Spherical dual group

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(G, M) hyperspherical $\rightsquigarrow$ $(\check{G}, \check{M})$ ($\check{G}$ as usual)

<u>Expectation</u>: There exists a bijection

$$\{(G, M)\} \longleftrightarrow \{(\check{G}, \check{M})\}$$

between anomaly-free hypersphericals over $\mathbb{C}$ w/ nice properties

e.g. for polarized hyperspherical

$$\begin{array}{ccc} Shv(LX/L^+G) & \simeq & QCoh^{\sharp}(\check{M}/\check{G}) \\ \circlearrowright & & \circlearrowright \\ \text{Hecke} & \xleftrightarrow[\text{derived Satake}]{} & D^{\sharp}_{perf}(\check{g}/\check{G}) \end{array}$$

When $M = \mathrm{Whit\,Ind}(H, sl_2, S)$

polarized + eigenmeasure

$\rightsquigarrow (\check{G}_X, sl_2 \to \check{g}, S_X)$

<u>Thm</u>. Can construct S_X & it is self-dual

<u>Conj</u>. S_X supports a sympl. str. $\rightsquigarrow \check{M} = \mathrm{Whit\,Ind}(\check{G}_X, sl_2, S_X)$

§. Structure of spherical var.

Def A <u>spherical var.</u> is a var. $X \circlearrowleft G$ s.t.

$\forall B$ / fix B, there exists a (unique) open B-orbit.

Ex. If $G = B = T \twoheadrightarrow T'$ s.t. open T-orbit is a T'-torsor

$\Rightarrow$ toric var. for T'.

First want to understand $H = G_x$ for $x \in$ open B-orbit.

Fix $B \subset G$, $\quad \underbrace{X^\circ_B}_{\text{open } B\text{-orbit}} \subset \underbrace{X^\circ_G}_{\text{open } G\text{-orbit}} \subset X$

$P(X) :=$ stabilizer of $X^\circ_B \supset B$

Fact (Knop). $U(X) =$ unip. radical of $P(X)$ acts freely on X°_B.

$$L(X) = P(X)/U(X) \curvearrowright X^\circ_B/U(X)$$

$$L(X) \twoheadrightarrow \underset{\text{torus}}{A_X}, \qquad A_X \curvearrowright X^\circ_B/U(X) \text{ torsor}$$

$$\underset{P(X)\text{-equiv.}}{X^\circ_B \simeq} T_X \times U(X) \overset{\simeq\, \backslash P(X)}{\underset{L(X)}{}} \text{ for some embedding } L(X) \hookrightarrow P(X)$$

& A_X-torsor T_X.

$\underline{Ex}.$ $G = GL_n \curvearrowright X = \mathbb{A}^n$

$$\begin{gathered} X = \mathbb{A}^n \\ \cup \\ X^\circ_G = \mathbb{A}^n - \{0\} \\ \subset \\ X^\circ_B = \mathbb{A}^n - \{x_n \neq 0\} \end{gathered} \qquad B = \begin{pmatrix} * & & * \\ & * & \\ 0 & & * \end{pmatrix}$$

$P(X) = \Big\{$

$= P_{n-1,1}$

$L(X) = GL_{n-1} \times GL_1 \xrightarrow{pr_2} A_X = \mathbb{G}_m$

$\underline{Ex}.$ $G = GL_n \curvearrowright X = G/H = GL_{n-1} \cong \left\{ (x, V) : \begin{array}{l} x \in \mathbb{A}^n,\ V \subset \mathbb{A}^n \text{ codim } 1, \\ 0 \neq x,\ x \notin V \end{array} \right\}$

$\subset$

$X^\circ_B = \left\{ \begin{array}{l} x_n \neq 0 \\ e_1 \notin V \end{array} \right\}$

$P(X) = P_{1,n-2,1}$

$L(X) = GL_1 \times GL_{n-2} \times GL_1 \xrightarrow{pr_{1,3}} \mathbb{G}_m^2 = A(X)$

(G, M) hyperspherical $\rightsquigarrow$ $(\check{G}, \check{M})$

$\wr$ $\qquad\qquad\qquad\qquad$ $\hat{\wr}$ $\qquad\qquad$ $\Psi = S^+ \overset{HU'}{\times} G$, $\mathbb{G}_a$-torsor

$(H, \mathfrak{sl}_2 \to \mathfrak{g}, S) \rightsquigarrow (\check{G}_X, SL_2 \to \check{G}, S_X)$ $\quad$ modification $\qquad$ $U' = \ker(U \to \mathbb{G}_a)$

$M = T^*(X, \Psi)$.

$\wr$ $\qquad\qquad\qquad\qquad$ $\wr$

$X = S^+ \overset{HU}{\times} G$ spherical $\rightsquigarrow$ $(\check{G}_X, SL_2 \to \check{G}, S_X)$

§. Structure theory of spherical vars

$B \subset G \qquad X \supset X_G^\circ \supset X_B^\circ$ open orbits

$P(X) =$ stabilizer of $X_B^\circ \supset B$

$$X_B^\circ \simeq T(X) \times U(X)$$

$T(X)$: $A(X)$-torsor, $\circlearrowright$ by $L(X)$, Levi of $P(X)$

$U(X)$: unipotent radical of $P(X)$

respecting $P(X)$-action

$\mathbb{Q}$- valued valuations $K(X) \to \mathbb{Q} \cup \{-\infty\}$

$$K(X)^{(B)} = \{B\text{- eigen func. of } K(X)\} = \bigoplus_{\chi \in X^*(A(X))} \mathbb{C}$$

$$\begin{tikzcd} B \arrow[rr, "\chi"] \arrow[dr] & & \mathbb{G}_m \\ & T \arrow[ur] & \end{tikzcd}$$

$\chi(U(X)) = 1$ pulled back from $T(X)$

$$\{G\text{-inv valuations of } K(X)\} \longrightarrow \{\text{hom. } X^*(A(X)) \to \mathbb{Q}\} = X_*(A(X))_{\mathbb{Q}}$$

<u>Fact</u>. This map is injective.

$V(X) =$ image $=$ convex cone in $X_*(A(X))_{\mathbb{Q}}$ (negative Weyl chamber)

<u>Def</u>. A <u>spherical root</u> is an elt of $X^*(A(X))$ that annihilates a face of $V(X)$ s.t. $V(X)$ is negative & is primitive in $X^*(A(X)) \cap \mathbb{Z}\Delta$

$$\mathbb{Z}\Delta \subset X^*(T)$$

$\Delta(X) = \{$spherical roots$\}$

$\chi : A(X) \longrightarrow \mathbb{G}_m$ $\rightsquigarrow$ B- eig. func. f_χ (up to scalar)
$(P(X)\text{-})$

$d \log f_\chi \in H^0(X_B^\circ, T^* X_B^\circ)$

$\rightsquigarrow$ $\mathbb{C}$-linear combinations

$$\mathfrak{a}(X)^* \times X_B^\circ \longrightarrow T^* X_B^\circ$$

Start conjugating $P(X)$

$$\mathfrak{a}(X)^* \times \underbrace{(X_G^\circ \times \underset{\substack{\uparrow \\ \text{flag for } P(X)}}{\mathcal{P}})^\circ}_{\substack{\text{universal open} \\ \text{for } X^\circ_{P(X)}}} \xrightarrow{\varphi} T^* X_G^\circ$$

<u>Thm</u> (Knop) $\mathrm{Im}(\varphi)$ is dense & taking quotient by G induces an isom.

$$\mathfrak{a}(X)^* /\!/ W_X \xrightarrow{\sim} T^* X_G^\circ /\!/ G \quad \text{for some finite gp } W_X \curvearrowright \mathfrak{a}(X)^*.$$

$\Rightarrow$ define $\Delta^\vee(X)$ as well.

<u>Thm</u> $\left(X^*(A(X)), \Delta(X), X_*(A(X)), \Delta^\vee(X) \right)$ forms a root datum.

<u>Def</u> $\check{G}_X$ corresponds to $(X_*, \Delta^\vee, X^*, \Delta)$.

<u>Ex.</u> $X = \mathbb{A}^n \curvearrowleft G = GL_n$ (row vectors), B

$P(X) = P_{1,n-1}$

$L(X) = \mathbb{G}_m \times GL_{n-1} \xrightarrow{pr_1} \mathbb{G}_m = A(X)$

$X_G^\circ = \mathbb{A}^n - \{0\}$

$X_B^\circ = \{x_1 \neq 0\} \subset \mathbb{A}^n - \{0\}$ $\qquad (x_1, x_2, \dots, x_n) \mapsto \left(x_1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right)$

$\xrightarrow{\sim} \underset{T(X)}{\underset{\shortparallel}{\mathbb{G}_m}} \times \underset{U(X)}{\underset{\cong}{\mathbb{A}^{n-1}}}$

B- eigenfunctions: powers of x_1

G- inv. val'ns on $\mathbb{A}^n$

$\mathbb{P}^n \supset \mathbb{A}^n$ <u>boundary divisor</u> $\rightsquigarrow$ val. val_b $\quad \mathrm{val}_b(x_1) = -1$

$\circlearrowright GL_n$

$\mathbb{A}^n \leftarrow Bl_{\{0\}} \mathbb{A}^n$ except'l div. $\rightsquigarrow$ val $\quad \mathrm{val}_e$ $\quad \mathrm{val}_e(x_1) = 1$

$$\Rightarrow \mathcal{V}(X) = X_*(A(X))_{\mathbb{Q}}$$

$$\Delta(X) = \phi$$

Ex. $X = GL_{n-1} \backslash GL_n \supset G = GL_n$

$\quad \|$

$\{(x,H) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x \notin H\}$ $\rightarrow \exists! \ \xi$ linear s.t. $\xi(H)=0, \xi(x)=1$

$X_B^o = \{(x,H) : \begin{matrix} x \notin H \\ x_1 \neq 0 \end{matrix}, \ e_n \notin H\}$

$p(X) = P_{1,n-2,1}$

$\downarrow$

$\mathbb{G}_m^2 \times \mathbb{A}^{2n-3} \quad \left(x_1, \xi(e_n)^{-1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}, \frac{\xi(e_2)}{\xi(e_n)}, \dots, \frac{\xi(e_{n-1})}{\xi(e_n)}\right)$

$L(X) = GL_1 \times GL_{n-2} \times GL_1 \xrightarrow{pr_{1,3}} A(X)$

eigen func. $x_1, \xi(e_n)^{-1}$ $\qquad$ eval. eig. func.

G- inv. val'ns on $K(X)$

① div. @ ∞ for x $\qquad (-1,-1)$

② exceptional for x after blow up $\qquad (1, 1)$

③ div $\{x \in H\}$ $\qquad (0, 1)$

$X_B^o \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$

$$\mathcal{V}(X) = \{(\alpha,\beta) : \alpha \leq \beta\} \subset \mathbb{Q}^2$$

$\Delta(X) = \{(1,-1)\} \hookrightarrow X^*(T)$

Ψ

$\mapsto \delta_1 + \delta_2 + \cdots + \delta_{n-1}$ highest root

$(SL_2 \to GL_n) = \rho_{triv} \oplus Sym^{n-3} \rho_{std} \oplus \rho_{triv}$

Rmk If $G = PGL_4$, $H = PSp_4$, $X = H \backslash G$

$\Rightarrow \Delta(X) = \{\delta_1 + 2\delta_2 + \delta_3\} \not\subseteq \Phi$

$SL_2 \to \check{G}$:

$\Delta_{L(X)} \subset \Delta$: roots appearing in $L(X)$

$\exists\, SL_2 \to \check{G}$ s.t. max'l $\mathbb{G}_m$ maps as $2\rho_{L(X)} \in X^*(T) = X_*(\check{T})$

§. The dual gp.

Strategy : ① Construct a subgp $\hat{G}_X \subset \check{G}$ coming from sub root datum of Φ

② Construct $\check{G}_X \to \hat{G}_X$.

Prop. Let $\sigma \in \Delta(X) - \Phi$ be a spherical root. Then $\exists!$ a subset $\{\gamma_1, \gamma_2\} \subset \Phi_+$ s.t.

① $\gamma_1 + \gamma_2 = \sigma$

② $(\mathbb{Q}\gamma_1 + \mathbb{Q}\gamma_2) \cap \Phi = \{\pm\gamma_1, \pm\gamma_2\}$

③ $\gamma_1^\vee - \gamma_2^\vee$ is of the form $\delta_1^\vee - \delta_2^\vee$ for $\delta_1, \delta_2 \in \Delta$.

[e.g. $\delta_1 + 2\delta_2 + \delta_3 = (\delta_1 + \delta_2) + (\delta_2 + \delta_3)$]

For each $\sigma \in \Delta(X)$, define

$$\underset{\Phi^\vee}{\overset{\hat{\sigma}}{\cap}} = \begin{cases} \{\sigma^\vee\}, & \sigma \in \Phi_+ \\ \{\gamma_1^\vee, \gamma_2^\vee\}, & \sigma \notin \Phi_+ \end{cases} \qquad \hat{\Delta}(X) = \bigcup_{\sigma \in \Delta(X)} \hat{\sigma} \subset \Phi^\vee.$$

Thm. $\hat{\Delta}(X) \subset \Phi^\vee$ generates an additively closed subroot system.

Def $\hat{G}_X \subset \check{G}$ corresponding to $\hat{\Delta}(X)$.

Def For general root datum $(\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$

an involution is an involution $S: \Delta \longrightarrow \Delta$ s.t.

① $\langle \alpha, S(\alpha)^\vee \rangle = 0$, $\forall\, \alpha \neq S(\alpha)$

② $\langle \alpha - S(\alpha), \beta^\vee + S(\beta)^\vee \rangle = 0$ for all $\alpha, \beta \in \Delta$

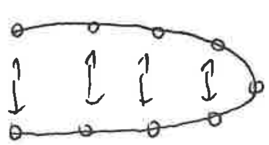
automatic if S comes from an autom. of Dynkin diagram.

Prop Every folding is a disjt union of

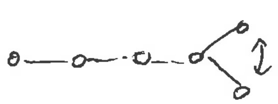
① two components exchanged by isom.

② component w/ triv involution

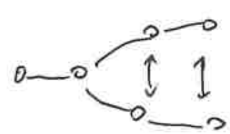
③



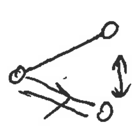
A_{2n+1}



D_n



E_6



B_3

Lemma. Let S be an involution of root syst. of G. Assume $\exists$ lattice Ξ &

$q: X^*(T) \to \Xi$ w/ fin. coker $\rightsquigarrow q^*: \Xi^* \hookrightarrow X_*(T)$

Assume $\tau(\alpha - s(\alpha)) = 0, \ \forall \alpha \in \Delta$

Assume $\bar{\alpha}^\vee := \begin{cases} \alpha^\vee & , \alpha = s(\alpha) \\ \alpha^\vee + s(\alpha)^\vee & , \alpha \neq s(\alpha) \end{cases} \in \tau^*(\Xi^*)$

$\Rightarrow \exists$ conn. red. H w/ root datum $(\Xi, \tau(\Delta), \underset{\underset{X_*(T)}{\cap}}{\Xi^*}, \{\bar{\alpha}^\vee\})$

together w/ $H \to G$ w/ finite central kernel.

$\hat{\Delta}(X) = \bigcup \hat{\sigma}$ where each $\hat{\sigma}$ has 1 or 2 elts $\Rightarrow$ canonical involution

Thm. This is an involution $\rightsquigarrow$ folding gives us $\check{G}_X \to \hat{G}_X$.

$\boxed{(\check{G}_X, SL_2 \to \check{G})}$ for Ψ trivial

$X = S^+ \overset{HU}{\times} G \leftarrow \Psi = S^+ \overset{HU'}{\times} G$

G_a-torsor

$P \to L$ parabolic for $sl_2 \to g$

$\cup$

U unipotent radical

$$\begin{array}{l} X_L = S^+ \overset{HU}{\times} P \subset X \\ \quad\quad \uparrow \quad\quad\quad \simeq S^+ \overset{H}{\times} L \\ \Psi_L = S^+ \overset{HU'}{\times} P \end{array}$$

Claim. X_L is a spherical var. for L.

Consider open $\bar{B}$-orbit

$$X = S^+ \overset{HU}{\times} G \overset{\text{open}}{\supset} \underbrace{S^+ \overset{HU}{\times} P\bar{U} \xrightarrow{\bar{U}\text{-torsor}} S^+ \overset{HU}{\times} P}_{} = X_L$$

preimage of open $\underset{\bar{B}/\bar{U}}{\overset{\shortparallel}{\bar{B}_L}}$-orbit $=$ open $\bar{B}$-orbit multiply by lift of $w_0 \in W$

$$\rightsquigarrow \quad P(X)/U \simeq P(X_L), \quad A(X) \simeq A(X_L)$$

$$U \subset B \subset C$$

Claim. For $\alpha \in \Delta - \Delta_L$, we have $\alpha \in X^*(A(X)) \subset X^*(T)$.

Consider
$$\begin{tikzcd} \hat{\Delta}(X_L) \cup (\Delta - \Delta_L) \arrow[r, squiggly] \arrow[d, "\text{folding}"', "\ ||"] & \hat{G}_{X,\Psi} \subset \check{G} \\ \Delta(X_L) \cup (\Delta - \Delta_L) \arrow[r, squiggly] & \check{G}_{X,\Psi} \arrow[u] \end{tikzcd}$$

Ex $X = \mathbb{A}^n \curvearrowleft G = GL_n$. $\quad \check{G}_X = \mathbb{G}_m \hookrightarrow \check{G} = GL_n$

$$\begin{pmatrix} x & & \\ & 1 & \\ & & \ddots \; 1 \end{pmatrix}$$

$$SL_2 \longrightarrow \check{G} = GL_n$$

$$\rho_{triv} \oplus \mathrm{Sym}^{n-2} \rho_{std}$$

Ex $X = GL_{n-1} \backslash GL_n \curvearrowleft G = GL_n$. $\quad \check{G}_X = GL_2 \xrightarrow{\begin{pmatrix} a & & b \\ & \ddots & \\ c & & d \end{pmatrix}} \check{G}$

(G, M)

$\rotatebox{90}{$\rightsquigarrow$}$

$(G, sl_2 \to \mathfrak{g}, H, S) \quad M = T^*(X, \Psi)$, $\; X = S^+ \times^{HU} G$, $\; \Psi = S^+ \times^{HU'} G$

$\rotatebox{90}{$\rightsquigarrow$}$

$(\check{G}, SL_2 \to \check{G}, \underbrace{\check{G}_X}_{\text{folding construction}} \subset \check{G})$

<u>Remains to construct S_X</u>.

Def. For $G \curvearrowright X$ spherical, affine ($X \supset X^\circ_G$) define the canonical open as $\mathbb{C}[X^\circ_G] = X^{can} \hookrightarrow X$

$$X^\circ_G \nearrow$$

In the twisted case, $X^{can} = \mathbb{C}[(X_L)^{\circ}_L] \overset{P}{\times} G \hookrightarrow X$

$H \subset L \subset P$ $\quad X_L = S^+ \overset{HU}{\times} P$

Levi

$\Psi_L = S^+ \overset{HU'}{\times} P$

spherical L-var.

In both cases, $\Psi^{can} = \Psi|_{X^{can}}$

① Define S_X when $(X, \Psi) = (X^{can}, \Psi^{can})$

② Define in general.

<u>Def</u>. A <u>color</u> is an irred. comp. of $X^{\circ}_G - X^{\circ}_B$ $\Rightarrow$ codim 1

prime Weil divisor $\quad$ smooth $\quad$ affine

Given D a colour $\Rightarrow$ assoc. valuation v, B-invariant ($\mathbb{Z}$-valued)

B-stable

restrict to B-eigenspaces $K(X)^{(B)} \simeq \mathbb{C}[X^*(A(X))]$

$v|_{K(X)^{(B)}}$ defines elt of $X_*(A(X))$

$Col(X) = \{\text{colours}\} \longrightarrow X_*(A(X)) = X^*(\check{A}(X))$

$\Delta \supset \Delta_{L(X)}$

simple roots of G

those appearing in $L(X)$

<u>Def</u>. A subset $R \subset Col(X) \times (\Delta - \Delta_{L(X)})$

$(D, \alpha) \in R \Leftrightarrow DP_\alpha \supset X^{\circ}_B$

$\Leftrightarrow D \cap X^{\circ}_B P_\alpha \neq \phi$.

P_α min. parabolic containing $U_{-\alpha}$.

$\rightsquigarrow X_B^\circ \underbrace{P_\alpha / R(P_\alpha)}_{PGL_2 \sim SL_2}$ radical of P_α $\circlearrowleft P_\alpha / R(P_\alpha)$ is spherical.

rank 1

$D R(P_\alpha) = D/R(P_\alpha)$

a colour

Classification of homogeneous spherical for PGL_2

- Type U: $RU \backslash PGL_2$, where $U \simeq \mathbb{G}_a$, R is fin. subgp $\subset \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$ has 1 colour
- type N: $N(\mathbb{G}_m) \backslash PGL_2$ X
- type T: $\mathbb{G}_m \backslash PGL_2$ has 2 colours
- Type G: $PGL_2 \backslash PGL_2$ X

$\Rightarrow$ fibers of $\begin{matrix} R \\ \downarrow \\ \Delta - \Delta_{L(X)} \end{matrix}$ has size 1 or 2; U or T

Def A parabolic $P \supset B$ is of even spherical type when $(P/R(P), X_B^\circ P / R(P))$ is isom. to either $(SO_{2n+1}, SO_{2n} \backslash SO_{2n+1})$ or $(G_2, SL_3 \backslash G_2)$

A colour is of even spherical type if it meets $X_B^\circ \cdot P$ for some even spherical P.

$C_X = \{\text{even spherical type colours}\}$

$\downarrow$

$X_*(A(X))$

Consider the image & take their dominant W_X-translates

$\rightarrow D_X \subset X_*(A(X))$.

Def. $D_X^{max} \subset D_X$ consisting of max'l elts in the Bruhat order (integral)

$$S_X = \bigoplus_{\lambda \in D_X^{max}} V_\lambda \circlearrowleft \check{G}_X .$$

What about $(X, \Phi) \neq (X^{can}, \Phi^{can})$?

Lemma $\exists$ a torus ext'n $1 \to T \to G' \to G \to 1$

$G' \curvearrowright Y \xrightarrow{} X$ T-torsor s.t. vals assoc. to colours freely generate a direct summand of $X_*(A(Y))$

$Col(Y) \simeq Col(X)$

Def $D^G(X) =$ G-inv. valuations corresponding to prime Weil divisors of $X - X^\circ_G$
$(X - X^{can})$

$\rightsquigarrow D^G(X) \to X_*(A(X))$

$$S_X := \left(S_Y\big|_{\check{G}_X} \oplus \bigoplus_{\lambda \in D^G(X)} T^* V_\lambda \right)$$

Ex 1 $X = \mathbb{A}^n \circlearrowleft G = GL_n \supset B = \{ \text{upper triangular} \}$

$X^\circ_G = \mathbb{A}^n - \{0\} \hookleftarrow X^\circ_B = \{ x_1 \neq 0 \}$

$\qquad\qquad\qquad\qquad \mathbb{G}_m \times \mathbb{A}^{n-1} \ni \left(x_1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) \leftarrow x$

$P(X) = P_{1,n-1}$

$\cup$

$L(X) = GL_1 \times GL_{n-1} \xrightarrow{m_1} A(X) = GL_1$

B-eig. valuations x_1^k $(k \in \mathbb{Z})$

G-inv. valuations ① $\mathbb{A}^n \hookrightarrow \mathbb{P}^n \hookleftarrow \mathbb{P}^{n-1}$

$v(x_1) = -1$

② $\mathbb{A}^n \leftarrow Bl_0 \mathbb{A}^n \hookleftarrow$ exceptional

$v(x_1) = 1$

$V_X = X^*(A(X))_{a}, \qquad \Sigma(X) = \phi$

$\check{G}_X = \mathbb{G}_m \xhookrightarrow{\left(\begin{smallmatrix} x & & \\ & 1 & \\ & & \ddots & \\ & & & 1\end{smallmatrix}\right)} \check{G} = GL_n \longleftarrow SL_2$

$\rho_{triv} \oplus Sym^{n-2} \rho_{std}$

Colours $X^{\circ}_{G} - X^{\circ}_{B} = \{x_1 \neq 0\}$ type U

$\Rightarrow \mathcal{C}_X = \phi \quad , D_X^{max} = \phi$

$\mathcal{S}_X = 0$