Modular representation theory

Calder Morton-Ferguson

Lecture 1 $\quad G$ conn'd reductive alg. gp over $k = \bar{k}$.

1950's: Chevalley = classification of simple modules.

$B \subset G$ be a Borel subgp, $T \subset B$ a max'l torus.

$\mathrm{Rep}(G)$ = the cat. of fin. dim'l alg rep'ns of G

For any $\lambda \in X^*(T)$, $N(\lambda) = \mathrm{Ind}_B^G(\lambda)$, character given by the Weyl char. formula.

Thm (Chevalley)

- We have $N(\lambda) = 0$ unless λ is dominant
- If $\lambda \in X^*(T)^+$, then it contains a unique simple submodule $L(\lambda) \subset N(\lambda)$.
- If char $k = 0$, then $L(\lambda) = N(\lambda)$
- The assignment $\lambda \mapsto L(\lambda)$ is a bijection between dominant weights and isom. classes of irred. rep'ns.

How do you actually describe $N(\lambda)$, or even the character of $N(\lambda)$? Lusztig had some conjecture, which is asymptotic, i.e. for large p.

Jantzen - Anderson 1970s, Lusztig 1980s, Kazhdan - Lusztig, Kashiwara - Tanisaki, Andersen - Jantzen - Soergel 1990s, proved Lusztig's conjecture for large p.

The conjecture was originally expected to be true for $p > h$, h Coxeter no.

However, in 2013 Williamson proved that for GL_n, Lusztig's formula can't be true under any assumption of the form $p > P(n)$ for a fixed poly. P.

How do we fix this? Kazhdan - Lusztig polynomials relate to the geom. of the flag var. G/B and the affine flag var. What we now have to do is to introduce a new combinatorial obj's called the p- Kazhdan-Lusztig polynomials, and try to understand the geometry underlying these combinatorial objects.

The second topic is on rep'n of $G(\mathbb{F}_p)$ on $\overline{\mathbb{F}_p}$. What do these look like? How does one "reduce" rep'ns of $G(\mathbb{F}_p)$ over $\mathbb{C}$ into a rep'n over $\overline{\mathbb{F}_p}$? The latter is a procedure due to Brauer - Nesbitt from the 1940s. Then we want to discuss formulae for modular reductions of char. zero irred. rep'ns of $G(\mathbb{F}_p)$, of the form

$$\mathrm{char}\ \bar{\rho} = \sum_{\lambda} c_{\rho,\lambda}\ \mathrm{char}\ \overline{N(\lambda)}.$$

Again, Lusztig conjectured such a formula, but it's not quite right.

<u>Weyl modules</u>.

Let $\mathbb{K}$ be an alg closed field of char. p, $G/\mathbb{K}$ conn'd reductive gp, and $B = T \ltimes U$ a Borel containing a max'l torus and its unipotent radical.

$\mathbb{X} = X^*(T)$, $\lambda \in \mathbb{X}$ extends uniquely to a morphism $\lambda : B \to \mathbb{G}_m$.

$R \subset \mathbb{X}$ the root system of (G,T). We also have $U^+ \subset B^+$ the opposite Borel & unip. rad. Then we have the positive roots $R^+ \subset R$ consisting of wts of $\mathrm{Lie}(U^+)$. We denote by $R^s \subset R^+$ the set of simple roots. We have

$\mathbb{X}^\vee = X_*(T)$, coroots $R^\vee \subset \mathbb{X}^\vee$, positive Weyl chamber

$$\mathbb{X}^+ = \{\lambda \in \mathbb{X} : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

Let $W = N_G(T)/T$, $S \subset W$ simple reflections, (W,S) Coxeter system.

For any $\mathbb{k}$-alg. gp H, $\mathrm{Rep}(H)$ = Cat. of f.d. alg. H-modules

$\cap$

$\mathrm{Rep}^\infty(H)$ = cat. of all alg. H-modules.

For $V \in \mathrm{Rep}(G)$, there is the dual V^* defined by the usual formula

$$(hf)(v) = f(h^{-1}v).$$

For any alg. subgp $K \subset H$, there is an induction functor

$$\mathrm{Ind}_K^H : \mathrm{Rep}^\infty(K) \longrightarrow \mathrm{Rep}^\infty(H)$$

$$(M, \rho) \longmapsto \{f : H \to M : f(hk) = \rho(k^{-1})(f(h)), \ h \in H, k \in K\}.$$

Def. For $\lambda \in \mathbb{X}$, $N(\lambda) := \mathrm{Ind}_B^G \mathbb{k}_B(\lambda) = \{f \in \mathcal{O}(G) : f(gb) = \lambda(b)^{-1} f(g), b \in B, g \in G\}$

These are called coWeyl modules. We define the Weyl module as $M(\lambda) = (N(-w_0\lambda))^*$.
$w_0 \in W$ longest elt.

This is fin. dim'l because it is the space of global sections of a l.b. on G/B, which is proj. Also $M(\lambda)$ has h.w. λ.

Def. For any alg. T-module M, we define

$$ch\, M = \sum_{\lambda \in X} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[X]$$

For any $M \in Rep(G)$, we see that $ch(M)$ is invt. under the action of W, because we can conjugate by lifts of elts of W. Then we really get

$$ch: Rep(G) \longrightarrow \mathbb{Z}[X]^W.$$

Lemma. For any $\lambda \in X$, we have $\mu \in wt(N(\lambda))$ implies $\mu \leq \lambda$.
Moreover, if $N(\lambda) \neq 0$, then $\dim N(\lambda)_\lambda = 1$.

Lecture 2. Classification of simple modules

Thm (Chevalley) $\forall \lambda \in X^+$, the G-module $N(\lambda)$ has a unique simple submod $L(\lambda)$

$\lambda \longmapsto L(\lambda)$ dominant wts $\longleftrightarrow$ simple alg. G-modules

- $\forall \lambda \in X^+$, $(N(\lambda))^{U^+} = N(\lambda)_\lambda$
- $\forall\, 0 \neq V \subset N(\lambda)$, $V^{U^+} \neq 0$. [$L(\lambda)$ simple socle]

- $w_0(\lambda)$ unique minimal elt in $wt(L(\lambda))$, so $L(\lambda)^* \simeq L(-w_0(\lambda))$, $\forall \lambda \in X^+$.

$\Rightarrow L(\lambda)$ is $\simeq$ unique simple quotient of $M(\lambda)$.

In char. $=0$, $L(\lambda) = N(\lambda)$, but in general no complete understanding of $L(\lambda)$.

__Def__. $M \in Rep(G)$, $\lambda \in X^+$, $[M : L(\lambda)]$ = mult. of $L(\lambda)$ in M

For $M \in Rep(G)$, $[M] = \sum_{\lambda \in X^+} [M : L(\lambda)] \cdot [L(\lambda)]$

SES $0 \to V' \to V \to V'' \to 0$ in $Rep(G)$ $\rightsquigarrow$ $0 \to (V')_\mu \to V_\mu \to (V'')_\mu \to 0$

$\rightsquigarrow$ $ch : K_0(Rep(G)) \to \mathbb{Z}[X]^W$.

__Prop__. $ch : K_0(Rep(G)) \xrightarrow{\sim} \mathbb{Z}[X]^W$ is an isom.

__Weyl's char. formula__

$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathbb{Q}X$.

Given $w \in W$, $\lambda \in X$, set $w \cdot \lambda = w(\lambda + \rho) - \rho$.

__Thm__ $\lambda \in X^+$, $ch(N(\lambda)) = \dfrac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0}} = ch(M(\lambda))$

$[M(\lambda)] = [N(\lambda)]$

__Cor__ $\dim N(\lambda) = \dfrac{\prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in R^+} \langle \rho, \alpha^\vee \rangle}$

$Z(G)$ center of G

$\cap$

T identifies w/ the gp scheme assoc. w/ $X / \mathbb{Z}R$

char $= p > 0$ not necessarily smooth

$\mathrm{Rep}(Z(G))$ is semisimple, simple objs $\longleftrightarrow \mathbb{X}/\mathbb{Z}R$

Every $V \in \mathrm{Rep}^\infty(G)$ is a rep'n of $Z(G)$ by restriction

$V_{Z=x}$, $x \in \mathbb{X}/\mathbb{Z}R$

$\hookrightarrow$ subspace of vectors where $Z(G)$ acts by x

$$V = \bigoplus_{x \in \mathbb{X}/\mathbb{Z}R} V_{Z=x}, \quad \mathrm{Rep}^\infty G = \bigoplus_{x \in \mathbb{X}/\mathbb{Z}R} \mathrm{Rep}^\infty G_{Z=x}.$$

<u>Affine Weyl group</u>

$W_{\mathrm{aff}} = W \ltimes \mathbb{Z}R$, $\lambda \in \mathbb{Z}R \rightsquigarrow t_\lambda \in W_{\mathrm{aff}}$

acts on $\mathbb{X}$ by $(wt_\lambda) \cdot_p \mu = w(\mu + p\lambda + \rho) - \rho$

extended affine Weyl gp

$W_{\mathrm{ext}} = W \ltimes \mathbb{X}$

<u>Thm</u> $\lambda, \mu \in \mathbb{X}^+$, $\mathrm{Ext}^1_{\mathrm{Rep}(G)}(L(\lambda), L(\mu)) \neq 0$

$\to W_{\mathrm{aff}} \cdot_p \lambda = W_{\mathrm{aff}} \cdot_p \mu$ (same orbit)

(Humphreys, Jantzen, Andersen)

<u>Cor.</u> $(M_c)_{c \in \mathbb{X}/(W_{\mathrm{aff}} \cdot_p)} \longmapsto \bigoplus_{c \in \mathbb{X}} M_c$

gives an equiv. of cat. $\prod_{c \in \mathbb{X}/(W_{\mathrm{aff}} \cdot_p)} \mathrm{Rep}^\infty(G)_c \xrightarrow{\sim} \mathrm{Rep}^\infty(G)$.

Lusztig's character formula

(W,S) Coxeter system, if admits a presentation by gen. S & relations

- $s^2 = e$, $\forall s \in S$
- $(st)^{m_{s,t}} = e$, $\forall s,t \in S$

$\downarrow$

braid relations

Hecke algebra assoc. to (W,S)

$\mathbb{Z}[v,v^{-1}]$-alg. $\mathcal{H}$ w/ basis $(H_w : w \in W)$

& mult. $(H_s + vH_e)(H_s - v^{-1}H_e) = 0$, $\forall s \in S$

$H_x \cdot H_y = H_{xy}$ if $\ell(xy) = \ell(x) + \ell(y)$

$\mathcal{H}_{fin} =: \mathcal{H}$, $\mathcal{H}_{aff} =: \tilde{\mathcal{H}}$, $H_s^{-1} = H_s + (v - v^{-1})$

Kazhdan-Lusztig basis

Def KL involution.

the unique ring involution $\iota : \mathcal{H} \longrightarrow \mathcal{H}$ s.t. $\iota(v) = v^{-1}$, $\iota(H_x) = (H_{x^{-1}})^{-1}$, $x \in W$

Thm $\forall w \in W$, $\exists!$ $\underline{H}_w \in \mathcal{H}$ s.t. $\iota(\underline{H}_w) = \underline{H}_w$

& $\underline{H}_w = H_w + \sum_{y \in W} v\mathbb{Z}[v] H_y$

$\{\underline{H}_w\} \longrightarrow$ KL basis.

Standard basis $\longleftrightarrow$ Weyl modules

KL basis $\longleftrightarrow$ Simple modules

In fact, $\underline{H}_w \in H_w + \sum_{\substack{y \in W \\ y < w}} v\mathbb{Z}[v] H_y$ "upper triangularity".

Write $\underline{H}_x = \sum_{y \in W} h_{y,x} H_y$

$h_{w,w} = 1$, $h_{y,w} = 0$ unless $y < w$

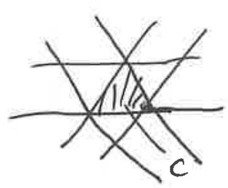
$h_{y,x}$ ⇝ KL polynomials.

Lecture 3 Lusztig's conjecture

$p \geq h$ Coxeter #, $h = n$ for $G = GL_n$

fix $\lambda \in C \cap X$

C ↳ fundamental p-alcove



Def ${}^f W_{aff} \subset W_{aff}$ subset of elements, minimal coset reps of $W_{fin} \backslash W_{aff}$.

Conj. $\forall w \in {}^f W_{aff}$ s.t. $\langle w \cdot_p \lambda + \rho, \alpha^\vee \rangle \leq p(p-h+2)$, $\forall \alpha \in R^+$

↳ Jantzen's condition

(*) $[L(w \cdot_p \lambda)] = \sum_{y \in {}^f W_{aff}} (-1)^{\ell(w)+\ell(y)} h_{w_0 y, w_0 w}(1) [N(y \cdot_p \lambda)]$

- Translation functor ⇝ choice of λ doesn't matter. So we just set $\lambda = 0$.
- The coefficients in (*) don't depend on p.

  = the characters of simple G mod don't depend on p!

- Jantzen's condition comes from "Steinberg tensor product formula"

$\mu = \mu_0 + p\mu_1, \quad L(\mu) \simeq L(\mu_0) \otimes Fr^*(L(\mu_1)), \quad Fr: G \to G$

μ satisfies this condition

$\Rightarrow \langle \mu_1, \alpha^\vee \rangle < p - h + 2$

$\langle \mu_1 + \rho, \alpha^\vee \rangle \le p \quad \Rightarrow \mu_1 \in C \quad \Rightarrow \underline{L(\mu_1) \simeq N(\mu_1)}$

- It's true for $p \gg ?$, but not just $p \geq h$!

<u>Lusztig's idea (early 90s)</u>

(subtle)

1) Show characters of <u>certain</u> simple G-mod are equal to similar characters for quantum gps at a root of unity

2) Compare quantum groups & cat. of rep. of affine Lie alg $/\mathbb{C}$ (KL, L)

3) build some localization theory relating their reps to some cat. of D-modules on an affine flag var. (Kashiwara - Tanisaki)

<u>Step I</u>. • Anderson - Jantzen - Soergel proved for $p \gg ?$ (not explicit at all)

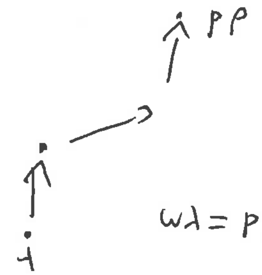
• Fiebig late 00s: reproved by using "combinatorial cat", some bounds, very large

• Williamson's counterexample

Soergel's modular category $\mathcal{O}$

Assume $p > h$, G ss, simply conn'd.

- $\mathcal{A}$ Serre subcat. of $\mathrm{Rep}(G)$ gen. by $L(\lambda)$ s.t. $\lambda \in X^+$ and $\lambda \uparrow p\rho$
- $\mathcal{B} \subset \mathcal{A}$ Serre subcat. of $\mathrm{Rep}(G)$ gen. by $L(\lambda)$, $\lambda \in X^+$ w/ $\lambda \uparrow p\rho$, $\lambda \notin \{(p-1)\rho + W_{fin} \cdot \rho\}$.



$w\lambda = p\rho$

Def. Soergel's modular cat. $\mathcal{O}$ is defined as $\mathcal{O}_{\mathbb{K}} = \mathcal{A}/\mathcal{B}$.

μ unique elt in $C \cap W_{aff} \cdot_p (p\rho)$

Then $w \longmapsto w \cdot_p \mu$ induces ${}^f W_{aff} \xrightarrow{\sim} X^+ \cap W_{aff} \cdot_p (p\rho)$ identifies the Bruhat order on ${}^f W_{aff}$ w/ "$\uparrow$"

$w \in W_{aff}$, $w \cdot_p \mu = p\rho$

Simple obj. in $\mathcal{A}$ $\longleftrightarrow$ $\{y \in {}^f W_{aff} : y \leq w\}$

$$\omega = w^{-1} t_\rho \subset W_{ext}$$

Lemma w is max'l in the coset $w W^\omega$ where $W^\omega = \omega W \omega^{-1}$.
We identify the poset W^ω w/ W, then for $x \in W$,

$$w \omega x \omega^{-1} \cdot_p \mu = t_\rho x \cdot_p 0 = (p-1)\rho + x\rho$$

So for $x \in W$, N_x, M_x, L_x image of modules

$N((p-1)\rho + x(\rho))$, $M(-)$, $L(-)$

$[N_y : L_x] = [M_y : L_x]$ composition factors.

Consequence of Lusztig's conjecture for $\mathcal{O}_{\mathbb{K}}$:

Prop $[N_y : L_x] = h_{y,x}(1)$.

Soergel bimodules

Q: How to construct interesting families of bimodules out of semisimple cpxes on flag var.?

G cpx reductive alg gp w/ Borel B, max'l torus T, $\mathcal{X} = G/B$ flag var.

Consider $D^b_B(\mathcal{X}; \mathbb{Q})$

$\mathcal{X} = \coprod_{w \in W} \mathcal{X}_w$ Bruhat decomposition

Simple objects in $D^b_B(\mathcal{X}; \mathbb{Q})^{\heartsuit} \longleftrightarrow IC_w = IC(\mathcal{X}_w; \underline{\mathbb{Q}})$

$IC_B(\mathcal{X}; \mathbb{Q}) \subset D^b_B(\mathcal{X}; \mathbb{Q})$ semisimple cpxes

$D^b_B(\mathcal{X}; \mathbb{Q})$ admits a convolution product $*$

Decomposition thm $\Rightarrow IC_B(\mathcal{X}; \mathbb{Q})$ closed under $*$.

Let $X^*(T)$ be the character lattice of T.

Λ- alg $R = S(\Lambda \otimes_{\mathbb{Z}} X^*(T))$

w grading $\downarrow$ deg 2

$R\text{-mod}^{\mathbb{Z}}\text{-}R$: ab. cat. of graded R-bimodules.

Next time: functor

$$\mathbb{H} : D^b_B(X; \Lambda) \longrightarrow D^b(R\text{-mod}^{\mathbb{Z}}\text{-}R)$$

Lecture 4. Soergel bimodules

1) Geometric motivation

2) Abstract construction

$IC_B(X, \Lambda) \subset D^b_B(X, \Lambda)$ $\qquad X = G/B$ (over $\mathbb{C}$)

$R := S(\Lambda \otimes_{\mathbb{Z}} X^*(T))$ $\circlearrowleft$ deg 2

$\mathbb{H} : D^b_B(X; \Lambda) \longrightarrow R\text{-Mod}^{\mathbb{Z}}\text{-}R$

abelian cat. of $\mathbb{Z}$-graded R-bimodules

Given $\mathcal{F} \in D^b_B(X; \Lambda)$, we have $H^\bullet_B(X, \mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} H^n_B(X, \mathcal{F})$

$R \circlearrowright \qquad \circlearrowleft R$

$B = T \ltimes U \leftarrow$ unipotent

$H^\bullet_B(pt, \Lambda) \xrightarrow{\sim} H^\bullet_T(pt, \Lambda) \simeq R$ $\qquad$ Similarly, $H^\bullet_{B\times B}(pt, \Lambda) \simeq R \otimes_{\Lambda} R$

We also have

$$H^\bullet_B(X, \mathbb{Q}) \xrightarrow{\sim} H^\bullet_{B\times B}(G, \mathbb{Q})$$

So this means we have a morphism of graded algebras

$$R \otimes_{\mathbb{Q}} R \longrightarrow H^\bullet_B(X; \mathbb{Q})$$

By construction, $\mathbb{H}(\mathcal{F})$ has a canonical action of $H^\bullet_B(X, \mathbb{Q})$, and so using this morphism it acquires an action of $R \otimes_{\mathbb{Q}} R$.

For $r \in \mathbb{Z}$, we write

$$(r) : R\text{-}\mathrm{Mod}^{\mathbb{Z}}\text{-}R \longrightarrow R\text{-}\mathrm{Mod}^{\mathbb{Z}}\text{-}R \quad \text{acts s.t.} \quad (M(r))^n = M^{n+r}$$

<u>So</u> $\mathbb{H} \circ [1] = (1) \circ \mathbb{H}$

<u>Prop.</u> $\mathbb{H} : IC_B(X, \mathbb{Q}) \longrightarrow R\text{-}\mathrm{Mod}^{\mathbb{Z}}\text{-}R$ is fully faithful.

What is essential image of $\mathbb{H}$?

- This is equiv. to describing $\mathbb{H}(IC_w)$

$\qquad\qquad\qquad\qquad IC$-sheaf on $X_w \subset X$

- Instead, we describe the image of a different family.

<u>Def</u> For any expression $\underline{w} = (s_1, \cdots, s_r)$, we set $IC_{\underline{w}} := IC_{s_1} * \cdots * IC_{s_r}$.

- If $w \in W$, and $\underline{w}$ is a reduced expr for w, then $IC_w \ominus IC_{\underline{w}}$.
- $IC_B(X, \mathbb{O})$ is the full subcat. of $D^b_B(X, \mathbb{O})$ whose objects are direct sums or shifts of direct summands of $IC_{\underline{w}}$ for expressions $\underline{w}$.

For any $s \in S$, consider the subalgebra

$R^s \subset R$ of s-invt elts, and set

$$B_s^{bim} := R \otimes_{R^s} R(1) \in R\text{-Mod}^{\mathbb{Z}}\text{-}R.$$

Given an expr. $\underline{w} = (s_1, \dots, s_r)$, we define

$$B_{\underline{w}}^{bim} := B_{s_1}^{bim} \otimes_R \cdots \otimes_R B_{s_r}^{bim} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_r}} R(r).$$

Prop. For any $\underline{w}$, there's a canonical isom. $\mathbb{H}(IC_{\underline{w}}) \simeq B_{\underline{w}}^{bim}$.

Def. The essential image of $IC_B(X, \mathbb{O})$ under $\mathbb{H}$ is the category of <u>Soergel bimods</u> $(SBim(W, V))$

assoc. to
- the Coxeter system (W, S)
- the representation $\mathbb{O} \otimes_{\mathbb{Z}} X_*(T)$ of W.

We will abstract this.

Def. Let $B_w^{bim} := \mathbb{H}(IC_w)$

indecomposable objects in $SBim(W, V)$

Let V be a finite dim'l rep'n of W over some field k.

Denote $T \subset W$ the set of reflections (i.e. conj. of elts of S)

(different from involutions)

Def. We say V is reflection faithful, if it's faithful and for any $x \in W$,

we have $\dim(V^x) = \dim(V) - 1$ iff $x \in T$.

Example. Let (W, S) be a Coxeter system and set $V = \mathbb{R}^S$ w/ basis $(e_s : s \in S)$

We define a sym. bilinear form $\langle , \rangle$ on V by
$$\langle e_s, e_t \rangle = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } s \neq t,\ \langle s,t \rangle \text{ infinite} \\ -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{otherwise} \end{cases}$$

$$\underbrace{st\cdots = ts\cdots}_{m_{s,t}}$$

Then the assignment
$$s \mapsto \left[x \mapsto x - 2\langle x, e_s \rangle e_s \right]$$
extends to a rep'n of W

on V. If W is finite, $\langle , \rangle$ is nondegenerate, and this rep'n is reflection faithful.

Soergel bimodules, abstract def'n

Fix a Coxeter system (W, S) and a fin. dim'l rep. V of W over any field k

s.t. $\mathrm{char}(k) \neq 2$

$SBim(W, V)$

We set $R = S(V^*)$

concentrated in deg 2

$\mathbb{Z}$-graded alg.

We can consider $R\text{-Mod}^{\mathbb{Z}}\text{-}R$. $B_s^{bim} = R \otimes_{R^s} R(1) \in R\text{-Mod}^{\mathbb{Z}}\text{-}R$.

$B_{\underline{w}}^{bim}$ is defined as before

$\hat{}$

Bott-Samelson bimodules

<u>Def</u> $SBim(W,V)$ is the full sub cat. of $R\text{-Mod}^{\mathbb{Z}}\text{-}R$ whose objects are direct sums of grading shifts of direct summands of $B_{\underline{w}}^{bim}$ (for $\underline{w}$ a word in S).

- There's no reason for $SBim(W,V)$ to be well-behaved for arbitrary V, but if V is reflection faithful, then many of the properties which hold in the "geom. origin" setting will still hold.

<u>Structure of $SBim(W,V)$ if V is reflection faithful.</u>

Assume V is reflection faithful.

<u>Thm</u> There exists a unique ring hom.

$$\varepsilon : \underset{\mathbb{Z}[v,v^{-1}]}{\mathcal{H}_{(W,S)}} \longrightarrow [SBim(W,V)]_{\oplus}$$

s.t. $\varepsilon(v) = [R(1)]$

$\varepsilon(\underline{H}_s) = [B_s^{bim}]$ $\quad$ $(\underline{H}_s = H_s + v)$

Kazhdan-Lusztig basis elt.

<u>Proof strategy</u>. RHS has a ring str. induced by tensor product.

$\longrightarrow$ upgrade this to a $\mathbb{Z}[v,v^{-1}]$-alg. str. by $v \mapsto [R(1)]$

Then one must prove that $[B_s^{bim}]-v$ satisfy quadratic relations & braid relations.

For the quadratic relations:

$$B_s^{bim} \otimes_R B_s^{bim} = R \otimes_{R^s} R \otimes_{R^s} R(2)$$

Recall: a refl'n of a f.dim vec. sp. is an endom. squaring to id which acts as the identity on a hyperplane.

Lemma. Assume s acts on V as a refl'n, and let $\alpha \in V^*$ be s.t. $s(\alpha) = -\alpha$, then as a graded R^s-module, R is graded free w/ basis $(1, \alpha)$

So $B_s^{bim} \simeq R(1) \oplus R(-1)$ as a R^s-bimodule.

$$\rightsquigarrow R \otimes_{R^s} R \otimes_{R^s} R(2) = R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R(0)$$

$$\simeq B_s^{bim}(1) \oplus B_s^{bim}(-1)$$

$\Rightarrow$ $[B_s^{bim}] \cdot [B_s^{bim}] = (v+v^{-1}) [B_s^{bim}]$ $\iff$ the quadratic rel'n for $[B_s^{bim}] - v$.

<u>Pf of lemma</u>. s acts as a refl'n on V $\to$ also on V^*

So $\exists\, H \subset V^*$ on which $s|_H = id$ and s.t. $V^* = H \oplus k \cdot \alpha$

Then $R = \bigoplus_{n \geq 0} S(H) \cdot \alpha^n$, and $R^s = \bigoplus_{\substack{n \geq 0 \\ n \text{ even}}} S(H) \cdot \alpha^n$

This implies that as graded left (or right) R^s-modules, we have $\boxed{R \simeq R^s \oplus R^s(-2)}$

$\Rightarrow$ quadratic relation + braid relations ---- $\longrightarrow$ Thm is true.

$\mathcal{E} : \mathcal{H}(W,S) \longrightarrow SBim(W,V)$

<u>Thm.</u> For any $w \in W$, $\exists!$ indecomp. obj. $B_w^{bim} \in SBim(W,V)$ s.t. for any reduced expr $\underline{w}$ for any w, B_w^{bim} is the unique indecomp. summand of $B_{\underline{w}}^{bim}$ which is <u>not</u> a direct summand of any $B_{\underline{y}}^{bim}(n)$ w/ $\underline{y}$ a reduced exp'n for some $y < w$ and $n \in \mathbb{Z}$.

Further, $(w,n) \longmapsto B_w^{bim}(n)$ gives a bijection between $(W, \mathbb{Z})$ and the set of isom. classes of indec. objects in $SBim(W,V)$.

<u>Lecture 5</u>.

Incarnations of "category $\mathcal{O}$"

- usual cat. $\mathcal{O}$
- Soergel's modular cat $\mathcal{O}_{\mathbb{K}}$

subquotient

- Algebraic definition
- Soergel bimodules
- Geometry of G/B
- Diagrammatics (Elias – Williamson) $\longleftarrow$ Williamson's counterexample

$\}$ categorify the Hecke algebra

	std basis H_w,	$^{(p-)}$Kazhdan – Lusztig basis $\underline{H}_w$.
algebraic:	Δ_w or $N(w)$	$L(w)$
Soergel:	$B_{\underline{w}}^{bim}$	B_w^{bim} $\longleftarrow$ biggest indecomposable direct summand of $B_{\underline{w}}^{bim}$.

Soergel's conjectures

$$\varepsilon : \mathcal{H}_{(W,S)} \longrightarrow K_0(\mathrm{SBim}(W,V))_{\oplus}$$

Soergel conjectured "at least if $\mathbb{K} = \mathbb{C}$" that $\varepsilon(\underline{H}_w) = [B_w^{bim}]$. $(*)$

Soergel also defined a map $h_\Delta : [\mathrm{SBim}(W,V)]_{\oplus} \to \mathcal{H}_{(W,S)}$

which is left-inverse to ε.

If we write $h_\Delta([M])$ in the basis $(H_w : w \in W)$, the coeff. are nonnegative.

The statement $(*)$ can be refined to the claim that

$$h_\Delta([B_w^{bim}]) \in H_w + \sum_{y<w} \mathbb{Z}_{\geq 0}[v,v^{-1}] \cdot H_y$$

Elias-Williamson: $(*)$ holds if V is a reflection faithful rep. over $\mathbb{R}$

satisfying one additional condition.

This condition holds for V built from any Coxeter system (W,S).

$\Rightarrow$ Soergel's conjecture $\Rightarrow$ Kazhdan – Lusztig positivity.

Geometry of G/B motivated Soergel bimodules. in $\mathbb{K} = \mathbb{C}$.

How do we properly work w/ the geometry of G/B when $\mathrm{char}(\mathbb{K}) > 0$?

<u>Motivation</u>: Bott-Samelson sheaves on flag varieties.

$G \supset B \supset T$, $X = G/B$, (W, S)

The Bruhat decomposition:

$$X = \bigsqcup_{w \in W} X_w, \quad X_w = BwB/B \simeq \mathbb{A}_{\mathbb{C}}^{\ell(w)}$$

Let $\mathbb{K}$ be any field, then $D^b_{(B)}(X, \mathbb{K})$ is the derived cat. of Bruhat constructible complexes of $\mathbb{K}$-sheaves on X

$\Leftrightarrow$ The full sub cat. of $D^b(X, \mathbb{K})$ consisting of complexes s.t.

$H^i(\mathcal{F}|_{BwB/B})$ is constant, $\forall w \in W$, $i \in \mathbb{Z}$.

From $\mathcal{F} \in D^b_{(B)}(X, \mathbb{K})$, we can construct an elt. of $\mathcal{H}_{(W,S)}$.

<u>Def</u>. Let

$$ch(\mathcal{F}) = \sum_{\substack{w \in W \\ k \in \mathbb{Z}}} \dim\left(H^{-\ell(w)-k}(\mathcal{F}_{wB})\right) v^k H_w \in \mathcal{H}_{(W,S)}$$

where $\mathcal{F}_{wB}$ is the stalk of $\mathcal{F}$ at any pt in X_w.

Note that $ch(\mathcal{F}[1]) = v\, ch(\mathcal{F})$ for any $\mathcal{F} \in D^b_{(B)}(X, \mathbb{K})$.

For any $s \in S$, let $P_s \subset G$ be the minimal parabolic corresponding to s

Let $X^s = G/P_s$, $W^s = \{w \in W : \ell(ws) > \ell(w)\}$ $\leftarrow$ set of reps for $W/\langle s \rangle$ ($\langle s \rangle = \{e, s\}$).

$$X^s = \bigsqcup_{w \in W^s} X^s_w, \quad X^s_w := BwP_s/P_s \simeq \mathbb{A}_{\mathbb{C}}^{\ell(w)}, \quad w \in W^s$$

$D^b_{(B)}(X^s, \Bbbk)$ defined as before.

We have a morphism $\pi_s : X \to X^s$ which induces derived functors

$$(\pi_s)_* : D^b_{(B)}(X, \Bbbk) \longrightarrow D^b_{(B)}(X^s, \Bbbk)$$

$$(\pi_s)^* : D^b_{(B)}(X^s, \Bbbk) \to D^b_{(B)}(X, \Bbbk)$$

$$\left[(\pi_s)_* = (\pi_s)_! , \quad (\pi_s)^! = (\pi_s)^* [2] \right]$$

For $(s_1, \dots, s_n)$, $s_i \in S$, we set

$$\mathcal{E}(s_1, \dots, s_n) = (\pi_{s_n})^* (\pi_{s_n})_* \cdots (\pi_{s_1})^* (\pi_{s_1})_* \underline{\Bbbk}_{X_e} [n]$$

Bott-Samelson sheaf.

<u>Prop</u>. For any $s_1, \dots, s_n \in S$, $w \in W$,

$$H^i(\mathcal{E}(s_1, \dots, s_n)_{wB}) = 0 \quad \text{unless } i \equiv n \pmod 2.$$

Moreover, we have

$$ch(\mathcal{E}(s_1, \dots, s_n)) = \underline{H}_{s_1} \cdots \underline{H}_{s_n} = (H_{s_1} + v) \cdots (H_{s_n} + v)$$

<u>Lemma</u> Let $\mathcal{F} \in D^b_{(B)}(X, \Bbbk)$ s.t. $H^k(\mathcal{F}) = 0$ unless k is even, then $H^k((\pi_s)^*(\pi_s)_* \mathcal{F}) = 0$ unless k even, and

$$ch((\pi_s)^*(\pi_s)_* \mathcal{F}) = ch(\mathcal{F}) \cdot v^{-1} \underline{H}_s .$$

<u>category $\mathcal{O}_0$</u>

To each $w \in W$, we have

$\Delta_w, \nabla_w, L_w, P_w, T_w$

$$D^b(\mathcal{O}_0) \xleftrightarrow{\text{Koszul}} D^b(\mathcal{O}_0)$$

$$L_w \longleftrightarrow T_w$$

$$\Delta_w \longleftrightarrow \Delta_w$$

<u>Proof</u>. For $y \in W$, $H^k(((\pi_s)^*(\pi_s)_* \mathcal{F})_{yB}) = H^k(((\pi_s)_* \mathcal{F})_{yP_s})$

$$= H^k(\pi_s^{-1}(yP_s), \mathcal{F}|_{\pi_s^{-1}(yP_s)})$$

Two cases:

$ys > y$, $ys < y$.

If $ys > y$, $yxB \in \begin{cases} BysB/B, & \text{if } x \notin B \\ ByB/B, & \text{if } x \in B \end{cases}$

$$\pi_s^{-1}(yP_s) = \{ygB : g \in P_s\} \simeq P_s/B \simeq \mathbb{P}^1_{\mathbb{C}}$$

$$B/B \xrightarrow{i} P_s/B \xleftarrow{j} (P_s/B \setminus B/B) \qquad 0 \to \mathbb{P}^1 \hookleftarrow \mathbb{A}^1$$

we have a dist. $\triangle$

$j_! j^* \to \mathrm{id} \to i_* i^* \to$, So we get an exact sequence

$$\cdots \to H_c^k(\mathbb{A}^1, \mathcal{F}|_{\mathbb{A}^1}) \to H^*((\pi_s)^*(\pi_s)_*\mathcal{F}|_{yB}) \to H^*(pt, \mathcal{F}|_{pt}) \to \cdots$$

$\mathcal{F}|_{\mathbb{A}^1}$ constant w/ value $\mathcal{F}_{ysB}$, $\mathcal{F}|_{pt} - \mathcal{F}_{yB}$

we have $H_c^k(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}|_{\mathbb{A}^1_{\mathbb{C}}}) \simeq H^{k-2}(\mathcal{F}_{ysB})$

(bc $H_c^k(\mathbb{A}^1, \mathbb{k}_{\mathbb{A}^1}) = \begin{cases} \mathbb{k}, & k=2 \\ 0, & \text{otherwise} \end{cases}$)

$$H^k(pt, \mathcal{F}|_{pt}) \simeq H^k(\mathcal{F}_{yB})$$

So $\dim H^k((\pi_s)^*(\pi_s)_*\mathcal{F}|_{yB}) = \begin{cases} 0 & \text{if } k \text{ odd} \\ \dim H^{k-2}(\mathcal{F}_{ysB}) + \dim H^k(\mathcal{F}_{yB}) & \text{if } k \text{ is even} \end{cases}$

<u>Application</u>. Let $\mathbb{k} = \mathbb{Q}$, For $w \in W$, let

$$IC_w := j_{w!*}(\underline{\mathbb{Q}}_{X_w}[\ell(w)]) \in \mathrm{Perv}_{(B)}(X, \mathbb{Q})$$

Thm we have $H^k(IC_w) = 0$ unless $k \equiv \ell(w) \mod 2$

Moreover, $ch(IC_w) = \underline{H}_w$.

Proof sketch . The def of IC has a condition on stalks which implies

$$ch(IC_w) \in H_w + \sum_{y<w} v\,\mathbb{Z}[v]\, H_y$$

- Remains to show parity vanishing, and $ch(IC_w)$ is self-dual under Kazhdan - Lusztig involution.

- $(\pi_s)^*[1]$ sends IC cpxes to IC cpxes.

$(\pi_s)_*$ sends IC cpxes to direct sums of shifts of IC cpxes (Decomp. Thm)

Write $w = s_1 \cdots s_n$

$\Rightarrow \mathcal{E}(s_1, \ldots, s_n)$ is a direct sum of ICs

$\mathcal{E}(s_1, \ldots, s_n)\big|_{X_w} = \underline{\mathbb{Q}}_{X_w}[\ell(w)]$ supported on $\overline{X_w}$.

So IC_w is a direct summand $\Rightarrow$ the parity claim.

Finally, we need $ch(IC_w)$ is self-dual.

$\mathcal{E}(s_1, \ldots, s_n) = IC_w \oplus \bigoplus_{y<w} IC_y$

Induction: if $w = e$ it's obvious

induction: $ch(IC_w)$ is self dual.

$\mathbb{D} \circ (\pi_s)_* = (\pi_s)_! \circ \mathbb{D} = (\pi_s)_* \circ \mathbb{D}$

$\mathbb{D} \circ (\pi_s)^* = (\pi_s)^! \circ \mathbb{D} = (\pi_s)^*[2] \circ \mathbb{D}$

$\Rightarrow \mathbb{D}(\mathcal{E}(s_1, \ldots, s_n)) \simeq \mathcal{E}(s_1, \ldots, s_n)$

Lecture 6 Parity sheaves

- Decomposition thm fails in positive characteristic
  - ⟿ IC sheaves aren't as nice to work with.
- Multiplicities of IC in Δ or ∇ are no longer always given by KL polynomials.
- Parity sheaves are often IC sheaves, but in general are not necessarily perverse.

p-Kazhdan-Lusztig polynomials.

- We defined SBim(W,V) in all char.
- We can already define p-Kazhdan-Lusztig polynomials by mult. of B_w^{bim} in $\underline{B}_y^{bim}$.

Eg. Type B_2

$$ {}^p\underline{H}_{sts} = \begin{cases} \underline{H}_{sts} + \underline{H}_s & \text{if } p=2 \\ \underline{H}_{sts} & \text{if } p \neq 2 \end{cases} $$

- when $p=0$
  ${}^p\underline{H}_w = \underline{H}_w$
- for any fixed type, for $p \gg 0$,
  ${}^p\underline{H}_w = \underline{H}_w$.

In type A, ${}^p\underline{H}_w = \underline{H}_w$ for $A_1, \cdots, A_6$.

In A_7, ${}^p\underline{H}_w = \underline{H}_w$ unless $p=2$, in which case they differ.

p-canonical bases in Type A_m for $m \geq 8$ are not known completely.

Eg. 3-canonical basis $\neq$ KL basis in type A_{11}

Open Q. Is this the first m for which they differ for $p=3$?

↪ Later we will discuss a way to produce pairs (p,m) s.t. ${}^p\underline{H}_w \neq \underline{H}_w$ in type A_m.

Let $\mathbb{k}$ be a field, D be a $\mathbb{k}$-linear triangulated cat. w/ a bounded t-str. w/ heart $\mathcal{A}$. (we use H for cohomology functors w.r.t. this t-str.)

Let $X \in \mathcal{A}$. $\langle X \rangle_\Delta$ be the triangulated subcat. of D gen. by X.

<u>Lemma</u>. Assume that $\mathrm{End}(X) = \mathbb{k}$ and $\mathrm{Hom}(X, X[1]) = 0$, then for $Y \in D$, TFAE:

1) $Y \in \langle X \rangle_\Delta$

2) $\forall n \in \mathbb{Z}$, $H^n(Y)$ is isom. to a direct sum of copies of X. (Lemma 2.1 in the book)

<u>Lemma 2.2</u>. Assume that $\mathrm{End}(X) = \mathbb{k}$, and $\mathrm{Hom}(X, X[2n+1]) = 0$ for any $n \in \mathbb{Z}_{\geq 0}$,

then $\forall Y \in D$, TFAE:

1) $Y \in \langle X \rangle_\Delta$ and $H^m(Y) = 0$ for any m odd.

2) There exist even integers $n_1, \cdots, n_r$ and an isom. $Y \simeq \bigoplus_{i=1}^{r} X[n_i]$

<u>Geometric setting</u>

Let $\mathbb{F}$ be an alg. closed field, and an $\mathbb{F}$-alg. var. X.

We assume we're given a decomp. $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ where Λ finite, each X_λ smooth conn'd locally closed, and s.t. $\forall \lambda$, $\overline{X_\lambda}$ is a union of X_μ for $\mu \in \Lambda$.

Write $j_\lambda : X_\lambda \hookrightarrow X$, we'll consider $D(X ; \mathbb{k})$

1) Analytic : $\mathbb{F} = \mathbb{C}$, $\Bbbk$ arbitrary.

$D(X;\Bbbk)$ is constr. derived cat. of $\Bbbk$-sheaves w.r.t. analytic topology

2) Étale : $\mathbb{F}$ arbitrary, $\Bbbk$ is [a finite field w/ $\mathrm{char}(\Bbbk) \neq \mathrm{char}(\mathbb{F})$; a finite ext'n of $\mathbb{Q}_\ell$, $\ell \neq \mathrm{char}(\mathbb{F})$]

$D(X;\Bbbk)$ is constr. der. cat. of étale $\Bbbk$-sheaves

3) & 4) Equivariant analytic & equivariant étale.

In these settings, we have derived functors $(j_\lambda)_*, (j_\lambda)_! : D(X_\lambda;\Bbbk) \to D(X;\Bbbk)$

$(j_\lambda)^*, (j_\lambda)^! : D(X;\Bbbk) \to D(X_\lambda;\Bbbk)$

Given for any $\lambda \in \Lambda$, a local system $\mathcal{L}_\lambda$ on X_λ s.t.

$\mathrm{End}_{D(X_\lambda;\Bbbk)}(\mathcal{L}_\lambda) = \Bbbk$, $\quad \mathrm{Hom}(\mathcal{L}_\lambda, \mathcal{L}_\lambda[2n+1]) = 0$, $\forall n \in \mathbb{Z}_{\geq 0}$

then we set

$\underset{\text{std}}{\Delta_\lambda} = (j_\lambda)_! \mathcal{L}_\lambda[\dim X_\lambda]$, $\underset{\text{costd}}{\nabla_\lambda} = (j_\lambda)_* \mathcal{L}_\lambda[\dim X_\lambda]$

We will also assume that for any λ, μ, $(j_\mu)^* \nabla_\lambda \in \langle \mathcal{L}_\mu \rangle_\Delta$

<u>Def</u> Let $D_\Lambda(X,\Bbbk)$ be the triangulated subcat. of $D(X;\Bbbk)$ consisting of objs $\mathcal{F}$ s.t. $\mathcal{F}|_{X_\mu} \in \langle \mathcal{L}_\mu \rangle_\Delta$ for any $\mu \in \Lambda$.

<u>Parity complexes</u> <u>Def.</u> Let $\mathcal{F} \in D_\Lambda(X;\Bbbk)$

1) We call $\mathcal{F}$ $*$-even (odd) if $\forall \lambda \in \Lambda$, $H^n(j_\lambda^* \mathcal{F}) = 0$ unless n even (odd)

2) !-even / odd $H^n(j_\lambda^! \mathcal{F}) = 0$ unless n even / odd

3) We call $\mathcal{F}$ <u>even</u> if it's $*$-even & $!$-even.

4) We call $\mathcal{F}$ a <u>parity cpx</u> if it's isom. to a direct sum of an even and an odd object.

<u>Lemma</u> Let $\mathcal{F} \in D_\Lambda(X; \Bbbk)$

(1) If $|\Lambda| = 1$, then TFAE

(a) $\mathcal{F}$ is $*$-even ; (b) $\mathcal{F}$ is $!$-even; (c) $\mathcal{F}$ is even

(d) $\mathcal{F}$ is a direct sum of $\mathcal{L}_\lambda[n]$ for n even.

Further, if $\mathcal{F}, \mathcal{G}$ are even, $\forall n \in \mathbb{Z}$, $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[n]) = 0$ unless n even.

(2) $\mathcal{F}$ is $!$-even iff $\mathbb{D}(\mathcal{F})$ is $*$-even.

So $\mathcal{F}$ is parity iff $\mathbb{D}(\mathcal{F})$ is parity.

(3) $\mathcal{F}$ is even iff $\mathcal{F}[1]$ is odd, so $\mathcal{F}$ is parity iff $\mathcal{F}[1]$ parity.

There is a Verdier duality functor $\mathbb{D}: D_\Lambda(X, \Bbbk) \rightleftarrows D_{\Lambda, \mathrm{dual}}(X; \Bbbk)$ and we'll consider cases where $\Lambda \simeq (\Lambda, \mathrm{dual})$.

(4) $\mathcal{F}$ is even iff $\mathcal{H}^n(\mathcal{F}) = \mathcal{H}^n(\mathbb{D}(\mathcal{F})) = 0$, $\forall$ odd n.

<u>Pf of (4)</u>. A sheaf $\mathcal{G}$ on X is zero iff $j_\lambda^!(\mathcal{G}) = 0$, $\forall \lambda \in \Lambda$.

Since $H^n(j_\lambda^! \mathcal{F}) = j_\lambda^! \mathcal{H}^n(\mathcal{F})$ for any n, $\mathcal{F}$ is $!$-even iff $\mathcal{H}^n(\mathcal{F}) = 0$ $\forall$ odd integers n

Prop. Let $\mathcal{F}, \mathcal{G} \in D(X; \mathbb{k})$. If $\mathcal{F}$ is a direct sum of a $*$-even and $*$-odd object, and $\mathcal{G}$ for $!$, then

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}(j_\lambda^* \mathcal{F}, j_\lambda^! \mathcal{G}).$$

Proof goes by induction on $|\Lambda|$.

Let Y be the supp. of $\mathcal{F}$. $X_\mu \subset Y$ be an open stratum. Let $j : X \setminus (Y \setminus X_\mu) \hookrightarrow X$ be an open embedding, and i the complementary closed embedding. then

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{[1]}$$

$\rightsquigarrow$ LES associated to the functor $\mathrm{Hom}(-, \mathcal{G})$.

<u>Corollary</u>.

Let $\mathcal{F}, \mathcal{G} \in D_\Lambda(X; \mathbb{k})$

(1) If $\mathcal{F}$ is $*$-even, $\mathcal{G}$ $!$-odd, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

(2) If $\mathcal{F}, \mathcal{G}$ parity. Let $U \subset X$ be an open union of strata, $j : U \hookrightarrow X$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(j^* \mathcal{F}, j^* \mathcal{G})$ is <u>surj</u>.

(3) Let $\mathcal{F}$ be indecomposable, and parity, then w/ $j : U \hookrightarrow X$ as in (3), $j^* \mathcal{F}$ is either indecomp. or 0.

Proof. (1) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}(j_\lambda^* \mathcal{F}, j_\lambda^! \mathcal{G}) = 0$

(2) We can assume $\mathcal{F}, \mathcal{G}$ are even.

$i : X \setminus U \hookrightarrow X$. Then $\mathcal{G} \to j_* j^* \mathcal{G} \to i_! i^! \mathcal{G}[1] \xrightarrow{[1]}$

So we get an exact seq.

vanishes by (1)

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) & \to & \mathrm{Hom}(\mathcal{F}, j_* j^* \mathcal{G}) & \to & \mathrm{Hom}(\mathcal{F}, i_! i^! \mathcal{G}[1]) \\ & & \wr\vert & & \\ & & \mathrm{Hom}(j^* \mathcal{F}, j^* \mathcal{G}) & & \end{array}$$

(3) By (2), $\underset{\substack{\uparrow \\ \text{local ring}}}{\mathrm{End}(\mathcal{F})} \twoheadrightarrow \underset{\substack{\uparrow \\ \text{either } 0 \text{ or local ring.}}}{\mathrm{End}(j^* \mathcal{F})}$

Thm For each $\lambda \in \Lambda$, there exists at most 1 indecomposable parity complex $\mathcal{E}_\lambda$ supp. on $\overline{X_\lambda}$ and s.t. $\mathcal{E}_\lambda |_{X_\lambda} \simeq \mathcal{L}_\lambda [\dim X_\lambda]$.

Moreover, any indecomp. parity cpx is isom. to $\mathcal{E}_\lambda[n]$ for some $\lambda \in \Lambda$, $n \in \mathbb{Z}$.

Lecture 7 Thm. For each Λ, $\exists$ at most 1 indecomp. parity complex $\mathcal{E}_\lambda$ supp. on $\overline{X_\lambda}$ s.t. $\mathcal{E}_\lambda |_{X_\lambda} \simeq \mathcal{L}_\lambda [\dim X_\lambda]$.

Moreover, any indecomp. parity cpx is isom. to $\mathcal{E}_\lambda[n]$ for some $\lambda \in \Lambda$, $n \in \mathbb{Z}$.

Proof. Suppose $\mathcal{E}_\lambda, \mathcal{E}_\lambda'$ are indecomp., supp. on $\overline{X_\lambda}$, s.t. $\mathcal{E}_\lambda|_{X_\lambda} \simeq \mathcal{L}_\lambda[\dim X_\lambda] \simeq \mathcal{E}_\lambda'|_{X_\lambda}$.

$\mathrm{Hom}(\mathcal{E}_\lambda, \mathcal{E}_\lambda') \twoheadrightarrow \mathrm{Hom}(\mathcal{E}_\lambda|_{X_\lambda}, \mathcal{E}_\lambda'|_{X_\lambda}) \simeq \mathbb{k}$.

There is $f: \mathcal{E}_\lambda \to \mathcal{E}_\lambda'$ s.t. $f|_{X_\lambda}$ is an isom.

$g: \mathcal{E}_\lambda' \to \mathcal{E}_\lambda$ s.t. $g|_{X_\lambda}$ is an isom.

$g\circ f \in \mathrm{End}(\mathcal{E}_\lambda)$ ← local ring $\Rightarrow$ $g\circ f$ invertible

not nilpotent (since restriction isn't)

Similarly, $f\circ g$ is invertible $\Rightarrow$ f & g are both isomorphism $\Rightarrow \mathcal{E}_\lambda \simeq \mathcal{E}_\lambda'$.

Now let $\mathcal{F}$ be some indec. parity complex, let Y be its supp.

There is a unique λ s.t. X_λ is open in Y.

(otherwise if $X_\lambda \cup X_\mu$ open in Y, $\mathcal{F}|_{X_\lambda \cup X_\mu} \simeq \mathcal{F}|_{X_\lambda} \oplus \mathcal{F}|_{X_\mu}$).

So $Y = \overline{X_\lambda} \Rightarrow \mathcal{F}|_{X_\lambda} \simeq \mathcal{L}_\lambda[\dim(X_\lambda)+n]$ for some n $\Rightarrow \mathcal{F} \simeq \mathcal{E}_\lambda[n]$.

Rmk. In cases we'll consider, $\mathcal{E}_\lambda$ exists.

If $\mathbb{F} = \mathbb{C}$, and if each X_λ is contractible (which also implies $\mathcal{L}_\lambda \simeq \underline{\mathbb{k}}_{X_\lambda}$), then existence is guaranteed.

The case of affine flag varieties

Let $\mathbb{F}$ be alg. closed, and G is a conn'd red. alg. gp over $\mathbb{F}$.

$G \supset B \supset T$.

To G, one associate two functors,

LG and L^+G from the cat. of $\mathbb{F}$-algebras to Set by setting

$LG(R) = G(R((z)))$, $L^+G(R) = G(R[[z]])$

$\rightsquigarrow$ L^+G is representable by a gp scheme over $\mathbb{F}$

$\rightsquigarrow$ LG is representable by a gp ind-scheme over $\mathbb{F}$

To each subset of $\underbrace{S_{\text{aff}}}_{\text{affine simple roots}}$, we can associate a parahoric subgp.

$Q_A \subset LG$, and consider the functor $R \mapsto LG(R)/Q_A(R)$. (Fl_A be the fppf sheafification)

If $A = \phi$, then Q_A = the std Iwahori subgp of LG, and we write Fl.

If $A = S$, then Q_S is L^+G, and we write $Fl_S = Gr$ the affine grassmannian.

Affine grassmannian & geometric Satake

$L^+G \curvearrowright Gr$, L^+G-orbits on Gr are parametrized by $X_*(T)^+$ (dominant cochars of T)

We call Gr^λ the L^+G-orbit assoc. to $\lambda \in X_*(T)^+$, so $(Gr^\lambda ; \lambda \in X_*(T)^+)$ is a

strat. of Gr. $\dim(Gr^\lambda) = \langle 2\rho, \lambda \rangle$. ($2\rho$ = sum of pos. roots)

and $\overline{Gr^\lambda} \subset \overline{Gr^\mu}$ iff $\mu - \lambda$ is a sum of pos. coroots.

Conn'd components of Gr are in canonical bijection w/ $``\pi_1(G)" = X_*(T)/\mathbb{Z}R^\vee$.

<u>Lemma</u> If $\lambda, \mu \in X_*(T)^+$, and Gr^λ, Gr^μ are in the same conn'd component, then $\dim(Gr^\lambda)$, $\dim(Gr^\mu)$ have the same parity.

We'll consider $D^b_{L^+G}(Gr; \Bbbk)$

There is a conv. product

$$(-) \underset{L^+G}{*} (-) : D^b_{L^+G}(Gr, \Bbbk) \times D^b_{L^+G}(Gr; \Bbbk) \longrightarrow D^b_{L^+G}(Gr; \Bbbk)$$

and so $D^b_{L^+G}(Gr; \Bbbk)$ is monoidal.

Let $Perv_{L^+G}(Gr; \Bbbk)$ be the heart of the perverse t-structure. Then if $A, B \in Perv_{L^+G}(Gr; \Bbbk)$,

$A * B \in Perv_{L^+G}(Gr; \Bbbk)$.

So $\left(Perv_{L^+G}(Gr; \Bbbk), \underset{L^+G}{*}\right)$ is monoidal.

To each $\lambda \in X_*(T)^+$, one can assoc. 3 nat'l objs in $Perv_{L^+G}(Gr; \Bbbk)$

Let $j^\lambda : Gr^\lambda \to Gr$ be the embedding, then we let

$$J_!(\lambda) = {}^pH^0(j^\lambda_! \underline{\Bbbk}_{Gr^\lambda}[\langle 2\rho, \lambda\rangle]), \quad J_*(\lambda) = {}^pH^0(j^\lambda_* \underline{\Bbbk}_{Gr^\lambda}[\langle 2\rho, \lambda\rangle])$$

There's a canonical morphism $j^\lambda_! \underline{\Bbbk}_{Gr^\lambda}[\langle 2\rho, \lambda\rangle] \longrightarrow j^\lambda_* \underline{\Bbbk}_{Gr^\lambda}[\langle 2\rho, \lambda\rangle]$

$\rightsquigarrow J_!(\lambda) \longrightarrow J_*(\lambda)$, and let $J_{!*}(\lambda)$ be its image.

Let's now denote by $G^\vee_{\mathbb{k}}$ the conn'd reductive gp over $\mathbb{k}$ whose root datum is

$(X_*(T), R^\vee, X^*(T), R)$

So $G^\vee_{\mathbb{k}}$ has a max torus $T^\vee_{\mathbb{k}}$ whose character lattice is $X_*(T)$.

<u>Thm</u> (Geom. Satake) $\left(\mathrm{Perv}_{L^+G}(\mathrm{Gr};\mathbb{k}), *_{L^+G}\right) \simeq \left(\mathrm{Rep}(G^\vee_{\mathbb{k}}), \otimes\right)$

This sends, for any $\lambda \in X_*(T)^+$,

$$J_!(\lambda) \mapsto M(\lambda)$$
$$J_*(\lambda) \mapsto N(\lambda)$$
$$J_{!*}(\lambda) \mapsto L(\lambda).$$

We've just defined $\mathcal{E}_\lambda$ for any $\lambda \in X_*(T)^+$

<u>Q</u>. • Are they perverse?

• If they are, where do they go under geom. Satake?

<u>Thm</u>. (Juteau - Mautner - Williamson) If $\mathrm{char}(\mathbb{k})$ is <u>good</u> for G, then $\mathcal{E}^\lambda$ is perverse, for any $\lambda \in X_*(T)^+$.

<u>Prop</u>. Let $\lambda \in X_*(T)^+$, If $\mathcal{E}^\lambda$ is perverse, then its image in $\mathrm{Rep}(G^\vee_{\mathbb{k}})$ is <u>tilting</u> and denoted $T(\lambda)$.

(and indecomp.)

<u>Proof</u>. $\mathrm{Ext}^1(A,B) = \mathrm{Hom}(A, B[1])$

$\Rightarrow$ we need to show

$\mathrm{Hom}(\mathcal{E}^\lambda, J_*(\mu)[1]) = 0 \quad (*)$

$\mathrm{Hom}(J_!(\mu), \mathcal{E}^\lambda[1]) = 0 \quad (!)$

$\mathrm{Ext}^1(T, J_*(\mu)) = 0$

$\mathrm{Ext}^1(J_!(\mu), T) = 0$

for all $\mu \in X_*(T)^+$.

We can assume $\langle 2\rho, \lambda\rangle$ and $\langle 2\rho, \mu\rangle$ have the same parity. (assume WLOG both even)

$j_!^\lambda \underline{\Bbbk}_{Gr^\lambda}[\langle 2\rho,\lambda\rangle]$ is concentrated in nonpositive degrees.

$$A \to j_!^\mu \underline{\Bbbk}_{Gr^\mu}[\langle 2\rho,\mu\rangle] \to J_!(\mu)$$

concentrated in neg. degrees

So $\underset{\overset{\|}{0}}{\mathrm{Hom}(A, \mathcal{E}^\lambda)} \to \mathrm{Hom}(J_!(\mu), \mathcal{E}^\lambda[1]) \to \mathrm{Hom}(j_!^\mu \underline{\Bbbk}_{Gr^\mu}[\langle 2\rho,\mu\rangle], \mathcal{E}^\lambda[1])$

then $j_!^\mu \underline{\Bbbk}_{Gr^\mu}[\langle 2\rho,\mu\rangle]$ is $*$-even, and $\mathcal{E}^\lambda[1]$ is odd (therefore $!$-odd)

so $= 0$

So $\mathcal{E}^\lambda$ is tilting

Rmk even if $\mathrm{char}(\Bbbk)$ is bad for G, ${}^pH^0(\mathcal{E}^\lambda)$ these are tilting (and they give all the tiltings).

What are tilting modules in $\mathrm{Rep}(G)$?

Prop. Let $M \in \mathrm{Rep}(G)$,

(1) TFAE

(a) M admits a costd filtration

(b) For any $\lambda \in X^+$ and any $n > 0$, $\mathrm{Ext}^n(M(\lambda), M) = 0$

(c) For any $\lambda \in X^+$, we have $\mathrm{Ext}^1(M(\lambda), M) = 0$

(2) TFAE

(a) —"— std filt'n

(b) —"— $\mathrm{Ext}^n(M, N(\lambda)) = 0$

(c) —"— $\mathrm{Ext}^1(M, N(\lambda)) = 0$

(3) TFAE

(a) M is tilting

(b) both previous (b)s

(c) both previous (c)s

Rmk In literature: costd filt. $\Leftrightarrow$ "good filt"

std filt $\Leftrightarrow$ "Weyl filt".

Def. $\mathrm{Tilt}(G) \subset \mathrm{Rep}(G)$ subcat. of tilting objects.

If $M \in \mathrm{Tilt}(G)$, then write $(M : N(\lambda))$ = # of occurences of $N(\lambda)$ as a subqt in a costd filtration

$$(M : M(\lambda)) = \text{—"—} \; M(\lambda) \; \text{—"—}$$

$$(M : N(\lambda)) = \dim \mathrm{Hom}(M(\lambda), M).$$

$$\Rightarrow [M] = \sum_{\lambda \in X^+} (M : N(\lambda)) \cdot [N(\lambda)]$$

coeff. are nonneg!

Recall $[M(\lambda)] = [N(\lambda)]$ in $K_0(\mathrm{Rep}(G))$.

$$(M : N(\lambda)) = (M : M(\lambda))$$

So if M, N are tilting,

$$\dim \mathrm{Hom}(M, N) = \sum_{\lambda \in X^+} (M : N(\lambda)) \cdot (N : N(\lambda))$$

<u>Lecture 8</u> • Tilting modules for G $\longleftrightarrow$ simple characters

$\hookrightarrow$ $G_1 T$-modules

$\hookrightarrow$ baby Vermas

$\hookrightarrow$ ...

• A taste of Williamson's counterexamples and conjectures

$\hookrightarrow$ geometric side

$\hookrightarrow$ diagrammatic side

• Reps of $G(\mathbb{F}_q)$ and reduction mod p.

Std objs $M(\lambda)$

costd objs $N(\lambda)$

$\boxed{T(\lambda)}$ $\longleftarrow$ exists b/c of general theory of h.w. categories

$\hookrightarrow$ indecomp. tilting assoc. to λ

KD

$L(\lambda)$

Finkelberg - Mirković conjecture: understanding $\mathrm{Rep}_0(G)$ very explicitly in terms of geometry

Thm. For any $\lambda \in X^+$, there exists a unique indecomp. tilting G-module $T(\lambda)$ s.t. $(T(\lambda) : N(\lambda)) = 1$, and $(T(\lambda) : N(\mu)) = 0$ unless $\mu \leq \lambda$.

Moreover, $\lambda \mapsto T(\lambda)$ is a bijection between X^+ and the iso. classes of indecomp. tilting G-modules.

__Goal__: Understand why these modules are relevant to computing characters of simples. (Result of Jantzen)

__Representations of the group scheme G_1.__ (suppose $\mathrm{char}(\mathbb{k}) = p > 0$)

For any $\mathbb{k}$-scheme X, the Frobenius twist of X is the fiber product

$$X^{(1)} := \mathrm{Spec}(\mathbb{k}) \times_{\mathrm{Spec}(\mathbb{k})} X$$ where $\mathrm{Spec}(\mathbb{k}) \to \mathrm{Spec}(\mathbb{k})$ is induced by $x \mapsto x^p$.

If $X = \mathrm{Spec}\, A$ for A a $\mathbb{k}$-algebra, then $X^{(1)}$ is $\mathrm{Spec}\, \bar{A}$, but $\bar{A}$ is a $\mathbb{k}$-alg. w/ $\lambda \cdot a = \lambda^{1/p} a$, where $(-)^{1/p}$ is the inverse of $x \mapsto x^p$.

We have morphism of $\mathbb{k}$-schemes $Fr_X : X \to X^{(1)}$.

$G^{(1)} \supset B^{(1)} \supset T^{(1)}$

$Fr_G : G \to G^{(1)}$

Given $V \in \mathrm{Rep}(G^{(1)})$, we can consider $Fr_G^*(V) \in \mathrm{Rep}(G)$.

$Fr_T^* : X^*(T^{(1)}) \longrightarrow \mathbb{X}$ injective w/ image $p.\mathbb{X}$.

The classification of simples holds for $G^{(1)}$.

For $\lambda \in X^*(T^{(1)})^+$, we write $L^{(1)}(\lambda)$ the corresp. simple $G^{(1)}$-module.

Set $\mathbb{X}^+_{res} = \{\lambda \in \mathbb{X} : \forall \alpha \in R^s, \ 0 \le \langle \lambda, \alpha^\vee \rangle < p\}$

(Steinberg tensor product theorem)

Thm For any $\lambda \in \mathbb{X}^+_{res}$, and $\mu \in X^*(T^{(1)})^+$, we have

$$L(\lambda + Fr_T^*(\mu)) \simeq L(\lambda) \otimes Fr_G^*(L^{(1)}(\mu)).$$

Usually, fix isom. of $\Bbbk$-alg-gps $G^{(1)} \simeq G$, identifying $B^{(1)} \simeq B$, $T^{(1)} \simeq T$ so that

Fr_G^* identifies w/ mult. by p.

$\hookrightarrow$ Then it becomes $L(\lambda + p\mu) \simeq L(\lambda) \otimes Fr_G^*(L(\mu))$

Reps of G_1

$Fr : G \longrightarrow G^{(1)}$

Def. The Frobenius G_1 is the scheme theoretic kernel of Fr.

G_1 is a finite affine gp scheme over $\Bbbk$, so $\mathcal{O}(G_1)$ is a fd. Hopf alg over $\Bbbk$.

$\hookrightarrow$ has a concrete description.

Let $\mathfrak{g}$ be the Lie alg. of G, $x \longmapsto x^{[p]}$ nonlinear map from $\mathfrak{g} \to \mathfrak{g}$.

For Ug, elements of the form $X^P - X^{[P]}$ w/ $X \in g$ are central. They give a subalg.

Z_{Fr} canonically isom. to $\mathcal{O}((g^*)^{(1)})$.

The restricted univ. enveloping algebra $U_0 g$ of g is the quotient of Ug by the ideal gen. by elts of the form $X^P - X^{[P]}$ w/ $X \in g$

$U_0 g$ is a fin. dim. alg. of dim $p^{\dim G}$.

$$\mathcal{O}(G_1) \simeq (U_0 g)^*.$$

Let $\mathrm{Rep}(G_1)$ be the cat. of fin. dim'l G_1-modules.

Reps of G_1 $\longleftrightarrow$ comods over $\mathcal{O}(G_1)$ $\longleftrightarrow$ comods over $(U_0 g)^*$ $\longleftrightarrow$ mods over $U_0 g$.

$\mathrm{Soc}_{G_1}(M)$ (largest semisimple submod)

$\mathrm{top}_{G_1}(M)$ (largest semisimple quotient)

Each simple N admits an injective hull, i.e. the unique inj. I_N s.t. $\mathrm{soc}(I_N) \simeq N$

projective cover proj. P_N $\mathrm{top}(P_N) \simeq N$.

$G_1 \subset G$, $\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(G_1)$

$M \longmapsto M|_{G_1}$.

$Fr_G : G/G_1 \simeq G^{(1)}$, a G-module is of the form $Fr_G^*(V)$ for some $G^{(1)}$-mod V iff its restriction to G_1 is trivial.

How do we classify simple G_1-modules?

Weyl modules $\rightsquigarrow$ baby Verma modules

B^+ oppo. Borel subgroup to B (w.r.t. T)

$\mathfrak{b}^+$ its Lie alg.

For any $\lambda \in \mathbb{X}$, the 1-dim B^+-mod $\Bbbk_{B^+}(\lambda)$ $\rightsquigarrow$ 1-dim $U_0\mathfrak{b}^+$-module $\Bbbk_{\mathfrak{b}^+}(\lambda)$

which depends only on $\bar{\lambda} \in \mathbb{X}/p\mathbb{X}$

<u>Def</u> The <u>baby Verma module</u> assoc. to λ is $Z(\lambda) := U_0\mathfrak{g} \otimes_{U_0\mathfrak{b}^+} \Bbbk_{\mathfrak{b}^+}(\lambda)$.

<u>Theorem</u> (Jantzen) For any $\lambda \in \mathbb{X}$, the top $L_1(\lambda)$ of $Z(\lambda)$ is simple.

Moreover, $L_1(\lambda)$ only depends on $\bar{\lambda} \in \mathbb{X}/p\mathbb{X}$, and $\lambda \mapsto L_1(\lambda)$ is a bijection

between $\mathbb{X}/p\mathbb{X}$ and isom. classes of simple G_1-modules.

<u>Theorem</u> (Curtis) For any $\lambda \in \mathbb{X}^+_{\text{res}}$, the G_1-module $L(\lambda)|_{G_1} \simeq L_1(\lambda)$.

<u>Remark</u> The set of labels $\mathbb{X}/p\mathbb{X}$ has no partial order w/ any rep-theoretic meaning

eg. For any $w \in W$, it's known that $Z(\lambda)$ and $Z(w \cdot_p \lambda)$ have the same composition

factors. One way to fix this is to work w/ G_1T-modules.

For $\lambda \in \mathbb{X}$, write $Q(\lambda)$ for the injective hull of $L_1(\lambda)$.

$\rightsquigarrow$ a general result on finite group schemes implies that $\mathcal{O}(G_1) \simeq \mathcal{O}(G_1)^*$

$\rightsquigarrow$ G_1-mod is injective $\Leftrightarrow$ proj.

$\Rightarrow$ $Q(\lambda)$ is also the proj. cover of $L_1(\lambda)$.

<u>Rep G_1T</u>: Let G_1T be the subgp scheme gen. by G_1 and T

The datum of a G_1T-module str. on a $\Bbbk$-vector space V is equiv. to a $U_0\mathfrak{g}$-module str. w/ a T-mod structure ($\mathbb{X}$-grading) such that $U_0\mathfrak{t}$ acts on V_λ by the character $U_0\mathfrak{t} \to \Bbbk$ defined by diff'l of $\lambda \in \mathbb{X}$.

In particular, each G_1T-module has an action of T, so its "T-weights" makes sense.

We have restriction functors

$$\operatorname{Rep} G \longrightarrow \operatorname{Rep} G_1 T \quad , \quad \operatorname{Rep} G_1 T \longrightarrow \operatorname{Rep} G_1$$

forgetting $\mathbb{X}$-grading.

- $\operatorname{soc}_{G_1}(M|_{G_1}) \simeq \operatorname{soc}_{G_1T}(M)|_{G_1}$.

For any $\lambda \in \mathbb{X}$, the baby Verma $Z(\lambda)$ can be "lifted" to a G_1T-mod:

$$\hat{Z}(\lambda) = U_0\mathfrak{g} \underset{U_0\mathfrak{b}^+}{\otimes} \Bbbk_{\mathfrak{b}^+}(\lambda)$$

T acts here by λ.

Now $\hat{Z}(\lambda)$ really depends on λ.

In fact, for any $\lambda \in \mathbb{X}$ and $\mu \in X^*(T^{(1)})$, there's a canonical isom.

$$\hat{Z}(\lambda + \mathrm{Fr}_T^*(\mu)) \simeq \hat{Z}(\lambda) \otimes \Bbbk_{T^{(1)}}(\mu)$$

$$\uparrow$$

G_1T-mod. via $G_1T \to T^{(1)}$

Theorem. For any $\lambda \in \mathbb{X}$, the top $\hat{L}(\lambda)$ of $\hat{Z}(\lambda)$ is simple, and for any $\lambda \in \mathbb{X}$ and $\mu \in X^*(T^{(1)})$, $\hat{L}(\lambda + \mathrm{Fr}_T^*(\mu)) \simeq \hat{L}(\lambda) \otimes \Bbbk_{T^{(1)}}(\mu)$.

and $\lambda \mapsto \hat{L}(\lambda)$ is bij.

(2) $\lambda \in \mathbb{X} \rightsquigarrow \hat{L}(\lambda)|_{G_1} \simeq L_1(\lambda)$

(3) For any $\lambda \in \mathbb{X}^+_{res}$, $L(\lambda)|_{G_1T} \simeq \hat{L}(\lambda)$.

Cor. For any $\lambda \in \mathbb{X}^+_{res}$, we have $\hat{L}(\lambda)^* \simeq \hat{L}(-w_0(\lambda))$

Lemma. For any $\lambda \in \mathbb{X}$, $\hat{L}(2(p-1)\rho - \lambda)^*$ is a composition factor of $\hat{Z}(\lambda)$ w/ mult. 1.

Pf. By construction, one can check that $\hat{Z}(\lambda)$ admits $\lambda - 2(p-1)\rho$ as a minimal weight.

So it's a max'l weight of $\hat{Z}(\lambda)^*$ $\to \exists$ nonzero morphism of $(U_0\mathfrak{b}^+, T)$-modules

$\Bbbk_{\mathfrak{b}^+}(2(p-1)\rho - \lambda) \longrightarrow \hat{Z}(\lambda)^*$. Inducing to $U_0\mathfrak{g}$, we get

$\hat{Z}(2(p-1)\rho - \lambda) \longrightarrow \hat{Z}(\lambda)^*$ $\Rightarrow$ top of $\hat{Z}(2(p-1)\rho - \lambda)$ must appear as a composition factor of $\hat{Z}(\lambda)^*$ w/ mult. 1. Dualizing we get the claim.

$\forall\ \lambda \in \mathbb{X}$, write $\hat{Q}(\lambda)$ for the inj. hull of $\hat{L}(\lambda)$ in $\mathrm{Rep}(G_1T)$

- $\hat{Q}(\lambda)|_{G_1} \simeq Q(\lambda)$
- $\hat{Q}(\lambda + \mathrm{Fr}_T^*(\mu)) \simeq \hat{Q}(\lambda) \otimes \mathbb{k}_{T^{(1)}}(\mu)$

$\hat{Q}(\lambda)$ is the proj. cover of $\hat{L}(\lambda)$.

<u>Prop</u> (Humphreys) For every $\lambda \in \mathbb{X}$, the G_1T-mod $\hat{Q}(\lambda)$ admits a filt'n w/ subquotients of the form $\hat{Z}(\mu)$ w/ $\mu \in \mathbb{X}$. The multiplicity is equal to $[\hat{Z}(\mu) : \hat{L}(\lambda)]$

"BGG reciprocity"

<u>Lecture 9</u>.

<u>Prop</u>. For any $\lambda \in \mathbb{X}$, the G_1T-module $\hat{Q}(\lambda)$ admits a $\hat{Z}(\mu)$-filtration.

Moreover, # of occurences of some $\hat{Z}(\mu)$ in such a filt. is $[\hat{Z}(\mu) : \hat{L}(\lambda)]$

1. Sketch a completion of the argument for why G_1T-modules & $\hat{Q}(\lambda)$ help us connect simple $\longleftrightarrow$ tiltings.
2. Why BGG reciprocity? $\longrightarrow$ HW cats

G-module structure on injective hulls.

<u>Def</u>. We say a G-module V is p-bounded if for any weight μ of V and any dominant short root α, we have $\langle \mu, \alpha^\vee \rangle \leq \langle (2p-1)\rho, \alpha^\vee \rangle$. $\rightsquigarrow$ $\mathrm{Rep}_b(G) \subset \mathrm{Rep}(G)$. $\rightarrow$ Serre subcat. gen. by $L(\mu)$ for μ p-bdd

- $X_b^+ \subset X$ p-bounded dominant weights

- $\mathrm{Rep}_b(G)$ has a h.w. str., each block has fin. many simples.

<u>Def</u>. Let $R(\lambda)$ be the injective hull of $L(\lambda)$ in $\mathrm{Rep}_b(G)$

<u>Thm</u>. Assume $p \geq 2h-2$, For any $\lambda \in X^+_{res}$, we have an isom. of G_1-modules

$$R(\lambda)|_{G_1} \simeq Q(\lambda).$$

technical input

In particular, $Q(\lambda)$ admits a str. of a G-module.

<u>Cor</u>. Assume $p \geq 2h-2$. For any $\lambda \in X^+_{res}$, there's an isom. of G_1T-modules

$$R(\lambda)|_{G_1T} \simeq \hat{Q}(\lambda)$$

<u>Relation w/ tilting modules</u>

For any $\lambda \in X_b^+$, since $R(\lambda)$ is inj. in the HW cat. $\mathrm{Rep}_b(G)$, it admits a costd filt., and satisfies the reciprocity formula

$$(R(\lambda) : N(\mu)) = [M(\mu) : L(\lambda)] \quad \text{for any } \mu \in X_b^+.$$

$X_b^+ \subset X^+$ is stable under the operation $\mu \mapsto -w_0\mu$

$\rightsquigarrow$ $\mathrm{Rep}_b(G)$ is stable under duality $V \mapsto V^*$.

$\boxed{p \geq 2h-2}$

<u>Lemma</u> Let $M \in \mathrm{Rep}_b(G)$ and $\mu \in X^+_{res}$, and assume $M|_{G_1T} \simeq \hat{Q}(\mu)$

Then $M \simeq R(\mu)$ as G-modules

Proof. Suppose $M|_{G_1T} \simeq \hat{Q}(\mu)$

Then $M|_{G_1} \simeq Q(\mu) \Rightarrow soc_{G_1}(M)$ is simple.

Since M is injective as a G_1-module, the embedding $L(\mu) \hookrightarrow R(\mu)$

induces a surj. $\quad \mathrm{Hom}_{G_1}(R(\mu), M) \longrightarrow \mathrm{Hom}_{G_1}(L(\mu), M)$

$\rightarrow \mathrm{Hom}_G(R(\mu), M) \twoheadrightarrow \mathrm{Hom}_G(L(\mu), M)$

since the socle of M as a G_1T-mod is $\hat{L}(\mu) = L(\mu)|_{G_1T}$

$\rightarrow$ there is a unique simple sub G-module, isom. to $L(\mu)$

The embedding factors through

$$\begin{tikzcd} L(\mu) \arrow[r, hook] \arrow[dr] & M \\ & R(\mu) \arrow[u, dashed] \end{tikzcd}$$

$\leftarrow$ since this morphism is inj. on the unique simple submod of $R(\mu)$, it must be injective

And since we understand $M|_{G_1T}$, we know $\dim M = \dim R(\mu)$

$\Rightarrow$ this injection is an isom.

Corollary For $\lambda \in X^+_{res}$, $R(\lambda)^* \simeq R(-w_0\lambda)$.

Prop For any $\lambda \in X^+_{res}$, the G-module $R(\lambda)$ is tilting and is isom. to

$T(2(p-1)\rho + w_0\lambda)$

Proof Fix $\lambda \in X^+_{res}$, $R(\lambda)$ has a costd filt, but so does $R(\lambda)^*$. so $R(\lambda)$ must also have a std filt. What's its h.w.?

All the weights μ of $R(\lambda)$ satisfy $\mu \leq 2(p-1)\rho + w_0\lambda$,

Further, $R(\lambda)|_{G_1T} \simeq \hat{Q}(\lambda)$

The baby Verma $\hat{Z}(2(p-1)\rho + w_0\lambda)$ admits $\underset{\substack{\text{is} \\ \hat{L}(\lambda)}}{\hat{L}(-w_0\lambda)^*}$ as a comp. factor.

By reciprocity, we deduce that

$\hat{Z}(2(p-1)\rho + w_0\lambda)$ appears as subqt of $\hat{Q}(\lambda) \Rightarrow 2(p-1)\rho + w_0\lambda$ is a T-wt of $\hat{Q}(\lambda)$. $\Rightarrow 2(p-1)\rho + w_0\lambda$ is a T-wt of $R(\lambda)$, and its the h.w. □

Now we know $T(2(p-1)\rho + w_0\lambda)|_{G_1T} \simeq \hat{Q}(\lambda)$

<u>Theorem</u> (Donkin's tensor product theorem) For any $\lambda \in X^+_{res}$ and any $\mu \in X^*(T^{(1)})$ dom., the G-module

$$T((p-1)\rho + \lambda) \otimes Fr_G^*(T^{(1)}(\mu)) \quad \text{is tilting of h.w.}$$

$(p-1)\rho + \lambda + Fr_T^*(\mu)$.

If $T((p-1)\rho+\lambda)$ is indecomposable as a G_1-module, then

$$T((p-1)\rho + \lambda) \otimes Fr_G^*(T^{(1)}(\mu)) \simeq T((p-1)\rho + \lambda + Fr_T^*(\mu)).$$

<u>How do we actually compute simple chars from tilting ones?</u>

Assume we know the characters $ch(T(2(p-1)\rho + w_0\lambda))$ for any $\lambda \in X^+_{res}$. Then we know the characters of $\hat{Q}(\lambda)$ for any $\lambda \in X^+_{res}$, hence for any $\lambda \in X$.

Since characters of baby Vermas are easy (Exercise 4.8 in the text), our problem is really understanding $\left(\left[\hat{Z}(\lambda) : \hat{L}(\mu)\right], \lambda, \mu \in X\right)$.

Now we claim if we know all $\left[\hat{Z}(\lambda) : \hat{L}(\mu)\right]$, then we know chars of all $\hat{L}(\lambda)$.

We can reduce to understanding chars of $\hat{L}(\lambda)$ for $\lambda \in X^{+}_{res}$.

We want $\dim \hat{L}(\lambda)_\mu$, $\mu \in X$.

For any $\nu \in X$, weights of $\hat{Z}(\nu)$ are $\leq \nu$. So $\left[\hat{Z}(\nu) : \hat{L}(\eta)\right]$ vanishes unless $\eta \leq \nu$ (and $=1$ if $\eta = \nu$)

So we get some expr.

$$\mathrm{ch}(\hat{L}(\lambda)) = \sum_{\nu \in X_\lambda} m_\nu \cdot \mathrm{ch}(\hat{Z}(\nu)) + \sum_{\nu \in Y_\lambda} m'_\nu \cdot \mathrm{ch}(L(\nu))$$

there is no dom. μ s.t. $\mu \leq \nu$ for $\nu \in Y_\lambda$.

$$\Rightarrow \dim\left(\hat{L}(\lambda)_\mu\right) = \sum_{\nu \in X_\lambda} m_\nu \dim\left(\hat{Z}(\nu)_\mu\right).$$

HW cat. essentials.

$\mathbb{k}$ field, $\mathcal{A}$ finite length $\mathbb{k}$-linear abelian cat. s.t. $\mathrm{Hom}_{\mathcal{A}}(M, N)$ f.d.

Let $\mathcal{S}$ be the set of isom. classes of irr obj. in $\mathcal{A}$. Assume $(\mathcal{S}, \leq)$ partial.

Call L_s irred. corresp. to $s \in \mathcal{S}$.

Assume $\forall s$ we have objects Δ_s, ∇_s s.t. $\Delta_s \to L_s$, $L_s \to \nabla_s$

For any $J \subset \mathcal{S}$, write $\mathcal{A}_J$ for Serre subcat. gen. by $\{L_t\}_{t \in J}$.

$\mathcal{A}_{\leq s}$. $\mathcal{A}_{<s}$.

Def. $\mathcal{A}$ w/ this data is a h.w. cat. if

(1) $\forall s \in \mathcal{S}$, $\{t \in \mathcal{S} : t \leq s\}$ is finite.

(2) $\forall s \in \mathcal{S}$, $\mathrm{Hom}(L_s, L_s) = \mathbb{k}$

(3) $\forall s \in \mathcal{S}$ and any ideal $J \subset \mathcal{S}$ s.t. $s \in J$ is max

$\Delta_s \to L_s$ is a proj. cover, $L_s \to \nabla_s$ inj. envelope in $\mathcal{A}_J$.

(4) The kernel of $\Delta_s \to L_s$ and coker of $L_s \to \nabla_s$ lie in $\mathcal{A}_{<s}$.

(5) $\mathrm{Ext}^2(\Delta_s, \nabla_t) = 0$, $\forall s, t \in \mathcal{S}$.

We call $(\mathcal{S}, \leq)$ the weight poset $\Rightarrow \Delta_s, \nabla_s \in \mathcal{A}_{\leq s}$, and $[\Delta_s : L_s] = [\nabla_s : L_s] = 1$.

Thm Assume $\mathcal{S}$ is finite, Then $\mathcal{A}$ has enough proj. and any proj. admits a Δ-filt'n

Further, if P_s is the proj. cover of L_s, we have $(P_s : \Delta_t) = [\nabla_t : L_s]$.

Proof idea. Both sides are equal to $\dim \mathrm{Hom}(P_s, \nabla_t)$. $=$ RHS

$$\begin{gathered} \dim \mathrm{Hom}(P_s, \nabla_t) \\ \parallel \\ \sum_i \dim \mathrm{Hom}(\Delta_i, \nabla_t) \\ \parallel \\ \#\text{ of } \nabla_t. \end{gathered}$$

Cor. $\mathrm{Ext}^i(\Delta_s, \nabla_t) = \begin{cases} \mathbb{k} & \text{if } s = t,\ i = 0 \\ 0, & \text{o/w} \end{cases}$

Prop. If $\mathcal{A}$ is a h.w. cat.

TFAE (1) M admits a ∇-filt

(2) $\mathrm{Ext}^i(\Delta_s, M) = 0$, $i \in \mathbb{Z}_{>0}$, $\forall s$

(3) $\mathrm{Ext}^1(\Delta_s, M) = 0$, $\forall s$.

Lecture 10.

- Sketch geometric proof for character formula for tilting modules
- "Torsion explosion": geometric reason for failure of Lusztig's original conjecture.

Tilting character formula & antispherical p- Kazhdan Lusztig polynomials

${}^f W_{aff} \subset W_{aff}$ elts in W_{aff} minimal in their right coset relative to $W \subset W_{aff}$.

For $y, w \in {}^f W_{aff}$, $\quad {}^p n_{y,w} = \sum_{z \in W} (-1)^{\ell(z)} \, {}^p h_{zy, w}$

"antispherical p- Kazhdan-Lusztig polynomials"

Assume $p \geq h$. $\quad \lambda \in C \cap X$

Conjecture. For any $y, w \in {}^f W_{aff}$, $\quad (T(w \cdot_p \lambda) : N(y \cdot_p \lambda)) = {}^p n_{y,w}(1)$
$\lambda = 0$

The Finkelberg - Mirković conjecture

$\mathrm{Sat} : (\mathrm{Perv}_{L^+G}(Gr; \Bbbk), *) \xrightarrow{\sim} (\mathrm{Rep}(G^\vee_{\Bbbk}), \otimes)$

$\Bbbk$ alg closed field of char p

$G^\vee_{\Bbbk}$ split conn'd reductive gp over $\Bbbk$ w/ max. torus $T^\vee_{\Bbbk}$ whose lattice of characters is $X_*(T)$, w/ root datum of $(G^\vee_{\Bbbk}, T^\vee_{\Bbbk})$ dual to that of (G,T).

So now let's suppose we choose data s.t.

$$\mathbb{G}^{(1)} = G^\vee_{\Bbbk}, \quad \mathbb{B}^{(1)} = B^\vee_{\Bbbk}, \quad \mathbb{T}^{(1)} = T^\vee_{\Bbbk}$$

$\mathbb{X} = X^*(\mathbb{T}) \simeq X^*(\mathbb{T}^{(1)}) \simeq X_*(T)$ s.t. pullback under $\mathbb{T} \to \mathbb{T}^{(1)}$ corresp. to $\lambda \mapsto p\lambda$.

$$Gr' := L^+G \backslash LG \circlearrowleft LG$$

$$Gr' \simeq Gr, \quad g \mapsto g^{-1}$$

Let I_u = prounipotent radical of I.

$\mathrm{Perv}_{I_u}(Gr'; \Bbbk)$ ⟵ h.w. cat. w/ weight poset W_{ext}.

⟵ Δ_w, ∇_w, IC_w, $w \in W_{ext}$
std, costd, simple

$$D^b_{L^+G}(Gr; \Bbbk) \times D^b_{I_u}(Gr'; \Bbbk) \to D^b_{I_u}(Gr'; \Bbbk)$$

This is t-exact, so it gives an action of $(\mathrm{Perv}_{L^+G}(Gr; \Bbbk), *)$ on $\mathrm{Perv}_{I_u}(Gr'; \Bbbk)$.

(Gaitsgory nearby cycles)

<u>Conjecture (Finkelberg–Mirković)</u> Assume $p \geq h$. There is an equiv. of cats

$$FM : \mathrm{Perv}_{I_u}(Gr'; \Bbbk) \xrightarrow{\sim} \mathrm{Rep}_0(\mathbb{G}).$$

which satisfies :

- for any $w \in {}^f W_{aff}$, we have

$$FM(IC_w) \simeq L(w \cdot_p \lambda)$$
$$FM(\Delta_w) \simeq M(w \cdot_p \lambda)$$
$$FM(\nabla_w) \simeq N(w \cdot_p \lambda)$$

- For $\mathcal{F} \in Perv_{I_u}(Gr'; \mathbb{k})$ and $\mathcal{G} \in Perv_{L^+G}(Gr; \mathbb{k})$

$$FM(\mathcal{G} * \mathcal{F}) \simeq FM(\mathcal{F}) \otimes Fr^*(Sat(\mathcal{G}))$$

<u>Impact on characters</u>:

Suppose Conj. is true,

$$K_0(Perv_{I_u}(Gr'; \mathbb{k})) \xrightarrow{\sim} K_0(Rep_0(G))$$

in here

$$[\mathcal{F}] = \sum_{y \in {}^f W_{aff}} (-1)^{\ell(y)} \chi_y(\mathcal{F}) [\Delta_y]$$

Euler char. of stalk of $\mathcal{F}$ at y.

FM isom. implies

$$[L(w \cdot_p \lambda)] = \sum_{y \in {}^f W_{aff}} (-1)^{\ell(y)} \chi_y(IC_w) \cdot [M(y \cdot_p \lambda)]$$

For any fixed w, if $p \gg 0$, the dimensions of the stalks of IC_w at y are given by coeffs of $h_{w_0 y, w_0 w}$ which implies $\chi_y(IC_w) = (-1)^{\ell(w)} h_{w_0 y, w_0 w}(1)$

$\Rightarrow$ Lusztig's conj. for large p.

Singular version

Let $\mu \in X \cap \bar{C}$. There is an equiv. of cats

$$FM_\mu : Perv_{(I_u^A, \chi_A)}(Gr'; \Bbbk) \xrightarrow{\sim} Rep_\mu(G)$$

satisfying similar properties.

Here $A \subset S_{aff}$ is the subset of elts fixing μ.

I_u^A, some local system χ_A. $\leftarrow$ what if $\mu = -\rho$, $A = S_{aff}$.

The Iwahori-Whittaker model of the Satake cat.

$$FM_{-\rho} : Perv_{(I_u^S, \chi_S)}(Gr'; \Bbbk) \xrightarrow{\sim} Rep_{-\mu}(G)$$

Let $\psi_I : I_u \to G_a$ be given by

$$\psi_I(g g^{\leq 0}) = \Psi(g) \quad \text{for } g \in I_u \cap N,\ g^{\leq 0} \in I_u \cap B^-.$$

$$\Psi(n) = \operatorname{Res}\left(\frac{dt}{t} \sum_{\substack{\alpha \\ \text{simple roots}}} u_\alpha(n)\right)$$

Consider (I_u, ψ)-equiv. objects in $D^b(Gr'; \Bbbk)$

There is an equiv

$$Perv_{L^+G}(Gr; \Bbbk) \xrightarrow{\sim} Perv_{(I_u, \psi)}(Gr'; \Bbbk)$$

"spherical - antispherical isomorphism"

Proof of the tilting character formula via Koszul duality

Idea: Rephrase the question in terms of functors

$$\begin{tikzcd} & D^b \mathrm{Coh}^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[dl, "F"'] \arrow[dr, "\sim"] & \\ D^b \mathrm{Rep}_0 G & & D^{\mathrm{mix}}_{(I_u, \psi)}(\mathcal{F}\ell; \Bbbk) \end{tikzcd}$$

$\leftarrow$ coherent model for antispherical category

$\uparrow$ AB equivalence

The variety $\tilde{\mathcal{N}}$ is the Springer resol'n of $G^\vee_{\Bbbk} = G^{(1)}$, i.e. cotangent bundle of $G^\vee_{\Bbbk} / B^\vee_{\Bbbk}$.

$G^\vee_{\Bbbk}$- action, $\mathbb{G}_m$- action by dilation along the cotangent direction, $z \in \Bbbk^\times$ acts by mult. by z^{-2} on each cotangent fiber.

There's an auto-equivalence given by "shift"

$$\langle 1 \rangle : D^b \mathrm{Coh}^{G^\vee_{\Bbbk}, \mathbb{G}_m}(\tilde{\mathcal{N}}) \circlearrowleft$$

The functor F is not an equiv, but it's "close":

There's a canonical isom. $F \circ \langle 1 \rangle [1] \simeq F$ s.t.

- $\forall\, \mathcal{F}, \mathcal{G}$ in $D^b \mathrm{Coh}^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}})$, F induces isom.

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}\langle n \rangle [n]) \xrightarrow{\sim} \mathrm{Hom}(F(\mathcal{F}), F(\mathcal{G}))$$

- the essential image of F generates $D^b \mathrm{Rep}_0 G$ as a triangulated cat.

- For $\mathcal{F} \in D^b \mathrm{Coh}^{G^\vee_{\Bbbk} \times \mathbb{G}_m}(\tilde{\mathcal{N}})$, $V \in \mathrm{Rep}(G^\vee_{\Bbbk})$,

$$F(\mathcal{F} \otimes V) \simeq F(\mathcal{F}) \otimes \mathrm{Fr}_G^*(V).$$

Other ingredients

$D^{mix}_{Iu}(Gr', \Bbbk)$ "mixed derived cat. of Iu-equiv. sheaves on Gr'".

has a perverse t-str. and heart $\mathrm{Perv}^{mix}_{Iu}(Gr'; \Bbbk)$, and a Tate twist autoequiv. $\langle 1 \rangle$.

There's an equiv. of triangulated cats

$$\Phi : D^b \mathrm{Coh}^{G^\vee_{\Bbbk} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \xrightarrow{\sim} D^{mix}_{Iu}(Gr'; \Bbbk)$$ "spherical coherent - constructible correspondence".

$$\Phi \circ \langle 1 \rangle \simeq \langle 1 \rangle [-1] \circ \Phi.$$

There's a bifunctorial isom.

$$\Phi(\mathcal{F} \otimes \mathrm{Sat}(\mathcal{G})) \simeq \Phi(\mathcal{F}) * \mathcal{G}.$$

Combining these functors, we'll get a map

$D^{mix}_{Iu}(Gr'; \Bbbk) \longrightarrow D^b \mathrm{Rep}_0(G)$ which is "de-grading" wrt. $\langle 1 \rangle$.

It sends std objects to Weyl modules,

costd objects to coWeyl modules.

Lecture 11. § 3.2 of Riche's book

Achar - Riche "Reductive gps, the loop grassmannian, & the Springer resol'n".

$$\begin{tikzcd}
 & D^b \mathrm{Coh}^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[dl, "F"'] \arrow[dr, "\Phi"] & \\
D^b \mathrm{Rep}_0(G) & & D^{mix}_{I^u}(\mathrm{Gr}'; \Bbbk)
\end{tikzcd}$$

$$F \circ \langle 1 \rangle [1] \simeq F$$

$$\Phi \circ \langle 1 \rangle \simeq \langle 1 \rangle [-1] \circ \Phi$$

<u>Thm (Achar - Makisumi - Riche - Williamson)</u>

$$D^{mix}_{I_u}(\mathrm{Gr}'; \Bbbk) \xrightarrow[\sim]{\Psi} D^{mix}_{(I_u, \psi)}(\mathcal{F}\ell; \Bbbk)$$

$$\Delta_w \longleftrightarrow \Delta_w$$

$$\nabla_w \longleftrightarrow \nabla_w$$

$$\Psi \circ \langle 1 \rangle = \langle 1 \rangle [1] \circ \Psi$$

indecomp. tilting perverse sheaves $\longrightarrow$ indecomp. parity complexes

$\to$ mult. of standard objects in indecomp. tilting objects $D^{mix}_{I_u}(\mathrm{Gr}'; \Bbbk)$

$\rightsquigarrow$ dimensions of stalks of parity complexes in $D^b_{(I_u, \psi)}(\mathcal{F}\ell; \Bbbk)$

known to be given by p-Kazhdan-Lusztig polynomials.

$$D^{mix}_{I_u}(\mathrm{Gr}; \Bbbk) \xrightarrow{\sim} D^b \mathrm{Coh}^{\check{G}^{(1)} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \underbrace{\xrightarrow[F]{} D^b_{Stein}(\mathbb{B}) \xrightarrow[\sim]{R\mathrm{Ind}_{\mathbb{B}}^{G}} D^b \mathrm{Rep}_0 G}_{F}$$

$D^b_{Stein}(\mathbb{B})$ = derived cat. of complexes of $\mathbb{B}$-modules whose cohomology is trivial on $\mathbb{B}_1 \subset \mathbb{B}$.

1. The "Formality Theorem" gives that $D^b Coh^{G^{(1)} \times G_m}(\tilde{\mathcal{N}})$ is a graded version of $D^b_{Stein}(\mathbb{B})$.

2. "Induction Theorem" says that $R\mathrm{Ind}_{\mathbb{B}}^G$ is an equiv. of cats.

1. F is a degrading functor wrt. $\langle 1 \rangle [1]$ s.t. for any $V \in Rep(G^{(1)})$

$$F(\mathcal{F} \otimes V) \simeq F(\mathcal{F}) \otimes Fr^*(V)$$

2. $R\mathrm{Ind}_{\mathbb{B}}^G$ is an equiv. and for any $V \in Rep(G^{(1)})$,

$$R\mathrm{Ind}_{\mathbb{B}}^G(M \otimes Fr^*(V)) \simeq R\mathrm{Ind}_{\mathbb{B}}^G(M) \otimes Fr^*(V)$$

<u>Alternative proof via "Smith-Treumann theory"</u>

Geometric Satake + Koszul duality gives an equiv.

$$Rep(G_k^\vee) \simeq Perv_{(I_u, \psi)}(Gr'; \mathbb{k})$$

"Smith-Treumann theory" gives a localization functor relating sheaves on Gr' to sheaves on the fixed points under the group of pth roots of unity in $\mathbb{F}$ (via loop rotation)

⟶ These fixed pts identify with a disjoint union of partial flag varieties, for some "p-dilated" loop group of G.

⟶ This localization functor is fully faithful on tilting modules

→ This allows us to compute dimensions of morphisms between tiltings

→ Exercise 7.10 (Riche) ⇝ if we understand these dimensions, we understand multiplicities of stds/costds in tiltings.

<u>Geometric example</u>

Let's suppose G is defined over $\mathbb{Z}$, w/ $\mathrm{char}(\mathbb{k}) = p$

<u>Achar - Riche, Fiebig</u>: Lusztig's conj. for $G_{\mathbb{k}}$ is equiv. to the absence of p-torsion in the stalks and costalk of $IC(\overline{Gr_x} ; \mathbb{Z})$ for any $x \in {}^f W_{aff}$ satisfying "Jantzen's condition".

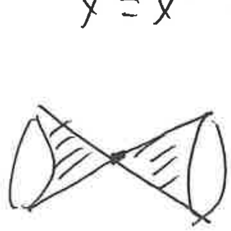
How does this torsion show up?

$$X = \left\{ x = \begin{pmatrix} c & -a \\ b & -c \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \ : \ x \text{ nilpotent} \right\}$$

$$= \mathrm{Spec}\ \mathbb{C}[a,b,c]/(ab-c^2)$$

$$= \mathrm{Spec}\ \mathbb{C}[x^2, xy, y^2]$$

$$X = X^{reg} \sqcup \{0\}$$



Suppose $\mathcal{F}$ is a perverse sheaf on X w.r.t. the given stratification.

Where can $H^i(\mathcal{F}|_{X'})$ be nonzero ?
$\in \{X^{reg}, 0\}$

	-3	-2	-1	0	1
X_{reg}	0	$*$	0	0	0
$\{0\}$	0	$*$	$*$	$*$	0

To compute $\quad IC(X;\Bbbk) = IC(\overline{X_{reg}};\Bbbk)$

$$IC(X;\Bbbk) = \tau_{<0}\, j_* \underline{\Bbbk}_{X^{reg}}[2] \qquad j : X^{reg} \to X$$

First compute $\left(j_* \underline{\Bbbk}_{X^{reg}}[2]\right)_0 = \lim_{\varepsilon \to 0} H^{*+2}(B(0,\varepsilon) \cap X^{reg};\Bbbk)$

$B(0,\varepsilon) \cap X^{reg}$ is homotopic to $S^3_\varepsilon \cap X^{reg} \simeq S^3/(\pm 1) \simeq \mathbb{RP}^3$

$H^*(\mathbb{RP}^3;\mathbb{Z}) =$

0	1	2	3
$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}$

$H^*(\mathbb{RP}^3;\Bbbk) =$

0	1	2	3
$\Bbbk$	$(\Bbbk)_2$	$(\Bbbk)_2$	$\Bbbk$

$$(\Bbbk)_2 = \begin{cases} \Bbbk, & 2=0 \\ 0, & \text{else} \end{cases}$$

Stalks of $j_* \underline{\Bbbk}_{X_{reg}}[2]$

	-2	-1	0	1	2
X^{reg}	$\Bbbk$	0	0	0	0
$\{0\}$	$\Bbbk$	$(\Bbbk)_2$	$(\Bbbk)_2$	$\Bbbk$	0

$\xrightarrow{\tau_{<0}}$

	-2	-1	0	1	2
X^{reg}	$\Bbbk$	0	0	0	0
$\{0\}$	$\Bbbk$	$(\Bbbk)_2$	0	0	0

Stalks of $IC(X;\Bbbk)$

Similarly, we have

$IC(X;\mathbb{Z}) = \underline{\mathbb{Z}}[2]$ stalks of $IC(X;\mathbb{Z})$

	-2	-1	0
X^{reg}	$\mathbb{Z}$	0	0
$\{0\}$	$\mathbb{Z}$	0	0

stalks of $\mathbb{D}(IC(X;\mathbb{Z}))$

	-2	-1	0	1
X^{reg}	$\mathbb{Z}$			
$\{0\}$	$\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	

$IC(X,\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathbb{k}$ is simple if char $(\mathbb{k})$ odd.

o/w it has composition factor $IC(X,\mathbb{k})$ and $IC(0;\mathbb{k})$.

$G_{\mathbb{Z}}$ we could study

1. reps of $G_{\mathbb{C}}$ over $\mathbb{C}$

2. reps of $G_{\mathbb{k}}$ over $\overline{\mathbb{k}}$ (char $\mathbb{k} > 0$) "modular rep. theory"

3. reps of $G(\mathbb{F}_q)$ over $\mathbb{C}$
   - ↳ Naively, it's just a finite gp!
   - ↳ Deligne - Lusztig in 70s.

$V \longrightarrow \overline{V}$

Brauer - Nesbitt 1940s "modular reduction" $\rho \longmapsto \overline{\rho}$

4. reps of $G(\mathbb{F}_q)$ over $\overline{\mathbb{F}_q}$.
   - ↳ Jantzen

Lusztig in 2021 (building off much older work)

conjectured that if you start w/ a nice rep. $\rho \in (3)$, then

$$\overline{\rho} = \sum_{\lambda \in ?} c_\lambda \overline{V_\lambda} \quad \text{for } V_\lambda \text{ a Weyl module w/ h.w. } \lambda$$

$$= \sum_{\lambda \in ?} d_\lambda \overline{L_\lambda}$$

<u>Lecture 12</u>. Reps of $G(\mathbb{F}_p)$ over $\mathbb{C}$ $\rightsquigarrow$ over $\overline{\mathbb{F}_p}$

Unipotent irreducible reps of $G(\mathbb{F}_p)$ in type A

This week: $G = GL_n$.

<u>Recall</u>: We are going to explain a conj. (by G. Lusztig) about how to write

$$\overline{\rho} = \sum_\lambda c_{\rho,\lambda} \overline{V_\lambda} \quad \text{for } \rho \text{ a } \underline{\text{unip. irr. rep}} \text{ of } G(\mathbb{F}_p).$$

in type A is easy to describe

Deligne - Lusztig theory is needed outside of type A.

<u>Let $G = SL_n$ or GL_n</u>

<u>Thm / Def</u>. A unipotent irred. rep. is an irred. rep. of $G(\mathbb{F}_p)$ s.t. $V^{B(\mathbb{F}_q)} \neq 0$.

R_w^θ, $w \in W$, θ char. of T_w, <u>unipotent</u>: $\langle \rho, R_w^1 \rangle \neq 0$ for some w

<u>unipotent principal series</u>: $\langle \rho, R_1^1 \rangle \neq 0$.

$G = GL_n(\mathbb{F}_p)$, $B = B(\mathbb{F}_p)$ finite gps.

$$V^B \neq 0 \iff \mathrm{Hom}_B(V, \mathbb{C}) \neq 0.$$

$$\begin{aligned} &\qquad \shortparallel \\ &(V^*)^B = (V^B)^* \\ &= \mathrm{Hom}_G(V, \mathbb{C}[B\backslash G]) \end{aligned}$$

$$V \longmapsto \mathrm{Hom}_B(V, \mathbb{C}) \simeq \mathrm{Hom}_G(V, \mathbb{C}[B\backslash G])$$

gives a bijection between the set of $V \in \mathrm{Irr}(G)$ in $\mathbb{C}[B\backslash G]$ and $\mathrm{Irr}(\mathrm{End}_G(\mathbb{C}[B\backslash G]))$

Now let $\mathbb{C}[B\backslash G]^B$ be the B-invariants.

Define a convolution product

$$\mathbb{C}[B\backslash G]^B \otimes \mathbb{C}[B\backslash G] \longrightarrow \mathbb{C}[B\backslash G]$$

$$(F * f)(g) = \frac{1}{|B|} \sum_{h \in G} F(h) f(h^{-1}g)$$

$F * - : \mathbb{C}[B\backslash G] \longrightarrow \mathbb{C}[B\backslash G]$ is G-equivariant.

so $\mathbb{C}[B\backslash G/B] := \mathbb{C}[B\backslash G]^B$ is an assoc. alg. w.r.t. convolution, acting on $\mathbb{C}[B\backslash G]$ by G-equiv. endomorphisms. This gives a map

$$\mathbb{C}[B\backslash G/B] \longrightarrow \mathrm{End}_G(\mathbb{C}[B\backslash G]).$$

<u>Lemma</u> This map is an isom.

Pf. As vec. sp., $\mathrm{End}_G(\mathbb{C}[B\backslash G]) \simeq \mathrm{Hom}_B(\mathbb{C}, \mathbb{C}[B\backslash G]) \simeq \mathbb{C}[B\backslash G]^B$,

so injectivity $\Rightarrow$ isomorphism.

$\uparrow$ apply $F * -$ to characteristic functions on B-orbits

$\Rightarrow F(g) = 0$ (using B-equiv. of F)

$G = \coprod_{w\in W} BwB$, $\tilde{T}_w = \chi_{BwB}$.

Prop $\tilde{T}_s \tilde{T}_w = \tilde{T}_{sw}$ if $\ell(sw) = \ell(w) + 1$

$s\in S$ $\tilde{T}_s\tilde{T}_w = p\tilde{T}_{sw} + (p-1)\tilde{T}_w$ if $\ell(sw) = \ell(w) - 1$

$w\in W$

$\Rightarrow \mathbb{C}[B\backslash G/B] \simeq \mathcal{H}_p$.

We know $\mathcal{H}_v$ "deforms" $\mathbb{C}[W]$, $\mathcal{H}_1 \simeq \mathbb{C}[W]$

$\Rightarrow$ Irreps of $\mathcal{H}_p$ are in bij. w/ irreps of $\mathbb{C}[W]$.

$E \mapsto E(v)$

Kazhdan - Lusztig cells

Combinatorial theory which will allow us to understand how irreps "live" in $\mathcal{H}_v \curvearrowright \mathcal{H}_v$.

Let $A = \mathbb{C}[v, v^{-1}]$, $\mathcal{H}_v$ is an A-algebra. Let T_w be such that

$$(T_s - v)(T_s + v^{-1}) = 0, \ \forall s\in S$$

Let $a \mapsto \bar{a} : A \to A$ be the $\mathbb{C}$-alg. involution defined by $v \mapsto v^{-1}$.

There is an $(A, \bar{\ })$- semilinear ring hom. $x \mapsto \bar{x} : \mathcal{H}_v \to \mathcal{H}_v$

defined by $T_s \mapsto T_s^{-1}$, $T_w \mapsto T_{w^{-1}}^{-1}$.

<u>Def</u> For $w, y \in W$, define $r_{w,y} \in A$ by

$$\overline{T_w} = \sum_{y \in W} \overline{r_{y,w}} \, T_y$$

Note that $r_{w,w} = 1$.

For $n \in \mathbb{Z}$, define $A_{\le n} = \bigoplus_{m \le n} \mathbb{C} v^m$, and $A_{\ge n}$, $A_{<n}$, $A_{>n}$ idem.

Define $\mathcal{H}_{\le 0} = \bigoplus_{w} A_{\le 0} T_w$, $\mathcal{H}_{<0}$ idem.

<u>Theorem.</u> Let $w \in W$, there exists a unique element $C_w \in \mathcal{H}_{\le 0}$ s.t.

$\overline{C_w} = C_w$ and $C_w \equiv T_w \mod \mathcal{H}_{<0}$.

Additionally, $C_w \in T_w + \sum_{y < w} A_{<0} T_y$, and $\{C_w\}_{w \in W}$ is an A-basis for $\mathcal{H} = \mathcal{H}_v$.

<u>Lemma.</u> 1. For any $x, z \in W$, $\sum_{y \in W} \overline{r_{x,y}} \, r_{y,z} = \delta_{x,z}$

2. For any $x, y \in W$, let $s \in S$ be s.t. $y > sy$, then

$$r_{x,y} = \begin{cases} r_{sx,sy} & \text{if } sx < x \\ r_{sx,sy} + (v - v^{-1}) \, r_{x,sy} & , \ sx > x \end{cases}$$

3. If $r_{x,y} \neq 0$, then $x \leq y$.

Pf. 1. follows from $\bar{\ }$ being an involution.

2. Follows from the formula for $T_s T_w$, using $\bar{\ }$ is multiplicative.

3. Induction on length of y.

Existence & uniqueness of KL basis

Fix $w \in W$. For any $x \leq w$, we'll construct $u_x \in A_{\leq 0}$ s.t.

1. $u_w = 1$
2. For $x < w$, $u_x \in A_{<0}$, and $\overline{u_x} - u_x = \sum_{x<y\leq w} r_{x,y} u_y$.

Proof. Induct on $l(w) - l(x)$. for $l(x) = l(w)$, $u_x = u_w = 1$.

Now, assume u_y defined for all $y \leq w$ s.t. $l(y) > l(x)$ & above properties

then $a_x = \sum_{x<y\leq w} r_{x,y} u_y$ is defined.

By Lemma from before, one can show $a_x + \overline{a_x} = 0$

$\Rightarrow$ $a_x = \sum_{n\in\mathbb{Z}} c_n v^n$ for $c_n + c_{-n} = 0$

Define $u_x = -\sum_{n<0} c_n v^n$, then u_x satisfies the properties 1 & 2.

Then we define $C_w := \sum_{y\leq w} u_y T_y \in \mathcal{H}_{\leq 0}$.

$$\overline{C_w} = \sum_{y\le w} \overline{u_y}\, \overline{T_y} = \sum_{y \le w} \overline{u_y} \sum_{x \le y} \overline{r_{x,y}}\; T_x$$

$$= \sum_{x \le w} \Big(\sum_{x \le y \le w} \overline{r_{x,y}}\; \overline{u_y} \Big) T_x$$

$$= \sum_{x \le w} (\overline{a_x} + u_x) T_x = \sum_{x \le w} u_x T_x = C_w$$

<u>Uniqueness</u>.

<u>Claim</u>. If $h \in \mathcal{H}_{<0}$ satisfies $\overline{h} = h$, then $h = 0$.

<u>Pf</u>. Since $h \in \mathcal{H}_{<0}$, write $\sum f_y T_y$, $f_y \in A_{<0}$.

Suppose f_y not all zero, then choose $f_{y_0} \neq 0$, y_0 max'l.

Then $\sum_y f_y T_y = \sum_y \overline{f_y}\, \overline{r_{x,y}}\, T_x$

Since $r_{y_0,y_0} = 1$, $r_{y_0,y} = 0$, $\forall\, y < y_0$, coeff of T_{y_0} in LHS is f_{y_0}

and on RHS is $\overline{f_{y_0}}$, which can't be equal. Contradiction. $\square$

<u>Cells & cell representations</u>.

<u>Def</u>. Let $\mathcal{A}$ be an associative alg. w/ basis $\{a_w\}_{w \in W}$ indexed by a Coxeter gp W.

We say an ideal in $\mathcal{A}$ is <u>based</u> if it's spanned by basis elements.

For $x \in W$, we define $I_{x,L}$, $I_{x,R}$, $I_{x,LR}$ the left, right, & two-sided based ideal generated by a_x.

Define the pre orders $\leq_L, \leq_R, \leq_{LR}$ as

$x \leq_L y$ iff $a_x \in I_{y,L}$

Let $\sim_L$ be the corresponding equiv. relations

Call the corresp. equiv. classes in W the "left cells".

<u>Remark</u>. The map $w \longleftrightarrow w^{-1}$ switches left $\leftrightarrow$ right cells

b/c $C_w \mapsto C_{w^{-1}}$ is an anti-involution

<u>Def</u>. Let $w \in W$. Define $\mathcal{H}_{\leq_L w} = \bigoplus_{x \leq_L w} A \cdot C_x$

$\uparrow$

left ideal

<u>Def</u>. The left cell module assoc. to a cell e

L_e is $L_e = \mathcal{H}_{\leq_L e} / \mathcal{H}_{<_L e}$.

$\mathcal{H}_V \simeq \bigoplus_e L_e$.

In type A

standard Yong tableaux.

$\mathfrak{S}_n \xleftrightarrow{RSK}$ pairs of SYT w/ the same shape

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & \\ \cline{1-3} 8 & & & \\ \cline{1-1} 9 & & & \\ \cline{1-1}\end{array}$$

left cells $\longleftrightarrow$ fix a SYT on the left

two sided cells $\longleftrightarrow$ pick a shape

Lecture 13. Conj. (Lusztig, 2021) in Type A

If we

Let ρ be a complex irred. unip. rep. of $G(\mathbb{F}_q)$

For any $w \in W$ for which $w^2 = 1$, there exists a character M_w of $G(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$ s.t.

$$\rho = \sum_{\substack{w \in C_\rho \\ w^2 = 1}} M_w$$

for $C_\rho \subset W$ the two sided cell "attached to ρ".

Further, the M_w satisfy some nice properties:

(i) For any involution w, M_w is a linear combination of $\overline{V_\lambda}$ for λ "very close" to $(p-1)\omega_{I(w)}$

a region (x) does not depend on p

(iii) These dimension polynomials satisfy some nice symmetry: $\forall$ invol'n w, $\exists$ invol'n w' s.t. $x^k d_w(1/x) = d_{w'}(x)$

(ii) can define a "dimension polynomial" which is some $d_w(x)$ s.t. $d_w(p) = \dim M_w$ when we use p.

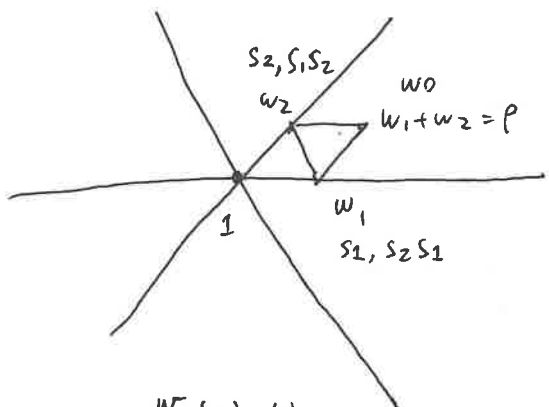
(iv) M_w are positive

Def For any $w \in W$, let $I(w) \subset S$ be defined by

$$I(w) = \left\{ s \in S : \ \ell(ws) < \ell(w) \right\}, \text{ and let } \overline{I(w)} = S \sim I(w)$$

Def. For $s \in S$, ω_s is a fundamental weight.

For $S' \subset S$, $\omega_{S'} = \sum_{s \in S'} \omega_s$.



$w \mapsto \omega_{I(w)}$

Type A_3. $V_{a,b,c} = \overline{V_\lambda}$ for $\lambda = a\omega_1 + b\omega_2 + c\omega_3$

$M_1 = V_{0,0,0}$

$M_{s_1} = V_{p-1,0,0}$

$M_{s_2} = V_{0,p-1,0}$

$M_{s_3} = V_{0,0,p-1}$

$M_{s_1s_3} = V_{p-1,0,p-1} - V_{p-2,0,p-2}$ $L_{p-1,0,p-1}$

$M_{s_2s_1s_3s_2} = V_{0,p-1,0} + V_{0,p-3,0}$

$M_{s_1s_2s_1} = V_{p-1,p-1,0}$

$M_{s_1s_3s_2s_3s_1} = V_{p-1,0,p-1} + V_{p-3,0,p-2}$

$M_{s_2s_3s_2} = V_{0,p-1,p-1}$

$M_{w_0} = V_{p-1,p-1,p-1}$

<u>Thm</u> (Bezrukavnikov, Finkelberg, Kazhdan, MF)

- Lusztig's elts M_w exist
- There is an explicit formula for them

- $\Rightarrow$ we can explicitly write any f as a linear comb. of Weyl chambers
- However, properties (iii) & (iv) are false.

$$\begin{array}{cc} \downarrow & \downarrow \\ \text{dimension polynomial reciprocity} & \text{positivity} \end{array}$$

- We know why they "should" be false.

<u>Goal</u>: reduce the construction of the elts M_w to the construction of a "nice" basis of

$\underset{\mathcal{O}(T)}{\overset{\parallel}{\mathbb{C}[T]}}$ over $\underset{\mathcal{O}(T/\!\!/W)}{\overset{\parallel}{\mathbb{C}[T]^W}}$.

Let $[p] : T \to T$ given by
$z \mapsto z^p$

Let $\pi: T \to T/\!\!/W$ be the projection.

Let $\overline{[p]} : T/\!\!/W \to T/\!\!/W$ be the unique morphism s.t. $\pi \circ [p] = \overline{[p]} \circ \pi$.

$(T/\!\!/W)_p$ be the fixed pt scheme of $\overline{[p]}$.

$\iota_p : (T/\!\!/W)_p \hookrightarrow T/\!\!/W$ the inclusion.

The sheaf $\pi_* \mathcal{O}_T$ carries an action of W, while $\iota_p^* \pi_* \mathcal{O}_T$ carries an aut. ϕ defined by $\phi(f) = [p]^* f$.

The semisimple conj. classes of G biject w/ $T/\!/W$

The semisimple conj. classes of $G(\mathbb{F}_p)$ biject w/ $(T/\!/W)_p$.

The Brauer character β is a map to $\mathcal{O}((T/\!/W)_p)$

<u>Prop</u> (Jantzen 1986)

$$\beta(\rho_x) = \frac{1}{\dim x} \operatorname{tr}(\phi, [\iota_p^* \pi_* \mathcal{O}_T : x])$$

$\uparrow$

cpx unip. irr.
of $G(\mathbb{F}_p)$ assoc.
to irrep x of W

If $\{f_w\}_{w\in W}$ is a basis, then define $\tilde{\phi}(f_w) = [p]^* f_w$, extend $\mathcal{O}(T)^W$-linearly

equiv. to a choice of basis of $\mathcal{O}(T)$ over $\mathcal{O}(T)^W$.

<u>Cor.</u> Suppose $\tilde{\phi}$ is some aut. of $\pi_* \mathcal{O}_T$ commuting w/ the action of W s.t. $\iota_p^* \tilde{\phi} = \phi$.

Then let $\tilde{\rho}_x = \frac{1}{\dim x} \operatorname{tr}(\tilde{\phi}, [\pi_* \mathcal{O}_T : x])$

(a character of alg. gp G over $\overline{\mathbb{F}}_p$)

Then $\rho_x = \tilde{\rho}_x \big|_{G(\mathbb{F}_p)}$

If $[\pi_* \mathcal{O}_T : x]$ = span of some subset C of basis elts, then

$$\text{RHS} = \sum_{w\in C} \underbrace{\langle [p]^* f_w, f_w^* \rangle}_{\overset{\shortparallel}{M_w}}$$

Lecture 14

$A = \mathcal{O}(T/\!\!/W) = k[T]^W$ $\qquad$ char $k > h$

$N = \mathcal{O}(T) = k[T]$

N is an A-module

$A_p = \mathcal{O}((T/\!\!/W)_p)$

Prop Suppose $\tilde{\Phi}$ is an automorphism of $\pi_*(\mathcal{O}_T)$ commuting w/ the action of W s.t.

$c_p^* \tilde{\Phi} = \phi$, then let $\tilde{V}_x = \frac{1}{\dim x} \operatorname{tr}(\tilde{\Phi}, [\pi_*(\mathcal{O}_T) : x]) \in A$

then $\underline{V}_x = \tilde{V}_x \big|_{G(\mathbb{F}_p)}$

reduction mod p of the

irrep. of $G(\mathbb{F}_p)$ corresp. to the irrep. x of W.

If $\{f_w\}_{w \in W}$ is a basis for N over A, then if we define

$\tilde{\Phi}(f_w) = [p]^* f_w$, then extend A-linearly, this gives some

well-defined $\tilde{\Phi}$, $\tilde{\Phi}$ W-invariant $\Leftarrow$ the $\mathbb{C}$-span of $\{f_w\}_{w \in W}$ is W-invariant

The goal is

- Define some basis $\{f_w\}_{w \in W}$ of N over A s.t.
  - $*$ some W-invariance property is satisfied
  - $*$ isotypic components of N as a W-rep. to be spanned by basis elements. two sided KL cell picture is respected.

$\Rightarrow$ Define $\quad \langle\, ,\, \rangle : \mathbb{C}[T] \times \mathbb{C}[T] \to \mathbb{C}[T]^W$

$$M_w = \langle [p]^* f_w, f_w^* \rangle$$

Def. For any $w \in W$, let $\quad E_w = w \cdot \exp(\omega_{I(w)})$

$$\omega_{I(w)} = \sum_{\substack{i \text{ s.t.} \\ \ell(w s_i) < \ell(w)}} \omega_i$$

this is the <u>Steinberg basis</u>.

This basis doesn't satisfy W-invariance

i.e. $\mathbb{C}\{E_w\}_{w \in W}$ is not W-invariant.

We could define $\quad E^w = \left(w \cdot \exp\left(\omega_{\overline{I(w)}}\right)\right) \exp(-\rho)$

In small ranks, $\{E_w\}$, $\{E^w\}$ are dual bases under $\langle\, ,\, \rangle$.

In SL_4, $\langle E_w, E^1 \rangle \neq 0$ whenever w is the product of two permutations indexing singular Schubert varieties.

<u>The Kazhdan-Lusztig-Steinberg basis</u>

Since our goal is to compute $\mathrm{tr}(\tilde{\varphi}, [\pi_*(\mathcal{O}_T) : x])$, we can relax the cond'n that the $\mathbb{C}$-span of $\{f_w\}_{w \in W}$ is W-invariant ...

Let $\leq$ be an partial order on W which refines the partial orders on two-sided cells of W.

For any $w \in W$, let $N_{\leq w}$ and $N_{<w}$ be the A-modules generated by $\{f_y\}_{y \leq w}$ and $\{f_y\}_{y<w}$ respectively, with $gr_w(N) = N_{\leq w}/N_{<w}$.

write $\bar{f}_y$ for the image in $gr_w(N)$

<u>Lemma</u> Suppose the following conditions are satisfied:

(i) For every $w \in W$, the submodules $N_{\leq w}$ and $N_{<w}$ are W-invariant.

(ii) For any $w \in W$, the $\mathbb{C}$-span of $\{\bar{f}_y\}_{y \leq w}$ in $gr_w(N)$ is W-inv't.

(iii) For every irrep χ of W, there exists a unique equiv. class of $w \in W$ s.t. χ appears in $gr_w(N)$ w/ nonzero multiplicity.

THEN,
the endomorphism $\tilde{\phi}$ preserves $N_{\leq w}$ and $N_{<w}$ and the induced automorphism of $gr_w(N)$ commutes w/ W.

Further, $\quad tr(\tilde{\phi}, [gr_w(N) : \chi]) = tr(\tilde{\phi}, [N : \chi])$

<u>Def</u>. For any $w \in W$, let $f_w = \frac{1}{|W_{I(w)}|} C'_w(\exp(w_{\overline{I(w)}})) \in \mathbb{Z}[T]$

$\mathbb{Z}[W]$ $\quad T_s \quad IC_s$

$C'_w = \sum a_{w,y}\, y \qquad 0 = (T_s + v^{-1})(T_s - v)$

$C'_s \qquad C_s$

$$f^w = \frac{1}{|W_{\overline{I(w)}}|} w_0 C'_{w_0 w}\left(\exp(\omega_{I(w)})\right).$$

<u>Def.</u> Let $\langle , \rangle : N \times N \longrightarrow A$ be defined by

$$\langle f, g\rangle = \underbrace{\frac{1}{\delta}}_{\text{Weyl denominator}} \left(\sum_{w \in W} (-1)^{\ell(w)} \, w(fg) \right) \in A$$

For $w \in W$, let $h(w) = (\rho, \omega_{I(w)}) \in \mathbb{Z}_{\geq 0}$.

<u>Lemma.</u> For $w, v \in W$ w/ $h(v) \leq h(w)$,

$$\langle f_w, f^v \rangle = \delta_{w,v}.$$

In rank 2, $\langle f_w, f^v\rangle = \delta_{w,v}$, $\forall\, w, v \in W$.

<u>Cor.</u> The set $\{f_w\}_{w \in W}$ is a basis of $N_{\mathbb{Z}} = \mathbb{Z}[T]$ over $A_{\mathbb{Z}} = \mathbb{Z}[T]^W$.

<u>Def'n.</u> For c a two-sided KL cell, let $N_{\mathbb{Z}}^{\leq c} := \mathbb{Z}[W]^{\leq c} N_{\mathbb{Z}}$

$$= \bigoplus_{\substack{y \in c' \\ c' \leq c}} A_{\mathbb{Z}} f_y.$$

$N_{\mathbb{Z}}^{<c}$ similarly defined.

Let $N_{\mathbb{Z}}^{c} = N_{\mathbb{Z}}^{\leq c} / N_{\mathbb{Z}}^{<c} \simeq \bigoplus_{y \in c} A_{\mathbb{Z}} f_y.$

<u>Cor.</u> For $G = SL(n)$ and any irrep χ of $W = \mathfrak{S}_n$, we have

$$\tilde{V}_x = \sum_{y \in c} \langle [q]^* f_y, f_y^* \rangle$$ for c the two-sided cell corresp. to x.

Proof. We have

$$\tilde{V}_x = \mathrm{tr}(\tilde{\varphi}, [gr_w(N) : x])$$ where w is some elt of c.

In type A, $gr_w(N)$ breaks up into a direct sum of x as a $\mathbb{C}[W]$-module.

(In fact, we have $gr_w(N) \simeq x^{\oplus \dim x}$ as W-reps).
Each has the form $\mathbb{C}[W] f_y$ for some y in the same left KL cell as w
So $\frac{1}{\dim x} \mathrm{tr}(\tilde{\varphi}, gr_w(N))$

$$= \frac{1}{\dim x} \sum_y \mathrm{tr}(\tilde{\varphi}, \mathbb{C}[W] f_y) = \sum_y \langle [p]^* f_y, f_y^* \rangle$$

$\Rightarrow$ In type A, this gives that V_x is a sum of $\langle [p]^* f_y, f_y^* \rangle$ terms.
Each of these is a lin. comb. of Weyl characters by the def'n of $\langle , \rangle$.

$\langle [p]^* f_w, f_w^* \rangle$	v.s. $\langle [p]^* f_w, f^w \rangle$
- actually computes the correct trace	- easier to compute
often	agree
- Counter example in Type A_4	- satisfies the conjectured symmetries among dimension polynomials
- nonpositive (sometimes)	- positivity

To do.

- define M_w in general & state Lusztig's full conjecture, w/ proof sketch given by asymptotic Hecke alg.

- Uniqueness of the M_w.

Lecture 15

Def In each left Kazhdan–Lusztig cell, there is a unique almost invol'n of minimal length, called the Duflo invol'n.

Def. If $w \in W$ has w & w^{-1} lying in the same left cell, we call w an "almost-invol'n". We call the set of almost invol'ns $\mathcal{J}$.

Def. For $w \in \mathcal{J}$, let $M_w = \langle [p]^* b_{w_0 d}, b^*_{w_0 w} \rangle$ where d is the Duflo invol'n in the same left cell as w.

Lemma For any $w \in \mathcal{J}$, there exists some subset $\Lambda_w \subset \Lambda$ indep. of p and some coeffs $r_\lambda \in \mathbb{Z}$ s.t.

$$M_w = \sum_{\lambda \in \Lambda_w} r_\lambda V_{(p-1)\varpi_{I(w)} + \lambda} \qquad (*)$$

Pf. Let $w \in \mathcal{J}$, let d be the Duflo invol'n in the same left cell.

First, note that $[p]^* b_{w_0 d}$ is a linear combination of monomials from the W-orbit of $\exp(p\varpi_{I(w)})$ (since $\overline{I(w_0 d)} = I(d) = I(w)$)

The pairing of such monomials $\exp(\nu)$ with another $\exp(\mu)$ (appearing w/ nonzero coeff in $f^*_{w_0 w}$)

equals $V_{y(\nu+\mu)-\rho}$ for $y \in W$ s.t. $y(\nu+\mu)$ is dominant.

$$\begin{array}{c} \shortparallel \\ V_{(p-1)\omega_{I(w)}+\lambda} \end{array} \quad \text{for some } \lambda.$$

We can write $[p]^* f_{w_0 d} = \sum_{\nu \in W \cdot p\omega_{I(w)}} a_\nu \exp(\nu)$

$f^*_{w_0 w} =: \sum_{\mu \in \Lambda'_w} b_\mu \exp(\mu)$

$\uparrow$ set of μ s.t. $\exp(\mu)$ has nonzero coeff in $f^*_{w_0 w}$. (indep. of p).

$\Rightarrow$ $(*)$ holds where Λ_w is the set of all λ s.t. $\mu+\nu$ and $(p-1)\omega_{I(w)}+\rho+\lambda$ lie in the same W-orbit, for $\mu \in W\cdot q\omega_{I(w)}$, $\nu \in \Lambda'_w$ (For every such λ, $z_\lambda = \sum_{\mu,\nu} \pm a_\nu b_\mu$)

Recall $\{C_w\}_{w\in W}$ is the KL basis of $\mathcal{H}_v$ or of $\mathbb{C}[W]$.

Def. Let $h_{x,y,z} \in \mathbb{Z}[v,v^{-1}]$ be the structure constant for multiplication of $\{C_w\}_{w\in W}$ in $\mathcal{H}_v$ for $x,y,z \in W$. $C_x C_y = \sum_{z\in W} h_{x,y,z} C_z$

We define $\gamma_{x,y,z} \in \mathbb{Z}$ to be the top degree coeff. of $h_{x,y,z}$.

Def. Let J be the free abelian gp generated by $\{t_w\}_{w \in W}$ equipped w/ ring structure given by $t_x t_y = \sum_{z \in W} \gamma_{x,y,z}\, t_z$.

This multiplication is assoc. The identity under multiplication is $1_J = \sum_{d \text{ Duflo invol's}} t_d$

Given any two-sided cell c of W, let $J_c = \{t_w : w \in c\}$

There is a direct sum decomposition $J = \bigoplus_c J_c$ where each J_c has unit elt

$$1_{J_c} = \sum_{d \in c \cap D} t_d.$$

There's a nat'l correspondence between irred. modules over $\mathbb{C}[W]$, H_v, and J.

Prop. For a two-sided cell $c \subset W$, there is a $(\mathbb{Z}[W]_c, J_c)$-bimodule B_c (the "regular" bimodule) w/ basis b_w, $w \in c$. It's defined so that

$$\mathbb{Z}[W]_c \longrightarrow B_c \qquad , \quad J_c \longrightarrow B_c$$
$$C_w \longmapsto b_w \qquad \quad t_w \longmapsto b_w$$

are each isomorphisms as a left $\mathbb{Z}[W]_c$-module and a right J_c-mod. respectively.

The action of the algebras $\mathbb{Z}[W]_c$ and J_c on B_c are mutual centralizers for one another.

$A = \mathbb{C}[T]^W$, $N = \mathbb{C}[T]$.

(subscript: two-sided cell piece wrt. C_w
superscript: two-sided cell piece wrt. C_w')

Recall $\xi: A[W]^c \xrightarrow{\sim} N_{\mathbb{Z}}^c$

Cor. The map $\eta: A[B_c] \to N^{w_0 c}$ given by $b_w \mapsto f_{w_0 w}$

is an isom. of A-modules intertwining the W-action (after twisting by sgn).

As a result, there's a well-defined action of $A[J_c]$ on $N^{w_0 c}$.

Prop. a) For any two-sided cell c, the endomorphism $\tilde{\Phi}|_{gr_c(N)}$ coincides w the right action of some elt $h \in A \otimes_{\mathbb{Z}} J_c$.

b) $h = \sum_{w \in c} M_w t_w$. "Intrinsic description of M_w".

Proof. Since $\sum_{d \in D \cap c} t_d$ is the identity element of J_c, we have

$$h = (\eta^{-1} \circ \tilde{\varphi} \circ \eta)\Big(\sum_{d \in D\cap c} t_d\Big) = (\eta^{-1} \circ \tilde{\Phi})\Big(\sum_{d \in D \cap c} f_{w_0 d}\Big)$$

$$= \eta^{-1}\Big(\sum_{d \in D\cap c} \sum_{w \sim_L d} \langle [p]^* f_{w_0 d}, f_{w_0 w}\rangle f_{w_0 w}\Big).$$

$$= \sum_{d \in D\cap c} \sum_{w \sim_L d} M_w \eta^{-1}(f_{w_0 w}) = \sum_{d \in D \cap c} \sum_{w \sim_L d} M_w t_w = \sum_{w \in c} M_w t_w.$$

Nonabelian Fourier transform

For any $x \in \mathrm{Irr}(W)$, there exists a "principal series module" $U_x \in \mathrm{Irr}(G(\mathbb{F}_p))$ lives in $\mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}(\mathrm{triv})$.

Also, there is a virtual rep. V_x assoc. to a character sheaf on G.

There exists an involution on $\mathrm{Irr}(G(\mathbb{F}_p))$ called the "nonabelian Fourier transform" sending $U_x \longleftrightarrow V_x$.

For any $y \in W$, there exists some virtual character

R_y^1 "unipotent Deligne - Lusztig character".

comes from geometry.

<u>Def.</u> A unipotent irrep. ρ of $G(\mathbb{F}_p)$ is s.t. $(\rho, R_y^1) \neq 0$ for some $y \in W$.

For any $w \in J$, let
$$R_{\alpha_w} = \frac{1}{|W|} \sum_{\substack{x \in \mathrm{Irr}(W) \\ y \in W}} c_{w,x} \, \mathrm{tr}(y, x) R_y^1$$

trace of t_w in the rep. of J corresp. to x

$$\overset{\text{Lusztig}}{=} \sum_{x \in \mathrm{Irr}(W)} c_{w,x} V_x$$

$$= FT\Big(\sum_{x \in \mathrm{Irr}(W)} c_{w,x} U_x \Big).$$

Lusztig's actual conj. in arbitrary type:

Conj. For every $w \in \mathcal{J}$, there is a nonzero character M_w of $G(\mathbb{F}_p)$ over $\overline{\mathbb{F}_p}$

s.t. $\forall$ irred. unip. reps ρ, we have $f = \sum_{w \in \mathcal{J}} (\rho : R_{\alpha_w}) M_w$.

Corollary to our "intrinsic def. of M_w":

For χ an irrep. of W,

$$\underline{V}_\chi = \sum_{w \in \mathcal{J}} (V_\chi, R_{\alpha_w}) M_w.$$

Proof. By Jantzen's thm, & our nice choice of basis

$$\underline{V}_\chi = \tilde{V}_\chi \big|_{G(\mathbb{F}_p)} \quad \text{where} \quad \tilde{V}_\chi = \frac{1}{\dim \chi} \operatorname{tr}\left(\Phi, [\mathrm{gr}_w(N) : \chi]\right)$$

By "intrinsic description", we have

$$\tilde{V}_\chi \big|_{G(\mathbb{F}_p)} = \frac{1}{\dim \chi} \operatorname{tr}\Big(\sum_{w \in c} M_w t_w, [\mathbb{C}[W]_c : \chi]\Big)$$

$$= \frac{1}{\dim \chi} \sum_{w \in c} \operatorname{tr}(t_w, [\mathbb{C}[W]_c : \chi]) M_w$$

$$= \sum_{w \in c} c_{w,\chi} M_w.$$

$$= \sum_{w \in \mathcal{J}} (V_\chi, R_{\alpha_w}) M_w.$$

Lecture 16

- "Uniqueness" & properties of M_w
- alg. geom: - exceptional collections
  - attempts to categorify f_w & M_w.

Prop. $\exists$ examples of M_w which are non-positive

For example:

- in Type B_3 for $w = s_1$
- in Type A_4 for $w = s_2 s_3 s_2$.

$$M_{s_2 s_3 s_2} = V_{0,p-1,p-1,0} - V_{0,p-2,p-2,0} = L_{0,p-1,p-1,0} - L_{p-2,0,0,p-2}$$

Def. Let γ be a dominant weight. We say λ_1, λ_2 are γ-close if $\lambda_1 - \lambda_2$ is in the W-orbit of some dominant weight $\gamma' < \gamma$.

- We say $\{M'_w\}_{w \in W}$ is a γ-close coll'n of candidate elements if

1) $\forall$ irr. unip. reps of $G(\mathbb{F}_p)$ $\quad \rho = \sum_{w \in J} (\rho : R_{\lambda_w}) M'_w$

2) $\forall w \in W$, M'_w is a lin. comb. of V_λ for λ γ-close to $(p-1)\omega_{I(w)}$.

Thm For $\gamma = kp$ where $k = 2(\rho, \rho) + 2$, our M_w is a γ-close coll'n of cand. elts.

Prop. If $G = SL(n)$, $n \geq 3$, $\exists$ some γ s.t. $\forall p$, the coll'n $\{M_w\}$ is NOT the unique γ-close cce.

Pf. (For $G = SL(3)$) write $A = V_{1,1} + V_{0,0}$.

note that $A = ch(V_{1,0} \otimes V_{0,1})$ and $L_{1,0} = V_{1,0}$, $L_{0,1} = V_{0,1}$

By the Steinberg tensor product formula, $A = L_{p,1} = L_{1,p}$

$$\left(\begin{array}{l} L_{p,1} = L_{0,1} \otimes F_r^*(L_{1,0}) \\ \text{after restricting to } G(\mathbb{F}_p) \\ \qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{G^{F_r}}, \\ L_{p,1} = L_{0,1} \otimes L_{1,0} \end{array} \right)$$

we can write $(p \gg 0)$
$$A = V_{p,1} - V_{p-2,2} + V_{p-4,0}$$
$$= V_{1,p} - V_{2,p-2} + V_{0,p-4}$$

Now let

$$M'_{s_1} = M_{s_1} + A, \quad M'_{s_2} = M_{s_2} - A, \quad M'_w = M_w \text{ for } w \notin \{s_1, s_2\}$$

Let $\gamma = 4p$

Lemma Let $G = SL(n)$, $1 \leq n \leq 5$. If C is some two-sided cell, and $w, w' \in C$ are involutions, then $w = w'$ iff $I(w) = I(w')$.

($\Leftrightarrow$ any two standard Young tableaux of the same shape w/ identical descent sets must be equal if they have ≤ 5 boxes).

Prop. Suppose $G = SL(n)$, $n \leq 5$, For any fixed γ, there exists $p' > 0$ s.t. if $p > p'$ and $\{M'_w\}_{w \in W}$ is a γ-close cce, then $\forall w \in W$, $M'_w - M_w$ is a lin. comb. of V_λ for λ which are $(\gamma + 2\rho)$-close to 0.

Pf. Let γ be a fixed dominant weight. We can choose p' large s.t. if $p > p'$, then $\forall$ subsets $I, I' \subseteq S$, $\nexists$ a weight λ, both $(\gamma+2\rho)$-close to $(\gamma+2\rho)$-close to $(p-1)\omega_I$ and $(p-1)\omega_{I'}$. Since $\{M_w\}_{w\in W}$ and $\{M_w'\}_{w\in W}$ satisfy the same equation, (1)

we have $\sum_{w\in J\cap C} (M_w' - M_w) = 0$

For any $w \in W$, $M_w' - M_w$ is a linear comb. of V_λ w/ λ being γ-close to $(p-1)\omega_{I(w)}$

Exercise. Any such lin. comb. can be written as a lin. comb. of L_λ for λ $(\gamma+\rho)$-close to $(p-1)\omega_{I(w)}$

By Steinberg tensor product formula, this can be rewritten as a lin comb. of L_λ for λ p-restricted and $(\gamma+2\rho)$-close to either $(p-1)\omega_{I(w)}$ or to 0.

We now claim they're all close to 0. Suppose otherwise, then some L_λ for λ $(\gamma+2\rho)$-close to $(p-1)\omega_{I(w)}$ would also have to appear in $M_y' - M_y$, $y \neq w$, $y \in C$

But, all L_λ terms in $M_y' - M_y$ must be $(\gamma+2\rho)$-close to $(p-1)\omega_{I(y)}$ or 0. Since $p > p'$ $\Rightarrow$ $\omega_{I(y)} = \omega_{I(w)} \overset{\text{Lemma}}{\Rightarrow} y = w$.

Def. Consider a family of collections $\{M_w\}_{w\in W}$ defined simultaneously across all large p. (we write M_w^p) We say M_w is a lin. comb. of Weyl char. <u>depending on p</u>

if $\exists$ polynomials $f_i(t) \in \mathbb{Z}[t][X^*(T)]$ s.t. $M_w^p = \sum_i c_i V_{f_i(p)}$ for all p large.

Corollary For $G = SL(5)$ and any dominant wt γ, there exists no γ-close cce $\{M_w'\}_{w\in W}$ w/ each M_w' a lin. comb. of Weyl chars dep. on p which satisfies Lusztig's properties.

Proof The failure of positivity in Type A_4 can't be resolved by any lin. comb. of V_λ for λ close to 0, if $p \gg 0$, so there's no $\{M_w'\}$ satisfying all our assumptions, since it would have to differ from $\{M_w\}$ by such a lin. comb.

$\boxed{k = \overline{\mathbb{F}_q}}$

Lemma Let X be a proj. var. $/\mathbb{F}_q$, w/ an action of an alg gp G. Then we can consider the rep. $k[X(\mathbb{F}_q)]$ of $G(\mathbb{F}_q)$. Suppose we have a decomposition

$$(*)\ [\mathcal{O}_\Delta] = \sum_i [\mathcal{F}_i \boxtimes \mathcal{F}_i'] \text{ in } K_0(\mathrm{Coh}^G(X^2)), \quad \mathcal{F}_i \in \mathrm{Coh}^G(X)$$

then the virtual k-rep. of G given by $V = \sum_i R\Gamma(\mathcal{F}_i \otimes \mathrm{Fr}^*(\mathcal{F}_i'))$, then $V|_{G(\mathbb{F}_q)} = k[X(\mathbb{F}_q)]$.

Pf sketch Let $\Delta : X \to X\times X$ be the diagonal morphism.

Let $g : X \to X\times X$ be the graph of Fr.

Δ is equiv. for the diag. action of $G \curvearrowright X^2$, g is equiv. for $G \curvearrowright X^2$ by std action on the first factor, std action twisted by Frob on the second factor.

$\Rightarrow$ Both are equiv for the diag action of $G(\mathbb{F}_q)$.

Apply g^* to two sides of $(*)$.

on RHS, $g^*(\mathcal{F}_i \boxtimes \mathcal{F}_i') \simeq \mathcal{F}_i \otimes \mathrm{Fr}^*(\mathcal{F}_i')$

and on LHS, $g^*(\mathcal{O}_\Delta) \simeq \mathcal{O}_{\Delta(X) \cap g(X)}$

disj't union of pts in $X(\mathbb{F}_q)$

So $[\mathcal{O}_{X(\mathbb{F}_q)}] = \sum_i [\mathcal{F}_i \otimes \mathrm{Fr}^* \mathcal{F}_i']$ in the $G(\mathbb{F}_q)$-equiv. Grothendieck gp of X

apply $\Gamma \Rightarrow$ result.

Let D be a tri. cat. over a field k

<u>Def</u>. $\mathcal{F} \in D$ is <u>exceptional</u> if there's an isom. of graded k-alg.

$$\mathrm{Hom}^\bullet_D(\mathcal{F}, \mathcal{F}) \simeq k.$$

We say a coll'n $(\mathcal{F}_1, \cdots, \mathcal{F}_n)$ of exceptional objects is exceptional if

$1 \le i < j \le n$, $\mathrm{Hom}^\bullet_D(\mathcal{F}_j, \mathcal{F}_i) = 0$

It's a <u>full</u> exceptional coll'n if $\mathcal{F}_i$ gen. D as a Δ-cat.

A collection $(\mathcal{G}_n, \cdots, \mathcal{G}_1)$ is the dual exceptional collection to $(\mathcal{F}_1, \cdots, \mathcal{F}_n)$ if

$$\begin{aligned} R\mathrm{Hom}(\mathcal{F}_i, \mathcal{G}_j[l]) &\simeq \\ R\mathrm{Hom}(\mathcal{G}_i, \mathcal{F}_j[l]) &= \end{aligned} \begin{cases} k, & i=j,\ l=0 \\ 0, & \text{o/w} \end{cases}$$

In this case, $(\mathcal{G}_n, \cdots, \mathcal{G}_1)$ is also exceptional.

Lecture 17

$k = \overline{\mathbb{F}_q}$

Lemma. Let X be a proj. var. over $\mathbb{F}_q$ w/ an action of the alg gp G. then we can consider two virtual reps of $G(\mathbb{F}_q)$:

1) $k[X(\mathbb{F}_q)]$

2) If we have a decomp. $[\mathcal{O}_\Delta] = \sum_i [\mathcal{F}_i \boxtimes \mathcal{F}_i']$ in $K_0(\mathrm{Coh}^G(X^2))$

then $\sum R\Gamma(\mathcal{F}_i \otimes Fr^*(\mathcal{F}_i'))$

2) restricts to 1) under restriction from G to $G(\mathbb{F}_q)$.

We will let $X = G/P$ for P some parabolic subgp of G.

For any $\mathcal{F} \in D^b\mathrm{Coh}^G(G/P)$, let $\mathcal{F}^\vee$ be its image under the Grothendieck duality $R\mathcal{H}om(-, \mathcal{O}_{G/P})$ (G/P is Calabi-Yau), so for any $\mathcal{G} \in D^b\mathrm{Coh}^G(G/P)$,

$R\mathrm{Hom}_{G/P}(\mathcal{F}, \mathcal{G}) = R\Gamma(G/P, \mathcal{F}^\vee \overset{L}{\otimes} \mathcal{G})$.

Prop. Suppose $C = \{\mathcal{F}_w\}_w$ is an exceptional collection generating $D^b\mathrm{Coh}(G/P)$, and $\{\mathcal{F}^w\}_w$ is the dual collection. We define a virtual k-rep of G by

$V_C = \sum_{w \in W} R\mathrm{Hom}(\mathcal{F}_w, Fr^*(\mathcal{F}^w))$, then $V_C|_{G(\mathbb{F}_q)} \simeq k[(G/P)(\mathbb{F}_q)]$

Pf sketch

First, we claim that

$$(*)\ [\mathcal{O}_\Delta] = \sum_w [\mathcal{F}_w^\vee \boxtimes \mathcal{F}^w] \quad \text{in } K_0(\mathrm{Coh}^G((G/P)^2))$$

There's a pairing $([\mathcal{F}],[\mathcal{G}]) = \chi(R\Gamma(\mathcal{F} \overset{L}{\otimes} \mathcal{G}))$

and to show $(*)$, it's enough to check that the pairing of each side w/ some elt of the form $[\mathcal{F}_{y_1}^\vee \boxtimes \mathcal{F}^{y_2}]$ are equal.

Exercise. Both sides pair w/ to give δ_{y_1,y_2}

Then: $\sum_w R\mathrm{Hom}(\mathcal{F}_w, \mathrm{Fr}^*(\mathcal{F}^w)) = \sum_w R\Gamma(\mathcal{F}_w^\vee \otimes \mathrm{Fr}^* \mathcal{F}^w)$.

By the Lemma, we are done.

Fact. In type A, all irred unip. reps lie inside $k[(G/B)(\mathbb{F}_q)]$

Further, $k[(G/P)(\mathbb{F}_q)]$ span the space of virtual characters as P varies.

Samokhin - van der Kallen constructed an except'l coll'n on each G/P

↳ originally we thought that via this V_C construction for C = this coll'n, we get the same lift. But this is only true in type A_2, B_2, G_2, A_3.

Admissible subcat.

For $\mathcal{C} \subset D$ $\leftarrow$ triangulated additive cat, write $\mathcal{C}^{\perp}$ (resp. ${}^{\perp}\mathcal{C}$)

for the strictly full subcat. in D consisting of objects X s.t.

$\mathrm{Hom}(A, X) = 0$ (resp. $\mathrm{Hom}(X, A) = 0$), $\forall A \in \mathcal{C}$.

Def $\mathcal{C} \subset D$ is called right admissible if

a) $\mathcal{C} \to D$ has a right adjoint

b) $D = \langle \mathcal{C}, \mathcal{C}^{\perp} \rangle$

c) $[D] = [\mathcal{C}] * [\mathcal{C}^{\perp}]$

set of isom. classes

subset of $[D]$ consisting of all Z s.t. there's a distinguished triangle $X \to Z \to Y$ w/ $X \in \mathcal{C}$, $Y \in \mathcal{C}^{\perp}$.

Prop Let $\nabla \subset \mathrm{Ob}(D)$ be a finite exceptional collect'n.

a) the triangulated subcat. of D gen. by ∇ is both left & right admissible

b) The dual except'l collect'n Δ exists.

Let $\nabla = (\nabla^i)$ be a ~~n~~ full except'l set in a finite type triangulated k-linear cat. D

Prop. There exists a unique t-str. $(D^{\geq 0}, D^{<0})$ on D s.t.

$\nabla^i \in D^{\geq 0}$, $\Delta^i \in D^{\leq 0}$ w/

$I = \{1, \dots, n\}$.

a) $D^{\geq 0} = \langle \nabla^i[d] \quad , i \in I, d \leq 0 \rangle$ gen. under ext'n $\quad D^{<0} = \langle \Delta_i[d], i \in I, d > 0 \rangle$ gen. under ext'n

b) The t-str. is bounded

c) For $X \in D$, we have $X \in D^{\geq 0} \iff \mathrm{Ext}^{<0}(\Delta_i, X) = 0$, $\forall i$

$X \in D^{<0} \iff \mathrm{Ext}^{\leq 0}(X, \nabla^i) = 0$, $\forall i$

d) Let A be the heart of this t-str., then every obj. of A has finite length.

Further, the image L_i of $\underbrace{\tau_{\geq 0}(\Delta_i) \to \tau_{\leq 0}(\nabla^i)}_{\text{canonical arrow}}$ is irreducible.

These L_i are distinct; they're all the irreducibles in A.

If $D_i = \langle \nabla_1, \dots, \nabla_i \rangle$, then there's a well-defined induced t-str. on D_i, and

$A_i = D_i^{\heartsuit}$ = the Serre subcat. of A gen. by $\{L_1, \dots, L_i\}$.

$\tau_{\geq 0}(\Delta_i) \to L_i$ is a proj. cover of L_i in A_i

$L_i \to \tau_{\leq 0}(\nabla^i)$ is an inj. hull of L_i in A_i

Mutation of an exceptional set

Suppose $\preceq$ is another order on I.

Then we can write $D_{\preceq i} = \langle \nabla^j : j \preceq i \rangle$

$$D_{\prec i} = \langle \nabla^j : j \prec i \rangle$$

Lemma a) For $i \in I$, there's a unique up to unique isom. obj. ∇^i_{mut} s.t.

$$\nabla^i_{mut} \in D_{\preceq i} \cap D_{\prec i}^{\perp} \quad \text{and} \quad \nabla^i_{mut} = \nabla^i \bmod D_{\prec i}$$

b) The objects ∇^i_{mut} form an exceptional set indexed by $(I, \preceq)$

c) We have $D_{\preceq i} = \langle \nabla^j_{mut} : j \preceq i \rangle$.

Pf sketch. Let $\pi_i : D_{\preceq i} \to D_{\preceq i}/D_{\prec i}$ be the projection functor.

Let π_i^r denote the right adjoint functor. We then set $\nabla^i_{mut} = \pi_i^r \pi_i (\nabla^i)$

One can check this gives an except'l coll'n.

c) Follows by induction.

Exceptional coll'n on G/B explicitly.

For a weight $\chi \in X(T)$, let $\mathcal{L}(\chi)$ denote the corresponding G-equiv. line bundle on G/B.

Define the vector bundle $\Psi_1^{w_i}$ as the kernel of the canonical morphism

coWeyl mod of h.w. w_i

$$\nabla_{w_i}\otimes\mathcal{L}(0)\longrightarrow\mathcal{L}(w_i)$$

$$0\longrightarrow\Psi_1^{w_i}\longrightarrow\nabla_{w_i}\otimes\mathcal{L}(0)\longrightarrow\mathcal{L}(w_i)\longrightarrow 0$$

For any simple root α_i, let P_{α_i} be the minimal parabolic gen. by B and $U_{-\alpha_i}$.

and denote $Y_i := G/P_{\alpha_i}$. $\pi_{\alpha_i}: G/B\longrightarrow G/P_{\alpha_i}$.

Given a reduced expr. $w = s_{i_1}\cdots s_{i_n}$, we build an endofunctor of $D^b(G/B)$

$$D_w = D_{\alpha_1}\circ\cdots\circ D_{\alpha_n},\qquad D_{\alpha_i} := \pi_{\alpha_i}^*\pi_{\alpha_i *}.$$

Here's a full except'l coll'n. in $D^b Coh(SL_3/B)$:

$$\begin{array}{cccc}
\mathcal{A}_{-3}, & \mathcal{A}_{-2}, & \mathcal{A}_{-1} & \mathcal{A}_0 \\
 & D_{\alpha_1}(\Psi_1^{w_1}) & \Psi_1^{w_1} & \\
\mathcal{L}(-\rho) & & & \mathcal{L}(0) \\
 & D_{\alpha_2}(\Psi_1^{w_2}) & \Psi_1^{w_2} &
\end{array}$$

Look at their classes in $K_0(D^b Coh^G(G/B))$

It turns out $D_{\alpha_1}(\Psi_1^{w_1})$ is the kernel of a map $\Psi_1^{w_1}\longrightarrow\mathcal{L}(w_1-\alpha_1)$

and thus one can deduce $D_{\alpha_1}(\Psi_1^{w_1})\simeq\mathcal{L}(-w_2)$. Idem $D_{\alpha_2}(\Psi_2^{w_2})\simeq\mathcal{L}(-w_1)$

The weights of $\Phi_1^{w_1}$ are $w_1 - \alpha_1$ and $-w_2$, and similarly $\Phi_1^{w_2}$ gives weights $w_2 - \alpha_2$ and $-w_2$

One can compare these explicit computations to f_w & f^w (the Kazhdan-Lusztig-Steinberg basis).