Derived Satake with modular coefficients

Dima Arinkin, Roman Bezrukavnikov

Talk 1 (Dima)

Fact [Boyarchenko-Drinfeld]

G group. $(\mathrm{Shv}(G), *)$ is rigid if G is projective

↳ constructible / holonomic D-mod

Example D-modules

$G = \mathbb{G}_a$, $(\mathrm{Shv}(\mathbb{G}_a), *) = (\text{D-mod}(\mathbb{A}^1), \otimes)$

$G =$ Elliptic curve, $(\mathrm{Shv}(G), *) = (\mathrm{Perf}(\text{smooth}), \otimes)$

Observation $H \subset G$, $(\mathrm{Shv}(H\backslash G/H), *)$ rigid if G/H is projective.

Observation': $G \supset \tilde{H} \supset H$, $G/\tilde{H}$ is projective, $\tilde{H}/H$ is a torus.

$\left(\mathrm{Shv}(H\backslash G/H)^{\tilde{H}/H\text{-monodromic}}, *\right)$ rigid.

Formalism. X scheme / stack, $\mathrm{Shv}(X)$
$$\mathrm{Shv}(X) = \mathrm{Ind}(\text{constructible sheaves})$$

Functors f_*, $f^!$, $\overset{!}{\otimes}$

$G =$ group. G-space. $\underset{G\text{-Sch}}{\text{schematic over } BG}$ ← 2-category

$X, Y \in G\text{-Sch}$, $\mathrm{Mor}(X, Y) = \mathrm{Shv}(X \times Y/G)$, composition = convolution.

Duality: ① ~~on objects~~

② Duality on 1-morphisms

does $f \in \mathrm{Mor}(X,Y)$ have a right adjoint $f^R \in \mathrm{Mor}(Y,X)$?

Fact. If $X = G/H$ – projective, then any $f \in \mathrm{Mor}(X,Y)^c$ has a right adjoint $f^R \in \mathrm{Mor}(Y,X)^c$. any $f \in \mathrm{Mor}(Y,X)^c$ has a ~~left~~ right adjoint $f^L \in \mathrm{Mor}(X,Y)^c$

condition on X.

Theorem. For any Y, Z

$$Y \to X \to Z$$

$$\mathrm{Mor}(X,Z) \otimes_{\mathrm{End}(X)} \mathrm{Mor}(Y,X) \xrightarrow{*} \mathrm{Mor}(Y,Z)$$

$*$ is fully faithful

Talk 2 (Rome)

Recall $\{G\text{-schemes}\}$

$$\mathrm{Mor}(X,Y) = \mathrm{Shv}(X \times Y/G)$$

$$\mathrm{Mor}(X,Z) \otimes_{\mathrm{End}(X)} \mathrm{Mor}(Y,X) \xrightarrow{f.f.} \mathrm{Mor}(Y,Z) \qquad (X = G/H \text{ proj.})$$

Apply this to get derived Satake.

$$\mathrm{Gr} = G_F/G_O$$

$$D_{G(O)}(\mathrm{Gr}) \xleftarrow{\sim} D({}_{G_O}\backslash G_F/(I_0,\psi)) \otimes_{D((I_0,\psi)\backslash G_F/(I_0,\psi))_{\mathrm{unip}}} D((I_0,\psi)\backslash G_F/G_O)$$

$$I_0 \overset{\text{radical}}{\subset} I \quad \text{Iwahori.}$$

$$\begin{tikzcd} & U \arrow[dr] & \\ \psi: I_0 \arrow[ur] \arrow[rr] & & \mathbb{G}_a \end{tikzcd}$$

(I_0, ψ) - equivariant - baby Whittaker

Hom $D\big((I_0,\psi)\backslash G_F/(I_0,\psi)\big)_{unip} \subset D\big((I_0,\psi)\backslash G_F/(I_0,\psi)\big)$

full subcat. gen. by the image of the averaging functor from

$D(I^-\backslash G_F/(I_0,\psi))$. This behaves as sheaves on an ind-projective quotient.

Derived Satake

$$D_{G(O)}(Gr) \simeq DCoh^{\check{G}}\big(\{1\} \times_{\check{\mathfrak{g}}} \{1\}\big)$$

1) Thm $D\big(G_O\backslash G_F/(I_0,\psi)\big) \simeq D^b Rep(\check{G})$

"Geometric Casselman - Shalika"

Frenkel Gaitsgory. Vilonen 2000 w/ $\mathbb{C}$- coeff.

B. - Gaitsgory - Mirković - Riche - Rider. w/ modular coeff.

$$Perv\big(G_O\backslash G_F/(I_0,\psi)\big) \simeq Rep(\check{G})$$

2) (w/. Riche, Rider, in progress)

$$D\big((I_0,\psi)\backslash G_F/(I_0,\psi)\big)_{unip} \simeq D^b Coh^{\check{G}}_{\mathcal{U}}(\check{G})$$

unipotent cone

Recall Gaitsgory's central functor $Z: \mathrm{Rep}(\check{G}) \longrightarrow D_I(\mathcal{F}\ell)$

$$D_I(\mathcal{F}\ell) = D(I \backslash G_F / I)$$

Defined by nearby cycles, carries a tensor automorphism (monodromy), allowing to extend it to a functor from $\mathrm{Coh}^{\check{G}}_{\text{free}}(\check{\mathcal{N}})$.

$$\mathcal{F} = V \otimes \mathcal{O}$$

Version $\hat{Z} : \mathrm{Rep}(\check{G}) \longrightarrow D(\acute{I} \backslash G_F / \acute{I})$

monodromic w/ unip. monodromy

Hope: version of this works for $D((I_0, \psi) \backslash G_F / (I_0, \psi))_{\text{unip}}$

Not worked out.

<u>Another approach</u>.

Consider $D(I^- \backslash G_F / I^-) = \mathcal{H}$ $\quad I, I^- \subset G(\mathcal{O})$ opposite

Have obvious functors

$$\begin{tikzcd} & D(I^- \backslash G_F / (I_0, \psi)) \arrow[dr] & \\ \mathcal{H} \arrow[ur] \arrow[rr, "Av_{wh}"'] & & D((I_0, \psi) \backslash G_F / (I_0, \psi))_{\text{unip}} \end{tikzcd}$$

$$D((I_0, \psi) \backslash G_F / (I_0, \psi))_{\text{unip}} = Wh$$

<u>Observations</u>. It's easy to describe the kernel of Av_{wh}.

$\underset{\mathcal{H}}{\mathrm{Irr}(\mathrm{Perv})} = \{ L_w : w \in W_{aff} \}$

$\ker(Av_{wh}) = \langle L_w : w \notin {}^f W^f \rangle$, ${}^f W^f$ - minimal in its 2-sided coset.

$\ker(\mathrm{Av}_{Wh})^{\perp} =: R$

R can be described using $\hat{Z}$ & its properties.

Now R is almost equiv. to Wh.

More precisely,

Wh is nat'lly a cat. over $T^\vee/W$ (set. th. supp. at 1)

Prop. $R \simeq Wh \underset{Coh(T^\vee/W)}{\otimes} Coh(\check{T} \times_{\check{T}/W} \check{T})$

$W \times W$ acts on both sides

Thm. (w/ Riche, Rider)
Then $R \simeq Coh^{\check{G}}_{\check{U}}(\check{\mathfrak{g}} \underset{\check{T}/W}{\times} \check{T} \underset{\check{T}/W}{\times} \check{T})$ $W \times W$-equivariently

$\Rightarrow Wh \simeq Coh^{\check{G}}_{\check{U}}(\check{\mathfrak{g}})$

Prop $\Leftarrow$ formalism of last lecture
+ work in progress w/ S. Deshpande
— modular version of a result of Ginzburg (for D-modules)

We show $D((U,\psi)\backslash G/(U,\psi)) \subset D(\frac{T}{W})$

G - reductive gp

explicit subcat.

$$D((U,\psi)\backslash G/(U,\psi))_{unip} \simeq DCoh_1(\check{T}/\!/W) \subset DCoh(\check{T}/\!/W)$$

$$D(X/(U,\psi))_{unip} \underset{D((U,\psi)\backslash G/(U,\psi))_{unip}}{\otimes} D((U,\psi)\backslash G/U^-)_{unip} \longrightarrow D(X/U^-)$$

$$- \underset{Coh(\check{T}/W)}{\otimes} Coh(\check{T})$$

lands in the right orthogonal to the kernel of the averaging functor.

(and is an equiv. w the kernel at least in our case)

Proof of the Thm:

Recall $D(B\backslash G/U) \to D(B\backslash G/(U,\psi))$

The kernel is gen. by L_w, $w \neq 1$.

Its orthogonal is gen. by Ξ - big projective - projective cover of L_1

(in the monodromic setting - proobject $\hat{\Xi}$)

Our R is gen. by

1) $R = \langle Z(V) * \hat{\Xi} \rangle$

2) in modular setting, can restrict to $V = T_\lambda$ - tilting in $Rep(\check{G})$

Lemma. $\Xi * Z(T_\lambda)$ tilting

$\hat{\Xi} * \hat{Z}(T_\lambda)$ - free-monodromic tilting.

$Ext^{>0}(-,-) = 0$

$Hom(\quad) = Hom(T_\lambda \otimes O, T_\mu \otimes O)$

- proved by analysing the quotient $D(I\backslash G_F/I) / L_w$, $w \neq 1$

$\leftrightarrow Coh(\check{\mathfrak{g}}_{reg})$ (or sheaves on Kostant slice)

Talk 3 (Roma)

$G_F, \quad F = k((t)), \quad Gr$

$$D_{G(O)}(Gr) = D(G_O \backslash G_F / G_O)$$
$$\cup$$
$$\mathrm{Perv}_{G(O)}(Gr) \underset{MV}{\simeq} \mathrm{Rep}(\check{G})$$

Thm (suggested by Drinfeld)

$$D_{G(O)}(Gr) \simeq D\mathrm{Coh}^{\check{G}}(\{1\} \underset{\check{G}}{\times} \{1\})$$

(in char 0, follows from Ginzburg (B.- Finkelberg))

the method uses, eg. weights, doesn't work w/ coefficients k - field of char $l > 0$.

(for l is not small)

Idea comes from R.K. Gordin

$\check{G}/\check{G}$ - stack of L-parameters

$$(\tilde{D}(I^\circ,\psi \backslash G_F/G_O))$$

End $\tilde{D}(I^\circ,\psi \backslash G_F / I^\circ,\psi)$

Idea of proof. Arinkin's talk

$$\tilde{D}_{G(O)}(Gr) = \tilde{D}(G_O \backslash G_F / (I^\circ,\psi)) \underset{\tilde{D}((I^\circ,\psi) \backslash G_F / (I^\circ,\psi))_u}{\otimes} \tilde{D}((I^\circ,\psi) \backslash G_F / G_O)$$

$D(G_O \backslash G_F / (I^\circ,\psi)) \simeq D(\mathrm{Rep}\,\check{G})$ Gaitsgory - Frenkel - Vilonen in char 0.

geometric Casselman Shalika

over k of char $l > 0$

(some assump) BGMRR

$D(I^0,\psi \backslash G_F / I^0,\psi)$

I^0 radical of Iwahori, ψ Whit. char.

$\tilde{D}$ is the large cat. — constructible.

Rmk. "small" version follows.

Fact. $G \curvearrowright X$, $Av_{U,\psi}: D_{B^-}(X) \longrightarrow D_{U,\psi}(X)$

commutes w/ duality

Prove that $D(I^0,\psi \backslash G_F / I^0,\psi)_u \simeq D\,Coh^{\check{G}}_{u}(\check{G})$ — set theoretic supp. on $U \subset \check{G}$ (unipotent cone)

full subcat. gen. by the image of averaging

$$Av: D(I_- \backslash G_F / I_0,\psi) \longrightarrow \underline{\qquad}$$

$I, I_- \subset G(O)$

Lemma $D^b_{perf}\left(Coh^{\check{G}}_{u}\left(\check{G} \times_{\check{T}/\!/W} \check{T}\right)\right) \xhookrightarrow{f.f.} D(I_- \backslash G_F / I_0,\psi)$

$$\downarrow Av_{I_0,\psi}$$

$$D(I_0,\psi \backslash G_F / I_0,\psi)_u$$

s.t. the image $= \underbrace{\ker}_{\langle L_w,\, w \notin {}^fW \rangle}(Av_{I_0,\psi})^\perp$.

Digression: $St = \tilde{g} \times_{g} \tilde{g}$

$\pi_* O = O$

π^* is fully faithful

$$\begin{tikzcd} & St \arrow[dl] \arrow[dr] & \\ \tilde{g} \arrow[d, "\pi"'] \arrow[drr] & & \tilde{g} \arrow[d] \\ g \times_{t/\!/W} t \arrow[rr, "affine"] & & g \end{tikzcd}$$

Proof of Lemma: use central functors

$$\begin{tikzcd} \hat{Z} : \mathrm{Rep}(\check{G}) \arrow[r] \arrow[d] & D(I_-\backslash G/I_-) \arrow[d] \\ V\otimes\mathcal{O} \quad \mathrm{Coh}^{\check{G}}_{fr}(\check{G}\times_{\check{T}/\!/W}\check{T}) \arrow[r] \arrow[ur] & D(I_-\backslash G/I_0,\psi) \end{tikzcd}$$

V

$\int$

building a functor from $\mathrm{Coh}^{\check{G}}(\check{G})$ from $\mathrm{Rep}(\check{G})$ use monodromy automorphism of $\hat{Z}$ + Tannakian form.

Claim. $\mathrm{Rep}(\check{G}) \longrightarrow \mathrm{Perv}(I_-\backslash G/I,\psi)$ (See B. Rider Riche) sends tilting to tilting. $\Rightarrow$ full faithfulness

This + description of the quotient cat.

$$\mathrm{Perv}(I_-\backslash G_F/I_0,\psi) \Big/ \langle L_w,\ \ell(w)>0\rangle \Rightarrow \text{full faithfulness}$$

$$\mathrm{Perv}(I_-\backslash G_F/I_-)$$

in (BRR) such quotient of $\mathrm{Perv}(I^-\backslash G_F/I^-) \simeq \mathrm{Rep}(Z_{\check{G}}(u))$

u - regular.

Lemma $\Rightarrow$ the equiv. $D(I,\psi\backslash G_F/I,\psi)_u \simeq D\mathrm{Coh}^{\check{G}}_u(\check{G})$

(partly relies on a result w/ T. Deshpande)

Rmk passing from $\check{G}\times_{\check{T}/\!/W}\check{T}$ to $\check{G}$ is similar to passing from

$D(B\backslash G/U)\big/(\ker \text{Transl. to the wall}) \simeq$ sing. block for $\tilde{U} = U\otimes_{\mathbb{C}[t/\!/W]}\mathbb{C}[t]$

to $D(B\backslash G/U,\psi)$

Lemma $\Rightarrow$ equiv.

Claim. $G \curvearrowright X$, $\quad D(U,\psi\backslash X) \supset D(U,\psi\backslash X)_u$

span of the image

$$\begin{array}{c} D(U,\psi\backslash X) \\ \uparrow \mathrm{Av} \\ k[\check{T}] \curvearrowright D(B_-\backslash X) \end{array}$$

monodromy

$$D(B_-\backslash X) \supset \ker(\mathrm{Av})^{\perp} \simeq D(U,\psi\backslash X) \otimes_{k[\check{T}/\!/W]} k[\check{T}] \curvearrowleft W$$

$$\frac{D(B_-\backslash X)}{\ker(\mathrm{Av})} \curvearrowleft W \quad \text{(related to Kazhdan-Laumon action)}$$

$$\simeq \mathrm{Coh}_1(\check{T})$$

$$D(B_-\backslash G/U,\psi) \underset{D(U,\psi\backslash G/U,\psi)}{\otimes} D(U,\psi\backslash X) \xhookrightarrow{\text{f.f. by Arinkin}} D(B_-\backslash X)$$

$D(U_-\backslash G/U,\psi)$

$\wr\!\wr$

$D(T) \qquad\qquad D(U,\psi\backslash G/U,\psi)_u \simeq \mathrm{Coh}_1(\check{T}/\!/W)$

So $\tilde{D}(B_-\backslash X) \supset \tilde{D}(U,\psi\backslash X)_u \underset{k[\check{T}/\!/W]}{\otimes} k[\check{T}]$

In our case want to describe $D(U,\psi\backslash X)_u$. $X = G_+(O)\backslash G_F/I,\psi$

$G_+(O) = \ker(G(O) \to G)$

first get $D(B_-\backslash X) = ? \underset{k[\check{T}/\!/W]}{\otimes} k[\check{T}]$

Check W-actions match

Question Can one run the construction of Z directly for $D(I_{0,\psi} \backslash G_F / I_{0,\psi})_u$?

Summary.

$$Coh^{\check{G}}(\check{\mathcal{N}}) \xrightarrow{\hat{Z}} D(I_- \backslash G_F / I_-)$$