On vanishing cycles and duality, after A. Beilinson

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S strictly local trait, $s, \eta, \bar\eta$, $I = \mathrm{Gal}(\bar\eta|\eta)$.

$\ell \neq \mathrm{char}(s)$, $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$, $\nu \geq 1$

X/S

$$\begin{tikzcd} & & X_{\bar\eta} \arrow[dl, "\bar j"'] \arrow[d, "\bar\eta"] \\ X_s \arrow[r, "i"] & X & X_\eta \arrow[l, "j"'] \end{tikzcd}$$

$L \in D^+(X, \Lambda)$

$$R\Psi(L) = i^* R\bar j_*(L|X_{\bar\eta}) \in D^+(X_s, \Lambda[I])$$

$M \in D^+(X,\Lambda)$, $R\Phi(M) := \mathrm{Cone}\,(i^*M \to R\Psi(M)) \in D^+(X_s, \Lambda)$.

Deligne: $R\Psi : D^b_c \longrightarrow D^b_c$

$\Psi L := i_* R\Psi L[-1] \in D^+(X, \Lambda[I])$

$\Phi L := i_* R\Phi L[-1] \in D^+(X, \Lambda[I])$

$K_S = \Lambda_S(1)[2]$, $a_X : X \to S$, $K_X = a_X^! K_S$

$\mathbb{D} = R\mathcal{H}om(-, K_X)$

Gabber: ~1982 • $(\Psi \mathbb{D} L)(-1) \xrightarrow{\sim} \mathbb{D}\Psi L$ $\quad (L \in D^b_c(X_\eta, \Lambda))$

$(\Rightarrow \Psi$ t-exact$)$

• $\Phi : \mathrm{Perv}(X) \longrightarrow \mathrm{Perv}(X)$

$\Phi D = D\Phi(\ldots)$?

1986, Beilinson "maximal extension". M. Saito

Duality for φ

$$1 \to P' \to I \to \mathbb{Z}_\ell(1) \to 1$$

$\mathcal{F}$ I-module, $\mathcal{F}_t = \mathcal{F}^{P'}$ "ℓ-tame" $\qquad$ $\mathbb{Z}_\ell(1)$ sheaf.

$\mathcal{F}^{P'} = k\mathcal{F}$, $\quad k = \frac{1}{(P')} \sum_{g \in P'} g \quad$. $\mathcal{F} = \mathcal{F}_t \oplus \mathcal{F}_{nt}$

$\Psi = \Psi_t \oplus \Psi_{nt}$, $\quad \varphi = \varphi_t \oplus \varphi_{nt}$.

$$\begin{tikzcd} & & X_{\eta_t} \arrow[dll, "j_t"'] \arrow[r] \arrow[d] & \eta_t = \varprojlim \eta[\pi^{\ell^{-n}}] \arrow[d, "q"] \\ X_s \arrow[r, hook, "i"] & X & X_\eta \arrow[l, hook', "j"'] \arrow[r] & \eta \end{tikzcd}$$

π unif.

$J = q_* \Lambda$ "infinite Jordan block"

$L \in D^+(X_\eta, \Lambda)$, $\quad \varphi_t L = i_* i^* j_{t*}(L | X_{\eta_t})[-1] = i_* i^* j_*(J \otimes L)[-1]$

$\in D^+(X, R)$

$R = \Lambda[\![\mathbb{Z}_\ell(1)]\!]$ $\left(\simeq \Lambda[\![t]\!], t = 1 - \sigma, \sigma \in \mathbb{Z}_\ell(1)\right)$

$\varphi_t(M) \in D^+(X, R)$

$\left(J_{\eta_t} = R[t^{-1}]/R\right)$

$$0 \to \underset{(\simeq tR)}{R(1)^\tau} \to R \to \Lambda \to 0$$

$$R(n)^\tau := \left(R(1)^\tau\right)^{\otimes n}, \quad n \in \mathbb{Z}$$

$$\begin{array}{cc} t^n & R(n)^\tau \\ \uparrow & \wr\uparrow \\ 1 & R \end{array}$$

$n \geq m$, $\quad R(n)^\tau \subset R(m)^\tau$

$R \hookrightarrow R(-1)^\tau$

$\mathcal{F}$ R-module, $\quad \mathcal{F}(n)^\tau := \mathcal{F} \otimes_R R(n)^\tau$

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\beta"] \arrow[dr, bend right, "t"'] & \mathcal{F}(-1)^\tau \arrow[d, "\wr"] \\ & \mathcal{F} \end{tikzcd}$$

monodromy

Th. 1 For $M \in D^b_c(X, \Lambda)$, there exists a canonical functorial isom.

$$(\psi_t \mathbb{D} M)(1)^{\tau}(-1) \xrightarrow{\sim} \mathbb{D}\psi_t M.$$

Cor $\mathbb{D}\psi M \simeq (\psi_t \mathbb{D}M)(1)^{\tau}(-1) \oplus \psi_{nt} \mathbb{D} M_\eta(-1)$

$$\overset{\text{choose } t}{\approx} (\psi \mathbb{D} M)(-1)$$

Cor. ψ t-exact.

ψ_t and Beilinson's Ξ

$A \longrightarrow B$, $\operatorname{cocone}(A \to B) = \operatorname{cone}(A \to B)[-1]$

$$0 \to \Lambda \to J \to J(-1)^{\tau} \to 0$$

$L \in D^+(X_\eta, \Lambda)$, $\quad J_! L = \operatorname{cocone}(j_!(J \otimes L) \to j_!(J \otimes L)(-1)^{\tau})$

$\qquad J_* L = \operatorname{cocone}(j_*(J \otimes L) \to j_*(J \otimes L)(-1)^{\tau})$

$$\psi_t L = \operatorname{cocone}(j_!(J \otimes L) \to j_*(J \otimes L))$$

$$\begin{tikzcd} j_*(J\otimes L) \arrow[r, "\beta"] & j_*(J\otimes L)(-1)^{\tau} \\ j_!(J\otimes L) \arrow[u] \arrow[r, "\beta"] \arrow[ur, "\gamma"] & j_!(J\otimes L)(-1)^{\tau} \arrow[u] \end{tikzcd}$$

Def. $\Xi L = \operatorname{cocone}(\gamma : j_!(J \otimes L) \to j_*(J \otimes L)(-1)^{\tau})$

"max'l extension" $\quad \Xi : D^+(X_\eta, L) \to D^+(X, R)$

(1) $j_! L \to \xi L \to \Psi_t L(-1)^{\tau} \longrightarrow$

(2) $\Psi_t L \to \xi L \to j_* L \longrightarrow$

$\xi : D^b_c \longrightarrow D^b_c$, perverse $\to$ perverse

$X = S$, $L = \underline{\Lambda}$: $\xi \Lambda_\eta \longleftrightarrow c \in H^2_s(S, \Lambda(1))$, $c = cl(s)$

$$\mathrm{Im}\left(\beta : \xi L(1)^{\tau} \to \xi L\right) = \Psi_t L$$

$\xi \longleftrightarrow \Psi_t$: $M \in D^+(X, \Lambda)$,

$$b(M) = \begin{pmatrix} M & \longrightarrow & j_* M_\eta \\ \uparrow & & \uparrow \\ j_! M_\eta & \longrightarrow & \xi M_\eta \end{pmatrix} \qquad [0,1] \times [-1,0]$$

Prop (Beilinson): $\underline{\underline{s}}\, b(M) = \Psi_t M$.

$$\begin{array}{ccccc} i^* M & \longrightarrow & R\Psi_t M & & \\ \uparrow & & \uparrow & & \\ M & \longrightarrow & j_*(J \otimes M_\eta) & \longrightarrow & j_*(J \otimes M_\eta)(-1)^{\tau} \\ \uparrow & & \uparrow & & \| \\ j_! M & \longrightarrow & j_!(J \otimes M_\eta) & \longrightarrow & j_*(J \otimes M_\eta)(-1)^{\tau} \end{array}$$

Th. 2 For $L \in D^b_c(X_\eta, \Lambda)$, there exists a canonical functorial isom

$$(\xi \mathbb{D} L)(1)^{\tau}(-1) \xrightarrow{\sim} \mathbb{D} \xi L$$

(compatible w/ (1), (2), $\Psi_t \mathbb{D} L(-1) \simeq \mathbb{D} \Psi_t L$)

Th 2 $\Rightarrow$ Th 1.

$N \in D^+(X, \Lambda)$, $\quad b^-(N) = \begin{pmatrix} \xi N_\eta & \longrightarrow & j_* N \\ \uparrow & & \uparrow \\ j_! N_\eta & \longrightarrow & N \end{pmatrix}$ $\quad [-1,0] \times [0,1]$

$\xi\, b^-(N) = \Psi_t N \qquad \mathbb{D}\, b(M) = b^{-1}(\mathbb{D}M)\,(1)^\tau(-1) \Rightarrow \xi\,\mathbb{D} = \cdots$

$R(1)^\tau / R(2)^\tau = \Lambda(1) \qquad \mathcal{F}(n)^\tau = \mathcal{F}(n)$ if trivial $\mathbb{Z}_\ell(1)$-action.

$$\begin{array}{l} \Psi_t M_\eta \xrightarrow{\text{can}} \Psi_t M \longrightarrow i_* i^* M \longrightarrow \\ \mathbb{D} \\ i_* i^! M(1) \longrightarrow \Psi_t(M)(1)^\tau \xrightarrow{\text{var}} \Psi_t M_\eta \longrightarrow \end{array}$$

$\Psi_t(M)(1)^\tau = \text{cocone}(0, \beta)$

Proof of th. 2

$J = q_* \Lambda, \quad q : \eta_t \to \eta, \quad J = \varinjlim_m J_m,$

$J_m = q_{m*} \Lambda, \quad \eta_t \to \eta_m \xrightarrow{q_m} \eta, \quad \eta_m = \eta[\pi^{\ell^{-m}}] \qquad J_m = \check{J}_m$

(joint w. Zheng)

Lemma. $R\,\mathcal{H}om(J, \Lambda) = J(-1)[-1]$ canonically.

$R\,\mathcal{H}om(J, \Lambda) = R\varprojlim J_m$

$R\,\mathcal{H}om(J, \Lambda)_{\eta_t} = \varinjlim_n R\varprojlim_m R\Gamma(\eta_n, J_m)$

$``\varprojlim_m" H^0(\eta_n, J_m) = 0, \quad ``\varprojlim_m" H^1(\eta_n, J_m) \xrightarrow{\sim} R_n(-1)$

$$J \otimes J \longrightarrow \Lambda(1)[1]$$

Y/η f.t. $\quad L \in D^+(Y, \Lambda)$, $\quad (J \otimes L) \otimes (J \otimes \mathbb{D}L) \longrightarrow K_Y(1)[1] \qquad (*)$

X/S, $\quad L \in D^+(X_\eta, \Lambda) \quad (**)\ j_!(J \otimes L) \otimes j_*(J \otimes \mathbb{D}L) \longrightarrow K_X(1)[1]$.

Th. 3. For $L \in D_c^b$, $(*)$, $(**)$ perfect.

Proof $(*)$ perfect, reduce to $Y = \eta$.

$$(**) \left[\begin{array}{lll} (a) & j_*(J \otimes \mathbb{D}L)(-1)[-1] \longrightarrow \mathbb{D} j_!(J \otimes L) & \simeq ? \\ (b) & j_!(J \otimes \mathbb{D}L)(-1)[-1] \longrightarrow \mathbb{D} j_*(J \otimes L) & \simeq ? \end{array} \right.$$

$(*) + \mathbb{D} j_! = j_* \mathbb{D} \Rightarrow (a)$ isom.

For (b), combine (a) w/ Gabber's $\Psi_t \mathbb{D}L(-1) \xrightarrow{\sim} \mathbb{D} \Psi_t L$

$$\Psi_t L = \mathrm{cone}\left(j_!(J \otimes L) \to j_*(J \otimes L)\right)$$

Th. 3. $\Rightarrow$ $\xi\, \mathbb{D}\, (1)^c(-1) = \mathbb{D}\, \xi$ ...

$f : \mathrm{Spec}(\overline{\mathbb{F}}_q) \longrightarrow \mathrm{Spec}(\mathbb{F}_q)$

$J = f_* \Lambda$, $\quad R\underline{\mathrm{Hom}}(J, \Lambda) \overset{?}{=} J[-1]$

$\eta_{\mathrm{proet}} \xrightarrow{\nu} \eta_{\mathrm{et}}$, $\quad \varprojlim_m \nu^* J_m =: \check{J}$

$R\nu_*(\check{J})$ $\quad$ R-torsion

Applications

$$\mathbb{D}(LA) = LA$$

$$SS(\mathbb{D}\mathcal{F}) = SS(\mathcal{F})$$ X/k smooth

$$CC(\mathbb{D}\mathcal{F}) = CC(\mathcal{F})$$ $\mathcal{F} \in D_{ctf}(X, \Lambda)$

Gabber: can generalize over general bases

$$LA = ULA$$

$$\mathbb{D}(LA) = LA$$

S regular