Level rank dualities from d-Harish-Chandra series

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§1. d-HC series.

G conn. red $/\overline{\mathbb{F}_q}$, split $/\mathbb{F}_q$, $F: G \to G$ Frobenius

Deligne-Lusztig induction / restriction $\mathbb{Z}\,\mathrm{Irr}(L^F) \underset{{}^*R_L^G}{\overset{R_L^G}{\rightleftarrows}} \mathbb{Z}\,\mathrm{Irr}(G^F)$

L : F-stable Levi

Unip irreps $\mathrm{Uch}(G^F) = \{\rho : (\rho : R_T^G(1)) \neq 0$, T : F-stable max. torus$\}$

Lusztig: $\mathrm{Uch}(G^F)$ is indep of q, only depends on W_G. "generic"

Broué-Malle-Michel: define "generic" finite reductive gps

Write $\mathrm{Uch}(\mathbb{G})$ = set of unip chars of $\mathbb{G}$.

<u>Thm</u> (BMM'93) $\forall d \in \mathbb{Z}_{>0}$, $\exists$ a partition

$$\mathrm{Uch}(\mathbb{G}) = \coprod_{(\mathbb{L},\lambda)} \mathrm{Uch}(\mathbb{L},\lambda)_d \leftarrow \{\rho : (\rho : R_{\mathbb{L}}^{\mathbb{G}}\lambda) \neq 0\}$$

$\mathbb{L}$-d-split Levi $= Z_{\mathbb{G}}(\mathbb{T})$ $\qquad$ $\mathbb{T}$: d-split $\iff |\mathbb{T}^F| = \Phi_d$

$\lambda \in \mathrm{Uch}(\mathbb{L})$ cuspidal $\left({}^*R_{\mathbb{M}}^{\mathbb{L}}(\lambda) = 0, \ \forall\, d\text{-split } \mathbb{M} \subset \mathbb{L}\right)$

If $d = 1$, HC series

Moreover, $\exists$ $\varphi^{BMM}_{(L,\lambda)_d} : \mathrm{Uch}(\mathbb{L},\lambda)_d \xleftrightarrow{\sim} \mathrm{Irr}(W_{(\mathbb{L},\lambda)_d})$

$\hookrightarrow$ complex reflection group

& $\varepsilon_{(L,\lambda)_d} : \mathrm{Uch}(\mathbb{L},\lambda)_d \longrightarrow \{\pm 1\}$

Rmks 1) d-HC series non trivial only if $\Phi_d(q) \,\Big|\, |G^F|$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{f(q)}$

2) $\exists$ d-split max. torus $\Leftrightarrow$ d regular number of G.

<u>Broué - Malle '93.</u> (conj.) $W_{(L,\lambda)_d} \rightsquigarrow H_{(L,\lambda)_d}(x)$ generic cyclotomic Hecke alg.

s.t. 1) $H_{(L,\lambda)_d}(x)\big|_{x \mapsto \zeta_d} = $ gp alg of $W_{(L,\lambda)_d}$

2) $H_{(L,\lambda)_d}(x)\big|_{x \mapsto q} = \mathrm{End}_{G^F}\left(H_c^*(Y^G_{L\subset P})[\lambda_q]\right)$

$\sim$

§2. Conjectual level-rank duality from d-HC series

fix $\ell \neq m$, $\ell, m \in \mathbb{Z}_{>0}$.

$G = GL_N$

$W_{(L,\lambda)_d} \cong G_{d,1,2}$

$\simeq \mathfrak{S}_2 \ltimes (\mathbb{Z}/d)^2$

$$\begin{tikzcd}
 & \mathrm{Uch}(L,\lambda)_\ell \cap \mathrm{Uch}(M,\mu)_m \arrow[dl, "\varphi_L^{BMM}"'] \arrow[dr, "\varphi_M^{BMM}"] \arrow[dddl, bend left, "\Phi_L"] \arrow[dddr, bend right, "\Phi_M"'] & \\
\mathrm{Irr}(W_{(L,\lambda)_\ell})_{(M,\mu)_m} \arrow[d] \arrow[rr, leftrightarrow, "U_{\ell,m}"] & & \mathrm{Irr}(W_{(M,\mu)_m})_{(L,\lambda)_\ell} \arrow[d] \\
\mathrm{Irr}(H_{(L,\lambda)}(x)) \arrow[d, "d_{\zeta_m}"', "x \mapsto \zeta_m"] & & \mathrm{Irr}(H_{(M,\mu)}(x)) \arrow[d, "d_{\zeta_\ell}"', "x \mapsto \zeta_\ell"] \\
K_0(H_{(L,\lambda)}(\zeta_m)) & & K_0(H_{(M,\mu)}(\zeta_\ell))
\end{tikzcd}$$

Conj. (Trinh – Xi)

1) The image of Φ_ℓ (resp. Φ_m) is a union of blocks of $H_{(L,\lambda)_\ell}(\zeta_m)$ (resp. $H_{(M,\mu)_m}(\zeta_\ell)$)

2) The bijection $V_{\ell,m}$ categorifies to a derived equiv between highest wt covers of blocks of $\mathrm{Rep}\, H_{(L,\lambda)_\ell}(\zeta_m)$ (resp. $\mathrm{Rep}\, H_{(M,\mu)_m}(\zeta_\ell)$).

Thm (TX) For $G = GL_n / GU_n$, (ℓ, m) coprime, conjecture holds.

$V_{\ell,m} \longleftrightarrow$ Uglov's level-rank duality

§3. Connections w/ character sheaves on graded Lie algebras / affine Springer fibers

m regular elliptic # of G $\overset{RLYG}{\longleftrightarrow}$ $\theta : G \to G$ order m

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$$

$G^\theta := G_0 \curvearrowright \mathfrak{g}_1$ has GIT-stable vectors

$\mathrm{Char}_{G_0}(\mathfrak{g}_1) := \mathrm{Four}(\mathrm{SPerv}_{G_0}(\mathfrak{g}_{-1}^{nil}))$

Vinberg: $\mathfrak{g}_1 \xrightarrow{f} \mathfrak{g}_1 /\!/ G_0 \simeq \mathfrak{e}/C$, $C = N_{G_0}(\mathfrak{e}) / Z_{G_0}(\mathfrak{e})$ cplx refl. gp.

$\Psi_f : \mathrm{Perv}_{G_0}(f^{-1}(\bar{a})) \to \mathrm{Perv}_{G_0}(f^{-1}(0))$, $\bar{a} \in \mathfrak{e}^{rs}/C$

Thm (Vilonen - X., Grinberg - Vilonen - X.)

$$\mathrm{Four}(P) \simeq IC(\mathfrak{g}_1^{rs}, \mathcal{H}_c)$$

$\uparrow$

$\psi \in \mathbb{C}$

$\hookrightarrow$ cyclotomic Hecke alg.

principal 1 - HC series

$W_{(A,1)_1} = W_G$

$\mathcal{O}^{\mathrm{rat}}_{\frac{1}{m}}(W_G)$

$\downarrow$

$\mathrm{Rep}\, H_{(A,1)}(\zeta_m)$

principal m - HC series

$W_{(T,1)_m} = C$

$H_{(T,1)_m}(1)$

$\wr\|$

$\mathcal{H}_c$

$$\begin{array}{ccc} \mathrm{Char}^{\mathrm{cusp}}_{G_0}(\mathfrak{g}_1)_{0,st} & \xrightarrow{\mathrm{Four}} & \mathrm{Spera}^{\mathrm{cusp}}_{G_0}(\mathfrak{g}^{\mathrm{nil}}_{-1})_{0,st} \\ \wr\downarrow \text{ GVX + VX} & & \wr\downarrow \text{ Lusztig - Yun + Wille Liu, Etingof} \\ \mathrm{Irr}_0(\mathcal{H}_c) & \xleftrightarrow{\sim} & \mathrm{Irr}^{f.d.}_0(H^{\mathrm{rat}}_{\frac{1}{m}}(W_G)) \end{array}$$

$\uparrow$

conjecturally induced by $(1,m)$- duality

Homogeneous affine Springer fibers

$\gamma \in \mathfrak{g}(\mathbb{C}((t)))^{rs}$ homogeneous of slope $\nu = \frac{1}{m}$ $\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$

Let $\mathrm{Sp}_\gamma \subset \mathrm{Fl}$ affine Springer fiber

Thm (Oblomkov-Yun '16) $\mathrm{Br}_C \curvearrowright \mathrm{Gr}^P_* H^*_{\varepsilon=1}(\mathrm{Sp}_\gamma)_{st} \curvearrowleft H^{rat}_{\frac{1}{m}}(W_a)$

Conj (Trinh-X.)

1) The action of Br_C factors through $\mathcal{H}_C \simeq H_{(T,1)_m}(1)$

2) Let $[\mathcal{E}_{\gamma,\nu}] = \sum_{i,j} (-1)^i t^j [\mathrm{Gr}^P_j H^i_{\varepsilon=1}(\mathrm{Sp}_\gamma)_{st}]$

Then $[\mathcal{E}_{\gamma,\nu}] = \sum_{\rho \in \mathrm{Uch}_{1,\mathrm{pr}} \cap \mathrm{Uch}_{m,\mathrm{pr}}} \varepsilon(\rho) [\Delta_\nu(\varphi_1(\rho)) \otimes S(\varphi_m(\rho))]$

std of $H^{rat}_{\frac{1}{m}}(W_a)$ Specht of $\mathcal{H}_C$ mod

Rmk Also expect $[\mathcal{E}_{\gamma,\nu}] = \sum_{\tau \leftrightarrow \sigma} \varepsilon_{\tau,\sigma} [L_\nu(\tau) \otimes D_\sigma]$

(1, m) duality $\mathrm{Irr}^{f.d.}(H^{rat}_{\frac{1}{m}}(W))$ $\mathrm{Irr}(\mathcal{H}_C)$