On the derived category of the Iwahori-Hecke algebra

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Let F fin $/\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, k res field, $q = |k|$.

C = (alg closed) field of char 0, fix $q^{1/2} \in C$.

<u>Recall</u> LLC for $GL_n(F)$

$$\left\{ \begin{matrix} \text{irred smooth rep'ns} \\ \pi \text{ of } GL_n(F) \text{ on } C\text{-v.s.} \end{matrix} \right\} \Big/ \simeq \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{matrix} n\text{-dim'l Frobenius semisimple} \\ \text{Weil-Deligne rep'ns} \\ \text{on } C\text{-v.s.} \end{matrix} \right\} \Big/ \simeq$$

<u>here</u>.

$$\begin{tikzcd} \mathrm{Gal}(F^{sep}/F) \arrow[d] & \underset{\text{Weil gp}}{W_F} \arrow[l, hook'] \arrow[d] & I_F \arrow[l, hook'] \arrow[d] \\ \mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbb{Z}} & \mathbb{Z} \arrow[l, hook'] & \{0\} \arrow[l, hook'] \end{tikzcd}$$

WD-rep'n: $\rho: W_F \longrightarrow GL_n(C)$ s.t. $\rho|_{I_F}$ factors through a fin. quot

$+ \; N: C^n \longrightarrow C^n$ nilpotent s.t. $q^{\|\sigma\|} \rho(\sigma) N = N \rho(\sigma)$.

Frob. semisimple: ρ semisimple.

<u>Easiest special case of LLC</u>: $\left\{ \begin{matrix} \text{irred rep'ns } \pi \\ \text{s.t. } \pi^I \neq 0 \end{matrix} \right\} \Big/ \simeq \;\; \simeq \left\{ \begin{matrix} \text{simple modules} \\ \text{over } \mathcal{H}_G \end{matrix} \right\} \Big/ \simeq$

Iwahori $I \subset GL_n(\mathcal{O}_F)$

$\mathcal{H}_G$ = Iwahori-Hecke alg $= \mathrm{End}_G(\text{c-ind}_I^G \mathbb{1})$

$\overset{1:1}{\longleftrightarrow}$ $\{ (\rho, N)$ WD-rep'ns, $\rho|_{I_F} = \mathrm{triv} \}/\simeq$

$= \{ (\varphi, N) \in GL_n(\mathbb{C}) \times \mathrm{Lie}(GL_n) : \varphi \text{ ss}, \ N\varphi = q\varphi N \} /_{\mathrm{conj}}$.

Aim lift this bijection to a fully faithful embedding of (derived) categories.

Rmk. more generally, $G|_F$ split reductive group, $G = G(F)$.

$\rightsquigarrow \mathcal{H}_G$ Iwahori-Hecke alg for some choice of Iwahori $I \subset G$.

$\{\text{simple } \mathcal{H}_G\text{-mod}\}/\simeq \longrightarrow \left\{ \begin{matrix} (\varphi, N) \in \check{G}(\mathbb{C})^{ss} \times \mathrm{Lie}\, \check{G} \\ \mathrm{Ad}(\varphi) N = q^{-1} N \end{matrix} \right\}/\simeq$ $\quad$ $\check{G}/\mathbb{C}$ dual of G

surj. w/ finite fibers, $\Big(\text{fiber over } (\varphi, N)\Big) \longrightarrow \Big(\text{irred rep'ns of centralizer of } (\varphi, N)\Big)$

depending on additional choice: Whitaker datum

— back to GL_n:

built using parabolic induction

(a) $(\varphi, N) = Sp(\lambda, n) = \left(\varphi = \begin{pmatrix} \lambda & & & \\ & q^{-1}\lambda & & \\ & & \ddots & \\ & & & q^{-(n-1)}\lambda \end{pmatrix}, N = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \right)$

$\rightsquigarrow LL(\varphi, N) = St(\lambda, n)$ generalized Steinberg

$=$ unique irred quotient of $i_B^G \big(\overbrace{\mathrm{unr}_\lambda \otimes \mathrm{unr}_{q^{-1}\lambda} \otimes \cdots \otimes \mathrm{unr}_{q^{-(n-1)}\lambda}}^{\text{unram. rep'n of } T}\big)$ $\quad$ $i_B^G = \mathrm{Ind}_B^G(\delta_B^{1/2} \otimes -)$

(b) $(\varphi, N) = \bigoplus_{i=1}^{s} Sp(\lambda_i, r_i)$

"modified" Langlands corresp.

$LL(\varphi,N)$ = unique irred quot of $LL^{mod}(\varphi, N)$

$= i_P^G\big(Sp(\lambda_1, r_1) \otimes \cdots \otimes Sp(\lambda_s, r_s) \big)$

for "correct" ordering of $\lambda_1, \dots, \lambda_s$.

<u>try to upgrade this to a functor</u>:

Rep G — cat of smooth rep's of G $(= GL_n(F))$

$\cup$ direct factor (Bernstein block)

$\mathrm{Rep}^I G$ — cat. of rep's π generated (as a G-rep'n) by π^I

$\wr\!\wr$

H_G-mod

$\rightsquigarrow$ $D^+(\mathrm{Rep}^I G) = D^+(H_G\text{-mod})$ (bdd below) derived cats

$\check{G}/\mathbb{C}$ dual gp

$X_{\check{G}} = \{ (\varphi, N) \in \check{G} \times \mathrm{Lie}\,\check{G} : \mathrm{Ad}(\varphi) N = q^{-1} N \}$

$\circlearrowleft$ $\subset$ $\check{G} \times \mathrm{Lie}\,\check{G}$ finite type scheme

$\check{G}$ closed

$\mathrm{QCoh}^{\check{G}}(X_{\check{G}})$ = cat. of $\check{G}$-equiv. q. coh. sheaves on $X_{\check{G}}$

quotient $[X_{\check{G}}/\check{G}]$

$\rightsquigarrow$ $D^+_{\mathrm{QCoh}}([X_{\check{G}}/\check{G}])$ (bdd below) derived cat of qcoh sheaves on the stack

$\mathrm{Rep}^I G \simeq \mathcal{H}_G\text{-mod}$ $\qquad\qquad$ $\mathrm{QCoh}^{\check{G}}(X_{\check{G}})$

$\circlearrowright$ $\qquad\qquad$ $\circlearrowright$

$\mathfrak{Z}_G =$ center of $\mathrm{Rep}^I G$ $\qquad\qquad$ ring of $\check{G}$-inv't functs on $X_{\check{G}}$

$=$ center of $\mathcal{H}_G$ $\qquad\qquad$ $\|$

$\Gamma(\check{T}/W, \mathcal{O}_{\check{T}/W})$

$\|$ $\qquad$ $W =$ Weyl gp

$\mathbb{C}[X^*(\check{T})]^W$ $\qquad$ $\check{T} \subset \check{G}$ dual torus

$\|$

$\mathfrak{Z}_G = \mathbb{C}[X_*(T)]^W$

<u>Conj.</u> For all (G, B, T) as above, (+ choice of Whittaker datum ψ)

$\exists$ $\mathfrak{Z}_G$-linear fully faithful functor

$$R_G^{\psi} : D^+(\mathrm{Rep}^I G) \longrightarrow D^+_{\mathrm{QCoh}}([X_{\check{G}}/\check{G}])$$

Rt.

- if $G = T$ torus, R_T induced by

$$\begin{aligned}
\mathrm{Rep}^{T^0}(T) &= \mathbb{C}[T/T^0]\text{-mod} \\
&= \mathbb{C}[X_*(T)]\text{-mod} \\
&= \mathbb{C}[X^*(\check{T})]\text{-mod} \\
&= \mathrm{QCoh}(X_{\check{T}}), \quad X_{\check{T}} = \check{T}
\end{aligned}$$

$+$ trivial $\check{T}$-action.

- for $B \subset P \subset G$ parabolic w/ Levi M,

$$\xi^G_P : R_G \circ Z^G_P \xrightarrow{\sim} R\beta_{P,*} \circ L\alpha^*_P \circ R_M$$

$$\alpha_P : [X_{\check{\mathfrak{p}}} / \check{P}] \longrightarrow [X_{\check{\mathfrak{m}}} / \check{M}] \quad \text{induced by } \check{P} \to \check{M}$$

$$\beta_P : [X_{\check{\mathfrak{p}}} / \check{P}] \longrightarrow [X_{\check{\mathfrak{g}}} / \check{G}] \quad \text{induced by } \check{P} \to \check{G}.$$

+ compatibilities among the various ξ^G_P.

Rmk - Similar conj. / results in ongoing work of

- Ben-Zvi - Harrison - Helm - Nadler

- X. Zhu

- Why derived?

Compare flat structures w/ non-flat structures?

e.g. $\mathcal{H}_G$ flat / $\mathfrak{z}_G$

$$X_{\check{G}} \to \check{G} \to \check{T}/W$$

$$(\varphi, N) \mapsto \varphi \mapsto \text{char poly of } \varphi \qquad \underline{\text{not flat}}.$$

irred comp's of $X_{\check{G}}$ $\xleftrightarrow{1:1}$ Jordan canonical forms of N

$N \neq 0$. $N\varphi = q\varphi N$ enforces relations between eigenvalues of φ.

Pages

- in fact: "compatibility of various ζ^G_P"

forces us to define $X_{\check{P}} \subset \check{P} \times \mathrm{Lie}\,\check{P}$ as <u>the derived</u> zero locus of

$$\mathrm{Ad}(\varphi) N = q^{-1} N.$$

needed for GL_n if $n \geq 6$

- dependence on Whittaker datum

$\psi : N \longrightarrow \mathbb{C}^\times$ generic char. $\quad N \subset B$ unip. radical

$$R^\psi_G \big(\underbrace{(\text{c-ind}^G_N \psi)_{[T,1]}}_{} \big) = \mathcal{O}_{[X_{\check{G}}/\check{G}]}.$$

image of $\text{c-ind}^G_N \psi$

in the block $\mathrm{Rep}^I G$

<u>Results</u>.

<u>Thm</u> Conj. true for GL_2.

<u>Thm</u> ($G = GL_n$)

$\exists$ candidate $\quad R_G = - \overset{\mathbb{L}}{\underset{\mathcal{H}_G}{\otimes}} M_G$

$$D^+(\mathcal{H}_G\text{-mod}) \longrightarrow D^+_{\mathrm{Coh}}([X_{\check{G}}/\check{G}])$$

s.t.
- R_G is Z_G-linear
- satisfies compatibilities w/ parabolic induction over $X^{\mathrm{reg}}_{\check{G}} \subset X_{\check{G}}$ large open subset
- $R_G\big((\text{c-ind}^G_N \psi)_{(T,1)} \big) = \mathcal{O}_{[X_{\check{G}}/\check{G}]}$

– φ reg semisimple,

$$R_G\left(LL^{mod}(\varphi, N)\right) = \mathcal{O}\left[\overline{\check{G}(\varphi,N)/\check{G}}\right]$$

– " Zelevinsky involution $\simeq$ Serre duality (up to a small modification)"

How to construct $\mathcal{M}_G$?

A) comp. w/ parabolic induction enforces

$$\mathcal{M}_G = R\beta_* \, \mathcal{O}_{[X_{\check{B}}/\check{B}]} \qquad \beta : [X_{\check{B}}/\check{B}] \to [X_{\check{G}}/\check{G}]$$

$$\circlearrowleft$$

$$\mathcal{O}_{X_{\check{G}}} \underset{\mathfrak{z}}{\otimes} \mathcal{H}_T \qquad \mathcal{H}_T = \Gamma(\check{T}, \mathcal{O}_{\check{T}})$$

+ extend to $\mathcal{H}_G$- action.

B) " R_G expected to satisfy local-global compatibility"

$\rightarrow$ $\mathcal{M}_G$ = I – invts in $\mathcal{V}_G$

" family of G– rep'ns on $X_{\check{G}}$ interpolating modified LLC "

(Emerton – Helm)

i.e. $\left(c\text{-ind}_N^G \psi\right)_{[T,1]} \underset{\mathfrak{z}_G}{\otimes} \mathcal{O}_{X_{\check{G}}} \twoheadrightarrow \mathcal{V}_G$ + at generic pts (conjecturally everywhere) $x = (\varphi, N)$

$$\left(V_G \otimes k(x)\right)^\vee = \quad LL^{mod}\left(\left(\varphi^{ss}_{,N}\right)^\vee\right)$$

<u>Idea</u> use B) to extend the H_T-action in A) to an H_G-action.

- identify

$$V_G^I = R\beta_* \mathcal{O}[X_{\check{B}}/\check{B}] \quad \text{as} \quad \mathcal{O}[X_{\check{G}}/\check{G}] \otimes H_T\text{-modules}$$

works fine over regular locus $X_{\check{G}}^{reg} = \{(\varphi, N) \text{ stabilizes only finitely many flags}\}$

<u>Expect</u>: $j : X_{\check{G}}^{reg} \hookrightarrow X_{\check{G}}$

complement codim ≥ 2

$R\beta_* \mathcal{O}[X_{\check{B}}/\check{B}]$ should sit in deg 0 + max. Cohen-Macaulay module

(Ok GL_2, GL_3)

$\to$ $R\beta_* \mathcal{O}[X_{\check{B}}/\check{B}] = j_*\left(R\beta_* \mathcal{O}[X_{\check{B}}/\check{B}] \big|_{X_{\check{G}}^{reg}}\right)$

$\rightsquigarrow$ H_G-action extends.