Matroids and the integral Hodge conjecture for abelian varieties

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Let X be a sm. proj. var. $/\mathbb{C}$, $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ "Hodge decomposition"

Conj. (Hodge 1950) Any cohomology class $\alpha \in H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q})$ is rep'td by a $\mathbb{Q}$-linear comb. of fundamental classes of subvarieties of X of codim p.

Original version (IHC, integral Hodge conj.) $\alpha \in H^{p,p}(X) \cap H^{2p}(X,\mathbb{Z})$ is an alg. cycle class.

Disproven in 1962 (Atiyah - Hirzebruch). $[\alpha \in H^{2p}(X,\mathbb{Z})_{\text{tors}}]$

(Kollár 1992) Nontorsion counterexamples.

§ Regular matroids

Def A regular matroid R is a set of vectors $\vec{x}_s \in \mathbb{Z}^g$, $s \in S$ s.t.

(1) $\vec{x}_s$ generate $\mathbb{Z}^g$

(2) Any subset of $\vec{x}_s$ generate a saturated sublattice. (cokernel torsion free)

Ex 1. $\{(1,0), (0,1), (1,1)\} \subset \mathbb{Z}^2$.

Non-Ex 2. $\{(1,0), (0,1), (1,1), (1,-1)\} \subset \mathbb{Z}^2$.

Ex 3 Let G be an oriented graph. $H_1(G,\mathbb{Z}) \subset \mathbb{Z}^E$.

$\{e_i^* \in H^1(G,\mathbb{Z})\} \subset H^1(G;\mathbb{Z})$ cographic matroid $M^*(G)$.

<u>Ex 4</u> (Graphic matroid) $H_1(G,\mathbb{Z}) \subset \mathbb{Z}^E$

$$M(G) := \{\overline{e_i} \in \mathbb{Z}^E / H_1(G,\mathbb{Z})\} \quad \text{graphic matroid}$$

<u>Ex 5</u> (R_{10})

$$\{e_i,\ e_i - e_{i+1} + e_{i+2}\}_{i \bmod 5} \subset \mathbb{Z}^5$$

§ Degenerations of PPAVs

<u>Def</u> A PPAV (X,Θ) is a cplx torus $X = \mathbb{C}^g/\Lambda$, $\Theta \in H^2(X,\mathbb{Z})$ ample class s.t.

$$\Theta = \sum_{i=1}^{g} e_i^* \wedge f_i^* \in \Lambda^2 H_1(X,\mathbb{Z})^*.$$

<u>Thm</u> (E.-daF-S.) Let R be a regular matroid in $\mathbb{Z}^g$ on S, $\exists$ a degeneration

$f: X = X(R) \to \Delta^S$ of PPA g-folds. s.t.

polydisk

(1) f is <u>nodal</u>.

(2) <u>vanishing cycles</u> over s^{th}- coord. hyperplane $V(u_s) \cong \Delta^{S\setminus\{s\}} \subset \Delta^S$

$\leftrightarrow \vec{x}_s \in \mathbb{Z}^g \subset H_1(X_t;\mathbb{Z})$

generic fiber $\boxed{\mathbb{Z}^g = W_{-2}H_1(X_t;\mathbb{Z})}$

(3) $K_X \sim \mathcal{O}_X$.

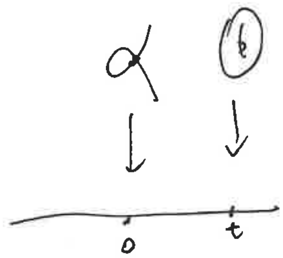
<u>Explanations</u> A morphism f is <u>nodal</u> if it is (étale) analytically-locally of the following form

$$\prod \{x_i y_i = u_i\} \times \Delta^{j^{th}} \to \prod \Delta_{u_i} \times \Delta^j$$

smooth morphism.

product of node smoothings

$$\begin{array}{c}\{y^2 = x^3 + x^2 + u\} \\ \downarrow \\ \Delta_u\end{array}$$



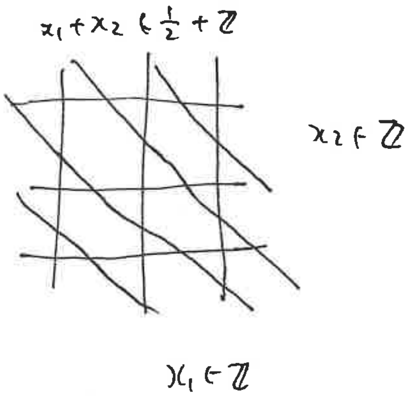
Vanishing cycle over $V(\omega_s)$:

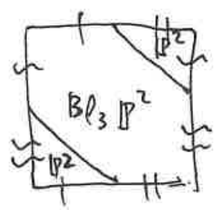
$\alpha \in H_1(X_t; \mathbb{Z})$ so that α is null homologous over $V(U_s)$

Proof $\quad \vec{x}_s \in \mathbb{Z}^g \rightsquigarrow H_s \subset (\mathbb{Z}^g)^* \otimes \mathbb{R}$

$$H_s = \{ v : \vec{x}_s(v) \in \mathbb{Z} + \varepsilon_s \} \qquad \varepsilon_s \in \mathbb{Q}/\mathbb{Z} \quad \text{random}$$

Ex $\quad R = \{(1,0), (0,1), (1,1)\}$, $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = 1/2$.





$$\begin{array}{ccc} = X_0 & \longrightarrow & X(R) \leftarrow \text{family of PPA surfaces} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Delta^3 \ \text{Mumford construction} \end{array}$$

§ Main results

Thm (E-AGF-S) $\exists\, d(R) \in \mathbb{N}$ inv't of regular matroids s.t.

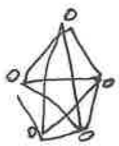
(1) if $C \subset X_t$ is a curve on the very general fiber of $X(R) \to \Delta^S$ and $[C] = m \cdot \frac{\theta^{g-1}}{(g-1)!}$,

then $d(R) \mid m$

(2) $d(R) = 1$ iff R is cographic.

<u>Cor 1</u>. IHC is false for $X \in \mathcal{A}_g$ very general PPAV, $g \geq 4$.

<u>Proof</u>. $R = M(K_5)$ <u>not</u> cographic. $X(R) \to \Delta^{10}$



$g = 4$ So $\frac{\theta^3}{3!} \in H^6(X_t, \mathbb{Z})$

Hodge, not algebraic.

$d(M(K_5)) = 2$.

<u>Cor 2</u>. Very general cubic 3-fold is stably irrat'l.

<u>Pf</u> $Y_0 = \{x_0 + \dots + x_5 = x_0^3 + \dots + x_5^3 = 0\}$

"Segre cubic"

$$\begin{array}{ccc} Y_0 & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & \Delta^{10} \end{array}$$

univ. def.

$$\begin{array}{l} X(R_{10}) = X = IJ(Y/\Delta^{10}) \\ \quad\quad\quad\ \downarrow \\ \quad\quad\quad \Delta^{10} \end{array}$$

$H^3(Y_t, \mathbb{Z}) \rightsquigarrow (0\,5\,5\,0)$

X degen. of PPA 5-folds

$d(R_{10}) = 2$

R_{10} not cographic $\Rightarrow$ $\frac{\theta^4}{4!}$ is not alg on X_t (v. gen'l fiber)

$\xRightarrow{\text{Voisin 2017}}$ Y_t has no "decomposition of the diagonal".

$\Rightarrow$ Y_t is stably irrat'l ($\nexists\, N > 0$ s.t. $Y_t \times \mathbb{P}^N \dashrightarrow \mathbb{P}^{N+3}$)

<u>Proof sketch</u>. Let $C_t \subset X_t$ curve on the <u>generic fiber</u> representing $m \cdot \frac{\theta^{g-1}}{(g-1)!}$

$$\begin{array}{ccc} \mathcal{C}_U & \hookrightarrow & X(\mathbb{R}) \\ \downarrow & & \downarrow \\ U & \subset & \Delta^s \end{array}$$

Zar.-open

$\overline{\mathcal{C}_U} =: \mathcal{C}$

p prime, $p \nmid m$

$$\begin{tikzcd} H_1(\mathcal{C}_t, \mathbb{Z}) \arrow[r, "i_*"] \arrow[d, "\theta_{\mathcal{C}_t}"'] & H_1(X_t; \mathbb{Z}) \arrow[l, dashed, bend left] \arrow[d, "\theta_{X_t}"] \\ H^1(\mathcal{C}_t, \mathbb{Z}) & H^1(X_t; \mathbb{Z}) \arrow[l, "i^*"'] \end{tikzcd}$$

$$0 \to K \to H_1(\mathcal{C}_t, \mathbb{Z}_{(p)}) \twoheadrightarrow H_1(X_t, \mathbb{Z}_{(p)}) \to 0$$

<u>Moving lemma</u> replace $\mathcal{C}$ w/ $\mathcal{C}$ transverse to X_0, $\mathcal{C}_0 = \mathcal{C} \cap X_0$.

$g \geq 0$

$G = \Gamma(\mathcal{C}_0)$

$$0 \to U' \to H_1(G, \mathbb{Z}_{(p)}) \rightleftarrows (\mathbb{Z}_{(p)}^g)^\vee \to 0$$

$\rightsquigarrow$ splitting $H^1(G, \mathbb{Z}_{(p)}) \simeq (\mathbb{Z}_{(p)}^g)^\vee \oplus U'$ $(*)$

s.t. $Q_s = \sum_{i,\, s(i)=s} (e_i^*)^2 \in$ Quad Form on $H_1(G, \mathbb{Z}_{(p)})$.

satisfies (1) $Q_s \big|_{(\mathbb{Z}_{(p)}^g)^\vee} = m \cdot \vec{x}_s^{\,2}$

(2) $(*)$ is Q_s orthogonal, $\forall s \in S$.