Localization for affine Lie algebras

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How to study / control $\mathrm{Shv}(\mathrm{Bun}_G)$

- Sam's : Whittaker coeff.

$$\mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Rep}(\check{G}_{\mathrm{Ran}})$$

not yet enough for GLC for general G

Conj. $\mathrm{coeff}_{\mathrm{Whit}} : \mathrm{Shv}(\mathrm{Bun}_G)_{\mathrm{cusp}} \longrightarrow \mathrm{Rep}(\check{G}_{\mathrm{Ran}})$ is fully faithful.

Thm (Færgeman – Raskin) This is conservative (de Rham setting).

In the de Rham setting, another way to do $\mathrm{DMod}(\mathrm{Bun}_G)$

localization theory.

- classical story $\quad \mathrm{Loc} : \mathfrak{g}\text{-mod}_\chi \rightleftarrows \mathrm{DMod}(G/B)_\lambda : \Gamma$

$$\chi \in \mathrm{Spec}\, Z(U(\mathfrak{g})) \simeq \mathfrak{t}^*/\!/W$$

$$\chi \longleftarrow\!\!\!\shortmid\ [\lambda]$$

In general, $G \curvearrowright X$

$$\mathfrak{g}\text{-mod} \rightleftarrows \mathrm{DMod}(X)$$

$K \subset G$, $\mathfrak{g}\text{-mod}^K \rightleftarrows \mathrm{DMod}(X/K)$

- Back to $\mathrm{Bun}_G = \{G\text{-torsors on } X\}$

$x \in |X|$, $\quad \mathrm{Bun}_G^{\mathrm{level}_\infty, x} = \{G\text{-torsors on } X,\ \text{trivialization on } D_x\}$

$L_x^+ G \curvearrowright \mathrm{Bun}_G^{\mathrm{level}\infty, x}$, quotient is Bun_G

$L_x G \curvearrowright \mathrm{Bun}_G^{\mathrm{level}\infty, x}$ via glueing.

$\mathcal{P}$ on X, $\quad \mathcal{P} = \mathcal{P}|_{X-x} \underset{\mathcal{P}|_{D_x^\circ}}{\sqcup} \overset{\text{trivialized}}{\mathcal{P}|_{D_x}}$

$\mathcal{V}$: $L_x G \curvearrowright \mathrm{Bun}_G^{\mathrm{level}\infty, x}$ changes $\mathcal{P}_G$

Loc_x: $L_x\mathfrak{g}\text{-mod} \longrightarrow \mathrm{DMod}(\mathrm{Bun}_G^{\mathrm{level}\infty, x})$

$$\begin{array}{ccc} \mathrm{Loc}_x: L_x\mathfrak{g}\text{-mod}^{L_x^+G} & \longrightarrow & \mathrm{DMod}(\mathrm{Bun}_G) \\ \| & & \\ \mathrm{KL}_G & & \end{array}$$

$$\begin{array}{ccc} L_x\mathfrak{g}\text{-mod}_\kappa^{L_x^+G} & \longrightarrow & \mathrm{DMod}(\mathrm{Bun}_G)_\kappa \\ \| & & \\ \mathrm{KL}_{G,\kappa} & & \end{array}$$

There is a central ext'n of $L\mathfrak{g}$: $0 \to \mathbb{C} \to \widetilde{L\mathfrak{g}}_\kappa \to L\mathfrak{g} \to 0$

$$L\mathfrak{g} \simeq \mathfrak{g}\otimes k((t))$$

classified by 2-cocycles:

$(\alpha\otimes f, \beta\otimes g) \mapsto \kappa(\alpha,\beta)\,\mathrm{Res}(g\,df)$

$\kappa: \mathfrak{g}\otimes\mathfrak{g} \to k$ Ad-inv. sym. bilinear form.

$\widetilde{L\mathfrak{g}}_\kappa\text{-mod} = \{\text{rep'ns of } \widetilde{L\mathfrak{g}}_\kappa \text{ w/ } \mathbb{C} \text{ acts by } 1\}$

Exer.

$$\begin{array}{ccc} \mathrm{triv} \in \mathrm{Rep}(L^+G) & \xrightarrow[\mathrm{Loc}_x^{\mathrm{QCoh}}]{\text{pullback along } \mathrm{Bun}_G \to \mathbb{B}L_x^+G} & \mathrm{QCoh}(\mathrm{Bun}_G) \ni \mathcal{O} \\ \mathrm{ind} \downarrow & & \downarrow \mathrm{ind} \\ \mathrm{Vac}_\kappa \in \mathrm{KL}_{\kappa,x} & \xrightarrow[\mathrm{Loc}]{} & \mathrm{DMod}(\mathrm{Bun}_G)_\kappa \ni \mathrm{Diff}_\kappa \end{array}$$

$\kappa_{crit} = -\frac{1}{2}\text{Kil}$

Rmk. $U(L\mathfrak{g})_\kappa = U(\widetilde{L\mathfrak{g}_\kappa}) / (\mathbb{1} - 1)$

If $\kappa = -\frac{1}{2}\text{Kil}$, it has a big center $\mathfrak{z} = \mathfrak{z}_\mathfrak{g}$

$\text{Spec}\, \mathfrak{z}_\mathfrak{g} \simeq \text{Op}_{\check{G}}(D)$

B-D used this to do GLC.

$x, y \in X$, $\text{Loc}_{x,y} : KL_{\kappa, x} \otimes KL_{\kappa, y} \longrightarrow \text{DMod}(\text{Bun}_G)_\kappa$

$\left[\text{Bun}_G^{\text{level } \infty\ x,\ \infty y} \curvearrowleft L_x G \times L_y G \right]$

$\text{Loc}_{\underline{x}} : \bigotimes_i KL_{\kappa, x_i} \longrightarrow \text{DMod}(\text{Bun}_G)_\kappa$

$\underline{x} = (x_1, \cdots, x_d)$

$KL_{\kappa, \overset{\circ}{X}^d} \longrightarrow \text{DMod}(\text{Bun}_G)_\kappa$

s.t. $KL_{\kappa, \overset{\circ}{X}^d} \otimes_{\text{DMod}(\overset{\circ}{X}^d)} \text{Vect}_{\underline{x}} \simeq KL_{\kappa, \underline{x}}$

$\rightsquigarrow KL_{\kappa, \text{Ran}} \longrightarrow \text{DMod}(\text{Bun}_G)_\kappa$

$\underline{x} \in \text{Ran}$ recover $\text{Loc}_{\kappa, \underline{x}}$.

Rmk. In general, things about D, $\overset{\circ}{D}$ $\rightsquigarrow$ things over Ran w/ structures.

- Spaces (schemes, stacks ...) $\rightsquigarrow$ $\mathcal{Y}_{\text{Ran}} \to \text{Ran}$
- Vect spaces $\rightsquigarrow$ $\mathcal{F} \in \text{Shv}(\text{Ran})$
- categories $\rightsquigarrow$ $\text{Shv}(\text{Ran})\text{-Mod}$

  KL_κ $\qquad$ $\boxed{KL_{\kappa, \text{Ran}}}$

Thm. (GLC2) For any quasi-compact $U \subset \mathrm{Bun}_G$,

$$\mathrm{KL}_{\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{Loc}} \mathrm{DMod}(\mathrm{Bun}_G)_\kappa \xrightarrow{j^*} \mathrm{DMod}(U)_\kappa$$

is a localization functor (admits a f.f. right adjoint)

Thm (critical FLE) $\mathrm{KL}_{\mathrm{crit},\mathrm{Ran}} \simeq \mathrm{IndCoh}_*\Big(\mathrm{Op}_{\check{G}}(\mathring{D}) \underset{\mathrm{LS}_{\check{G}}(\mathring{D})}{\times} \mathrm{LS}_{\check{G}}(D)\Big)_{\mathrm{Ran}}$

$$\mathrm{Op}_{\check{G}}(\mathring{D}) \underset{\mathrm{LS}_{\check{G}}(\mathring{D})}{\times} \mathrm{LS}_{\check{G}}(D) = \mathrm{Op}^{\mathrm{unr}}_{\check{G}}$$

<u>Other properties</u>.

- Loc is compatible w/ inserting vacuums

$$\mathrm{Loc}_{x,y}(\mathrm{Vac}_x \boxtimes V) \simeq \mathrm{Loc}_y(V)$$

- Loc is compatible w/ duality.

$$\begin{tikzcd}
\mathrm{KL}_{\kappa,\mathrm{Ran}} \arrow[r, "\mathrm{Loc}"] \arrow[d, leftrightarrow, "\mathrm{dual}"] & \mathrm{DMod}(U)_\kappa \arrow[d, leftrightarrow, "\mathrm{dual}"] \\
\mathrm{KL}_{-\kappa,\mathrm{Ran}} \arrow[r, "\mathrm{Loc}"] & \mathrm{DMod}(U)_{-\kappa}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Rep}(L^+_x G) \arrow[r, "\mathrm{Loc}^{\mathrm{QCoh}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \\
\mathrm{KL}_{\kappa,x} \arrow[u] \arrow[r, "\mathrm{Loc}"'] \arrow[ur, phantom, "\Searrow"] & \mathrm{DMod}(\mathrm{Bun}_G)_\kappa \arrow[u]
\end{tikzcd}$$

lax unital functor

$\Downarrow$ universal way

unital functor

$$\begin{tikzcd}
\mathrm{Rep}(L^+G)_{\mathrm{Ran}_x} \arrow[rr, "\mathrm{pullback}"] & & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{DMod}(\mathrm{Ran}_x) \arrow[d, "\int"] \\
\mathrm{KL}_{\kappa,\mathrm{Ran}_x} \arrow[u, "\mathrm{oblv}"] & /// & \mathrm{QCoh}(\mathrm{Bun}_G) \\
& \mathrm{KL}_{\kappa,x} \arrow[ul, "\mathrm{ins.\,vac.}"] \arrow[r, "\mathrm{Loc}"] & \mathrm{DMod}(\mathrm{Bun}_G)_\kappa \arrow[u, "\mathrm{oblv}"']
\end{tikzcd}$$

$\mathrm{Ran}_x = \{ I \subset X/k,\ x \in I \}$

$$\begin{tikzcd}
\mathrm{KL}_{\kappa,G} \arrow[r, "\mathrm{Loc}"] \arrow[d, "C^{\frac{\infty}{2}}(n((t)),-)\ \mathrm{BRST}"'] \arrow[dr, phantom, "\Nearrow"] & \mathrm{DMod}(\mathrm{Bun}_G)_\kappa \arrow[d, "\mathrm{CT}_*"] \\
\mathrm{KL}_{\kappa+\mathrm{shift},T} \arrow[r, "\mathrm{Loc}"] & \mathrm{DMod}(\mathrm{Bun}_T)_{\kappa+\mathrm{shift}}
\end{tikzcd}$$

does not commute

Prop. But it commutes if $\int_{\mathrm{Ran},c} \circ \cdots \circ$ ins.vac.