Tilting property of Iwahori - Whittaker average of central sheaves

Vaughan McDonald

1) Introduction
2) Multiplicities
3) Propagation via $\otimes$, minuscule case
4) Quasi-minuscule case

Last time: defined a functor

$\Delta_0^{IW} \star^I (-) = Av_{IW} : D^b_I(Fl_G) \longrightarrow D^b_{IW}(Fl_G)$

t-exact, induces a fully faithful functor on a quotient

$$\begin{tikzcd} P_I \arrow[r] \arrow[dr] & P_I^{asph} \arrow[r] \arrow[d, equal] & P_{IW} \\ & P_I / \langle IC_w^I ;\ w \notin {}^f W \rangle \arrow[ur] & \end{tikzcd}$$

${}^f W = \{w : w \text{ is minimal in } W_f w\}$

Goal of talk:

Study the composite

$Z^{IW} = Av_{IW} \circ Z : Rep(\check{G}) \longrightarrow P_{IW}$

has standard Δ_λ^{IW}, $\lambda \in X^\vee$
costandard ∇_λ^{IW}

endows P_{IW} w/ the structure of a "highest wt cat".

$\rightsquigarrow$ robust notion of a <u>tilting object</u>.

$\mathcal{F} \in P_{IW}$ is <u>tilting</u> if it has a filtration w/ subqts of form Δ_λ^{IW}

+ filtration ∇_λ^{IW}.

<u>Some facts</u>: For any $\lambda \in X^\vee$, $\mathcal{F}$ tilting, the # of occurrences of Δ_λ^{IW} (resp. ∇_λ^{IW})

is indep. of choice of filtration.

Denote these #'s by $(\mathcal{F} : \Delta_\lambda^{IW})$ resp. $(\mathcal{F} : \nabla_\lambda^{IW})$.

$$\begin{array}{ccc} (\mathcal{F} : \Delta_\lambda^{IW}) & & (\mathcal{F} : \nabla_\lambda^{IW}) \\ \| \text{ easy fact} & & \| \\ \dim_{D^b_{IW}} (\mathcal{F}, \nabla_\lambda^{IW}) & & \dim_{D^b_{IW}} (\Delta_\lambda^{IW}, \mathcal{F}) \end{array}$$

<u>Main thm</u>: For all $V \in \mathrm{Rep}(G^\vee)$, $Z^{IW}(V)$ is tilting, and moreover

$$(Z^{IW}(V) : \Delta_\lambda^{IW}) = (Z^{IW}(V) : \nabla_\lambda^{IW}) = \dim (\underset{\substack{\uparrow \\ \text{eigenspace}}}{V_\lambda}) \qquad (\text{mult})$$

To show tilting, we will use

<u>Lemma</u>. $\mathcal{F} \in P_{IW}$ is tilting $\iff$ $\forall \lambda \in X^\vee$, $(j_\lambda^{IW})^* \mathcal{F}$, $(j_\lambda^{IW})^! \mathcal{F}$ are

concentrated in deg $-\deg(Fl_{G,\lambda}^{IW})$.

§2. Multiplicities

Key prop^{n} is as follows:

Prop. $V \in Rep(G^\vee)$, $\mu \in X^\vee$, then

$$\chi_\mu(Z^{IW}(V)) = \sum_{i \geq 0} (-1)^i \dim Hom_{D^b_{IW}}(\Delta^{IW}_\mu, Z^{IW}(V)[i])$$

$$= \dim V_\mu$$

Similar formula for costd.

Prop $\Rightarrow$ (Mult) since tilting $\Rightarrow$ $\forall i > 0$, $\dim Hom_{D^b}(\Delta^{IW}_\mu, Z^{IW}(V)[i]) = 0$

$$\chi_\mu(Z^{IW}(V)) = (\mathcal{F} : \nabla^{IW}_\lambda).$$

Proof of Prop^n. $\mathcal{F} \longmapsto \chi_\mu(\mathcal{F})$ factors through the Grothendieck gp $K_0(D^b_{IW}(\mathcal{F}l_G))$.

Want to compute $[Z^{IW}(V)]$.

Recall for perverse sheaves $\mathcal{F} \in P_I$ admitting a Wakimoto filtration by $J_\lambda = J_\lambda(\bar{\mathbb{Q}}_\ell)$, there is an assoc. graded functor : $Grad : P_I \longrightarrow Vect$

$= \bigoplus_{\lambda \in X^\vee} Grad_\lambda$ s.t. $Grad_\lambda(J_\mu(M)) = \begin{cases} M & \text{if } \lambda = \mu \\ 0, & \text{else} \end{cases}$

Fact: $Grad \circ Z = F$ fiber functor on $Perv_{L^+G}(Gr_G)$

$Grad_{w_0(\lambda)} \circ Z = F_\lambda$ $\Rightarrow$ $[Z(V)] = \sum_{\lambda \in X^\vee} \dim(V_{w_0(\lambda)}) [J_\lambda(\bar{\mathbb{Q}}_\ell)]$ in $[D^b_I]$

Notation: $t(\lambda) \in W_{ext} = W \ltimes X^\vee$, $t(\lambda) = (e, \lambda)$

Recall by def'n of Wakimoto, $[J_\lambda(\bar{\mathbb{Q}}_\ell)] = [\Delta_{t(\lambda-\mu)} \overset{I}{*} \nabla_{t(\mu)}] = [\nabla_{t(\lambda)}]$

$\Rightarrow [Z(V)] = \sum_{\lambda \in X^\vee} \dim(V_\lambda)\, [\nabla^I_{t(\lambda)}]$

Applying Av_{IW} $\rightsquigarrow$ $[Z^{IW}(V)] = \sum_{\lambda \in X^\vee} \dim(V_\lambda)\, [\nabla^{IW}_\lambda]$

apply χ_μ, $\chi_\mu(Z^{IW}(V)) = \sum \dim(V_\lambda)\, \chi_\mu(\nabla^{IW}_\lambda) = \dim(V_\mu)$. $\square$

§3. Propagation and minuscule.

Prop $Z^{IW}(V)$, $Z^{IW}(V')$ tilting $\Rightarrow$ $Z^{IW}(V \otimes V')$ tilting.

Lemma $\forall\, x, y \in W_{ext}$, $\Delta^I_x \overset{I}{*} \Delta^I_y$ lies in the full subcat. gen'd under ext'ns by objs of the form $\Delta^I_z[n]$ for $z \in W_{ext}$, $n \leq 0$

$\nabla^I_x \overset{I}{*} \nabla^I_y$ ... $\nabla^I_z[n]$, $n \leq 0$.

Proof of lemma. If $\ell(xy) = \ell(x) + \ell(y)$, then ✓.

o/w, proceed by induction on $\ell(y)$. If $\ell(y) = 0$, ✓

Otherwise, if $\ell(x) + \ell(y) > \ell(xy)$, for $s \in S$ s.t. $sy < y$

Then $\Delta^I_x \overset{I}{*} \Delta^I_y = \Delta^I_x * (\Delta_s \overset{I}{*} \Delta^I_{sy}) = (\Delta^I_x * \Delta^I_s) * \Delta^I_{sy}$

Suffices to show lemma for $\Delta^I_x * \Delta^I_s$. If $\ell(xs) = \ell(x) + 1$, ✓.

Otherwise, use exact seq. of perverse sheaves

$$IC_e \hookrightarrow \Delta^I_s \twoheadrightarrow IC_s$$
$$IC_s \hookrightarrow \nabla^I_s \twoheadrightarrow IC_e.$$

Apply $\Delta_x \overset{I}{*} (-)$.

Get 2 dist Δ's
$$\Delta_x \to \Delta_x * \Delta_s \to \Delta_x \overset{I}{*} IC_s$$
$$\Delta_x * IC_s \to \underset{\substack{\wr\wr \\ \Delta_{xs}}}{\Delta_x * \nabla_s} \to \Delta_x.$$

Splicing sequences $\Rightarrow$ Lemma for $\Delta_x * \Delta_s$.

Proof of Prop'n $\quad Z^{IW}(V \otimes V') \simeq Z^{IW}(V) \overset{I}{*} Z(V')$

Admits filtration w/ qts of form

$$\Delta_{\mu}^{IW} \overset{I}{*} Z(V') \simeq \Delta_0^{IW} * \underset{\substack{\uparrow \\ \text{any } w_\mu \text{ s.t. } \; w_f w_\mu = w_f t(\mu)}}{\Delta_{w_\mu}^{I}} * Z(V')$$

$$\simeq \Delta_0^{IW} * \underbrace{Z(V') * \Delta_{w_\mu}^{I}}_{\text{admits further filt'n } \Delta_\lambda^{IW} * \Delta_{w_\mu}^{I}} \simeq Z^{IW}(V') * \Delta_{w_\mu}^{I}$$

$$\simeq Av_{IW}(\Delta_{w_\lambda}^{I} * \Delta_{w_\mu}^{I}).$$

Prop follows from lemma, $j_\lambda^{*!}(Av_{IW}(\Delta_{w_\lambda}^{I} * \Delta_{w_\mu}^{I}))$ is concentrated in deg $\geq -\dim Fl_{G,\lambda}$.

$\square$

For main thm reduce to G semisimple: idea is to use the map

$Gr_G \longrightarrow Gr_{G/Z(G)^\circ}$

surjective, univ. homeom. on each comp. of Gr_G.

Proof

G semisimple, say $\lambda \in X^\vee_+$ is

1) <u>minuscule</u> if $\forall$ roots $\alpha \in \Phi$, $|\langle \lambda, \alpha \rangle| \leq 1$ $\iff$ Gr^λ_G closed

2) <u>quasi-minuscule</u> if it's minimal in $X^\vee_+$ & non-minuscule $\iff$ $\overline{Gr^\lambda_G} = Gr^\lambda_G \cup Gr^0_G$

$\Rightarrow \exists !$ root γ s.t. $\langle \lambda, \gamma \rangle \geq 2$.

$\lambda = \gamma^\vee$

<u>Fact</u>. Any simple $G^\vee$-module is a direct summand of $\otimes$-powers of minuscule & quasi-minuscule.

<u>Fact</u> + <u>Prop'n</u> reduces Main Thm to showing $Z^{IW}(N(\lambda))$ λ 1) or 2).

<u>Minuscule case</u>.

<u>Lemma</u>. $\forall \lambda \in X^\vee, x \in W_f, n \in \mathbb{Z}$, $\mathrm{Hom}_{D^b_{IW}}(\Delta^{IW}_\lambda, Z^{IW}(V)[n])$

$$\cong \mathrm{Hom}_{D^b_{IW}}(\Delta^{IW}_{x(\lambda)}, Z^{IW}(V)[n])$$

Similar for ∇^{IW}_λ.

<u>Proof</u>. WLOG $\lambda \in X^\vee_+$. Then recalling $w_\lambda :=$ elt of minimal length in $W_f(t(\lambda))$.

$w_\lambda = \underset{\underset{(e,\lambda)}{\parallel}}{t(\lambda)}$ <u>and</u> $w_{x(\lambda)} = t(\lambda) \cdot y$ where $\ell(w_{x(y)}) = \ell(w_\lambda) - \ell(y)$

y of min'l length s.t. $y(\lambda) = \lambda$.

Then $\Delta^{IW}_\lambda \cong \Delta^{IW}_0 * \Delta^I_{w_\lambda} \simeq \Delta^{IW}_0 * \Delta^I_{w_{x(\lambda)}} * \Delta^I_{y^{-1}}$
$\simeq \Delta^{IW}_{x(\lambda)} * \Delta^I_{y^{-1}}$) invertible

Then $\mathrm{Hom}(\Delta^{IW}_{x(\lambda)}, Z^{IW}(V)[n]) \simeq \mathrm{Hom}(\Delta^{IW}_{x(\lambda)} * \Delta^{I}_{y^{-1}}, Z^{IW}(V) \overset{I}{*} \Delta^{I}_{y^{-1}}[n])$

$$\simeq \mathrm{Hom}(\Delta^{IW}_{y}, Z^{IW}(V) * \Delta^{I}_{y^{-1}}[n])$$

$$Z^{I}(V) * \Delta^{I}_{y^{-1}} \simeq \Delta^{FW}_{0} * \underset{\substack{\uparrow \\ W_f}}{\Delta_{y^{-1}}} * Z(V) \simeq \Delta^{IW}_{0} * Z(V).$$

<u>Minuscule case</u>. If $\lambda \in X^\vee_+$ is minuscule, then $Fl^{IW}_{G,\mu} \subset \overline{Fl^{IW}_{G,\lambda}}$

$\Rightarrow \mu \in W_f(\lambda)$ and $Z^{IW}(N(\lambda))$ is always supported on $\overline{Fl^{IW}_{G,\lambda}}$ — true for any λ.

For $\mu \in W_f(\lambda)$, perversity of $j^!_\mu Z^{IW}(N(\lambda))$, $j^*_\mu Z^{IW}(N(\lambda))$ reduce to the case $\mu = \lambda$ by Lemma. But then $Fl^{IW}_{G,\lambda} \subset \overline{Fl_{G,\lambda}}$ open ✓.

§4. <u>Quasi-minuscule case</u>

If λ is quasi-minuscule, $Fl_{G,\mu} \subset \overline{Fl_{G,\lambda}}$ $\Rightarrow \mu \in W_f(\lambda) \cup \{0\}$

$W_f(\lambda)$: handled by Lemma

reduced to $Fl^{IW}_{G,0}$ case.

<u>Key estimate in the quasi-minuscule case</u>.

<u>Lemma</u>. There exists a regular nilp. $n_0 \in \mathfrak{g}^\vee$ s.t.

$$\dim(\mathrm{Hom}_{P_{IW}}(\Delta^{IW}_0, Z^{IW}(V))) \le \dim(\underbrace{V^{n_0}}_{n_0\text{-inv.}})$$

and $\dim(V^{n_0}) = \sum_{\substack{\mu \in X^\vee \\ \langle \mu, 2\rho\rangle \in \{0,1\}}} \dim(V_\mu)$

$V = N(\lambda)$, λ quasi-minuscule

$\Rightarrow \dim(V^{n_0}) = \dim(V_0)$

Proof of tilting, assuming key estimate

$$i : \overline{Fl_{G,0}^{IW}} \longrightarrow Fl_G \xleftarrow[\text{open}]{j} Fl_G \setminus \overline{Fl_{G,0}^{IW}}$$

$\rightsquigarrow$ dist Δ

$$j_! j^* Z^{IW}(V) \to Z^{IW}(V) \to i_* i^* Z^{IW}(V)$$

$$i_* i^! Z^{IW}(V) \to Z^{IW}(V) \to j_* j^* Z^{IW}(V)$$

<u>Know</u>: $j^* Z^{IW}(V)$ perverse $\to$ $j_! j^* Z^{IW}(V)$, $j_* j^* Z^{IW}(V)$ perverse.

$\Rightarrow$ $i^* Z^{IW}(V)$ in perv. degs $0, -1$

$i^! Z^{IW}(V)$ $0, 1$

$$\Rightarrow \dim \mathrm{Hom}_{P_{IW}}(\Delta_0^{IW}, Z^{IW}(V)) \leq \chi_0(Z^{IW}(V)) = \dim \mathrm{Hom}(\Delta_0^{IW}, Z^{IW}(V)) - \dim \mathrm{Hom}(\Delta_0^{IW}, Z^{IW}(1)[1])$$

$\Rightarrow$ $i^! Z^{IW}(V)$ perverse.

Input to estimate:

Let $P_I^\circ = P_I / \langle IC_w^I : \ell(w) > 0 \rangle$

$$P_I \xrightarrow{\pi^\circ} P_I^\circ \qquad Z^\circ = \pi^\circ \circ Z$$

P_I° has a natural monoidal str.

Z° has structure of central functor on P_I°.

$\tilde{P}_I^\circ$ full ab. subcat. gen. by $Z^\circ(V)$ and subquotients.

To any $V \in \mathrm{Rep}(G^\vee)$, there is a nilp. endom. $n_V : Z(V) \longrightarrow Z(V)$

functorial in $Z^0(V) \to Z^0(V')$.

Hard Prop^n (Tannakian reconstruction of $\widetilde{P}^0_I$)

$\exists\, n_0$ a nilp. regular elt of $\mathfrak{g}^\vee$ and a closed subgp $H \subset Z_{G^\vee}(n_0)$ and equiv. of monoidal cats $\Phi^0 : (\widetilde{P}^0_I, *) \xrightarrow{\sim} (\mathrm{Rep}(H), \otimes)$ + isom. of functors

$\eta : \Phi^0 \circ \widetilde{Z}^0 \cong \mathrm{For}^{G^\vee}_H : \mathrm{Rep}(G^\vee) \to \mathrm{Rep}(H)$ + moreover, $\forall\, V \in \mathrm{Rep}(G^\vee)$,

$\eta(\Phi^0(n_V)) = n_0$.