Hitchin moduli spaces and wildly ramified geometric Langlands

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<u>Lecture 1</u>. Bezrukavnikov, Boixeda-Alvarez, McBreen - Y.

"Non-abelian Hodge ..." (PAMQ 2025)

Plan: I. Homogeneous affine Springer fibers

II. Hitchin moduli and their quantization

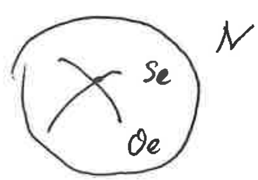
III. Betti moduli spaces, (wildly) ramified geom. Langlands

$/\mathbb{C}$. $\mathfrak{g} \supset \mathcal{N} \ni e \rightsquigarrow (e, h, f)$ $\mathfrak{sl}_2$-triple $\qquad [h, e] = 2e$

$\mathrm{Ad}(h(s))\cdot e = s^2 e$

$S_e^{\mathfrak{g}} = e + \mathfrak{g}^f$ $\pitchfork$ slice to the adj. orbit of e.

$S_e := S_e^{\mathfrak{g}} \cap \mathcal{N}$



$$\begin{tikzcd} \widetilde{S_e} \arrow[r, hook] \arrow[d, "\pi_e"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathcal{N}} = \{(x, B) \text{ s.t. } x \in \mathfrak{n}_B\} \arrow[d, "\text{Springer resol'n}"] \\ S_e \arrow[r] & \mathcal{N} \end{tikzcd}$$

π_e is a symplectic resol'n. $\quad \widetilde{S_e}$ smooth symplectic var.
$\downarrow$
S_e Poisson

conic structure: $\mathbb{C}^\times \curvearrowright S_e$ contracting to $\{e\}$

$h : \mathbb{G}_m \to G$

$s^2 \cdot \mathrm{Ad}(h(s^{-1}))$

$$\begin{tikzcd} B_e \arrow[r, hook, "\text{Lagrangian}"] \arrow[d] & \tilde{S}_e \arrow[d] \\ \{e\} \arrow[r, phantom, "\in"] & S_e \end{tikzcd}$$

$\rightsquigarrow$ constructible quantization

$Sh((U, \psi_e)\backslash \mathcal{B})$ $\hookrightarrow G/B$

Affine analogue: $\mathfrak{g} \rightsquigarrow L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$.

nilp. elts $\rightsquigarrow$ topologically nilp. elts

e.g. $\mathfrak{g} = \mathfrak{sl}_n$, $\gamma \in L\mathfrak{g} \rightsquigarrow \text{eigen}(\gamma) = \{\lambda_1, \cdots, \lambda_n\}$

$\lambda_i \in \overline{\mathbb{C}((t))}$

$\text{val}(\lambda_i) \in \mathbb{Q}$

γ is <u>top. nilp</u> if $\text{val}(\lambda_i - \lambda_j) > 0$

In general, $\gamma \underset{ad}{\curvearrowright} L\mathfrak{g}$, γ is top. nilp if $\text{ad}(\gamma) \in \text{End}_{\mathbb{C}((t))}(L\mathfrak{g})$ is top. nilp.

Assume γ • top. nilp.
• regular semisimple $\in L\mathfrak{g}$

analog of sl_2-triple $(\gamma, \overset{h}{?}, ?)$

$\mathbb{G}_m \xrightarrow{h} LG$

$\text{Ad}(h(s)) \cdot \gamma = s^{?\neq 0} \cdot \gamma$

compute char. pol. on both sides

$\text{char}(\gamma) \neq \text{char}(s^? \cdot \gamma)$

Instead, $h: \mathbb{G}_m \longrightarrow \underset{G(\mathbb{C}((t)))}{\overset{\shortparallel}{LG}} \rtimes \mathbb{G}_m^{\text{rot}}$ scales t

Ex. $\gamma = \gamma_0 \cdot t^n$, $\gamma_0 \in \mathfrak{g}^{rs}$

$\mathrm{rot}(s) \cdot \gamma = s^n \cdot \gamma$

We call γ homogeneous of slope n.

Def. Let $\gamma \in L\mathfrak{g}$ be regular s.s.

say γ is homogeneous of slope $\nu = \frac{d}{m}$ if $\mathrm{rot}(s^m) \cdot \gamma \underset{LG_{ad}}{\sim} s^d \cdot \gamma$, $\forall s \in \mathbb{G}_m$

Eg. $\mathfrak{g} = \mathfrak{sl}_n$

$$\gamma = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix}$$

homog. slope

cha. pol. $X^n \pm t = 0$

homog. slope $\frac{1}{n} \longleftrightarrow$ w = cyclic perm $\in \mathfrak{S}_n$

$\mathrm{rot}(s^n) \cdot \gamma \rightsquigarrow X^n \pm s^n \cdot t = 0$

$s \cdot \gamma \rightsquigarrow X^n \pm s^n \cdot t = 0$

γ^d homog. slope $\frac{d}{n}$

Construct homog. elts $T \subset G$ max'l torus

$h = (\lambda, m) : \mathbb{G}_m \longrightarrow T \times \mathbb{G}_m^{\mathrm{rot}}$

$\lambda \in X_*(T)$, $m \in \mathbb{Z}_{\geq 0}$

$\mathrm{Ad}(h)$ gives $\mathbb{G}_m \curvearrowright L\mathfrak{g}$

$L\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} (L\mathfrak{g})(d)$

$\gamma \in (L\mathfrak{g})(d)$ satisfies

$\mathrm{Ad}(s^\lambda)\,\mathrm{rot}(s^m)\gamma = s^d \gamma$

if r.s., it is homog. of slope $\frac{d}{m}$

Fact. There exists homog. elt of slope $\nu = \frac{d}{m}$

$\Leftrightarrow$ m is the order of a <u>regular elt</u> in W. [Reeder – Yu]

Regular elt in W (Springer)

$w \in W \curvearrowright \mathfrak{h}$ ($\mathbb{C}$ - refl. rep.)

w is regular if it has an eigenvector $\in \mathfrak{h}^{reg}$.

Fact. $\{\text{regular elts in } W\}/\underset{\sim}{\text{conj.}}$ is classified by their orders.

Ex. $W = \mathfrak{S}_n$.

w regular $\Leftrightarrow$ all cycles of w has equal length

(ii) $\underbrace{(\cdots)(\cdots)\cdots(\cdots)}_{\text{equal length}}$

Their orders

$m \mid n$

$m \mid n-1$

γ homog. of slope $\frac{d}{m}$ $\rightsquigarrow$ $T_\gamma = C_{LG}(\gamma)$ max torus in $G_{\mathbb{C}((t))=F}$

$\{\text{max tori in } G/F\}/\underset{\sim}{G(F)} \longleftrightarrow H^1(F, W)$

$\updownarrow$

Conj. classes in W

$\Downarrow$

$T_\gamma \longleftarrow\!\!\rightsquigarrow [w]$ regular order m

Take sl_n $\quad h = (\check{\rho}, n) \quad : \mathbb{G}_m \longrightarrow T_{ad} \times \mathbb{G}_m^{rot}$

$\uparrow$ Coxeter # in general

$$(Lg)(1) = \bigoplus_{i=0}^{n-1} (Lg)_{\alpha_i}$$

$\{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$ affine simple roots

$\alpha_0 = 1-\theta$ $\qquad (Lg)_{\alpha_0} = t \cdot g_{-\theta}$

$\gamma \in (Lg)(1)$ nonzero in each $(Lg)_{\alpha_i}$.

(general G, $n \rightsquigarrow$ Coxeter no. (longest ord. of reg. elt)) $\quad \frac{1}{\text{Coxeter}}$

General slope $\nu = \frac{d}{m}$ $\qquad (\check{\rho}, m) : \mathbb{G}_m \longrightarrow T_{ad} \times \mathbb{G}_m^{rot}$

$\gamma \in (Lg)(d)$

if m is regular, then $(Lg)(d)$ contains a r.s. elt.

any such gives homog. slope $\frac{d}{m}$.

<u>Affine Springer fibers</u> (Kazhdan-Lusztig, late 80s)

γ homog. slope $\nu = \frac{d}{m} > 0$

$$Fl_\gamma = \{ gI \in LG/I : Ad(g^{-1}) \cdot \gamma \in \text{Lie } I^+ \}$$

compare

$$B_e = \{ gB \in G/B : Ad(g^{-1}) \cdot e \in n_B \}$$

<u>Fact</u>. Fl_γ is finite-dim'l. $\dim Fl_\gamma = \dfrac{\nu \cdot |\Phi| - \dim (\mathfrak{h}/\mathfrak{h}^W)}{2}$

$T_\gamma \times \mathbb{G}_m \curvearrowright Fl_\gamma$

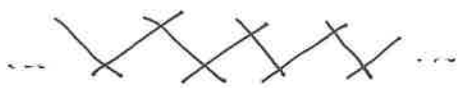
$Ad(s^{\check{\rho}}) \circ rot(s^m)$ preserves Fl_γ

$$\begin{tikzcd} B_e \arrow[r] \arrow[d] & \widetilde{S}_e \arrow[d] \circlearrowleft \mathbb{C}^\times \\ \{e\} \in & S_e \circlearrowleft \mathbb{C}^\times \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} Fl_\gamma \arrow[r, leftrightarrow, "\substack{\text{conic} \\ \text{Lagrangian}}"] \arrow[d] & \overset{\text{sympl.}}{M_\gamma} \arrow[d, "\substack{\text{completely} \\ \text{int. system}}"] \circlearrowleft \mathbb{C}^\times \\ \{a_\gamma\} \arrow[r] & A_\gamma \circlearrowleft \mathbb{C}^\times \end{tikzcd}$$

Geom. of Fl_γ

- Slope $\dfrac{1}{\text{Coxeter } \#}$, $Fl_\gamma = \{*\}$ $(G = \text{s.c.})$

- sl_2, slope 1, $\gamma = \gamma_0 \cdot t$



∞-chain of $\mathbb{P}^1$s

- sl_2, slope $\frac{3}{2}$, $\begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}$

  $Fl_\gamma \simeq \mathbb{P}^1 \vee \mathbb{P}^1$

- sl_3, slope $\frac{2}{3}$.

  $Fl_\gamma = \mathbb{P}^2 \vee \mathbb{P}^1 \vee \mathbb{P}^1$

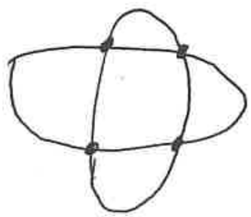
Bernstein's example

Sp_6, γ homog. slope $\frac{1}{2}$ $\left[\begin{array}{c|c} 0 & A \\ \hline tB & 0 \end{array}\right]$, $A, B \in Sym^2(\mathbb{C}^3)$ generic

$(Fl_\gamma)^{\mathbb{G}_m}$ $\underline{\underline{\text{general}}}$ $\coprod$ smooth proj. var. (Hessenberg var.)

In this example, one of the Hessenberg var. $\simeq$ elliptic curve.

$$
\begin{array}{c|c}
\mathbb{P}^2 & \check{\mathbb{P}}^2 \\
\cup & \cup \\
Q(B) & Q(A) \\
 & \\
 & Q(B)^*
\end{array}
\qquad Q(B) \xleftrightarrow{\text{dual curve}} Q(B)^*
$$



$Q(A) \cap Q(B)^*$

elliptic curve $\xrightarrow{2:1} Q(A)$

ramified over $Q(A) \cap Q(B)^*$.

Lecture 2

Hitchin moduli $\mathcal{M}_\gamma$, $\mathfrak{g} \simeq \mathfrak{g}^*$

γ : homog. elt $\in L\mathfrak{g}$, slope $\nu = \frac{d}{m}$

$$
\begin{tikzcd}
Fl_\gamma \arrow[r, hook, "\text{Lag}"] \arrow[d] & \mathcal{M}_\gamma = \text{symplectic} \arrow[d, "\text{comp. int. sys}"] \\
\{a_\gamma\} \arrow[r, hook] & \mathcal{A}_\gamma
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
\mathcal{B}_e \arrow[r, hook, "\text{Lag.}"] \arrow[d] & \widetilde{S}_e \arrow[d] \\
\{e\} \arrow[r, hook] & S_e
\end{tikzcd}
$$

e, h, f $\quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ $\quad$ grading by h , $e \in \mathfrak{g}(2)$

$$\overset{C_{G_0}(e)}{\curvearrowright}\quad G(\le -2) \underset{\text{free}}{\curvearrowright} \overset{C_G(e)}{\overset{\curvearrowright}{e}} + \mathfrak{g}(\le 0) \qquad , \qquad C_G(e) \subset G(\ge 0)$$

$$S_e^{\mathfrak{g}} \simeq G(\le -2) \backslash (e + \mathfrak{g}(\le 0)) .$$

$$\begin{tikzcd} & \tilde{S}_e \arrow[dl] \arrow[dr] & \\ S_e^{g} \arrow[dr] & & \tilde{\mathcal{N}}/G \arrow[dl] \\ & g/G & \end{tikzcd}$$

$Lg = \widehat{\bigoplus_{i\in\mathbb{Z}}} Lg(i)$ (complete in $i \to \infty$) grading by $(\check{\rho}, m) : G_m \longrightarrow T_{ad} \times G_m^{rot}$

$\gamma \in (Lg)(d)$

$\underbrace{(LG)(\leq -d)} \curvearrowright \gamma + \underbrace{(Lg)(\leq 0)}_{\text{parahoric subalg of } L_\infty g}$

$\approx$ Moy-Prasad gp

$T_\gamma = C_{LG}(\gamma)$ loop gp of a max torus $\subset G/F$

$Lg \supset t_\gamma = \widehat{\bigoplus_i} t_\gamma(i)$

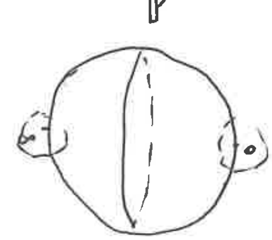
$T_\gamma(\leq 0) \curvearrowright \gamma + (Lg)(\leq 0)$

LG	$G(\mathbb{Q}_p)$	
$(LG)(\geq 0)$	$G_{\check{\rho}/m, 0}$	parahoric
$(LG)(\geq i)$	$G_{\check{\rho}/m, i/m}$	

$\underbrace{(LG)(\leq -d)\cdot T_\gamma(\leq 0)}_{\text{a group}} \curvearrowright \gamma + (Lg)(\leq 0)$

$\underbrace{((LG)(\leq -d)\cdot T_\gamma(<0))}_{K_\gamma}$

$\mathbb{P}^1$



Think of (complete in $i \to -\infty$) $K_\gamma \subset L_\infty G = G((t^{-1}))$

$Bun_G(I_0, K_{\gamma,\infty})$

$=$ moduli stack of G-bundles $\mathcal{E}$ on $\mathbb{P}^1$ w/ Iwahori level at 0

$K_{\gamma,\infty}$ level at ∞

choose a full flag of $\mathcal{E}|_0$

Ex $\gamma = \gamma_0 \cdot t$ slope 1

$$K_{\gamma,\infty} = G[[t^{-1}]]_1 = \ker\left(G[[t^{-1}]] \xrightarrow[t^{-1}=0]{} G\right)$$

$K_{\gamma,\infty}$ level at ∞ $\Leftrightarrow$ a trivialization of $\mathcal{E}|_\infty$.

Ex. $\gamma = \gamma_0 \cdot t^2$, $\gamma_0 \in \mathfrak{h}^{rs}$

$$K_{\gamma,\infty} = \underset{\substack{\| \\ 1+\mathcal{O}(t^{-2})}}{G[[t^{-1}]]_2} \cdot (1 + t^{-1}\mathfrak{h})$$

<u>Higgs bundle</u> $(\mathcal{E}, \varphi)$ on $\mathbb{P}^1$

$\mathcal{E}$: G-bundle on $\mathbb{P}^1$

φ: Section of $Ad(\mathcal{E}) \otimes \omega_{\mathbb{P}^1}$

(may have poles)

GL_n: $\mathcal{E}$ rk n v.b.

$\varphi : \mathcal{E} \to \mathcal{E} \otimes \omega_{\mathbb{P}^1}$

Sp_{2n} $(\mathcal{E}, \underset{\text{sympl. form}}{\langle \cdot, \cdot \rangle})$

$\varphi : \mathcal{E} \to \mathcal{E} \otimes \omega_{\mathbb{P}^1}$

$\langle \varphi x, y \rangle + \langle x, \varphi y \rangle = 0$ $\forall$ local sections x, y of $\mathcal{E}$.

$\mathcal{M}_\gamma$ classifies $(\mathcal{E}, \varphi)$

- $\mathcal{E} \in Bun_G(I_0, K_{\gamma,\infty})$
- φ : rat'l section of $Ad(\mathcal{E}) \otimes \omega_{\mathbb{P}^1}$ 1) regular over $\mathbb{P}^1 \setminus \{0, \infty\}$.

$\mathcal{E}_{0,B} \subset \mathcal{E}_0$ B reduction of $\mathcal{E}_0$

2) at 0, simple pole, $res_0(\varphi) \in \mathfrak{n}(\mathcal{E}_{0,B})$
($res_0(\varphi)$ strictly upper Δ w.r.t. full flag)

3) at ∞. Choose a triv. of $\mathcal{E}|_{D_\infty}$ w/ $K_{\gamma,\infty}$-level under which require

$$\varphi|_{D_\infty} \in \left(\gamma + (Lg)(\leq 0)\right) \frac{dt^{-1}}{t^{-1}}$$

Ex. $\gamma = \gamma_0 \cdot t$ $\qquad G = GL_n$

$K_{\gamma,\infty} = G[[t^{-1}]]_1$

$K_{\gamma,\infty}$ - level means a basis for $\mathcal{E}|_\infty$.

φ has $\leq 2^{nd}$ order pole at ∞

$\varphi_{-2} \in \mathfrak{gl}_n$ under the chosen basis

require $\varphi_{-2} = \gamma_0$

$$\begin{tikzcd}
 & \tilde{S}_e \arrow[dl] \arrow[dr] & \\
\boxed{G(\leq -2) \backslash e + \mathfrak{g}(\leq 0)} \arrow[dr] \arrow[dd, leftrightarrow, "\cong"'] & & \Pi/B \arrow[dl] \\
 & \mathfrak{g}/G & \\
U_e \backslash e + \mathfrak{u}_e^\perp & &
\end{tikzcd}$$

"Higgs bundle on a degenerate curve"

$\langle e, [x,y] \rangle$

$\overbrace{\mathfrak{g}(-1) \text{ symplectic}}$

$G(\leq -2) \subset U_e \subset G(\leq -1)$

Lagrangian

$\tilde{S}_e = T^*B /\!\!/_e \, U_e$

$(e \in \mathfrak{u}_e^*)$

$$\begin{array}{ccccc}
LG(\le -d)\cdot T_\gamma(<0) & \curvearrowright & \gamma + (Lg)(\le 0) \\
\downarrow \quad\quad \| & & \downarrow \\
\underbrace{\text{"}LG(\lesssim -\tfrac{d}{2})\text{"}\cdot T_\gamma(0)}_{J_\gamma} & \curvearrowright & \gamma + (\mathrm{Lie}\, J_\gamma)^\perp
\end{array}$$

$$\overbrace{\underbrace{LG(<-\tfrac{d}{2}) \subset LG(\lesssim -\tfrac{d}{2})}_{\text{Lagrangian modulo } t_\gamma(-\frac{d}{2})} \subset LG(\le -\tfrac{d}{2})}^{(Lg)(-\frac{d}{2}) \text{ has alt. form } \langle \gamma, [\cdot,\cdot]\rangle}$$

$\downarrow$ kernel $t_\gamma(-\frac{d}{2})$

$\mathcal{M}'_\gamma$: replace $K_{\gamma,\infty}$ by $J_{\gamma,\infty}$.

$$\varphi|_{D_\infty} \in \left(\gamma + (\mathrm{Lie}\, J_\gamma)^\perp\right)\frac{dt^{-1}}{t^{-1}}$$

$$\mathcal{M}_\gamma \simeq \mathcal{M}'_\gamma \simeq T^*Fl /\!\!/_\gamma J_\gamma$$

$$J_\gamma \curvearrowright Fl = LG/I_0$$

$$\gamma \in (\mathrm{Lie}\, J_\gamma)^*$$

Hitchin base

$\mathcal{A}_\gamma$ affine space, parametrizing all possible char. poly. of φ from $\mathcal{M}_\gamma$.

$(f_1, \cdots f_r)$ generators of $\mathbb{C}[g]^G$, deg $d_1, \cdots d_r$.

$f_i(\varphi)$ rat'l section of $\omega_{\mathbb{P}^1}^{\otimes d_i}$

$$f_i(\varphi) = \underbrace{f_i(\gamma)}_{\substack{\shortparallel \\ (*)\cdot t^{d_i\nu} \\ 0 \text{ if } d_i\nu \notin \mathbb{Z}}} + \left(\begin{array}{l}\text{poly. in } t \text{ of} \\ \text{deg} \le (d_i - 1)\cdot\nu\end{array}, \text{ no constant term}\right) \rightsquigarrow \mathcal{A}_\gamma$$

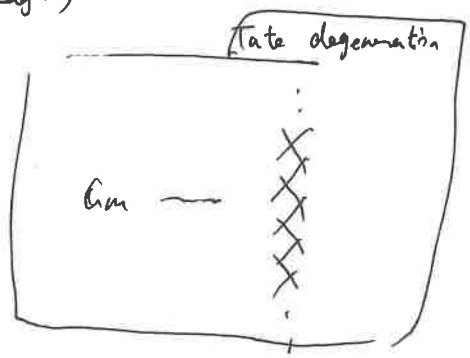
$$\dim \mathcal{A}_\gamma = \tfrac{1}{2}\dim \mathcal{M}_\gamma = \dim Fl_\gamma = \tfrac{1}{2}\left(|\Phi|\cdot \nu - \dim(\mathfrak{h}/\mathfrak{h}^w)\right).$$

<u>Thm</u> (BBAMY)

1) $\mathcal{M}_\gamma$ is a smooth alg. space (locally of f.t.) w/ canonical sympl. str.

2) $f: \mathcal{M}_\gamma \longrightarrow \mathcal{A}_\gamma$ is a completely int. system (fibers are Lag.)

sl_2, $\nu = 1$



3) $\mathbb{C}^\times \curvearrowright \mathcal{M}_\gamma$, $\mathbb{C}^\times \curvearrowright \mathcal{A}_\gamma$ contracting to $\{a_\gamma\}$

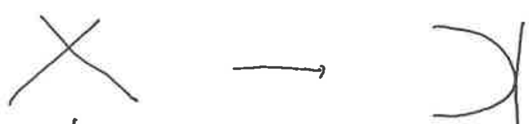
4) $Fl_\gamma \xrightarrow{\text{homeom.}} f^{-1}(a_\gamma)$

$Fl_\gamma \hookrightarrow M_\gamma$ $\cong$ on H_*

5) When γ is elliptic, $\Rightarrow$ f is proper

gen'l fibers are abelian var.

<u>Ex</u>. sl_2, $\nu = \frac{3}{2}$



$\shortparallel$

Fl_γ

<u>Lecture 3</u>.

$\tilde{S}_e$

nilp. orbit

$\mathcal{M}_\gamma$

homog. $\gamma \in L\mathfrak{g}$

$\nu = \frac{d}{m}$ + continuous moduli of γ's

$$\mathcal{M}_\gamma \simeq T^*Fl /\!/_{\gamma} J_\gamma \text{ ``}\cong\text{'' } T^*\left(Bun_G(I_0, G_{\gamma,\infty}, \gamma)\right)$$

$$\begin{tikzcd}
\mathrm{Bun}_G(I_0, J^{\ker}_{r,\infty}) \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & \mathbb{G}_a \\
\mathrm{Bun}_G(I_0, J_{r,\infty}) &
\end{tikzcd}$$

$J^{\ker}_{r,\infty} \hookrightarrow \ker(r : J_{r,\infty} \to \mathbb{G}_a)$

$$\left(T^* \mathrm{Bun}_G(I_0, J^{\ker}_{r,\infty})\right) /\!/_1 \mathbb{G}_a$$

constructible quantization of $\mathcal{M}_r$

"$\mathrm{Sh}(\mathrm{Bun}_G(I_0, (J_{r,\infty}, r)))$"

space / char p, $\mathrm{Sh}_{(\mathbb{G}_a, AS)}(\mathrm{Bun}_G(I_0, J^{\ker}_{r,\infty}))$ $\overline{\mathbb{Q}}_\ell$-sheaves

space / char. 0, $D\text{-mod}_{(\mathbb{G}_a, \exp)}(\text{----})$

In general: Kirillov model (Gaitsgory)

$$X \circlearrowleft \mathbb{G}_a \rtimes \mathbb{G}_m$$

$$\mathrm{Kir}(X) = \mathrm{Sh}_{\mathbb{G}_m}(X) / \mathrm{Sh}_{\mathbb{G}_a \rtimes \mathbb{G}_m}(X)$$

Non-abelian Hodge companions of $\mathcal{M}_r$

(Simpson correspondence)

$$\begin{tikzcd}
\mathcal{M}_r^{Dol} \arrow[r] \arrow[d] & \mathcal{M}_r^{Hod} \arrow[d] & \mathcal{M}_r^{dR} \arrow[l, hook'] \arrow[d] \arrow[r, "\text{analytic map}", "RH"'] & \mathcal{M}_r^{Bet} \\
0 \in & \mathbb{A}^1 \ni \lambda & \ni 1 &
\end{tikzcd}$$

$(\mathcal{E}, \varphi)$ $\quad$ $\{\lambda\text{-connections}\}$ $\quad$ $(\mathcal{E}, \nabla)$ $\quad$ $\left(\begin{array}{l}\text{top. loc. sys} \\ + \text{extra data}\end{array}\right)$

$$\mathcal{M}_\gamma^{dR} = \Big\{ (\mathcal{E}, \nabla) : \quad \mathcal{E} \in \mathrm{Bun}_G(I_0, K_{\gamma,\infty}) \Big.$$

$$\overset{\shortparallel}{(L\mathfrak{n})(\leq -d) \cdot T_\gamma(<0)}$$

∇ : G-connection

- $\nabla|_{D_0}$ simple pole $\quad \mathrm{res}_0(\nabla) \in \mathfrak{n}(\mathcal{E}_{0,B})$
- $\nabla|_{D_\infty}$ (under trivialization)

$$\Big. \nabla \in d + \left(\gamma + (L\mathfrak{g})(\leq 0)\right) \frac{dt^{-1}}{t^{-1}} \quad \Big\}$$

$\mathcal{M}_\gamma^{dR}$ smooth symplectic alg sp.

$\mathcal{M}_\gamma^{Hod} \circlearrowleft \mathbb{G}_m$

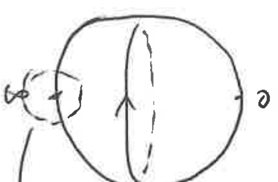
contracting to $Fl_\gamma \subset \mathcal{M}_\gamma^{Dol}$

<u>Thm</u>

$$H^*(\mathcal{M}_\gamma^{Dol}) \xleftarrow{\sim} H^*(\mathcal{M}_\gamma^{Hod}) \xrightarrow{\sim} H^*(\mathcal{M}_\gamma^{dR})$$

<u>Betti moduli space</u>

Classifies G-local sys. on $\mathbb{P}^1 \setminus \{0,\infty\}$ + Stokes data at ∞.



top monodromy $\overset{g}{\in} G/\mathrm{Ad}(G)$

$(\mathcal{E}^\nabla$, filtered by decay rate$)$

Singular direction

SL_3, $\gamma = \gamma_0 t$, slope 1

$\left(\frac{\gamma_0}{\tau^2} + \dots\right) d\tau \qquad (\tau = t^{-1})$

$\beta \in B_W^+ = \langle s_1, \cdots s_r : \text{braid rel.} \rangle$

$\beta = s_{i_1} s_{i_2} \cdots s_{i_n}$, $\quad l(\beta) = n$

(reduced)

part of data

$$\mathcal{M}(\beta) = \left\{ B_0 \overset{s_{i_1}}{—} B_1 \overset{s_{i_2}}{—} \cdots \overset{s_{i_n}}{—} B_n = {}^{g}B_0 \right\} \Big/ Ad(G)$$

- Lusztig c.s.
- Shende - Treumann - Zaslow: relation to Stokes data
- Minh-Tam Trinh's thesis

Alternate def

$$M(\beta) = \left\{ \begin{array}{l} E_0, E_1, \cdots, E_n, \quad B\text{-torsors} \\ E_0 \xrightarrow{s_{i_1}} E \overset{s_{i_2}}{—} \cdots \overset{s_{i_n}}{—} E_n \underset{\text{isom.}}{\simeq} E_0 \end{array} \right\}$$

an isom. of

of B-torsors

$E_0 \overset{B}{\times} G \simeq E_1 \overset{B}{\times} G$

(G-torsors)

$$\begin{array}{c} B \\ \downarrow \\ T \end{array} = \text{univ. Cartan}$$

$$\begin{tikzcd} & M(\beta) \arrow[dl, "\text{formal monodromy}"'] \arrow[dr, "\text{top. monodromy}"] & \\ T/Ad_w(T) & & G/Ad(G) \end{tikzcd}$$

w = image of β in W

$\gamma : \nu = \frac{d}{m}$. $[w]$ reg. conj. class in W, order m

$\rightsquigarrow \beta_\gamma = \beta_\nu = \overbrace{\dot{w}\,\dot{w} \cdots \dot{w}}^{d} \in B^+_W$

(γ elliptic) $\quad \uparrow$ min. length rep. from $[w]$

Ex. $\nu = \frac{1}{h}$, $\gamma = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix}$

β = Coxeter elt $= s_1 s_2 \cdots s_r$

$\nu = 1 = \frac{h}{h}$,

$\beta = w_0 \cdot w_0$ "full twist"

Ex. G , $\nu = 1$.

$$\mathcal{M}(\beta) = \{ B_0 \underset{opp}{—} B_1 \underset{opp}{—} B_2 = {}^g B_0 \} / G$$

$$\underset{w_0 \cdot w_0}{\parallel}$$

$$= \{ \underbrace{B^+ — B^-}_{\text{fixed opp. pair}} — B_2 = {}^g B^+ \} / T$$

$$= \frac{B^+ B^-}{Ad(T)}$$

$$\begin{tikzcd} & \frac{B^+B^-}{Ad(T)} \arrow[dl] \arrow[dr, "\text{top. monodromy}"] & \\ \frac{T}{Ad(T)} & & \frac{G}{Ad(G)} \end{tikzcd}$$

(analytic stack)

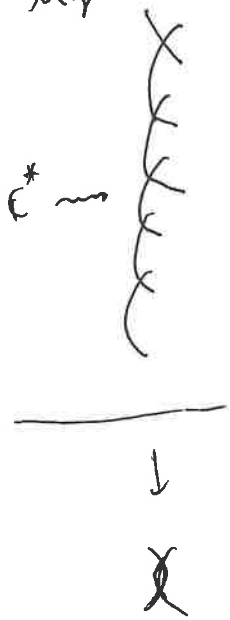
$$\begin{tikzcd} \mathcal{M}^{\text{Betti}}_\gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \frac{\widetilde{G}}{G} \times \mathfrak{t}_\gamma(0) \sim \mathfrak{h}^w \arrow[d, "\exp"] \\ \mathcal{M}(\beta) \arrow[r] & \frac{G}{G} \times \frac{T}{Ad_w T} \end{tikzcd}$$

Thm. $\exists$ analytic map $RH : \mathcal{M}^{dR}_\gamma \longrightarrow \mathcal{M}^{\text{Betti}}_\gamma$

Conj. RH is an analytic $\simeq$.

Ex. SL_2, $\nu = 1$.

$\mathcal{M}_\gamma$



$\circlearrowleft \mathbb{Z}$

$\uparrow$ translate $\mathbb{P}^1$ 2 points!

$$\begin{array}{c} \mathcal{M}^{\text{Betti}}_\gamma \\ \big\downarrow \ \mathbb{Z}\text{-torsor} \quad (\mathbb{Z} \text{ comes from } \ker \exp) \\ \widetilde{B^+ B^- \cap U} \\ \| \\ T^* \mathbb{P}^1 \setminus \Gamma\left(\frac{dt}{t}\right) \end{array}$$

homotopy equiv.

Wildly ram. geom. Langlands conj.

γ homog. slope ν

$$\underset{\substack{FR_\gamma \\ \uparrow \\ \text{Singular Support}}}{Sh}\left(\mathrm{Bun}_G(I_0, (J_{r,\infty}, \gamma))\right) \xhookrightarrow{\text{f.f.}} \mathrm{IndCoh}\left(\mathcal{M}^{\text{Betti}}_{\check{G}, \gamma^\vee}\right)$$

same slope ν

Evidence: $\nu = 1$.

$\nu = \frac{1}{m}$ in progress.