Coherent sheaves on Springer resolution

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(G, B, T) conn. reductive split $/k$

$\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$

$$\begin{array}{lll} X = X^*(T), & X^\vee = X_*(T) & \\ \cup & \cup & \\ \Phi \text{ roots} & \Phi^\vee \text{ coroots} & \rightsquigarrow (G^\vee, B^\vee, T^\vee)/\mathbb{k} \\ \cup & \cup & \\ \text{simple roots } \alpha & \text{simple coroots } \alpha^\vee & \end{array}$$

$\check{h}$, $\bar{\mathbb{k}} = \mathbb{k}$, $\operatorname{char} \mathbb{k} = 0$

$\mathfrak{g}^\vee \supset \mathfrak{b}^\vee = \mathfrak{n}^\vee \oplus \mathfrak{t}^\vee$

Springer resolution $\tilde{\mathcal{N}} := G^\vee \overset{B^\vee}{\times} \mathfrak{n}^\vee \hookrightarrow \mathcal{B} \times \mathfrak{g}^\vee$. $\mathcal{B} = G^\vee/B^\vee$

$$(g, x) \longmapsto (gB^\vee, g.x)$$

Plan: Construct

$$\begin{tikzcd} & D^b_{Coh}\left(G^\vee \backslash \tilde{\mathcal{N}} \overset{L}{\underset{\mathfrak{g}^\vee}{\times}} \tilde{\mathcal{N}}\right) \arrow[d, "\cong\ \text{Bez.}"] \\ \Phi_{diag} : D^b Coh(G^\vee \backslash \tilde{\mathcal{N}}) \arrow[ur, "\Delta^*"] \arrow[r, "*"] \arrow[dr, "\sim"', "\text{(Arkhipov-Bezrukavnikov)}"] & D^b_c(I \backslash Fl_G) = D^b_c(I \backslash LG/I) \arrow[d, "Av_\psi"', "\text{Next time}"] \\ & D(I_W \backslash Fl_G) \end{tikzcd}$$

How do we construct Φ_{diag}:

Starting point: Central functor: $Z : Rep(\check{G}) \simeq Perv(L^+G \backslash LG / L^+G) \longrightarrow Perv(I \backslash Fl_G) =: P_I$

Wakimoto sheaves: $\lambda \in X^\vee \rightsquigarrow J_\lambda = \begin{cases} j_{\lambda *}(\mathbb{k}[\langle 2\rho, \lambda \rangle]), & \lambda \in X^\vee_+ \\ j_{\lambda !}(\mathbb{k}[\langle -2\rho, \lambda\rangle]), & \lambda \in -X^\vee_+ \\ J_{\lambda_1} * J_{\lambda_2}, & \lambda = \lambda_2 - \lambda_1,\ \lambda_i \in X^\vee_+ \end{cases}$

$J_\lambda * J_\mu \simeq J_{\lambda+\mu}$.

$\rightsquigarrow \mathrm{Rep}(T^\vee) \longrightarrow P_I^{Wak}$

$P_I^{Wak} \subset P_I$ consisting of objs w/ Wakimoto filtration

$\cup$

$\mathrm{Im}\, Z$.

$\rightsquigarrow \mathrm{Rep}(G^\vee \times T^\vee) \longrightarrow P_I^{Wak}$ starting point

$$
\begin{tikzcd}
\mathrm{Rep}(G^\vee \times T^\vee) \arrow[r] & P_I^{Wak} \\
\mathrm{Coh}(G^\vee \times T^\vee \backslash *) \arrow[u, phantom, "\wr"] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\hat{\mathcal{N}} \arrow[r, hook, "\text{closed}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G^\vee/U^\vee \times \mathfrak{g}^\vee \arrow[d, "T^\vee\text{-torsor}"] \\
\tilde{\mathcal{N}} = G^\vee \times^{B^\vee} \mathfrak{n}^\vee \arrow[r, hook, "\text{closed}"] & \mathcal{B} \times \mathfrak{g}^\vee
\end{tikzcd}
$$

Want to extend to $\mathrm{Coh}(G^\vee \backslash \tilde{\mathcal{N}}) \simeq \mathrm{Coh}(G^\vee \times T^\vee \backslash \hat{\mathcal{N}})$

$G^\vee/U^\vee$ quasi-affine

$T^\vee \downarrow$

$\mathcal{B}$ proj var.

e.g. $G^\vee = SL_2$, $\mathcal{B} = \mathbb{P}^1$,

$G^\vee/U^\vee = \mathbb{A}^2 \setminus \{0\}$.

$\boxed{G^\vee/U^\vee}$. $\lambda \in X^\vee \rightsquigarrow B^\vee\text{-rep.}$ $\Bbbk(\lambda)$: $B^\vee \longrightarrow T^\vee \xrightarrow{\lambda} \mathbb{G}_m$

$V_\lambda := \mathrm{Ind}_{B^\vee}^{G^\vee} \Bbbk(\lambda)$ char. 0 $\begin{cases} = 0 & \text{if } \lambda \notin X_+^\vee \\ \text{irred.} & \text{if } \lambda \in X_+^\vee \end{cases}$

h.w. rep., $v_\lambda \in V_\lambda$ h.w. vector

Stabilizer of $v_\lambda = U^\vee$ if v_λ is regular dominant

$G^\vee/U^\vee \hookrightarrow V_\lambda$ $\Rightarrow G^\vee/U^\vee$ quasi-affine

orbit of v_λ $g \longmapsto g v_\lambda$

$$\mathcal{O}(G^\vee/U^\vee) = \bigoplus_{\lambda\in X^\vee_+} V_\lambda \otimes (V_\lambda^*)^{U^\vee} = \bigoplus_{\lambda\in X^\vee_+} V_\lambda \otimes \Bbbk(-w_0\lambda)$$

One can check $\mathcal{O}(G^\vee/U^\vee) \otimes \mathcal{O}(G^\vee/U^\vee) \xrightarrow{m} \mathcal{O}(G^\vee/U^\vee)$

$$V_\lambda \otimes \Bbbk(-w_0\lambda) \otimes V_\mu \otimes \Bbbk(-w_0\mu) \mapsto V_{\lambda+\mu} \otimes \Bbbk(-w_0(\lambda+\mu))$$

$$G^\vee/U^\vee \xhookrightarrow{\text{open}} \mathcal{X} = \operatorname{Spec} \mathcal{O}(G^\vee/U^\vee)$$

$G^\vee/U^\vee \hookrightarrow \mathcal{X}$

$$\begin{tikzcd}
\hat{\mathcal{N}} = G^\vee \times^{U^\vee} \mathfrak{n}^\vee \arrow[r, hook] \arrow[d, hook, "\text{open}"'] \arrow[dr, phantom, "\ulcorner\ \text{not dense in general}"] & G^\vee/U^\vee \times \mathfrak{g}^\vee \arrow[d, hook, "\text{open}"] \\
\exists\ \hat{\mathcal{N}}_{\mathcal{X}} \arrow[r] & \mathcal{X} \times \mathfrak{g}^\vee
\end{tikzcd}$$

$\mathfrak{g}^\vee \curvearrowright \mathcal{X}$

$\mathcal{X} \times \mathfrak{g}^\vee$ $\quad V_{taut}$ tautological vector field $\left(\begin{matrix}\text{infinitesimal} \\ \text{universal} \\ \text{stabilizer}\end{matrix}\right)$

(x, v)

$$V_{taut}\big|_{(x,v)} = (v_x, 0)$$

$\rightsquigarrow$ Derivation V_{taut} on $\mathcal{O}(\mathcal{X} \times \mathfrak{g}^\vee)$

Vanishing locus of $\langle \mathrm{Im}\, V_{taut} \rangle =: \hat{\mathcal{N}}_{\mathcal{X}}$.

We'll construct

$$\begin{tikzcd} \mathrm{Coh}\left(G^\vee\times T^\vee \backslash \widehat{\tilde{\mathcal N}} * \right) \arrow[r] & P_I^{\mathrm{Wak}} \quad \text{first.} \\ \mathrm{Coh}\left(G^\vee\times T^\vee \backslash \tilde{\mathcal N}\times \mathfrak g^\vee\right) \arrow[u, "\text{restriction}"] \arrow[ur] & \end{tikzcd}$$

<u>Recall</u>: Tannakian formalism $G/\bar{\Bbbk}=\Bbbk$ char $\Bbbk=0$

$g\in G(A) \rightsquigarrow$ $\otimes$-auto functor of $A\otimes \mathrm{For}(-) : \mathrm{Rep}(G)\longrightarrow \mathrm{Mod}_A$

$G(A)\simeq \mathrm{Aut}^{\otimes}(A\otimes\mathrm{For}(-))$ $\quad g(v\otimes w)=gv\otimes gw$

$\mathfrak g(A)\simeq \mathrm{Der}^{\otimes}(A\otimes\mathrm{For}(-))$ $\quad X: V\otimes W\longrightarrow V\otimes W$

$X(v\otimes w)=Xv\otimes w+v\otimes Xw$

① $F:\mathrm{Rep}(G^\vee\times T^\vee)\longrightarrow(\mathcal C,*)$ $\Bbbk$-linear monoidal cat.

F + data of a $\otimes$-derivation of $F|_{\mathrm{Rep}(G^\vee)}$ $\quad N\in\mathrm{Der}^{\otimes}(F|_{\mathrm{Rep}\,G^\vee})$

<u>Claim</u>: $\exists!$ lifting of F to $\tilde F:\mathrm{Coh}_{fr}(G^\vee\times T^\vee\backslash\mathfrak g^\vee)\longrightarrow\mathcal C$

$\subset \mathrm{Coh}(G^\vee\times T^\vee\backslash\mathfrak g^\vee)$

$V\otimes\mathcal O_{\mathfrak g^\vee}\longrightarrow F(V)$

consisting of objects
of the form $V\otimes\mathcal O_{\mathfrak g^\vee}$, $V\in\mathrm{Rep}(G^\vee\times T^\vee)$

Proof (sketch) First consider special case $\mathcal{C} = A\text{-mod}^{\check{G}\times\check{T}}_{fr} = \mathrm{Coh}_{fr}(\check{G}\times T^\vee \backslash \operatorname{Spec} A)$

s.t. $F : V \longmapsto V \otimes A$

If this is the case, $N \in \mathrm{Der}^{\otimes}(F|_{\mathrm{Rep}(\check{G})})$

$\rightsquigarrow \operatorname{Spec} A \longrightarrow g^\vee \iff \mathcal{O}(g^\vee) \longrightarrow A.$

$$\begin{array}{ccc} \mathrm{Hom}_{\check{G}\times\check{T}}(V \otimes \mathcal{O}_{g^\vee}, W \otimes \mathcal{O}_{g^\vee}) & \cong & (V^* \otimes W \otimes \mathcal{O}_{g^\vee})^{\check{G}\times T^\vee} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\check{G}\times T^\vee}(F(V), F(W)) & \cong & (V^* \otimes W \otimes A)^{\check{G}\times T^\vee} \end{array}$$

General case.

$F : \mathrm{Rep}(\check{G}\times T^\vee) \longrightarrow \mathcal{C}$, $A := \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, F(\mathcal{O}(\check{G}\times\check{T})))$

$\mathcal{O}(G) \curvearrowright \mathcal{C} \otimes_{\mathrm{QCoh}(BG)} \mathrm{QCoh}(pt) = \mathcal{C}_{deeq}$ obj. = obj. in $\mathcal{C}$

linear str.

deequivariantization morphism : $\mathrm{Hom}(c, c') = \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(c, c' * F(\mathcal{O}(\check{G}\times\check{T})))$

idea is to reduce to $\mathcal{C}_{deeq}$.

General idea: X stack $(\mathrm{QCoh}(X), \otimes)$ $X \longmapsto \mathrm{Mod}_{\mathrm{QCoh}(X)}$ satisfies descend

e.g. $\mathrm{Mod}_{\mathrm{QCoh}(\operatorname{Spec} A/G)} \xrightarrow{\sim} \lim(\mathrm{Mod}_{\mathrm{Mod}_A} \rightrightarrows \mathrm{Mod}_{\mathrm{Mod}\dots} \rightrightarrows \cdots)$

$$\begin{tikzcd} \mathrm{Coh}_{fr}(\check{G}\times T^\vee \backslash \check{\mathfrak{g}}) \arrow[r] & \mathcal{C} \\ & \mathrm{Rep}(\check{G}) \arrow[u, "Z"] \end{tikzcd}$$

. In our case $\mathcal{C} = \mathcal{P}_I^{Wak}$.

Need to find N

$Z(V) \circlearrowleft m_V$ unipotent monodromy

$\log m_V$ is a $\otimes$-derivation.

Next:

$$\begin{tikzcd} \mathrm{Coh}_{fr}(\check{G}\times T^\vee \backslash \mathcal{X}) \arrow[rr, "\tilde{F}"] & & \mathcal{C} \\ & \mathrm{Coh}(\check{G}\times T^\vee \backslash *) \arrow[ul] \arrow[ur, "F"] & \end{tikzcd}$$

Claim: The data we need is

$\lambda \in X_+^\vee$

$b_\lambda : F(V_\lambda) \to F(\Bbbk(w_0\lambda))$

s.t.

$$\begin{tikzcd} F(V_\lambda) * F(V_\mu) \arrow[r] \arrow[d] & F(\Bbbk(w_0\lambda)) * F(\Bbbk(w_0\mu)) \arrow[d] \\ F(V_{\lambda+\mu}) \arrow[r] & F(\Bbbk(w_0(\lambda+\mu))) \end{tikzcd}$$

Drinfeld - Plücker relation.

$$\mathcal{O}(\mathcal{X}) = \bigoplus_{\lambda \in X_+^\vee} V_\lambda \otimes \Bbbk(-w_0\lambda)$$

$$V_\lambda \otimes \Bbbk(-w_0\lambda) \otimes \mathcal{O}(\mathcal{X}) \xrightarrow{m} \mathcal{O}(\mathcal{X})$$

$$\Rightarrow V_\lambda \otimes \mathcal{O}(\mathcal{X}) \xrightarrow{b_\lambda} \Bbbk(w_0\lambda) \otimes \mathcal{O}(\mathcal{X})$$

Special case:

$$\mathrm{Coh}_{fr}(\check{G}\times T^\vee \backslash \mathcal{X}) \longrightarrow \mathcal{C} = A\text{-mod}_{fr}^{\check{G}\times\check{T}}$$

$$\underbrace{V_\lambda \otimes \Bbbk(\mu)}_{V} \otimes \mathcal{O}_{\mathcal{X}} \longmapsto F(\underbrace{V_\lambda \otimes \Bbbk(\mu)}_{V})$$

$$\begin{array}{ccc} \mathrm{Hom}(V\otimes \mathcal{O}_{\mathcal{X}}, W\otimes \mathcal{O}_{\mathcal{X}}) & \longrightarrow & \mathrm{Hom}(V\otimes A, W\otimes A) \\ \| & & \| \\ (W\otimes V^*\otimes \mathcal{O}_{\mathcal{X}})^{\check{G}\times\check{T}} & \longrightarrow & (V^*\otimes W\otimes A)^{\check{G}\times\check{T}} \end{array}$$

$$\begin{array}{ccc} b_\lambda : F(V_\lambda) & \longrightarrow & F(\Bbbk(w_0\lambda)) \\ \| & & \| \\ A\otimes V_\lambda & \longrightarrow & A\otimes \Bbbk(w_0\lambda) \end{array}$$

$$V_\lambda \otimes \Bbbk(-w_0\lambda) \longrightarrow A.$$

Pro

$\mathcal{C} = P_I^{Wak}$

$\lambda \in X_+^\vee$

$$Z(V_\lambda) \longrightarrow J_\lambda = j_{\lambda *}\left(\Bbbk[\langle 2\rho, \lambda\rangle]\right)$$

$$\mathrm{Hom}(Z(V_\lambda), J_\lambda) \cong \mathrm{Hom}(j_\lambda^* Z(V_\lambda), \Bbbk[\langle 2\rho,\lambda\rangle]) \simeq \Bbbk$$

$$\begin{array}{c} \| \text{ fact} \\ \Bbbk[\langle 2\rho,\lambda\rangle] \end{array}$$

$$\begin{tikzcd}
Z(V_\lambda) * Z(V_\mu) \arrow[r] \arrow[d] & J_\lambda * J_\mu \arrow[d, "\wr"] \\
Z(V_{\lambda+\mu}) \arrow[r] & J_{\lambda+\mu}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Coh}_{fr}(G^\vee \times T^\vee \backslash \mathfrak{g}^\vee \times *) \arrow[r] \arrow[d] & \mathcal{C} \\
\mathrm{Coh}_{fr}(G^\vee \times T^\vee \backslash \hat{\mathcal{N}}_*) \arrow[ur] &
\end{tikzcd}$$

" $A\text{-mod}_{fr}^{G^\vee \times T^\vee}$

__Claim__. To factor through $\hat{\mathcal{N}}_*$, the data we need is $f_\lambda \circ N_{V_\lambda} = 0$.

$\uparrow$

true in our case

$\Leftrightarrow f_\lambda \circ m_\lambda = f_\lambda$

By Tannakian,

$$\begin{tikzcd}
V_\lambda \otimes \mathcal{O}_{\mathfrak{g}^\vee} \arrow[r, "N_{taut}"] \arrow[d] & V_\lambda \otimes \mathcal{O}_{\mathfrak{g}^\vee} \arrow[r, "f_\lambda \otimes N"] \arrow[d] & \Bbbk(\lambda) \otimes A \\
V_\lambda \otimes A \arrow[r, "N_{V_\lambda}"] \arrow[urr, bend right=60, "0"'] & V_\lambda \otimes A \arrow[ur, "f_\lambda \otimes A"'] &
\end{tikzcd}$$

$$\mathrm{Coh}_{fr}\left({}_{\check G\times T^\vee}\backslash\hat{\mathcal N}_*\right) \longrightarrow P_I^{Wak} \subset D^b_c(I\backslash Fl_G)$$

$$\mathrm{Perf}\left(\frac{\hat{\mathcal N}_*}{\check G\times\check T}\right) \supset \mathrm{Perf}\left({}_{\check G\times T^\vee}\backslash\hat{\mathcal N}_*\right) \qquad \partial\hat{\mathcal N}_* = \hat{\mathcal N}_*\setminus\hat{\mathcal N}$$

$$\mathrm{Perf}\left(\check G\times\check T\backslash\hat{\mathcal N}\right) \simeq \mathrm{Perf}\left(\check G\times\check T\backslash\hat{\mathcal N}_*\right)\Big/\mathrm{Perf}\left(\check G\times\check T\backslash\hat{\mathcal N}_*\right)_{\partial\hat{\mathcal N}_*}$$

IS $\tilde{\mathcal N}$ smooth

$$D^b\mathrm{Coh}\left(\check G\times\check T\backslash\hat{\mathcal N}\right)$$

$$gr: K^- P_I^{Wak} \longrightarrow K^-\mathrm{Rep}(T^\vee) \xrightarrow{Oblv} K^-\mathrm{Vect}.$$

"Conservativity": $\mathcal F$ is acyclic $\Leftrightarrow$ $gr\,\mathcal F$ is acyclic.

$\mathcal F$ supp. on $\partial\hat{\mathcal N}_*$.

$gr\circ\tilde{\mathcal F}$ the same as pullback to $(1,0)\in \underbrace{\check G\times^{\check U}\mathfrak n^\vee}_{=\hat{\mathcal N}} \hookrightarrow \hat{\mathcal N}_*$