On the Gaiotto conjecture and its Iwahori version

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G reductive gp $/ \mathbb{C}$.

$$\begin{array}{ll} G(F) & \text{loop group} \\ \cup & \\ G(\mathcal{O}) & \text{arc grp} \\ \cup & \\ I & \text{Iwahori subgp} \end{array}$$

$Gr_G = G(F)/G(\mathcal{O})$, $Fl_G = G(F)/I$, $\check{G}$ Langlands dual gp.

Gaiotto conjecture.

Def (hyperspherical) $G \curvearrowright M$ hamiltonian, is hyperspherical if $\mathbb{C}[M]^G$ is Poisson commutative + technical assumptions

spherical var. : $G \curvearrowright X$ w/ open dense B-orbit.

e.g. (spherical): toric variety. $\mathbb{G}_m \curvearrowright \mathbb{A}^1$

symmetric space $G \curvearrowright G/G^\theta$

grp case $G \times G \curvearrowright G \rightsquigarrow T^*G$

pt case $G \curvearrowright pt \rightsquigarrow pt$

Flag $G \curvearrowright G/U \rightsquigarrow T^*_+(G/U)$

S-duality

$\{G\text{- hyper spherical var.}\} \longleftrightarrow \{\check{G}\text{- hyperspherical}\}$

$G \curvearrowright M \qquad\qquad M^\vee \curvearrowleft \check{G}$

e.g. 1) grp case $\longleftrightarrow$ grp case

$G \times G \curvearrowright T^*G \longleftrightarrow T^*\check{G} \curvearrowleft \check{G} \times \check{G}$

2) Whittaker case $\longleftrightarrow$ pt case

$G \curvearrowright T^*_\psi(G/U) \longleftrightarrow pt \curvearrowleft G^\vee$

<u>BZSF Conjecture</u>.

$$\mathcal{O}_{M(F)} \overset{\text{def. quant.}}{\rightsquigarrow} QM(F)$$

assume $M = T^*_\psi(G/H) \rightsquigarrow QM(F) = \{\psi\text{-TDO on } (G/H)(F)\}$

$$QM(F)\text{-mod} = D^{H(F),\psi}(G(F))$$

$$D^{H(F),\psi}(Gr_G) = QM(F)\text{-mod}^{G(O)} \simeq QCoh^{\check{G}}(M^{\vee,\varnothing})$$

eg. 1) grp case $\longleftrightarrow$ grp case

$$M = T^*(G \times G/G)$$

$$D^{G(F)}(Gr_G \times Gr_G) \simeq QCoh^{\check{G}\times\check{G}}(T^*\check{G}[2])$$

$$\text{\rotatebox{90}{$\simeq$}} \qquad\qquad\qquad \text{\rotatebox{90}{$=$}}$$

$$D^{G(O)}(Gr_G) \simeq \operatorname{Sym}\check{\mathfrak{g}}[-2]\text{-mod}^{\check{G}}$$

derived Satake, B-F.

eg. 2) $D^{U(F),\psi}(Gr_G) \simeq QCoh^{\check{G}}(pt) = \operatorname{Rep}(\check{G})$

geometric Casselman–Shalika, Frenkel - Gaitsgory - Vilonen.

<u>Deformations</u>. $Gr_G \rightsquigarrow \mathcal{P}_{\det}$

$D_K(Gr_G)$ = K-twisted D-modules on Gr_G

$D^{G(O)}(Gr_G) \rightsquigarrow D^{G(O)}_K(Gr_G)$

$D^{U(F),\psi}(Gr_G) \rightsquigarrow D^{U(F),\psi}_K(Gr_G)$

e.g. $k \notin \mathbb{Q}$, $\# \mathrm{Irr}(D_k^{G(O)}(\mathrm{Gr}_G)) = 1$

but for any k, $\# \mathrm{Irr}(D^{U(F),\psi}(\mathrm{Gr}_G)) = \# \mathrm{Irr}(D_k^{U(F),\psi}(\mathrm{Gr}_G))$

$$
\begin{array}{ccc}
D^{U(F),\psi}(\mathrm{Gr}_G) & \simeq & \mathrm{Rep}(\check{G}) \\
\wr & & \wr \\
D_k^{U(F),\psi}(\mathrm{Gr}_G) & \overset{?}{\simeq} & \mathrm{Rep}_q(\check{G})
\end{array}
\quad \underline{\mathrm{Thm}} \text{ (FLE)},
$$

$q = \exp(\pi i k)$

$k = 0$, FGV

$k \notin \mathbb{Q}$, Gaitsgory

$k \in \mathbb{Q}$, Campbell-Dhillon-Raskin

Gaiotto conj. $\approx$ quantum BZSV when it has a reasonable deformation

$\approx$ super "FLE".

<u>Reasonable deformation.</u>

1) $P_{\det}$ has a $H(F)$-equiv. str.

2) almost all irred. obj. have deformations.

$\exists\, G_S$ s.t. $(G_S)_{\bar{0}} = \check{G}$, $(\mathfrak{g}_S)_{\bar{1}} = M^\vee$

$$D^{H(F),\psi}(\mathrm{Gr}_G) \simeq \mathrm{QCoh}^{G_{\bar{0}}}(\mathfrak{g}_{\bar{1}}) = \mathrm{Rep}(G_S)$$

<u>BZSV</u>: $D^{H(F),\psi}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(G_S)$

<u>Gaiotto</u>: if k generic, there exists a braided monoidal eq $D_k^{H(F),\psi}(\mathrm{Gr}_G) \simeq \mathrm{Rep}_q(G_S)$

Basic classical supergroups (V. Kac)

$GL(M|N)$

$SOSp(k|2l)$

$D(2,1,\alpha)$

$F(4)$

$G(3)$

e.g. $GL(M|N)$

$M = N-1$, $\quad G_{\bar 0} = GL_M \times GL_N$

$$M^\vee = \mathfrak{g}_{\bar 1} = T^* \operatorname{Hom}(\mathbb{C}^M, \mathbb{C}^N)$$

$G = GL_M \times GL_N$

$M = T^* (GL_M \times GL_N / GL_M)$

<u>BZSV</u>: $\quad D^{GL_{N-1}(\mathcal{O})}(Gr_{GL_N}) \simeq Rep(\underline{GL}(N-1|N))$

Braverman - Finkelberg - Ginzburg - Travkin

<u>Gaiotto</u>. $\quad \kappa$ generic

$$D_\kappa^{GL_{N-1}(\mathcal{O})}(Gr_{GL_N}) \simeq Rep_q(GL(N-1|N))$$

$M < N-1$. $\quad U_{M,N} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \subset GL_N$, $\quad \psi : U_{M,N} \to \mathbb{A}^1$

$G_{\bar 0} = GL_M \times GL_N$

$\mathfrak{g}_{\bar 1} = T^* \operatorname{Hom}(\mathbb{C}^M, \mathbb{C}^N)$

$G = GL_M \times GL_N$

$M = T^*_\psi (GL_N / U_{M,N}) = T^*_\psi (GL_M \times GL_N / GL_M \ltimes U_{M,N})$

<u>BZSV</u>:

$$D^{GL_M(O)\ltimes U_{M,N}(F),\psi}(Gr_N) \simeq Rep(\underline{GL}(M|N))$$

$M=0$, FGV

$0<M<N-1$, Travkin–Y.

<u>Gaiotto</u>

$$D_k^{GL_M(O)\ltimes U_{M,N}(F),\psi}(Gr_N) \simeq Rep_q(GL(M|N))$$

$M=0$, Gaitsgory

$0<M<N-1$ Travkin–Y.

$\boxed{G(3)}$. $e\in\mathfrak{g}_2$ short root vector $\rightsquigarrow (e,f,h)$

$SL_2 \subset Z_{G_2}(e)$

$h \curvearrowright \mathfrak{g}_2$, $\mathfrak{u}_5 = (\mathfrak{g}_2)_{\leq -1} \rightsquigarrow U_5$

$G = SL_2\times G_2$ ----- $G_{\bar 0} = SL_2 \times G_2$

$\mathfrak{g}_{\bar 1} = \mathbb{C}^2\otimes\mathbb{C}^7$

$M = (T^*G_2\times\mathbb{C}^2)/\!/U_5$

<u>BZSV</u>: $$\left(D(Gr_{G_2})\otimes W\text{-mod}(F^2)\cdot\right)^{SL_2(2,0)\ltimes U_5(F)} \simeq Rep(G(3))$$

<u>Iwahori BZSV</u>, $$D^{H(F),\psi}(Fl_G) \overset{?}{\simeq} QCoh^{\check G}\left(M^{\vee,\theta}\times_{\check{\mathfrak g}^{*,\theta}} \tilde{\check{\mathfrak g}}^{*,\theta}\right)$$

eg. 1) grp $\leftrightarrow$ grp, Безрукавников equivalence

2) Whittaker $\leftrightarrow$ pt, Архипов – Безрукавников equiv.

$$D^{U(F),\psi}(Fl_G) \simeq QCoh^{\check G}(\tilde{\mathcal N})$$

$$\begin{tikzcd}
D_k^{U(F),\psi}(Fl_G) \arrow[rr, "\substack{k=0\ \text{A-B} \\ k\neq 0,\ \text{Y.}}", "\simeq"'] \arrow[dr, "\wr", "\text{C-D-R}"'] & & \mathcal{O}_q^{\text{mixed}} \arrow[dl, "\text{C-F}", "\wr"'] \\
& \hat{\mathfrak{g}}\text{-}\kappa^\vee\text{-mod} &
\end{tikzcd}$$

Iwahori FLE

Iwahori Gaiotto

For any k (mild assumption)

$$D_k^{H(F),\psi}(Fl_G) \text{ "}\simeq\text{" } \mathcal{O}_q^{\text{mixed}}(G_s)$$

Thm (Travkin–Y., 2025 ??) The above is true.