Symplectic singularities and vertex algebras

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Lecture 1

Beem - Lemos - Liendo - Peelaers - Rastelli - van Rees (BL^2PRvR)

$$
\begin{tikzcd}
\{4d\ \mathcal{N}=2\ \text{SCFTs}\} \arrow[r, "\mathbb{V}"] \arrow[dr, "\text{Schur index}"'] & \{\text{VOAs}\} = \{\text{2D chiral CFT}\} \arrow[d] \\
& \{q\text{-series}\}
\end{tikzcd}
$$

$$V = \bigoplus_{d \geq 0} V_d \longmapsto \chi_V(q) = q^{-\frac{c}{24}} \sum_{d \geq 0} (\dim V_d)\, q^d$$

Beem - Rastelli

- $\mathrm{Higgs}(T) \cong X_{\mathbb{V}(T)} = \mathrm{Specm}\, R_{\mathbb{V}(T)}$

  ass. var.

$$\left(\begin{array}{l} \bullet\ \mathrm{gr}\, \mathbb{V}(T) \cong \mathbb{C}[J_\infty \tilde{X}_{\mathbb{V}(T)}] \\ \qquad\qquad \text{arc space} \\ \\ \tilde{X}_V = \mathrm{Spec}\, R_V \quad \text{assoc. scheme} \end{array} \right)$$

R_V Zhu's C_2-alg.

$$V \longleftrightarrow \mathrm{span}\{ :(\partial^{n_1} a_1(z)) \cdots (\partial^{n_r} a_r(z)): \}$$

$$R_V = V / C_2(V), \quad C_2(V) = \{ \quad " \quad ; \textstyle\sum n_i \geq 1 \}$$

$$\overline{f(z)}\,\overline{g(z)} := \overline{:f(z)g(z):}$$

$$\{\overline{f(z)}, \overline{g(z)}\} = \mathrm{Res}_{w=z}\, f(w)g(z)$$

$$\mathrm{Higgs}(T) \underset{S^1 \text{ compactification}}{=} \mathrm{Higgs}(T_{3D}) = \mathrm{Coulomb}(\check{T}_{3D})$$

Costello – Creutzig – Gaiotto, Beem – Ferrari

$\forall$ 3D $\mathcal{N}=4$ gauge theory T

$\exists$ VA V_T (Gaiotto's boundary VOA)

s.t. $\mathrm{Higgs}(T) \simeq X_{V_T}$

$\mathrm{Coulomb}(T) \simeq \mathrm{Spec}\left(\mathrm{Ext}^\bullet(V_T, V_T)\right)$

Higgs(T) is a hyperkähler cone, in particular, symplectic (w/ singularities)

(The normalization of)

* Higgs(T) should be a symplectic singularity.

Beauville. X is a symplectic singularity (ss)

$\overset{\text{def}}{\iff}$ 1) X is normal

2) $\exists\, \omega$ symplectic 2-form on X_{reg}

3) for any $\pi: \tilde{X} \longrightarrow X$ res. sing., $\pi^*\omega$ extends to a holo. 2-form on $\tilde{X}$.

Rmk. 1) " for any $\tilde{X} \to X$ " can be replaced by " a $\tilde{X} \to X$ "

2) X ss $\Rightarrow$ X Poisson

$\{f, g\}$ = unique ext. of $\{f|_{reg}, g|_{reg}\}$

Eg (1) X smooth symp. var.

(2) $\mathfrak{g}$ simple, $X = \mathcal{N} = \{ x \in \mathfrak{g} : (\mathrm{ad}\, x)^n = 0,\ n \gg 0 \} \subset \mathfrak{g} = \mathfrak{g}^*$

nilp. cone

$T^*(G/B) \longrightarrow \mathcal{N}$ Springer resolution

$\mathcal{N} = \overline{\mathcal{O}_{prin}}$

(3) More generally, the normalization of nilpotent orbit closure $\overline{\mathbb{O}}$.

(4) (Beauville) X s.s.

$G \subset \mathrm{Aut}(X)$ preserving ω, G finite

$\Rightarrow X/G$ s.s.

(4)' Γ cpx reflection gp

$\Gamma = \langle r_\alpha \rangle \subset GL(\mathfrak{h}_\Gamma)$, $\mathfrak{h}_\Gamma = \mathbb{C}^{\mathrm{rk}\,\Gamma}$

$T^* \mathfrak{h}_\Gamma = \mathfrak{h}_\Gamma \oplus \mathfrak{h}_\Gamma^* \circlearrowleft \Gamma$ diagonal

$M_\Gamma := T^* \mathfrak{h}_\Gamma / \Gamma$ s.s.

(5) (Affine) Nakajima quiver varieties

Kaledin.

s.s. $\Rightarrow$ has only finitely many symplectic leaves

(and leaves are algebraic)

(Brown - Gordon)

So $X_{W(T)}$ should have finitely many s. leaves.

i.e. $W(T)$ is quasi-lisse (A - Kawasetsu)

(lisse if $X_V = pt$)

Example $\mathfrak{g}$ simple

$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ affine KM

$k \in \mathbb{C}$, $V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \underset{U(\mathfrak{g}[t] \oplus \mathbb{C}K)}{\otimes} \mathbb{C}_k$

$\uparrow$
$K = k$
$\mathfrak{g}[t] = 0$

$R_{V^k(\mathfrak{g})} \cong S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$

$L_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$

$\mathbb{C}[\mathfrak{g}^*] = R_{V^k(\mathfrak{g})} \longrightarrow R_{L_k(\mathfrak{g})} = \dfrac{\mathbb{C}[\mathfrak{g}^*]}{I_k}$

gen. by image of null vectors

Remark

$X_{L_k(\mathfrak{g})}$ = zero locus of $I_k \subset \mathfrak{g}^*$

G-inv. conic

Thm $X_{L_k(\mathfrak{g})} = \{0\} \iff L_k(\mathfrak{g})$ is integrable $(\iff k \in \mathbb{Z}_{\geq 0})$

proof of $\Leftarrow$: $\quad :e_\theta(z)^{k+1}: \quad \theta$ highest root

$\Rightarrow e_\theta^{k+1} \in I_k \quad \Rightarrow e_\theta \in \sqrt{I_k} \ni \mathfrak{I}$

$\underset{\mathfrak{g}\text{ simple}}{\Longrightarrow} \mathfrak{g} \subset \sqrt{I_k} \quad \Rightarrow \quad X_{L_k(\mathfrak{g})} = 0. \quad \square$

$k \in \mathbb{Z}_{\geq 0}$ $\quad L_k(\mathfrak{g})$ is lisse (quasi-lisse)

but it does not come from 4D $\quad (c_{2D} = -12\, c_{4D})$

{integrable reps} $\subsetneq$ {adm. reps}

completely reducible $\quad$ "

modular inv. $\quad$ "

$L_k(\mathfrak{g})$ is admissible $\iff k + h^\vee = \frac{p}{q} \in \mathbb{Q}_{>0}$, $(p,q) = 1$, $p \geq \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1 \\ h & \text{if } (q, r^\vee) = r^\vee \end{cases}$

$L(\lambda)$ is adm. $\iff$ 1) λ is regular dom.

$\Rightarrow \operatorname{ch} L(\lambda) = \sum_{w \in \hat{W}(\lambda)} (-1)^{\ell(w)} \frac{e^{w\circ\lambda}}{R}$

2) $\hat{W}(\lambda)$ is infinite gp containing translation part

Thm (Feigin–Malikov, A)

$L_k(\mathfrak{g})$ adm. $\Rightarrow$ $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_q}$ $\exists\, \mathbb{O}_q$ nilp orbit

$$= \begin{cases} \{x \in \mathfrak{g} : (\operatorname{ad} x)^{2q} = 0\} & \text{if } (r^\vee, q) = 1 \\ \{x \in \mathfrak{g} : \pi_{\theta_s}(x)^{\frac{2q}{r^\vee}} = 0\} & \text{if } (r^\vee, q) = r^\vee \end{cases}$$

θ_s highest short root

Proof for $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ for $\mathfrak{g} = \mathfrak{sl}_2$

$$\mathcal{N} = \{\lambda \in \mathfrak{g}^* : \Omega(\lambda) = 0\}, \quad \Omega = ef + fe + \tfrac{1}{2}h^2$$

$$L_k(\mathfrak{g}) = \frac{V^k(\mathfrak{g})}{\langle v \rangle}$$

$0 \neq \bar{v} \in R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$ singular vector for $\operatorname{ad}\mathfrak{g}$-action

Kostant $\bar{v} =$ nonzero constant $\times\, \Omega^m \times e^n$

$$0 = \Omega^m(\lambda)\, e^n(\lambda)$$

$\Omega(\lambda) = 0 \Rightarrow \lambda \in \mathcal{N}$

$e(\lambda) = 0 \Rightarrow x(\lambda) = 0, \ \forall x \in \operatorname{ad}\mathfrak{g}$-submod. gen. by $e = \mathfrak{g}$

$\Rightarrow \lambda = 0$!

Rmk 1) (Song–Xie–Yan, Xie–Yan–Yau) adm. $L_k(\mathfrak{g})$ comes from some Argyres–Douglas theory, if k is boundary admissible (p is as small as possible)

2) $X_{L_k(\mathfrak{g})} = \mathrm{Specm}\, H^*(u_\zeta(\mathfrak{g}), \mathbb{C})$

if q is odd and not bad prime for $\mathfrak{g}$

where $u_\zeta(\mathfrak{g})$ is small quantum gp, $\zeta = \sqrt[q]{1}$

Deligne exceptional series

$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$

<u>Thm</u> (A.- Moreau) $\mathfrak{g} \in$ DES, $k = -\frac{h^\vee}{6} - 1. \Rightarrow X_{L_k(\mathfrak{g})} \cong \overline{\mathbb{O}_{min}}$

<u>Rmk</u> If $\mathfrak{g} \in$ DES, simply laced, they are exactly the examples VAs coming from 4D in [BL²PRvR]

<u>Lecture 2</u>. <u>Recall</u>. Γ cpx reflection gp

$\Gamma \subset GL(\mathfrak{h}_\Gamma)$

$\mathcal{M}_\Gamma = T^* \mathfrak{h}_\Gamma / \Gamma$

$\Gamma = W(\mathfrak{g})$ Weyl gp of $\mathfrak{g}$, $\mathcal{M}_\Gamma = \mathrm{Higgs}(\mathrm{SYM}_{\mathfrak{g}})$

$\mathcal{N}=4$ super Yang-Mills w/ gauge algebra $\mathfrak{g}$

large family of crystrographic gp $\Leftrightarrow$ $\mathcal{N}=3$;

Conj: (Bonneti – Mengheli – Rastelli)

$\forall\ \Gamma$ cpx reflection gp, $\exists\ V_\Gamma$ VsA s.t.

i) $X_{V_\Gamma} \cong \mathcal{M}_\Gamma = \mathrm{Spec}\,(\mathbb{C}[T^*\mathfrak{h}_\Gamma]^\Gamma)$

In particular, V_Γ is quasi-lisse.

ii) Conformal, central charge $= -3\sum_{i=1}^{\mathrm{rk}\,\Gamma}(2p_i - 1)$

where $p_1, \cdots, p_{\mathrm{rk}\,\Gamma}$ are the degrees of fundamental invariants of Γ

$\mathbb{C}[\mathfrak{h}_\Gamma]^\Gamma = \mathbb{C}[f_1, \cdots, f_{\mathrm{rk}\,g}]$, $p_i = \deg f_i$

iii) V_Γ has $\begin{cases} \mathcal{N}=2 & \\ \mathcal{N}=4 & \text{if } \Gamma \text{ Coxeter gp} \end{cases}$

Supersymmetry

$V_\Gamma \xleftarrow{\text{Conformal}} \begin{cases} \mathrm{Vir}_{\mathcal{N}=2} \\ \mathrm{Vir}_{\mathcal{N}=4} \\ \quad \text{small } \mathcal{N}=4 \end{cases}$

iv) $V_\Gamma \underset{\substack{\text{free field}\\ \text{realization}}}{\hookrightarrow} (\beta\gamma - bc)^{\otimes \mathrm{rk}\,\Gamma}$

affinization of Weyl alg $\otimes$ Clifford alg.

☆ $\Gamma = W(\mathfrak{sl}_N) = \mathfrak{S}_N$

$$\mathfrak{h}_{\mathfrak{S}_N} = \mathbb{C}^{N-1} \circlearrowleft \mathfrak{S}_N$$

$$\mathcal{M}_{\mathfrak{S}_N} = \frac{\mathbb{C}^{N-1} \oplus \mathbb{C}^{N-1}}{\mathfrak{S}_N}$$

$$\text{c.c.} = -3 \sum_{i=1}^{N-1} (2i-1) = -3(N^2-1)$$

$\star$ $N=2$, $\mathcal{M}_{\mathfrak{S}_2} = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/2}$ $x \longleftrightarrow -x$, $y \longleftrightarrow -y$

$$= \operatorname{Spec} \underbrace{\mathbb{C}[x^2, y^2, xy]}_{\shortparallel}$$

$$\frac{\mathbb{C}[a,b,c]}{(ab-c^2)} = \mathcal{N}(sl_2)$$

c.c. $= -9$

$$\mathrm{Vir}^{-9}_{N=4} \supset V^{-\frac{3}{2}}(sl_2)$$

simple quotient

Adamovic $\mathrm{Vir}_{N=4,\, c=-9}$

- $\mathrm{Vir}_{N=4,\, c=-9} \hookrightarrow \beta\gamma\, bc$

- $X_{\mathrm{Vir}_{N=4,\, c=-9}} \cong \mathcal{N}(sl_2)$

$$\Rightarrow V_{\mathfrak{S}_2} = \mathrm{Vir}_{N=4,\, c=-9}$$

* How to understand this?

Springer resolution

$$T^*\mathbb{P}^1 \longrightarrow \mathcal{N}$$

$$\mathbb{C}[\mathcal{N}] = \Gamma(T^*\mathbb{P}^1, \mathcal{O}_{T^*\mathbb{P}^1})$$

$\Omega^{ch}_{\mathbb{P}^1}$: chiral de Rham cpx on $\mathbb{P}^1$ (Malikov - Schechtman - Vaintrob)

locally $\beta\gamma$ bc system

<u>Question</u> $\Gamma(\mathbb{P}^1, \Omega^{ch}_{\mathbb{P}^1}) \cong Vir_{N=4, c=-9}$?

<u>No</u> because $SL_2 \curvearrowright \mathbb{P}^1$ induces $\widehat{sl_2}$ on $\Gamma(\mathbb{P}^1, \Omega^{ch}_{\mathbb{P}^1})$ at level 0.

<u>Gorbounov - Malikov - Schechtman</u>

$\exists\ \Omega^{ch}_{\mathbb{P}^1, \alpha}$ twisted chiral de Rham cpx $(\alpha \in \frac{1}{2}\mathbb{Z})$

(locally $\beta\gamma$ bc)

$\widehat{sl_2}$ - action is at level $2\alpha^2 - 2$

<u>Thm</u> [A- Kuwabara - Möller]

$$\Gamma(\mathbb{P}^1, \Omega^{ch}_{\mathbb{P}^1, \frac{1}{2}}) \cong Vir_{N=4, c=-9}.$$

* For $N > 2$, $\mathcal{M}_{\Gamma_{G_N}}$ is not resolved by some T^*M.

* More convenient to work on $\mathfrak{gl}_N$ setting

$$\frac{\mathbb{C}^N \oplus \mathbb{C}^N}{\mathfrak{S}_N} = (\mathbb{C}^2)^N / \mathfrak{S}_N \quad \text{symmetric power of } \mathbb{C}^2$$

$$\cong \mathcal{M}_{\mathfrak{S}_N} \times T^* \mathbb{C}$$

Well-known that $(\mathbb{C}^2)^N / \mathfrak{S}_N$ is resolved by Hilbert schemes $\mathrm{Hilb}^N(\mathbb{C}^2)$ of

N-point in $\mathbb{C}^2$.

$$\begin{array}{ccc} \mathrm{Hilb}^N(\mathbb{C}^2) = & \{ I \subset \mathbb{C}[x,y] : \dim \frac{\mathbb{C}[x,y]}{I} = N \} \\ \big\downarrow & \big\updownarrow \\ (\mathbb{C}^2)^N / \mathfrak{S}_N & \mathrm{supp} \frac{\mathbb{C}[x,y]}{I} \end{array}$$

<u>Model:</u>

<u>Kashiwara-Rouquier</u>

$\exists$ $\mathcal{A}_\hbar$ sheaf of $\hbar$-adic alg on $\mathrm{Hilb}^N(\mathbb{C}^2)$ s.t.

- $\mathcal{A}_\hbar / \hbar \mathcal{A}_\hbar \cong \mathcal{O}_{\mathrm{Hilb}^N(\mathbb{C}^2)}$

$\Gamma(\mathrm{Hilb}^N(\mathbb{C}^2), \mathcal{A}_\hbar)^{\mathbb{C}^*} \cong$ spherical rational Cherednik alg

$\mathrm{gr}\, \Gamma(\mathrm{Hilb}^N(\mathbb{C}^2), \mathcal{A}_\hbar)^{\mathbb{C}^*} \cong \mathcal{O}(\mathrm{Hilb}^N(\mathbb{C}^2)) = \mathbb{C}[\mathcal{M}_{\mathfrak{S}_N}]$

* $\mathrm{Hilb}^N(\mathbb{C}^2)$ = Nakajima quiver variety

for

$$V = \mathrm{End}(\mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^N)$$

$$\ni (x, r)$$

$$V^* = \mathrm{End}(\mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C})$$

$$\ni (Y, \beta)$$

$$\mu : T^*V = V \oplus V^* \longrightarrow \mathfrak{gl}_N(\mathbb{C}) \qquad \text{moment map}$$

$$(X, Y, r, \beta) \longmapsto [X, Y] + \beta r$$

$$\mu^{-1}(0) /\!/ GL(N) \cong (\mathbb{C}^2)^N / \mathfrak{S}_N$$

$$\mu^{-1}(0) \cap \divideontimes /\!/ GL(N) \cong \mathrm{Hilb}^N(\mathbb{C}^2)$$

$\divideontimes$ = stable subspace

<u>KR</u>:

$T^*V \rightsquigarrow$ Weyl alg.

Hamiltonian reduction $\rightsquigarrow$ quantum Hamiltonian reduction.

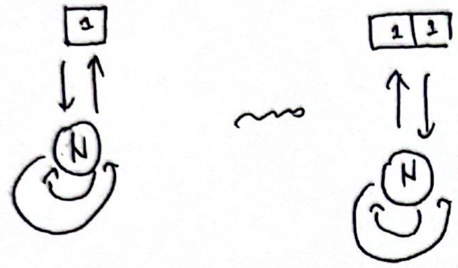
<u>Chiralization</u>

Weyl alg. $\rightsquigarrow$ $\beta\gamma$ system

QHR $\rightsquigarrow$ BRST reduction

But this does not work because BRST op Q is not nilpotent.

★ BMR says we should get vertex super alg.



$\mathbb{C}^{1,1}$

$$\begin{array}{c} \mathrm{Hilb}^N(\mathbb{C}^2)_{sup} \\ \downarrow \pi \\ \mathrm{Hilb}^N(\mathbb{C}^2) \end{array}$$

<u>Thm</u> (AKM)

$\exists\ \mathcal{V}_N$ sheaf of $\hbar$-adic VSA on $\mathrm{Hilb}^N(\mathbb{C}^2)$ s.t.

1) $\mathcal{V}_N / \hbar \mathcal{V}_N \simeq \pi_* \mathcal{O}_{J_\infty \mathrm{Hilb}^N(\mathbb{C}^2)_{sup}}$

2) $\mathcal{V}_N$ is locally $(\beta\gamma\, bc)^{\otimes N}$

3) $X_{\Gamma(\mathrm{Hilb}^N(\mathbb{C}^2), \mathcal{V}_N)^*} \cong (\mathbb{C}^2)^N / \mathfrak{S}_N$

Moreover, a) $\Gamma(\mathrm{Hilb}^N(\mathbb{C}^2), \mathcal{V}_N)^{\mathbb{C}^*} \simeq V_N \otimes \beta\gamma \otimes SF$
symplectic fermions

b) $\exists\ \mathrm{Vir}_{N=4} \longrightarrow V_N$ c) $X_{V_N} \simeq \mathcal{M}_{\mathfrak{S}_N}$

Rmk

1) $N=2$, $V_2 = \mathrm{Vir}_{\kappa=4,\, c=-9}$

2) $N=3$, V_3 is as described in BMR

3) $\mathcal{U} \underset{\text{open}}{\subset} \mathrm{Hilb}^N(\mathbb{C}^2)$

$$\Gamma(\mathrm{Hilb}^N(\mathbb{C}^2), \mathcal{V}_N) \hookrightarrow \Gamma(\mathcal{U}, \mathcal{V}_N)$$

choosing appropriate $\mathcal{U}$, we get $V_N \hookrightarrow (\beta\gamma bc)^{\otimes N-1}$

4) $\mathcal{M}_{\mathfrak{S}_N}$ is self-dual in 3D

CCG conj. says
$$\mathcal{M}_{\mathfrak{S}_N} = \mathrm{Spec}\ \mathrm{Ext}^\bullet(V_N, V_N)$$
$$\mathcal{M}_{\mathfrak{S}_N} \cong \frac{\mathbb{C}^{N-1} \oplus \mathbb{C}^{N-1}}{\mathfrak{S}_N}$$
, open even for $N=2$.

5) Coman-Shim-Yamazaki-Zhou : This construction works for any Nakajima quiver variety, and get boundary VOA for the corresponding quiver gauge theory.

Lecture 3 Class $\overset{=6}{S}$ theory (Gaiotto - Moore - Neitzke)

6D $(2,0)$ SCFT on $\Sigma \times \mathbb{R}^4$

Σ: punctured Riemann surface

$\rightsquigarrow$

4D theory $S_G(\Sigma)$

Coulomb $(S_G(\Sigma))$ on $\mathbb{R}^3 \times S^1$

$= \{$ Higgs bundles on $\Sigma \}$

Higgs $(S_G(\Sigma))$

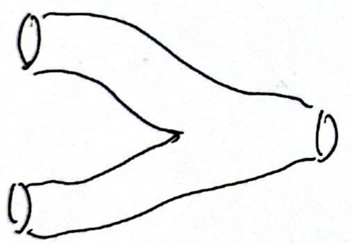
Moore - Tachikawa , Ginzburg - Kazhdan , Braverman - Finkelberg - Nakajima

" 2D TQFT w/ targets in symplectic varieties "

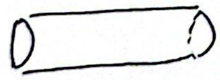
$\mathbb{B}_2$ cat. of 2-bordisms

obj : cpt oriented 1-mfds (disjt unions of S^1)

morphism: $\mathrm{Hom}(B_1, B_2) = \{$ cpt oriented 2-mfds Σ w/ $\partial\Sigma = B_1 \cup (-B_2) \}/\sim$



B_1 B_2

$id_{S_1} =$ 

$\mathbb{B}_2$ is a symmetric monoidal cat. $\otimes$ = disjt union

$\mathbb{H}$ cat. of holo. sym. varieties

objs = G cpx semisimple gp ; morphism $\mathrm{Hom}(G_1, G_2) = \left\{ \begin{array}{l} X \text{ sympl. var.} : \\ X \circlearrowleft G_1 \times G_2 \\ \text{Hamiltonian} \end{array} \right\}$

+...

$\mu_i : X \longrightarrow \mathfrak{g}_i^*$, $i = 1, 2$, $\mathfrak{g}_i = \mathrm{Lie}(G_i)$

<u>Composition</u>. $\mathrm{Hom}(G_1, G_2) \ni X$ $\mathrm{Hom}(G_2, G_3) \ni Y$

$$X \circ Y = (X \times Y) /\!/\!/_{\Delta} (G_2)$$

sympl. reduction

<u>Example</u>. $\mathrm{id}_G = T^*G = G \times \mathfrak{g}^*$

$$G \times \mathfrak{g}^* \xrightarrow{\mu = \text{proj.}} \mathfrak{g}^*$$

$$T^*G \circ Y = \{ (g, x, y) : x = \mu_Y(g) \} / G = G \times Y / G = Y$$

$\mathbb{B}$ monoidal, $\otimes$ = product

<u>MT, GK, BFN</u>

$\forall G$ semisimple, $\exists\ \eta_G : \mathbb{B}_2 \longrightarrow \mathbb{B}$ monoidal

s.t. 1) $\eta_G(S^1) = G$

2) $\eta_G(\;) = T^*G$

3) $\eta_G(\;) = G \times S$

Kostant – Slodowy slice

$S = e + \mathfrak{g}^f$, $\{e, h, f\}$ principal $\mathfrak{sl}_2$-triple

Rmk $\quad G \times S \overset{\text{Kostant}}{\simeq} G \times_N (f + \mathfrak{n}^\perp)$ $\quad$ twisted cotangent bundle

$$\begin{array}{c} G \times S \\ \downarrow \\ G/N \end{array}$$

$N \subset G$ max. unipotent

$\mathfrak{n} = \mathrm{Lie}(N)$

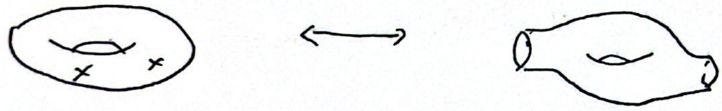
$$T^*(G/N) = G \times_N \mathfrak{n}^\perp$$

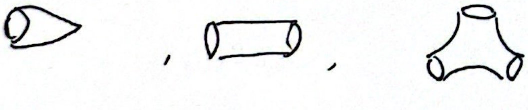
MT: $\quad \mathrm{Higgs}(S_G(\Sigma)) = \eta_G(B)$

$\Sigma \longleftrightarrow B$

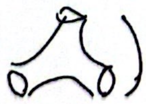
same genus

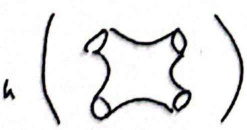
punctures = # bdries

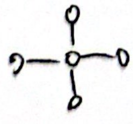


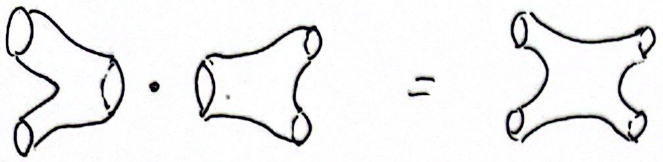
Rmk  generates $\mathbb{B}_2$,

$\eta_G($  $)$ determines everything.

Ex. $\quad G = SL_2$, $\quad \eta_G($  $) = (\mathbb{C}^2)^{\otimes 3}$ $\qquad \mathbb{C}^2 \curvearrowleft SL_2$

$\eta_G($  $) = \overline{\mathbb{O}_{min}(D_4)}$

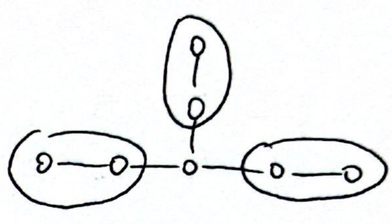




$$(\mathbb{C}^2)^{\otimes 3} \times (\mathbb{C}^2)^{\otimes 3} /\!\!/\!\!/ SL_2 \cong \overline{\mathbb{O}_{min}(D_4)}$$

$\uparrow$

ADHM construction for

$G = SL_3$,  $\longrightarrow \overline{\mathbb{O}_{min}(E_6)}$ 

In general, there is no simple description of $\underbrace{\eta_G(\text{trinion})}_{}$

Moore - Tachikawa variety.

Rmk. In type A, $\eta_G(\;B\;) =$ Coulomb$_{3D}$(Star shaped quiver)

(BFN)

$\Rightarrow \eta_G(\;B\;)$ is a symplectic singularity (Weeks)

How about VOAs?

$\mathbb{V}(S_G(\Sigma)) = ?$

Beem - Peelaers - Rastelli - van Rees $\mathbb{V}(S_G(\Sigma))$ has a similar description.

$\mathbb{B} \rightsquigarrow \mathcal{V}_A$ cat. of VAs

obj: semisimple gp G

morphism: $\mathrm{Hom}(G_1, G_2) = \{ V \text{ VA} : V \xleftarrow{\text{VA hom.}} V^{k_1}(\mathfrak{g}_1) \otimes V^{k_2}(\mathfrak{g}_2) \}$, $k_1 = -h^\vee_{G_1}$, $k_2 = -h^\vee_{G_2}$

$X_V \longrightarrow X_{V^{k_i}(\mathfrak{g}_i)} = \mathfrak{g}_i^*$ moment map for G_i

<u>Composition</u>: $V \in \mathrm{Hom}(G_1, G_2)$, $W \in \mathrm{Hom}(G_2, G_3)$,

$$V \circ W := H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_2, \mathfrak{g}_2, \underbrace{V \otimes W}_{\text{diagonal } \hat{\mathfrak{g}}_2\text{-action}})$$

level $= k_1 + k_2 = -2h^\vee_{G_2}$

$X_{V \circ W} \simeq (X_V \times X_W) /\!/\!/_{\Delta(G_2)}$ when moment maps are flat

$\mathcal{V}_A$ monoidal, $\otimes = \otimes$

<u>BPRvR, A</u>: $\forall G$ ss gp, $\exists!\ \eta^{ch}_G : \mathbb{B}_2 \to \mathcal{V}_A$ monoidal functor

1) $\eta^{ch}_G(S^1) = G$

2) $\eta^{ch}_G(\quad) = \mathcal{D}^{ch}_{G,-h^\vee}$ cdo on G $\left(X_{\mathcal{D}^{ch}_{G,-h^\vee}} \simeq T^*G \right)$

3) $\eta^{ch}_G(\quad) = H^0_{DS, f_{prin}}(\mathcal{D}^{ch}_{G,-h^\vee}) =: W_{G,-h^\vee}$

$(G \times S \xleftarrow[\text{Kostant reduction}]{} T^*G)$ equiv. W-alg. $X_{W_{G,-h^\vee}} \simeq G \times S$

Moreover,

a) $\eta_G^{ch}(B)$ simple (for genus zero B)

b) " conformal

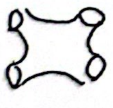
c)
$$\begin{tikzcd} \mathbb{B}_2 \arrow[r, "\eta_G^{ch}"] \arrow[dr, "\eta_G"'] \arrow[r, phantom, shift right=6, "\circlearrowright"] & V_A \arrow[d] \\ & \mathbb{S} \end{tikzcd} \qquad V \mapsto X_V$$

$$\eta_G^{ch}(B) = \mathbb{V}(S_G(\Sigma))$$

$B \mapsto \Sigma$ as before

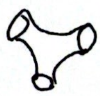
<u>Examples</u> $G = SL_2$

 $\longrightarrow \; \beta\gamma((\mathbb{C}^2)^{\otimes 3})$

 $\longrightarrow \; L_{-2}(D_4)$

$\overline{\mathbb{O}_{min}(D_4)}$

$G = SL_3$

 $\longrightarrow \; L_{-3}(E_6)$

<u>Rmk</u>. $\mathbb{V}(S_G(\Sigma)) = \eta_G(B)$ is physically true only for simply laced G.

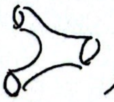
Non-simply laced cases were worked out Beem-Nair.

- Why $\mathcal{D}^{ch}_{G,-h^\vee}$ should be identity functor?

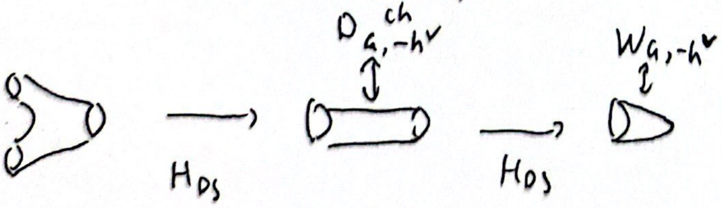
<u>Arkhipov – Gaitsgory</u>: $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathfrak{g}, \mathcal{D}^{ch}_{G,-h^\vee} \otimes M) \cong M, \quad \forall M \in KL_{-h^\vee}(\mathfrak{g})$

$$
\begin{array}{ccccc}
\mathcal{D}^{ch}_{G,-h^\vee}\text{-mod} & \simeq & D\text{-mod}_{LG,-h^\vee} & & \\
\uparrow & & \uparrow & & \\
\mathcal{D}^{ch}_{G,-h^\vee}\text{-mod}^{L^+G} & \simeq & D\text{-mod}^{L^+G}_{LG,-h^\vee} & & \\
 & & \| & & \\
\uparrow & & D\text{-mod}_{-h^\vee}(Gr_G) & & \\
 & & \uparrow & & \\
\mathcal{D}^{ch}_{G,-h^\vee}\text{-mod}^{L^+G\times L^+G} & \simeq & \left(D\text{-mod}_{-h^\vee}(L^+G\backslash Gr_G)\right)^{*)} & \simeq & Rep(\check{G}) \\
\mathcal{D}^{ch}_{G,-h^\vee} & \longleftarrow\!\shortmid & \delta_e & \longleftarrow\!\shortmid & \mathbb{C}
\end{array}
$$

<u>Idea of construction</u>

- enough to construct $\eta^{ch}_G($  $)$

- $\eta_G(B \circ$  $) = H_{DS, f_{prin}}(\eta_G(B))$



So we need inverse to DS reduction.

$$\mathcal{D}^{ch}_{G,-h^\vee} \circlearrowleft \hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$$

$$W_{G,-h^\vee} \circlearrowleft \hat{\mathfrak{g}}$$

$$\begin{tikzcd}
W_{G,-h^\vee} \otimes W_{G,-h^\vee} & * & \mathcal{D}^{ch}_{G,-h^\vee} \arrow[d, leftrightarrow] \\
V^{-h^\vee}(\mathfrak{g}) \otimes V^{-h^\vee}(\mathfrak{g}) \arrow[u, hook] \arrow[urr] \arrow[rr] & & V^{-h^\vee}(\mathfrak{g}) \underset{\mathfrak{z}(\hat{\mathfrak{g}})}{\otimes} V^{-h^\vee}(\mathfrak{g})
\end{tikzcd}$$

$\uparrow$

Feigin-Frenkel center

$$H_{BRST}\left(\mathfrak{z}(\hat{\mathfrak{g}}), W_{G,-h^\vee} \otimes W_{G,-h^\vee}\right) \simeq \mathcal{D}^{ch}_{G,-h^\vee}$$