Ramification bound

$$\begin{array}{ccccc} \mathcal{O}_L & & \mathcal{O}_K & & \\ \cap & & \cap & & \\ L & | & K & | & \mathbb{Q}_p \end{array}$$ tower of finite ext'ns, $G = \mathrm{Gal}(L|K)$

π_K uniformizer of K, v val'n on K s.t. $v(\pi_K) = 1$.

$\mathcal{O}_L = \mathcal{O}_K[\alpha]$, π_L uniformizer of L. v extends uniquely to a val'n of L

& $v(\pi_L) = \dfrac{1}{e_{L|K}}$ ramification index

Inertia subgroup $I = \{\sigma \in G : \sigma(x) \equiv x \pmod{\pi_L},\ \forall x \in \mathcal{O}_L\} \triangleleft G$

$$= \{\sigma \in G : v(\sigma(x) - x) > 0,\ \forall x \in \mathcal{O}_L\}$$

$$= \{\sigma \in G : v(\sigma(\alpha) - \alpha) > 0\}$$

<u>Ramification filtration</u>: $G \supset I \supset \cdots$

<u>Lower numbering</u>. $G_{(i)} = \{\sigma \in G : v(\sigma(x) - x) \geq i,\ \forall x \in \mathcal{O}_L\}$

$$= \{\sigma \in G : v(\sigma(\alpha) - \alpha) \geq i\}$$

$i \leq 0$, $G_{(i)} = G$; $i > 0$, $G_{(i)} \subset I$, $0 < i \leq \dfrac{1}{e_{L|K}}$, $G_{(i)} = I$.

For $\sigma \in G$, let $i(\sigma) = v(\sigma(\alpha) - \alpha) \in \dfrac{1}{e_{L|K}}\mathbb{Z} \cup \{+\infty\}$, $i_{L|K} := \max\limits_{\sigma \neq id} i(\sigma)$

<u>Famous problem</u> lower numbering does not play well w/ quotient

$$\mathrm{Gal}(\tilde{L}|K) \twoheadrightarrow \mathrm{Gal}(L|K) \quad \text{for } \tilde{L}|L|K.$$

<u>Upper numbering</u>: $\phi_{L|K}(i) = \sum\limits_{\sigma \in G} \min(i, i(\sigma)) : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$

Fact. $\phi_{L|K}$ is a piecewise linear, monotone increasing function.

Define $G^{(\phi_{L|K}(i))} = G_{(i)}$; $G^{(u)} = G(\phi^{-1}_{L|K}(u))$

Def $u_{L|K} := \phi_{L|K}(i_{L|K})$, $u_{L|K}$ is the largest u for which $G^{(u)} \neq \{1\}$.

Calculate. $u_{L|K} = \phi_{L|K}(i_{L|K}) = \sum_{\sigma \in G} \min(i_{L|K}, i(\sigma))$

$$= i_{L|K} + \sum_{\sigma \neq id} i(\sigma)$$

$$= i_{L|K} + \sum_{\sigma \neq id} v(\sigma(\alpha) - \alpha)$$

$$= i_{L|K} + v\Big(\underbrace{\prod_{\sigma \neq id} (\sigma(\alpha) - \alpha)}_{\mathcal{D}_{L|K} \text{ different}}\Big).$$

$$N_{L|K}(\mathcal{D}_{L|K}) = \Delta_{L|K} \quad \text{discriminant}$$

$v(\mathcal{D}_{L|K}) = u_{L|K} - i_{L|K}$, $v(\Delta_{L|K}) = [L:K](u_{L|K} - i_{L|K})$

$< [L:K]\, u_{L|K}.$

$L|K$ unramified $\iff u_{L|K} = 0$

tamely ram. $\iff u_{L|K} = 1$

wildly ram. $\iff u_{L|K} > 1$.

Ex. $L = \mathbb{Q}_p(\zeta_{p^n}) \,|\, K = \mathbb{Q}_p$, p odd $(p^n > 2)$, $G = Gal(L|K) \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$ $\quad \sigma \longleftrightarrow m$ $\quad \alpha = \zeta_{p^n}$.

$v(\zeta_{p^n} - 1) = \frac{1}{(p-1)p^n}$

$$i(\sigma) = v(\sigma(\zeta_{p^n}) - \zeta_{p^n}) = v\left(\frac{\sigma(\zeta_{p^n})}{\zeta_{p^n}} - 1\right) = v(\zeta_{p^n}^{m-1} - 1) = p^{v(m-1)}\, v(\zeta_{p^n} - 1)$$

$$= \frac{p^{v(m-1)}}{(p-1)\,p^n} \Rightarrow i_{L|K} = \frac{p^n}{(p-1)p^n} = \frac{1}{p-1}$$

$$\frac{p^s}{(p-1)p^{n-1}} < i \leq \frac{p^{s+1}}{(p-1)p^{n-1}} \qquad (s+1 \leq n-1)$$

$$G_{(i)} = \{\sigma \in G : \sigma \equiv 1 \pmod{p^{s+1}}\}$$

$$\phi(i) = \sum_{\sigma \in G} \min(i(\sigma), i) \quad . \quad \phi(i_{L|K}) = i_{L|K} + \sum_{\sigma \neq 1} i(\sigma)$$

$$= \frac{1}{p-1} + \sum_{1 \neq m \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \frac{p^{v(m-1)}}{(p-1)p^n} = n$$

$$v(D_{L|K}) = u_{L|K} - i_{L|K} = n - \frac{1}{p-1} \quad , \quad v(\Delta_{L|K}) = p^n(p-1)\left(n - \frac{1}{p-1}\right)$$

Fontaine's thm

Suppose a finite flat $\overset{\text{comm}}{\vee}$ gp scheme Γ over $\mathcal{O}_K \supset \mathbb{Z}_p$ is killed by p^n.

Let $e_K = e_{K|\mathbb{Q}_p}$, L = field obtained by adjoining the points of Γ to K.

$G = \mathrm{Gal}(L|K)$, then $G^{(u)} = \{1\}$ for $u > e_K \cdot \left(n + \frac{1}{p-1}\right)$

$\Gamma = \mathrm{Spec}\, A$. $A \otimes_{\mathcal{O}_K} K$ is a finite dim'l étale K-alg, so $\simeq \prod_i L_i$, $L_i \overset{\text{sep.}}{\supset} K$. L = compositum of L_i's.

Cor. 1) $u_{L|K} \leq e_K \cdot \left(n + \frac{1}{p-1}\right)$

2) $v(D_{L|K}) = u_{L|K} - i_{L|K} < u_{L|K} \leq e_K \cdot \left(n + \frac{1}{p-1}\right)$.

Eg. $K = \mathbb{Q}_p$, $\Gamma = \mu_{p^n}$, $L = \mathbb{Q}_p(\zeta_{p^n})$

$$e_K\left(n + \frac{1}{p-1}\right) = n + \frac{1}{p-1} . \quad v(D_{L|K}) = n - \frac{1}{p-1} .$$

$L|K|\mathbb{Q}_p$

Main Prop Let A be a finite flat $\mathcal{O}_K$-alg of the form

$A = \mathcal{O}_K [\![x_1, \dots, x_m]\!] / \langle f_1, \dots, f_m \rangle$. Suppose $\exists$ $0 \neq a \in \mathcal{O}_K$ annihilating $\Omega^1_{A/\mathcal{O}_K}$, so that $\Omega^1_{A/\mathcal{O}_K}$ is a flat A/aA-module.

i) Suppose S is a finite flat $\mathcal{O}_K$-alg and I a top. nilp. PD ideal, then

$$\mathrm{Hom}_{\mathcal{O}_K}(A, S) = \mathrm{Im}\left(\mathrm{Hom}_{\mathcal{O}_K}(A, S/aI) \to \mathrm{Hom}_{\mathcal{O}_K}(A, S/I) \right)$$

ii) $L :=$ K adjoining $\bar{k}$-pts of $Y = \mathrm{Spec}\, A$, then $u_{L|K} \leq v(a) + \frac{e_K}{p-1}$.

<u>PD ideal</u>: finite flat $\mathcal{O}_K$-alg. S, ideal $I \subset S$ is a <u>PD ideal</u> if

$\forall x \in I$, $n \in \mathbb{Z}_{\geq 1}$, $\frac{x^n}{n!} = \gamma_n(x) \in I$.

$I^{[m]} :=$ ideal gen. by $\gamma_{n_1}(x_1) \cdots \gamma_{n_r}(x_r)$, $n_1 + \dots + n_r \geq m$.

I is <u>top. nilp</u> if $\bigcap_{m \geq 1} I^{[m]} = 0$.

<u>Pf of i) of Main Prop</u> $m_A :=$ max'l ideal of A, $J = \langle f_1, \dots, f_m \rangle \subset \mathcal{O}_K [\![x_1, \dots x_m]\!]$

$\Omega^1_{A/\mathcal{O}_K}$ is finite free A/aA-mod $\Rightarrow$ $\frac{\partial f_i}{\partial x_j} = a\, p_{ij}$ for some $p_{ij} \in A$.

also $a\, dx_i =$ linear comb. of $df_j = \frac{\partial f_j}{\partial x_k} dx^k$ $\Rightarrow$ the mat. (p_{ij}) is invert.

<u>WTS</u>. $\forall$ $\mathcal{O}_K$-hom. $\phi: A \to S/aI$, $\underline{\exists !}$ lift to $\phi: A \to S$.

Inductively $\overset{!\text{-ly}}{\text{lift}}$ $\phi: \underset{\underset{\mathcal{O}_K [\![x_1, \dots x_m]\!] / \langle f_1, \dots, f_m \rangle}{\parallel}}{A} \to S/aI^{[n]}$ to $\phi: A \to S/aI^{[n+1]}$.

Given $u_1, \dots, u_m \in S$ s.t. $f_i(u_1, \dots, u_m) \in aI^{[n]}$, want to find $\varepsilon_i \in I^{[n]}$ unique modulo $I^{[n+1]}$, s.t. $f_i(u_1 + \varepsilon_1, \dots, u_m + \varepsilon_m) \in aI^{[n+1]}$.

Taylor expansion. $$f_i(\underline{u}+\underline{\varepsilon}) = f_i(\underline{u}) + \underbrace{\frac{\partial f_i}{\partial \underline{x}}(\underline{u})\cdot\underline{\varepsilon}}_{\parallel} + \overbrace{\left(\sum_{|\alpha|\geq 2} \frac{\partial^2 f_i}{\partial \underline{x}^2}(\underline{u})\frac{\underline{\varepsilon}^2}{2!}\right)}^{\text{mult. of } a}$$

$$\left(a\widetilde{p_{ij}}(\underline{u}) + r_{ij}(\underline{u})\right)\varepsilon_j \qquad \left(I^{[n]}\right)^{[2]} \subset I^{[n+1]}$$

lift of p_{ij} $\quad r_{ij} \in J$

$$r_{ij}(\underline{u})\,\varepsilon_j \in a I^{[n]} I^{[n]} \subset a I^{[n+1]}$$

$$\equiv f_i(\underline{u}) + a\, \overset{\text{inv.}}{p_{ij}}(\underline{u})\,\varepsilon_j$$

$\rightsquigarrow \underline{\varepsilon} = (\varepsilon_j)$ exists, and unique.

$\square$

Proof of Main Prop (ii).

Lemma $E|K$ finite Galois, $t \in \mathbb{R}_{>0}$: $m_E^t = \{x \in \mathcal{O}_E : v(x) \geq t\}$.

(i) If $t > u_{L|K}$, any $\mathcal{O}_K$-alg hom. $\mathcal{O}_L \to \mathcal{O}_E/m_E^t$ lifts to an $\mathcal{O}_K$-alg. hom. $\mathcal{O}_L \to \mathcal{O}_E$.

(ii) If every $\mathcal{O}_K$-alg. hom. $\mathcal{O}_L \to \mathcal{O}_E/m_E^t$ lifts to an $\mathcal{O}_K$-alg hom. $\mathcal{O}_L \to \mathcal{O}_E$, then $t > u_{L|K} - \frac{1}{e_{L|K}}$

Pf (i) $f(X) \in \mathcal{O}_K[X]$ minimal poly. of π_L. $\mathcal{O}_L \to \mathcal{O}_E/m_E^t$ is determined by the image $\beta \in \mathcal{O}_E$ of π_L, $f(\pi_L) = 0 \Rightarrow f(\beta) \in m_E^t$, i.e. $v(f(\beta)) \geq t > u_{L|K}$.

On the other hand, $v(f(\beta)) = \phi_{L|K}\left(\sup_{g\in G} v(\beta - g\pi_L)\right)$: suppose g_0 attains max.,

$$\forall g \in G,\ v(\beta - g\pi_L) = \min\left(v(\beta - g_0\pi_L),\ v(g_0\pi_L - g\pi_L)\right) = \min\begin{pmatrix} v(\beta - g_0\pi_L) \\ i_{L|K}(g_0^{-1}g)\end{pmatrix}$$

$$f(\beta) = \prod_{g\in G}(\beta - g\pi_L).$$

Now

$\phi_{L|K}(v(\beta - g_0\pi_L)) > u_{L|K} = \phi_{L|K}(i_{L|K}) \Rightarrow v(\beta - g_0\pi_L) > i_{L|K}$

$\geq v(gg_0\pi_L - g_0\pi_L), \ \forall_{1\neq} g \in G$

Krasner's lemma $\Rightarrow$ $K(g_0\pi_L) \subset K(\beta) \subset E \rightsquigarrow O_L \to O_E$.

$\underset{L}{\shortparallel}$

(ii) ETS: for $t = u_{L|K} - \frac{1}{e_{L|K}}$, $\exists$ O_K-alg hom $O_L \to O_E/m_E^t$ which does not lift.

Step 1 $L|\overbrace{K'|K}^{\text{max'l unramified}}$, $E \otimes_K K' = \prod_i E_i$, $E_i|E$ unram.

any $O_L \to O_E/m_E^t$ (O_K-map) extends to $O_{K'}$-map $O_L \to O_{E_i}/m_{E_i}^t$.

$u_{L|K} = u_{L|K'}$, $e_{L|K} = e_{L|K'}$, so can assume $L|K$ totally ram. $(L \neq K)$

Step2 $L|K$ tame. $i_{L|K} = \frac{1}{e_{L|K}}$, $u_{L|K} = 1$. $t = 1 - \frac{1}{e_{L|K}}$.

$E|K$ any totally ram. ext'n of deg $d < e_{L|K}$. No O_K-alg map $O_L \to O_E$

OTOH, $O_L \to O_E/m_E^t$, $\pi_L \to \pi_E$ unif. of E is well-defined: (ram. indices)

$$v\Big(\prod_{g\in G}(g\pi_L - \pi_E)\Big) \geq \frac{[L:K]}{e_{L|K}} = 1$$

Step3 $L|K$ wild. $\forall\, 1\neq g \in G$, $e_{L|K}\, i_{L|K}(g) \geq 1$, $p \mid [L:K]$, $p-1$ of $g \in G\setminus\{1\}$ satisfies $e_{L|K}\, i_{L|K}(g) \geq 2 \Rightarrow t > 1$.

$e_{L|K}t \in \mathbb{Z} =: e_{L|K}r + s$, $r,s \in \mathbb{Z}_{\geq 0}$, $s < e_{L|K}$.

Let $f(x) \in O_K[x]$ be the minimal poly. of π_L, $g(x) = f(x) - \pi_K^r x^s$.

$e_{L|K} > s \Rightarrow g$ is monic. f Eisenstein. $s > 0 \to g$ Eisenstein

$s = 0 \Rightarrow r \geq 2$, g Eisenstein

β a root of $g(x)$, $E = K(\beta)$. $v(\beta) = \frac{1}{e_{L|K}}$,

g Eisenstein $\Rightarrow$ $E|K$ totally ram. $O_L \to O_E/m_E^t$, $\pi_L \mapsto \beta$: $f(\beta) = \pi_K^r \beta^s$

$\Rightarrow v(f(\beta)) = t$

No $\mathcal{O}_K$ - alg. map $\mathcal{O}_L \to \mathcal{O}_E$: o/w $L \subset E$. $[L:K] = [E:K] \Rightarrow L = E$

$\Rightarrow v(g\pi_L - \beta) \in \frac{1}{e_{L|K}}\mathbb{Z}$, $\forall g \in G$, OTOH, $v(f(\beta)) = t$

$$\Rightarrow \underbrace{e_{L|K} \sup_{g \in G} v(g\pi_L - \beta)}_{\in \mathbb{Z}} = e_{L|K}\, \phi_{L|K}^{-1}\left(v(f(\beta))\right) = e_{L|K}\, \phi_{L|K}^{-1}(t)$$

$d =$ slope of the left segement of $\phi_{L|K}$ at $i_{L|K}$. i.e. $d = |G(i_{L|K})|$

$$e_{L|K}\,\phi_{L|K}^{-1}\left(t = u_{L|K} - \frac{1}{e_{L|K}}\right) = e_{L|K}\left(i_{L|K} - \frac{1}{e_{L|K}\, d}\right) = e_{L|K}\, i_{L|K} - \frac{1}{d} \in \mathbb{Z}$$

$\to d = 1$, Contradiction.

Rmk. After passing to arbitrarily large tame ext'n of L, can show that $t \geq u_{L|K}$.

Proof of Main Prop (ii). $L|K$ tame, $u_{L|K} \leq 1 \leq v_K(a) \leq v_K(a) + \frac{e_K}{p-1}$ ✓

$L|K$ wild. By Lemma, WTS:

Claim For $t > v_K(a) + \frac{e_K}{p-1}$ and finite Galois $E|K$, any $\mathcal{O}_K$-alg. hom. $\mathcal{O}_L \to \mathcal{O}_E / m_E^t$ lifts to an $\mathcal{O}_K$-alg hom. $\mathcal{O}_L \to \mathcal{O}_E$.

Pf of claim. $|Y(\mathcal{O}_E)| \leq |Y(\bar{K})| = |Y(\mathcal{O}_L)|$ (by def'n of L)

w/ equality iff $L \subset E$ iff $\exists$ $\mathcal{O}_K$-alg map $\mathcal{O}_L \to \mathcal{O}_E$.

Note $m_E^t = a\, m_E^{t - v(a)}$, $t - v(a) > \frac{e_K}{p-1} \Rightarrow m_E^{t-v(a)}$ is a top. nilp. PD ideal.

$m_L^{t-v(a)}$ is also a top. nilp. PD ideal. By part (i) of main prop,

$$Y(\mathcal{O}_E) = \mathrm{Im}\left(\mathrm{Hom}_{\mathcal{O}_K}(A, \mathcal{O}_E / a m_E^{t-v(a)}) \to \mathrm{Hom}_{\mathcal{O}_K}(A, \mathcal{O}_E / m_E^{t-v(a)})\right)$$

$$Y(\mathcal{O}_L) = \mathrm{Im}\left(\mathrm{Hom}_{\mathcal{O}_K}(A, \mathcal{O}_L / a m_L^{t-v(a)}) \to \mathrm{Hom}_{\mathcal{O}_K}(A, \mathcal{O}_L / m_L^{t-v(a)})\right)$$

$$\begin{tikzcd}
O_L \arrow[r] & O_E/m_E^t \arrow[r, rightsquigarrow] & Y(O_L) \arrow[r] \arrow[d, leftrightarrow] \arrow[rrr, bend left, "\mathrm{id}"] \arrow[rr, bend right, "\text{injective}"'] & Y(O_E/m_E^t) \arrow[r, hook] & Y(O_E) \arrow[r] & Y(O_L) \\
& & Y(O_L/m_L^t) & & &
\end{tikzcd}$$

$\square$

<u>Proof of Thm</u>. Γ finite flat comm. gp scheme $/O_K$, $L = K(\Gamma(\bar{K}))$,

$G^{(u)} = 1$ for $u > e_K(n + \frac{1}{p-1})$.

Let $\Gamma = \operatorname{Spec} A$. $A = \prod A_i$, $A_i = O_{K_i}[[x_1, \dots, x_m]] / (f_1^{(i)}, \dots, f_m^{(i)})$

$K_i | K$ unram. ext'n

For each A_i, apply Main Prop (ii), $a = p^n$.