The v-topology

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§1

Def. R perfectoid Tate ring, the <u>tilt</u>

$$R^\flat := \varprojlim_{x \mapsto x^p} R, \quad \text{equipped w/ the inverse limit topology.}$$

will define addition on $R^\flat$ via

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) = (z^{(0)}, z^{(1)}, \dots)$$

where $z^{(i)} = \lim_{n \to \infty} \left(x^{(i+n)} + y^{(i+n)}\right)^{p^n}$

Lemma 1. The above converges & defines a ring str. on $R^\flat$, which makes it a topological $\mathbb{F}_p$-alg., and a perfect complete Tate ring.

The power bdd elts are given by $R^{\flat,\circ} \simeq \varprojlim_{x \mapsto x^p} R^\circ \overset{\text{ring isom.}}{\simeq} \varprojlim_{\Phi} R^\circ/p$

Moreover, $\exists$ a pseudo uniformizer $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots)$ for $\varpi \in R^\circ$ a p.u.

and $R^\flat = R^{\flat,\circ}[\frac{1}{\varpi^\flat}]$.

Pf (sketch). By construction, $R^\flat$ is perfect. Let $\varpi_0 \in R$ be a p.u.

need to show any sequence $(\bar{x}_0, \bar{x}_1, \cdots) \in \varprojlim_{\Phi} R^\circ/p$ lifts uniquely to

$(x_0, x_1, \cdots) \in \varprojlim_{\Phi} R^\circ$.

Take any lifts x_i of $\bar{x}_i$ $\rightsquigarrow$ $(x^{(0)}, x^{(1)}, \cdots)$ by $x^{(i)} = \lim_{n\to\infty} x_{n+i}^{p^n}$.

The limit existing boils down to : $x \equiv y \mod p^n$, then $x^p \equiv y^p \mod p^{n+1}$.

Passing the addition str. on $\varprojlim_{\Phi} R^\circ/p$ to $\lim_{x \mapsto x^p} R^\circ$ gives our defined addition.

Arrange first that $\varpi_0^p | p$, $R^{\flat,\circ} \to \varprojlim_{\Phi} R^\circ/\varpi_0^p \to R^\circ/\varpi_0^p$

ϖ_0

Taking the preimage of ϖ_0 under this map yields the desired $\varpi^\flat$. $\square$

Example. $\mathbb{Q}_p^{cyc} := \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ perfectoid field.

Then $(\mathbb{Q}_p^{cyc})^\flat$ has pseudo uniformizer given by $t = (1, \zeta_p, \zeta_{p^2}, \cdots) - 1$

fix a compatible system of p^nth roots of 1.

$\left[(1-\zeta_p)^{p-1} = p \cdot \text{unit in } \mathcal{O}_{\mathbb{Q}_p(\zeta_p)} \right]$

Then in fact $(\mathbb{Q}_p^{cyc})^\flat \cong \mathbb{F}_p((t^{1/p^\infty}))$.

$\cong$

$\left(\widehat{\mathbb{Q}_p(p^{1/p^\infty})}\right)^\flat$

non isom. perfectoid fields can have same tilt

$(-)^\sharp : R^\flat \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} R \xrightarrow[\pi_0]{} R$ cts multiplicative, but not additive

This projection induces a ring isom $R^{\flat,\circ}/\varpi^\flat \xrightarrow[\sim]{(-)^\sharp} R^\circ/\varpi$.

rewrite as $\varprojlim_{\Phi} R^\circ/\varpi$.

(Need $\varpi^p \mid p$, admits p^nth roots, and

$\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \dots)$)

<u>Thm</u> (R, R^+) perfectoid Huber pair, tilt $(R^\flat, R^{\flat +})$, then

$\exists$ homeomorphism. $X = \mathrm{Spa}(R, R^+) \simeq X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$

$$x \longmapsto \underbrace{x^\flat}$$

defined by $f(x^\flat) = f^\sharp(x)$

This homeom. preserves rat'l subsets, and for any rat'l subset $U \subset X$ w/ image $U^\flat \subset X^\flat$, $\mathcal{O}_X(U)$ is perfectoid w/ tilt $\mathcal{O}_{X^\flat}(U^\flat)$.

$\rightsquigarrow$ (R, R^+) stably uniform $\Rightarrow$ sheafy.

<u>Def</u>. We say a perfectoid Huber pair $(R^\sharp, R^{\sharp +})$ is an <u>untilt</u> of a Huber pair (R, R^+) if $\exists$ an isom. $R^{\sharp\flat} \simeq R$, identifying $(R^{\sharp +})^\flat$ and R^+.

<u>Prop</u>. Let $(R^\sharp, R^{\sharp +})$ be an untilt of (R, R^+), then $\exists$ canonical surj. ring hom. $\theta : W(R^+) \longrightarrow R^{\sharp,+}$; $\sum [r_n] p^n \longmapsto \sum r_n^\sharp p^n$.

$\uparrow$ Witt vectors

is kernel principal ideal (ξ) where $\underset{\substack{\uparrow \\ \text{non zero divisor}}}{\xi} = p + [\varpi]\alpha$ $\quad \alpha \in W(R^+)$

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§ v-topology

Let Perfd = cat. of all perfectoid spaces

$\cup$

Perf = cat. of perf. spaces in char. p.

Def. The v-topology on Perfd is the topology generated by open covers and all surjective maps of affinoids, i.e. $\{f_i : X_i \to Y\}_{i \in I}$ is a cover $\Leftrightarrow$ $\forall\ V \subset Y$ quasi cpt open, $\exists\ I_V \subset I$ finite + quasi cpt opens $U_i \subset X_i$ for $i \in I_V$ s.t. $V = \bigcup_{i \in I_V} f_i(U_i)$

A convenient basis for the v-topology comes as follows:

Ex 1. $S = \mathrm{Spa}(A, A^+)$ affinoid perfectoid, $\varpi \in A^+$ p.u.,

$x \in |\mathrm{Spa}(A, A^+)| \rightsquigarrow x : (A, A^+) \to (k_x, k_x^+) \to (k(x), k(x)^+)$

Let $\varpi_x = x(\varpi) \in k_x$. $k(x)^+$ ϖ_x-adic completion.

perfectoid Huber pair.

Set $R^+ = \prod_{x \in |S|} k(x)^+$ $\quad$ (ϖ'-adic topology) p.u. $\varpi' := (\varpi_x)_x$

$R = R^+[1/\varpi']$, $\quad \tilde{S} := \mathrm{Spa}(R, R^+)$ again perfectoid, and $\tilde{S} \to S$ is a v-cover.

" "

We call an aff. perf. S a <u>product of points</u> if $S = \mathrm{Spa}(R, R^+)$

$R^+ = \prod_i K_i^+$, $R = R^+[1/\varpi]$, and each (K_i, K_i^+) is an affinoid perfectoid field.

<u>Def</u>: A perfectoid space is (strictly) totally disconn'd if it's qcqs and every (étale) open cover of it admits a splitting.

<u>Lemma</u>. A product of pts is totally disconn'd.

Affinoid
Totally disconnected spaces form a basis for v-topology.

For totally disconn'd spaces, flatness is automatic.

<u>Prop</u>. $X = \mathrm{Spa}(R, R^+)$ totally disconn'd perfectoid, $f^*: (R, R^+) \to (S, S^+)$ to any Huber pair, $\varpi \in R$ p.u., then S^+/ϖ is <u>flat</u> over R^+/ϖ.

Moreover, if $f: |\mathrm{Spa}(S, S^+)| \to |\mathrm{Spa}(R, R^+)|$ is surjective, then

S^+/ϖ is <u>faithfully flat</u> over R^+/ϖ

This is the key fact to show:

<u>Thm</u>: The functors $\mathrm{Perfd} \to \mathrm{Ab}$,
$$X \mapsto H^0(X, \mathcal{O}_X)$$
$$X \mapsto H^0(X, \mathcal{O}_X^+)$$
are sheaves on the v-topology.

Moreover, when X is affinoid, $H^i_v(X, \mathcal{O}_X) = 0$, $\forall i > 0$

$$H^i_v(X, \mathcal{O}_X^+)^a = 0, \quad \forall i > 0.$$

Key pt of the proof. Reduce to the case X totally disconn'd, show the Čech cpx of a v-cover $Y \to X$ is acyclic, using faithful flatness.

For sheafiness, ✓. For acyclicity, use Čech-to-derived ss.

Cor. Representable presheaves $h_X : Y \mapsto \mathrm{Hom}(Y, X)$ are sheaves on the v-site

X adic space.

Pf. Reduce to case $Y = \mathrm{Spa}(S, S^+)$, $X = \mathrm{Spa}(R, R^+)$,

a v-cover $Y_i \to Y$ $\rightsquigarrow$ $Y_i \to X$ $\rightsquigarrow$ $(R, R^+) \to (\mathcal{O}(Y_i), \mathcal{O}^+(Y_i))$

agreeing on overlaps

$\mathcal{O}, \mathcal{O}^+$ v-sheaves $\Rightarrow$ glue to get maps

$$(R, R^+) \to (S, S^+). \quad \square$$

Thm The fibered cat. $\mathrm{Perfd} \to \mathrm{Gpd}$ is a stack for

$$X \mapsto \{\text{locally free } \mathcal{O}_X\text{-modules}\}$$

the v-topology.

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§3. v-sheaves associated to spaces.

Want to define a functor $\left\{ \begin{matrix} \text{adic spaces} \\ \text{over } \mathbb{Z}_p \end{matrix} \right\} \xrightarrow{\diamond} \left\{ \begin{matrix} \text{(small)} \\ v\text{-sheaves} \end{matrix} \right\}$

$X \longmapsto X^\diamond : \mathrm{Perf} \longrightarrow \mathrm{Set}$

$$X^\diamond(S) = \coprod_{S^\sharp,\, \iota: (S^\sharp)^\flat \xrightarrow{\sim} S} \mathrm{Hom}_{\mathbb{Z}_p}(S^\sharp, X)$$

e.g. $X = \mathrm{Spa}\, \mathbb{F}_p$, $X^\diamond$ is the trivial functor, $S \mapsto pt$.

$X = \mathbb{Z}_p$, $X^\diamond =: \mathrm{Spd}\, \mathbb{Z}_p : S \mapsto \{$ isom. classes of untilts of $S^\sharp / \mathbb{Z}_p \}$.

Lemma. The presheaf $X^\diamond$ is a v-sheaf.

Key point in proof. need $\mathrm{Spd}\, \mathbb{Z}_p$ is a v-sheaf.

this result + representable presheaves are v-sheaves $\Rightarrow$ Lemma.

This construction also gives a functor

$$\begin{tikzcd} \{\text{schemes}/\mathbb{Z}_p\} \arrow[rr, "\diamond"] \arrow[dr] & & \text{v-sheaves} \\ & \{\text{adic spaces}/\mathbb{Z}_p\} \arrow[ur, "\diamond"'] & \end{tikzcd}$$

$\mathrm{Spec}\, R \longmapsto \mathrm{Spa}(R, R)$

Q: What information does $X^\diamond$ carry?

If X scheme in char. p, then in fact $X^\diamond = (X_{\mathrm{perf}})^\diamond$

Prop. $X \longmapsto X^\diamond$, $\left\{\begin{matrix}\text{perfect schemes in}\\ \text{char. } p.\end{matrix}\right\} \longrightarrow \{v\text{-sheaves}\}$

is fully faithful.

For formal <u>schemes</u>, call a formal scheme $\mathfrak{X}$ <u>nice</u> if $\mathfrak{X}$ is locally formally of finite type, flat, normal / $\mathcal{O}_E$. $E/\mathbb{Q}_p$ complete discretely valued ext'n, perf. res. field.

$$\left\{\begin{matrix}\text{nice formal schemes}\\ /\mathcal{O}_E\end{matrix}\right\} \xrightarrow{\diamond} \left\{\begin{matrix}\text{(small)}\\ v\text{-sheaves over } \operatorname{Spd}\mathcal{O}_E\end{matrix}\right\} \text{ is fully faithful.}$$

$$\operatorname{Spd}\mathcal{O}_E = \operatorname{Spa}(\mathcal{O}_E)^\diamond$$

$$\left\{\begin{matrix}\text{Seminormal}\\ \text{rigid space}\\ /K\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}v\text{-sheaves}\\ \text{over } \operatorname{Spd} K\end{matrix}\right\} \text{ fully faithful.}$$

K n.a. field $/\mathbb{Q}_p$