# Classification of irreducible representations of the Iwahori-Hecke algebra

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<u>Recall</u>: $\mathcal{H}_I \rightsquigarrow \mathcal{H}_a$ , $a = (s,t) \in G \times \mathbb{C}^\times$

$\mathcal{H}_I \simeq K^{G \times \mathbb{C}^\times}(Z)$

$\mathcal{H}_a \simeq \mathrm{Ext}^\bullet(\mathbb{L}, \mathbb{L})$ , $\mathbb{L} = \mu_* \mathbb{C}_{\tilde{N}^a} \simeq \bigoplus_\phi L_\phi \otimes IC_\phi$

$\phi$ are parametrized by nilp. orbits

& an irred. rep. $\chi$ of $\pi_0(G(y))$

$\rightsquigarrow$ a local sys. on the orbit.

To understand $L_\phi$ concretely, we exhibited them as the image of a map between std & costd modules.

The Std module $\rightsquigarrow H_\bullet^{BM}(\mathcal{B}_x^s)$

costandard module in terms of transversal slice ...

$H^\bullet(i_{\mathbb{O}}^! \mathbb{L}) \longrightarrow H^\bullet(i_{\mathbb{O}}^* \mathbb{L})$ The image is $L_{a,x,\chi} \simeq L_\phi$

$\uparrow$ std $\uparrow$ costd.

<u>Theorem 1.</u> For any semisimple elt $a = (s,t) \in G \times \mathbb{C}^\times$, and any $x \in \mathcal{N}^a$, $\chi \in C(s,x)^\wedge$, the $\mathcal{H}_a$-module $L_{a,x,\chi}$ is simple (provided that it's nonzero).

Two such modules $L_{a,x,\chi}$, $L_{a,x',\chi'}$ isom. iff $(x,\chi)$ and $(x',\chi')$ are $G(s)$-conjugate.

<u>Theorem 2</u> (Deligne - Langlands Conj.) Irred. $\mathcal{H}$-modules are parametrized by $G$-conj. classes of triples. $(s,x,\chi)$ where $sxs^{-1} = q\,x$, and $\chi \in C(s,x)^\wedge$ (i.e. $\chi$ shows up in $H_\bullet^{BM}(\mathcal{B}_x^s)$).

not a root of unity.

<u>Non-vanishing theorem</u>, (Kazhdan - Lusztig - Grojnowski - Ginzburg)

<u>Proposition A</u>. Assume $t \in \mathbb{C}^\times$ is not a root of unity. there exists a $G(s)$-stable union of conn'd compoments $\hat{\mathbb{O}}$ of $\mathcal{N}^a$ s.t. $\mu(\hat{\mathbb{O}}) = \overline{\mathbb{O}}$.

Let $\hat{\mathcal{B}}_x^s = \mathcal{B}_x^s \cap \hat{\mathbb{O}}$, (For any $G(s)$-orbit $\mathbb{O}$)

<u>Proposition B</u>. Assume $a = (s,t)$, $t$ not a root unity, then any simple $C(s,x)$-module occuring in $H^\bullet(\mathcal{B}_x^s)$ w/ nonzero mult. also occurs in $H^\bullet(\hat{\mathcal{B}}_x^s)$ w/ nonzero multiplicity.

<u>Theorem (non-vanishing theorem)</u> If $\chi$ shows up in $H_\bullet(\mathcal{B}^s_x)$, then $L_{a,x,\chi}$ is nonzero.

<u>Proof</u> Recall we have:

$$\mu_* \mathbb{C}_{\tilde{\mathcal{N}}^a} = \bigoplus_{\mathbb{O}, \chi} L_{a,x,\chi} \otimes IC(\mathbb{O}, \chi)$$

- The complex $\mu_* \mathbb{C}_{\tilde{\mathcal{N}}^a}$ contains $\mu_* \mathbb{C}_{\hat{\mathbb{O}}}$ as a direct summand.

$$\mu_* \mathbb{C}_{\hat{\mathbb{O}}} = \left(\bigoplus_{\chi} \hat{L}_\chi \otimes IC(\mathbb{O}, \chi)\right) \oplus B$$

supp. on the boundary $\bar{\mathbb{O}} \setminus \mathbb{O}$.

and $L_\chi$ are exactly the mult. of $\chi$ in $H^\bullet(\hat{\mathcal{B}}^s_x)$.

<u>Sketch of Proof of Prop A</u>:

<u>Lemma 1.</u> a) The group $G(s)$ is a conn'd reductive gp, and each conn'd component of $\mathcal{B}^s$ is a submanifold of $\mathcal{B}$ which is $G(s)$-equiv. isom. to the flag variety for $G(s)$.

use $DG$ simply connected.

b) If $t \in \mathbb{C}^\times$ is not a root of unity, then $\mathfrak{g}^a = \mathcal{N}^a$

consists only of nilp. elements.

<u>Lemma 2</u>. a) There is an embedding $sl_2(\mathbb{C}) \xhookrightarrow{\gamma} \mathfrak{g}$ associated to $x$ s.t.

$s = s_\gamma \cdot s_0$ where $s_0$ is a semisimple element commuting w/ the image of $\gamma$,

and $s_\gamma = \gamma\begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1}\end{pmatrix}$, $\tau^2 = t$.

b) There is a group homomorphism $v: \mathbb{C}^\times \to \mathbb{R}$ w/ $v(\tau) > 0$.

Let $\gamma, \tau, s_0$ be as in Lemma 2.

We have a wt space decomposition of $\mathfrak{g}$ by conjugation by $s_0$.

$\mathfrak{g} = \bigoplus_{\alpha\in\mathbb{C}^\times} \mathfrak{g}_\alpha$, and we define $\mathfrak{p} = \bigoplus_{v(\alpha)\leq 0} \mathfrak{g}_\alpha$, $\mathfrak{l} = \bigoplus_{v(\alpha)=0} \mathfrak{g}_\alpha$, $\mathfrak{u} = \bigoplus_{v(\alpha)<0} \mathfrak{g}_\alpha$

One can check that $x \in \mathfrak{l}$, and $s \in L$.

Let $\mathcal{P} \supset \mathcal{B}$ be the subvar. of all Borel subalgs $\mathfrak{b} \subset \mathfrak{p}$ (the flag var. for $L$)

Let $\mathcal{P}_x^s = \mathcal{B}_x^s \cap \mathcal{P} \neq \phi$, (It's the $s$ and $\exp(x)$-fixed pt set of the closed subset $\mathcal{P}$)

<u>Def</u> Let $\hat{\mathbb{O}}$ be the union of all connected components of $\tilde{\mathcal{N}}^a$ which have nonempty intersection w/ $\mathcal{P}_x^s$.

↳ It's easy to see that $\hat{\mathbb{O}}$ is $G(s)$-stable, and that the image of any conn'd component of $\hat{\mathbb{O}}$ contains $x$ ⇒ contains the $G(s)$-orbit $\mathbb{O}$.

Checking that $\varphi(\hat{\textcircled{I}}) = \bar{\textcircled{I}}$ reduces to:

<u>Lemma</u>: Let $\mathfrak{b} \in \mathcal{P}^s_x$ and $\mathfrak{n}$ is its nilradical, then

$$\mathrm{Ad}\, G(s) \cdot (\mathfrak{n} \cap \mathfrak{g}^a) = \bar{\textcircled{I}}. \qquad [\text{Very technical}]$$

## Sketch of Proof of Prop B.

idea. $[H^\bullet(\mathcal{B}^s_x) : \chi] \neq 0 \to [H^\bullet(\mathcal{P}^s_x) : \chi] \neq 0 \xrightarrow{(2)} [H^\bullet(\hat{\mathcal{B}}^s_x) : \chi] \neq 0.$

Let $Z = Z^\circ(L)$ be the identity component of the center of $L$, then $Z$ is a complex torus which commutes w/ both $x$ and $s$ (since $x \in \mathfrak{l}$, $s \in L$), so

$\mathcal{B}^s_x$ is a $Z$-stable variety of $\mathcal{B}$. Let $T$ be the max'l cpt of $Z$,
$\rightsquigarrow (\mathcal{B}^s_x)^Z = (\mathcal{B}^s_x)^T$

fixed pt reduction $\Rightarrow [H^\bullet(\mathcal{B}^s_x).] = [H^\bullet((\mathcal{B}^s_x)^{Z^\circ(L)})]$ in the Grothendieck gp of $L(s,x)$-modules $\rightsquigarrow$ this is also true in the Grothendieck gp of $L(s,x)/L(s,x)^\circ$-modules. $\rightsquigarrow$ holds as $G(s,x)/G(s,x)^\circ$-modules.

If $\chi$ is s.t. $[H^\bullet(\mathcal{B}^s_x) : \chi] \neq 0 \Rightarrow [H^\bullet((\mathcal{B}^s_x)^{Z^\circ(L)}) : \chi] \neq 0$.

then one can show that $(\mathcal{B}^s_x)^{Z^\circ(L)}$ is a disj't union of pieces isom. to $\mathcal{B}(L)^s_x \simeq \mathcal{P}^s_x$.

Finally, $P^s_x \subset \hat{B}^s_x \rightsquigarrow H^\bullet(P^s_x)$ is canonically a direct summand

of $H^\bullet(\hat{B}^s_x)^{Z^\circ(L)}$.

So we know (2) for $L(s,x)$-modules

$\overset{(2)\text{ for}}{\rightsquigarrow}$ $G(s,x)/G(s,x)^\circ$-modules $x$,

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$\mu_* \mathbb{C}_M \simeq \bigoplus L_\phi \otimes IC_\phi \quad (*)$

Thm. The multiplicity of the simple $\mathcal{H}_a$-module $L_\phi$ in the standard module

$H_\bullet(M_x)_\psi$ is given by the following formula:

$$[H_\bullet(M_x)_\psi : L_\phi] = \sum_k \dim H^k(i_x^! IC_\phi)_\psi.$$

Proof. Recall that if $\mathbb{L} = \mu_* \mathbb{C}_M$, this is the same as finding the mult. of the $R\mathrm{Hom}(\mathbb{L},\mathbb{L})$ module $L_\phi$ in the module $H^\bullet(i_x^! \mathbb{L})_\psi$.

Apply the functor $H^\bullet i_x^!$ to the decomposition. $(*)$, and we get

$$H_\bullet^{BM}(M_x) = H^\bullet(i_x^! \mu_* \mathbb{C}_M) = \bigoplus_\phi L_\phi \otimes H^\bullet(i_x^! IC_\phi).$$

For any $j,k$, we have $\mathrm{Ext}^k(\mathbb{L},\mathbb{L}) : \bigoplus_\phi L_\phi \otimes H^j(i_x^! IC_\phi)$

$$\to \bigoplus_\phi L_\phi \otimes H^{j+k}(i_x^! IC_\phi)$$

Define $F^p H^\bullet(i_x^! \mathbb{L}) = \bigoplus_{j \geq p} \left(\bigoplus_\phi L_\phi \otimes H^j(i_x^! IC_\phi)\right)$, it is $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-stable.

$\rightsquigarrow$ we can consider $gr^F H^\bullet(i_x^! \mathbb{L})$. Here the action factors through projection to $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L}) \simeq \bigoplus_\phi \mathrm{End}(L_\phi)$

As a vector space, $gr^F H^\bullet i_x^! \mathbb{L} \simeq \bigoplus_\phi L_\phi \otimes H^\bullet(i_x^! IC_\phi)$.

$\psi$ components $\rightsquigarrow$ $(gr^F H^\bullet i_x^! \mathbb{L})_\psi = \bigoplus_\phi L_\phi \otimes (H^\bullet(i_x^! IC_\phi))_\psi$

So $L_\phi$ occurs exactly $\dim (H^\bullet(i_x^! IC_\phi))_\psi$.

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$$\begin{tikzcd}
K^{\check{G}\times\mathbb{C}^\times}(Z) \arrow[r, phantom, "\simeq"] \arrow[d, squiggly, dash] & \mathcal{H}_I \arrow[r, phantom, "\simeq"] & \mathbb{C}[I\backslash G(k)/I] \arrow[d, squiggly, dash] \\
D^b Coh^{\check{G}}(Z) \arrow[rr, phantom, "\simeq"] & & D^b_I(Fl)
\end{tikzcd}$$