Finite flat group schemes

Hikari 岩崎光宇

Tate's thm $\rightsquigarrow$ Torsions $E[2]$

Abelian var. A $\qquad \underbrace{A[m]} = \ker(A \xrightarrow{m} A)$

finite flat gp scheme m^{2g}, g rel dim.

Torsion $\sim$ rank

Over $\mathbb{Z}$, $\mathbb{Z}_p$

§ Hopf algebra

Def. An affine gp scheme G over R is a gp obj. in the cat. of affine R-schemes.

$$\left.\begin{array}{l} c: A \to A \otimes_R A \\ e: A \to R \\ \mathrm{inv}: A \to A \end{array}\right] \text{ comm. Hopf } R\text{-alg.}$$

$I = \ker(e: A \to R)$ augmentation ideal.

ex. finite free of rank 2.

$$0 \to I \to A \xrightarrow{e} R \to 0$$
$$A \cong R \oplus R$$

$$I = (\underbrace{e(0,1)\cdot(1,0) - e(1,0)(0,1)}_{\alpha})$$

$A \simeq R[\alpha]/(\alpha^2 + p\alpha + q)$

$\simeq R[X]/(X^2 + aX)$

$e(X) = 0$.

$$R[T]/(T^2+aT) \longrightarrow R[X,X']/(X^2+aX, X'^2+aX')$$

$$T \longmapsto f(X,X') = \alpha + \beta X + \gamma X' + \delta X X'$$

① $X' \to 0$

$T \mapsto f(X,0) = X \quad \Rightarrow \alpha = 0,\ \beta = 1$

② $X \to 0 \qquad \alpha = 0,\ \gamma = 1$

$$f(X,X') = X + X' + \delta X X'.$$

$f(X,X')^2 + a f(X,X') = 0$ in RHS

$$\rightsquigarrow 0 = (2 - a\delta)(1 - a\delta)$$

inv: $1 - a\delta$ is a unit $\Rightarrow 2 = a\delta$.

Upshot. free rank 2 gp scheme is of the form $\mathrm{Spec}\left(R[X]/(X^2+aX)\right)$

$T \mapsto X + X' + \delta X X'$

w/ $a\delta = 2$

§ Cartier duality

Thm (Cartier)
A finite flat Hopf alg. over (noeth) local R.

$\rightsquigarrow A^\vee = \mathrm{Hom}_R(A,R)$ is also a Hopf alg.

$m : A \otimes A \to A \qquad \rightsquigarrow c^\vee : A^\vee \to A^\vee \otimes A^\vee$
$a \otimes b \mapsto ab$

The dual gp scheme $G^\vee$ of $G = \mathrm{Spec}\,A$ is $G^\vee = \mathrm{Spec}\,(A^\vee)$

$$G^\vee(R') = \mathrm{Hom}_{R'\text{-gp scheme}}(G_{R'}, \mathbb{G}_m)$$

$$= \mathrm{Hom}_R^{\text{comm Hopf}}\left(R'[T, \tfrac{1}{T}], A \otimes_R R'\right)$$

(e) finite flat rank 2 gp scheme $G_{a,b} =$ $\mathrm{Spec}\ R[X]/(X^2+aX)$

$T \mapsto X + X' + bXX'$

$$G_{a,b}^\vee = \mathrm{Spec}\left(\mathrm{Hom}\left(R[T,\tfrac{1}{T}],\ R[X]/(X^2+aX)\right)\right)$$

$T \longmapsto p(X)$

$p(X)\, p(X') = p(X + X' + bXX')$ $\rightsquigarrow$ $G_{a,b}^\vee \simeq G_{b,a}$

$p(X) = 1 - eX$, $e^2 + be = 0$

§. Deligne's Thm

Thm. (Deligne) If $G \overset{\text{Spec } A}{=}$ finite flat comm. gp sch. $/R$ of rank m $(= \mathrm{rank}_R A)$

then $[m]$ annihilates G.

[Proof idea]. Reduce to show $[m]$ annihilates $G(R)$. R local

$$N : \underset{\text{f. free } R\text{-alg}}{S} \to R \qquad \text{norm} \quad s \mapsto \det(s : S \xrightarrow{R\text{-linear}} S)$$

$N : A \to R$

$\tilde{N} : A^\vee \otimes A \to A^\vee$

$u \in G(R) = A^\vee$ "translation by u"

$\tilde{N}(id_A) = \tilde{N}(\tau_u(id_A))$

$= N(u)\tilde{N}(id_A) = u^m \tilde{N}(id_A)$

$\boxed{u^m = 1}$

Prop If $[m]$ annihilates G, then $m\Omega^1_{A/R} = 0$.

§ $\Omega^1_{A/R}$ $\qquad G \underset{I}{\xrightarrow{\Delta}} G\times G$

Prop R noeth, A R-Hopf alg. then $\Omega^1_{A/R} = A \otimes_R I/I^2$

$$\begin{tikzcd}
(g,h) \arrow[r, mapsto] & (g, gh) \\
G\times G \arrow[r, "\sim"] & G\times G \\
 & G \arrow[u, "\Delta"] \arrow[ul, bend left, "\mathrm{id}\times e"]
\end{tikzcd}$$

Prop If $[m]\cdot G = 0$, then $m\,\Omega^1_{A/R} = 0$

Lem $c : A \longrightarrow A \otimes_R A$ $\quad$ if $i \in I$

$c(i) = i\otimes 1 + 1 \otimes i \mod I \otimes I$

[Proof] $A \simeq R \oplus I$, $\quad A \otimes_R A = R \oplus (R\otimes I) \oplus (I \otimes R) + I \otimes_R I$

[Proof] In $\Omega^1_{A/R}$, $[m]\cdot i \equiv m i \mod I^2$.

Thm ((Cartier) If K is of char 0, then every finite flat gp scheme is étale.

Proof idea. $\hat{A} = \varprojlim A/I^n$, $\quad J = \bigcap_n I^n$

$A \simeq A/J \times A/J'$

$\Omega^1_{A/R} \simeq \Omega^1_{(A/J)/R} \times \Omega^1_{(A/J')/R}$.