Symplectic duality and the Tutte polynomial

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(jt w/ Ben Davison)

Matroid

Def A matroid is a pair $M = (E, \mathcal{I})$ where E is a finite set,

$\mathcal{I}$ is a family of subsets of E s.t.

1) $\phi \in \mathcal{I}$
2) $A \in \mathcal{I}$, $B \subset A \Rightarrow B \in \mathcal{I}$
3) $A, B \in \mathcal{I}$, $|A| > |B|$, $\Rightarrow \exists\, \lambda \in A \setminus B$ s.t. $B \cup \lambda \in \mathcal{I}$.

Ex. Let V be a finite dim'l v.s. $/k$

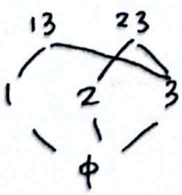
Let $\{v_1, \cdots, v_n\}$ be a spanning set.

$\{v_1, \cdots, v_n\} = E$

$\mathcal{I}$ = linearly indep. subsets.

Ex $E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\quad\; v_1 \qquad v_2 \qquad v_3$

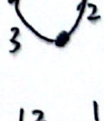
$\mathcal{I}$ =



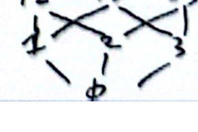
Ex. Let $G = (E, V)$ be a graph. E = edges,

$\mathcal{I}$ = forests

G =



$\mathcal{I}$ =



~~Def~~: Def. a base of M is an element $B \in I$ of max'l size.

Lemma. all bases have same size $(= rk(M))$.

Def. a flat of M is a set $S \subset E$ max. among sets of given rank.

Polynomial invariants

Def. Let G be a graph,

$\chi_G(q) = \#$ q-colorings of the vertices

e.g. $\chi_{\triangle} = q(q-1)(q-2)$

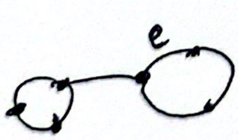
Conj. (Reed, 1968) $\chi_G(q) = a_n q^n - a_{n-1} q^{n-1} + \cdots + (-1)^n a_0$

then $\{a_i\}$ are unimodular.

$a_0 \leq a_1 \leq \cdots \leq a_i \geq \cdots \geq a_n$

Proof (Huh '11) uses algebraic geometry.

Why is $\chi_G(q)$ polynomial?

$G =$ 

$\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$

$G \setminus e$

deletion.

G/e contraction

Universal Tutte-Grothendieck invariant

T: matroids $\longrightarrow \mathbb{Z}[x,y]$

1) $T_M(x,y) = 1$ for $M = \cdot$

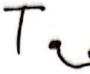
2) $T_M(x,y) = T_{M/e}(x,y) + T_{M \setminus e}(x,y)$ e not a loop or a coloop

$T_M(x,y) = x\, T_{M/e}(x,y)$ e coloop

$T_M(x,y) = y\, T_{M \setminus e}(x,y)$ e loop

Prop?

T  $=$ T  $+$ T 

$= x^2 +$ T  $+$ T 

$= x^2 + x + y$

Matroid duality $\cup$ (planar graph duality)

$$M \longrightarrow M^\vee$$
$$M = (E, \mathcal{I}) \qquad M^\vee = (E, \mathcal{I}^\vee)$$

base b $\rightarrow$ complement $E \setminus b$

$\{$bases$\}$ $\longleftrightarrow$ $E \setminus b \in$ bases

$$T_M(x,y) = T_{M^\vee}(y,x)$$

Q. find a geometric avatar of $T_M(x,y)$

restrict to regular matroids $\supset$ graphical matroids, cographical matroids

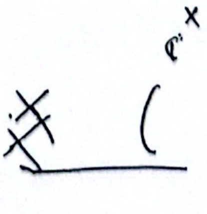
Fix such an $M = (E, \mathcal{I})$. choose a tot. unimodular representation $E \longrightarrow \mathfrak{t}/\mathbb{C}$

- Hypertoric variety $\quad \mathfrak{g} \longrightarrow \mathbb{C}^E \longrightarrow \mathfrak{t} \quad$ SES

$G = \mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{C}^\times, \quad G \curvearrowright \mathbb{C}^E$

Pick $\eta \in \mathfrak{g}_{\mathbb{Z}}^\vee$ reg. Define $\mathcal{M} = T^*\mathbb{C}^E /\!\!/\!\!/\!\!/_\eta \, G$

Ex.  $\rightsquigarrow$ $\mathcal{M} = T^*\mathbb{P}^2$

 $\rightsquigarrow$ $\mathcal{M} = \widetilde{\mathbb{C}^2/\mathbb{Z}_3} =$ 

Thm $\mathcal{M}$ is a smooth alg. sympl var.

Def $(\mathcal{M}, \Omega)$ is a conical symplectic resol'n if

$\downarrow \pi$ is proper & birat'l

$\mathcal{M}_0 = \operatorname{Spec} \mathbb{C}[\mathcal{M}]$

& $\mathbb{C}^\times \curvearrowright \mathcal{M}$ scaling Ω & contracting $\mathcal{M}$ to a pt.

geometry of $\mathcal{M} \longleftrightarrow$ Combinatorics of M

$T = t \otimes_{\mathbb{Z}} \mathbb{C}^\times \curvearrowright \mathcal{M}$

$\mathcal{M}^T \longleftrightarrow$ bases of M

$H^2_T(\mathcal{M}, \mathbb{C}) \simeq \mathbb{C}^E$

$|E| = n$

Konno: $H^*_T(\mathcal{M}) = \mathbb{C}[u_1, \cdots, u_n] / \prod_{i \in S} u_i = 0$ if $S \notin I$

Ex.  $\rightsquigarrow \mathcal{M} = T^*\mathbb{P}^2 \rightsquigarrow H_T(\mathcal{M}) = \mathbb{C}[u_1, u_2, u_3] / (u_1 u_2 u_3 = 0)$

$\overset{HS}{\Rightarrow} \quad P_{\mathcal{M}}(x^{-1}) x^d = T_M(x, 1)$

Consider $\mathcal{M}^!$ associated to $M^\vee$

$P_{\mathcal{M}^!}(y^{-1}) y^{d^!} = T_M(1, y)$

Want to combine $H_T(\mathcal{M})$ & $H_{G^\vee}(\mathcal{M}^!)$

to get Tutte.

Stable envelope

Given $\sigma: \mathbb{C}^\times \to T$ s.t. $\mathcal{M}^{\mathbb{C}^\times} = \mathcal{M}^T$ & $\varepsilon: \mathcal{M}^T \to \pm 1$

$\exists!$ $\operatorname{Stab}: H_T(\mathcal{M}^T) \to H_T(\mathcal{M})$ s.t. 1) $\operatorname{Stab}(x)$ is supp. on the attracting locus of x

2) $\operatorname{Stab}(x)|_x = \varepsilon(x) eu(T_x \mathcal{M})$, 3) $\operatorname{Stab}(x)|_y = 0$ for $y \neq x \in \mathcal{M}^T$

Stab is an isom. after specializing to generic $\sigma \in \mathfrak{t}$
$H_T^\bullet(pt)$

<u>Def.</u> $M : H_T^\bullet(\mathcal{M})_\sigma \xrightarrow{\text{Stab}^{-1}} H_T^\bullet(\mathcal{M}^T)_\sigma$

$\Big\downarrow$ use bijection on fixed pts $b \mapsto b^c$

$H_{G^\vee}(\mathcal{M}^{!\,G^\vee})_{-\eta}$

$M : H^\bullet(\mathcal{M})_\sigma \xrightarrow{\sim} H^\bullet(\mathcal{M}^!)_{-\eta}$ is an isom. of $\mathbb{C}$-vec. sp.

each side is filtered by coh. degree

$\Rightarrow$ get bifiltered v.s. $H^{\bullet,\bullet}(\mathcal{M}, \mathcal{M}^!)$

<u>Thm</u> [Davison - M.] $T_M(x,y) = \sum_{i,j} \dim H^{i,j}(\mathcal{M}, \mathcal{M}^!)\, x^{d-i} y^{d^\vee - j}$

<u>Cor.</u> $t_{i,j} \geq t_{i+k, j+k}$ for $0 \leq k \leq d-i, d^!-j$. (*)

<u>Pf.</u> apply Hausel's Lefschetz operator to both sides at once, and note that they <u>coincide</u> up to scalar.

<u>Thm.</u> M expresses $Z = H^{top}(\mathcal{M} \times_{\mathcal{M}_0} \mathcal{M})$ as the commutant of $Z^! = H^{top}(\mathcal{M}^! \times_{\mathcal{M}_0} \mathcal{M})$

More generally, can define for any symplectic dual CSR's $\mathcal{M}, \mathcal{M}^!$ a bicharacteristic poly. satisfying (*)