Equivalence (I)

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§1. Review

G connected reductive gp $/\mathbb{k}$

$\check{G}$

$$St_{Nilp} = \tilde{\mathcal{N}} \overset{\mathbb{L}}{\underset{\check{\mathfrak{g}}^*}{\times}} \tilde{\mathcal{N}}$$

$$St' = \tilde{\mathfrak{g}} \underset{\check{\mathfrak{g}}^*}{\times} \tilde{\mathcal{N}}$$

$$St = \tilde{\mathfrak{g}} \underset{\check{\mathfrak{g}}^*}{\times} \tilde{\mathfrak{g}}$$

Done:

1) $\Phi_{diag} : \mathrm{Perf}(\tilde{\mathcal{N}}/\check{G}) \longrightarrow \mathrm{Shv}_c(I\backslash LG/I)$

2) anti-spherical projector Ξ

3) $\Phi_{perf} : \mathrm{Perf}(St_{Nilp}/\check{G}) \longrightarrow \mathrm{Shv}_c(I\backslash LG/I)$

$$\begin{array}{c} \mathrm{Perf}(St_{Nilp}/\check{G}) \\ \wr\!\wr \ [\mathrm{BZFN}] \\ \mathrm{Perf}(\tilde{\mathcal{N}}/\check{G}) \underset{\mathrm{Perf}(\check{\mathfrak{g}}^*/\check{G})}{\otimes} \mathrm{Perf}(\tilde{\mathcal{N}}/\check{G}) \end{array}$$

$$(M, N) \longmapsto \Phi_{diag}(M) * \Xi * \Phi_{diag}(N)$$

Φ_{perf} is fully faithful.

4) Passing to right adjoint

$\Phi : \mathrm{Shv}_c(I\backslash LG/I) \longrightarrow \mathrm{QCoh}(\mathrm{St}_{Nilp}/\check{G})$

Goal: Show Φ is monoidal equiv. onto Coh.

Prop. Assume that

1) Φ lands in Coh

2) Φ has finite cohomological amplitude.

3) $\exists\, d \in \mathbb{Z}$ s.t. $\forall\, \mathcal{F} \in \mathrm{Shv}_c(I\backslash LG/I)$, if $H^i(\Phi(\mathcal{F})) = 0$ for $i \in [-d, d]$,

$\Rightarrow {}^pH^0(\mathcal{F}) = 0$.

Then Φ is an $\simeq$ into Coh.

§a. Coherence

$\mathcal{F} \in \mathrm{Shv}_c(I\backslash LG/I)$, want:

(A) $\forall\, n$, $\mathrm{Hom}^\circ(-, \mathcal{F})\big|_{\mathrm{Perf}^{\geq n}}$ is rep. by an object in Perf.

(B) $\exists\, m$, $\mathrm{Hom}^\circ(-, \mathcal{F})\big|_{\mathrm{Perf}^{\leq m}} = 0$.

$\mathrm{St}_{Nilp}/\check{G} \xrightarrow{\text{proper}} \check{\mathfrak{g}}/\check{G}$, $\mathcal{L} = \mathcal{O}(\lambda, \mu)$

$\widetilde{\Phi}(-) := \bigoplus_n H^0\, \mathrm{Hom}^{\check{G}}_{\mathrm{deeq}}(\Phi_{\mathrm{perf}}(\mathcal{L}^{\otimes -n}), -)$

(A) $\Leftrightarrow$ $\forall\, \mathcal{F} \in Shv_c(I\backslash LG/I)$,

(1) $\tilde{\Phi}(\mathcal{F}[n])$ is f.g. over $R^0\Gamma(\mathcal{L}^{\otimes}) := \bigoplus_{m\geq 0} R^0\Gamma(\mathcal{L}^{\otimes m})$, $\forall\, n$

(2) $\tilde{\Phi}(\mathcal{F}[n]) = 0$, $\forall\, n \gg 0$.

Lemma. (1) is true.

Proof ① $\mathcal{F}$ is in the image of Φ_{perf}, then (1) is true for $\mathcal{F}$.

$\Rightarrow \mathcal{F} = J_\alpha * \Xi * J_\beta$, then (1) is true for $\mathcal{F}$

② $Shv_c(I\backslash LG/I)$ is gen. by $J_\alpha * \mathcal{G} * J_\beta$, $\mathcal{G} \in Perv(B\backslash G/B)$

Prop.
③ Want to show $\tilde{\Phi}(\mathcal{F} * J_{\check{\rho}}(m)) = 0$, $\forall\, \mathcal{F} \in Perv(B\backslash G/B)$, $\forall\, m \neq 0$.

Prop implies $\tilde{\Phi}(\Xi * J_{\check{\rho}})$ is f.g. $\Rightarrow$ $\tilde{\Phi}(\mathcal{F} * J_{\check{\rho}})$ is f.g. for $\mathcal{F} \in Perv(B\backslash G/B)$

$\Uparrow$ everything above needs to be replaced by monodromic sheaves

Proof. Check for $\mathcal{F} = \Delta_w^{mon}, \nabla_w^{mon}$.

$\mathrm{Ext}^i(J_{-\alpha} * \Xi * Z_\nu * J_{-\beta}, \mathcal{F}) = 0$, $\forall\, i \neq 0$, $\forall\, \alpha, \beta, \nu \in X_*(T)^+$, $\beta \in X_*(T)^{++}$.

- $\ell(\alpha \cdot w) = \ell(\alpha) + \ell(w)$, $\quad w \in W_{fin}$ $\quad \Rightarrow \quad \nabla_{\alpha \cdot w} \simeq \nabla_\alpha * \nabla_w$
- $\ell(w\beta) = \ell(\beta) - \ell(w)$ $\quad \Delta_w * \nabla_\beta \simeq \nabla_{w \cdot \beta}$

$$\mathrm{Ext}^i(J_{-\alpha} * \Xi * Z_V, \nabla)$$

$$\mathrm{Ext}^i(\Xi * Z_V * J_{-\beta}, \nabla) \quad , i \neq 0.$$

$\square$

Thm (Lusztig)

$\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(I \backslash LG / I)$, $\quad$ Xinwen: convol. of $L^+G \backslash LG / L^+G$ t-exact

maybe gives an estimate

$\mathcal{F} * \mathcal{G}$ has perverse degrees bounded in $[-\dim G/B, \dim G/B]$.

Cor $\exists\ d \in \mathbb{Z}$ s.t.

$$\mathrm{Ext}^k(J_{-\lambda} * Z_V * \Xi * J_{-\mu}, \mathcal{F}) = 0, \ \forall \mathcal{F} \in \mathrm{Perv}(I \backslash LG / I), \ \forall i \notin [-d, d]$$

Proof of Cor

Lusztig's Thm $\Rightarrow$ $J_\lambda * \mathcal{F} * J_\mu$ has perv deg in $[-2\dim G/B, 2\dim G/B]$

Need to prove boundedness for $\mathrm{Ext}^k(Z_V * \Xi, \mathcal{F})$

$\mathcal{F}$: perverse

$$\mathrm{Map}(Z_V * \Xi, \mathcal{F}) = \mathrm{Map}(Av_! Av_*(Z_V), \mathcal{F}) \simeq \mathrm{Map}(Av_*(Z_V), Av_*(\mathcal{F}))$$

$$\simeq \mathrm{Map}(\underline{V_V}, \Phi_{AB}^{-1}(Av_*(\mathcal{F})))$$

$\Phi_{AB}^{-1}(\mathcal{G})$ ($\mathcal{G}$: perverse) has bounded degree $[0, \dim \tilde{\mathfrak{g}}^*]$ (maybe $[-\dim \tilde{\mathfrak{g}}^*, 0]$)

§3. Equivalence

Prop $\exists d > 0$, s.t. $\forall \mathcal{F} \in Shv_c(I \backslash LG / I)$ satisfying

$$\mathrm{Hom}^i(\Phi_{perf}(\mathcal{O}(\lambda, \mu) \otimes V), \mathcal{F}) = 0,$$

$\forall i \in [-d, d]$, $\forall \lambda, \mu \in X_*(T)$, $\forall V \in Rep(\check{G})$

$\Rightarrow {}^pH^0(\mathcal{F}) = 0$

Pf · Claim for $\lambda \in X_*(T)$ large enough,

$\mathcal{F} * J_\lambda$ is concentrated in deg $\geq d - 2\dim \tilde{\mathfrak{g}}$ & $\leq -d + 2\dim \tilde{\mathfrak{g}}$

perverse cohomology

Granted this claim, $\mathcal{F} * J_\lambda * J_{-\lambda}$ $\geq d - 2\dim\tilde{\mathfrak{g}} - \dim G/B$

$\leq -d + 2\dim\tilde{\mathfrak{g}} + \dim G/B$

proof of claim

test on $\Delta_w = j_{w!}$

$!\text{-supp}(\mathcal{F} * J_\lambda) \subset S \subset W_{ext}$

finite set

① $!\text{-supp}(\mathcal{F} * J_\lambda) \subset (!\text{-supp}(\mathcal{F})) * (!\text{-supp}(J_\lambda))$

② if λ is large enough,

$S \cdot \lambda \subset W_{fin} \cdot X_*(T)_+$

each element is minimal in its right coset W_{ext}/W_{fin}

Lemma. $d > 2\dim\tilde{\mathfrak{g}} + \dim G/B$

$\mathcal{F}$ as above, λ large enough, then for $\forall w \in W_{ext}$,

we have either $\quad \mathrm{Hom}(\Delta_w, \mathcal{F} * J_\lambda) = 0$

or $\mathrm{Hom}(\Delta_w, \mathcal{F} * J_\lambda) \simeq \mathrm{Hom}(\Delta_w * \Xi, \mathcal{F} * J_\lambda)$

Granted this lemma $\Rightarrow$ Claim

$$\begin{gathered} \wr\wr \\ \mathrm{Hom}(Av_* \Delta_w, Av_*(\mathcal{F} * J_\lambda)) \\ \simeq \mathrm{Hom}(\Phi_{AB}^{-1}(Av_* \Delta_w), \Phi_{AB}^{-1}(-)) \end{gathered}$$

- Φ_{AB}^{-1} bounded $[0, \dim\tilde{\mathfrak{g}}]$.
- Assumption $\Rightarrow$ vanishing for $\mathcal{F} * J_\lambda$
- $\mathrm{Coh}(\tilde{\mathfrak{g}}/\check{h})$ has bounded coh. dim $\dim\tilde{\mathfrak{g}}$.

w min in right coset

- $\Delta_w * \Xi$ has a filtration w/ graded being

$$\bigoplus_{w_f \in W_{fin}} \Delta_{ww_f}.$$

- $\mathrm{Hom}(\Delta_{ww_f}, \mathcal{F} * J_\lambda) = 0, \quad \text{if } w_f \neq id.$

Rmk. (1) Ξ tilting property, convolution property.

(2) Φ_{diag} (Φ_{perf}), $\mathrm{Perf}(X \times_Y X) \longrightarrow \mathrm{Coh}(X \times_Y X)$

$$\begin{gathered} \wr\wr \\ \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(Y)} \mathrm{Perf}(X) \end{gathered}$$

$(M, N) \mapsto (\Delta_* M) * \mathcal{O} * (\Delta_* N)$