Hamiltonian dynamics from the perspectie of holomorphi curves

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Lecture 1 (M, ω) sympl mfd, most time cpt

$H: M \to \mathbb{R}$ Hamiltonian $\rightsquigarrow$ vector field X_H,

$$\omega(X_H, V) = \underset{\uparrow \text{ convention}}{-} dH(V)$$

Check. X_H tangent to the level sets of H.

integrate X_H $\rightsquigarrow$ $\varphi: M \times \mathbb{R} \to M$, $(\varphi^t)^* \omega = \omega$

$$\rightsquigarrow \varphi^1 \in \underbrace{\mathrm{Ham}(M, \omega)}_{\substack{\text{Special property} \\ \text{in dynamics}}} \subset \mathrm{Symp}(M, \omega)$$

More generally, consider $H_t: M \times \mathbb{R} \to \mathbb{R}$ time dependent

$$\rightsquigarrow X_{H_t} \rightsquigarrow \varphi^t \qquad \frac{d\varphi^t}{dt}(x) = X_{H_t}(\varphi^t(x))$$

Note. critical pt of H is rest pt of X_H, i.e. $\varphi^t(x) = x$, $\forall t$

but in general for time dependent no such thing

Q. Fixed pts of φ^1?

Fixed pts of φ^1

$\updownarrow$

periodic orbits of H_t

i.e. $\gamma: S^1 \to M$, $\gamma'(t) = X_{H_t}(\gamma(t))$

- set time $t \in \mathbb{R}/\mathbb{Z}$

Refined Q (Arnold)

- \# fixed pts $\geq$ min \# crit. pts ?
- φ^t non-deg \# fixed pts $\geq$ min \# crit. of Morse

Non-degeneracy. $\Delta_\varphi = \{(x, \varphi(x)) \in M \times M\} \pitchfork \Delta$

i.e. $d\varphi: T_xM \to T_xM$ at a fixed pt no eigenvector w/ $\lambda = 1$

- min \# Cr (f: Morse) can be extracted from $H_*(M;\mathbb{Z})$ when $\pi_1 M = \{1\}$

(when dim $M \geq 5$, this is essentially Smale: rk $H_*(M;\mathbb{Z})$ + 2 min \# finite cyclic summands of $H_*(M;\mathbb{Z})$)

Degenerate case: Lusternek-Schnizelman theory

No complete alg. invt.

- min \# crit. = min \# crit. value
- cup length gives lower bound

Status. Non-deg.: almost there (difference between cpxes / $\mathbb{Z}$ & $\mathbb{Z}/2$)

Deg. case: We know nothing except ≥ 1 !

Plan:
- Floer's work on non-deg case
- How to reduce tech. assumptions
- Degenerate case (Floer, Hofer, ...)
- Periodic orbits (Conley-Zehnder, etc...)
- closing lemmas?

Morse theory. Witten: $CM_*(f)$ for Morse-Smale function f

filtered by f

$$CM_*^{[a,b]}(f) = CM_*^{(-\infty,b]}(f) / CM_*^{(-\infty,a]}(f)$$

$$H_*(CM_*^{[a,b]}(f)) \cong H_*(M_b, M_a)$$

compatible w/ $M_a \hookrightarrow M_{a'}$ $(a<a')$

Floer: Similar alg. str.

$CF_*^{[a,b]}(H)$ generated by "lifts" of periodic orbits

(i.e. periodic orbits on some cover of $\mathcal{L}M$)

Action functional. $A(\gamma) = \int H_t(\gamma)\,dt - \int \gamma^*(\lambda)$ (exact case: $\omega = d\lambda$)

locally makes sense; can define on cylinders

$u: S^1\times[0,1] \longrightarrow M \rightsquigarrow \int u^*\omega \in \mathbb{R}$

$\left(A(u(0)) - A(u(1))\right.$ when exact$\left.\right)$. "dA" on $\mathcal{L}M$.

In practice.

- restrict to contractible cpts of $\mathcal{L}M$

fix a lift by choosing a "capping disk"

$$p: S^1 \to M \quad \rightsquigarrow \quad \tilde{p}: D^2 \to M, \quad \tilde{p}|_{S^1} = p$$

$$A(\tilde{p}) = \int H_t(p(t))\, dt - \int \tilde{p}^* \omega$$

Output of Floer homology ('87)

- $\omega(\pi_2 M) = 0 \rightsquigarrow$ trivial cover (strong & too restrictive)

$CF_*^{[a,b]}(H)$ persistent system

$HF_*^{(-\infty,+\infty)}(H) \cong H_*(M)$ (originally over $\mathbb{Z}/2$. Hofer over $\mathbb{Z}$)

Lecture 3. Defined Morse cpx $CM^*(f)$

cont. map [Constructed map $CM^*(f_1) \to CM^*(f_2)$ assoc. to interpolation between ∇f_1 & ∇f_2.

Modification of cont. map gives $CM^*(f_1) \to CM^*(f_2)$ assoc. to any cycle $Z \subset M \times M$.

① To see map on CM^* is h.e., first look at case $f_1 = f_2$, then we are considering the identity continuation map.

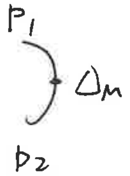
The space of trajectories

in this map is exactly same as space of gradient trajectories (except, don't mod out by $\mathbb{R}$ translation).

All isolated such trajectories are constant.

Because non-constant trajectories arise in 1-dim families.

Picture. $Z = \Delta_M \subset M \times M$

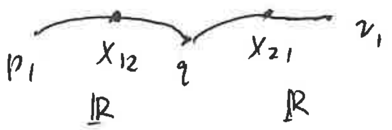


$\rightsquigarrow$ The map $CM^*(f) \longrightarrow CM^*(f)$ is the identity.

Consider a pair X_{12} & X_{21} of continuations between $\nabla f_1 \overset{X_{12}}{\rightsquigarrow} \nabla f_2 \overset{X_{21}}{\rightsquigarrow} \nabla f_1$

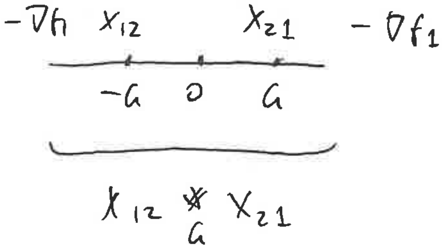
The composite map $\underbrace{CM^*(f_1) \longrightarrow CM^*(f_2) \longrightarrow CM^*(f_1)}$

is counting "broken continuation". $\sum_q$

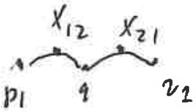


defines the map $\mathcal{O}_{p_1} \longrightarrow \mathcal{O}_{r_1}$.

Gluing Thm. If $0 \ll G$, then # of sol. to ODE.



agrees w/ $\sum_q$ # Sol.. $= \sum_q$



a map $\mathbb{R} \longrightarrow C^\infty$ (v. fields on M)

Proof. Look near q.

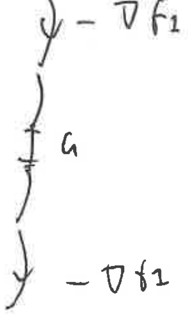
par. by $[-G, G]$

Choose a 1-par family of vec. fields on $\mathbb{R}$ (par. by $G \in [0,\infty)$) So that at $G=0$, get $-\nabla f_1$ (indep. of t), and for $0 \ll G$, given by $X_{12} \underset{G}{\ast} X_{21}$.

Define $\mathcal{H}(f_1, f_2, f_1)$ to be the union over all $G \in [0, \infty)$ of the sol'ns to these ODE's.

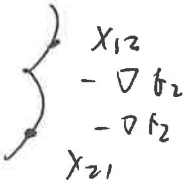
This has a "Morse-Gromov-Floer" compactification $\overline{\mathcal{H}}(f_1, f_2, f_1)$ where you add

①



② $G \to +\infty$

(Gluing Thm)



(Note $G = 0$ forms a boundary stratum as well.)

Define a map $CM^*(f_1) \longrightarrow CM^{*-1}(f_1)$ by counting the elts of $\overline{\mathcal{H}}(f_1, f_2, f_1)$ which are rigid (ie the dim. of the corresponding transverse intersection in 0)

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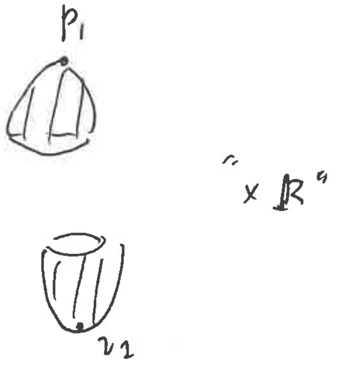


$\Rightarrow \deg v_1 = \deg p_1 - 1.$

Main result This map defines a homotopy from composition to identity

$$dH = C_{21} \circ C_{12} - Id_{CM^*(f_1)}$$

Pf. Identify these terms w/ ∂ codim 1 strata.

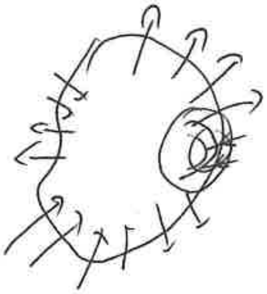
$$H : CM^*(p_1) \longrightarrow CM^*(f_1)$$

$$(dH)(p_1) = H(dp_1) \pm dH(p_1)$$

If one studies Morse theory on mfd w/ ∂, get map $CM^*(f_1) \to CM^*(f_2)$ that are not htpy equiv.

Simplest version

$$\begin{array}{ccc}\partial^{0,f} M & \subset & \partial M \\ \parallel & & \parallel \\ \partial^{+,f} M \cap \partial^{-,f} M & & \partial^{+,f} M \cup \partial^{-,f} M\end{array}$$



preclude this. $\Rightarrow$ $CM^*(f)$ is well-defined.

Naive. If $\partial^{(+,f_1)} M = \partial^{(+,f_2)} M$, then $CM^*(f_1) \longrightarrow CM^*(f_2)$ which is a h.e.

In fact, we can formulate the existence of a map under the assumption that

$$\boxed{\partial^{(+,f_1)} M \subset \partial^{(+,f_2)}(M)}$$

non symmetric. (holds for cohomological version)

Go back to cpt case.

Use these ideas to construct h.e. $CM^*(f) \longrightarrow \boxed{C^*(M)}$ use simplicial cochain of a triangulation.

$C^*(M)$

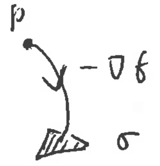
Every smooth mfd may be triangulated so that every simplex $\sigma : \Delta^n \subset X$ is a

Smooth submfd w/ corners.

Define a map $CM^*(f) \longrightarrow C^*(M)$

same as $CM^*(f) \otimes C_*(M) \longrightarrow \mathbb{Z}$

By counting the # elts of moduli spaces



Sol'n to ODE

$\gamma : (-\infty, 0] \to M$

$\gamma(0) \in \sigma$

To make sense of this, either

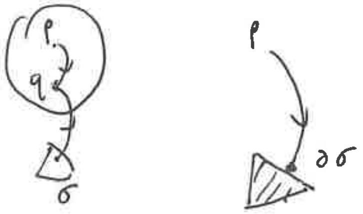
① Assume triangulation is $\pitchfork$ to gradient flow

② perturb the eq'n near $t=0$.

To prove that this is a chain map, consider 1 dim moduli spaces

The ∂ of these moduli spaces is

d on $CM^*(f)$



→ Chain map

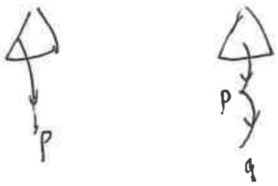
Write a map in the opposite direction.

$C_*(M) \longrightarrow CM^*(f)$

↑

w/ respect to a cellular subdiv.

dual to the triangulation we started w/.



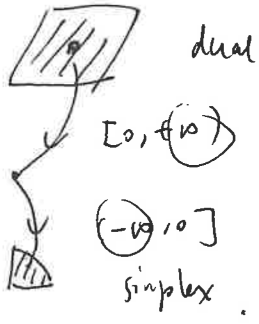
The duality induces an isom. $C_*(M) \cong C^*(M)$

If M is non-orientable, need to work w/ "co-oriented" chains

gen. as pairs (σ, α) α is orientation of $\sigma^*(TM)$

$\Delta^n \xrightarrow{\sigma} M$.

Now, we need to prove that the two composites are htpic to identity:



glue to get a moduli space for each $a \in [0,\infty]$ consisting of

When $a = 0$, get intersection pt between elt of dual subdiv. and simplex.

i.e.

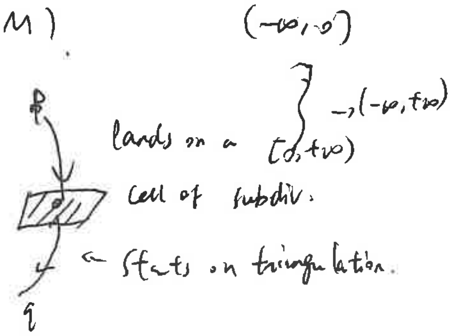
$$\begin{tikzcd} C^*(M) \otimes C_*(M) \arrow[d] \arrow[dr, bend left, "\text{duality pairing}"] \arrow[dr, phantom, "\text{htp}\ \Uparrow", near start] & \\ CM_*(f) \otimes C_*(M) \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Conclude that map $C^*(M) \to CM_*(f) \to C^*(M)$ is htpic to id.

$\Rightarrow$ Rk of $CM_*(f)$ is longer than rk of $C^*(M)$

To prove equality, try to glue in the other direction.

Need to introduce further refinement of dual subdivision...



For mfds w/ ∂, get $CM^*(f) \cong C^*(M, \partial^{(-,f)}M)$

This is consistent w/ existence of map $CM^*(f_1) \longrightarrow CM^*(f_2)$

when $\partial^{(+,f_1)}M \subset \partial^{(+,f_2)}M$.

$$\Updownarrow$$

$$\partial^{(-,f_1)}M \supset \partial^{(-,f_2)}M$$

Extreme case. $f_+ \rightsquigarrow$ pts outward $\sim C^*(M) \simeq \overset{\text{(proper submfd)}}{C_*(M, \partial M)}$

$f_- \sim$ pts inward $\rightsquigarrow \underset{\text{" closed submfd "}}{C^*(M, \partial M)} \cong C_*(M)$

Lecture 4. Hamiltonian HF^*

Recall M closed symplectic mfd, $H : M \times S^1 \longrightarrow \mathbb{R}$ Hamiltonian function

$\sim$ time independent v field X_{H_t}

The flow of X_H is a diffeo $\varphi : M \to M$ (preserves ω)

Interested in $\mathrm{Fix}(\varphi) = \{x : \varphi(x) = x\}$

φ is non-deg of graph of φ is $\pitchfork$ to diagonal in $M \times M$.

Alternatively describe $\mathrm{Fix}(\varphi)$ as time-1 closed orbits of X_H, i.e.

Map $S^1 \xrightarrow{p} M$ s.t. $\frac{dp}{dt}(t) = X_{H_t}(p(t))$

Note that $\varphi(p(0)) = p(1)$. and $p(0) = p(1)$

So $d\varphi : T_{p(0)}M \rightleftarrows T_{p(1)}M$.

Non degeneracy $\iff$ $d\varphi_{p(0)}$ does not have 1 as an eigenvalue.

i.e. $(Id - d\varphi_{p(0)})$ is invertible.

Can have either + or − det.

(Corresponds to the intersection $\#$ between Δ_M & $\underset{\substack{\uparrow \\ \text{graph of } \varphi}}{\Delta_\varphi}$)

(Toy case: H is autonomous, in which case this mod 2 invariant is the reduction of deg. of a critical pt mod 2).

To obtain a $\mathbb{Z}$-grading, we need more data.

Minimal assumption:

$$\begin{tikzcd} \pi_2(M) \arrow[r] \arrow[rr, bend right, "0"'] & H_2(M) \arrow[r, "{\langle\, ,\, c_1(M)\rangle}"] & \mathbb{Z} \end{tikzcd}$$

allows one to define graded groups generated by contractible orbits.

Concretely, this means that for any map $S^2 \xrightarrow{u} M \longrightarrow B\underset{\substack{\uparrow \\ \text{"real sympl. gp"}}}{Sp(2n)}$

the pull back of $u^* TM$ is a trivial symplectic vector bundle on S^2.

To see this, think in terms of classifying spaces.

$$\begin{tikzcd} \text{max'l cpt.} \arrow[r] & \subset Sp(2n) \subset GL(2n;\mathbb{R}) \\ U(n) & \subset GL(n,\mathbb{C}) \subset \end{tikzcd}$$

So we are considering a map $S^2 \longrightarrow BU(n)$.

Case 1. $n=1$, $BU(1) \cong \mathbb{CP}^\infty$, $S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty$

Fibration induces a LES on homotopy gps

$$\Rightarrow \pi_i(\mathbb{CP}^\infty) \simeq \pi_{i-1}(S^1)$$

$$\underset{\text{maps } S^i \to \mathbb{CP}^\infty}{\overset{\shortparallel}{\text{htpy classes of}}}$$

$$\pi_2(\mathbb{CP}^\infty) \cong \pi_1(S^1) \cong \mathbb{Z}$$

When compute $H^*(\mathbb{CP}^\infty, \mathbb{Z})$ get a generator in deg. 2 (c_1 of the tautological l.b.)

$\Rightarrow$ This generator induces isom. $\pi_2(\mathbb{CP}^\infty) \xrightarrow{\sim} \mathbb{Z}$.

Extend computation to higher dim. The spaces

$BU(i)$ are related to each other by fibrations involving S^{2i-1}.

Apply LES again, get $\pi_{i+1}(BU(i)) \cong \mathbb{Z}$, $\forall i$

Compute H^* get that c_1 detects this.

Cor. If c_1 vanishes on $\pi_2(M)$, then u^*TM is trivial, $\forall\, u: S^2 \to M$.

(Go further, $\pi_3(BU(n)) \sim \pi_2(U(n)) = 0$)

$\Rightarrow$ One trivialization up to htpy.

Say that $p: S^1 \to M$ is a contractible orbit of X_H.

Choose a cap $D^2 \xrightarrow{u} M$ and note that u^*TM is <u>trivial</u>.

The map $d\varphi^t$ defines a path in $Sp(2n)$ (φ^t is flow of X_H)

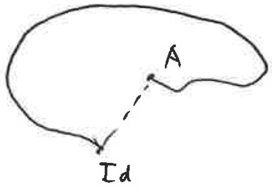
under this trivialization.

$$\begin{tikzcd}
\mathbb{R}^{2n} & & \mathbb{R}^{2n} \\
T_{p(t)}M \arrow[u, phantom, sloped, "\cong"] & & T_{p(0)}M \arrow[u, phantom, sloped, "\cong"] \arrow[ll, "d\varphi_t"']
\end{tikzcd}$$

Computation of $\pi_2(BU(n)) \cong \pi_1(U(n)) \cong \pi_1(Sp(2n)) \cong \mathbb{Z}$

Maslov index is a <u>canonical</u> ext'n of this iso. to identification of

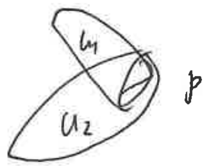
$$\left.\begin{array}{l} \text{htpy classes of paths from id to } A \\ (\text{w/ property } \det(Id - A) \neq 0) \end{array}\right\} \cong \mathbb{Z}$$



$\Rightarrow$ Assign to every <u>capped orbit</u> (p,u) an integer which is its <u>Maslov index</u>.

(This can always be done)

If $c_1(M)$ vanishes on $\pi_2(M)$, then this integer is independent of capping



$\rightsquigarrow$

$u: S^2 \to M$ $\begin{pmatrix} u_+ \,\&\, u_- \\ \text{induce} \\ \text{different} \\ \text{Maslov index} \end{pmatrix} \longleftrightarrow \left\{\begin{array}{l} \text{triv. of } p^*(TM) \\ \text{assoc. to } u_\pm \\ \text{differ by } S^1 \to U(n) \end{array}\right\}$

$\xleftrightarrow{\text{clutching}} \left\{ \begin{array}{c} u^* TM \text{ over } S^2 \text{ has} \\ \text{non trivial } c_1 \end{array} \right\}$

How to obtain $\mathbb{Z}$-gradings for arbitrary orbits ?

Most general: require vanishing of c_1 on tori.

replace capping by fixing a basept in each component of $\mathcal{L}M$, and trivialize TM over that component, replace capping by path to base pt.

Instead. Assume that c_1 vanishes (as a class in $H^2(M;\mathbb{Z})$).

$\rightarrow$ $\underset{\underset{\mathbb{C}}{\wr\wr}}{\Lambda^n TM}$ $\leftarrow$ Top exterior power of (TM, J) cpx vector bdle w/ almost cpx str. J

And choose a trivialization

(Graded symplectic manifold).

Trivializations form an affine space $M \to U(1) \simeq S^1$

$\Leftrightarrow$ Classes in $H^1(M;\mathbb{Z})$.

(so if $H^1 M = 0$, then no choice)

Once this choice is fixed, for any loop $p: S^1 \to M$.

$p^*(\Lambda^n TM)$ has an induced trivialization.

(But for cpx vector bundle on circles $\pi_1\big(\overset{\overset{\mathbb{Z}}{\wr\wr}}{U(n)}\big) \xrightarrow[\sim]{\det} \pi_1(U(1)) \simeq \mathbb{Z}$)

$\Rightarrow$ fixed trivialization of $p^*(TM)$.

In fact, it suffices to assume c_1 is torsion.

$$\Updownarrow$$

$$(\Lambda^n TM)^{\otimes d} \cong \mathbb{C}.$$

Choose a trivialization.

$\Rightarrow$ get a $\mathbb{Z}$-grading.

(A better thing to do is to introduce a $\mathbb{Z}[\frac{1}{d}]$ grading)

Back to general (M, ω), want to produce a $\mathbb{Z}/2$ graded chain cpx

$CF^*(H)$.

What are the coefficients?

① Need to set up signs.

② Unlike in Morse theory, no "naive" finite count.

Imagine $M \longrightarrow S^1$

$\rightsquigarrow$ makes map generic

(non-deg. critical pts)

Want to define a "Morse theory". generated by critical pts.

Differential counting flow lines of "vfield dual" to $f^*(dt)$

Can turn this into "Equivt Morse theory" By passing to $\tilde{M} \xrightarrow{\tilde{f}} \mathbb{R}$.

No problem ensuring compactness of $\mathcal{C}(\tilde{p}, \tilde{q})$ for fixed lifts.

What we actually need is cptness of $\mathcal{C}(p, q) = \coprod_{\tilde{q} \text{ lift } q} \mathcal{C}(\tilde{p}, \tilde{q})$

Associate $\tilde{f}(\tilde{p}) - \tilde{f}(\tilde{q}) \in [0, \infty)$ to every pair of lifts.

Introduce the "Universal Novikov ring".

over any chosen ground ring $\textcircled{\mathbb{K}}$.

$\mathbb{F}_q, \mathbb{Z}, \mathbb{Q}, \dots$

elts are "formal" power series $\sum_{\lambda \in [0,\infty)} a_\lambda T^\lambda$, $a_\lambda \in \mathbb{K}$

s.t. the set $\{\lambda : a_\lambda \neq 0\}$ is discrete.

i.e. $a_{\lambda_0} T^{\lambda_0} + a_{\lambda_1} T^{\lambda_1} + \cdots$

$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$

$\lim_{i \to \infty} \lambda_i = +\infty$.

So in the case of $M \xrightarrow{f} S^1$, we can define a Novikov - Morse cpx. generated by Crit. in $M(df)$

$$\partial_p \xrightarrow{d_{pq}} \partial_q$$

$$d_{pq} = \sum_\lambda \textcircled{d_{pq}^\lambda} \quad \text{and} \quad d_{pq}^\lambda = \begin{pmatrix} \# \text{ of elts of } \mathcal{C}(\tilde{p}, \tilde{q}) \\ \text{where } \tilde{f}(\tilde{p}) - \tilde{f}(\tilde{q}) = \lambda \end{pmatrix} T^\lambda.$$

Notation

$$\begin{tikzcd} & \Lambda_0 \arrow[d, "T \mapsto 0"] & \leftarrow \text{Universal Novikov ring} \\ \boxed{\Lambda \cong \Lambda_0[T^{-1}]} & \mathbb{k} & \end{tikzcd}$$

invert T

Novikov field ← misnomer

because $\mathbb{k} = \mathbb{Z}$, then not field

How to recover ordinary Morse theory from this

$$M \xrightarrow[\text{null htpic}]{} S^1$$

$\Rightarrow \tilde{M} = \coprod M$

⇒ In fact for each pair (p, q) a chosen lift $\tilde{p}$ determine $\tilde{q}$ s.t. $e(\tilde{p}, \tilde{q}') = \phi$ if $\tilde{q}'$ is another lift.

→ Morse – Novikov cpx can be defined polynomially $\mathbb{k}[T^{[0,\infty)}] = \dfrac{\mathbb{k}[T^\lambda : \lambda \in [0,\infty)]}{T^\lambda \cdot T^{\lambda'} = T^{\lambda+\lambda'}}$

$$\begin{tikzcd} \mathbb{k}[T^{[0,\infty)}] \arrow[d, "T^\lambda \to 1"] \\ \mathbb{k} \end{tikzcd}$$

$$\left\{ \begin{array}{l} CMN^*(\tilde{f}, \mathbb{k}[T^\lambda]/\sim) \\ \quad \cap \\ CMN^*(\tilde{f}; \Lambda_0) \\ \\ \displaystyle\bigotimes_{\mathbb{k}[T^\lambda]/\sim} \mathbb{k} \cong CM^*(f). \end{array} \right.$$

Lecture 5 (M, ω) symplectic, $H : M \times S^1 \to \mathbb{R}$ nondegenerate Hamiltonian

periodic orbit $p : S^1 \to M$ of X_H

Want, assign an orientable line (i.e. free ab gp of rk 1)

O_p to p

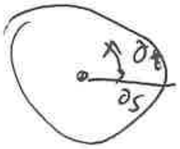
(in general, only a $\mathbb{Z}/2\mathbb{Z}$ graded line)

This will be used to define "signs" in the differential coming from Floer equation

$$(du - X_H \otimes dt)^{0,1} = 0$$

i.e. pick J_t an S^1-dep. complete almost cpx str. on M

Consider cylinder $\Sigma \simeq \mathbb{R} \times S^1_t$ w/ almost cpx $j\partial_s = \partial_t$



$$J(du - X_H \otimes dt) = (du - X_H \otimes dt) \circ j$$

as 1-form on Σ valued in TM

This is determined by value on ∂_s

$$\boxed{J \partial_s u = \partial_t u - X_H}$$

Notice that

$$\begin{array}{ccc} \Sigma & \longrightarrow & M \\ (s,t) & \longmapsto & p(t) \end{array}$$

time-1 orbit of X_H

gives a solution to this equation.

This corresponds to a "constant gradient traj." at a critical pt.

Deformations of solutions to this equation are controlled by an elliptic operator

$$C^\infty(\Sigma, u^*TM) \longrightarrow \Omega^{0,1}(\Sigma)$$

$$u^*TM \underset{\substack{\uparrow \\ \text{choice}}}{\cong} \mathbb{C}^n$$

Under this trivialization, can write the linearization as maps $v: \Sigma \to \mathbb{C}^n$ solving an equation of the form $\bar\partial v = Y(s,t)$

$\uparrow$ matrix that depends on where we are on Σ.

Consider only solutions which have finite energy.

Non-degeneracy $\Rightarrow$ Exponential convergence at the ends for any solution u of

$$(du - X_H \otimes dt)^{0,1} = 0$$

i.e. $\exists\ p_\pm(t)$ s.t. $\lim_{s \to \pm\infty} u(s,t) = p_\pm(t)$

and rate of convergence is $e^{-c(p_\pm t)\cdot|s|}$

This can be used to express $Y(s,t)$ near $\pm\infty$ in terms of a ~~loop~~ $B_t \in \mathfrak{u}(n)$ $(\mathfrak{sp}(2n))$

$\mathfrak{u}(n) = \text{Lie}\, U(n)$

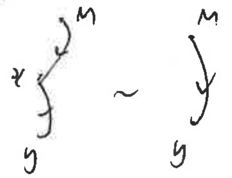
$\}$ generates the path of unitary matrices A_t whose end pt is the Poincaré return map of the trajectories $p_\pm$.

This depends on the choice of trivialization.

<u>Thm</u> (Floer - Hofer) Can associate to B_t^p a $\mathbb{Z}/2\mathbb{Z}$ graded line that is indep. up to canonical isom. ~~of~~ the choice of trivialization.

(Aside: In Morse theory, think of $\mathcal{O}_x$ as assoc. to $\overline{\mathcal{E}}(M,x) \sim$

Compactification of a <u>Ball</u>



Extend the loop B_t^p to a map $B_z^p : \mathbb{C} \longrightarrow U(n)$

Consider the inhomogeneous eq'n $\boxed{\bar\partial v = B_z^p \cdot v}$ ~ D_p on the space of maps $\mathbb{C} \xrightarrow{v} \mathbb{C}^n$.

Because of the condition of non-deg. at ∞. This is an elliptic equation w/ associated Fredholm problem.

Define: $O_p \cong \left(\ker D_p\right)^{\wedge top} \otimes \left(\mathrm{coker}\, D_p\right)^{-\wedge top}$

$$\underset{\substack{\uparrow \\ \text{orientation line}}}{o}(\ker D_p) \otimes o(\mathrm{coker}\, D_p)^{-1}$$

(in some degenerate case, set $B_z^p = e$, restrict to constant v, $O_p \simeq O_{\mathbb{C}^n}$)

To see that trivialization doesn't matter, consider a loop $S^1 \xrightarrow{\phi} U(n)$

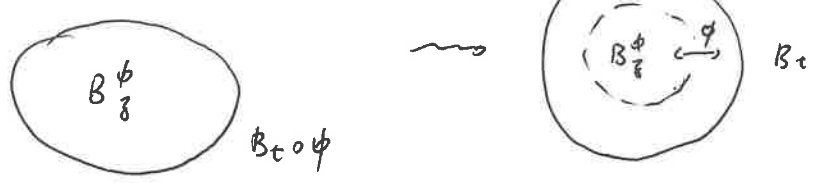
Want to compare solutions to $\bar\partial v = B_z^p v$

and $\bar\partial v = \boxed{B_z^{p,p}}\, v$ ↪ extension of $\phi \circ B_t^p : S^1 \longrightarrow U(n)$

If data is fixed at infinity, then diff. choices of extensions are connected by a family of Fredholm operators.

→ Iso. associated to paths which are unique by contractibility of space of hermitian matrices.

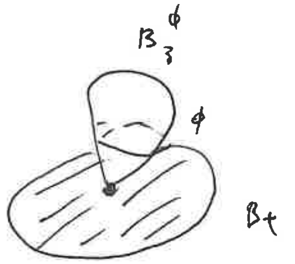
Compare w/ a picture.



describe a vbundle $E_\phi \downarrow \mathbb{C}$

Then $B_z \sim \mathrm{End}(E_\phi)$

Deform the eq'n to



$\mathbb{P}^1$ v bundle E'_ϕ on $\mathbb{P}^1$ w/ trivialization at origin

$\mathbb{C}$

Deform further to the case where inhomogeneous term on $\mathbb{P}^1$ is identically 0, and vbundle on $\mathbb{P}^1$ is holomorphic.

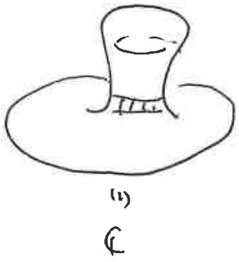
cpx linear operator

operator associated to initial triv ~~of~~ p^*TM

We are considering dets of kernels of these operators w/ the same value at the two origins that are glued together.

Since cpx vspaces have canonical orientations, conclude that the operator on the broken surface has orientation line canonically iso. to (Op) defined using first triv. (only as a $\mathbb{Z}/2\mathbb{Z}$-graded line)

To compare to the other one, use "pre-glue".



to produce approx. solution to $\bar{\partial} v =$ "Inhomogeneous" on glued surface.

Appeal to implicit function thm to conclude that kernels are the same before and after gluing.

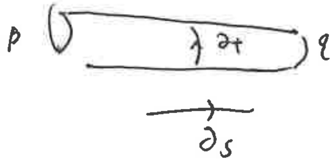
Outcome: Assign orientation lines to each time-1 periodic orbit of X_H.

How to get $CF^*(H)$ from this

Ideally, consider for each pair (p,q) the moduli spaces

$\mathcal{M}(p,q)$ of solutions to Floer equation $u: S^1 \times \mathbb{R} \to M$

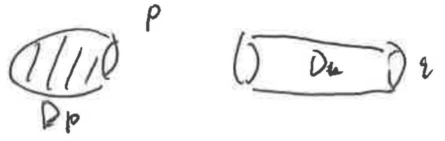
$$(du - X_H \otimes dt)^{0,1} = 0$$



(consider modulo $\mathbb{R}$-translations)

Consider along those solutions w/ the property that the linearized equation has no cokernel, and has kernel of dim 1 (must be "∂_s")

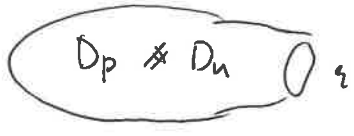
Start w/ a representative of o_p



$$\begin{tikzcd} o_p \otimes o_{D_u} \arrow[d] \\ o_q \end{tikzcd}$$

trivialized by ∂_t

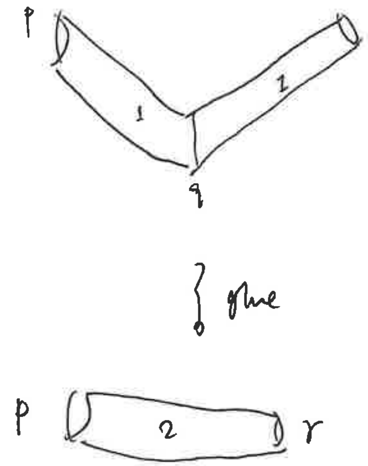
glue $\rightsquigarrow$



some "cap" for q $\rightsquigarrow$

$$\begin{tikzcd} o_p \arrow[d] \\ o_q \end{tikzcd}$$

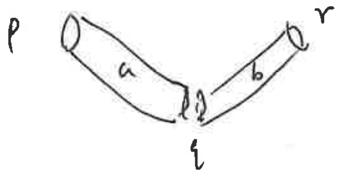
Problem. Can't prove $d^2 = 0$ w/ this setup for arbitrary M.



label is by <u>index</u>

(one more than)

dim M.



$a+b=2$

Can prove that these don't appear generically here

either a or $b<1$.

$\Rightarrow$ The dim. of that moduli space is negative...

Problem. Sphere bubbling

If $\|du\|^2$ blows up on a sphere of pts in Σ

get

know that $0 \leq a$

($a=0 \Rightarrow$ constant)

Issue is about whether we can have $b \leq 1$.

This is not a problem if the sphere is simple (i.e. $\exists$ a pt $z \in S^2$ $\exists$ $dv|_z \neq 0$ <u>and</u> v is injective at this pt) $v: S^2 \to M$

So need some condition so that no multiple covers w/ index in the wrong range exist.

<u>McDuff - Salamon</u> "Semi-positive"

① $\exists\ 0 \leq \lambda$ s.t. the homomorphisms $\pi_2(M) \to \mathbb{Z}$ given by $\langle v, [w] \rangle$ and $\langle v, c_1(M) \rangle$ are proportional, $c_1 = \lambda \cdot [\omega]$

② There is no class α in $\pi_2(M)$ w/ $\langle \alpha, [\omega] \rangle > 0$

and $\langle \alpha, c_1 \rangle \leq n-2$. (check) $(\dim M = 2n)$.

By now, we can define CF^* over $\mathbb{Z}$ w/o any assumption (Fukaya - Ono, Bai - Xu, Rezchikov)

Lecture 6.

Weakly monotone. (M, ω)

$$\begin{tikzcd}
\pi_2(M) \arrow[r, "\langle -, \alpha \rangle"] \arrow[d, "\langle -, c_1(M) \rangle"] & \mathbb{R} \\
\mathbb{Z} &
\end{tikzcd}$$

Require that if $\langle \beta, \omega \rangle > 0$, then $\langle \beta, c_1(M) \rangle \notin [n+3, 0)$.

Consider the moduli space $\mathcal{M}_{0,0}(M, J; \beta)$ of genus 0 Riemann surfaces w/ no marked pts in class β.

$\{S^2 \xrightarrow{u} M : J \circ du = du \circ j\} / PSL_2(\mathbb{C})$.

$\mathcal{M}^{simple}_{0,0}(M, J; \beta) \quad \subset \mathcal{M}_{0,0}(M, J; \beta)$

Comes w/ the property that $\exists$ pt $z \in S^2$ w/ $du|_z \neq 0$ and

$u^{-1}(u(z)) = z$.

Thm. (Classical, see McDuff – Salomon) For generic J, $\mathcal{M}^{simple}_{0,0}(M, J; \beta)$ is a smooth mfd. of real dim

$$2n + 2\langle \beta, c_1(M) \rangle - 6. = 2\left(n - 3 + \langle \beta, c_1(M) \rangle\right)$$

Consequences

If $c_1(M) = 0$, then no spheres generically if $\dim M < 6$.

For $\dim M = 6$, obtain a 0-dim'l space.

~~Example~~. From this, one derive the formula that

$$\dim \mathcal{M}^{simple}_{0,1}(M, J, \beta) = 2(n - 2 + \langle \beta, c_1(M) \rangle)$$

evaluation up to M

Codim of the cycle is $2\left(\langle \beta, c_1(M) \rangle - 2\right)$.

e.g. When the target is $\mathbb{CP}^1$, $c_1(\mathbb{CP}^1) \cong 2 \cdot$ generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$

$\Rightarrow$ if we set $\beta =$ generator of $H_2(\mathbb{CP}^1, \mathbb{Z})$

Looking at "linear maps": $(\mathbb{CP}^1, 0) \to \mathbb{CP}^1$

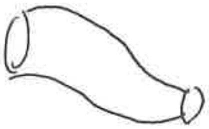
Weak monotonicity says:

no spheres w/ positive area and $\langle \beta . c_1 \rangle \in [3-n, 0)$ (This interval is ϕ if $n \leq 3$).

Below this range; get generically $\mathcal{M}_{0,0}(M, J; \beta) = \phi$.

Above this case, have that codim $\mathcal{M}^{\text{simple}}_{0,1}(M, J; \beta)$ in M positive.

Want to define Hamiltonian Floer homology;



Arrange for orbits to be disjoint from all spheres of index ≤ 1.

Consider the cylinders that appear in the Floer differential.

These sweep a 2-dim'l cycle.

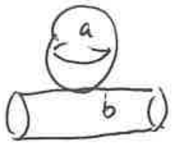
(might not be disjoint from $c_1 = 1$ sphere).

Consider Floer cylinders appearing in the proof of $d^2 = 0$.

Want. (label by index)

A priori nice cylind pictures

$\boxed{a+b=2}$



$n \leq 2$.

Say $\boxed{b=1}$ X

Next. $\boxed{b=2}$ $\Rightarrow a=0.$

$\downarrow$

This moduli space sweeps a dim 3 cycle.

But codim $\mathcal{M}^{\text{simple}}_{0,0}(M, J; \beta)$ is $\underline{\underline{4}}$ if $\langle c_1, \beta \rangle = 0$.

Issue. No simple curve in ∂, but need to argue for arbitrary elts of $\overline{\mathcal{M}_{0,0}}(M, J, \beta)$

Toy case. $c_1(M) = 0$.

Every elt. of $\overline{\mathcal{M}}_{0,0}(M, J, \beta)$ has components each of which covers a simple curve.

$\rightsquigarrow$ cycle swept by $\overline{\mathcal{M}}_{0,1}(M, J, \beta)$ is also a cycle of simple curves.

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,1}(M, J, \beta) \arrow[d, "ev"] \\ M \end{tikzcd}$$

Generalize the argument to case no sphere has $c_1 \in [3-n, 0)$.

If $\langle c_1, \beta \rangle \in [3-n, 0)$ and $u \in \mathcal{M}^{\text{simple}}(M, J; \beta)$,

$S^2 \xrightarrow{d:1} S^2 \longrightarrow M$

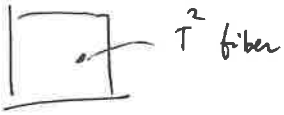
branch cover

This will be class $d \cdot \beta$ hence for d large enough,

$\langle c_1, d\beta \rangle < 3-n \Rightarrow$ moduli space is regular

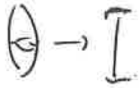
As soon as this happens, we don't know how to ensure that moduli spaces which define d & prove $d^2=0$ are manifolds w/ ∂.

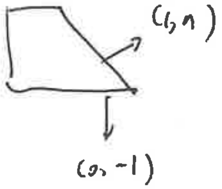
We know this happens for "Lagrangian disc counts" for tori in cpx surfaces.



$\mathbb{P}^1 \times \mathbb{P}^1$

(toric)





Hirzebruch surfaces.

Holo. structure has edges that holomorphic rat'l curves.

Try to count discs (of Maslov index 2.) w/ ∂ on torus fiber $T^2 \subset X$.

For $n=2$, $\exists$ a Chern 0 sphere.

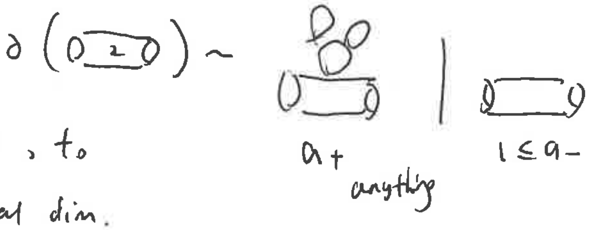
In fact, it's not just about "defining" d or proving $d^2=0$.

An even bigger problems is the existence of

total index is negative.

Not only about

But also about



To deal w/ this, need to make some choice, to elliminate all moduli spaces of negative virtual dim.

This choice will affect the resulting cpx.

Example. Morse theory for surfaces

index 0 solutions

(NOT generic)

Next time: Sketch proof of Arnold for non-deg.

Starting pt $HM^\bullet(f) \simeq H^\bullet(M)$.

Lecture 7 (M, ω), $H : M \times S^1 \longrightarrow \mathbb{R}$ non deg. Hamiltonian

Goal: Prove that it has more fixed pts than the minimal # of generator of a chain cpx of free ab gps computing the homology of M (as a $\mathbb{Z}/2\mathbb{Z}$-graded cpx)

In this form, this is proved by Rezchikov & Bai-Xu ('21)

Strategy: show that

$$\begin{tikzcd} H^*(M) \arrow[r] \arrow[rr, bend right, "\text{iso}"'] & HF^\bullet(H) \arrow[r] & H^*(M) \end{tikzcd}$$

over "Novikov" field" Λ w/ $\mathbb{Z}$- coeff." $\sum\limits_{\mathbb{R} \ni \lambda \to +\infty} a_\lambda T^\lambda$, $a_\lambda \in \mathbb{Z}$

Construction done at the chain level:

Use Morse theory as a model for $H^*(M)$, w/ fixed Morse function $f : M \to \mathbb{R}$, and a gradient to define $CM^*(f)$.

Recall Floer cpx is defined by counting sol'ns to Floer's eqn over the cylinder $\mathbb{R}\times S^1$

$$(du - X_H \otimes dt)^{0,1} = 0$$

w.r.t. family J_t of almost cpx str.

$$J_t\,(du - X_H \otimes dt) = (du - X_H \otimes dt)\circ j$$

$$J_t \frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - X_H$$

Additionally, "eliminates" contribution of negative virtual dim. moduli spaces.

Equip the plane $\mathbb{C}$ w/ cylindrical condition $(s,t) \longmapsto e^{s+2\pi i t}$

$$[0,\infty)\times S^1 \longrightarrow \mathbb{C}$$

Obtain an eq'n for maps $\mathbb{C} \longrightarrow M$ by extending the choice of

① Almost cpx str. J_z s.t. $J_{e^{s+2\pi i t}} = J_t$

② 1-form valued in TM, α , $\alpha|_{\text{cylindrical end}} = X_H \otimes dt$

Let $\mathcal{M}(J_z, \alpha)$ denote the space of solutions to the eq'n $(du-\alpha)^{0,1} = 0$

$\left(J_z\,(du-\alpha) = (du-\alpha)\circ j\right)$ which have finite energy $\int \|du-\alpha\|^2 < \infty$.

$\Rightarrow$ At ∞, finite energy + non-deg $\Rightarrow$ every elt of $\mathcal{M}(J_z, \alpha)$ converges to a periodic orbit of X_H. $\left(\int_{[R,+\infty)\times S^1} \|du - \underset{\underset{dt\otimes X_H}{\wr}}{\alpha}\|^2 \le \varepsilon \text{ for } R \gg 0\right)$ "Ellipticity" $\exists$ pointwise estimate $du - X_H\otimes dt$ is small

$$\begin{tikzcd}
M & \mathcal{M}(J_z, \alpha_z) \arrow[l, "\text{ev at } 0"'] \arrow[r, "\text{ev at } \infty"] & \text{periodic orbits } (H) \\
\coprod_{p\in \text{periodic orbit } (H)} & \mathcal{M}(M,p) \arrow[u, phantom, "\cong" sloped] &
\end{tikzcd}$$

Need ① Compactness. Introduce $\overline{\mathcal{M}}(M,p)$ "Kontsevich" "Gromov – Floer" compactification.

Two prototypical cases for failure of compactness (consider cases w/ $\int \|du_t - X_t \otimes \alpha\|^2 < \textcircled{E}$)

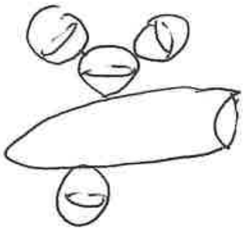
① sequence $z_i \to z \in \mathbb{C}$ w/ $\|du\| \to \infty$.

Rescale. $v_i(w) = u_i\left(z + \frac{w}{|du_i(z)|}\right)$ $w \in \mathbb{C}$

Get a sequence $v_i : D^2(\underbrace{|du_i(z)|}_{\text{get to } \infty}) \to M$

w/ a well-defined limit $v : \mathbb{C} \to M$.

$(\underbrace{du}_{\text{rescale}} - \underbrace{\alpha}_{\text{not being rescaled}})^{0,1} = 0$



Outcome: Limit is holomorphic plane in M for J_z w/ finite energy.

"Removable singularity". extend to $\mathbb{P}^1 \to M$ which is J_z holomorphic.

Warning. Following only "analysis" leads to a singular space in general.

To get a nice compactification, have to think about stable maps.

eg.

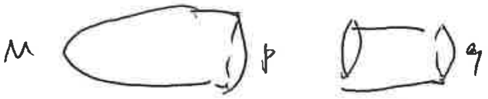
Say $\int \|dv_0\|^2 = 0 \Downarrow v_0$ constant

Cross-ratio between 4 pts

If we have a sequence z_i w/ $\varepsilon < \|du - X_H \otimes dt\|$.

$z_i \to \infty$, Then rescale "at ∞"

$\Rightarrow$ Breaking of a sol'n to Floer's eqn. $(\partial \overline{\mathcal{M}}(M,q) \leftarrow \overline{\mathcal{M}}(M,p) \times \overline{\mathcal{M}}(p,q))$



Piunikhin – Salamon – Schwarz.

Proved under favorable circumstances,

Key pt:

should have $\underline{\dim 2}$

looks like a cpx divisor.

$\overline{\mathcal{M}}(M,q)$ is a mfd w/ bdry whose bdry has codim 1 strata given by $\overline{\mathcal{M}}(M,p) \times \overline{\mathcal{M}}(p,q)$

Now we can define moduli spaces $\overline{\mathcal{M}}(x,q)$ for · x a crit. pt of $f: M \to \mathbb{R}$

· q periodic orbit of $H: M \times S^1 \to \mathbb{R}$

as compactification of $\boxed{T(x,M)} \underset{M}{\times} \mathcal{M}(M,q)$

gradient "trajectories"
par by $(-\infty, 0]$ w/ end pt x

(compactify the two sides separately)

__Thm__ The count of rigid (i.e. virtual dim 0) elts of these moduli spaces define a cochain map

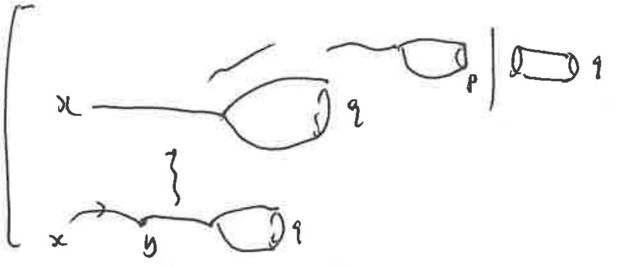
$$CM^*(f) \xrightarrow{PSS} CF^*(H)$$
$$x \longmapsto \sum_q \# \overline{\mathcal{M}}(x,q) \cdot q$$

(in terms of signed counts $\mathfrak{o}_x \xrightarrow{\oplus PSS_u} \bigoplus_{\substack{u \\ \in \\ \overline{\mathcal{M}}(x,q)}} \mathfrak{o}_q$)

For this to be well-defined, need to use the Novikov variable, assign every elt of $\overline{\mathcal{M}}(M,q)$ a real number ~~an elt of~~ "Topological energy". (This is NOT $\int \|du - \alpha\|^2$) "$E^{geom}(u)$"

__One step ahead__: To prove cochain property, look at 1-dim moduli spaces.

homotopy inv. of "Topo. energy"



$$\text{"}\cancel{A}(q)\text{"} = \int_{S^1} H_t(q)\,dt - \int_{\partial u = q} u^*\omega$$
$E^{top}(u)$

"= induced map on orientation"

This gives the correct wt $u \to T^{E^{top}(u)}$

$E^{top}(u) \neq E^{geom}(u)$ and can be negative.

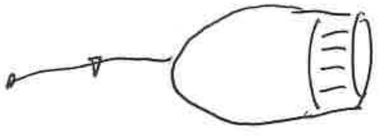
(but difference is bounded)

Lecture 8 **Last time:**

PSS map

$$\begin{tikzcd} CM^2(f) \arrow[r] & CF^2(H) \\ \text{a Morse function} \arrow[u] & \text{non-deg. Hamiltonian} \arrow[u] \end{tikzcd}$$

$$\frac{d\gamma}{dt} = -\nabla f$$



$\gamma : (-\infty, 0] \longrightarrow M$ $\quad \mathbb{C} \xrightarrow{u} M$ $\quad$ at ∞, u satisfies

$\gamma(0) = u(0)$ $\qquad (du - X_H \otimes dt)^{0,1} = 0$

Have to take Gromov-Floer compactification. Including codim 2 bubblings

Goal: Show that this induces an isom. on H^* (stronger than what is needed for Arnold)

Construct a "dual" or inverse map. Since M is orientable, we have an isom. of

(max of f $\longleftrightarrow$ min of $-f$)

chain complexes $(CM^*(f))^\vee \simeq CM^{2n-*}(-f)$.

Similarly, on the Floer side, $H : M \times S^1 \longrightarrow \mathbb{R}$, define $\bar{H} : M \times S^1 \longrightarrow \mathbb{R}$

by $\bar{H}(x,t) = -H(x,-t)$

This involution on $C^\infty(M \times S^1, \mathbb{R})$ satisfies the property that there's an 1-1 correspondence

$$\mathrm{Orbit}(X_{H_t}) \longleftrightarrow \mathrm{Orbit}(X_{\bar{H}_t})$$

$$p \longleftrightarrow \bar{p}(t) = p(-t)$$

If we define $CF^*(H)$ using a time dependent family J_t of almost cpx structures, then get

$$\underset{\substack{\downarrow\ \downarrow \\ \text{orbits of } X_H}}{\mathcal{M}(p,q)} \rightleftarrows \underset{\substack{\uparrow\ \nearrow \\ \text{orbits of } X_{\bar{H}}}}{\mathcal{M}(\bar{q},\bar{p})}$$

$$u : \mathbb{R}\times S^1 \longrightarrow M, \longleftrightarrow \bar{u} : \mathbb{R}\times S^1 \longrightarrow M$$

$$\bar{u}(s,t) = u(-s,-t)$$

This suggests that diff. on $CF^*(\bar{H})$ is the dual to the differential on $CF^*(H)$

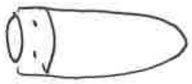
Need some work to establish w/ signs.

D_p

$D_{\bar{p}}$

We can draw this as

D_p



$D_{\bar{p}}$

(Trade the cylindrical end)

$(s,t) \to e^{s+it}$, $t\in[0,2\pi]$, $s\in[0,\infty)$

(positive cylindrical end)

For the negative cylindrical end $\underbrace{(s,t) \to e^{-s-it}}_{\text{pull back the data used for } D_{\bar{p}}}$, $s\in(-\infty,0]$, $t\in[0,2\pi]$

I get $(du - \boxed{X_H}\otimes dt)^{0,1} = 0$
$\nwarrow$ NOT $\bar{H}$.

<u>Glue</u>: Cauchy - Riemann operator where domain is S^2.

Deform to homogeneous operator.

Depending on whether we impose vanishing or not, get a surjective operator w/

either $\left[\begin{array}{l} 0 \text{ kernel} \\ \text{kernel} \simeq T_{p(0)} M \leftarrow \text{oriented} \end{array} \right.$

$$\rightarrow \left[\begin{array}{l} \deg p + \deg \bar{p} = 2n, \\ o_p \otimes o_{\bar{p}} \cong o_{TM} \leftarrow \text{trivialize by } \omega^n \end{array} \right.$$

$$\begin{tikzcd} CF^*(H) \arrow[r, phantom, "\simeq"] & \left(CF^{2n-*}(\bar{H})\right)^\vee \arrow[d, "(PSS^{\bar{H}}_{-f})^\vee"] & \\ & \left(CM^{2n-*}(-f)\right)^\vee \arrow[r, phantom, "\simeq"] & CM^*(f) \end{tikzcd}$$

Concretely, this means that we equip the plane w/ negative cylindrical end, On the image,

use the eq'n $(du - X_H \otimes dt)^{0,1} = 0$ $\left(dt = -d\theta \right)$

extend dt to a 1-form β on $\mathbb{C}$. valued in C^∞ v.fields on M.

For each periodic orbit p of H, we get a moduli space

$\mathcal{M}(p, M)$ of sol'n to $(du - \beta)^{0,1} = 0$

$u : \mathbb{C} \to M$, $\lim_{s \to \infty} e^{-s-it} = p(t)$.

Evaluate at 0.

$$\mathcal{M}(p, M) \longrightarrow M$$

Take fiber product w/ a flow line $\gamma : [0, +\infty) \longrightarrow M$

$$\frac{d\gamma}{dt} = -\nabla f, \quad \lim_{t \to +\infty} \gamma(t) = x.$$

$$\mathcal{M}(p, x) = \mathcal{M}(p, M) \times \ell(M, x)$$

Define $\overline{\mathcal{M}}(p, x)$ to be the Gromov-Floer compactification.

Now: define $CF^*(H) \xrightarrow{PSS^\vee} CM^*(f)$ to count elts of these moduli spaces.

Look at composite $CM^*(f) \to CF^*(H) \to CM^*(f)$

Want this to be homotopic to the identity.

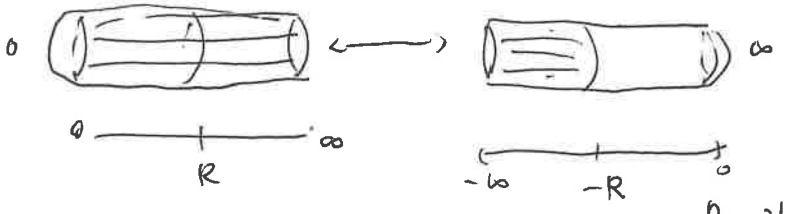
(This would arise from a cobordism between

$$\coprod_p \overline{\mathcal{M}}(x_1, p) \times \overline{\mathcal{M}}(p, x_2) / \sim \quad \longleftrightarrow \quad \mathcal{I}(x_1, x_2)$$

$$\uparrow$$

"continuation" for constant Hamiltonian.

Do this by glueing. For each large $R \geq 0$,



Deforming an equation

$v : S^2 \longrightarrow M$

1-form α & β valued in TM agree in the gluing region

get 1-form $\alpha \#_R \beta$ on S^2 valued in TM, $(dv - \alpha \#_R \beta)^{0,1} = 0$

We have evaluation from moduli space of solutions to that equation to $M \times M$
(at 0 & ∞)

Take fiber product w/ ascending & descending mfds

$\rightsquigarrow \bar{\mathcal{M}}(x_1, x_2; R)$

As $R \to +\infty$, show that $\coprod_{R \geq 0} \bar{\mathcal{M}}(x_1, x_2, R)$ can be compactified by adding

$\coprod_p \bar{\mathcal{M}}(x_1, p) \times \bar{\mathcal{M}}(p, x_2)$ (Gromov compactification)

For $R < 0$, we can deform the ~~equation~~ $\alpha \#_R \beta$ to the 0 1 form. So that we have

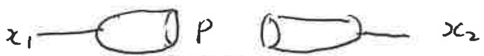
the homogeneous eq'n $\bar{\partial} v = 0$ (for a fixed J rather than time dependent)

Outcome Moduli space

$$\bar{\mathcal{H}}(x_1, x_2) \xrightarrow[\text{energy}]{\text{a topological}} \mathbb{R}$$ proper map.

$\partial \bar{H}(x_1, x_2)$ has the following components:

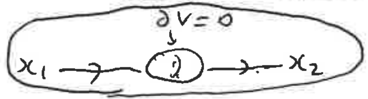
- As $R \to +\infty$,



corresponds to identity on $CM^*(f)$

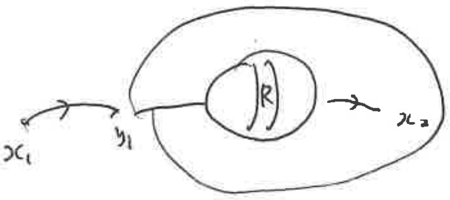
Except when the sphere is constant.

- As $R \to -\infty$.



← This moduli space can never be 0 dim'l.

Additional boundary associates to "breaking" of a graded traj. at input or output

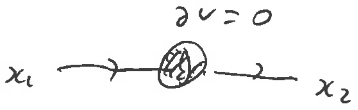


⇒ Use the 0-dim'l elts of $\mathcal{H}(x_1, x_2)$ to define a map $CM^*(f_1) \xrightarrow{G} CM^{*-1}(f_2)$

Then $dG(x_1) - G(dx_1) = PSS^\vee \circ PSS(x_1) - x_1$.

Back to end pt $R \to -\infty$. Replace Id on $CM^*(f)$ by an iso $\Phi : CM^*(f) \circlearrowright$

which counts



Uses the fact that contribution of each configuration is multiplication by $T^{E_{top}(u)}$.

For PSS map (and $PSS^\vee$) & For htpy

Only know that Etop takes a discrete set of values & is bounded below

But For $R \to -\infty$, have homog. eq'n ⇒ Energy is ≥ 0 and 0-energy sol'n are constant.

$CM^*(f;\Lambda_0) \circlearrowright \quad Id + o(T)$

$CM^*(f;\Lambda_0) \otimes_{\Lambda_0} \Lambda \quad$ → invertible map

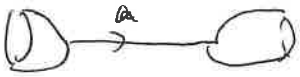
This is a trick, and expectation that in the setting of ordinary H^* w/ $\mathcal{O}$-coeff., this map is always the identity.

If action is free, expect to get 0 in bordism, But if NOT free ...?

We can also consider

$$CF^*(H) \longrightarrow CM^*(f) \longrightarrow CF^*(H)$$

This is htpic to identity.



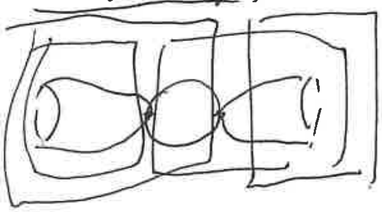
finite length trajectory.

Have to specify which compactification is being used at $a = 0$.

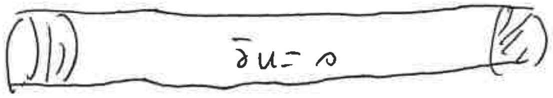
let $a \to 0$.

$$\begin{tikzcd}
 & \overline{\mathcal{M}}(p,M) \times_M \overline{\mathcal{M}}(M,q) \arrow[dl] \arrow[dr] & \\
\overline{\mathcal{M}}(p,M) \arrow[dr] & & \overline{\mathcal{M}}(M,q) \arrow[dl] \\
 & M &
\end{tikzcd}$$

NOT quite right.

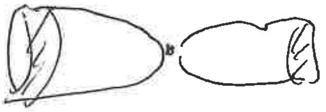


Want to see this as limit of



$(X_H \otimes dt)^{0,1} = 0 \qquad R \to \infty \qquad (X_H \otimes dt)^{0,1} = 0$

The top strata of link



But we don't have two separate components in codim 2.

In the other limit $R \to 0$, get identity continuation map for $CF^*(H)$

Hard to show that this gives identity, Easy to get $Id + \alpha T$.

Lecture 9 Arnol'd If $M \circlearrowleft \varphi$ is a Hamiltonian diffeo of a cpt symplectic mfld M, then $\mathrm{Fix}(\varphi) \geq \min \mathrm{Crit}(f)$

$\uparrow$ Smooth funct. on M

Known if M is symplectically aspherical.

(i.e. $\langle [\omega], - \rangle$ vanishes on $\pi_2(M)$)

Otherwise the problem is completely open except for S^2.

i.e. the minimal known bound is 1.

If $\langle c_1, - \rangle = 0$ on $\pi_2(M)$, then can prove that $\exists$ at least 2.

Paper of Schwarz in Inventiones gives "quantitative estimates" in terms of

$$\int_0^1 \max H_t - \min H_t \, dt$$ for Hamiltonian generating φ.

If this goes to ∞, Schwarz bound goes to 1.

Focus on Classical side

M closed smooth mfd. $f: M \longrightarrow \mathbb{R}$ smooth function, w/ isolated critical pt.

Goal: Bound $\mathrm{Crit}(f)$ from below using topological invariants.

Aside: Contribution of $\pi_1(M)$ to this bound. We will ignore it.

In the non-deg. version, we know that index 1 crit pts give generators of $\pi_1(M)$
index 2 crit pts give relations for $\pi_1(M)$

$\Rightarrow$ 2+g invariants of $\pi_1(M)$, give bound for min # of crit pts.

Smale $\Rightarrow$ Can achieve this bound on manifolds of dim > 5.

Correct way to do this is to look at $H^*(M, \underset{\uparrow \text{ rep'n of } \pi_1(M)}{\rho})$

The minimal rk of a cpx depends on whether one allows generators in arbitrary degrees,

OR only in deg ≥ 0.

This means that for $\pi_1(M) \neq 0$, we don't know whether the correct bound is given by the classical invariant.

There is a way to create "negative" index crit. pts in Morse setting by "stabilization".

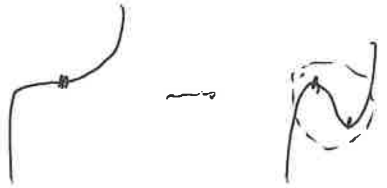
$M \rightsquigarrow M \times D^n$ (look at functions w/ gradient pointing out at ∂)

$M \rightarrow M \times \mathbb{R}^{2n}_{(x,y)}$ (look at functions which agree w/ x^2-y^2 at ∞.)

M. Damian proved that after this stab., get minimal # prescribed by algebra.

It seems difficult to expect a bound for degenerate one that is better than

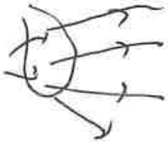
$\min\limits_{F: M\times D^n \to \mathbb{R}} \text{Crit}(F)$. or $\min\limits_{F: M\times \mathbb{R}^{2n} \to \mathbb{R}} \text{Crit}(F)$.



Problem. Can always find an embedded ball $D^n \hookrightarrow M$ containing all the crit pts.

Need to use the properties of gradient flow.

Take small nbhds $(U_1, \dots, U_k)$ of the crit pts, and consider all pts in M which flow to U_i for some positive time. $\rightsquigarrow W_i$



Can show that the inclusion $W_i \to M$ is null htpic.

These sets cover M. Use them to compute $\check{C}^*(M)$ $\boxed{\text{Compute } \check{H}^*(M)}$

$$\bigoplus \check{C}^*(W_i) \to \bigoplus \check{C}^*(\underset{\substack{\shortparallel \\ W_i \cap W_j}}{W_{ij}}) \to \dots \to \bigoplus C(W_1 \cap \dots \cap W_k)$$

(need to force an ordering $i_1 < \dots < i_\ell$)

This is a differential graded algebra

Define cup product by $\alpha \in W_I$, $\beta \in W_J$ $\rightsquigarrow$ restrict to $W_{I \cup J}$ and multiply.

Since we are moving to the right in every step $\Rightarrow$ can have at most k-many nontrivial multiplication $\alpha_1 \cup \dots \cup \alpha_k$.

Cor. The cup length of M (i.e. max'l length of a sequence $(\alpha_1, \dots, \alpha_k)$ of elts w/ $|\alpha_i| \neq 0$ and $\alpha_1 \cup \dots \cup \alpha_k \neq 0$) is a lower bound for $\min_f \text{Crit}(f)$

Can generalize to other H^* theories. More sophisticated H^* op (like Steenrod...)

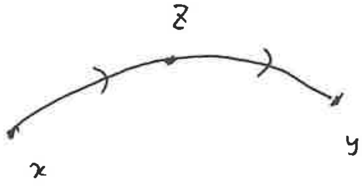
Morse theoretic perspective

Perturb f to a Morse function $g: M \to \mathbb{R}$

Decomposition of $\text{Crit}\, g = \bigcup_{x \in \text{Crit} f} \boxed{\text{Crit}\, g_x}$

$\uparrow$ all gradient flow lies between these lie in a contractible set.

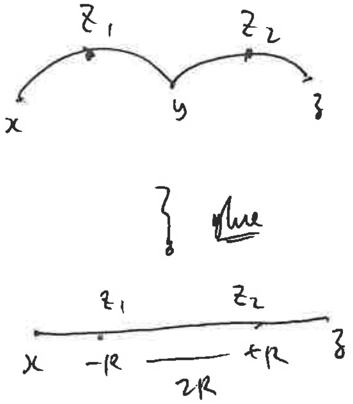
Consider $CM^{\bullet}(g)$ as a module over $C^{\bullet}(M)$



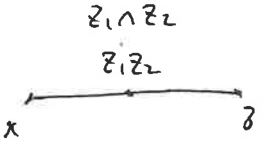
Count gradient flow lines "rigidified" by z

Def. $H^*(M) \otimes HM^*(g) \longrightarrow HM^*(g)$

Check that this makes $HM^*(g)$ into a module.



Let $R \to 0$



"If we can" represent cycles by manifolds, then get module structure for intersection product on H^*.

Can resolve this by

① Work w/ a simplicial model for $C^*(M)$ (i.e. associated to triangulation)

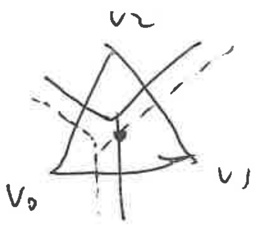
Formula for cup product of cochains (strictly associative)

This agrees w/ intersection product of dual cycles.

σ an n- chain

$$(\varphi^k \cup \psi^{d-k})(\sigma) = \varphi^k(v_0, \dots, v_k) \cdot \psi^{d-k}(v_k, \dots, v_d)$$

Pass to dual subdivision



use the ordering to perturb

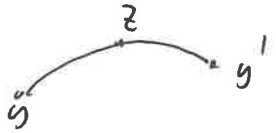
get intersection product.

Can then improve the H^* level statement to chain level.

(i.e. $CM^*(g)$ is a dg module over dga $C^*(M)$ w/ simplicial cup.)

② Introduce Morse model for $C^*(M)$ as an algebra ...

<u>Key fact</u>.

If $Z \neq M$, then the moduli space


is empty

(i.e. codim $Z \geq 1$)

if $y, y' \in$ Crit$_x g$.

Can filter $CM^2(g)$ by crit pts of f.

Filtration has length the # of Crit f.

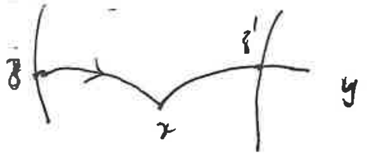
& every time we act by an elt of $H^\circ(M)$, go strictly deeper in filt'n level.

<u>Last ingredient</u>.

$CM^\circ(g) \simeq C^\circ(M)$ as an A_∞ module over $C^\circ(M)$

From this, we could conclude an iso. $H^*(M) \simeq HM^*(g)$ as modules over $H^*(M)$

In fact, our comparison map $H^*(M) \longrightarrow HM^*(g)$ gives such an isom.



Conclusion We have a Morse theory proof that the cup length of M is a lower bound for # Crit of any smooth function.

Lecture 10

More comments about minimal # of Crit pts of functions $M \xrightarrow{f} \mathbb{R}$.

Traditional way of proving estimates uses "category #" of M.

(and "strong category")

Strong category # of M is the minimal # of elts of a cover of M by contractible subsets. (category # is min # of elts of a cover by subsets U_i s.t. the map

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & \text{up to homotopy} & M \\ & * \arrow[ur] & \end{tikzcd}$$

(Manifold example ?)

Known to differ at most by 1 for spaces of the htpy type of finite CW cpxes.

(Book ~ 2000, Lusternik - Schnirelman Theory).

Idea If $f : M \to \mathbb{R}$ is a smooth function, equip M w/ metric

$\Rightarrow$ gradient.

Pick contractible nbhds V_p of all crit pts. Then closure of V_p under descending gradient flow

$$\boxed{U_p} = \bigcup_{\substack{t\in(-\infty,0] \\ x\in V_p}} \varphi^t(x) \qquad \curvearrowleft \varphi^t = \text{gradient flow}$$

This forms a cover of M

$\Rightarrow$ min # crit pts $\geq$ cat. #.

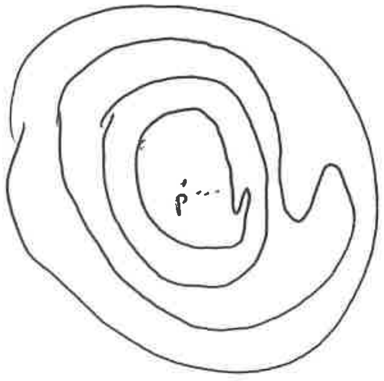
Problem In all known examples, can arrange for U_p to be contractible by a careful choice of V_p.

If we impose local assumption (e.g. $f \sim$ real analytic function near each crit. pt)

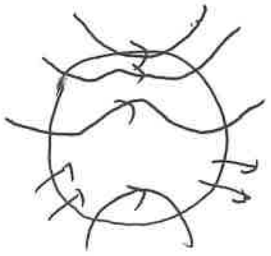
Then get min # crit pts $\geq$ strong cat. #.

This result also holds if dim M is 2 or 3 (Takens)

Goresky example



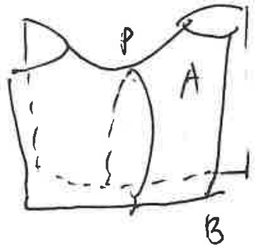
Picture of gradient flow



(Conley index)

Conley: Associated to any crit. pt is a (ptd) htpy type of the A/B where A is a nbd of p & B is some subset of A

If crit pt is non-deg, then take $\underset{A}{\text{nbd}}$ & mod out by B ← smallest" values of f in A.



Conj. If p is not a min, then the Conley index is a suspension of a space of dim $\leq n-2$.

$(n = \text{dim of } M)$

<u>Example</u>.

$$X \hookrightarrow S^{n-1}$$

↑ inclusion of a subcomplex.

Let νX be a nbhd of X which def. retracts to X <u>and</u> which is a (smooth) mfd w/ bdry. Then pick $g: S^{n-1} \longrightarrow \mathbb{R}$ which is 0 on ∂V_X

strictly negative on Int V_X, strictly positive outside + no crit pts on ∂V_X.

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be $f(x) = \|x\|^2 g\left(\frac{x}{\|x\|}\right)$

<u>Check</u> The only crit pt is the origin.

<u>Pf</u>. look at $f^{-1}(0)$. which is <u>cone</u> on $g^{-1}(0)$.

where $f \neq 0$, radial direction has non-zero derivative.

<u>Stronger Conj</u>.

Conley index is the suspension of a subset of S^{n-1}.

Back to category *

Use Čech cpx assoc to cover to show that cup length gives a bound for the category *.

The strong cat. * can be bounded using more refined homotopical invariants.

Intermediate Step. Cone length.

- Cone length of pt is 0.
- If X is h.e. to ΣY, then cone length $(X) = 1$.

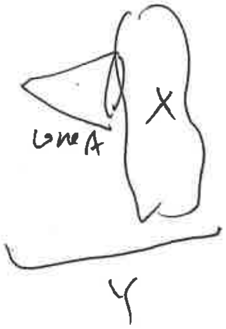
~~If we have filtration $X \to Y \to \Sigma X/Y$,~~

~~cone length (Y) =~~

take min over all such filtrations w/ assoc. graded given by ΣA.

If I attach cone A to X to get Y, $X \to Y \to \Sigma A$,

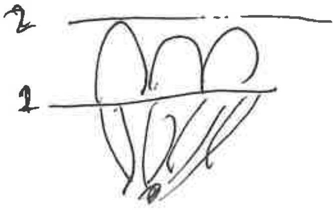
then set cone length $Y \leq$ cone length $X + 1$.



Example If M is a mfd of dim n, then Cone $M \leq n$

$(6 \leq n)$

Idea. Use self indexing Morse function.



Relation with cup length $\Bbbk$ ring

(A is based)

The cup product on $H^p(\Sigma A)$ vanishes

Comes from looking at diagonal in $\Sigma A \wedge \Sigma A \sim \Sigma^2 (A \wedge A)$.

In suspension factor, get $S^1 \to S^2$.  which is null.

$\Rightarrow$ Vanishing for any generalized cohomology theory.

Because vanishing is unstable, get vanishing of "unstable H^* operations".

Recall. define spaces $H(\Bbbk, n)$ s.t. $H^n(\underset{\substack{\uparrow \\ \text{based}}}{X}; \Bbbk) \simeq [X, H(\Bbbk, n)]$

There are maps $H(\Bbbk, n) \wedge H(\Bbbk, m) \to H(\Bbbk, n+m)$ which give rise to cup products.

$$\begin{tikzcd}
X \arrow[r, "\alpha \cup \beta"] \arrow[d] \arrow[dr, phantom, "\text{commutes up to htpy}"] & H(\Bbbk, n+m) \\
X \wedge X \arrow[r, "(\alpha, \beta)"] & H(\Bbbk, n) \wedge H(\Bbbk, m) \arrow[u]
\end{tikzcd}$$

Suspension iso.

$$\begin{tikzcd}
H^i(A; \Bbbk) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, sloped, "\simeq"] & H^{i+1}(\Sigma A; \Bbbk) \arrow[d, phantom, sloped, "\simeq"] \\
{[A, H(\Bbbk, i)]} & {[\Sigma A, H(\Bbbk, i+1)]}
\end{tikzcd}$$

$$\text{Maps}(S^1 \times A, Y) \simeq \text{Maps}(A, \text{Maps}(S^1, Y))$$

In based case,

$$[\Sigma A, Y] \simeq [A, \overset{\text{based loop space}}{\Omega} Y]$$

In the case of Eilenberg-Mclane space, we have a htpy equiv.

$$H(\mathbb{k}, i) \simeq \Omega H(\mathbb{k}, i+1).$$

(To check, $H(\mathbb{k}, i)$ is charac. by $\underset{[S^i, H(\mathbb{k}, i)]}{\underset{\|}{\pi_i(H(\mathbb{k}; i))}} \simeq \mathbb{k}$, all other vanish

<u>Key observation.</u>

Every class $\gamma \in H^*(H(K, n))$ of deg. d. induces a map

$$H^n(X; \mathbb{k}) \xrightarrow{\gamma} H^{n+d}(X; \mathbb{k}) \qquad \text{for all spaces } X$$

$$\alpha \longmapsto \gamma(\alpha)$$

which is natural in $X \xrightarrow{f} Y$ $\qquad \gamma(f^*\alpha) = f^*(\gamma(\alpha)).$

$\gamma(\alpha)$ is obtained by composing α w/ γ.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[rr, bend right, "\gamma \circ \alpha"'] & H(k, n) \arrow[r, "\gamma"] & H(k, d) \end{tikzcd}$$

Basic comp. $\tilde{H}^d(H(k, n); \mathbb{Z}) = 0$ if $d < n$.

Suspension iso gives $H^d(H(\mathbb{k}, n)) \simeq H^{d+1}(\Sigma H(\mathbb{k}, n))$

Want to relate to $H(\mathbb{k}, n+1)$. $\Sigma H(\mathbb{k}, n) \to H(\mathbb{k}, n+1)$

$\Omega H(\mathbb{k}, n+1) \simeq H(\mathbb{k}, n)$

Lecture 12.

If M closed & symplectic aspherical, then

$$\# \text{ Fix } \varphi \leq \text{cuplength } M + 1$$

$\uparrow$

Hamilt. diff

We followed Floer's proof, started discussing Hofer's proof.

Today

Symplectically aspherical $\Rightarrow$

① $\mathcal{A}$ on $\mathcal{L}M$ is single-valued

② No sphere bubbling.

If we assume ② but not ①, no strategy for pf.

i.e. assume M admits a tame a.c. str. w/ no J-spheres.

This happens

① If M is negatively monotone (i.e. $c_1(\alpha) \leq -n+2$, $\forall \alpha \in \pi_2(M)$)

i.e. All surfaces w/ $c_1 \leq 0$ satisfy this.

e.g. $H(d) \subset \mathbb{CP}^n$, smooth of degree d, then holds whenever

$$d \geq 2n+3.$$

$$c_1(\mathbb{CP}^n) \sim (n+1) \cdot u \in H^2(\mathbb{CP}^n)$$

(Warning. These are not the right examples, because monotonicity allows for specialized arguments)

Want: Examples with $\alpha \in \pi_2(M)$, $\exists$ $c_1(\alpha)=0$ & $\omega(\alpha)>0$.

If we allow "sphere bubbling", then lots of examples "CY manifolds".

eg. $H^d \subset \mathbb{CP}^n$, $d=n+1$

The examples which are Kähler fall in classes one of which "hyper kähler".

Differential geom. def'n. is existence of 3 integrable almost cpx structures (I, J, K)

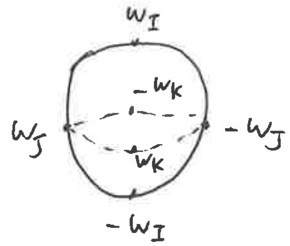
Satisfying Quaternion axioms $(IJ=K, \dots)$

For each of these have Kähler forms $(\omega_I, \boxed{\omega_J, \omega_K})$.

$\uparrow$ kähler for I.

$\sin\theta\, \omega_J + \cos\theta\, \omega_K$ cpx symplectic form.

$\rightsquigarrow$ $\overset{S^2}{\overset{=}{\mathbb{P}^1}}$ family of symplectic forms.



Also an S^2 family of almost cpx structures.

Fix ω_I to be preferred symp. form. get that "upper hemisphere" of cpx. structures are all ω_I-tame.

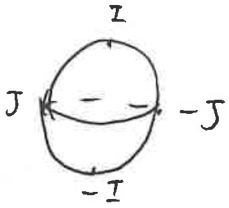
eg. If (Q, I) cpx. mfds. then T^*Q almost cpx

"Hope" use $J : T(TQ) \circlearrowleft$ coming from a choice of metric ⇒ HK metric

Verbitsky ⇒ works in nbd of 0-section.

If $Q \simeq G/K$, then get global structure.

Basic computation with periods shows that for generic J_α cpx. str for



∃ no holomorphic curves.

Key fact. If $\omega_{\mathbb{C}}$ is a "holomorphic" symplectic form on (M, J_α). $\Sigma \xrightarrow{u} M$

$du \circ j = J_\alpha \circ du$

then $u^*\omega_{\mathbb{C}} = 0$.

HK mfd ⇒ Examples of (M, ω) w/ $c_1 \equiv 0$ on $\pi_2(M)$, $\omega \not\equiv$ on $\pi_2(M)$

and No J-sphere.

Very few cpt examples

① K3 surfaces, can be realized as degree 4 hypersurface in $\mathbb{CP}^3$.

② Examples derived from the above as "moduli spaces of sheaves".

③ Sporadic examples, ... O'Grady.

In this case, Floer homology is really a Morse-Novikov homology group assoc to $(\mathcal{L}M, A)$.

f.d. model

- M closed smooth mfds
- $\alpha \in H^1(M, \mathbb{R})$

Novikov Bound (from below) the # of zeroes of a closed 1-form representing this class.

$\underline{\Lambda}$. In the non-deg. case, can define $HN^*(\eta)$, module over Novikov ring.

$$\sum_{\lambda \in \mathbb{R}} a_\lambda T^\lambda$$

$\{\lambda : a_\lambda \neq 0\}$ discrete, bounded below

Show ① Chain cpx computing this generated over Λ by crit pts.

② $HN^*(\eta) \simeq HN^*(\eta')$ if $[\eta] = [\eta']$.

Note. The groups $HN^*(\eta)$ encode information about $\pi_1(N)$ and action on htpy type of $\tilde{N}$

e.g. If $M \simeq T^n$, then all these groups are 0.

Idea. Use flat metric, represent α by {harmonic 1 forms}

(translation invt $\Rightarrow$ no zero)

Specialize. $\alpha \in H^1(M; \mathbb{Z})$ correspond to a htpy class of maps $M \to S^1$.

No critical pt

$\Updownarrow$

fiber bundle over S^1

(Stallings)

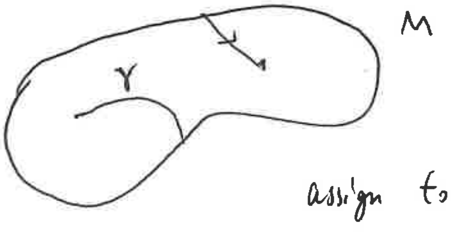
In this case, we can discretize Novikov ring, work w/ $\mathbb{Z}[t,t^{-1}]$, i.e. group ring of $\pi_1(S^1)\simeq\mathbb{Z}$, t is trivialization on

$$\begin{tikzcd}\tilde{M} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ M \arrow[r] & S^1\end{tikzcd}$$

& the gp $HN^*(\eta)$ is a "decompletion" of $H^*(\tilde{M})$.

(No difference in completion for H_c).

In principle.



assign to γ

$$\int_{\gamma} \gamma^*(\eta)$$

Discretize to intersection # w/ a such hypersurface representing class of $[\eta]$.

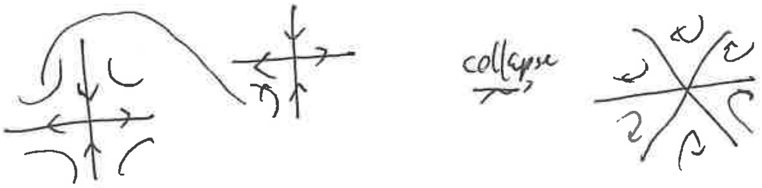
Thm (M. Farber ...) $\dim M \geq 6$ The min. # of crit pts of closed 1-forms in a given homlgy class is either 0 or 1 (if class $\neq 0$)

Two facts needed

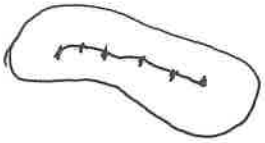
① Smale "handle move"

② Takens result on "collapsing" critical pts of smooth functions.

<u>Taken</u>. If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function w/ finite isolated crit pts (lying on $f^{-1}(0)$, then if $f^{-1}(0)$ is connected ($\&$ $n \geq 3$), can find smooth $f' : \mathbb{R}^n \to \mathbb{R}$ agrees w/ f outside cpt set & has a unique crit pt.



Taken strategy



Push pts together ⇝ collapse

Get C^∞ functions outside the crit pt.

Then approximate by smooth.

Want to deform π to minimize # of crit pts

$$\begin{tikzcd} M \arrow[d, "\pi"'] \\ S^1 \end{tikzcd}$$

Conclusion: Proving Arnold by usual strategy fails unless can "prevent" isotopies around loop in $\mathcal{L}M$. First place to look for counterexample is K3.

Lecture 13. The Arnold conj. on min no. of isolated fixed pts holds for $\mathbb{CP}^n$

(method does not currently work for $\mathbb{CP}^k \times \mathbb{CP}^m$).

Spectral invts (min-max invariants)

Say that M is a manifold equipped w/ a function $f: M \to \mathbb{R}$

- bounded above
- proper

Associate to each $a \in \mathbb{R}$ two different spaces

$$M^{\geq a} = \{x \in M : f(x) \geq a\}$$

$$M^{(-\infty, a)} = M / M^{\geq a}$$

$$M^{\geq a} \to M \to M^{(-\infty, a)}$$

Induces maps on H^*

$$H^*(M^{\geq a}) \leftarrow H^*(M) \leftarrow H^*(M^{(-\infty, a)})$$

Part of LES of $(M, M^{>a})$

So $\alpha \in H^\bullet(M)$ lies in image of pullback iff its restriction to

$M^{\geq a}$ vanishes.

Def. The spectral invariant $c(\alpha; f)$ of class $\alpha \in H^\bullet(M)$ is the infimum over $a \in \mathbb{R}$

s.t. α vanishes on $M^{\geq a}$.

Fundamental properties. (M conn'd)

① $C(1; f) = \max_{x \in M} f(x)$

② If M is cpt, then $C([M]; f) = \min_{x \in M} f(x)$.

(This is an instance of Poincaré duality).

Def The spectral norm of f is the difference $\max f - \min f$.

③ If $f+g \geq h$, ~~then~~ and α, β are elts of $H^\bullet(M)$, then

$$c(\alpha \cup \beta, h) \leq c(\alpha, f) + c(\beta, g).$$

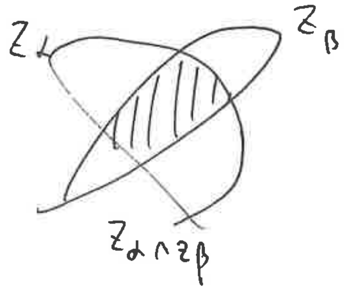
By Poincaré duality, we can represent any class $\alpha \in H^*(M)$ by a "co-oriented cycle"

$Z_\alpha = \{ \sum a_i \sigma_i \}$, $\sigma_i : \Delta^k \to M$

Interpret $C(\alpha, f)$ as $\min_{[Z]=\alpha} \max_{z \in Z \hookrightarrow M} f(z)$

Note: If $f(z) < a$, $\forall z \in Z$, then $Z \cap M^{\geq a} = \phi$.

Now, interpret cup product as an intersection product.



Want to estimate h on every pt of $Z_\alpha \cap Z_\beta \subset Z_\alpha$

$h \leq \textcircled{f} + g$. For good choice of Z_α, get

$f(z) \leq C(\alpha, f)$ and $g(z) \leq C(\beta, g)$

hence $h(z) \leq c(\alpha, f) + c(\beta, g)$, $\forall z \in Z_\alpha \cap Z_\beta$

① Using $C(1,0)$ is 0, get $C(\alpha,h) \leq C(\alpha,f)$ if $h \leq f$

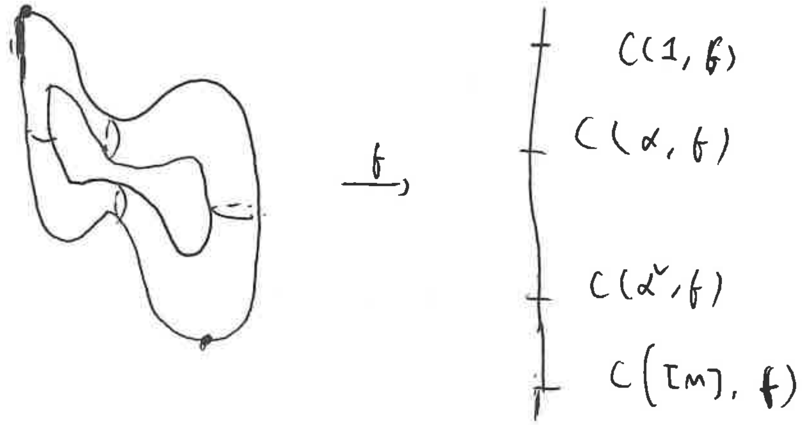
"monotonicity".

② Use $C(\alpha,0)=0,\ \forall \alpha$ to get $C(\alpha \cup \beta, f) \leq C(\beta, f)$

In other direction, $C(\alpha \cup \beta) \leq C(\alpha, f)$

$\Rightarrow C(\alpha \cup \beta, f) \leq \min\left(C(\alpha,f), C(\beta,f)\right)$

Picture.



Key fact we will use

Observed by Entov & Polterovich

Floer theoretic Spectral norm of any Hamiltonian on $\mathbb{CP}^n$ is uniformly bounded by $\sim \frac{n}{n+1}$.

Want to apply these ideas to Floer homology. Consider the Floer cpx gen. by "capped" periodic orbits.

$$A(\tilde{p}) = \int \tilde{p}^*(\omega) - \int_{S^1} p^* H_t \, dt$$

<u>Convention</u>. Floer differential decreases action & increases degree by 1.

So Floer cpx $CF^*(H)$ is filtered, w/ subcpx $CF^*_{\geq a}(H)$ of orbits of action $\geq a$.

and quotient cpx $CF^*_{\geq a}(H) \to CF^*(H) \to CF^*_{(-\infty,a)}(H)$

Can do the same as in the finite dim'l case & assign to each class $\alpha \in HF^*(H)$

a spectral invt $c(\alpha)$.

We know that we have an iso $H^*(M,\Lambda) \xrightarrow{\sim} HF^*(H)$ as modules over the

Novikov ring ($\sum a_i T^{\lambda_i}$)

So want to think of these as invts assoc. to elts of $H^*(M,\Lambda) \hookleftarrow H^*(M;\mathbb{k})$

That required a fixed choice of map $H^*(M) \longrightarrow HF^*(H)$.

Cap action of $H^*(M)$ on $HF^*(H)$ is compatible w/ spectral invts in the sense that for $\beta \in H^*(M)$, $c(\beta \cap \alpha) \leq c(\alpha)$

& inequality is strict if $\beta \neq 1$.

If we use PSS, get some invts assoc. to each class in $H^*(M)$.

<u>Problem</u>. Need to extract invariants of time -1 map φ gen. by H, rather than path H_t. Quasi-isomorphism types of $CF^*(H)$ as a filtered <u>cpx</u> does not depend on choices because it can be interpreted as (a subcpx of) $CF^*(\Delta_{id}, \Delta_\varphi)$, $\Delta_\varphi \subset M \times M$.

Problem lives in the PSS map: This gives a submodule

$$H^*(M;\Bbbk) \subset \underset{p}{HF^*}(H)$$

which generates Λ- module.

<u>Simplest problem.</u> H & $H + \overset{\text{const}}{C}$ gen. the same flow.

So $\mathcal{A}_H(\tilde{p}) + c = \mathcal{A}_{H+c}(\tilde{p})$.

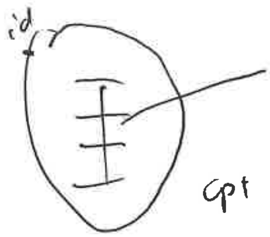
Resolved by normalization: eg. $\int H \, d\mathrm{vol}\, M = 0$

② We can have a non-trivial H which generates the identity

Set $H_{\frac{1}{2}+t} = \varepsilon H_{\frac{1}{2}-t} \circ \varphi$.

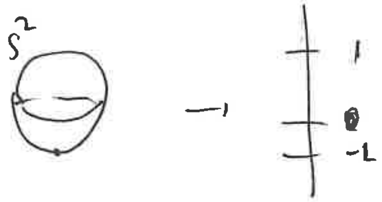
This can also be accounted for by vanishing of Act..

③



not null hlgy in $\mathrm{Ham}(M.)$

H_t determines a path in $\mathrm{Ham}(M)$



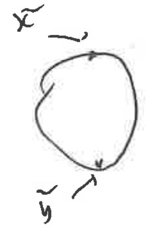
Normalize symplectic form so that φ^t is rotation by 2π.

Loop is non-trivial in $\mathrm{Ham}(S^2)$, because the maps

$$\begin{tikzcd}
S^1 \times S^2 \arrow[rr] \arrow[dr, "\text{up to htpy}"] & & S^2 \\
& S^3 \arrow[ur, "\text{Hopf}"'] &
\end{tikzcd}$$

Related to $\pi_2(SO(3)) \simeq \mathbb{Z}/2$.

φ^{ε} v.s. $\varphi^{1+\varepsilon}$



$\int H\,dt$ $\quad A_{H_\varepsilon}(\tilde{x}) \sim \varepsilon$

$A_{H_{1+\varepsilon}}(\tilde{x}) \sim 1+\varepsilon$

Outcome: Assoc. to each elt of $\widetilde{\mathrm{Ham}}(M)$ a filtered cpx so that the assoc. cpx up to shifting the filtration by a const. is an invt of the proj. to Ham.

In general, "Seidel invt" of a loop of Hamilt. diffeomorphism

$\gamma : S^1 \longrightarrow \mathrm{Ham}$ based at identity.

Associate to γ a Hamiltonian fiber bundle over S^2 w/ fiber M

$$\begin{tikzcd}
M \arrow[r] & E_\gamma \arrow[d, dash] \\
& S^2
\end{tikzcd}$$

$D_+^2 \times M$ glue $D_-^2 \times M$

by $(t, x) \leftrightarrow (t, \gamma(t).x)$

Assoc. to γ an elt of $QH^*(M)$ by counting pseudoholomorphic sections

$$\begin{tikzcd}
E_\gamma \arrow[d] \\
S^2 \arrow[u, bend left]
\end{tikzcd}$$

$u : S^2 \longrightarrow E_\gamma$ $\quad \pi \cdot u(0) = 0$, $\pi \cdot u(\infty) = 1$ $\Rightarrow \mathcal{M}_1(E_\gamma) \xrightarrow{ev} M$

~~Count of elts of $\mathcal{M}_1(E_\gamma)$ passing through~~

Image of $[\mathcal{M}_1(E_\gamma)]$ in $H'(M)$ defines a cycle $S(\gamma)$.

Key pt: If φ is a Hamilt. diffeo gen. by H_t, then spectral invts assoc. to H_t & $H_t \# H_{\gamma(t)}$ are related by cup w.r.t. $S(\gamma)$.

Lecture 14.

$QH^*(M, \omega)$ (   )

Define a deformation of $H^*(M)$ ← sympl. mfd

Commutative ring over Λ_0 ← Novikov (in general alg. $\mathbb{Z}/2$-graded)

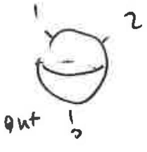
$\Lambda_0 = \{ \sum a_i T^{\lambda_i},\ 0 \le \lambda_i \to \infty \}$ $a_i \in \mathbb{Q}$ (can work integrally)

T-adic ideal consisting of elts w/ $0 < \lambda_0 \sim \Lambda_+$

$\Lambda_0 / \Lambda_+ \simeq \mathbb{k}$ ← ring where a_i live (e.g. $\mathbb{Q}$)

Defined by a "count" of stable J-holo. spheres w/ 3 marked pts. $2n = \dim_{\mathbb{R}} M$

$\Rightarrow \overline{\mathcal{M}_3}(M; \beta)$, $\beta \in H_2(M; \mathbb{Z})$ ← carries a "virtual fundamental class" in hlgy of degree $2(c_1(\beta) + n) = d_\beta$



$$\begin{tikzcd} & \overline{\mathcal{M}_3}(M;\beta) \arrow[dl, "in"'] \arrow[dr, "out"] & \\ M^2 & & M \end{tikzcd}$$

$$H^*(M) \otimes H^*(M) \xrightarrow{\text{Künneth}} H^*(M^2) \xrightarrow{\text{pullback}}$$
$$H^*(\overline{\mathcal{M}_3}(M;\beta)) \xrightarrow{\cap [\overline{\mathcal{M}_3}]} H_{\cancel{d_\beta *}}(\overline{\mathcal{M}_3}(M;\beta))$$
$$\to H_{d_\beta - *}(M) \overset{PD}{\simeq} H^{*+n-d_\beta}(M).$$

To implement this, replace $\overline{\mathcal{M}}_3(M,\beta)$ by some "thickening" which is an orbifold

$\mathcal{C}_3(M,\beta)$ equipped w/ a v bundle.

$\mathcal{O} \leftarrow$ "obstruction" of a section

arrange for evaluation map to extend to

$$\begin{tikzcd} & \mathcal{C}_3(M,\beta) \arrow[dl] \arrow[dr] & \\ M^2 & & M \end{tikzcd}$$

Define $[\overline{\mathcal{M}}_3(M,\beta)] \in H_{d_\beta}(\mathcal{C}_3(M:\beta))$ to be the cap of Euler class of $\mathcal{O}$

(defined rel choice of section) $\Rightarrow$ lies in cptly supported H^* with $\mathcal{C}_3(M,\beta)$

which lies "locally finite hlgy".

This requires $\mathcal{C}$ & $\mathcal{O}$ to be appropriately oriented.

Define $a_i \in H^*(M)$, $\beta \in H_2(-)$

$$a_1 \underset{\beta}{*} a_2 \simeq T^{\omega(\beta)} \text{ "pull push } (a_1, a_2)\text{"}$$

Positivity $\Rightarrow$ operation defined over Λ_0.

If $\beta = 0$, then $\overline{\mathcal{M}}_3(M,0) \simeq M$.

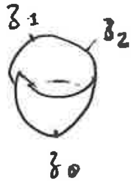
$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ M^2 & & M \end{tikzcd}$$

diagonal

The operation $a_1 \underset{0}{*} a_2$ is the intersection product. (usual cup).

This operation is commutatie because 

$\xrightarrow{u} M$

$(\mathbb{P}^1, (0,1,\infty)) \cong$ 

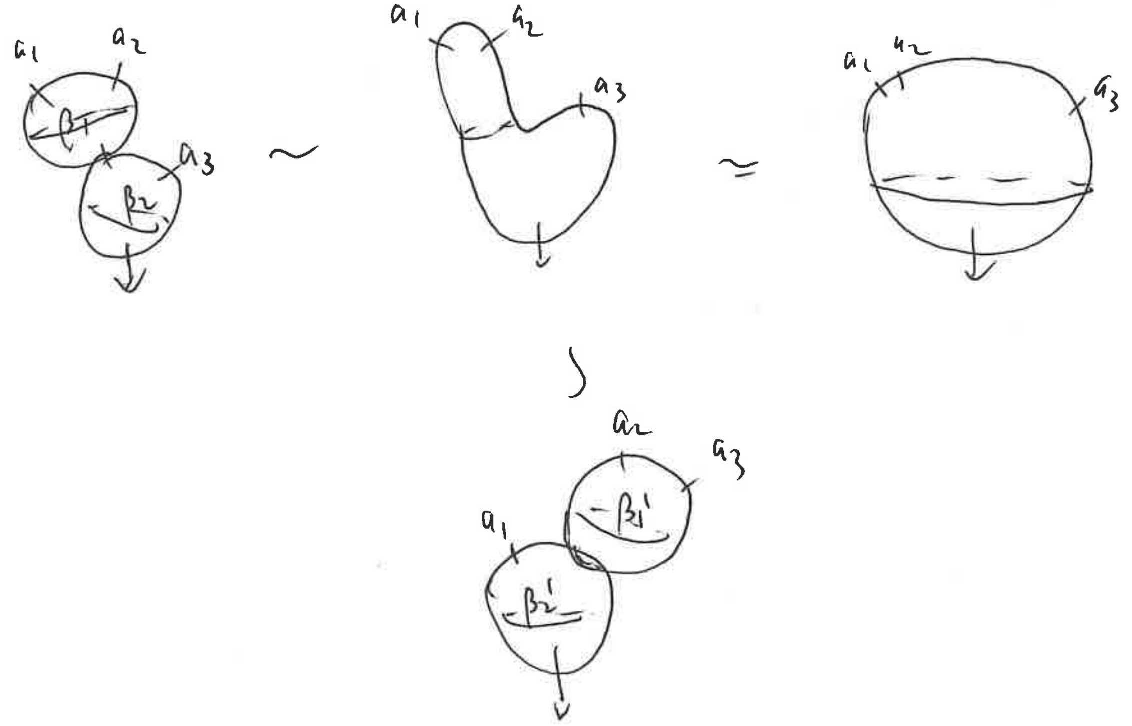
$\xrightarrow{u} M$ $\quad du \circ j = J \circ du$

$\uparrow \uparrow \uparrow$
$z_1 \; z_0 \; z_2$

Precompse u and z_0 fixed $\quad z \to \frac{1}{z} \quad (z_1, z_2) \to (z_2, z_3)$

<u>Associativity</u>

"WDVV" equation



For each $\beta \in H_2(M;\mathbb{Z})$, we have

$$\sum_{\beta_1+\beta_2=\beta} (a_1 \underset{\beta_1}{*} a_2) \underset{\beta_2}{*} a_3 = \sum_{\beta_1'+\beta_2'=\beta} a_1 \underset{\beta_1'}{*} (a_2 \underset{\beta_1'}{*} a_3) \rightarrow \text{associativity}$$

Compute For $M = \mathbb{CP}^n$

Ordinary H^* is $\mathbb{Z}[u] / u^{n+1} = 0$

$1 \sim$ "fundamental class"

$u \sim$ "hyperplane class" class of $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$

$u^2 \sim$ transverse intersection of a hyperplane w/ itself

$\vdots$

$u^n \sim pt$

$$u * u = u^2 + \sum_{1 \leq d} u \underset{d}{\textcircled{*}} u$$

$\beta \in H_2(\mathbb{CP}^n; \mathbb{Z})$

$\updownarrow$

detected by degree $d \in \mathbb{Z}$

Counting J-spheres in $\mathbb{CP}^n$ of deg. d

passing through a pair of hyperplanes.

$c_1(\mathbb{CP}^n) \simeq (n+1) \cdot u$

$\rightarrow \dim \overline{\mathcal{M}}_3(\mathbb{CP}^n, d) = 2(n + d(n+1))$

We will not see any contribution of any curve of degree $d > 1$.

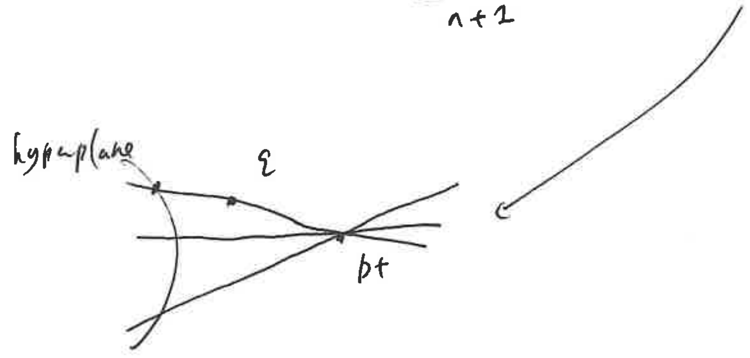
$H_{\dim \overline{\mathcal{M}}_3 - x}(\mathbb{CP}^n)$

$x \in [0, 2n]$

<u>Claim</u>. Whenever $i < n+1$, $\underbrace{u * \cdots * u}_{i \text{ times}} = u^i$

because the dimension of the moduli space of J-spheres w/ these constraints is too big.

So we need to compute $\underbrace{u * \cdots * u}_{n+1} = u * u^n$



$$H^2(\mathbb{CP}^n) \otimes H^{2n}(\mathbb{CP}^n) \longrightarrow H^{2n+2}(\overline{\mathcal{M}}_3(\mathbb{CP}^n))$$

$$\longrightarrow H_{2n}(\overline{\mathcal{M}}_3(\mathbb{CP}^n)) \longrightarrow H_{2n}(\mathbb{CP}^n) \to H^0(\mathbb{CP}^n)$$

$\Rightarrow \underbrace{u * u \cdots * u}_{n+1} = T \cdot 1$ (if we set ω to have the property that line has area 1).

How this gets used in proof that minimal # of fixed pts of Ham. diffeo on $\mathbb{CP}^n$ is $n+1$.

① Seidel homomorphism

Map $\pi_1(\mathrm{Ham}(M)) \longrightarrow (QH(M))^{\overset{\text{invertible elts}}{\times}}$

loop η defining
$$\begin{tikzcd} M \arrow[r] & E_\eta \arrow[d] \\ & S^2 \end{tikzcd}$$

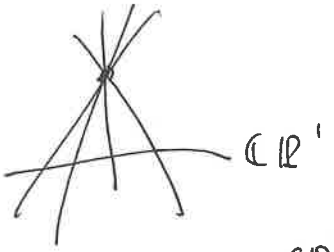
Map is a count of J-holo. sections

Compute for $\mathbb{P}^1$.

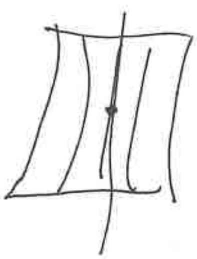
For η given by rotation (weight 1).

E_η is NOT the trivial bundle.

Can be identified w/ $Bl_{pt}\mathbb{CP}^2$.



Blow up at $*$ ~



$$\begin{tikzcd}\mathbb{CP}^1 \arrow[r] & Bl_{pt}\mathbb{CP}^2 \arrow[d] \\ & \mathbb{CP}^1 \simeq S^2\end{tikzcd}$$

Count pseudoholomorphic sections.

Contribution for each class $\gamma \in H_2(Bl.\mathbb{CP}^2;\mathbb{Z})$

All such classes ~~form~~ an affine space over ker; which is the class of a fibre.

Two distinguished classes

① Except'l fiber of blowup ~ σ_0

② Class of a line in $\mathbb{CP}^2$ σ_1

(related by $\sigma_1 = \sigma_0 + F$)

Note, ∃ exactly one curve in class σ_0. This gives the Seidel elt. of this loop as

$T^{\omega(\sigma_0)} \cdot u$

Same computation works for the loop in $\text{Ham}(\mathbb{C}P^n)$ coming from the circle action

$[e^{2\pi i\theta}\cdot z_0 : z_1 : \cdots : z_n]$ gives Seidel elt. $T^{\boxed{\omega(\sigma_0)}}\cdot u \in QH^*(\mathbb{C}P^{n+1})$

normalize.

Seidel Thm $\Rightarrow$ $(T^{\omega(\sigma_0)}\cdot u)^2 = 1$

$$T^{2\omega(\sigma)}\cdot(u'\cdot u) = T^{2\omega(\sigma)}\cdot T^{\text{Ham}(S^2)}\cdot 1.$$

$$\Rightarrow \quad u^{*\,n+1} = T^n\cdot 1.$$

$\pi_1(PU(n+1)) \simeq \mathbb{Z}/n+1$

$\downarrow$

acts by Hamiltonians on $\mathbb{C}P^n$.

Combine w/ spectral invt of $H : M\times S^1 \to \mathbb{R}$.

Chosen iso assoc. to H $\quad H^*(M) \xrightarrow{\Psi} HF^*(H) \supseteq$ filtered

$\Downarrow$

Given class $a \rightsquigarrow c(a,H) \in \mathbb{R}$

Need If η is a Hamilt. loop & $H \# H_\eta$

then $c(a\cup S(\eta), H) = c(a, H \# H_\eta)$ + "correction" depending only on η.

$\uparrow$

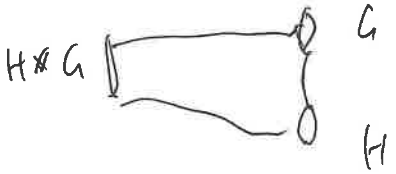
cap product.

Proof uses the fact that Ham is a gp.

$\eta : S^1 \to G, \quad \gamma : [0,1) \to G, \quad \gamma(0) = id$

Concatenation $\eta \# \gamma \overset{or}{=} \eta(t)\cdot\gamma(t)$ these are htpic (Eckmann-Hilton)

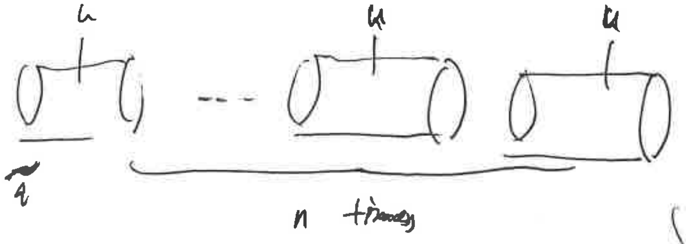
Not true for concatenation of paths.



$c(\alpha * \beta, H*G) \le c(\alpha, H) + c(\beta, G)$

Idea of pf. ① statement that $\gamma(H) < 1$ for $H : \mathbb{CP}^n \times S^1 \to \mathbb{R}$

$(c(1,H) - c([\mathbb{CP}^n]; H) =: \gamma(H))$



$\tilde{p}$ represents 1 in $HF^*(H)$

represents u^n in HF^*

action difference is < 1.

⇒ No two orbits can be the same.

$\Rightarrow \mathcal{A}(\tilde{q}) \ge c(u^n; H)$

Pick $\tilde{p} \ni \mathcal{A}(\tilde{p}) = c(1, H)$