Iwahori - Hecke algebra

<u>Notation:</u> F nonarchimedean local field,

$\cup$

$\mathcal{O}$ valuation ring

$\cup$

$\mathfrak{m} = (\pi)$ maxm'l ideal

$\uparrow$

uniformizer

$k = \mathcal{O}/\mathfrak{m}$ residue field, $q = |k|$.

G connected split reductive group $/\mathcal{O}$ $\supset T$ split maxm'l torus$/\mathcal{O}$

$\supset$ $\quad$ $\supset$

B Borel subgp containg T

$B = TN$, N unipotent radical $/\mathcal{O}$.

$K := G(\mathcal{O}) \subset G(F)$, a maxm'l compact subgroup

$ev: G(\mathcal{O}) \longrightarrow G(k) \supset B(k)$, $I := ev^{-1}(B(k)) \subset K$ Iwahori subgroup

<u>Iwahori factorization.</u> $I = N(\mathcal{O})T(\mathcal{O})(I \cap \bar{N}(F))$, $\bar{N}$ opposite unipotent

More precisely, the multiplication $N(\mathcal{O}) \times T(\mathcal{O}) \times (I \cap \bar{N}(F)) \longrightarrow I$ is a bijection,

and we can change the order of $N(\mathcal{O}), T(\mathcal{O}), I \cap \bar{N}(F)$.

$$\widetilde{W} = N_{G(F)}(T(F))/T(\mathcal{O}) \quad \text{extended affine Weyl group}$$

$$\cup \qquad\qquad \supset$$

$$W = N_K(T(F))/T(\mathcal{O}) \qquad T(F)/T(\mathcal{O}) \cong X_*(T) \quad \text{coweight lattice}$$

finite Weyl group

$$\pi^\mu \mapsfrom \mu$$

$$\widetilde{W} = W \ltimes X_*(T).$$

<u>Iwahori - Matsumoto decomposition</u> $\quad G(F) = \coprod_{x \in \widetilde{W}} IxI$

<u>Bruhat decomposition</u> $\quad K = \coprod_{w \in W} IwI = \coprod_{w \in W} B(\mathcal{O}) w I$

<u>Iwasawa decomposition</u> $\quad G(F) = B(F)K = \coprod_{\mu \in X_*(T)} N(F)\pi^\mu K = \coprod_{\mu \in X_*(T)} \pi^\mu N(F) K$

<u>Iwahori - Hecke algebra</u>: $\mathcal{H} = C_c^\infty(I \backslash G(F)/I)$, algebra structure given by

Convolution:
$$(\varphi * \psi)(g) = \int_{G(F)} \varphi(gh^{-1})\psi(h)\, \underbrace{dh}$$

right Haar measure s.t. $\mathrm{vol}(I) = 1$.

By Iwahori - Matsumoto decomposition, $\mathcal{H}$ has a $\mathbb{C}$-basis

$$\{T_x\}_{x \in \widetilde{W}}, \quad T_x = \mathbb{1}_{IxI}.$$

<u>Goal</u> ① Bernstein presentation of $\mathcal{H}$. ② Center of $\mathcal{H}$

Auxiliary module M. $\quad M := C_c^\infty(T(\mathcal{O})N(F)\backslash G(F)/I)$.

M is a right $\mathcal{H}$-module by convolution.

Notice that $G(F) \overset{\text{Iwasawa}}{=} \dot\bigcup_{\mu \in X_*(T)} \pi^\mu N(F) K$

$$\overset{\text{Bruhat}}{=} \dot\bigcup_{\substack{\mu \in X_*(T) \\ w \in W}} \pi^\mu N(F) B(\mathcal{O}) w I$$

$$= \bigcup_{\substack{\mu \in X_*(T) \\ w \in W}} \pi^\mu N(F) T(\mathcal{O}) w I$$

$$= \bigcup_{\substack{\mu \in X_*(T) \\ w \in W}} T(\mathcal{O}) N(F) \pi^\mu w I.$$

$$= \bigcup_{x \in \tilde{W}} T(\mathcal{O}) N(F) x I.$$

Moreover, we can see that this is a disjoint union, so M has a $\mathbb{C}$-basis

$\{v_x\}_{x \in \tilde{W}}$, $\quad v_x = \mathbb{1}_{T(\mathcal{O})N(F)xI}$.

Let $R = C_c^\infty(T(F)/T(\mathcal{O})) = \mathbb{C}[X_*(T)]$, we can make M a left R-module

$\pi^\mu \cdot v_x = q^{-\langle \rho, \mu \rangle} v_{\pi^\mu \cdot x}$. This is basically left translation,

$q^{-\langle \rho, \mu \rangle}$ comes from the modular quasi-character.

Left translation commutes with right convolution, so M is an $(R, \mathcal{H})$-bimodule.

Prop. Via the map $\mathcal{H} \xrightarrow{\Xi} M$, $h \mapsto v_1 \cdot h$, M is a free right $\mathcal{H}$-module of rank 1.

Cor: $\mathcal{H} \simeq \mathrm{End}_{\mathcal{H}}(M)$, $h' \mapsto [v_1 h \mapsto v_1 h' h]$.

Proof. I have to show that written in terms of the basis $\{T_x\}_{x\in\tilde{W}}$, $\{v_x\}_{x\in\tilde{W}}$, Ξ is an upper triangular matrix w/ nonzero diagonal elements. This is a direct computation:

$$\begin{aligned} \mathrm{Supp}(v_1 * T_x) &= T(\mathcal{O})N(F)K\, I x I = T(\mathcal{O})N(F)IxI \\ &= N(F)T(\mathcal{O})IxI \\ &= N(F)IxI \end{aligned}$$

Claim For $y \in \tilde{W}$, $y \in N(F)IxI \Rightarrow y \le x$ in the Bruhat order.

Also notice that $x \in N(F)IxI$, so claim $\Rightarrow$ proof.

Proof of claim. $\exists\, n \in N(F)$ s.t. $ny \in IxI$. We can find $\mu \in X_*(T)$ sufficiently dominant s.t. $\pi^{\mu} n \pi^{-\mu} \in N(\mathcal{O}) \subset I$

$$I\pi^\mu y I = I\cdot \pi^\mu n \pi^{-\mu}\cdot \pi^\mu y I$$

$$= I\cdot \pi^\mu n y I$$

$$\subset I\cdot\pi^\mu\cdot I x I$$

$$= (I\pi^\mu I)\cdot(IxI) \underset{\text{IM decomp.}}{\overset{\text{part of}}{=}} \coprod_{x'\le x} I\pi^\mu x' I$$

$$\Rightarrow \pi^\mu y \le \pi^\mu x \Rightarrow y \le x. \qquad \square.$$

<u>Prop.</u>

- For $w\in W$, $v_1 T_w = v_w$.
- For $w\in W$, $\mu\in X_*(T)$, $v_{\pi^\mu} T_w = v_{\pi^\mu w}$
- For dominant $\mu\in X_*(T)$, $v_1 T_{\pi^\mu} = v_{\pi^\mu}$.

<u>Proof.</u> (1) I check the support condition. The coefficient is not difficult to determine.

$$\mathrm{supp}\,(v_1\cdot T_w) = N(F) I w I$$

$$\underset{\text{Iwahori}}{=} N(F) N(\mathcal{O}) T(\mathcal{O}) \left(I\cap \bar{N}(F)\right) w I$$

$$= N(F) T(\mathcal{O}) w \left[\underset{\substack{\cap \\ I}}{w^{-1}(I\cap\bar{N}(F)) w}\right] I$$

$$= T(\mathcal{O}) N(F) w I$$

(2) Use the left R-module structure: let π^μ act on both sides of (1).

(3) Again I check the support condition.

$$\mathrm{Supp}(v_1 T_{\pi^\mu}) = N(F)\, I\, \pi^\mu I$$

$$= N(F)\, N(\mathcal{O})\, T(\mathcal{O})\, (I \cap \bar{N}(F))\, \pi^\mu I$$

$$= N(F)\, T(\mathcal{O})\, \pi^\mu \underbrace{\left[\pi^{-\mu}(I \cap \bar{N}(F))\, \pi^\mu\right]}_{\subset\, I \cap \bar{N}(F) \ (\mu \text{ dominant})} I$$

$$= T(\mathcal{O})\, N(F)\, \pi^\mu I.$$

<u>Decomposition of $\mathcal{H}$</u>. Let $\mathcal{H}_0 := C_c^\infty(I\backslash K/I)$ be the finite Hecke algebra,

$\mathcal{H}_0$ is a subalgebra of $\mathcal{H}$ via "extension by zero".

Since M is an $(R, \mathcal{H})$-bimodule, we have a morphism of algebra

$$R \longrightarrow \mathrm{End}_{\mathcal{H}}(M) \simeq \mathcal{H} \qquad v_1 \cdot \Theta_\mu = \pi^\mu \cdot v_1$$

$$\pi^\mu \longmapsto \Theta_\mu ,$$

<u>Prop</u>. Multiplication in $\mathcal{H}$ induces a <u>vector space</u> isomorphism

$$R \otimes_{\mathbb{C}} \mathcal{H}_0 \xrightarrow{\sim} \mathcal{H}.$$

<u>Proof</u>. The composition
$$R \otimes_{\mathbb{C}} \mathcal{H}_0 \longrightarrow \mathcal{H} \longrightarrow M$$
$$\pi^\mu \otimes T_w \mapsto \Theta_\mu T_w \mapsto v_1 \cdot \Theta_\mu T_w = \pi^\mu \cdot v_1 T_w = q^{-\langle \rho, \mu\rangle} v_{\pi^\mu w}$$
is an isomorphism. Also $\mathcal{H} \to M$ is an isomorphism,

so $R \otimes_{\mathbb{C}} \mathcal{H}_0 \xrightarrow{\sim} \mathcal{H}$ is an isomorphism.

Intertwiners. Let us vary B in the set $\mathcal{B}(T)$ of Borel subgps containing T. For $B = TN \in \mathcal{B}(T)$, put

$$M_B = C_c^\infty\left(T(\mathcal{O})\,N(F) \backslash G(F) / I\right).$$

Let J be a set of coroots that is a subset of some system of positive coroots,

$\mathbb{C}[J] :=$ $\mathbb{C}$- subalg. of $R = \mathbb{C}[X_*(T)]$ generated by J

$\mathbb{C}[J]^\wedge :=$ completion of $\mathbb{C}[J]$ w.r.t. the ideal generated by J.

(our assumption guarantees that $\mathbb{C}[J]^\wedge \neq 0$).

$R_J = \mathbb{C}[J]^\wedge \otimes_{\mathbb{C}[J]} R$ = functions on $X_*(T)$ supported on a finite union of sets of the form $x + C_J$, $x \in X_*(T)$, $C_J = \mathbb{Z}_{\geq 0} J$.

$M_{B,J} := R_J \otimes_R M_B$ = functions f on $G(F)$, $(T(\mathcal{O})N(F), I)$ – biinvariant, supported on $\{ T(\mathcal{O}) N(F) \pi^\nu K \}$, ν runs through a finite union of sets of the form $x + C_J$, $x \in X_*(T)$.

$M_{B,J}$ is an $(R_J, \mathcal{H})$ - bimodule.

$B = TN \in \mathcal{B}(T)$, $B' = TN' \in \mathcal{B}(T)$, $J = \{$ coroots that are positive for B' and negative for B $\}$

We have an intertwiner $I_{B',B} : M_{B,J} \longrightarrow M_{B',J}$

$$\varphi \mapsto \left[\varphi' : g \mapsto \frac{1}{\mathrm{vol}(N'(F) \cap \bar{N}(F) \cap K)} \int_{N'(F)\cap \bar{N}(F)} \varphi(n'g)\, dn' \right]$$

<u>Claim</u>: $I_{B',B}$ is well-defined because the integrand is smooth & compactly supported on $N' \cap \bar{N}$.

<u>Cor.</u> We get an $(R_J, \mathcal{H})$-bimodule homomorphism $I_{B',B} : M_{B,J} \longrightarrow M_{B',J}$.

The proof of the claim is based on the following Lemma:

<u>Lemma</u>. For $\nu \in X_*(T)$, set $C_\nu = N'(F) \cap \bar{N}(F) \cap \pi^\nu N(F) K$.

(1) $C_\nu \neq \phi \Rightarrow \nu \in C_J = \mathbb{Z}_{\geq 0} J$.

(2) C_ν is compact.

<u>Proof of lemma (1)</u>. By induction on $|J|$, basically we only need to treat the adjacent case, that is, $|J| = 1$. Now by Jacobson–Morozov theorem things break down into an SL_2-calculation:

$$\underset{\pi^\nu}{\underset{\shortparallel}{\begin{pmatrix} \pi^a & 0 \\ 0 & \pi^{-a} \end{pmatrix}}} \underset{N(F)}{\underset{\uparrow}{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}}} \underset{SL_2(\mathcal{O})}{\underset{\Uparrow}{\begin{pmatrix} x & y \\ z & w \end{pmatrix}}} = \begin{pmatrix} \pi^a x + \pi^a \lambda z & \pi^a y + \pi^a \lambda w \\ \pi^{-a} z & \pi^{-a} w \end{pmatrix}$$

In particular, we want $\pi^{-a} w = 1$, $\pi^a = w \in \mathcal{O} \Rightarrow a \in \mathbb{Z}_{\geq 0}$.

(2) It suffices to show that $\bar{N}(F) \cap N(F) C$ is compact for any compact $C \subset G(F)$.

This is equivalent to saying that the map $\bar{N}(F) \to N(F) \backslash G(F)$ is proper (in the topological sense). But in fact, $\bar{N} \to N \backslash G$ is a closed immersion (in the sense of algebraic geometry). By the Cartesian diagram,

$$\begin{tikzcd} N \cdot \bar{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G \arrow[d] \\ \bar{N} \arrow[r] & N \backslash G \end{tikzcd},$$

suffices to show $N \cdot \bar{N} \hookrightarrow G$ closed. This is known from structure theory of G. $\square$

Now suppose we have three Borel subgroups $B_1 = TN_1$, $B_2 = TN_2$, $B_3 = TN_3 \in \mathcal{B}(T)$. Let $J_{ij} = \{$ coroots that are positive for B_i, negative for B_j $\}$,

and assume $J_{31} = J_{21} \sqcup J_{32}$.

$I_{B_2, B_1} : M_{B_1, J_{21}} \longrightarrow M_{B_2, J_{21}}$

$I_{B_3, B_2} : M_{B_2, J_{32}} \longrightarrow M_{B_3, J_{32}}$

can be extended to

$I_{B_2, B_1} : M_{B_1, J_{31}} \longrightarrow M_{B_2, J_{31}}$; $I_{B_3, B_2} : M_{B_2, J_{31}} \longrightarrow M_{B_3, J_{31}}$

via $R_{J_{31}} \otimes_{R_{J_{21}}} -$; $R_{J_{31}} \otimes_{R_{J_{32}}} -$. And we can see directly that

$I_{B_3, B_1} = I_{B_3, B_2} \circ I_{B_2, B_1}$.

Rephrasing. Now we fix $B = TN \in \mathcal{B}(T)$. For each $w \in W$,

we define an intertwiner $I_w : M_{B, w^{-1}J} \longrightarrow M_{B,J}$

as the composition $M_{B,w^{-1}J} \xrightarrow{\sim} M_{wB,J} \xrightarrow{I_{B,wB}} M_{B,J}$

Using formula, $I_w(\varphi)(g) = \frac{1}{Vol(N_w(F)\cap K)} \int_{N_w(F)} \varphi(w^{-1}ng)\,dn$, $N_w(F) = N(F) \cap w\bar{N}(F)w^{-1}$

Prop. (i) $I_w \circ \pi^{\mu} = \pi^{w\mu} \circ I_w$, $\forall \mu \in X_*(T)$

(ii) $I_{w_1w_2} = I_{w_1} \cdot I_{w_2}$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$

(iii) I_w is right $\mathcal{H}$-module homomorphism

Rephrasing

$I_{B,wB}$ is left R_J-module homomorphism

$I_{B,w_1w_2B} = I_{w_2B, w_1w_2B} \circ I_{B, w_2B}$

$I_{B,wB}$ is right $\mathcal{H}$-module homomorphism.

SL_2 calculation. α unique positive root, s_α corresponding simple reflection.

I would like to calculate $I_{s_\alpha}(v_1)$

$$I_{s_\alpha}(v_1)(g) = \frac{1}{Vol(\bar{N}(F)\cap K)} \int_{\bar{N}(F)} v_1(s_\alpha ng)\,dn$$

Here $s_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T(\mathcal{O})N(F)I = \begin{pmatrix} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{pmatrix}\begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ (\pi) & \mathcal{O}^\times \end{pmatrix}$

$$= \begin{pmatrix} F & F \\ (\pi) & \mathcal{O}^\times \end{pmatrix}$$

Set $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in F$, $\pi^{j\alpha^\vee} = \begin{pmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{pmatrix}$.

$$s_\alpha n \pi^{j\alpha^\vee} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{pmatrix} = \begin{pmatrix} 0 & -\pi^{-j} \\ \pi^j & x\pi^{-j} \end{pmatrix}$$

$$s_\alpha n \pi^{j\alpha^\vee} s_\alpha = \begin{pmatrix} 0 & -\pi^{-j} \\ \pi^j & x\pi^{-j} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\pi^{-j} & 0 \\ x\pi^{-j} & -\pi^j \end{pmatrix}$$

So $$I_{s_\alpha}(v_1)(\pi^{j\alpha^\vee}) = \frac{1}{\mathrm{Vol}(\mathcal{O})} \int_F \mathbb{1}_{\{j \geq 1,\ x\pi^{-j} \in \mathcal{O}^\times\}} \, dx$$

$$= \delta_{\{j \geq 1\}} \frac{\mathrm{vol}(\pi^j \mathcal{O}^\times)}{\mathrm{Vol}(\mathcal{O})} = \delta_{\{j \geq 1\}} q^{-j}(1-q^{-1})$$

$$I_{s_\alpha}(v_1)(\pi^{j\alpha^\vee} s_\alpha) = \frac{1}{\mathrm{vol}(\mathcal{O})} \int_F \mathbb{1}_{\{j=0,\ x \in (\pi)\}} \, dx$$

$$= \delta_{\{j=0\}} \frac{\mathrm{vol}((\pi))}{\mathrm{vol}(\mathcal{O})} = \delta_{\{j=0\}} q^{-1}$$

So we conclude that

$$I_{s_\alpha}(v_1) = q^{-1} v_{s_\alpha} + \sum_{j \geq 1} q^{-j}(1-q^{-1}) v_{\pi^{j\alpha^\vee}}$$

$$= q^{-1} v_{s_\alpha} + (1-q^{-1}) \sum_{j \geq 1} \pi^{j\alpha^\vee} \cdot v_1 .$$

Similarly, $I_{s_\alpha}(\mathbb{1}_{T(\mathcal{O})N(F)K}) = q^{-1} \mathbb{1}_{T(\mathcal{O})N(F)K} + \sum_{j \geq 0} q^{-j}(1-q^{-1}) \mathbb{1}_{T(\mathcal{O})N(F)\pi^{j\alpha^\vee}K}$

$$= \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} \mathbb{1}_{T(\mathcal{O})N(F)K}.$$

The calculation for intertwiners in the general case can be reduced to the SL_2-calculation above by Jacobson - Morozov theorem:

<u>Prop.</u> Let α be a simple root, s_α be the corresponding simple reflection, then

- $I_{s_\alpha}(v_1) = q^{-1} v_{s_\alpha} + (1-q^{-1}) \sum_{j\geq 1} \pi^{j\alpha^\vee} v_1$

- $I_{s_\alpha}(v_1 + v_{s_\alpha}) = \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} (v_1 + v_{s_\alpha})$

- $I_{s_\alpha}(\mathbb{1}_{T(\mathcal{O})N(F)K}) = \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} \mathbb{1}_{T(\mathcal{O})N(F)K}.$

<u>Cor.</u> (Gindikin - Karpelevich formula) For $w\in W$, set R_w be the set of positive roots α s.t. $w^{-1}\alpha$ is negative. then

$$I_w(\mathbb{1}_{T(\mathcal{O})N(F)K}) = \left(\prod_{\alpha\in R_w} \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} \right) \mathbb{1}_{T(\mathcal{O})N(F)K}.$$

<u>Proof.</u> $w = \prod_{\alpha\in R_w} s_\alpha$ is a reduced expression of w, so

$$I_w = \prod_{\alpha\in R_w} I_{s_\alpha}.$$

<u>Cor.</u> $T_{s_\alpha}^2 = (q-1)T_{s_\alpha} + q$

<u>Proof.</u> Use two ways to compute $I_{s_\alpha}(v_{s_\alpha})$

- $I_{s_\alpha}(v_{s_\alpha}) = I_{s_\alpha}(v_1) \cdot T_{s_\alpha}$
- $I_{s_\alpha}(v_{s_\alpha}) = I_{s_\alpha}(v_1 + v_{s_\alpha}) - I_{s_\alpha}(v_1)$

To eliminate the denominator, set

$$J_w = \Big(\prod_{\alpha \in R_w} (1-\pi^{\alpha^\vee}) \Big) \cdot I_w$$

By GK formula, we can check that J_w maps the subspace $M \subset M_{B,w^{-1}J}$ into $M \subset M_{B,J}$, so $J_w \in \mathrm{End}_{\mathcal{H}}(M) \cong \mathcal{H}$ can be viewed as elements in $\mathcal{H}$.

For a simple root α,

$$\begin{aligned} J_{s_\alpha} \cdot v_1 &= (1-\pi^{\alpha^\vee})\, I_{s_\alpha}(v_1) \\ &= q^{-1}(1-\pi^{\alpha^\vee})\, v_{s_\alpha} + (1-q^{-1})(1-\pi^{\alpha^\vee}) \sum_{j=1}^{\infty} \pi^{j\alpha^\vee} v_1 \\ &= q^{-1}(1-\pi^{\alpha^\vee})\, v_1 T_{s_\alpha} + (1-q^{-1})\, \pi^{\alpha^\vee} v_1 \end{aligned}$$

So as elements in $\mathcal{H} \cong R \otimes_{\mathbb{C}} \mathcal{H}_0$,

$$J_{s_\alpha} = q^{-1}(1-\pi^{\alpha^\vee})\, T_{s_\alpha} + (1-q^{-1})\, \pi^{\alpha^\vee} \qquad (*)$$

Now $J_{s_\alpha} \cdot \pi^\mu = \pi^{s_\alpha(\mu)} \cdot J_{s_\alpha}$. Pludge $(*)$ in, we get

<u>Thm</u> (Bernstein presentation) $T_{s_\alpha} \cdot \pi^\mu = \pi^{s_\alpha(\mu)} \cdot T_{s_\alpha} + (1-q)\, \dfrac{\pi^{s_\alpha(\mu)} - \pi^\mu}{1-\pi^{-\alpha^\vee}}$.

Center of $\mathcal{H}$

Prop. $R^W \subset Z(\mathcal{H})$.

Proof. Let $r \in R^W$, then r commutes w/ R. Since $\mathcal{H} \simeq R \otimes_{\mathbb{C}} H_0$,

I have to show that r commutes w/ T_{s_α} for any simple root α.

By the intertwining property of J_{s_α}, r does commute with

$$J_{s_\alpha} = (1-q^{-1})\pi^{\alpha^\vee} + q^{-1}(1-\pi^{\alpha^\vee})T_{s_\alpha}$$, so r commutes with

$(1-\pi^{\alpha^\vee})T_{s_\alpha}$. But $\mathcal{H}$ is a free R-module, so r commutes w/ T_{s_α}.

Normalized intertwiner Let $L = \mathrm{Frac}(R)$, then $L^W = \mathrm{Frac}(R^W)$, Consider

the generic fiber $\mathcal{H}_{gen} = L^W \otimes_{R^W} \mathcal{H}$, $M_{gen} = L \otimes_R M = L^W \otimes_{R^W} M$,

$(L, \mathcal{H}_{gen})$-bimodule.

Define normalized intertwiner
$$K_w = \left(\prod_{\alpha \in R_w} \frac{1}{1-q^{-1}\pi^{\alpha^\vee}}\right) \cdot J_w$$
$$= \left(\prod_{\alpha \in R_w} \frac{1-\pi^{\alpha^\vee}}{1-q^{-1}\pi^{\alpha^\vee}}\right) \cdot I_w .$$

$K_w \in \mathrm{End}_{\mathcal{H}_{gen}}(M_{gen}) \cong \mathcal{H}_{gen}$.

The virtue of K_w is that

- $K_{s_\alpha}^2 = 1$, α simple root
- $K_{w_1 w_2} = K_{w_1} \cdot K_{w_2}$.

So we get a homomorphism

$$L[W] \longrightarrow \mathrm{End}_{\mathcal{H}_{gen}}(M_{gen}) \cong \mathcal{H}_{gen}$$

$$w \longmapsto K_w$$

Prop. The above homomorphism is an isomorphism.

Proof. $L[W]$ is a matrix algebra over L^W, so it's simple, whence the map is injective. Comparing dimension $/L$, we see that the map is an isomorphism.

Cor. $Z(\mathcal{H}_{gen}) \cong L^W$. $Z(\mathcal{H}) \cong R^W$.

Proof. We see that $\mathcal{H}_{gen}$ is a matrix algebra over L^W, so $Z(\mathcal{H}_{gen}) \cong L^W$.

Then $Z(\mathcal{H}) \subseteq L^W \cap \mathcal{H} = R^W$. (notice that $\mathcal{H}$ is torsion-free as an R^W-module)