Hitchin moduli spaces and ramified geometric Langlands

恽之玮

L1. Homogeneous affine Springer fibers (local)

L2. Hitchin moduli spaces (global)

L3. Examples, some proofs

L4. Betti moduli spaces

L5. Ramified geometric Langlands + evidence

Lecture 1. $G/\mathbb{C}$ conn'd reductive gp

$(G, \nu = \frac{d}{m}) \rightsquigarrow$ sympl. alg var. $\mathcal{M}$

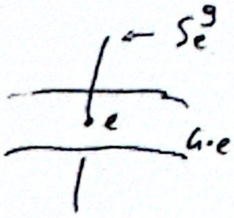
ν contain pos. rat'l no.

and a Lagrangian $\Lambda \subset \mathcal{M}$ is A.S.F.

Classical story: $\mathcal{N} \subset \mathfrak{g}$ nilp. cone

$e \in \mathcal{N} \rightsquigarrow (e, h, f)$ $\mathfrak{sl}_2$-triple in $\mathfrak{g}$

Slodowy slice $S_e^{\mathfrak{g}} := e + \mathfrak{g}^f$



$S_e := S_e^{\mathfrak{g}} \cap \mathcal{N}$ singular

$e = 0$, $S_e^{\mathfrak{g}} = \mathfrak{g}$, $S_e = \mathcal{N}$

$$\begin{tikzcd}
\mathcal{B}_e \subset \widetilde{S_e} \arrow[r] \arrow[d, "\pi_e"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathcal{N}} \arrow[d, "\pi"] & = \{ (x, \underset{\text{Borel subgp of } G}{B}) \;:\; x \in \mathcal{N},\ x \in \mathfrak{n}_B = \text{nil. rad. of } B \} \cong T^*\mathcal{B} \arrow[d] \\
e \in S_e \arrow[r] & \mathcal{N} \arrow[rr, bend right, "\text{Killing form}"] & & \mathfrak{g}^*
\end{tikzcd}$$

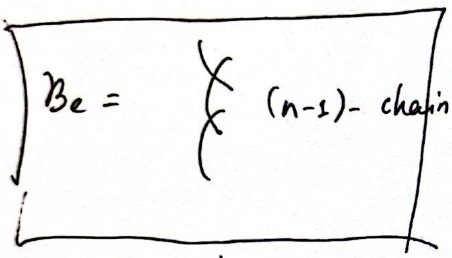
Fact: • π_e is a resol'n.

• $\widetilde{S_e}$ is a sympl. alg. var.

• $\mathcal{B}_e$ = Springer fiber of e = $\{ B : e \in \mathfrak{n}_B \}$ is a Lagrangian in $\widetilde{S_e}$.

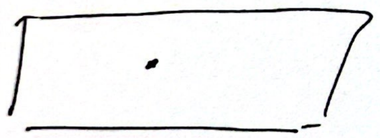
Ex. $\mathfrak{g} = \mathfrak{sl}_n$, e = subreg $\begin{pmatrix} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} & \\ & 0 \end{pmatrix}$

$\widetilde{S_e} =$



$\downarrow$

$S_e =$



A_{n-1} surface sing

$\mathbb{C}^\times$- action: $\quad h : \mathbb{G}_m \to G_{ad}$

$S_e^{\mathfrak{g}} = e + \mathfrak{g}^f$

$s \in \mathbb{G}_m$ acts by $s^2 \cdot \mathrm{Ad}(h(s^{-1}))$

e has wt 2 under $h(s)$

$\mathfrak{g}^f$ has ≤ 0 weights under $h(s)$

D.. -

$\Rightarrow$ This $\mathbb{G}_m$-action contracts $S_e^{\mathfrak{g}}$ to $\{e\}$.

$$\begin{array}{ccc} \mathbb{G}_m & \curvearrowright & \widetilde{S_e} \\ \| & & \downarrow \\ \mathbb{G}_m & \curvearrowright & S_e \end{array}$$

$\Rightarrow$ B_e is a conical Lagr. in $\widetilde{S_e}$.

($\mathbb{G}_m$-action is nontrivial on B_e)

contracting to $\{e\}$

$\mathfrak{g}$	$L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$
e	nilp. elts / top. nilp. elts
(e, h, f)	$\mathfrak{sl}_2$-triple ?
B_e	A.S.F.
$\widetilde{S_e}$	μ

Ex. of top. nilp. elts

$\mathfrak{sl}_n$

$$\gamma = \begin{pmatrix} a_1 t^{e_1} & & & \\ & a_2 t^{e_2} & & \\ & & \ddots & \\ & & & a_n t^{e_n} \end{pmatrix} \qquad e_i > 0$$

$$\gamma = \begin{pmatrix} 0 & & & t \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \qquad \gamma^N \to 0 \text{ in the } t\text{-adic top.} \quad (N \to \infty)$$

reg. ss.

$\Leftrightarrow$ eigenval. of γ have >0 val'n.

$\bigcup_{m \geq 1} \mathbb{C}((t^{1/m}))$

General G, $\gamma \in L\mathfrak{g}$,

$[\gamma, -] : L\mathfrak{g} \to L\mathfrak{g}$ (end. of v.s. $/ \mathbb{C}((t))$)

γ is top. nilp. $\Leftrightarrow$ eigenval's of $[\gamma, -]$ have val'n > 0.

$(e, h, f) \rightsquigarrow (\gamma, ?, ?)$

Require γ to be regular semisimple as an elt of $L\mathfrak{g} / \mathbb{C}((t)) = F$.

finite dim'l case:

$h : \mathbb{G}_m \to G_{ad}$

$Ad(h(s)).e = s^2 \cdot e$

$\theta : \mathbb{G}_m \to LG_{ad} = G_{ad}(\mathbb{C}((t)))$

$Ad(\theta(s)) \cdot \gamma = s^d \cdot \gamma$

Compare char. poly.

$P_\gamma(x) = P_{s^d \cdot \gamma}(x) = s^{nd} P_\gamma(s^{-d} x)$

$\Rightarrow P_\gamma(x) = x^n$. NOT what we want.

$LG_{ad} \rtimes \mathbb{G}_m^{rot} \curvearrowright L\mathfrak{g}$

scaling t

Look for $\theta : \mathbb{G}_m \to LG_{ad} \rtimes \mathbb{G}_m^{rot} \to \mathbb{G}_m^{rot}$, $s \mapsto s^m$

Want $\theta(s) \cdot \gamma = s^d \cdot \gamma$. sln : $P_\gamma(x) \big|_{t \mapsto s^m t} = s^{nd} P_\gamma(s^{-d} x)$

Ex. $\gamma = \begin{pmatrix} 1 & & t \\ & \ddots & \\ & & 1 \end{pmatrix}$, $P_\gamma(x) = x^n \pm t$

$$\left.\begin{array}{l} P_\gamma(x)\big|_{t \mapsto s^m t} = x^n \pm s^m t \\ s^{nd} P_\gamma(s^{-d}x) = x^n \pm s^{nd} \cdot t \end{array}\right] \text{same} \qquad m = nd \qquad \boxed{\frac{1}{n} = \frac{d}{m}}.$$

Def. A reg. s.s. $\gamma \in Lg$ is <u>homogeneous</u> if

$\exists\ \theta : G_m \longrightarrow LG_{ad} \rtimes G_m^{rot}$ s.t. $\theta(s)\cdot\gamma = s^d \gamma$, $\forall s \in G_m$.

Fact: (enough to check char. pol.)

$\mathfrak{C} = \mathfrak{g}/\!/G$, $\chi : \mathfrak{g} \to \mathfrak{C}$

γ is homog iff $\exists\ m, d \in \mathbb{Z}$ s.t.
(reg. s.s.)

$$\underset{\substack{\uparrow \\ \mathfrak{C}(\mathbb{C}((t)))}}{\chi(\gamma)}\Big|_{t \mapsto s^m t} = \underset{\substack{\downarrow \\ \text{weighted action}}}{s^d \cdot} \chi(\gamma)$$

Ex. sl_n : any γ w/ $P_\gamma(x) = x^n + \underset{\substack{\uparrow \\ \mathbb{C}^\times}}{a}\, t^d$

$m = n, d = d$

$\frac{d}{m}$ is called the <u>slope</u> of γ.

- Construct homog. elts ?
- How many are there ? $\frac{d}{m}$ can't be arbitrary.

$T_{ad} \subset G_{ad}$ max. torus

$\theta : \mathbb{G}_m \to T_{ad} \times \mathbb{G}_m^{rot}$

$s \longmapsto (s^{\lambda}, s^{m})$, $\lambda \in X_*(T_{ad})$

$$\mathbb{G}_m \underset{\theta}{\curvearrowright} L\mathfrak{g} = \widehat{\bigoplus_{i \in \mathbb{Z}}} (L\mathfrak{g})_{\frac{i}{m}}$$

$$(L\mathfrak{g})_{\frac{i}{m}} = \{ \gamma \in L\mathfrak{g} : \theta(s)\cdot\gamma = s^{i}\cdot\gamma \}$$

$(L\mathfrak{g})^{r.s.s.}_{\frac{i}{m}}$ consists of homog. elts of slope $\frac{i}{m}$.

$$L\mathfrak{g} = \underset{\substack{\| \\ \mathrm{Lie}\,T}}{\mathfrak{h}} \oplus \Big(\widehat{\bigoplus_{\alpha \in \Phi_{aff}}} (L\mathfrak{g})_{\alpha} \Big) \qquad (L\mathfrak{g})_{\alpha} = \mathfrak{g}_{\bar{\alpha}} \cdot t^{n} , \ \alpha = \bar{\alpha} + n\delta$$

$$(L\mathfrak{g})_{n\delta} = \mathfrak{h} \cdot t^{n}$$

$$(L\mathfrak{g})_{\frac{i}{m}} = \bigoplus (L\mathfrak{g})_{\alpha}$$

$i \neq 0$

$$\underset{\substack{\circlearrowleft \\ \mathrm{Ad}(s^{\lambda})}}{\theta(s) \curvearrowright \mathfrak{g}_{\bar{\alpha}}} \cdot \underset{\substack{\circlearrowleft \\ t \mapsto s^{m} t}}{t^{n}} \quad \text{has wt} \quad \langle \lambda, \bar{\alpha} \rangle + m \cdot n \overset{?}{=} i$$

$$\langle \tfrac{\lambda}{m}, \bar{\alpha} \rangle + n = \frac{i}{m} \iff \alpha\big(\tfrac{\lambda}{m}\big) = \frac{i}{m}$$

view α as an aff. linear func. on $X_*(T)_{\mathbb{R}}$

$$(Lg)_{\frac{i}{m}} = \bigoplus_{\alpha(\frac{d}{m}) = \frac{i}{m}} (Lg)_\alpha$$

<u>Constraint on m.</u>

<u>Def</u> (Springer) $W \curvearrowright \mathfrak{h}_{\mathbb{C}}$ Weyl gp of G

$w \in W$ is <u>regular</u> if $w \curvearrowright \mathfrak{h}$ has an eigenvector not lying on any wall.

$$\left\{ \begin{matrix} \text{regular conj. classes} \\ \text{in } W \end{matrix} \right\} \hookrightarrow \mathbb{N} \qquad \mathrm{Im} \overset{\text{def}}{=} \text{regular no. for } W$$

$$[w] \longmapsto \mathrm{ord}(w)$$

<u>Ex.</u> sl_n. $W = \mathfrak{S}_n$

complete list of reg. conj. classes

- $(1\,2 \ldots n)$ reg
- $(1\,2 \ldots n)^k$ reg. (equal length cycles)
- $(1\,2 \ldots n{-}1)$ reg $(1, \zeta, \ldots, \zeta^{n-2}, 0)$, $\zeta \in \mu_{n-1}$
- $(1\,2 \ldots n{-}1)^k$ reg

<u>Ex</u> G simple, w/ simple refl'ns $\{s_1, s_2, \ldots, s_r\} \subset W$

$W \ni w_{cox} = s_1 s_2 \cdots s_r$ is regular, $\mathrm{ord}(w_{cox}) = h$ Coxeter no.

$G = E_8$, $h_G = 30 = \dfrac{|\Phi|}{2}$

$(w_{cox})^k \longrightarrow$ 12 reg. conj. classes

<u>Thm</u> (Reeder - Yu, cf. Oblomkov - Y.)

1) Lg has a homog. elt of slope $\nu = \frac{d}{m}$ (lowest terms)

$\Leftrightarrow$ m is a reg. no. of W.

2) m reg. no. of W, $(d, m) = 1$

$$\theta : \mathbb{G}_m \longrightarrow T_{ad} \times \mathbb{G}_m^{rot}$$
$$s \longmapsto (s^{\rho^\vee}, s^m)$$

$$\rho^\vee = \frac{1}{2} \sum_{\alpha^\vee > 0} \alpha^\vee \in X_*(T_{ad})$$

$$(Lg)^{r.s.s.}_{\frac{d}{m}} \neq \phi.$$

Aside: w_0 longest elt in W.
suppose $w_0 \curvearrowright \mathfrak{h}$ by -1. $\Rightarrow$ w_0 is regular
$\Rightarrow$ $m = 2$ is a reg no.

<u>Ex</u> $G = Sp_{2n}$, $\nu = \frac{1}{2}$

$$\gamma = \left[\begin{array}{c|c} & tB \\ \hline C & \end{array}\right]$$

B, C const matrices
+ genericity cond'n

$\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$

$\Rightarrow \gamma$ is homog. of slope $1/2$

γ has slope ν $\Leftrightarrow$ after diag. γ in F'/F. $\alpha(\gamma) \in \mathbb{C}^\times \cdot t^\nu$

$(\gamma, \theta, \cancel{\chi})$

Affine Springer fibers

(Kazhdan - Lusztig, 80s)

$$\begin{array}{ccc} LG \supset L^+G & \supset & I = \text{Iwahori} \\ t \mapsto 0 \downarrow & & \downarrow \\ G & \supset & B \end{array}$$

Affine flag var. $Fl = LG/I = \bigcup_{\omega}$ proj. schemes

$\mathcal{B} = \{\text{Borel subgps of } G\}$.

If G is simply conn'd, $\Rightarrow$ $Fl = \{\text{Iwahori subgps of } LG\}$

$C_G(e) \curvearrowright \mathcal{B}_e = \{g \in G/B : Ad(g^{-1}).e \in \mathfrak{n}_B\}$

Def $Fl_\gamma = \{g \in LG/I : Ad(g^{-1})\gamma \in \mathrm{Lie}\, I^+\}$

$\gamma \in L\mathfrak{g}$

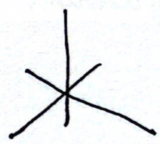
$I^+ = \ker(I \to T)$ pro-unip. rad. of I.

$Fl_\gamma \neq \emptyset \iff \gamma$ is top. nilp.

Fact (KL) If γ is reg. s.s. top. nilp., then Fl_γ is finite dim'l.

Ex $\mathfrak{sl}_2$, $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, $\nu = 1$  ∞-chain of $\mathbb{P}^1$s.

$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$, $\nu = \frac{3}{2}$ 

Ex $\mathfrak{sl}_3$, $\gamma = \begin{pmatrix} & & 1 \\ t & & \\ & t & \end{pmatrix}$, $\nu = \frac{2}{3}$.  3 $\mathbb{P}^1$s

Ex $\mathfrak{g}$ simple. $\nu = \frac{1}{h_G}$.

$$\gamma = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ t & & & \end{pmatrix} = \sum_{i=0}^{r} x_i, \quad 0 \neq x_i \in \mathfrak{g}_{\alpha_i}, \quad 0 \neq x_0 \in t\,\mathfrak{g}_{-\theta}$$

$$\left[\begin{array}{l} \sum n_i \alpha_i = 0 \\ \sum n_i = h_G \end{array}\right.$$

$G =$ s.c., $Fl_\gamma =$ base pt

Fact (KL, Bezrukavnikov) Special case of homog. γ slope ν,

$$\dim Fl_\gamma = \frac{|\Phi| \cdot \nu - c_w}{2}.$$

Here $\nu = \frac{d}{m}$, $m = \mathrm{ord}(w)$, w regular $\in W$.

$c_w = \dim(\mathfrak{h}/\mathfrak{h}^w)$

$T_\gamma = C_{G_F}(\gamma)$ torus $/ F = \mathbb{C}((t))$.

LT_γ loop group of T_γ.

<u>Fact.</u> $\{\text{max. tori in } G_F\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{conj. classes} \\ \text{in } W\end{matrix}\right\}$

$$T_\gamma \longleftrightarrow [w]$$

<u>Ex</u> $\gamma = \begin{pmatrix} & & t^d \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathfrak{sl}_n$, $(d,n) = 1$

$\nu = \frac{d}{n}$

$$T_\gamma(F) = \underbrace{F[\gamma]^\times}_{E^\times} \cap SL_n(F) \quad , [E:F] = n$$

$LT_\gamma \curvearrowright Fl_\gamma$

same infinitness (no. of components)

$$\pi_0(LT_\gamma) \simeq X_*(T)_w$$

$\curvearrowright$

$\mathrm{Irr}(Fl_\gamma)$ free, finitely many orbits.

Fl_γ f.t. $\Leftrightarrow$ w is elliptic, ie $\mathfrak{h}^w = \{0\}$.

<u>Lecture 2</u>. homog. γ of slope $\nu = \frac{d}{m}$

(λ, m): $\mathbb{G}_m \xrightarrow{\theta} T_{ad} \times \mathbb{G}_m^{rot} \curvearrowright L\mathfrak{g}$

$L\mathfrak{g} = \widehat{\bigoplus_{i \in \mathbb{Z}}} (L\mathfrak{g})_{i/m}$, $\gamma \in (L\mathfrak{g})_\nu^{r.s.s.}$

$$\mathfrak{g} = \bigoplus_{\frac{i}{m} \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}} \mathfrak{g}_{i/m}$$

$t=1 \uparrow$

$$\supset \bigoplus_{i \in \mathbb{Z}} (L\mathfrak{g})_{i/m} \circlearrowleft \mathbb{C}[t, t^{-1}]$$

$(Lg)_{i/m} \xrightarrow[\sim]{t} (Lg)_{i/m+1}$

$\bar{\gamma} \in g_{\bar{\nu}}$, $\bar{\nu} \equiv \nu \mod \mathbb{Z}$

γ is r.s.s. in Lg $\Longleftrightarrow$ $\bar{\gamma}$ is r.s.s. in g

Ex. $g = sp_{2n}$, $d_x = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right)$, $m=2$

$$g = g_0 \oplus g_{1/2}$$
$$\begin{array}{ccc} g_0 & & g_{1/2} \\ \wr\!\wr & & \wr\!\wr \\ gl_n & & Sym^2(St) \oplus Sym^2(St^*), \end{array}$$
St = standard rep'n

$$\left(\begin{array}{c|c} g & Sym \\ \hline Sym & {}^t g^{-1} \end{array}\right).$$

$\gamma \rightsquigarrow [w]$ reg. conj. class in W, $\mathrm{ord}(w) = m$.

To see $[w]$, $g = \bigoplus_{\frac{1}{m}\mathbb{Z}/\mathbb{Z}} g_{i/m}$, $\bar{\gamma} \in g_{\bar{\nu}}^{r.s.}$, $H = C_G(\bar{\gamma})$ max. torus in G.

$$h = \bigoplus_{\frac{1}{m}\mathbb{Z}/\mathbb{Z}} h_{i/m}$$

$\longleftrightarrow$ order m autom. of H

$\uparrow$

this is the action of w.

$\uparrow$

up to conj.

$Fl_\gamma = \{ gI \in LG/I : \ Ad(g^{-1}) \cdot \gamma \in \mathrm{Lie}\, I^+ \}$

$\mathbb{G}_m$-action on Fl_γ

θ induces $\mathbb{G}_m$-action on LG, on LG/I, by s^1. rot (s^m) . it stabilizes Fl_γ.

$\mathbb{G}_m \curvearrowright Fl_\gamma$ = ind-proj. scheme

$(Fl_\gamma)^{\mathbb{G}_m}$

$$Fl = \bigcup_{w \in \tilde{W} = X_*(T) \rtimes W} IwI/I$$

$\circlearrowleft$
I

$(Lg)_{\geq 0}$ parahoric subalg.

$\parallel$

$\mathrm{Lie}\, P$

$P = P_{\frac{\Lambda}{m}} \subset LG$ conn'd subgp (parahoric subgp)

The G_m-action on P contracts it to L_P, $\mathrm{Lie}\, L_P = (Lg)_0$

$$Fl = \bigsqcup_{\bar{w} \in W_P \backslash \tilde{W}} P\bar{w}I/I$$

W_P — Weyl gp of L_P

$P\bar{w}I/I$ contracts to $L_P\bar{w}I/I = L_P/L_P \cap {}^{\bar{w}}I \simeq$ flag variety of L_P

$$Fl_\gamma = \bigsqcup_{\bar{w} \in W_P \backslash \tilde{W}} Fl_\gamma \cap (P\bar{w}I/I)$$

$\Big\}$ contracting

$$\mathrm{Hess}_\gamma(\bar{w}) := Fl_\gamma \cap (L_P\bar{w}I/I)$$

Hessenberg variety (Goresky - Kottwitz - MacPherson)

$G_{\bar{0}} \simeq L_P$

$\mathrm{Ad} \circlearrowright g_{\bar{\nu}} \ni \gamma$

$\mathrm{Hess}_\gamma(\bar{w})$ can be defined using the cyclic grading on g.

$$\mathrm{Hess}_\gamma(\bar{w}) \subset L_P\bar{w}I/I \simeq L_P/L_P \cap {}^{w}I$$

$\parallel$

$$\Big\{ l \in L_P/L_P \cap {}^{w}I : \mathrm{Ad}(l^{-1}) \cdot \gamma \in \underbrace{g_{\bar{\nu}} \cap {}^{\bar{w}}\mathrm{Lie}(I^+)}_{\text{analog of } n_B} \Big\}$$

generalization of Springer fibers

$(L_P \curvearrowright g_{\bar{\nu}} \supset g_{\bar{\nu}} \cap {}^{w}\mathrm{Lie}\, I^+)$

$(G \curvearrowright g \supset n_B)$

Bernstein's example (appendix to KL)

Sp_6, $\nu = \frac{1}{2}$, $\mathrm{Hess}_\gamma(\bar{w})$ for a specific $\bar{w}$ $\simeq$ elliptic curve.

Note, it is expected : $\mathcal{B}_e$ should be paved by affine spaces

DeConcini – Lusztig – Procesi : $H^*(\mathcal{B}_e)$ spanned by alg. cycles. (in part., no H^1)

Hitchin moduli stacks

$*$ $T^* \mathrm{Bun}_G$ $X/\mathbb{C}$ complete smooth curve

Bun_G = moduli stack of G-torsors on X

Bun_G is a smooth Artin stack (locally $\underset{\text{smooth}}{M}/H$)

of dim $= (\dim G)(g-1)$, g = genus of X.

$$T_{[\mathcal{E}]}\mathrm{Bun}_G = H^1(X, \mathrm{Ad}(\mathcal{E}))$$

$$\mathrm{Ad}(\mathcal{E}) = \mathcal{E} \overset{G}{\times} \mathfrak{g}$$ v.b. locally modelled on $\mathfrak{g}$.

$$T^*_{[\mathcal{E}]}\mathrm{Bun}_G \underset{\text{Serre duality}}{=} H^0(X, \underbrace{\mathrm{Ad}^*(\mathcal{E})}_{= \mathcal{E}\overset{G}{\times}\mathfrak{g}^*} \otimes \omega_X)$$

G semisimple. identify $\mathfrak{g} \simeq \mathfrak{g}^*$ (Killing)

$\Rightarrow$ A point in $T^*\mathrm{Bun}_G$ is a pair $(\mathcal{E}, \varphi)$: $\mathcal{E}$ G-bundle on X

G-Higgs bundle $\varphi \in H^0(X, \mathrm{Ad}(\mathcal{E}) \otimes \omega_X)$

Higgs field

$\mathcal{M} = T^*\mathrm{Bun}_G$ is the Hitchin moduli stack.

local $\rightsquigarrow$ global

Fl_γ $\rightsquigarrow$?

$$Gr_\gamma = \{ g \in \underset{\substack{\| \\ Gr}}{LG/L^+G} : Ad(g^{-1})\gamma \in L^+\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[[t]] \}$$

$\cap$

$Gr = \{ (\mathcal{E}, \tau) :$ $\mathcal{E}$: G-bun. on $D = \mathrm{Spec}\, \mathbb{C}[[t]]$,
τ : a triv. of $\mathcal{E}$ on $D^\times = \mathrm{Spec}\, \mathbb{C}((t))$ $\}$

$(\mathcal{E}, \tau)$: choose τ_0 : triv. of $\mathcal{E}$ on D

$\downarrow$

$\tau \cdot \tau_0^{-1} \in G(F)$

well-defined in $G(F)/G(\mathcal{O}_F)$

$Gr_\gamma = \{ (\mathcal{E}, \tau) :$ $\mathcal{E}, \tau$ as before, τ transform γ to a section of $Ad(\mathcal{E})$ on $D \}$

$$\gamma \in Ad(\mathcal{E}_{triv}|_{D^\times}) \\ \| \\ L\mathfrak{g} = \mathfrak{g} \otimes F$$

$Gr_\gamma \longrightarrow \{ (\mathcal{E}, \varphi) :$ $\mathcal{E}$ G-bundle on D
$\varphi \in H^0(D, Ad(\mathcal{E}))$ $\}$

$(\mathcal{E}, \tau) \longmapsto (\mathcal{E}, \tau \cdot \gamma \cdot \tau^{-1})$

torsor for LT_γ local Higgs bundle

$$[LT_\gamma \backslash Gr_\gamma] \simeq \left\{ \begin{array}{l} \text{local Higgs bundles.} \\ \text{w/ the same char. poly. as } \gamma \end{array} \right\}$$

$\cap$

Hitchin fibration

$$G = GL_n$$

$$\begin{array}{ccc} \mathcal{M} & (\mathcal{E}, \varphi) & \longleftrightarrow & (V, \ \varphi : V \longrightarrow V \otimes \omega_X) \\ \downarrow & \rotatebox{90}{$\mapsto$} & & \text{rk } n \text{ v.b.} \qquad \rotatebox{90}{$\mapsto$} \\ A & \text{char. poly. of } \varphi & & \end{array}$$

$$\mathrm{char}(\varphi) = y^n + a_1 y^{n-1} + \cdots + a_n$$

$$-a_1 = \mathrm{tr}(\varphi) \in H^0(X, \omega_X)$$

$$\pm a_i = \mathrm{tr}(\Lambda^i \varphi) \in H^0(X, \omega_X^{\otimes i})$$

General G, $\mathcal{O}(\mathfrak{g}/\!\!/G) = \mathbb{C}[\mathfrak{g}]^G \underset{\text{choose}}{=} \mathbb{C}[f_1, f_2, \cdots, f_r]$, $r = \mathrm{rk}\, G$.

deg $d_1, d_2, \cdots, d_r$

$$A = \prod_{i=1}^{r} H^0(X, \omega_X^{\otimes d_i})$$

$f : \mathcal{M} \to A$,

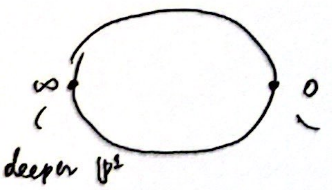
$(\mathcal{E}, \varphi) \mapsto (f_1(\varphi), f_2(\varphi), \cdots, f_r(\varphi))$.

__Thm__ (Ginzburg, Laumon) $\underset{\substack{\shortparallel \\ \text{symplectic}}}{f : \mathcal{M}} \to A$ is a Lagrangian fibration.

$$(\Rightarrow \dim A = \tfrac{1}{2} \dim \mathcal{M} = \dim \mathrm{Bun}_G = (\dim G)(g-1))$$

__Exer.__ Check $\dim A = (\dim G)(g-1)$ directly $(g \geq 2)$.

Our situation



$(X, 0)$ $\mathrm{Bun}_G(I_0)$

$$= \{(\mathcal{E}, \mathcal{E}_0^B) : \mathcal{E}_0^B \text{ is a } B\text{-reduction of the fiber } \mathcal{E}_0\}$$

$$T^*\mathrm{Bun}_G(I_0) = \{(\mathcal{E}, \mathcal{E}_0^B, \varphi) : (\mathcal{E}, \mathcal{E}_0^B) \in \mathrm{Bun}_G(I_0),$$

$$\varphi \in H^0(X, \mathrm{Ad}(\mathcal{E}) \otimes \underbrace{\omega_X(0)}_{\text{simple pole at } 0})$$

$$\mathrm{res}_0 \varphi \in (\mathcal{E}_0^B) \times^B \mathfrak{n}_B$$

Level at ∞.

Recall Slodowy slices. h gives $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$

$$S_e^{\mathfrak{g}} = \underbrace{\underset{\substack{| \\ \text{wt } 2}}{e} + \mathfrak{g}^f \simeq (e + \mathfrak{g}_{\leq 0}) / \underbrace{G_{\leq -2}}_{\text{unip. gp}}}_{\text{has affine analogue}}$$

$$\begin{tikzcd} & \widetilde{S}_e \arrow[dl] \arrow[dr] & \\ S_e^{\mathfrak{g}} \arrow[dr] & & \widetilde{N} \arrow[dl] \\ & \mathfrak{g} & \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} & \mathcal{M}_\gamma \arrow[dl, "ev_\infty"'] \arrow[dr, "res_0"] & \\ \text{affine analog. of } S_e^{\mathfrak{g}} & & \mathfrak{n}_B/B \end{tikzcd}$$

Moy-Prasad group.

$$\gamma + (L_\infty \mathfrak{g})_{\leq 0} \Big/ \underbrace{(L_\infty G)_{\leq -\nu} \cdot (L_\infty T_\gamma)_{<0}}_{K_\gamma} \qquad , \ \overset{\mathbb{G}_m}{\curvearrowright} LT_\gamma = C_{LG}(\gamma)$$

$$Lt_\gamma = \hat{\bigoplus} (Lt_\gamma)_{i/m}$$

$(L\mathfrak{g})_{\leq 0}$ is completed t^{-1}-adically. $\subset \mathfrak{g} \otimes \mathbb{C}((t^{-1})) =: L_\infty \mathfrak{g}$

parahoric of $L_\infty \mathfrak{g}$ $\qquad$ unif. at ∞

Def'n of $\mathcal{M}_\gamma$ moduli stack of $(\mathcal{E}, \mathcal{E}_0^B, \mathcal{E}_\infty^{K_\gamma}, \varphi)$

$\mathcal{E}$: G-bun. on X

$\mathcal{E}_0^B$: B-red. of $\mathcal{E}$,

$\mathcal{E}_\infty^{K_\gamma}$: K_γ-level str. at ∞.

$\varphi \in H^0(\mathbb{P}^1 \setminus \{0, \infty\}, \mathrm{Ad}\,\mathcal{E} \otimes \omega_X)$

s.t.

1) φ has at most simple pole at 0.

$\quad res_0 \varphi \in (\mathcal{E}_0^B) \overset{B}{\times} \mathfrak{n}_B$

2) $\varphi|_{D_\infty^\times} \sim \left(\gamma + (L_\infty \mathfrak{g})_{\leq 0}\right) \frac{dt}{t}$

under a trivialization

$\psi\ (\mathcal{E}|_{D_\infty}, \mathcal{E}_\infty^{K_\gamma})$ ambiguity $\in K_\gamma$

$\underline{Ex}$ $\nu = 1$, $\gamma = \gamma_0 \cdot t$, $\gamma_0 \in \mathfrak{h}^{rs}$, $K_\gamma = \ker\left(G\llbracket t^{-1}\rrbracket \xrightarrow{\text{mod } t^{-1}} G\right)$

M_γ classifies $(\mathcal{E}, \mathcal{E}_0^B, \tau_\infty: \text{triv. of } \mathcal{E}_\infty, \varphi)$

$$\varphi|_{D_\infty^\times} \underset{\substack{\uparrow \\ \text{using } \tau_\infty}}{\sim} \left(\frac{-\gamma_0}{t^{-2}} + \cdots\right) d(t^{-1})$$

<u>Lecture 3</u>. $\nu = 1$ case. $\gamma = \gamma_0 t$, $\gamma_0 \in \mathfrak{g}^{rs}$

Def of M_γ: defines $(\mathcal{E}, \mathcal{E}_0^B, \tau_\infty, \varphi)$

$\mathcal{E}$: G-bundle on $\mathbb{P}^1$

$\mathcal{E}_0^B$: B-reduction of $\mathcal{E}_0$

τ_∞: trivialization of $\mathcal{E}_\infty$

φ: section of $\mathrm{Ad}(\mathcal{E}) \otimes \omega_{\mathbb{P}^1}$ over $\mathbb{P}^1 \setminus \{0, \infty\}$.

s.t. • φ has simple pole at 0 w/ $\mathrm{res}_0 \varphi$ strictly upper Δ w.r.t. $\mathcal{E}_0^B$

$$\left(\begin{array}{l} GL_n: \mathcal{E}^B \text{ gives } F_1 \subset F_2 \subset \cdots \subset F_n = \mathcal{E}_0, \\ \qquad\qquad (\mathrm{res}_0 \varphi)(F_i) \subset F_{i-1} \end{array}\right)$$

• φ at ∞ takes the form

$$\left(\frac{-\gamma_0}{t^{-2}} + \text{h.o. terms of } t^{-1}\right) \frac{dt^{-1}}{t^{-1}} \qquad (\text{under } \tau_\infty)$$

<u>Questions</u>: • symplectic structure on M_γ

• Fl_γ and M_γ

• Hitchin fibration for M_γ

Hitchin base $\nu = 1$. $f_1, \cdots, f_r$ homog. deg $d_1, \cdots, d_r$ generators of $\mathbb{C}[\mathfrak{g}]^G$.

$$f_i(\varphi) \in H^0(\mathbb{P}^1, \omega(2 \cdot \underline{\infty} + 1 \cdot \underline{0})^{\otimes d_i}) \qquad \omega_{\mathbb{P}^1}(\underline{\infty} + \underline{0}) \simeq \mathcal{O}_{\mathbb{P}^1}$$

$$\quad \text{\rotatebox{90}{$\simeq$}} \qquad\qquad\qquad\qquad \frac{dt}{t} \leftrightarrow 1$$

$$H^0(\mathbb{P}^1, \mathcal{O}(\underline{\infty})^{\otimes d_i})$$

$f_i(\varphi)$ is a poly. in t of deg $\leq d_i$

- leading coeff $= f_i(-\gamma_0)$ ($\Leftarrow$ condition at ∞)
- constant coeff $= 0$. ($\Leftarrow$ res$_0 \varphi$ is nilp)

$$\mathcal{A}_\gamma(\mathbb{C}) = \prod_{i=1}^{n} \underbrace{\mathbb{C}[t] \begin{array}{l} \deg \leq d_i.\ \text{leading } f_i(-\gamma_0) \\ \text{const } \cancel{\text{coeff}} = 0 \end{array}}_{\text{affine space of dim} = d_i - 1}$$

$$f_\gamma : \mathcal{M}_\gamma \longrightarrow \mathcal{A}_\gamma$$

affine space of dim $= \sum_{i=1}^{n} \underbrace{(d_i - 1)}_{\text{exponents of } G} = \dim \mathcal{B}$

- Hitchin fibers via spectral curves $G = GL_n$

$\mathcal{M}_\gamma$: ($\mathcal{V}$, $F_\bullet$ at 0, basis of $\mathcal{V}_\infty$, φ)

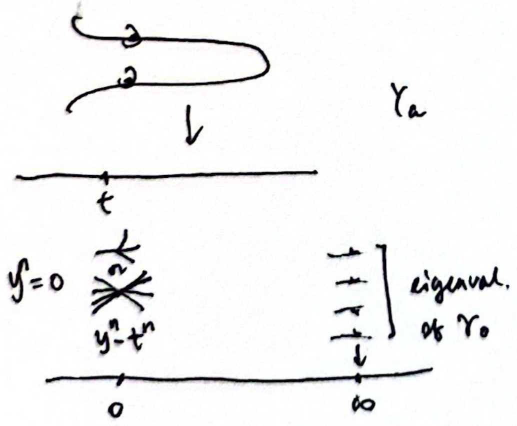
$$\varphi : \mathcal{V} \longrightarrow \mathcal{V} \otimes \omega(2 \cdot \underline{\infty} + 1 \cdot \underline{0}) \simeq \mathcal{V}(\underline{\infty})$$

Spectral curve attached to $a \in \mathcal{A}_\gamma$.

$$a = (a_1, a_2, \cdots, a_n)$$

$a_i \in H^0(\mathbb{P}^1, \mathcal{O}(d_i))$

$y^n + a_1 y^{n-1} + \cdots + a_n = 0$ defines a curve $\subset \mathrm{Tot}(\mathcal{O}(1))$



at ∞, eqn $=$ char. pol. (γ_0)

Fact. $f_\gamma^{-1}(a) = \{ (\mathcal{L}, F_\bullet, \tau_\infty) :$ • $\mathcal{L} \in \overline{\mathrm{Pic}}(Y_a)$ (torsion free, gen. line bundle)

• $0 \subset F_1 \subset \cdots \subset F_n = \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathbb{C}_0$

each F_i must be $\mathcal{O}_{Y_a} \otimes_{p_a^{-1}(0)} \mathbb{C}_0$-mod

$p_a : Y_a \to \mathbb{P}^1$

• τ_∞ : a basis of $\mathcal{L}|_{\infty'}$, $\forall \infty'$ above ∞ $\}$

$(\mathcal{L}, F_\bullet, \tau_\infty) \rightsquigarrow \big(V = p_{a*}\mathcal{L}$. equipped w/ $\underset{y}{\overset{\shortparallel}{\varphi}} \curvearrowright V$ (really $V \to V \otimes \mathcal{O}(1)$) $F_\bullet, \tau_\infty \big)$

$G = SL_n$. $(V, F_\bullet, \tau_\infty, \varphi$, $\iota : \det(V) \simeq \mathcal{O}$, compatible w/ τ_∞, $\mathrm{tr}\,\varphi = 0)$

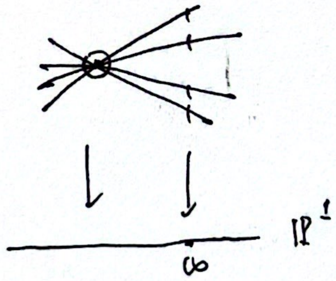
$\mathcal{A}_\gamma \ni a_\gamma = \left(b_1(-\gamma_0)t^{d_1}, b_2(-\gamma_0)t^{d_2}, \dots \right)$

$\exists\ \mathbb{G}_m \curvearrowright \mathcal{A}_\gamma$ contracting to a_γ.

$s \in \mathbb{G}_m$ acts by $s \cdot \mathrm{rot}(s^{-1})$

$\mathbb{G}_m \curvearrowright \mathcal{M}_\gamma$ $(\varphi \mapsto s\varphi) \cdot \mathrm{rot}(s^{-1})$

GL_n, Y_{a_γ}



$\mathrm{Tot}(\mathcal{O}(1))$

$Fl_\gamma \longrightarrow f^{-1}(a_\gamma)$

$\|$ bijection on $\mathbb{C}$-points

(For $\gamma = \frac{d}{n}$, $(d,n) = 1$, Y_{a_γ} is irred.)

$\{ \Lambda_\bullet$ $\mathcal{O}_F$-lattice in F^n $: \gamma\Lambda_i \subset \Lambda_{i-1} \}$

$\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_n \overset{\shortparallel}{\underset{}{}} t^{-1}\Lambda_0$

glue $\Lambda_\bullet$ w/ $\Big(\mathcal{O}_{Y_{a_\gamma}}|_{\underbrace{\mathbb{P}^1 - 0}_{= \coprod_n \mathbb{A}^1}}$, can. triv. above $\infty \Big)$ to get $(\mathcal{L}, F_\bullet, \tau_\infty) \in f^{-1}(a_\gamma)$

$\mathcal{O}_F[\gamma] \curvearrowright \Lambda_i$

view Λ_i as a torsion free rank 1 sheaf on $D_{Y_{a_\gamma},0}$.

Sympl. str. on $\mathcal{M}_\gamma$

"$T^*\mathrm{Bun}_G$"

Hamiltonian reduction $\quad H \curvearrowright X, \quad \mu: T^*X \xrightarrow{H\text{-eq}} \mathfrak{h}^*$

$$\begin{array}{c} \mathfrak{h}^* \\ \Psi \\ \zeta : H\text{-invt} \end{array}$$

$$T^*X /\!\!/_\zeta H := \mu^{-1}(\zeta)/H$$

$$\zeta = 0, \quad T^*X /\!\!/_0 H = T^*(X/H)$$

$$K_1 = \ker\left(G[\![t^{-1}]\!] \xrightarrow{\bmod t^{-1}} G\right)$$

$$K_2 = \ker\left(G[\![t^{-1}]\!] \xrightarrow{\bmod (t^{-1})^2} G\left(\mathbb{C}[t^{-1}]/(t^{-2})\right)\right)$$

$\mathfrak{g} \curvearrowright \mathrm{Bun}_G(I_0, K_2) = \left\{(\mathcal{E}, \mathcal{E}^B_0, \tau_{2\cdot\infty}) : \begin{array}{l}\tau_{2\cdot\infty} \text{ triv.} \\ \mathcal{E}|_{\mathrm{Spec}\, \mathbb{C}[t^{-1}]/(t^{-2})}\end{array}\right\}$

$\mathfrak{g}$ is K_1/K_2; K_2 at ∞

$\downarrow$ quotient by $K_1/K_2 \simeq \mathfrak{g}\cdot t^{-1}$

$\mathrm{Bun}_G(I_0, K_1)$

$\rightarrow$ changing $\tau_{2\cdot\infty}$ while keeping τ_∞.

$$T^*\mathrm{Bun}_G(I_0, K_2) \xrightarrow{\mu} \mathfrak{g}^* \simeq \mathfrak{g}$$

for $K_1/K_2 \simeq \mathfrak{g}$-action

$$\begin{array}{c} \mathfrak{g}^* \\ \Psi \\ \gamma_0 \end{array}$$

$e_1 + t^{-1}v_1$

$e_2 + t^{-1}v_2$

$\vdots$

$e_n + t^{-1}v_n$

<u>Fact</u>. $T^*\mathrm{Bun}_G(I_0, K_2) /\!\!/_{\gamma_0} \mathfrak{g} \simeq \mathcal{M}_\gamma \quad \Rightarrow$ Sympl. str. on $\mathcal{M}_\gamma$

General slope ν.

Recall how to see sympl. str. on $\widetilde{S}_e$.

Want $\quad \widetilde{S}_e = T^*X /\!\!/_{\xi} H.$

$H \circlearrowright X = \mathcal{B}$

└ Premet group attached to (e, h, f)

$$e + \mathfrak{g}_{\le 0} \circlearrowleft G_{\le -2}$$

If e is even (i.e. $\mathfrak{g}_{odd} = 0$), Take $H = G_{\le -2} \twoheadrightarrow \mathfrak{g}_{-2} \xrightarrow{\langle e, - \rangle} \mathbb{C}$

thus $e \in \mathfrak{h}^*$ inv't under H

$$\mathfrak{g}_{\le 0} = \mathfrak{g}_{\le -2}^{\perp} \quad , \quad e + \mathfrak{g}_{\le -2}^{\perp} / G_{\le -2}.$$

$$\begin{array}{c} T^*\mathcal{B} = \widetilde{\mathcal{N}} \\ \downarrow \\ e + \mathfrak{g}_{\le -2}^{\perp} \subset \mathfrak{g}^* \\ \downarrow \\ e \in \mathfrak{g}_{\le -2}^* \end{array}$$

$$\Rightarrow \widetilde{S}_e \simeq T^*\mathcal{B} /\!\!/_e G_{\le -2}$$

General e, $H = G_{\le -2} \cdot (\text{half of } \mathfrak{g}_{-1})$

Lagr. under $(\cdot,\cdot)_e$

$$(\cdot,\cdot)_e : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \xrightarrow{\text{symplectic}} \mathbb{C}$$

$$(x, y) \mapsto \langle e, [x, y] \rangle$$

e extends to $H \to \mathbb{C}$, $\quad e \in (\mathfrak{h}^*)^H$

$$\begin{array}{ccc} T^*\mathcal{B} /\!\!/_e H & \longrightarrow & e + \mathfrak{h}^{\perp}/H \simeq e + \mathfrak{g}_{\le 0} / G_{\le -2} \\ \wr\!\wr & & \\ \widetilde{S}_e & & \end{array}$$

Affine analogue. $\quad \gamma + (L_\infty \mathfrak{g})_{\le 0} \Big/ \overbrace{\left((L_\infty G)_{\le -\nu} \cdot (L_\infty T_\gamma)_{<0} \right)}^{K_\gamma}$

$$\wr\!\wr$$

look for $J_\gamma \supset K_\gamma$ s.t. $\quad \gamma + (\mathrm{Lie}\, J_\gamma)^{\perp} / J_\gamma$

$\gamma : J_\gamma \to \mathbb{C}$ (homo.)

$$\text{“} T^*\mathrm{Fl} /\!\!/_{\gamma} J_\gamma \text{”} = \mathcal{M}_\gamma$$

$$\downarrow$$

Pick $J_\gamma' \triangleleft J_\gamma$ s.t. $\gamma : \overbrace{J_\gamma / J_\gamma'}^{\text{finite dim'l}} \longrightarrow \mathbb{C}$

$$\mathrm{Bun}_G(I_0, J_\gamma') \circlearrowleft J_\gamma / J_\gamma'$$

$$T^*\mathrm{Bun}_G(I_0, J_\gamma') /\!\!/_{\gamma} J_\gamma / J_\gamma' \simeq \mathcal{M}_\gamma .$$

$$J_\gamma = (L_\infty G)_{\leq -\frac{\nu}{2}} \, (L_\infty J_\gamma)_{<0}$$

$\uparrow$ ok if $(L_\infty \mathfrak{g})_{-\frac{\nu}{2}} = 0$

In general.

$$\begin{tikzcd} J_\gamma \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (L_\infty G)_{\leq -\frac{\nu}{2}} (L_\infty T_\gamma)_{<0} \arrow[d] \\ L \arrow[r, phantom, "\underset{\text{Lag}}{\subset}"] & (L_\infty \mathfrak{g})_{-\frac{\nu}{2}} / (L_\infty \mathfrak{t}_\gamma)_{-\frac{\nu}{2}} \end{tikzcd}$$

$\leftarrow$ has a sympl str.

$\gamma \in \mathrm{Lie}\,T$, $(T \subset G)$

Ex. $\nu = 2$,

$$\begin{aligned} K_\gamma &= K_2 \cdot (LT)_{\leq -1} \quad \leftarrow \text{1st cong. subgp of } T[[t^{-1}]] \\ &\;\;\cap \qquad \| \\ &\;\; (L_\infty G)_{\leq -2} \\ J_\gamma &= \{ 1 + b \cdot t^{-1} + \cdots \} \qquad b \in \mathfrak{g}/\mathrm{Lie}T \\ &\;\;\cap \\ K_1 &= (L_\infty G)_{\leq -1} \end{aligned}$$

Hitchin base for general ν. $\qquad \omega(0 + \infty) \simeq \mathcal{O}$

$$A_\gamma \subset \prod H^0(\mathbb{P}^1, \mathcal{O}([d_i \cdot \nu]))$$

$$\begin{aligned} &\Psi \\ a &= (a_1, a_2, \ldots, a_r) \end{aligned}$$

$a_i = (-)\, t^{[d_i \nu]} + \cdots + 0$ $\leftarrow$ const. term

if $\overset{?}{d_i \nu} \in \mathbb{Z}$, then this coeff $=$ $t^{d_i \nu}$ coeff of $f_i(\gamma)$

Thm γ homog. of slope $\nu > 0$

1) $\mathcal{M}_\gamma$ is a smooth alg. space w/ a canonical sympl. str.

2) $f_\gamma : \mathcal{M}_\gamma \longrightarrow \mathcal{A}_\gamma$ is a Lagrangian fib.

$(\Rightarrow \dim \mathcal{A}_\gamma = \frac{1}{2} \dim \mathcal{M}_\gamma)$

3) If γ is elliptic $(\Leftrightarrow \nu = \frac{d}{m},\ m = \text{ord (elliptic) reg. } w)$

f_γ is proper.

4) $\mathbb{G}_m$-action on $\mathcal{M}_\gamma, \mathcal{A}_\gamma$ compatibly. Contracting $\mathcal{A}_\gamma$ to $a_\gamma = (f_i(\gamma))$.

5) Nat'l map $\quad Fl_\gamma \to f_\gamma^{-1}(a_\gamma)$ homeomorphism.

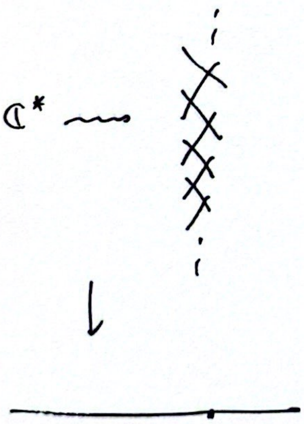
6) $H^*(\mathcal{M}_\gamma) \xrightarrow{\sim} H^*(Fl_\gamma)$

Ex. SL_2, $\nu = \frac{3}{2}$

$Fl_\gamma \qquad\qquad f^{-1}(a_\gamma)$

Ex. SL_2, $\nu = 1$

$\mathbb{C}^* \rightsquigarrow$



$a_\gamma \qquad \mathcal{A}_\gamma \simeq \mathbb{A}^1$

Lecture 4. $\quad f_\gamma : \mathcal{M}_\gamma \to \mathcal{A}_\gamma$

Correction: $\mathcal{A}_\gamma = \prod_{i=1}^{r} \left\{ \begin{array}{l} \text{poly. in } t, \text{ const term} = 0 \\ \text{leading term} = f_i(-\gamma) = (*)\, t^{d_i \nu} \\ \text{other terms have deg} \leq (d_i - 1)\nu \end{array} \right\}$

$$\dim A_\gamma = \sum_{i=1}^{r} [(d_i - 1)\nu]$$

f_γ is a Lag. fibration, central fiber $\underset{\text{homeo.}}{\simeq}$ Fl_γ.

<u>Check</u>. $\dim Fl_\gamma \overset{?}{=} \dim A_\gamma$

$$\begin{array}{ccc} \dim Fl_\gamma & \overset{?}{=} & \dim A_\gamma \\ \| \text{B} & & \| \\ \dfrac{|\Phi|\cdot\nu - c_w}{2} & & \sum_{i=1}^{r} [(d_i-1)\nu] \end{array}$$

$$\frac{1}{2}|\Phi| = \sum_{i=1}^{r} (d_i - 1)$$

Nonabelian Hodge theory /curve (Simpson correspondence)

$\mathcal{M}_{Dol}$	$\mathcal{M}_{dR}$	$\mathcal{M}_{Bet}$
moduli of Higgs bundles $(\mathcal{E}, \varphi)$	moduli of v.b. w/ connections $(\mathcal{E}, \nabla)$	moduli of homo. $\pi_1(X) \to G$ GL_n

$(\mathcal{E}, \varphi) \rightsquigarrow (\mathcal{E}, \nabla) \overset{RH}{\longmapsto} \mathcal{E}^\nabla$ loc. sys

$\varphi = A\,dt \qquad \nabla = d + A\,dt \qquad \leftrightarrow (\pi_1(X) \to GL_n)$

Riemann–Hilbert map

λ-connection $(\mathcal{E}, \nabla)$, $\nabla = \lambda d + A\,dt$

$\nabla(f \cdot s) = \lambda\, df \cdot s + f \nabla(s)$

$\lambda = 1$, usual conn.

$\lambda = 0$, Higgs field

$$\begin{array}{ccc} \mathcal{M}_{Hod} & \{(\mathcal{E},\nabla) : \lambda\text{-conn. } \nabla \text{ on } \mathcal{E}\} & \mathcal{M}_{Dol} \quad \mathcal{M}_{Hod} \quad \mathcal{M}_{dR} \\ \downarrow & \downarrow & \downarrow \qquad\quad \downarrow \qquad\qquad \\ \mathbb{C} & \ni \lambda & 0 \in \mathbb{C} \ni 1 \end{array}$$

M_{Dol}, M_{dR}, M_{Bet} are diff. cpx str. of the same hyperkähler mfd.

e.g. GL_1, $M^{0}_{Dol} \simeq Jac \times H^0(X,\omega) = T^*Pic$

topologically $\mathbb{C}^{2g}/\mathbb{Z}^{2g}$

$$\begin{array}{c} \qquad\qquad M^{0}_{Dol} \\ \qquad\qquad \uparrow{\scriptstyle \lambda \to 0} \\ 1 \to H^0(X,\omega) \to M^{0}_{dR} \to Jac \to 1 \end{array}$$

$$M^{0}_{Bet} \simeq Hom(\pi_1(X), \mathbb{G}_m) \simeq H^1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{G}_m$$

$$M_{Hod,\gamma} = \{ (\lambda, \underbrace{\mathcal{E}, \mathcal{E}_0^B}_{\text{same as in } M_\gamma}, \mathcal{E}_\infty^{K_\gamma}, \nabla): \qquad \nabla: \lambda\text{-conn. on } \mathcal{E}$$

- at 0, ∇ has a simple pole w/ $res_0 \in$ str upp Δ wrt. $\mathcal{E}_0^B$
- at ∞, $\nabla \sim \lambda d + (\gamma + (L_\infty \mathfrak{g})_{\le 0})\frac{dt}{t}$ under trivialization of $(\mathcal{E}, \mathcal{E}_\infty^{K_\gamma})$

$$K_\gamma = (L_\infty G)_{\le -\nu}(L_\infty T_\gamma)_{\kappa_0}$$

<u>Fact</u>. $M_{Hod,\gamma} \xrightarrow{\lambda} \mathbb{C}$ is smooth, relatively symplectic.

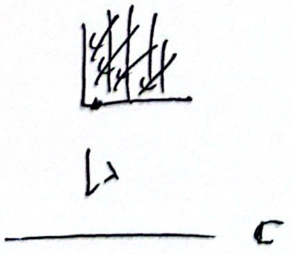
$M_{dR,\gamma} = \lambda^{-1}(1)$ sympl. alg space. same dim. as M_γ

$\mathbb{C}^\times \curvearrowright M_{Hod,\gamma}$ scales λ with wt d, restricts to previous $\mathbb{C}^\times \curvearrowright M_\gamma$.

$$\Rightarrow H^*(M_\gamma) \xleftarrow{\sim} H^*(M_{Hod,\gamma}) \xrightarrow{\sim} H^*(M_{dR,\gamma})$$

$$\swarrow \sim$$

$H^*(Fl_\gamma)$

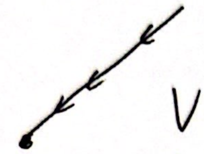


$M_{Bet,r}$ should parametrize G-local sys. on $\mathbb{P}^1 \setminus \{0,\infty\}$ w/

- B-reduction at 0
- <u>Stokes data</u> at ∞

τ loc. coord. at ∞, $\quad f'(\tau) = A(\tau) f(\tau) \qquad f(\tau) : D_\infty^\times \to \mathbb{C}^n$

$\overset{\shortparallel}{t^{-1}}$



∞

$\tau = 0$

V n-dim'l v.s.

|

has a filt.

max. decay. $C \;\underline{\quad\quad}\; C \cdots$

$$f(\tau) = \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_n(\tau) \end{pmatrix}$$

$r = 1$.

$$f'(\tau) = \begin{pmatrix} -\frac{a_1}{\tau^2} & & & \\ & -\frac{a_2}{\tau^2} & & \\ & & \ddots & \\ & & & -\frac{a_n}{\tau^2} \end{pmatrix} f(\tau)$$

$$\left(e^{\frac{a_1}{\tau}}, 0, \dots, 0\right)$$
$$\vdots$$
$$\left(0, 0, \dots, 0, e^{\frac{a_n}{\tau}}\right)$$

$\tau = r \cdot e^{i\theta}$

$$\left| e^{\frac{a_i}{\tau}} \right| = e^{\frac{1}{r} \mathrm{Re}(a_i e^{-i\theta})}$$

order $\underline{\mathrm{Re}(a_i e^{-i\theta})}$

when distinct, get a full flag.

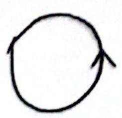
Singular directions: θ s.t. $\exists\, i \neq j$ s.t. $\mathrm{Re}\left((a_i - a_j) e^{-i\theta}\right) = 0$

(i,j) starting sector

$$0 \subset V_1 \subset \cdots \subset V_{k-1} \overset{i}{\subset} V_k \overset{j}{\subset} V_{k+1} \subset \cdots$$

$\Big\}$ cross (i,j) direction

$$0 \subset V_1 \subset \cdots \subset V_{k-1} \overset{j}{\subset} V_k' \overset{i}{\subset} V_{k+1} \subset \cdots$$



top. monod.

$\rightsquigarrow$ positive braid $\beta \in B_W^+ = \langle s_i : \text{braid rel.} \rangle$

$\theta_1, \theta_2, \cdots$ singular direction.

$\beta = s_{\theta_1} s_{\theta_2} \cdots \in B_W^+$

$\mathcal{M}(\beta)$ $\quad\quad \beta = s_{i_1} s_{i_2} \cdots s_{i_N}$ reduced

GL_n: $\mathcal{M}(\beta) = \left\{ V_\bullet^{(0)} \xrightarrow{s_{i_1}} V_\bullet^{(1)} \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_N}} V_\bullet^{(N)} \simeq V_\bullet^{(0)} \right\}$

full flag $\quad\quad$ isom of full flags

General G, $\quad \beta = s_{i_1} s_{i_2} \cdots s_{i_N}$

$$\mathcal{M}(\beta) = \left\{ \begin{array}{c} \mathcal{F}^{(0)} - \mathcal{F}^{(1)} - \cdots - \mathcal{F}^{(N)} \simeq \mathcal{F}^{(0)} \\ \text{B-bundles} \\ \text{B-bundles over a pt} \\ \alpha_j : \mathcal{F}^{(j-1)} \overset{B}{\times} G \simeq \mathcal{F}^{(j)} \overset{B}{\times} G \\ \text{rel. pos}(\mathcal{F}^{(j-1)}, \mathcal{F}^{(j)}) = s_{i_j} \end{array} \right\}$$

(compare Deligne - Lusztig variety)

top. mono.: $\mathcal{M}(\beta) \longrightarrow \{(\mathcal{E}, \alpha) : \mathcal{E} : G\text{-bundle on pt}, \ \alpha \in \mathrm{Aut}_G(\mathcal{E})\} = \frac{G}{G}$

$$\underbrace{\mathcal{F}^{(0)} \overset{B}{\times} G \simeq \mathcal{F}^{(1)} \overset{B}{\times} G \simeq \cdots \simeq \mathcal{F}^{(N)} \overset{B}{\times} G \simeq \mathcal{F}^{(0)} \overset{B}{\times} G}_{\text{auto. of } \mathcal{F}^{(0)} \overset{B}{\times} G \text{ as a } G\text{-bundle}}$$

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(\beta) & \longrightarrow & \mathcal{M}(\beta) \\ \downarrow & \lrcorner & \downarrow \\ \frac{\widetilde{\mathcal{U}}}{G} & \longrightarrow & \frac{G}{G} \end{array}$$

$\widetilde{\mathcal{U}} \to \mathcal{U}$ Spr res. of unip. cone $\mathcal{U} \subset G$.

$\mathcal{M}_{Bet,\gamma} \longrightarrow \widetilde{\mathcal{M}(\beta)}$ (= when γ is elliptic)

$\gamma \rightsquigarrow \beta$

(

elliptic • $\nu = \frac{d}{m}$, $m = \mathrm{ord}(w)$, $w \in W$

choose w to have minimal length within conj. class

$$\ell(w) = \frac{|\Phi|}{m} \qquad (\langle w \rangle \curvearrowright \Phi \text{ freely})$$

$\tilde{w} \in B_W^+$ cano. lift of w, $\beta = \tilde{w}^d \in B_W^+$

Ex. $\nu = \frac{d}{h}$. $w = s_1 s_2 \dots s_r$

$$\beta = \underbrace{\underbrace{s_1 s_2 \cdots s_r}\ \underbrace{s_1 \cdots s_r} \cdots \underbrace{s_1 \cdots s_r}}_{d \text{ times}}$$

Works for $\nu = 1 = \frac{h}{h}$. $\beta = (s_1 s_2 \cdots s_r)^h = \tilde{w}_0^2$ full twist $\in B_W^+$.

$$\mathcal{M}(\text{full twist}) = \left\{ \mathcal{F}^{(0)} \underset{\text{gen.}}{\text{----}} \mathcal{F}^{(1)} \underset{\text{gen}}{\text{----}} \mathcal{F}^{(2)} \simeq \mathcal{F}^{(0)} \right\}$$

$$= \{g, B_0 \xrightarrow{\text{gen}} B_1 \xrightarrow{\text{gen}} B_2 = {}^g B_0\} / G$$

$$\underset{\text{Fix } B, B^{op}}{=} \{(g, B, B^{op} \underset{\text{gen}}{\text{---}} B_2 = {}^g B_0)\} / B \cap B^{op} = T$$

$$= (B^{op} \cdot B) / \mathrm{Ad}(T)$$

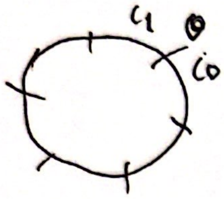
$\gamma \rightsquigarrow \beta \in B_W^+$

$\gamma(\tau) = \mathbb{C}^\times \to \mathfrak{g}^{rs} \to \mathfrak{g}^{rs} /\!/ G = \mathfrak{h}^{rs} /\!/ W$

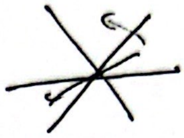
$\mathbb{Z} = \pi_1(\mathbb{C}^\times) \longrightarrow \pi_1(\mathfrak{h}^{rs} /\!/ W) = B_W \rightsquigarrow$ gives conj. class of β.

$\nu \in \mathbb{Z}$, get pos. braid

$$\begin{array}{ccccc} \gamma : \mathbb{C}^\times & \longrightarrow & \mathfrak{h}^{rs} & \longrightarrow & \mathfrak{h}_{\mathbb{R}} \\ \cup & & & & \\ S^1_\varepsilon & & \longrightarrow & & \mathfrak{h}_{\mathbb{R}} \end{array}$$



singular θ $\longleftrightarrow$ preimages of walls



$$\mathrm{pos}(c_0, c_1) \in W$$

$$\beta = \widetilde{\mathrm{pos}(c_0, c_1)}\ \widetilde{\mathrm{pos}(c_1, c_2)} \cdots$$

Riemann–Hilbert map. (analytic map)

$$\begin{array}{ccccc} \mathcal{M}_{dR,\gamma} & \longrightarrow & \mathcal{M}_{Bet,\gamma} & \longrightarrow & \widetilde{\mathcal{M}(\beta)} \\ (\mathcal{E}, \cdots, \nabla) & & \longmapsto & & \text{Stokes data} \end{array}$$

$$\begin{tikzcd} & & \mathcal{M}(\beta) \arrow[dl] \arrow[dr] & (Lt_\gamma)_0^{\mathcal{F}^{(i)}} \arrow[d, "\mathfrak{h}^w"] \arrow[dr] & \\ \frac{\widetilde{G}}{G} \arrow[r] & \frac{G}{G} & & \frac{T}{Ad_w(T)} & \mathcal{F}^{(i)}_T \end{tikzcd}$$

$$\mathcal{F}^{(0)}_T = \mathcal{F}^{(0)}_T$$

w-twisted

Thm. Analytic map

$$\mathrm{RH} : \mathcal{M}_{dR,\gamma} \longrightarrow \mathcal{M}_{Bet,\gamma} = \frac{\widetilde{G}}{G} \underset{\frac{G}{G}}{\times} \mathcal{M}(\beta) \underset{\frac{T}{Ad_w(T)}}{\times} (Lt_\gamma)_0 .$$

Conj. This is an analytic $\simeq$.

Question, $\exists$? hyperkähler str. on $\mathcal{M}_\gamma$?

$$\mathcal{M}_{dR,\gamma}/\mathbb{Z} \longrightarrow \widetilde{B^- B \cap U}$$

$$\begin{array}{c} \| \\ T^*\mathbb{P}^1 \setminus \Gamma\left(\frac{dt}{t}\right) \end{array}$$

$Fl_{\gamma}/\mathbb{Z} \hookrightarrow \mathcal{M}_{\gamma}/\mathbb{Z}$

 $\rightsquigarrow E.$

Lecture 5

$$\begin{array}{ccccccc} \mathcal{M}_\gamma & \hookrightarrow & \mathcal{M}_{Hod,\gamma} & \longleftarrow & \mathcal{M}_{dR,\gamma} & \xrightarrow{RH} & \mathcal{M}_{Bet,\gamma} \\ \| \\ \mathcal{M}_{Dol,\gamma} \end{array}$$

(braid var., wild char. var.)

Ramified geometric Langlands conjecture (specific case)

connections (de Rham)
rep. of π_1 (Betti)

$Shv(Bun_G(X))$ $IndCoh_{nilp}(Loc_{\check{G}}(X))$

automorphic side   spectral side (Galois)

Betti (Ben-Zvi, Nadler)

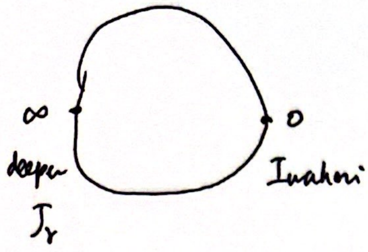
$$Sh_{\mathcal{N}_{glob}}(Bun_G(X)) \simeq IndCoh_{\mathcal{N}}\left(Loc^{Bet}_{\check{G}}(X) = Hom(\pi_1(X), \check{G})/\check{G}\right)$$

$$(s.s. \subset \mathcal{N}_{glob} \subset T^*Bun_G(X))$$

$$\begin{array}{ccc} & & \| \\ & & \text{Hitchin moduli} \\ \downarrow & & \downarrow \\ 0 & \in & A \end{array}$$

$\{F : SS(F) \subset G \times \mathcal{N}^* \subset T^*G\}$

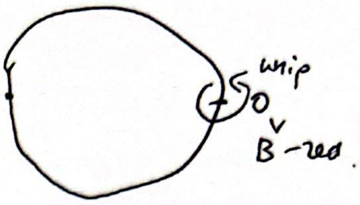
Finite analog: $CS(G) \subset D_h(G)$



$\mathrm{Bun}_{G,\,pt}(I_0, J_{\gamma,\infty})$

autom.

$\gamma \rightsquigarrow \beta \in B_W^+$



$\check{G}$- loc. sys.

dR: irregular conn.

Betti: Stokes data

Spectral $\widetilde{\mathcal{M}_{\check{G}}(\beta)}$ $\beta \in B_W^+$

$$\begin{array}{ccc} Fl_\gamma & \subset & \mathcal{M}_\gamma \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{G}_m & & \mathbb{G}_m \end{array}$$

$\mu Sh_{Fl_\gamma}(\mathcal{M}_\gamma)$

microsheaves

$S = \coprod S_\alpha$ (stratified cpx mfd)

$T^*S \underset{\text{Lag.}}{\supset} \bigcup_\alpha T^*_{S_\alpha} S = \Lambda$

$T^*S \circlearrowleft$ $\mathbb{G}_m$ scaling fibers; $\Lambda \circlearrowleft \mathbb{G}_m$

$\mu Sh_\Lambda(T^*S) \simeq D_{\{S_\alpha\}}(S)$

Recall $\mathcal{M}_\gamma = T^*\mathrm{Bun}_G(I_0, J'_{\gamma,\infty}) /\!\!/_{\overline{\gamma}} \overline{J}_\gamma$

$$\begin{tikzcd} J_{\gamma,\infty} \arrow[r, "\gamma"] \arrow[d, two heads] & \mathbb{G}_a \\ J_{\gamma,\infty}/J'_{\gamma,\infty} \arrow[ur, "\overline{\gamma}"'] & \end{tikzcd}$$

$\overline{J}_\gamma = J_{\gamma,\infty}/J'_{\gamma,\infty}$

$J_\gamma = (L_\infty G)_{\leq -\frac{\nu}{2}} (L_\infty T_\gamma)_{<0}$

Take $J'_{\gamma,\infty} = \ker(\gamma : J_{\gamma,\infty} \to \mathbb{G}_a)$, $\overline{J}_\gamma \simeq \mathbb{G}_a$.

$$\mathcal{M}_\gamma = T^*\mathrm{Bun}_G(I_0, J'_{\gamma,\infty}) /\!\!/_1 \mathbb{G}_a$$

$M = T^*\tilde{S} /\!\!/_1 G_a$, $G_a \curvearrowright \tilde{S}$ = cpx mfd. | $T^*\tilde{S} /\!\!/_0 G_a = T^*(S)$

$S = \tilde{S}/G_a$

want a sheaf category on $\tilde{S}$ or S

that can be microlocalized to M.

ℓ-adic setting. char $k = p > 0$.

$G_a \curvearrowright \tilde{S}$

$D_{G_a}(\tilde{S}) = D(S)$

$D_{(G_a, AS_\psi)}(\tilde{S})$

$\llcorner$ non-trivial

$G_a \xrightarrow{\mathbb{F}_p\text{-torsor}} G_a$

$x \longmapsto x - x^p$

$\alpha_* \overline{\mathbb{Q}}_\ell = \bigoplus AS_\psi$ rank 1 loc-sys. on G_a

$\psi: \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$

char. sheaf str. on AS_ψ:

add: $G_a \times G_a \longrightarrow G_a$,

$\mathrm{add}^* AS_\psi \simeq AS_\psi \boxtimes AS_\psi$

An object in $D_{(G_a, AS_\psi)}(\tilde{S})$ is $\mathcal{F} \in D(\tilde{S})$ s.t. act: $G_a \times \tilde{S} \longrightarrow \tilde{S}$

$\mathrm{act}^* \mathcal{F} \simeq AS_\psi \boxtimes \mathcal{F}$.

<u>Gaitsgory : Kirillov model</u>.

k = any. $A_{\mathrm{ff}} = G_a \rtimes G_m \curvearrowright \tilde{S}$

$D_{(G_a, AS_\psi)}(\tilde{S})$ indep. of ψ (nontrivial)

$\mathrm{Kir}(\tilde{S}) = D_{G_m}(\tilde{S}) / D_{A_{\mathrm{ff}}}(\tilde{S})$

when $\mathrm{char}(k) = p$.

$$\mathrm{Kir}(\tilde{S}) \xrightarrow[\sim]{\mathrm{Av}(\mathbb{G}_a, AS_\psi)} D_{(\mathbb{G}_a, AS_\psi)}(\tilde{S})$$

" $\mathrm{Kir}(\tilde{S})$ microlocalizes to $M = T^*\tilde{S} /\!/_1 \mathbb{G}_a$ "

$D_{\mathbb{G}_m}(\tilde{S})$ microlocalize to $T^*(\tilde{S}/\mathbb{G}_m)$

$$\begin{tikzcd}
(\ ) \arrow[r, phantom, "\subset"] \arrow[d] & T^*\tilde{S} \arrow[d, "\mu_{\mathbb{G}_m}"] \\
0 \arrow[r, phantom, "\in"] & (\mathrm{Lie}\,\mathbb{G}_m)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu_{\mathbb{G}_a}^{-1}(\neq 0) \arrow[r, phantom, "\subset"] \arrow[dd] & T^*\tilde{S} \arrow[d, "\mu_{\mathrm{Aff}}"] \arrow[dd, bend left=60, "\mu_{\mathbb{G}_a}"] \\
& (\mathrm{Lie}\,\mathrm{Aff})^* \arrow[d] \\
(\mathrm{Lie}\,\mathbb{G}_a)^* \setminus \{0\} \arrow[r] & (\mathrm{Lie}\,\mathbb{G}_a)^*
\end{tikzcd}$$

Exer.
$$\mu_{\mathrm{Aff}}^{-1}(\neq 0, 0)/\mathbb{G}_m = M$$
with $(\neq 0, 0) \in (\mathrm{Lie}\,\mathbb{G}_a)^* \oplus (\mathrm{Lie}\,\mathbb{G}_m)^*$, and $M = \mu_{\mathbb{G}_a}^{-1}(1)/\mathbb{G}_a$.

$\mathrm{Kir}(\tilde{S})$ microlocalizes to $\mu_{\mathrm{Aff}}^{-1}(\neq 0, 0)/\mathbb{G}_m \simeq M$.

$M_\gamma = T^*\mathrm{Bun}_G(I_0, J'_{\gamma,\infty}) /\!/_1 \mathbb{G}_a$.

$\tilde{S} = \mathrm{Bun}_G(I_0, J'_{\gamma,\infty})$

$\circlearrowleft$

$\mathbb{G}_a \rtimes \mathbb{G}_m$

$\mathcal{F} \in \mathrm{Kir}(\tilde{S})$

has $SS(\mathcal{F}) \subset M_\gamma$

automorphic:

$\mathrm{Kir}_{Fl_\gamma}(\mathrm{Bun}_G(I_0, J'_{\gamma,\infty}))$ is defined

spectral:

$$\widetilde{M_{\check{G}}(\beta)} = M_{\check{G}}(\beta) \times_{\check{\mathfrak{g}}/\check{G}} \widetilde{\check{\mathfrak{u}}}/\check{G}$$

Conj. Full embedding

$$\mathrm{Kir}_{Fl_\gamma}(\mathrm{Bun}_G(I_0, J_{\gamma,\infty}^!)) \hookrightarrow \mathrm{IndCoh}(\widetilde{\mathcal{M}_{G^\vee}(\beta)}).$$

Slope $\nu = 1$.

$$\mathrm{Bun}_G(I_0, J_{\gamma,\infty}) = \mathrm{Bun}_G(I_0, K_{1,\infty})$$

$$\mathcal{M}_{G}^\vee(\tilde{w}_0^2) = \check{B}^{op}\check{B} / \mathrm{Ad}(\check{T})$$

$$\widetilde{\mathcal{M}_{G}^\vee(\tilde{w}_0^2)} = (\widetilde{B^{\vee,op}\check{B} \cap \mathcal{U}}) / \mathrm{Ad}(\check{T})$$

① Satake action on autom. side

$$\mathrm{Rep}(\check{G}) \curvearrowright \mathrm{Kir}_{Fl_\gamma}(\cdots)$$

$$\widetilde{\mathcal{M}_{\check{G}}(\beta)} \longrightarrow \mathcal{M}_{\check{G}}(\beta) \longrightarrow \frac{G^\vee}{\check{G}} \longrightarrow \frac{pt}{\check{G}}$$

$$\Rightarrow \mathrm{Rep}(\check{G}) = \mathrm{QCoh}\left(\frac{pt}{\check{G}}\right) \curvearrowright \mathrm{IndCoh}(\widetilde{\mathcal{M}_{\check{G}}(\beta)})$$

② at 0.

$$\begin{array}{ccc} \mathrm{Kir}(\cdots) & \simeq & \mathrm{IndCoh}(\cdots) \\ \circlearrowleft & & \circlearrowleft \\ \mathcal{H}_{aff,0} & \underset{\mathrm{Bez}.}{\simeq} & \mathrm{IndCoh}^{\check{G}}(\tilde{\mathcal{U}}^\vee \times_{\check{G}} \tilde{\mathcal{U}}^\vee) \\ \wr\wr & & \\ \mathcal{D}(I_0\backslash L_0G/I_0) & & \end{array}$$

③ $L_\infty T_\gamma / (L_\infty T_\gamma)_{<0} \curvearrowright \mathcal{M}_\gamma \supset Fl_\gamma$

$$\uparrow$$

$$\Lambda_\gamma \equiv \pi_0(L_\infty T_\gamma) \simeq X_*(T)_w$$

$$\rightsquigarrow \Lambda_\gamma \curvearrowright \mathrm{Kir}_{Fl_\gamma}(\cdots)$$

$$\begin{array}{ccc} \widetilde{\mathcal{M}_G(\beta)} & & \\ \downarrow & & \\ \mathcal{M}_G(\beta) & & \\ \downarrow & & \\ \dfrac{T^\vee}{\mathrm{Ad}_w T^\vee} & \longrightarrow & \dfrac{pt}{(T^\vee)^w} \end{array}$$

$$\mathrm{Rep}((\check{T})^w) \curvearrowright \mathrm{IndCoh}(\widetilde{\mathcal{M}_{\check{G}}(\beta)})$$

Result in slope 1 case

Thm.

$$\mathrm{Kir}\left(\mathrm{Bun}_G(K'_{1,\infty})\right) \simeq \mathrm{Vect}_{X_*(T)}$$

$\gamma = \gamma_0 t$

$\gamma_0 \in \mathfrak{h}^{rs} \subset \mathfrak{g}^{rs}$

$$\gamma : K_{1,\infty} \twoheadrightarrow \mathfrak{g} \xrightarrow{\langle \gamma_0, - \rangle} \mathbb{G}_a$$

$$K'_{1,\infty} = \ker(\gamma : K_{1,\infty} \to \mathbb{G}_a)$$

$\overline{\mathcal{M}_\gamma}$ version of $\mathcal{M}_\gamma$ w/o I_0.

$$\overline{\mathcal{M}_\gamma} = \left\{ (\mathcal{E}, \mathcal{T}_\infty, \varphi) \right\}, \quad \varphi \in H^0(\mathbb{A}^1, \mathrm{Ad}(\mathcal{E}) \otimes \omega)$$

$= \text{discrete } X_*(T)$

$$\varphi|_{D_\infty^\times} \sim \left(-\frac{\gamma_0}{t} + \cdots\right)\frac{dt}{t}$$

$$\downarrow$$

$\mathrm{pt} = \overline{\mathcal{A}}_\gamma \ni (a_i)$, $\quad a_i \in H^0(\mathbb{P}^1, \omega(2\cdot\infty)^{\otimes d_i}) = \mathbb{C}$

$$a_i(\infty) = f_i(-\gamma_0)$$

$$X_*(T) \simeq \{ g \in \mathrm{Gr} : \mathrm{Ad}(g^{-1}) \cdot \overset{t\cdot\gamma_0}{\overset{\|}{\gamma}} \in t\,\mathfrak{g}[[t]] \} \quad \text{Spaltenstein fiber.}$$

$$\|$$

$$\{ g \in \mathrm{Gr} : \mathrm{Ad}(g^{-1}) \cdot \gamma_0 \in \mathfrak{g}[[t]] \}$$

$$\mathcal{D}_\gamma \underset{\text{full}}{\subset} \underset{\mathrm{Fl}_\gamma}{\mathrm{Kir}}\left(\mathrm{Bun}_G(I_0, J'_{\gamma,\infty})\right)$$

$\|$

generated by the image of pullback along $\mathrm{Bun}_G(I_0, K'_{1,\infty}) \longrightarrow \mathrm{Bun}_G(K'_{1,\infty})$

under $\mathcal{H}_{\mathrm{aff},0}$-action.

Thm. $D_\gamma \simeq \mathrm{Coh}^{\check{T}}(\tilde{\mathcal{N}}^\vee)_{B^\vee}$ ← top. supp. on zero section $B^\vee \hookrightarrow \tilde{\mathcal{N}}^\vee = T^* B^\vee$

$$\begin{array}{c} \mathrm{Coh}^{\check{T}}(\tilde{\mathcal{N}}^\vee)_{B^\vee} \\ \wr\wr \\ \mathrm{Coh}^{\check{T}}\left(\widetilde{B^{\vee,op} B^\vee \cap \mathcal{U}}\right)_{B^\vee} \\ \downarrow \\ \text{spectral side in conj.} \end{array}$$

auto. side of conj.

① $\mathrm{Rep}\,\check{G} \curvearrowright D_\gamma$

modif. at any $x \in \mathbb{P}^1 \setminus \{0, \infty\}$

factors through $\mathrm{Rep}\,\check{T}$

$$\mathrm{Rep}\,\check{T} \curvearrowright \mathrm{Coh}^{\check{T}}(\cdots)$$

② $D_\gamma \circlearrowleft \mathcal{H}_{aff,0}$

$\mathrm{Coh}^{\check{T}}(\tilde{\mathcal{N}}^\vee)_{B^\vee} \circlearrowleft \mathrm{Coh}^{\check{G}}(St)$

③ $\mathrm{char}(k) = p$

$$D_\gamma \subset D(\mathrm{Bun}_G(I_0, (K_{1,\infty}, \gamma))) = D_{(G_a, AS_\psi)}(\mathrm{Bun}_G(I_0, K_{1,\infty}'))$$

$$\circlearrowleft$$

$$\mathcal{H}_\infty = D\left((K_{1,\infty}, \gamma) \backslash L_\infty G / (K_{1,\infty}, \gamma)\right)$$

Thm (Kamgarpour – Schedler, ...) $\mathcal{H}_\infty \simeq \mathrm{Rep}(\check{T}) \underset{\circledast}{\otimes} \underset{\text{conv.}}{D(T)}$

Reason. $K_{1,\infty} \backslash L_\infty G / K_{1,\infty}$ $\quad \{K_{1,\infty}\, t^\lambda \cdot T\, K_{1,\infty}\}_{\lambda \in X_*(T)}$ are the only relevant double cosets.

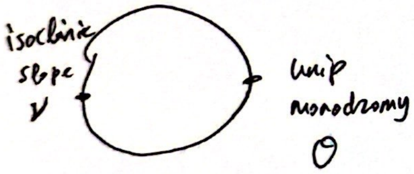
$\mathrm{Rep}\,\check{T} \longrightarrow \mathcal{H}_\infty$

$\lambda \longmapsto$ univ. loc. sys. on $t^{\lambda} T$

$$\rightsquigarrow \quad \begin{array}{ccc} \mathrm{Rep}\,\check{T} & \curvearrowright & D_\gamma \\ \| & & \updownarrow \\ \mathrm{Rep}\,\check{T} & \curvearrowright & \mathrm{Coh}^{\check{T}}(\tilde{\mathcal{N}}^\vee)_{B^\vee} \end{array}$$

Many cases for $\frac{1}{m}$. ✓

Jakob – Y. Use $\mathcal{M}_{Hod,\gamma}$ to solve Deligne–Simpson problem



For which $(\nu, \mathcal{O})$ does there exist a G-conn. satisfying local cond. at $0, \infty$.

Yes / No?

$$\Big[L_\nu(\mathrm{triv}) : \underbrace{E_{\mathcal{O}}}_{\text{irrep. of } W} \Big] \text{ rat'l DAHA}$$

Yes $\iff$

$\neq 0$

Kashiwara – Schapira: $\Omega \overset{\text{open}}{\subset} T^*X$.

$$\mu Sh(\Omega) = D(X) / D_{T^*X \setminus \Omega}(X)$$