Hochschild homology of algebraic varieties in characteristic $p > 0$

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§1. Hochschild homology

k field, A assoc. k-alg.

A is an $A \otimes A^{op}$-module

<u>Def</u> The Hochschild homology and cohomology of A/k are

$$HH_\bullet(A/k) = \mathrm{Tor}_\bullet^{A\otimes A^{op}}(A, A) \quad ; \quad HH^\bullet(A/k) = \mathrm{Ext}^\bullet_{A\otimes A^{op}}(A, A).$$

<u>Rmks</u> 1) derived "trace" and "center"

$$HH_0(A) = A/[A, A] \quad , \quad HH^0(A) = Z(A)$$

2) $$A \leftarrow A^{\otimes 2} \leftleftarrows A^{\otimes 3} \leftleftarrows\!\!\!\leftarrow A^{\otimes 4} \cdots$$ "bar resolution"

$\rightsquigarrow$ $HH^\bullet(A/k)$ controls assoc. deformations of A

$$HH^2(A/k) \simeq \{\text{first order deformations}\}/\simeq$$

3) dg-Morita invariant

Consider R comm. k-alg.

<u>Thm</u> (Hochschild – Kostant – Rosenberg 1962) If R/k is smooth, then there are nat'l isoms

$$HH_i(R/k) \simeq \Omega^i_{R/k} \quad ; \quad HH^i(R/k) \simeq \Lambda^i T_{R/k} \simeq \Lambda^i \mathrm{Der}(R/k)$$

§ Algebraic varieties

X/k alg. var. (previously $X = \mathrm{Spec}\, R$)

$\Delta : X \longrightarrow X \times X$

Def. The <u>Hochschild homology sheaf</u> is $\underline{HH}(X/k) = L\Delta^* \Delta_* \mathcal{O}_X \in D_{qc}(X)$.

The <u>Hochschild homology</u> of X/k is $HH(X/k) = R\Gamma(X, \underline{HH}(X/k))$.

$\underset{X \text{ smooth}}{HKR \Rightarrow} \; \mathcal{H}^{-i}\left(\underline{HH}(X/k)\right) \simeq \Omega^i_{X/k}$

hypercohomology spectral sequence $E_2^{s,t} = H^s(X, \Omega^{-t}_{X/k}) \Rightarrow HH_{-s-t}(X/k)$

<u>HKR spectral sequence</u>.

Thm (Swan '96, Kontsevich, Yekutieli '02) If char $k = 0$, then HKR spectral sequence $[(\dim X)! \in k^\times]$ <u>canonically</u> degenerates.

There are canonical isom's $HH_i(X/k) \simeq \bigoplus_s H^s(X, \Omega^{s+i}_{X/k})$. [adding columns of Hodge diamond!]

Q. char $k = p > 0$?

Thm (Antieau - Bhatt - Mathew, 2019) (For each prime p) There exists smooth proj. $X/\mathbb{F}_p$ of $\dim X = 2p$ s.t. HKR spectral seq. is nondegenerate. In fact, $d_p \neq 0$.

It follows $\sum_{s,t} \dim H^s(X, \Omega^{-t}_{X/k}) > \sum_i \dim HH_i(X/k) \Rightarrow$ no $\bigoplus$ as in char 0

Q: describe differentials in the HKR spectral seq.

- for which r are $d_r \equiv 0$?
- if r is least s.t. $d_r \not\equiv 0$ $(E_r^{st} \simeq E_2^{st})$, what is a formula for d_r ?

Formula involves lifting X/k.

If k is perfect, then $\exists\ W_2(k)$, ring flat $\mathbb{Z}/p^2\mathbb{Z}$-alg, $W_2(k)/pW_2(k) \simeq k$.

Def A lift of X/k is a flat $\tilde{X}/W_2(k)$ s.t. $\tilde{X} \times_{W_2(k)} k \simeq X$.

Bockstein C complex of $\mathbb{Z}/p^2\mathbb{Z}$-modules.

$$0 \to \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

$$\rightsquigarrow \quad C/^{\mathbb{L}}p \to C \to C/^{\mathbb{L}}p$$

Def. $\mathrm{Bock}_C : C/^{\mathbb{L}}p \to C/^{\mathbb{L}}p\,[1]$ is the boundary morphism of

$$C \overset{\mathbb{L}}{\otimes} (\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p)$$

$$\rightsquigarrow H^i(C/^{\mathbb{L}}p) \to H^{i+1}(C/^{\mathbb{L}}p).$$

Main Theorem. (M '24) If X/k is smooth (char $k = p$)

1) $d_r = 0$, $r < p$
2) (Formula) There exists a nat'l map $V : \Omega^i[i] \to \Omega^{i+p-1}[i+p-1]$ in D_{qc}

s.t. if k is perfect, X admits a lift to $W_2(k)$,

$$d_p = [V, \mathrm{Bock}_{\tilde{X}}] : H^s(\Omega^i) \to H^{s+p}(\Omega^{i+p-1})$$

Rmk. V is a p^{th} power operation on $T_X[-1] = \mathrm{Lie}(\mathcal{L}X = X \times_{X \times X} X)$

If G/k is a group, $\mathrm{Lie}(G)$ is a restricted Lie algebra.

$\partial \in \mathrm{Lie}(G) \rightsquigarrow \partial^p \in \mathrm{Lie}(G)$.

Ex. $X = BG$, $\quad L\Omega^1_{BG/k} = \mathrm{coLie}(G)[-1] = e^* L\Omega^1_{G/k}[-1]$

$$V: \mathrm{coLie}(G)[-1][1] \longrightarrow \Lambda^p(\mathrm{coLie}(G)[-1])[p] = \mathrm{Sym}^p(\mathrm{coLie}(G))$$

linearly dual to $\partial^{\otimes p} \longmapsto \partial^p \in \mathrm{Lie}(G)$

Ex (Antieau-Bhatt-Mathew)

$X = B\mu_p$, $\quad \mu_p = \ker(\mathbb{G}_m \xrightarrow{p} \mathbb{G}_m)$

$$\mathrm{coLie}(\mu_p) = \left(\overset{c}{\frac{dt}{t}} \xrightarrow{\ p\ } \overset{d}{\frac{dt}{t}}\right)$$

$$H^\bullet(L\Omega^\bullet_{B\mu_p}) = k[c, d]$$

$c \in H^1(L\Omega^1)$, $\quad d \in H^0(L\Omega^1)$

$\mathrm{Bock}(d) = c$

$$\left(t\frac{d}{dt}\right)^p \cdot t^n = n^p t^n = n t^n = \left(t\frac{d}{dt}\right)\cdot t^n$$

$\Rightarrow \quad V(c) = c^p$

In fact, $V(d) = 0$. $\quad d_p(d) = V(\mathrm{Bock}(d)) - \underbrace{\mathrm{Bock}(V(d))}_{0} \quad \Rightarrow d_p \neq 0.$

$= V(c) = c^p$

§3. Sketch of proof.

$LX = X \times_{X \times X} X = \underline{\mathrm{Maps}}(S^1, X)$

$$\begin{tikzcd} \bullet \sqcup \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} \rightsquigarrow \quad \underline{\mathrm{Maps}}(S^1, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

$\underline{\text{Q}}$. How to see HKR universally?

$\underline{\text{Thm}}$ (Moulinos - Robalo - Toën, Raksit 2019)

$\exists \quad S^1_{fil} \longrightarrow \mathbb{A}^1/\mathbb{G}_m$ s.t.

$\underline{\mathrm{Maps}}(S^1_{fil}, X) \longrightarrow \mathbb{A}^1/\mathbb{G}_m$ has ring of funcs $\underline{HH}(X)$ w/

HKR filtration