Iwahori block

Main goal:

$$\mathrm{Rep}(G(F))^{[I]} \cong \mathrm{Rep}(\mathcal{H}_I) \cong \mathrm{Rep}(G(F))_{[T,\mathrm{unr}]}$$

$\uparrow$ rep's gen. by I-fixed vectors

§1. Unramified principal series (For simplicity, everything is split)

Defn. A character $\chi : T(F) \longrightarrow \mathbb{C}^\times$ is unramified if $\chi|_{T(\mathcal{O})}$ is trivial.

$$X^{\mathrm{unr}}(T) = \mathrm{Hom}_{gp}(T(F)/T(\mathcal{O}), \mathbb{C}^\times) \cong \mathrm{Hom}_{gp}(X_*(T), \mathbb{C}^\times)$$

Defn $\quad I(\chi) = i_B^G(\chi) = \mathrm{Ind}_{B(F)}^{G(F)}(\chi\, \delta_B^{\frac{1}{2}})$

$$\delta_B : B(F) \longrightarrow \mathbb{C}^\times \quad , \quad b \mapsto \det(\mathrm{Ad}(b)|_{\mathrm{Lie}(N)}).$$

Construction.

$$P_\chi : \underbrace{C_c^\infty(G(F))}_{= \mathcal{H}(G)} \longrightarrow I(\chi)$$

$$f \mapsto \left[P_\chi(f) : g \mapsto \int_{B(F)} \chi^{-1}\delta^{\frac{1}{2}}(b)\, f(bg)\, db \right]$$

$w \in W_{\mathrm{fin}}, \quad \phi_{w,\chi} = P_\chi(\mathbb{1}_{IwI})$

$\phantom{w \in W_{\mathrm{fin}},} \quad \phi_{G(\mathcal{O}),\chi} = P_\chi(\mathbb{1}_{G(\mathcal{O})})$

__Prop.__ (1) $I(\chi)^I$ has a $\mathbb{C}$-basis given by $\phi_{w,\chi}$ $(w \in W_{fin})$

$$\Rightarrow \dim I(\chi)^I = |W_{fin}|$$

(2) $I(\chi)^{G(\mathcal{O})}$ has a basis given by $\phi_{G(\mathcal{O}),\chi}$.

__Proof:__ (1) $\mathrm{supp}(\phi_{w,\chi}) = B(F)wI$,

$$G(F) = \coprod_{w \in W_{fin}} B(F)wI$$

(2) $G(F) = B(F)G(\mathcal{O})$.

__Rmk.__ $M = C_c^\infty(T(\mathcal{O})N(F)\backslash G(F)/I, \mathbb{C})$

$$R = \mathbb{C}[X_*(T)] \cong \mathbb{C}[T(F)/T(\mathcal{O})] = C_c^\infty(T(F)/T(\mathcal{O}), \mathbb{C})$$

$$= \text{c-ind}_{T(\mathcal{O})}^{T(F)} \text{triv.} \circlearrowleft T(F)$$

$$i_B^G(R) = \text{c ind}_{N(F)T(\mathcal{O})}^{G(F)}(\text{triv}) = C_c^\infty(N(F)T(\mathcal{O})\backslash G(F), \mathbb{C})$$

$$(i_B^G(R))^I \xrightarrow{\sim} M.$$

$$\chi : T(F)/T(\mathcal{O}) \longrightarrow \mathbb{C}^\times$$

$$R \longrightarrow \mathbb{C}$$

$$M \otimes_{R,\chi} \mathbb{C} \cong I(\chi)^I. \qquad M \cong H_I \cong H_{fin} \otimes_{\mathbb{C}} R$$

Rmk. $\dim_{\mathbb{C}} r_B(I(\lambda))^{T(\mathcal{O})} = |W_{fin}|$.

Thm. $I(\chi)^I \xrightarrow{\sim} r_B(I(\chi))^{T(\mathcal{O})}$ is an isom. of vec. sp.

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§2. Jacquet's Lemma

Setting $P = MN \subset G(F)$ parabolic, K cpt open

$\bar{P} = M\bar{N}$ opposite parabolic

$\Lambda :=$ a lift of $T(F)/T(\mathcal{O})$ inside $T(F)$,

$\Lambda^+ := \{\lambda \in \Lambda : \mathrm{Ad}(\lambda)|_N$ is contracting $\}$

Defn. (1) (K, P) is in good position

if $K = K_{\bar{N}} K_M K_N$, $K_{\bar{N}} = K \cap \bar{N}, \dots$

(2) (K, P) is dominated by Λ^+ if

1) K_M is stable under the $\mathrm{Ad}\,\Lambda$-action

2) K_N is stable under Ad_{Λ^+}

3) $K_{\bar{N}}$ is stable under $(\mathrm{Ad}\,\Lambda^+)^{-1}$.

(3) $\lambda \in \Lambda^+$ is strictly dominating (K, P), if $P = P_\lambda$

Fact (Bruhat) P fixed, $\exists$ arbitrarily small K s.t.

(K, P) satisfies (1), (2).

V be an admissible rep^n of $G(F)$.

(K, P) satisfies (1), (2)

$$p : V^K \longrightarrow r_P(V)^{K_M}.$$

<u>Lemma</u> (Jacquet) p is a surjection.

$\exists$ linear operator $A \curvearrowright V^K$

$$V^K = V_0 \oplus V_*, \quad A|_{V_0} \text{ nilpotent}, \quad A|_{V_*} \text{ invertible}.$$

$$V_* \twoheadrightarrow r_P(V)^{K_M} \text{ under } p.$$

<u>Proof</u>. $\lambda \in \Lambda_{++} \leftarrow$ Strictly dominating

$$A_0 = \mathbb{1}_K * \mathbb{1}_\lambda * \mathbb{1}_K \in \mathcal{H}(G, K)$$

$$\begin{tikzcd} V^K \arrow[r] & r_P(V)^{K_M} \end{tikzcd}$$

$A_0 \curvearrowright V^K$, $\quad \mathbb{1}_{K_M} * \mathbb{1}_\lambda * \mathbb{1}_{K_M} = A_0 \curvearrowright r_P(V)^{K_M}$

<u>Claim</u>: $r_P(V)^{K_M}$ is the localization of V^K w.r.t. A_0-action.

$\forall\, v \in r_P(V)^{K_M}$, choose a lift $\tilde{v} \in V^{K_M}$, $\exists\, K'$ stabilizes $\tilde{v}$.

$\tilde{v}_n := A_0^n \tilde{v}$ is stabilized by $\lambda^n K' \lambda^{-n}$.

$K_{\bar{N}} = \bigcup_n (\lambda^n K' \lambda^{-n}) \cap K_{\bar{N}}$. $\exists$ large n, $\tilde{v}_n$ is stable under $K_{\bar{N}}$

$v' := \mathbb{1}_{K_N} * \tilde{v}_n \in V^K$

Because

$p(v') = A_0^n v.$

$\forall\, v \in r_p(V)^{K_m}$, $\exists\, n \in \mathbb{Z}$, $A_0^n v \in \mathrm{Im}(p)$, $\exists$ uniform n_0 s.t.

$A_0^{n_0} v \in \mathrm{Im}(p)$. $\forall\, v \in r_p(V)^{K_m}$.

$A := A_0^{n_0}$, $V^k = V_0 \oplus V_*$

$V^I \twoheadrightarrow r_B(V)^{T(0)}$

$\mathbb{1}_{I\lambda I}.$

<u>Lemma</u> $\forall\, w \in W_{ext}$, $\mathbb{1}_{IwI}$ is always invertible in $\mathcal{H}_I$.

<u>Pf.</u> 1) $\mathbb{1}_{I s_\alpha I}$ is always invertible by previous talk

2) $W_{ext} = W_{aff} \rtimes \Omega$

$T_\alpha T_w = T_{\alpha w}$, $\forall\, \alpha \in \Omega$, $w \in W_{ext}$

$T_\alpha T_{\alpha^{-1}} = 1.$

<u>Cor.</u> Any V admissible rep^n of $G(F)$, $V^I \xrightarrow{\sim} r_B(V)^{T(0)}$ is an isom. as v.s.

<u>Cor.</u> V irred. rep^n of G, if $V^I \neq 0$, then $\exists\, \chi$, $V \hookrightarrow I(\chi)$

Conversely, if $V \hookrightarrow I(\chi)$, then $V^I \neq 0$.

Pf. $\mathrm{Hom}_G(V, I(\chi)) = \mathrm{Hom}_{T(F)}(r_B(V), \chi\delta^{\frac{1}{2}})$

unramified

$= \mathrm{Hom}_{T(F)}(r_B(V)_{T(\mathcal{O})}, \chi\delta^{\frac{1}{2}})$

$= \mathrm{Hom}_{T(F)}(r_B(V)^{T(\mathcal{O})}, \chi\delta^{\frac{1}{2}})$ ($T(\mathcal{O})$ cpt group, coinv. = inv.)

$\Downarrow$ vs.

$V^{I} \neq 0$.

Cor. $I(\chi)$ is gen. by $I(\chi)^I$.

Pf. $W := \mathcal{H}_I \cdot I(\chi)^I$

$0 \to W \to I(\chi) \to I(\chi)/W \to 0$

$(I(\chi)/W)^I = 0$. $(-)^I$ exact because I cpt.

$\widetilde{I(\chi)/W} \hookrightarrow \widetilde{I(\chi)} = I(\chi^{-1})$ $\sim$ smooth dual.

(Fact: $\widetilde{i_P^G V} = i_P^G \widetilde{V}$)

$W = I(\chi)$. $\square$

Rmk. $V^I \cong r_B(V)^{T(\mathcal{O})}$.

$V = C_c^\infty(G(F)/I)$, $V^I = C_c^\infty(I\backslash G(F)/I) = \mathcal{H}_I$

$r_B(V)^{T(\mathcal{O})} = C_c^\infty(T(\mathcal{O})N(F)\backslash G(F)/I, \mathbb{C}) \cong M \Rightarrow M \cong \mathcal{H}_I$.

Prop. Let E be an admissible repn of G.

Assume ① E is gen. by E^I

② $0 \neq E' \subset E$, then $(E')^I \neq 0$ (e.g. $I(\chi)$ satisfies ①, ②)

then 1) given SES $0 \to E \to V \to Q \to 0$ $(*)$

$\&$ $Q^I = 0$ $\Rightarrow$ $(*)$ splits

2) $E' \subset E$, E' is gen. by $(E')^I$.

Proof. 1) $\Rightarrow$ 2) $E' \neq 0$, E' satisfies ①, ②, $W = \mathcal{H}(G) \cdot E'^I$

$0 \to W \to E' \to E'/W \to 0$ W satisfies ①, ②

$E'/W \hookrightarrow E' \hookrightarrow E$. $(E'/W)^I = 0$. By assumption on E, $E'/W = 0$

$\Rightarrow E' = W$.

1) r_B is exact. $0 \to r_B(E) \to r_B(V) \to r_B(Q) \to 0$,

$$r_B(E)^{T(0)} \cong r_B(V)^{T(0)}$$

$$r_B(V) = r_B(E)^{T(0)} \oplus V' \quad \text{(decomp. w.r.t. } T(0))$$

get a map $r_B(V) \to E^I$, by adjunction $\tilde{\beta} : V \to i_B^G E^I$.

Claim: a) $E \cap \ker(\tilde{\beta}) = 0$

b) $\tilde{\beta}(V) = \tilde{\beta}(E)$,

thus, $V \cong E \oplus \ker(\tilde{\beta})$.

Pf of claim a) $\left(E \cap \ker(\tilde{\beta}) \right)^I = \ker\left(\tilde{\beta}\big|_{V^I}\right) = 0$ by injectivity

of $E^I \hookrightarrow i_B^G(E^I)$.

b) $\tilde{\beta}(V)/\tilde{\beta}(E) = 0$. $\square$

Apply to $E = I(\chi)$,

- any subquotient E of $I(\chi)$ is gen. by E^I.
- if E is gen. by E^I, then E subquotient of direct sum of $i_B^G(\chi)$'s.

- $Rep(G)^{[I]} = \{ V \in Rep^{sm}(G) : V \text{ is gen. by } V^I \}$

$\Rightarrow$
- $Rep(G)^{[I]}$ is gen. by $\{I(\chi)\}_{\chi \in X(T)^{unr}}$.
- $Rep(G)^{[I]}$ is closed under taking subquotient.

Generality, K open subgroup of G,

$Rep(G)^{[K]} = \{ V \in Rep(G) : V \text{ is gen. by } V^K \}$

$Rep(\mathcal{H}(G,K))$, $\mathcal{H}(G,K) = C_c^\infty(K \backslash G(F) / K; \mathbb{C})$

$$\mathrm{Rep}(G)^{\{K\}} = \{ V \in \mathrm{Rep}^{sm}(G(F)) : V^K \neq 0 \text{ if } V \neq 0 \}$$

$$m_K : \mathrm{Rep}(G)^{\{K\}} \longrightarrow \mathrm{Rep}(\mathcal{H}(G,K)) : F_K$$

$$V \longmapsto V^K$$

$$W \otimes_{\mathcal{H}(G,K)} C_c^\infty(K \backslash G(F)) \longleftarrow\!\shortmid W$$

<u>Fact</u> ① m_K is exact

② (F_K, m_K) is an adjunction.

③ $m_K \circ F_K = id$

④ F_K is fully-faithful

<u>Warning</u> : Not equiv. in general.

<u>Fact</u>: $\mathrm{Irr}(\mathrm{Rep}(G)^{[K]}) \cong \mathrm{Irr}(\mathcal{H}(G,K))$ under m_K,

(but F_K is not the inverse !)

<u>Lemma</u>. m_K and F_K are inverse to each other iff

$\mathrm{Rep}(G(F))^{[K]}$ is stable under taking subobjects.

<u>Pf of Lemma</u>. (If) need to check counit of the adjunction is an isom.

$C_c^\infty(G(F)/K) \otimes_{\mathcal{H}(G,K)} W^K \twoheadrightarrow W$ is an isom. By assumption, surjection.

Taking K- inv., isom. $\Rightarrow$ isom.

<u>Rmk.</u> $\mathrm{Rep}(G(F))^{[G(\mathcal{O})]} \not\cong \mathrm{Rep}(\mathcal{H}(G(F), G(\mathcal{O})))$

$$0 \to St \to \text{c-ind}_{G(\mathcal{O})}^{G(F)} \text{triv.} \to \text{triv.} \to 0$$