Construction of the Igusa stack

Quanlin Chen

§1. Recollections and the goal.

Fix prime p. $(B, *, V, (\cdot,\cdot), \mathcal{O}_B, \Lambda)$ PEL datum type AC, unramified at p

$G/\mathbb{Q}$, $\Lambda = \Lambda_0 \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$

(G, X) PEL Shimura datum, reflex field E_0. $\mathfrak{p} | p$. $E = E_{0,\mathfrak{p}}$, $\mathbb{F}_q$ residue field of E.

$K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ and let $K^p \subset G(\mathbb{A}_f^p)$ cpt open.

$K = K_p K^p \subset G(\mathbb{A}_f)$.

ν_h a Hodge cocharacter. $T \subset B \subset G_{\overline{\mathbb{Q}}_p}$. Let μ dominant cocharacter representing

$G(\overline{\mathbb{Q}}_p)$-conj. class of ν_h^{-1}.

Good reduction locus at ∞-level at p

S°_{Kp}: Perf $\longrightarrow$ Sets

analytic sheafification of $\mathrm{Spa}(R, R^+) = S \longmapsto \{(S^\#, \mathcal{A}, \beta)\}$

- $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ untilt of S over $\mathrm{Spa}\, E$ i.e. perfectoid space $S^\#$, $S^\# \to \mathrm{Spa}(E, \mathcal{O}_E)$.

$S^{\#\flat} \xrightarrow{\sim} S$

- $\mathcal{A}$ formal abelian scheme w/ G-str. at level K. / $\mathrm{Spf}\, R^{\sharp +}$.

- $\beta \in \underline{\mathrm{Isom}}\,(T_p \mathcal{A}[p^\infty], \underline{\Lambda})\,(R^\sharp)$

$\mathcal{A}[p^\infty]$ p-div. gp / $\mathrm{Spf}\, R^{\sharp +}$ $\longrightarrow$ a p-div. gp / $\mathrm{Spa}\, R^{\sharp +}$.

base change $\rightsquigarrow$ a p-div. gp / $\mathrm{Spa}\, R^\sharp$.

$T_p \mathcal{A}[p^\infty]$ $\mathbb{Z}_p$-loc. sys. / $R^\sharp$.

<u>Goal</u>: construct v-stack Igs (roughly parametrize ab. sch. w/ G-str. up to p-power q-isog.)

$$\begin{tikzcd}
S^\diamond_{KP} \arrow[r, "\pi_{HT}^\diamond"] \arrow[d, "\mathrm{red}"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Gr}_{G,\mu} \arrow[d, "\mathrm{BL}"] \\
\mathrm{Igs} \arrow[r, "\bar{\pi}_{HT}^\diamond"] & \mathrm{Bun}_G
\end{tikzcd}$$

§2. Inputs from p-adic Hodge theory.

<u>Classifying p-div. gps</u>. 3 ways.

$S = \mathrm{Spa}\,(R, R^+) \in \mathrm{Perf}$, $S^\sharp = \mathrm{Spa}\,(R^\sharp, R^{\sharp +})$ char. 0

$\varpi \in R^+$ pseudo. unif. s.t. $(\varpi^\sharp)^p \mid p$. $\left(R^+/\varpi \simeq R^{\sharp +}/\varpi^\sharp\right)$.

① Crystalline Dieudonné modules.

$\{p\text{-div. gp} / R^+/\varpi\}$

$$\{p\text{-div. gp over } R^+/\varpi\} \longrightarrow \{p\text{-div. gp over } R^+/\varpi\}/q\text{-isog.}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \text{ fully faithful.}$$

$$\left\{\begin{array}{l}\text{integral crystalline Dieudonné mod}\\ (M/\underbrace{A_{crys}(R^+/\varpi)}, F, V)\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{rat'l crys. Dieudonné mod.}\\ (M/B^+_{crys}, \varphi)\end{array}\right\}$$

p-adic completion of PD hull of $W(R^+) \to R^{\#+}/\varpi^{\#} \cong R^+/\varpi$

M finite proj. $= A_{crys}[1/p]$

$\downarrow$ faithful [actually fully faithful when $R^+ = R^0$]

$$\left\{\begin{array}{l}\text{vector bundle on}\\ X^{alg}_{FF,S}\end{array}\right\}$$

$$X \longmapsto M_{crys}[1/p] \longmapsto \mathrm{Proj}\left(\bigoplus_{d\geq 0} (M_{crys}[1/p])^{\varphi=p^d}\right)$$

$$X^{alg}_{FF,S} = \mathrm{Proj}\left(\bigoplus_{d\geq 0} (B^+_{crys})^{\varphi=p^d}\right)$$

② Breuil-Kisin-Fargues module

$$\{p\text{-div gps}/R^{\#+}\} \xrightarrow[\sim]{\text{equiv}} \left\{\begin{array}{l}\text{Breuil-Kisin-Fargues module}\\ (M/W(R^+), \varphi_M)\end{array}\right\} \to \{\text{v.b. on } X_{FF,S}\}$$

$$\mathcal{X} \longmapsto (M(\mathcal{X}), \varphi_M) \longmapsto \Sigma(\mathcal{X})$$

③ Scholze – Weinstein classification $(R, R^+) = (C, C^+)$ C complete alg. closed.

$$\{\text{p-div gp} / \mathcal{O}_{C^\#}\} \xrightarrow{\sim} \left\{ (T, W) : \begin{array}{l} T \text{ free } \mathbb{Z}_p\text{-module} \\ W \subset T \otimes_{\mathbb{Z}_p} C^\#(-1) \\ \text{sub } - C^\#\text{-v.s.} \end{array} \right\}$$

$$\mathcal{H} \longmapsto (T_p \mathcal{H}, \operatorname{Lie} \mathcal{H} \otimes_{\mathcal{O}_{C^\#}} C^\#)$$

Thm (Faltings, Scholze – Weinstein) $\mathcal{X} = \mathfrak{X} \times_{R^{\#+}} R^{\#+}/\varpi^\#$

$\exists$ natural isom. $\Sigma(\mathfrak{X})^{alg} \simeq \Sigma(\mathcal{X})$

Moreover, when T generic fiber Tate module of $\mathfrak{X}$ is trivial,

$$\Sigma(\mathcal{X})\Big|_{X^{alg}_{FF,S} - S^{\#,alg}} \simeq T \otimes_{C^0(\operatorname{Spec} R^\#, \mathbb{Z}_p)} \mathcal{O}_{X^{alg}_{FF,S} - S^{\#,alg}}$$

In particular, $\underset{\substack{\text{BKF module} \\ M(\mathfrak{X})}}{M} \otimes_{W(R^+)} B_{dR}(R^\#) \simeq \underset{M_{crys}(\mathcal{X})}{M_{crys}} \otimes_{A_{crys}(R^+/\varpi)} B_{dR}(R^\#) \simeq T \otimes_{C^0(\cdot, R^\#)} B_{dR}(R^\#)$

$(R, R^+) = (C, C^+)$, $\quad Gr_{G,\mu} \simeq Fl^{\diamond}_{G,\mu}$ (PEL)

$\{\operatorname{Spa}(C, C^+)\text{-point of } Gr_{G,\mu}\}$

$\simeq \{(C^\#, C^{\#+}), \mathcal{O}_B\text{-stable max'l isotropic subspace } W \subset V \otimes_{\mathbb{Q}} C^\#\}$

(Λ, w) $\qquad$ $\mathcal{G}$ p-div gp $/\mathcal{O}_{C^\#}$ G-str on $\mathcal{G}$ together w/ $\beta : T_p \mathcal{G} \xrightarrow{\sim} \Lambda$

$\mathcal{G}/\mathcal{O}_{C^\#} \longrightarrow \mathcal{E}(\mathcal{G})$ on $X_{FF,\,\mathrm{Spa}(C,\mathcal{O}_C)}$

(later we'll prove this $\mathcal{E}(\mathcal{G})$ defines a G-bundle)

$j : X_{\mathrm{Spa}(C,\mathcal{O}_C)} \longrightarrow \underline{X_{\mathrm{Spa}(C,C^+)}}$ $\quad j^*$: equiv. of $\{$v. bundles$\}$

natural $\mathcal{E}(\mathcal{G}) \xrightarrow{\sim} j^*(\mathcal{BL}(S^\#, \mathcal{G}, \beta))$

---

§3. Hodge-Tate period map.

$\pi^\circ_{HT} : S^\circ_{KP} \longrightarrow Gr_G$ (lands in $Gr_{G,\mu}$)

$S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$

S-point : $(S^\#, A, \beta)$ $\qquad \mathcal{A} = A[p^\infty]$

$\beta : T \longrightarrow \underline{\Lambda}$ $\quad \mathbb{Z}_p$-local system on $R^\#$.

$$c : T \otimes_{C^0(\mathrm{Spec}\, R^\#, \mathbb{Z}_p)} B_{dR}(R^\#) \xrightarrow{\sim} M(\mathcal{A}) \otimes_{W(R^+)} B_{dR}(R^\#)$$

compatible w/ G-str. $\qquad \mathcal{F}$ étale sheaf on $X = \mathrm{Spec}\, B^+_{dR}(R^\#)$

$$\underline{\mathrm{Isom}}_G(M \otimes_{W(R^+)} \mathcal{O}_X,\ \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X)$$

on $\mathrm{Spec}\, B_{dR}$, $\alpha := (\beta \otimes \mathrm{id}) \circ c^{-1}$. , It is a G-torsor as a consequence for the Kottwitz (Rapport-Zink)

$(\mathcal{F}, \alpha) \rightsquigarrow$ S-point on Gr_G.

Pass

§4. Igusa stacks

Perf / Spd $\mathbb{F}_q$ v-topology

<u>Def</u>: $\mathrm{Igs} := \mathrm{Igs}^{\bullet}_{K^p}$ be the stackification of Igs^{pre}

$$S = \mathrm{Spa}(R, R^+) \longmapsto \mathrm{Igs}^{pre}(S)$$

<u>objects</u>: abelian schemes w/ G-str, w/ level K over R^+/ϖ

<u>Isom</u>: quasi-isog. $f: A \to A'$ s.t. $f^\vee \circ \lambda' \circ f = c\lambda$, $c \in \underline{\mathbb{G}}^\times(S)$,

f commutes w/ $\mathcal{O}_B$-action & level structure.

<u>Rmk</u>. Doesn't depend on ϖ by Serre-Tate theory.

Natural reduction map $((\varpi^\sharp)^p | p)$

$$\mathrm{red}: S^{\bullet}_{K^p} \to \mathrm{Igs}$$

$$(S^\sharp, A, \beta) \mapsto (S^\sharp, \underset{R^{\sharp +}}{A \times} \overset{A_0}{\overset{\shortparallel}{R^{\sharp +}}}/\varpi^\sharp)$$

$$\overline{\pi}^{\bullet}_{HT}: \mathrm{Igs} \to \mathrm{Bun}_G$$

Consider the sheaf $\underline{\mathrm{Isom}}_G(\mathcal{E}(A_0), \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S})$

This is a G-torsor.

$$\mathcal{E}(A_0)\big|_{X_S^{alg} - S^{\sharp,alg}} \xrightarrow{\sim} T \otimes_{\mathbb{Z}_p} \mathcal{O} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$$

in the completion of $S^\# \hookrightarrow X_S$, boils down to the Kottwitz condition.

$\rightsquigarrow \mathrm{Bun}_G(S)$

$$\bar{\pi}^\circ_{HT} : (S^\#, A_0) \longmapsto \underline{\mathrm{Isom}}_G(\mathcal{E}(A_0), \Lambda \otimes \mathcal{O}_{X_S})$$

---

§5. Fiber product formula.

Thm

$$\begin{tikzcd} S^\circ_{Kp} \arrow[r, "\pi^\circ_{HT}"] \arrow[d, "red"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Gr}_{G,\mu} \arrow[d, "BL"] \\ \mathrm{Igs} \arrow[r, "\bar{\pi}^\circ_{HT}"] & \mathrm{Bun}_G \end{tikzcd}$$

is 2-Cartesian.

$f : X \to Z$, $g : Y \to Z$ morphisms between small v-stacks,

$X \times_Z Y(S)$ obj.: $(x, y, \varphi : f(x) \xrightarrow{\sim} g(y))$

mor: $(x, y, \varphi) \to (x', y', \varphi')$

$\alpha : x \to x'$

$\beta : y \to y'$

$\varphi' \circ f(\alpha) = g(\beta) \circ \varphi$.

This is again a small v-stack.

Proof. $S^{\circ,pre}_{Kp} \simeq \mathrm{Igs}^{pre} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$

(on a basis of v-topology: products of points (We'll check it on geometric pts)).

① <u>2-commutativity</u>

$$\bar{\pi}^{\circ}_{HT} \circ \mathrm{red}\,(S^{\#}, \mathcal{A}, \beta) = \bar{\pi}^{\circ}_{HT}(S^{\#}, A_0) = \Sigma(A_0) \quad (G\text{-torsor})$$

$$\parallel$$

$$BL \circ \pi^{\circ}_{HT}\,(S^{\#}, \mathcal{A}, \beta) = BL\,(\mathcal{F}, \alpha)$$

$$\downarrow$$

$$M \otimes_{W(R^+)} B_{dR}^{+}(R^{\#})$$

② $\mathrm{Igs}^{\mathrm{pre}} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}(S)$ discrete.

$$\Sigma(\cdot): \{p\text{-div} / R^+/\varpi\} \xrightarrow{\text{faithful.}} \{\text{v. bundle}\}_{X_S}$$

③ Only need to check bijection on their objects.

Check on geom. pts $\quad S = \mathrm{Spa}(C, C^+) \quad (C = \bar{C})$

What is an S-pt of fiber product $\quad \mathrm{Gr}_{G,\mu} \simeq \mathrm{Fl}^{\diamond}_{G,\mu}$

A_0 ab. scheme $\quad G$-str. $/ C^+/\varpi$.

$S^{\#}$ untilt

$\mathcal{G}$ p-div. gp $/ \mathcal{O}_{C^{\#}}$

$\beta: T_p\mathcal{G} \xrightarrow{\sim} \Lambda$

$\phi: \Sigma(A_0^{[p^\infty]}) \xrightarrow{\sim} BL(S^{\#}, \mathcal{H}, \beta)$

$$\mathrm{Fl}^{\diamond}_{G,\mu}(S) = \mathrm{Gr}_{G,\mu}(S)$$

"$j^*\phi$": $\Sigma(A_0^{[p^\infty]} \times_{C^+/\varpi} \mathcal{O}_C/\varpi) \longrightarrow \Sigma(\mathcal{G} \times_{\mathcal{O}_{C^{\#}}} \mathcal{O}_C/\varpi)$

$\Leftrightarrow \quad \rho: A_0^{[p^\infty]} \times_{C^+/\varpi} \mathcal{O}_C/\varpi \dashrightarrow \mathcal{G} \times_{\mathcal{O}_{C^{\#}}} \mathcal{O}_C/\varpi$

$k := \mathcal{O}_C / m_C$ $\quad \overline{C^+} \subset k$ image of C^+ valuation ring.

$p^n \rho$: isogeny $\quad B := (A_0[p^\infty])_{\overline{C^+}}$

$\ker(p^n \rho_k) \subset B_k$. $\quad B/\overline{\ker} \xrightarrow{\sim} B'/\overline{C^+}$

<u>Milnor square:</u>

$$\begin{tikzcd} \mathcal{H} \quad C^{\#+} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \overline{C^{\#+}} = \overline{C^+} \arrow[d] \\ A_0[p^\infty] \, \mathcal{O}_{C^{\#}} \arrow[r, two heads] & k \end{tikzcd}$$

. $B'[p^\infty]$

$$\begin{tikzcd} \mathcal{A} \quad \{\text{ab. scheme}/R\} \arrow[r] \arrow[d] & \mathcal{H} \quad \{p\text{-div.}\} \arrow[d] \\ [A_0] \, \{\text{ab. sch.}/R\}/p\text{-power isog.} \arrow[r] \arrow[d] & \{p\text{-div}\}/q \text{ isog.} \\ R^{\#}/p^n & \end{tikzcd}$$

Serre-Tate $\rightsquigarrow$ formal ab. scheme $/R^{\#+}$.