# Hecke algebras with unequal parameters

Cédric Bonnafé

## Lecture 1

**Generalities** $R$ comm. ring, $A = \bigoplus_{i \in I} R e_i$ $R$-algebra

- $i \xleftarrow{L} j$ if $\exists\, a \in A$ s.t. $e_i \mid a e_j$ (nonzero coeff. of $e_i$)
- $i \xleftarrow{R} j$ ———————— $e_i \mid e_j a$
- $i \xleftarrow{LR} j$ —— $a, b$ —— $e_i \mid a e_j b$

$\leq_?$ transitive closure of $\xleftarrow{?}$, $? \in \{L, R, LR\}$.

This is a preorder $\rightsquigarrow$ assoc. eq. rel.

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$x \sim_? y$ if $x \leq_? y$ and $y \leq_? x$

If $C$ is a class for $\sim_?$, you can define $A^{\leq_? C} = \bigoplus_{i \leq_? C} R e_i$, ?- ideal of $A$

$V^C = A^{\leq_? C} / A^{<_? C}$ is a ?- module

$\simeq \bigoplus_{i \in C} R \tilde{e_i}$

Examples ① $A = R[G] = \bigoplus_{g \in G} Rg$

$g \overset{?}{\longleftarrow} h$ always

$\rightsquigarrow$ there is only one ?- equiv class, not interesting.

② $A = R[T] = \bigoplus_{n \geq 0} R \cdot T^n$

$m \overset{?}{\longleftarrow} n \iff m \geq n$, $\leq_? = \geq$

$V^m \cong R[T]/\langle T \rangle$

If $(a_n)_{n \geq 0}$ is a sequence in $R$, let $P_n = (T-a_1) \cdots (T-a_n)$

$$A = \bigoplus_{n \geq 0} R P_n, \quad V^m = R[T]/(T-a_m)$$

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Hecke algebras

$(W, S)$ Coxeter group. $W = \langle S : \begin{array}{l} \forall s \in S,\ s^2 = 1. \\ \forall s \neq t \in S,\ \underbrace{sts\cdots}_{m_{st} \geq 2} = \underbrace{tst\cdots}_{m_{st} \geq 2} \end{array} \rangle$

$\Gamma$ group (abelian)

$R = \mathbb{Z}[\Gamma] = \bigoplus_{\gamma \in \Gamma} R \cdot e^{\gamma}$, $e^{\gamma} \cdot e^{\gamma'} = e^{\gamma + \gamma'}$.

Let $\varphi : S \longrightarrow \Gamma$ s.t. $\varphi(s) = \varphi(t)$ if $s \sim_W t$

$$\mathcal{H} = \mathcal{H}(W, S, \varphi) = \bigoplus_{w \in W} R\text{-}T_w$$

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \qquad \ell : W \to \mathbb{Z}_{\geq 0} \\ \left(T_s - e^{\varphi(s)}\right)\left(T_s + e^{-\varphi(s)}\right) = 0 \end{cases}$$

Example $W_n = W(B_n)$

$$\begin{array}{ccccccc} t & & s_1 & & s_2 & & s_{n-1} \\ \circ & = & \circ & - & \circ & - \cdots - & \circ \\ b & & a & & a & a \cdots & a \\ \| & & & & \| & & \\ \varphi(t) & & \varphi(s_1) & = & \varphi(s_2) & \cdots & \varphi(s_{n-1}) \end{array}$$

$Q = e^{b}$, $q = e^{a}$.

$$\begin{cases} (T_t - Q)(T_t + Q^{-1}) = 0 \\ (T_{s_i} - q)(T_{s_i} + q^{-1}) = 0 \end{cases}$$

Two particular cases.

① If $\Gamma = \mathbb{Z}$, $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[v, v^{-1}]$, $v = e^{1}$

$$\begin{cases} (T_t - v^{b})(T_t + v^{-b}) = 0 \\ (T_{s_i} - v^{a})(T_{s_i} + v^{-a}) = 0 \end{cases}$$

② $\Gamma = \mathbb{Z}^2$, $b = (1,0)$, $a = (0,1)$.

$$\mathbb{Z}[\Gamma] = \mathbb{Z}[Q, Q^{-1}, q, q^{-1}]$$

<u>Kazhdan-Lusztig basis</u>

- $\bar{\ }: \mathcal{H} \longrightarrow \mathcal{H}$ is an automorphism of ring
  $e^{\gamma} \longmapsto e^{-\gamma}$ (involution)
  $T_w \longmapsto T_{w^{-1}}^{-1}$

- Let $\leq$ be an order on $\Gamma$ s.t. $\Gamma$ is a totally ordered abelian group.

$$R_{<0} = \bigoplus_{\gamma<0} \mathbb{Z} e^{\gamma}$$

$$\mathcal{H}_{<0} = \bigoplus_{w \in W} R_{<0} T_w$$

Previous examples. $\Gamma = \mathbb{Z}$
① : only "one" order

② $\Gamma = \mathbb{Z}^2$ : orders on $\mathbb{Z}^2$ : * lexicographic order

* $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{Z}^2 \longrightarrow \mathbb{R}$ $\rightsquigarrow \leq_\theta$
$(m,n) \longmapsto n\theta + m$ order on $\mathbb{Z}^2$.

<u>Thm</u> (Kazhdan - Lusztig '79) If $w \in W$, there exists a unique $C_w \in \mathcal{H}$ s.t.

$$\left[\begin{array}{l} \overline{C_w} = C_w \\ C_w \equiv T_w \mod \mathcal{H}_{<0} \end{array}\right.$$

"Proof" * <u>uniqueness</u>. If $C = \overline{C}$ and $C \in \mathcal{H}_{<0}$, then $C = 0$

Why? Write $C = \sum_{w \in W} a_w T_w$. Assume that $C \neq 0$. Take $w$ the max'l

s.t. $a_w \neq 0$: $\overline{C} = \overline{a_w} T_{w^{-1}}^{-1} + \sum_{x \not> w} \overline{a_x} T_{x^{-1}}^{-1}$

Write $w = s_1 \cdots s_l$ reduced exp.

$T_{w^{-1}} = T_{s_l} T_{s_{l-1}} \cdots T_{s_1}$

$$T_{w^{-1}}^{-1} = T_{s_1}^{-1} T_{s_2}^{-1} \cdots T_{s_l}^{-1}$$

$$= \left(T_{s_1} + e^{\varphi(s_1)} - e^{-\varphi(s_1)}\right) \cdots \left(T_{s_l} + e^{\varphi(s_l)} - e^{-\varphi(s_l)}\right)$$

$$= T_w + \sum_{x < w} r_{x,w} T_x$$

$\Rightarrow \overline{a_w} = a_w$, contradiction.

* <u>Existence</u>.

$t \quad s = s_1$

$0 = 0$

$b \quad a$

$b, a > 0$

$Q = e^b, \; q = e^a$

$$C_1 = T_1$$

$$C_s = T_s + e^{-a} = T_s + q^{-1}$$

$$C_t = T_t + Q^{-1}$$

$$C_s C_t = T_{st} + Q^{-1} T_s + q^{-1} T_t + Q^{-1} q^{-1} = C_{st}$$

$$C_t C_{st} = T_{tst} + q^{-1} T_t^2 \quad \text{modulo } \mathcal{H}_{<0}$$

$T_t^2 = 1 + (\alpha - \alpha^{-1}) T_t$

So $C_t C_{st} = T_{tst} + \underset{\underset{e^{b-a}}{\shortparallel}}{Q q^{-1}} T_t \mod H_{<0}$

- If $b < a$, $C_t C_{st} = C_{tst}$
- If $b = a$, $\underbrace{C_t C_{st} - C_t}_{C_{tst}} \equiv T_{tst} \mod H_{<0}$
- If $b > a$, $\underbrace{C_t C_{st} - (Q q^{-1} + Q^{-1} q) C_t}_{C_{tst}} \equiv T_{tst} \mod H_{<0}$

$C_s C_w = T_{sw} + \cdots$

$\rightsquigarrow C_{sw}$

- Write $C_y = \sum_{x \in W} p^*_{x,y} T_x$
- If $sy > y$, write $C_s C_y = C_{sy} + \sum_{x \in W} \lambda^s_{x,y} C_x$

Thm. (a) $p^*_{y,y} = 1$

$p^*_{x,y} \neq 0 \Rightarrow x \leq y$

(b) If $\lambda^s_{x,y} \neq 0$, then $sx < x < y < sy$

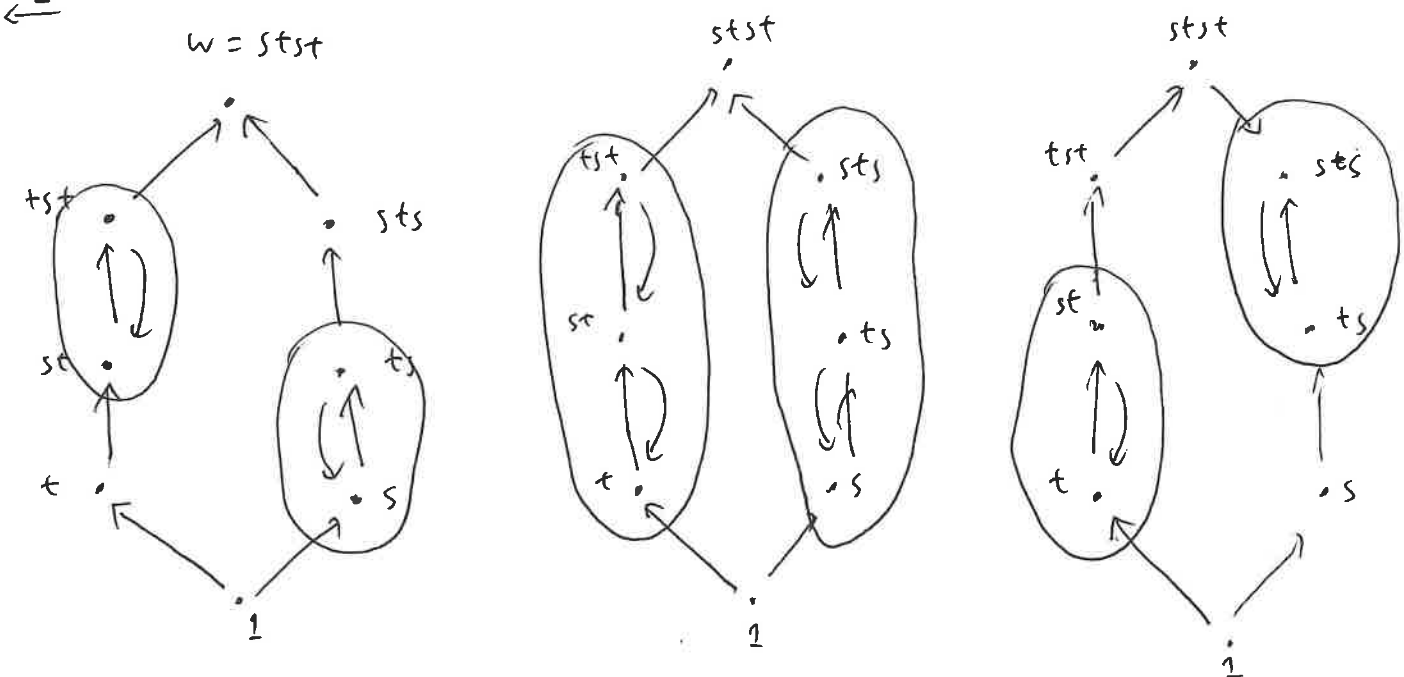
© If $sw < w$, $C_s C_w = \left(e^{\varphi(s)} + e^{-\varphi(s)}\right) C_w$

Now we can define $\leq_L$, $\leq_R$, $\leq_{LR}$ by using the Kazhdan–Lusztig basis:

$x \xleftarrow{L} y$ if $\exists\, h \in \mathcal{H}$ s.t. $C_x \mid h C_y$.

$\sim_L$, $\sim_R$, $\sim_{LR}$ : equivalence classes are called the ? -cells.

$\boxed{B_2}$ $\xleftarrow{L}$




$b<a$ $b=a$ $b>a$

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Lecture 2. $\boxed{\forall s \in S,\ \varphi(s) > 0}$

$$C_y = \sum_{x \in W} P^*_{x,y} T_x$$

* $P^*_{y,y} = 1$
* $P^*_{x,y} \neq 0 \Leftrightarrow x \leq y$
* If $sy < y$, $C_s C_y = \left(e^{\varphi(s)} + e^{-\varphi(s)}\right) C_y$.
* If $sy > y$, $C_s C_y = C_{sy} + \sum_{sx < x < y} \lambda^s_{x,y} C_x$

* If $ys<y$, $\quad C_y C_s = \left(e^{\varphi(s)} + e^{-\varphi(s)}\right) C_y$

* If $ys>y$, $\quad C_y C_s = C_{ys} + \sum_{xs<x<y} \rho^{s}_{x,y}\, C_x$.

Consequence: $\quad \bigoplus_{ys<y} R C_y = \left\{ h\in H : h C_s = \left(e^{\varphi(s)} + e^{-\varphi(s)}\right) h \right\}$

is a left ideal.

Proof. Let $\mathcal{E} = \{ y\in W : ys<y \}$. $\qquad \boxed{C_s = T_s + e^{-\varphi(s)}}$

Then $W = \mathcal{E} \sqcup \mathcal{E}s$.

Let $\mu : H \longrightarrow H$

$h \longmapsto h\Big(C_s - \overbrace{\left(e^{\varphi(s)} + e^{-\varphi(s)}\right)}^{a_s}\Big)$

$$\begin{array}{c|c} & \begin{array}{cc} \mathcal{E} & \mathcal{E}s \end{array} \\ \begin{array}{c} \mathcal{E} \\ \mathcal{E}s \end{array} & \left(\begin{array}{c|c} 0 & * \\ \hline 0 & \begin{matrix} -a_s & & 0 \\ & \ddots & \\ 0 & & -a_s \end{matrix} \end{array}\right) \end{array} \qquad \Rightarrow \ker\mu = \bigoplus_{y\in\mathcal{E}} R\cdot C_y$$

Let $* : H \longrightarrow H$

$e^{\gamma} \longmapsto e^{\gamma}$ $\qquad$ anti-automorphism of R-alg. (involution)

$T_w \longmapsto T_{w^{-1}}$

$(\overline{h})^* = \overline{(h^*)}$. $\quad \Rightarrow \quad C_w^* = C_{w^{-1}}$.

$\Rightarrow \quad \lambda^{s}_{x,y} = \rho^{s}_{x^{-1},y^{-1}}$.

* $x \xleftarrow{L} y$ if $\exists\ h \in \mathcal{H}$ s.t. $C_x \mid h C_y$.

$\leq_L$, $\sim_L$

$\rightsquigarrow$ $\leq_R$, $\sim_R$

$\leq_{LR}$, $\sim_{LR}$

We have $x \leq_L y \iff x^{-1} \leq_R y^{-1}$

$x \sim_L y \iff x^{-1} \sim_R y^{-1}$

* Motivation.

Equal parameter case: $\Gamma = \mathbb{Z}$, $\varphi(s) = 1$, $\forall s$.

- Geometry of Schubert varieties

$$K_0\left(D^b_G(G/B \times G/B) + \text{action of Frobenius}\right) \simeq \mathcal{H}$$

$$\rightsquigarrow\ P^*_{x,y} \in \mathbb{N}[\Gamma]$$

- link with the theory of primitive ideals of enveloping algebras
- Special Unipotent Class $(G) \overset{\sim}{\longleftrightarrow}$ Two sided cells of $W$

Unipotent Classes $(G) \overset{\sim}{\longleftrightarrow}$ Two sided cells of $\tilde{W}$.

* General case Let $K = \operatorname{Frac}(R)$.
  * lead to construction of "small" $\mathcal{H}$-modules

$KH$ is split semisimple, If $C$ is a left cell, it's "interesting" to understand the structure $KV_c$.

* Even if $KV^c$ is not irreducible they are important in the representation theory of $G(k)$, where $k$ is finite or $p$-adic.

* In type $B_n$, the modules $V^c$ should give an intepretation of Ariki's Thm. (Fock space).

"Compute $\sim_L, \sim_R, \sim_{LR}$"

<u>Proposition</u> * If $x \leq_L y$, then $R(y) \subset R(x)$

where $R(y) = \{s \in S : ys < y\}$

* If $x \sim_L y$, then $R(x) = R(y)$.

<u>Proof</u>. We can assume that $x \overset{L}{\leftarrow} y$, so $C_x \mid hC_y$, $h \in H$

If $s \in R(y)$, then $\bigoplus_{ws<w} RC_w$ is a left ideal

$$\begin{array}{c} \bigoplus_{ws<w} RC_w \\ \uplus \\ C_y \end{array}$$

$$\Rightarrow hC_y \in \bigoplus_{ws<w} RC_w$$

Since $C_x \mid hC_y \Rightarrow xs < x \iff s \in R(x)$. $\square$

<u>Parabolic subgroups</u> If $I \subset S$, $W_I = \langle I \rangle$.

$$\mathcal{H}_I \subset \mathcal{H}$$
$$\parallel$$
$$\bigoplus_{w \in W_I} R \cdot T_w$$

$$X_I = \{x \in W : xs > x, \ \forall s \in I\}$$

$$= \{x \in W : x \text{ has min. length in } xW_I\}$$

$$\begin{array}{ccc} X_I & \ni & x \\ \downarrow \wr & & \downarrow \\ W/W_I & \ni & xW_I \end{array}$$

$$\begin{array}{ccc} X_I \times W_I & \xrightarrow{\sim} & W \\ (x, w) & \longmapsto & xw \end{array}$$

$$\xi_I : W \to X_I, \quad xw \mapsto x$$
$$\pi_I : W \to W_I, \quad xw \mapsto w$$

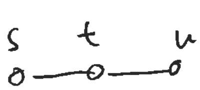
<u>Thm (check)</u> Let $x, y \in W$

- If $x \leq_L y$, then $\pi_I(x) \leq_L^I \pi_I(y)$.

  (so if $x, y \in W_I$, $x \leq_L y \iff x \leq_L^I y$)

- If $x \sim_L y$, then $\pi_I(x) \sim_L^{(I)} \pi_I(y)$

- If $C$ is a left cell of $W_I$, then $X_I \cdot C$ is a union of left cells of $W$.

<u>Rmk</u>. 

, $I = \{s, u\}$, then $s \sim_{LR} u$, but $s \not\sim_{LR}^I u$

**Theorem** (Lusztig) If $x \in X_I$, $w, w' \in W_I$, and $w \leq_L w'$, then

$wx^{-1} \leq_L w'x^{-1}$.

**Rmk.** $\{1\}$ is ?-cell.

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**Lecture 3**

Decomposition into left, right & two-sided cells:

**Equal par.**

- Type A (KL, 1979)
- $\tilde{B}_2, \tilde{G}_2$ (Lusztig, 1986)
- B, D (Garfinkle, Barbasch - Vogan, 90)
- $\tilde{A}$

**Unequal**

- Quasi-split case (Lusztig)
- $B_n$, $b > (n-1)a$ (Iancu - B., 2003)
- $F_4$ (Geck, 2004)
- $\tilde{G}_2$, $\tilde{B}_2$ (All choices of parameters, Guilhot, 2009)

A subset $\Sigma$ of $W$ is <u>left-conn'd</u> if, for all $x, y \in \Sigma$, $\exists$ a sequence $s_1, \dots, s_r \in S$

s.t. $y = s_r \cdots s_2 s_1 x$ and $s_i \cdots s_2 s_1 x \in \Sigma$ for all $i$

<u>Conj:</u> (Lusztig)

Every left cell is left-conn'd.

Left cells are left-conn'd components of two-sided cells.

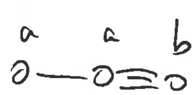
Assume here that $\Gamma = \mathbb{R}$. Let $V$ be the vec. sp. of maps $\varphi : S/\sim \longrightarrow \mathbb{R}$,

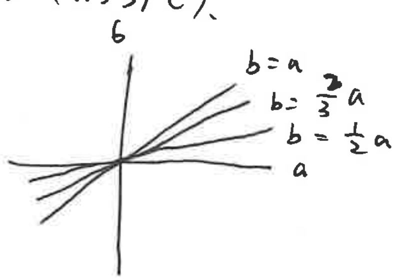
and let $V^{\#}$ be the subset of maps $\varphi : S/\sim \longrightarrow \mathbb{R}_{>0}$

<u>Conj:</u> (B.)

$\exists$ a finite set of (linear) rat'l hyperplanes $\mathcal{A}$ in $V$ s.t.

- If $\varphi$ & $\varphi'$ belong to the same $\mathcal{A}$-facet in $V^{\#}$, then the left (right, two-sided) cells for $(W, S, \varphi)$ and $(W, S, \varphi')$ coincide.
- If $\varphi \in V^{\#}$, then a left (resp. right, two-sided) $\varphi$-cell is a <u>minimal</u> subset $X$ of $W$ s.t. For each $\mathcal{A}$ chamber $\mathcal{C}$ s.t. $\varphi \in \overline{\mathcal{C}}$, $X$ is a union of left (resp. right, two-sided) cells for $(W, S, \mathcal{C})$.

$\tilde{G}_2$ : 

Conj:

Every left cell contains at least one "involution".

True for finite Coxeter gp:

- Equal par. case: Lusztig (nice proof, not involving the classification, using the geometry of Schubert varieties.)
- Geck ($F_4$), Iancu - B. 2003 and B. 2008 (type B)

## Lusztig's a-function

From now on, $W$ finite or affine

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$$

w/ $h_{x,y,z} \in R = \mathbb{Z}[\Gamma]$ and $\overline{h_{x,y,z}} = h_{x,y,z}$.

$$\deg : R \to \Gamma \cup \{-\infty\} \qquad \deg\left(\sum a_\gamma e^\gamma\right) = \max\{\gamma : a_\gamma \neq 0\}$$

$$\mathrm{val} : R \to \Gamma \cup \{+\infty\}$$

Let $a(z) = \max\limits_{x,y \in W} \deg(h_{x,y,z})$

Lusztig proved that $a(z) \leq \max\limits_{I \subset S,\, W_I \text{ finite}} \varphi(w_I)$ $\qquad C_y = \sum_{x \in W} P^*_{x,y} T_x$

Write $h_{x,y,z} \in \gamma_{x,y,z^{-1}} e^{a(z)} + R_{<a(z)}$ and $P^*_{1,z} \in n_z e^{-\Delta(z)} + R_{<-\Delta(z)}$

w/ $\Delta(z) \in \Gamma_{\geq 0}$, $\gamma_{x,y,z} \in \mathbb{Z}$ and $n_z \in \mathbb{Z}$

- $a(z) \geq 0$
- If $z \in W \setminus \{1\}$, $a(z) > 0$.
- Finally, let $D = \{z \in W : a(z) = \Delta(z)\}$.

<u>Conjectures (Lusztig)</u>

P1. If $z \in W$, then $a(z) \leq \Delta(z)$

P2. If $d \in D$ and if $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.

P3. If $y \in W$, then there exists a unique $d \in D$ s.t. $\gamma_{y^{-1},y,d} \neq 0$.

P4. If $z' \leq_{LR} z$, then $a(z) \leq a(z')$. Therefore, if $z \sim_{LR} z'$, then $a(z) = a(z')$.

P5. If $d \in D$ and $y \in W$ satisfies $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.

P6. If $d \in D$, then $d^2 = 1$

P7. If $x, y, z \in W$, then $\gamma_{x,y,z} = \gamma_{y,z,x}$.

P8. If $x, y, z \in W$ satisfy $\gamma_{x,y,z} \neq 0$, then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$, and $z \sim_L x^{-1}$.

P9. If $z' \leq_L z$ and $a(z') = a(z)$, then $z' \sim_L z$.

P10. If $z' \leq_R z$ and $a(z') = a(z)$, then $z' \sim_R z$.

P11. If $z' \leq_{LR} z$ and $a(z') = a(z)$, then $z' \sim_{LR} z$.

P12. If $I \subset S$ and $z \in W_I$, then $a_{W_I}(z) = a_W(z)$

P13. Every left cell $C$ of $W$ contains a unique element $d \in D$.
If $y \in C$, then $\gamma_{y^{-1},y,d} \neq 0$.

P14. If $z \in W$, then $z \sim_{LR} z^{-1}$.

P15. If $x, x', y, w \in W$ are such that $a(y) = a(w)$, then

$$\sum_{y' \in W} h_{w,x',y'} \otimes_{\mathbb{Z}} h_{x,y',y} = \sum_{y' \in W} h_{y',x',y} \otimes_{\mathbb{Z}} h_{x,w,y'}.$$

True. Equal par., quasi-split case, $W = W(B_n)$ and $b > (n-1)a$

(Geck - Iancu - 13.)

<u>Asymptotic algebra</u>

Let $J = \bigoplus_{w \in W} \mathbb{Z} t_w$ and if $x, y \in W$, we set

$$t_x \cdot t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

<u>Thm</u> (Lusztig)

If Lusztig's conjectures $P_1, P_2, \ldots, P_{15}$ hold, then the no. of left cells is finite

(so $|D| < \infty$), and $(J, \cdot)$ is a <u>unital</u> <u>assoc.</u> alg.

The unit is $\sum_{d \in D} n_d t_d$. We have $t_d t_e = n_d \delta_{d,e} t_d$ for all $d, e \in D$.

If $X$ is a subset of $W$, we set $J_X = \bigoplus_{w \in X} \mathbb{Z} t_w$ and $b_X = \sum_{d \in D \cap X} n_d t_d$

<u>Thm</u> (Lusztig) If Lusztig's conjectures $P_1, P_2, \ldots, P_{15}$ hold, then

- If $C$ is a two-sided cell, then $J_C = J b_C$ and $b_C$ is a central idempotent

- $J \simeq \prod_{c \text{ two-sided cell}} J_c$

- If $C$ is a left cell, then $J_C = J b_C$. In particular, it is a left ideal projective as a $J$-module.

- If $C$ is a left cell, then $J_{C \cap C^{-1}} = b_C J b_C$, so $J_{C\cap C^{-1}}$ is a ring (w/ unit $b_C$) isom. to $\mathrm{End}_J(J_C)^{op}$.

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<u>Lecture 4</u>. $W$ finite or affine. Assume that $P_1, \dots, P_{15}$ hold.

$$J = \bigoplus_{w\in W} \mathbb{Z}\cdot t_w, \qquad t_x \cdot t_y = \sum_{z\in W} \gamma_{x,y,z^{-1}} t_z .$$

unit $\sum_{d\in D} n_d t_d$

$$J = \prod_{c \text{ two-sided cells}} J_c, \qquad J_X = \bigoplus_{w\in X} \mathbb{Z}\cdot t_w, \qquad b_X = \sum_{d\in D\cap X} n_d t_d$$

$b_c$ is a central idempotent.

$$J_c = b_c J = J b_c.$$

<u>Lusztig's morphism</u>. $\bar{\ }, {}^*$.

$$\dagger : \mathcal{H} \to \mathcal{H}, \quad e^\gamma \mapsto e^\gamma, \quad T_w \mapsto (-1)^{\ell(w)} T_{w^{-1}}$$

$R$-linear involutive automorphism of $\mathcal{H}$.

$$\phi : \mathcal{H} \to R \otimes_{\mathbb{Z}} J, \qquad c_x^\dagger \mapsto \sum_{z\in W,\, d\in D,\, a(d)=a(z)} h_{x,d,z}\, \hat{n}_z t_z .$$

where $\hat{n}_z = n_{d_0}$, where $d_0$ is the unique element of $D$ belonging to the right cell of $z$.

$\phi$ is a morphism of algebras.

From now on, $W$ finite.

$K = \mathrm{Frac}\left(\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]\right)$, $K\mathcal{H}$ is split semisimple.

By Tits deformation theorem, $K\mathcal{H} \simeq KW$.

$$\mathrm{Irr}\, W \xleftrightarrow{\sim} \mathrm{Irr}\, K\mathcal{H}$$

$$\chi \longmapsto \chi_{gen}$$

We have $\chi(w) = \mathrm{aug}\left(\chi_{gen}(T_w)\right)$

where $\mathrm{aug}: \mathbb{Z}[\Gamma] \to \mathbb{Z},\ e^{\gamma} \mapsto 1$

Let $\tau: \mathcal{H} \to R$ is a symmetrizing form,
$T_w \mapsto \delta_{1,w}$

$\tau(T_x T_y) = \delta_{x, y^{-1}}$.

We can write $\tau_K = \sum_{\chi \in \mathrm{Irr}\, W} \frac{\chi_{gen}}{s_\chi}$, $s_\chi \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] = \mathbb{R}[\Gamma]$

Schur elements

Write $s_\chi = \underbrace{f_\chi}_{\text{alg. int. over } \mathbb{Z}} e^{-a_\chi} + \mathbb{R}[\Gamma_{>-a_\chi}]$

## J-induction

**Prop** If $I \subset S$ and $\psi \in \mathrm{Irr}\, W_I$, $\mathrm{Ind}_{W_I}^{W} \psi = \sum_{\chi \in \mathrm{Irr}\, W} m_\chi \cdot \chi$

then $\mathrm{Ind}_{\mathcal{H}_I}^{\mathcal{H}} \psi_{gen} = \sum_{\chi \in \mathrm{Irr}\, W} m_\chi \cdot \chi_{gen}$.

If $m_\chi \neq 0$, $a_\chi \geqslant a_\psi$.

Let $J_I^S(\psi) = \sum_{\substack{\chi \in \mathrm{Irr}\, W \\ a_\chi = a_\psi}} m_\chi \, \chi$

J-induction is transitive.

$\mathrm{Const}(W) = \{\text{constructible characters}\}$

by induction: $\mathrm{Const}(W_\phi = 1) = \{1\}$.

Assume that $\mathrm{Const}(W_I)$ are constructed for $I \subsetneq S$.

$\mathrm{Const}(W) = \{ J_I^S(\psi) \text{ or } J_I^S(\psi) \otimes \mathrm{sgn} : I \subsetneq S,\ \psi \in \mathrm{Const}(W_I) \}$

**Lusztig families.** Let $\mathcal{G}_W$ be the graph defined by:

- vertices $\mathrm{Irr}\, W$
- edges $\chi - \chi'$ if $\chi, \chi'$ occur in the same constructible character.

Lusztig families $=$ conn'd components of $\mathcal{G}_W$.

Theorem (Lusztig, Rouquier, Geck)

Assume P1, ⋯, P15.

ⓐ Lusztig's morphism $\phi_K : KH \xrightarrow{\sim} KJ$ is an isom.

$$\mathcal{R} = R[(1+R_{>0})^{-1}] \qquad \text{"Rouquier ring"}$$

In fact, $\phi_{\mathcal{R}} : \mathcal{R}H \xrightarrow{\sim} \mathcal{R}J$.

$$e_d \longmapsto n_d t_d, \quad d \in D$$

ⓑ Const (W) are the characters of the left cell modules.

ⓒ Lusztig families = "two sided cells"

$$kH \simeq \prod_{C \text{ two-sided cell}} kH_C.$$

ⓓ $e_d$ is primitive

ⓔ If $C$ is a two-sided cell, then $e_C := \sum_{d \in D \cap C} e_d$ is a primitive central idempotent of $\mathcal{R}H$.

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Finite reductive groups.

$G$ conn'd reductive alg. gp / $\overline{\mathbb{F}}_q$, $F : G \to G$ Frobenius / $\mathbb{F}_q$

$G^F = \{ g \in G : F(g) = g \}$ finite reductive group

$W$ Weyl gp of $G$

If $w \in W$, let $G^F \curvearrowright X(w) = \{ B \in \mathcal{B} : B \xrightarrow{w} F(B) \}$ (flag var. $\mathcal{B}$)

$$R(w) = \sum_{i \geq 0} (-1)^i \left[ H_c^i (X(w), \bar{\mathbb{Q}}_\ell) \right] \in \mathbb{Z}\,\mathrm{Irr}\, G^F.$$

$\mathrm{Unip}(G^F)$ = set of irred. char. of $G^F$ occurring in some $R(w)$.

$\subset \mathrm{Irr}\, G^F$.

If $\chi \in (\mathrm{Irr}\, \mathcal{W})^F$ let $R_\chi := \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \tilde{\chi}(wF)\, R(w) \in \bar{\mathbb{Q}}_\ell \mathrm{Irr}\, G^F$

extends to $\tilde{\chi} \in \mathrm{Irr}(\mathcal{W} \rtimes F)$

$$\langle R_\chi, R_{\chi'} \rangle = \delta_{\chi, \chi'}.$$

Let $\mathcal{G}_{G^F}$ be the graph defined by

- vertices $\mathrm{Unip}(G^F)$
- edges $\rho - \rho'$ if $\rho, \rho'$ occur in some $R_\chi$.

Unipotent Lusztig families = conn'd components of $\mathcal{G}_{G^F}$.

- If $P = L \cdot U$ is an $F$-stable parabolic subgroup of $G$, $L$ $F$-stable Levi subgp.
  $U = \mathrm{Rad}_u(P)$. and if $\lambda \in \mathrm{Unip}(L^F)$ w/ $\lambda$ "cuspidal"

$$\mathrm{End}_{G^F}\left( \mathrm{Ind}_{P^F}^{G^F} (\tilde{\lambda}) \right) = \mathcal{H}(W, S, \varphi)_{v \mapsto q} \leftarrow \text{alg. over } \mathbb{Z}[z] = \mathbb{Z}[v, v^{-1}].$$

$W \hookrightarrow \mathcal{W}$ "canonically"

$\varphi(s) = \ell_{\mathcal{W}}(s) \in \mathbb{Z}$

$$\begin{tikzcd}
\mathcal{E}(G^F, L, \lambda) = \{\text{irr. components of } \mathrm{Ind}_{P^F}^{G^F} \tilde{\lambda}\} \arrow[d] \arrow[r, phantom, ""] & \\
\mathrm{Unip}(G^F) & 
\end{tikzcd}$$

$$\{\text{irr. components of } \mathrm{Ind}_{P^F}^{G^F} \tilde{\lambda}\} \overset{\wr}{\longleftrightarrow} \mathrm{Irr}\, W$$

<u>Thm (Lusztig)</u> If $\mathcal{F} \subset \mathrm{Unip}(G^F)$ is a unipotent Lusztig family, then

$\mathcal{F} \cap \mathcal{E}(G^F, L, \lambda)$ (viewed inside $\mathrm{Irr}\, W$) is a Lusztig family (assoc. to $\varphi$)