Classical limit of category $\mathcal{O}$ via positive characteristic

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$\mathfrak{g}/\mathbb{C}$ ss Lie alg. BGG cat. $\mathcal{O}_\lambda$

$\mathfrak{g}\text{-mod}_\lambda$ $\qquad\qquad G \supset B \supset U$

$\uparrow$
integral reg. char

BB $\qquad \mathcal{O}_\lambda \simeq D\text{-mod}^\lambda_U(G/B) \simeq D\text{-mod}_U(G/B)$

$D\text{-mod}(X)$ — modules over $\mathrm{Diff}(X)$ - quantization of T^*X

$$D\text{-mod}(X) \rightsquigarrow QCoh(T^*X)$$

$$D\text{-mod}_U(G/B) \rightsquigarrow Coh^B\left(T^*G/B \times_{\mathrm{Lie}(U)^*} \{0\}\right)$$

Claim a) LHS $\hookrightarrow$ RHS. "classical approximation for cat. $\mathcal{O}$
full subcat. $\qquad$ is exact".

Versions. $\quad$ Rmk $\quad \mathcal{O} \simeq (\mathfrak{g}\oplus\mathfrak{g}, G)\text{-mod}_{\hat{\mu},\lambda} \subset (\mathfrak{g}\oplus\mathfrak{g}, G)\text{-mod}_{\hat{\lambda},\hat{\mu}}$

"^" - max ideal acts nilp.

$\Uparrow$

$$B\backslash G/B = \frac{G/B \times G/B}{G}$$

b) $(\mathfrak{g}\oplus\mathfrak{g}, G)\text{-mod}_{\hat{\mu},\hat{\lambda}} \hookrightarrow Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$, $\quad \tilde{\mathfrak{g}} = T^*(G/U)/T$
$= \{(x,\mathfrak{b}) \in \mathfrak{g} \times G/B : x \in \mathfrak{b}\}$

also inducing a full embedding of triangulated cats

c) same for $(\mathfrak{g}, K)$- mod

$K = G^\theta$ (θ- quasisplit) (jt w/ A. Ionov, D. Davis)

Connection to Soergel bimodules.

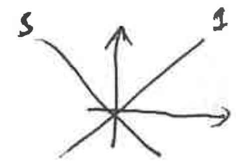
<u>Soergel</u>: – description of cats in a), b) via $\mathrm{Coh}(\check{\mathfrak{t}} \times_{\check{\mathfrak{t}}/\!/W} \check{\mathfrak{t}})$

union of graphs of all $w \in W$ acting on $\mathfrak{t}^*$

$$\mathbb{S}: \mathcal{O} \longrightarrow \mathrm{Coh}(\check{\mathfrak{t}} \times_{\check{\mathfrak{t}}/\!/W} \check{\mathfrak{t}})$$

HC

fully faithful on projections



Ex. $\mathfrak{g} = \mathfrak{sl}_2$

in c) was generalized by B.- Vilonen.

$$\begin{tikzcd}
\tilde{\mathfrak{g}} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{t}/\!/W \arrow[dl, "k"] \\
\tilde{\mathfrak{g}}^{reg} \arrow[u, phantom, "\cup"] \arrow[r] & \mathfrak{g}^{reg} \arrow[u, phantom, "\cup"] & \\
\mathfrak{t} \arrow[u] \arrow[r] & \mathfrak{t}/\!/W \arrow[u, "k"'] &
\end{tikzcd}$$

k - Kostant section

ramified $W-1$ cover

$$\begin{tikzcd}
\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \arrow[d] & \mathfrak{t} \times_{\mathfrak{t}/\!/W} \mathfrak{t} \arrow[l, hook] \arrow[d] \\
\mathfrak{g} & \mathfrak{t}/\!/W \arrow[l]
\end{tikzcd}$$

$\mathbb{S}$ is the composition of the full embedding & restriction to Kostant slice.

(In c) Kostant - Rallis slices).

Rmk 1) At the level of Grothendieck groups,

$$\begin{array}{ccc} K\left(\mathrm{Coh}^G\left(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}\right)\right) & \leftrightarrow & K\left((\mathfrak{g}\oplus\mathfrak{g}, G)\text{-mod}_{\hat{\lambda},\hat{\mu}}\right) \\ \| & & \| \\ \mathbb{Z}[W_{aff}] & \longleftarrow & \mathbb{Z}[W] \end{array}$$

2) Using Soergel's theory, can check $(\mathfrak{g}\oplus\mathfrak{g}, G)\text{-mod}_{\hat{\mu},\hat{\lambda}} \simeq (\check{\mathfrak{g}}\oplus\check{\mathfrak{g}}, \check{G})\text{-mod}_{\check{\hat{\lambda}},\check{\hat{\mu}}}$

(same for a)

$$D\text{-mod}_{\check{U}}(\check{G}/\check{B}) \hookrightarrow D\text{-mod}_{\check{I}^\circ}\underbrace{(L\check{G}/\check{I})}_{\check{F\ell}} \overset{\text{derived}}{\simeq} D^b\mathrm{Coh}^G\left(T^*(G/B) \times_{\mathfrak{g}} \tilde{\mathfrak{g}}\right)$$

"local geom. Langlands"

Rmk X smooth proj. var. Hodge theorem says

$$H^*_{dR}(X) = \mathrm{Ext}_{D\text{-mod}(X)}(\mathcal{O}_X, \mathcal{O}_X)$$

$$\wr$$

$$\bigoplus H^i(\Omega^j_X) = \mathrm{Ext}_{\mathrm{Coh}(T^*X)}(\mathcal{O}_X, \mathcal{O}_X)$$

$$H^*_{DR}(X) \simeq \bigoplus H^i(\Omega^j_X) = \mathrm{Ext}_{\mathrm{Coh}(T^*X)}(\mathcal{O}_X, \mathcal{O}_X)$$

$X \hookrightarrow T^*X$ zero section

$\mathcal{O}_X \leftrightsquigarrow (\mathcal{O}, \nabla = d)$

Can be proved algebraically (Deligne - Illusie, 1987) by reduction to char $p>0$.

Plan of proof use Hodge D-modules + positive char.

A Hodge D-module is a D-module w/ extra structure.

incl.: a good filtration.

$$D_H\text{-mod}(X) \xrightarrow{gr_H} Coh(T^*X)$$

It has strong functorial properties, imply:

gr_H is compatible w/ convolution.

Need to check for specific $\mathcal{M}, \mathcal{N} \in D_H\text{-mod}^G(G/U \times G/B)$, etc.

Compare $\mathrm{Hom}_{D\text{-Mod}}(\mathcal{M}, \mathcal{N})$
$\mathrm{Ext}_{D\text{-mod}}(\mathcal{M}, \mathcal{N})$ to $\mathrm{Ext}(gr_H(\mathcal{M}), gr_H(\mathcal{N}))$

Key Lemma. Compare $gr_H(\mathcal{M})$, $gr_H(\mathcal{N})$ to Cartier transform of $\mathcal{M}, \mathcal{N}$

– coh. sheaves on $T^*(-)$ defined via positive char.

Compatibility of gr_H w/ convolution.

$$D\text{-mod}_H^G(G/U \times G/U)_{T\times T\text{-mon}} \xrightarrow{\mathcal{M} \mapsto gr_H(\mathcal{M})(-\rho,-\rho)} Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$$

Both derived cats have convolution, is compatible w/ convolution.

Cartier transform. work over k of char p

$\mathrm{Diff}(X)$ — Azumaya alg. over $T^*X^{(1)}$. — Frob. twist

If $Z \subset T^*X^{(1)}$ s.t. AA splits on Z.

then for $\mathcal{M} \in$ D-mod, $\mathrm{supp}(\mathcal{M}) \subset Z$, get $C(\mathcal{M}) \in \mathrm{Coh}(Z) \subset \mathrm{Coh}(T^*X)$

Rmk The class of AA on $T^*(G/B)$ is pulled back under $\pi : T^*(G/B) \to \mathcal{N} \subset \mathfrak{g}^*$
nilp cone

Cor The AA on $T^*(G/B) \underset{\mathfrak{g}^*}{\times} T^*(G/B)$ splits. b/c left & right pull-backs cancel.

Claim The AA on $\dfrac{T^*(G/B) \times_{\mathfrak{g}^*} T^*(G/B)}{G}$ (or U's formal nbhd in $\dfrac{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}{G}$)

splits canonically.

Same for $(\mathfrak{g}, K)$ situation (θ-quasi-split)

Sketch of proof. Pick proj. generators P for the cat. endow w/ Hodge module structure.

$$\bigoplus_n \mathrm{Hom}_{\text{D-mod}_H}(P, P(n)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Coh}_{T^*X}}(\mathrm{gr}_H(P), \mathrm{gr}_H(P))$$
$$\| $$
$$\bigoplus_n \mathrm{Hom}_{\mathrm{Coh}^{\mathbb{G}_m}_{T^*X}}(\mathrm{gr}_H(P), \mathrm{gr}_H(P)(n))$$

Key step, (for almost all p)

$$C(P) \simeq \mathrm{gr}_H(P).$$ Fontaine–Laffaille modules

Proof Do by hand for specific objects, use comp. w/ convolution.

The claim reduces to degeneration of a spectral seq. (bound on dim)

The cotangent stack is cohomologically finite $\quad \mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ is finite dim'l.

$$\Uparrow$$

$\mathcal{F} \in \mathrm{Coh}^G(\mathcal{N})$, $\quad \Gamma(\mathcal{F})^G$ is finite dim'l

Ex. $\underset{\mathcal{F}}{\underset{\shortparallel}{\mathcal{O} \otimes V}}$, $\quad \dim \Gamma(\mathcal{F})^G = \dim V(0)$.

$\dim \begin{matrix} \mathrm{Hom} \\ \mathrm{Ext} \end{matrix}$ over a char p field

= dim —— over $\mathbb{C}$

for almost all p

true for $\mathrm{Coh}(T^*-)$

& for D-mod

Rmk. $(G/B \times G/B)_w \overset{}{\underset{j_w}{\hookrightarrow}} (G/B)^2$

$$\nabla_w = j_{w*}(\mathcal{O})$$

Can show $\quad \mathrm{gr}_H(\nabla_w)(-\rho, -\rho) = \mathcal{O}_{Z_w}$

For nonintegral twist, harder to describe

<u>Dream</u>: apply p-adic Hodge theory to RT in pos. char.

reprove positivity of grading w/o using Langlands duality

<u>Potential application to Vogan's character duality</u>.

$$D(K \backslash G/B) \hookrightarrow D^b\mathrm{Coh}^K\left(T^*(G/B) \underset{\mathfrak{g}^*}{\times} \mathfrak{k}^\perp\right) \overset{KD}{\longleftrightarrow} D^b\mathrm{Coh}^K\left(\tilde{\mathfrak{g}} \underset{\mathfrak{g}}{\times} \mathfrak{k}\right)$$

linear KD of Mirkovic-Riche

$$\begin{array}{c} \mathcal{E}_1 \subset \mathcal{E} \supset \mathcal{E}_2 \\ \downarrow \\ X \end{array} \qquad \mathcal{E}_1 \times_{\mathcal{E}} \mathcal{E}_2 \longleftrightarrow \mathcal{E}_1^{\perp} \times_{\mathcal{E}^*} \mathcal{E}_2^{\perp}$$

Rmk. The cohomology functor goes to $(\mathcal{F} \longmapsto \Gamma(\mathcal{F})^{\kappa})$

$\simeq$ restriction to Kostant slice in k.