Moduli space of Langlands parameters

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Γ finitely gen. group, $\exists$ moduli stack $/k$ $\mathcal{X}_{\Gamma,n}$ classifying
fix k base ring (Noeth.) n-dim'l rep'n of Γ.

Ex. 1) $\Gamma = \mathbb{Z}$, $\mathcal{X}_{\Gamma,n} = G/G$, $G = GL_n$
Free(n) G^n/G

2) $\Gamma = \pi_1(\Sigma) = \Gamma_g$ $\mathcal{X}_{\Gamma_g,n} = \{(A_i, B_i) \in (G^2)^g : \prod [A_i, B_i] = 1\}/G$

3) $\Gamma_q = \langle \sigma, \tau : \sigma\tau\sigma^{-1} = \tau^q \rangle$, $\mathcal{X}_{\Gamma_q,n} = \{A, B : ABA^{-1} = B^q\}/G$
$q = p^r$

4) $\Gamma = \mathbb{Z}/d$, $\mathcal{X}_{\Gamma,1} = \mathbb{G}_m[d] = \mu_d/\mathbb{G}_m$

Want: construct $X_{\Gamma,n}$ parametrizing continuous rep'ns of Galois gp (locally profinite group)

Two issues to address:

(1) Some derived geometry

e.g. $\mathcal{X}_{\Gamma_1,1} = \{a, b \in \mathbb{G}_m : aba^{-1}b^{-1} = 1\}/\mathbb{G}_m$
$\uparrow$
derived stack

Similarly, $\mathcal{X}_{\mathbb{Z}/d,1}$ has derived structure.

(2) Continuity. $\left[k = \mathbb{F}_\ell \ , \mathbb{Z}/\ell^n \ , \mathbb{Z}_\ell , \ \mathbb{Q}_\ell \right]$

$k = \mathbb{F}_\ell \quad (\mathbb{Z}/\ell^n)$

$$\mathfrak{X}^{cts}_{\Gamma, n} := \varinjlim_{\substack{U \text{ open normal} \\ \text{subgps of } \Gamma}} \mathfrak{X}_{\Gamma/U, n} \quad \leftarrow \text{ind-stack}$$

$U_1 \subset U_2, \quad \Gamma/U_1 \twoheadrightarrow \Gamma/U_2 \ , \ \mathfrak{X}_{\Gamma/U_2} \hookrightarrow \mathfrak{X}_{\Gamma/U_1}.$

Ex. $\Gamma = \hat{\mathbb{Z}}$

$$\mathfrak{X}^{cts}_{\hat{\mathbb{Z}}, 1} = \varinjlim_d \mathfrak{X}_{\mathbb{Z}/d, 1} = \varinjlim_d \mathbb{G}_m[d]/\mathbb{G}_m = \coprod_{\substack{x \text{ closed} \\ \text{pts of } \mathbb{G}_m}} \hat{\mathbb{G}}_{m,x} / \mathbb{G}_m$$

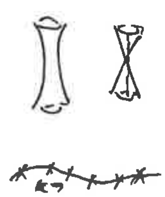
$\overline{\mathbb{F}}_\ell$

$d = \ell^n \ , \ \mathbb{G}_m[d] = \mu_{\ell^n}$

Ex. $\Gamma = \hat{\mathbb{Z}} \ , \quad \mathfrak{X}^{cts}_{\hat{\mathbb{Z}}, 2} = \varinjlim_d \mathfrak{X}_{\mathbb{Z}/d, 2} = \coprod_{\substack{x \text{ closed} \\ \text{pts of } GL_2/\!/GL_2}} (GL_2/GL_2) \times (GL_2/\!/GL_2)^{\wedge}_{x}$

$GL_2/\!/GL_2$

$$\begin{array}{c} GL_2 \\ \downarrow \\ GL_2/\!/GL_2 \end{array}$$



$\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \quad \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$$\mathbb{A}^1 \longrightarrow X_{\mathbb{Z}/\ell, 2} \hookrightarrow \mathfrak{X}_{\hat{\mathbb{Z}}, 2}$$

$$c \longmapsto \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}$$

Suppose F na local field or $\mathbb{F}_q(C)$, C curve

$\Gamma = \mathrm{Gal}(\overline{F} | F)$

$$\begin{tikzcd} & & & \mathrm{Gal}(\overline{\mathbb{F}}_q | \mathbb{F}_q) \arrow[d, phantom, "\parallel"] & \\ 1 \arrow[r] & \Gamma_0 \arrow[r] & \Gamma_F \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\ 1 \arrow[r] & \Gamma_0 \arrow[r] \arrow[u, equal] & W_F \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 1 \end{tikzcd}$$

$F = \mathbb{F}_q(C)$, $U \underset{\text{open}}{\subset} C$

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{geom}(U) \arrow[r] \arrow[d, equal] & W(U) \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \pi_1^{geom}(U) \arrow[r] & \pi_1(U) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

<u>Thm</u> (Z.) Let $\Gamma = \begin{cases} \Gamma_F & F \text{ local} \quad (\ell \neq p) \\ W(U) & , F \text{ global} \end{cases}$

$\exists$ an <u>algebraic</u> stack $\mathfrak{X}_{\Gamma,n} / \operatorname{Spec} \mathbb{Z}_\ell$ parametrizing cts n-dim'l rep'n of Γ.

$\mathfrak{X}_{\Gamma,n} = \coprod_V \mathfrak{X}_{\Gamma,V,n}$ — of finite type $/ \mathbb{Z}_\ell$

$\operatorname{Spec} A_{\Gamma,V,n}/G$

In addition,
- when F is local, $A_{\Gamma,V,n}$ are flat, l.c.i $/\mathbb{Z}_\ell$, generically reduced of rel dim $= \dim G$ $(= n^2)$
- When F is global, $A_{\Gamma,V,n}$ are (derived) l.c.i.

<u>Rmk</u>. local case, independently proved by Fargues - Scholze, Dat - Helm - Kurinczuk - Moss

global case / $\mathbb{C}_\ell$, AGKRRV

<u>Idea of proof</u>. Local case: $\Gamma_F / P_F =:$ tame inertia $= \Gamma_F^t$

$$\begin{tikzcd}
1 \arrow[r] & \prod_{(\ell',p)=1} \mathbb{Z}_{\ell'}(1) \arrow[r] & \Gamma_F^t \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & \prod_{(\ell',p)=1} \mathbb{Z}_{\ell'}(1) \arrow[r] \arrow[u, equal] & W_F^t \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

Γ_q

$$GL_2/GL_2 \times_{GL_2/\!/GL_2} \coprod (GL_2/\!/GL_2)^\wedge_x \hookrightarrow GL_2/GL_2$$

$$\mathcal{X}^{cts}_{W_F^t, n} \simeq \mathcal{X}_{\Gamma_q, n}$$ (Grothendieck's local monodromy theorem)

F global,

$$1 \to \pi_1^{geom}(U) \to W(U) \to \mathbb{Z} \to 1$$

$\mathcal{F} = \mathcal{X}^{cts}_{W(U), n} / \mathbb{F}_\ell$ try to verify Artin's axioms of representability

$= \varinjlim \mathcal{X}_{W(U)/H, n}$

(A, m) complete noetherian local $\mathbb{F}_\ell$-alg.

$$\varprojlim_r \mathcal{F}(A/m^r) \xrightarrow{\sim} \mathcal{F}(A)$$

$$\begin{tikzcd} W(U) \arrow[r, "cts"'] & GL_n(A) \\ & \text{m-adic top.} \arrow[u] \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} W(U) \arrow[r, "cts"] & GL_n(A) \\ \pi_1^{geom}(U) \arrow[u, hook] \arrow[ur] \arrow[r, two heads] & \text{finite quotient} \arrow[u] \end{tikzcd}$$

Given $\rho : W(U) \to GL_n(A)$ cts w.r.t. m-adic top.,

$\rho(\pi_1^{geom}(U))$ is finite

$A = \mathbb{F}_\ell[[t]]$: de Jong's conjecture, proved by Gaitsgory.