Wakimoto sheaves

As usual, G conn'd reductive gp $/ k = \bar{k}$, w/ fixed pinning $B, T, \ldots$

I Iwahori. We have the Iwahori-Hecke algebra

$$\mathcal{H} = \mathrm{Fun}_c\left(I \backslash G(k((t)))/I\right), \quad \text{basis} \quad T_w = \mathbb{1}_{I\dot{w}I}, \quad w \in \widetilde{W} \text{ ext. affine Weyl gp.}$$

- $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$
- $T_s^2 = (q-1)T_s + q$, s affine simple reflection

  i.e. $(T_s - q)(T_s + 1) = 0$

Better: $v = q^{-1/2}$, $h_w = v^{\ell(w)} T_w$

- $h_w h_{w'} = h_{ww'}$
- $(h_s - v^{-1})(h_s + v) = 0$, s affine simple reflection

$\mathcal{H}$	$D = D(I \backslash LG / I)$
h_w	Δ_w standard sheaf
$h_{w^{-1}}^{-1}$	∇_w costandard sheaf
KL-involution $v \longleftrightarrow v^{-1}$, $h_w \longleftrightarrow h_{w^{-1}}^{-1}$	$\mathbb{D}$ Verdier duality
Bernstein element Θ_λ, $\lambda \in X_*(T)$	J_λ Wakimoto sheaf
KL basis c_w	IC_w IC sheaf

Standard & costandard sheaves

Recall $Fl_G = LG/I$ has a stratification

$$(Fl_G)_{red} = \bigsqcup_{w \in \widetilde{W}} \underset{\substack{\| \\ Fl_w}}{I\dot{w}I/I},$$

Fl_w is a smooth $\underline{\underline{\text{affine}}}$ variety of dim $\ell(w)$.

$\uparrow$

Fl_w is an orbit of I^+ pro-unipotent radical of I.

$Fl_w \subset \overline{Fl_{w'}} \iff w \leq w'$ in the Bruhat order for $\widetilde{W}$.

$j_w : Fl_w \hookrightarrow Fl_G$

Standard sheaf. $\Delta_w := j_{w!}(\underline{e}[\ell(w)])$ e field of coefficient

Costandard sheaf $\nabla_w := j_{w*}(\underline{e}[\ell(w)])$

Prop. $\Delta_w, \nabla_w \in P_I(Fl_G)$

Proof. j_w is affine $\Rightarrow j_{w!}, j_{w*}$ are perverse t-exact (Artin vanishing)

Convolution properties

Prop. 1) For $w_1, w_2 \in \widetilde{W}$ s.t. $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$,

$\exists$ a canonical isom.

$$\theta_{w_1, w_2} : \Delta_{w_1} * \Delta_{w_2} \xrightarrow{\sim} \Delta_{w_1 w_2}$$

Moreover, if $w_1, w_2, w_3 \in \widetilde{W}$ satisfy $\ell(w_1 w_2 w_3) = \ell(w_1) + \ell(w_2) + \ell(w_3)$,

the following diagram commutes

$$\begin{tikzcd}
\Delta_{w_1} * \Delta_{w_2} * \Delta_{w_3} \arrow[r, "\theta_{w_1,w_2} * \mathrm{id}", "\sim"'] \arrow[d, "\mathrm{id} * \theta_{w_2,w_3}"', "\wr"] & \Delta_{w_1 w_2} * \Delta_{w_3} \arrow[d, "\theta_{w_1 w_2, w_3}", "\wr"'] \\
\Delta_{w_1} * \Delta_{w_2 w_3} \arrow[r, "\theta_{w_1, w_2 w_3}", "\sim"'] & \Delta_{w_1 w_2 w_3}
\end{tikzcd}$$

2) Analogous statement for costandard sheaves

3) For any $w \in \widetilde{W}$, $\exists$ isom. $\Delta_w * \nabla_{w^{-1}} \cong \nabla_{w^{-1}} * \Delta_w \cong \underbrace{\delta}_{\text{monoidal unit}}$

Rmk. 1) Think about the corresponding statements for $\mathcal{H}$.

2) By [Deligne, Action du groupe des tresses sur une catégorie], these data give rise to an action of the affine braid group B_{aff} on D.

3) The isom. in 3) is not completely canonical, involves a choice of a reduced expr. for w. Can be made canonical.

Proof?

Proof. 1) $q : LG \longrightarrow Fl = LG/I$

The affine Bruhat decomposition: $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$

$$\Rightarrow \quad q^{-1}(Fl_{w_1}) \overset{I}{\times} Fl_{w_2} \xrightarrow{\sim} Fl_{w_1 w_2}$$

$$\Rightarrow \Delta_{w_1} * \Delta_{w_2} = m_! \circ (j_{w_1,w_2})_! \left(\underline{e} \,\tilde{\boxtimes}\, \underline{e} \right) [\ell(w_1) + \ell(w_2)]$$

where $j_{w_1,w_2} : q^{-1}(Fl_{w_1}) \overset{I}{\times} Fl_{w_2} \longrightarrow LG \overset{I}{\times} Fl_G$ $\qquad = j_{w_1 w_2,!}\, \underline{e}[\ell(w_1 w_2)]$

$m : \quad LG \overset{I}{\times} Fl_G \longrightarrow Fl_G$ $\qquad = \Delta_{w_1 w_2}$.

To check higher compatibilities, draw larger diagrams ...

2) Idem. $m_! = m_*$ because m is ind-proper.

3) By induction on $\ell(w)$, it suffices to treat the case when

$w \in \Omega$ (length 0) or $w \in S$ affine simple reflection.

$w \in \Omega$ easy: Fl_w is just a point.

$w \in S$: $\Delta_s * \nabla_s$ supp. on $\overline{Fl_s} = Fl_s \sqcup Fl_e$

First show that the restriction of $\Delta_s * \nabla_s$ to Fl_s is zero.

$$q^{-1}(Fl_s) \overset{I}{\times} Fl_s \xrightarrow{u} q^{-1}(Fl_s) \overset{I}{\times} \overline{Fl_s} \xrightarrow{m} \overline{Fl_s}$$

$$\Delta_s * \nabla_s = m_! \left(\underline{e}[1] \,\tilde{\boxtimes}\, \nabla_s \right) = m_!\, u_* \left(\underline{e}[1] \,\tilde{\boxtimes}\, \underline{e}[1] \right)$$

For any geom. pt x of Fl_s, $m^{-1}(x) \cong \mathbb{A}^1$, $(m \circ u)^{-1}(x) \cong \mathbb{G}_m$

So $(\Delta_s * \nabla_s)_x = H_c^*(\mathbb{A}^1, j_*\underline{e}[1])$

$$= R\Gamma(\mathbb{P}^1, \underline{e}_{0*,\infty!}[1]) = 0.$$

With a bit more effort, can determine the stalk at $1 = e$

$\Rightarrow$ $\Delta_s * \nabla_s \cong \delta$. Idem $\nabla_s * \Delta_s \cong \delta$.

Prop. For any $w, y \in \tilde{W}$, $\Delta_w * \nabla_y$, $\nabla_w * \Delta_y \in P_I(Fl_G)$.

Proof.
$$q^{-1}(Fl_w) \overset{I}{\times} Fl_y \xrightarrow{a} q^{-1}(Fl_w) \overset{I}{\times} \overline{Fl_y} \xrightarrow{b} Fl_G$$

$$\Delta_w * \nabla_y = b_! a_* \left(\underline{e}_{q^{-1}(Fl_w) \overset{I}{\times} Fl_y} [\dim] \right)$$

a open affine embedding $\Rightarrow$ a_* t-exact

Claim: $b: q^{-1}(Fl_w) \overset{I}{\times} \overline{Fl_y} \longrightarrow Fl_G$ is affine $\Rightarrow$ $m_!$ is left t-exact

$\Rightarrow$ $\Delta_w * \nabla_y \in {}^pD^{\geq 0}$

On the other hand,

$$q^{-1}(Fl_w) \overset{I}{\times} Fl_y \xrightarrow{c} q^{-1}(\overline{Fl_w}) \overset{I}{\times} Fl_y \xrightarrow{d} Fl_G$$
$$\Delta_w * \nabla_y = d_* c_! (\underline{e}_{q^{-1}(Fl_w) \overset{I}{\times} Fl_y} [\dim])$$

c open affine embedding $\Rightarrow$ $c_!$ t-exact

Claim. $d: q^{-1}(\overline{Fl_w}) \overset{I}{\times} Fl_y \longrightarrow Fl_G$ is affine $\Rightarrow$ d_* is right t-exact

$\Rightarrow$ $\Delta_w * \nabla_y \in {}^pD^{\leq 0}$. $\square$ $\nabla_w * \Delta_y$ idem

Lemma. For any $w \in \widetilde{W}$ and any closed finite union of I-orbits X, the morphisms

$$q^{-1}(Fl_w) \overset{I}{\times} X \to Fl_G, \quad q^{-1}(X) \overset{I}{\times} Fl_w \to Fl_G$$

are affine.

Proof $I_w :=$ stabilizer in I of $\dot{w} I \in Fl_G$

(so $I/I_w \xrightarrow{\sim} Fl_w$)

Let Y be a closed finite union of I-orbits containing $\dot{w} X$, then

$$q^{-1}(Fl_w) \overset{I}{\times} X \xrightarrow{\sim} I \overset{I_w}{\times} \dot{w} X \xhookrightarrow{\text{closed im.}} I \overset{I_w}{\times} Y \xrightarrow{\sim} \underbrace{I/I_w}_{\text{affine}} \times Y \to Y \xhookrightarrow{\text{closed im.}} Fl_G$$

Idem for $q^{-1}(X) \overset{I}{\times} Fl_w \to Fl_G$.

Wakimoto sheaves

$\lambda \in X_*(T)$, find $\mu, \mu' \in X_*(T)^+$ s.t. $\lambda = \mu - \mu'$.

Bernstein element $\theta_\lambda := h_{\mu'}^{-1} h_\mu \in \mathcal{H}$ (does not depend on the choice of μ, μ')

$$Z(\mathcal{H}) = e[X_*(T)]^W = e[z_\lambda]_{\lambda \in X_*(T)^+}, \quad z_\lambda = \sum_{w \in W_{fin}} \theta_{w(\lambda)}$$

Wakimoto sheaves. $J_\lambda := \nabla_{-\mu'} * \Delta_\mu \in P_I(Fl_G)$

Better: $J_\lambda = \varinjlim_{\mu \in \lambda + X_*(T)^+} \nabla_{\lambda - \mu} * \Delta_\mu \in P_I(Fl_G)$

Goal: Central sheaves admit Wakimoto filtration.

Prop. For $\mathcal{F} \in P_I(\mathcal{F}\ell_G)$, if

i) for any $\nu \in -X_*(T)_+$, we have an isom.

$$\Delta_\nu * \mathcal{F} \cong \mathcal{F} * \Delta_\nu$$

and moreover these objects are perverse

ii) for any $\nu \in X_*(T)_+$, $\nabla_\nu * \mathcal{F}$ is perverse.

then $\mathcal{F}$ admits a Wakimoto filtration.

$$Z : P_{L^+G}(Gr_G) \to P_I(\mathcal{F}\ell_G)$$

Admitting prop, for any $\mathcal{G} \in P_{L^+G}(Gr_G)$, $Z(\mathcal{G})$ admits Wakimoto filtration.

Proof of prop: Some quick observations:

- $J_\lambda * J_{\lambda'} \cong J_{\lambda+\lambda'}$ + compatibilities
- If $\lambda \in X_*(T)^+$, then $J_\lambda \cong \Delta_\lambda$

  If $\lambda \in -X_*(T)^+$, then $J_\lambda \cong \nabla_\lambda$

For $\mathcal{F} \in D^{\cdot}$, set
$$*\text{-}Supp(\mathcal{F}) := \{ w \in \widetilde{W} : j_w^*(\mathcal{F}) \neq 0 \}$$
$$!\text{-}Supp(\mathcal{F}) := \{ w \in \widetilde{W} : j_w^!(\mathcal{F}) \neq 0 \}$$

We say $\mathcal{F} \in P_I(\mathcal{F}\ell_G)$ has a dominant Wakimoto filtration, if it admits a finite filtration whose subqts are of the form J_λ, $\lambda \in X_*(T)_+$.

<u>Observation</u>. A perverse sheaf $\mathcal{F} \in P_I(\mathcal{F}\ell_G)$ admits a Wakimoto filtration iff $\exists\ \lambda \in X_*(T)_+$ s.t. $\nabla_\lambda * \mathcal{F}$ is perverse and admits a dominant W. filtration.

<u>Lemma</u>. Let $\mathcal{F} \in D$. TFAE

1) $\mathcal{F}$ is a perverse sheaf admitting a dominant Wakimoto filtration

2) ${}^p\mathcal{H}^n(j_w^!(\mathcal{F})) \neq 0 \Rightarrow n = 0$ and $w \in X_*(T)_+$.

<u>Proof</u>. 1) $\Rightarrow$ 2) easy

2) $\Rightarrow$ 1) Do induction on $\#(!\text{-}\mathrm{Supp}(\mathcal{F}))$.

$!\text{-}\mathrm{supp}(\mathcal{F}) = \phi \Rightarrow \mathcal{F} = 0$

perverse by assumption

$!\text{-}\mathrm{supp}(\mathcal{F}) \neq \phi$, pick a max'l elt $v \in !\text{-}\mathrm{supp}(\mathcal{F})$, then $\mathcal{F}\ell_v$ is open in the support of $\mathcal{F}$.

$$\text{Set } M = \mathrm{Hom}_{D_{(I)}(\mathcal{F}\ell_G)}(\Delta_v, \mathcal{F}) \cong \mathrm{Hom}_{D_{Loc}(\mathcal{F}\ell_v)}(\underline{e}[\ell(v)], \underbrace{j_v^! \mathcal{F}}_{= j_v^*})$$

$\cong \mathrm{Hom}_{\mathrm{Lisse}\,P(\mathcal{F}\ell_v)}(\underline{e}[\ell(v)], j_v^! \mathcal{F})$

$$\Rightarrow \mathcal{F}|_{\mathcal{F}\ell_v} \cong \underline{M}_{\mathcal{F}\ell_v}[\ell(v)], \text{ and a canonical map } \theta: \mathcal{F} \longrightarrow \underbrace{\nabla_v(M)}_{= j_{v*}(\underline{M}[\ell(v)])}$$

Dist. $\triangle$ $\mathcal{F}' \longrightarrow \mathcal{F} \xrightarrow{\theta} \nabla_v(M) \xrightarrow{+1}$. Since θ is an isom. over $\mathcal{F}\ell_v$,

$$j_w^! \mathcal{F}' \cong \begin{cases} j_w^! \mathcal{F}, & w \neq v \\ 0, & w = v \end{cases}$$

So $\mathcal{F}'$ satisfies (2) as well, but $!$-$\mathrm{Supp}(\mathcal{F}')$ is smaller.

By induction, $\mathcal{F}'$ is a perverse sheaf admitting a dom. W. filtration.

By assumption, $v \in X_*(T)_+ \Rightarrow \nabla_v(M) \cong J_v(M)$.

So the dist. Δ $\mathcal{F}' \longrightarrow \mathcal{F} \xrightarrow{\theta} J_v(M) \xrightarrow{+1}$ shows that $\mathcal{F}$ is a perverse sheaf admitting a dom. Wakimoto filtration.

<u>Lemma</u> Let $X \subset Fl_a$ be a finite union of I-orbits, then there exists a finite subset $A_X \subset \tilde{W}$ s.t. $\forall x \in \tilde{W}$, we have

$$m\left(q^{-1}(Fl_x) \overset{I}{\times} X\right) \subset \bigcup_{y \in A_X} Fl_{xy},$$

$$m\left(q^{-1}(X) \overset{I}{\times} Fl_x\right) \subset \bigcup_{y \in A_X} Fl_{yx}$$

Pf. WLOG $X = Fl_w$ for some $w \in \tilde{W}$. Do induction on $\ell(w)$.

Algebro-geometric version of the corresponding Bruhat decomposition.

Cor. For any $\mathcal{F} \in D$, $\exists$ a finite subset $A_{\mathcal{F}} \subset \tilde{W}$ s.t. $\forall x \in \tilde{W}$,

$*$-$\mathrm{Supp}(\Delta_x * \mathcal{F}) \subset x \cdot A_{\mathcal{F}}$, $!$-$\mathrm{supp}(\nabla_x * \mathcal{F}) \subset x \cdot A_{\mathcal{F}}$

$*$-$\mathrm{supp}(\mathcal{F} * \Delta_x) \subset A_{\mathcal{F}} \cdot x$, $!$-$\mathrm{supp}(\mathcal{F} * \nabla_x) \subset A_{\mathcal{F}} \cdot x$

Proof

Pf. Let $X \subset Fl_a$ be a closed finite union of I-orbits s.t. $\mathcal{F}$ is supp. on X.

Take $A_{\mathcal{F}} = A_X$.

By base change,

$$\begin{tikzcd} & Fl_w \arrow[d, "j_w"] \\ q^{-1}(Fl_x) \overset{I}{\times} X \arrow[r, "m"] & Fl_a \end{tikzcd}$$

$$j_w^*(\Delta_x * \mathcal{F}) = 0$$

unless $Fl_w \subset m\left(q^{-1}(Fl_x) \overset{I}{\times} X\right)$.

so $*$-Supp$(\Delta_x * \mathcal{F}) \subset x \cdot A_X = x \cdot A_{\mathcal{F}}$.

Other statements idem.

Lemma. For any $\mathcal{F} \in D$,

- $\mathcal{F}$ belongs to the subcat. of D generated under extension by the objects

  $j_{w!} j_w^* \mathcal{F}$ where w runs over $*$-Supp$(\mathcal{F})$.

- $\mathcal{F}$ belongs to the subcat. of D generated under extension by the objects

  $j_{w*} j_w^! \mathcal{F}$ where w runs over $!$-supp$(\mathcal{F})$.

Pf. Do induction on $*$-supp$(\mathcal{F})$ (resp. $!$-supp$(\mathcal{F})$)

Now we are ready to prove the criterion for Wakimoto filtration.

Let $A_{\mathcal{F}} \subset \widetilde{W}$ be the finite set associated to $\mathcal{F}$.

Pick $\nu \in -X_*(T)_+$, $\nu \ll 0$ s.t.

$$\nu \cdot A_{\mathcal{F}} \subset (-X_*(T)_+) \cdot W_{fin}$$

$$A_{\mathcal{F}} \cdot \nu \subset W_{fin} \cdot (-X_*(T)_+)$$

$$(\nu \cdot A_{\mathcal{F}}) \cap (A_{\mathcal{F}} \cdot \nu) \subset (-X_*(T)_+) \cdot W_{fin} \cap W_{fin} \cdot (-X_*(T)_+) \overset{\text{fact}}{=} -X_*(T)_+$$

$$\nu \in -X_*(T)_+ \Rightarrow J_\nu \cong \Delta_\nu$$

$$\Rightarrow *\text{-Supp}(J_\nu * \mathcal{F}) = *\text{-Supp}(\Delta_\nu * \mathcal{F}) \subset \nu \cdot A_{\mathcal{F}}$$

$$= *\text{-Supp}(\mathcal{F} * \Delta_\nu) \subset A_{\mathcal{F}} \cdot \nu$$

$$\Rightarrow *\text{-supp}(J_\nu * \mathcal{F}) \subset -X_*(T)_+$$

On the other hand, $J_\nu * \mathcal{F} \cong \Delta_\nu * \mathcal{F}$ is perverse,

$\Rightarrow j_w^*(J_\nu * \mathcal{F})$ is concentrated in degrees $\leq -\ell(w)$.

$\Rightarrow J_\nu * \mathcal{F}$ belongs to the full subcat. of D gen. under ext'n by

$\Delta_\mu[n]$, $\mu \in -X_*(T)_+$, $n \in \mathbb{Z}_{\geq 0}$.
$\;\;\;\;\wr\wr$
$J_\mu[n]$

$\Rightarrow \mathcal{F} \simeq J_{-\nu} * (J_\nu * \mathcal{F})$ can be obtained from some objects

$(J_{\mu_i}[n_i])_{i \in I}$, I finite index set, $\mu_i \in X_*(T)$, $n_i \in \mathbb{Z}_{\geq 0}$

by taking successive extensions.

Take $\eta \in X_*(T)_+$, $\eta >> 0$ s.t. $\mu_i + \eta \in X_*(T)_+$, $i \in I$.

Then $J_\eta * \mathcal{F}$ belongs to the full subcat. of D gen. under ext'ns by $J_\mu[n]$, $\mu \in X_*(T)_+$, $n \in \mathbb{Z}_{\geq 0}$.

$$\begin{array}{l} J_\mu[n] \\ \parallel \\ \nabla_\mu[n] \end{array}$$

$\therefore$!-supp $(J_\eta * \mathcal{F}) \subset X_*(T)_+$, and for each $\mu \in X_*(T)_+$,

$j_\mu^!(J_\eta * \mathcal{F})$ is concentrated in deg $\leq -\ell(\mu)$.

On the other hand, $J_\eta * \mathcal{F}$ is perverse by assumption

$\Rightarrow j_\mu^!(J_\eta * \mathcal{F})$ is concentrated in deg $\geq \ell(\mu)$

$\therefore$ $j_\mu^!(J_\eta * \mathcal{F})$ is concentrated in deg $\ell(\mu)$, it satisfies the cond'n of previous lemma $\Rightarrow$ $J_\eta * \mathcal{F}$ is a perverse sheaf admitting dom. W. filt'n

$\Rightarrow$ $\mathcal{F}$ admits Wakimoto filtration.