S-dual of Hamiltonian G spaces and relative Langlands duality

Hiraku Nakajima

<u>Lecture 1</u>. <u>Plan</u>

1. relative Langlands (Ben-Zvi - Sakellaridis - Venkatesh)

   from TQFT pt of view

2. def'n of Coulomb branches and S-dual

3. Examples (quantum symmetric pair)

<u>Warning</u> (Tachikawa) Quantum field theory is <u>not</u> theory of quantum fields.

LC from TQFT point of view Kapustin-Witten

G reductive gp $/\mathbb{C}$, G_c: cpt Lie gp

$\mathcal{A}_G, \mathcal{B}_G$: topologically twisted 4d $\mathcal{N}=4$ SYM theories

$\mathcal{J} = \mathcal{J}_G$

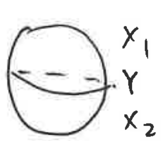
Atiyah-Segal axiomatic way to understand $\mathcal{J}$:

- $\mathcal{J}(X^4)$: number $= \underset{A:\, G_c\text{-connection}}{\int} DA\, e^{\cdots}$

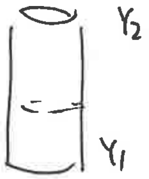
- $\mathcal{J}(Y^3)$: vector space

  $\mathcal{J}(Y_1 \sqcup Y_2) = \mathcal{J}(Y_1) \otimes \mathcal{J}(Y_2)$

  $\partial X^4 = Y \rightsquigarrow \underset{\text{vector}}{\mathcal{J}(X)} \in \underset{\text{vector sp.}}{\mathcal{J}(Y)}$

$X =$ 

$\mathcal{J}(X) = \langle \underset{\mathcal{J}(Y)}{\mathcal{J}(X_1)}, \underset{\mathcal{J}(-Y) = \mathcal{J}(Y)^*}{\mathcal{J}(X_2)} \rangle$

$X =$ 

$\mathcal{J}(X) \in \mathrm{Hom}(\mathcal{J}(Y_1), \mathcal{J}(Y_2))$

extended TQFT

- $\mathcal{J}(\Sigma^2)$: $\mathbb{C}$-linear category

  $\partial Y = \Sigma^2 \rightsquigarrow \mathcal{J}(Y) \in \mathcal{J}(\Sigma)$

  object category

$Y =$ 

$$\mathcal{J}(Y) = \mathrm{Hom}_{\mathcal{J}(\Sigma)}(\mathcal{J}(Y_1), \mathcal{J}(Y_2))$$

$\uparrow$ $\mathbb{C}$-vec. sp.

- $\mathcal{J}$(1-mfd) : 2-category etc.

<u>KW</u> claim:

(1) $\mathcal{A}_G(\Sigma)$ – category rel. for GLC (automorphic side)

$\uparrow$ fake cpx str.

$\mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$

$\mathcal{B}_G(\Sigma)$ – —— " —— (Galois side)

$\mathrm{IndCoh}_!(\mathrm{Loc}_G(\Sigma))$

(2) (S-duality) $\mathcal{A}_G(-) \simeq \mathcal{B}_{G^\vee}(-)$

$\check{G}$ = Langlands dual gp

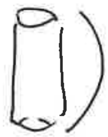
This is an explanation of GLC

Rmk [BZSV]

$C/\mathbb{F}_q$ should be regarded as a 3-mfd

curve $\circlearrowright_{Fr}$ regarded as mapping cylinder

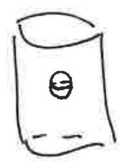
$\updownarrow$ analog.

$f : \Sigma \circlearrowleft \quad Y^3 = \Sigma \times I/\sim$ 

$(x,0) \sim (f(x), 1)$

So $A_G(C/\mathbb{F}_q)$, $B_{G^\vee}(C/\mathbb{F}_q)$ $\leftarrow$ vector spaces $= \{$Functions on moduli spaces $/\mathbb{F}_q\}$

operators

$M = \Sigma \times [0,1] \setminus B$  $\partial M = \Sigma \cup \Sigma \cup S^2$

$\uparrow$ small ball

$$J(\Sigma) \times J(S^2) \longrightarrow J(\Sigma)$$

$\uparrow$ Operators

In particular, $J(S^2)$ is a monoidal cat.

and $J(\Sigma)$ is its module.

$J = A_G \rightsquigarrow$ Hecke operators $\in A_G(S^2)$

$B_{G^\vee} \rightsquigarrow$ Wilson operators $\in B_{G^\vee}(S^2)$

$\updownarrow$ (derived) geometric Satake

$$\mathrm{Perv}_{G(O)}(\mathrm{Gr}) \simeq \mathrm{Rep}(G^\vee)$$

[Bezrukavnikov - Finkelberg] $\cap$ $\cap$

$$D_{G(O)}(\mathrm{Gr}_G) \simeq D^{G^\vee}(\mathrm{Sym}\, \mathfrak{g}^\vee)$$

$\uparrow$ dg-alg.

Rmk (technical) Strictly speaking, $G(\mathcal{O})\backslash Gr_G = Bun_G$ (ravioli space)

instead of S^2

$$D = \operatorname{Spec} \mathbb{C}[[z]] = \operatorname{Spec} \mathcal{O} \qquad\qquad \|$$

$$D^* = \operatorname{Spec} \mathbb{C}((z)) = \operatorname{Spec} K \qquad D \coprod_{D^\times} D \qquad D \coprod_{S^1} D$$

Interfaces [Gaiotto – Witten]

$\hookleftarrow$ relative Langlands Functoriality

G, H reductive groups

$\mathcal{J}_G(-)$, $\mathcal{J}_H(-)$

Interface is a "homomorphism"

$\mathcal{I} = \mathcal{I}_{H,G}$ from $\mathcal{J}_H(-)$ to $\mathcal{J}_G(-)$.

Namely, $\mathcal{J}_H(Y^3) \xrightarrow{\mathcal{I}(Y)} \mathcal{J}_G(Y^3)$

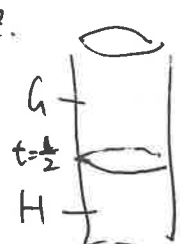
homomorphism of vec. sps.

$\mathcal{J}_H(\Sigma^2) \xrightarrow{\mathcal{I}(\Sigma)} \mathcal{J}_G(\Sigma^2)$

functor between categories

Picture.

$\Sigma \times [0,1]$ or $Y \times [0,1]$



Special case. $H = \{1\}$, $\mathcal{I}_{\{1\},G}(Y) \in \mathcal{J}_G(Y)$

vector — vector sp.

function — $= \{\text{functions}\}$

$\mathcal{I}_{\{1\},G}(\Sigma) \in \mathcal{J}_G(\Sigma)$

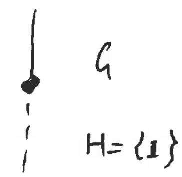
object — cat.

$\mathcal{I}_{\{1\},G}(S^1) \underset{\text{cat.}}{\in} \mathcal{J}_G(S^1)$ 2-cat.

Example. (old material in differential geometry)

G_c : cpt Lie gp.

Nahm's equation, ODE for $\mathfrak{g}_c$-valued functions T_0, T_1, T_2, T_3 on $[0, 1/2)$



$$\frac{dT_1}{dt} + [T_0, T_1] = [T_2, T_3] + \text{cyclic perm.}$$

T_0 has no pole

$T_{1,2,3}$ pole at $t = 1/2$ up to gauge transf.

$$T_i = \frac{a_i}{t-1/2} + \text{regular} \qquad \text{Nahm's eqn} \Rightarrow \quad a_1 = [a_2, a_3] \text{ etc.}$$

$\rightsquigarrow \rho : su(2) \longrightarrow \mathfrak{g}_c$ Lie alg. hom.

$\rho : sl_2 \longrightarrow \mathfrak{g}$ sl_2-triple $\langle e, h, f \rangle$

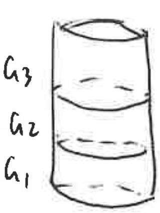
Thm [Bielawski '95] Fix ρ. Sol. of Nahm's eqn / gauge transf. $\simeq T^*G /\!/\!/_{(U,\psi)} = G \times S_e^{\mathfrak{g}}$

hyperkähler mfd

symp. mfd

equivariant Slodowy slice

It should be possible to compose interfaces.



<- can regard as an interface between $\mathcal{J}_{G_1}$ and $\mathcal{J}_{G_3}$.

<- This picture appears in bow varieties

Claim [GW]

a (top. twisted) 3d $\mathcal{N}=4$ SUSY QFT with $G\times H$-symmetry gives an interface between

$\mathcal{A}_H(-) \overset{\theta}{-} \mathcal{A}_G(-)$ & $\mathcal{B}_H(-) \overset{L}{-} \mathcal{B}_G(-)$.

$$\begin{tikzcd}
\mathcal{A}_H(\ ) \arrow[r, "\theta"] \arrow[d, "LC"', "\wr"] & \mathcal{A}_G(\ ) \arrow[d, "\wr", "LC"'] \\
\mathcal{B}_{H^\vee}(\ ) \arrow[r, "L'"] & \mathcal{B}_{G^\vee}(\ )
\end{tikzcd}$$

Claim. This commutes if $L' = \theta^\vee$ (S-dual).

Lecture 2.

G-Hamiltonian space (M, ω)
↑ cpx symplectic mfd

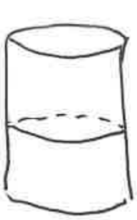
$G \curvearrowright M$ preserving ω

$\mu : M \to \mathfrak{g}^*$ moment map, G-equiv.

$$d\langle \underset{\substack{\Uparrow \\ \mathfrak{g}}}{\mu, \xi} \rangle = \underbrace{L_{\xi_M}}_{\text{vector field on } M \text{ generated by } \xi} \omega$$

Symplectic reduction (hamiltonian) $\quad M/\!\!/\!\!/G := \mu^{-1}(0)/\!\!/G = \mathrm{Spec}\left(\mathcal{O}[\mu^{-1}(0)]^G\right)$

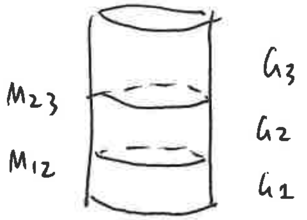
or $\left[M \overset{G}{\underset{\mathfrak{g}^*}{\times}} \{0\} \right]$ dg-stack



interface I, $\quad I(\ ) : I_H(-) \to I_G(-)$
↑ 3d TQFT w $H\times G$-symmetry

For Y^3, linear map
Σ, functor

Composite of interfaces



$$I_{G_2,G_3}(-) \circ I_{G_1,G_2}(-)$$

$\uparrow$

given by "gauging" w.r.t. G_2

$$\underbrace{Gr_{G_1} \times Gr_{G_2}}_{A_{M_{12}}} \times Gr_{G_3} \qquad \underbrace{Gr_{G_2} \times Gr_{G_3}}_{A_{M_{23}}}$$

$/\!/\!/ G_2$ mixture of hamiltonian reduction

ring objects and $\int_A DA \cdots$ $A \leftarrow G_2$-connections

$$P_{13*}\left(A_{M_{12}} \overset{!}{\otimes} A_{M_{23}}\right)$$

$\uparrow$

ring object in $D_{G_1(\mathcal{O}) \times G_3(\mathcal{O})}\left(Gr_{G_1} \times Gr_{G_3}\right)$

We have many examples of 3d TQFT defined by $H \times G$-hamiltonian space M

$\rightsquigarrow I^M = I^M_{H,G}$ interface

two versions corresp. to A_G, B_G for 4d TQFT (GL)

θ-func.

$A \nearrow \Theta^M : A_H(-) \to A_G(-)$

$B \nearrow L^M : B_H(-) \to B_G(-)$

L-func.

Θ^M, L^M are given (very roughly)

$$\mathrm{Map}(-,[pt/H]) \swarrow \mathrm{Map}(-,[M/G\times H]) \searrow \mathrm{Map}(-,[pt/G])$$

Secretly, We (I) replace $M = T^*N$ by N

$$\mathrm{Map}(-,[H\times G/H/H\times G])^{H \subset G}$$

$$\mathrm{Map}(-,[pt/H]) \swarrow \qquad \searrow \mathrm{Map}(-,[pt/G])$$

e.g. $\mathrm{Bun}_H(\Sigma^2)$

e.g. $H \subset G$ subgp $M = T^*(H\times G/H) = {}^{H \curvearrowright} T^*G^{\curvearrowleft G}$

Langlands functoriality: What is the dual of M?

$$\begin{array}{ccc} \mathcal{A}_H(-) & \xrightarrow{\theta^M} & \mathcal{A}_G(-) \\ \text{LC}\ \wr\downarrow & & \downarrow\wr\ \text{LC} \\ \mathcal{B}_{H^\vee}(-) & \xrightarrow{\mathcal{L}^{M^\vee}} & \mathcal{B}_{G^\vee}(-) \end{array}$$

If $M^\vee$ is S-dual of M, then this is commutative.

?: S-dual is well-defined on the level of 3d TQFT

But the S-dual may <u>NOT</u> come from $G^\vee$- hamil. space.

But one can always approximate arbitrary 3d TQFT $\mathcal{J}$ by Hamiltonian space $M^\vee$ (possibly singular)

(effective field theory)

$M^\vee = \text{Higgs}(\mathcal{J})$

$\mathcal{J} \xmapsto{\text{Higgs}} \text{Higgs}(\mathcal{J})$

affine alg. var,
sympl str on smooth locus

If $\mathcal{J}$ has G-symmetry, then $G \curvearrowright \text{Higgs}(\mathcal{J})$

$$\text{Higgs}\left(\begin{array}{l} \mathcal{J}_{12} \circ \mathcal{J}_{23} \\ = \mathcal{J}_{12} \times \mathcal{J}_{23} /\!/\!/ G_2 \end{array}\right) = \text{Higgs}(\mathcal{J}_{12}) \times \text{Higgs}(\mathcal{J}_{23}) /\!/\!/ G_2$$

<u>Non-example</u>. interface $\mathcal{J}[G]$ in [BFN3]

$$\mathcal{A}_G(-) \xrightarrow{\mathcal{OI}_G(-)} \mathcal{B}_{G^\vee}(-)$$

interface
between $\mathcal{A}_G$ & $\mathcal{B}_{G^\vee}$
$G \times G^\vee$ - symmetry

For $G = GL_n$, such interface is <u>known</u>.

<u>Goal today</u>

Assuming $\underset{G \curvearrowright}{M} = T^* \underset{\text{smooth, affine var.}}{\underline{N}}$, we propose a definition of $\underset{\check{G} \curvearrowright}{M^\vee}$, approximating S-dual.

<u>Hope</u>. (1) If $M^\vee$ is smooth, this S-dual is coming from $M^\vee$.

(2) happens if M is hyperspherical [BZSN]

Take $\Sigma = S^2$ or $\underset{\text{ravioli space}}{D \underset{D^\times}{\sqcup} D}$

$\mathcal{A}_G(S^2)$, $\mathcal{B}_{\check{G}}(S^2)$ are monoidal cats

$$\begin{array}{ccc} \mathcal{A}_G(S^2) & & \mathcal{B}_{\check{G}}(S^2) \\ \| & & \| \\ D_{G(O)}(Gr_G) & \underset{\substack{\text{derived geom} \\ \text{satake}}}{\simeq} & D^{\check{G}}(\mathrm{Sym}^{\Pi} \mathfrak{g}^\vee) \end{array}$$

$$\begin{array}{ccc} \text{object } \Psi & [BF(M)G] & \Psi \\ \mathcal{O}^M(S^2) & \longleftrightarrow & \mathcal{L}^{M^\vee}(S^2) \end{array}$$

They are ring objects in the respective monoidal cat.

They are coming from

$$\mathrm{Map}(S^2, [\underset{\substack{\| \\ T^*N}}{M}/G]) \longrightarrow \mathrm{Map}(S^2, [pt/G])$$

or $M^\vee$

- $\mathcal{B}$-side $\mathcal{O}(M^\vee)$ = coord. ring of $M^\vee$ viewed as $\mathrm{Sym}\, \mathfrak{g}^\vee$-module via $M^\vee \xrightarrow{\mu} \mathfrak{g}^{\vee *}$.

- $\mathcal{A}$-side, $\mathrm{Map}(S^2, [N/G]) \xrightarrow[p]{1^{st} \text{ projection}} \mathrm{Map}(S^2, [pt/G]) = G(O) \backslash Gr_G$

$$\mathrm{Map}(S^2, [N/G]) = \{(P_G, s) : P_G : G\text{-bdle on } S^2,\ s \in \Gamma(P_G \overset{G}{\times} N)\}$$

$$\theta^M(S^2) = p_* \omega_{Map(S^2, [N/G])}$$

dualizing sheaf.

<u>Proposal</u>: $M^\vee \overset{def}{=} Spec\left(\text{der.} \# \left(\theta^M(S^2)\right)\right)$

<u>Rmk</u>. [BFN] $\mathcal{M}_C(G \curvearrowright M = T^*N) = Spec\ H_*(Map(S^2, [N/G]))$

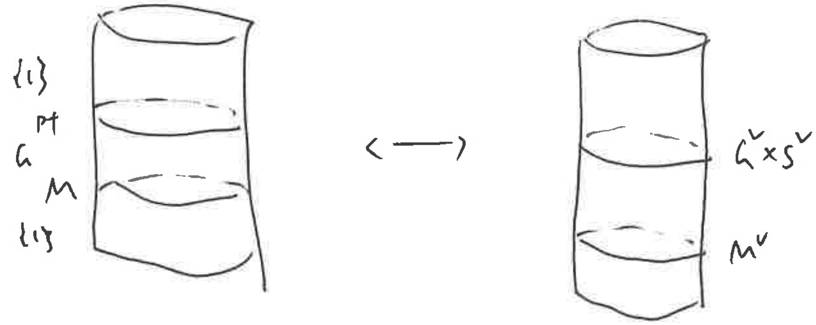
ring str. by convolution.

By [BF], $\mathcal{M}_C = M^\vee \times G^\vee \times S^\vee /\!/\!/ G^\vee$

Kostant reduction of S-dual.

$$\mathcal{M}_C(G \curvearrowright M) = [\{1\} \sim M /\!/\!/ G]^\vee$$

$$= M^\vee \circledcirc (G^\vee \times S^\vee)$$



So $[G \to \{pt\}]^\vee = G^\vee \curvearrowright G^\vee \times S^\vee$

slice in $g^\vee$ at $e_{regular}$.

Universal centralizer $= Spec\ H_*^{G(O)}(Gr_G)$

Take $M = pt$, $M^\vee = G^\vee \times S^\vee$.

Lecture 3. Recall: $M = T^* N$

$G \curvearrowright$ $\underline{N}$ smooth affine G-mfd

$$S^2 = D \sqcup_{D^*} D$$

$$\begin{array}{c} \mathrm{Map}(S^2, [N/G]) \\ \downarrow p \\ \mathrm{Map}(S^2, [pt/G]) = G(O)\backslash Gr_G \end{array}$$

$$\Theta^M(S^2) := p_* \omega_{\mathrm{Map}(S^2,[N/G])}$$

$$\begin{array}{ccc} \Theta^M(S^2) & & \\ \cap & & \\ D_{G(O)}(Gr_G) & \longrightarrow & D^G(\mathrm{Sym}^{\mathbb{Z}} g^\vee) \end{array}$$

$$\star$$

$$M^\vee := \mathrm{Spec}\left(\overbrace{H^*(\text{der.}\ \mathcal{H}}(\Theta^M(S^2)))\right)$$

This is an $\underline{\text{approximation}}$ of 3d TQFT, S-dual to Θ^M.

$$M^\vee = \mathrm{Higgs}((\Theta^M)^\vee)$$

$\underline{\text{Rmk}}$. Sympl. str. on $M^\vee$ is defined via deformation quantization of $M^\vee$.

given $\underset{\substack{\uparrow \\ \text{loop rotation}}}{H^*_{\mathbb{G}_m}}$ $\qquad \mathbb{G}_m \curvearrowright D$ etc.

$\underline{\text{Rmk}}$ BDFRT, Teleman $\quad M$: symplectic rep, anomaly condition.

In order to compute $M^\vee$, $\star \ \underset{\overline{\overline{\quad}}}{\boxed{\text{BFN3}}}$ $H^*_{G(O)}(\Theta^M(S^2) \overset{!}{\otimes} \underbrace{\mathcal{A}_{reg}}_{\substack{\text{reg. sheaf on } Gr_G \\ \text{(perverse)}}}) = (p_{Gr_G \to Gr_{\{1\}}})_*(\Theta^M(S^2) \overset{!}{\otimes} \underbrace{\mathcal{A}_{reg}}_{\substack{\Theta^{\mathcal{A}_{reg}} \\ \text{3d TQFT} \\ \text{lift of } \mathcal{A}_{reg}}})$ $\quad \not\!\!/\!\!/ G$ $\quad Gr_{\{1\}} = pt$

where $\mathcal{A}_{reg} = (\text{der.}\ \mathcal{H})^{-1}(\mathbb{C}[T^* G^\vee])$

$$= \mathcal{H}^{-1}(\mathbb{C}[G^\vee]) = \bigoplus_\lambda V_{G^\vee}(\lambda)^* \otimes IC(Gr_G^\lambda)$$

$$\mathcal{L}^{M^\vee} = \left(\underset{\substack{\uparrow \\ G}}{\mathcal{O}^M} \times \underset{\substack{\uparrow \\ G \times G^\vee\text{-sym.}}}{\mathcal{O}^{A_{reg}}} /\!\!/\!\!/_G \right)^! \quad , \ ! = \text{3d mirror symmetry}$$

J 3d TQFT $\rightsquigarrow$ Higgs(J)

$\rightsquigarrow$ Coulomb(J)

$$\text{Higgs}(\mathcal{O}^{A_{reg}}) = \mathcal{N}_G$$

$$\text{Coulomb}(\mathcal{O}^{A_{reg}}) = \mathcal{N}_{G^\vee}$$

$\Rightarrow$ $J^!$ 3d TQFT, $\quad \text{Higgs}(J^!) = \text{Coulomb}(J)$

$$\text{Coulomb}(J^!) = \text{Higgs}(J)$$

In this sense, $\quad A_G \overset{(OL)}{\longrightarrow} B_G \ — \ \mathcal{O}^{A_{reg}} + !$

Consider

$$\text{Coulomb}(\mathcal{L}^{M^\vee}) = \text{Higgs}(\mathcal{O}^M \times \mathcal{O}^{A_{reg}} /\!\!/\!\!/_G)$$

$$= M \times \underset{\substack{| \\ \text{nilpotent cone}}}{\mathcal{N}_G} /\!\!/\!\!/ G$$

<u>Expectation</u>. If $\mathcal{L}^{M^\vee}$ is really coming from $M^\vee$ (e.g. if smooth), we should have

$$\text{Coulomb}(\mathcal{L}^{M^\vee}) = \text{“}pt\text{''}$$

? hyperspherical

<u>Examples</u>

Yesterday: $[pt \leftarrow G]^\vee = [\check{G} \times \check{S} \leftarrow \check{G}]$

<u>Rmk</u>. $(\mathcal{O}^M)^\vee = \mathcal{L}^{M^\vee}$

$$[\check{G} \times \check{S} \leftarrow \check{G}]^\vee \overset{?}{=} [pt \leftarrow G]$$

not clear

twisted cotangent

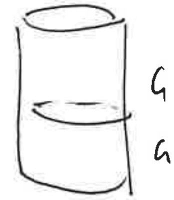
current def. cannot apply.

Exercises

(0) $G \curvearrowright T^*G \curvearrowleft G$

$\uparrow$

identity interface



$M \times T^*G /\!\!/\!\!/ G = M$

$\overset{\text{S-dual}}{\longleftrightarrow} \quad G^\vee \curvearrowright T^*G^\vee \curvearrowleft G^\vee$

(1) $G = \mathbb{G}_m, \quad \mathrm{Gr}_{\mathbb{G}_m} \simeq \mathbb{Z}$

$\mathcal{A}_{reg} = \omega_{\mathrm{Gr}_{\mathbb{G}_m}}$

$\bullet \overset{!}{\otimes} \mathcal{A}_{reg} = \bullet$

$\mathbb{G}_m \curvearrowright M = pt \quad \longleftrightarrow \quad M^\vee = T^*\mathbb{G}_m = \mathbb{G}_m \times \mathbb{A}^1$

$\curvearrowleft$

$\mathbb{G}_m^\vee = \mathbb{G}_m$

(2) (Iwasawa - Tate)

$\mathbb{G}_m \curvearrowright T^*\mathbb{A}^1 = M$

$\mathrm{wt}\,(1,-1)$

$\mathrm{Map}(S^2, [N/\mathbb{G}_m])$

$\longleftrightarrow \quad \mathbb{G}_m^\vee \curvearrowright T^*\mathbb{A}^1$

$\mathrm{wt}\,(1,-1)$

$N = \mathbb{A}^1$

type A_1-singularity

(3) $\mathbb{G}_m \curvearrowright T^*\mathbb{A}^2 = M \quad \longleftrightarrow \quad M^\vee = \mathbb{C}[x,y,w]/xy = w^2$

$\mathrm{wt}\,(1,1,-1,-1)$

$[T^*\mathbb{A}^2 /\!\!/ \mathbb{G}_m]$

$\mathbb{C}[w] = H^*_{\mathbb{G}_m}(pt)$

x = fund. class of fiber over $1 \in \mathbb{Z}$

y = fund. class of fiber over -1

(3)' $\mathbb{G}_m \curvearrowright T^*\mathbb{A}^1$ $\rightsquigarrow$ $M'^\vee = \{xy = w^2\}$ $[\mathbb{A}^2/\pm 1]$

wt $(2,-2)$

Thm [BFN3]

$\mathcal{A}^{reg}$ is realized as follows:

when $G = GL_n$

$$1 \to 2 \to \cdots \to (n-1) \to \boxed{n}$$

$$M = T^*N, \quad N = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1}), \quad G = \prod_{i=1}^{n-1} GL(i)$$

$$\tilde{G} = G \times GL_n$$

$$\begin{array}{ccc} \theta^{M = T^*N} & \in & D_{\tilde{G}(\theta)}(Gr_{\tilde{G}}) \\ \downarrow & & \downarrow \\ p_* \theta^M & & D_{GL_n(\theta)}(Gr_{GL_n}) \\ \| & & \\ \mathcal{A}^{reg} & & \end{array}$$

$GL_n^\vee = GL_n$ equiv. structure is not manifested.

Application. $M^\vee = \mathcal{M}_c(M \times 1 - 2 - \cdots - \boxed{n} /\!\!/\!\!/ GL_n)$

Gannon last week: $GL_n \curvearrowright \overline{T^*(GL_n / U_{n_1,\ldots,n_r})}$ ← affine closure $\curvearrowleft GL_{n_1} \times \cdots \times GL_{n_r}$

$n = n_1 + \cdots + n_r$

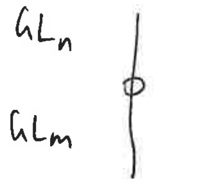
$$= \mathcal{M}_c\left(\begin{array}{l} 1 - 2 - \cdots - (n-1) - (n_1) - (n_1 - 1) - \cdots - 1 \\ \qquad\qquad\qquad\quad\;\; (n-1) - (n_2) - (n_2 - 1) - \cdots - 1 \\ \qquad\qquad\qquad\qquad\qquad \vdots \\ \qquad\qquad\qquad\quad\;\; (n-1) - (n_r) - \cdots \end{array} \right)$$

$$= [GL_n \curvearrowright T^*GL_n \curvearrowleft GL_{n_1} \times \cdots \times GL_{n_r}]^\vee$$

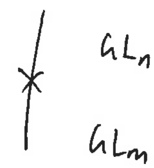
More generally, we expect [Ginzburg – Riche ?]

$$\left[G \curvearrowright T^*G \curvearrowleft L \right]^\vee = \left(G^\vee \curvearrowright T^*\left((G^\vee \times L^\vee)/P^\vee\right) \curvearrowleft L^\vee \right)$$

<u>Example</u>.



S-dual $\longleftrightarrow$



$$\| \qquad\qquad\qquad\qquad \|$$

$$GL_m \curvearrowright T^*\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \curvearrowleft GL_n$$

$$\underset{\text{Bielawski}}{=} \left[\begin{array}{ll} GL_m \times S(m-n, 1^n) & \text{if } m > n \\ m \leftrightarrow n & \text{if } m < n \\ T^*(GL_m \times \mathbb{C}^m) & \text{if } m = n \end{array} \right.$$

space of solutions of Nahm's equations.

$\longleftrightarrow$

$$\{1\} \curvearrowright \mathcal{N} \curvearrowleft GL_n \qquad = \{1\} \curvearrowright T^*GL_n \curvearrowleft GL_n$$