Coda

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<u>Thm</u> (Fontaine) There are no abelian schemes $/\mathbb{Z}$ $(\dim g = 1)$

Obstruction: existence of $A/\mathbb{Z}$ by the finite flat gp schemes $\overset{G/\mathbb{Z}}{\overset{\shortparallel}{A[p^n]}}$, $n \geq 1$

p prime no order $p^{2g \cdot n}$

<u>Fontaine</u>. uses $p \in \{3, 5, 7, 11, 13, 17\}$

<u>Schoof</u>. $p = 2$ $\longleftarrow$ We'll follow this.

<u>Strategy</u>. Let $L = \mathbb{Q}(G(\bar{\mathbb{Q}})) \,|\, \mathbb{Q}$ finite

① $L|\mathbb{Q}$ unramified outside p

② $L|\mathbb{Q}$ not too ram. @ p (Fontaine ram. bound)

i.e. $|\Delta_L|^{\frac{1}{[L:\mathbb{Q}]}} < p^{\frac{p}{p-1}}$

③ (Global input) $\rightsquigarrow$ bound on $[L:\mathbb{Q}]$.

Hermite-Minkowski

Minkowski, Odlyzko

over $\mathbb{Z}$ – ④ Classify finite flat gp schemes $/\mathbb{Z}$, of <u>p-prime order</u>, <u>simple</u>

⑤ Filter G by simple objects $\Rightarrow$ lower bound on # pts in G

$\Big\}$ contradiction

ab. scheme – ⑥ Weil pairing + Weil bound $\Rightarrow$ upper bound on # pts in G

Fontaine: (ram. bound) If $K | \mathbb{Q}_p$ finite, ram. index e, Γ fin. flat comm. gp scheme $/ \mathcal{O}_K$, $L = K(\Gamma(\bar{K}))$, then $\mathrm{Gal}(L|K)^{(u)} = 1$ for $u > e(n + \frac{1}{p-1})$,

$$v(\mathcal{D}_{L|K}) < e(n + \frac{1}{p-1})$$

Global consequence: $\Gamma / \mathbb{Z}$ fin. flat. comm., order dividing p^n. $E = \mathbb{Q}(\Gamma(\bar{\mathbb{Q}}))$.

Then: E unram. outside p ①, $|\Delta_E|^{\frac{1}{[E:\mathbb{Q}]}} < p^{n + \frac{1}{p-1}}$ ②

Pf. ① $\Gamma = \mathrm{Spec}\, A$, p^n kills A $\Rightarrow$ p^n kills I/I^2. $\Rightarrow$ p^n kills $\Omega^1_{\Gamma/\mathbb{Z}} = A \otimes I/I^2$

Fixing a prime $\ell \neq p$, $p^n \in \mathbb{F}_\ell^\times$, $\underbrace{\Omega^1_{\Gamma_{\mathbb{F}_\ell}/\mathbb{F}_\ell}}$ killed by a unit $= 0$

$\Rightarrow \Gamma_{\mathbb{F}_\ell}$ (so $\Gamma_{\mathbb{Z}_\ell}$) étale $\rightarrow$ $A \otimes \mathbb{Z}_\ell = \prod \underbrace{\mathcal{O}_{E_{\ell,i}}}_{\text{unram. } / \mathbb{Z}_\ell}$

$\Rightarrow \bigcup E_{\ell,i} = E_\ell$

② If $(p) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ in $\mathcal{O}_E$, $\xrightarrow{\text{ram. bds}}$ $v(\mathcal{D}_{E_{\mathfrak{p}_i}/\mathbb{Z}_p}) < (n + \frac{1}{p-1})$

$E | \mathbb{Q}$ Galois

$\Rightarrow v_p(\Delta_E) = v_p(N_{E|\mathbb{Q}}(\mathfrak{p}_1 \cdots \mathfrak{p}_r))$

$< [E:\mathbb{Q}] \, (n + \frac{1}{p-1})$

Specialize to $p=2$.

If $G/\mathbb{Z}$ fin. flat gp scheme killed by 2.

Global consequence $\Rightarrow$ $L = \mathbb{Q}(G(\bar{\mathbb{Q}}))$ unram. outside 2, and $|\Delta_E|^{\frac{1}{[E:\mathbb{Q}]}} < 2^{\frac{2}{2-1}} = 4$

Minkowski bds

$$|\Delta_E| > \left(\frac{\pi}{4}\right)^{2r_2}\left(\frac{n^n}{n!}\right)^2$$

$r_1 = \#$ real places

$r_2 = \#$ pairs of cplx places

$[E:\mathbb{Q}] = r_1 + 2r_2$

Odlyzko.

$$|\Delta_E| > c_1^{r_1} c_2^{r_2}$$

$c_1 \approx 60$

$c_2 \approx 22$

Use this to construct tables.

Totally imag. fields

$[L:\mathbb{Q}]$	$\Delta_L^{\frac{1}{[L:\mathbb{Q}]}}$
2	1.732
4	3.289
6	4.622 > 4

Ramification bounds + Odlyzko $\Rightarrow$ $[L:\mathbb{Q}] \leq 4$

Prop^n The only simple finite étale comm. 2-power torsion $\Gamma/\mathbb{Z}$ are μ_2, $\mathbb{Z}/2$.

Rmk. For $17 \geq p > 2$, this Thm follows from Raynaud's classification + "descent" classifies f.f. gp sch. p-torsion $/\mathbb{Z}_p$, $p > 2$

Pf. ~~Define~~ If Γ simple, $\Gamma[2] \simeq \Gamma$

Let $G_{-1} = \operatorname{Spec}\left(\mathbb{Z}[x]/(x^2-1) \times \mathbb{Z}[x]/(x^2+1)\right) \simeq \mu_4$

$\mathbb{Q}(G_{-1}) = \mathbb{Q}(i)$. For every $G/\mathbb{Z}$ 2-power order, let $G' = G \times G_{-1}$ killed by 2

$\deg \leq 4$ $\mathbb{Q}(G'(\bar{\mathbb{Q}})) \supset \mathbb{Q}(i)$

$\cup$

$L = \mathbb{Q}(G(\bar{\mathbb{Q}}))$

$\Rightarrow [L:\mathbb{Q}] = 1, 2,$ or 4 (not 3)

Γ simple $\operatorname{Gal}(L/\mathbb{Q}) \curvearrowright \Gamma(\bar{\mathbb{Q}})$, $\#\Gamma(\bar{\mathbb{Q}}) = 2^b$

$\Rightarrow \Gamma(\bar{\mathbb{Q}})^{\operatorname{Gal}(L/\mathbb{Q})}$ nontrivial.

$\hat{\Gamma}(\bar{\mathbb{Q}})$

$\Gamma_{\mathbb{Q}}$ étale $\Rightarrow \exists$ subgp $S \subset \Gamma_{\mathbb{Q}}$ order 2 $\Rightarrow \exists S_{\mathbb{Z}} \subset \Gamma$ order 2

Uses that R Dedekind $\left\{ \text{flat closed subgroup } S \subset G/R \right\} \overset{1:1}{\longleftrightarrow} \left\{ \text{flat closed subgp } S_K \subset G/K \right\}$

$\Downarrow$

torsion free $\Leftrightarrow$ flat

$$S \longmapsto S_K$$

$$\overline{S} \cap G_R \longleftarrow\!\shortmid S$$

$\Rightarrow \Gamma \simeq S_{\mathbb{Z}}$ order 2

Order 2 ffgs / $\mathbb{Z}$ classified by $G_{a,b} = \operatorname{Spec} \mathbb{Z}[x]/(x^2 - ax)$

$ab = -2$ comult. given by $y * z = y + z + byz$

$G_{-2,1} \simeq \mu_2$, $G_{1,-2} \simeq \mathbb{Z}/2$ $\leftarrow$ all the possibilities.

$A/\mathbb{Z} \rightsquigarrow A[2^n]$ filtered by simple objects $(\simeq \mu_2, \mathbb{Z}/2)$

<u>Thm.</u> Any $\Gamma/\mathbb{Z}$ finite flat 2-power order is an ext'n

$$0 \to \Gamma_{\text{diag}} \to \Gamma \to \Gamma_{\text{const}} \to 0$$

, Γ_{const} constant gp sch

Γ_{diag} diagonalizable gp sch.

<u>Pf</u> reduces to $\operatorname{Ext}^1_{\mathbb{Z}}(\mu_2, \mathbb{Z}/2) = 0$

Mayer-Vietoris type seq. $\mathbb{Z}_2$, $\mathbb{Z}[\frac{1}{2}]$.

<u>Finishing the proof</u> Suppose scheme $A/\mathbb{Z}$, $0 \to D_n \to A[2^n] \to C_n \to 0$

diag. constant

For almost all q, $C_n(\mathbb{F}_q) \hookrightarrow A/D_n(\mathbb{F}_q)$. Weil conjectures $\Rightarrow |C_n(\mathbb{F}_q)| \leq (\sqrt{q}+1)^{2g}$

$$0 \longrightarrow C_n^\vee \longrightarrow A[2^n]^\vee \longrightarrow D_n^\vee \longrightarrow 0$$

$$\begin{matrix} A[2^n]^\vee \\ \text{\rotatebox{90}{$\cong$}} \\ A^\vee[2^n] \end{matrix}$$

Same argument $\Rightarrow$ $\# D_n = \# D_n^\vee \leq (\sqrt{q}+1)^{2g}$

$\Rightarrow$ $\# A[2^n] \leq \# C_n \, \# D_n = (\sqrt{q}+1)^{4g}$

$\# A[2^n] = 2^{2g \cdot n}$

$2^{2g \cdot n} >> (\sqrt{q}+1)^{4g}$

contradiction. !

$\Rightarrow A/\mathbb{Z}$ doesn't exist.