

Introduction to Fontaine's thm

Def. $A \rightarrow S$ abelian scheme : smooth proper scheme over S

which is a group object over S s.t. fibers are conn'd.

\Rightarrow all fibers are abelian varieties (over $\mathbb{C} : \simeq \mathbb{C}^g/\Lambda$)

group of points is abelian.

Compact complex Lie group is abelian.

$A[n] = \ker(A \xrightarrow{\times n} A)$ finite over S of degree n^{2g} , $g = \dim A_S$

étale group scheme over S away from points of char. dividing n .

Thm. $\#A \rightarrow \operatorname{Spec} \mathbb{Z}$ abelian scheme

Cor. The only curve over \mathbb{Z} is $\mathbb{P}_{\mathbb{Z}}^1$

($f: C \rightarrow \operatorname{Spec} \mathbb{Z}$ rel dim 1 smooth proper w/ conn'd fibers)

Proof $e \rightsquigarrow \operatorname{Jac}(C) := \operatorname{Pic}_{C/S}^0$ (def. theory of $\in H^2(X, \mathcal{O})$)

dimension = genus of C

If $g > 0$, contradicts Fontaine's thm.

$g=0$ = "conics"

C/K genus 0 curve, Riemann-Roch ω_C^{-2} has 3 sections

$C \hookrightarrow \mathbb{P}_K^2$ as a smooth conic.

C has good reduction at all primes.

Say that $C|C_1$ has good reduction at p if there is some \mathcal{C}/\mathbb{Z}_p and an isom.

$$\mathcal{C}|_{\mathbb{A}^1_p} \simeq C|_{\mathbb{A}^1_p}.$$

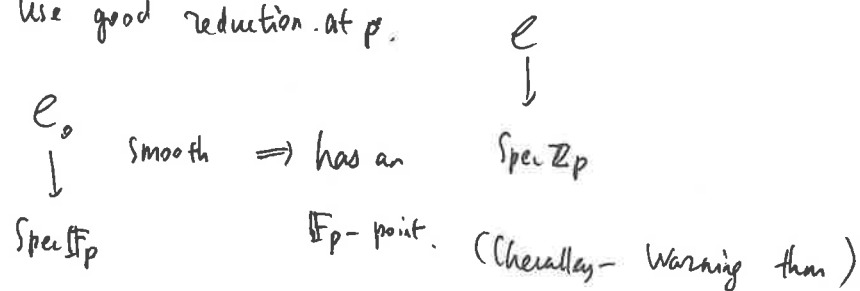
Upshot: "good reduction is unique" unless $g=0$.

$g=0$: want to show $C/C_1, C \simeq \mathbb{P}^1_{C_1}$.

fact. $C \simeq \mathbb{P}^1 \Leftrightarrow C(\mathbb{A}^1_p) \neq \emptyset$.

Idea, Hasse principle says suffices to show $C(\mathbb{A}^1_p) \neq \emptyset$

Use good reduction at p .



Use smoothness to lift to a \mathbb{Z}_p -point of \mathcal{C} . thus $C(\mathbb{A}^1_p) \neq \emptyset$

If $K \neq \mathbb{Q}$, \exists conics $\xrightarrow{\text{w/ good reduction}}$ unram. @ all finite primes $\neq \mathbb{P}^1$

[eg. $K = \mathbb{Q}(\sqrt{2})$, and $\{x^2 + y^2 + z^2 = 0\}$ in \mathbb{P}^2_K : good reduction at all finite places

$C(\mathbb{A}^1_p) \neq \emptyset$, $\left(\frac{-1, -1}{K}\right) = K\langle i, j \rangle / (i^2+1, j^2+1, ij+ji)$ split at all primes

$0 \rightarrow Br(K) \rightarrow \bigoplus_v Br(K_v) \xrightarrow{\sum \text{inv}_v} \mathbb{Q}/\mathbb{Z} \rightarrow 0$ (but not so).

conics \Leftrightarrow 2-torsion elts.



Proof of Fontaine's Thm for $g=1$

$p \nmid \#A/K$ dim 1, curve of genus 1. cubic in \mathbb{P}^2_K . $\mathcal{O}(3P_0)$, $E \hookrightarrow \mathbb{P}^2_K$

Claim. $z^2 y^2 + a_1 zxy + a_3 z^2 y = x^3 + a_2 zx^2 + a_4 z^2 x + a_6 z^3$ (*)

Same as choice of coords on \mathbb{P}^2 s.t.

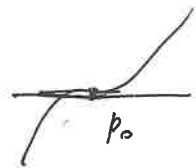
(1) $p_0 \mapsto [0:1:0]$

(2) tangent line to E at p_0 is line $\{z=0\}$

(3) $\{z=0\}$ is a flex (tangent of order 3 at p_0)

$2p_0 + p' \sim H|_E \sim 3p_0 \quad p_0 \sim p' \Rightarrow p_0 = p'$

\Rightarrow tangent at p_0 is a flex line.



Abstract Weierstrass Models

$x \mapsto u^2 x, y \mapsto u^3 y$

Let equations are modified by $a_i \mapsto u^i a_i$ over any B

then new (x,y) satisfies equation for $u^i a_i$.

$\mathbb{P}_B (\underbrace{\mathcal{O}_B \oplus \mathcal{L}^{-2}x \oplus \mathcal{L}^{-3}y}_{\Sigma})$

$z \in H^0(B, \mathcal{E})$

$x \in H^0(B, \mathcal{E} \otimes \mathcal{L}^2)$

$\Sigma = \pi^* \mathcal{O}(1)$

$y \in H^0(B, \mathcal{E} \otimes \mathcal{L}^3)$

$a_i \in H^0(B, \mathcal{L}^{\otimes i})$

$W \subset \mathbb{P}(\mathcal{E})$ cut out by $(*) \in H^0(\mathbb{P}, \mathcal{O}(3) \otimes \pi^* \mathcal{L}^6)$

Prop. $W \xrightarrow{\sim} W'$
 $\downarrow \quad \downarrow$
 $s \quad B$

then claim: (1) $\varphi: \mathcal{I} \xrightarrow{\sim} \mathcal{I}'$

(2) $\psi: \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \xrightarrow{\sim} \mathcal{O} \oplus \mathcal{L}'^{-2} \oplus \mathcal{L}'^{-3}$

that preserves the flag

$\mathcal{O}_B \subsetneq \mathcal{O}_B \oplus \mathcal{L}^{-2}y \subsetneq \mathcal{E}$

$\begin{pmatrix} 1 & a & b \\ \varphi^2 & c \\ \varphi^3 \end{pmatrix}$

Proof $\frac{dx}{y} \mapsto u^{-1} \frac{dx}{y}$

$$\omega_{W/B} = \omega_{P/B} \otimes \mathcal{N}_{W/P}$$

$$0 \rightarrow \Omega_{P/B} \rightarrow \mathcal{E}(-1) \rightarrow 0 \rightarrow 0$$

$$\begin{aligned} \det \Omega_{P/B} &= \det \mathcal{E} \otimes \mathcal{O}(-3) \\ &= \mathcal{O}(-3) \otimes \mathcal{L}^{-5} \end{aligned}$$

$$\mathcal{N} = \mathcal{O}(3) \otimes \mathcal{L}^6 \leadsto \omega_{W/B} \xrightarrow{\pi^*} \mathcal{L}.$$

$$\omega = \mathcal{L}$$

$$\mathcal{E} \simeq \pi_* \mathcal{O}(3S)$$

$$\omega \rightarrow \mathbb{P}(\pi_* \mathcal{O}(3S))$$

$\Delta \in H^0(B, \mathcal{L}^{\otimes 12})$ whose zeroes are the singular fibers.

integral poly. in a_i

For R a DVR, $K = \text{Frac}(R)$ and $E|K$. Consider all integral Weierstrass models $/R$.

$a_i \in R$, $\Delta \in R$, ask to find $\min \text{val}(\Delta)$.

$\min = 0 \iff$ good reduction.

Thm. min Weierstrass models $/ \hat{\mathcal{O}_{C,x}}$ are unique

If C is a Dedekind scheme, $\exists!$ $W \rightarrow C$ whose completion at $x \in C$ is minimal

Weierstrass. * But $\mathcal{L} \neq \mathcal{O}$ in general.

Upshot: K no. field, \mathcal{O}_K ring of integers. If $\text{cl}(\mathcal{O}_K) = 1$ (eg. \mathbb{Z})

$\mathcal{L} \simeq \mathcal{O} \Rightarrow$ Weierstrass eq'n w/ $a_i \in \mathbb{Z}$

s.t. $\Delta(a_i)$ is minimal at all p .

Proof. If \mathcal{L} good reduction at every $p \Rightarrow \Delta = \pm 1$

