)^2$. Λ finite $/\mathbb{Z}_\ell$

$$R_{\pi_1^c(H), \mathbb{G}_m}(\Lambda) = \{ \text{cts } \pi_1^c(H) \longrightarrow \Lambda^\times \}$$

$$= \{ \text{cts } \underset{\substack{\| \\ I_F^t}}{\hat{\mathbb{Z}}(1)} \longrightarrow \hat{H}(\Lambda) \} = R_{I_F^t, \hat{H}}(\Lambda)$$

Cor $R_{\pi_1^c(H), \mathbb{G}_m} \simeq R_{I_F^t, \hat{H}}$

$R_{\pi_1^c(H), \mathbb{G}_m}(\Lambda) \simeq CS(H, \Lambda)$

$\left(\begin{array}{l} H\text{-tors} \quad \wr\!\wr \\ \qquad R_{I_F^t, \hat{H}}(\Lambda) \end{array} \right)$

$R_{\mathbb{Z}_\ell, \mathbb{G}_m} = \bigcup Z$

$Z \subset \mathbb{G}_m$ closed

$Z \to \operatorname{Spec} \mathbb{Z}_\ell$ finite

$Z \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \subset \hat{\mathbb{G}}_m$

$\exists$ an exact functor, fully faithful

$$\operatorname{Coh}(R_{\pi_1^c(H), \mathbb{G}_m})^\heartsuit \longrightarrow \operatorname{Shv}(H)^\heartsuit$$

$$\left(\operatorname{Spec}\Lambda \to R_{\pi_1^c(H), \mathbb{G}_m}\right)_* \mathcal{O} =: \mathcal{O}_x \longmapsto \operatorname{Ch}_x$$

The image is the thick abelian subcat. of $\operatorname{Shv}(H)^\heartsuit$ gen. by character sheaves, denoted by $\operatorname{Shv}_{\text{mon}}(H)^{w,\heartsuit}$

Rmk $\operatorname{Coh}(\cdots)^\heartsuit = \left\{ \begin{array}{l} \text{cts } \pi_1^c(H)\text{-mod} \\ \text{on f. } \mathbb{Z}_\ell\text{-modules} \end{array} \right\}$

Lemma Let $f: H_1 \to H_2$ be surj.

Let $\mathcal{F} \in \operatorname{Shv}(H_2)^\heartsuit$ s.t. $f^*\mathcal{F} \in \operatorname{Shv}_{\text{mon}}(H_1)^{w,\heartsuit}$, then $\mathcal{F} \in \operatorname{Shv}_{\text{mon}}(H_2)^{w,\heartsuit}$.

Proof $\mathcal{F}$ is a local system on $H_2 \iff$

$$\begin{tikzcd} \pi_1^{\text{ét}}(H_2) \arrow[r] \arrow[d] & \operatorname{Aut}(\mathcal{F}_1) \\ \pi_1^c(H_2) \arrow[ur, dashed] & \end{tikzcd}$$

Case 1
$\ker f$ is finite étale

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(H_1) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(H_2) \arrow[r] \arrow[d] & \ker f \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1^{c}(H_1) \arrow[r] & \pi_1^{c}(H_2) \arrow[r] & \ker f \arrow[r] & 1
\end{tikzcd}$$

Case 2. $H = \ker f$ is conn'd.

$$\begin{tikzcd}
 & \Gamma_1 \arrow[r] \arrow[d] & \Gamma_2 \arrow[d] & \\
\pi_1^{\text{ét}}(H) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(H_1) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(H_2) \arrow[r] \arrow[d] & 1 \\
\pi_1^{c}(H) \arrow[r] & \pi_1^{c}(H_1) \arrow[r] & \pi_1^{c}(H_2) \arrow[r] & 1
\end{tikzcd}$$

$\chi : \pi_1^c(H) \longrightarrow \Lambda^\times \quad \longleftrightarrow \quad Ch_\chi$

Let $Sh_{\chi\text{-mon}}(H)$ be the presentable stable ∞-cat. $\subset Shv(H)$ gen. by Ch_χ.

Let $Shv_{mon}(H)$ be the cat. gen. by all $\{Ch_\chi\}_\chi$.

$$Shv_{mon}(H) = Ind(Shv_{mon}(H)^\omega)$$

$$Shv_{mon}(H)^\omega = \{ F \in Shv(H)^\omega : \mathcal{H}^i F \in Shv_{mon}(H)^{\omega, \heartsuit} \}$$

$$\begin{array}{ccccccc}
\Lambda & \longmapsto & & Ch_\chi & & & \\
Mod_\Lambda & \longrightarrow & Shv_{\chi\text{-mon}}(H) & \subset & Shv_{mon}(H) & \subset & Shv(H)
\end{array}$$

All functors admit cts right adj. Right adj: $Av^{mon} : Shv(H) \to Shv_{mon}(H)$.

Prop. $Av^{mon} : Shv(H) \longrightarrow Shv_{mon}(H)$ is a monoidal functor.

Lecture 9. Lemma

$$1 \to H_1 \to H_2 \to H_3 \to 1 \quad \text{SES of conn'd gps}$$

$$\Rightarrow \quad \pi_1^c(H_1) \to \pi_1^c(H_2) \to \pi_1^c(H_3) \to 1$$

Pf

$$\begin{tikzcd} \widetilde{H}_1 \arrow[dr] & & & \\ & \ker(\widetilde{\pi}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{H}_2 \arrow[r, "\widetilde{\pi}"] \arrow[d] & \widetilde{H}_3 \arrow[d] \\ & H_1 \arrow[r] & H_2 \arrow[r, "\pi"] & H_3 \end{tikzcd}$$

$$\widetilde{\widetilde{H}} := \ker(\widetilde{\pi})^\circ$$

$$1 \to \ker\left(\pi_1^c(H_2) \to \pi_1^c(H_3)\right) \to \ker(\widetilde{\pi}) \to H_1 \to 1$$

$$\ker(\widetilde{\pi})^\circ \hookrightarrow \ker(\widetilde{\pi})$$

$$\widetilde{H}_2 / \ker(\widetilde{\pi})^\circ \twoheadrightarrow \widetilde{H}_2 / \ker(\widetilde{\pi}) = \widetilde{H}_3$$

Universal property of $\widetilde{H}_3$ $\Rightarrow$ $\overset{\text{iso.}}{\uparrow}$ $\ker(\widetilde{\pi})^\circ = \ker(\widetilde{\pi})$.

H conn'd alg. gp, $\mathbb{R}_{\pi_1^c(H), \mathbb{G}_m}(\Lambda) = \{ \text{cts } \pi_1^c(H) \xrightarrow{\chi} \Lambda^\times \}$

ind-scheme, ind-finite$/\mathbb{Z}_\ell$.

Λ finite$/\mathbb{Z}_\ell$, $\mathcal{O}_\ell$, ...

$$= \left\{ \mathrm{Ch}_\chi \quad \begin{array}{l} \Lambda\text{-linear character sheaf} \\ m^* \mathrm{Ch}_\chi \simeq \mathrm{Ch}_\chi \boxtimes \mathrm{Ch}_\chi \end{array} \right\}$$

e.g. H torus, $\hat{H}/\mathbb{Z}_\ell$. I_F^t = tame inertia of $F = k((\varpi))$

$\simeq \hat{\mathbb{Z}}^p(1)$

$$R_{\pi_1^c(H), \mathbb{G}_m} \simeq R_{I_F^t, \hat{H}}$$

$\wr$ choosing a top. generator of I_F^t.

$\hat{H}$

Fix Λ alg. $/\mathbb{F}_\ell$, $\mathbb{Q}_\ell$, or finite ext. of $\mathbb{Z}_\ell$ (regular noetherian local ring)

$\mathrm{Shv}(H, \Lambda)$ = unbounded ∞-derived cat. of Λ-étale sheaves on H.

$\mathrm{Mod}_\Lambda \ni \Lambda \mapsto \mathrm{Ch}_\chi$

$\mathrm{Shv}_{\chi\text{-mon}}(H, \Lambda) \subset \mathrm{Shv}_{\mathrm{mon}}(H, \Lambda) \subset \mathrm{Shv}(H, \Lambda)$

- $\mathrm{Shv}_{\chi\text{-mon}}(H, \Lambda)$ is the full Λ-linear (presentable, stable, tensored over Mod_Λ) subcat. gen. by

  Ch_χ. $\chi : \pi_1^c(H) \to \Lambda^\times$

- $\mathrm{Shv}_{\mathrm{mon}}(H, \Lambda)$ full Λ-linear subcat. gen. by $\{\mathrm{Ch}_{\chi'}\}_{\chi' : \pi_1^c(H) \to (\Lambda')^\times}$

  Λ' finite Λ-alg.

$$\mathrm{Shv}_{\mathrm{mon}}(H, \Lambda)^\omega = \{ \mathcal{F} \in \underset{\substack{\shortparallel \\ D_c^b(H, \Lambda)}}{\mathrm{Shv}(H, \Lambda)^\omega} : \mathcal{H}^i\mathcal{F} \in \mathrm{Shv}_{\mathrm{mon}}(H, \Lambda)^{\omega, \heartsuit} \}$$

thick abelian subcat. gen. by $\{\mathrm{Ch}_{\chi'}\}_{\chi'}$

$$\mathrm{Ch} : \mathrm{Coh}(R_{\pi_1^c(H), \mathbb{G}_m})^\heartsuit \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{mon}}(H, \Lambda)^{\omega, \heartsuit}$$
$$\mathcal{O}_\chi \longmapsto \mathrm{Ch}_\chi$$

Shv(H) has a natural monoidal str.

$$\mathcal{F} * \mathcal{G} = m_*(\mathcal{F} \boxtimes \mathcal{G}) \quad , \quad m: H \times H \longrightarrow H$$

unit: $\delta_1 = (\{1\} \to H)_* \Lambda$

<u>Prop</u> (1) $Shv_{mon}(H) \subset Shv(H)$ is a bimodule

(2) $Shv_{mon}(H)$ has a monoidal unit $\widetilde{Ch}$.

(3) The right adjoint of $Shv_{mon}(H) \subset Shv(H)$

$$Av^{mon}: Shv(H) \longrightarrow Shv_{mon}(H) \qquad \text{is monoidal.}$$

There are similar statements for $Shv_{\chi\text{-}mon}(H)$.

<u>Lemma</u>. $Shv_{mon}(H) \otimes Shv_{mon}(H) \xrightarrow{\sim} Shv_{mon}(H \times H)$

(Rmk: $Shv(H) \otimes Shv(H) \hookrightarrow Shv(H \times H)$ is just a f. faithful embedding)

<u>Pf</u> Ch is a character sheaf on $H \times H$.

$Ch_{\chi_1} = Ch_\chi |_{H \times \{1\}}$, $\quad Ch_{\chi_2} = Ch_\chi |_{\{1\} \times H}$.

$Ch_\chi = Ch_{\chi_1} \boxtimes Ch_{\chi_2}$.

$$\begin{tikzcd} H \times H \arrow[r, "\sim"] \arrow[rrr, bend right, "id"'] & (H \times \{1\}) \times (\{1\} \times H) \arrow[r, hook] & (H \times H) \times (H \times H) \arrow[r, "m_{H \times H}"] & H \times H \end{tikzcd}$$

Lemma. Let $\mathcal{F} \in Shv(H)$ s.t. $m^*\mathcal{F} \simeq Ch_x \boxtimes \mathcal{F}$ for some x,

then $\mathcal{F} \in Shv_{mon}(H)$.

Pf.

$$\begin{tikzcd} Ch_x \otimes i^*\mathcal{F} & Ch_x \boxtimes \mathcal{F} & \mathcal{F} \\ H \times \{1\} \arrow[r, hook] & H \times H \arrow[r, "m"] & H \end{tikzcd}$$

$\Rightarrow \mathcal{F} \in Shv_{mon}(H)$. $\square$

Lemma $Ch_x * \mathcal{F} \in Shv_{mon}(H)$.

Pf

$$\begin{tikzcd} H \times H \times H \arrow[r, "m \times id"] \arrow[d, "id \times m"'] & H \times H \arrow[d, "m"] \\ H \times H \arrow[r, "m"'] & H \end{tikzcd}$$

$$m^*(Ch_x * \mathcal{F})$$
$$= (1 \times m)_*(Ch_x \boxtimes Ch_x \boxtimes \mathcal{F})$$
$$= Ch_x \boxtimes (Ch_x * \mathcal{F})$$

$$\Rightarrow Ch_x * \mathcal{F} \in Shv_{mon}(\mathcal{F}).$$

Prop (1) ✓.

Lemma $Av^{mon}(\mathcal{F}) = \underset{\underset{Av^{mon}(\delta)}{\shortparallel}}{\widetilde{Ch}} * \mathcal{F}$, $\forall \mathcal{F} \in Shv(H)$.

$\boxed{\text{Cor. } \widetilde{Ch} * \widetilde{Ch} = \widetilde{Ch}.}$

Pf Need to show $\mathcal{G} \in Shv_{mon}(H)$

$$Hom(\mathcal{G}, Av^{mon}(\mathcal{F})) = Hom(\mathcal{G}, \mathcal{F}) = Hom(\mathcal{G}, \widetilde{Ch} * \mathcal{F})$$

$$Hom(Ch_x, \widetilde{Ch} * \mathcal{F}) = Hom(Ch_x \boxtimes Ch_x, \widetilde{Ch} \boxtimes \mathcal{F})$$
$$= Hom(Ch_x, \widetilde{Ch}) \otimes Hom(Ch_x, \mathcal{F})$$
$$= Hom(Ch_x, \delta_1) \otimes Hom(Ch_x, \mathcal{F}) = Hom(Ch_x, \mathcal{F}) = Hom(Ch_x, Av^{mon}(\mathcal{F}))$$

Next:

$$\mathrm{Av}_V^{mon}(\mathcal{F} * \mathcal{G}) \overset{?}{=} \mathrm{Av}_V^{mon}(\mathcal{F}) * \mathrm{Av}_V^{mon}(\mathcal{G})$$

$$\mathrm{Av}_V^{mon}(\mathcal{F}) \boxtimes \mathrm{Av}_V^{mon}(\mathcal{G}) \longrightarrow \mathrm{Av}_V^{mon}(\mathcal{F} \boxtimes \mathcal{G})$$

$\mathcal{E}$ char. sheaf.

$$\begin{aligned}
\mathrm{Hom}(\mathcal{E}, \mathrm{Av}_V^{mon}(\mathcal{F} * \mathcal{G})) &= \mathrm{Hom}(\mathcal{E}, \mathcal{F} * \mathcal{G}) \\
&= \mathrm{Hom}(\mathcal{E} \boxtimes \mathcal{E}, \mathcal{F} \boxtimes \mathcal{G}) \\
&= \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{G}) \\
&= \mathrm{Hom}(\mathcal{E}, \mathrm{Av}_V^{mon}(\mathcal{F})) \otimes \mathrm{Hom}(\mathcal{E}, \mathrm{Av}_V^{mon}(\mathcal{G})) \\
&= \mathrm{Hom}(\mathcal{E} \boxtimes \mathcal{E}, \mathrm{Av}_V^{mon}(\mathcal{F}) \boxtimes \mathrm{Av}_V^{mon}(\mathcal{G})) \\
&= \mathrm{Hom}(\mathcal{E}, \mathrm{Av}_V^{mon}(\mathcal{F}) * \mathrm{Av}_V^{mon}(\mathcal{G}))
\end{aligned}$$

Ex. $H = \mathbb{G}_a / \mathbb{F}_p$, $\pi_1^c(\mathbb{G}_a) \longrightarrow \mathbb{G}_a(\mathbb{F}_p) \xrightarrow{\phi \text{ nontrivial}} \Lambda^\times$

Ch_ϕ Artin-Schreier sheaf on $\mathbb{G}_a$

$\mathrm{Ch}_{\phi\text{-mon}} = \mathrm{Ch}_\phi$

$$\mathrm{Mod}_\Lambda \xrightarrow{\sim} \mathrm{Shv}_{\phi\text{-mon}}(\mathbb{G}_a) \hookrightarrow \mathrm{Shv}(\mathbb{G}_a)$$

Ex. $H = \mathbb{G}_m$, $\pi_1^c(\mathbb{G}_m) \longrightarrow \mathbb{F}_q^\times \xrightarrow{\chi} \Lambda^\times$

$\rightsquigarrow \mathrm{Ch}_\chi$ Kummer local system on $\mathbb{G}_m$

$$\mathrm{Mod}_\Lambda \xrightarrow{\ast} \mathrm{Shv}_{\chi\text{-mon}}(\mathbb{G}_m). \qquad \mathrm{Ch}_\chi \neq \mathrm{Ch}_{\chi\text{-mon}}.$$

Ex. $\chi = u$ trivial,

$\mathrm{Ch}_{u\text{-mon}} = \varinjlim_n \mathcal{L}_n$, $\mathcal{L}_n$ unip. local system on $\mathbb{G}_m$

w/ monodromy $\begin{pmatrix} 1 & \ddots & 0 \\ 1 & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+1} \hookrightarrow \cdots$$

$\mathrm{Ch}_{\chi\text{-mon}} = \mathrm{Ch}_\chi \otimes \mathrm{Ch}_{u\text{-mon}}$.

$\Lambda = \overline{\mathbb{Q}}_\ell$. $\widetilde{\mathrm{Ch}} = \bigoplus_\chi \mathrm{Ch}_{\chi\text{-mon}}$.

Prop. $f : H_1 \to H_2$ hom. Then

(1) $f^* : \mathrm{Shv}(H_2) \to \mathrm{Shv}(H_1)$ sends $\mathrm{Shv}_{\mathrm{mon}}(H_2) \to \mathrm{Shv}_{\mathrm{mon}}(H_1)$

f^* admits a cts right adj. $f_*^{\mathrm{mon}} = \mathrm{Av}^{\mathrm{mon}} \circ f_*$

(2) If f is surjective, (a) $f_*^{\mathrm{mon}} = f_*$.

(b) $f_! = f_*$ up to shift, restricted to $\mathrm{Shv}_{\mathrm{mon}}(H_1)$.

Pf (1) easy to check.

(2) WTS $f_* \mathrm{Ch}_\chi \in \mathrm{Shv}_{\mathrm{mon}}(H_2)^\omega$

ETS $\mathcal{H}^i(f_* \mathrm{Ch}_\chi) \in \mathrm{Shv}_{\mathrm{mon}}(H_2)^{\omega, \heartsuit}$.

ETS $\quad f^* \mathcal{H}^i(f_* Ch_x) \in Shv_{mon}(H_1)^{\omega, B}$

$$\begin{tikzcd}
\underset{Ch_x \boxtimes (Ch_x|_{\ker f})}{H_1 \times \ker f} \arrow[r] \arrow[d, "pr"'] & \underset{Ch_x \boxtimes Ch_x}{H_1 \times H_1} \arrow[r, "m"] & \underset{Ch_x}{H_1} \arrow[d, "f"] \\
H_1 \arrow[rr, "f"'] & & H_2
\end{tikzcd}$$

$$Ch_x \otimes H^i \underset{(c)}{R\Gamma}(Ch_x|_{\ker f}) = f^* H^i(\underset{(!)}{f_*} Ch_x)$$

(2) (b) Can assume either (b1) f is finite

(b2) $\ker f$ is conn'd.

(b1) f finite, $f_* = f_!$

(b2) B Borel of $\ker f$

$$H_1 \longrightarrow H_1/B \longrightarrow H_2$$

reduce to show $\quad R\Gamma_c(B, Ch_x) = R\Gamma(B, Ch_x)[d]$

Reduce to $\mathbb{G}_a$, $\mathbb{G}_m$.

__Prop__. Let H be a torus, $\exists$ t-exact, monoidal equiv.

$$Ch: \operatorname{Ind}(\operatorname{Coh}(R\, I_F^t, \hat{H})) \xrightarrow{\sim} Shv_{mon}(H).$$

Let $f: H_1 \to H_2$, $\hat{f}: \hat{H}_2 \to \hat{H}_1$, $\quad f^* \longleftrightarrow \hat{f}^{\operatorname{IndCoh}}_*$

$f_*^{mon} \longleftrightarrow \hat{f}^{\operatorname{IndCoh},!}$

Lecture 10

H torus, $\hat{H}$ dual torus $/\Lambda$ $(\mathbb{F}_\ell, \mathbb{Q}_\ell, \mathbb{Z}_\ell)$, $F = k((\varpi))$

Prop. $Ch : \mathrm{IndCoh}(R_{I_F^t, \hat{H}}) \xrightarrow{\sim} \mathrm{Shv}_{mon}(H)$

fix a top. generator

$\mathrm{QCoh}(\hat{H})$

- t-exact
- monoidal
- $f : H_1 \to H_2$. $\hat{f} : \hat{H}_2 \to \hat{H}_1$

$$\begin{tikzcd}
\hat{f}^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(R_{I_F^t, \hat{H}_2}) \arrow[r, shift left] \arrow[d, phantom, "\wr"] & \mathrm{IndCoh}(R_{I_F^t, \hat{H}_1}) : \hat{f}^{\mathrm{IndCoh}, !} \arrow[l, shift left] \arrow[d, phantom, "\wr"] \\
f^* : \mathrm{Shv}_{mon}(H_2) \arrow[r, shift left] & \mathrm{Shv}_{mon}(H_1) : f_*^{mon} \arrow[l, shift left]
\end{tikzcd}$$

Pf. $\mathrm{Coh}(R_{I_F^t, \hat{H}})^{\heartsuit} \xrightarrow{\sim} \mathrm{Shv}_{mon}(H)^{\omega, \heartsuit}$

$\mathcal{O}_x \longmapsto Ch_x$

$x : \mathrm{Spec}\,\Lambda \to R_{I_F^t, \hat{H}}$

$\Updownarrow$

$\pi_1^t(H) \to \Lambda^\times$

$$\begin{tikzcd}
R\mathrm{Hom}_{\mathrm{Coh}}(\mathcal{F}_1, \mathcal{F}_2) \arrow[r, "\sim"] \arrow[d, Rightarrow] & R\mathrm{Hom}_{\mathrm{Shv}(H)}(Ch(\mathcal{F}_1), Ch(\mathcal{F}_2)) \arrow[d, Leftrightarrow, "\wr"', "K(\pi,1)\text{ property of } H"] \\
R\mathrm{Hom}_{\pi_1^t(H)\text{-mod}}(\mathcal{F}_1, \mathcal{F}_2) \arrow[r, "\sim"] & R\mathrm{Hom}_{\pi_1^{et}(H)}^{cts}(\mathcal{F}_1, \mathcal{F}_2)
\end{tikzcd}$$

$\mathrm{Coh}(R_{I_F^t, \hat{H}}) \xrightarrow{\sim} \mathrm{Shv}_{mon}(H)^{\omega}$. Take Ind completion. □

$\omega_{R_{I_F^t,\hat{H}}} \longleftrightarrow \widetilde{Ch}$ units.

Let $\chi \in R_{I_F^t,\hat{H}}(\Lambda)$

$\mathrm{Spec}\,\Lambda \longrightarrow R_{I_F^t,\hat{H}}$

$\hat{\chi}$ formal completion of $R_{I_F^t,\hat{H}}$ along χ.

$\mathrm{Ind\,Coh}(\hat{\chi}) \xrightarrow{\simeq} \mathrm{Shv}_{\chi\text{-mon}}(H)$

$\omega_{\hat{\chi}} \longleftrightarrow Ch_{\chi\text{-mon}}$.

$\chi = u$ trivial local system

$u = \chi \hookrightarrow R_{I_F^t,\hat{H}} \hookrightarrow \hat{H} \cong \Lambda[x_i^{\pm 1}]$

χ defined by $\Lambda[x_i^{\pm 1}]/(x_1-1, \dots, x_n-1)$

$$\hat{\chi} = \varinjlim_d \mathrm{Spec}\underbrace{\left(\Lambda[x_i^{\pm 1}]/((x_1-1)^d, \dots, (x_n-1)^d)\right)}_{\chi_d}$$

$$\omega_{\chi_d} = R\mathrm{Hom}_{\Lambda[x_i^{\pm 1}]}\left(\Lambda[x_i^{\pm 1}]/((x_1-1)^d, \dots, (x_n-1)^d), \omega_{\hat{H}}\right)$$

$$\simeq \Lambda[x_i^{\pm 1}]/((x_1-1)^d, \dots, (x_n-1)^d)\left(\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}\right)$$

$$\omega_{\hat{\chi}} = \varinjlim_d \omega_{\chi_d}$$

$$\mathrm{End}_{\mathrm{Shv}_{mon}(H)}(\widetilde{Ch}) = \mathrm{End}_{\mathrm{IndCoh}}(R_{I_F^t}, \hat{H})^{\omega}$$

$$\| $$

$$\left(\left(\{1\} \to H\right)^* \widetilde{Ch}\right)^{\vee} \quad \left(= \mathrm{Hom}(\widetilde{Ch}, \delta_1)\right)$$

Lemma Let $\mathcal{X}$ be an (ind-scheme)

$$\mathrm{End}(\omega_{\mathcal{X}}) = \mathrm{Fun}(\mathcal{X}).$$

Pf $\mathcal{X} = \varinjlim_i X_i$, X_i scheme f.t. $/\Lambda$

$$\mathrm{Hom}(\omega_{\mathcal{X}}, \omega_{\mathcal{X}}) = \mathrm{Hom}\left(\varinjlim_i \omega_{X_i}, \varinjlim_i \omega_{X_i}\right)$$

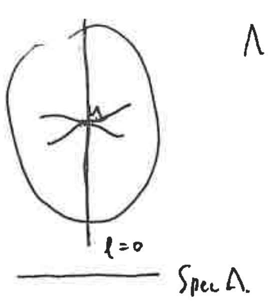
$$= \varprojlim_i \mathrm{Hom}(\omega_{X_i}, \omega_{\mathcal{X}})$$

$$= \varprojlim_i \mathrm{Hom}(\omega_{X_i}, \omega_{X_i})$$

(everything is derived)

$$\overset{\text{Grothendieck-Serre}}{=\!=} \varprojlim_i \mathrm{Hom}(\mathcal{O}_{X_i}, \mathcal{O}_{X_i})$$

$$= \varprojlim_i \mathrm{Fun}(X_i) = \mathrm{Fun}(\mathcal{X})$$



$\Lambda = \mathbb{Z}_l$

$$\omega_{R_{I_F^t}, \hat{H}} = \bigoplus_{\bar{x}} \omega_{(R_{I_F^t}, \hat{H})_{\bar{x}}}$$

$\bar{x} : \pi_1^c(H) \to E^*$ $\quad E/\mathbb{F}_l$ finite.

$$\mathrm{End}\left(\omega_{R_{I_F^t},\, \hat{H}}\right) = \prod_{\substack{\bar{x} : \pi_1^c(H) \to E^\times \\ E \mid \mathbb{F}_\ell \text{ finite}}} \mathrm{End}\left(\omega_{(R_{I_F^t},\, \hat{H})_{\bar{x}}}\right)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad \mathfrak{X}.$$

Claim. $\mathrm{End}\left(\omega_{(R_{I_F^t},\, \hat{H})_{\bar{u}}}\right) = \mathbb{Z}_\ell[[x_1 - 1, \dots, x_n - 1]]$

$\|$

$\varprojlim \mathrm{Fun}(X_i) = \varprojlim_i \varprojlim_d \left(\mathrm{Fun}(X_i)/\ell^d\right)$

$X_i \hookrightarrow \hat{H}$

X_i mod ℓ set theoretically supported at $\bar{u}$ $\quad = \underline{\varprojlim_d \varprojlim_i \left(\mathrm{Fun}(X_i)/\ell^d\right)}$

formal completion of $\bar{u}$ in $\hat{H}$ mod ℓ^n

Cor. $\Lambda = \mathbb{Z}_\ell$ $\qquad R = R_{\bar{u}}$

(1) $\mathrm{IndCoh}(R_{I_F^t}, \hat{H})_{\bar{x}} \to \overbrace{\mathbb{Z}_\ell[[x_1 - \bar{x}(x_1), \dots, x_n - \bar{x}(x_n)]]}^{R_{\bar{x}}}\text{-mod}$ $\qquad \mathbb{Z}_\ell/\ell^d[[x_1 - 1, \dots, x_n - 1]]$

$\mathcal{F} \longmapsto R\mathrm{Hom}\left(\omega_{(R_{I_F^t},\, \hat{H})_{\bar{x}}}, \mathcal{F}\right)$ $\qquad$ <u>not continuous</u>

<u>Monoidal functor</u>

(2) $\mathrm{IndCoh}(R_{I_F^t}, \hat{H})_{\bar{u}} \otimes \mathrm{IndCoh}(R_{I_F^t}, \hat{H})_{\bar{u}} \longrightarrow (R \hat{\otimes} R)\text{-mod}$

$\|$

$\mathrm{IndCoh}(R_{I_F^t}, \hat{H} \times \hat{H})_{\bar{u}}$

$\mathcal{F} \longmapsto R\mathrm{Hom}(\omega_{\dots}, \mathcal{F})$

monoidal

$\mathcal{A} \curvearrowright \mathrm{Shv}_{\bar{u}\text{-mon}}(H)$, $\mathcal{A} \longrightarrow \mathrm{IndCoh}\left((R^2_{I_F^t}, \hat{H})_{\bar{u}}\right)$

$\mathbb{V} : \mathcal{A} \longrightarrow (R \hat{\otimes} R)\text{-mod}$, $\quad a \otimes b \mapsto \mathbb{V}(a) \otimes_R \mathbb{V}(b)$

Spoiler: later, $\mathcal{A}$ = affine Hecke cat.

Monodromic & equivariant categories

$H \curvearrowright X \qquad H$ conn'd alg. gp.

$Shv(H) \curvearrowright Shv(X)$

$$H \times X \xrightarrow{a} X$$
$$\mathcal{F} \quad \mathcal{G} \longmapsto a_*(\mathcal{F} \boxtimes \mathcal{G}).$$

Def (1) $Shv_{H\text{-mon}}(X) := Shv_{mon}(H) \otimes_{Shv(H)} Shv(X)$ is a $Shv_{mon}(H)$-mod.

$$Shv_{H,\chi\text{-mon}}(X) := Shv_{\chi\text{-mon}}(H) \otimes_{Shv(H)} Shv(X)$$

$$\begin{tikzcd} (Mod_\Lambda)_\chi \arrow[r] & Shv_{\chi\text{-mon}}(H) \arrow[l, bend right, "Hom(\chi,-)"'] \subset Shv_{mon}(H) \subset Shv(H) \arrow[ll, bend right] \end{tikzcd}$$

$$\Lambda \longmapsto Ch_\chi$$

(2) $Shv((H,\chi)\backslash X) := (Mod_\Lambda)_\chi \otimes_{Shv(H)} Shv(X)$

Rmk (1) $Shv_{H,(\chi\text{-})mon}(X)$ is a full subcat. of $Shv(X)$

(2) $Shv((H,\chi)\backslash X)$ is not a subcat. of $Shv(X)$ in general.

But if $H = \mathbb{G}_a$ (unipotent),

$Shv((H,\chi)\backslash X) \simeq Shv_{H,\chi\text{-mon}}(X)$.

$$\begin{tikzcd} Shv_{mon}(H) \otimes_{Shv(H)} Shv(X) = Shv_{H\text{-mon}}(X) \arrow[d, shift right] \\ Shv(H) \otimes_{Shv(H)} Shv(X) = Shv(X) \arrow[u, shift right] \end{tikzcd}$$

(3) $H = \mathbb{G}_m$.

$\circlearrowright \mathrm{Shv}_{H,\, u\text{-mon}}(X)$

$\mathrm{Shv}_{u\text{-mon}}(\mathbb{G}_m)$

$\mathrm{Ch}_{u\text{-mon}} \curvearrowright \mathcal{F}$ is identity functor.

$$\mathrm{End}(\mathrm{Ch}_{u\text{-mon}}) \longrightarrow \mathrm{End}(\mathcal{F}) \qquad \mathcal{F} = \mathrm{Ch}_{u\text{-mon}} * \mathcal{F}$$

$$\wr\!\wr$$

$$\Lambda[\![x-1]\!] \quad \curvearrowright \mathcal{F}$$

Verdier's monodromy action on monodromic sheaves.

<u>Prop</u> $X = u$ trivial.

$$\mathrm{Shv}\big((H,u)\backslash X\big) = \mathrm{Mod}_\Lambda \underset{\mathrm{Shv}(H)}{\otimes} \mathrm{Shv}(X)$$

$$\wr\!\wr$$

$$\mathrm{Shv}(\underbrace{H}\backslash X)$$

quotient stack

<u>Lecture 11</u>. $H \curvearrowright X$, $\mathrm{Shv}(H) \curvearrowright \mathrm{Shv}(X)$

$$H \times X \xrightarrow{act} X$$

$$(\mathcal{F}, \mathcal{G}) \rightsquigarrow act_*(\mathcal{F} \boxtimes \mathcal{G})$$

$$\begin{array}{ccccccc} \Lambda & & \mathrm{Ch}_{x\text{-mon}} & & \mathrm{Ch}_{mon} & & \delta_1 \\ \mathrm{Mod}_\Lambda & \rightleftarrows & \mathrm{Shv}_{x\text{-mon}}(H) & \rightleftarrows & \mathrm{Shv}_{mon}(H) & \rightleftarrows & \mathrm{Shv}(H) \end{array}$$

$$\mathrm{Hom}(\mathrm{Ch}_x, \mathcal{F}) \longleftarrow\!\shortmid \mathcal{F}$$

Def. $\underset{H,\chi\text{-mon}}{\mathrm{Shv}_{H\text{-mon}}}(X) := \underset{\chi\text{-mon}}{\mathrm{Shv}_{\mathrm{mon}}}(H) \otimes_{\mathrm{Shv}(H)} \mathrm{Shv}(X)$

$$\mathrm{Shv}\left({}_{(H,\chi)}\backslash X\right) = (\mathrm{Mod}\,\Lambda)_\chi \otimes_{\mathrm{Shv}(H)} \mathrm{Shv}(X) = (\mathrm{Mod}\,\Lambda)_\chi \otimes_{\mathrm{Shv}_{\mathrm{mon}}(H)} \mathrm{Shv}_{H\text{-mon}}(X)$$

- $$\begin{tikzcd} \mathrm{Shv}_{H\text{-mon}}(X) \arrow[r, hook, "\text{full subcat.}"] & \mathrm{Shv}(X) \\ \mathrm{Shv}\left({}_{(H,\chi)}\backslash X\right) \arrow[u, shift left] & \end{tikzcd}$$

Ex. $\chi = u$ trivial, $\mathrm{Shv}_{H,u\text{-mon}}(X)$ (unipotent) monodromic sheaves on X.

$$\mathrm{Shv}_{H,u\text{-mon}}(X) \overset{..}{=} \mathrm{Shv}(H \overset{..}{\backslash} X)$$

$\mathcal{F} \in \mathrm{Shv}(H \overset{..}{\backslash} X)$

$\mathrm{Ch}_{u\text{-mon}} * \mathcal{F} \simeq \mathcal{F}$

$\mathrm{End}(\mathrm{Ch}_{u\text{-mon}}) \curvearrowright \mathcal{F}$

$H = \mathbb{G}_m$ $\mathrel{\|}\!\wr$

$\Lambda[\![x-1]\!] \curvearrowright \mathcal{F}$

Ex. $H = \mathbb{G}_a$, or more generally unipotent, $\phi : H(\mathbb{F}_q) \longrightarrow \Lambda^\times$

$$\mathrm{Shv}\left({}_{(H,\phi)}\backslash X\right) \simeq \mathrm{Shv}_{H,\phi\text{-mon}}(X).$$

Prop. $\mathcal{X} = \mathcal{L}$ trivial

$$\mathrm{Shv}\big((H,\mathcal{L})\backslash X\big) \cong \mathrm{Shv}(H\backslash X)$$

Pf.

$$\mathrm{LHS} = (\mathrm{Mod}\,\Lambda)_{\mathcal{L}} \underset{\mathrm{Shv}_{\mathrm{mon}}(H)}{\otimes} \mathrm{Shv}_{H\text{-mon}}(X)$$

$$\underset{\text{Bar resolution}}{=\!=} \Big(\cdots \rightrightarrows \underbrace{\mathrm{Shv}_{\mathrm{mon}}(H) \otimes \mathrm{Shv}_{\mathrm{mon}}(H)}_{=\ \mathrm{Shv}_{\mathrm{mon}}(H\times H)} \rightrightarrows \mathrm{Shv}_{\mathrm{mon}}(H) \Big) \underset{\mathrm{Shv}_{\mathrm{mon}}(H)}{\otimes} \mathrm{Shv}_{H\text{-mon}}(X)$$

$$\cdots \rightrightarrows H \times H \times X \rightrightarrows H \times X \underset{pr}{\overset{act}{\rightrightarrows}} X$$

$$\Big(\cdots \leftleftarrows \mathrm{Shv}(H\times X) \leftleftarrows \mathrm{Shv}(X) \Big) \longleftarrow \mathrm{Shv}(H\backslash X)$$

$$\uparrow \text{fully faithful} \qquad\qquad \uparrow$$

$$= \Big(\cdots \leftleftarrows \mathrm{Shv}_{\mathrm{mon}}(H) \otimes \mathrm{Shv}_{\mathrm{mon}}(X) \leftleftarrows \mathrm{Shv}_{\mathrm{mon}}(X) \Big)$$

limit preserves fully faithful

(colimit doesn't)

$$\mathrm{Shv}\big((H,\mathcal{L})\backslash X\big) \xrightarrow{\text{fully faithful}} \mathrm{Shv}(H\backslash X)$$

Essential surjectivity omitted. [fully faithful embedding w/ conservative right adjt is an equiv.]

Rmk $\mathrm{Shv}_{mon}(H)$ is a right $\mathrm{Shv}(H)$-mod admitting a dual given by $\mathrm{Shv}_{mon}(H)$ as left $\mathrm{Shv}(H)$-mod

$$\mathrm{Mod}_\Lambda \xrightarrow{\text{unit}} \underset{Ch_{mon}\,\boxtimes\, Ch_{mon}}{\mathrm{Shv}_{mon}(H) \otimes_{\mathrm{Shv}(H)} \mathrm{Shv}_{mon}(H)}, \quad \mathrm{Shv}_{mon}(H) \otimes \mathrm{Shv}_{mon}(H) \xrightarrow{\text{counit}} \mathrm{Shv}(H)$$

$$\begin{tikzcd} \mathrm{Shv}_{mon}(H) \otimes \mathrm{Shv}_{mon}(H) \arrow[dr] \arrow[rr, "\text{counit}"] & & \mathrm{Shv}(H) \\ & \mathrm{Shv}_{mon}(H) \arrow[ur] & \end{tikzcd}$$

Informally, $\mathrm{Shv}_{mon}(H) \cong \mathrm{Fun}^L(\mathrm{Shv}_{mon}(H), \mathrm{Mod}_\Lambda) \simeq \mathrm{Fun}^L_{\mathrm{RMod}_{\mathrm{Shv}(H)}}(\mathrm{Shv}_{mon}(H), \mathrm{Shv}(H))$

$\mathrm{Shv}(H)$

$$\Rightarrow \mathrm{Shv}_{H\text{-}mon}(X) = \mathrm{Fun}^L_{\mathrm{Shv}(H)\text{-mod}}(\mathrm{Shv}_{mon}(H), \mathrm{Shv}(X))$$

Similarly, $\mathrm{Shv}((H,\chi)\backslash X) = \mathrm{Fun}^L_{\mathrm{Shv}(H)\text{-mon}}((\mathrm{Mod}_\Lambda)_\chi, \mathrm{Shv}(X))$

(G, B, T)

$LG \supset I \supset I^u$, $\quad I/I^u \xrightarrow{\sim} T$.

$\mathrm{Shv}(I\backslash LG/I) \longleftarrow$ affine Hecke category, monoidal

$$I\backslash LG/I \times I\backslash LG/I \longleftarrow I\backslash LG \overset{I}{\times} LG/I \xrightarrow{m} I\backslash LG/I$$

$$\mathcal{F} \quad \mathcal{G} \longmapsto \mathcal{F} \tilde{\boxtimes} \mathcal{G} \longmapsto \mathcal{F} * \mathcal{G} = m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

<u>unit</u>. $I\backslash I/I \xrightarrow{\Delta_1} I\backslash LG/I$. $(\Delta_1)_* \delta_{I\backslash I/I}$, δ is the unit of $\mathrm{Shv}(I\backslash I/I)$ (w/ its symmetric monoidal str.)

$m_* = m_!$ since m is ind-projective.

$Shv(I\backslash LG/I)$ categorical analogue of $H_I = \mathrm{End}(\text{c-ind}_I^{G(F)} \mathbb{C})$

$$Shv(I^u\backslash LG/I^u) \longleftrightarrow \mathrm{End}(\text{c-ind}_{I^u}^{G(F)} \mathbb{C})$$

$$I^u\backslash LG/I^u \times I^u\backslash LG/I^u \longleftarrow I^u\backslash LG \overset{I^u}{\times} LG/I^u \xrightarrow{m^u} I^u\backslash LG/I^u$$

$$\mathcal{F}, \mathcal{G} \qquad \mathcal{F}\tilde{\boxtimes}\mathcal{G} \qquad \mathcal{F} \overset{u}{*} \mathcal{G} := (m^u)_*(\mathcal{F}\tilde{\boxtimes}\mathcal{G})$$

w/ a unit.

problem: $(m^u)_* \neq (m^u)_!$

$$\mathrm{End}(\text{c-ind}_{I^u}^{G(F)} \mathbb{C}) \simeq \bigoplus_{\chi, \chi'} {}_\chi H_{\chi'}$$

$$T \curvearrowright I^u\backslash LG/I^u \curvearrowleft T$$

Let $Shv_{mon}(I^u\backslash LG/I^u) \subset Shv(I^u\backslash LG/I^u)$ be the full subcat. of $(T\times T)$-monodromic sheaves.

(1) $\chi, \chi' : \quad \pi_1^c(T) \longrightarrow \Lambda^\times$

$$Shv_{\chi,\chi'\text{-mon}}(I^u\backslash LG/I^u) =: Shv((I,\chi)\backslash LG/(I,\chi'))$$

If $\Lambda = \bar{\mathbb{F}}_\ell, \bar{\mathbb{Q}}_\ell$, $\quad \mathrm{Shv}_{\mathrm{mon}}(I^u\backslash LG/I^u) \simeq \bigoplus_{\chi,\chi'} \mathrm{Shv}\big((I,\chi)\backslash LG/(I,\chi')\big)$

(2) $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\mathrm{mon}}(I^u\backslash LG/I^u)$, $\quad \mathcal{F} \overset{u}{*} \mathcal{G} \in \mathrm{Shv}_{\mathrm{mon}}(I^u\backslash LG/I^u)$

$$\boxed{\begin{array}{c} X \longrightarrow Y \\ \circlearrowright_H \circlearrowleft \\ \mathrm{Shv}_{H\text{-mon}}(X) \rightleftarrows \mathrm{Shv}_{H\text{-mon}}(Y) \end{array}}$$

<u>Unit</u>: $\quad I^u\backslash I^u/I^u \xrightarrow{\Delta_1^u} I\backslash LG/I$

$$\mathrm{Av}^*\big((\Delta_1^u)_* \, \delta_{I^u\backslash I^u/I^u}\big) = \big(I^u\backslash I/I^u \xrightarrow{\Delta_1} I^u\backslash LG/I^u\big)_* \mathrm{Ch}_{\mathrm{mon}}$$

$$\begin{array}{ccc} I^u\backslash I^u/I^u & \hookrightarrow & I^u\backslash I/I^u \\ \{1\} & \longrightarrow & T \\ \delta_1 & \longrightarrow & \mathrm{Ch}_{\mathrm{mon}}. \end{array}$$

Similarly, χ, χ'

$$\underset{(\Delta_1)_*\mathrm{Ch}_{\chi\text{-mon}}}{\mathrm{Shv}\big((I,\chi)\backslash LG/(I,\chi)\big)} \sim \mathrm{Shv}\big((I,\chi)\backslash LG/(I,\chi')\big) \sim \underset{(\Delta_1)_*\mathrm{Ch}_{\chi'\text{-mon}}}{\mathrm{Shv}\big((I,\chi')\backslash LG/(I,\chi')\big)}$$

<u>Rmk</u>. χ, χ'. also have $\mathrm{Shv}\big((I,\chi)\backslash LG/(I,\chi')\big)$

$$= (\mathrm{Mod}_\Lambda)_\chi \otimes_{\mathrm{Shv}_{\mathrm{mon}}(T)} \mathrm{Shv}_{\mathrm{mon}}(I^u\backslash LG/I^u) \otimes_{\mathrm{Shv}_{\mathrm{mon}}(T)} (\mathrm{Mod}_\Lambda)_{\chi'}.$$

In particular, $\chi = \chi' = u \rightsquigarrow \mathrm{Shv}(I\backslash LG/I)$.

(Kac-Moody)
<u>Central extension.</u>

$$\mathrm{Shv}\left({}_I\backslash LG/I\right)$$

$$\begin{array}{ccc} \mathrm{End}(\mathrm{Chu\text{-}mon}) \circlearrowleft & \mathcal{F} & \circlearrowright \mathrm{End}(\mathrm{Chu\text{-}mon}) \\ \wr\wr & & \wr\wr \\ \Lambda[\![x_1-1, x_2-1, \dots, x_n-1]\!] & & \Lambda[\![x_1-1, \dots, x_n-1]\!] \end{array}$$

$$1 \to \mathbb{G}_m \to \widehat{LG} \to LG \to 1$$

If G is simple & simply conn'd, all possible central exts of LG by $\mathbb{G}_m$ are classified by $\mathbb{Z}$.

$G = GL_n$.

Gr_G affine Grassmannian of G

$$= LG/L^+G = \left\{ \begin{array}{c} \Lambda \subset k(\!(\varpi)\!)^n \\ \text{lattices} \end{array} \right\}$$

$$\exists \text{ a line bundle } \quad \begin{array}{c} \mathcal{L}_{det} \\ \downarrow \\ \mathrm{Gr}_G \end{array} \qquad \begin{array}{c} \det\left(\dfrac{\overset{=:\Lambda_0}{k[\![\varpi]\!]^n}}{k[\![\varpi]\!]^n \cap \Lambda}\right) \otimes \det\left(\dfrac{\Lambda}{k[\![\varpi]\!]^n \cap \Lambda}\right)^{-1} \\ \downarrow \qquad\qquad\qquad\qquad \uparrow \\ \ni \Lambda \qquad\qquad\qquad k\text{-line.} \end{array}$$

$$\begin{array}{c} \mathcal{L}_{det} \\ \downarrow \\ LGL_n \curvearrowright \mathrm{Gr}_{GL_n} \end{array}$$

The action of LGL_n on Gr_G does not lift to an action on $\mathcal{L}_{det}$.

$$\det\left(\frac{\Lambda_0}{\Lambda_0 \cap \Lambda}\right) \otimes \det\left(\frac{\Lambda}{\Lambda_0 \cap \Lambda}\right)^{-1} \qquad \mathcal{L}_{\det, \Lambda}$$

$$\uparrow$$

$$\det\left(\frac{\Lambda_0}{\Lambda_0 \cap g\Lambda}\right) \otimes \det\left(\frac{g\Lambda}{\Lambda_0 \cap g\Lambda}\right)^{-1} \qquad \mathcal{L}_{\det, g\Lambda}$$

$$\widehat{LG} \longrightarrow LG \longrightarrow 1$$

$$\begin{array}{ccc} \mathcal{L}_{\det, g\Lambda} \otimes \mathcal{L}_{\det, \Lambda}^{-1} & \longmapsfrom & \begin{array}{c}\in\\ g\end{array} \\ \cong & & \\ \det(g\Lambda_0 \mid \Lambda_0) & & \end{array}$$

<u>Lecture 12</u>. <u>Goal</u>: $1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1$.

$V = k((\varpi))^n = F^n$

$\Lambda_0 = k[\![\varpi]\!]^n = \mathcal{O}^n$

$\Lambda_1, \Lambda_2 \subset V$ $\qquad \mathcal{O}$-lattices

$\det(\Lambda_1 \mid \Lambda_2)$ $\qquad$ 1-dim'l v.s.

$$:= \det\left(\frac{\Lambda_1}{\Lambda_1 \cap \Lambda_2}\right) \otimes \det\left(\frac{\Lambda_2}{\Lambda_1 \cap \Lambda_2}\right)^{-1}$$ (can replace $\Lambda_1 \cap \Lambda_2$ by any lattice $\Lambda \subset \Lambda_1 \cap \Lambda_2$)

$\Lambda_1, \Lambda_2, \Lambda_3$. $\exists$ canonical isom.

$$\det(\Lambda_1|\Lambda_2) \otimes \det(\Lambda_2|\Lambda_3) \xrightarrow{\sim} \det(\Lambda_1|\Lambda_3)$$

satisfying natural compatibility.

<u>Rmk</u>. This works in family.

R/k. $\quad \Lambda_1 \subset R((\varpi))^n \quad$ proj.
$\quad \Lambda_2 \quad R[[\varpi]]$-mod

$\rightsquigarrow \det(\Lambda_1|\Lambda_2) \in \mathrm{Pic}_R$.

$G = GL_n$

$Gr_{GL_n} = \{ \mathcal{O}\text{-lattices } \Lambda \subset F^n \}$

$$\begin{array}{ccc} \mathcal{L}_{\det} & & \det(\Lambda_0|\Lambda) \\ | & & | \\ Gr_{GL_n} & \ni & \Lambda \end{array}$$

(This is the ample one)

$$\begin{array}{ccc} \widehat{LGL_n} & \circlearrowright & \mathcal{L}_{\det} \\ \pi \big| & & \big| \\ LGL_n & \circlearrowright & Gr_{GL_n} \\ \Psi & & \\ g & & \end{array}$$

If you want to lift to an LGL_n action on $\mathcal{L}_{\det}$,

need canonical isom. $g^* \mathcal{L}_{\det} \xrightarrow{\sim} \mathcal{L}_{\det}$

But $(g^*\mathcal{L}_{\det})_\Lambda = \det(\Lambda_0|g\Lambda)$

$\mathcal{L}_{\det,\Lambda} = \det(\Lambda_0|\Lambda) = \det(g\Lambda_0|g\Lambda)$

No canonical choice.

$$\pi^{-1}(g) = \det(g\Lambda_0 \mid \Lambda_0)^{\times}. \qquad \square$$

Rmk. This is indeed a non-trivial central ext'n.

Choose $G \overset{\rho}{\hookrightarrow} GL_n$ faithful

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & (\widehat{LG})_\rho \arrow[r] \arrow[d] & LG \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widehat{LGL_n} \arrow[r] & LGL_n \arrow[r] & 1
\end{tikzcd}$$

Rmk. $(\widehat{LG})_\rho$ is non-trivial.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & (\widehat{LT})_\rho \arrow[r] \arrow[d] & LT \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & (\widehat{LG})_\rho \arrow[r] \arrow[d] & LG \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widehat{LGL_n} \arrow[r] & LGL_n \arrow[r] & 1
\end{tikzcd}$$

One can compute the Commutator: $LT \times LT \longrightarrow \mathbb{G}_m$

$f, g \in k((\varpi))^{\times}$ $\quad (\lambda(f(\varpi)), \mu(g(\varpi))) \mapsto (-1)^{\cdots}\{f, g\}^{\cdots}$

tame symbol.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widehat{LGL_n} \arrow[r] & LGL_n \arrow[r] & 1 \\
& & & L^+GL_n \arrow[u, hook] \arrow[ul, "\exists \text{ canonical}"] &
\end{tikzcd}$$

$g \in L^+GL_n$,

$\det(g\Lambda_0 | \Lambda_0)^\times = \det(\Lambda_0|\Lambda_0)^\times$

$\rightsquigarrow \rho: G \hookrightarrow GL_n$ defined $/\mathbb{Q}$.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widehat{LG}_\rho \arrow[r] & LG \arrow[r] & 1 \\
& & & L^+G \arrow[u, hook] \arrow[ul] & \\
& & & \cup & \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \hat{I} \arrow[r] & I \arrow[r] & 1 \\
& & \wr & \cup & \\
& & I \times \mathbb{G}_m^{cen} & I^+ &
\end{tikzcd}$$

Rmk. (1) If G semisimple, $\exists!$ splitting $L^+G \to \widehat{LG}_\rho$.

(2) The splitting over I is <u>Not</u> unique. We use the one defined by the pinning.

$$I^+\backslash\widehat{LG}/I^+ \xrightarrow{\mathbb{G}_m^{cen}\text{-torsor}} I^+\backslash LG/I^+$$

$$\hat{I}/I^+ = \tilde{T} = T \times \mathbb{G}_m^{cen}$$

$$Shv_{\underline{\tilde{T}}\text{-mon}}(I^+\backslash\widehat{LG}/I^+) =: \mathcal{H}_{mon}$$

χ, χ' character sheaves on $\tilde{T} = T \times \mathbb{G}_m^{cen}$

$$\chi\text{-mon}\,\mathcal{H}_{\chi'\text{-mon}} := \mathrm{Shv}\left((\hat{I}, \chi) \backslash \widehat{LG} / (\hat{I}, \chi') \right)$$

These are monoidal categories. $(\chi = \chi')$

$$\begin{tikzcd}
I^+ \backslash \widehat{LG} / I^+ \arrow[d, "\frac{\tilde{T} \times \tilde{T}}{\Delta(\mathbb{G}_m^{cen})}\text{-torsor}"'] & I^+ \backslash \hat{I} w \hat{I} / I^+ \arrow[l, "i_w"'] \arrow[d, "\frac{\tilde{T} \times \tilde{T}}{\Delta(\mathbb{G}_m^{cen})}\text{-torsor}"] \arrow[loop right, "\tilde{T}"] \arrow[d, phantom, "\tilde{T}"', "\simeq"] \\
I \backslash LG / I & I \backslash I w I / I \arrow[l, "i_w"']
\end{tikzcd} \qquad w \in \tilde{W}$$

$$I \backslash LG / I \xleftarrow{\text{closed}} I \backslash \overline{IwI} / I \xleftarrow{\text{open}}$$

i_w is a locally closed affine embedding

$$\mathcal{H}(w)_{mon} = \mathrm{Shv}_{mon}\left(I^+ \backslash \hat{I} w \hat{I} / I^+ \right)$$

Similarly $\chi\text{-mon}\,\mathcal{H}(w)_{\chi'\text{-mon}}$.

<u>Lemma</u>. Let $\dot{w}$ be a lifting of w to $\widehat{LG}$,

Then $\alpha_{\dot{w}} : \mathcal{H}(w)_{mon} \xleftarrow{\sim} \mathrm{Shv}_{mon}(\tilde{T})$

<u>Pf</u>. $I \backslash IwI / I = * / I \cap wIw^{-1}$.

$$\begin{tikzcd}
I^+ \backslash \hat{I} w \hat{I} / I^+ \arrow[d] & \simeq & \tilde{T} \times \mathbb{B}(I^+ \cap wI^+w^{-1}) \arrow[d] \\
I \backslash IwI / I & \simeq & \mathbb{B}T \times \mathbb{B}(I^+ \cap wI^+w^{-1})
\end{tikzcd}$$

$$t \cdot \dot{w} \longleftarrow\!\shortmid (t, \cdot)$$

Compatible w/ left $\tilde{T}$-action. $I^+ \cap w I^+ w^{-1}$ unipotent $\square$

<u>Def</u>. $\Delta^{mon}_{\dot{w}} : Shv_{mon}(\tilde{T}) \longrightarrow \mathcal{H}_{mon}$ $\quad (2w)_! \circ \alpha_{\dot{w}}$

$\nabla^{mon}_{\dot{w}} : Shv_{mon}(\tilde{T}) \longrightarrow \mathcal{H}_{mon}$ $\quad (2w)_* \circ \alpha_{\dot{w}}$

$w=1$, $\dot{w}=1$. $\quad \Delta^{mon}_1 = \nabla^{mon}_1$

Unit of $\mathcal{H}_{mon}$ is $\Delta^{mon}_1(\tilde{Ch})$.

There is also χ, χ' version.

$\tilde{W} \curvearrowright \tilde{T}$

$\tilde{W} = N_{LG}(T)/L^+T \longrightarrow N_{LG}(T)/LT = W \curvearrowright T$

$= N_{LG}(\tilde{T})/L^+T \curvearrowright \tilde{T}$

This action does not factors through the action of W.

In fact, $\tilde{W} \hookrightarrow End(\tilde{T})$

$$\begin{array}{c} \tilde{W} \\ \| \\ \lambda \quad X_*(T) \rtimes W \\ \swarrow \quad \swarrow \\ \lambda(\varpi) \quad LT \end{array}$$

$T \times \mathbb{G}_m^{cen} = \tilde{T}$

$\widetilde{\lambda(\varpi)}\left((t,c)\right)\widetilde{\lambda(\varpi)}^{-1}$

$= \left(t, c\, Comm(\lambda(\varpi), t)\right)$

$f(\varpi) = a_n\varpi^n + \dots, g(\varpi) = b_m \varpi^m + \dots$

$\{f(\varpi), g(\varpi)\}$

$= (-1)^{nm} \dfrac{a_n^m}{b_m^n}$.

$\underset{a \in k^\times}{Comm(\lambda(\varpi), \mu(a))} = (-1)^{\cdots} \{\varpi, a\}^{B(\lambda,\mu)}$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\left(\frac{1}{a}\right)}^{B(\lambda,\mu)}$

Lemma (1) $w, v \in \widetilde{W} \curvearrowleft \widetilde{T}$, $\ell(w) + \ell(v) = \ell(wv)$

$$\Delta^{mon}_{\dot{w}}(\mathcal{L}) \overset{u}{\star} \Delta^{mon}_{\dot{v}}(\mathcal{L}') \simeq \Delta^{mon}_{\dot{w}\dot{v}}(\mathcal{L} \star w(\mathcal{L}'))$$

$$\nabla^{mon}_{\dot{w}}(\mathcal{L}) \overset{u}{\star} \nabla^{mon}_{\dot{v}}(\mathcal{L}') \simeq \nabla^{mon}_{\dot{w}\dot{v}}(\mathcal{L} \star w(\mathcal{L}'))$$

$$IwI \overset{I}{\times} IvI/I \xrightarrow{\sim} IwvI/I \qquad \text{when } \ell(wv) = \ell(w) + \ell(v)$$

$$\hat{I} w \hat{I} \overset{I^+}{\times} \hat{I} v \hat{I}/\hat{I}$$

$$\|$$

$$\widetilde{T} \times I^+ \dot{w} I^+ \overset{I^+}{\times} \widetilde{T} \times I^+ \dot{v} I^+/I^+$$

Lemma $\Delta^{mon}_{\dot{w}}(\mathcal{L}) \star \nabla^{mon}_{\dot{w}^{-1}}(\mathcal{L}') = \Delta^{mon}_{1}(\mathcal{L} \star \dot{w}(\mathcal{L}'))$

Lecture 13 $\mathcal{H}_{mon} = Shv_{mon}(I^+ \backslash \widetilde{LG}/I^+)$

$\curvearrowleft w \in \widetilde{W}$

$\widetilde{T} = \mathbb{G}_m^{cen} \times T$

$$\begin{tikzcd} I^+ \backslash \widetilde{LG}_w/I^+ \arrow[r, "j_w"] \arrow[rr, bend right, "i_w"'] & I^+ \backslash \widetilde{LG}_{\leq w}/I^+ \arrow[r, "i_{\leq w}"] & I^+ \backslash \widetilde{LG}/I^+ \end{tikzcd}$$

Choose a lifting $\dot{w}$

$$\begin{tikzcd} Shv_{mon}(I^+ \backslash \widetilde{LG}_w/I^+) \arrow[r, shift left, "(i_w)_!{[-\ell(w)]}"] \arrow[r, shift right, "(i_w)_*{[-\ell(w)]}"'] & \mathcal{H}_{mon} \\ Shv_{mon}(\widetilde{T}) \arrow[u, "\alpha_{\dot{w}}", "\wr"'] \arrow[ur, bend right, "\Delta^{mon}_{\dot{w}}", "\nabla^{mon}_{\dot{w}}"'] & \end{tikzcd}$$

<u>Prop.</u> (1) $\ell(wv) = \ell(w) + \ell(v)$

$$\Delta^{mon}_{\dot w}(\mathcal{L}) \overset{u}{\star} \Delta^{mon}_{\dot v}(\mathcal{L}') \simeq \Delta^{mon}_{\dot w \dot v}(\mathcal{L} \star w(\mathcal{L}'))$$

Similarly for ∇

(2) $$\Delta^{mon}_{\dot w}(\mathcal{L}) \overset{u}{\star} \nabla^{mon}_{\dot w^{-1}}(\mathcal{L}') \xrightarrow{\sim} \Delta^{mon}_{e}(\mathcal{L} \star w(\mathcal{L}'))$$

(3) We have fiber sequence (s simple reflection)

$$\begin{matrix} \Delta^{mon}_{\dot s}(\mathcal{L} \star s(\mathcal{L}')) \\ \oplus \\ \Delta^{mon}_{\dot s}(\mathcal{L} \star s(\mathcal{L}'))[-1] \end{matrix} \longrightarrow \Delta^{mon}_{\dot s}(\mathcal{L}) \overset{u}{\star} \Delta^{mon}_{\dot s}(\mathcal{L}') \longrightarrow \Delta^{mon}_{e}(\mathcal{L} \star s\mathcal{L}')$$

<u>Sketch of proof (3)</u>

$$\begin{tikzcd} \widetilde{\Delta}^{mon}_{\dot s} \arrow[r] \arrow[d, equal] & \widetilde{\nabla}^{mon}_{\dot s} \arrow[r] \arrow[d, equal] & \mathcal{F} \\ \Delta^{mon}_{\dot s}(\widetilde{Ch}) & \nabla^{mon}_{\dot s}(\widetilde{Ch}) & \end{tikzcd}$$

$$\mathcal{F} = (i_e)_* (i_e)^* \mathcal{F} = \Delta^{mon}_{e}(\mathcal{L}'')$$

$$\begin{tikzcd} \Delta^{mon}_{\dot s}(\mathcal{L}) \overset{u}{\star} \widetilde{\Delta}^{mon}_{\dot s} \arrow[r] \arrow[d, equal] & \Delta^{mon}_{\dot s}(\mathcal{L}) \star \widetilde{\nabla}^{mon}_{\dot s} \arrow[r] \arrow[d, equal] & \Delta^{mon}_{\dot s}(\mathcal{L}) \star \mathcal{F} \arrow[d, equal] \\ \Delta^{mon}_{\dot s}(\mathcal{L}) \overset{u}{\star} \widetilde{\Delta}^{mon}_{s} \arrow[r] & \Delta^{mon}_{e}(\mathcal{L}) \arrow[r] & \Delta^{mon}_{\dot s}(\mathcal{L} \star s(\mathcal{L}'')) \end{tikzcd}$$

$$\Delta^{mon}_{s}(\mathcal{L}) \overset{u}{\star} \Delta^{mon}_{s}(\widetilde{Ch})\Big|_{I^+ \backslash \overline{LG}_s / I^+} \simeq (\mathcal{L} \oplus \mathcal{L}[-1])[1]$$

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & LG_s \overset{I}{\times} LG_s / I \arrow[d] \\ s \arrow[r, hook] & LG/I \end{tikzcd}$$

$\to \mathcal{L}'' = \widetilde{Ch} \oplus \widetilde{Ch}[-1]$.

Block decomposition of $\mathcal{H}_{mon}$.

(For simplicity, assume Λ is an alg. closed field)

$$\mathrm{Shv}_{mon}(\widetilde{T}) = \mathrm{IndCoh}\left(R_{I_F^t, \hat{\widetilde{T}}}\right)$$

$$\downarrow$$

$$\chi \in \hat{\widetilde{T}}$$

$\uparrow$ order prime to p

$$\hat{\chi} \hookrightarrow R_{I_F^t, \hat{\widetilde{T}}}$$

$$\mathrm{Shv}_{\chi\text{-}mon}(\widetilde{T}) \simeq \mathrm{IndCoh}(\hat{\chi})$$

$$\mathrm{Shv}_{mon}(\widetilde{T}) = \bigoplus_{\chi} \mathrm{Shv}_{\chi\text{-}mon}(\widetilde{T})$$

$$\chi \in \hat{\widetilde{T}} \circlearrowleft \widetilde{W}$$

$${}_{\chi}\widetilde{W}_{\chi'} = \{ w \in \widetilde{W} : w^{-1}\chi = \chi' \}$$

$${}_{\chi}\widetilde{W}^{\circ}_{\chi} \subset {}_{\chi}\widetilde{W}_{\chi} \longrightarrow \Omega_{\chi}$$

$\uparrow$ Coxeter gp

$\|$

$\langle s_\alpha : \alpha^\vee$ affine coroot $\mathbb{G}_m \to \widetilde{T}$, $(\alpha^\vee)^* Ch_\chi$ is trivial $\rangle$

$${}_x\widetilde{W}^{\beta}_{x'} \leftarrow \text{preimage of } \beta$$

$${}_x\widetilde{W}^{\beta}_{x'} \subset$$

$$\begin{tikzcd}
\widetilde{W}^\circ_x \backslash {}_x\widetilde{W}_{x'} \arrow[r, phantom, "\simeq"] \arrow[dr, phantom, "\Vert"] & {}_x\widetilde{W}_{x'} / \widetilde{W}^\circ_{x'} \arrow[d, phantom, "\Vert"] & \\
 & {}_x\Omega_{x'} \arrow[d] & \beta \arrow[d, maps to] \\
 & {}_x\widetilde{W}_{x'} & w^\beta
\end{tikzcd}$$

$\leftarrow$ minimal length elt in ${}_x\widetilde{W}^\circ_x \cdot w^\beta$ & in $w^\beta{}_{x'}\widetilde{W}^\circ_{x'}$

<u>Def</u>. Let ${}_x\mathcal{H}_{x'} \subset \mathcal{H}_{mon}$ be the full subcat. gen. by

$$\Delta^{mon}_{\dot{w}}(Ch_x), \quad w \in {}_x\widetilde{W}_{x'}$$

${}_x\mathcal{H}^{\beta}_{x'} \subset {}_x\mathcal{H}_{x'}$ full subcat. gen. by $\Delta^{mon}_{\dot{w}}(Ch_x)$

$w \in {}_x\widetilde{W}^{\beta}_{x'}$.

<u>Prop</u> (Thm)

$$\mathcal{H}_{mon} = \bigoplus_{x, x', \beta} {}_x\mathcal{H}^{\beta}_{x'}$$

<u>Lemma</u>. Fix x. Let s be a simple reflection. If $s \notin {}_x\widetilde{W}^\circ_x$,

then $\Delta^{mon}_{\dot{s}}(Ch_x) \cong \nabla^{mon}_{\dot{s}}(Ch_x)$

<u>Proof</u>. $SL_2 \hookrightarrow \widetilde{LG}$ $\quad \alpha_s : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \to \widetilde{T}$,

$\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \to U_\alpha$.

$$(s,\mu) \mapsto (\mu, s\mu) = (x,y) = (\lambda t, \lambda) \leftarrow (t,\lambda)$$

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathbb{G}_m \arrow[r, hook] \arrow[d, "\Lambda_{\mathbb{A}^1} \boxtimes Ch_x"'] & \mathbb{A}^2 - \{(0,0)\} \arrow[d, "\mathbb{G}_m"] & \mathbb{A}^1 \times \mathbb{G}_m \arrow[l] \arrow[d] & \mathbb{G}_m \arrow[l] \arrow[d] \\
\mathbb{P}^1 - \{0\} \arrow[r, "j_s"'] & \mathbb{P}^1 & \mathbb{A}^1 = \mathbb{P}^1 - \{\infty\} \arrow[l, hook'] & 0 \arrow[l, hook']
\end{tikzcd}$$

$\mathbb{P}^1 - \{0\} = \mathbb{A}^1$

<u>Claim</u> If Ch_x is non-trivial, then $j_{s!}(\Lambda_{\mathbb{A}^1} \boxtimes Ch_x)$

$$\xrightarrow{\sim} j_{s*}(\Lambda_{\mathbb{A}^1} \boxtimes Ch_x)$$

$$\begin{array}{ccccc}
\Lambda \boxtimes Ch_x & & Ch_x \boxtimes Ch_x & & \\
\mathbb{G}_m \times \mathbb{G}_m & \xleftarrow{\sim} & \mathbb{G}_m \times \mathbb{G}_m & \xhookrightarrow{j} & \mathbb{A}^1 \times \mathbb{G}_m \\
(s,\mu) & & (t,\lambda) & &
\end{array}$$

$s = \frac{1}{t}$, $\mu = t\lambda$ $\qquad\qquad$ $j_! Ch_x \xrightarrow{\sim} j_* Ch_x$

$\ulcorner \; ({}^xT_s^x)^2 = c \cdot \delta_e \; \lrcorner$

<u>Lemma</u>. $(-) * \Delta_{\dot{s}}(\widetilde{Ch})$ sends ${}_x\mathcal{H}^{\beta}_{x'} \to {}_x\mathcal{H}^{\beta'}_{sx'}$, $[\beta'] = [\beta]\cdot s$

$${}_x\widetilde{W}^{\beta}_{x'} \cdot s = {}_x\widetilde{W}^{\beta'}_{sx'}$$

<u>Pf</u>. $\underset{\in\, {}_x\mathcal{H}^{\beta}_{x'}}{\Delta^{mon}_{\dot{w}}(Ch_x)} * \Delta^{mon}_{\dot{s}}(\widetilde{Ch})$

$$= \begin{cases} \Delta^{mon}_{\dot{w}\dot{s}}(Ch_x), & \ell(ws) = \ell(w) + \ell(s) \\ \\ \Delta^{mon}_{\dot{w}\dot{s}} * \Delta^{mon}_{\dot{s}}(\widetilde{Ch}) * \Delta^{mon}_{\dot{s}}(\widetilde{Ch}) & \end{cases}$$

$$= \begin{cases} {}_x\widetilde{W}^{\beta}_{x'}, & \text{if } s \in {}_x\widetilde{W}^{\circ}_{x'} \\ \cdots & s \notin {}_x\widetilde{W}^{\circ}_{x'} \\ & \Updownarrow \\ & s \in {}_x\Omega_{sx'} \end{cases}$$

First case, $s \in {}_{x'}\widetilde{W}^{\circ}_{x'}$.

Second case $s \notin {}_{x'}\widetilde{W}^{\circ}_{x'}$,

$$\widetilde{\Delta}^{mon}_{\dot{w}}(Ch_x) * \widetilde{\Delta}^{mon}_{\dot{s}}(Ch_{x'-mon})$$

$$= \begin{cases} \widetilde{\Delta}^{mon}_{ws}(Ch_x * Ch_{wx'-mon}), & l(ws) = l(w)+1 \\ \qquad\qquad \overset{\shortparallel}{Ch_x} \quad (wx' = x) & \\ \widetilde{\Delta}^{mon}_{ws}(Ch_x). & l(ws) = l(w)-1 \\ \qquad (\widetilde{\Delta}^{mon}_{\dot{s}}(Ch_{x'-mon}) = \widetilde{\nabla}^{mon}_{s}(Ch_{x'-mon})) & \end{cases}$$

$\square$

Proof of the thm. If $\underset{\displaystyle F}{\overset{\displaystyle \Delta^{mon}_{\dot{v}}(Ch_{x_1})}{\shortparallel}} \in {}_{x_1}\mathcal{H}^{\beta}_{x_1'}$, $w \in {}_{x_2}W^{\beta_2}_{x_2'} \neq {}_{x_1}W^{\beta_1}_{x_1'}$

Then Need to show $RHom(F, \Delta^{mon}_{\dot{w}}(Ch_{x_2})) = 0$.

Prove by induction on $l(w)$.

(1) If $l(w) = 0$, $RHom(\Delta^{mon}_{\dot{v}}(Ch_{x_1}), \Delta^{mon}_{\dot{w}}(Ch_{x_2}))$

$$= RHom(\imath_w^*(\Delta^{mon}_{v}(Ch_{x_1})), Ch_{x_2})$$

$RHom \neq 0 \Rightarrow \begin{matrix} w = v \\ x_1 = x_2 \end{matrix} \Rightarrow$ contradiction.

(2) $w = xs$, $\ell(w) = \ell(x)+1$,

$$\Delta^{mon}_{w}(Ch_{x_2}) = \Delta^{mon}_{x}(Ch_{x_2}) * \Delta^{mon}_{s}(\widetilde{Ch})$$

$$LHS = RHom\left(\Delta^{mon}_{v}(Ch_{x_1}) * \nabla^{mon}_{s}(\widetilde{Ch}), \Delta^{mon}_{x}(Ch_{x_2})\right) \underset{\substack{\uparrow \\ \text{by induction} \\ \text{hypothesis.}}}{=} 0$$

Lecture 14

Correction $\Delta^{mon}_{s}(\mathcal{L} * s(\mathcal{L}') * \widetilde{Ch}_s) \to \Delta^{mon}_{s}(\mathcal{L}) \overset{u}{*} \Delta^{mon}_{s}(\mathcal{L}') \to \Delta^{mon}_{e}(\mathcal{L} * s(\mathcal{L}')) \xrightarrow{+1}$

Here s simple reflection in $\widetilde{W}$

$\updownarrow$

$\alpha_s : \mathbb{G}_m \to \widetilde{T}$ affine simple coroot

$$\hat{\alpha}_s : R_{I_F^t, \hat{\widetilde{T}}} \to R_{I_F^t, \mathbb{G}_m}$$

$$\ker \hat{\alpha}_s = \{u\} \underset{R_{I_F^t, \mathbb{G}_m}}{\times} R_{I_F^t, \hat{\widetilde{T}}}$$

$$Ch : \mathrm{IndCoh}(R_{I_F^t, \hat{\widetilde{T}}}) \xrightarrow{\sim} Shv_{mon}(\widetilde{T})$$

$$\widetilde{Ch}_s := Ch(\omega_{\ker \hat{\alpha}_s})$$

Ex.

$$\mathbb{G}_m \overset{j}{\hookrightarrow} \mathbb{A}^1 \overset{i}{\longleftarrow} \{0\}$$

$$j_! Ch_\chi \to j_* Ch_\chi \to \mathcal{F} = i_* \mathcal{F}'$$

$$\left.\begin{array}{l} H^0 \mathcal{F}' = \Lambda \\ H^1 \mathcal{F}' = \Lambda \end{array}\right] \text{ if } \chi \text{ is trivial}$$

$$H^0 \mathcal{F}' = H^1 \mathcal{F}' = 0 \quad \text{if } \chi \text{ is non-trivial.}$$

$$0 \to j_! \widetilde{Ch} \to j_* \widetilde{Ch} \to i_* \Lambda \to 0 \quad \text{in } Shv(\mathbb{A}^1)^\heartsuit$$

$$\widetilde{Ch} = \bigoplus_\chi \widetilde{Ch}_\chi \quad (\text{say } \Lambda = \bar{\Lambda} \text{ field})$$

$$\left[\begin{array}{l} j_! \widetilde{Ch}_\chi \xrightarrow{\sim} j_* \widetilde{Ch}_\chi \quad \text{if } \chi \neq u \\ j_! \widetilde{Ch}_u \to j_* \widetilde{Ch}_u \to i_* \Lambda \end{array}\right.$$

$$\widetilde{Ch}_u \longleftrightarrow \text{rep of } \pi_1(\mathbb{G}_m) \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \ddots \\ & & & \ddots \end{pmatrix}$$

infinite Jordan block

inv. 1 dim

coinv. 0 !!

Dually.

$$\nabla_e^{mon}(\mathcal{L} * s(\mathcal{L}')) \to \nabla_{\dot{s}}^{mon}(\mathcal{L}) \overset{u}{*} \nabla_{\dot{s}}^{mon}(\mathcal{L}') \to \nabla_{\dot{s}}^{mon}(\mathcal{L} * s(\mathcal{L}') * \widetilde{Ch}_s[1])$$

Last time:

$$\mathcal{H}_{mon} = \bigoplus_{\chi, \chi', \beta} {}_\chi\mathcal{H}_{\chi'}^\beta \quad (\Lambda = \bar{\Lambda} \text{ field})$$

Def. An object in $\mathcal{H}_{mon}$ is said to admit a Δ-flag if it is a finite successive extensions of objects of form $\Delta_{\dot{w}}^{mon}(\mathcal{L})$, $\mathcal{L} \in Shv_{mon}(\widetilde{T})^\heartsuit$.

Similarly for the notion of ∇-flag.

An object is called a tilting object if it admits both Δ-flag & ∇-flag.

Ex. $\mathrm{Shv}_{mon}(U\backslash SL_2/U)$, $U=\begin{pmatrix}1&*\\0&1\end{pmatrix}$

$$\begin{tikzcd}
U\backslash SL_2/U & SL_2/U \arrow[l] \arrow[d] & = \mathbb{A}^2\setminus\{(0,0)\} \\
BsB/B \sqcup B/B = & SL_2/B & = \mathbb{P}^1
\end{tikzcd}$$

$$\tilde{\Delta}_{\dot s}^{mon} := \Delta_{\dot s}^{mon}(\widetilde{Ch})$$
$$\tilde{\nabla}_{\dot s}^{mon} := \nabla_{\dot s}^{mon}(\widetilde{Ch})$$

$$\begin{tikzcd}
\Delta_e^{mon}(Ch_u) \arrow[r] \arrow[d] & \tilde{\Delta}_{\dot s}^{mon} \arrow[r] \arrow[d] & \tilde{\nabla}_{\dot s}^{mon} \arrow[d, equal] \\
\tilde{\Delta}_e^{mon} = \tilde{\nabla}_e^{mon} \arrow[r] \arrow[d] & \tilde{T}_{\dot s}^{mon} \arrow[r] \arrow[d] & \tilde{\nabla}_{\dot s}^{mon} \\
\tilde{\Delta}_e^{mon} \arrow[r, equal] & \tilde{\Delta}_e^{mon} &
\end{tikzcd}$$

Lemma. $\mathcal{F}\in \mathcal{H}_{mon}$ admits a Δ-flag iff its $*$-restriction to each stratum

$I^+\backslash \widetilde{Ch}_w/I^+ \simeq \tilde{T}\times \mathbb{B}(I^+\cap wI^+w^{-1})$ is $\underset{\substack{\uparrow \\ \mathrm{Shv}_{mon}(\tilde T)^{\heartsuit}}}{\mathcal{L}} \boxtimes \Lambda[\ell(w)]$, is zero for all but finitely many strata

admits a ∇-flag iff !-restriction ...

Basic fact:

$$\mathrm{Hom}\left(\Delta^{mon}_{\dot v}(\mathcal{L}),\ \nabla^{mon}_{\dot w}(\mathcal{L}')\right) = \begin{cases} 0 & \text{if } v \neq w \\ \mathrm{Hom}_{Shv_{mon}(\tilde T)}(\mathcal{L}, \mathcal{L}') & \text{if } v = w \end{cases}$$

If $\mathcal{F}$ is an ext. $\left\{\Delta^{mon}_{\dot w_i}(\mathcal{L}_i)\right\} \longrightarrow \nabla^{mon}_{v}(\widetilde{Ch})$

$\mathrm{Hom}(\mathcal{L}_i, \widetilde{Ch})$

Conversely if $\mathcal{F}$ satisfies this condition, pick a maximal w s.t.

$$\mathcal{F}\Big|_{*\text{-restr. } I^+ \backslash L\tilde G_w / I^+} \cong \mathcal{L} \boxtimes \ldots$$

$\Delta^{mon}_{\dot w}(\mathcal{L}) \to \mathcal{F} \to \mathcal{F}'$, $\mathcal{F}'$ still satisfies the condition;

then do induction!

Def. An obj. in $\mathcal{H}_{mon}$ is called a cofree χ-monodromic tilting obj. if it is a finite successive ext'n $\left\{\Delta^{mon}_{\dot w}(\widetilde{Ch}_\chi)\right\}_w$

as well as a finite successive ext'n $\left\{\nabla^{mon}_{\dot v}(\widetilde{Ch}_\chi)\right\}_v$

Note: $\widetilde{Ch}_\chi$ is indecomposable in $Shv_{mon}(\tilde T)$.

$$R\,\mathrm{End}(\widetilde{Ch}_\chi) = R_\chi = \widehat{\mathcal{O}}_{\check T, \chi} \overset{\chi = u}{\simeq} \Lambda[\![x_1 - 1, \cdots, x_n - 1]\!]$$

$H^1 R\mathrm{End}(\widetilde{Ch}_\chi) = 0$, no nontrivial idempotents!

- If $\mathcal{F} \subset \widetilde{Ch}_x^{\oplus n}$ is a direct summand, then $\mathcal{F} \simeq \widetilde{Ch}_x^{\oplus m}$.

<u>Lemma</u> $\mathcal{J} \in \mathcal{H}_{mon}$ is a cofree χ-mon tilting obj. iff its

$*$-restriction to each strata $I^+ \backslash L\widetilde{G}_w / I^+ \simeq \widetilde{T} \times \mathbb{B}(I^+ \cap wI^+w^{-1})$

is a finite direct sum of $\widetilde{Ch}_x \boxtimes \Lambda[\ell(w)]$

& the similar statement for $!$-restriction.

<u>Lemma</u> $\mathcal{J}_1, \mathcal{J}_2$ cofree χ-mon tilting,

$$H^i \operatorname{Hom}_{\mathcal{H}_{mon}}(\mathcal{J}_1, \mathcal{J}_2) = 0, \quad i \neq 0.$$

<u>Pf.</u> $\operatorname{Hom}(\mathcal{J}_1, \mathcal{J}_2)$ admit a filtration w/ assoc. graded

$$\operatorname{Hom}(\Delta_w^{mon}(\widetilde{Ch}_x), \nabla_w^{mon}(\widetilde{Ch}_x)) \in \operatorname{Mod}_\Lambda^\heartsuit$$

Suppose w is the longest element s.t. $\mathcal{J}_1|_{I^+\backslash L\widetilde{G}_w/I^+} \neq 0$, $\mathcal{J}_2|_{I^+\backslash L\widetilde{G}_w/I^+} \neq 0$.

$$\operatorname{Hom}(\mathcal{J}_1, \mathcal{J}_2) \twoheadrightarrow \operatorname{Hom}(\underset{\widetilde{Ch}_x^{\oplus m}}{\underset{\shortparallel}{i_w^* \mathcal{J}_1}}, \underset{\widetilde{Ch}_x^{\oplus n}}{i_w^! \mathcal{J}_2})$$

$$\begin{array}{ccc} \Delta_w^{mon}(i_w^* \mathcal{J}_1) & \longrightarrow & \mathcal{J}_1 \\ & & \downarrow \\ \nabla_w^{mon}(i_w^! \mathcal{J}_1) & \longleftarrow & \mathcal{J}_2 \end{array}$$

Lemma $T \in \mathcal{H}_{mon}$ is a cofree χ-mon tilting obj., then T is dualizable obj. w.r.t. the monoidal str.,

Pf $\Delta^{mon}_{\tilde{w}}(\widetilde{Ch}_\chi)$ dualizable.

w/ dual $\nabla^{mon}_{w^{-1}}(\widetilde{Ch}_\chi)$

$$C_1 \to C_2 \to C_3$$

$$C_1^\vee \leftarrow C_2^\vee \leftarrow C_3^\vee$$

Thm (I) For each $w \in \widetilde{W}^\chi$, $\exists!$ (up to nonunique isom.) $T_{w,\chi}$ cofree χ-mon tilting object s.t. which is

(1) indecomposable

(2) T_w is supported on $I^+ \backslash \widetilde{LG}_{\le w} / I^+$, & $T_w|_{I^+\backslash \widetilde{LG}_w / I^+} \simeq \widetilde{Ch}_\chi$

(II) Let $\mathcal{H}^{Tilt}_{mon}$ be the full subcat. containing cofree monodromic tilting objects. Then every obj. in $\mathcal{H}^{Tilt}_{mon}$ is isom to a finite direct sum of $T_{w,\chi}$ (some χ)

(III) $\mathcal{H}^{Tilt}_{mon}$ is closed under $\overset{u}{*}$.

In particular, $\mathcal{H}^{Tilt}_{mon}$ is an ordinary additive monoidal cat.

w/ unit $\Delta^{mon}_e(\widetilde{Ch}) = \bigoplus_\chi \Delta_e(\widetilde{Ch}_\chi)$

$\mathcal{H}^{tilt}_{mon}$ completely determine $\mathcal{H}_{mon}$.

Lecture 15

Thm. (I) For each χ, $w \in \widetilde{W}$, $\exists!$ (up to non-unique isom) $T^{mon}_{w,\chi}$,

cofree χ-mon tilting sheaf, which is

(1) indecomposable

(2) $\mathrm{supp}\,(T^{mon}_{w,\chi}) \subset I^+ \backslash \widetilde{Lh}_{\leq w} / I^+$

$$T^{mon}_{w,\chi} \big|_{I^+ \backslash \widetilde{Lh}_w / I^+} \simeq ch_{\chi\text{-mon}}$$

(II) Let $\mathcal{H}^{Tilt}_{mon}$ be the full subcat. containing cofree monodromic tilting objects,

Then every obj. is a finite direct direct sum of $T^{mon}_{w,\chi}$.

(III) $\mathcal{H}^{Tilt}_{mon}$ is an additive monoidal subcat. of $\mathcal{H}_{mon}$.

Pf. Step 1. $\ell(w) = 0$, $T^{mon}_{\dot{\omega},\chi} = \Delta^{mon}_{\dot{\omega}}(ch_{\chi\text{-mon}}) = \nabla^{mon}_{\dot{\omega}}(ch_{\chi\text{-mon}})$

$\ell(w) = 1$, $w = s$ simple reflection.

$$\begin{tikzcd}
\Delta^{mon}_e(\widetilde{Ch}_s) \arrow[r] \arrow[d] & \Delta^{mon}_s(\widetilde{Ch}) \arrow[r] \arrow[d] & \nabla^{mon}_s(\widetilde{Ch}) \arrow[d, equal] \\
\Delta^{mon}_e(\widetilde{Ch}) \arrow[r] \arrow[d] & T^{mon}_s \arrow[r] \arrow[d] & \nabla^{mon}_s(\widetilde{Ch}) \\
\Delta^{mon}_e(\widetilde{Ch}) \arrow[r, equal] & \Delta^{mon}_e(\widetilde{Ch}) &
\end{tikzcd}$$

$R_{I^t_F,\hat{T}} \xrightarrow{\hat{\alpha}_s} R_{I^t_F,\mathbb{G}_m}$ ← coroot assoc. to s

$\omega_{\ker(\hat{\alpha}_s)} \in \mathrm{IndCoh}(R_{I^t_F,\hat{T}})$

$\widetilde{Ch}_s := Ch(\omega_{\ker(\hat{\alpha}_s)})$

$$0 \to \omega_{\ker(\hat{\alpha}_s)} \to \omega_{R_{I^t_F,\hat{T}}} \xrightarrow{x-1} \omega_{R_{I^t_F,\hat{T}}} \to 0$$

Rmk.

$s \notin \widetilde{W}^{\circ}_{\chi}$, $\quad T^{mon}_{\dot{s},\chi} \simeq \nabla^{mon}_{\dot{s}}(Ch_{\chi\text{-mon}}) \simeq \Delta^{mon}_{s}(Ch_{\chi\text{-mon}})$

Lemma. If $\mathcal{F}$ admits a finite filtration by cofree $\overset{(co)}{\text{standard}}$ objects, so is

$T^{mon}_{\dot{s},\chi} * \mathcal{F}$.

Pf. Enough to deal w/ $\mathcal{F} = \Delta^{mon}_{\dot{w}}(Ch_{\chi\text{-mon}})$

$T^{mon}_{\dot{s},\chi} \overset{u}{*} \Delta^{mon}_{\dot{w}}(Ch_{\chi\text{-mon}})$

$$\begin{cases} \Delta^{mon}_{sw}(Ch_{\chi\text{-mon}}) \to \square \to \Delta^{mon}_{w}(Ch_{\chi\text{-mon}}), & \ell(sw) = \ell(w)+1 \\ \Delta^{mon}_{w}(Ch_{\chi\text{-mon}}) \to \square \to \Delta^{mon}_{sw}(Ch_{\chi\text{-mon}}), & \ell(sw) = \ell(w)-1 \end{cases}$$

$(\nabla_s * \Delta_w = \Delta_{sw})$

Cor. $w = s_{i_1} s_{i_2} \cdots s_{i_n} \omega$, $\ell(\omega)=0$

$T^{mon}_{s_{i_1},\chi} * T^{mon}_{s_{i_2}, s_{i_1}(\chi)} * \cdots * T^{mon}_{\omega, (s_{i_1}\cdots s_{i_n})(\chi)} =: \mathcal{J}'$

is a cofree monodromic tilting, supported on $I^+ \backslash \widetilde{LG}_{\leq w} / I^+$.

$\mathcal{J}' \big|_{I^+ \backslash \widetilde{LG}_w / I^+} \simeq Ch_{\chi\text{-mon}}$ Let $S = \{ \underset{\text{direct summand}}{\mathcal{J}'' \hookrightarrow \mathcal{J}'} : \mathcal{J}'' \big|_{I^+ \backslash \widetilde{LG}_w / I^+} \simeq Ch_{\chi\text{-mon}} \}$

Choose a minimal $J'' \in S$, then it is indecomposable. $\square$

Suppose J_1, J_2 cofree χ-mon tilting, supported on $I^+ \backslash \widetilde{LG}_{\leq w} / I^+$,

$\mathrm{Hom}(J_1, J_2)$ admits a filtration w/ associated graded being

$$\mathrm{Hom}\left((\iota_{w'})^* J_1, (\iota_{w'})^! J_2\right) \Big] \text{ finite free over } R_\chi = \mathrm{End}(Ch_{\chi\text{-mon}})$$

$$\mathrm{Hom}(J_1, J_2) \twoheadrightarrow \underbrace{\mathrm{Hom}\left((\iota_w)^* J_1, (\iota_w)^! J_2\right)}_{\text{finite free over } R_\chi = \mathrm{End}(Ch_{\chi\text{-mon}})}$$

Cor.

$$\mathrm{End}(J) \twoheadrightarrow \mathrm{End}\left(J|_{I^+ \backslash \widetilde{LG}_w / I^+}\right)$$

$$J = T^{mon}_{\tilde{w}, \chi} \rightsquigarrow \mathrm{End}(T^{mon}_{w,\chi}) \text{ complete local ring.}$$

Now let T be a cofree χ-mon tilting supported on $I^+ \backslash \widetilde{LG}_{\leq w} / I^+$,

$$T^{mon}_{\tilde{w}, \chi}\big|_{\cdots} \to J|_{I^+ \backslash \widetilde{LG}_w / I^+} \to T^{mon}_{\tilde{w}, \chi}\big|_{\cdots}$$

$$\begin{array}{ccc} \left(T^{mon}_{w,\chi} \to J \to T^{mon}_{\tilde{w},\chi}\right) & \in & \mathrm{End}(T^{mon}_{w,\chi}) \text{ local ring} \\ \downarrow & & \downarrow \\ id & \in & R_\chi \end{array}$$

so $T^{mon}_{w,\chi} \xhookrightarrow{\text{direct summand}} J$.

Cor. If T is another indecomposable supported on $I^+ \backslash \widetilde{LG}_{\leq w} / I^+$

$\rightsquigarrow T_{w,x}^{mon} \simeq T$.

$K^b(\mathcal{H}_{mon}^{tilt})$ (DG version, $Hom(C^\bullet, D^\bullet)^i = \bigoplus_j Hom(C^j, D^{j+i})$)

monoidal stable cat.

Lemma. $\exists$ natural fully faithful monoidal cat $K^b(\mathcal{H}_{mon}^{tilt}) \xrightarrow{G} \mathcal{H}_{mon}$

$$T_{w,x}^{mon} \longmapsto T_{w,x}^{mon}$$

Pf. $F: \mathcal{A} \longrightarrow \mathcal{C}$ ← cocomplete stable

$$Ch^b(\mathcal{A})$$
$$\cap$$
$$Ch(\mathcal{A})^{\leq 0} \simeq Fun(\Delta^{op}, \mathcal{A}) \longrightarrow Fun(\Delta^{op}, \mathcal{C}) \xrightarrow{colim} \mathcal{C}.$$

$$Hom_{\mathcal{A}}(a, b) = Hom_{\mathcal{C}}(F(a), F(b))$$

Thm. $\mathcal{H}_{mon} \underset{G}{\overset{F}{\rightleftarrows}} Ind\, K^b(\mathcal{H}_{mon}^{tilt})$ G admits a left adj. F that is fully faithful.

Proof next time.

Thm. $\exists$ a nat'l action of $\mathcal{H}_{mon} \curvearrowright Shv_{mon}(\tilde{T})$

$$\begin{tikzcd} Shv_{mon}(\tilde{T}) \arrow[d, phantom, "\wr"] \\ \mathrm{IndCoh}(R_{I^t_F, \hat{T}}) \end{tikzcd}$$

not fully faithful

$$\Big(\Leftrightarrow \ \mathbb{V} : \mathcal{H}_{mon} \longrightarrow \mathrm{IndCoh}\big(R_{I^t_F, \hat{T}} \times R_{I^t_F, \hat{T}}\big) \Big)$$

s.t.

$$\begin{tikzcd} \mathbb{V}\big|_{{}_x\mathcal{H}^{tilt}_{mon,x}} : {}_x\mathcal{H}^{tilt}_{mon,x} \arrow[r] \arrow[dr, "\text{fully faithful}"'] & \mathrm{IndCoh}(R_{I^t_F,\hat{T}} \times R_{I^t_F,\hat{T}}) \arrow[d] \\ & (R_x \hat{\otimes} R_x)\text{-mod} \end{tikzcd}$$

$$T^{mon}_{\dot{w},x} \longmapsto \mathbb{V}(T^{mon}_{w,x})$$

Construction of $\mathcal{H}_{mon} \curvearrowright Shv_{mon}(\tilde{T})$

$$\begin{tikzcd} & & \mathcal{F}_v \arrow[d] & \\ \mathcal{F}_v \arrow[r, "\mathbb{G}_m\text{-torsor}"] \arrow[dd, dash] & Bun_G(\mathbb{P}^1)_{(0, I^+), (\infty, I^{++}_\infty)} \arrow[d, "(T \times I^+_\infty / I^{++}_\infty)\text{-torsor}"] & LG/I^+ \arrow[l] & \\ G & & & \\ \mathcal{L}_v \arrow[r] \arrow[d, dash] & Bun_G(\mathbb{P}^1)_{(0, I), (\infty, I^+_\infty)} \arrow[d] & & \\ I_v \arrow[r] & Bun_G(\mathbb{P}^1) & & \end{tikzcd}$$

$$Bun_G(\mathbb{P}^1) \simeq (\tilde{W}, \text{ opposite Bruhat order})$$

$G \hookrightarrow GL(V)$

$$\begin{tikzcd}
 & & & \mathcal{L}_{det} \arrow[dll] \arrow[d] \\
\mathcal{L}_V \arrow[r] \arrow[d] & \mathcal{L}_{det} \arrow[d] & & Gr_{GL_n} \arrow[dll] \\
Bun_G(\mathbb{P}^1) \arrow[r] & Bun_{GL(V)}(\mathbb{P}^1) & &
\end{tikzcd}$$

$Bun_n(C)$, $\mathcal{O}$

$\det R\Gamma(Bun_n(C), \mathcal{O})$

Choose affine pinning, ψ additive character as before

$$\mathcal{H}_{mon} \curvearrowright Shv\left({}_{(\tilde{T}, \chi\text{-mon})}\backslash Bun_G(\mathbb{P}^1)_{(0, I^+), (\infty, I_\infty^{++})} / (I_\infty^+/I_\infty^{++}, \psi) \right)$$

$\cong$

$Shv_{mon}(\tilde{M})$

$$1 \to T \to \frac{N_{LG}(T)}{L^{>0}T} \to \frac{N_{LG}(T)}{L^+T} = \tilde{W} \to 1$$

$$1 \to \tilde{T} \to \tilde{M} \to \Omega$$

$\cup$

Lecture 16

Thm 1. $\mathcal{H}_{mon} \underset{G}{\overset{F}{\rightleftarrows}} \operatorname{Ind} K^b(\mathcal{H}_{mon}^{Tilt})$

$\left(\begin{matrix} Shv_{mon}(\tilde{T}) \\ \cong \end{matrix} \right)$

Thm 2. $\exists$ action $\mathcal{H}_{mon} \curvearrowright Shv(\tilde{T} \backslash \tilde{T})$

$$\left(\Leftrightarrow \mathbb{V} : \mathcal{H}_{mon} \to \operatorname{Ind}Coh\left(R_{I_F^t, \hat{\tilde{T}}} \times R_{I_F^t, \hat{\tilde{T}}}\right)\right)$$

s.t. $\mathbb{V}|_{\mathcal{H}_{mon}^{tilt}} : \mathcal{H}_{mon}^{tilt} \to \operatorname{Ind}Coh\left(R_{I_F^t, \hat{\tilde{T}}} \times R_{I_F^t, \hat{\tilde{T}}}\right)^{\heartsuit}$ is fully faithful.

$$\begin{tikzcd}
\widetilde{\mathrm{Bun}_G(\mathbb{P}^1)}_{I_0^+, I_\infty^{op,++}} \arrow[d] \\
\mathrm{Bun}_G(\mathbb{P}^1)_{I_0^+, I_\infty^{op,++}} \arrow[d] \\
\mathrm{Bun}_G(\mathbb{P}^1)_{I_0, I_\infty^{op,+}}
\end{tikzcd}$$

$I_\infty^{op,+}/I_\infty^{op,++} = (\widetilde{T} \times V_\infty)$-torsor

$G \hookrightarrow GL(V)$

$$\mathrm{Shv}(\widetilde{T} \backslash \widetilde{M})$$

$$\wr|$$

$$\mathcal{H}_{mon} \curvearrowright \mathrm{Shv}\left(\widetilde{T} \backslash \widetilde{\mathrm{Bun}_G(\mathbb{P}^1)}_{I_0^+, I_\infty^{op,++}} /\!/ (V_\infty, \phi)\right) \curvearrowleft \mathrm{Shv}(\widetilde{M_\phi})$$

$$1 \to T \to M = \frac{N_{L_\infty(G)}(T) \cap N_{L_\infty(G)}(I^{op})}{L_\infty^{>0} T} \to \Omega \to 1$$

$$1 \to Z_G \to M_\phi = Z_M(\phi) \to \Omega \to 1$$

Assume Z_G is conn'd. Can choose a splitting $\Omega \subset M_\phi$.

$$\mathrm{Shv}\left(\widetilde{T} \backslash \widetilde{\mathrm{Bun}_G(\mathbb{P}^1)} /\!/ (V_\infty, \phi)\right) \otimes_{\mathrm{Shv}(\Omega)} \mathrm{Mod}_\Lambda$$

$$\wr|$$

$$\mathrm{Shv}(\widetilde{T} \backslash \widetilde{T}).$$

For each $w \in \Omega$, $I_0 \backslash I_0 w I_\infty^{op,+} / I_\infty^{op,+} \simeq *$

$$\begin{tikzcd}
& I_0 \backslash I_0 w I_\infty^{op,+} / I_\infty^{op,+} \simeq * \\
& \mathrm{Shv}\left(\widetilde{T} \backslash \widetilde{T} \times V_\infty /\!/ (V_\infty, \phi)\right) \simeq \mathrm{Shv}(\widetilde{T} \backslash \widetilde{T}) \arrow[u] \arrow[d] \\
& \mathrm{Shv}\left(\widetilde{T} \backslash \widetilde{\mathrm{Bun}_G(\mathbb{P}^1)} /\!/ (V_\infty, \phi)\right)
\end{tikzcd}$$

Δ^ϕ

Lemma.

$$\Delta^{mon}_{\dot{w}}(\mathcal{L}) \overset{u}{*} \Delta^{\phi}_{e}(\mathcal{L}') \xrightarrow{\sim} \Delta^{\phi}_{e}(\mathcal{L} * w(\mathcal{L}'))$$

Similarly $\Delta \to \nabla$

Pf. ETS $\quad \Delta^{mon}_{\dot{s}}(\mathcal{L}) \overset{u}{*} \Delta^{\phi}_{e}(\mathcal{L}') \xrightarrow{\sim} \Delta^{\phi}_{e}(\mathcal{L} * s(\mathcal{L}'))$

Reduce to SL_2.

SL_2/k

$u \backslash SL_2 / u \qquad u \backslash SL_2 / (u^{\phi}, \phi) \qquad$ Check it!

Cor. $\quad \Psi(\Delta^{mon}_{w}(\widetilde{Ch})) \simeq \Psi(\nabla^{mon}_{w}(\widetilde{Ch})) \simeq \omega_{\Gamma_w}$

$$\underset{\widetilde{\Delta}^{mon}_{w}}{\parallel}$$

Γ_w is the graph of $w \curvearrowright R_{I^t_F, \hat{\check{T}}}$.

Lemma. $\quad Hom(\Delta^{mon}_{e}(Ch_{\chi\text{-mon}}), \nabla^{mon}_{w}(Ch_{\chi\text{-mon}})) = \begin{cases} 0, & e \neq w \\ R_{\chi}, & e = w \end{cases}$

$w\chi = \chi \qquad R_{\chi} = \hat{\mathcal{O}}_{R_{I^t_F, \check{T}}, \chi}$

$$\simeq \tau^{\leq 0} \underset{IndCoh((R_{I^t_F, \hat{\check{T}}})^2)}{Hom}(\omega_{\Gamma_e}, \omega_{\Gamma_w})$$

$w = e: \quad = \quad Hom_{\hat{\chi}}(\omega, \omega) = R_{\chi}$

$w \neq e$, $\tau^{\leq 0} \mathrm{Hom}(w_{\Gamma_e}, w_{\Gamma_w})$

$\mathrm{Ind\,Coh}(\hat{x}^2) \longrightarrow (R_x \hat{\otimes} R_x)\text{-mod}$

$\simeq \tau^{\leq 0} \mathrm{Hom}_{R_x \hat{\otimes} R_x}(R_x, R_x(w))$

$$\varphi(a) = a \cdot \varphi(1)$$

$$= \varphi(1)\, w(a)$$

$\Rightarrow (a - w(a))\, \varphi(1) = 0 \qquad \Rightarrow \varphi(1) = 0.$

$\neq 0$ for some a

Cor $\mathrm{Hom}(T^{mon}_{w_1, x}, T^{mon}_{w_2, x}) = \mathrm{Hom}(\mathbb{V}(T^{mon}_{w_1,x}), \mathbb{V}(T^{mon}_{w_2,x}))$

Pf $\mathrm{Hom}(T^{mon}_{w_1, x}, T^{mon}_{w_2, x})$

$= \mathrm{Hom}(\Delta^{mon}_e(\mathrm{Ch}_{x\text{-mon}}), {}^{\vee}T^{mon}_{w_1,x} * T^{mon}_{w_2,x})$

x, $\tilde{W} \supset \tilde{W}_x \supset \tilde{W}^{\circ}_x$

Let $r \in \tilde{W}^{\circ}_x$ be a simple reflection.

$r : \hat{x} \longrightarrow \hat{x}$

$\hat{x} \times_{\hat{x}/\!/\{1,r\}} \hat{x} \hookrightarrow \hat{x} \times \hat{x}$

Prop. $\mathbb{V}(T^{mon}_{r,x}) \simeq \omega_{\hat{x} \times_{\hat{x}/\!/\langle 1,r\rangle} \hat{x}}$

$\mathcal{H}^{tilt}_{mon}$ gen. by $\{ T^{mon}_{w,x} , T^{mon}_{s,x} \}$

$w \in \Omega$, s simple reflection

as idempotent complete additive monoidal cat

Lemma. ${}_x\mathcal{H}^{tilt}_{mon,x}$ gen. by $\{ T^{mon}_{w,x} , T^{mon}_{r,x} \}$

$w \in \Omega_x$, r simple reflection of $\tilde{W}^\circ_x$

as idempotent complete additive monoidal cat.

Fix x

Def Let $SBim_x \subset IndCoh(\hat{x}^2)^\heartsuit$ be the idempotent complete additive monoidal cat. gen. by

ω_{Γ_w}, $\omega_{\hat{x} \times_{\hat{x}/\langle 1,r\rangle} \hat{x}}$

$w \in \Omega_x$

Thm $\mathbb{V}: {}_x\mathcal{H}^{tilt}_{mon,x} \xrightarrow{\sim} SBim_x$

Lecture 17. Fix $x \in R_{\mathbb{F},\hat{T}}^{lt}(\Lambda) \subset \hat{T}(\Lambda)$

Thm. ${}_x\mathcal{H}^{tilt}_{mon,x} \xrightarrow{\sim} SBim_x$ $\qquad 1 \to \tilde{W}^\circ_x \to \tilde{W}_x \to \Omega_x \to 1$, $\quad w^\beta \leftarrow \beta$

$\left[\hat{T}/_\Lambda \circlearrowleft \tilde{W} \right]$

Recall. $SBim_x \subset IndCoh(\hat{x}^2)^\heartsuit$ idempotent complete additive monoidal cat. gen. by

$\omega_{\Gamma_{w^\beta}}$, $\beta \in \Omega_x$, $\omega_{\hat{x} \times_{\hat{x}//W_r} \hat{x}}$, r simple reflection of $\tilde{W}^\circ_x$

$W_r = \{1, r\}$

Lemma (1) (a) $\mathbb{V}\left(T^{mon}_{s,x}\right) = w_{\hat{x} \times_{\hat{x}/\!\!/W_s} \hat{x}}$, (b) $\mathbb{V}\left(T^{mon}_{w^\beta,x}\right) = w_{\Gamma_{w^\beta}}$.

s simple reflection in $\tilde{W}^\circ_x$

Lemma 2. ${}_x\mathcal{H}^{tilt}_{mon,x}$ is gen. by $T^{mon}_{w^\beta,x}$, $T^{mon}_{s,x}$ as idempotent complete monoidal additive cat.

Lemma 3. $\left(\tilde{W}^\circ_x, \le_x, \ell_x\right)$

$$\wr$$

$$\left(\tilde{W}^\beta_x := \tilde{W}^\circ_x w^\beta, \le_\beta, \ell_\beta\right)$$

$$= w^\beta \tilde{W}^\circ_x$$

$${}_x\tilde{W}_{x'} \longrightarrow x\Omega x'$$

$$w^\beta \longleftarrow\!\shortmid \beta$$

$$\underbrace{\tilde{W}^\circ_x w^\beta = w^\beta \tilde{W}^\circ_{x'}}_{{}_x\tilde{W}^\beta_{x'}}$$

Let $w \in \tilde{W}^\beta_x$, then only $\left\{\Delta^{mon}_v(Ch_{x\text{-mon}})\right\}_{v \le_\beta w}$

(resp. $\left\{\nabla^{mon}_v(Ch_{x\text{-mon}})\right\}_{v\le_\beta w}$)

will appear in the assoc. graded of the standard (resp. costandard) filt'n. of $T^{mon}_{w,x}$.

Lemma 4 For $w = w^\beta$, minimal length in $\tilde{W}^\beta_x$
$$T^{mon}_{w,x} = \Delta^{mon}_w(Ch_{x\text{-mon}}) = \nabla^{mon}_w(Ch_{x\text{-mon}})$$

Pf of Lemma 4. Induction on $\ell(w^\beta) = \ell(w)$

$\ell(w) = \ell(w^\beta) = 0$, ✓

Otherwise, $w = vs$, $\ell(w) = \ell(v) + 1$, s simple reflection in $\tilde{W}$.

$\tilde{W}^\circ_{sx} \not\ni s \notin \tilde{W}^\circ_x \Rightarrow \Delta^{mon}_s(Ch_{x\text{-mon}}) = \nabla^{mon}_s(Ch_{x\text{-mon}})$ "cleaness"

$$\Rightarrow \Delta_s^{mon}(Ch_{sx\text{-}mon}) * (-) \to \Delta_s^{mon}(Ch_{x\text{-}mon})$$

$${}_x\mathcal{H}_{mon,x} \xrightarrow{\sim} {}_{sx}\mathcal{H}_{mon,sx}$$ preserves std obj. & costd obj.

$${}_x\mathcal{H}^{tilt}_{mon,x} \xrightarrow{\sim} {}_{sx}\mathcal{H}^{tilt}_{mon,sx}$$

$$T^{mon}_{w,x} \longmapsto T^{mon}_{sws,sx}$$

$$\begin{array}{ccc} {}_x\widetilde{W}_{x'} & \rightleftarrows & {}_x\Omega_{x'} \\ w^{\beta} & \longleftarrow & \beta \end{array}$$

<u>Lemma 4</u>. If $w = w^{\beta}$, then $T_{w,x}^{mon} \simeq \Delta_w^{mon}(Ch_{x\text{-}mon}) \simeq \nabla_w^{mon}(Ch_{x\text{-}mon})$

$$\begin{array}{c} T_{w,x}^{mon} \\ \cap \\ {}_x\mathcal{H}^{tilt}_{mon,w^{-1}x} \end{array}$$

<u>Pf</u>:

✓ $\ell(w) = 0$

$\Rightarrow v$ minimal length in ${}_{x}\widetilde{W}_{sx}$

$w = vs$, $\ell(w) = \ell(v) + 1 \Rightarrow s \notin \widetilde{W}^{\circ}_x$

s minimal length

$${}_x\widetilde{W}_{sx} \rightleftarrows {}_x\Omega_{sx}$$

$$\Delta_s^{mon}(Ch_{x\text{-}mon}) = \nabla_s^{mon}(Ch_{x\text{-}mon})$$

$$(-) * \Delta_s^{mon}(Ch_{x\text{-}mon}) : {}_{x'}\mathcal{H}^{tilt}_{mon,x} \xrightarrow{\sim} {}_{x'}\mathcal{H}^{tilt}_{mon,sx}$$

$$T^{mon}_{w,x'} \longrightarrow T^{mon}_{v,x'}$$

Proof of Lemma 3. $w \in \widetilde{W}_\chi^\beta$

$T_{w,\chi}^{mon}$

$\ell_\beta(w) = 0 \quad \Leftarrow \quad$ Lemma 4

$w = v r$, $\quad r$ simple reflection in $\widetilde{W}_\chi^\circ$.

$T_{v,\chi}^{mon} * T_{r,\chi}^{mon}$

Lemma 5. $r \in \widetilde{W}_\chi^\circ$ simple reflection.

$$0 \to \nabla_e^{mon}(Ch_{\chi\text{-mon}}) \to T_{r,\chi}^{mon} \to \nabla_r^{mon}(Ch_{\chi\text{-mon}}) \to 0$$

$$0 \to \Delta_r^{mon}(Ch_{\chi\text{-mon}}) \to T_{r,\chi}^{mon} \to \Delta_e^{mon}(Ch_{\chi\text{-mon}}) \to 0$$

Pf. $r = s$ is a simple reflection in $\widetilde{W}$, ✓

r is not a simple reflection,

$r = s r' s$, $\ell(r) = \ell(r') + 2$, s simple reflection in $\widetilde{W}$

$\Rightarrow s \notin \widetilde{W}_\chi^\circ$

Use $\Delta_s(Ch_{\chi\text{-mon}}) * (-) * \Delta_s(Ch_{\chi\text{-mon}})$ $\square$

<u>Proof of Lemma 1</u>

(b) ✓

(a)

$$\begin{tikzcd}
 & \omega_{\Gamma_\tau} \arrow[d] \arrow[dr, equal] & \\
\omega_{\Gamma_e} \arrow[r] \arrow[dr] & \nabla(T^{mon}_{\tau,x}) \arrow[r] \arrow[d] & \omega_{\Gamma_\tau} \\
 & \omega_{\Gamma_e} &
\end{tikzcd}$$

Recall $\tau = s$ simple reflection in $\widetilde{W}$

$$\begin{tikzcd}
0 \arrow[r] & \Delta_e(\widetilde{Ch}_s) \arrow[r] \arrow[d] & \Delta^{mon}_s(\widetilde{Ch}) \arrow[r] \arrow[d] & \nabla^{mon}_s(\widetilde{Ch}) \arrow[r] \arrow[d, equal] & 0 \\
 & \Delta_e(\widetilde{Ch}) \arrow[r] \arrow[d, "x-1"] & \widetilde{T}^{mon}_s \arrow[r] \arrow[d] & \nabla^{mon}_s(\widetilde{Ch}) & \\
 & \Delta_e(\widetilde{Ch}) \arrow[r, equal] & \Delta_e(\widetilde{Ch}) & &
\end{tikzcd}$$

$$\begin{tikzcd}
\hat{\mathcal{T}}_s \arrow[d] & \hat{\mathcal{T}} \arrow[d, "\alpha_s^\vee"] \\
1 \in & \mathbb{G}_m
\end{tikzcd}$$

$$\mathrm{IndCoh}(\hat{X}^2) \longrightarrow (R_x \hat{\otimes} R_x)\text{-mod}$$

$$\omega_{\Gamma_w} \longmapsto R_x(w)$$

$\circlearrowright$ R_x usual action $\qquad$ $\circlearrowleft$ w-twisted action

$$\nabla(T^{mon}_{\tau,x}) \longrightarrow R_x(e) \oplus R_x(s)$$

$\circlearrowright$

$R_x \hat{\otimes} R_x$

Lecture 18.

Lemma 1 . ${}_x\mathcal{H}^{tilt}_{mon,x}$ is gen. by

$$\{T^{mon}_{w^\beta,x}\} \qquad \tilde{W}^\circ_x \curvearrowright \tilde{W}_x \underset{\longleftarrow}{\overset{\tilde{W}^\circ_x \backslash}{\longrightarrow}} \Omega_x$$

$$w^\beta \longleftarrow\!\shortmid \beta$$

$$\{T^{mon}_{r,x}\} \qquad r \text{ simple reflection of } \tilde{W}^\circ_x$$

as idempotent complete monoidal additive cat.

Lemma 2 . ${}_x\mathcal{H}^{tilt}_{mon,x'}$

$$\ni$$

$$T^{mon}_{w,x}$$

If $\Delta^{mon}_v(Ch_{x\text{-}mon})$ appears in the standard filtration of $T^{mon}_{w,x}$, then $v \le_\beta w$.

$(\Rightarrow v \le w$ in $\tilde{W})$

Similarly for $\nabla^{mon}_v(Ch_{x\text{-}mon})$

$$\begin{array}{ccc} {}_x\tilde{W}_{x'} & \longrightarrow & {}_x\Omega_{x'} \\ \ni & & \ni \\ w & \longmapsto & \beta \end{array}$$

Pf of Lemma 2 . If $w = w^\beta$, done last time.

$${}_x\tilde{W}^\beta_{x'} = {}_x\tilde{W}^\circ_x \cdot w^\beta = w^\beta {}_{x'}\tilde{W}^\circ_{x'}$$

$w = us$, $\ell(w) = \ell(u) + 1$, s simple reflection in $\tilde{W}$.

(1) If $s \notin {}_{x'}\tilde{W}^\circ_{x'}$, $\Rightarrow$ $s = w^\beta \longmapsto \beta$

$$\beta \in {}_{sx'}\Omega_{x'}$$

$$\Delta^{mon}_s(Ch_{sx'\text{-}mon}) = \nabla^{mon}_s(Ch_{sx'\text{-}mon})$$

$$T^{mon}_{w,x} = T^{mon}_{u,sx'} * \Delta^{mon}_s(Ch_{sx'-mon})$$
$$\nabla^{mon}_s(-)$$

$${}_x\widetilde{W}^{\beta\gamma}_{sx'} \cdot s = {}_x\widetilde{W}^{\beta}_{x'}$$

$$\leq_{\beta\gamma} \qquad \leq_\beta$$

The lemma holds for $u \Rightarrow$ holds for w

(2) $s \in {}_{x'}\widetilde{W}^{\circ}_{x'}$, simple reflection.

$T^{mon}_{w,x}$ appears as a direct summand of $T^{mon}_{u,sx'} * T^{mon}_{s,x'}$

Proof of Lemma 1. Enough $w = uv$ in ${}_x\widetilde{W}^{\beta}_x$

$$\ell_\beta(w) = \ell_\beta(u) + 1, \quad (\Leftrightarrow \; \ell(w) = \ell(u) + \ell(v))$$

Then $T^{mon}_{w,x}$ appears as a direct summand of $T^{mon}_{u,x} * T^{mon}_{v,x}$.

If $v = s$ is simple reflection in $\widetilde{W}$, this is OK.

$v = sts$, s simple reflection in $\widetilde{W}$,

t ------ in ${}_{sx}\widetilde{W}^{\circ}_{sx}$

$\ell(v) = \ell(t) + 2$ $\qquad s \longmapsto \gamma \in {}_x\Omega_{sx}$

$(\Leftrightarrow s \notin {}_x\widetilde{W}^{\circ}_x)$ $\quad \Delta^{mon}_s(Ch_{sx-mon}) * (-) * \Delta^{mon}_s(Ch_{x-mon})$

then do induction on $\ell(v)$. ${}_xH^{tilt,\beta}_{mon,x} \xleftarrow{\sim} {}_{sx}H^{tilt,\gamma\beta\gamma}_{mon,sx}$, $T^{mon}_{v,x} \longleftarrow T^{mon}_{t,sx}$

Thm

$$\mathbb{V} : {}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x} \xrightarrow[\sim]{\text{additive monoidal}} \mathrm{SBim}_x \qquad \widetilde{W} \curvearrowright \hat{\widetilde{T}}$$

$$\cap$$

$$\mathrm{IndCoh}(\hat{x}^2)^{\heartsuit}$$

$$\left(\begin{array}{cc} {}^{w}\Gamma_{w}\beta & , \\ {}^{w}\hat{x} \times & \hat{x} \\ \hat{x} /\!/ (1,v) & \end{array} \right)$$

$$G_0 : K^b({}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x}) \xrightarrow[\text{monoidal}]{\text{fully faithful}} {}_x\mathcal{H}_{\text{mon},x}$$

$$F \qquad G$$

Thm $\mathrm{Ind}\left(K^b\left({}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x}\right)\right) \xrightarrow{\text{monoidal}} {}_x\mathcal{H}_{\text{mon},x}$

admits a left adjoint F, $\quad G \circ F \simeq \mathrm{id}$

F is fully faithful embedding.

Def. For each $w \in {}_x\widetilde{W}_x$, let $K^b({}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x})_{\leq w}$ be the full subcat (idem. complete)

$\beta \quad {}_x\Omega_x$ $\qquad (< w)$

generated by

$$T^{\text{mon}}_{v,x}, \quad v \leq_\beta w \quad (v <_\beta w)$$

Then $K^b({}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x})_{<w} \xrightarrow{\iota_{<w}} K^b({}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x})$ admits left & right adjoints

$$(\iota_{<w})^*, \quad (\iota_{<w})^!$$

$$(\iota_{<w})_*(\iota_{<w})^! T^{\text{mon}}_{w,x} \longrightarrow T^{\text{mon}}_{w,x} \longrightarrow \nabla^{\text{mon}}_{w,x}$$

$\Rightarrow$ (cofree) (co) standard one in $G_0(K^b({}_x\mathcal{H}^{\text{tilt}}_{\text{mon},x}))$

Lemma

$$\left(_\chi \mathcal{H}_{\mathrm{mon},\chi}\right)^{w} \subset G_0\left(K^b(-)\right).$$

$$\| $$

$$\left\langle \Delta_w^{\mathrm{mon}}(\mathcal{L}) \right\rangle \quad \mathcal{L} \in \mathrm{Ch}(\widetilde{T})$$

$$\begin{tikzcd}
K^b\left(_\chi \mathcal{H}_{\mathrm{mon},\chi}^{\mathrm{tilt}}\right) \arrow[r, hook, "G_0"] & _\chi \mathcal{H}_{\mathrm{mon},\chi} \\
_\chi \mathcal{H}_{\mathrm{mon},\chi}^{w} \arrow[u, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
_\chi \mathcal{H}_{\mathrm{mon},\chi} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Ind}\, K^b\left(_\chi \mathcal{H}_{\mathrm{mon},\chi}^{\mathrm{tilt}}\right) \arrow[r, "G"] & _\chi \mathcal{H}_{\mathrm{mon},\chi}
\end{tikzcd}$$

$\square$

$G /_{F = k((\varpi))}$ split, $k = \bar{k}$

Z_G connected

χ character sheaf on T

$$\widetilde{T} = T \times \mathbb{G}_m$$

$$\widetilde{\chi} \quad (\chi, u)$$

$$\begin{tikzcd}
_{\tilde\chi} \mathcal{H}_{G,\mathrm{mon},\tilde\chi} \arrow[r, leftrightsquigarrow] & \widetilde{W} \curvearrowright \hat{\widetilde{T}} \\
_\chi \mathcal{H}_{\mathrm{mon},\chi}^{\mathrm{tilt}} \arrow[u, rightsquigarrow] & \widetilde{W}_\chi \curvearrowright \hat{\widetilde{\chi}} \\
& \mathrm{SBim}_\chi \arrow[ul]
\end{tikzcd}$$

$x \in \hat{T}$, $\hat{H} = Z_{\hat{G}}(x)$ conn'd, (since Z_G is connected)

$$\begin{array}{ccl} \tilde{W}_H & \curvearrowright \hat{\tilde{T}} & H \text{ endoscopic group of } (G, x) \\ \| & & \cup \\ \tilde{W}_x & \curvearrowright & T \end{array}$$

$${}_{\tilde{x}}\mathcal{H}_{H,\mathrm{mon},\tilde{x}}$$

<u>Cor.</u> $\exists$ a monoidal equiv. (depending on some auxiliary choices)

$${}_{\tilde{x}}\mathcal{H}_{G,\mathrm{mon},\tilde{x}} \simeq {}_{\tilde{x}}\mathcal{H}_{H,\mathrm{mon},\tilde{x}} \simeq {}_{u}\mathcal{H}_{H,\mathrm{mon},u}$$

Sends (co)std to (co)std.

perverse t-exact.

Kill the central monodromy

$$\tilde{T} = T \times \mathbb{G}_m \qquad \underset{\mathrm{Shv}_{u\text{-mon}}(\mathbb{G}_m)}{\otimes} (\mathrm{Mod})_u$$

$$\begin{array}{ccc} \mathrm{Shv}\left((I^+, x)\backslash LG / (I^+, x)\right) & \simeq & \mathrm{Shv}\left((I^+, u)\backslash LH / (I^+, u)\right) \\ \| & & \wr\| \\ \mathrm{IndCoh}\left(\left(\frac{B\hat{G}}{B\hat{G}}\right)_{\hat{x}} \underset{\hat{G}/\hat{G}}{\times} \left(\frac{B\hat{G}}{B\hat{G}}\right)_{\hat{x}}\right) & \simeq & \mathrm{IndCoh}\left(\left(\frac{B\hat{H}}{B\hat{H}}\right)_{\hat{u}} \underset{\hat{H}/\hat{H}}{\times} \left(\frac{B\hat{H}}{B\hat{H}}\right)_{u}\right) \end{array}$$

$$\begin{array}{ccccc} \frac{B\hat{H}}{B\hat{H}} & \longrightarrow & \frac{\hat{T}}{\hat{T}} & \to \hat{T} = & \frac{\hat{T}}{\hat{T}} \\ & & & \downarrow & \\ \left(\frac{B\hat{H}}{B\hat{H}}\right)_{\hat{u}} & & \longrightarrow & \hat{u} & \end{array}$$

(In fact, $\left(\frac{B\hat{G}}{B\hat{G}}\right)_{\hat{x}} \underset{\hat{G}/\hat{G}}{\times} \left(\frac{B\hat{G}}{B\hat{G}}\right)_{\hat{x}} \simeq \left(\frac{B\hat{H}}{B\hat{H}}\right)_{\hat{u}} \underset{\hat{H}/\hat{H}}{\times} \left(\frac{B\hat{H}}{B\hat{H}}\right)_{u}$)

$\mathrm{Shv}_{\kappa}\left(I^+\backslash \widetilde{LG}/I^+\right)$ ↙ decategorify to get Hecke algebra for metaplectic gp

$$\mathrm{Shv}_{\kappa}\left(I^+\backslash \widetilde{LG}/I^+\right) \xleftrightarrow{\sim} \mathrm{Shv}_{\check{\kappa}}\left(I^+\backslash \widetilde{LG^\vee}/I^+\right)$$

$\widetilde{T} = T \times \mathbb{G}_m$

$\quad\;\; u \qquad \kappa$

$G, G^\vee / \mathbb{C} = k = \Lambda$

$$\widetilde{W}_{G,\kappa} \simeq \widetilde{W}_{G^\vee, \kappa^\vee}$$

$$\circlearrowright$$

$$(\hat{\widetilde{T}}, \kappa) \hookrightarrow (\hat{\widetilde{T}}^\vee, \hat{\kappa})$$