Fermat's last theorem

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Lecture 1

<u>Goal</u> (FLT) If $\ell \geq 5$ is a prime, then $a^\ell + b^\ell = c^\ell$ has no non-zero integer sol'n.

<u>Step 1</u>. (Frey) Suppose that there's a sol'n (a, b, c) to the Fermat eq'n, $abc \neq 0$.

Consider the Frey curve $y^2 = x(x - a^\ell)(x + b^\ell)$ E elliptic curve $/ \mathbb{Q}$

satisfying (1) E has semistable reduction w/ discriminant $\Delta = 16\,(abc)^{2\ell}$.

& conductor N = product of prime factors p of abc.

(2) $\bar{\rho}_{E,\ell} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[\ell]) \simeq GL_2(\mathbb{F}_\ell)$

$$E[\ell] = \left\{ x \in E(\bar{\mathbb{Q}}) : \frac{x + \cdots + x}{\ell} = 0_E \right\}$$

is <u>unramified</u> at odd primes p dividing abc.

<u>Step 2</u>. (Taniyama - Shimura - Weil conj.) proved by Andrew Wiles

E is "associated" w/ a cuspidal modular eigenform.

$$f_E(q) = \sum_{n \geq 1} a_n q^n \quad \text{of weight 2 and level } \Gamma_0(N)$$

i.e. $a_1 = 1$, and $\forall\, p \nmid N$, $a_p = p + 1 - \#E(\mathbb{F}_p)$

$$\#E(\mathbb{F}_p) = \{\infty\} \cup \left\{ \begin{array}{c} \text{sol'n } (x,y) \in \mathbb{F}_p^2 \text{ satisfying} \\ y^2 = x(x - a^\ell)(x + b^\ell) \end{array} \right\}$$

$\boxed{\text{Main Target}}$

Step 3. (Serre, Ribet's level-lowering) Since $\bar{\rho}_{E,\ell}$ is unramified at odd primes $p \mid N$,

$\exists$ cusp. modular eigenform f' of level $\Gamma_0(2)$ & wt 2 s.t. $\forall (n,N)=1$ $a_n(f) \equiv a_n(f')$ mod ℓ

BUT there's NO cusp. modular form of level $\Gamma_0(2)$ and wt 2.

b/c $X_0(2) \simeq \mathbb{P}^1$, $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$. $\square$

Idea for step 2 By Eichler-Shimura, wt 2 cusp. eigenform f $\rightsquigarrow$ $\mathbb{Q}$-coeff.

$\rightsquigarrow$ Galois rep'n $\rho_f : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}_\ell)$.

$\longleftarrow$ Wile's Thm

$\rightsquigarrow$ E elliptic curve,

Main idea

$\exists ?\ f \xleftrightarrow{\text{modularity lifting}} \rho$

$\bar{f} = f \bmod \ell$ $\equiv$ mod ℓ $\equiv$ mod ℓ $\bar{\rho} = \rho \bmod \ell$

$f' \longleftrightarrow \rho'$

Universal deformation of $\bar{\rho}$

T = universal deformation of $\bar{f}$

= Hecke algebra

R = universal deformation ring

Thm $R \cong T$.

Langlands-Tunnell: solvable base change.

When image of $\bar{\rho}$ is $\underbrace{GL_2(\mathbb{F}_3)}_{\text{solvable}}$.

§1. ℓ-adic representations (ℓ prime no.)

<u>Notation</u>. Let L/K be a Galois ext'n of fields

× When $L|K$ finite, $\mathrm{Gal}(L|K)$ = Galois gp

In general, $\mathrm{Gal}(L|K) = \varprojlim_{\substack{L|M|K \\ M|K \text{ finite Galois}}} \mathrm{Gal}(M|K)$

profinite gp ($\Rightarrow$ compact, totally disconnected)

<u>Galois theory</u>. $\left\{ \begin{array}{l} \text{Intermediate (normal) fields} \\ M: L \supset M \supset K \end{array} \right\} \longleftrightarrow \left\{ \underline{\text{closed}} \text{ (normal) subgp of } \mathrm{Gal}(L|K) \right\}$

$$M \longmapsto \mathrm{Gal}(L|M)$$

$$L^H \longleftarrow\!\shortmid H$$

In the normal case, $\mathrm{Gal}(M|K) \simeq \mathrm{Gal}(L|K) / \mathrm{Gal}(L|M)$.

- K^{sep} = separable closure of K,

  $\bar{K}$ = alg closure

$G_K = \mathrm{Gal}(K^{sep}|K)$

<u>Def'n</u>. A <u>Galois rep'n</u> is a cts gp hom. $\rho: G_K \longrightarrow GL_n(R)$

where R is a topological ring. $\qquad G_K \curvearrowright R^{\oplus n} = V$

eg. $R = \mathbb{F}_\ell, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{C}, \overline{\mathbb{Q}_\ell}$.

Example $\chi_{cycl}: G_K \longrightarrow \mathbb{Z}_\ell^\times$ $\qquad \zeta_{\ell^n} \in \bar{K}$ ℓ^nth primitive root of unity

for $g \in G_K$, $g(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\chi_{cycl}(g)}$ $\qquad (\zeta_{\ell^{n+1}})^\ell = \zeta_{\ell^n}$

Eg. $K = \mathbb{F}_p$, $p \neq \ell$. $\quad G_{\mathbb{F}_p} \simeq \hat{\mathbb{Z}}$

$\mathrm{Fr}_p \longleftarrow 1$

$\parallel$

inverse of arithmetic Frob $x \mapsto x^p$

$\mathrm{Fr}_p(\zeta_{\ell^n}) = \zeta_{\ell^n}^{1/p}$, $\quad \chi_{cycl}(\mathrm{Fr}_p) = \frac{1}{p} \in \mathbb{Z}_\ell^\times$

If $\rho: G_K \to GL_n(R) = GL(V)$ for R a $\mathbb{Z}_\ell$-algebra.

write $\rho(m)$ or $V(m)$ for $\rho \otimes \chi_{cycl}^m$ for $m \in \mathbb{Z}$

$$\rho(m)(g) = \chi_{cycl}(g)^m \rho(g).$$

Lemma Let Γ be a cpt top. gp. (e.g. Galois gp), and let

$\rho: \Gamma \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$ be a cts rep'n. then there exists a finite ext'n L of $\mathbb{Q}_\ell$

s.t. $\rho(\Gamma) \subset GL_n(L)$ & up to conjugation, we may assume that $\rho(\Gamma) \subset GL_n(\mathcal{O}_L)$.

Rmk. Easier to work w/ L or $\mathcal{O}_L$ as it is ℓ-adically complete, but $\bar{\mathbb{Q}}_\ell$ is not.

Pf $\rho(\Gamma)$ is cpt & Hausdorff. By <u>Baire Category Theorem</u> $\Rightarrow$

(*) intersection of countably many open dense subset of $\rho(\Gamma)$ is still dense in $\rho(\Gamma)$.

Now $\rho(\Gamma) = \bigcup_{L \mid \mathbb{Q}_\ell \text{ finite}} \big(\underbrace{\rho(\Gamma) \cap GL_n(L)}_{\text{closed subgp}}\big)$

<u>Claim</u> $\exists L'$ s.t. $\rho(\Gamma) \cap GL_n(L')$ contains an open subset.

<u>Pf.</u> If not $\Rightarrow$ for every L', $\left(\rho(\Gamma) \cap GL_n(L')\right)^c$ is open <u>dense</u>.

$$\phi = \rho(\Gamma)^c = \bigcap_{L'} \underbrace{\left(\rho(\Gamma) \cap GL_n(L')\right)^c}_{\text{open dense}}$$

contradiction. $\square$.

Claim $\Rightarrow$ $\rho(\Gamma) \cap GL_n(L')$ has finite index.

By enlarging L' to finite ext'n, containing entries of a soset of

$\Rightarrow$ $\rho(\Gamma) \subset GL_n(L)$.

To conjugate $\rho(\Gamma)$ into $GL_n(\mathcal{O}_L)$, need to find an $\mathcal{O}_L$-lattice $\Lambda \subset L^{\oplus n}$ stable under Γ-action. (then if $\Lambda = g\,\mathcal{O}_L^{\oplus n}$ for some $g \in GL_n(L)$,

$$\rho : \Gamma \longrightarrow g\, GL_n(\mathcal{O}_L)\, g^{-1} \quad \square)$$

<u>Note</u>: $\underbrace{\rho^{-1}(GL_n(\mathcal{O}_L))}_{\text{open}} \subset \Gamma$ has finite index.

Pick a set of representatives of this subgp $\{g_1, \cdots, g_r\}$.

Take $\Lambda = \sum_{i=1}^{r} g_i \cdot \mathcal{O}_L^{\oplus n}$.

<u>Def'n</u>. If $\rho : \Gamma \longrightarrow GL_n(\underset{\text{field}}{k}) = GL(V)$ is a rep'n, then there exists a Γ-stable filtration $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$ s.t. V_{i+1}/V_i is irreducible.

Then $\rho^{ss} := \bigoplus_i V_{i+1}/V_i$, called the <u>semisimplification of ρ</u>. (indep. of choice of the filtration)

Ex $\rho(g) = \begin{pmatrix} a(g) & b(g) \\ 0 & d(g) \end{pmatrix}$, $\rho^{ss} = a \oplus d$.

When $k = L$ a fin. ext'n of $\mathbb{Q}_\ell$, may take a Γ-stable $\mathcal{O}_L$-lattice Λ

$\rightsquigarrow$ rep'n $\rho : \Gamma \to GL_n(\mathcal{O}_L)$

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr, "\bar{\rho}_\Lambda"'] & GL_n(\mathcal{O}_L) \arrow[d, "\bmod \varpi_L"] \\ & GL_n(k) \end{tikzcd}$$

Exer. The semisimplification of $\bar{\rho}_\Lambda$ is indep. of the choice of Λ, denoted by $\bar{\rho}^{ss}$.

Prop. (Brauer–Nesbitt) Let $\rho_1, \rho_2 : \Gamma \to GL_n(k)$ be two rep'ns
w/ discrete top

Suppose that either (1) $\forall g \in \Gamma$, $\det(x \cdot I_n - \rho_1(g)) = \det(x \cdot I_n - \rho_2(g))$ in $k[x]$

or (2) char $k = 0$ (or $> n$), $\forall g \in \Gamma$, $\mathrm{tr}(\rho_1(g)) = \mathrm{tr}(\rho_2(g))$,

Then $\rho_1^{ss} \simeq \rho_2^{ss}$.

Proof. May assume that ρ_1 & ρ_2 are semisimple.

Let $V_1, \cdots, V_r$ be all distinct irred. factors of ρ_1 & ρ_2 w/ multiplicities $m_i^{(1)}, m_i^{(2)}$ resp.

WTS $m_i^{(1)} = m_i^{(2)}$.

Claim. Fix i, $\exists\, e_i \in k[\Gamma] = \{\sum_{\gamma \in \Gamma} k[\gamma]\}$ s.t. e_i acts by identity on V_i & 0 on other V_j's.

Pf. Let $R = \mathrm{Im}\left(k[\Gamma] \to \prod_i \mathrm{End}(V_i)\right)$
semisimple k-alg, i.e. nilp. ideal $= 0$.

$\Rightarrow$ R semisimple fin. dim'l k-alg.

By <u>Artin–Wedderburn thm</u>, $R = \prod_{j=1}^{r} \underbrace{M_{n_j}(D_j)}_{\text{simple module } D_j^{\oplus n}} \sim V_1, \dots, V_r$ simple modules

f.d. division k-alg

$\Rightarrow$ after permuting, each V_i is a $D_j^{\oplus n_j}$ $\Rightarrow$ take $e_i = (0, \dots, 0, \underset{j\text{th}}{1}, 0, \dots, 0) \in \prod M \simeq R$

take a preimage

<u>Case (1)</u>. <u>Fix i</u>

$$\det(x \cdot I - e_i ; V_j) = \begin{cases} (x-1)^{\dim V_i}, & j = i \\ x^{\dim V_j}, & j \neq i \end{cases}$$

$$\det(x \cdot I - \rho_*(e_i)) = (x-1)^{m_i^{(*)} \dim V_i} \, x^{n - m_i^{(*)} \dim V_i}$$

$$\Rightarrow m_i^{(1)} = m_i^{(2)}. \qquad \square$$

<u>Case (2)</u>

$$\begin{array}{l} \operatorname{tr}(\rho_1(e_i)) = m_i^{(1)} \dim V_i \\ \quad \| \\ \operatorname{tr}(\rho_2(e_i)) = m_i^{(2)} \dim V_i \end{array} \quad \text{(-k} \quad \Rightarrow \quad m_i^{(1)} = m_i^{(2)}.$$

<u>§ 2. Galois gp of local fields</u>

<u>local field</u> K fin. ext'n of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

$$K \supset \mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/(\varpi_K) \simeq k_K \simeq \mathbb{F}_q, \quad q = p^r$$

Fix a system of roots of unity. $\zeta = (\zeta_n)_{p \nmid n}$ s.t. ζ_n is a nth prim. root of unity.

and $\zeta_m^{m/n} = \zeta_n$ if $n \mid m$.

$$
\begin{array}{l}
K^{sep} \\
\quad | \qquad \Big) \; P_K = \text{wild inertia, pro-}p\text{ gp} \\
I_K \Big(K^{tame} = K^{ur}(\sqrt[n]{\varpi_K} : p \nmid n) \\
\quad | \qquad\qquad\qquad\qquad \Big) \; \hat{\mathbb{Z}}^{(p)}(1) \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \\
\quad K^{ur} = K(\mu_n : p \nmid n) \\
\quad | \qquad \Big) \; G_K^{ur} \simeq G_{k_K} \simeq \hat{\mathbb{Z}} \\
\quad K
\end{array}
$$

$$G_K^{ur} \simeq G_{k_K} \simeq \varprojlim_b \mathrm{Gal}(\mathbb{F}_{q^b} \mid \mathbb{F}_q) \simeq \varprojlim_b \mathbb{Z}/b\mathbb{Z} \simeq \hat{\mathbb{Z}}$$

$$\phi = \phi_K \mapsto F_K \longleftarrow\!\!\!

Lecture 2

§1. Grothendieck's local monodromy theorem

$L|\mathbb{Q}_\ell$ finite. $L \supset \mathcal{O}_L \twoheadrightarrow \mathcal{O}_L/(\varpi_L) \simeq k_L \leftarrow$ finite ext'n of $\mathbb{F}_\ell$

$\rho : G_K \longrightarrow GL_n(\mathcal{O}_L)$

$$I_K \left(\begin{array}{c} K^{sep} \\ | \\ K^{tame} \\ | \\ K^{un} \\ | \\ K \end{array} \right. \begin{array}{l} \\ ) P_K \\ \\ ) \hat{\mathbb{Z}}^{(p)}(1) \\ \\ ) \hat{\mathbb{Z}} \\ \\ \\ \end{array}$$

<u>Step 1</u>. $\rho(P_K)$ is finite.

<u>Pf</u>. Consider $\rho|_{P_K} : P_K \longrightarrow GL_n(\mathcal{O}_L)$, $GL_n(\mathcal{O}_L) \supset 1 + \varpi_L M_n(\mathcal{O}_L)$

$\downarrow$ reduction mod ϖ_L

$\bar{\rho}|_{P_K} : P_K \longrightarrow GL_n(k_L)$

Then $\underset{\text{pro-}p}{\rho(P_K)} \cap \underset{\text{pro-}\ell}{\left(1 + \varpi_L M_n(\mathcal{O}_L)\right)} = \{1\}$

$\Rightarrow \rho(P_K) = \bar{\rho}(P_K)$ is finite.

$\Rightarrow \exists$ finite Galois ext'n K'' of K s.t. $\rho(P_{K''})$ is trivial.

<u>Step 2</u>. $\rho : G_{K''} \longrightarrow G_{K''}/P_{K''} \simeq \hat{\mathbb{Z}}^{(p)}(1) \rtimes \hat{\mathbb{Z}} \longrightarrow GL_n(\mathcal{O}_L)$

$\hat{\mathbb{Z}}^{(p)}(1) \ni \underbrace{\tau_{K''}}_{\text{generator of tame inertia subgp}}$, $\hat{\mathbb{Z}} \ni \underbrace{\phi_{K''}}_{\text{geom. Frob}}$

(*) $\phi_{K''}^{-1} \tau_{K''} \phi_{K''} = \tau_{K''}^{q_{K''}}$

<u>Step 3</u>. $\exists K'|K$ finite Galois s.t. $\rho(I_{K'})$ are unipotent (i.e. all eigenvalues are 1)

<u>Claim</u>. all eigenvalues of $\tau_{K''}$ are roots of unity.

$\Rightarrow$ after replacing K'' by a finite ext'n K' $\rightsquigarrow$ $\tau_{K'} = \tau_{K''}^{N}$ $\Rightarrow$ $\rho(\tau_{K'})$ is unipotent.

By $\circledast$, $\{\text{eigenvalues of } \tau_{K''}\} = \{(\text{eigenvalues of } \tau_{K''})^{q_{K''}}\}$

This is a finite set. $\Rightarrow$ all eig. val. of $\tau_{K''}$ are roots of unity.

<u>Corollary</u>. $\exists!\ N \in \mathrm{End}(V)$ nilpotent & $\exists\ K'|K$ s.t.

for $\sigma \in I_{K'}$, $\rho(\sigma) = \exp(t_{\zeta,\ell}(\sigma)\, N)$

$$\begin{tikzcd}
\mathrm{Gal}(K^{\mathrm{tame}}|K^{\mathrm{ur}}) \arrow[r, "t_{\zeta,\ell}"] & \mathbb{Z}_\ell(1) & \ni e^{\mathrm{tame}}_{K'|K} \\
\mathrm{Gal}(K'^{\mathrm{tame}}|K'^{\mathrm{ur}}) \arrow[u] \arrow[r, "t'_{\zeta,\ell}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathbb{Z}_\ell(1) \arrow[u, "\times e^{\mathrm{tame}}_{K'|K}"'] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
\tau_{K'} & 1 &
\end{tikzcd}$$

$$N = \frac{1}{e^{\mathrm{tame}}_{K'|K}} \log(\rho(\tau_{K'}))$$

Moreover, if $\tilde{\Phi}_K$ is a lift of Frobenius to G_K,

$$\rho(\tilde{\Phi}_K)^{-1}\, N\, \rho(\tilde{\Phi}_K) = q_K \cdot N$$

<u>Upshot</u>. Describe the representation ρ more algebraically (& hopefully, indep. of ℓ)

<u>Weil gp</u>.

$$\begin{tikzcd}
1 \arrow[r] & I_K \arrow[r] \arrow[d, equal] & G_K \arrow[r, "\psi"] & G_K \simeq \hat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_K \arrow[r] & W_K \arrow[u] \arrow[r, "\psi"] \arrow[d, equal] & \mathbb{Z} \arrow[u, hook] \arrow[r] & 1 \\
 & & \psi^{-1}(\mathbb{Z}) & &
\end{tikzcd}$$

Thm. (Grothendieck's monodromy thm) Fix an isom. $\iota: \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$. Fix ζ as above.

Fix a lift of Frob $\widetilde{\Phi_K}$ of Φ_K in W_K. Then, there's an equiv. of cats (depending on ι, ζ, $\widetilde{\Phi_K}$),

$$\left\{ \begin{array}{l} \rho_K : W_K \longrightarrow GL_n(\bar{L}) \\ \text{cts rep'n} \end{array} \right\} \xleftrightarrow{\sim} \left\{ (r, N) : \begin{array}{l} r : W_K \longrightarrow GL_n(\mathbb{C}) \text{ cts rep'n} \\ N \in M_n(\mathbb{C}) \text{ s.t. } r(g) N r(g)^{-1} = q^{\psi(g)} N \end{array} \right\}$$

$\uparrow$ need $\rho_K(\Phi_K)^N$ is unipotent

top. Frob eig. vals are ℓ-adic units

$$\left\{ \begin{array}{l} \rho_K : G_K \longrightarrow GL_n(\bar{L}) \\ \text{cts rep'n} \end{array} \right\}$$

$\uparrow$ Weil-Deligne rep'n

$$\rho \longmapsto N = \frac{1}{m} \log(\tau_K^m) \quad \text{for } m \text{ suff. large}$$

$$r(\widetilde{\Phi_K}^a \cdot \underset{\substack{\uparrow \\ I_K}}{x}) = \rho(\widetilde{\Phi_K}^a \cdot x) \exp(-t_{\zeta,\ell}(x) N)$$

$a \in \mathbb{Z}$

<u>Upshot</u>. $r(I_K)$ is finite.

<u>Pf</u>. When $\widetilde{\Phi}^a x \widetilde{\Phi}^b y = \widetilde{\Phi}^{a+b} (\widetilde{\Phi}^{-b} x \widetilde{\Phi}^b) y$

<u>Check</u>. $\rho(\widetilde{\Phi}^a x) \exp(-t_{\zeta,\ell}(x) N) \rho(\widetilde{\Phi}^b y) \exp(-t_{\zeta,\ell}(y) N)$

$$\overset{?}{=} \rho(\widetilde{\Phi}^a x \widetilde{\Phi}^b y) \exp(-t_{\zeta,\ell}(\widetilde{\Phi}^{-b} x \widetilde{\Phi}^b y) N)$$

$$\Leftrightarrow \underbrace{\rho(\widetilde{\Phi}^b y)^{-1} \exp(-t_{\zeta,\ell}(x) N) \rho(\widetilde{\Phi}^b y)}_{= \exp(-t_{\zeta,\ell}(x) \underbrace{\rho(\widetilde{\Phi}^b y)^{-1} N \rho(\widetilde{\Phi}^b y)}_{= q^b N})} \overset{?}{=} \exp(-\underbrace{\boxed{t_{\zeta,\ell}(\widetilde{\Phi}^{-b} x \widetilde{\Phi}^b)}}_{q^b t_{\zeta,\ell}(x)} N)$$

§2. Higher ramification group

K finite ext'n of $\mathbb{Q}_p$ / $\mathbb{F}_p((t))$, $L|K$ finite Galois ext'n, $\ell|k$ residue fields

<u>Normalize</u> the valuation $v_L(\varpi_L) = 1$

For $u \geq -1$, <u>define</u>

$$G_{L|K,u} = \{ g \in G_{L|K} : v_L(g(x) - x) \geq u+1, \ \forall x \in \mathcal{O}_L \}$$

enough to check on a generator α s.t. $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

$$G_{L|K,u} = G_{L|K,\lceil u \rceil}.$$

In particular,

$$G_{L|K} = \underbrace{G_{L|K,-1} \supset I_{L|K}}_{G_{\ell|k}} = G_{L|K,0} \supset P_{L|K} = G_{L|K,1} \supset G_{L|K,2} \supset \cdots$$

quotient of $\hat{\mathbb{Z}}^{(p)}(1)$

$$gr_0 G_{L|K} = \frac{G_{L|K,0}}{G_{L|K,1}} \hookrightarrow \ell^\times$$

$$g \longmapsto \frac{g\varpi_L}{\varpi_L} \bmod \varpi_L \in \ell^\times$$

$\boxed{\text{lower numbering ramification filtration}}$

For $n \geq 1$,

$$gr_n G_{L|K} = \frac{G_{L|K,n}}{G_{L|K,n+1}} \hookrightarrow \frac{(1+\varpi_L^n \mathcal{O}_L)^\times}{(1+\varpi_L^{n+1}\mathcal{O}_L)^\times} \simeq \varpi_L^n \cdot \ell \simeq \ell \cong (\mathbb{Z}/p\mathbb{Z})^{\square}$$

p-group

<u>injective homomorphisms</u>

$$g \longmapsto \frac{g\varpi_L}{\varpi_L} \in 1 + \varpi_L^n \mathcal{O}_L$$

$\Rightarrow$ $P_{L|K}$ is a p-group

<u>Key fact</u>. If

$$\begin{pmatrix} L \\ | \\ F \\ | \\ K \end{pmatrix} \begin{matrix} \text{Galois} \\ \\ \text{Galois} \end{matrix}$$

$$G_{L|F,u} = G_{L|K,u} \cap G_{L|F}$$

$v(g(x)-x) \geq u+1$

<u>BWT</u>. $G_{F|K,u}$ & $G_{L|K,u}$ are not directly related.

<u>Def'n</u>. For $L|K$ above, define the <u>Herbrand function</u> as follows:

for $u \geq -1$,

$$\phi_{L|K}(u) = \int_0^u \frac{dt}{[G_{L|K,0} : G_{L|K,t}]}$$

The integrand is step-function, $\phi_{L|K}$ is piece-wise linear function, strictly increasing, concave.

Take $\psi_{L|K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be the inverse function.

& define $G^{v}_{L|K} := G_{L|K, \psi_{L|K}(v)}$, or $G^{\phi_{L|K}(u)}_{L|K} = G_{L|K,u}$.

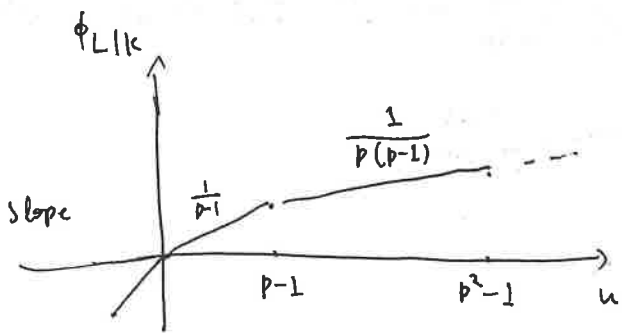
E.g. $L|K = \mathbb{Q}_p(\zeta_{p^n}) \,|\, \mathbb{Q}_p$,

$$
\begin{array}{ccc}
L = \mathbb{Q}_p(\zeta_{p^n}) & & 1 \\
| & & \cap \\
\vdots & & \vdots \\
| & & \cap \\
\mathbb{Q}_p(\zeta_{p^2}) & & (1+p^2\mathbb{Z}_p)/(1+p^n\mathbb{Z}_p) = G_p = G_{p+1} = \cdots = G_{p^2-1} = G^2 \\
| & & \cap \\
\mathbb{Q}_p(\zeta_p) & & (1+p\mathbb{Z}_p)/(1+p^n\mathbb{Z}_p) = G_1 = G_2 = \cdots = G_{p-1} = G^1 \\
| & & \cap \\
\mathbb{Q}_p & & (\mathbb{Z}/p^n\mathbb{Z})^\times = G_0 = G_{-1} = G^0
\end{array}
$$

$\zeta_{1+p}(\zeta_{p^n}) - \zeta_{p^n}$

$= \zeta_{p^n}^{1+p} - \zeta_{p^n} = \zeta_{p^n}(\zeta_{p^n}^{p} - 1)$

val $= p$



Key Theorem. If $\begin{array}{c} L \\ | \\ F \\ | \\ K \end{array}$ is a tower of Galois ext'ns, then the image of $G^{v}_{L|K}$ in $G_{F|K}$ is $G^{v}_{F|K}$. i.e. $G^{v}_{F|K} = G^{v}_{L|K}\, G_{L|F} / G_{L|F}$

Cor. For $v \geq -1$, define $G^{v}_{K} = \varprojlim_{L|K} G^{v}_{L|K}$ $\overset{\text{then}}{\Longrightarrow}$ $G^{v}_{L|K} \simeq G^{v}_{K}\, G_L / G_L$.

Def $G_K^{v+} = \overline{\bigcup_{w>v} G_K^{w}}$ closure

Def. Artin & Swan conductor:

For a rep'n $\rho : W_K \to GL(V)$ (s.t. $\rho(I_K)$ is finite), V/field of char. 0

$$\star \ \underline{\underline{\mathrm{Art}(\rho)}} := \sum_{v \geq 0} v \dim\left(V^{G_K^{v}} / V^{G_K^{v+}} \right) \qquad \mathrm{Art}(\rho) = \mathrm{Art}(\rho|_{G_{K^{ur}}})$$

$$\mathrm{Swan}(\rho) := \sum_{v \geq 1} (v-1) \dim\left(V^{G_K^{v}} / V^{G_K^{v+}} \right) \qquad \mathrm{Swan}(\rho) = e_{K'|K} \, \mathrm{Swan}(\rho|_{G_{K'}})$$

$K'|K$ tame

Thm (Hasse - Art) $\mathrm{Art}(\rho), \mathrm{Swan}(\rho) \in \mathbb{Z}$

Cor. $\begin{array}{c} L \\ | \\ K \end{array}$ abel ext'n, $\quad G_{L|K} \supsetneq G_{L|K}^{b_1} \supsetneq G_{L|K}^{b_2} \supsetneq \cdots$

$$b_1 < b_2 < \cdots$$

Take a rep'n $\chi_{b_i} : G_{L|K} / G_{L|K}^{b_i} \to \mathbb{C}^\times$ that doesn't factors through

$$G_{L|K} / G_{L|K}^{b_i - 1}$$

Then $\mathrm{Art}(\chi_{b_i}) = b_i$

Hasse - Art Thm $\Rightarrow$ b_i are integers

$\square$

§3. Local class field theory

<u>K = local field</u>

(1)

$$\begin{tikzcd}
 & \varpi_K \arrow[r, mapsto] & \text{geom. Frob. } \phi_K \\
\mathrm{Art}_K : & K^\times \arrow[r, "\sim"'] & W_K^{ab} \ni \\
\varpi_K^{\mathbb{Z}} \times \mathcal{O}_K^\times = \arrow[d] & \arrow[d] & \arrow[d] \\
\cap \quad = & \hat{K}^\times \arrow[r, "\sim"'] & G_K^{ab} \\
\varpi_K^{\hat{\mathbb{Z}}} \times \mathcal{O}_K^\times & \cup & \cup \\
 & \mathcal{O}_K^\times \arrow[r, "\sim"'] & I_{K^{ab}|K} \\
 & \cup & \cup \\
 & (1+\varpi_K^i \mathcal{O}_K)^\times \arrow[r, "\sim"'] & G^i_{K^{ab}|K} \quad , i \geq 1
\end{tikzcd}$$

$$G^i_{K^{ab}|K} = \overline{G_K^i [G_K, G_K]} \Big/ \overline{[G_K, G_K]}$$

$$\begin{tikzcd}
K^{ab} \arrow[d, dash] \\
K^{ur} \arrow[d, dash] \\
K
\end{tikzcd} \quad I_{K^{ab}|K}$$

(2) If $L|K$ is a finite ext'n,

$$\begin{tikzcd}
L^\times \arrow[r, "\mathrm{Art}_L", "\sim"'] \arrow[d, "\text{norm}"] \arrow[dr, phantom, "///"] & W_L^{ab} \arrow[d, "\text{nat'l}"'] \\
K^\times \arrow[r, "\mathrm{Art}_K", "\sim"'] \arrow[u, bend left, "\text{inclusion}"] & W_K^{ab} \arrow[u, bend right, "\underline{\text{transfer map}}"']
\end{tikzcd}$$

if $H \subset G$ of finite index,

$$\exists \; G^{ab} \longrightarrow H^{ab}$$
$$\| \qquad\qquad \|$$
$$H_1(G, \mathbb{Z}) \xrightarrow{\mathrm{Res}} H_1(H, \mathbb{Z})$$

(3) If $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}_p} | \mathbb{Q}_p)$, then

$$\mathrm{Art}_{\tau K}(\tau\alpha) = \tau \underbrace{\mathrm{Art}_K(\alpha)}_{W_K^{ab}} \tau^{-1} \in W_{\tau K}^{ab}$$

<u>$K = \mathbb{R}$</u>

$$W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \{1, j\} \quad , j^2 = 1, \; j z j = \bar{z}$$
$$\downarrow$$
$$\mathrm{Gal}(\mathbb{C} | \mathbb{R})$$

Check.

$$\begin{tikzcd}
G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z} & W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \{1, j\} \arrow[l] \arrow[r, two heads] & W_{\mathbb{R}}^{ab} \simeq \mathbb{R}^\times_{>0} \times \{1, j\} & \mathbb{R}^\times \arrow[l, "\mathrm{Art}_{\mathbb{R}}"'] \\
G_{\mathbb{C}} = \{1\} \arrow[u] & W_{\mathbb{C}} = \mathbb{C}^\times \arrow[l] \arrow[r, two heads] \arrow[u] & W_{\mathbb{C}}^{ab} \simeq \mathbb{C}^\times \arrow[u] & \mathbb{C}^\times \arrow[l] \arrow[u, "\mathrm{norm}"']
\end{tikzcd}$$

Lecture 3. K local field.

$$\mathrm{Art}_K : \widehat{K^\times} \xrightarrow{\sim} G_K^{ab}$$

LLC (a preview)

$$\left\{ \begin{array}{l} \text{Irred smooth adm.} \\ \text{rep'ns of } GL_n(K) \text{ over } \mathbb{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} n\text{-dim'l Frob-semisimple} \\ \text{Weil-Deligne rep'ns of } W_K \end{array} \right\}$$

(r, N) : $r : W_K \to GL_n(\mathbb{C})$

$N \in M_n(\mathbb{C})$

$\phi_K^{-1} N \phi_K = q_K N$

$n=1$. $\{\text{characters of } K^\times\}$

$K^\times \to \mathbb{C}^\times$ $\longleftrightarrow$ $\{r : W_K \to \mathbb{C}^\times\}$

$$\begin{tikzcd}
 & W_K \arrow[r] \arrow[d] & \mathbb{C}^\times \\
 & W_K^{ab} \arrow[ur] &
\end{tikzcd}$$

$K^\times \xrightarrow[\sim]{\mathrm{Art}_K} W_K^{ab}$

$\cup$ $\qquad\qquad$ $\cup$

$\left\{ \text{chars } K^\times / (1 + \varpi_K^m \mathcal{O}_K)^\times \to \mathbb{C}^\times \right\}$ $\qquad$ $\left\{ \text{those } r \text{ s.t. } \mathrm{Art}(r) \le m \right\}$ $\longleftrightarrow$ $r|_{I^m_{K^{ab}|K}} = \text{triv.}$

$n=2$ $\left\{ \begin{array}{l} \text{irred. sm. adm. rep'n} \\ \pi \text{ of } GL_2(K)/\mathbb{C} \end{array} \right\}$ $\longleftrightarrow$ $\left\{ \begin{array}{l} 2\text{-dim'l Frob-ss} \\ \text{WD-rep'n } (r, N) \end{array} \right\}$

$c(\pi) = \min\{n : \exists v \in \pi \text{ s.t.}$ $\longleftrightarrow$ $\mathrm{Art}(r, N) = \mathrm{Art}(r) + \dim r^{I_K} - \dim r^{I_K, N=0}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \underbrace{\omega_\pi(d)}_{\text{central char.}} \cdot v, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ \varpi^n \mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \Big\}$

$\pi \longleftrightarrow$ modular form of level $\Gamma_1(p^{c(\pi)} N)$, $p \nmid N$

§1. Global class field theory

F = no. field

$\mathbb{A}_F$ = adeles of F

<u>Exer.</u> $\mathbb{A}_F^\times$ can be identified topologically as $\{(x,y) \in \mathbb{A}_F \times \mathbb{A}_F : xy = 1\}$

$$\mathbb{A}_F^\times = \prod_v{}' (F_v^\times, \mathcal{O}_{F_v}^\times)$$

Similarly, $GL_n(\mathbb{A}_F) = \{(g,c) \in M_n(\mathbb{A}_F) \times \mathbb{A}_F : \det g \cdot c = 1\}$

$$= \prod_v{}' \left(GL_n(F_v), GL_n(\mathcal{O}_{F_v})\right)$$

Main Theorems of global class field theory

(1)

$$\begin{tikzcd}
\mathrm{Art}_F : & F^\times \backslash \mathbb{A}_F^\times / (F_{\mathbb{R}}^\times)^\circ \arrow[r, "\sim"] & \mathrm{Gal}(F^{ab} | F) \\
& ((F \otimes_{\mathbb{Q}} \mathbb{R})^\times)^\circ \arrow[u, hook] \arrow[d, phantom, "///"] & \\
\forall v, & F_v^\times \arrow[u] \arrow[r, "\mathrm{Art}_{F_v}"] & \mathrm{Gal}^{ab}_{F_v} \arrow[uu]
\end{tikzcd}$$

<u>Equivalent</u>: intersection of all fin. index open subgp of $F^\times \backslash \mathbb{A}_F^\times$ is $(F_{\mathbb{R}}^\times)^\circ$

<u>Variant</u>. $\{$finite index open subgp of $F^\times \backslash \mathbb{A}_F^\times\} \longleftrightarrow \{$finite abel. ext'n E of $F\}$

$$\mathrm{Nm}_{E|F}(E^\times \backslash \mathbb{A}_E^\times) \longleftarrow\!\shortmid \; E$$

(2) $E|F$ finite ext'n

$$\begin{tikzcd}
E^\times \backslash \mathbb{A}_E^\times \arrow[r, "\mathrm{Art}_E"] \arrow[d, "\text{norm}"] & G_E^{ab} \arrow[d, "\text{nat'l}"'] \\
F^\times \backslash \mathbb{A}_F^\times \arrow[r, "\mathrm{Art}_F"] \arrow[u, bend left, "\text{inclusion}"] & G_F^{ab} \arrow[u, bend right, "\text{transfer}"']
\end{tikzcd}$$

(3) If $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, then $\mathrm{Art}_{\tau F}(\tau\alpha) = \tau\, \mathrm{Art}_F(\alpha)\, \tau^{-1}$ for $\alpha \in F^\times \backslash \mathbb{A}_F^\times$.

<u>Example</u>. $F = \mathbb{Q}$.

<u>Know</u>

$$\mathrm{Gal}_{\mathbb{Q}}^{ab} \xrightarrow{\text{Kronecker-Weber}} \varprojlim_N \mathrm{Gal}(\mathbb{Q}(\mu_N)|\mathbb{Q}) \simeq \varprojlim_N (\mathbb{Z}/N\mathbb{Z})^\times \simeq \hat{\mathbb{Z}}^\times$$

$$(p^{-n}, p^{-n}, \dots, \alpha, p^{-n}, \dots, p^{-n}, \dots) \longmapsto$$

$$\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{R}_{>0}^\times \xrightarrow[\sim]{\mathrm{Art}_{\mathbb{Q}}} \hat{\mathbb{Z}}^\times$$

$$\mathbb{A}_{\mathbb{Q},f}^\times = \frac{\mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times}{\{\pm 1\}}$$

$$\begin{tikzcd}
p^n\alpha \arrow[uuuu

Claim. $\forall \ell$ prime, $\mathrm{Hom}_{\text{cts gp}}(G_{F,S}, \mathbb{F}_\ell)$ is fin. dim'l. $/\mathbb{F}_\ell$.

$$\simeq H^1(G_{F,S}, \mathbb{F}_\ell)$$

(or equivalently, $G_{F,S} \Big/ \overline{\langle [G_{F,S}, G_{F,S}], G_{F,S}^{\ell} \rangle}$ is finite)

Proof This is also equiv. to show $G_{F,S}^{ab}/\ell$ is finite.

By GCFT, $G_{F,S}^{ab} \simeq G_F^{ab} \Big/ \text{image of } \prod_{v \notin S} I_{F_v}$

$$\begin{array}{c} F^{ab} \\ \Big| \\ F^{ab,S} \\ \Big| \\ F \end{array} \Big) \text{ image of } I_{F_v},\ v \notin S \qquad \simeq F^\times \backslash \mathbb{A}_F^\times / F_{\mathbb{R}}^{\times,0} \prod_{v \notin S} \mathcal{O}_{F_v}^\times$$

Recall. $1 \to \mathcal{O}_F^\times \to F^\times \to \mathbb{A}_F^\times \Big/ \prod_v \mathcal{O}_{F_v}^\times \to cl(F) \to 1$

$$\mathbb{A}_F^\times \Big/ \prod_v \mathcal{O}_{F_v}^\times \cong \bigoplus_v \left(\frac{F_v^\times}{\mathcal{O}_{F_v}^\times} \right) \cong \bigoplus_v v^{\mathbb{Z}}$$

$$\frac{\dfrac{F_{\mathbb{R}}^\times}{F_{\mathbb{R}}^{\times,0}}}{\ell} \times \frac{\prod_{v \in S} \mathcal{O}_{F_v}^\times}{\ell} \longrightarrow \frac{G_{F,S}^{ab}}{\ell} \longrightarrow \underbrace{\frac{cl(F)}{\ell}}_{\text{finite}} \longrightarrow 0$$

Application 2 Given a finite set S of places of F and for each $v \in S$, a finite Galois ext'n F_v' of F_v, Then there exists a finite <u>solvable</u> Galois ext'n $E|F$.

s.t. $\forall \underset{\substack{\uparrow \\ \text{place of } E}}{w} | v \in S$, $E_w \simeq F_v'$.

Philosophy: Can always find finite solvable ext'n to satisfy a finite list of local conditions.

Proof. Note $F_v' | F_v$ is solvable for every v.

So by induction, we may reduce to the case when each $F_v' | F_v$ is cyclic.

$$\exists\ \chi_v : \mathrm{Gal}(F_v' | F_v) \hookrightarrow \mathbb{C}^\times$$

$$\begin{tikzcd} \mathrm{Gal}(F_v' | F_v) \arrow[r, hook] & \mathbb{C}^\times \\ F_v^\times / \text{finite index subgp} \arrow[u, "\wr"] \arrow[ur, "\chi_v"] & \end{tikzcd}$$

(thus $\ker \chi_v$ = finite index subgp of $F_v^\times \longleftrightarrow$ ext'n F_v' under LCFT)

$$\chi_S := \prod_{v \in S} \chi_v : \prod_{v \in S} F_v^\times \longrightarrow \mathbb{C}^\times$$

Claim. χ_S extends to a finite image char. $\chi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$.

Claim $\Rightarrow$ App 2: $\ker \chi$ is a finite index open subgp of $F^\times \backslash \mathbb{A}_F^\times \longleftrightarrow E | F$

$$\begin{tikzcd} F^\times \backslash \mathbb{A}_F^\times / \ker \chi \arrow[r, "\sim"] & \mathrm{Gal}(E|F) \\ F_v^\times / \ker \chi_v \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{Gal}(F_v' | F_v) \end{tikzcd}$$

$\ker \chi_v = F_v^\times \cap \ker \chi$

$$\begin{tikzcd} F^\times \backslash \mathbb{A}_F^\times \arrow[rr] \arrow[d] & & \mathrm{Gal}_F^{ab} \arrow[d] \\ F^\times \backslash \mathbb{A}_F^\times / \ker \chi \arrow[rr, "\sim"] & & \mathrm{Gal}(E|F) \\ F_v^\times / \ker \chi_v \arrow[u] \arrow[rr] \arrow[d] & & \mathrm{Gal}(F_v' | F_v) \arrow[u] \\ & & \mathrm{Gal}_{F_v}^{ab} \arrow[u] \arrow[d] \\ & & \end{tikzcd}$$

image should be $\mathrm{Gal}(E_w | F_v)$

$\Rightarrow E_w \simeq F_v'$.

Proof of the claim: Enough to take a small enough open cpt subgp $U \subset \mathbb{A}_F^{(S),\times}$ s.t.

$$\begin{tikzcd} \prod_{v \in S} F_v^\times \arrow[dr, "\chi_S"'] & U \times \prod_{v \in S} F_v^\times \arrow[l] \arrow[d, "\chi", "\sim"'] \arrow[r] \arrow[rr, dashed] & (U \cap F^\times) \backslash U \times \prod_{v \in S} F_v^\times & F^\times \backslash \mathbb{A}_F^\times / F_{\mathbb{R}}^{\times,0} \arrow[l, hook, "\text{open}"] \\ & \mathbb{C}^\times & & \end{tikzcd}$$

Key: $\tilde{\chi}$ is trivial on $U \cap F^\times$.

Note $U \cap F^\times \subset \left(\prod_{v \notin S} \mathcal{O}_{F_v}^\times\right) \cap F^\times = \underbrace{\mathcal{O}_F[\tfrac{1}{S}]^\times}_{\text{f.g. abel. gp (Dirichlet unit thm)}}$

WTS $\tilde{\chi}$ is trivial on $U \cap F^\times \subset \underbrace{U}_{\tilde{\chi} \text{ trivial here}} \times \underbrace{\prod_{v \in S} F_v^\times}_{\chi_S}$

need χ_S trivial on $(U \cap F^\times) \subset \prod_{v \in S} F_v^\times$

$\mathcal{O}_F[\tfrac{1}{S}]^\times \subset \bigcup_{\ker \chi_S}$ finite index

Choose U small enough

But $\ker \chi_S \cap \mathcal{O}_F[\tfrac{1}{S}]^\times$ is a fin. index subgp of $\mathcal{O}_F[\tfrac{1}{S}]^\times$ and thus is a congruence subgp

§3. Basics of group cohomology

For a (discrete) group G acting on an abel. gp M (w/ discrete top.) ← call M a G-module

$\rightsquigarrow$ cohomology $H^i(G, M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.

① $H^0(G, M) = M^G$

② If G acts trivially on M, $H^1(G, M) \simeq \mathrm{Hom}_{gp}(G, M) = \mathrm{Hom}_{gp}(G^{ab}, M)$.

③ When $G = \mathbb{Z}/n\mathbb{Z}$, can identify $\mathbb{Z}[G] \simeq \mathbb{Z}[x]/(x^n - 1)$

$$\sigma = [1] \longmapsto x$$
$$[0] \longmapsto 1$$

Take a resol'n of $\mathbb{Z}$ as $\mathbb{Z}[G]$-module:

$$\cdots \xrightarrow{x-1} \frac{\mathbb{Z}[x]}{(x^n-1)} \xrightarrow{1+x+\cdots+x^{n-1}} \frac{\mathbb{Z}[x]}{(x^n-1)} \xrightarrow{x-1} \frac{\mathbb{Z}[x]}{(x^n-1)} \xrightarrow{ev_{x=1}} \mathbb{Z} \to 0$$

So $H^\bullet(G, M) = H^\bullet\left(M \xrightarrow{\sigma-1} M \xrightarrow{1+\sigma+\cdots+\sigma^{n-1}} M \xrightarrow{\sigma-1} M \xrightarrow{N_{m\sigma}} \cdots \right)$

So $H^0(G, M) = M^{\sigma=1}$

$H^{odd}(G, M) = M^{N_{m\sigma}=0} / (\sigma-1)M$

$H^{even, >0}(G, M) = M^{\sigma=1} / N_{m\sigma}(M)$

④ (Shapiro Lemma) If $H \subset G$ is a finite index subgp & M an H-module, then

$$H^i(H, M) \simeq H^i(G, \mathrm{Ind}_H^G M)$$

$$\| $$

$$\{ f: G \to M : f(hg) = h.f(g) \}$$

$$(g * f)(x) = f(xg^{-1})$$

⑤ If G is a finite gp, $H^{>0}(G, M)$ is killed by $\# G$.

In particular, if G is a p-group & M is ℓ^∞-torsion, $\ell \neq p$, then $H^{>0}(G, M) = 0$.

⑥ (Hochschild-Serre spectral sequence) If $H \triangleleft G$ is a normal subgp, and M is a G-mod, then we have an isom. of functors $\{G\text{-mods}\} \longrightarrow Ab$

$$(-)^G = (-)^{G/H} \circ (-)^H$$

$$\Rightarrow E_2^{i,j} = H^i(G/H, H^j(H, M)) \Rightarrow H^{i+j}(G, M).$$

E_2 page

$$\begin{tikzcd}
H^0(G/H, H^2(H,M)) & H^1(G/H, H^2(H,M)) & \\
H^0(G/H, H^1(H,M)) \arrow[rr, "d_2"] & H^1(G/H, H^1(H,M)) & H^2(G/H, H^1(H,M)) \\
H^0(G/H, H^0(H,M)) \arrow[rr, "d_2"] & H^1(G/H, H^0(H,M)) & H^2(G/H, H^0(H,M))
\end{tikzcd}$$

$$H^0(G,M) = H^0(G/H, H^0(H,M)) \quad , \quad M^G = (M^H)^{G/H}$$

$$\begin{tikzcd}
0 \arrow[r] & H^1(G/H, H^0(H,M)) \arrow[r, "\mathrm{Inf}"] & H^1(G,M) \arrow[r] \arrow[d, "\mathrm{Res}"] & \ker d_2^{1,0} \arrow[r] & 0 \\
& & H^1(G/H, H^1(H,M)) \arrow[d, "d_2"] & & \\
& & H^2(G/H, H^0(H,M)) \arrow[d] & & \\
& & H^2(G,M) & &
\end{tikzcd}$$

five term exact seq.

assoc. to Hochschild-Serre spec. seq.

<u>Lecture 4</u>. Last time, group cohomology $H^i(G,M)$

⑥ $H \triangleleft G$, $\rightarrow$

$$0 \to H^1(G/H, M^H) \to H^1(G,M) \to H^0(G/H, H^1(H,M)) \xrightarrow{d_2} H^2(G/H, M^H) \to H^2(G,M).$$

$$\begin{array}{ccccc}
H^n(M) & \to & H^0(H^n) & \qquad & H^n(H^0) \to H^n(G,M) \\
\| & & \| & & \| \qquad\qquad \| \\
H^n(G,M) & \xrightarrow{\mathrm{res}} & H^n(H,M)^{G/H} & & H^n(G/H, M^H) \xrightarrow{\mathrm{inf}} H^n(G,M)
\end{array}$$

④ <u>Shapiro Lemma</u>. if $H \subset G$ finite index subgp, then $H^i(H, M) \simeq H^i(G, \mathrm{Ind}_H^G M)$.

$\Downarrow$

⑤ (degree shifting trick). <u>E.g.</u> G finite $\curvearrowright M$.

$$M \hookrightarrow \mathrm{Ind}_{\{*\}}^G \underline{M} \quad \Rightarrow \text{coker} =: Q$$

$\llcorner$ just abel gp

$$\mathrm{Ind}_{\{*\}}^G \underline{M} = \{f : G \to M\}$$

$$m \longmapsto [g \mapsto gm]$$

For $i \geq 1$, LES

$$H^i(G, \mathrm{Ind}_{\{1\}}^G \underline{M}) \to H^i(G, Q) \xrightarrow{\sim} H^{i+1}(G, M) \to H^{i+1}(G, \mathrm{Ind}_{\{1\}}^G \underline{M})$$

$$H^i(G, \mathrm{Ind}_{\{1\}}^G \underline{M}) = H^i(\{1\}, \underline{M}) = 0$$

$$H^{i+1}(G, \mathrm{Ind}_{\{1\}}^G \underline{M}) = H^{i+1}(\{1\}, \underline{M}) = 0$$

<u>Variant</u>. $\mathrm{ind}_{\{1\}}^G \underline{M} := \underline{M} \otimes_{\mathbb{Z}[\{1\}]} \mathbb{Z}[G]$, $\mathbb{Z}[\{1\}] = \mathbb{Z}$

$$0 \to K \to \mathrm{ind}_{\{1\}}^G M \to M \to 0$$

$$\rightsquigarrow H^i(G, M) \simeq H^{i+1}(G, K).$$

Now, assume G is a profinite gp (e.g. Galois gp), but M still w/ discrete top.

(usually a finite length $\mathbb{Z}_\ell$-mod).

<u>Def</u>. $H^i(G, M) = \varinjlim_{\substack{H \triangleleft G \\ \text{open normal}}} H^i(G/H, M^H)$

Then ①, ②, ③ still hold here.

(H is closed normal subgp for G).

③ $\mathbb{Z}/n\mathbb{Z} \curvearrowright M$, $\quad H^0(\mathbb{Z}/n\mathbb{Z}, M) = M^\sigma$

$\mathbb{Z}/n\mathbb{Z} = \langle\sigma\rangle$

$$H^{odd}(\mathbb{Z}/n\mathbb{Z}, M) = M^{Nm_\sigma = 1} / (\sigma - 1)M$$

$$H^{even, >0}(\mathbb{Z}/n\mathbb{Z}, M) = M^\sigma / Nm_\sigma(M).$$

Now, $G = \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ $\quad$ (variant $\hat{\mathbb{Z}}^{(p)}$)

Say M is N-torsion for some N $\quad$ (variant assume $p \nmid N$)

The connecting map for $n | n'$ & $N | n$:

$$H^i(\mathbb{Z}/n, M) \longrightarrow H^i(\mathbb{Z}/n', M) \quad \text{is mult. by } \left(\frac{n'}{n}\right)^{\lfloor i/2 \rfloor}$$

e.g. $i=2$ $\quad M^\sigma/- \longrightarrow M^\sigma/-$

So <u>in the limit</u>, $H^{\geq 2}(\hat{\mathbb{Z}}, M) = 0$.

So $H^0(\hat{\mathbb{Z}}, M) = M^{\hat{\mathbb{Z}}}$, $\quad H^1(\hat{\mathbb{Z}}, M) \simeq M/(\sigma-1)M$, where σ is a top. gen of $\hat{\mathbb{Z}}$.

this depends on the choice of σ.

$\| $

$H^1(C^\bullet_{cts}(\hat{\mathbb{Z}}, M))$ evaluation of the cocycle at σ.

<u>An important case of ⑥: Hochschild-Serre spec. seq.</u>

K local field (fin. ext'n of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$), coeffs $E \supset \mathcal{O}_E \to \mathcal{O}_E/\varpi_E \simeq k_E \simeq \mathbb{F}_{\ell^r}$ s.t. $\ell \neq \operatorname{char} K$.

M a finite length $\mathcal{O}_E$-module, w/ cts G_K-action.

(Assume $\ell \neq p$) $\quad 1 \to I_K \to G_K \to G_{k_K} \to 1$

$$0 \to \underbrace{H^1(G_{k_K}, M^{I_K})}_{\substack{\| \\ H^1_{ur}(G_K, M) \\ \text{unramified part}}} \to H^1(G_K, M) \to \underbrace{H^1(I_K, M)^{G_K}}_{\substack{\| \\ H^1_{sing}(G_K, M) \\ \text{singular part}}} \to \underbrace{H^2(G_{k_K}, M^{I_K})}_{\substack{\cong \\ \| \\ 0}}$$

$$(*) \quad 0 \to H^1_{ur}(G_K, M) \to H^1(G_K, M) \to H^1_{sing}(G_K, M) \to 0$$

<u>Rmk.</u> Assume G_K-action on M is unramified, M vec. sp. $/\mathbb{F}_\ell$

$$[c] \in H^1(G_K, M) = \mathrm{Ext}^1_{\mathbb{F}_\ell[G_K]}(\mathbb{F}_\ell, M)$$

$$0 \to M \to \underset{\substack{\circlearrowleft \\ G_K}}{E_c} \to \mathbb{F}_\ell \to 0$$

$\Rightarrow$ E_c is unramified $\Leftrightarrow$ $[c] \in H^1_{ur}(G_K, M)$.

* <u>Pontryagin dual</u>

For N an fin. length $\mathcal{O}_E$-mod, $N^* := \mathrm{Hom}_{\mathcal{O}_E}(N, E/\mathcal{O}_E)$.

<u>Theorem</u> (local Tate duality) Let M be a finite length $\mathcal{O}_E$-mod w/ cts G_K-action.

(0) For $i = 0, 1, 2$, $H^i(G_K, M)$ is a fin. length $\mathcal{O}_E$-mod, and $H^{>2}(G_K, M) = 0$.

(1) For $i = 0, 1, 2$, there's a canonical & functorial perfect pairing ($\mathcal{O}_E$-linear)

$$H^i(G_K, M) \times H^{2-i}(G_K, \underbrace{M^*(1)}_{\text{Tate twist}}) \xrightarrow{\cup} H^2(G_K, E/\mathcal{O}_E(1)) \simeq E/\mathcal{O}_E.$$

So $H^i(G_K, M) \simeq H^{2-i}(G_K, M^*(1))^*$. $(*)$

(2) When $\ell \neq p$, under the local duality, the exact seq. $(*)$ is "self dual":

$$\begin{tikzcd}
0 \arrow[r] & H^1_{un}(G_K, M) \arrow[r] \arrow[d, equal] & H^1(G_K, M) \arrow[r] \arrow[d, equal] & H^1_{sing}(G_K, M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^1_{sing}(G_K, M^*(1))^* \arrow[r] & H^1(G_K, M^*(1))^* \arrow[r] & H^1_{un}(G_K, M^*(1))^* \arrow[r] & 0
\end{tikzcd}$$

(3) (Euler char.)

$$\chi(G_K, M) := \sum_{i=0}^{2} (-1)^i \operatorname{length}_{\mathcal{O}_E}(H^i(G_K, M)) = \begin{cases} 0 & \text{if } \ell \neq p \\ -[K:\mathbb{Q}_p] \operatorname{length}_{\mathcal{O}_E}(M) & \text{if } \ell = p \\ & \& \ K \mid \mathbb{Q}_p \end{cases}$$

<u>Example</u>: Unramified calculation

Assume that G_K-action on M factors through G_{k_K} (and $\ell \neq p$).

$$0 \to H^1_{unr}(G_K, M) \to H^1(G_K, M) \to H^1_{sing}(G_K, M) \to 0$$

- $H^1_{un}(G_K, M) = H^1(\underset{\phi^{\hat{\mathbb{Z}}}}{G_{k_K}}, M) = M/(\phi - 1)M$

- $H^1_{sing}(G_K, M) = H^1(I_K, M)^{\phi=1} = \operatorname{Hom}(I_K, M)^{\phi=1} = \operatorname{Hom}(\overset{\hat{\mathbb{Z}}^{(p)}(1)}{\overset{\shortparallel}{I_K/P_K}}, M)^{\phi=1}$

$$1 \to \underbrace{P_K}_{\text{pro-}p} \to I_K \to \hat{\mathbb{Z}}^{(p)}(1) \to 1$$

$$= \operatorname{Hom}(\mathbb{Z}_\ell(1), M)^{\phi=1}$$

$$= (M(-1))^{\phi=1}.$$

WTS $\quad H^1_{ur}(G_K, M) \simeq H^1_{sing}(G_K, M^*(1))^*$

$$H^1_{sing}(G_K, M^*(1))^* = \left(M^*(1)(-1)\right)^{\phi=1} = M^{*,\phi=1}$$

$$\begin{tikzcd}
M \arrow[r, "\phi-1"] \arrow[d, leftrightarrow] & M \arrow[r] \arrow[d, leftrightarrow] & M/(\phi-1)M \arrow[r] \arrow[d, leftrightarrow] & 0 \\
M^* & M^* \arrow[l, "\phi-1"'] & M^{*,\phi=1} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

<u>Proof of the local Tate duality</u>:

Input from LCFT:

$$H^i(G_K, K^{sep,\times}) = \begin{cases} K^\times & , i=0 \\ 0 & , i=1 \quad \text{(Hilbert 90)} \\ \mathbb{Q}/\mathbb{Z}, & i=2 \quad \text{(LCFT)} \\ 0 & , i\geq 3 \end{cases}$$

res. = mult. by $[L:K]$.

& when $L|K$ is a finite ext'n, $\quad \underset{\mathbb{Q}/\mathbb{Z}}{\underset{\wr\wr}{H^2(L, L^{sep,\times})}} \xrightarrow[\sim]{\text{Cores}} \underset{\mathbb{Q}/\mathbb{Z}}{\underset{\wr\wr}{H^2(K, K^{sep,\times})}}$

By Kummer theory, $\quad 1 \to \mu_{\ell^n} \to K^{sep,\times} \xrightarrow{x \mapsto x^{\ell^n}} K^{sep,\times} \to 1$

$$H^1(G_K, K^{sep,\times}) = 0$$

$$H^2(G_K, \mu_{\ell^n}) \to \underset{\mathbb{Q}/\mathbb{Z}}{\underset{\wr\wr}{H^2(G_K, K^{sep,\times})}} \xrightarrow{\ell^n} \underset{\mathbb{Q}/\mathbb{Z}}{\underset{\wr\wr}{H^2(G_K, K^{sep,\times})}}$$

$H^2(G_K, \mu_{\ell^n}) \simeq \mathbb{Z}/\ell^n\mathbb{Z}$, $H^{\geq 3}(G_K, \mu_{\ell^n}) = 0$

<u>Take limit</u> & $\otimes \mathcal{O}_E$: $H^2(G_K, E/\mathcal{O}_E(1)) \simeq E/\mathcal{O}_E$. $H^{\geq 3}(G_K, E/\mathcal{O}_E(1)) = 0$

$$H^i(G_K, M) \times H^{2-i}(G_K, M^*(1)) \xrightarrow{\cup} H^2(G_K, E/\mathcal{O}_E(1)) \simeq E/\mathcal{O}_E$$

<u>Note</u> for an exact seq $0 \to M_1 \to M \to M_2 \to 0$ $\Rightarrow$ a canonical map

$$H^i(G_K, M) \to H^{2-i}(G_K, M^*(1))^*$$

$(*)$ of M_1 & M_2 $\Rightarrow$ $(*)$ for M.

So we may assume that M is a k_E-vector space.

<u>Special case</u> $K = K(\mu_\ell)$, and $M = k_E \approx k_E(1)$ w trivial G_K-action

$\rightsquigarrow$ reduce to $M = \mathbb{F}_\ell \approx \mu_\ell$

$$\begin{array}{ccc} H^0(G_K, \mathbb{F}_\ell) & \times & H^2(G_K, \mathbb{F}_\ell(1)) \xrightarrow{\cup} H^2(G_K, \mathbb{F}_\ell(1)) = \mathbb{F}_\ell \\ \| & & \wr\| \\ \mathbb{F}_\ell & & \mathbb{F}_\ell. \end{array}$$

$$\begin{array}{ccc} H^1(G_K, \mathbb{F}_\ell) & \xrightarrow{?} & (H^1(G_K, \mathbb{F}_\ell))^* \\ \| & & \| \text{ via Kummer seq.} \\ \mathrm{Hom}_{gp}(G_K, \mathbb{F}_\ell) & & (K^\times/(K^\times)^\ell)^* \\ \| & & \| \\ \mathrm{Hom}_{gp}(G_K^{ab}, \mathbb{F}_\ell) & & \mathrm{Hom}(K^\times/(K^\times)^\ell, \mathbb{F}_\ell) \\ & \overset{\text{LCFT}}{=} & \| \\ & & \mathrm{Hom}(K^\times, \mathbb{F}_\ell) \end{array}$$

$$H^{\geq 3}(G_K, \mathbb{F}_\ell) \approx H^{\geq 3}(G_K, \mu_\ell) = 0.$$

<u>Input</u> (Exer 1.4) $L|K$ fin. ext'n, N a G_L-module.

$$\begin{array}{ccc} H^i(G_L, N) & \times & H^{2-i}(G_L, N^*(1)) \xrightarrow{\cup} H^2(L, E/\mathcal{O}_E(1)) \\ \wr\| & & \wr\| \qquad\qquad \downarrow \text{Cores} \\ H^i(K, \mathrm{Ind}_{G_L}^{G_K} N) & \times & H^{2-i}(K, \mathrm{Ind}_{G_L}^{G_K} N^*(1)) \xrightarrow{\cup} H^2(K, E/\mathcal{O}_E(1)) \end{array}$$

<u>General case</u> Let M be a $k_E[G_K]$-mod of f. dim.

$\exists$ finite Galois ext'n $L|K$ s.t. G_L acts trivially on M

$$L = L(\mu_\ell)$$

$$\rightsquigarrow \quad 0 \to M \to \mathrm{Ind}_{G_L}^{G_K} M|_{G_L} \to Q \to 0$$

$$\begin{tikzcd}
 & & H^0(G_L, M) \arrow[d, dash] & & \\
0 \arrow[r] & H^0(G_K, M) \arrow[r] \arrow[d, "①"'] & H^0(G_K, \mathrm{Ind}_{G_L}^{G_K} M) \arrow[r] \arrow[d, "\wr"] & H^0(G_K, Q) \arrow[r] \arrow[d, "②"'] & H^1(G_K, M) \arrow[r] \arrow[d, "③"'] & \cdots \\
\cdots \arrow[r] & H^2(G_K, M^*(1))^* \arrow[r] & H^2(G_K, \mathrm{Ind}_{G_L}^{G_K} M^*(1))^* \arrow[r] \arrow[d, "\wr"] & H^2(G_K, Q^*(1))^* \arrow[r] & H^1(G_K, M^*(1))^* \arrow[r] & \\
 & & H^2(G_L, M^*(1))^* & & &
\end{tikzcd}$$

① is injective & <u>true</u> for all M.

$\Rightarrow$ ② must be injective as well.

$\Rightarrow$ ③ is injective & true for all M.

Dually, $0 \to K \to \mathrm{ind}_{G_L}^{G_K} M \to M \to 0 \rightsquigarrow$ surjectivity.

Lecture 5

<u>Euler characteristic formula</u> M fin. length $\mathcal{O}_E$-module, w/ cts G_K-action

$$\chi(G_K, M) = \sum_{i=0}^{2} (-1)^i \, \mathrm{length}_{\mathcal{O}_E} H^i(G_K, M) \overset{?}{=} \begin{cases} 0 & \text{if } \ell \neq p \\ -[K : \mathbb{Q}_p] \cdot \mathrm{length}_{\mathcal{O}_E} M & \text{if } \ell = p,\ K|\mathbb{Q}_p \end{cases}$$

<u>Rmk.</u> Both sides of the equality are additive in M.

Reduce to the case when M is a k_E-vec. sp. $\rightsquigarrow$ reduce to $\mathbb{F}_\ell$-vec. sp.

Blackbox. Let G be a finite gp. Write for a field $k = \mathbb{F}_\ell, \mathbb{Q}_\ell$

$R_k(G)$ = Grothendieck gp of fin. dim'l rep'ns of $k[G]$.

For each subgp $H \subset G$, we have $\mathrm{Ind}_H^G : R_k(H) \longrightarrow R_k(G)$.

WTS $\chi(G_K, -)$ & RHS are the same functional on $R_{\mathbb{F}_\ell}(G)$.

Blackbox thm.

$$\begin{tikzcd}
\bigoplus_{\substack{H \subset G \\ H \text{ cyclic} \\ \ell \nmid |H|}} R_{\mathbb{F}_\ell}(H) \otimes \mathbb{Q} \arrow[r, two heads, "\oplus \mathrm{Ind}_H^G"] \arrow[dr, dashed] & R_{\mathbb{F}_\ell}(G) \otimes \mathbb{Q} \arrow[d, "\chi(G_K, -)"] \\
& \mathbb{Q}
\end{tikzcd}$$

is surj.

Sketch.

$$\begin{tikzcd}
\bigoplus_{\substack{H \subset G \\ H \text{ cyclic}}} R_{\mathbb{Q}_\ell}(H) \otimes \mathbb{Q} \arrow[r, two heads, "\oplus \mathrm{Ind}_H^G"] \arrow[d] \arrow[dr, phantom, "\text{Artin's thm.}"] & R_{\mathbb{Q}_\ell}(G) \otimes \mathbb{Q} \arrow[d, two heads, "{[V] \mapsto [\bar{V}^{ss}]}", "\text{Thm}"'] \\
\bigoplus_{\substack{H \subset G \\ H \text{ cyclic}}} R_{\mathbb{F}_\ell}(H) \otimes \mathbb{Q} \arrow[r, two heads] & R_{\mathbb{F}_\ell}(G) \otimes \mathbb{Q}
\end{tikzcd}$$

the contribution of characters of ℓ-group H is trivial b/c the only $\mathbb{F}_\ell$-coeff. irred. rep. of such H is the trivial one.

But then $[\mathrm{Ind}_H^G \mathbb{F}_\ell] = \frac{1}{|H|}[\mathrm{Ind}_H^G \mathbb{F}_\ell[H]] = \frac{1}{|H|}[\mathrm{Ind}_{\{1\}}^G \mathbb{F}_\ell]$

Now. given an $\mathbb{F}_\ell[G_K]$-mod M $\rightsquigarrow$ $G_K \twoheadrightarrow \underset{\text{finite ext'n}}{G_{L|K}} \longrightarrow GL_{\mathbb{F}_\ell}(M)$. **WLOG** $L = L(\mu_\ell)$.

$$G \left(\begin{array}{c} L \\ | \\ F \\ | \\ K \end{array} \right) H$$

By Thm, reduce to the case for $M = \mathrm{Ind}_H^G N$ for a subgp $H \subset G$, cyclic of order prime to ℓ.

Now $H^i(G_K, M) = H^i(G_K, \mathrm{Ind}_{G_F}^{G_K} N) \simeq H^i(G_F, N)$

$$0 \to G_L \to G_F \to H \to 0$$

$$H^i(\underbrace{H}_{\substack{\text{order} \\ \text{prime to } \ell}}, \underbrace{H^j(G_L, N)}_{\mathbb{F}_\ell\text{-vec. sp.}}) \Rightarrow H^{i+j}(G_F, N)$$

$\Rightarrow H^{>0}(H, H^j(G_L, N)) = 0$

$\Rightarrow H^0(H, H^i(G_L, N)) = H^i(G_F, N)$

$$\| $$

$$H^i(G_L, N)^H = \left(H^i(G_L, \mu_\ell) \otimes N(i)\right)^H$$

$\uparrow$ G_L acts trivially

$H^0(G_L, \mu_\ell) = \mu_\ell$

$H^2(G_L, \mu_\ell) = \mathbb{F}_\ell$ (LCFT) w/ trivial H-action

$$1 \to \mu_\ell \to L^{\text{sep.}\times} \xrightarrow{\cdot \ell} L^{\text{sep.}\times} \to 1$$

$$L^\times \to L^\times$$

$\Rightarrow H^1(G_L, \mu_\ell) \simeq L^\times / (L^\times)^\ell$

$\to H^1(G_L, \mu_\ell) \to 1$

$$1 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{\text{val}} \mathbb{Z} \to 0$$

$$\Rightarrow 1 \to \mathcal{O}_L^\times / (\mathcal{O}_L^\times)^\ell \to L^\times / (L^\times)^\ell \to \mathbb{F}_\ell \to 0$$

<u>Pretend</u> $\mathcal{O}_L^\times \approx \mu(L) \times \mathcal{O}_L \approx \mu(L) \times \mathcal{O}_F[H]$. Then $\mathcal{O}_L^\times / (\mathcal{O}_L^\times)^\ell \simeq \underbrace{\mu(L)/\ell}_{\substack{\text{is as } H\text{-mod} \\ \mu_\ell}} \times \begin{cases} \{1\} & \text{if } \ell \neq p \\ \mathcal{O}_F/(p)[H], & \ell = p \end{cases}$

$$\chi(G_K, M) = \sum_{i=0}^{2} (-1)^i \dim\left(H^i(G_L, \mu_\ell) \otimes N(-1)\right)^H$$

$$= \dim\left(\mu_\ell \otimes N(-1)\right)^H + \dim\left(\mathbb{F}_\ell \otimes N(-1)\right)^H$$

$$- \dim\left(\mathbb{F}_\ell \otimes N(-1)\right)^H - \dim\left(\mu_\ell \otimes N(-1)\right)^H - \begin{cases} 0 & \ell \neq p \\ \dim\left(\mathcal{O}_F/(p)[H] \otimes N(-1)\right)^H, & \ell = p \end{cases}$$

$$\shortparallel$$

$$\dim \mathcal{O}_F/(p) \cdot \dim N$$

$$\shortparallel$$

$$[F : \mathbb{Q}_p] \dim N$$

$$\shortparallel$$

$$[K : \mathbb{Q}_p] \dim M$$

<u>Useful statement</u> if V is a $k[G]$-module,

$$\underset{G}{k[G] \otimes_k V} \simeq k[G]^{\oplus \dim V}$$

as G-mod

$\rightsquigarrow$

<u>Global duality for Galois cohomology</u>

* Local duality at a real place (only matter if $\ell = 2$)

If M is a finite length $\mathcal{O}_E$-module w/ $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$-action, then

① $H^i(G_\mathbb{R}, M) \simeq H^{i+2}(G_\mathbb{R}, M)$ when $i \geq 1$ $\rightsquigarrow$ write H^{odd} & H^{even} instead

② For $* =$ odd or even,

$$H^*(G_\mathbb{R}, M) \times H^*(G_\mathbb{R}, M^*(1)) \xrightarrow{\cup} H^{2*}(G_\mathbb{R}, E/\mathcal{O}_E(1)) \simeq \tfrac{1}{2}\mathcal{O}_E/\mathcal{O}_E$$

is a perfect pairing. $\hookrightarrow E/\mathcal{O}_E$

Let F be a no-field, $S =$ finite set of places containing all arch. places & places dividing ℓ.

Thm. Let M be a finite length $\mathcal{O}_E$-module w/ cts $G_{F,S}$-action. Set $M^* = \mathrm{Hom}(M, E/\mathcal{O}_E)$.

(0) All $H^i(G_{F,S}, M)$ are of finite length over $\mathcal{O}_E$.

(1) We have a LES (Poitou-Tate) if $v = \mathbb{R}/\mathbb{C}$, use $H^0_{Tate}(G_{F_v}, M) = H^{even}(G_{F_v}, M)$

$$0 \to H^0(G_{F,S}, M) \to \bigoplus_{v \in S} H^0(G_{F_v}, M) \xrightarrow{\oplus \mathrm{Res}_v^*} H^2(G_{F,S}, M^*(1))^* \to$$

extremely non-trivial!

$$\to H^1(G_{F,S}, M) \xrightarrow{\oplus \mathrm{Res}_v} \bigoplus_{v \in S} H^1(G_{F_v}, M) \xrightarrow{\oplus \mathrm{Res}_v^*} H^1(G_{F,S}, M^*(1))^* \to$$

$$\to H^2(G_{F,S}, M) \xrightarrow{\oplus \mathrm{Res}_v} \bigoplus_{v \in S} H^2(G_{F_v}, M) \xrightarrow{\oplus \mathrm{Res}_v^*} H^0(G_{F,S}, M^*(1))^* \to 0$$

(2) When $i \geq 3$, $H^i(G_{F,S}; M) \simeq \bigoplus_{v \text{ real}} H^i(G_{F_v}, M)$.

In particular, if $l \geq 3$, these are all zero.

(3) $$\chi(G_{F,S}; M) = \sum_{i=0}^{2} (-1)^i \, \mathrm{length}_{\mathcal{O}_E} H^i(G_{F,S}; M) = \sum_{v \in S} \chi(G_{F_v}, M)$$

use usually Euler char. for degs in $i = 0, 1, 2$.

$$= -[F:\mathbb{Q}] \, \mathrm{length}_{\mathcal{O}_E} M + \sum_{v \text{ arch}} \mathrm{length}_{\mathcal{O}_E} M^{G_{F_v}}$$

Rmk 1 (Poitou-Tate for M)* = Poitou-Tate for $M^*(1)$.

Rmk 2 For $i = 1, 2$, let $Ш^i(G_{F,S}, M) := \ker\left(H^i(G_{F,S}, M) \to \bigoplus_{v \in S} H^i(G_{F_v}, M)\right)$

then there's a perfect pairing

$$Ш^1(G_{F,S}, M) \times Ш^2(G_{F,S}, M^*(1)) \to E/\mathcal{O}_E.$$

Input from GCFT

For a finite Gal ext'n L of F, we have

$$H^1(\mathrm{Gal}(L|F),\ L^\times\backslash\mathbb{A}_L^\times) = 0$$

$$H^2(\mathrm{Gal}(L|F);\ L^\times\backslash\mathbb{A}_L^\times) \simeq \frac{1}{[L:F]}\mathbb{Z}/\mathbb{Z}$$

Take limit over sub ext'ns L inside $F^S|F$

require that degree of such $L|F$ is sufficiently ℓ-divisible

$$\rightsquigarrow\ H^2(G_{F,S}:\ F^{S,\times}\backslash\mathbb{A}_{F^S}^\times)\otimes\mathbb{Z}_\ell \simeq \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

OK here b/c $F(\zeta_{\ell^n})\subset F^S$.

$$H^i(G_{F,S}:\ F^{S,\times}\backslash\mathbb{A}_{F^S}^\times)\otimes\mathbb{Z}_\ell = \begin{cases}(F^\times\backslash\mathbb{A}_F^\times)\otimes\mathbb{Z}_\ell, & i=0\\ 0, & i=1 \text{ or } i\geq 3\end{cases}$$

A subtle technical point

$$F^{S,\times}\backslash\mathbb{A}_{F^S}^\times\Big/\prod_{v\notin S}(\mathcal{O}_{F_v}\otimes F^S)^\times \simeq \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big\backslash\prod_{v\in S}(F_v\otimes F^S)^\times$$

Lecture 6. F no. field, S = finite set of places including those dividing $\ell\infty$.

M = finite length $\mathcal{O}$-mod w/ cts $G_{F,S}$ = Gal$(F^S|F)$-action. $\mathcal{O}\subset E|\mathbb{Q}_\ell$.

$$0\to H^0(G_{F,S},M)\longrightarrow\bigoplus_{v\in S}\underbrace{H^0(G_{F_v},M)}_{\text{Tate coh. when } v \text{ real}}\longrightarrow H^2(G_{F,S},M^*(1))^*\ -$$

$$\to H^1(G_{F,S},M)\longrightarrow\bigoplus_{v\in S}H^1(G_{F_v},M)\longrightarrow H^1(G_{F,S},M^*(1))^*\ -$$

$$\to H^2(G_{F,S},M)\longrightarrow\bigoplus_{v\in S}H^2(G_{F_v},M)\longrightarrow H^0(G_{F,S},M^*(1))^*\longrightarrow 0$$

Goal. Prove this when $\ell \neq 2$.

$$- \quad F^{S,\times} \backslash \mathbb{A}_{F^S}^{\times} \Big/ \prod_{v \notin S} (\mathcal{O}_{F_v \otimes F^S})^{\times} = \mathcal{O}_{F^S}[\tfrac{1}{S}]^{\times} \Big\backslash \prod_{v \notin S} (F_v \otimes F^S)^{\times}$$

Pf. If L is a fin. subext'n of $F^S | F$,

$$\begin{tikzcd}
\prod_{v \notin S_L} L_v^{\times} \arrow[r] & L^{\times} \backslash \mathbb{A}_L^{\times} \Big/ \prod_{v \notin S_L} \mathcal{O}_{L_v}^{\times} \arrow[r, "\text{coker}"] & L^{\times} \backslash \mathbb{A}_L^{(S_L),\times} \Big/ \prod_{v \notin S_L} \mathcal{O}_{L_v}^{\times} \simeq \underbrace{cl(\mathcal{O}_L[\tfrac{1}{S_L}])}_{\text{finite}} \arrow[r] \arrow[d] & 0 \\
\mathcal{O}_L[\tfrac{1}{S_L}]^{\times} \arrow[u] & & cl(\mathcal{O}_L) &
\end{tikzcd}$$

$S_L = \{\text{all places of } L \text{ over } S\}$

Take $\varinjlim$. Fact $\varinjlim_L cl(\mathcal{O}_L[\tfrac{1}{S_L}]) = 0$ (in Exer.) $\square$

Now. $$1 \to \prod_{v \notin S} (\mathcal{O}_{F_v \otimes F^S})^{\times} \to F^{S,\times} \backslash \mathbb{A}_{F^S}^{\times} \to \underbrace{\boxed{\mathcal{O}_{F^S}[\tfrac{1}{S}]^{\times} \Big\backslash \prod_{v \notin S} (\mathcal{O}_{F_v \otimes F^S})^{\times}}}_{\substack{\| \\ Z_{F,S}}} \to 1$$

$$1 \to \prod_{v \notin S} \mathcal{O}_{F_v}^{\times} \otimes \mathbb{Z}_\ell \to F^{\times} \backslash \mathbb{A}_F^{\times} \otimes \mathbb{Z}_\ell \to H^0(G_{F,S}, Z_{F,S}) \otimes \mathbb{Z}_\ell$$

$$\boxed{\begin{array}{c} L|K \text{ local, unram.} \\ H^1(L/K, \mathcal{O}_L^{\times}) \\ = 0 \end{array}} \to \prod_{v \notin S} \underbrace{H^1(\overset{\text{unramified}}{F_w^S / F_v}, \mathcal{O}_{F_w^S}^{\times})}_{= 0} \otimes \mathbb{Z}_\ell \to 0 \to H^1(G_{F,S}, Z_{F,S}) \otimes \mathbb{Z}_\ell$$

$$\to 0 \to \mathbb{Q}_\ell / \mathbb{Z}_\ell \to H^2(G_{F,S}, Z_{F,S}) \otimes \mathbb{Z}_\ell \to 0$$

$$\Rightarrow \quad H^i\Big(G_{F,S},\ \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \Big\backslash \prod_{v\in S}(F_v\otimes F^S)^\times\Big)\otimes \mathbb{Z}_\ell = \begin{cases} F^\times\backslash \mathbb{A}_F^\times / \prod_{v\notin S}\mathcal{O}_{F_v}^\times, & i=0 \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell, & i=2 \\ 0, & i=1 \text{ or } i\geq 3 \end{cases}$$

<u>Rmk.</u> Using this, can compute the cohomology of $\mathcal{O}_{F^S}[\frac{1}{S}]^\times$.

& further compute $H^i(G_{F,S},\ \mu_\ell)$

$$1 \to \mu_\ell \to \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \xrightarrow{x\mapsto x^\ell} \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \to 1$$

⌈ Why not using $1\to \mu_\ell \to F^S \xrightarrow{x\mapsto x^\ell} F^S$: not surj. !
$p\notin S,\ v_p(a)=1$ ⌋

<u>Homological algebra technique</u>

M is a finite length $\mathcal{O}_E$-module $\circlearrowleft$ cts action by a profinite gp Γ

Define $R\Gamma(\Gamma, M) = \Big[\underset{\substack{\uparrow \\ \text{deg } 0}}{M} \to C_{cont}(\Gamma, M) \to C_{cont}(\Gamma^2, M) \to \cdots\Big]$

cts cochain for group cohomology. $H^i(R\Gamma(\Gamma,M)) = H^i(\Gamma, M)$.

<u>$\ell\geq 3$</u>. $\widetilde{R\Gamma}_{S_f}(G_{F,S}, M) := \mathrm{Fib}\Big(R\Gamma(G_{F,S}, M) \xrightarrow{\oplus \mathrm{res}_v} \bigoplus_{v\in S_f} R\Gamma(G_{F_v}, M)\Big)$, $S_f = S\backslash\{\text{arch. places}\}$

- Define $\widetilde{H}^i_{S_f}(G_{F,S}, M) = H^i(\widetilde{R\Gamma}_{S_f}(G_{F,S}, M)) \leftarrow$ special case of Selmer complex

$$0 \to \tilde{H}^0_{S_f}(G_{F,S}; M) \to H^0(G_{F,S}, M) \to \bigoplus_{v \in S_f} H^0(G_{F_v}, M) \to \tilde{H}^1_{S_f}(G_{F,S}, M)$$

$$\to H^1(G_{F,S}, M) \to \bigoplus_{v \in S_f} H^1(G_{F_v}, M) \to \tilde{H}^2_{S_f}(G_{F,S}, M) \to \cdots$$

<u>Need to show</u>. $\tilde{H}^{3-i}_{S_f}(G_{F,S}, M) \simeq H^i(G_{F,S}, M^*(1))^*$

* $\tilde{H}^0_{S_f}(G_{F,S}, M) = 0$ because $M^{G_{F,S}} \hookrightarrow M^{G_{F_v}}$

* Euler char. formula. $\chi(G_{F,S}, M) \overset{?}{=} \sum_{v \in S} \chi(G_{F_v}, M)$

$$= \sum_{v \in S_f} \chi(G_{F_v}, M) + \sum_{v \mid \infty} \text{length } M^{G_{F_v}}$$

Need to show $\tilde{\chi}(G_{F,S}, M) = \sum_{v \mid \infty} \text{length } M^{G_{F_v}}$

<u>Rmk</u>. when $\ell = 2$,

$$\widetilde{R\Gamma}_S(G_{F,S}, M) = \mathrm{Cone}\left(R\Gamma(G_{F,S}, M) \to \bigoplus_{v \in S_f} R\Gamma(G_{F_v}, M) \oplus \bigoplus_{v = \mathbb{R}} R\Gamma_{Tate}(G_{F_v}, M)\right)[-1]$$

<u>Then</u>. $\tilde{H}^i_S(G_{F,S}, M) \simeq H^{3-i}(G_{F,S}, M^*(1))^*$. $i \leq 3$

$\boxed{\ell \geq 3}$

<u>Step 2</u> Construct a nat'l hom. $\tilde{H}^i_{S_f}(G_{F,S}, M) \to H^{3-i}(G_{F,S}, M^*(1))^*$

$$\mathrm{Cone}\left(R\Gamma(G_{F,S}, M) \to \bigoplus_{v \in S_f} R\Gamma(G_{F_v}, M)\right)[-1] \otimes R\Gamma(G_{F,S}, M^*(1))$$

$$\xrightarrow{\cup} \mathrm{Cone}\left(R\Gamma(G_{F,S}, M \otimes M^*(1)) \to \bigoplus_{v \in S_f} R\Gamma(G_{F_v}, M \otimes M^*(1))\right)[-1]$$

$$\xrightarrow{M \otimes M^*(1) \to E/\mathcal{O}_E(1)} \mathrm{Cone}\left(R\Gamma(G_{F,S}, E/\cancel{\mathcal{O}_E(1)}\,\mu_{\ell^\infty}) \to \bigoplus_{v \in S_f} R\Gamma(G_{F_v}, E/\cancel{\mathcal{O}_E(1)}\,\mu_{\ell^\infty})\right)[-1] \otimes_{\mathcal{O}_E} \mathcal{O}$$

$$\xrightarrow{\text{nat'l}} \quad \mathrm{Cone}\left(R\Gamma(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times) \to \bigoplus_{v\in S_f} R\Gamma(G_{F_v}, \bar{F}_v^\times)\right)[-1] \otimes_{\mathbb{Z}_\ell} \mathcal{O}$$

$$\simeq R\Gamma\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v\in S} (F_v \otimes F^S)^\times\Big)[-1] \otimes_{\mathbb{Z}_\ell} \mathcal{O}$$

But $H^3(\quad) = H^2\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v\in S} (F_v \otimes F^S)^\times\Big) \otimes_{\mathbb{Z}} \mathcal{O} \simeq E/\mathcal{O}$

Get a pairing $\tilde{H}^i_{S_f}(G_{F,S}, M) \times H^{3-i}(G_{F,S}, M^*(1)) \to E/\mathcal{O}$.

<u>Step 2</u>. Show that $\tilde{H}^i \to (H^{3-i})^*$ is an isom.

By exactly the same argument as in the local Tate duality case, just need to prove this when $F = F(\mu_\ell)$, $M \simeq \mathbb{F}_\ell \simeq \mu_\ell$.

In this case, $\widetilde{R\Gamma}_{S_f}(G_{F,S}, \mu_\ell) = \mathrm{Cone}\left(R\Gamma(G_{F,S}, \mu_\ell) \to \bigoplus_{v\in S_f} R\Gamma(G_{F_v}, \mu_\ell)\right)[-1]$

$$\overset{\text{Kummer}}{\simeq} \mathrm{Cone}\left(R\Gamma\left(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \xrightarrow{\cdot \ell} \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\right) \to \bigoplus_{v\in S_f} R\Gamma\left(G_{F_v}, \bar{F}_v^\times \xrightarrow{\cdot \ell} \bar{F}_v^\times\right)\right)[-1]$$

$$\simeq R\Gamma\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v\in S} (F_v \otimes F^S)^* \xrightarrow{\cdot \ell} \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v\in S} (F_v \otimes F^S)^*\Big)[-1]$$

But $H^i\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v\in S} (F_v \otimes F^S)^\times\Big) \otimes \mathbb{Z}_\ell = \begin{cases} F^\times \backslash \mathbb{A}_F^\times / \prod_{v\in S} \mathcal{O}_{F_v}^\times, & i=0 \\ \mu_\ell/\mathbb{Z}_\ell, & i=2 \\ 0, & i=1,\ i\geq 3 \end{cases}$

So

$$0 \to \tilde{H}^1_{S_f}(G_{F,S}, \mu_\ell) \to \underset{\substack{\| \\ F^\times \backslash \mathbb{A}_F^\times / \prod_{v\in S} \mathcal{O}_{F_v}^\times}}{H^0(G_{F,S}, \mathbb{Z}_{F,S})} \xrightarrow{\cdot \ell} F^\times \backslash \mathbb{A}_F^\times / \prod_{v\in S} \mathcal{O}_{F_v}^\times$$

$$\to \tilde{H}^2_{S_f}(G_{F,S}, \mu_\ell) \to 0 \longrightarrow 0$$

$$\to \tilde{H}^3_{S_f}(G_{F,S}, \mu_\ell) \to \mu_\ell/\mathbb{Z}_\ell \xrightarrow{\times \ell} \mu_\ell/\mathbb{Z}_\ell \to 0.$$

$$\Rightarrow \tilde{H}^3_{S_f}(G_{F,S}, \mu_\ell) \simeq \mathbb{F}_\ell \simeq H^0(G_{F,S}, \mathbb{F}_\ell)^*$$

$$H^1(G_{F,S}, \mathbb{F}_\ell)^* = \mathrm{Hom}(G_{F,S}, \mathbb{F}_\ell)^* = \mathrm{Hom}\left(G^{ab}_{F,S}/\ell, \mathbb{F}_\ell\right)^* \simeq G^{ab}_{F,S}/\ell$$

$$\overset{\text{GCFT}}{\simeq} \left(\overline{F^\times F^{\circ,\times}_{\mathbb{R}}} \backslash \mathbb{A}_F^\times \Big/ \prod_{v \notin S} \mathcal{O}_{F_v}^\times\right)/\ell \overset{\ell \geq 3}{\simeq} \tilde{H}^2_{S_f}(G_{F,S}, \mu_\ell)$$

Return to <u>Poitou-Tate</u>. Use self-duality, $\mathbb{F}_\ell \simeq \mu_\ell$, get

$$H^2(G_{F,S}, \mu_\ell)^* \simeq \tilde{H}^1_{S_f}(G_{F,S}, \mathbb{F}_\ell). \qquad \square$$

<u>Step 3</u> <u>Euler char.</u>

By exactly the same reduction, we'll consider the following

$$G \left(\begin{array}{c} L \\ | \\ K \\ | \\ F \end{array} \right) H$$

H cyclic of order prime to ℓ, $L = L(\mu_\ell)$

$M = \mathrm{Ind}_H^G N$

$$\chi(\widetilde{R\Gamma}_{S_f}(G_{F,S}, M)) = \chi(\widetilde{R\Gamma}_{S_K,f}(G_{K,S_K}, N))$$

$$= \sum_{i=1}^{3} (-1)^i \dim_{\mathbb{F}_\ell} \left(\tilde{H}^i_{S_L,f}(G_L, S_L, \mu_\ell) \otimes N(-1)\right)^H$$

$$\tilde{H}^i_{S_L,f}(G_L, S_L, \mu_\ell) = \begin{cases} \ker\left(L^\times \backslash \mathbb{A}_L^\times \Big/ \prod_{v \notin S_L} \mathcal{O}_L^\times \xrightarrow{\times \ell}\right) & i=1 \\ \mathrm{coker}\left(\text{———}\right) & i=2 \\ \mathbb{F}_\ell & i=3 \end{cases}$$

$\tilde{H}^1 = \ker(Z_{L,S_L} \xrightarrow{\ell} Z_{L,S_L})$

$\tilde{H}^2 = \mathrm{coker}(Z_{L,S_L} \xrightarrow{\ell} Z_{L,S_L})$

$\tilde{H}^3 = \mathbb{F}_\ell$

$$1 \to \mathcal{O}_L^\times \to \prod_{v \text{ and}} L_v^\times \times \prod_{v \in S_{L,f}} \mathcal{O}_{L_v}^\times \to L^\times \backslash \mathbb{A}_L^\times / \prod_{v \notin S_L} \mathcal{O}_{L_v}^\times \to \underbrace{Cl(\mathcal{O}_L)}_{\text{fin.}} \to 1$$

no contribution.

$$\tilde{\chi}_{S_f}(G_{F,S}, M) = \chi\Big(H; \Big[\prod_{v \in S_{L,f}} \mathcal{O}_{L_v}^\times \xrightarrow{\cdot \ell} \mathcal{O}_{L_v}^\times\Big] \cdots\cdots$$

Lecture 7. Euler char. formula

$$G \left(\begin{array}{c} L \\ | \\ k \\ | \\ F \end{array} \right) H$$

Reduce to the case

$$L = L(\mu_\ell), \quad M = \mathrm{Ind}_H^G N, \quad \ell \nmid |H|.$$

$$\chi\left(\widetilde{R\Gamma}_{S_f}(G_{F,S}, M)\right) = \sum_{i=1}^{3} (-1)^i \dim_{\mathbb{F}_\ell}\left(\tilde{H}^1_{S_{L,f}}(G_L, S_L, \mu_\ell) \otimes N(-1)\right)^H$$

$$\overset{?}{=} \sum_{v \in S_\infty} \dim_{\mathbb{F}_\ell} H^0(G_{F_v}, \mathrm{Ind}_H^G N) = \sum_{w \in S_{K,\infty}} \dim H^0(G_{F_w}, N)$$

<u>Introduce notation</u>: For an $\mathbb{Z}_\ell[H]$-mod map $P \to Q$ s.t. ker & coker killed by ℓ.

Define $\chi_N(P \to Q) = \dim_{\mathbb{F}_\ell}(\ker(P \to Q) \otimes N(-1))^H - \dim_{\mathbb{F}_\ell}(\mathrm{coker}(P \to Q) \otimes N(-1))^H$

$\chi_N(P) = \chi_N(P \to 0)$

<u>Note</u> ① if P is finite length $\mathbb{Z}_\ell$-mod, $\chi_N(P \to P) = 0$.

② if P & Q ($\circlearrowleft H$) are fin. free $\mathbb{Z}_\ell$-mod s.t. $P \otimes \mathbb{Q}_\ell \simeq Q \otimes \mathbb{Q}_\ell$ as $\mathbb{Q}_\ell[H]$-module

then $\chi_N(P \xrightarrow{\ell} P) = \chi_N(Q \xrightarrow{\ell} Q)$

③ if $0 \to P \to Q \to R \to 0$,

$\chi_N(Q \xrightarrow{\ell} Q) = \chi_N(P \xrightarrow{\ell} P) + \chi_N(R \xrightarrow{\ell} R)$

<u>Recall</u>

$$\tilde{H}^i_{S_{L,f}}(G_L, S_L, \mu_\ell) = \begin{cases} \ker\Big(L^\times \backslash \mathbb{A}_L^\times / \prod_{v \notin S_L} \mathcal{O}_{L_v}^\times \xrightarrow{\ell} \text{same}\Big) & i = 1 \\ \mathrm{coker}(\text{same} \xrightarrow{\ell} \text{same}) & i = 2 \\ \mathbb{F}_\ell & i = 3 \end{cases}$$

$$\Rightarrow \chi\left(\widetilde{R\Gamma}_{Sf}(G_{F,S}, M)\right) = -\chi_N\left(L^\times \backslash \mathbb{A}_L^\times / \prod_{v \notin S_L} \mathcal{O}_{L_v}^\times \xrightarrow{\ell} \text{same}\right) - \chi_N(\mathbb{F}_\ell)$$

$$1 \to \overline{\mathcal{O}_L^\times} \to \prod_{v \in S_{L,\infty}} L_v^\times \times \prod_{v \in S_{L,f}} \mathcal{O}_{L_v}^\times \to L^\times \backslash \mathbb{A}_L^\times / \prod_{v \notin S_L} \mathcal{O}_{L_v}^\times \to Cl(\mathcal{O}_L) \to 1$$

$$= \chi_N\left(\overline{\mathcal{O}_L^\times} \xrightarrow{\ell} \overline{\mathcal{O}_L^\times}\right) - \sum_{v|\infty} \chi_N\left(L_v^\times \xrightarrow{\ell} L_v^\times\right) - \sum_{v \in S_{L,f}} \chi_N\left(\mathcal{O}_{L_v}^\times \xrightarrow{\ell} \mathcal{O}_{L_v}^\times\right)$$

only when $v|\ell$ matters

$$= \dim_{\mathbb{F}_\ell}\left(\frac{\mathcal{O}_K[H]}{\ell} \otimes N(-1)\right)^H$$

$$= \dim N \cdot [L : \mathbb{Q}]$$

$$- \chi_N\left(Cl(\mathcal{O}_L) \xrightarrow{\ell} Cl(\mathcal{O}_L)\right) - \chi_N(\mathbb{F}_\ell)$$

$$\| \quad 0$$

Framed deformations

Let E be a fin. ext. of $\mathbb{Q}_\ell$, $E \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \simeq \mathbb{F}$, $\mathbb{F}$ fin. ext'n of $\mathbb{F}_\ell$.

Fix G a profinite gp satisfying the following equiv. conditions:

$(\star)_1$ for all fin. length $\mathcal{O}$-mod M w/ cts G-action, $H^1(G, M)$ is finite [resp. $\forall H^i$]

$(\star)_2$ for all open cpt subgps $H \subset G$, $H^1(H, \mathbb{F}_\ell)$ is finite. [resp. $\forall H^i$]

(Exer. check equiv.)

Example. ① If K is a local field (allowing $\mathbb{Q}_\ell$ but not $\mathbb{F}_\ell((t))$), $G = G_K$

② If F is a no. field, S finite set of places, $G = G_{F,S}$.

Consider $CNL_{\mathcal{O}}$ = cat. of complete noetherian local $\mathcal{O}$-algs (A, m)

for which the structure map $\mathcal{O} \to A$ induces an isom. $\mathbb{F} \xrightarrow{\sim} A/m$.

(Morphisms are local maps $A \xrightarrow{f} B$, $f(m_A) \subset m_B$).

Given $\bar{\rho} : G \longrightarrow GL_n(\mathbb{F}) = GL(V)$, consider possible lifts

$$\begin{tikzcd} \rho : G \arrow[r] \arrow[dr, "\bar{\rho}"'] & GL_n(\mathcal{O}) \arrow[d] \\ & GL_n(\mathbb{F}) \end{tikzcd}$$

<u>Want</u>: universal lift.

<u>Consider the functors</u>

(framed deformation) $\mathrm{Def}^{\square}_{\bar{\rho}} : CNL_{\mathcal{O}} \longrightarrow \mathrm{Set}$

$$(A, m) \mapsto \left\{ \begin{array}{l} \text{cts rep'n } \rho : G \longrightarrow GL_n(A) \\ \quad \text{s.t. } \rho \bmod m = \bar{\rho} \end{array} \right\}$$

(unframed deformation) $\mathrm{Def}_{\bar{\rho}} : CNL_{\mathcal{O}} \longrightarrow \mathrm{Set}$

$$(A, m) \mapsto \frac{\left\{ \begin{array}{l} \text{cts rep'n } \rho : G \longrightarrow GL_n(A) \\ \quad \text{s.t. } \rho \bmod m = \bar{\rho} \end{array} \right\}}{\rho \sim \rho' \text{ if } \exists g \in 1 + M_n(m)}$$

$$\text{s.t. } \rho(x) = g\rho'(x)g^{-1}, \ \forall x \in G$$

$$= \left\{ \begin{array}{l} \text{free } A\text{-mod } M \text{ of rk } n \text{ w/ cts } A\text{-linear} \\ G\text{-action, together w/ a } G\text{-isom. } M/mM \simeq V \end{array} \right\}$$

<u>Thm</u>. Assume that G is a profinite group satisfying $(\star)$

(a) $\mathrm{Def}^{\square}_{\bar\rho}$ is rep'd by some $R^{\square}_{\bar\rho} \in CNL_{\mathcal{O}}$

(b) If $\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$, then $\mathrm{Def}_{\bar\rho}$ is rep'd by some $R_{\bar\rho} \in CNL_{\mathcal{O}}$.

<u>Meaning of (a)</u>. there's a universal rep'n $\rho^{univ} : G \to GL_n(R^{\square}_{\bar\rho})$

s.t. for any $(A, m) \in CNL_{\mathcal{O}}$, every rep'n $\rho : G \to GL_n(A)$ lifting $\bar\rho$ comes

from a <u>unique ring hom.</u> $R^{\square}_{\bar\rho} \xrightarrow{\phi} A$ s.t.

$$\begin{tikzcd} G \arrow[r, "\rho^{univ}"] \arrow[rr, bend right, "\rho"'] & GL_n(R^{\square}_{\bar\rho}) \arrow[r, "\phi"] \arrow[d, phantom, "///"] & GL_n(A) \end{tikzcd}$$

$R^{\square}_{\bar\rho}$ is called universal framed deformation ring.

In other words, $\mathrm{Def}^{\square}_{\bar\rho}(A) = \mathrm{Hom}_{CNL_{\mathcal{O}}}(R^{\square}_{\bar\rho}, A)$ or $\mathrm{Def}^{\square}_{\bar\rho}(-) \simeq \mathrm{Hom}_{CNL_{\mathcal{O}}}(R^{\square}_{\bar\rho}, -)$

<u>Rmk</u>. Take $A = R^{\square}_{\bar\rho}$, $\mathrm{Def}^{\square}_{\bar\rho}(R^{\square}_{\bar\rho}) = \mathrm{Hom}(R^{\square}_{\bar\rho}, R^{\square}_{\bar\rho})$

$$\rho^{univ} \longleftarrow\!\shortmid \quad \mathrm{id}$$

<u>Pf</u>. First, assume G is finite, say $G = \langle g_1, \cdots, g_s : r_1(g_1, \cdots g_s) = 1, \cdots, r_t(g_1, \cdots, g_s) = 1 \rangle$

Then a rep'n of G = assigning a matrix to each g_i satisfying the relations.

<u>Def'n</u> $R = \mathcal{O}[x^{(k)}_{i,j} : i,j = 1, \cdots, n, \ k = 1, \cdots, s] \Big/ \left(r_1(X^1, \cdots, X^s) - I_n, \cdots, r_t(X^1, \cdots, X^s) - I_n\right)$

write $X^{(k)} = (x^{(k)}_{i,j})$.

<u>Claim</u>. R parametrizes all possible rep'n of G.

Eg. $G = D_{2n} = \langle x, y : x^n = 1,\ xyxy = 1,\ y^2 = 1 \rangle$

$$R = \mathcal{O}[X_{11}, X_{12}, X_{21}, X_{22}, Y_{11}, Y_{12}, Y_{21}, Y_{22}] \Big/ (\cdots)$$

Then. $\tilde{\bar{\rho}} : R \longrightarrow \mathbb{F}$

$$X_{ij}^{(k)} \mapsto (i,j)^{th}\text{-entry of } \bar{\rho}(g_k).$$

Let $J := \ker(\tilde{\bar{\rho}})$. max'l ideal of R

Claim. The completion of R at $J = R_J^{\wedge}$ rep. the functor $\mathrm{Def}_{\bar{\rho}}^{\square}$.

Pf $\rho^{univ} : G \longrightarrow GL_n(R) \longrightarrow GL_n(R_J^{\wedge})$ is a cts gp hom. lifting $\bar{\rho}$.

$$g_k \longmapsto (X_{ij}^{(k)})_{1 \le i,j \le n}$$

For any $(A, m) \in CNL_{\mathcal{O}}$ and any morphism $R_J^{\wedge} \to A$ in $CNL_{\mathcal{O}}$,

get an induced rep'n $G \xrightarrow{\rho^{univ}} GL_n(\hat{R}_J) \longrightarrow GL_n(A)$.

Conversely, given $\rho_A \in \mathrm{Def}_{\bar{\rho}}^{\square}(A)$, then form $\tilde{\rho} : R \longrightarrow A$

$$X_{ij}^{(k)} \mapsto (i,j)^{th}\text{-entry of } \rho_A(g_k)$$

$$\tilde{\rho} \searrow A/mA \simeq \mathbb{F}$$

$\tilde{\rho}(J) \subset m$ induces $\hat{R}_J \longrightarrow A$.

So $\mathrm{Hom}(R_J^{\wedge}, A) = \mathrm{Def}_{\bar{\rho}}^{\square}(A)$.

Rmk

$$\begin{array}{ccc} \rho^{univ} & & \bar{\rho} \\ | & & | \\ \mathrm{Spec}\, R & \xleftarrow[R \xrightarrow{\tilde{\bar{\rho}}} \mathbb{F}]{} & \mathrm{Spec}\, \mathbb{F} \\ \nwarrow \rho & & \end{array}$$

$\mathrm{Spec}\, \hat{R}_J$ = completion at the pt defined by $\mathbb{F}$.

In general, $G = \varprojlim G/H_i$ over open normal subgps $H_i \subset G$.

(may assume that all $H_i \subset \ker(\bar\rho)$, so that $\bar\rho : G \to GL_n(\mathbb{F})$ induces $\bar\rho_i : G/H_i \to GL_n(\mathbb{F})$)

For each i, get a universal pair $(R^{\square}_{\bar\rho_i}, \rho_i^{univ} : G/H_i \to GL_n(R^{\square}_{\bar\rho_i}))$

$$\begin{tikzcd} G/H_i \arrow[r, "\rho_i^{univ}"] \arrow[dr, "\bar\rho_i"'] & GL_n(R^{\square}_{\bar\rho_i}) \arrow[d] \\ & GL_n(R^{\square}_{\bar\rho_i}/m^{\square}_{\bar\rho_i}) = GL_n(\mathbb{F}) \end{tikzcd}$$

when $H_j \subset H_i$, then

$$\begin{tikzcd} G/H_i \arrow[r, "\rho_i^{univ}"] & GL_n(R^{\square}_{\bar\rho_i}) \\ G/H_j \arrow[u] \arrow[r, "\rho_j^{univ}"] & GL_n(R^{\square}_{\bar\rho_j}) \arrow[u, dashed, "\exists!\ R^{\square}_{\bar\rho_j} \to R^{\square}_{\bar\rho_i}"'] \end{tikzcd}$$

<u>Define</u> $(R^{\square}_{\bar\rho}, \rho^{univ}) = \varprojlim_i (R^{\square}_{\bar\rho_i}, \rho_i^{univ})$.

(1) Show that $R^{\square}_{\bar\rho} \in CNL_{\mathcal{O}}$ (noetherian) later

(2) Show that $R^{\square}_{\bar\rho}$ rep. the functor $\mathrm{Def}^{\square}_{\bar\rho}$.

* For (1), $R^{\square}_{\bar\rho}$ is clearly a complete local ring

Suffices to show $m^{\square}_{\bar\rho}/(m^{\square\,2}_{\bar\rho}, \varpi)$ is finite $\Rightarrow R^{\square}_{\bar\rho} = \mathcal{O}[[z_1, \dots z_n]]/I$

* For (2). Given a morphism $R^{\square}_{\bar\rho} \to A$ in $CNL_{\mathcal{O}}$, get $\rho_A \in \mathrm{Def}^{\square}_{\bar\rho}(A)$.

Conversely, given a rep'n $\rho_A : G \to GL_n(A)$ lifting $\bar\rho$.

For each k, $\rho_A^{(k)} : G \to GL_n(A) \to \underbrace{GL_n(A/m^k)}_{\text{finite}}$

$\ker \rho_A^{(k)}$ is open.

Then $\exists\, i_k$ s.t. $H_{i_k} \subset \ker \rho_A^{(k)}$

$$\begin{tikzcd} \rho_A^{(k)} : G/H_{i_k} \arrow[r] \arrow[d, "\rho_{i_k}^{univ}"] & GL_n(A/m^k) \\ GL_n(R_{\bar\rho_{i_k}}^{\square}) \arrow[ur, dashed, "\exists!"] & \end{tikzcd} \qquad R_{\bar\rho_{i_k}}^{\square} \to A/m^k$$

Moreover,

$$\begin{tikzcd} R_{\bar\rho_{i_{k+1}}}^{\square} \arrow[r] \arrow[d] \arrow[dr, phantom, "///"] & A/m^{k+1} \arrow[d] \\ R_{\bar\rho_{i_k}}^{\square} \arrow[r] & A/m^k \end{tikzcd} \quad \to \varprojlim R_{\bar\rho_{i_k}}^{\square} \to A$$

$$\varprojlim R_{\bar\rho_{i_k}}^{\square} = R_{\bar\rho}^{\square}$$

(b) $\mathrm{Def}_{\bar\rho}(A) = \mathrm{Def}_{\bar\rho}^{\square}(A) \big/ \mathrm{Ad}\left((1+M_n(m_A))^\times / (1+m_A)^\times\right)$

$$\boxed{\mathrm{Def}_{\bar\rho} = \mathrm{Def}_{\bar\rho}^{\square} \big/ \widehat{PGL_n}} \Rightarrow \mathrm{Spf}\, R_{\bar\rho} = \mathrm{Spf}\, R_{\bar\rho}^{\square} \big/ \widehat{PGL_n}.$$

Σ

<u>Tangent space</u> Let $\mathbb{F}[\varepsilon] = \mathbb{F}[x]/(x^2)$. $\quad 0 \to \mathbb{F} \to \mathbb{F}[\varepsilon] \to \mathbb{F} \to 0$

Then every rep'n $\rho : G \to GL_n(\mathbb{F}[\varepsilon]) = GL(V_{\mathbb{F}[\varepsilon]})$ lifting $\bar\rho$

$$0 \to V \to V_{\mathbb{F}[\varepsilon]} \to V \to 0.$$

$$\updownarrow$$

$$\mathrm{Def}_{\bar\rho}(\mathbb{F}[\varepsilon]) \overset{\text{v.s.}}{\longleftrightarrow} \mathrm{Ext}^1_{\mathbb{F}[G]}(V,V) = H^1(G, \mathrm{Ad}\, V)$$

$$\mathrm{Ad}\, V = V \otimes V^* = \mathrm{Hom}_{\mathbb{F}}(V,V)$$

When G satisfies $(\star)$

<u>Prop</u> (1) There's a canonical isom.

$$\left(m_{\bar\rho}^{\square} / (m_{\bar\rho}^{\square 2}, \varpi)\right)^* \simeq \mathrm{Def}_{\bar\rho}^{\square}(\mathbb{F}[\varepsilon]) \simeq \underbrace{Z^1(G, \mathrm{Ad}\,\bar\rho)}_{\text{finite}}$$

(2) When $\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$,

$$\text{——— w/o } \square \simeq H^1(G, \mathrm{Ad}\,\bar\rho)$$

<u>Lecture 8</u>

$E/\mathbb{Q}_\ell$, $E \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \simeq \mathbb{F}$

G profinite gp satisfying $(\star)$, $H^1(G, M) < \infty$ for any $\mathbb{F}[G]$-mod M

$\bar\rho : G \to GL_n(\mathbb{F}) = GL(V)$

<u>Last time</u>: $\mathrm{Def}_{\bar\rho}^{\square} : CNL_{\mathcal{O}} \longrightarrow \mathrm{Set}$

$$(A, m) \longmapsto \left\{ \begin{array}{l} (+) \text{ rep'n } \rho : G \to GL_n(A) \\ \text{s.t. } \rho \bmod m = \bar\rho \end{array} \right\}$$

is representable by $(R_{\bar\rho}^{\square}, m_{\bar\rho}^{\square})$, provided knowing $m_{\bar\rho}^{\square} / (m_{\bar\rho}^{\square 2}, \varpi)$ is fin. dim'l.

$\quad\quad\quad

Proof $\boxed{1}$. $\mathrm{Def}^{\square}_{\bar\rho}(\mathbb{F}[\varepsilon]) = \mathrm{Hom}_{CNL_{\mathcal{O}}}(R^{\square}_{\bar\rho}, \mathbb{F}[\varepsilon])$

$$\begin{tikzcd} R^{\square}_{\bar\rho} \arrow[r] \arrow[dr, two heads] & \mathbb{F}[\varepsilon] \arrow[d] \\ & \mathbb{F} \end{tikzcd}$$

$$= \mathrm{Hom}_{CNL_{\mathcal{O}}}\left(R^{\square}_{\bar\rho}/(m^{\square\,2}_{\bar\rho}, \varpi), \mathbb{F}[\varepsilon]\right)$$

$$= \mathrm{Hom}_{\mathbb{F}\text{-v.s.}}\left(m^{\square}_{\bar\rho}/(m^{\square\,2}_{\bar\rho}, \varpi), \mathbb{F}\cdot\varepsilon\right)$$

$\boxed{2}$. Let $\rho: G \longrightarrow GL_n(\mathbb{F}[\varepsilon])$ lifting $\bar\rho$

$$g \longmapsto (1 + A_g\cdot\varepsilon)\,\bar\rho(g), \quad \text{for some } A_g \in M_n(\mathbb{F}).$$

$$\rho(g)\cdot\rho(h) = \rho(gh) \Rightarrow (1 + A_g\cdot\varepsilon)\bar\rho(g)\,(1 + A_h\cdot\varepsilon)\bar\rho(h) = (1 + A_{gh}\cdot\varepsilon)\bar\rho(gh)$$

$$\Rightarrow \bar\rho(g)\bar\rho(h) + A_g\bar\rho(g)\bar\rho(h)\varepsilon + \bar\rho(g)A_h\bar\rho(h)\varepsilon = \bar\rho(gh) + A_{gh}\bar\rho(gh)\varepsilon$$

$$\Rightarrow A_g + \bar\rho(g)A_h\bar\rho(g)^{-1} = A_{gh}$$. This is a 1-cocycle in $Z^1(G, \mathrm{Ad}\,\bar\rho)$

$$(\Rightarrow \mathrm{Def}^{\square}_{\bar\rho}(\mathbb{F}[\varepsilon]) \Rightarrow Z^1(G, \mathrm{Ad}\,\bar\rho))$$

For $\mathrm{Def}_{\bar\rho}(\mathbb{F}[\varepsilon])$: $\rho \sim \rho'$ iff $\rho'(g) = (1+\varepsilon X)\rho(g)(1+\varepsilon X)^{-1}$ for some $X \in M_n(\mathbb{F})$

$$\rho'(g) = (1 + A'_g\varepsilon)\bar\rho(g)$$

$$\rho'(g) = (1+\varepsilon X)(1 + A_g\varepsilon)\bar\rho(g)(1+\varepsilon X)^{-1}$$

$$= (1+\varepsilon X)(1 + A_g\varepsilon)(1 - \varepsilon\bar\rho(g)X\bar\rho(g)^{-1})\bar\rho(g)$$

$$= \Big(1 + \varepsilon\underbrace{(X + A_g - \bar\rho(g)X\bar\rho(g)^{-1})}_{= A'_g}\Big)\bar\rho(g)$$

This means that A_g' & A_g are differed by a 1-boundary.

$\Rightarrow \mathrm{Def}_{\bar\rho}(\mathbb{F}[\varepsilon]) \Rightarrow H^1(G, \mathrm{Ad}\,\bar\rho)$

Representability of $\mathrm{Def}_{\bar\rho}$ when

$\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$

① Schlessinger criterion

② makes sense of $\mathrm{Spf}\, R^{\square}_{\bar\rho} / \widehat{PGL_n}$

SGA 3 VII B. Thm 1.4

<u>Theorem</u> Let $S = \mathrm{Spec}\, k$ w/ k local ring

Let $G = \mathrm{Spec}\, R$ be a flat group scheme $/S$

(eg. $PGL_n/\mathcal{O}$) s.t. R is a free k-module

Suppose that G acts on a k-scheme $Y = \mathrm{Spec}\, B$ <u>freely</u>

i.e. $G \times Y \xrightarrow{act} Y$ [freeness: $G \times_S Y \xrightarrow{act \times pr_2} Y \times_S Y$ is a closed immersion]

Then there exists a geom. quotient of Y by G, $Y/G = X$

More precisely, $X = \mathrm{Spec}(B^G)$, w/ $B^G = \{x \in B : pr_2^*(x) = act^*(x)\}$

$$G \times Y \underset{pr_2}{\overset{act}{\rightrightarrows}} Y$$

$$R \otimes_k B \underset{pr_2^*}{\overset{act^*}{\leftleftarrows}} B$$

Moreover, the image of $act \times pr_2$ is

$$\begin{tikzcd} Y \times_X Y \arrow[r, phantom, "\overset{\text{closed}}{\subset}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \times_S Y \arrow[d] \\ X \arrow[r, "\text{closed}"] & X \times_S X \end{tikzcd}$$

- $Y \to X$ is smooth if G is smooth $/S$.

<u>Proof</u>. $G \curvearrowright Y \xleftarrow[\text{faithfully flat}]{pr_2} G \times Y \curvearrowleft G$

$\mathrm{id} \times act \Big\| \Big\downarrow$ $\quad g.(h,y) = (hg^{-1}, gy)$

$\boxed{G \times Y}$ $\quad (h, hy) \xmapsto{g} (hg^{-1}, hy)$

easy to see how to quotient by G

$$\begin{tikzcd}
Y & G\times Y \arrow[l] & G\times G\times Y \arrow[l, shift right] \arrow[l, shift left] \\
 & G\times Y \arrow[u, "\wr"] \arrow[d, "pr_2"'] & G\times G\times Y \arrow[u, "\wr"'] \arrow[l, shift right, "pr_{23}"'] \arrow[l, shift left, "act"] \arrow[d] \\
Y^G & Y \arrow[l] & G\times Y \arrow[l, shift right, "pr_2"'] \arrow[l, shift left, "act"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & (g,hy) & (g,h,hy) \arrow[l, shift right] \arrow[l, shift left] \arrow[d, mapsto] \\
 & (h,y) & (g,h,y) \arrow[l, mapsto] \arrow[d, mapsto] \\
(hg^{-1},gy) \arrow[d, mapsto] & y & (g,y) \arrow[l, mapsto] \arrow[ll, mapsto] \\
gy & & \arrow[ll, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & R\otimes_k B \arrow[r, shift left] \arrow[r, shift right] & R\otimes_k R\otimes_k B \\
0 \arrow[r] & B^G \arrow[r] \arrow[u, dashed] & B \arrow[u, "pr_2^*"] \arrow[r, shift left, "pr_2^*"] \arrow[r, shift right, "act^*"'] & R\otimes_k B \arrow[u]
\end{tikzcd}$$

<u>Want</u> $B\otimes_{B^G} B \simeq R\otimes_k B$.

The freeness condition:

$$\begin{tikzcd}
B\otimes_k B \arrow[r, two heads, "act^*\otimes pr_2"] & R\otimes_k B \\
V\otimes_k B \arrow[u] \arrow[ur, "\sim"] &
\end{tikzcd}$$

R is a free k-module (co-rank)

Then can find a free k-module $V\subset B$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & R\otimes_k B \arrow[r, shift left] \arrow[r, shift right] & R\otimes_k R\otimes_k B \\
0 \arrow[r] & V\otimes_k B^G \arrow[r] \arrow[u] & V\otimes_k B \arrow[u, "\wr", "act^*\otimes pr_2^*"'] \arrow[r, shift left] \arrow[r, shift right] & V\otimes_k R\otimes_k B \arrow[u, "\wr"']
\end{tikzcd}$$

$\therefore\ V\otimes_k B^G \simeq B$

$$B\otimes_{B^G} B \simeq V\otimes_k B^G\otimes_{B^G} B \simeq V\otimes_k B \simeq R\otimes_k B.$$

$$\Rightarrow\ Y\times_{Y/G} Y \simeq G\times_S Y.$$

$$\begin{tikzcd} G\times Y \simeq Y \times_X Y \arrow[r] \arrow[dr, "sm"'] & Y \arrow[d] \arrow[r, "sm"] & \\ & Y \arrow[r] & Y/G = X \end{tikzcd}$$

faithfully flat descent.

<u>Existence of unframed deformation</u>.

<u>Thm</u> When $\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$, $\mathrm{Def}_{\bar\rho}$ is rep'ble.

<u>Examples</u> ① $\bar\rho$ is <u>absolutely irreducible</u> $\Rightarrow \mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$, i.e. $\bar\rho : G \to GL_n(\mathbb{F}) \to GL_n(\bar{\mathbb{F}})$ is irred.

<u>Example of irred. but not abs. irred.</u> $\bar\rho : G \to \mathbb{F}_{\ell^2}^\times \subset GL_2(\mathbb{F}_\ell)$ is irred.

as an $\mathbb{F}_\ell$-rep'n, but not absolutely. $\bar\rho \otimes \mathbb{F}_{\ell^2} = \bar\rho \oplus \bar\rho$.

In this case, $\mathrm{End}_{\mathbb{F}_\ell[G]}(\bar\rho) = \mathbb{F}_{\ell^2}$.

② If $\bar\rho = \begin{pmatrix} \bar\rho_1 & & * \\ & \bar\rho_2 & \\ 0 & & \ddots \ \bar\rho_2 \end{pmatrix}$ w/ each $\bar\rho_i$ abs irred. & $\bar\rho_i \neq \bar\rho_j$ and $\underline{\underline{* \neq 0}}$.

<u>Proof of the theorem</u>. Have shown that $\mathrm{Def}^\square_{\bar\rho}$ is rep'ble by $R^\square_{\bar\rho} \in CNL_{\mathcal{O}}$.

<u>Step 1</u>. Explain that $\widehat{PGL}_n$ acts on $\mathrm{Spf}\, R^\square_{\bar\rho}$.

For each $A \in CNL_{\mathcal{O}}$, $\widehat{PGL}_n(A) = (1+M_n(m_A))^\times / (1+m_A)^\times \sim \mathrm{Def}^\square_{\bar\rho}(A) = \mathrm{Hom}_{CNL_{\mathcal{O}}}(R^\square_{\bar\rho}, A)$

$x : \rho \mapsto x\rho x^{-1}$

$$\begin{tikzcd} \mathrm{out}_A : \widehat{PGL}_n(A) \times \mathrm{Def}^\square_{\bar\rho}(A) \arrow[r] & \mathrm{Def}^\square_{\bar\rho}(A) \\ & (\mathrm{Spf}\, R^\square_{\bar\rho})(A) \arrow[u, "\sim"'] \end{tikzcd}$$

$$\| $$

$$(\widehat{PGL}_n \times \mathrm{Spf}\, R^\square_{\bar\rho})(A) \to (\mathrm{Spf}\, R^\square_{\bar\rho})(A).$$

Take $A = \mathcal{O}_{\widehat{PGL_n}} \hat{\otimes} R_{\bar\rho}^{\square}$, LHS $\ni$ id $\longmapsto$ an element in

$$\mathrm{Hom}\left(R_{\bar\rho}^{\square}, \mathcal{O}_{\widehat{PGL_n}} \hat{\otimes} R_{\bar\rho}^{\square}\right)$$

$$\mathrm{act}^*: R_{\bar\rho}^{\square} \longrightarrow \mathcal{O}_{\widehat{PGL_n}} \hat{\otimes} R_{\bar\rho}^{\square}.$$

<u>In explicit terms</u>. $R_{\bar\rho}^{\square} = \varprojlim_{H \triangleleft G} \underbrace{\mathcal{O}[[x_{ij}^{(k)}]]/\sim}_{R_i^{\square}}$

$$\mathrm{act}^*: R_i^{\square} \longrightarrow R_i^{\square} \hat{\otimes} \mathcal{O}_{\widehat{PGL_n}} \subset \mathcal{O}[[z_{ij}]]$$

$$x_{ij}^{k} \longmapsto \sum z_{ii'} x_{i'j'}^{(k)} (z^{-1}_{j'j})$$

<u>Step 2</u>. Show $\widehat{PGL_n} \times \mathrm{Spf}\, R_{\bar\rho}^{\square} \xrightarrow{\mathrm{act} \times pr_2} \mathrm{Spf}\, R_{\bar\rho}^{\square} \times \mathrm{Spf}\, R_{\bar\rho}^{\square}$ is a closed immersion.

This is b/c $\forall A \in CNL_{\mathcal{O}}$, $\widehat{PGL_n}(A) \times \mathrm{Spf}\, R_{\bar\rho}^{\square}(A) \longrightarrow \mathrm{Spf}\, R_{\bar\rho}^{\square}(A) \times \mathrm{Spf}\, R_{\bar\rho}^{\square}(A)$

is injective. i.e. $(g, \rho) \longmapsto (\mathrm{Ad}_g(\rho), \rho)$ $\quad g\rho g^{-1}$

if $(g,\rho) \longmapsto$ same, $(g',\rho) \longmapsto$ $\Rightarrow g = g'$

$$g\rho g^{-1} = g'\rho g'^{-1} \overset{?}{\Rightarrow} g/g' \text{ is a scalar mat.}$$

WLOG, $g' = 1$. $\quad g = 1 + \varepsilon_g$ for $\varepsilon_g \in M_n(m_A)$

$$(1+\varepsilon_g)\rho(1+\varepsilon_g)^{-1} = \rho$$

mod m_A^2 $\Rightarrow \forall x \in G$, $\rho(x) + \varepsilon_g \rho(x) - \rho(x)\varepsilon_g = \rho(x)$ mod m_A^2

But $\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F} \Rightarrow \varepsilon_g$ is a scalar mod m_A^2.

<u>General statement</u>. If $B \xrightarrow{\varphi} C$ is a morphism in $CNL_{\mathcal{O}}$ s.t. for every $A \in CNL_{\mathcal{O}}$,

$\mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(B, A)$ is injective, then φ is injective.

<u>Note</u>: Claim + general statement $\Rightarrow \widehat{PGL_n} \times \mathrm{Spf}\, R_{\bar\rho}^{\square} \longrightarrow (\mathrm{Spf}\, R_{\bar\rho}^{\square})^2$ is a closed immersion. (action is free)

Proof. Suppose $\varphi: B \to C$ is not surjective.

Take $A = C \otimes_B C$

$$\mathrm{Hom}(C, C\otimes_B C) \to \mathrm{Hom}(B, C\otimes_B C)$$

$$\begin{array}{c} (a \mapsto a\otimes 1) \\ \| \\ (a \mapsto 1\otimes a) \end{array} \quad \begin{array}{c} \mapsto \\ \mapsto \end{array} \text{ same}$$

$\therefore \forall a \in C,\ a\otimes 1 = 1\otimes a \Rightarrow C\otimes_B C = C$

If $B \xrightarrow{\varphi} C$ is not surj, then $\underset{\mathbb{F}}{\underset{\|}{B/m_B}} \to \underset{\mathbb{F}\text{-vec. sp.}}{\underset{\uparrow}{C/m_B C}}$ is not surj.

$$C\otimes_B C/m_B = C/m_B C \underset{B/m_B B}{\otimes} C/m_B C \neq C/m_B C, \text{ contradiction.}$$

Step 3. Define $R_{\bar\rho} := \ker\left(R^{\square}_{\bar\rho} \underset{\mathrm{aut}^*}{\overset{pr_2^*}{\rightrightarrows}} \mathcal{O}_{\widehat{PGL}_n} \hat\otimes R^{\square}_{\bar\rho}\right)$

& $R_{\bar\rho} \underset{R_{\bar\rho}}{\hat\otimes} R^{\square}_{\bar\rho} \simeq \mathcal{O}_{\widehat{PGL}_n} \hat\otimes R^{\square}_{\bar\rho}$

Also, $R^{\square}_{\bar\rho}$ is formally smooth / $R_{\bar\rho} \to R^{\square}_{\bar\rho} \simeq R_{\bar\rho}[[z_1, \cdots z_{n^2-1}]]$

Claim. $R_{\bar\rho}$ rep. $\mathrm{Def}_{\bar\rho}$.

Pf Given $A \in CNL_{\mathcal{O}}$.

Step 1 Show $\mathrm{Def}^{\square}_{\bar\rho}(A) = \mathrm{Hom}(R^{\square}_{\bar\rho}, A) \to \mathrm{Hom}(R_{\bar\rho}, A)$

factors through the quotient by $\widehat{PGL}_n(A)$. $\Big] \Rightarrow$

$\mathrm{Def}_{\bar\rho}(A) \to \mathrm{Hom}(R_{\bar\rho}, A)$

$\phi \in \mathrm{Hom}(R^{\square}_{\bar\rho}, A)$, $g \in \widehat{PGL}_n(A)$.

WTS ϕ & $g\circ\phi$ have the same image.

$$g\circ\phi : R^{\square}_{\bar\rho} \xrightarrow{act^*} \mathcal{O}\hat{[GL_n]} \otimes R^{\square}_{\bar\rho} \xrightarrow{g^*\otimes\phi} A \hat{\otimes} A \xrightarrow{mult} A$$

$$\begin{tikzcd}
R_{\bar\rho} \arrow[r] \arrow[rrr, bend left, "\text{image of } g\circ\phi"] \arrow[rrr, bend right, "\text{image of } \phi"'] & R^{\square}_{\bar\rho} \arrow[r, shift left, "act^*"] \arrow[r, shift right, "pr_2^*"'] & \mathcal{O}\hat{[GL_n]} \otimes R^{\square}_{\bar\rho} \arrow[r, "g^*\otimes\phi"] & A
\end{tikzcd}$$

Lecture 9. Relations for deformation rings

$E \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \simeq \mathbb{F}$. $\quad E \mid \mathbb{Q}_\ell$

G pro-finite gp satisfying $(*)$

$\bar\rho : G \to GL_n(\mathbb{F})$

$\mathrm{Def}^{\square}_{\bar\rho} : CNL_{\mathcal{O}} \to \mathrm{Set}$

$$A \mapsto \left\{ \begin{array}{l} \text{cts rep'n } \rho : G \to GL_n(A) \\ \rho \bmod m_A = \bar\rho \end{array} \right\}$$

rep'td by $R^{\square}_{\bar\rho}$.

<u>Last time</u> $\quad t_{R^{\square}_{\bar\rho}/(\varpi)} = \left(m^{\square}_{\bar\rho} / (m^{\square\,2}_{\bar\rho}, \varpi) \right)^* = \mathrm{Def}^{\square}_{\bar\rho}(\mathbb{F}[\varepsilon]) = Z^1(G, \mathrm{Ad}\,\rho)$

Set $t = \dim Z^1(G, \mathrm{Ad}\,\rho)$.

Then $\exists$ surjection $\mathcal{O}[[z_1, \dots z_t]] \twoheadrightarrow R^{\square}_{\bar\rho}$.

Let J be the kernel.

<u>General fact</u>

Over a noe. local ring (A, m), a f.g. A-mod M is gen. by $\dim M/mM$ elts

<u>Q</u>. What's the minimal no. of generators of J? $\Leftrightarrow \dim J/m_{\underline{z}} J$, $m_{\underline{z}} = (z_1, \dots z_t, \varpi)$

Theorem. J is gen. by $h^2 = \dim_{\mathbb{F}} H^2(G, \mathrm{Ad}\,\bar\rho)$ elts.

More precisely, there's a canonical injection

$$(J/m_{\underline{3}} J)^* \hookrightarrow H^2(G, \mathrm{Ad}\,\bar\rho)$$

Rmk. In all applications, h^2 is finite.

(Rmk: same result holds for $R_{\bar\rho}$, when $\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$)

Proof. $\rho^{univ}: G \longrightarrow GL_n(R^{\square}_{\bar\rho}) = GL_n(\mathcal{O}[[\underline{3}]]/J)$

(Philosophy: this rep'n "extends no more")

$\boxed{\text{Should not extend}}$

$$GL_n(\mathcal{O}[[\underline{3}]]/m_{\underline{3}} J)$$

Lift element-wise each $\rho^{univ}(g)$ to $\tilde\rho(g) \in GL_n(\mathcal{O}[[\underline{3}]]/m_{\underline{3}} J)$

For $g_1, g_2 \in G$, $c(g_1, g_2) := \tilde\rho(g_1 g_2)\,\tilde\rho(g_2)^{-1}\,\tilde\rho(g_1)^{-1} - 1_n \in M_n(J/m_{\underline{3}} J)$

$\mathbb{F}$-vec. sp.

$$\xrightarrow{\text{mod } J} \rho^{univ}(g_1 g_2)\,\rho^{univ}(g_2)^{-1}\,\rho^{univ}(g_1)^{-1} - 1_n = 0$$

$M_n(J/m_{\underline{3}} J) \cong M_n(\mathbb{F}) \otimes J/mJ$

G-acts by $\mathrm{Ad}\,\bar\rho$

Check. $c(g_1, g_2)$ defines a 2-cocycle, i.e. for $g_1, g_2, g_3 \in G$,

$$\bar\rho(g_1)\,c(g_2, g_3)\,\bar\rho(g_1)^{-1} - c(g_1 g_2, g_3) + c(g_1, g_2 g_3) - c(g_1, g_2) = 0.$$

[direct calculation].

Fact. For a different lift $\tilde\rho'$ of ρ^{univ} $\rightarrow$ $c'(g_1, g_2)$ is differed from $c(g_1, g_2)$ by a 2-boundary. So get canonical class $[c] \in H^2(G, J/mJ \otimes \mathrm{Ad}\,\bar\rho) = H^2(G, \mathrm{Ad}\,\bar\rho) \otimes J/mJ$

Thus for every $\mathbb{F}$-linear map $f: J/mJ \to \mathbb{F}$, get a class $[c_f] \in H^2(G, \mathrm{Ad}\,\bar{\rho})$.

$$c_f(g_1, g_2) = f(c(g_1, g_2))$$

$\rightsquigarrow$ get a map $(J/m_{\underline{\xi}} J)^* \longrightarrow H^2(G, \mathrm{Ad}\,\bar{\rho})$.

Show that this is injective $f \mapsto 0 = [c_f]$. Suppose not, $f \neq 0$,

$f: J/m_{\underline{\xi}} J \to \mathbb{F}$, $\mathrm{Ker}(f)$ gives $J_f \subset J$ s.t. $J/J_f \simeq \mathbb{F}$.

$$J_f \subset m_{\underline{\xi}} J$$

Exer. J_f is an ideal of $R^{\square}_{\bar{\rho}}$.

$$R^{\square}_{\bar{\rho}}/J_f \twoheadrightarrow R^{\square}_{\bar{\rho}}/J$$

By how $c_f(g_1, g_2)$ is defined, can take a lift $\tilde{\rho}_f(g)$ of $\rho_f(g)$ s.t.

$$\tilde{\rho}_f(g_1 g_2) = \tilde{\rho}_f(g_1)\,\tilde{\rho}_f(g_2) + \underbrace{c_f(g_1, g_2)}_{=0}\,\tilde{\rho}_f(g_1)\,\tilde{\rho}_f(g_2)$$

get a hom. $\tilde{\rho}_f: G \to GL_n(\mathcal{O}[[\underline{\xi}]]/J_f)$

By universal property of ρ^{univ}, $\exists$

$$\begin{tikzcd} R^{\square}_{\bar{\rho}} = \mathcal{O}[[\underline{\xi}]]/J \arrow[r, "\phi"] \arrow[rr, bend left, "id"] & \mathcal{O}[[\underline{\xi}]]/J_f \arrow[r, two heads] & \mathcal{O}[[\underline{\xi}]]/J \end{tikzcd}$$

s.t. $\tilde{\rho}_f: G \xrightarrow{\rho^{univ}} GL_n(R^{\square}_{\bar{\rho}}) \xrightarrow{\phi} GL_n(\mathcal{O}[[\underline{\xi}]]/J_f)$ NOT possible.

Cor. When $H^2(G, \mathrm{Ad}\,\bar{\rho}) = 0$ (often the case when $G = G_K$ for K local),

$R^{\square}_{\bar{\rho}}$ & $R_{\bar{\rho}}$ is a formal power series ring $\mathcal{O}[[\xi_1, \dots, \xi_t]]$.

In this case, we say the deformation is <u>unobstructed</u>.

$J = \ker\left(\mathcal{O}[[z_1, \dots, z_t]] \to R_{\bar\rho}^{\square}\right)$.

<u>Thm</u> $(J/m_{\underline{z}} J)^* \hookrightarrow H^2(G, \mathrm{Ad}\,\bar\rho)$, so J is gen. by $\dim H^2(G, \mathrm{Ad}\,\bar\rho)$ elts.

<u>Cor</u> When $\mathrm{End}_{\mathbb{F}[G]}(\bar\rho) = \mathbb{F}$, Krull dim $R_{\bar\rho} \geq \underset{\dim H^0(G, \mathrm{Ad}\,\bar\rho)}{\underset{\shortparallel}{1}} + \dim H^1(G, \mathrm{Ad}\,\bar\rho) - \dim H^2(G, \mathrm{Ad}\,\bar\rho)$

<u>Rmk</u>. Mazur conjectured that when $G = G_{F,S}$ w/ F no. field, S finite (includ. $|\infty$)

$\geq$ should be an equality.

When $G = G_{F,\{v|\ell\infty\}}$, this conj. $\Leftrightarrow$ Leopoldt conjecture.

"<u>Better way of thinking</u>"

" correct deformation stack should be $X_{\bar\rho} = \left[\mathrm{Spec}\, R_{\bar\rho}^{\square} / \widehat{GL}_n\right]$

$$\text{"}\dim_{\mathcal{O}} X_{\bar\rho}\text{"} = \text{Krull dim } R_{\bar\rho}^{\square} - \dim \widehat{GL}_n - 1$$

$$\geq 1 + \dim Z^1(G, \mathrm{Ad}\,\bar\rho) - \dim H^2(G, \mathrm{Ad}\,\bar\rho) - \dim \widehat{GL}_n - 1$$

$$= -\chi(G, \mathrm{Ad}\,\bar\rho)$$

b/c. $0 \to H^0(G, \mathrm{Ad}\,\bar\rho) \to \overset{C^0}{\overset{\shortparallel}{\mathrm{Ad}\,\bar\rho}} \xrightarrow{d} Z^1(G, \mathrm{Ad}\,\bar\rho) \to H^1(G, \mathrm{Ad}\,\bar\rho) \to 0$

$x \longmapsto [g \mapsto gx - x]$

<u>A variant</u> Fix a lift $\chi : G \to \mathcal{O}^\times$ of $\det \bar\rho$.

<u>Consider</u> $\mathrm{Def}_{\bar\rho}^{\square,\chi} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$

$$(A, m_A) \mapsto \left\{ \begin{array}{l} \text{cts rep } \rho : G \to GL_n(A) \\ \text{s.t. } \rho \bmod m_A = \bar\rho,\ \det \rho = \chi \end{array} \right\}$$

Can show: $\mathrm{Def}^{\square,\chi}_{\bar\rho}$ is rep'td by $R^{\square,\chi}_{\bar\rho} \in \mathrm{CNL}_{\mathcal{O}}$.

and $\left(\mathfrak{m}^{\square,\chi}_{\bar\rho} \big/ \left((\mathfrak{m}^{\square,\chi}_{\bar\rho})^2, \varpi\right)\right)^* \simeq Z^1(G, \mathrm{Ad}^0_{\bar\rho})$ $\quad \mathrm{Ad}^0_{\bar\rho} \subset \mathrm{Ad}\,\bar\rho$, $\quad \mathrm{Ad}^0_{\bar\rho} = M_n(\mathbb{F})^{\mathrm{tr}=0}$, $\quad \mathrm{Ad}\,\bar\rho = M_n(\mathbb{F})$

$$\text{Krull-dim } R^{\square,\chi}_{\bar\rho} \geq 1 + \dim Z^1(G, \mathrm{Ad}^0_{\bar\rho}) - \dim H^2(G, \mathrm{Ad}^0_{\bar\rho})$$

"$X^{\chi}_{\bar\rho} := [\mathrm{Spec}\, R^{\square,\chi}_{\bar\rho} / \widehat{PGL}_n]$"

$$\dim_{\mathcal{O}} X^{\chi}_{\bar\rho} \geq -\chi(G, \mathrm{Ad}^0\bar\rho)$$

Examples. When $G = G_K$ for K finite $|\mathbb{Q}_p$.

If $p \neq \ell$, $\dim_{\mathcal{O}} X^{\chi}_{\bar\rho} \geq -\chi(G_K, \mathrm{Ad}^0_{\bar\rho}) = 0$ (means Krull dim $R^{\square,\chi}_{\bar\rho} \geq n^2$)

If $p = \ell$, $\dim_{\mathcal{O}} X^{\chi}_{\bar\rho} \geq -\chi(G_K, \mathrm{Ad}^0_{\bar\rho}) = [K:\mathbb{Q}_p]\, n^2$

Rmk (1) if $\bar\rho : G \to H(\mathbb{F}_\ell)$ for some "nice" alg. gp, expect similar formula holds.

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & H(\mathbb{F}_\ell) \\ & H(R) \arrow[u] \end{tikzcd}$$

(2) When $p \neq \ell$ (maybe ok for $p = \ell$), equality holds. In fact, all irred. cpnts of $\mathrm{Spec}\, R^{\square,\chi}_{\bar\rho}[\frac{1}{\ell}]$ are smooth.

(Heuristic reason: for a closed pt $x \in \mathrm{Spec}\, R^{\square,\chi}_{\bar\rho}[\frac{1}{\ell}]$,

$\mathrm{Def}^{\square,\chi}_{\rho^{\mathrm{univ}}_x}$ is rep'td by $\left(R^{\square,\chi}_{\bar\rho}[\frac{1}{\ell}]\right)^{\wedge}_x$

smoothness of this completion $\Leftrightarrow H^2(G_K, \mathrm{Ad}^0\rho_x) = 0 \Leftarrow H^0(G_K, \mathrm{Ad}^0\rho_x^*(1)) = 0$ holds very often.

When $G = G_{\mathbb{R}} = \{1, c\}$. $\bar{\rho} : G_{\mathbb{R}} \longrightarrow GL_2(\mathbb{F})$, $\ell \neq 2$

$$c \longmapsto \begin{cases} I_2 \text{ or } -I_2 & \text{even} \\ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \text{odd} \end{cases} \qquad \det(\bar{\rho}(c)) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\leftarrow$ modular form case

Fix χ lifting $\det(\bar{\rho})$. $\chi(c) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\rightsquigarrow R^{\square, \chi}_{\bar{\rho}}$ deformation ring. $\qquad$ Q: What is $\dim_{\mathcal{O}} X^{\chi}_{\bar{\rho}}$?

* <u>even case</u>. Any deformation $\rho : G_{\mathbb{R}} \longrightarrow GL_2(A)$ of $\bar{\rho}$

$$c \longmapsto I_2 \text{ or } -I_2$$

So $R^{\square, \chi}_{\bar{\rho}} = \mathcal{O}$.

$$\dim_{\mathcal{O}} X^{\chi}_{\bar{\rho}} = \text{Krull dim } \mathcal{O} - 1 - \dim \widehat{PGL_2} = -3$$

When $\bar{\rho}(c) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, any $\rho : G \longrightarrow GL_2(A)$ has two eigenvalues $1, -1$

So $\rho(c) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1}$, $x, y \in m_A$

$\rho(c) v_1 = v_1$

$\rho(c) v_{-1} = v_{-1}$

$v_1 \bmod m_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow v_1 = \begin{pmatrix} 1 \\ x \end{pmatrix}$

$\rightsquigarrow v_2 = \begin{pmatrix} y \\ 1 \end{pmatrix}$

So $R^{\square, \chi}_{\bar{\rho}} \simeq \mathcal{O}[[x, y]]$, and $\dim_{\mathcal{O}} X^{\chi}_{\bar{\rho}} = -1$.

<u>Compare this w/ Euler char.</u>

$$-\chi(G_{\mathbb{R}}, \mathrm{Ad}^0 \bar{\rho}) = -\dim H^0(G_{\mathbb{R}}, \mathrm{Ad}^0 \bar{\rho}) = -\dim (\mathrm{Ad}^0 \bar{\rho})^{c=1}$$

$$= \begin{cases} -3 & \text{even} \\ -1 & \text{odd} \end{cases} \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

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$K \mid \mathbb{Q}_p$ residue field k_K, $\# k_K = q$, $p \neq \ell$.

$\bar{\rho} : G_K \to G_{k_K} \to \mathbb{F}^\times$ unram., say $\bar{\rho}(\phi_K) = \bar{\alpha} \in \mathbb{F}^\times$.

lift it arbitrarily to $\alpha \in \mathcal{O}^\times$.

Def $R^\square_{\bar{\rho}}(A) = \{\rho : G_K \to A^\times$ s.t. $\rho \bmod m_A = \bar{\rho}\}$.

$$\begin{tikzcd} G_K \arrow[r] \arrow[dr] & A^\times \\ & G_K/P_K \arrow[u] \end{tikzcd}$$

<u>Claim</u>. $\rho(P_K) = \text{triv.}$ $\cong$ b/c $\bar{\rho}(P_K) = 0 \Rightarrow \underbrace{\rho(P_K)}_{\text{pro-}p} \subset \underbrace{(1+m_A)^\times}_{\text{pro-}\ell}$ $\square$

$$\begin{array}{c} \mathbb{Z}_\ell^{(p)}(1) \rtimes \hat{\mathbb{Z}} \\ \downarrow \qquad \downarrow \\ \tau_K \qquad \phi_K \end{array}$$

So the universal rep'n $\rho^{univ} : G_K \twoheadrightarrow G_K/P_K \to \left(\mathcal{O}[[X,Y]]/(\ ?\ ?\ )\right)^\times$

$$\phi \longmapsto \alpha + X$$

$$\tau \longmapsto 1 + Y.$$

$$\phi^{-1}\tau\phi = \tau^q \Rightarrow 1+Y = (1+Y)^q \Rightarrow (1+Y)^{q-1} = 1$$

$$\ell^m \| q-1 \Rightarrow (1+Y)^{\ell^m} = 1$$

$$\Rightarrow R^\square_{\bar{\rho}} = \mathcal{O}[[X,Y]] / \left((1+Y)^{\ell^m} - 1\right)$$

The relation is trivial $\Leftrightarrow m = 0 \Leftrightarrow q \not\equiv 1 \pmod{\ell}$

$\Updownarrow$?

$H^2(G_K, \mathrm{Ad}\,\bar{\rho}) = 0 \Leftrightarrow H^2(G_K, \mathbb{F}_\ell) = 0 \Leftrightarrow \mathbb{F}_\ell(1)^{G_K} = 1$

ϕ^{-1} acts by mult. by q

$\Leftrightarrow q \not\equiv 1 \bmod \ell$

Lecture 10

(Taylor - Wiles deformation) $K | \mathbb{Q}_p$, $p \neq \ell$. $\# k_K = q$

Same setup as before $\bar{\rho}: G_K \twoheadrightarrow G_{k_K} \longrightarrow GL_2(\mathbb{F})$

$$\phi \longmapsto \begin{pmatrix} \bar{\alpha} & \\ & \bar{\beta} \end{pmatrix}$$

Fix $\alpha, \beta \in \mathcal{O}^\times$ lifting $\bar{\alpha}, \bar{\beta}$. We assume that $\bar{\alpha} \neq \bar{\beta} \cdot q \pm 1$, $\bar{\alpha} \neq \bar{\beta}$

As in last time, any lift $\rho: G_K \longrightarrow GL_2(A)$ of $\bar{\rho}$, $A \in \mathcal{CNL}_{\mathcal{O}}$, $1 + \varpi M_n(\mathcal{O}) \subset GL_2(A)$

$$\begin{tikzcd} G_K \arrow[rr] \arrow[dr] & & GL_2(A) \\ & G_K/P_K \arrow[ur] & \end{tikzcd}$$

$$G_K/P_K \cong \hat{\mathbb{Z}}^{(p)}(1) \rtimes \hat{\mathbb{Z}}$$

$$\tau \in \hat{\mathbb{Z}}^{(p)}(1), \quad \phi \in \hat{\mathbb{Z}}$$

<u>Claim</u>. $\exists$ eigenvectors of ϕ: $v_1 = \begin{pmatrix} 1 \\ X \end{pmatrix}$ w/ eig. val. $\alpha + S$

$v_2 = \begin{pmatrix} Y \\ 1 \end{pmatrix}$ w/ eig val. $\beta + T$ for $X, Y, S, T \in m_A$

<u>Pf</u>. (Hensel Lemma type argument)

Say we have computed X & S mod m_A^N, will determine them module m_A^{N+1}.

<u>need</u>

$$\rho(\phi) = \begin{pmatrix} \alpha + A & B \\ C & \beta + D \end{pmatrix} \begin{pmatrix} 1 \\ X \end{pmatrix} = (\alpha + S) \begin{pmatrix} 1 \\ X \end{pmatrix}$$

$A, B, C, D \in m_A$

<u>Need</u>

$$\begin{cases} \cancel{\alpha} + A + BX = \cancel{\alpha} + S \\ C + (\beta + D)X = (\alpha + S)X \end{cases}$$

← knowing X mod m_A^N gives S mod m_A^{N+1}

$$= (\alpha + A + BX)X$$

$$\underbrace{(\beta + D - \alpha - A)}_{\neq 0 \bmod m_A \text{. so it's a unit}} X = -C + BX^2$$

When X is determined modulo m_A^N,

BX^2 ——— m_A^{N+1}

Set $X = (\beta + D - \alpha - A)^{-1}\left(-C + B(X \bmod m_A^N)^2\right) \bmod m_A^{N+1}$. $\square$

So $\rho(\phi) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} \alpha+S & \\ & \beta+T \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

$$\rho(T) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} (I_2 + P + Q) \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}, \quad P = \begin{pmatrix} p_{11} & \\ & p_{22} \end{pmatrix}$$

$$Q = \begin{pmatrix} & Q_{12} \\ Q_{21} & \end{pmatrix}$$

<u>Know</u> $\phi^{-1}\tau\phi = \tau^q$

$$\Rightarrow \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} (\alpha+S)^{-1} & \\ & (\beta+T)^{-1} \end{pmatrix} (I_2 + P + Q) \begin{pmatrix} \alpha+S & \\ & \beta+T \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} (I_2 + P + Q)^q \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

<u>Claim</u> $Q = 0$.

<u>Pf</u> Suppose that we have proved that $Q \in (m_A^N)^{\oplus 2}$

$$I_2 + \begin{pmatrix} (\alpha+S)^{-1} & \\ & (\beta+T)^{-1} \end{pmatrix} P \begin{pmatrix} \diagdown \end{pmatrix} + \begin{pmatrix} \diagdown \end{pmatrix}^{-1} Q \begin{pmatrix} \diagdown \end{pmatrix}$$

$$= (I_2 + P)^q + q(I_2 + P)^{q-1} \cdot Q + \binom{q}{2} (I_2 + P)^{q-2} Q^2 + \cdots \quad \bmod m_A^{N+1}$$

$$\begin{pmatrix} & Q_{12} \dfrac{\cancel{(\alpha+S)^{-1}}}{\cancel{(\beta+T)^{-1}}} \\ Q_{21} \dfrac{\cancel{(\beta+T)^{-1}}}{\cancel{(\alpha+S)^{-1}}} & \end{pmatrix} \equiv q \begin{pmatrix} & Q_{12} \\ Q_{21} & \end{pmatrix} \bmod m_A^{N+1}$$

$\dfrac{\bar{\alpha}}{\bar{\beta}}$ $\qquad \Rightarrow Q_{12} \cdot \dfrac{\bar{\beta}}{\bar{\alpha}} \equiv q\, Q_{12} \bmod m_A^{N+1}$

$\Rightarrow Q_{12} \equiv 0 \bmod m_A^{N+1}$. Idem. $Q_{21} \equiv 0 \bmod m_A^{N+1}$.

$\square$

Back to $\rho(T) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+u & \\ & 1+v \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

$$\rho(\phi) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} \alpha+S & \\ & \beta+T \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

univ. rep.

$$\phi^{-1}\tau\phi = \tau^q \Rightarrow (1+u)^{q-1} = (1+v)^{q-1} = 1$$

- If $\ell^m \| q-1 \iff (1+u)^{\ell^m} = 1$, $(1+v)^{\ell^m} = 1$.

<u>Summary</u> $R^{\square}_{\bar\rho} = \mathcal{O}[[X, Y, S, T, u, v]] \Big/ \left((1+u)^{\ell^m} - 1,\ (1+v)^{\ell^m} - 1\right)$

$$\frac{\mathbb{Z}_\ell[u]}{(1+u)^{\ell^m} - 1} \in (u, \ell)^m$$

<u>Variant</u>. When $\det\bar\rho = \chi_{cycl}$ & $\ell^m \| q-1$, $m > 0$.

Assume $\underbrace{\bar\alpha \neq \bar\beta}_{\bar\alpha \neq \pm 1}$ $(\bar\alpha \cdot \bar\beta = 1)$

Fix the lift χ of $\det\bar\rho$ to be χ_{cycl}. Then $R^{\square,\chi}_{\bar\rho} \simeq \mathcal{O}[[X, Y, S, u]] \Big/ ((1+u)^{\ell^m} - 1)$

$$\rho^{univ}(\phi) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} \alpha+S & \\ & q^{-1}(\alpha+S)^{-1} \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

$$\rho^{univ}(\tau) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+u & \\ & (1+u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

<u>Rmk</u>. Let $R^{\square,\chi,m}_{\bar\rho}$ denote the deformation ring when viewing $\bar\rho$ as a rep. of G_{k_k}

Then $R^{\square,\chi}_{\bar\rho} \longrightarrow R^{\square,\chi,m}_{\bar\rho}$

$$\frac{\mathcal{O}[[X, Y, S, u]]}{((1+u)^{\ell^m} - 1)} \xrightarrow{\text{mod } u} \mathcal{O}[[X, Y, S]]$$

Geometrically, $\operatorname{Spec} R_{\bar\rho}^{\square,\chi,u} \not\hookrightarrow \operatorname{Spec} R_{\bar\rho}^{\square,\chi}$ is a closed subspace def'd by $u=0$.

$$\boxed{\begin{aligned} \widehat{PGL}_n : CNL_{\mathcal{O}} &\longrightarrow Grp \\ A &\longmapsto \frac{(1+M_n(m_A))^\times}{(1+m_A)^\times} \end{aligned}}$$

§. Relative deformation problem

Let F be a no. field, S a finite set of places of F, including those dividing $\ell\infty$.

Fix $\bar\rho : G_{F,S} \longrightarrow GL_n(\mathbb{F})$ abs. irred. & a lift $\chi : G_{F,S} \longrightarrow \mathcal{O}^\times$ of $\det\bar\rho$.

For each $v \in S$, write $\bar\rho_v := \bar\rho|_{G_{F_v}} : G_{F_v} \longrightarrow GL_n(\mathbb{F})$, $\chi_v := \chi|_{G_{F_v}}$.

(Caveat: very often $\bar\rho_v$ is reducible)

Then, we have nat'l transformations of functors

$$\begin{aligned} \mathrm{Def}_{\bar\rho}^{\square} &\longrightarrow \mathrm{Def}_{\bar\rho_v}^{\square} \qquad \forall v \in S. \\ \rho &\longmapsto \rho|_{G_{F_v}} \end{aligned}$$

$X_{\bar\rho}^{\square} := \operatorname{Spf} R_{\bar\rho}^{\square}$, $X_{\bar\rho_v}^{\square} := \operatorname{Spf} R_{\bar\rho_v}^{\square}$

$$\rightsquigarrow X_{\bar\rho}^{\square} \longrightarrow X_{\bar\rho_v}^{\square}.$$

$X_{\bar\rho} =$ " $X_{\bar\rho}^{\square} / \widehat{PGL}_n$ ", $X_{\bar\rho_v} :=$ " $X_{\bar\rho_v}^{\square} / \widehat{PGL}_n$ "

Want. $\boxed{X_{\bar\rho} \longrightarrow \prod_{v\in S} X_{\bar\rho_v}}$

Def. For a subset $T \subset S$ (containing all places dividing $\ell\infty$), define

$$\mathrm{Def}_{\bar\rho}^{\square_T,\chi} : CNL_{\mathcal{O}} \longrightarrow Set, \quad A \longmapsto \left\{ \begin{array}{l} (\rho, h_v)_{v\in T} : \rho : G_F \longrightarrow GL_n(A) \text{ cts rep'n } (\det\rho = \chi) \\ \text{s.t. } \rho \bmod m_A = \bar\rho,\ h_v \in \widehat{PGL}_n(A) = \frac{(1+M_n(m_A))^\times}{(1+m_A)^\times} \end{array} \right\}$$

Say $(\rho, (h_v)_{v\in T}) \sim (\rho', (h'_v)_{v\in T})$ if $\exists\ X \in \widehat{PGL}_n(A)$ s.t.

$\rho' = X\rho X^{-1}$, $h'_v = h_v X^{-1}$, $\forall v \in T$.

$$= \left\{ \begin{array}{l} \rho : G_{F,S} \to GL_n(A) \text{ cts rep'n s.t. } \rho \bmod m_A = \bar\rho \\ \text{and } h_1, \dots, h_{|T|-1} \in \widehat{PGL}_n(A) \qquad \det \rho = \chi \end{array} \right\}$$

$$\Rightarrow R^{\square_T, \chi}_{\bar\rho} \simeq R^{\square, \chi}_{\bar\rho} \hat\otimes \left(\mathcal{O}_{\widehat{PGL}_n}\right)^{\otimes(|T|-1)}$$

$$\simeq R^{\chi}_{\bar\rho}[[z_1, z_2, \dots, z_{(n^2-1)|T|}]]$$

$$\mathrm{Def}^{\square_T, \chi}_{\bar\rho} \longrightarrow \prod_{v\in T} \mathrm{Def}^{\square, \chi_v}_{\bar\rho_v}$$

$$(\rho, (h_v)_{v\in T}) \longmapsto (h_v \rho|_{G_{F_v}} h_v^{-1})$$

$G_{F_v} \xrightarrow{\text{well-defined up to conjugation}} G_{F,S}$

$$\begin{array}{ccc} F^S & & \bar v \\ | & & | \\ F & & v \end{array}$$

$$[h'_v \rho'_v h'^{-1}_v = (h_v x^{-1})(x\rho_v x^{-1})(h_v x^{-1})^{-1} = h_v \rho_v h_v^{-1}]$$

Get $\mathrm{Spf}\, R^{\square_T, \chi}_{\bar\rho} \longrightarrow \underbrace{\mathrm{Spf}\left(\hat\otimes_{v\in T} R^{\square, \chi_v}_{\bar\rho_v}\right)}_{\substack{!! \\ R^{\square_T}_{loc}}}$ $\qquad R^{\square_T}_{loc} \longrightarrow R^{\square_T, \chi}_{\bar\rho}$

<u>Rmk.</u> In later application, will use the case $T = S$ (base case)

or $S' = S \sqcup Q$, $T' = S$ (Taylor–Wiles)

<u>Theorem.</u> We can write $R^{\square_T, \chi}_{\bar\rho}$ as a quotient of $R^{\square_T}_{loc}[[x_1, \dots, x_{h^1}]]/(f_1, \dots, f_{h^2})$

where $h^1 = \dim \tilde H^1_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)$, $h^2 = \dim \tilde H^2_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)$.

Recall $\widetilde{H}^i_T(G_{F,S}, -)$ is the cohom. of the relative Galois cohomology

$$\widetilde{R\Gamma}_T(G_{F,S}, \mathrm{Ad}^0\bar\rho) = \mathrm{Cone}\Big(R\Gamma(G_{F,S}, \mathrm{Ad}^0\bar\rho) \to \bigoplus_{v\in T} R\Gamma(G_{F_v}, \mathrm{Ad}^0\bar\rho)\Big)[-1].$$

Proof $\boxed{\text{For } h^1}$, we compute the relative tangent space

$$= \ker\Big(\mathrm{Def}^{\square_T, x}_{\bar\rho}(\mathbb{F}[\varepsilon]) \to \prod_{v\in T} \mathrm{Def}^{\square, x_v}_{\bar\rho_v}(\mathbb{F}[\varepsilon])\Big)$$

$$\ker\Big(\text{tangent}_{R^{\square_T, x}_{\bar\rho}} \longrightarrow \prod_{v\in T} \text{tangent}_{R^{\square, x_v}_{\bar\rho_v}}\Big)$$

$$= \ker\left(\left(\frac{m^{\square_T, x}_{\bar\rho}}{\left((m^{\square_T, x}_{\bar\rho})^2, \varpi\right)}\right)^* \to \prod_{v\in T}\left(\quad\right)^*\right)$$

$$= \mathrm{coker}\left(\bigoplus_{v\in T} \frac{m^{\square, x_v}_{\bar\rho_v}}{(m^{\square, x_v}_{\bar\rho_v}{}^2, \varpi)} \longrightarrow \frac{m^{\square_T, x}_{\bar\rho}}{(m^{\square_T, x}_{\bar\rho}{}^2, \varpi)}\right)^*$$

$$\mathrm{Def}^{\square_T, x}_{\bar\rho}(\mathbb{F}[\varepsilon]) \xrightarrow{\sim} \frac{Z^1_{\mathrm{cts}}(G_{F,S}, \mathrm{Ad}^0\bar\rho) \oplus (\mathrm{Ad}^0\bar\rho)^{|T|}}{\partial\ \mathrm{Ad}^0\bar\rho \ \text{diagonal}}$$

$$(\rho, 1+h_v\varepsilon) \longmapsto (g \mapsto Ag, \ h_v)$$

$\rho(g) = (1+Ag\varepsilon)\bar\rho(g)$

$$(1+x\varepsilon)^{-1}\rho(1+x\varepsilon), (1+h_v\varepsilon)(1+x\varepsilon) \mapsto (g \mapsto Ag - x + gxg^{-1}, \ h_v + x)$$

$= 1 + (h_v + x)\varepsilon$

$$\prod_{v\in T} \mathrm{Def}^{\square, x}_{\bar\rho_v}(\mathbb{F}[\varepsilon]) \xrightarrow{\sim} \prod_{v\in T} Z^1(G_F, \mathrm{Ad}^0\bar\rho_v)$$

$$(1+h_v\varepsilon)\rho(1+h_v\varepsilon)^{-1} \longmapsto (g_v \mapsto Ag_v + h_v - g_v h_v g_v^{-1})$$

On the other hand, $\widetilde{H}^1_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)$ can be computed as cohom. of

$$\begin{tikzcd}
\mathrm{Ad}^0\bar\rho \arrow[r, "d"] \arrow[dr] & C^1_{cts}(G_{F,S}, \mathrm{Ad}^0\bar\rho) \arrow[r, "d"] \arrow[dr, "res"] & C^2_{cts}(G_{F,S}, \mathrm{Ad}^0\bar\rho) \arrow[r] \arrow[dr, "res"] & \cdots \\
 & \bigoplus_{v\in T} \mathrm{Ad}^0\bar\rho \arrow[r, "-d"] & \bigoplus_{v\in T} C^1_{cts}(G_{F_v}, \mathrm{Ad}^0\bar\rho) \arrow[r, "-d"] & \\
\end{tikzcd}$$

$$\widetilde{H}^1_T = \ker\left(\begin{tikzcd}
Z^1(G_{F,S}, \mathrm{Ad}^0\bar\rho) \arrow[dr] \\
\bigoplus_{v\in T} \mathrm{Ad}^0\bar\rho \arrow[r] & \bigoplus Z^1(\cdots)
\end{tikzcd}\right)$$

<u>Lecture 11</u> Recall F, $S \supset T$

For a cts $G_{F,S}$-module M, define

$$\widetilde{R\Gamma}_T(G_{F,S}, M) = \mathrm{Cone}\left(R\Gamma(G_{F,S}, M) \longrightarrow \bigoplus_{v\in T} R\Gamma(G_F, M)\right)[-1]$$

Let $\widetilde{H}^i_T(G_{F,S}, M) = i^{th}$-coh. of $\uparrow$

<u>Theorem</u>. Let $\bar\rho : G_{F,S} \longrightarrow GL_n(\mathbb{F})$ be an abs. irred. rep'n,

$R^{\square_T, \chi}_{\bar\rho}$:= deformation ring of $\bar\rho$ w/ det χ & framed at T

$$\begin{tikzcd}
R^{\square_T, \chi}_{\bar\rho} \\
\hat\bigotimes_{v\in T} R^{\square_T, \chi_v}_{\bar\rho_v} \arrow[u] \\
R^{\square_T}_{loc} \arrow[u, equal]
\end{tikzcd}$$

Then we can write $R^{\square_T, \chi}_{\bar\rho} = R^{\square_T}_{loc}[[x_1, \dots, x_t]]/(f_1, \dots, f_s)$

where $t = \widetilde{h}^1_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)$, $s = \widetilde{h}^2_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)$.

<u>Recall the Euler char. formula</u>, $\ell \geq 3$

$$\chi(G_{F,S}, \mathrm{Ad}^0\bar\rho) = \sum_{v\in S} \chi(G_{F_v}, \mathrm{Ad}^0\bar\rho) \Rightarrow \widetilde{\chi}_T(G_{F,S}, \mathrm{Ad}^0\bar\rho) = \chi(G_{F,S}, \mathrm{Ad}^0\bar\rho) - \sum_{v\in T} \chi(G_{F_v}, \mathrm{Ad}^0\bar\rho)$$

$$= \sum_{v \in S \setminus T} \chi(G_F, \mathrm{Ad}^0 \bar\rho) \overset{T \text{ contains all}}{\underset{\text{places} \mid \ell\infty}{=\!=}} 0$$

<u>Fact</u> $\tilde H^0_T(G_{F,S}, \mathrm{Ad}^0\bar\rho) = \tilde H^3_T(G_{F,S}, \mathrm{Ad}^0\bar\rho) = 0$

$\Rightarrow \tilde h^1_T(G_{F,S}, \mathrm{Ad}^0\bar\rho) = \tilde h^2_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)$

<u>Cor.</u> Assume $\bar\rho$ abs. irred., & T contains all places $\mid \ell\infty$,

$\mathrm{Krull\ dim}\, R^{\square_T, \chi}_{\bar\rho} \geq \mathrm{Krull\ dim}\, R^{\square_T}_{loc}$. (or "$\dim_0 X_{\bar\rho}$" $\geq \sum_{v \in T} \dim_0 X_{\bar\rho_v}$")

<u>Pf of the fact</u> $0 \to \tilde H^0_T(G_{F,S}, \mathrm{Ad}^0\bar\rho) \to \underbrace{H^0(G_{F,S}, \mathrm{Ad}^0\bar\rho)}_{\substack{\| \\ (\mathrm{Ad}^0\bar\rho)^{G_{F,S}} = 0}} \to \bigoplus_{v \in T} \cdots$

$(\mathrm{Ad}\bar\rho)^{G_{F,S}} = \mathrm{End}_{G_{F,S}}(\bar\rho) = \mathbb{F}$.

if $S = T$

First $(\tilde H^3_T(G_{F,S}, \mathrm{Ad}^0\bar\rho))^* \simeq H^0(G_{F,S}, \mathrm{Ad}^0\bar\rho(1))$ $\quad (\mathrm{Ad}^0\bar\rho)^* = \mathrm{Ad}^0\bar\rho$

$= 0$

<u>Claim</u> $\tilde H^i_T(G_{F,S}, M) \simeq \tilde H^{3-i}_{S\setminus T}(G_{F,S}, M^*(1))$.

not quite correct b/c of $H^0(G_{\mathbb{R}}, M)$ & $H^0(G_{\mathbb{R}}, M^*(1))$.

$\|$

coh. of cone$\left(R\Gamma(G_{F,S}, M^*(1)) \to \bigoplus_{v \in S\setminus T} R\Gamma(G_{F_v}, M^*(1))\right)[-1]$.

$\Rightarrow \tilde H^3_T(G_{F,S}, \mathrm{Ad}^0\bar\rho)^* = \tilde H^0_{S\setminus T}(G_{F,S}, \mathrm{Ad}^0\bar\rho(1)) = 0$.

Overview of the Taylor – Wiles argument (Galois patching)

Eichler – Shimura ($k=2$), Deligne ($k \geq 2$), Deligne – Serre ($k=1$)

Assoc. to each normalized cusp. eigenform $f(q) = \sum_{n \geq 1} a_n q^n$ of level $\Gamma_0(N)$, there's an irred. Galois rep'n $\rho_f : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_p(f))$

$\cong$ ext'n of $\mathbb{Q}_p$ containing all a_n's.

s.t. $\forall \ell \nmid N$, ρ_f is unram. @ ℓ, and

char. poly $(\rho_f(\mathrm{Frob}_\ell)) = X^2 - a_\ell X + \ell^{k-1}$.

$E \supset \mathbb{Q}_\ell$

$E \supset \mathcal{O} \to \mathcal{O}/\varpi = \mathbb{F}$

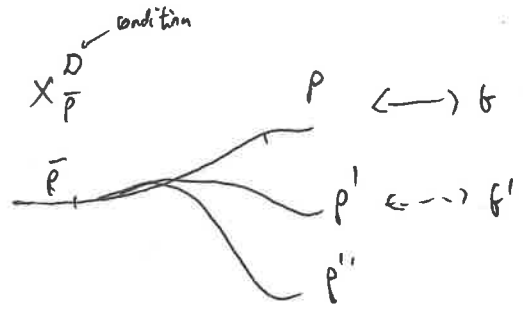
<u>Target Theorem</u> (Rough form) Some technical assumptions. $S = \{p \mid N\ell\} \cup \{\infty\}$

Suppose that $\rho, \rho' : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathcal{O})$ are two rep's satisfying some nat'l conditions, s.t. $\rho = \rho_f$ for some mod. eigenform f of wt 2, level $\Gamma_0(N)$. (and $\bar{\rho} = \rho \bmod m$ is abs. irred.)

Then $\exists$ modular eigenform f' of wt 2 level $\Gamma_0(N)$ s.t. $\rho_{f'} = \rho'$.

<u>Pf</u>

In practice, $\dim_{\mathbb{F}} X^D_{\bar{\rho}} = 0$

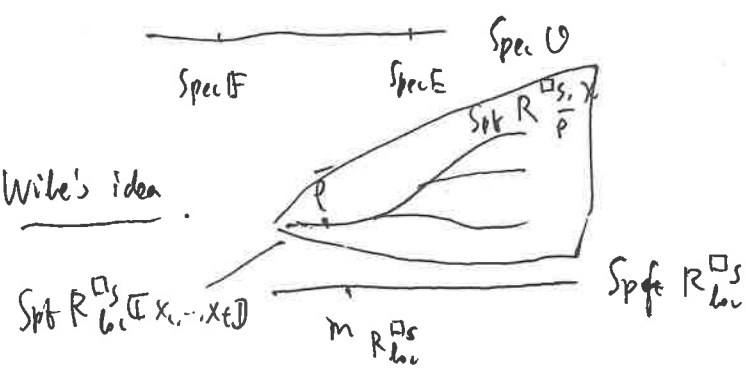


Can view $S_2(\Gamma_0(N), \mathbb{Z}_\ell)$ as a module over $R^D_{\bar{\rho}}$.

The question is: what's

$\mathrm{Supp}(S_2(\Gamma_0(N), \mathbb{Z}_\ell))$?

<u>Wiles's idea</u>.



$R^{\square_S, \chi}_{\bar\rho, Q_M}$:= deformation of $\bar\rho$ as a rep'n of $G_{Q, S \cup Q_M}$

$\|$ (for a finite set of places Q_M)

$R^{\square_S}_{loc}[[X_1, \dots, X_t]] / (f_1^{(M)}, \dots, f_t^{(M)})$

But "$\lim_M$" $(f_1^{(M)}, \dots, f_t^{(M)}) = (0)$.

$Q_M = \{q_1^{(M)}, \dots, q_t^{(M)}\}$, $S_2(\Gamma_1(N q_1^{(M)} \cdots q_2^{(M)}))$

$$\text{"}\varprojlim\text{"} R^{\square_S, \chi}_{\bar\rho, Q_M} = \text{"}\lim\text{"} \operatorname{Supp} S_2(\Gamma_0(N) \cap \Gamma_1(\prod q))$$

Toy model : $\operatorname{Spec} \dfrac{\mathbb{Z}_\ell[[x,y]]}{(xy)} \hookrightarrow \operatorname{Spec} \dfrac{\mathbb{Z}_\ell[[X,Y]]}{(X^2 Y^2)} \to \cdots \to \operatorname{Spec} \mathbb{Z}_\ell[[X,Y]]$

Choice of Taylor - Wiles's primes

Theorem Let F, S be as before. $\ell \geq 3$

Assume that $\bar\rho : G_{F(\sqrt{(-1)^{(\ell-1)/2}\ell})} \to GL_2(\mathbb{F})$ is abs irred. & $\det \bar\rho = \overline{\chi_{cycl}}$.

Write $t = \tilde h^1_S(G_{F,S}, \mathrm{Ad}^0 \bar\rho) = \tilde h^2_S(G_{F,S}, \mathrm{Ad}^0 \bar\rho)$. Then for each $N \in \mathbb{Z}_{\geq 1}$, there exists a finite set of places $Q = \{v_1, \dots, v_t\}$ s.t. $Q \cap S = \phi$

(1) For each $v \in Q$, $\# k_v \equiv 1 \mod \ell^N$

$$\bar\rho(\mathrm{Frob}_v) = \begin{pmatrix} \bar\alpha_v & \\ & \underbrace{q_v^{-1}}_{=1} \bar\alpha_v^{-1} \end{pmatrix} \qquad \bar\alpha_v \neq \bar\alpha_v^{-1}$$

(2) $R^{\square_S, \chi}_{\bar\rho, Q}$ is a quotient of $R^{\square}_{loc}[[x_1, \dots x_t]]$

Rmk. (2) $\Leftrightarrow$ $\tilde{h}^1_S(G_{F,S\cup Q}, \mathrm{Ad}^0\rho) = 0$

$\Leftrightarrow$ $\mathrm{Spec}\, R^{\square_S, x}_{\bar\rho} \longrightarrow \mathrm{Spec}\, R^{\square}_{loc}$

$\mathrm{Spec}\, R^{\square_S, x}_{\bar\rho, Q} \longrightarrow \mathrm{Spec}\, R^{\square}_{loc}$ have the same tangent space

$\Leftrightarrow$ nat'l map $\tilde{H}^1_S(G_{F,S}, \mathrm{Ad}^0\bar\rho) \hookrightarrow \tilde{H}^1_S(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho)$ is an isom.

Proof. Under assumption that $\bar\rho(G_{F,S}) \supset SL_2(\mathbb{F}_\ell)$, $\ell \geq 5$.

$*$ injectivity

$$\begin{tikzcd}
\bigoplus_{v\in S} H^0(G_{F_v}) \arrow[r] \arrow[d, equal] & \tilde{H}^1_S(G_{F,S}, \mathrm{Ad}^0\bar\rho) \arrow[r] \arrow[d] & H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho) \arrow[r] \arrow[d] & \bigoplus_{v\in S} H^1(G_{F_v}, \mathrm{Ad}^0\bar\rho) \arrow[d, equal] \\
\bigoplus_{v\in S} H^0(G_{F_v}) \arrow[r] & \tilde{H}^1_S(G_{F,S\cup Q}) \arrow[r] & H^1(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho) \arrow[r] & \bigoplus_{v\in S} H^1(G_{F_v})
\end{tikzcd}$$

Need $\tilde{h}^1_S(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho) = \tilde{h}^1_S(G_{F,S}, \mathrm{Ad}^0\bar\rho)$.

Recall $\tilde{\chi}_S(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho) = \tilde{\chi}_S(G_{F,S}, \mathrm{Ad}^0\bar\rho) = 0$, so $\tilde{h}^1_S = \tilde{h}^2_S$

Enough to show this for $\tilde{h}^2_S$.

Poitou-Tate duality

$$\tilde{H}^2_S(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho)^* \simeq \tilde{H}^1_Q(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho(1))$$

$$\tilde{H}^2_S(G_{F,S}, \mathrm{Ad}^0\bar\rho)^* \simeq H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1))$$

same dimension?

$$\begin{tikzcd}
(\mathrm{Ad}^0\rho(1))^{G_{F,S\cup Q}} \arrow[r] \arrow[d, equal] & \bigoplus_{v\in Q} H^0(G_{F_v}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r] & \tilde{H}^1(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r] & \bigoplus_{v\in Q} H^1(G_{F_v}, \mathrm{Ad}^0\bar\rho(1)) \\
0 \arrow[r] & 0 \arrow[r] & H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r, "\mathrm{Inf}"] & H^1(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho(1))
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
 & & 0 \arrow[d] & & \overbrace{\bigoplus_{v\in Q} H^1(G_{k_v}, \mathrm{Ad}^0\bar\rho(1))}^{t\text{-dim'l}} \arrow[d] \\
(\mathrm{Ad}^0\bar\rho(1))^{G_{F,S\cup Q}} \arrow[r] \arrow[d, equal] & \overbrace{\bigoplus_{v\in Q} H^0(G_{F_v}, \mathrm{Ad}^0\bar\rho(1))}^{t\text{-dim'l}} \arrow[r] & \tilde H^1_Q(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r] \arrow[d, hook] & H^1(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r] \arrow[d, equal] & \bigoplus_{v\in Q} H^1(G_{F_v}, \mathrm{Ad}^0\bar\rho(1)) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r, "\mathrm{Inf}"] & H^1(G_{F,S\cup Q}, \mathrm{Ad}^0\bar\rho(1)) \arrow[r, "\mathrm{Res}"] & \bigoplus_{v\in Q} H^1(I_{F_v}, \mathrm{Ad}^0\bar\rho(1))
\end{tikzcd}$$

$\forall\, k_v \equiv 1 \pmod{\ell^N}$.

$$\bar\rho(\mathrm{Frob}_v) = \begin{pmatrix} \bar\alpha_v & \\ & \bar\alpha_v^{-1} \end{pmatrix}$$

$$\mathrm{Ad}^0\bar\rho(1) = \begin{pmatrix} \bar\alpha_v^2 & \neq 1 & \\ & \bar\alpha_v^{-2} & \neq 1 \\ & & 1 \end{pmatrix}$$

$\cong$

$\mathrm{Ad}^0\bar\rho$

Need to find Q s.t.

$H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1)) \longrightarrow \bigoplus_{v\in Q} H^1(G_{k_v}, \mathrm{Ad}^0\bar\rho(1))$ is injective.

<u>NTS</u>

$\forall$ class $[c] \in H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1))$, $\exists$ place $v \notin S$ s.t.

$\neq 0$

(1) $\forall\, k_v \equiv 1 \bmod \ell^N$

(2) $\bar\rho(\mathrm{Frob}_v)$ has distinct eigenvals

(3) $c(\mathrm{Frob}_v) \notin (\mathrm{Frob}_v - 1)\,\mathrm{Ad}^0\bar\rho$

<u>Lecture 12</u>. <u>Choice of Taylor-Wiles primes</u>

<u>Thm</u>. $\bar\rho : G_{F,S} \to GL_2(\mathbb{F})$ abs. irred.,

$\ell \geq 3$.

<u>Corrections</u>. Whenever talking about deformation w/ fixed determinant, need $\ell \nmid n$

$\Rightarrow \mathrm{Ad}\,\bar\rho = \mathrm{Ad}^0\bar\rho \oplus \mathbb{1}$.

$\mathrm{Ad}^{\mathrm{tr}=0}\bar\rho \longrightarrow \mathrm{Ad}\,\bar\rho / \text{scalar}$

Assume that $\bar\rho\Big|_{G_{F(\sqrt{(-1)^{(\ell-1)/2}\ell})}, S_{F(-)}}$ is abs. irred. $\Leftarrow$ $\mathrm{Im}(\bar\rho)$ contains a conjugate of $SL_2(\mathbb{F}_\ell)$ for $\ell \geq 5$.

$\det\bar\rho = \overline{\chi_{cycl}}$, $\chi = \chi_{cycl}$ lifts $\det\bar\rho$.

Let $t = \tilde h^1_S(G_{F,S}, \mathrm{Ad}^0\bar\rho) = \tilde h^2_S(G_{F,S}, \mathrm{Ad}^0\bar\rho)$

Then for each $N \in \mathbb{Z}_{>0}$, there exists a finite set of places $Q_N = \{v_1^{(N)}, \dots, v_t^{(N)}\}$ s.t. $Q_N \cap S = \phi$

(1) For each $v \in Q_N$, $|k_v| \equiv 1 \mod \ell^N$.

(2) $\bar\rho(\mathrm{Frob}_v) = \begin{pmatrix} \bar\alpha_v & \\ & \bar\alpha_v^{-1} \end{pmatrix}$, $\bar\alpha_v \neq \bar\alpha_v^{-1}$

(3) $R^{\square_S, \chi}_{\bar\rho, Q_N}$ is a quotient of $R^{\square}_{loc}[[x_1, \dots, x_t]]$

$\Updownarrow$

deformation of $\bar\rho : G_{F,S\cup Q_N} \to G_{F,S} \xrightarrow{\bar\rho} GL_2(\mathbb{F})$ framed at S.

$\tilde H^1_S(G_{F,S}, \mathrm{Ad}^0\bar\rho) \xrightarrow{\sim} \tilde H^1_S(G_{F,S\cup Q_N}, \mathrm{Ad}^0\bar\rho)$.

$$\Leftrightarrow \underbrace{H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1))}_{t\text{-dim'l}} \xrightarrow{\sim} \underbrace{\bigoplus_{v\in Q_N} H^1(k_v, \mathrm{Ad}^0\bar\rho(1))}_{t\text{-dim'l}}$$

$\simeq \mathbb{F}$

$\mathbb{Q}(\zeta_{\ell^N})$

$\mathbb{Q}$ $\quad (\mathbb{Z}/\ell^N\mathbb{Z})^\times \ni \phi_p$, $p^{-1} \mapsto$

$\phi_p(\zeta_{\ell^N}) = \phi_{\ell^N}^{p^{-1}}$

Today prove the existence using Chebotarev density.

Need to show for every class $[c] \neq 0$ in $H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1))$, $\exists$ ∞'ly many places v

s.t. (1)' $|k_v| \equiv 1 \mod \ell^N \Leftrightarrow \phi_v \in \mathrm{Gal}(F(\zeta_{\ell^N}))$ (3)'' $\mathrm{res}_v([c]) \neq 0$ in $H^1(k_v, \mathrm{Ad}^0\bar\rho(1))$.

(2)' $\bar\rho(\mathrm{Frob}_v)$ has distinct eigenvalues $\Leftrightarrow \bar\rho(\phi_v) \neq 1$ & order prime to ℓ.

$c \mapsto c(\phi_v)$ $\quad \mathrm{Ad}^0\bar\rho(1)/(\mathrm{Frob}_v - 1)\mathrm{Ad}^0\bar\rho(1)$, $\mathrm{Frob}_v = \phi_v$

$\Leftrightarrow c(\phi_v) \notin (\phi_v - 1)\mathrm{Ad}^0\bar\rho(1)$

Once this is done, get $H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1)) \longrightarrow H^1(k_{v_1}, \mathrm{Ad}^0\bar\rho(1))$

$$\oplus\ H^1(k_{v_2}, \mathrm{Ad}^0\bar\rho(1))$$

$$\oplus\ \cdots$$

Write $D = \bar F^{\ker \mathrm{Ad}^0\bar\rho(1)}$

$$\mathrm{Ad}^0\bar\rho(1) : G_F \longrightarrow GL_2(\mathbb{F})$$

$$G_F \to \mathrm{Gal}(D|F) \to GL_2(\mathbb{F})$$

$$\begin{tikzcd}
 & \bar F \arrow[d, dash] \arrow[dl, dash, bend right, "\ker \mathrm{Ad}^0\bar\rho(1)"'] \\
D \arrow[r, dash] \arrow[d, dash, "\mathrm{Im}\,\mathrm{Ad}^0\bar\rho(1)"'] & D(\zeta_{\ell^N}) \arrow[d, dash] \\
F \arrow[r, dash] & F(\zeta_{\ell^N})
\end{tikzcd}$$

<u>Step 1</u>. $H^1(\mathrm{Gal}(D(\zeta_{\ell^N})|F), \mathrm{Ad}^0\bar\rho(1)) = 0$ [prove later]

<u>Step 2</u>

$$\begin{tikzcd}
 & & H^1(G_{F,S}, \mathrm{Ad}^0\bar\rho(1)) \arrow[d] & & \\
0 \arrow[r] & H^1(\mathrm{Gal}_{D(\zeta_{\ell^N})|F}, \mathrm{Ad}^0\bar\rho(1)^{\cancel{G_{D(\zeta_{\ell^N})}}}) \arrow[r] & H^1(\mathrm{Gal}_F, \mathrm{Ad}^0\bar\rho(1)) \arrow[r, hook] & H^1(\mathrm{Gal}_{D(\zeta_{\ell^N})}, \mathrm{Ad}^0\bar\rho(1))^{\mathrm{Gal}_{D(\zeta_{\ell^N})|F}} \\
 & 0 \arrow[u, phantom, "="] & [c] \neq 0 \arrow[u, phantom, "\in"] & [c] \neq 0. \arrow[u, phantom, "\in"]
\end{tikzcd}$$

<u>Step 3</u>. Pick any non trivial element $\phi \in \mathrm{Gal}(D(\zeta_{\ell^N})|F(\zeta_{\ell^N}))$ that has order rel. prime to ℓ.

$c \in \mathrm{Hom}_{\mathrm{Gal}_{D(\zeta_{\ell^N})|F}}(\mathrm{Gal}_{D(\zeta_{\ell^N})}, \mathrm{Ad}^0\bar\rho(1))$ surj.

τ image not in $(\phi-1)\mathrm{Ad}^0\bar\rho(1)$

Lift it arbitrarily to $\tilde\phi \in \mathrm{Gal}(F|F(\zeta_{\ell^N}))$

Then $\bar\rho(\tilde\phi) = \bar\rho(\phi)$ order rel. prime to ℓ.

For (3), if $c(\tilde\phi) \notin (\phi-1)\mathrm{Ad}^0\bar\rho(1)$, Done!

Suppose $c(\tilde\phi) \in (\phi-1)\mathrm{Ad}^0\bar\rho(1)$, want to take $\tilde\phi' = \tau\tilde\phi$ for $\tau \in \mathrm{Gal}\,D(\zeta_{\ell^N})$

$c(\tilde\phi') = c(\tau) + \tau c(\tilde\phi) = c(\tau) + c(\tilde\phi)$.

$$\mathrm{Gal}_{D(\mathfrak{Z}_{\ell N})|F} \longrightarrow \mathrm{Gal}_{D|F} \longrightarrow \mathrm{Aut}(\mathrm{Ad}^0\bar\rho(1))$$

$$\wr\wr$$

$$SL_2(\mathbb{F}_\ell) \longrightarrow \mathrm{Aut}\big(\underbrace{\mathrm{End}^0(\mathbb{F}^{\oplus 2})}_{\text{irred.}}\big)$$

<u>Step 1</u>. $H^1\big(\mathrm{Gal}_{D(\mathfrak{Z}_{\ell N})|F}, \mathrm{Ad}^0\bar\rho(1)\big) = 0$

$$0 \to H^1(\mathrm{Gal}(D|F), \mathrm{Ad}^0\bar\rho(1)) \to H^1(\mathrm{Gal}(D(\mathfrak{Z}_{\ell N})|F), \mathrm{Ad}^0\bar\rho(1)) \to H^1(\mathrm{Gal}(D(\mathfrak{Z}_{\ell N})|D), \mathrm{Ad}^0\bar\rho(1))^{\mathrm{Gal}_{D|F}}$$

$$\mathrm{Gal}_F \xrightarrow{\mathrm{Ad}^0\bar\rho(1)} GL_3(\mathbb{F})$$

$$\searrow \quad \mathrm{Gal}(D|F) \quad \nearrow$$

$$\mathrm{Hom}_{\mathrm{Gal}_{D|F}}(\underbrace{\mathrm{Gal}(D(\mathfrak{Z}_{\ell N})|D)}_{\text{abel gp}}, \underbrace{\mathrm{Ad}^0\bar\rho(1)}_{\text{irred. rep. of } \mathrm{Gal}_{D|F}}) = 0$$

$\mathrm{Gal}(D|F)$ = projective image of $\bar\rho$

$\approx PSL_2(\mathbb{F})$

$$H^1(PSL_2(\mathbb{F}), \mathrm{End}^0(\mathbb{F}^{\oplus 2})) \overset{?}{=} 0.$$

$$= H^1\left(\begin{pmatrix}\mathbb{F}^\times & \mathbb{F} \\ & \mathbb{F}^\times\end{pmatrix}/\mathbb{F}^\times, \mathrm{End}^0(\mathbb{F}^{\oplus 2})\right) = H^1\left(\begin{pmatrix}1 & \mathbb{F} \\ 0 & 1\end{pmatrix}, \mathrm{End}^0(\mathbb{F}^{\oplus 2})\right)^{\begin{pmatrix}\mathbb{F}^\times & \\ & \mathbb{F}^\times\end{pmatrix}}$$

$$\left(\mathbb{F}\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\right)^{\begin{pmatrix}\mathbb{F}^\times & \\ & \mathbb{F}^\times\end{pmatrix}} = 0.$$

<u>Local deformation conditions</u>.

- $E|\mathbb{Q}_\ell$, $E \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \simeq \mathbb{F}$
- Γ profinite gp satisfying the finiteness cond. $(*)$.
- $\bar\rho : \Gamma \longrightarrow GL_n(\mathbb{F})$ residual rep'n

$\rightsquigarrow R^\square_{\bar\rho}$ = universal deformation ring

$\mathrm{Spf}\, R^\square_{\bar\rho} \circlearrowleft \widehat{PGL_n}$

Def A <u>local cond.</u> is a $\widehat{PGL}_n$- stable (closed) ideal I of $R^{\square}_{\bar\rho}$.

(or equivalently sub formal scheme $\mathrm{Spf}(R^{\square}_{\bar\rho}/I) \hookrightarrow \mathrm{Spf}\, R^{\square}_{\bar\rho}$ that is stable under $\widehat{PGL}_n$ - action)

① <u>Pointwise version</u>. Every element $a \in \widehat{PGL}_n(R^{\square}_{\bar\rho}) = \dfrac{(1+M_n(m^{\square}_{\bar\rho}))^\times}{(1+m^{\square}_{\bar\rho})^\times}$

$a\rho^{univ}a^{-1}$: a different lift of $\bar\rho$.

$\rightsquigarrow$ $f_a : R^{\square}_{\bar\rho} \longrightarrow R^{\square}_{\bar\rho}$ defines an action of $\widehat{PGL}_n(R^{\square}_{\bar\rho})$ on $R^{\square}_{\bar\rho}$.

<u>Pointwise stability cond</u>: for all such a, $f_a(I) \subset I$.

② <u>Scheme theoretic version</u>. $\mathrm{act}: \widehat{PGL}_n \times \mathrm{Spf}\, R^{\square}_{\bar\rho} \longrightarrow \mathrm{Spf}\, R^{\square}_{\bar\rho}$

$\rightsquigarrow$ $\mathrm{act}^* : R^{\square}_{\bar\rho} \longrightarrow R^{\square}_{\bar\rho} \hat\otimes \mathcal{O}_{\widehat{PGL}_n}$

<u>Condition</u> $\mathrm{act}^*(I) \subset I \hat\otimes \mathcal{O}_{\widehat{PGL}_n}$.

② $\Rightarrow$ ① Given an elt $a \in \widehat{PGL}_n(R^{\square}_{\bar\rho}) \Leftrightarrow a^* : \mathcal{O}_{\widehat{PGL}_n} \longrightarrow R^{\square}_{\bar\rho}$.

<u>Rmk</u>. ① $\Rightarrow$ ② needs $R^{\square}_{\bar\rho}$ to be reduced.

$$\begin{tikzcd}
\widehat{PGL}_n \times \mathrm{Spf}\, R^{\square}_{\bar\rho} \arrow[r] & \mathrm{Spf}\, R^{\square}_{\bar\rho} \\
\mathrm{Spf}\, R^{\square}_{\bar\rho} \arrow[u, "a"] \arrow[r, "\Delta"] \arrow[ur, bend right, "\text{action by } a"'] & \mathrm{Spf}\, R^{\square}_{\bar\rho} \times \mathrm{Spf}\, R^{\square}_{\bar\rho} \arrow[u]
\end{tikzcd}$$

$f_a^*(I) = \Delta^*((a\otimes 1)^*(\mathrm{act}^*(I)))$

<u>Typical example</u>: $\bar\rho : G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to GL_2(\mathbb{F})$

$\phi \longmapsto \begin{pmatrix} 1 & \\ & p \end{pmatrix}$

$p \not\equiv 1 \bmod \ell$

$R^{\square}_{\bar\rho} \simeq \mathcal{O}[[X,Y,S,T,B]]/(B\cdot(T-pS))$

looks like $\mathcal{O}[[x,y]]/(xy)$

$\rho^{univ}(\phi_p) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+S & \\ & p+T \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

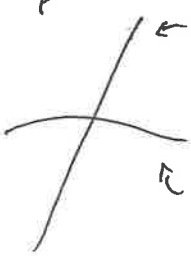
$\rho^{univ}(\tau) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

$\phi_p^{-1} \tau \phi_p = \tau^p$ $\quad \begin{pmatrix} (1+S)^{-1} & \\ & (p+T)^{-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} \begin{pmatrix} 1+S & \\ & p+T \end{pmatrix} = \begin{pmatrix} 1 & pB \\ & 1 \end{pmatrix}$

$\frac{p+T}{1+S} B = pB \quad \Rightarrow \quad B(T-pS) = 0$

$\widehat{PGL_2}$-action on $\operatorname{Spf} R^{\square}_{\bar\rho}$ is purely on X, Y var. & possibly scales B.

$\operatorname{Spf} R^{\square}_{\bar\rho}$



$\leftarrow B=0 \leftarrow R^{\square}_{\bar\rho}/(B) = R^{\square, ur}_{\bar\rho} \longleftrightarrow$ mod. forms w level N, $p \nmid N$

$T = pS$ & typically $B \neq 0$

Steinberg $\longleftrightarrow$ mod. forms w level pN, $p \nmid N$.

$R^{\square}_{\bar\rho}/\underline{(T-pS)} = R^{\square, St}_{\bar\rho}$ & it is <u>p-new</u>.

two Frob. eig. vals have ratio $= p$

<u>Lecture 13</u>. <u>Last time</u>. $\bar\rho : \Gamma \longrightarrow GL_n(\mathbb{F})$ residual rep'n

$\rightsquigarrow R^{\square}_{\bar\rho}$, $\operatorname{Spf} R^{\square}_{\bar\rho} \circlearrowleft \widehat{PGL_n}$

<u>Def</u> A <u>local cond.</u> is a $\widehat{PGL_n}$-stable ideal I of $R^{\square}_{\bar\rho}$.

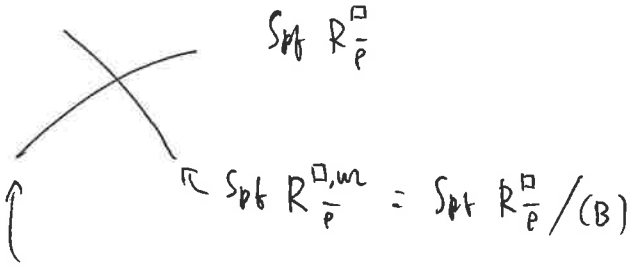
<u>Example</u> $\bar\rho : G_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{F}_p} \longrightarrow GL_2(\mathbb{F})$

$\phi_p \longmapsto \begin{pmatrix} 1 & \\ & p \end{pmatrix}$

$\rho^{univ}(\phi_p) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+S & \\ & p+T \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

$\rho^{univ}(\tau) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

$R^{\square}_{\bar\rho} = \mathcal{O}[\![X, Y, S, T, B]\!]/(B(T-pS))$.



<u>Rmk</u> $\mathrm{Spec}\left(R^{\square}_{\bar\rho}[\tfrac{1}{\ell}]\right)(\bar{\mathbb{Q}}_\ell) = \mathrm{Spec}\left(R^{\square,m}_{\bar\rho}[\tfrac{1}{\ell}]\right)(\bar{\mathbb{Q}}_\ell) \cup \mathrm{Spec}\left(R^{\square,st}_{\bar\rho}[\tfrac{1}{\ell}]\right)(\bar{\mathbb{Q}}_\ell)$

$$\begin{array}{ccc} \mathrm{Spec}\left(R^{\square}_{\bar\rho}[\tfrac{1}{\ell}]\right)(\bar{\mathbb{Q}}_\ell) & = & \{(x,y,s,t) \in \bar{\mathbb{Q}}_\ell^4 : |x|_\ell<1, |y|_\ell<1, |s|_\ell<1, |t|_\ell<1\} \\ \| & & \| \\ \{(x,y,s,t) \in \bar{\mathbb{Q}}_\ell^4 : |x|_\ell<1, |y|_\ell<1, |s|_\ell<1, |t|_\ell<1\} & & \end{array}$$

<u>Def</u> A <u>deformation problem</u> $\mathcal{D}$ is a collection $\{(A,\rho) : A \in CNL_{\mathcal{O}},\ \rho : \Gamma \to GL_n(A)$ lifting $\bar\rho\}$

(e.g. $\mathcal{D}^{st}_{m} = \{(A,\rho) : \rho \in \mathrm{Def}^{\square}_{\bar\rho}(A),\ \rho$ st unram.$\}$)

s.t. (1) $(\mathbb{F}, \bar\rho) \in \mathcal{D}$

(2) $f : A \to B$ a morphism in $CNL_{\mathcal{O}}$, $(A,\rho) \in \mathcal{D} \Rightarrow (B, f\circ\rho) \in \mathcal{D}$.

(i.e. $\mathrm{Def}^{\square,\mathcal{D}}_{\bar\rho}(A) = \{\rho : (A,\rho) \in \mathcal{D}\} \subset \mathrm{Def}^{\square}_{\bar\rho}(A)$ is a subfunctor)

(3) $f : A \hookrightarrow B$ injective mor. in $CNL_{\mathcal{O}}$, then $(A,\rho) \in \mathcal{D} \Leftrightarrow (B, f\circ\rho) \in \mathcal{D}$.

(4) $A_1, A_2 \in CNL_{\mathcal{O}}$ w/ ideals $\underset{A_1}{\overset{\cap}{I_1}}, \underset{A_2}{\overset{\cap}{I_2}}$, $(A_1,\rho_1), (A_2,\rho_2) \in \mathcal{D}$. Suppose that there's an isom. $f : A_1/I_1 \xrightarrow{\sim} A_2/I_2$

and $f(\rho_1 \bmod I_1) = \rho_2 \bmod I_2$

Then $(A_1 \times_{A_1/I_1} A_2,\ \rho_1 \oplus \rho_2) \in \mathcal{D}$

$\| $

$\{(a_1,a_2) \in A_1 \times A_2 \text{ s.t. } f(a_1 \bmod I_1) = a_2 \bmod I_2\}$

(5) If $I_1 \supset I_2 \supset \cdots$ is a nested sequence of ideals of $A \in CNL_{\mathcal{O}}$ s.t. $\bigcap_i I_i = \{0\}$,

and if (A, ρ) lifts $\bar{\rho}$ s.t. $(A/I_i, \rho \bmod I_i) \in \mathcal{D}$, then $(A, \rho) \in \mathcal{D}$.

(6) $(A, \rho) \in \mathcal{D}$ and $a \in \widehat{PGL}_n(A) = \dfrac{(1+M_n(\mathfrak{m}_A))^\times}{(1+\mathfrak{m}_A)^\times}$, then $(A, a\rho a^{-1}) \in \mathcal{D}$

<u>Thm</u> There is a bijection

$$\left\{ \begin{array}{c} \text{scheme-theoretic } \widehat{PGL}_n\text{-stable} \\ \text{ideals } I \subset R^{\square}_{\bar{\rho}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Deformation problem } \mathcal{D} \\ \text{for } \bar{\rho} : \Gamma \to GL_n(\mathbb{F}) \end{array} \right\}$$

if I is radical

$$I \longmapsto \mathcal{D}(I) = \left\{ (A, \rho) : A \in CNL_{\mathcal{O}},\ \rho : \Gamma \to GL_n(A) \text{ lifting } \bar{\rho} \right\}$$

$$I(\mathcal{D}) \longleftarrow\!\shortmid \mathcal{D}$$

w/ assoc. hom.

$$\begin{tikzcd} R^{\square}_{\bar{\rho}} \arrow[r, "t_\rho"] \arrow[dr, two heads] & A \\ & R^{\square}_{\bar{\rho}}/I \arrow[u] \end{tikzcd}$$

$$\left\{ \widehat{PGL}_n(R^{\square}_{\bar{\rho}}) \text{-stable ideals } I \subset R^{\square}_{\bar{\rho}} \right\}$$

<u>Check (4)</u>

$$\begin{tikzcd} & & A_1 \arrow[r, two heads] & A_1/I_1 \\ R^{\square}_{\bar{\rho}} \arrow[r, two heads] & R^{\square}_{\bar{\rho}}/I \arrow[ur, "t_{\rho_1}"] \arrow[r, dashed] \arrow[dr, "t_{\rho_2}"'] & A_1 \times_{A_1/I_1} A_2 \arrow[u] \arrow[d] & \wr\| \\ & & A_2 \arrow[r, two heads] & A_2/I_2 \end{tikzcd}$$

$\mathrm{Def}^{\square,\mathcal{D}}_{\bar{\rho}}(A) = \{\rho : (A, \rho) \in \mathcal{D}\}$ is rep'd by $R^{\square}_{\bar{\rho}}/I(\mathcal{D})$.

(6) $(A, \rho) \in \mathcal{D} \rightsquigarrow$

$$\begin{tikzcd} R^{\square}_{\bar{\rho}} \arrow[r] \arrow[rr, bend right, "t_\rho"'] & R^{\square}_{\bar{\rho}}/I \arrow[r] & A \end{tikzcd}$$

<u>$a\rho a^{-1}$</u> ? We can construct $t_{a\rho a^{-1}}$ as follows:

$$\begin{tikzcd} R^{\square}_{\bar{\rho}} \arrow[r, "act^*"] \arrow[rrr, bend right, "t_{a\rho a^{-1}}"'] & U_{\widehat{PGL}_n} \hat{\otimes} R^{\square}_{\bar{\rho}} \arrow[r, "a^* \otimes t_\rho"] & A \hat{\otimes} A \arrow[r, "mult"] & A \end{tikzcd}$$

Conversely, given a deformation problem $\mathcal{D}$, define

$$\mathcal{I} = \{\text{all ideals } I \subset R^{\square}_{\bar\rho} \text{ s.t. } (R^{\square}_{\bar\rho}/I,\ \rho^{univ} \bmod I) \in \mathcal{D}\}$$

<u>Goal</u>. Will take $I(\mathcal{D})$ <u>the</u> minimal member of $\mathcal{I}$.

<u>Check</u> $\mathcal{I}$ is closed under finite intersection.

Say $I_1, I_2 \in \mathcal{I}$, then $(R^{\square}_{\bar\rho}/I_1,\ \rho^{univ} \bmod I_1), (R^{\square}_{\bar\rho}/I_2,\ \rho^{univ} \bmod I_2) \in \mathcal{D}$.

<u>Note</u>
$$0 \to R^{\square}_{\bar\rho}/I_1 \cap I_2 \to R^{\square}_{\bar\rho}/I_1 \oplus R^{\square}_{\bar\rho}/I_2 \to R^{\square}_{\bar\rho}/I_1 + I_2 \to 0$$

Thus, $R^{\square}_{\bar\rho}/I_1 \cap I_2 = R^{\square}_{\bar\rho}/I_1 \times_{R^{\square}_{\bar\rho}/I_1+I_2} R^{\square}_{\bar\rho}/I_2$ By (4) ✓

<u>Global deformation ring w/ local conditions</u>

F no. field, S finite set of places $\supset \{v \mid \ell\infty\}$ $T \subset S$, $\ell \nmid n$

Let $\bar\rho : G_{F,S} \to GL_n(\mathbb{F})$ be an abs. irred. rep'n and $\chi : G_{F,S} \to \mathcal{O}^\times$ lifting $\det\bar\rho$.

At $v \in T$, specify a local deformation problem $\mathcal{D}_v$ for $\bar\rho_v : G_{F_v} \to GL_n(\mathbb{F})$

<u>Define</u>, $\mathrm{Def}^{\square_T, \chi, \mathcal{D}}_{\bar\rho} : CNL_{\mathcal{O}} \to \mathrm{Set}$

$$A \mapsto \left\{ (\rho, (h_v)_{v\in T}) : \begin{array}{l} \rho : G_{F,S} \to GL_n(A) \text{ cts rep'n lifting } \bar\rho, \\ \det\rho = \chi,\ h_v \in \widehat{PGL}_n(A),\ \forall v \in T \\ (A, \rho|_{G_{F_v}}) \in \mathcal{D}_v,\ \forall v \in T \end{array} \right\} \Big/ \sim$$

$\leftrightarrow (A, h_v \rho|_{G_{F_v}} h_v^{-1}) \in \mathcal{D}_v$

Then $\mathrm{Def}^{\square_T, \chi, \mathcal{D}}_{\bar\rho}$ is rep'd by $R^{\square_T, \chi, \mathcal{D}}_{\bar\rho}$

& we have a Cart. diagram

$$\begin{tikzcd}
\mathrm{Spf}\, R_{\bar\rho}^{\square_T, \chi, D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spf}\, R_{\bar\rho}^{\square_T, \chi} \arrow[d] \\
\prod_{v \in T} \mathrm{Spf}\, R_{\bar\rho_v}^{\square, \chi, D_v} \arrow[r] & \prod_{v \in T} \mathrm{Spf}\, R_{\bar\rho_v}^{\square, \chi}
\end{tikzcd}$$

$$R_{\bar\rho_v}^{\square, \chi, D_v} = R_{\bar\rho_v}^{\square, \chi} / I(D_v)$$

<u>Key Cor</u>: Set $t = \tilde{h}_T^1(G_{F,S}, \mathrm{Ad}^0 \bar\rho)$. Then for <u>any</u> local deformation problem D_v, $v \in T$,

$$R_{\bar\rho}^{\square_T, \chi, D} \simeq \frac{\hat{\bigotimes}_{v \in T} R_{\bar\rho_v}^{\square, \chi, D_v} [\![x_1, \dots, x_t]\!]}{(f_1, \dots, f_s)} \quad \text{w/ } s = \tilde{h}_T^2(G_{F,S}, \mathrm{Ad}^0 \bar\rho).$$

Moreover, can use the same Taylor-Wiles <u>primes</u>.

§ Local deformation problems when $\ell \neq p$

- K is a fin. ext'n of $\mathbb{Q}_p$, $\ell \neq p$. $q = |k_K|$.

<u>Grothendieck's ℓ-adic monodromy thm</u>

Fix geom. Frob ϕ_K, isom $\iota_\ell : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, and a compatible system $(\zeta_{\ell^n})_n$ of ℓ^nth roots of unity.

$$\left\{ \begin{array}{l} \ell\text{-adic cts rep'n} \\ \rho : W_K \to GL_n(\overline{\mathbb{Q}_\ell}) \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{ll} \text{Weil-Deligne rep'n } (r, N, V)/\mathbb{C} & \\ r : W_K \to GL(V) & (\Rightarrow r(I_K) \text{ is finite}) \\ N \in \mathrm{End}(V) & \text{s.t. } r(\phi_K)^{-1} N r(\phi_K) = q \cdot N \end{array} \right\}$$

$$\rho \longmapsto N = \frac{1}{m} \log \rho(\tau^m) \quad \text{for } \tau \text{ a tame generator, and } m \text{ sufficiently divisible.}$$

$$r(\phi_K^m x) = \iota_\ell(\rho(\phi_K^m x) \exp(-t_\zeta(x) \cdot N)) \quad m \in \mathbb{Z},\ x \in I_K$$

The <u>inertia type</u> of ρ is the pair $(\overset{\psi}{\overset{\shortparallel}{\tau|_{I_K}}}, N)$

<u>Eg.</u> $\rho^{univ}(\phi) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+S & \\ & p+T \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

$\rho^{univ}(\tau) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$

Fix a pt $x \in \operatorname{Spec} R^{\square}_{\bar\rho}[\frac{1}{\ell}]\,(\overline{\mathbb{Q}}_\ell)$

ρ_x — $x \in$ unram. comp., $\rightsquigarrow N = 0$, τ is unram.

type = unramified

— if $Bx \neq 0$, $X = Y = 0$. $N = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. τ unram.

$\tau(\phi_K) = \begin{pmatrix} 1+S & \\ & p+pS \end{pmatrix}$

<u>Big Theorem</u> Let $\bar\rho : \mathrm{Gal}_K \to GL_n(\mathbb{F}) \rightsquigarrow R^{\square}_{\bar\rho}$, $X^{\square}_{\bar\rho,E} = \operatorname{Spec} R^{\square}_{\bar\rho}[\frac{1}{\ell}]$

For each <u>inertia type</u> (ψ, N), consider the subset

$$X^{\square,(\psi,N),o}_{\bar\rho,E} = \left\{ x \in \operatorname{Spec} R^{\square}_{\bar\rho}[\tfrac{1}{\ell}]\,(\overline{\mathbb{Q}}_\ell) : \rho^{univ} : \mathrm{Gal}_K \to GL_n(R^{\square}_{\bar\rho}) \xrightarrow{x} GL_n(\overline{\mathbb{Q}}_\ell) \text{ has inertia type } (\psi, N) \right\}$$

<u>Then</u> (1) each $X^{\square,(\psi,N),o}_{\bar\rho,E}$ is a locally closed subspace of $X^{\square}_{\bar\rho,E}$,

(2) Its closure $X^{\square,(\psi,N)}_{\bar\rho,E}$ is <u>smooth</u> & irred. of dim n^2 over E.

<u>Cor.</u> Krull dim $R^{\square}_{\bar\rho}$ is n^2+1.

Consider an E-pt of $X^{\square}_{\bar\rho,E}$

$\rightsquigarrow \rho_x : \mathrm{Gal}_K \xrightarrow{\rho^{univ}} GL_n(R^{\square}_{\bar\rho}) \xrightarrow{x^*} GL_n(\mathcal{O}_E) \subset GL_n(E)$

$x^*(m^{\square}_{\bar\rho}) \subset m_E$ (a), so ρ_x is a cts rep'n.

$CNL_E = \{$complete noeth. local E-algebra A s.t. $E \xrightarrow{\sim} A/m_A\}$

Consider $\mathrm{Def}^{\square}_{\rho_x} : CNL_E \longrightarrow \mathrm{Set}$

$$A \longmapsto \{\rho : \mathrm{Gal}_K \longrightarrow GL_n(A) \text{ s.t. } \rho \bmod m_A = \rho_x\}$$

<u>Thm</u> The completion $R^{\square}_{\bar\rho}[\frac{1}{\ell}]^{\wedge}_x$:= completion of $R^{\square}_{\bar\rho}[\frac{1}{\ell}]$ at the max'l ideal det. by x represents the functor $\mathrm{Def}^{\square}_{\rho_x}$.

<u>Proof</u> $\mathrm{Hom}_{CNL_E}(R^{\square}_{\bar\rho}[\frac{1}{\ell}]^{\wedge}_x, A) = \mathrm{Def}^{\square}_{\rho_x}(A) = \{\rho : \mathrm{Gal}_K \xrightarrow{\text{cts}} GL_n(A) \text{ lifting } \rho_x\}$.

- Given a homomorphism $R^{\square}_{\bar\rho}[\frac{1}{\ell}]^{\wedge}_x \longrightarrow A$ (mapping m_x to m_A)

$$\rho : \mathrm{Gal}_K \longrightarrow GL_n(R^{\square}_{\bar\rho}) \longrightarrow GL_n(R^{\square}_{\bar\rho}[\tfrac{1}{\ell}]^{\wedge}_x) \longrightarrow GL_n(A)$$

- Conversely, given $\rho : \mathrm{Gal}_K \longrightarrow GL_n(A)$ lifting ρ_x.

$$\begin{tikzcd} \mathrm{Gal}_K \arrow[r] \arrow[dr, bend right, "\rho_r"'] & GL_n(A) \arrow[d] \\ & GL_n(\underbrace{A/m_A^r}_{A_r}) \end{tikzcd}$$

<u>Def</u> A_r° := $\mathcal{O}$-subalg. gen. by entries of $\rho_r(\gamma)$ for $\gamma \in \mathrm{Gal}_K$.

<u>Note</u> A_r is fin. dim'l $/E$ & Gal_K is cpt

$\Rightarrow A_r^{\circ}$ is a finite $\mathcal{O}$-module

$$\begin{tikzcd} \rho_r^{\circ} : \mathrm{Gal}_K \arrow[r] \arrow[dr, bend right, "\bar\rho"'] \arrow[drr, bend left, "\rho_x"] & GL_n(A_r^{\circ}) \arrow[r] \arrow[d] & GL_n(A_r) \arrow[d, "\bmod m_A"'] \\ & GL_n(A_r^{\circ}/m_A \cap A_r^{\circ}) \arrow[r, hook] \arrow[d, "\bmod \varpi"] & GL_n(A/m_A) \\ & GL_n(\mathbb{F}) & E \arrow[u, equal] \end{tikzcd}$$

So $R^{\square}_{\bar\rho} \longrightarrow A_r^{\circ}$

$\rightsquigarrow R^{\square}_{\bar\rho}[\frac{1}{\ell}] \to A_r^{\circ}[\frac{1}{\ell}] \to E$

Composition sends m_x to m_A

$\Rightarrow R^{\square}_{\bar\rho}[\frac{1}{\ell}]^{\wedge}_x \longrightarrow A_r$

Take inverse lim over r.

Cor. A pt $x \in \operatorname{Spec} R_{\bar\rho}^{\square}[\frac{1}{\ell}](E)$ is a smooth pt

if $H^2(\mathrm{Gal}_K, \mathrm{Ad}\rho_x) = 0$ $\iff$ $\mathrm{Ad}\rho_x(1)^{\mathrm{Gal}_K} = 0$

Example Say $\bar\rho : \mathrm{Gal}_K \to \mathrm{Gal}_{k_K} \to GL_n(\mathbb{F})$ is unram.

$$\phi_K \longmapsto \begin{pmatrix} \bar\alpha_1 & & \\ & \ddots & \\ & & \bar\alpha_n \end{pmatrix} \qquad \text{Say } \bar\alpha_i \neq \bar\alpha_j, \text{ if } i \neq j$$

An unram. lift of $\bar\rho$ to E-coeff. rep'n

$$\rho_x(\phi_K) \sim \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$$

Then $\mathrm{Ad}\rho_x(1)(\phi_K)$ has eigenvalues $\alpha_i \alpha_j^{-1} q^{-1}$

So $\mathrm{Ad}\rho_x(1)^{\mathrm{Gal}_K}$ is nonzero iff some $\alpha_i/\alpha_j = q$.

(& in this case, there's a Steinberg component through x).

If say $\bar\alpha_1 : \bar\alpha_2 : \bar\alpha_3 = 1 : q_v : q_v^2$ mod ϖ_E

& <u>Steinberg lift</u> $\rho_x(\phi_K) = \begin{pmatrix} 1 & & \\ & q & \\ & & q^2 \end{pmatrix}$, $\rho_x(\tau_K) = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

<u>Q</u>: $\mathrm{Ad}\rho_x(1)^{\mathrm{Gal}_K} \overset{?}{=} 0$

$r_x : \mathrm{Gal}_{\mathbb{Q}_p} \to GL_2(E)$

$\phi_K \longmapsto \begin{pmatrix} 1 & \\ & q \end{pmatrix}$ $\qquad \rho_x \simeq \mathrm{Sym}^2 r_x$

$\tau_K \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

<u>ϕ_K - eigvals:</u>

$\vdots$

$\Rightarrow \mathrm{Ad}\rho_x(1)^{\mathrm{Gal}_K} = 0.$

$$\mathrm{Ad}\rho_x(1) \simeq \mathrm{Sym}^4 r_x \otimes \det{}^{-2}(1) \oplus \mathrm{Sym}^2 \otimes \det{}^{-1}(1) \oplus E(1)$$

Lecture 14

§1. Quick introduction to p-adic Hodge theory

- Let X be a proper smooth var. of dim d over a field K.

$\rightsquigarrow$ $\Omega^1_{X/K}$ = sheaf of differentials, locally free of rk d

$\Omega^i_{X/K} := \Lambda^i_{\mathcal{O}_X} \Omega^1_{X/K}$, rank $\binom{d}{i}$

<u>De Rham cpx</u> $\Omega^\bullet_{X/K} := \left[\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/K} \xrightarrow{d} \Omega^2_{X/K} \to \cdots \right]$

Define the <u>algebraic de Rham coh.</u> to be $H^*_{dR}(X/K) := \mathbb{H}^*(X, \Omega^\bullet_{X/K})$.

<u>Spectral seq.</u>

$$E_1^{pq} = \begin{array}{|l} H^2(X, \mathcal{O}_X) \to \\ H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_X) \to H^1(X, \Omega^2_X) \\ H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X) \end{array} \Rightarrow H^*_{dR}(X/K)$$

<u>Fact</u>. When Char $K = 0$, this s.s. degenerates at E_1

$$\Rightarrow \text{e.g. } H^2_{dR}(X/K) = \underset{\substack{\uparrow \\ \text{sub}}}{H^0(X, \Omega^2_{X/K})} - H^1(X, \Omega^1_{X/K}) - \underset{\substack{\uparrow \\ \text{quotient}}}{H^2(X, \mathcal{O}_X)}$$

or in another way,

$$\mathrm{Fil}^i H^2_{dR}(X/K) = \begin{cases} H^2_{dR}(X/K) & i \le 0 \\ H^0(X, \Omega^2_{X/K}) - H^1(X, \Omega^1_{X/K}), & i = 1 \\ H^0(X, \Omega^2_{X/K}) & i = 2 \\ 0 & i > 2 \end{cases}$$

In general, we have a decreasing filtration $\mathrm{Fil}^i H^n_{dR}(X/K) = \mathbb{H}^n(X, \Omega_X^{\geq i})$

If $K = \mathbb{R}$, X proper sm. var. $/\mathbb{R}$, we have an isom.

$$\begin{array}{ccccc} H^i_B(X(\mathbb{C})^{an}, \mathbb{Q}) & \otimes_{\mathbb{Q}} & \mathbb{C} & \simeq & H^i_{dR}(X/\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\ \circlearrowleft c & & \circlearrowleft c & & \circlearrowleft c \end{array} \qquad (*_{\mathbb{C}})$$

$\mathrm{Gal}(\mathbb{C}|\mathbb{R}) = \{1, c\}$

$\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ - equiv. isom.

- Now, let K be a CDVF of char. 0, s.t. the residue field k is perfect of char. $p > 0$.

($\mathcal{O}_K$ = ring of integers)

$\tilde{K} \simeq \mathbb{C}$

(Grothendieck - Fontaine) $\quad H^i_B(X(\mathbb{C})^{an}, \mathbb{Q}_p)$

$$\begin{array}{ccccccc} & & \wr\| & & & & \\ H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) & \otimes_{\mathbb{Q}_p} & ? & \simeq & H^i_{dR}(X/K) & \otimes_K & ? \\ \circlearrowleft & & \mathbb{B}_{dR} & & \uparrow & & \mathbb{B}_{dR} \\ \mathrm{Gal}_K & & \circlearrowleft \mathrm{Gal}_K \ \mathrm{Fil} & & \mathrm{Fil} & & \circlearrowleft \mathrm{Gal}_K \ \mathrm{Fil} \end{array} \qquad (*)_{dR}$$

$$H^i_{dR}(X/K) \simeq \left(H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR} \right)^{\mathrm{Gal}_K}$$

Fontaine constructed $\mathbb{B}_{dR}$, comparison thm is proved by Faltings, Tsuji, Scholze, ...

- Assume that K is abs. unramified, i.e. p is a uniformizer. e.g. $K = \mathbb{Q}_p$

and that X has a good reduction, i.e. $\exists\ \mathcal{X}$ proper smooth $/\mathcal{O}_K$ s.t. $\mathcal{X}_K \simeq X$

$$\begin{array}{ccccccc} H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) & \otimes_{\mathbb{Q}_p} & \mathbb{B}_{crys} & \simeq & H^i_{cris}(\mathcal{X}_k/\mathcal{O}_K)[\tfrac{1}{p}] & \otimes_K & \mathbb{B}_{crys} \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \mathrm{Gal}_K & & \mathrm{Gal}_K, \phi & & \phi & & \mathrm{Gal}_K, \phi \end{array}$$

$(*_{cris})$

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_k \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spa}\, K & \longrightarrow & \mathrm{Spa}\, \mathcal{O}_K & \longleftarrow & \mathrm{Spa}\, k \end{array}$$

What is $H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K)$?

- $H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \simeq H^i_{dR}(X/K)$
- $H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K)$ depends intrinsically only on $\mathfrak{X}_k$
- $H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K)$ carries an action of Frob. ϕ.

<u>Fact</u> $\mathbb{B}_{cris} \subset \mathbb{B}_{dR}$

<u>Fact</u> $(\mathbb{B}_{dR})^{Gal_K} = (\mathbb{B}_{cris})^{Gal_K} = K$

<u>Fact</u> $Fil^0(\mathbb{B}_{cris}^{\phi=1}) = \mathbb{Q}_p$

<u>Combine the two comparisons</u>.

$$\underset{\substack{\cup \\ Gal_K}}{H^i_{\acute{e}t}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p)} \otimes_{\mathbb{Q}_p} \underset{\substack{\cup \\ Gal_K,\phi \quad Fil}}{\mathbb{B}_{cris}} \simeq \underset{\substack{\cup \\ \phi,\ Fil}}{H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K)[\tfrac{1}{p}]} \otimes_K \underset{\substack{\cup \\ Gal_K,\phi \quad Fil}}{\mathbb{B}_{cris}}$$

If we take Gal_K - inv. $\rightsquigarrow$ $H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K) = \left(H^i_{\acute{e}t}(\mathfrak{X}_{\bar{K}}; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \overset{\phi,\ Fil}{\mathbb{B}_{cris}}\right)^{Gal_K}$

If we take ϕ-inv
Fil^0 $\rightsquigarrow$ $H^i_{\acute{e}t}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p) \simeq \left(H^i_{cris}(\mathfrak{X}_k/\mathcal{O}_K)[\tfrac{1}{p}] \otimes_K \underset{\substack{\cup \\ Gal_K}}{\mathbb{B}_{cris}}\right)^{\phi=1,\ Fil^0}$

<u>Upshot</u> In this case, étale coh. & crystalline + de Rham coh. determine each other.

§2. Definition of p-adic period rings

- K CDVF, w/ res. field k perfect of char $p>0$
  char $K=0$
- $\mathbb{C}_p := \widehat{\overline{K}^{alg}}$ > $\mathcal{O}_{\mathbb{C}_p} := \underbrace{\varprojlim_{\phi} \frac{\mathcal{O}_{\mathbb{C}_p}}{p\cdot\mathcal{O}_{\mathbb{C}_p}}}_{\text{ring str. given by termwise } +,\times} = \left\{ (x_0, x_1, \cdots) : \begin{array}{l} x_i \in \mathcal{O}_{\mathbb{C}_p}/p \\ x_i^p = x_{i-1} \end{array} \right\}$ $\boxed{\text{tilting process}}$

Set $\mathcal{O}^b_{\mathbb{C}_p}$ (Fontaine)

$(1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \cdots) =: \varepsilon$. ζ_{p^n} primitive p^n-th root of unity. $\zeta_{p^n}^p = \zeta_{p^{n-1}}$

$\mathcal{O}_{\mathbb{C}_p}^b$: – perfect $\quad (x_0, x_1, \cdots) = (x_1, x_2, \cdots)^p$

– valuation $\quad v_p^b(x_0, x_1, \cdots) := p^m v_p(x_m)$ picking m s.t. $x_m \notin p\mathcal{O}_{\mathbb{C}_p}$.

Eg $\varepsilon - 1 = (0, \zeta_p - 1, \zeta_{p^2} - 1, \cdots)$

$$v_p^b(\varepsilon - 1) = p \cdot v_p(\zeta_p - 1) = \frac{p}{p-1}$$
$$= p^2 \cdot v_p(\zeta_{p^2} - 1)$$

– char. p.

– complete

$$\mathcal{O}_{\mathbb{C}_p}^b \simeq \underbrace{\varprojlim_{p} \mathcal{O}_{\mathbb{C}_p}}_{\text{no explicit ring str.}} \qquad (x_0, x_1, \cdots) \mapsto (x^{(0)}, x^{(1)}, x^{(2)}, \cdots)$$

w/ $x^{(n)} = \lim_{m \to \infty} \tilde{x}_{n+m}^{p^m} \in \mathcal{O}_{\mathbb{C}_p}$

$\mathbb{C}_p^b := \mathcal{O}_{\mathbb{C}_p}^b[\frac{1}{\varepsilon - 1}]$. $\quad \mathbb{C}_p^b :=$ tilt of $\mathbb{C}_p$.

<u>Def</u>. $\underset{\ker\theta}{\underset{\cup}{\mathbb{A}_{inf}}} = W(\mathcal{O}_{\mathbb{C}_p}^b)$ & there's a nat'l hom. $\theta : \mathbb{A}_{inf} = W(\mathcal{O}_{\mathbb{C}_p}^b) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}$

$$\sum_{n \geq 0} p^n [a_n] \longmapsto \sum_{n \geq 0} p^n a_n^{(0)}$$

<u>Fact</u> $\ker\theta = \left(\frac{[\varepsilon] - 1}{[\varepsilon]^{1/p} - 1}\right)$ $\qquad$ <u>note</u> $\theta([\varepsilon]) = 1$.

<u>Def'n</u> ① $\mathbb{A}_{cris}^+ :=$ completed divided power envelope of $\ker\theta$ in $\mathbb{A}_{inf}$.

$\mathbb{A}$ p-integral $\quad = \left\{ \sum_{n \geq 1} a_n \frac{\left(\frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1}\right)^n}{n!} : a_n \in \mathbb{A}_{inf} \right\}$ $\quad \circlearrowleft \phi$-action

$\mathbb{B}$ p-inverted $\quad$ & $a_n \to 0$ $\quad \circlearrowleft \mathrm{Gal}_K$

② $\mathbb{B}_{dR}^+ :=$ Completion of $W(\mathbb{C}_p^b)[\frac{1}{p}]$ at $\ker(\theta : W(\mathbb{C}_p^b)[\frac{1}{p}] \twoheadrightarrow \mathbb{C}_p)$

$$= \left\{ \sum_{n \geq 0} a_n \left(\frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1}\right)^n : a_n \in W(\mathbb{C}_p^b)[\tfrac{1}{p}] \right\} \circlearrowleft \mathrm{Gal}_K$$

complete discrete valuation ring w/ unif. $\frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1}$ & residue field $\mathbb{C}_p$

$v_{\mathbb{B}_{dR}^+}\left(\frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1}\right) = 1, \qquad v_{\mathbb{B}_{dR}^+}(a) = 0 \quad \text{if } \theta(a) \neq 0.$

$\mathrm{Fil}^i \mathbb{B}_{dR} = \{x \in \mathbb{B}_{dR} : v_{\mathbb{B}_{dR}^+}(x) \geq i\}$

$\mathbb{B}_{dR} := \mathrm{Frac}\, \mathbb{B}_{dR}^+ = \mathbb{B}_{dR}^+\left[\frac{1}{\left(\frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1}\right)}\right]$

③ $t = \log[\varepsilon] = ([\varepsilon]-1) - \frac{1}{2}([\varepsilon]-1)^2 + \frac{1}{3}([\varepsilon]-1)^3 - \cdots \in \mathbb{A}_{cris}^+ \subset \mathbb{B}_{dR}^+$

So t is a uniformizer of $\mathbb{B}_{dR}^+$.

☆: t is the "p-adic $2\pi i$".

<u>Claim</u> $gt = \chi_{cycl}(g)t$ b/c $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{cycl}(g)}$ $\Rightarrow g([\varepsilon]) = [\varepsilon^{\chi_{cycl}(g)}]$

for $g \in \mathrm{Gal}_K$ $\qquad = [\varepsilon]^{\chi_{cycl}(g)}$

$\mathbb{Q}_p \cdot t = \mathbb{Q}_p(1)$ $\qquad \phi([\varepsilon]) = [\varepsilon^p] = [\varepsilon]^p$

$\phi(t) = pt$

$\mathrm{Fil}^i \mathbb{B}_{dR} = t^i \mathbb{B}_{dR}^+$

$\mathrm{gr}^i \mathbb{B}_{dR} = t^i \mathbb{B}_{dR}^+ / t^{i+1} \mathbb{B}_{dR}^+ = t^i \frac{\mathbb{B}_{dR}^+}{t\mathbb{B}_{dR}^+} = t^i \mathbb{C}_p = \mathbb{C}_p(i)$

So $\mathbb{B}_{dR}$ is the ext'n of $\mathbb{C}_p(i)$'s. NONTRIVIAL EXT. $\mathbb{B}_{dR} \neq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$

④ $\mathbb{B}_{cris}^+ = \mathbb{A}_{cris}^+[\frac{1}{p}]$, $\mathbb{B}_{cris} = \mathbb{B}_{cris}^+[\frac{1}{t}]$

<u>Facts</u>. $(\mathbb{B}_{dR}^+)^{\mathrm{Gal}_K} = (\mathbb{B}_{dR})^{\mathrm{Gal}_K} = K$, $(\mathbb{A}_{cris}^+)^{\mathrm{Gal}_K} = W(k_K)$, $(\mathbb{B}_{cris})^{\mathrm{Gal}_K} = W(k_K)[\frac{1}{p}]$

$(\mathbb{B}_{cris}^{\varphi=1}) \cap \mathrm{Fil}^0(\mathbb{B}_{cris}) = \mathbb{Q}_p$.

Lecture 15

<u>Recall</u>: the comparison thm in the good reduction case.

* K CDVF of mixed char. $(0,p)$, abs. unram. & w/ perfect residue field

* $\mathcal{X}$ proper smooth $/\mathcal{O}_K$

$$\begin{array}{ccccc} H^i_{et}(\mathcal{X}_{\bar{K}}; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} & \simeq & H^i_{cris}(\mathcal{X}_k/\mathcal{O}_K)[\tfrac{1}{p}] \otimes_K B_{cris} & \cong & H^i_{dR}(\mathcal{X}_K/K) \\ \circlearrowleft \mathrm{Gal}_K \quad \circlearrowleft \phi, \mathrm{Gal}_K, \mathrm{Fil} & & \circlearrowleft \phi \quad \circlearrowleft \phi, \mathrm{Gal}_K, \mathrm{Fil} & & \end{array}$$

* Taking Gal_K-inv. gives $H^i_{cris}(\mathcal{X}_k/\mathcal{O}_K)[\frac{1}{p}] \circlearrowleft \phi, \mathrm{Fil}$

* Taking $\phi = 1$, Fil^0 gives $H^i_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}_p)$

* <u>Purely Galois theoretic version</u> (K CDVF, abs. unram.)

For $\rho: \mathrm{Gal}_K \longrightarrow GL_n(\mathbb{Q}_p)$ a cts p-adic rep'n $\mathrm{Gal}_K \curvearrowright V$

[<u>Rmk</u>: Typically, $\rho(I_K)$ is huge in $GL_n(\mathbb{Q}_p)$]

<u>Define</u> $\mathbb{D}_{cris}(V) := (V \otimes_{\mathbb{Q}_p} B_{cris})^{\mathrm{Gal}_K} \circlearrowleft \phi, \mathrm{Fil}$ decreasing filtration

ϕ-action is semilinear: $\phi(am) = \phi(a)\phi(m)$ for $a \in K$, $m \in \mathbb{D}_{cris}(V)$.

<u>Fact</u>: $\mathbb{D}_{cris}(V)$ is a fin. dim'l K-v.s. of dim $\leq \dim_{\mathbb{Q}_p} V = n$

<u>Def</u>. We say that V is <u>crystalline</u> if $\dim_K \mathbb{D}_{cris}(V) = n$.

In this case, the <u>Hodge-Tate wts</u> $HT(V)$ of V are the integers i s.t. $gr^i \mathbb{D}_{cris}(V) \neq 0$

counted w/ mult. $\dim gr^i \mathbb{D}_{cris}(V)$.

Fact. If A is an abelian var. $/K$, then TFAE

(a) A has good reduction, i.e. $\exists$ proper smooth abel. var. $\mathcal{A}/\mathcal{O}_K$ s.t. $\mathcal{A} \times_{\mathcal{O}_K} K \simeq A$

(b) For some /any $\ell \neq p$, $H^1_{\acute{e}t}(A_{\bar{K}}; \mathbb{Q}_\ell)$ is unramified

(c) $H^1_{\acute{e}t}(A_{\bar{K}}; \mathbb{Q}_p)$ is crystalline.

$\dim = 2n$, $n = \dim A$

Rmk. When $\mathcal{X}$ is proper sm. $/\mathcal{O}_K$,

$$\mathbb{D}_{cris}\left(H^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}; \mathbb{Q}_p)\right) \simeq H^i_{cris}(\mathcal{X}_k/\mathcal{O}_K)[\tfrac{1}{p}].$$

Thm. If V is a crystalline rep'n.

$$V \simeq \left(\mathbb{D}_{cris}(V) \otimes \mathbb{B}_{cris}\right)^{\phi=1} \cap \boxed{\mathrm{Fil}^0\left(\mathbb{D}_{cris}(V) \otimes \mathbb{B}_{cris}\right)}$$

$$\sum_{i+j=0} \mathrm{Fil}^i \mathbb{D}_{cris}(V) \otimes \mathrm{Fil}^j \mathbb{B}_{cris}.$$

Moreover, there's an equiv. of cats (cptble w/ tensors & duals)

$$\left\{\begin{matrix}\text{crystalline rep'ns} \\ \text{of } G_{alK}\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\underline{\text{admissible}} \text{ filtered} \\ \varphi\text{-mods}\end{matrix}\right\}$$

$$\parallel$$

$$\left\{ (D, \mathrm{Fil}^\bullet D) : \begin{matrix} D \text{ f-dim'l } K\text{-v.s. w/ a semilinear } \varphi\text{-action} \\ \mathrm{Fil}\, D \text{ is a decreasing filtration satisfying} \\ \text{a } \underline{\text{weakly adm. cond.}} \end{matrix} \right\}$$

(& φ is bijective)

<u>weakly admissibility</u>, when $\dim D = 1$, $D = K \cdot e$, $\phi(e) = \alpha \cdot e$

define $t_N(D) := v_p(\alpha)$. If $e' = \beta e$, $\phi(e') = \alpha \frac{\phi(\beta)}{\beta} e'$

Newton

$v_p\left(\alpha \frac{\phi(\beta)}{\beta}\right) = v_p(\alpha)$.

In general, $t_N(D) := t_N\left(\Lambda^{\dim D}_K D\right)$

Define $t_H(D) = \sum i \dim \mathrm{gr}^i D$ = sum of HT wts (w/ mult.)

$\uparrow$ Hodge

Say D is <u>weakly adm.</u> if $t_H(D) = t_N(D)$ and for any φ-stable sub K-v.s. $D' \subset D$,

$\underbrace{t_H(D')} \le t_N(D')$ using the induced filtration, i.e. $\mathrm{Fil}^* D' = D' \cap \mathrm{Fil}^* D$.

<u>E.g.</u> $V = \chi_{cycl}^{\otimes n}$ i.e. $V = \mathbb{Q}_p \cdot e$ w/ $g(e) = \chi_{cycl}(g)^n e$.

$$\mathbb{D}_{cris}(V) = \left(\mathbb{Q}_p \cdot e \otimes \mathbb{B}_{cris}\right)^{\mathrm{Gal}_K}$$

$$= K \cdot e \otimes t^{-n}$$

$$\boxed{\begin{array}{l} t \in \mathbb{B}_{cris} \\ g(t) = \chi_{cycl}(g) \cdot t \end{array}}$$

$$\phi(e \otimes t^{-n}) = p^{-n} \cdot e \otimes t^{-n}$$

$$t_N(\mathbb{D}_{cris}(V)) = -n.$$

<u>Recall</u> $\mathbb{B}_{cris} \subset \mathbb{B}_{dR} \leftarrow$ CDVF w/ t as the uniformizer

$$v_{\mathbb{B}_{dR}}(t^{-n}) = -n$$

So $\mathrm{Fil}^i(\mathbb{D}_{cris}(V)) = \begin{cases} K \cdot e \otimes t^{-n} & \text{if } i \le -n \\ 0 & \text{if } i > -n \end{cases} \Rightarrow t_H(\mathbb{D}_{cris}(V)) = -n.$

<u>Rmk</u> HT wts of a cyclotomic char. is -1.

<u>E.g.</u> $f(q) = \sum a_n q^n$ normalized cuspidal eigen cusp. form of level $\Gamma_0(N)$, wt k.

Assume $p \nmid N$. For simplicity, assume all $a_n \in \mathbb{Z}$.

Let α, β be zeroes of $X^2 - a_p X + p^{k-1} = 0$. (say $\alpha \ne \beta$). Then

$v_p(\alpha), v_p(\beta) \in [0, k-1]$.

$$\rightsquigarrow \rho_f : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_p)$$
$$\cup \qquad \nearrow \rho_{f,p}$$
$$\mathrm{Gal}_{\mathbb{Q}_p}$$

<u>Fact</u>. $\rho_{f,p}$ is a crystaline rep'n.

$$\mathbb{D}_{cris}(\rho_{f,p}) = \mathbb{Q}_p e_\alpha \oplus \mathbb{Q}_p e_\beta . \qquad \begin{array}{l} \varphi(e_\alpha) = \alpha \cdot e_\alpha \\ \varphi(e_\beta) = \beta \cdot e_\beta \end{array} \rightarrow t_N(\mathbb{D}_{cris}(\rho_{f,p})) = v_p(\alpha\beta) = k-1$$

$$\mathrm{Fil}^i \mathbb{D}_{cris}(\rho_{f,p}) = \begin{cases} \mathbb{D}_{cris}(\rho_{f,p}), & i \le 0 \\ \underline{\mathbb{Q}_p e_\alpha} \text{ or } \underline{\mathbb{Q}_p e_\beta} \text{ or } \underline{\mathbb{Q}_p(e_\alpha + e_\beta)} & ; i = 1, \cdots, k-1 \\ 0, & i > k-1 \end{cases} \Rightarrow t_H = k-1 .$$

<u>Admissibility</u>. Two φ-stable subspaces $\mathbb{Q}_p e_\alpha$ & $\mathbb{Q}_p e_\beta$.

$$t_N(\mathbb{Q}_p \cdot e_\alpha) = v_p(\alpha), \quad t_H(\mathbb{Q}_p \cdot e_\alpha) = \begin{cases} k-1, & \text{if } \mathrm{Fil}^1(\mathbb{D}_{cris}(\rho_{f,p})) = \mathbb{Q}_p \cdot e_\alpha \\ 0, & \text{o/w} \end{cases}$$ called critical case.

In the "strange case" when $\mathrm{Fil}^1 = \mathbb{Q}_p e_\alpha$, admissibility cond. $\Rightarrow k-1 \le v_p(\alpha)$

$\Rightarrow v_p(\alpha) = k-1$

$v_p(\beta) = 0.$

In the other cases, adm. cond. $\Rightarrow 0 < v_p(\alpha)$ <u>automatic</u>.

<u>Fontaine-Laffaille theory</u> (integral version of crystalline rep'ns)

<u>Rmk</u>. will work w/ $\mathbb{Z}_\ell$-coeff. & rep'ns of $\mathrm{Gal}_{\mathbb{Q}_\ell}$.

Will only discuss $K = \mathbb{Q}_\ell$, allowing coeffs to be $\mathcal{O} \subset E / \mathbb{Q}_\ell$.

Def'n Let MF (modules filtré) denote the following cat:

Objects. $(M, \mathrm{Fil}^\bullet M, (\Phi^i)_i)$ called a <u>Fontaine-Laffaille module</u>, where

- M f.g. $\mathcal{O}$-mod
- $\mathrm{Fil}^\bullet M$ is a decreasing filtration by $\mathcal{O}$-submod of M

  s.t. each $\mathrm{Fil}^i M$ is an $\mathcal{O}$-module direct summand of M (i.e. no $\ell\mathcal{O} \subset \mathcal{O}$)

  $\mathrm{Fil}^0 M = M$, $\mathrm{Fil}^{\ell-1} M = (0)$ $\Rightarrow$ $\boxed{\text{HT wts} \in [0, \ell-2]}$

- <u>$\mathcal{O}$-linear</u> map $\Phi^i : \mathrm{Fil}^i M \longrightarrow M$ s.t.

  linear action b/c working w/ $K = \mathbb{Q}_\ell$

  "Φ/ℓ^i"

  $$\Phi^i\big|_{\mathrm{Fil}^{i+1} M} = \ell \cdot \Phi^{i+1}, \text{ and } M = \sum_i \Phi^i(\mathrm{Fil}^i M)$$

Morphisms $M \longrightarrow M'$ compatible w/ all additional str.s

Rmk. Key constraints here are ① K needs to be abs. unram. ② HT wts must be in $[0, \ell-2]$) or HT wts $\subset [0, \frac{\ell-2}{e_K}]$

Fact: MF is an abelian cat.

Thm There's a nat'l exact, fully faithful, covariant functor

$$\mathbb{D}_{FL} : MF \longrightarrow \mathcal{O}[G_{\mathbb{Q}_\ell}]\text{-mod}^{cts} = \{\text{fin. gen. } \mathcal{O}\text{-mods w/ cts } G_{\mathbb{Q}_\ell}\text{-action}\}$$

$$M \longmapsto M^* = \mathrm{Hom}(M, E/\mathcal{O})\underbrace{\{\ell-2\}}_{\substack{\text{twist } \Phi\text{-action by } \ell^{\ell-2} \\ \text{\& shifts the filtration by } \mathrm{Fil}^{\ell-2}}} \mapsto \mathrm{Hom}(M^*, A_{cris} \otimes E/\mathcal{O})^{\Phi\text{-equiv., } \mathrm{Fil}^\bullet}$$

(same as $\otimes\, \mathbb{D}_{cris}(\chi_{cycl}^{2-\ell})$)

s.t. the essential image is closed under taking subobjects & quotients.

<u>Exer</u> If $M \in \mathcal{MF}$ is a finite free $\mathcal{O}$-mod, then $M \otimes E \circlearrowleft \Phi^0$, & filtration defines an <u>admissible</u> filtered φ-module.

<u>Rmk</u> When $M \in \mathcal{MF}$ is finite free $\mathcal{O}$-mod, each Φ^i is determined by Φ^0.

So the existence of $\Phi^i \Leftrightarrow \Phi^0 \big|_{\mathrm{Fil}^i M}$ is divisible by ℓ^i.

<u>Fact</u> In this case, $\mathbb{D}_{FL}(M) = \left(M \otimes A_{cris}[\frac{1}{\ell}]\right)^{\varphi = \ell^{2-\ell},\ \mathrm{Fil}^{2-\ell}}$

crystalline rep'n attached to filtered φ-mod.

So $\mathbb{D}_{FL}(M) \otimes \chi_{cycl}^{2-\ell} \underset{\mathbb{D}_{cris}}{\longleftrightarrow} M[\frac{1}{\ell}]$

In general, can always write M as the cokernel of $M_1 \to M_2$ in $\mathcal{MF}$ w/ both M_i $\mathcal{O}$-free. Then $\mathbb{D}_{FL}(M) = \mathrm{Coker}\left(\mathbb{D}_{FL}(M_1) \to \mathbb{D}_{FL}(M_2)\right)$.

Conversely, for every adm. filtered φ-mod V w/ HT wts $\subset [0, \ell-2]$, there exists a φ-stable lattice $M \subset V$ s.t.

$\mathrm{Fil}^i M = M \cap \mathrm{Fil}^i V$,

and $\varphi \big|_{\mathrm{Fil}^i M} \to \ell^i M \subset M$, so that we may define $\Phi^i = \varphi / \ell^i$

$\rightsquigarrow M \in \mathcal{MF}$.

<u>Examples</u> (1) $\chi_{cycl}^{\otimes -r}$ for $0 \le r \le \ell-2$

$M = \mathcal{O} \cdot e$, $\mathrm{Fil}^i M = \begin{cases} \mathcal{O}e, & i \le r \\ 0, & i > r \end{cases}$

$\phi / \ell^r : e \mapsto e$

$\Phi^r : \mathrm{Fil}^r M \to M$

$\phi : e \mapsto \ell^r e$

$\Phi^0 : M \to M$, $e \mapsto \ell^r e$

$\bar{M} = M \otimes \mathbb{F}$, $\Phi^0 = \cdots = \Phi^{r-1} = 0$, $\Phi^r : e \mapsto e$.

Typical examples for ℓ-adic Tate modules of elliptic curve $\mathcal{E}/\mathbb{Z}_\ell$

$$T_\ell(\mathcal{E}) = \varprojlim_n \mathcal{E}[\ell^n] \simeq \mathbb{Z}_\ell^2 \circlearrowleft \mathrm{Gal}\,\mathbb{Q}_\ell$$

$$M = \mathbb{Z}_\ell e_1 \oplus \mathbb{Z}_\ell e_2, \quad \mathrm{Fil}^i M = \begin{cases} \mathbb{Z}_\ell e_1 \oplus \mathbb{Z}_\ell e_2 & , i=0 \\ \mathbb{Z}_\ell \cdot e_1 & , i=1 \\ 0 & , i=2 \end{cases}$$

① <u>$\mathcal{E}$ ordinary at ℓ</u> $\quad \Phi^0 : M \longrightarrow M$ looks like $\begin{pmatrix} \ell\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha \in \mathcal{O}^\times$

$$\Phi^1 : \mathbb{Z}_\ell \cdot e_1 \longrightarrow M$$
$$e_1 \longmapsto \alpha \cdot e_1$$

② <u>$\mathcal{E}$ supersingular reduction @ ℓ</u> $\quad \Phi^0 = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}$, $\Phi^1(e_1) = e_2$

$\mathcal{E}$. $\quad X_{\bar\rho} \supset X_{\bar\rho}^{\mathrm{cris}}$

$\bar\rho \hookrightarrow \bar{M}$

<u>Deformation of Fontaine-Laffaille modules</u>

<u>Idea</u>: Instead of deforming $\bar\rho|_{G_{\mathbb{Q}_\ell}}$, we deform the corresponding Fontaine-Laffaille module.

- Let $(\bar{M}, \mathrm{Fil}^\bullet \bar{M}, \bar\Phi^i)$ be a FL-module s.t. $\bar{M}$ is an $\mathbb{F}$-v.s. of dim n

[<u>Assume</u> that for each i, $\dim \mathrm{gr}^i \bar{M} \le 1$

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ denote the different Hodge-Tate weights.]

Choose a basis of $\bar{M}$ so that

$$\mathrm{Fil}^{\lambda_n} \bar{M} = \mathbb{F} e_n$$
$$\mathrm{Fil}^{\lambda_{n-1}} \bar{M} = \mathbb{F} e_n \oplus \mathbb{F} e_{n-1}$$
$$\vdots$$
$$\bar{M} = \mathrm{Fil}^{\lambda_1} \bar{M} = \mathbb{F} e_n \oplus \cdots \oplus \mathbb{F} e_1$$

Consider $\mathrm{Def}_{\bar{M}}^{\square} : \mathrm{CNL}_{\mathcal{O}} \longrightarrow \mathrm{Set}$

$$A \longmapsto \left\{ \begin{array}{l} (M, \mathrm{Fil}^\bullet M, \Phi^\bullet) \in \mathcal{MF} \text{ w/ HT wts } \lambda_1 < \cdots < \lambda_n \text{ of the form} \\ \mathrm{Fil}^{\lambda_i} M = A e_n \oplus \cdots \oplus A e_i \\ \Phi^{\lambda_i} \big|_{\mathrm{Fil}^{\lambda_i} M \bmod m_A} = \bar{\Phi}^{\lambda_i} \end{array} \right\}$$

<u>Thm</u> $\mathrm{Def}_{\bar{M}}^{\square}$ is rep'td by $R_{\bar{M}}^{\square} \simeq \mathcal{O}[[x_1, \ldots, x_{n^2}]]$. $\leftarrow$ still true w/o hypotheses on $\dim \mathrm{gr}^i M$.

<u>Proof</u> Only data for $(M, \mathrm{Fil}^\bullet M, \Phi) \in \mathrm{Def}_{\bar{M}}^{\square}(A)$ is the Φ^{λ_i}'s.

We need to record $\Phi^{\lambda_n}(e_n) = \begin{pmatrix} \vdots \end{pmatrix}$. $\Phi^{\lambda_{n-1}}$ is defined on $\underbrace{\boxed{Ae_n}}_{p^{\lambda_n - \lambda_{n-1}} \Phi^{\lambda_n}} \oplus Ae_{n-1}$

$$\Phi^{\lambda_{n-1}}(e_{n-1}) = \begin{pmatrix} \vdots \end{pmatrix}$$

$$\Phi^{\lambda_{n-2}}(e_{n-2}) = \begin{pmatrix} \vdots \end{pmatrix}$$

Set $F = \left(\Phi^{\lambda_n}(e_n), \Phi^{\lambda_{n-1}}(e_{n-1}), \ldots, \Phi^{\lambda_1}(e_1) \right) \in M_n(A)$

- $F \bmod m_A = \bar{F}$
- <u>Claim:</u> condition $\sum \Phi^i(\mathrm{Fil}^i M) = M \iff F \in GL_n(A). \iff \bar{F} \in GL_n(A/m_A)$

$\Phi^i(\mathrm{Fil}^i M)$ spanned by $\Phi^{\lambda_n}(e_n), \Phi^{\lambda_{n-1}}(e_{n-1}), \ldots$

$\Updownarrow$

$\boxed{\sum \bar{\Phi}^i(\mathrm{Fil}^i \bar{M}) = \bar{M}}$ ✓

This is automatic!

So all we need to do is to lift $\bar{F}$ <u>arbitrarily</u>.

$\Rightarrow$ rep'd by $\mathcal{O}[[x_1, \ldots, x_{n^2}]]$.

There's a nat'l action of $\begin{pmatrix} 1+m_{\bar{M}}^{\square} & & m_{\bar{M}}^{\square} \\ & \ddots & \\ 0 & & 1+m_{\bar{M}}^{\square} \end{pmatrix}$ on $R_{\bar{M}}^{\square}$.

action of $B = \left(\triangledown\right)$, $\hat{B}_1 \curvearrowright \operatorname{Spf} R_{\bar{M}}^{\square}$ changing $e_n \mapsto e_n' = * e_n$

$e_{n-1} \mapsto e_{n-1}' = * e_{n-1} + * e_n$

$\vdots$

Say we take a new basis of the form $(e_n', \dots, e_1') = (e_n, \dots, e_1) P$.

Then $\Phi^0(e_n, \dots, e_1) = (e_n, \dots, e_1)\left(\Phi^0(e_n), \Phi^0(e_{n-1}), \dots, \Phi^0(e_1)\right)$

$$\Phi^0(e_n) = \ell^{\lambda_n} \Phi^{\lambda_n}(e_n), \quad \Phi^0(e_{n-1}) = \ell^{\lambda_{n-1}} \Phi^{\lambda_{n-1}}(e_{n-1}), \quad \Phi^0(e_1) = \ell^{\lambda_1} \Phi^{\lambda_1}(e_1)$$

$$= (e_n, \dots, e_1) F \begin{pmatrix} \ell^{\lambda_n} & & \\ & \ell^{\lambda_{n-1}} & \\ & & \ddots \\ & & & \ell^{\lambda_1} \end{pmatrix}$$

$$\Phi^0(e_n', \dots, e_1') = (e_n', \dots, e_1') F' \begin{pmatrix} \ell^{\lambda_n} & & \\ & \ddots & \\ & & \ell^{\lambda_1} \end{pmatrix}$$

$$\Phi^0(e_n', \dots, e_1') = \Phi(e_n, \dots, e_1) P = (e_n, \dots, e_1) F \begin{pmatrix} \ell^{\lambda_n} & & \\ & \ddots & \\ & & \ell^{\lambda_1} \end{pmatrix} P$$

$$(e_n, \dots, e_1) P F' \begin{pmatrix} \ell^{\lambda_n} & & \\ & \ddots & \\ & & \ell^{\lambda_1} \end{pmatrix}$$

$$\Rightarrow F' = P^{-1} F \begin{pmatrix} \ell^{\lambda_n} & & \\ & \ddots & \\ & & \ell^{\lambda_1} \end{pmatrix} P \begin{pmatrix} \ell^{-\lambda_n} & & \\ & \ddots & \\ & & \ell^{-\lambda_1} \end{pmatrix}$$

So it's nat'l to consider $X_{\bar{M}} =$ "$\left[\operatorname{Spf} R_{\bar{M}}^{\square} / \hat{B}_1\right]$"

$\ell_{\dim_G} = n^2 - \dim B = \frac{n(n-1)}{2}$

Expectedly, get a nat'l embedding $X_{\bar{M}} \hookrightarrow X_{\bar{P}} = \mathbb{D}_{FL}(\bar{M})$ $\left[\operatorname{Spf} R_{\bar{M}}^{\square} / \hat{B}_1\right] \hookrightarrow \left[\operatorname{Spf} R_{\bar{P}}^{\square} / \widehat{PL_n}\right]$.

When $n=2$

$$\begin{tikzcd} X_{\bar{\mu}} \arrow[r, hook] & X_{\bar{\rho}}^{\det = \chi} \\ \dim_\theta = 1 \arrow[u, hook] & \dim_\theta = 3 = \chi(G_{\mathbb{Q}_\ell}, \mathrm{Ad}^0 \bar{\rho}) \arrow[u, hook] \end{tikzcd}$$

$\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{F})$

$$\begin{tikzcd} \overset{\text{rel dim } 0}{X_{\bar{\rho}}^{\det, FL@p}} \arrow[r] \arrow[d] & X_{\bar{\rho}}^{\det} \arrow[d, "\text{rel dim} \geq 0"] \\ \prod_{v \nmid p} X_{\bar{\rho}_v}^{\times} \times X_{\bar{\rho}_p}^{FL} \arrow[r] & \prod_v X_{\bar{\rho}_v} \end{tikzcd}$$

$\det \bar{\rho}(\text{cpx conj.}) = -1$ $\qquad$ dim 2

Lecture 16. Galois rep'ns appearing in geometry.

Let k be a field.

X = proper smooth var. $/k$.

For every $\ell \neq \mathrm{char}\, k$, $\rightsquigarrow$ $H^i_{\text{ét}}(X_{k^{sep}}, \mathbb{Q}_\ell)$ = ith étale coh.

$\circlearrowleft \mathrm{Gal}(k^{sep}/k)$ $\qquad$ ($\neq 0$ only when $0 \leq i \leq 2\dim X$)

f.dim. $\mathbb{Q}_\ell$-vec-sp.

When $\mathrm{char}\, k = 0$ (assuming that there's no issue w/ cardinality of k) $\iota : k \hookrightarrow \mathbb{C}$

Then $H^i_{\text{ét}}(X_{\bar{\text{ét}}}, \mathbb{Q}_\ell) \simeq H^i_{\text{sing}}((X \times_{k,\iota} \mathbb{C})^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$

Eg. If X geom. conn, then $H^0_{\text{ét}}(X_{k^{sep}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ w/ trivial Galois action

$H^{2\dim}_{\text{ét}}(X_{k^{sep}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\dim X)$, so Gal_k acts by $\chi_{cycl}^{-\dim X}$

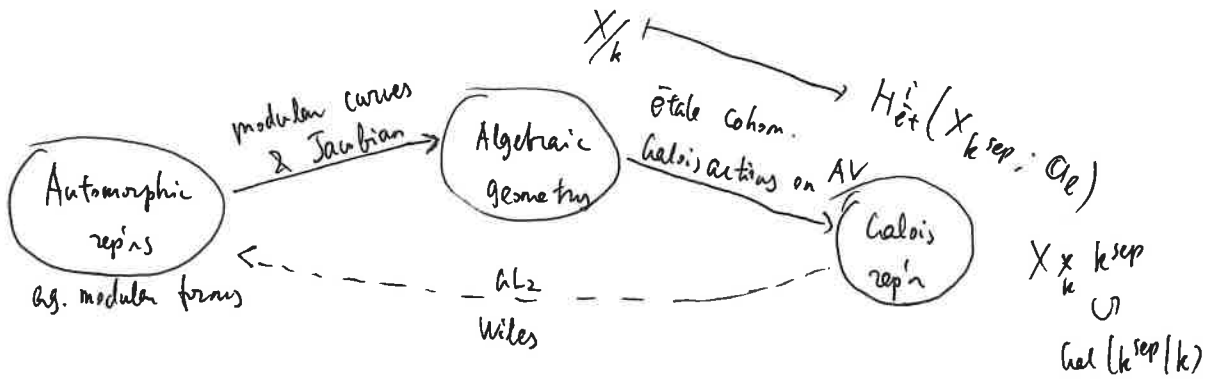
If X is a geom. conn. curve of genus g, then $\dim H^1_{\acute{e}t}(X_{k^{sep}}, \mathbb{Q}_\ell) = 2g$.

<u>Difference w/ singular cohomology</u>:

* $H^i_{sing}(-)$ can allow $\mathbb{Z}$-coeffs, whereas $H^i_{\acute{e}t}$ only allows $\mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$ coeff.
  $\mathbb{Q}$-

* $H^i_{\acute{e}t}(-)$ has Galois action.

$$\mathrm{Gal}_k \to GL(H^i_{\acute{e}t}) = GL_n(\mathbb{Q}_\ell)$$



* When k is a finite field, $k = \mathbb{F}_q$, $\ell \neq p$.

$$\underbrace{H^i_{\acute{e}t}(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)}_{\text{fin. dim'l}} \circlearrowleft \mathrm{Gal}_{\mathbb{F}_q} \ni \varphi_q \text{ — geom. Frob.}$$

<u>Deligne's Weil Conjecture</u>. The char. poly. $\det(x \cdot id - \varphi_q \mid H^i_{\acute{e}t}(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell))$

<u>Thm</u>

$$= x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

① has $\mathbb{Z}$-coeff. i.e. $a_i \in \mathbb{Z}$ $\Rightarrow$ roots are algebraic integers

② is indep. of the prime ℓ.

③ for any zero $\alpha \in \overline{\mathbb{Q}}$ and <u>any</u> embedding $\overline{\mathbb{Q}} \overset{\iota_\infty}{\hookrightarrow} \mathbb{C}$, $|\alpha|_\infty = q^{i/2}$.

(call such α a <u>Weil $q^{i/2}$-number</u>)

Conj. The φ_q-action on $H^i_{\acute{e}t}(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ is <u>semisimple</u> (i.e. no Jordan block).

Rmk <u>Lefschetz trace formula</u>: if $\alpha_1^{(i)}, \cdots, \alpha_{2_i}^{(i)}$ denote the φ-eig. val. of $H^i_{\acute{e}t}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, $\forall i$,

e.g. Hartshorne appendix

Then $\forall\, m$, $\# X(\mathbb{F}_{q^m}) = \sum_{i=0}^{2\dim X} \sum_{j=1}^{v_i} (-1)^i \left(\alpha_j^{(i)}\right)^m$.

Eg. X curve of genus g, $\Rightarrow$ $H^0(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) = \mathbb{Q}_\ell \Rightarrow \alpha_1^{(0)} = 1$

$H^2(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-1) \Rightarrow \alpha_1^{(2)} = q$

$H^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ has $2g$-dim. $\Rightarrow \alpha_1^{(1)}, \cdots, \alpha_{2g}^{(1)}$.

$|\alpha_i^{(1)}| = q^{1/2}$

$$|X(\mathbb{F}_{q^m})| = 1 + q^m - \sum_{j=1}^{2g} \left(\alpha_j^{(1)}\right)^m$$

Cor $\left| \# X(\mathbb{F}_{q^m}) - (1+q^m) \right| \le 2g\, q^{m/2}$.

- When $k \overset{=K}{=}$ fin. ext'n of $\mathbb{Q}_p$, <u>note</u>: we allow $\ell = p$. residue field of $K = \mathbb{F}_q$

<u>Case 1</u>. $\ell \neq p$. & assume that X admits a proper smooth model $/\mathcal{O}_K$ (<u>good reduction</u>)

i.e.
$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "\text{proper smooth}"] & \mathcal{X}_{\mathbb{F}_q} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} \mathcal{O}_K & \operatorname{Spec} \mathbb{F}_q \arrow[l, hook']
\end{tikzcd}$$

Then
$$\begin{tikzcd}
H^i_{\acute{e}t}(X_{\overline{K}}; \mathbb{Q}_\ell) \arrow[d, phantom, "\circlearrowleft"] & \simeq & H^i_{\acute{e}t}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \arrow[d, phantom, "\circlearrowleft"] \\
\operatorname{Gal}_K \arrow[rr] & & \operatorname{Gal}_{\mathbb{F}_q}
\end{tikzcd}$$

$\Rightarrow$ Gal_K action on $H^i_{\acute{e}t}(X_{\overline{K}}; \mathbb{Q}_\ell)$ is unramified.

& geom. Frob. eig. vals are Weil $q^{i/2}$-no.

& eig. val. are indep. of ℓ.

Case 2. $\ell \neq p$, X general projective (proper ?) smooth $/\mathrm{Spec}\, K$.

$\mathrm{Gal}_K \curvearrowright H^i_{\acute{e}t}(X_{\bar{K}}; \mathbb{Q}_\ell)$

("usually ramified")

(Corollary of) Weight-monodromy conjecture

Let (r, N, V) denote the Weil-Deligne rep'n attached to $H^i_{\acute{e}t}(X_{\bar{K}}; \mathbb{Q}_\ell)$.

$$\boxed{\begin{array}{l} r: W_K \longrightarrow GL(V) \quad \leftarrow r(I_K) \text{ finite} \\ N: V \longrightarrow V \text{ s.t. } r(\phi)^{-1} N r(\phi) = q \cdot N \end{array}}$$

① (r, N, V) is indep. of ℓ.

② As $\varphi^{-1} N \varphi = q N \Rightarrow N$ sends $\varphi = \alpha$-eig. space to $\varphi = \frac{\alpha}{q}$-eig. space.

$\Rightarrow$ N must be nilpotent $\qquad N = \bigoplus_{m=0}^{m_0} \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{m+1}$

General fact (Jacobson-Morozov) V can be decomposed as the direct sum of the following

$$\overset{\dim 2m_0}{\vee} P_{i+m_0} \xrightarrow[\sim]{N} N P_{i+m_0} \xrightarrow[\sim]{N} N^2 P_{i+m_0} \xrightarrow[\sim]{N} \cdots \xrightarrow{N} N^{m_0-1} P_{i+m_0} \xrightarrow[\sim]{N} N^{m_0} P_{i+m_0} \longrightarrow 0$$

$$P_{i+m_0-1} \xrightarrow[\sim]{N} N P_{i+m_0-1} \xrightarrow[\sim]{N} N^2 P_{i+m_0-1} \cdots N^{m_0-2} P_{i+m_0-1} \xrightarrow[\sim]{N} N^{m_0-1} P_{i+m_0-1} \longrightarrow 0$$

$$P_{i+m_0-2} \xrightarrow[\sim]{N} N P_{i+m_0-2} \xrightarrow[\sim]{N} \cdots \longrightarrow N^{m_0-2} P_{i+m_0-2} \longrightarrow 0$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{wt } i+m_0 & \text{wt } i+m_0-1 & \text{wt } i+m_0-2 \end{array} \qquad \underset{\text{wt } i}{P_i} \xrightarrow{N} 0$$

s.t. each P_i is φ-stable.

The WMC says that the φ-eig. val. on each $N^j P_{i+m}$ are $q^{\frac{1}{2}(i+m-2j)}$-Weil nos.

i.e. all N-chains are "centered at weight i" $\longleftarrow$ pure of wt i

Rmk When X has good reduction, $WD(H^i_{et}(X_{\bar{k}}; \mathbb{Q}_\ell)) = H^i_{et}(X_{\bar{k}}; \mathbb{Q}_\ell) \circlearrowleft \varphi$, $N=0$

φ-eig vals are $q^{i/2}$-Weil nos.

Case 3. $k = \mathbb{Q}_p$, $\ell = p$, $\mathcal{X}$ proper smooth over $\mathbb{Z}_p$.

$$\mathrm{Gal}_{\mathbb{Q}_p} \curvearrowright H^i_{et}(\mathcal{X}_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_p) =: H^i$$

but $H^i_{et}(\mathcal{X}_{\overline{\mathbb{F}}_p}; \mathbb{Q}_p)$ is badly behaved.

$$\mathbb{D}_{cris}(H^i) \circlearrowleft \boxed{\varphi} \qquad \dim \mathbb{D}_{cris}(H^i) = \dim H^i.$$

Crystalline Frob.

Thm. ① $\det(x\cdot id - \varphi; \mathbb{D}_{cris}(H^i)) \in \mathbb{Z}[x]$.

& this is equal to $\det(x\,id - \varphi_q; H^i_{et}(X_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_\ell))$ for any $\ell \neq p$.

- All φ-eig. values are $q^{i/2}$-Weil nos.
- It's a conj. that $\varphi \curvearrowright \mathbb{D}_{cris}(H^i)$ is semisimple.

Q: What can one say about $v_p(\varphi\text{-eig. val.})$? $\in [0, i]$

② There's a canonical isom. $\mathbb{D}_{cris}(H^i) \simeq \underbrace{H^i_{dR}(X)}_{\text{filtred } \mathbb{Q}_p\text{-v.s.}}$

The HT wts of $\mathbb{D}_{cris}(H^i)$ are precisely j w/ mult. $\dim H^j(X, \Omega^{i-j})$

(note $gr^j H^i_{dR}(X) = H^j(X, \Omega^{i-j})$)

Eg. If $A/\mathbb{Z}_p$ is an abel. var. of dim n (or curve C of genus n), for $H^1 = H^1_{et}(A_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_p)$,

$\varphi \curvearrowright \mathbb{D}_{cris}(H^1) =$ $2g$-dim'l, HT wts $= \{0^{n\text{ times}}, 1^{n\text{ times}}\}$

$|\varphi\text{-eig. val.}| = \sqrt{p}$

Case K general K fin. $/\mathbb{Q}_p$, X proper sm $/k$, $\ell = p$

$\mathrm{Gal}_k \curvearrowright H^i_{\acute{e}t}(X_{\bar{k}} ; \mathbb{Q}_p)$

There's a way (p-adic Hodge theory) to attach a WD-rep'n to H^i.

& it is conj. to be equal to $WD(H^i(X_{\bar{k}}, \mathbb{Q}_\ell))$ for $\ell \neq p$.

Now, let $k = \mathbb{Q}$, X projective sm. var. $/\mathbb{Q}$

$\forall \ell$, $H^i_\ell(X) := H^i_{\acute{e}t}(X_{\bar{\mathbb{Q}}} ; \mathbb{Q}_\ell) \circlearrowleft \mathrm{Gal}_{\mathbb{Q}}$.

For each prime p, $H^i_\ell(X)\big|_{\mathrm{Gal}_{\mathbb{Q}_p}} \simeq H^i_{\acute{e}t}(X_{\bar{\mathbb{Q}}_p} ; \mathbb{Q}_\ell) \circlearrowleft \mathrm{Gal}_{\mathbb{Q}_p}$.

Then $\{H^i_\ell(X)\}_{\ell \text{ prime}}$ forms a compatible system of Galois rep'ns.

Def'n Let L be a no. field. and for each prime λ of L, let

$\rho_\lambda : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_n(\bar{L}_\lambda)$ be a cts semisimple rep'n

(a) We say that $\{\rho_\lambda\}$ is weakly compatible if $\exists$ a finite set of places S of $\mathbb{Q}$ (ram. set)

s.t. if $p \notin S$, and $\lambda \nmid p$, then ρ_λ is unramified @ p. and

$\mathrm{char}_{\rho_\lambda(\mathrm{Frob}_p)}(x) \in L[x] \subset \bar{L}_\lambda[x]$, indep. of λ.

(b) We call a weakly compatible system $\{\rho_\lambda\}$ compatible if

(i) $\forall p$ s.t. $\lambda | p$, the restriction $\rho_\lambda\big|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is de Rham (crystalline)

(ii) $\forall p$, s.t. $\lambda | p$, the set $HT(\mathbb{D}_{\mathbb{Q}_p}(\rho_\lambda|_{\mathrm{Gal}_{\mathbb{Q}_p}}))$ ← j must $\dim H^j(X, \Omega^{i-j})$

is indep of p.

(iii) If $p \notin S$, and $\lambda | p$, then $\rho_\lambda|_{\mathrm{Gal}\,\mathbb{Q}_p}$ is crystalline. and $\det(x \cdot id - \varphi ; \mathbb{D}_{cris}(\rho_\lambda|_{\mathrm{Gal}\,\mathbb{Q}_p}))$

$\ell = p$ case.

$$= \mathrm{char}\, \rho_{\lambda'}(\mathrm{Frob}_p)(x) \quad \text{for } \lambda' \nmid p.$$

$p = \infty$ (iv) let $c =$ a cplx conjugation, then $\mathrm{tr}\, \rho_\lambda(c) \in \mathbb{Z}$ is indep. of λ.

(c) A compatible system is <u>strictly compatible</u> if $\forall p$, the Frob. s.s. assoc. WD rep'n

$WD_p(\rho_\lambda)$ of $W_{\mathbb{Q}_p}$ is indep of λ.

<u>Thm</u> If X proj. sm. $/\mathbb{Q}$, then $\{H^i_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)^{ss}\}$ is a cptble system $/\mathbb{Q}$.

If X is an AV$/\mathbb{Q}$, then strictly compatible.

<u>Lecture 17</u>. <u>Recall</u> <u>compatible system of Galois rep'ns</u>

<u>Def'n</u> L no. field, $\{\rho_\lambda : \mathrm{Gal}\,\mathbb{Q} \longrightarrow GL_n(L_\lambda)\}_{\lambda \text{ prime of } L}$ cts s.s. rep'n

<u>Say</u> * <u>weakly compatible system</u> if $\mathrm{char}_{\rho_\lambda(\mathrm{Frob}_p)}(x) \in L[x]$ is indep. of λ

(if $p \notin S_{ram}$)

* <u>compatible system</u> if - $HT(\rho_\lambda|_{\mathrm{Gal}\,\mathbb{Q}_p})$ when $\lambda | p$ is indep. of λ

- $\rho_\lambda|_{\mathrm{Gal}\,\mathbb{Q}_p}$ is de Rham & crystalline if $p \notin S_{ram}$, $\mathrm{char}(\varphi) =$

- $\mathrm{tr}(\rho_\lambda(\text{cpx conj.}))$ is indep. of λ.

* <u>strictly compatible</u> $WD(\rho_\lambda|_{\mathrm{Gal}\,\mathbb{Q}_p})$ is indep. of λ.

<u>Thm</u> (1) If $X/\mathbb{Q}$ is proj. sm., then $\{H^i_{\acute{e}t}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)^{ss}\}$ is a cptble system.

(2) If $X/\mathbb{Q}$ is an abel. var. $/\mathbb{Q}$, then $\{H^1_{\acute{e}t}(X_{\bar{\mathbb{Q}}}; \mathbb{Q}_\ell)^{ss}\}$ is strictly compatible.

Conj. (1) Any <u>algebraic</u> ℓ-adic rep'n $\rho_\ell : \mathrm{Gal}_{\mathbb{Q}} \to GL_n(\overline{\mathbb{Q}}_\ell)$ is a part of a strictly compatible system.

means ① ρ_ℓ is unramified outside a finite set of places

② $\rho_\ell|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}$ is de Rham ($\Leftarrow$ $\rho_\ell|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}$ is crystalline)

(2) weakly compatible $\Rightarrow$ strictly compatible $\Rightarrow$ comes from summands of $\{H^i_\ell(X)\}$.

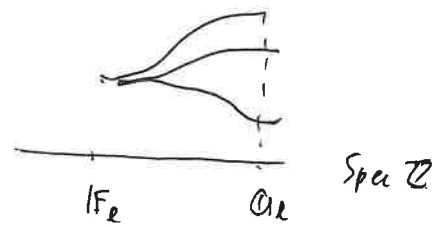
<u>Fontaine-Mazur Conj.</u> Every algebraic ℓ-adic rep'n comes from geometry.

<u>Dimension numerics</u> By Conj. $\Rightarrow$ any algebraic ℓ-adic rep'ns should come from geometry.

So fixing ramification set S and HT wts $\rightsquigarrow$ discrete set of such rep'ns $\longleftrightarrow M(\mathbb{Q})$

$\Rightarrow$ expectedly, the deformation space for such rep'ns should have discrete fibers $/\overline{\mathbb{Q}}_\ell$.

Fix $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_\ell)$,



$X^{\det = \chi}_{\bar{\rho}}$

$\downarrow$ rel. dim $\geq \chi(G_{\mathbb{Q},S}, \mathrm{Ad}^0\bar{\rho}) - \chi_{loc}(G_{\mathbb{Q},S}, \mathrm{Ad}^0\bar{\rho})$

$= 0$

$\boxed{\prod_v X^{\det = \chi_v}_{\bar{\rho}_v}} \leftarrow$ dim $= \begin{cases} 0 & \text{at } v \neq \ell\infty \\ 3 & \text{at } v = \ell \\ -\chi(G_{\mathbb{Q}_v}, \mathrm{Ad}^0\bar{\rho}_v) & \begin{cases} -3 & \det(\rho(c)) = 1 \\ -1 & \det(\rho(c)) = -1 \end{cases} \end{cases}$

$v = \infty$: $\det(\rho(c)) = 1$ — even; $\det(\rho(c)) = -1$ — odd

expected dim = 0

in odd case

$$\begin{tikzcd}
X^{\det=\chi,\, FL\otimes\ell}_{\bar\rho} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X^{\det=\chi}_{\bar\rho} \arrow[d] \\
X^{FL}_{\bar\rho_\ell} \times \prod_{v\nmid \ell} X^{\det=\chi_v}_{\bar\rho_v} \arrow[r] & \prod_{v} X^{\det=\chi_v}_{\bar\rho_v}
\end{tikzcd}$$

wt HT wts $\{0,1\}$

dim 1

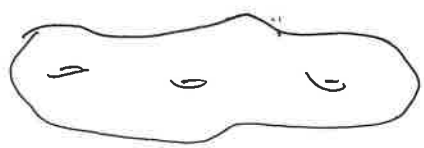
Jacobians of curves

§ story over $\mathbb{C}$

Let X be a proj. smooth curve $/\mathbb{C}$ of genus g.

$X^{an} := X(\mathbb{C})$ Riemann surface.

$H^0(X, \Omega^1_X) = g$-dim'l $/\mathbb{C}$



* <u>Pairing</u> $H^0(X, \Omega^1_X) \times H_1(X^{an}, \mathbb{Z}) \longrightarrow \mathbb{C}$

$(\omega, \gamma) \longmapsto \int_\gamma \omega$

Each loop γ gives a functional on $H^0(X, \Omega^1_X)$.

Define the Jacobian $\mathrm{Jac}_X = H^0(X, \Omega^1_X)^\vee / H_1(X^{an}; \mathbb{Z}) \simeq \mathbb{C}^g / \Lambda \simeq \mathbb{Z}^{2g}$

- This is a complex torus, & in fact an abelian var.

Fixing a point x_0 of X, there's a map

$$X^{an} \xrightarrow{\text{Abel-Jacobi map}} \mathrm{Jac}_X$$
$$x \longmapsto \left[\omega \mapsto \int_{x_0}^{x} \omega\right]$$

for a different choice of path, differed by some elt in $H_1(X^{an}; \mathbb{Z})$

Functorially for $h: X \to Y$ finite flat map of sm. proj. curves $/ \mathbb{C}$

- $h^* : H^0(Y, \Omega^1_Y) \longrightarrow H^0(X, \Omega^1_X)$

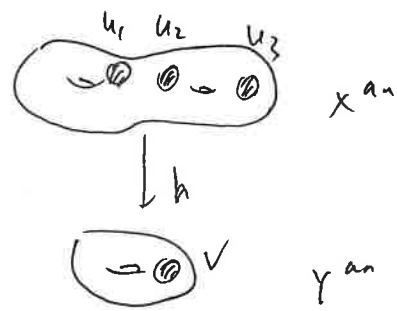
$$\int_\gamma h^* \omega = \int_{h \circ \gamma} \omega$$

So
$$\begin{array}{ccc} h_* : \dfrac{H^0(X, \Omega^1_X)^\vee}{H_1(X^{an}, \mathbb{Z})} & \longrightarrow & \dfrac{H^0(Y, \Omega^1_Y)^\vee}{H_1(Y^{an}, \mathbb{Z})} \\ \| & & \| \\ \mathrm{Jac}_X & \longrightarrow & \mathrm{Jac}_Y \end{array}$$

Going in the other direction. $\deg h = n$

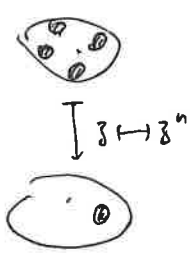
$$tr_h : H^0(X^{an}, \Omega^1_{X^{an}}) \longrightarrow H^0(Y^{an}, \Omega^1_{Y^{an}})$$

locally on $V \subset Y$
(away from ram.)
$h^{-1}(V) = \coprod_{i=1}^{n} U_i$



$$\omega \longmapsto tr_h(\omega)(V) = \sum_{i=1}^{n} h^{-1}(\omega(U_i))$$

<u>Eg.</u>



$$tr_h(dz) = dz + dz + \cdots + dz = n \cdot dz$$

<u>Rmk</u> tr_h is dual to $H^1(Y, \mathcal{O}_Y) \xrightarrow{h^*} H^1(X, \mathcal{O}_X)$

<u>By Serre duality</u> $H^0(Y, \Omega^1_Y)^\vee \longrightarrow H^0(X, \Omega^1_X)^\vee$

Similarly, for a loop γ on Y. $\int_\gamma tr_h(\omega) = \int_{h^{-1}(\gamma)} \omega$

$\rightsquigarrow$ get a map $h^*: \dfrac{H^0(Y, \Omega^1_Y)^\vee}{H_1(Y^{an}; \mathbb{Z})} \longrightarrow \dfrac{H^0(X, \Omega^1_X)^\vee}{H_1(X^{an}; \mathbb{Z})}$

$$\begin{array}{ccc} \| & & \| \\ \mathrm{Jac}_Y & & \mathrm{Jac}_X \end{array}$$

$$\mathrm{Jac}_X \underset{h^*}{\overset{h_*}{\rightleftarrows}} \mathrm{Jac}_Y \quad \Rightarrow \quad h_* \circ h^* = \text{mult. } \deg h \qquad \square$$

<u>Rmk.</u> There are nat'l isoms

$$H^1(X; \mathbb{Z}) \simeq H^1(\mathrm{Jac}_X; \mathbb{Z}) \quad \& \quad H^0(X, \Omega^1_X) \simeq H^0(\underset{\mathbb{C}^g/\mathbb{Z}^{2g}}{\mathrm{Jac}_X}, \Omega^1_{\mathrm{Jac}_X})$$

<u>Def'n</u> $\mathrm{Div}(X) = \{$group of divisors on $X\}$

$$\cup \qquad = \Big\{ \sum_{x \in X(\mathbb{C})} n_x [x], \quad n_x = 0 \text{ for all but finitely many } x \Big\}$$
$$n_x \in \mathbb{Z}$$

$\mathrm{Div}^0(X) = \{$group of $\deg = 0$ divisors on $X\}$

$\cup$

$\mathrm{Prin}(X) = \{ \delta \in \mathrm{Div}(X) : \delta = \mathrm{div}(f) = \mathrm{zero}(f) - \mathrm{pole}(f)$ for some $f \in \mathbb{C}(X)^\times \}$

$\mathrm{Pic}^{(0)}(X) = \mathrm{Div}^{(0)}(X) / \mathrm{Prin}(X) \simeq \{$line bundles (of deg 0) on $X\}$

$$D \longmapsto \mathcal{O}_X(D)$$

Pick a meromorphic section s of $\mathcal{L}$, $\mathrm{zero}(s) - \mathrm{pole}(s)$ $\longleftarrow\!\shortmid$ $\mathcal{L}$

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

Abel-Jacobi thm. There's an isom.

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X)$$

$$\sum_{x\in X} n_x [x] \longmapsto \sum_{x\in X} n_x \int_{x_0}^{x}$$

- Check that this is well-defined. for $f \in \mathbb{C}(X)\setminus\mathbb{C}$, WTS $\mathrm{im}(\mathrm{div}(f)) = 0$.

Upshot. such f defines a map $X \xrightarrow{f} \mathbb{P}^1$

But $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$

then $\displaystyle\sum_{x\in X} n_x \int_{x_0}^{x} \omega = \int_{\text{poles}(f)}^{\text{zeros}(f)} \omega = \int_{\infty}^{0} \mathrm{tr}_f(\omega)$ $\Rightarrow \mathrm{tr}_f(\omega) = 0$.

"Proof". Consider exact seq. in X^{an}

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{X^{an}} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^{\times}_{X^{an}} \to 0$$

$$\begin{tikzcd}
H^1(X^{an}; \underline{\mathbb{Z}}) \arrow[r] \arrow[d, equal] & H^1(X^{an}, \mathcal{O}_{X^{an}}) \arrow[r] \arrow[d, "\text{Serre duality}"] & H^1(X^{an}, \mathcal{O}^{\times}_{X^{an}}) \arrow[r] \arrow[d, equal] & H^2(X^{an}; \underline{\mathbb{Z}}) \arrow[d, equal] \\
H^1(X^{an}; \mathbb{Z}) \arrow[d, "\text{Poincaré duality}"] & & & H^2(X^{an}, \mathbb{Z}) \arrow[d, equal] \\
H_1(X^{an}; \mathbb{Z}) \arrow[r] & H^0(X, \Omega^1_X) \arrow[r] & \mathrm{Pic}(X^{an}) \arrow[r, "\deg"] & \mathbb{Z} \\
 & \mathrm{Coker} = \mathrm{Jac}_X \arrow[u] & \mathrm{Pic}(X) \arrow[u, equal] & \mathrm{Ker} = \mathrm{Pic}^0(X) \arrow[u]
\end{tikzcd}$$

Exer. $h: X \to Y$ finite flat of deg n

$$\begin{tikzcd}
h_*: \mathrm{Jac}_X \arrow[r] \arrow[d, equal] & \mathrm{Jac}_Y \arrow[d, equal] \\
\mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}^0(Y) \\
\mathcal{L} \arrow[r, mapsto] & \text{"}\mathrm{Nm}(\mathcal{L})\text{"} = \det h_*\mathcal{L}
\end{tikzcd}$$

$h_*\mathcal{L}$ loc. free of rk n

$$\begin{tikzcd}
h^*: \mathrm{Jac}_Y \arrow[r] \arrow[d, equal] & \mathrm{Jac}_X \arrow[d, equal] \\
\mathrm{Pic}^0(Y) \arrow[r] & \mathrm{Pic}^0(X) \\
\mathcal{L} \arrow[r, mapsto] & h^*\mathcal{L}
\end{tikzcd}$$

Kummer exact sequence

$$1 \to \mu_n \to \mathcal{O}_{X^{an}}^\times \xrightarrow{x \mapsto x^n} \mathcal{O}_{X^{an}}^\times \to 1$$

$$H^0(X^{an}, \mathcal{O}_{X^{an}}^\times) \xrightarrow{x \mapsto x^n} H^0(X^{an}, \mathcal{O}_{X^{an}}^\times)$$

$$\mathcal{O}_{X^{an}}^\times \simeq \mathbb{C}^\times, \quad \mathcal{O}_{X^{an}}^\times \simeq \mathbb{C}^\times$$

$$0 \to H^1(X^{an}, \mu_n) \to H^1(X^{an}, \mathcal{O}_{X^{an}}^\times) \to H^1(X^{an}, \mathcal{O}_{X^{an}}^\times)$$

$$H^1(X^{an}, \mathcal{O}_{X^{an}}^\times) \simeq \mathrm{Pic}(X), \quad H^1(X^{an}, \mathcal{O}_{X^{an}}^\times) \simeq \mathrm{Pic}(X)$$

$$\mathrm{Pic}(X) \xrightarrow{L \mapsto L^{\otimes n}} \mathrm{Pic}(X)$$

$$H^1(X^{an}, \mu_n) \simeq \mathrm{Pic}(X)[n] \simeq \mathrm{Pic}^0(X)[n] \simeq \mathrm{Jac}_X[n]$$

$$\varprojlim_n H^1(X^{an}, \mu_{\ell^n}) \simeq \varprojlim_n \mathrm{Pic}^0(X)[\ell^n]$$

§2. Story over schemes

Def'n Let S be a scheme and let $f: X \to S$ be a proper morphism.

Consider $\mathcal{P}^1_{X/S} : \mathrm{Sch}/_S^{fppf} \to \mathrm{Ab}$

$$T \longmapsto \mathrm{Pic}(X \times_S T)$$

defines a presheaf in "fppf topology" & let $\mathcal{P}_{X/S}$ be its sheafification.

Rmk. Say $\bigcup_i S_i$ open cover of S. it's very likely to have a nontrivial l.b. L on X that is trivial on each $X \times_S S_i$. $\Rightarrow \mathrm{Pic}(X) \to \prod_{i \in I} \mathrm{Pic}(X \times_S S_i)$ not injective.

Fact. Assume that - $\mathcal{O}_S \xrightarrow{\sim} f_* \mathcal{O}_X$ is an isom. $\Leftrightarrow$ all geom. fibers are reduced & conn'd

- $f: X \to S$ has a section $\sigma: S \to X$

Then for any T flat $/S$, we have a split exact sequence

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X \times_S T) \longrightarrow \mathcal{P}_{X/S}(T) \longrightarrow 0$$

i.e. $P_{X/S}(T) = \{ \mathcal{L} \in \mathrm{Pic}(X \times_S T) \} / \mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L} \simeq \mathcal{L}' \otimes f_T^* K$

for some $K \in \mathrm{Pic}(T)$

<u>Define</u> $\mathcal{P}^0_{X/S}(T) = \{ \mathcal{L} \in \mathcal{P}_{X/S}(T) :$ over each geom. pt $\bar{t}$ of T, $\mathcal{L}|_{X_{\bar{t}}}$ has deg. on each irred. comp. of $X_{\bar{t}} \}$

$\bar{t} : \mathrm{Spec}\, k^{sep} \to T$

<u>Thm</u> When $f: X \to S$ is a proper smooth curve of genus g s.t. each geom. fiber is conn.

(1) $\mathcal{P}^0_{X/S}$ is rept'd by an abelian scheme $j: J_{X/S} \to S$ of rel. dim g.

(2) There's a nat'l isom. $j_* \Omega^1_{J_{X/S}/S} \simeq f_* \Omega^1_{X/S}$

In particular, if $S = \mathrm{Spec}\, \mathbb{C}$, $\mathcal{P}_{X/S}(\mathrm{Spec}\, \mathbb{C}) = \mathrm{Pic}(X)$ in the usual sense.

$J_{X/\mathbb{C}} \simeq \mathrm{Jac}_X$ defined earlier.

<u>Rmk</u> A finite flat morphism $X \xrightarrow{h} Y$, $X \to S$, $Y \to S$ induces nat'l maps

$$h_* : \mathcal{P}^1_{X/S}(T) \to \mathcal{P}^1_{Y/S}(T), \qquad h^* : \mathcal{P}^1_{Y/S}(T) \to \mathcal{P}^1_{X/S}(T)$$

$$\mathcal{L} \text{ on } X \times_S T \longmapsto \det(h_{T*}\mathcal{L}) \qquad \mathcal{L} \longmapsto h_T^* \mathcal{L}$$

$\rightsquigarrow h_* : J_{X/S} \to J_{Y/S}$, $h^* : J_{Y/S} \to J_{X/S}$ s.t. $J_{Y/S} \xrightarrow{h^*} J_{X/S} \xrightarrow{h_*} J_{Y/S}$ is mult. by $\deg h$.

Assoc. Galois rep'ns

X proper smooth geom. conn'd curve $/\mathbb{Q}$ of deg g.

$\rightsquigarrow J_{X/\mathbb{Q}}$ ℓ-adic Tate module $T_\ell(J_{X/\mathbb{Q}}) \simeq \varprojlim_n J_{X/\mathbb{Q}}[\ell^n](\bar{\mathbb{Q}}) \circlearrowleft \mathrm{Gal}_{\mathbb{Q}}$, $\cong \mathbb{Z}_\ell^{\oplus 2g}$

$\varprojlim_n J_{X/\mathbb{Q}}[\ell^n] \simeq \varprojlim_n H^1_{et}(X_{\bar{\mathbb{Q}}}, \mu_{\ell^n}) = H^1_{et}(X_{\bar{\mathbb{Q}}}; \mathbb{Z}_\ell) \otimes \chi_{cycl}$

Lecture 18 Recall. Let S be a noe. scheme. X a proper variety $/S$, $f: X \to S$

Assume that - $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isom. $\Leftrightarrow$ all fibers are geom. conn'd

- $f: X \to S$ has a section $\sigma: S \to X$

$\cdot$ $P^{(0)}_{X/S}$: $Sch^{\text{flat. finite type}}_{/S} \longrightarrow Set$

(of deg 0 on each fiber)

$P_{X/S}(T) = \{ \mathcal{L} \text{ is a l.b. on } X \times_S T \text{ s.t. } \sigma^*\mathcal{L} \simeq \mathcal{O}_S \}$

$\mathcal{L} \otimes (f^*\sigma^*\mathcal{L})^{-1}$

$\uparrow$

$\mathcal{L}$ $= \{ \text{l.b. on } X \times_S T \} / \mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L} \simeq \mathcal{L}' \otimes f^* K$, K l.b. on S

(of deg 0 on each fiber)

If $X \to S$ proper smooth curve of genus g,

Fact. This $P^0_{X/S}$ is rep'd by an abelian scheme $\mathrm{Pic}^0_{X/S}$ over S.

Assoc. Galois rep'n

$X/\mathbb{Q}$ proper smooth curve of genus g.

$\rightsquigarrow \mathrm{Pic}^0_{X/\mathbb{Q}}$ ℓ prime

Tate module $T_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}}) := \varprojlim_n \mathrm{Pic}^0_{X/\mathbb{Q}}[\ell^n](\bar{\mathbb{Q}}) \approx \mathbb{Z}_\ell^{2g}$ (ℓ^n-torsion)

$\circlearrowleft$

$\mathrm{Gal}_{\mathbb{Q}}$

$V_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}}) = T_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}})[\frac{1}{\ell}]$

Fact. $V_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}}) \simeq H^1_{\acute{e}t}(\mathrm{Pic}^0_{X/\mathbb{Q}}, \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \otimes \chi_{cycl}$.

Lemma. If X has good reduction at $p \neq \ell$ then $V_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}})$ is unramified @ p.

Later. will consider $V_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}}) \otimes \chi_{cycl}^{-1}$.

Fact. geom. Frob. eig.vals on $V_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}})$ are $p^{-1/2}$ - Weil no's & are indep. of ℓ

When X has good reduction @ ℓ, $V_\ell(\mathrm{Pic}^0_{X/\mathbb{Q}})$ is crystalline of HT wts

$\{0 \text{ g times}, 1 \text{ g times}\}$

"Proof of the Lemma"

$$\begin{tikzcd} \exists\ \mathcal{X} \arrow[d, "\text{proper smooth}"'] & X \arrow[l] \arrow[d] \\ \mathrm{Spec}\,\mathbb{Z}_{(p)} & \mathrm{Spec}\,\mathbb{Q} \arrow[l] \end{tikzcd} \rightsquigarrow \begin{array}{c} \mathrm{Pic}^0_{\mathcal{X}/\mathbb{Z}_{(p)}} \\ | \\ \mathrm{Spec}\,\mathbb{Z}_{(p)} \end{array}$$

Fact. mult. by ℓ is étale on $J_{\mathcal{X}/\mathbb{Z}_{(p)}}$

$$\begin{tikzcd} \Rightarrow \mathrm{Pic}^0_{X/\mathbb{Q}_p}(\overline{\mathbb{Q}}_p) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cup"] & \mathrm{Pic}^0_{\mathcal{X}/\mathbb{Z}_p}(\overline{\mathbb{Z}}_p) \arrow[r, "\text{reduction}", "\text{mod } p"'] & \mathrm{Pic}^0_{\mathcal{X}_{\mathbb{F}_p}/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \arrow[d, phantom, "\cup"] \\ \mathrm{Pic}^0_{X/\mathbb{Q}_p}[\ell^n](\overline{\mathbb{Q}}_p) \arrow[rr] \arrow[u, phantom, "\text{properness of Picard}"] \arrow[d, phantom, "\circlearrowleft"] & & \mathrm{Pic}^0_{\mathcal{X}_{\mathbb{F}_p}/\mathbb{F}_p}[\ell^n](\overline{\mathbb{F}}_p) \arrow[d, phantom, "\circlearrowleft"] \\ \mathrm{Gal}_{\mathbb{Q}_p} & & \mathrm{Gal}_{\mathbb{F}_p} \end{tikzcd}$$

Bad reduction case.

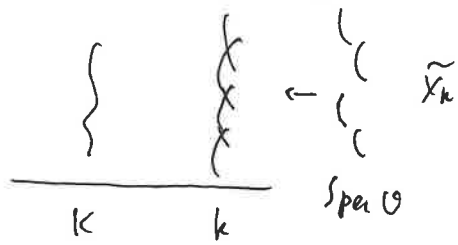
Consider $S = \operatorname{Spec} \mathcal{O}$ w/ $\mathcal{O}$ a DVR, $K = \operatorname{Frac}(\mathcal{O})$, k res. field.

$X \longrightarrow S$ a proper conn'd relative curve, w/ geom. conn'd fibers.

s.t. X_K is smooth of genus g, and completion at a closed pt $x \in X_k$ is either $\mathcal{O}'[[t]]$ or $\mathcal{O}'[[t,s]]/(ts - \varpi_K)$ for some $\mathcal{O}'/\mathcal{O}$ unramified.

$$\Big\} \otimes_{\mathcal{O}} k$$

$$k'[[t,s]]/(ts)$$



<u>Thm</u> $P^0_{X/S}$ is repd by a flat conn'd gp scheme $J_{X/S}$.

s.t. (1) $J_{X/S,K}$ is an abel. var.

(2) let $\tilde{X}_k = \coprod_{i=1}^{r} \tilde{X}_i \longrightarrow X_k$ be the normalization of the special fiber.

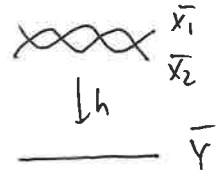
Then
$$1 \longrightarrow \text{torus} \longrightarrow \underset{\substack{\| \\ J_{X/S,k}}}{\overset{\text{Semi-abelian}}{J_{X_k/k}}} \longrightarrow \underbrace{\underbrace{\coprod_{i=1}^{r} J_{\tilde{X}_i/k}}_{\text{abel. var.}}}_{\text{abelian part of } J_{X_k/k}} \longrightarrow 1$$

Similar to last time, if $h: X \longrightarrow Y$ is a finite flat morphism of such rel. curves,

$$\rightsquigarrow h^*: J_{Y/S} \longrightarrow J_{X/S}, \quad h_*: J_{X/S} \longrightarrow J_{Y/S}$$

<u>A special case</u>

$$\begin{tikzcd}
* \arrow[d, "h"'] & * \arrow[l] \arrow[d] & \mathbb{F}_p = \bar{X}_1 \cup \bar{X}_2 \arrow[dl, "h_1"] \arrow[dl, "h_2"'] \\
\mathcal{Y} \arrow[d, dash, "\mathbb{Z}_p"'] & \mathcal{Y}_{\mathbb{F}_p} \arrow[l] & \\
\text{proper smooth} & &
\end{tikzcd}$$



$$\begin{tikzcd} J_{\mathcal{X}/S} \arrow[d, "h_*"] & J_{\mathcal{X}_{\mathbb{F}_p}/\mathbb{F}_p} \arrow[l] \arrow[d, "h_*"] \\ J_{\mathcal{Y}/S} & J_{\mathcal{Y}_{\mathbb{F}_p}/\mathbb{F}_p} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & \text{torus} \arrow[r] & J_{\mathcal{X}_{\mathbb{F}_p}} \arrow[r] \arrow[d] & J_{\bar{X}_1} \times J_{\bar{X}_2} \arrow[r] \arrow[dl, "h_{1*}"] \arrow[dll, "h_{2*}"] & 1 \\ & & J_{\mathcal{Y}_{\mathbb{F}_p}} \arrow[ul, dashed, "\text{affine}"] \arrow[ul, dashed, bend left, "\text{trivial}"] & & \end{tikzcd}$$

projective, geom. conn.

h_* factors through $J_{\bar{X}_1} \times J_{\bar{X}_2}$

Σ

Modular forms

Def'n For $N \geq 1$, $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ N \mid c \right\}$ $\qquad \dfrac{\Gamma_0(N)}{\Gamma_1(N)} \simeq (\mathbb{Z}/N\mathbb{Z})^\times$

$\cup$

$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0,\ d \equiv 1 \bmod N \right\}$

$\mathfrak{h} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \} \circlearrowleft SL_2(\mathbb{Z})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$ $\rightsquigarrow$ $Y_1(N) = \mathfrak{h}/\Gamma_1(N)$, $Y_0(N) = \mathfrak{h}/\Gamma_0(N)$

Def'n A modular form f of wt k, level N, nebentypus char. $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is a holomorphic func. $f : \mathfrak{h} \longrightarrow \mathbb{C}$ s.t. ① $f\left(\dfrac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

② For each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $(cz+d)^{-k} f\left(\dfrac{az+b}{cz+d}\right)$ is bounded as $z \to \infty$

Let $M_k(N;\chi)$ = space of such forms.

$\cup$

$S_k(N;\chi)$ = space of cusp. forms, i.e. $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $(cz+d)^{-k} f\left(\dfrac{az+b}{cz+d}\right) \to 0$ as $z \to \infty$.

Variants $\chi = 1$. $\rightsquigarrow$ $M_k(\Gamma_0(N)) := M_k(N;1)$

Say f has level $\Gamma_1(N)$ if $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$.

In this case, $(\mathbb{Z}/N\mathbb{Z})^\times \simeq \dfrac{\Gamma_0(N)}{\Gamma_1(N)} \curvearrowright M_k(\Gamma_1(N))$. For $\bar{d} \in (\mathbb{Z}/N\mathbb{Z})^\times$, pick $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ s.t. $d \equiv \bar{d} \bmod N$. define $\langle \bar{d} \rangle f(z) = (cz+d)^{-k} f\left(\dfrac{az+b}{cz+d}\right)$, indep. of choice of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

diamond operator

Thm. $S_k(\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} S_k(N;\chi)$. eigen span decomposition.

As $\begin{pmatrix}1&1\\0&1\end{pmatrix} \in \Gamma_1(N) \Rightarrow f(z) = f(z+1)$. Fourier analysis $\to$ q-expansion of modular forms

$f(q) = \sum_{n\geq 0} a_n q^n$, $q = e^{2\pi i z}$.

Hecke operator. When $p \nmid N$, $T_p : M_k(N:\chi) \to M_k(N;\chi)$

$$\begin{array}{ccc} M_k(N:\chi) & \to & M_k(N;\chi) \\ \cup & & \cup \\ S_k(N;\chi) & \to & S_k(N;\chi) \end{array}$$

$$T_p(f)(z) = \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right) + p^{k-1}\chi(p) f(pz)$$

$$\overset{f(q)=\sum a_n q^n}{=} \sum_{n\geq 0}\left(a_{np}\cdot q^n + p^{k-1}\chi(p) a_n q^{np}\right)$$

Fact. T_p sends $M_k(N;k)$ into $M_k(N;k)$

When $p \mid N$, $U_p : M_k(N;\chi) \to M_k(N;\chi)$, $U_p(f) = \sum_{n\geq 0} a_{np} q^n$.

We say that f is a normalized eigenform if $a_1 = 1$ and

for every $p \nmid N$, $T_p(f) = a_p \cdot f$, for every $p \mid N$, $U_p(f) = a_p \cdot f$.

$$\Rightarrow \begin{cases} a_{np^2} + p^{k-1}\chi(p) a_n = a_p \cdot a_{np} & \text{if } p \nmid N \\ a_{np} = a_p \cdot a_n & \text{if } p \mid N \end{cases}$$

$\to$ Hecke eig vals determine the q-expansion of eigen forms.

- When $f(q) = \sum_{n\geq 1} a_n q^n$ is a normalized cuspidal eigen form, define $L(f,s) = \sum_{n\geq 1}\frac{a_n}{n^s}$

Consider $\int_0^\infty f(iy) y^s \frac{dy}{y} \overset{\operatorname{Re} s \gg 0}{=\!=} \int_0^\infty \sum_{n=1}^\infty a_n e^{\overset{-2\pi n y}{2\pi i n \cdot iy}} y^s \frac{dy}{y}$

$$= \sum_{n=1}^\infty \int_0^\infty a_n e^{-2\pi n y} y^s \frac{dy}{y} \overset{w = 2\pi n y}{=\!=} \sum_{n=1}^\infty \int_0^\infty a_n e^{-w}\left(\frac{w}{2n\pi}\right)^s \frac{dw}{w}$$

$$= \sum_{n=1}^\infty \frac{a_n}{n^s}(2\pi)^{-s} \underbrace{\boxed{\int_0^\infty e^{-w}\cdot w^s \frac{dw}{w}}}_{\Gamma(s)} = (2\pi)^{-s}\Gamma(s) \underbrace{\boxed{\sum_{n=1}^\infty \frac{a_n}{n^s}}}_{L(f:s)}$$

Exer. If f is a normalized eigenform, we have

$$L(f,s) = \sum_{n=1}^\infty \frac{a_n}{n^s} = \prod_{p\nmid N} \overbrace{\frac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1-2s}}}^{L\text{-factor at } p} \prod_{p \mid N} \frac{1}{1 - a_p p^{-s}}$$

$L_\infty(f,s) = (2\pi)^{-s}\Gamma(s)$

Exer. Functional equation for $\Lambda(f,s) = \underbrace{L(f,s)}_{\xi} L_\infty(f,s)$.

Thm For each normalized eigen cusp form $f = \sum a_n q^n$ of level N, wt k, nebentypus $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

① $\mathbb{Q}(f) = \mathbb{Q}(a_n : n \geq 1) \supset \mathbb{Q}(\chi)$ is a no. field

② For every prime λ of $\mathbb{Q}(f)$, there's an irred cts rep'n

$$\rho_f : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}(f)_\lambda)$$

(1) when $p \nmid N \cdot Nm(\lambda)$, ρ_f is unram. @ p, & the char. poly. of Frob_p is

$$X^2 - a_p X + p^{k-1}\chi(p)$$

(2) When $p \mid Nm(\lambda)$, & $p \nmid N$, ρ is crystalline @ p of HT wts $\{0, k-1\}$.

& $\mathrm{char}(\phi, \mathbb{D}_{\mathrm{cris}}(\rho_f|_{\mathrm{Gal}_{\mathbb{Q}_p}})) = X^2 - a_p X + p^{k-1}\chi(p)$

(3) When $p^m \| N$ $(m \geq 1)$, $p \nmid Nm(\lambda)$, and $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is p-primitive. i.e. $\chi: (\mathbb{Z}/\frac{N}{p}\mathbb{Z})^\times \to \mathbb{C}^\times$

$\boxed{\text{Crystalline Frob. = Geom. Frob.}}$

Write $\chi = \chi_p \cdot \chi^p : (\mathbb{Z}/p^m\mathbb{Z})^\times \times (\mathbb{Z}/p^{-m}N\mathbb{Z})^\times \longrightarrow \mathbb{Q}(f)^\times$ $\quad \chi_p : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})|\mathbb{Q}_p) \simeq (\mathbb{Z}/p^m\mathbb{Z})^\times \xrightarrow{\chi_p} \mathbb{Q}(f)^\times$

Then $\rho|_{\mathrm{Gal}_{\mathbb{Q}_p}} = \begin{pmatrix} \mathrm{unr}(a_p) & \\ & \mathrm{unr}(p^{k-1}/a_p)\cdot\chi_p \end{pmatrix}$ where $\mathrm{unr}(\alpha)$ is the unram. char. of $G_{\mathbb{Q}_p}$ sending Frob_p to α.

Rmk. For a Galois rep'n $\rho: \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell) = GL(V)$ s.t. Frob. eig. vals are alg. integers, then we can associate an L-function ($\mathrm{Re}\, s \gg 0$)

$\Uparrow$

char (Frob) have $\mathbb{Q}$-coeffs

$\rho|_{\mathrm{Gal}\,\mathbb{Q}_\ell}$ de Rham

$$L(\rho, s) = \prod_{\substack{p \text{ prime} \\ p \neq \ell}} \underbrace{\frac{1}{\det\left(1 - \mathrm{Frob}_p \cdot p^{-s} : V^{I_{\mathbb{Q}_p}}\right)}}_{1 - a_p \cdot p^{-s} + p^{k-1}\chi(p)\, p^{-2s}} \cdot \text{factor at } \ell$$

$p \nmid N\ell$

$1 \longleftarrow$ if $\rho|_{\mathrm{Gal}\,\mathbb{Q}_\ell}$ is crystalline

$$\frac{1}{\det\left(1 - \phi_\ell \cdot \ell^{-s} : \mathbb{D}_{\mathrm{cris}}(V)\right)}$$

Upshot: It's always easier to extend $L(f, s)$ to the complex plane s.

If we can prove $\rho \longleftrightarrow f \Rightarrow L(\rho, s)$ extends meromorphically to complex plane

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Lecture 19. Geometric modular forms. $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$

Thm $Y_1(N) = \Gamma_1(N) \backslash \mathfrak{h}$ parametrizes all elliptic curves over $\mathbb{C}$ together w/ an N-torsion pt.

Explicitly, for $z \in \mathfrak{h} \longleftrightarrow \left(E_z = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z,\ \frac{1}{N} \in E_z\right)$

"Proof" Easy fact. Every elliptic curve $E/\mathbb{C}$ together w/ an N-torsion pt P is isom. to one of

On the other hand, if $(E_{z'}, \frac{1}{N}) \simeq (E_z, \frac{1}{N})$, then WTS $z' = \frac{az+b}{cz+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$.

$$\begin{array}{ccc} \| & & \| \\ \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z & \xrightarrow[\sim]{\varphi} & \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z \end{array}$$

φ must lift to

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z' \arrow[r] & \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z \end{tikzcd}$$

$\tilde{\varphi}$ = mult. by w

then $1 \longmapsto w = cz + d$

$z' \longmapsto wz' = az + b$

s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ $SL_2(\mathbb{Z})$

$\Rightarrow z' = \dfrac{az+b}{cz+d}$. $\operatorname{Im} z' > 0 \Rightarrow ad - bc > 0$

<u>Yet</u> $\frac{1}{N} \longmapsto \frac{1}{N} \mod \mathbb{Z} \oplus \mathbb{Z}z$, i.e. $\dfrac{w}{N} \equiv \dfrac{1}{N} \mod \mathbb{Z} \oplus \mathbb{Z}z$

$$\frac{w}{N} = \frac{cz+d}{N}$$

$\Rightarrow \begin{matrix} c \equiv 0 \\ d \equiv 1 \end{matrix} \pmod N$. Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$

Similarly, $Y_0(N) = \mathfrak{h}/\Gamma_0(N)$ parametrizes all elliptic curves over $\mathbb{C}$, together w/ a cyclic subgp of order N

$$z \in \mathfrak{h} \longleftrightarrow \left(E_z = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z,\ \tfrac{1}{N}\mathbb{Z}/\mathbb{Z} \subset E_z \right)$$

<u>Fact</u>: We have a universal family of elliptic curves

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ Y_1(N) = \Gamma_1(N) \backslash \mathfrak{h} \arrow[u, bend right, "e = \text{zero section}"'] \end{tikzcd} \qquad \begin{tikzcd} \left(E_z, \frac{1}{N}\right) \arrow[d, mapsto] \\ \ni z \end{tikzcd}$$

τ-plane

$\Omega^1_{\mathcal{E}/Y_1(N)}$ is locally free of rk 1

& $\omega = e^* \Omega^1_{\mathcal{E}/Y_1(N)}$

$= \pi_* \Omega^1_{\mathcal{E}/Y_1(N)}$

ω is a line bundle on $Y_1(N)$

$\Omega^1_{E_z/\mathbb{C}} = \mathcal{O}_{E_z} \cdot d\tau$

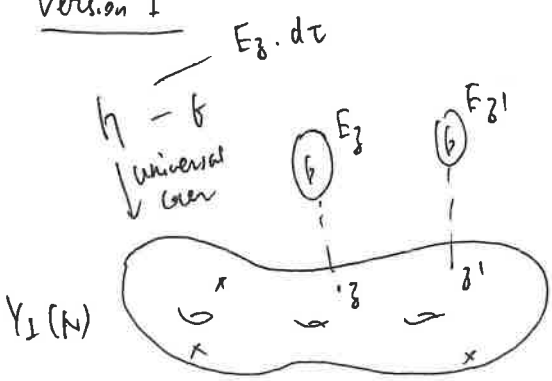
$\Omega^1_{E_z/\mathbb{C}}\big|_{\tau=0} = \mathbb{C} \cdot d\tau$

$\quad \| $

$H^0(E_z, \Omega^1_{E_z/\mathbb{C}})$

Geometric interpretation of modular forms

Version 1



On E_τ, the canonical differential is $d\tau$.

As we use the isom. $E_{\tau'} \xrightarrow{\sim} E_\tau$

$$\tau' \longmapsto \tau = w \cdot \tau'$$

$$= (c\tau + d)\tau'$$

thus $\quad (c\tau+d)\, d\tau' \longleftarrow\!\shortmid\; d\tau$

If we have a modular form f of wt k, level $\Gamma_1(N)$, define

$$E_\tau \longrightarrow \boxed{f(\tau) \otimes d\tau^{\otimes k}} \longleftarrow \text{this is indeed } \Gamma_1(N)\text{-invt}$$

$$\wr\wr$$

$$E_{\tau'} \longrightarrow f(\tau') \otimes (d\tau')^{\otimes k} = f\left(\frac{a\tau+b}{c\tau+d}\right) \frac{1}{(c\tau+d)^k}\, d\tau^{\otimes k} = f(\tau)\, d\tau^{\otimes k}$$

So $f(\tau) \otimes (d\tau)^{\otimes k}$ defines a section on $Y_1(N)$ of $\omega^{\otimes k}$

i.e. $f(\tau) \otimes d\tau^{\otimes k} \in H^0(Y_1(N), \omega^{\otimes k})$.

Version 2

When k is even,

note $\quad f(\tau)\, d\tau^{\otimes k/2}$ on $\mathfrak{h}$ is invt under $\Gamma_1(N)$-action

Indeed, $\quad f\left(\frac{a\tau+b}{c\tau+d}\right) d\left(\frac{a\tau+b}{c\tau+d}\right)^{\otimes k/2} = (c\tau+d)^k f(\tau) \left(\frac{a(c\tau+d) - c(a\tau+b)}{(c\tau+d)^2}\, d\tau\right)^{k/2}$

$$= f(\tau)\, d\tau^{\otimes k/2}$$

Thus $\quad f(\tau)\, d\tau^{\otimes k/2} \in H^0\left(Y_1(N), \left(\Omega^1_{Y_1(N)/\mathbb{C}}\right)^{\otimes k/2}\right)$

Rmk <u>Kodaira-Spencer isom.</u> There's a nat'l isom. $\omega^{\otimes 2} \simeq \Omega^1_{Y_1(N)/\mathbb{C}}$.

<u>Extension to cusps</u>. $X_1(N) = Y_1(N) \cup \underset{\overset{\shortparallel}{C}}{\{\text{cusp}\}} = \mathfrak{h}/\Gamma_1(N) \cup \mathbb{P}^1(\mathbb{Q})/\Gamma_1(N)$

$\mathfrak{h} \subset \mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$

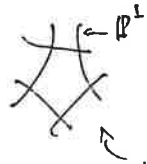
<u>Fact</u>

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, shift right, "\pi"'] & \bar{\mathcal{E}} \supset \mathcal{E}^{sm} \arrow[d, shift right, "\pi"'] \arrow[dl] \\ Y_1(N) \arrow[u, bend left, "e"] \arrow[r, hook] & X_1(N) \arrow[u, bend right, "e"'] \end{tikzcd}$$

$$\mathfrak{h}/SL_2(\mathbb{Z}) \cong \mathfrak{h}^{\pm}/GL_2(\mathbb{Z}) = \mathbb{C} - \mathbb{R} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$$

s.t. over each pt z of the cusp C,

we have "generalized elliptic curve".

$\bar{\mathcal{E}}_z = \pi^{-1}(z)$ looks like or 

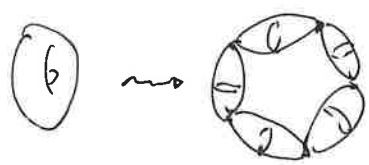
n-gon for some n

$\cup$

$\bar{\mathcal{E}}_z^{sm} = \bar{\mathcal{E}}_z \setminus$ Singular pts. $= \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$

is a gp scheme

$\bar{\mathcal{E}}^{sm}$ is a gp scheme $/X_1(N)$



& $\bar{\mathcal{E}}^{sm}$ acts on $\bar{\mathcal{E}}$.

$$\boxed{\underline{\omega} := e^* \Omega^1_{\bar{\mathcal{E}}^{sm}/X_1(N)}}$$

<u>Kodaira-Spencer isom.</u>

$$\omega^{\otimes 2} \simeq \Omega^1_{X_1(N)/\mathbb{C}}(\log C).$$

here, $\Omega^1_{X_1(N)}(\log C)$ = differential sheaf allowing log poles at cusps

i.e. if q_z is the local parameter at $z \in C$,

$$\Omega^1_{X_1(N)}(\log C)(U) = \Big\{ f(q)dq + \alpha \underbrace{\frac{dq}{q}}_{= d(\log q)} : f(q)dq \in \Omega^1_{X_1(N)}(U),\ \alpha \in \mathbb{C} \Big\}$$

<u>Rmk</u> in 1-dim'l case, $\Omega^1_{X_1(N)}(\log C) = \Omega^1_{X_1(N)} \otimes_{\mathcal{O}_{X_1(N)}} \mathcal{O}_{X_1(N)}(C)$.

<u>Geometric interpretation</u>

$$\begin{array}{ccc} M_k(\Gamma_1(N)) & = & H^0(X_1(N), \omega^{\otimes k}) \\ \cup & & \\ S_k(\Gamma_1(N)) & = & H^0(X_1(N), \omega^{\otimes k}(-C)) \end{array}$$

When $k=2$, KS-isom $\Rightarrow$

$$\begin{array}{ccc} M_2(\Gamma_1(N)) & = & H^0(X_1(N), \Omega^1_{X_1(N)/\mathbb{C}}(\log C)) \\ \cup & & \\ S_2(\Gamma_1(N)) & = & H^0(X_1(N), \Omega^1_{X_1(N)/\mathbb{C}}) \end{array}$$

$N \geq 4$

$\Gamma_0(N)$ - version if k is even.

<u>Hecke operators</u> Say $p \nmid N$

$$X(\Gamma_1(N) \cap \Gamma_0(p)) \xrightarrow{\pi_1} X_1(N), \qquad X(\Gamma_1(N) \cap \Gamma_0(p)) \xrightarrow{\pi_2} X_1(N)$$

$$\begin{tikzcd} & z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ z & & pz \end{tikzcd}$$

$$\left(E \xrightarrow[\text{order } p]{\text{cyclic isogeny}} E',\ P_N \in E \right)$$

pt of exact order N

cyclic subgp order p

$$\begin{tikzcd} & (E,\ P_N,\ C \subset E_z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (E,\ P_N) & & (E/C,\ \mathrm{Im}(P_N \in E \to E/C)) \end{tikzcd}$$

<u>Version 1</u> On $X(\Gamma_1(N) \cap \Gamma_0(p))$, we have $\varphi : \pi_1^* \mathcal{E} \longrightarrow \pi_2^* \mathcal{E}$.

$\Rightarrow \varphi^* : \pi_2^* \omega_2 \longrightarrow \pi_1^* \omega_1$

$$T_p^{\text{naive}} : H^0(X_1(N), \omega_2^{\otimes k}) \xrightarrow{\pi_2^*} H^0(X(\Gamma_1(N) \cap \Gamma_0(p)), \pi_2^* \omega_2^{\otimes k})$$
$$\xrightarrow{\varphi^*} H^0(X(\Gamma_1(N) \cap \Gamma_0(p)), \pi_1^* \omega_1^{\otimes k}) \xrightarrow[\text{Page 124}]{\mathrm{Tr}_{\pi_1}} H^0(X_1(N), \omega_1^{\otimes k}).$$

<u>Define</u> $T_p := \frac{1}{p} T_p^{naive} : S_k(\Gamma_1(N)) \to S_k(\Gamma_1(N))$

<u>Version 2</u>. $k=2$. realize $S_2(\Gamma_1(N)) \simeq H^0(X_1(N), \Omega^1_{X_1(N)})$

<u>Then</u>. $T_p : H^0(X_1(N), \Omega^1_{X_1(N)}) \xrightarrow{\pi_2^*} H^0(X(\Gamma_1(N) \cap \Gamma_0(p)), \Omega^1)$

$$\xrightarrow{Tr_\pi} H^0(X_1(N), \Omega^1_{X_1(N)})$$

<u>Exer</u>. check that this is compatible w/ version 1 under KS-isom.

$$\begin{tikzcd}
 & H_1(X_1(N)^{an}; \mathbb{Z}) \arrow[r, "\pi_1^*"] \arrow[d] & H_1(X(\Gamma_1(N) \cap \Gamma_0(p))^{an}; \mathbb{Z}) \arrow[d] \\
T_p^\vee : H^0(X_1(N), \Omega^1_{X_1(N)})^\vee \arrow[r, "(Tr_{\pi_1})^\vee"] \arrow[d, "(\pi_2^*)^\vee"] & H^0(X_1(\Gamma_1(N) \cap \Gamma_0(p)), \Omega^1)^\vee \\
H^0(X_1(N), \Omega^1_{X_1(N)})^\vee & \\
H_1(X_1(N)^{an}; \mathbb{Z}) \arrow[u] \arrow[uur, "\pi_{2*}"] &
\end{tikzcd}$$

$$\Rightarrow \quad T_p^\vee : \underset{\substack{\| \\ Jac(X_1(N))}}{J_1(N)} \to J(\Gamma_1(N) \cap \Gamma_0(p)) \to J_1(N)$$

<u>Upshot</u>.

$$\begin{array}{ccc}
S_2(\Gamma_1(N))^\vee & \twoheadrightarrow & J_1(N) \\
\circlearrowleft & & \circlearrowleft \\
(T_p)^\vee & & (T_p)^\vee
\end{array}$$

<u>Hodge theory</u>. For any smooth proj. curve $X/\mathbb{C}$, $H^1(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)}$

$\overline{H^0(X, \Omega^1_X)} \cong H^1(X, \mathcal{O}_X)$

So $H_1(X; \mathbb{Z}) \otimes \mathbb{C} \simeq H^0(X, \Omega^1_X)^\vee \oplus \overline{H^0(X, \Omega^1_X)^\vee}$.

In particular, $\underset{\substack{\| \\ H_1(Jac(X_1(N)); \mathbb{Z})}}{H_1(X_1(N); \mathbb{Z})} \otimes \mathbb{C} \simeq S_2(\Gamma_1(N))^\vee \oplus \overline{S_2(\Gamma_1(N))^\vee}$.

Cor. Eigenvalues of T_p are alg. integers ($k=2$)

Pf. This is because $T_p^\vee : H_1(X_1(N);\mathbb{Z}) \longrightarrow H_1(X_1(N);\mathbb{Z})$

$$\in \mathrm{End}(\mathbb{Z}^{\oplus 2})$$

So all eig. vals of $T_p^\vee$ on $H_1(X_1(N);\mathbb{Z})$ are alg. integers.

$\Rightarrow$ $T_p^\vee$ eig. vals on $S_2(\Gamma_1(N))$ are alg. integers.

Can similarly define U_p: when $p^m \| N$,

$$\left\{ \begin{array}{ccc} X(\Gamma_1(N)\cap\Gamma_0(p^{m+1})) & & (E, P_N, C \subset E) \\ \swarrow \quad \searrow & & \swarrow \quad \nwarrow \\ X_1(N) \qquad X_1(N) & & (E, P_N) \end{array} \right\}$$

cyclic of order p^{m+1}

$$\rightsquigarrow J_1(N) \xrightarrow{U_p} J_1(N)$$

$\forall \langle d\rangle \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\langle d\rangle : X_1(N) \longrightarrow X_1(N)$

$$(E,P) \longmapsto (E, d\cdot P)$$

$$\rightsquigarrow \langle d\rangle : J_1(N) \longrightarrow J_1(N)$$

More generally, consider the nat'l map

$$\mathbb{Z}\left[\begin{array}{ll} T_p : & p \nmid N \\ U_p : & p \mid N \\ \langle d\rangle : & d \in (\mathbb{Z}/N\mathbb{Z})^\times \end{array}\right] \longrightarrow \mathrm{End}(J_1(N)) \hookrightarrow \underbrace{\mathrm{End}(H_1(X_1(N)^{an};\mathbb{Z}))}_{\text{finite rank } /\mathbb{Z}}$$

Let $h_1(N) :=$ image of this map, which is finite $/\mathbb{Z}$.

Thm There's a perfect pairing $h_1(N)_{\mathbb{C}} \times S_2(\Gamma_1(N)) \longrightarrow \mathbb{C} \Rightarrow h_1(N)_{\mathbb{C}} \simeq S_2(\Gamma_1(N))^\vee$

$$(T, f) \longmapsto a_1(T(f))$$

Proof. non-deg. at S_2: If $f \in S_2(\Gamma_1(N))$ s.t. $\forall T \in h_1(N)$, $a_1(T(f)) = 0$.

then for $T = \underline{T_n}$, $\overset{0}{\overset{\shortparallel}{a_1(T_n(f))}} = a_n(f) \Rightarrow f(q) = 0 \Rightarrow f = 0$.

Here T_n is defined by $T_{nm} = T_n T_m$ if $\gcd(n,m) = 1$.

$T_{p^k} = U_p^k$ if $p \mid N$,

$T_{p^k} = T_p \cdot T_{p^{k-1}} - \chi(p) p^{k-1} T_{p^{k-2}}$ if $p \nmid N$

non-deg. @ $h_1(N)_{\mathbb{C}}$-side If $T \in h_1(N)_{\mathbb{C}}$ s.t. $\forall f \in S_2$, $a_1(Tf) = 0$,

Then $\underset{\overset{\shortparallel}{0}}{a_1(T(T_n f))} = a_1(T_n(Tf)) = a_n(Tf) \Rightarrow Tf(q) = 0 \Rightarrow Tf = 0$

This says that T kills all of $f \in S_2 \Rightarrow T = 0$. $\square$

Cor. $\underbrace{H_1(X_1(N)^{an}, \mathbb{Q})}_{\substack{\circlearrowleft \\ h_1(N) \otimes \mathbb{Q}\text{-mod} \\ \text{flat of rk } 2}} \otimes \mathbb{C} \xrightarrow{\sim} \underset{\substack{\circlearrowleft \\ h_1(N)_{\mathbb{C}}\text{-mod} \\ \text{free of rk } 1}}{S_2(\Gamma_1(N))^\vee} \oplus \underset{\substack{\circlearrowleft \\ h_1(N)_{\bar{\mathbb{C}}}\text{-mod} \\ \text{free of rk } 1}}{\overline{S_2(\Gamma_1(N))^\vee}}$

But $h_1(N) \otimes \mathbb{Q}$ is fin. dim'l

$\Rightarrow$ artinian $\Rightarrow$ free of rank 2

Rmk. The integral version might be badly behaved.

$$* \left\{ \begin{array}{l} \text{normalized eigen forms} \\ f \text{ in } S_2(N, \chi) \end{array} \right\} \qquad f = \sum a_n q^n$$

$$\Big\updownarrow \text{bijection}$$

$$\left\{ \text{homo. } \lambda_f : h_1(N) \twoheadrightarrow \underbrace{\mathbb{Q}(f)}_{\text{fin. dim'l}} \subset \mathbb{C} \right\} \qquad \mathbb{Q}(f) = \mathbb{Q}(a_n : n \in \mathbb{N})$$

$$\left[\begin{array}{ll} \lambda_f(T_p) = a_p & \text{if } p \nmid N \\ \lambda_f(U_p) = a_p & \text{if } p \mid N \\ \lambda_f(\langle d \rangle) = \chi(d), & d \in (\mathbb{Z}/n\mathbb{Z})^\times \end{array} \right.$$

$$\Big\downarrow$$

$$\left\{ \text{quotients of } J_1(N) \right\}$$

$$\Big\downarrow$$

Let $I_f := \ker(\lambda_f : h_1(N) \to \mathbb{C})$

Define $A_f = J_1(N) / I_f J_1(N)$

$$= \frac{S_2(\Gamma_1(N))[\lambda_f]^\vee}{H_1(X_1(N) : \mathbb{Z}) / I_f \cdot H_1}.$$

Then A_f is an abelian var. $(/\mathbb{Q})$ of dim $= [\mathbb{Q}(f) : \mathbb{Q}]$.

The Galois rep'n assoc. to f is $V_\ell(A_f) \otimes \chi_{cycl}^{-1}$.

<u>Rmk</u> $\mathbb{Q}(f)$ is always a totally real field or a <u>CM field</u>

$\parallel$

totally imaginary quadratic ext'n of a totally real field.

<u>Lecture 20</u> <u>Last time</u> $p \mid N$, $N = p^m M$, $p \nmid M$

$$X_1(N) \xleftarrow[\deg p]{\pi_1} \mathcal{M} \xrightarrow{\pi_2} X_1(N) \qquad \xrightarrow{\pi_2} \left\{ \left(E/C, \frac{P_M + C}{C}, \frac{P_{pm} + C}{C} \right) \right\}$$

$$\mathcal{M} = \left\{ (E, P_M, P_{pm}; C) : C \subset E[p] \text{ cyclic subgp of order } p,\ C \not\ni [p^{m-1}](P_{pm}) \right\}$$

$(E, P_M, P_{pm}) \longleftarrow$

P_M: a pt of exact order M; P_{pm}: a pt of exact order p^m

Today: Make constructions $/\mathbb{Z}$ & $\mathbb{Q}$

Prove Eichler–Shimura relation

$\boxed{N \geq 4}$ $Y_1(N)$ represents the functor $M_1(N): \mathrm{Sch}^{noe}_{\mathbb{Z}[\frac{1}{N}]} \longrightarrow \mathrm{Set}$

$$S \mapsto \left\{ (E, \iota) : \begin{array}{l} E \text{ elliptic curve } / S \\ \iota : \mu_N \hookrightarrow E[N] \end{array} \right\}$$

Fact. $M_1(N)$ is rep'd by a smooth affine curve $Y_1(N) / \mathrm{Spec}\, \mathbb{Z}[\frac{1}{N}]$

called the "open modular curve".

$X_1(N) :=$ "compactification" of $Y_1(N)$

$=$ proper smooth curve $/ \mathrm{Spec}\, \mathbb{Z}[\frac{1}{N}]$ "compactified modular curve"

- $Y_1(N)(\mathbb{C}) = \Gamma_1(N) \backslash \mathfrak{h}$, $X_1(N)(\mathbb{C}) = \Gamma_1(N) \backslash \mathfrak{h} \cup \Gamma_1(N) \backslash \mathbb{P}^1(\mathbb{Q})$

$C = X_1(N) - Y_1(N)$ cusp. finite flat $/ \mathrm{Spec}\, \mathbb{Z}[\frac{1}{N}]$.

Fact.

$$\begin{tikzcd}
\mathcal{E}_{Y_1(N)} \arrow[r] \arrow[d, "\text{universal elliptic curve}"] & \mathcal{E}_{X_1(N)} =: \mathcal{E} \arrow[d, "\text{universal generalized elliptic curves}"] & \mathcal{E}^{sm}_{X_1(N)} =: \mathcal{E}^{sm} \arrow[l] \arrow[dl, "\text{relative gp scheme}"] \\
Y_1(N) \arrow[r] & X_1(N) &
\end{tikzcd}$$

$\mathcal{E} \longrightarrow X_1(N)$ proper

$\mathcal{E}^{sm} \hookrightarrow \mathcal{E}$, $\mathcal{E}^{sm} \to X_1(N)$ gp scheme

Define $\omega = e^* \Omega^1_{\mathcal{E}^{an}/X_1(N)}$

$k \geq 1$ $S_k(\Gamma_1(N); \mathbb{Z}[\frac{1}{N}]) := H^0(X_1(N), \omega^{\otimes k}(-C))$.

Rmk. Kodaira–Spencer isom. holds: $\Omega^1_{X_1(N)/\mathbb{Z}[\frac{1}{N}]}(\log C) \simeq \omega^{\otimes 2} / \mathrm{Spec}\, \mathbb{Z}[\frac{1}{N}]$

In particular, $S_2(\Gamma_1(N)) \simeq H^0(X_1(N), \Omega^1_{X_1(N)/\mathbb{Z}[\frac{1}{N}]})$

$\circlearrowleft$ coming from Hecke operators

$$\begin{tikzcd} & X(\Gamma_1(N)\cap\Gamma_0(p)) \arrow[dl] \arrow[dr, "p\nmid N"] & \\ X_1(N) & & X_1(N)/\mathbb{Z}[\frac{1}{N}] \end{tikzcd}$$

$\rightarrow$ algebraically, $T_p, U_p, \langle d\rangle : J_1(N) \to J_1(N)$, $J_1(N) = \mathrm{Pic}^0_{X_1(N)/\mathbb{Z}[\frac{1}{N}]}$.

Let f be a normalized cuspidal eigenform of wt 2, level $\Gamma_1(N)$.

Define $\underline{\underline{A_f}} = J_1(N)/I_f J_1(N)$ for $I_f = \ker(h_1(N) \xrightarrow{\lambda_f} \mathbb{C})$

abelian var. of $\dim = [\mathbb{Q}(f):\mathbb{Q}] = \dim(h_1(N)/I_f \otimes \mathbb{Q})$

Then the Galois rep'n assoc. to f is $\rho_f = V_\ell(A_f) \otimes \chi_{cycl}^{-1}$ $\longleftarrow$ think of this as a 2-dim'l $\mathbb{Q}(f)\otimes\mathbb{Q}_\ell$ vec. sp.

$\mathbb{Q}(f)\otimes\mathbb{Q}_\ell = \prod_{\lambda|\ell} \mathbb{Q}(f)_\lambda$

<u>Rmk</u>. For Wiles' thm, we'll mostly need the one $\mathbb{Q}(f) = \mathbb{Q}$

i.e. $f = \sum a_n q^n$, all $a_n \in \mathbb{Q}$

This $f \longleftrightarrow A_f$ <u>elliptic curve</u>.

<u>Thm</u> (Eichler-Shimura) $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p)) = a_p$ if $p \nmid N$.

<u>Sketch of the proof</u> <u>Step 1</u> Understand Hecke correspondence T_p

$$\begin{tikzcd} X(\Gamma_1(N)\cap\Gamma_0(p)) = \{\text{parametrizes } (E, C, P_N): \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, mapsto] \arrow[dr, mapsto] & \\ X_1(N) & X_1(N) \quad (E,P) \quad (E/C, P+C/C) \end{tikzcd}$$

E generalized ell. curve

C subgp of order p

P_N : a pt of exact order N

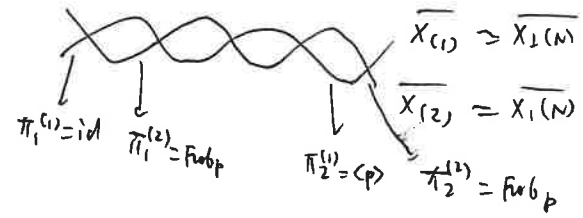
Then $T_p J_1(N) \xrightarrow{\pi_2^*} \mathrm{Jac}_{X(\Gamma_1(N)\cap\Gamma_0(p))} \xrightarrow{\pi_{1*}} J_1(N) \quad /\mathbb{Z}_p$ $\qquad p \nmid N$, $\mathbb{Z}[\frac{1}{N}]$

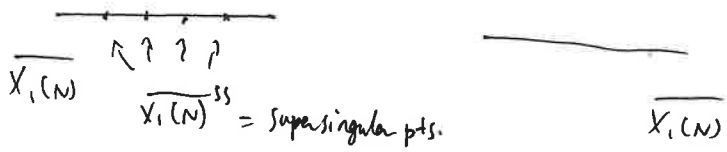
Special fiber of this?

Step 2

$$\overline{X(\Gamma_1(N)\cap\Gamma_0(p))} = \overline{X_{(1)}} \cup \overline{X_{(2)}}$$

$$\begin{tikzcd} & X(\Gamma_1(N)\cap\Gamma_0(p))_{\mathbb{F}_p} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \overline{X_1(N)} & & \overline{X_1(N)} \end{tikzcd}$$





Explanation Let $\overline{Y_1(N)}^{ord} = \overline{Y_1(N)} \setminus \overline{Y_1(N)}^{ss}$

$$\begin{tikzcd} \mathcal{E}^{ord} \arrow[d] \\ \overline{Y_1(N)}^{ord} \end{tikzcd}$$

The p-torsion subgp of $\mathcal{E}^{ord}$ sits in a canonical exact seq.

$$0 \to \underbrace{\mathcal{E}^{ord}[p]^{mult}}_{\text{family of } \mu_p,\ p^\infty} \to \underbrace{\mathcal{E}^{ord}[p]}_{p^\infty} \to \underbrace{\mathcal{E}^{ord}[p]^{\acute{e}t}}_{\text{family of } \mathbb{Z}/p\mathbb{Z},\ p^\infty} \to 0$$

s.t. at each $\overline{\mathbb{F}}_p$ pt $x \in \overline{Y_1(N)}^{ord}$, the exact seq. becomes

$$0 \to \underbrace{\mu_p}_{\mu_{p^\infty}} \to \underbrace{\mathcal{E}_x^{ord}[p]}_{p^\infty} \to \underbrace{\tfrac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p}_{\mathbb{Q}_p} \to 0$$

Rmk $\mathcal{E}_x^{ord} \xrightarrow{Frob} \mathcal{E}_x^{ord}$ has kernel μ_p

In family,

$$\begin{tikzcd} \mathcal{E}^{ord} \arrow[rr, "Fr_{\mathcal{E}^{ord}/\overline{Y_1(N)}}"] \arrow[dr] & & \mathcal{E}^{ord,(p)} \arrow[dl] \\ & \overline{Y_1(N)} & \end{tikzcd}$$

relative Frobenius.

$Fr_{\mathcal{E}^{ord}/\overline{Y_1(N)}} : \mathcal{E}^{ord} \to \mathcal{E}^{ord,(p)}$ has kernel precisely $\mathcal{E}^{ord}[p]^{mult}$.

Define $j_1 : \overline{Y_1(N)}^{ord} \longrightarrow \overline{Y(\Gamma_1(N)\cap\Gamma_0(p))}$ extends to $\overline{X_1(N)} \xrightarrow{j_1} \overline{X(\Gamma_1(N)\cap\Gamma_0(p))}$

$$(\mathcal{E}^{ord}, p) \longmapsto (\mathcal{E}^{ord}, p, \mathcal{E}^{ord}[p]^{mult})$$

$j_2 : \overline{Y_1(N)}^{ord} \longrightarrow \overline{Y(\Gamma_1(N)\cap\Gamma_0(p))}$ extends to $\overline{X_1(N)} \xrightarrow{j_2} \overline{X(\Gamma_1(N)\cap\Gamma_0(p))}$

$$(\mathcal{E}^{ord}, p) \longmapsto \left(\mathcal{E}^{ord}/\mathcal{E}^{ord}[p]^{mult},\ \operatorname{Im} p,\ \mathcal{E}^{ord}[p]/\mathcal{E}^{ord}[p]^{mult}\right).$$

Fact. $\overline{X(\Gamma_1(N)\cap\Gamma_0(p))} = \operatorname{Im} j_1 \cup \operatorname{Im} j_2$

Then

$$\begin{tikzcd}
\overline{Y_1(N)} \arrow[dr, "j_1"'] & & \overline{Y_1(N)} \arrow[dl, "j_2"] \\
& \overline{Y(\Gamma_1(N)\cap\Gamma_0(p))} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\overline{Y_1(N)} & & \overline{Y_1(N)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}^{ord} \arrow[d, bend right, "id"'] \arrow[dr, leftrightarrow] \arrow[ddr, bend left, "\mathrm{Frob}_{\overline{Y_1(N)}^{ord}}"] & & \\
& (\mathcal{E}^{ord}, \mathcal{E}^{ord}[p]^{mult}) \arrow[dl, mapsto] \arrow[dr, leftrightarrow] & \\
\mathcal{E}^{ord} & & \mathcal{E}^{ord}/\mathcal{E}^{ord}[p]^{mult} \\
& & \| \\
& & \mathcal{E}^{ord,(p)}
\end{tikzcd}$$

Step 3. Induced map on Jacobians.

$$\begin{tikzcd}
\mathcal{E}^{ord} \arrow[dl] \arrow[ddl, bend right, "\mathrm{Frob}_{\overline{Y_1(N)}^{ord}}"'] \arrow[ddr, bend left, "\langle p\rangle"] & \\
(\mathcal{E}^{ord}/\mathcal{E}^{ord}[p]^{mult}, \mathcal{E}^{ord}[p]/\mathcal{E}^{ord}[p]^{mult}) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathcal{E}^{ord}/\mathcal{E}^{ord}[p]^{mult} \arrow[d, equal] & \mathcal{E}^{ord}/\mathcal{E}^{ord}[p] \arrow[d, equal] \\
\mathcal{E}^{ord,(p)} & \mathcal{E}^{ord}
\end{tikzcd}$$

$$\begin{tikzcd}
& J_{X_1(N),\mathbb{F}_p} \arrow[d, "\pi_2^*"] \arrow[dr, bend left, "\langle p\rangle^*"] \arrow[ddr, bend right, "\mathrm{Frob}^*"'] & & \\
0 \arrow[r] & \text{torus} \arrow[r] & \boxed{J_{X(\Gamma_1(N)\cap\Gamma_0(p)),\mathbb{F}_p}} \arrow[r, "j_1^* \times j_2^*"] \arrow[d, "\pi_{1,*}"] & J_{\overline{X}(1)} \times J_{\overline{X}(2)} \arrow[r] \arrow[dl, "id"'] \arrow[dl, shift left=4, "\mathrm{Frob}\,*"] & 0 \\
& & J_{X_1(N),\mathbb{F}_p} & &
\end{tikzcd}$$

has semistable reduction

So $T_p = \langle p \rangle^* \mathrm{Frob}_* + \mathrm{Frob}^*$

$\Rightarrow (\mathrm{Frob}^*)^2 - T_p \cdot \mathrm{Frob}^* + p\langle p \rangle = 0.$. This gives the Eichler - Shimura rel'ns.

<u>Fact</u>. On $S_k(\Gamma_1(N); \mathbb{C}) \circlearrowleft T_p$ acts semisimply $(p \nmid N)$

but U_p -action $(p \mid N)$ may not.

$$\Rightarrow S_k(\Gamma_1(N); \mathbb{C}) = \bigoplus_{\{\lambda_p\}_{p \nmid N}} S_k(\Gamma_1(N))_{\{\lambda_p\}_{p \nmid N}}$$

$\leftarrow$ where T_p acts by mult. by λ_p

vec. sp. over $\mathbb{Q}_p$ $(\lambda_p : p \nmid N)$

$\cup$

$\mathbb{C}(f)$ f eigenform can assoc. a Galois rep'n

Let $S = \{p \text{ primes}, p \mid N\} \cup \{\infty, \ell\}$. $\boxed{\text{Assume } \ell \nmid N}$

Consider $\mathcal{O}[T_p : p \nmid N] \longrightarrow \mathrm{End}_{\mathcal{O}}(S_k(\Gamma_1(N); \mathcal{O}))$

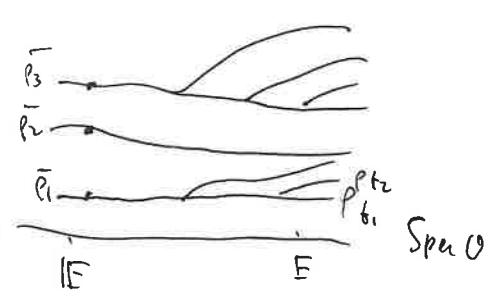
$\mathcal{O}/\mathbb{Z}_\ell$ $\qquad h_1^\circ(N) =$ image of this, called <u>Hecke algebra</u>.

finite flat over $\mathcal{O}$.

$\Rightarrow h_1^\circ(N)$ is a complete noetherian semi-local ring.

$$h_1^\circ(N) = \prod_{m \text{ max. ideal}} h_1^\circ(N)_m$$

Picture of $h_1^\circ(N)$



<u>Then</u> $h_1^\circ(N)_E = \mathrm{Im}\left(E[T_p : p \nmid N] \to \mathrm{End}_E(S_k(\Gamma_1(N); E))\right)$.

$$\mathrm{End}_E(S_k(\Gamma_1(N); E)) = \bigoplus_{\{\lambda_p\}_{p \nmid N}} S_k(\Gamma_1(N), E)_{\{\lambda_p\}}$$

enlarge E if necessary so that E contains all λ_p's

$$= \mathrm{Im}\left(E[T_p : p \nmid N] \to \bigoplus_{\{\lambda_p\}} \mathrm{End}\left(E^{\oplus \mathrm{mult}}\right) \right)$$

$$\circlearrowleft$$

$$T_p = \lambda_p$$

$$= \prod_{\{\lambda_p\}} E$$

$\rho : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(E)$ "appearing" in $S_k(\Gamma_1(M))$.

<u>Note</u>.

$$\begin{tikzcd}
\{\text{max'l ideal } P_f \text{ of } h_1^\circ(N)_E\} \arrow[r, "\text{"reduction mod } \ell\text{"}"] \arrow[d, leftrightarrow, "1:1"] & \{\text{max'l ideals of } h_1^\circ(N)\} \\
\{\text{system of eigenvalues } \{\lambda_p\}_{p \nmid N}\} \arrow[r] \arrow[d, leftrightarrow, "\|?"] & \{\text{systems of mod } \varpi \text{ eig. vals } \{\bar{\lambda}_p\}_{p \nmid N}\} \arrow[u] \arrow[d] \\
\left\{\begin{array}{c}\text{Galois rep'n } \rho : \mathrm{Gal}_{\mathbb{Q},S} \to GL_2(\mathcal{O}) \\ \text{crystalline of wt } (0, k-1) \otimes \ell\end{array}\right\} \arrow[r, "\rho \mapsto \bar{\rho}^{ss}", "\text{not injective}"'] & \left\{\begin{array}{c}\underline{\text{Semisimple}} \text{ residue Galois rep'n} \\ \bar{\rho} : \mathrm{Gal}_{\mathbb{Q},S} \to GL_2(\mathbb{F})\end{array}\right\}
\end{tikzcd}$$

$P_f \longmapsto$ unique max ideal containing ϖ and $P_f \cap h_1(N)$

<u>Prop</u> Let $\bar{\rho}_1, \cdots, \bar{\rho}_r$ be all semisimple residual rep'n "appearing" in $S_k(\Gamma_1(N); \mathbb{F}_\ell)$

For each $\bar{\rho}_i \rightsquigarrow m_{\bar{\rho}_i} := \left((\varpi, T_p - \widetilde{\mathrm{tr}(\bar{\rho}_i(\mathrm{Frob}_p))}), p \nmid N \right) \subset h_1^\circ(N)$

<u>Then</u> $h_1^\circ(N) = \prod_{i=1}^{r} h_1^\circ(N)_{m_{\bar{\rho}_i}} \sim e_i$

$$\wr$$

$$S_k(\Gamma_1(N), \mathcal{O}) = \bigoplus_{i=1}^{r} S_k(\Gamma_1(N), \mathcal{O})_{m_{\bar{\rho}_i}} \sim e_i S_k(-)$$

Moreover, $S_k(\Gamma_1(N), \mathcal{O})_{m_{\bar{\rho}_i}} \otimes E = \bigoplus_{\rho \text{ s.t. } \bar{\rho}^{ss} = \bar{\rho}_i} S_k(\Gamma_1(N); E)_{\{\lambda_p = \mathrm{tr}\,\rho(\mathrm{Frob}_p)\}_{p \nmid N}}$

<u>Rmk</u>. In fact, fixing N, there's only finitely many $\bar{\rho}_i$ needed for the decomp. of $S_k(\Gamma_1(N), E)$ $\forall$ any k.

Now, we fix an abs. irred. res. Galois rep'n $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F})$

$\rightsquigarrow R_{\bar{\rho}}$ = universal deformation ring for $\bar{\rho}$ as rep of $\mathrm{Gal}_{\mathbb{Q},S}$.

<u>* To-prove-in-next lecture</u> $\mathcal{O}[T_p : p \nmid N] \longrightarrow R_{\bar{\rho},S}$ has dense image.

$$T_p \longmapsto \mathrm{tr}(\rho^{univ}(\mathrm{Frob}_p))$$

$$\Rightarrow \mathcal{O}[[T_p - \widetilde{\mathrm{tr}\,\bar{\rho}(\mathrm{Frob}_p)} : p \nmid N]] \twoheadrightarrow R_{\bar{\rho},S}$$

Now,

$$\begin{tikzcd}
\mathcal{O}[[T_p - \widetilde{\mathrm{tr}\,\bar{\rho}(\mathrm{Frob}_p)} : p \nmid N]] \arrow[r] \arrow[d] & h_1^0(N)_{m_{\bar{\rho}}} \circlearrowleft S_k(\Gamma_1(N), \mathcal{O}) \text{ finite flat}/\mathcal{O} \arrow[d, leftrightarrow] \\
R_{\bar{\rho},S} \arrow[ur, dashed, "\text{next time}"] \arrow[dr, dashed, "\text{universal property of } R_{\bar{\rho},S}"'] & h_1^0(N)_{m_{\bar{\rho}}} \otimes E \\
& \cong \\
& \bigoplus_{\rho' \text{ appearing in } S_k(\Gamma_1(N)),\ \overline{\rho'} \simeq \bar{\rho}} E
\end{tikzcd}$$

$\rho' : \mathrm{Gal}_{\mathbb{Q},S} \to GL_2(\mathcal{O})$ lifts $\bar{\rho}$

So, can view $S_k(\Gamma_1(N), \mathcal{O})_{m_{\bar{\rho}}}$ as an $R_{\bar{\rho},S}$-mod.

<u>Lecture 21</u> Coeff. field $E / \mathbb{Q}_\ell$, $E \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \simeq \mathbb{F}$

Γ profinite gp satisfying the finiteness condition

$\bar{\rho} : \Gamma \longrightarrow GL_n(\mathbb{F})$ abs. irred. rep'n. $\ell \nmid n$.

$\rightsquigarrow R_{\bar{\rho}}$ (untramed) deformation ring.

Lemma Let $A \in CNL_{\mathcal{O}}$, $\rho : \Gamma \to GL_n(A)$ cts & $\rho \bmod m_A = \bar{\rho}$.

(1) (Schur's lemma for A-coeff) If $a \in GL_n(A)$, $a\rho a^{-1} = \rho$, then $a \in A^\times$.

(2) (Carayol's Lemma) If $B \subset A$ closed subring, $B \in CNL_{\mathcal{O}}$, and $\operatorname{tr}\rho \subset B$,

then $\exists\, a \in \widehat{PGL_n}(A)$ s.t. $a\rho a^{-1}(\Gamma)$ lands in $GL_n(B) \subset GL_n(A)$

Proof (1) Induction on $a \bmod m_A^N$.

(2) We will inductively conjugate ρ into $GL_n(B + m_A^N)$.

Assume that $\rho : \Gamma \to GL_n(B + m_A^N)$ w/ $\operatorname{tr}(\rho) \in B$

May replace A by $B + m_A^N / m_A^{N+1}$

& B by $B / m_A^{N+1} \cap B = B + m_A^{N+1} / m_A^{N+1}$

$\to$ A/B is a fin. dim'l $\mathbb{F}$-v.s. & $A = B + m_A^N / m_A^{N+1}$.

By a further induction, may assume $A/B \simeq \mathbb{F}$. or explicitly, $A = B \oplus \underline{\mathbb{F}\varepsilon}$.

m_A kills ε, $\varepsilon^2 = 0$.

So we may further modulo m_B. & thus reduces to the case

$A = \mathbb{F}[\varepsilon]/(\varepsilon^2)$ & $B = \mathbb{F}$.

This boils down to some explicit computation.

$\rho(\gamma) = \bar{\rho}(\gamma) + \theta(\gamma)\varepsilon$ for $\gamma \in \Gamma$

<u>We know</u>: $\operatorname{tr}\theta(\gamma) = 0$, $\forall\, \gamma \in \Gamma$

Want to find $1 + \varepsilon a \in 1 + M_n(\mathbb{F})\varepsilon$ s.t. $(1+\varepsilon a)(\bar{\rho}(\gamma) + \theta(\gamma)\varepsilon)(1+\varepsilon a)^{-1} \in M_n(\mathbb{F})$ $\forall \gamma$

$\Leftrightarrow$ $a\bar{\rho}(\gamma) - \bar{\rho}(\gamma)a + \theta(\gamma) = 0$, $\forall\, \gamma \in \Gamma$

Extend by linearity: $\bar{\rho}, \theta : \mathbb{F}[\Gamma] \longrightarrow M_n(\mathbb{F})$

(1) $\bar{\rho} : \mathbb{F}[\Gamma] \longrightarrow M_n(\mathbb{F})$ alg. hom. is <u>surjective</u> because $\bar{\rho}$ is abs irred.

(Artin - Wedderburn thm)

(2) for every $\gamma, \delta \in \Gamma$, $\rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$

$$\begin{array}{ccc} \rho(\gamma\delta) & = & \rho(\gamma)\rho(\delta) \\ \| & & \| \\ \bar{\rho}(\gamma\delta) + \theta(\gamma\delta)\varepsilon & = & (\bar{\rho}(\gamma) + \theta(\gamma)\varepsilon)(\bar{\rho}(\delta) + \theta(\delta)\varepsilon) \end{array}$$

$\Rightarrow \theta(\gamma\delta) = \bar{\rho}(\gamma)\theta(\delta) + \theta(\gamma)\bar{\rho}(\delta)$ $\leftarrow$ holds for all $\gamma, \delta \in \mathbb{F}[\Gamma]$

(3) $\operatorname{tr} \theta(\gamma) = 0$, $\forall \gamma \in \Gamma$ $\leftarrow$ holds for all $\gamma \in \mathbb{F}[\Gamma]$

<u>Claim</u>:

$$\begin{tikzcd} \mathbb{F}[\Gamma] \arrow[r, "\theta"] \arrow[d, "\bar{\rho}"] & M_n(\mathbb{F}) \\ M_n(\mathbb{F}) \arrow[ur, dashed, "\exists!\ \theta'"'] & \end{tikzcd}$$

uniquely factors as $\mathbb{F}$- linear maps.

<u>Pf</u> suffices to show that $\ker \bar{\rho} \subset \ker \theta$.

Pick $\delta \in \ker(\bar{\rho})$, then for any $\gamma \in \mathbb{F}[\Gamma]$,

$$\begin{array}{ccccc} \operatorname{tr}(\theta(\gamma\delta)) & = & \operatorname{tr}(\bar{\rho}(\gamma)\theta(\delta)) & + & \operatorname{tr}(\theta(\gamma)\bar{\rho}(\delta)) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

As $\bar{\rho}(\gamma)$ can be any matrix, $\Rightarrow \theta(\delta) = 0$

$\theta' : M_n(\mathbb{F}) \longrightarrow M_n(\mathbb{F})$

Want to find $a \in M_n(\mathbb{F})$ s.t. $a \cdot X - X \cdot a + \theta'(X) = 0$, $\forall X \in M_n(\mathbb{F})$

<u>Know</u>:
- θ' is $\mathbb{F}$-linear
- $\theta'(xy) = x\theta'(y) + \theta'(x)y$
- $\operatorname{tr} \theta'(x) = 0$, $\forall x \in M_n(\mathbb{F})$

$\Rightarrow$ θ' is a derivation in $\mathfrak{sl}_n$.

WTS that θ' is given by Lie bracket.

Cor. The universal deformation ring $R_{\bar\rho}$ is top. gen. over $\mathcal{O}$ by elts $\mathrm{tr}(\rho^{univ}(\gamma)) \in R_{\bar\rho}$ for $\gamma \in$ a dense subset $\Gamma_0 \subset \Gamma$.

Or equivalently, $\mathcal{O}[[T_\gamma : \gamma \in \Gamma_0]] \xrightarrow{\psi} R_{\bar\rho}$

$$T_\gamma \longmapsto \mathrm{tr}(\rho^{univ}(\gamma)) - \widetilde{\mathrm{tr}\,\bar\rho(\gamma)}$$

any lift to $\mathcal{O}$.

Proof. Let $R^1_{\bar\rho}$ denote the image of ψ. $R^1_{\bar\rho} \subset R_{\bar\rho}$

By above Lemma, can conjugate ρ^{univ} so that $\rho^{univ} : \Gamma \to GL_n(R^1_{\bar\rho}) \subset GL_n(R_{\bar\rho})$

$\rightsquigarrow \underbrace{R^1_{\bar\rho} \subset R_{\bar\rho}}_{\exists} \Rightarrow R^1_{\bar\rho} = R_{\bar\rho}$. $\square$

New form / old form theory & Γ_0/Γ_1 - theory

$N \geq 4$. q a prime. $q \nmid N$.
$q \equiv 1 \mod \ell$. $\ell^2 \| q-1$

$$\begin{array}{c} X(\Gamma_1(N) \cap \Gamma_1^*(q)) \\ \ell^2 \downarrow \pi_q \\ X(\Gamma_1(N) \cap \Gamma_0(q)) \\ pr_1 \downarrow \quad \downarrow pr_2 \\ X(\Gamma_1(N)) \end{array}$$

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : q \mid c \right\}$$

$$\ell^2 \ \cup$$

$$\Gamma_1^*(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q) : \left(\frac{a}{d}\right)^{q-1/\ell^2} \equiv 1 \mod q \right\}$$

$$\frac{q-1}{\ell^2} \ \cup$$

$$\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : q \mid c,\ q \mid a-1,\ d-1 \right\}$$

$$\begin{array}{ccc} & (E, P_N, \text{“}P_q\text{”}) & \\ & \downarrow & \\ & (E, P_N, \langle P_q \rangle) & \\ (E, P_N) \swarrow & & \searrow \left(E/\langle P_q\rangle, \frac{P_N + \langle P_q\rangle}{\langle P_q\rangle}\right) \end{array}$$

$$\begin{array}{ccc} & & S_k(\Gamma_1(N) \cap \Gamma_1^*(q); \mathcal{O}) \\ \text{Step 2} & & \uparrow \pi_q^* \\ & & S_k(\Gamma_1(N) \cap \Gamma_0(q), \mathcal{O}) \\ \text{Step 1} & & pr_1^* \uparrow \quad \uparrow pr_2^* \\ & & S_k(\Gamma_1(N), \mathcal{O}) \end{array}$$

Step 1 Classical newform / old form q-stabilization.

$$\begin{array}{ccc} S_k(\Gamma_1(N);\mathbb{C}) & \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} & S_k(\Gamma_1(N)\cap\Gamma_0(q);\mathbb{C}) \\ \circlearrowleft & & \circlearrowleft \\ T_q & & U_q \\ f(z) & \longmapsto & f(z) \\ f(z) & \longmapsto & f(qz). \end{array}$$

If f is a T_q-eig form w/ eig. val a_q, and also an eig form for all diamond operators

i.e. $\exists\ \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ s.t. $\langle d\rangle f(z) = \chi(d) f(z)$.

Let α, β be two roots of $x^2 - a_q x + q^{k-1}\chi(q) = 0$ (assuming that $\alpha \neq \beta$)

conjecturally true

Set $f_\alpha(z) = f(z) - \beta f(qz)$

$f_\beta(z) = f(z) - \alpha f(qz)$

Both U_q-eig form w/ eig vals α & β resp.

Check. $U_q(f(qz)) = U_q\left(\sum_{n\geq 1} a_n q^{qn}\right) = \sum_{n\geq 1} a_n q^n = f(z)$

$$U_q(f(z)) = \sum_{n\geq 1} a_{qn} q^n = T_q(f(z)) - q^{k-1}\chi(q) \underbrace{\sum a_n q^{qn}}_{f(qz)}$$

$\Rightarrow$ On basis $f(z), f(qz)$, U_q acts by $\begin{pmatrix} T_q & 1 \\ -q^{k-1}\chi(q) & 0 \end{pmatrix}$

Rmk. This process of turning T_q-eig forms into U_q-eig forms is called q-stabilization

Newform theory.

$$\begin{array}{ccc} S_k(\Gamma_1(N);\mathbb{C})^{\oplus 2} & \xrightarrow{pr_1^* + pr_2^*} & S_k(\Gamma_1(N)\cap\Gamma_0(q),\mathbb{C}) \\ (f,g) & \longmapsto & f(z) + g(qz) \end{array}$$

is injective.

Moreover, there's a natural complement space $S_k(\Gamma_1(N) \cap \Gamma_0(q), \mathbb{C})^{q\text{-new}}$.

- either def'd as the orthogonal complement for the Petersson inner product

On $S_k(\Gamma_1(N) \cap \Gamma_0(q), \mathbb{C})$, $\exists \langle\, , \rangle$

$$\langle f, g \rangle = \int_{\Gamma_1(N) \cap \Gamma_0(q) \backslash \mathfrak{h}} f(z)\, \bar{g}(z)\, y^k \frac{dx \wedge dy}{y^2}$$

eig-space decomp. by Hecke operators $T_{\ell'}$, $\ell' \nmid Nq$

- $$\underset{\substack{\cup \\ U_q}}{S_k(\Gamma_1(N) \cap \Gamma_0(q), \mathbb{C})} \simeq \underset{\substack{\| \\ S_k(\Gamma_1(N), \mathbb{C})^{\oplus 2}}}{S_k(\Gamma_1(N) \cap \Gamma_0(q), \mathbb{C})^{q\text{-old}}} \oplus \underset{\substack{\cup \\ U_q\text{-action satisfies} \\ U_q^2 = q^{k-2}\chi(q).}}{S_k(\Gamma_1(N) \cap \Gamma_0(q), \mathbb{C})^{q\text{-new}}}$$

<u>Now, recast the story / $\mathcal{O}$</u>

$\bar{\rho}: \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{F})$ abs. irred. $\det \bar{\rho} = \chi_{\mathrm{cycl}}^{1-k}\, \chi$

$\chi: \mathrm{Gal}_{\mathbb{Q},S} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_N) | \mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{F}^\times$

Let q be a prime $q \nmid N$, $q \equiv 1 \bmod \ell$.

Assume that $\bar{\rho}(\mathrm{Frob}_q)$ has distinct eig val. $\bar{\alpha}, \bar{\beta} \in \mathbb{F}^\times$ ← lift them to $\alpha, \beta \in \mathcal{O}^\times$ arbitrarily.

<u>Note</u> $\bar{\alpha} \cdot \bar{\beta} = \chi(q)\, q^{k-1} \equiv \chi(q)$ in $\mathbb{F}^\times$

$$\underset{\substack{\| \\ (m_{\bar{\rho}},\ T_q - \widetilde{\mathrm{tr}\,\bar{\rho}(\mathrm{Frob}_q)})}}{m_{\bar{\rho}}} = \text{ideal in } h_1(N) \text{ gen. by } \varpi,\ T_{q'} - \widetilde{\mathrm{tr}\,\bar{\rho}(\mathrm{Frob}_{q'})} \text{ for all } q' \nmid Nq.$$

$m_{\bar\rho}' =$ ideal in $h(\Gamma_1(N)\cap\Gamma_0(q))$ gen. by $\mathfrak{w}$, $T_{q'} - \widetilde{\mathrm{tr}\,\bar\rho(\mathrm{Frob}\,q')}$ for all $q' \nmid Nq$.

<u>Then</u>

$$\begin{tikzcd} S_k(\Gamma_1(N))_{m_{\bar\rho}} \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & S_k(\Gamma_1(N)\cap\Gamma_0(q))_{m_{\bar\rho}} \\ \circlearrowleft & \\ T_q - (\alpha+\beta) & \end{tikzcd}$$

acts topologically nilpotently

$\| \longleftarrow$ b/c over $S_k(-)^{q\text{-new}}$

$$S_k(\Gamma_1(N)\cap\Gamma_0(q))^{q\text{-old}}_{m_{\bar\rho}}$$

U_q acts by $\pm\sqrt{\chi(q)}$ $\neq \bar\alpha$ or $\bar\beta$

$\bar\alpha \neq \bar\beta$ $\|\wr$

$$S_k(\Gamma_1(N)\cap\Gamma_0(q))_{m'_{\bar\rho,\bar\alpha}} \oplus S_k(\Gamma_1(N)\cap\Gamma_0(q))_{m'_{\bar\rho,\beta}}$$

$m'_{\bar\rho,\bar\alpha} = (m'_{\bar\rho}, U_q - \alpha)$

So other $T_{q'}$ can't act in the same way as indicated by $\bar\rho$.

<u>Lemma</u>. Fix $\bar\alpha$. Then we have an isom.

$$\begin{tikzcd} S_k(\Gamma_1(N),\mathcal{O})_{m_{\bar\rho}} \arrow[r, "p_1^* - \tilde\beta p_2^*"] & S_k(\Gamma_1(N)\cap\Gamma_0(q),\mathcal{O})_{m'_{\bar\rho,\alpha}} \\ \circlearrowleft & \circlearrowleft \\ R_{\bar\rho,S} & R_{\bar\rho,S} \end{tikzcd}$$

where $\tilde\beta$ is the unique lift of $x^2 - T_q x + q^{k-1}\langle q\rangle$ in $h_1(N)_{m_{\bar\rho}}$

which reduces to $\bar\beta$ (exists by Hensel's lemma)

<u>Summary</u>: We may see the <u>same</u> $\bar\rho$-component of mod forms in $S_k(\Gamma_1(N)\cap\Gamma_0(q))$.

<u>Step 2</u>. Say $\ell^m \| q-1$, $\Gamma_1^*(q) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(q) : \left(\frac{a}{d}\right)^{(q-1)/\ell^m} \equiv 1 \bmod q\right\}$

$$X(\Gamma_1(N)\cap\Gamma_1^*(q)) \xrightarrow[\text{quotient by } \mathbb{Z}/\ell^m\mathbb{Z}]{\pi_q} X(\Gamma_1(N)\cap\Gamma_0(q))$$

$\subset \Gamma_0(q)$ index ℓ^m

$$\begin{tikzcd}
S_k(\Gamma_1(N)\cap\Gamma_1^*(q))^\vee_{m''_{\bar\rho,\alpha}} \arrow[r, "(\pi_q^*)^\vee"] & S_k(\Gamma_1(N)\cap\Gamma_0(q))^\vee_{m'_{\bar\rho,\alpha}} \arrow[d, phantom, "/\mathcal{O}"] \\
R^{\mathrm{crys},(0,k-1)}_{\bar\rho,\underbrace{S\cup\{q\}}_{\text{allowing ramification at } q}} \arrow[r, two heads] \arrow[u, phantom, "\circlearrowleft"] & R^{\mathrm{crys},(0,k-1)}_{\bar\rho,S} \arrow[u, phantom, "\circlearrowleft"]
\end{tikzcd}$$

<u>Recall</u> at q, $\bar\rho_q : \mathrm{Gal}\,\mathbb{Q}_q \longrightarrow GL_2(\mathbb{F})$

$$\phi_q \longmapsto \begin{pmatrix} \bar\alpha & \\ & \bar\beta \end{pmatrix}$$

χ lifts $\det \bar\rho_q$ & is unramified at q

The universal framed deformation ring $R^{\square,\chi}_{\bar\rho_q} = \mathcal{O}[[X,Y,S,Z]]/((1+Z)^{\ell^m}-1)$

& the universal framed deformation is

$$\rho_q^{\mathrm{univ}} : \mathrm{Gal}\,\mathbb{Q}_q \longrightarrow GL_2(R^{\square,\chi}_{\bar\rho_q})$$

$$\phi_q \longmapsto \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}\begin{pmatrix} \alpha+S & \\ & \chi(\phi_q)/(\alpha+S) \end{pmatrix}\begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1}$$

$$\tau_q \longmapsto \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}\begin{pmatrix} 1+Z & \\ & \frac{1}{1+Z} \end{pmatrix}\begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1}$$

& $R^{\square,\chi_m}_{\bar\rho_q} = R^{\square,\chi}_{\bar\rho_q}/(Z)$

Now, switch to the global deformation ring $R^{\mathrm{crys},(0,k-1)}_{\bar\rho,q}$

$\rho^{\mathrm{univ}}(\phi_q)$ has a unique eigenvector v_α s.t. the eigenvalue $\equiv \bar\alpha \bmod m_{\bar\rho,q}$

Let $Z = (\text{eigenvalue of } \rho^{\mathrm{univ}}(\tau_q) \text{ on } v_\alpha) - 1$

Then $R^{\mathrm{crys},(0,k-1)}_{\bar\rho,q}$ is an $\mathcal{O}[z]/((z+1)^{\ell^m}-1)$ - algebra

$$\mathcal{O}[z]/((z+1)^{\ell^m}-1) = \mathcal{O}[\mathbb{Z}/\ell^m\mathbb{Z}]$$

and $R^{\mathrm{crys},(0,k-1)}_{\bar\rho} \simeq R^{\mathrm{crys},(0,k-1)}_{\bar\rho,q}/(z)$.

Moreover, the compatibility of diamond operator & inertia gp action implies

$$\begin{tikzcd}
S_k(\Gamma_1(N)\cap\Gamma_1^*(q),\mathcal{O})^\vee_{m_{\bar\rho,\alpha}} \arrow[r, two heads] & S_k(\Gamma_1(N)\cap\Gamma_0(q),\mathcal{O})^\vee_{m'_{\bar\rho,\alpha}} \\
R^{\mathrm{crys},(0,k-1)}_{\bar\rho,q} \arrow[u, phantom, "\circlearrowleft"] \arrow[r, two heads, "/(z)"] \arrow[d, phantom, "\circlearrowleft"] & R^{\mathrm{crys},(0,k-1)}_{\bar\rho} \arrow[u, phantom, "\circlearrowleft"] \\
\mathcal{O}[\mathbb{Z}/\ell^m\mathbb{Z}] \arrow[u] \arrow[r, two heads, "/(z)"] \arrow[ur, phantom, "///"] & \mathcal{O} \arrow[u]
\end{tikzcd}$$

through diamond operators

$\mathbb{Z}/\ell^m\mathbb{Z} \to (\mathbb{Z}/q\mathbb{Z})^\times \curvearrowright S_k(-)$

<u>Fact</u>. $S_k(\Gamma_1(N)\cap\Gamma_1^*(q))^\vee_{m_{\bar\rho,\alpha}}$ is a free $\mathcal{O}[\mathbb{Z}/\ell^m\mathbb{Z}]$-mod

& $S_k(\Gamma_1(N)\cap\Gamma_1^*(q))^\vee_{m_{\bar\rho,\alpha}} \otimes_{\mathcal{O}[\mathbb{Z}/\ell^m\mathbb{Z}]} \mathcal{O}$

$$\simeq S_k(\Gamma_1(N)\cap\Gamma_0(q))^\vee_{m'_{\bar\rho,\alpha}}$$

$$\overset{\text{Step 1}}{\simeq} S_k(\Gamma_1(N))_{m_{\bar\rho}}.$$

Lecture 22. $E/\mathbb{Q}_\ell$, $E \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \simeq \mathbb{F}$, $\ell \geq 3$

N = square-free positive integer, $S = \{p \text{ primes} : p \mid N\} \cup \{\ell, \infty\}$

<u>Main Theorem</u> (Modularity lifting theorem)

Suppose that $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F})$ is an absolutely irred. rep'n s.t.

(1) $\bar{\rho}$ is <u>modular of level N</u> if $\exists$ some eigen cusp. form f of level $\Gamma_0(N)$ s.t. $\bar{\rho}_f = \bar{\rho}$.

(2) If $p \in S$ and $p \neq \ell$, then either $\bar{\rho}$ is unramified at p, or, after conjugation, $\bar{\rho}(I_{\mathbb{Q}_p}) = \begin{pmatrix} 1 & \mathbb{F}_\ell \\ & 1 \end{pmatrix}$

(3) $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is "semistable", i.e. assoc. to the ℓ-torsion points of an elliptic curve w/ good or multiplicative reduction.

(4) $\det \bar{\rho} = \chi_{cycl} \Rightarrow \det \bar{\rho}(c) = -1$

Then any lift $\rho : \mathrm{Gal}_{\mathbb{Q},S} \to GL_2(\mathcal{O})$ of $\bar{\rho}$ that is "<u>semistable</u>" is modular.

* <u>semistable</u> means $WD(\rho|_{G_{\mathbb{Q}_p}})$ is "unramified" but w/ a possibly nonzero N.
(r, N) r unram.

e.g. ρ comes from $T_\ell(E)$ for an elliptic curve $E/\mathbb{Q}$ w/ good & mult. reduction.

For simplicity, assume further

* $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is Fontaine-Laffaille & $\rho|_{G_{\mathbb{Q}_\ell}}$ is crystalline w/ HT wts $\{0, 1\}$.
(i.e. E has good reduction @ ℓ).

$\llcorner$ * If $p \in S$ and $p \neq \ell$, then $\bar\rho(I_{\mathbb{Q}_p}) \sim \begin{pmatrix} 1 & \mathbb{F}_\ell \\ 0 & 1 \end{pmatrix}$ & in $WD(\rho|_{G_{\mathbb{Q}_p}})$, $N \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

minimal modularity lifting

Write $\mathcal{D} = (\mathcal{D}_p)_{p \in S}$ for the needed condition.

- $p \in S$, $p \neq \ell$, $R^{\square, \mathcal{D}_p}_{\bar\rho_p}$ = framed deformation ring whose $\overline{\mathbb{Q}}_\ell$-pts x are the rep's for which $WD(\rho_x)$ is "unramified" w/ $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\det \rho_x = \chi_{cycl}$

$$\uparrow$$

$$R^{\square}_{\bar\rho_p} \simeq \mathcal{O}[[y_1^{(p)}, y_2^{(p)}, y_3^{(p)}]]$$

$h^1(G_{\mathbb{Q}_\ell}, \mathrm{Ad}^0 \bar\rho_\ell) = 3$

- $p = \ell$, $R^{\square}_{\bar\rho_\ell} \twoheadrightarrow R^{\square, \text{ord}, (0,1)}_{\bar\rho_\ell} = R^{\square, \mathcal{D}_\ell}_{\bar\rho_\ell}$

$\dim_{\mathcal{O}} \simeq \underline{\dim_{\mathcal{O}} \mathfrak{pgl}_n} + \underline{\dim_{\mathcal{O}} \mathfrak{sl}_n / \mathfrak{b}_n} = 4$

$$\simeq \mathcal{O}[[Y_1^{(\ell)}, Y_2^{(\ell)}, Y_3^{(\ell)}, Y_4^{(\ell)}]]$$

- $p = \infty$. $R^{\square}_{\bar\rho_\infty} = R^{\square, \mathcal{D}_\infty}_{\bar\rho_\infty} \simeq \mathcal{O}[[Y_1^{(\infty)}, Y_2^{(\infty)}]]$.

<u>Basic idea</u>: $R^{\mathcal{D}}_{\bar\rho}$:= deformation ring for $\bar\rho$ w/ these local conditions.

$$\begin{tikzcd}
R^{q}_{\bar\rho} \arrow[r] \arrow[d] & h_0(N)^{\circ}_{m_{\bar\rho}} \\
R^{\mathcal{D}}_{\bar\rho} \arrow[ur, dashed] &
\end{tikzcd}$$

$\mathrm{tr}(\mathrm{Frob}_q) \longmapsto T_q \in$

$\curvearrowright S_2(\Gamma_0(N); \mathcal{O})^{\vee}_{m_{\bar\rho}} = \mathrm{Hom}(S_2, \mathcal{O})$

(Another way to say this is that

$S_2(\Gamma_0(N); \mathcal{O})_{m_{\bar\rho}}$ is a module $/ R_{\bar\rho}$

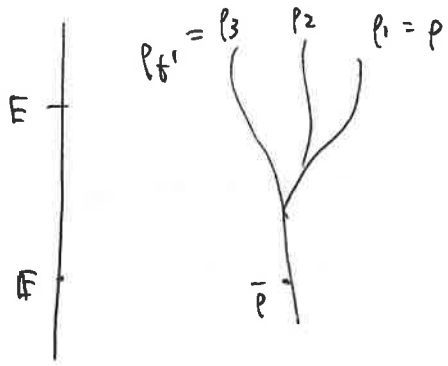
& $\mathrm{Supp}_{R_{\bar\rho}}(S_2(\Gamma_0(N), \mathcal{O})_{m_{\bar\rho}}) \subset \mathrm{Spec}\, R^{\mathcal{D}}_{\bar\rho} \subset \mathrm{Spec}\, R_{\bar\rho}$

The given rep'n ρ is a $\overline{\mathbb{Q}}_\ell$-pt x_ρ of $\operatorname{Spec} R^{D}_{\bar\rho}$.

Existence of f s.t. $\rho_f = \rho \iff \operatorname{Supp}\left(S_2(\Gamma_0(N),\mathcal{O})^{\vee}_{m_{\bar\rho}}\right) \ni x_\rho$

$$S_2(\Gamma_0(N))^{\vee,\square_S}_{m_{\bar\rho}} \twoheadleftarrow S_2(\Gamma_0(N)\cap\Gamma_1^*(Q_M))^{\vee,\square_S}_{m^1_{\bar\rho}}$$

$$\operatorname{Spec} R^{\square_S,D}_{\bar\rho} \subset \operatorname{Spec} R^{\square_S,D}_{\bar\rho,Q_M}$$



Need to prove that $\operatorname{Supp}\left(S_2(\Gamma_0(N),\mathcal{O})^{\vee}_{m_{\bar\rho}}\right) = \operatorname{Spec} R^{D}_{\bar\rho}$.

<u>Idea</u>: pass to a "local question"

<u>Problem</u>. No map $R^{D}_{\bar\rho} \not\hookrightarrow R^{\square,D_p}_{\bar\rho_p}$

<u>Recall</u>. $R^{\square_S,D}_{\bar\rho} :=$ global deformation ring w/ local conditions D & framed at S

$\cong R^{D}_{\bar\rho}[[X_1,\dots,X_{|S|}]]$ (n^2-1 "dummy variables")

$= \{(\rho,(h_v)_{v\in S}) : \rho$ lifts $\bar\rho$, $h_v \in \widehat{PGL}_n(A)\}/\sim$

Natural maps

$$\operatorname{Spf} R^{\square_S,D}_{\bar\rho}$$
$$\downarrow$$
$$\prod_{v\in S}\operatorname{Spf} R^{\square,D_v}_{\bar\rho_v} = \operatorname{Spf}\left(\hat{\bigotimes}_{v\in S} R^{\square,D_v}_{\bar\rho_v}\right)$$
$$\| $$
$$R^{\square,D}_{loc}$$

Define $S_2(\Gamma_0(N),\mathcal{O})^{\vee,\square_S}_{m_{\bar\rho}} = S_2(\Gamma_0(N),\mathcal{O})^{\vee}_{m_{\bar\rho}} \otimes_{R^{D}_{\bar\rho}} R^{\square_S,D}_{\bar\rho}$

$$\cup$$
$$R^{\square_S,D}_{\bar\rho} \longleftarrow R^{\square,D}_{loc}$$

Earlier thm $\Rightarrow$ $R_{\bar\rho}^{\square_S, D} = R_{loc}^{D}[[y_1, \dots, y_r]] / (b_1, \dots, b_r)$ for $r = \dim \tilde{H}_S^1(G_{\mathbb{Q},S} : \mathrm{Ad}^0 \bar\rho)$

Chebotarev density thm. for each m, there exists a collection of Taylor - Wiles primes

$Q_m = \{ q_1^{(m)}, \dots, q_r^{(m)} \}$ s.t. (1) each $q_i^{(m)} \equiv 1 \pmod{\ell^m}$ $\quad (\ell^{m_i} \| q_i^{(m)} - 1)$

$$\Delta_{Q_m} = \prod_{i=1}^{r} \mathbb{Z}/\ell^{m_i}\mathbb{Z}$$

(2) $\bar\rho(\mathrm{Frob}_{q_i^{(m)}})$ has distinct eigenvalues

(3)
$$\begin{tikzcd}
R_{loc}^{\square, D}[[y_1, \dots, y_r]] \arrow[r, two heads] \arrow[dr, dashed, "\exists"'] & R_{\bar\rho}^{\square_S, D} & \mathcal{O}[[x_1, \dots, x_{3|S|}]] \arrow[l] \\
& R_{\bar\rho, Q_m}^{\square_S, D} \arrow[u] & \mathcal{O}[\Delta_{Q_m}][[x_1, \dots, x_{3|S|}]] \arrow[l] \arrow[u, two heads, "\mathrm{mod}\, I_{\Delta_{Q_m}}"']
\end{tikzcd}$$

allowing ram. @ Q_m

Big Problem: There's no inverse system to take limit!

2 versions ① Wiles A sequence in a compact Hausdorff space has a converging subseq.

② Scholze Ultrafilter.

$$\mathrm{Spa}\, R_{loc}^{\square, D}[[y_1, \dots, y_r]]$$
$$\shortparallel$$
$$\mathcal{O}[[Y_1, \dots, Y_r, y_1^{(p)}, \dots]]$$

formal power series ring

Ultrafilter

Def Let S be a set. An ultrafilter $\mathcal{F}$ is a set of subsets of S

(1) $\phi \notin \mathcal{F}$

(2) If $I \in \mathcal{F}$ & $I \subset I' \subset S$, $\rightarrow I' \in \mathcal{F}$

(3) $I, J \in \mathcal{F}$, $I \cap J \in \mathcal{F}$.

(4) $\forall I \subset S$, exactly one of I and I^c belongs to $\mathcal{F}$.

Example. $\forall s \in S$, $\mathcal{F}_s = \{$subsets $I \subset S$ containing $s\}$

called a principal ultrafilter.

Conversely, if some $\{s\} \in \mathcal{F}$, then $\mathcal{F} = \mathcal{F}_s$.

Lemma. If S is infinite (e.g. $\mathbb{N}$), $\exists$ non-principal ultrafilters $\mathcal{F}$

(for such $\mathcal{F}$, $\bigcap_{I \in \mathcal{F}} I = \phi$)

Pf Start w/ $\mathcal{F}_{cofin} = \{I \subset S, \# (S \setminus I) < \infty\}$

Use Zorn's lemma $\square$

Basic properties

① If $S = S_1 \amalg \cdots \amalg S_n$, then exactly one S_i belongs to $\mathcal{F}$.

Pf. S_1 vs $(S_2 \amalg \cdots \amalg S_n) = S_1^c$. By (4), exactly one of them $\in \mathcal{F}$.

If $S_1 \in \mathcal{F}$, then other $S_j \notin \mathcal{F}$, o/w $S_1 \cap S_j = \phi \in \mathcal{F}$

Otherwise, $S_2 \amalg \cdots \amalg S_n \in \mathcal{F}$

S_2 vs S_2^c. If $S_2 \in \mathcal{F}$, we win

Otherwise, $S_2^c \in \mathcal{F}$

...

If all $S_j \notin \mathcal{F} \Rightarrow$ all $S_j^c \in \mathcal{F}$, $\bigcap_{j=1}^{n} S_j^c = \phi \in \mathcal{F}$. $\square$

② If $I \in \mathcal{F}$, then $\mathcal{F}_I := \{J \cap I : J \in \mathcal{F}\}$ is ultrafilter for I

$= \{J \subset I$ s.t. $J \in \mathcal{F}\}$.

Pf Check For any $J \subset I$, exactly one of J & $I \setminus J$ belongs to $\mathcal{F}$.

Say $J \notin \mathcal{F} \Rightarrow J^c \in \mathcal{F} \Rightarrow I \cap J^c \in \mathcal{F}$

③ An ultrafilter is non principal $\Leftrightarrow$ $\mathcal{F} \supset \mathcal{F}_{cofin}$.

Pf $\mathcal{F}$ non-principal $\Leftrightarrow$ $\forall s$, $\{s\} \notin \mathcal{F}$, i.e. $\{s\}^c \in \mathcal{F}$

$\overset{\text{by (3)}}{\Leftrightarrow}$ $\mathcal{F} \supset \mathcal{F}_{cofin}$

* $S = \mathbb{N}$, Fix $\mathcal{F}$ a non-principal ultrafilter

<u>Ultra product</u> (limit). Given $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ a collection of non-empty set

need $a_n = a_n'$ for $\mathcal{F}$-many n's

Define $\mathcal{U}(\mathcal{A}) := \mathcal{U}_{\mathcal{F}}(\mathcal{A}) := \left(\prod_{n=0}^{\infty} A_n\right) \Big/ (a_n) \sim (a_n')$ iff $a_n \overset{\mathcal{F}}{=} a_n'$

$\lim_{\mathcal{F}} \mathcal{A}$?

* <u>means</u>: $\exists\, I \in \mathcal{F}$ s.t. $a_n = a_n'$ for all $n \in I$.

(1) If $I \in \mathcal{F}$ $\longrightarrow$ $\mathcal{F}_I$, $\mathcal{A}_I = (A_i)_{i \in I}$

then $\mathcal{U}_{\mathcal{F}_I}(\mathcal{A}_I) \xrightarrow{\sim} \mathcal{U}_{\mathcal{F}}(\mathcal{A})$

$(a_n)_{n \in I} \longmapsto \left((a_n)_{n \in I}, (b_m)_{m \in I^c}\right)$ for your favorite fixed $b_m \in A_m$

$\| \mathcal{F}$

$\left((a_n)_{n \in I}, (b_m')_{m \in I^c}\right)$

In particular, can remove finitely many A_m's from $\mathcal{A}$.

(2) Assume that A_n's are finite sets, & $|A_n| \leq N$ is uniformly bounded.

Can compute $\mathcal{U}(\mathcal{A})$ as follows

<u>Step 1.</u> Group A_n's by cardinality. $\mathbb{N} = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_N$, where

$$I_r = \{ n \in \mathbb{N} : |A_n| = r \}$$

By ①, $\exists!\ s$ s.t. $I_s \in \mathcal{F}$.

$$\Rightarrow \mathcal{U}(\mathcal{A}) = \mathcal{U}_{\mathcal{F}_{I_s}}(\mathcal{A}_{I_s}) \quad \text{by (1)}$$

<u>Step 2</u> Let $A = \{c_1, \cdots c_s\}$ be an abstract set, & fix an arbitrary bijection

$\psi_i : A \xrightarrow{\sim} A_i$ for each $i \in I_s$

Then
$$\begin{aligned} A &\xrightarrow{\sim} \mathcal{U}_{\mathcal{F}_{I_s}}(\mathcal{A}_{I_s}) \\ c_j &\longmapsto (\psi_i(c_j))_{i \in I_s} \end{aligned}$$

<u>Proof of Surjectivity</u>. Given $(a_i)_{i \in I_s} \in \prod_{i \in I_s} A_i$, then $I_s = I_{s,c_1} \sqcup \cdots \sqcup I_{s,c_s}$

where $I_{s,c_j} = \{ i \in I_s : a_i = \psi_i(c_j) \}$

But exactly one $I_{s,c_j} \in \mathcal{F} \Rightarrow (a_i) \overset{\mathcal{F}}{=} (\psi_i(c_j))$.

$\mathcal{U}_{\mathcal{F}}(\mathcal{A})$ is a "universal way to take limit".

(3) If A_n's are alg objects (rings, mods, ...), so is $\mathcal{U}(\mathcal{A})$

(4) Let R be a <u>finite</u> local ring and set $\mathcal{R} = \prod_{n \in \mathbb{N}} R$.

An ultrafilter $\mathcal{F}$ gives a homomorphism $\mathcal{R} \xrightarrow{\phi_{\mathcal{F}}} \mathcal{U}(\{R\}_{n \in \mathbb{N}}) \simeq R$

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\phi_{\mathcal{F}}} & \mathcal{U}(\{R\}_{n \in \mathbb{N}}) \simeq R \\ \cup & & \cup \\ \text{max'l ideal} \quad \mathfrak{z}_{\mathcal{F}} := \phi_{\mathcal{F}}^{-1}(\mathfrak{m}) & \longrightarrow & \mathfrak{m} \end{array}$$

Exer. Every max'l / prime ideal of $\mathcal{R}$ arises this way!

& $\mathcal{R}_{\mathfrak{z}_{\mathcal{F}}} \simeq R$

$\uparrow$

localization.

Lecture 23. $\mathcal{F}$ a non-principal ultrafilter of $\mathbb{N}$

$\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ a collection of finite sets / rings / mods s.t. $|A_n| \le N$ for some fixed N

Then $U_{\mathcal{F}}(\mathcal{A}) = \left(\prod_{n\in\mathbb{N}} A_n\right) / (a_n) \overset{\mathcal{F}}{=} (a_n')$ if $a_n = a_n'$ for all n in an elt $I \in \mathcal{F}$

$\approx$

$\lim_{\mathcal{F}} A_n$

Patching argument $S_n = S_2(\Gamma_0(N) \cap \Gamma_1^*(Q_n))^{\vee,\square}_{m_{\bar\rho}''} \circlearrowleft R_n = R^{\square_S, D}_{\bar\rho, Q_n} \twoheadleftarrow R_{loc}[[Y_1, \dots, Y_2]]$

$\cup$

R^{∞}_{loc}

$\uparrow$

$\exists$ a factorization

$\mathbb{Z}_p[\Delta_{Q_n}] = \mathbb{Z}_p[\mathfrak{z}_1, \dots, \mathfrak{z}_r] / I_{Q_n}$, $I_{Q_n} = (\mathfrak{z}_i + 1)^{p^{n-1}}$, $i = 1, \dots, r$

$\Delta_{Q_n} \approx (\mathbb{Z}/p^n\mathbb{Z})^r$

$J_n = \mathcal{O}[[X_1, \dots, X_{3|S|}, \mathfrak{z}_1, \dots, \mathfrak{z}_r]] / I_{Q_n} \twoheadleftarrow J_\infty = \mathcal{O}[[\underline{X}, \underline{\mathfrak{z}}]]$

Moreover, S_n is a finite free J_n-module. & $S_n \otimes_{J_n} J_0 \simeq S_0 = S_2(\Gamma_0(N))^{\vee,\square}_{m_{\bar\rho}}$

$J_0 = \mathcal{O}[[\underline{X}, \underline{\mathfrak{z}}]] / (\mathfrak{z}_1, \dots, \mathfrak{z}_r)$

For each open ideal $\mathfrak{a} \subset J_\infty$, $J_\infty / \mathfrak{a}$ is a finite ring.

$S_n / \mathfrak{a} S_n \circlearrowleft R^{\infty}_{loc} / (m^{\infty}_{loc})^{k(\mathfrak{a})}$ for some $k(\mathfrak{a}) = \overbrace{\text{length } J_\infty / \mathfrak{a} J_\infty}^{\text{independent of } n}$.

finite free $\cup$

$\uparrow$

$\cdot \operatorname{rank}_{J_n} S_n$

$J_\infty / (I_{Q_n}, \mathfrak{a}) J_\infty \twoheadleftarrow J_\infty$

Take ultraproduct for $\mathcal{F}$.

$$\begin{tikzcd}
 & S_\infty/\mathfrak{a}J_\infty \text{ finite free } J_\infty/\mathfrak{a}J_\infty\text{-mod} & \\
U_{\mathcal{F}}\left((S_n/\mathfrak{a}S_n)_{n\in\mathbb{N}}\right) \arrow[r, circlearrow] & U_{\mathcal{F}}\left((R^\infty_{loc}/(m^\infty_{loc})^{k(\mathfrak{a})})_{n\in\mathbb{N}}\right) & \text{constant collection} \arrow[l] \\
U_{\mathcal{F}}\left((J_\infty/\underbrace{(\mathfrak{a}, I_{\mathfrak{a}_n})J_\infty}_{=\mathfrak{a},\, n\gg 0})_{n\in\mathbb{N}}\right) \arrow[u, circlearrow] & J_\infty \arrow[l] \arrow[u] & R^\infty_{loc}/(m^\infty_{loc})^{k(\mathfrak{a})} \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
 & J_\infty/\mathfrak{a}J_\infty \arrow[u, equal]
\end{tikzcd}$$

<u>note</u>: $n\gg 0$, $I_{\mathfrak{a}_n} \subset \mathfrak{a}$

Take <u>inverse limit on $\mathfrak{a}$ & $k(\mathfrak{a})$</u>:

$$\boxed{\begin{tikzcd}
S_\infty = \varprojlim_{\mathfrak{a}} U_{\mathcal{F}}(S_n/\mathfrak{a}S_n) \arrow[r, circlearrow] & R^\infty_{loc} \\
\text{finite free} \arrow[u, circlearrow] & \\
J_\infty \arrow[r, equal] & J_\infty \arrow[u]
\end{tikzcd}}$$

<u>Rmk</u> modulo $(x_1, \dots x_{3|S|}, z_1, \dots, z_2) \subset J_\infty$,

$$S_\infty/(\underline{x}, \underline{z}) \simeq S_2(\Gamma_0(N))^\vee_{m_{\bar\rho}}.$$

<u>Upshot</u>. S_∞ is a finite free mod $/J_\infty = \mathcal{O}[\![\underline{x}, \underline{z}]\!]$ ← $3|S|+2$ vars

$$R^\infty_{loc} = \left(\hat{\bigotimes}_{p\in S} R^{\square, \mathcal{O}_p}_{\bar\rho_p}\right)[\![y_1, \dots, y_2]\!] = \text{formal power series ring in } 2+\underbrace{\begin{cases} @\ p=\ell,\ 4 \\ @\ p\neq \ell, \infty,\ 3 \\ \quad + \chi(\mathrm{Gal}_p, \mathrm{Ad}^0_{\bar\rho}) \\ \qquad\qquad \| \\ \qquad\qquad 0 \\ @\ p=\infty,\ 2 \end{cases}}_{3|S|}$$

$\det\bar\rho(c) = -1$.

<u>Some commutative algebra</u>. R comm. noetherian local ring, M f.g. R-module,

$\mathrm{pdim}_R(M) :=$ projective dim of M

$= \min\{n : \mathrm{Ext}^{\geq n+1}(M, N) = 0 \text{ for any } R\text{-mod } N\}$

$\mathrm{pdim}_R(M) = 0 \iff M$ is projective R-mod $\iff M$ is a free R-mod

$\mathrm{depth}_R(M) := \max\{n : \exists\, x_1, \dots, x_n \in m_R$ s.t. $x_1 : M \to M$ is inj., $x_2 : M/x_1M \to M/x_1M$ inj. $\dots$ $x_n : M/(x_1, \dots, x_{n-1})M \to M/(x_1, \dots, x_{n-1})M$ is injective $M \neq (x_1, \dots, x_n)M\}$

Auslander - Buchsbaum Theorem $\quad \mathrm{pdim}_R M + \mathrm{depth}_R M = \mathrm{depth}_R(R)$.

e.g. R regular local ring, $\mathrm{depth}_R(R) = \mathrm{Krull\ dim}(R)$.

$$J_\infty \longrightarrow R_\infty \curvearrowright S_\infty$$

S_∞ f. free over J_∞

$$\underset{\substack{\| \\ J_\infty \oplus \cdots \oplus J_\infty}}{\mathrm{depth}_{J_\infty}(S_\infty)} = \underset{=2+3s+1}{\mathrm{K\,dim}(J_\infty)} \longrightarrow \mathrm{depth}_{R_\infty}(S_\infty) \geq 2+3s+1$$

By A-B theorem, $\underbrace{\mathrm{depth}_{R_\infty}(R_\infty)}_{=2+3s+1} = \mathrm{pdim}_{R_\infty}(S_\infty) + \underbrace{\mathrm{depth}_{R_\infty}(S_\infty)}_{\geq 2+3s+1}$ (irreducible)

$\Rightarrow \mathrm{pdim}_{R_\infty}(S_\infty) = 0 \Rightarrow S_\infty$ must be a f. free R_∞-module.

$$\Rightarrow \underset{\substack{\| \\ S_2(\Gamma_0(N))^\vee_{m_{\bar\rho}}}}{\boxed{S_\infty / (I_{\mathfrak{a}_0}, x_1, \cdots, x_{3|S|})}} \circlearrowleft R^D_{\bar\rho} \xleftarrow{\sim} \frac{R^\square_{loc}[[y_1, \cdots, y_2]]}{(z_1, \cdots, z_2, x_1, \cdots, x_{3|S|})}$$

is a fin. free $\leftarrow$ -mod

<u>Conclusion</u>: In this case, $S_2(\Gamma_0(N))^\vee_{m_{\bar\rho}}$ is a fin. free $R^D_{\bar\rho}$-module.

"minimal case" i.e. $R^\square_{loc}$ is a formal power series ring (regular ring).

depends on conditions like $\bar\rho(I_{\mathfrak{a}_p}) = \begin{pmatrix} 1 & \mathbb{F}_\ell \\ 0 & 1 \end{pmatrix}$ for $p \in S$, $p \neq \ell_\infty$.

In general, can't quite conclude S_2 free over $R^D_{\bar\rho}$.

<u>FLT</u>. $a^\ell + b^\ell = c^\ell$ for $\ell \geq 5$ prime has no nonzero integer solutions.

"<u>Proof</u>". Suppose there's a sol'n (a, b, c), coprime, b even, $a \equiv -1 \pmod 4$,

<u>Step 1</u>

$\rightsquigarrow E : y^2 = x(x-a^\ell)(x+b^\ell)$

<u>General theory.</u> if $y^2 = (x-e_1)(x-e_2)(x-e_3)$

$$\rightsquigarrow \Delta_E = 16(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2$$

$$\Delta_E = 16\, a^{2\ell} b^{2\ell}(a^\ell + b^\ell)^2 = 16\, a^{2\ell} b^{2\ell} c^{2\ell}.$$

<u>Key Lemma</u> (1) E has good or multiplicative reduction modulo any prime p

$(\Rightarrow N_E =$ conductor of E = 2 primes dividing $(abc))$

(2) $\bar{\rho}_{E,\ell} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[\ell]) \simeq GL_2(\mathbb{F}_\ell)$ is unramified at odd primes $p \mid N_E$, $p \nmid \ell$.

(b/c $v_p(\Delta_E)$ is divisible by ℓ)

<u>Step 2</u> Modularity thm (Wiles' work)

<u>Thm</u> E is modular, i.e. E is assoc. to a cuspidal eigenform $f \in S_2(\Gamma_0(N))$.

<u>Step 3</u> $E \longleftrightarrow f \in S_2(\Gamma_0(N))$, yet $\bar{\rho}_{E,\ell} = \bar{\rho}_{f,\ell}$ is unramified at odd primes $p \mid N$

Ribet's level-lowering thm $\Rightarrow \exists\, g \in S_2(\Gamma_0(2))$ s.t. $f \equiv g \mod \ell$.

<u>Step 4</u>. There's no cusp. form of level $\Gamma_0(2)$ & wt 2. $\square$.

$S = \{p \mid N\} \cup \{\infty\}$

<u>Step 2a</u>. Continuous odd irred rep. $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q},S\cup\{3\}} \longrightarrow GL_2(\mathbb{F}_3)$ is modular.

<u>Big Input</u> (Langlands-Tunnell) Suppose that $\rho : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{C})$ cts irred. rep'n whose image in $PGL_2(\mathbb{C})$ is a subgp of $\underbrace{\mathfrak{S}_4}_{\text{solvable}}$ & $\det(\rho(c)) = -1$.

Then there exists a normalized cusp. eigenform $g(q) = \sum b_n q^n$ of wt 1 & level $\Gamma_1(N)$

s.t. $b_q = \mathrm{trace}\,(\rho(\mathrm{Frob}_q))$ for $q \nmid N$.

<u>Coincidence for $\ell = 3$</u>. There's a faithful rep'n

$$\psi : GL_2(\mathbb{F}_3) \hookrightarrow GL_2(\mathbb{Z}[\sqrt{-2}]) \subset GL_2(\mathbb{C}) \quad \text{s.t.}$$

$$\mathrm{tr}\,(\psi(g)) = \mathrm{tr}\,(g) \mod (1+\sqrt{-2})$$

$$\det(\psi(g)) = \det(g) \mod 3.$$

$$\psi\underset{\substack{\uparrow \\ GL_2(\mathbb{F}_3)}}{\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \psi\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{-2} & 1 \\ 1 & 0 \end{pmatrix}$$

So, given $\bar{\rho}$, $\psi \circ \bar{\rho} : \mathrm{Gal}\,\mathbb{Q} \longrightarrow GL_2(\mathbb{F}_3) \hookrightarrow GL_2(\mathbb{Z}[\sqrt{-2}]) \subset GL_2(\mathbb{C})$

By Langlands - Tunnell $\Rightarrow \exists\ g(q)$ of wt 1 s.t. $b_q \equiv \mathrm{tr}\,(\bar{\rho}(\mathrm{Frob}_q)) \mod 1+\sqrt{-2}$

$q \nmid N$

(Moreover; as $\bar{\rho}$ is ram. @ 3 b/c $\det \bar{\rho} = \chi_{\mathrm{cycl}} \mod 3$ ← ram. @3

$\Rightarrow 3 \mid$ level N of g

Consider a weight 1 level $\Gamma_1(3)$ Eisenstein series

$$E(z) = 1 + \underbrace{6}_{\text{div. by } 3} \sum_{n=1}^{\infty} \sum_{d \mid n} \chi(d)\, e^{2\pi i z} \quad \text{where } \chi(d) = \begin{cases} 0 & \text{if } d \equiv 0\ (3) \\ 1 & \text{if } d \equiv 1\ (3) \\ -1 & \text{if } d \equiv -1\ (3) \end{cases}$$

$\Rightarrow g(z)E(z)$ has wt 2 & in $S_2(\Gamma_3(N))$

Moreover, $a_q(g(z)E(z)) \equiv b_q \equiv \mathrm{tr}\,(\bar{\rho}(\mathrm{Frob}_q)) \mod (1+\sqrt{-2})$

So $g(z)\,E(z) \in \underbrace{S_2(\Gamma_0(N))_{m_{\bar\rho}}}$

where $T_q - \tilde{\rho}(\widetilde{\mathrm{Frob}_q})$ acts topologically nilpotently.

$\Rightarrow \bar\rho$ is modular.

<u>Step 2b</u> If for our elliptic curve E, $\bar\rho_{E,3}$ is irred. , then E is modular

$\Updownarrow$ $\leftarrow$ o/w up to conj.

abs. irred.

By <u>Step 2a</u>, $\bar\rho$ modular,

so by MLT $\Rightarrow \rho_E$ is modular.

$$\begin{tikzcd}
\bar\rho_{E,3} : \mathrm{Gal}_{\mathbb{Q}} \arrow[r] & \mathbb{F}_9^\times \arrow[d, "\det"] \\
c \arrow[r, mapsto] \arrow[dr] & \mathrm{order}\ 2 \\
 & -1 \in \mathbb{F}_3^\times
\end{tikzcd}$$

$x \mapsto x^p$

<u>Step 2c</u>. Suppose that $\bar\rho_{E,3}$ is reducible

<u>Thm</u> (Wiles' 3-5 trick) In this case, we have the following

Either E is already reducible, or

(a) $\bar\rho_{E,5}$ irred.

and

(b) $\exists\, A \mid \mathbb{Q}$ another elliptic curve, semistable s.t. ① $\bar\rho_{E,5} \simeq \bar\rho_{A,5}$

② $\bar\rho_{A,3}$ irred.

<u>3-5 trick</u> $\Rightarrow$ <u>Modularity Thm</u> If E is not modular,

then Step 2a 2b $\Rightarrow$ A is modular $\Rightarrow$ $\bar\rho_{A,5}$ is modular $\overset{\text{MLT}}{\Longrightarrow}$ E is modular

$\bar\rho_{A,3}$ modular

$\bar\rho_{A,5} \cong \bar\rho_{E,5}$

<u>Proof</u> First assume that $\bar\rho_{E,5}$ is reducible. then $E[3]$ has a non-triv. subgp fixed by $\mathrm{Gal}_{\mathbb{Q}}$

$E[5]$ ——— $\mathrm{Gal}_{\mathbb{Q}}$

$\Rightarrow E$ defines a $\mathbb{Q}$-point on $X_0(15)$.

$X_0(15)$: $y^2 + xy + y = x^3 + x^2 - 10x - 10$

$X_0(15)(\mathbb{Q}) = \langle (-1,0), (8,18) \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

four of the rat'l pts are cusps

other four curves are known to be modular

$j \circ a_1$, $j \circ a_2$, $j \circ a_3$, $j \circ a_4$

If E is none of above. $\Rightarrow \bar{\rho}_{E,5}$ is abs. irred.

Find A? Consider $Y'(5) \subset X'(5)$ twisted version of modular curve

as moduli space, $R \longmapsto \left\{ (A/R, \phi);\ A/R \text{ elliptic curve } /R,\ E[5] \xrightarrow{\phi}_{\sim} A[5] \text{ preserving Weil pairing} \right\}$

This $X'(5)$ is an inner form of $X(\Gamma(5))$

(i.e. $X'(5)/\mathbb{Q}$ but $X'(5) \times_{\mathbb{Q}} \bar{\mathbb{Q}} \simeq X(\Gamma(5))_{\bar{\mathbb{Q}}}$)

assoc to $\bar{\rho}_{E,5}$: $\mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_5) \subset \mathrm{Aut}_{\bar{\mathbb{Q}}}(X(\Gamma(5))_{\bar{\mathbb{Q}}})$

Yet, genus $(X'(5))$ = genus $(X(\Gamma(5)) = 0$ & $E \in X'(5)(\mathbb{Q})$

$\Rightarrow X'(5)/\mathbb{Q} \simeq \mathbb{P}^1_{\mathbb{Q}}$

$\Rightarrow \exists$ ∞'ly many elliptic curves satisfying (b) @ $\bar{\rho}_{A,5} \simeq \bar{\rho}_{E,5}$

(picking any pt on $\mathbb{P}^1 \setminus \{\text{cusps}\}$)

$\downarrow$

A

Need to make sure that $\bar{\rho}_{A,3}$ is irreducible.

"bad ones"

$$\begin{array}{l} Y'(5,3) : R \mapsto \left\{ (A/R, \phi, C_3) : \begin{array}{l} (A/R, \phi) \in Y'(5)(R) \\ C_3 \subset A[3] \text{ subgp of order } 3 \end{array} \right\} \\ \quad \downarrow \\ \text{"good ones"} \; Y'(5) \end{array}$$

Yet, genus $(Y'(5,3)) > 1$

$$\left(Y'(5,3)_{\mathbb{C}} \simeq (\Gamma(5) \cap \Gamma_0(3)) \backslash \mathfrak{h} \right)$$

By Faltings' thm on Mordell's conjecture $\Rightarrow$ $\# Y'(5,3)(\mathbb{Q}) < \infty$

$\exists$ infinitely many elliptic curves A satisfying (1) ① ②

(Rmk. Semistable conditions are "open conditions")