q- Difference equations & p-curvature

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$$\nabla = \partial + A$$

$C_p(\nabla) = \nabla^p f \pmod p$ $\qquad \partial^p \equiv 0$

$p=2.$
$$\nabla^2 f = (\partial + A)(\partial f + Af)$$
$$= \underset{\substack{\parallel \\ 0}}{\partial^2 f} + (\partial A) f + \underset{\substack{\parallel \\ 0}}{2A\,\partial f} + A^2 f$$
$$= (\partial A + A^2) f$$

Claim. $C_p(\nabla) \in \mathrm{Mat}_N\big(\mathbb{F}_p(z_1, \dots, z_\ell)[s]\big)$

new geometric parameters

Th. (Etingof - Varchenko) The following connections are isospectral (mod p):

$$C_p(\nabla, s) \quad \text{and} \quad (s^p - s) - \nabla \circ F_p$$

Frobenius map

$$F_p : z \longmapsto z^p$$

Grothendieck, Katz 1970 Inv.

$y' = \frac{1}{az} y$, $a \in \mathbb{Z}$, $\overset{\text{over } \mathbb{C}}{\Longrightarrow}$ $y = c z^{1/a}$ doesn't lift

mod p: $y = z^b$, $p \nmid b$, $\quad b z^{b-1} = \frac{1}{a} z^{b-1}$, $\quad ab \equiv 1 \bmod p$

$$f(z q^{L_i}, a, q)\, L_i = M_{L_i}(z, a, q)\, f(z, a, q) \quad , \; L \in \underline{Pic}(X) \qquad (\dagger)$$

(Simplest: $f(qz) = M(z) f(z)$)

$q \to 1$

1) $q \to 1$ in $\mathbb{C}$-norm

2) $q \to 1$ in p-adic norm

z - Kähler parameters of X (quantum parameters)

a - Equivariant torus

$\hbar$ - character of cotangent fibers.

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q- Bianchi / Integrability.

$$M_{L_2 L_1}(z) = M_{L_2}(z q^{L_1}) \cdot M_{L_1}(z)$$

$$= M_{L_1}(z q^{L_2}) \cdot M_{L_2}(z)$$

$q \to 1$

$$M_{L_2}(z)\, M_{L_1}(z) = M_{L_1}(z)\, M_{L_2}(z)$$

$M \to$ Lax Matrix for XXZ spin chain.

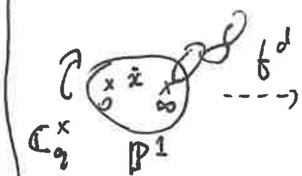
$$M_{L^p}(z) = M(z q^{(p-1)L}) \cdots M(z q^{2L})\, M(z q^{L})\, M_L(z)$$

$$[M_{L_1^p}, M_{L_2^p}] = 0.$$

$$M_L(z,a) = \lim_{q \to 1} M_L(z, a, q), \quad M_{L,\zeta_p}(z,a) = \lim_{q \to \zeta_p} M_{L^p}(z, a, q)$$

Quantum K-theory

(quasimaps)



X

$\cup$

$T^* \mathbb{P}^1$ $[u:v]$

$T^* Gr_{k,n}$

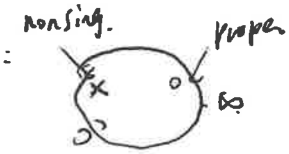
hyper-Kähler

$$\frac{u}{v} = \frac{(x - e_1) \cdots (x - e_d)}{(x - f_1) \cdots (x - f_d)}$$

$QM^d_{\text{nonsingular}}$, QM^d_{proper}

Capping operator

$\Psi(z, a, q)$:



$\downarrow$

Satisfies $(\dagger)$.

Th. 1 Let $\{\lambda_i(z,a)\}$ be the eigenvalues of M_L, then

$\{\lambda_i(z^p, a^p)\}$ – the eigenvalues of M_{L,ζ_p}.

Th. 2 $\Psi(z,a,q)\,\Psi^{-1}(z^p, a^p, q^{p^2})$ is finite as $q^p \to 1$

$X = T^* \mathbb{P}^0$, $\psi(zq) = \dfrac{1-z}{1-\hbar z}\psi(z)$

$$\psi(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1-\hbar^m}{1-q^m}\frac{z^m}{m}\right)$$

$$1-u^p = (1-u)(1+u+\cdots+u^{p-1})$$

$\mathbb{C}^2$ x, y

$xy = qyx$

$x^p y^p = q^{p^2} y^p x^p$

$$F_{\zeta_p}(z) = \lim_{q \to \zeta_p} \psi(z,a,q)\,\psi^{-1}(z^p, a^p, q^{p^2})$$

$$F_{\zeta_p}(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1-\hbar^m}{m} z^m \delta_m\right), \quad \delta_m = \begin{cases} \dfrac{1}{1-\zeta_p^m}, & p \nmid m \\ \dfrac{1-p}{2}, & p \mid m \end{cases}$$

Th 3. $M_L(z^p) = F_{\zeta_p}(z)^{-1} M_{L,\zeta_p}(z)\, F_{\zeta_p}(z).$

$\mathbb{Q}_p(\pi)$, $\pi^{p-1} = -p$.

Lemma $(1+\pi\alpha + O(\pi^2))^p = 1 + \underbrace{p\pi\alpha}_{-\pi^{p}} + \cdots + \pi^p\alpha^p = 1 + \pi^p(\alpha^p - \alpha)$

$q = 1 + \pi + O(\pi^2)$

$$\mathcal{M}_{L_i, z_p}(z) = \left(\mathcal{M}_{L_i}(z)\, q^{z_i \frac{\partial}{\partial z_i}}\right)^p$$

$$= 1 + \pi^p \left(\nabla_i^p - \nabla_i\right) \quad C_p(\nabla_T)$$

$$\nabla_i = \frac{\partial}{\partial z_i} - \frac{s}{z_i} C_i(z)$$

$$\mathcal{M}_{L_i}(z^p) = 1 + C_i(z^p)\,\pi^p + \dots$$

$$\frac{\mathcal{M}_{L_i}(z^p) - 1}{\pi^p} = (s^p - s)\, C_i(z^p) \qquad \bmod p$$