Bezrukavnikov's equivalence

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G split reductive / $F = k((\varpi))$

$G \supset B \supset T$ $\qquad$ $\check{G} \supset \check{B} \supset \check{N}$

$LG \supset L^+G \supset I \supset I^+$, $\mathcal{F}\ell = LG/I$ $\qquad$ $\check{\mathfrak{g}} \supset \check{\mathfrak{b}} \supset \check{\mathfrak{n}}$

$$\begin{array}{ccc} \check{\mathfrak{n}} & \longrightarrow & \check{\mathfrak{b}} \\ \downarrow & & \downarrow \\ 0 & \in & \check{\mathfrak{t}} \end{array}$$

$\Lambda = \mathbb{Q}_\ell$

$\tilde{\check{\mathcal{N}}} = \check{G} \overset{\check{B}}{\times} \check{\mathfrak{n}} \cong T^*(\check{G}/\check{B})$

$$D^b_c(I \backslash LG/I) = D^b_{I,c}(\mathcal{F}\ell) \overset{\text{Thm}}{\cong} \mathrm{Coh}_{\check{G}}\Big(\tilde{\check{\mathcal{N}}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\check{\mathcal{N}}}\Big) = \mathrm{Coh}\Big(\check{\mathfrak{n}}/\check{B} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \check{\mathfrak{n}}/\check{B}\Big)$$

as monoidal cat.

$\circlearrowright$ $\qquad\qquad$ $\circlearrowright$

$$D^b_c\big(I \backslash LG/(I^+, \psi)\big) \overset{AB}{\cong} \mathrm{Coh}\Big(\frac{\check{\mathfrak{n}}}{\check{B}}\Big)$$

$\mathcal{O}(\check{T}^\wedge_1)$ $\qquad$ $\mathcal{O}(\check{T}^\wedge_1)$

$$\circlearrowright D\big((I, \hat{u}) \backslash LG/(I, \hat{u})\big)^2 \overset{\text{Thm}}{\cong} \mathrm{Coh}\Bigg(\frac{\check{\mathfrak{n}}^\wedge}{\check{B}} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \frac{\check{\mathfrak{n}}^\wedge}{\check{B}}\Bigg)$$

$\cap$ gen. by pullbacks from

$D(I^+ \backslash LG/I^+)$ $\qquad$ $D^b_c(I \backslash LG/I)$

$\circlearrowright$ $\qquad$ $\circlearrowleft$

$\mathcal{O}(\check{\mathfrak{t}}^\wedge_0)$ $\qquad$ $\mathcal{O}(\check{\mathfrak{t}}^\wedge_0)$

compatible.

$\circlearrowright$ $\qquad\qquad$ $\circlearrowright$

$$D^b_c\big((I, \hat{u}) \backslash LG/(I^+, \psi)\big) \overset{\text{Thm}}{\simeq} \mathrm{Coh}\Big(\frac{\check{\mathfrak{n}}^\wedge}{\check{B}}\Big)$$

Strategy: to construct a monoidal functor $\mathrm{Coh}\Big(\frac{\check{\mathfrak{n}}^\wedge}{\check{B}}\Big) \xrightarrow{F} D\big((I, \hat{u}) \backslash LG/(I, \hat{u})\big)$

$\circlearrowright$

$$\Delta^{IW} \in D\big((I, \hat{u}) \backslash LG/(I^+, \psi)\big)$$

Recall the construction of F.

Inputs: (1) $\operatorname{Rep}(\check{G}) \xrightarrow{Z^{mon}} D((I,\hat{u})\backslash LG/(I,\hat{u}))$

(2) $\operatorname{Rep}(\check{T}) \xrightarrow{J^{mon}} D((I,\hat{u})\backslash LG/(I,\hat{u}))$

+ (3) compatibilities

$C^{(\simeq \mathbb{A}^1)}$ curve $/k \ni x$

$\mathcal{G}_C(1) \to \mathcal{G}_C(0)$ group scheme $\quad \mathcal{G}_C^{(i)}|_{C-x} \simeq G \times (C-x)$

$$L^+_{\hat{O}_{C,x}} \mathcal{G}_C(0) = I \supset L^+_{\hat{O}_{C,x}} \mathcal{G}_C(1) = I^+$$

$$\mathcal{L}^+_C \mathcal{G}(i) \backslash \mathcal{L}_C \mathcal{G}_C(i) / \mathcal{L}^+_C \mathcal{G}(i)$$

Want

$$\mathcal{H}k(0)|_{C-x} \simeq (L^+G\backslash LG/L^+G) \times (C-x)$$

$$\mathcal{H}k(0)|_x \simeq I\backslash LG/I$$

$$\mathcal{H}k(1)|_{C-x} \simeq (L^+G\backslash LG/L^+G) \times (C-x) \times I/I^+$$

$$\mathcal{H}k(1)|_x \simeq I^+\backslash LG/I^+$$

$$\begin{tikzcd}
 & \widetilde{Fl}_C \arrow[d] & LG/I^+ = \widetilde{Fl} \arrow[d] & \\
Gr \times G/B & Fl_C \arrow[d] & Fl & \left\{ y \in C,\ \mathcal{E}\ G\text{-torsor on } C,\ \beta : \mathcal{E}|_{C-y} \xrightarrow{\sim} \mathcal{E}^0|_{C-y},\ \epsilon \text{ reduction of } \mathcal{E}|_x \text{ to } B \right\} \\
C-x & C & &
\end{tikzcd}$$

x: fixed pt

$$\begin{tikzcd}
H \arrow[d] & \subset & \mathcal{G}(O)_x \arrow[d] \\
U & \subset & B
\end{tikzcd}$$

$V \in Rep(\check{G}) \simeq Perv(L^+G \backslash LG / L^+G)$

$V \longmapsto Sat_V$

$$R\Psi\left((Sat_V)_{C-x} \boxtimes \delta_{\hat{u}}\right)$$

$$\text{!!} \qquad \uparrow \text{ cofree unipotent mon. sheaf on } T = B/U \hookrightarrow G/U$$

$\pi_1^{tame}(\hat{O}_{C,x} - x) \curvearrowright \ Z^{mon}(V)$

$\pi_1(T) \curvearrowright \quad \uparrow$

$$D\left((I,\hat{u}) \backslash LG / (I,\hat{u})\right)$$

convolution no longer proper, but we work w/ monodromic sheaves

$$\begin{tikzcd}
I^+ \backslash LG / I^+ \arrow[d] & I^+ \backslash LG_w / I^+ \arrow[l, "\tilde{j}_{\tilde{w}}"'] \arrow[r, "\sim"] \arrow[d] & (*/I^+ \cap wI^+w^{-1}) \times T \arrow[r] & T \\
I \backslash LG / I & I \backslash LG_w / I \arrow[l] & &
\end{tikzcd}$$

$w \in \tilde{W}$

$\lambda \in X_*(T) \hookrightarrow \tilde{W}$

$\tilde{\Delta}_{\tilde{w}} = (\tilde{j}_w)_!(\delta_{\hat{u}})$

$\tilde{\nabla}_{\tilde{w}} = (\tilde{j}_w)_*(\delta_{\hat{u}})$

$J^{mon} : Rep(\check{T}) \to D((I,\hat{u}) \backslash LG / (I,\hat{u}))$

$\mathcal{O}_{e_\lambda} \longmapsto \tilde{\nabla}^{mon}_\lambda$ if λ anti-dom.

$\mathcal{O}_{e_\lambda} \longmapsto \tilde{\Delta}^{mon}_\lambda$ if λ dom.

Thm Every $Z^{mon}(V)$ admits a filtration w/ assoc. graded being J^{mon}_{μ},

μ appears in the weight of V.

If $V = V_\lambda$, irred. h.w. rep'n, h.w. λ,

$$\begin{tikzcd}
\pi_1(\hat{\mathcal{O}}_{C,x} - x) \arrow[r, phantom, "\circlearrowright"] & Z^{mon}(V) \arrow[r] & J^{mon}_{w_0(\lambda)} & \arrow[l, phantom, "\circlearrowleft"] \pi_1^{tame}(\mathbb{G}_m) \arrow[dl, "w_0(\lambda)"] \\
\pi_1^{tame}(\mathbb{G}_m) \arrow[u, phantom, "\parallel"] & \pi_1(T) \arrow[u, phantom, "\cup"] & \pi_1(T) \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

$$\begin{tikzcd}
Z^{mon}(V_\lambda) * Z^{mon}(V_\mu) \arrow[r] \arrow[d, "\wr"] & J^{mon}_{w_0(\lambda)} * J^{mon}_{w_0(\mu)} \arrow[d, "\wr"] \\
Z^{mon}(V_\lambda \otimes V_\mu) \arrow[r] \arrow[dr] & J^{mon}_{w_0(\lambda+\mu)} \\
& Z^{mon}(V_{\lambda+\mu}) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Rep}(\check{G}) \arrow[r] \arrow[d] & D((I,\hat{u})\backslash LG/(I,\hat{u})) \\
\mathrm{Perf}\left(\frac{\check{G}}{\check{G}}\right) \arrow[ur] &
\end{tikzcd}$$

$V \longmapsto Z^{mon}(V)$

$$V \longmapsto Z^{mon}(V) \longmapsto J^{mon}_{w_0(\lambda)}$$

$$\begin{tikzcd}
\mathrm{Rep}(\check{G}) \arrow[r] \arrow[d] & D((I,\hat{u})\backslash LG/(I,\hat{u})) \\
\mathrm{Perf}\left(\frac{\check{G}}{\check{G}}\right) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Rep}(\check{G} \times \check{T}) \arrow[r, "F"] \arrow[d] & \mathcal{C} & & \\
\mathrm{Perf}\left(\check{G}\backslash \overline{\check{G}/\check{N}}/\check{T}\right) \arrow[r] \arrow[ur, "\tilde{F}"] & \mathrm{Rep}(\hat{B}) \arrow[r] \arrow[u, "?"] & \mathrm{Perf}\left(\frac{\hat{B}}{B}\right) \arrow[ul] &
\end{tikzcd}$$

+ compatible family

$b_\lambda : F(V_\lambda) \to F(\mathcal{A}\ell_{w_0(\lambda)})$

general: $\mathcal{X} = X/H$,

$$\begin{tikzcd}
\mathrm{Rep}(H) \arrow[r, "F"'] \arrow[d] & \mathcal{C} \\
\mathrm{Perf}(\mathcal{X}) \arrow[ur] &
\end{tikzcd}$$

$\mathrm{Hom}(F(1), F(\mathrm{Reg}\, H)) =: B \circlearrowleft H$

Need $A \to B$, H-equiv.

If $\mathcal{C} = \mathrm{Vect}(S)$, $\tilde{F}$ factors through $\mathrm{Rep}(\hat{B})$

$\Leftrightarrow$ b_λ is surj. for every λ.

$$
\begin{tikzcd}
\mathrm{Perf}\left(\frac{\check{B}}{\check{B}}\right) \arrow[r, "\otimes"] \arrow[dr] & D((I,\hat{u})\backslash LG/(I,\hat{u})) \arrow[rr, "*\Delta_e^{Iw,mon}"] & & D\left((I,\hat{u})\backslash LG/((I^{opp})^+,\phi)\right) \\
& \mathrm{Coh}\left(\frac{\check{\tilde{N}}^\wedge}{\check{B}}\right) \arrow[u] \arrow[urr, "\widetilde{Av}^{Iw}", "\sim"'] & & \\
& \mathrm{Coh}\left(\frac{\check{\tilde{N}}}{\check{B}}\right) \arrow[u, "\iota_*"] \arrow[rr, "Av^{Iw}", "\sim"'] & & D(I\backslash LG/(I^{opp})^+,\phi) \arrow[uu, "\pi^*[\dim T]"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\frac{\check{\tilde{N}}}{\check{B}} \arrow[r] \arrow[d] & \frac{\check{B}}{\check{B}} \arrow[d] \\
1 & \check{T} \arrow[l, phantom, "\in"]
\end{tikzcd}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}(\iota_* F, G) & \overset{?}{\underset{\sim}{\to}} & \mathrm{Hom}(\widetilde{Av}^{Iw}(\iota_* F), \widetilde{Av}^{Iw} G) \\
\| & & \wr\| \\
\mathrm{Hom}(F, \iota^! G) & \underset{\sim}{\to} & \mathrm{Hom}(Av^{Iw}(F), Av^{Iw}(\pi^!(G)))
\end{array}
$$