No Enriques surfaces over $\mathbb{Z}$

Daniel Kim

Let $Y/\mathbb{Z}$ be an Enriques surface.

Thm In this case, $\mathrm{Pic}_{Y/\mathbb{Z}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\oplus 10}$ is a trivial scheme / sheaf.

$\rightsquigarrow \mathrm{Pic}_{Y_{\mathbb{F}_2}/\mathbb{F}_2} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\oplus 10}$ is trivial.

Prop (4.2 + ...) Let $Y_{\bar{\mathbb{F}}_2}/\bar{\mathbb{F}}_2$ be an Enriques surface s.t. $\mathrm{Pic}^{\tau}_{Y_{\bar{\mathbb{F}}_2}/\bar{\mathbb{F}}_2} \simeq \mathbb{Z}/2\mathbb{Z}$

"classical", $\Rightarrow \exists$ a genus-one fibration $\varphi : Y_{\bar{\mathbb{F}}_2} \to \mathbb{P}^1_{\bar{\mathbb{F}}_2}$ w/ exactly two multiple fibers $2F_1, 2F_2$ & $\omega_Y \simeq \mathcal{O}_{Y_{\bar{\mathbb{F}}_2}}(F_1 - F_2)$ (2-torsion, nontrivial)

& intersection form on $\mathrm{Num}_{Y_{\bar{\mathbb{F}}_2}/\bar{\mathbb{F}}_2}$ is unimodular

$\left(\simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8 \right)$

Def. A genus-one fibration is a proper map s.t. fibers are arithmetic genus 1. (base curve $\Longrightarrow$ gen. fib are geom. reduced)

Kodaira symbol

Y is a minimal surface

Let $Y_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ be a genus one fibration. Consider a fiber $C = m_1 C_1 + \dots + m_r C_r$ where C_i are integral curves / $\bar{k}$.

① $C_i \cdot C = 0$ for all i

④ $P_a(C) = 1$

② $C_i \cdot C_j \geq 0$ & >0 when they meet $\rightsquigarrow \sum_{i=1}^{r} m_i \left[1 - P_a(C_i) + \frac{1}{2}(C_i \cdot C_i) \right] = 0$

③ $C_i \cdot C_i \neq -1$. [always ≤ 0 & <0 if $r \geq 2$]

$\Rightarrow$ Either $r = 1$ or $C_i \cdot C_i \leq -2, \quad \forall i$

$$\begin{cases} C_i \cdot C_i = -2 \\ p_a(C_i) = 0 \end{cases} \quad \text{for all } i$$

$$C_i \simeq \mathbb{P}^1_{\bar{k}}$$

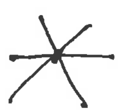
$r = 1$

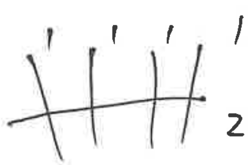
- I_0 $C = C_1$ elliptic curve
- I_1 $C = C_1$ nodal
- II $C = C_1$ cusp

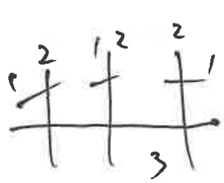
$r \geq 2$

I_r  n-gon

III  tangential double pt

IV 

I_0^*: 

IV^* 

§ Using triviality of Pic

Prop Let k be a perfect field, $Y_k \to \mathbb{P}^1_k$ a genus one fibration.
$\mathrm{Pic}_{Y/k}$ is trivial

Then ① for every closed pt $a \in \mathbb{P}^1_k$, the irred comp of Y_a are geom. irred. $/k(a)$

② If $a \in \mathbb{P}^1_k$ is not a k-pt $\Rightarrow$ Y_a is geom. irred. $/k(a)$

Pf. ② $(Y_a) \underset{k}{\otimes} \bar{k} \simeq \coprod_{k(a) \hookrightarrow \bar{k}} Y_a \underset{k(a)}{\otimes} \bar{k}$

If $C \subset Y_a \underset{k(a)}{\otimes} \bar{k}$ is an irred comp, $\forall\, \sigma \in \mathrm{Gal}_k$, $\sigma(C) \overset{\mathrm{Num}}{\sim} C$

$\Rightarrow \sigma(C) \cdot C = C \cdot C \leq 0$ w/ < 0 if $r \geq 2$

So either $\sigma(C) = C$ or $\sigma(C)$ & C are disjoint w/ $|C| = |Y_a \underset{k(a)}{\otimes} \bar{k}|$.

can't be true for al σ & C

① $C \subset Y_a \underset{k(a)}{\otimes} \bar{k}$ be an irred comp. $\Rightarrow \forall\, \sigma \in \mathrm{Gal}(\bar{k}|k(a))$,

$\sigma(C) \cdot C = C \cdot C \leq 0$ either $C \cdot C = 0$ & $r = 1$ done | or $C \cdot C < 0$ & $\sigma(C) = C$, $\forall\, \sigma \in \mathrm{Gal}(\bar{k}|k(a))$ $\Rightarrow$ also done.

char $k = 2$

Thm k perfect, Y/k Enriques surface s.t. $\mathrm{Pic}_{Y/k}$ trivial

① Every (-2)-curve on $Y_{\bar{k}}$ is defined $/k$. Moreover, it is $\simeq \mathbb{P}^1_k$ over k.

② Every gen-one fib $Y_{\bar{k}} \xrightarrow{\varphi} \mathbb{P}^1_{\bar{k}}$ is def $/k$

③ φ will have exactly two multiple fibers lying over k-pts of $\mathbb{P}^1_k$.

④ Each multiple fiber is either an ord. elliptic curve or not semistable

(i.e. $\frac{\text{fib}}{2}$ is not semistable)

Pf ① $\bar{E} \subset Y_{\bar{k}}$ a (-2)-curve.

$\forall \sigma \in \mathrm{Gal}(\bar{k}/k)$, $\sigma(\bar{E})\cdot \bar{E} = \bar{E}\cdot\bar{E} = -2 \Rightarrow \bar{E} = \sigma(\bar{E}) \rightsquigarrow$ descends to k.

(-2)-curve has $p_a(E) = 0 \Rightarrow E$ has to be a form of $\mathbb{P}^1_k$

$\mathcal{O}_{Y_k}(E) \in \mathrm{Pic}$ is primitive (-2 is square-free)

$\Rightarrow \exists\ \mathcal{L}$ l.b. $/Y$ s.t. $\deg_E(\mathcal{L}) = 1$. (unimodularity of Num)

② Consider $C \subset \varphi^{-1}(a)$, $a \in \mathbb{P}^1_{\bar{k}}$, $\rightsquigarrow C \sim \sigma(C) \Rightarrow \sigma(C)\cdot F = 0$

(F is the fiber class of φ)

$\Rightarrow \sigma(C)$ is in a fiber

$\Rightarrow \sigma$ sends fibers to fibers

$\Rightarrow$ can descend φ to $Y \longrightarrow B$, $B_{\bar{k}} \simeq \mathbb{P}^1_{\bar{k}}$.

Consider $G \subset Y_{\bar{k}}$ s.t. $2G$ is a fiber. Choose $\mathcal{L}/Y$ s.t. $\mathcal{L} \otimes \bar{k} \simeq \mathcal{O}_{Y_{\bar{k}}}(G)$.

If G contains (-2)-curve $\Rightarrow$ by ① contains k-pt $\Rightarrow G$ lies over a k-pt of B

Otherwise G is irred, $h^0(\mathcal{O}_G) = h^1(\mathcal{O}_G) = 1 \Rightarrow \exists\ \mathcal{L}'$ s.t. $\mathcal{L}\cdot\mathcal{L}' = 1$ ...

④ If $2F$ is such a mult. fiber $\Rightarrow \mathcal{O}_Y(F)|_F$ is nontriv. order 2.

($H^1(\mathcal{O}_Y) = 0$)

* S.S. elliptic curves don't have order 2 line bundles
* semistable curves $\mathrm{Pic} \simeq \mathbb{G}_m^k \Rightarrow$ no order 2 line bundles.

Combinatorics

Prop. $Y/\mathbb{F}_2$ as before (Enriques, triv Pic, ..)

$$Y(\mathbb{F}_2) = 1 + 10\cdot 2 + 2^2 = 25 \quad > 3.$$

Pf Betti #s: $b_0 = 1$, $b_2 = 10$, $b_4 = 1$

and $H^{2j}_{et}(Y_{\bar{\mathbb{F}}_2}, \mathbb{Q}_\ell)$ are gen. by cycles $/\mathbb{F}_2$

$\Rightarrow H^{2j}_{et}(Y_{\bar{\mathbb{F}}_2}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_j(-j)^{\oplus b_j}$. Lefschetz trace $\Rightarrow$ $\square$.

$\varphi: Y \to \mathbb{P}^1_{\mathbb{F}_2}$

Prop. (11.2) There is no $Y/\mathbb{F}_2$ s.t. the fibers over $0, 1, \infty$ are $\mathrm{III} + E_4 + I_8$.

multiple

elliptic curve w/ 4 pts ordinary

multiple

8-gon

simple

Pf. $\varphi^{-1}(0) = 2(D_0 + D_1)$

$\varphi^{-1}(1) = 2E_4$

$\varphi^{-1}(\infty) = C_0 + \cdots + C_7$

Fact. $\exists$ a second genus one fibration $\psi: Y \to \mathbb{P}^1_{\mathbb{F}_2}$ s.t. $\varphi^{-1}(*)\cdot\psi^{-1}(*) = 4$.

(general fact)

Consider $\psi^{-1}(0)$, $\psi^{-1}(1)$, $\psi^{-1}(\infty)$

mult

simple

say $\psi^{-1}(0) = 2\sum_{i=0}^{\ell-1} m_i \Theta_i$, $\Rightarrow \frac{1}{2}\varphi^{-1}(0)\cdot\frac{1}{2}\psi^{-1}(0) = (D_0 + D_1)(m_0\Theta_0 + \cdots + m_\ell\Theta_{\ell-1})$

$= 1$

Say $D_0 \cdot (m_0\Theta_0 + \cdots + m_{r-1}\Theta_{r-1}) = 0$, $D_1 \cdot (m_0\Theta_0 + \cdots + m_{r-1}\Theta_{r-1}) = 1$

$D_1 \xrightarrow[\sim]{\varphi} \mathbb{P}^1$

$D_1 \cdot \Theta_0 = 1$. $D_1 \cdot \Theta_i = 0$ for $i > 0$

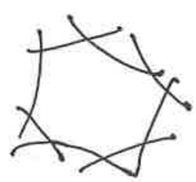
$D_0 \neq \Theta_0, \Theta_1, \ldots, \Theta_{r-1}$ because $D_0 \cdot D_1 = 2$.

$\Rightarrow D_0 \cdot \Theta_i \geq 0 \quad \Rightarrow D_0 \cdot \Theta_i = 0, \forall i \qquad m_0 = 1$.

$(D_0 + D_1) \cdot \Theta_i = 0$, $\forall\ 1 \leq i \leq r-1$, $\Rightarrow$ Θ_i is in a fiber of φ

but $\Theta_1 + \Theta_2 + \cdots + \Theta_{r-1}$ is connected or empty because look at the list of Kodaira symbols & take away a curve of mult 1 $\rightsquigarrow$ connected

$\to$ contained in single fiber of φ.

$\to \Theta_1, \ldots, \Theta_{r-1} \subset \varphi^{-1}(\infty) =$ 

$\Rightarrow$ it is a linear chain of $\mathbb{P}^1$'s. $\Rightarrow \psi^{-1}(0)$ is of the form II or III or IV.

Similarly for $\psi^{-1}(1)$ $\Rightarrow \ldots \Rightarrow$ fibers of ψ is also $\mathrm{III} + E_4 + I_8 \Rightarrow \ldots \Rightarrow X$.