Hyperspherical Hamiltonian spaces

Symplectic geometry

$/\mathbb{C}$. G alg. gp

$G \curvearrowright (M, \omega)$ smooth symplectic variety.

Def. This action is Hamiltonian if $\exists$ G-equivariant moment map (momentum map)

$\mu : M \longrightarrow \mathfrak{g}^*$ s.t. $\forall\ \xi \in \mathfrak{g}$,

$$d \langle \mu, \xi \rangle = \iota_{\rho(\xi)} \omega$$

$\langle \mu, \xi \rangle : M \longrightarrow \mathbb{C}$ function on M

$\rho : \mathfrak{g} \longrightarrow \Gamma(M, TM)$, $\rho(\xi)$ vector field on M

E.g. 1. $G \curvearrowright X$ smooth variety $\rightsquigarrow$ $G \curvearrowright T^*X$

T^*X has canonical symplectic form.

τ tautological 1-form on T^*X :

$\pi : T^*X \longrightarrow X$ $\rightsquigarrow$ $d\pi : T(T^*X) \longrightarrow TX$

Dual $\Rightarrow$ $T^*X \longrightarrow T^*(T^*X)$ $\rightsquigarrow$ 1-form τ on T^*X

$\omega = \pm d\tau$. Explicitly, $\tau = p_i dx^i$, $\omega = \sum dp_i \wedge dx^i$

momentum map: $\underset{\overset{\shortparallel}{M}}{\mu : T^*X} \longrightarrow \mathfrak{g}^*$ is just dual of $\mathfrak{g} \longrightarrow TX$.

E.g. 2. $SO(3) \curvearrowright T^*\mathbb{R}^3$, $\mu : T^*\mathbb{R}^3 \longrightarrow so(3)^*$
$(\vec{x}, \vec{p}) \longmapsto \vec{x} \times \vec{p}$ angular momentum

Hamiltonian reduction

M Hamiltonian G-space, $\mu: M \longrightarrow \mathfrak{g}^*$

<u>Def</u>. Hamiltonian reduction $M /\!/\!/ G := \mu^{-1}(0) / G$

<u>Rmk</u>. $\mu^{-1}(0)$ derived fiber product, $/G$ stack quotient $\rightsquigarrow$ "derived symplectic stack"

In practice, often $M /\!/\!/ G$ still produces a symplectic variety

More generally, $\mathcal{O} \subset \mathfrak{g}^*$ coadjoint orbit. $M /\!/\!/_{\mathcal{O}} G := \mu^{-1}(\mathcal{O}) / G$
$f \in$ $\qquad M /\!/\!/_f G := \mu^{-1}(G \cdot f) / G$.

<u>Eg</u>. $T^*X /\!/\!/ G = T^*(G \backslash X)$

<u>Eg. (Twisted cotangent bundles)</u> $\begin{matrix} \Psi \\ \downarrow \\ X \end{matrix}$ an equivariant $\mathbb{G}_a$-torsor over a G-variety X

(e.g. $\begin{matrix} \ker \psi \backslash G \\ \downarrow \\ U \backslash G \end{matrix}$, $U \overset{\text{rad}}{\subset} \overset{\text{Borel}}{B} \subset G$, $\psi: U \to \mathbb{G}_a$ generic additive character)

Twisted cotangent bundle $T^*_{\Psi} X := T^* \Psi /\!/\!/_1 \mathbb{G}_a$ $\qquad$ (e.g. $T^*_{\psi}(U \backslash G)$)

<u>Hamiltonian induction</u>. $H \subset G$ subgp. S Hamilt. H-space.

$$\text{h-ind}_H^G(S) := (S \times T^*G) /\!/\!/ H = \frac{S \times_{\mathfrak{h}^*} T^*G}{H} = (S \times_{\mathfrak{h}^*} \mathfrak{g}^*) \overset{H}{\times} G \xrightarrow{\text{fiber bundle}} H \backslash G$$

eg. $\text{h-ind}_H^G(T^*Y) = (T^*Y \times T^*G) /\!/\!/ H = T^*(Y \overset{H}{\times} G)$
$H \curvearrowright Y$

Frobenius reciprocity

Lagrangian correspondence

(M_1, ω_1), (M_2, ω_2) two symplectic manifolds, a <u>Lagrangian correspondence</u>

is a Lagrangian submanifold L_{12} of $X_1^{op} \times X_2$.

$\uparrow$ opposite symplectic form.

Composition of Lagrangian correspondence.

$$L_{12} \circ L_{23} = \pi_{13} \left(L_{12} \underset{X_2}{\times} L_{23} \right)$$

"higher cat. of Lagrangian correspondences of shifted symplectic stacks ..."

$$M = h\text{-}ind_H^G(S) = \underbrace{\left(S \underset{h^*}{\times} g^*\right)}_{\substack{\shortparallel \\ L}} \overset{H}{\times} G$$

$\rightsquigarrow$ $M^{op} \longleftarrow L \longrightarrow S$ Lagrangian correspondence.

Now, any Hamilt. G-space M equipped w/ an H-stable Lagrangian corr.

$$M^{op} \leftarrow L \rightarrow S$$

s.t. the compositions $L \rightarrow M \rightarrow g^* \rightarrow h^*$ & $L \rightarrow S \rightarrow h^*$ coincide.

$\rightsquigarrow$ induce to a Lag. corr. $M^{op} \hookleftarrow L \times^H G \longrightarrow h\text{-ind}_H^G(S)$

compatible w/ the moment maps of M & $h\text{-ind}_H^G(S)$.

Ideally, this should come from isom. of sympl spaces

$$M \xleftarrow{\sim} L \times^H G \xrightarrow{\sim} h\text{-ind}_H^G(S)$$

I don't see a general reason. In some cases, we can prove this.

<u>Rmk</u>. In many cases, there is an extra $\mathbb{G}_m$-symmetry (grading, $\mathbb{G}_{gr}$)

(e.g. T^*X, $\mathbb{G}_{gr}$ acts on fibers.)

<u>Graded Hamiltonian G-space</u>: M Hamilt. G-space, $\mathbb{G}_{gr} \curvearrowright M$

- $\mu: M \to \mathfrak{g}^*$, $\mathbb{G}_{gr}$-equiv.
- $\mathbb{G}_{gr} \curvearrowright \omega$ by <u>wt 2</u>.

All previous constructions have graded versions.

<u>Whittaker induction</u>. $H \times SL_2 \longrightarrow G$

$$\rightsquigarrow$$

(graded) Hamilt. H-spaces $\longrightarrow$ (graded) Hamilt. G-spaces

- When $SL_2 \to \{1\}$, reduces to Hamilt. induction.

<u>$\mathfrak{sl}_2$-pair (triple)</u> Fix invt identification $\mathfrak{g} \simeq \mathfrak{g}^*$

<u>$\mathfrak{sl}_2$-pair</u>: (ϖ, f) : $\varpi: \mathbb{G}_m \to [G,G]$, $f \in \mathfrak{g}^* \simeq \mathfrak{g}$, s.t. $(h = d\varpi(1), f)$ belongs to an <u>$\mathfrak{sl}_2$-triple</u> (e, h, f).

$H \subset G$: the centralizer of (e,h,f).

Decompose $\mathfrak{g} = \mathfrak{j} \oplus \bar{\mathfrak{u}} \oplus \mathfrak{u}^0 \oplus \mathfrak{u}$

$\mathfrak{j}$ = centralizer of sl_2, i.e. trivial sl_2-rep's

$\bar{\mathfrak{u}} \oplus \mathfrak{u}^0 \oplus \mathfrak{u}$ = sum of all nontrivial sl_2-rep's.

decomposed into the sum of negative, zero & positive wt spaces

$f \in \bar{\mathfrak{u}}$;

$\bar{U}, U$ assoc. unipotent subgroups.

$\varpi \rightsquigarrow \mathbb{G}_m$-action normalizes $U \rightsquigarrow$ treat $\mathfrak{u}$ as a graded Lie alg

$\mathfrak{u}_+ \subset \mathfrak{u}$: sum of h-eig. spaces of wt ≥ 2, U_+ assoc. unip. group.

Eg. If all the wts of the sl_2-action are even, i.e. $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in SL_2$ is central in G,

then $\boxed{\mathfrak{u}_+ = \mathfrak{u}}$

treat $f \in \bar{\mathfrak{u}}$ as $f : \mathfrak{u}_+ \to \mathbb{C}$

$$\mathfrak{u} \times \mathfrak{u} \to \mathbb{C}$$
$$(x,y) \mapsto \langle f, [x,y] \rangle$$

descends to an H-invariant sympl. form $(\mathfrak{u}/\mathfrak{u}_+) \times (\mathfrak{u}/\mathfrak{u}_+) \to \mathbb{C}$ $(*)$

$(\mathfrak{u}/\mathfrak{u}_+)_f$ = $(\mathfrak{u}/\mathfrak{u}_+)$ considered as a Hamiltonian HU-space

- H-action: adjoint action
- U-action: translation $U/U_+ \simeq \mathfrak{u}/\mathfrak{u}_+$

- moment map on the H- factor:

$$\mathfrak{u}/\mathfrak{u}_+ \longrightarrow \mathfrak{sp}(\mathfrak{u}/\mathfrak{u}_+)^* \longrightarrow \mathfrak{h}^*$$

$$m \longmapsto \left[X \mapsto \tfrac{1}{2}\langle Xm, m\rangle \right]$$

- moment on the U- factor

$$\mathfrak{u}/\mathfrak{u}_+ \xrightarrow[\sim]{\text{sympl. form}} (\mathfrak{u}/\mathfrak{u}_+)^* \xrightarrow{X \mapsto X+f} \mathfrak{u}^* .$$

S Hamilt. H-space $\rightsquigarrow$ $\tilde{S} = S \times (\mathfrak{u}/\mathfrak{u}_+)_f$ Hamilt. HU- space

(U acts trivially on S).

The Whittaker induction := h-ind$_{HU}^{G}(\tilde{S})$

$$= \left(\left(S \times (\mathfrak{u}/\mathfrak{u}_+)_f \right) \underset{(\mathfrak{h}+\mathfrak{u})^*}{\times} \mathfrak{g}^* \right) \overset{HU}{\times} G .$$

Eg. S trivial, $\mathfrak{sl}_2$ -wts are even (so $\mathfrak{u} = \mathfrak{u}_+$).

Whittaker induction $= HU \backslash \left((f + (\mathfrak{h}\mathfrak{u})^{\perp}) \times G \right)$

$\downarrow$ twisted cotangent bundle.

vector bundle (affine bundle).

$HU \backslash G$

In general,

When S is a symplectic H- <u>vector space</u>. Whittaker induction of S has a

base point: $\left(((0,0), f), \mathrm{id}_G \right)$.

Grading : shearing

$\varpi : G_{gr} \to \mathrm{Aut}(G)$ (e.g. conjugation by a cocharacter)

Def. A sheared Hamilt. G-space M is a Hamilt. G-space w/ G_{gr}-action compatible w/ the grading on G and $\mathfrak{g}^*$.

Concretely. $x \in M$, $g \in G$, $\lambda \in G_{gr}$

$$x \cdot g \cdot \lambda = x \cdot \lambda \cdot g^{\varpi(\lambda)}, \qquad \mu(x \cdot \lambda) = \lambda^2 \mu(x)^{\varpi(\lambda)}$$

e.g. when ϖ is the trivial action, this reduces the usual notion of graded Hamilt. G-space.

Eg. ① M graded Hamilt. G-space, $\varpi : \mathbb{G}_m \to G$ cocharacter.

can alter the G_{gr}-action by composing it w/ the (right) action of ϖ on M

$\rightsquigarrow$ sheared Hamilt. space, G is graded through the right inner action of ϖ.

② pt as a $\mathbb{G}_a$- Hamilt. space, but $\mu : pt \to \mathbb{C}$, $pt \mapsto 1$ is a sheared $\mathbb{G}_m$- Hamilt space: where $a^{\varpi(\lambda)} = a \cdot \lambda^{-2}$.

③ $(U/U_+)_f$ is a sheared U- Hamilt. space.

$G_{gr} \curvearrowright U$ by left conj. on U via the cocharacter ϖ in (ϖ, f).

$G_{gr} \curvearrowright U/U_+$ action = scaling by tautological char.

$\rightsquigarrow$ the f-shifted moment map is equiv. under the G_{gr}-action.

Grading of Whittaker induction

graded Hamilt. H-space $\xrightarrow{\times (U/U_+)_f}$ Sheared Hamilt. HU-Space $\xrightarrow{h\text{-ind}_{HU}^{G}}$ sheared Hamilt. G-space $\xrightarrow{\text{undo ①}}$ graded Hamilt G-space

Whit. induction of a sympl. vec. sp.

S : symplectic H-representation, equipped w/ scaling $\mathbb{G}_{gr}$-action.

Whit induction of $S =: M$

<u>Claim.</u> $M \simeq V \overset{H}{\times} G$, $\quad V = S \oplus (\mathfrak{h}^{\perp} \cap \mathfrak{g}^{*,e})$

$$\mathfrak{g}^{*,e} = \ker(e : \mathfrak{g}^* \to \mathfrak{g}^*) \simeq \text{centralizer of } e$$

<u>Rmk.</u> Isom as G-space ($\times \mathbb{G}_{gr}$) didn't mention symplectic structure

<u>Cor of Claim</u> If H is reductive, M is <u>affine</u>. [$H \backslash G$ affine b/c H reductive].

<u>Proof of claim</u> $M = \left(S \times (\mathfrak{u}/\mathfrak{u}_+)_f \underset{(\mathfrak{h}+\mathfrak{u})^*}{\times} \mathfrak{g}^* \right) \overset{HU}{\times} G$

$$S \times (\mathfrak{u}/\mathfrak{u}_+)_f \underset{(\mathfrak{h}+\mathfrak{u})^*}{\times} \mathfrak{g}^* \simeq \{ s \in S, t \in f + \mathfrak{u}_+^{\perp} : \mu(s) = t|_{\mathfrak{h}} \} \simeq [S \times_{\mathfrak{h}^*} \mathfrak{g}_e] \times U$$

<u>Slodowy slice</u>: $U \curvearrowright f + \mathfrak{u}_+^{\perp}$ is free,

& admits a transversal section $= f + \underbrace{\mathfrak{g}_e}_{\text{centralizer of } e}$

$$\rightsquigarrow M \simeq (S \times_{\mathfrak{h}^*} \underbrace{\mathfrak{g}_e}_{\simeq \mathfrak{g}^{*,e}}) \overset{H}{\times} G \simeq V \overset{H}{\times} G$$

<u>Hyperspecial Hamiltonian spaces.</u> A <u>hyperspherical</u> Hamiltonian G-space is a graded irreducible (smooth) Hamiltonian G-variety M s.t.

(1) M is affine

(2) $\mathbb{C}(M)^G$ is commutative w.r.t. Poisson bracket. (M is "coisotropic")

$(\omega^{-1})(df, dg)$

$\uparrow$

Poisson bivector.

(3) $\mu(M) \cap \mathcal{N} \neq \phi$

$\uparrow$

$\mathfrak{g}^*$ nilcone

(4) the stabilizer in G of a generic pt of M is conn'd.

(5) the G_{gr}-action is "neutral". (to be defined later)

Any Whittaker induction of a sympl. vector space satisfies (1), (3), (5);

Any Hyperspherical Hamilt. G-space comes this way

<u>Invariant moment map</u> $\mu_G : M \to \mathfrak{g}^* \to \mathfrak{c}^* := \mathfrak{g}^* /\!/ G$

Stein factorization : $M \xrightarrow{\tilde{\mu}_G} \mathfrak{c}^*_M \to \mathfrak{c}^*$, $\underbrace{\phantom{M \to \mathfrak{c}^*_M \to \mathfrak{c}^*}}_{\mu_G}$ $\tilde{\mu}_G$ dominant w/ conn'd fibers

$\boxed{\text{Condition (2)}}$

<u>Prop. TFAE</u>

(i) $\mathbb{F}(M)^G$ is commutative w.r.t. the Poisson bracket

(ii) the generic G-orbit on M is coisotropic

"spread out"

(iii) the generic fiber of $\tilde{\mu}_G$ contains an open G-orbit.

generic fiber contains a dense G-orbit [Losev]

For (iii):

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\tilde{\mu}_G"'] & M /\!/ G \arrow[d] \arrow[dl, dashed] \\ \mathfrak{c}^*_M \arrow[r] & \mathfrak{g}^* /\!/ G = \mathfrak{c}^* \end{tikzcd}$$

$\mathcal{O}_{\mathrm{Im}(\tilde{\mu}_G)} = \mathbb{C}[M]^G \cap \text{Poisson Center}(\mathbb{C}(M)^G)$

Rmk. $\mathrm{Im}(\tilde{\mu}_G) = M /\!/ G$.

$\boxed{\text{Condition (3)}}$ $\quad \mu_G : M \longrightarrow c^* = \mathfrak{g}^*/\!/G \simeq \mathfrak{t}^*/\!/W$

$\mathrm{Im}(\mu) \cap \mathcal{N} \neq \phi \Rightarrow 0 \in \mathfrak{t}^*/\!/W \in \mathrm{Im}(\mu_G)$

<u>Consequences</u>. (1) c_M^* contains a unique pt above $0 \in c^*$.

<u>Reason</u>: have compatible G_{gr}-action.

$c_M^* \xrightarrow{f} c^*$ $\quad f^{-1}(0)$ closed G_{gr}-orbit

f finite, $\mathbb{C}$ alg. closed $\Rightarrow f^{-1}(0)$ = single pt.

also denoted by 0.

(2) $\mathrm{Im}(\tilde{\mu}_G) = c_M^*$.

$\mathrm{Im}(\tilde{\mu}_G)$ open by a thm of Losev

Complement = closed set stable under G_{gr}-action

But cannot contain 0. (G_{gr}-action contacting to 0)

(3) $\exists !$ closed $G \times G_{gr}$-orbit $M_0 \subset M$;

$$\mathbb{C}[M]^{G \times G_{gr}} = \left(\mathbb{C}[M]^G\right)^{G_{gr}} = \mathbb{C}[c_M^*]^{G_{gr}} = \mathbb{C}$$

(4) M_0 is in fact a single G-orbit:

$M_0 \longrightarrow 0 \in c_M^* = M/\!/G$

whose fiber contains a <u>unique closed G-point</u>.

But by G_{gr}-transitivity, if one of those orbits is closed, all of them are

$\text{G-orb} \xrightarrow{G_{gr}} \text{G-orb}$

$\boxed{\text{Condition (5)}}$ neutrality.

Choose $x \in M_0$. $\quad f = \mu(x) \in \mathfrak{g}^*$

M_0 affine $\Rightarrow$ $H := G_x$ is reductive (expected to be connected).

$$\mu|_{M_0} : M_0 \simeq H\backslash G \longrightarrow \mathfrak{g}^*$$
$$Hg \longmapsto f^g = g^{-1} f g$$

$\mathbb{G}_{gr}$-action on M_0 commutes w/ G, so given by left mult. by a cocharacter

$\varpi : \mathbb{G}_m \to N(H)/H$, s.t. $f^{\varpi(\lambda)} = \lambda^2 f$.

Def. The $\mathbb{G}_{gr}$-action on M is neutral if

(i) the pair (ϖ, f) lifts to an sl_2-pair for G,

i.e. ϖ lifts to a cochar. $\lambda \mapsto \lambda^h$ for an sl_2-triple (h, e, f)

(ii) (i) implies the action of $(\lambda^{-h}, \lambda) \in G \times \mathbb{G}_{gr}$ stabilizes x

$\varpi_x : \lambda \mapsto (\lambda^{-h}, \lambda) \in G \times \mathbb{G}_{gr}$

Want: ϖ_x acts by the identity cocharacter on the fiber S of the symplectic normal bundle to the orbit $M_0 \subset M$.

Symplectic normal bundle: $\mathcal{O}$ G-orbit in a sympl. mfld M

$$\begin{array}{c}\square \\ \downarrow \\ \mathcal{O}\end{array}$$

. fiber over $x \in \mathcal{O}$ $= T_x\mathcal{O}^\perp / (T_x\mathcal{O}^\perp \cap T_x\mathcal{O}) = S$

S is a sympl. vector space, carries an action of G_x.
(Hamilt.)

Rmk. The sl_2-triple here is unique, Arthur-sl_2 attached to M.

Now we get $H \times SL_2 \to G$, S sympl. H-vetr space

Thm. There is a unique $G \times G_{gr}$-equivariant isom. of Hamilt. G-spaces

$M \simeq$ Whittaker induction of S from (H, sl_2)

which carries x to the base pt of the Whit. induction.

& induces there the identity on sympl. normal bundles.

Idea of proof. First construct maps by Frob. reciprocity, then prove isom.

When is a Whit. induced sympl. vec. sp. hyperspherical?

Prop. $M =$ Whit. induction of S from (H, sl_2)

- H is reductive $\Rightarrow$ $Y = HU \backslash G$ is quasi-affine.
- M is coisotropic (condition (2)) iff Y is spherical (dense B-orbit; B has nothing to do w/ U)

  & $\mathfrak{Z} = S \times (\mathfrak{u}/\mathfrak{u}_+)_f$ is coisotropic

  for the generic stabilizer of G on T^*Y
- M is hyperspherical iff, in addition, it satisfies (4).

Polarization.

Def. M admits a <u>distinguished polarization</u> if the wt-1 component $\mathfrak{u}_1 \subset \mathfrak{u}$ vanishes

& $\exists$ H-stable Lagrangian decomposition $S = S^+ \oplus S^-$.

In this case, $M \simeq T^*_{\Psi} X$, $X = S^+ \times^{HU} G$, $\Psi = S^+ \times^{HU'} G$

$U' = \ker(U \longrightarrow \mathbb{G}_a)$

$\uparrow$

additive char. induced by f.

Prop. Hyperspherical var. M admits a distinguished polarization $M \simeq T^*_{\Psi} X$, then

(a) X is a spherical G-var.

(b) the B-stabilizers of pts in the open B-orbit on X are conn'd.

Eigenmeasures $M = T^*(X, \Psi) = T^*_{\Psi} X$ has dist. polarization.

eigenmeasure: nowhere vanishing eigenvolume form. ω

$\exists$ char. $\eta : G \longrightarrow \mathbb{G}_m$ & $\gamma \in \mathbb{Z}$

$(g, \lambda)^* \omega = \eta(g) \lambda^{\gamma} \cdot \omega$

$\gamma = \dim(S^+) - \langle 2\rho, \varpi \rangle$

ϖ = cochar. assoc. to SL_2 for (G, M)

Let η be the char. of H acting on $\det(T_{(0,1)} X = S^+ \times^{HU} G)$

X admits eigenmeas. $\Leftrightarrow$ η extends to a char. of G.