Enriques surfaces (Cont'd)

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<u>Def</u> A family of Enriques surfaces over S is an algebraic space Y, $f: Y \to S$ flat proper, of finite presentation, and s.t. for each geom point $\operatorname{Spec} k \to S$, the base change Y_k is an Enriques surface $/k$.

$S = \operatorname{Spec}(R)$

<u>Prop 5.3</u> $Y \to S$ family of Enriques surfaces, then $\mathrm{Pic}_{Y/S}$ is a scheme, $\mathrm{Pic}^{\tau}_{Y/S}$ is representable by an open embedding, and $\mathrm{Num}_{Y/S}$ is representable by a local system of free abelian groups of rk 10.

$\mathrm{Num}_{Y/S} = \mathrm{Pic}_{Y/S} / \mathrm{Pic}^{\tau}_{Y/S}$

<u>Thm</u>. $Y \to S$ a family of Enriques surfaces, $\mathrm{Pic}^{\tau}_{Y/S}$ is flat group scheme of order 2 & $\mathrm{Pic}_{Y/S} / \mathrm{Pic}^{\tau}_{Y/S} = \mathrm{Num}_{Y/S}$ is a locally constant sheaf of torsion free f.g. abelian groups.

<u>Claim</u>. Only gp schemes $/\mathbb{Z}$ that are locally free of rank 2 are $\mathbb{Z}/2\mathbb{Z}$ & μ_2.

$G = \mathrm{Hom}(\mathrm{Pic}^{\tau}_{Y/S}, \mathbb{G}_{m,S})$

$\mathrm{Pic}^{\tau}_{Y/S} \subset \mathrm{Pic}_{Y/S}$ $\qquad R^1 f_*(G_Y)$ call this [X]

$$0 \to H^1(S,G) \to H^1(Y,G_Y) \to H^0(S, R^1 f_*(G_Y)) \xrightarrow{d} H^2(S,G) \to H^2(Y,G_Y)$$

$$d[X] = 0$$

G_Y-torsor

$X \to Y$ family of canonical coverings.

Prop 5-4 $X \to Y$ exists if one of the following holds:

(i) $H^2(S,G) \to H^2(Y,G_Y)$ injective

(ii) $Y \to S$ admits a section

(iii) $\mathrm{Pic}^{\tau}_{Y/S} \to S$ is étale & $\mathrm{Pic}(S) = 0$

Pf $S = \mathrm{Spec}(R)$ is noetherian $L = \Omega^2_{Y/\mathrm{Spec}(R)}$

$\forall a \in \mathrm{Spec}(R)$, $L|_{f^{-1}(a)}$ is the dualizing sheaf

this has order 2, $h^i(L^{\otimes 2}|_{f^{-1}(a)}) = h^i(\mathcal{O}_{f^{-1}(a)}) = 0$ for $i > 0$

then $f_*(L^{\otimes 2})$ is invertible

$$f^*(f_*(L^{\otimes 2})) \to L^{\otimes 2}$$

$$\varphi: \mathcal{O}_Y \nearrow$$

$A = \mathcal{O}_Y \oplus L^{-1}$ $\quad (a, b)(a', b') = (aa' + \varphi^{-1}(bb'), ab' + b'a)$

Set $X = \mathrm{Spec}(A)$

$\square$

$Y \to S$ family of Enriques

has constant local system if $\mathrm{Num}_{Y/S} \simeq \mathbb{Z}^{\oplus 10}$ $(= \mathrm{Pic}_{Y/S} / \mathrm{Pic}^{\tau}_{Y/S})$ constant

Picard scheme $\quad \mathrm{Pic}_{Y/S} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\oplus 10}$ split if

$$0 \to \mathrm{Pic}^{\tau}_{Y/S} \to \mathrm{Pic}_{Y/S} \to \mathrm{Num}_{Y/S} \to 0 \quad \text{splits.}$$

Prop 5.5 if $S = \mathrm{Spec}\, \mathbb{Z}$, then $f : Y \to \mathrm{Spec}\, \mathbb{Z}$ has constant Pic

Prop 5.6 If $Y \to S = \mathrm{Spec}(R)$ has constant Picard scheme & $\mathrm{Pic}(S)$, $\mathrm{Br}(S)$ vanish.

then $\exists$ a canonical covering $X \to Y$ & the set of isom. classes of these

principal homog. space for $R^{\times}/(R^{\times})^2$

for $\mathrm{Spec}\, K \to S$, Y_K, $Y_{\bar{K}}$ satisfy

(i) ω_{Y_K} has order 2

(ii) all $\mathrm{Num}_{Y_K/K}(\bar{K})$ come from $\mathcal{L}$ on Y_K, unique up to twisting by ω_{Y_K}.

(iii) every -2 curve $\bar{E} \subset Y_{\bar{K}}$ is the base change of a -2 curve $E \subset Y_K$

& $E \simeq \mathbb{P}^1_K$

(iv) every genus one fibration $Y_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ comes from a $Y_k \to \mathbb{P}^1_k$ which has exactly two multiple fibers, lying over $\bar{k}$-rat points

(v) $\exists$ a genus one $Y_k \to \mathbb{P}^1_k$.