Integrable many-body systems and gauge theories

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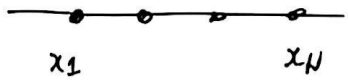
Lecture 1. Duality.

① Direct relation "gauge theory is equiv. to an int. system": $h_{\text{Gauge}} = h_{\text{int}}$

② Indirect relation: "Correlation functions of quantum gauge theory obey classical (or quantum) eqs of some int. sys." $h_{\text{Gauge}} \neq h_{\text{int}}$

Examples of int. systems, Lax representation $\longrightarrow$ path to gauge theory

Calogero - Moser - Sutherland many-body system



1-dim'l particles, indistinguishable, $\omega = \sum_{i=1}^{N} dp_i \wedge dx_i$

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \nu^2 \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

$$P = \left(T^*(\mathbb{R}^N \setminus \Delta)\right) / \mathfrak{S}_N$$

$$\left[\begin{array}{l} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{array}\right. \iff \dot{L} = [L, A]$$

$N \times N$ hermitian matrix

$$L = \| p_i \delta_{ij} + \sqrt{-1}\, \nu \frac{1}{x_i - x_j} (1 - \delta_{ij}) \| \Rightarrow H_k = \frac{1}{k} \operatorname{Tr} L^k \text{ integral of motion.}$$

$\rightarrow$ eigenvalues of L are constants of motion.

$\longrightarrow$ asymptotic momenta $p_1^{(\pm)}, \dots, p_N^{(\pm)}$

$t \longrightarrow \pm\infty$, $x_{\sigma(1)} \ll x_{\sigma(2)} \ll \cdots \ll x_{\sigma(N)}$

Large phase space

$$P_L = T^*\big(\mathfrak{u}(N) \times \mathbb{R}^N\big) \quad - \text{ sympl. vec. sp. of dim } 2(N^2+N).$$

$$\shortparallel$$

$$\left\{ \left(\sqrt{-1}\,P, \sqrt{-1}\,X;\ w, v\right), \quad \begin{matrix} v \in \mathbb{R}^N \\ w \in (\mathbb{R}^N)^* \end{matrix} \right\} \qquad \omega = \mathrm{Tr}\, dP \wedge dX + dw \wedge dv$$

$P^\dagger = P$, $X^\dagger = X$, $N \times N$

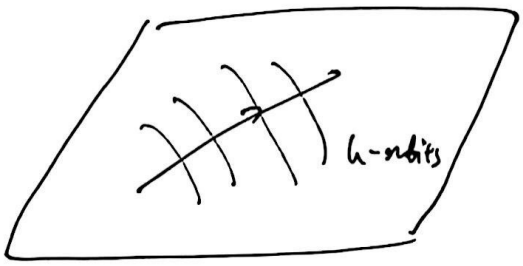
$$\mathcal{H}_2 := \frac{1}{2}\mathrm{Tr}\, P^2 \qquad \left\{ \begin{aligned} X(t) &= X(0) + t P(0) \\ P(t) &= P(0) \\ v(t) &= v(0) \\ w(t) &= w(0) \end{aligned} \right.$$

$$\mathcal{H}_k = \frac{1}{k}\mathrm{Tr}\, P^k \qquad \left\{ \begin{aligned} X(\vec{t}) &= X(0) + \sum_{k=1}^{N} t_k P^{k-1}(0) \\ P(\vec{t}) &= P(0) \\ v(\vec{t}) &= v(0) \\ w(\vec{t}) &= w(0) \end{aligned} \right.$$

<u>Symmetry</u>. $G = U(N)$

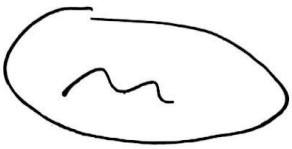
$$g \in G, \quad (P, X, Z) \longmapsto (g^{-1}Pg,\ g^{-1}Xg,\ g^{-1}Z)$$

$$Z = v + \sqrt{-1}\,w$$



Large phase space

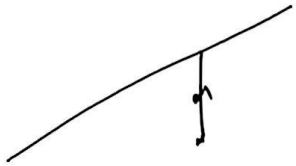
P_L



Small phase space

$$P_S = P_L /\!\!/ G = \mu^{-1}(0)/G$$

Symplectic mfd



$\mathbb{R}^3$

$\vec{r} = \vec{r}_0 + \vec{v} \cdot t$

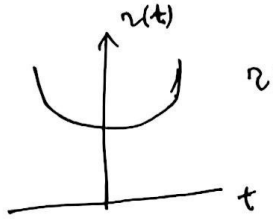
$T^* \mathbb{R}^3 = \{ (\vec{p}, \vec{r}) \}$

$\downarrow$

$\vec{L} = \vec{p} \times \vec{r}$

example of moment(um) map

$/ \, SO(3)$



$r(t) = |\vec{r}_0 + \vec{v} \cdot t|$

$\mu : P_L \longrightarrow \mathrm{Lie}(G)^*$

$\xi \in \mathrm{Lie}\, U(N)$

$g = 1 + \xi + \cdots$

$V_\xi (p, X, Z) = \left([p, \xi], [X, \xi], -\xi \cdot Z \right)$ $\qquad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}$

$H_{V_\xi} = \mathrm{Tr}\, \xi \left([p, X] + \sqrt{-1} Z \otimes Z^\dagger \right)$ $\qquad V_{H_\xi} = \omega^{-1} d\mu(\xi)$

$\qquad \| \qquad \qquad \| z_i \bar{z}_j \|_{i,j=1}^N$

$\mu(p, x, z)$

$\{ \mu(\xi_1), \mu(\xi_2) \} = \mu([\xi_1, \xi_2])$. G-equivariance of μ

$\mu(p, x, z) = [p, x] + \sqrt{-1} \left(Z \otimes Z^\dagger - \nu \cdot 1_N \right)$

$\mu = 0 \quad [p, x] = \sqrt{-1} \left(\nu \cdot 1_N - Z \otimes Z^\dagger \right)$ modulo G-symmetry

$\exists$ representative (p, X, Z) w/ $X = \mathrm{diag}(x_1, \cdots, x_N)$ $\qquad G \longrightarrow U(1)^N \rtimes \mathfrak{S}_N$

$[p, X]_{ij} = p_{ij} (x_i - x_j) = \sqrt{-1} \left(\nu \delta_{ij} - z_i \bar{z}_j \right)$

if $i = j$, RHS $= 0 \Rightarrow \nu = |z_i|^2, \ \forall i = 1, \cdots, N$ $\qquad (\nu > 0)$

use $g = \mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_N})$ to make $z_i = \sqrt{\nu}$, $i = 1, \cdots, N$

$G \longrightarrow \mathfrak{S}_N$

$p = \| p_i \delta_{ij} + \sqrt{-1} \nu \frac{\delta_{ij} - 1}{x_i - x_j} \|$

$(x_1(t), \dots, x_N(t))$ eigenvalues of $\left(x_i(0)\delta_{ij} + t_2 \left(p_i \delta_{ij} + \sqrt{-1}\, \nu \frac{\delta_{ij} - 1}{x_i - x_j} \right) \right)$

$X \mapsto g^{-1}(X + tP)g$

$P \mapsto g^{-1} P g$

$g = g(t)$

– <u>gauge transformation</u>

$G = U(N)$

$(p(t), X(t), z(t)) \mapsto \left(g(t)^{-1} P(t) g(t),\ g(t)^{-1} X(t) g(t),\ g(t)^{-1} z(t) \right)$

$(g(t)) \in \mathrm{Maps}(\mathbb{R}, G)$

$\boxed{A_t \longmapsto g^{-1} A_t g + g^{-1} \partial_t g}$ first glimpse of gauge invariance

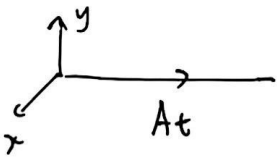
$$S = \int \mathrm{Tr}\left(P dX + z^t dz \right) - dt \frac{1}{2} \mathrm{Tr}\, P^2 + \mu(P, X, z)(A_t)$$

S – functional on $\left(\mathrm{Maps}(\mathbb{R}, \mathbb{P}_L) \times \Omega^1(\mathbb{R}) \otimes \mathfrak{g} \right)$

A_t – new field, 1-form on t valued in $\mathrm{Lie}(G) = \mathfrak{g}$

$\mathrm{Maps}(\mathbb{R}, G)$

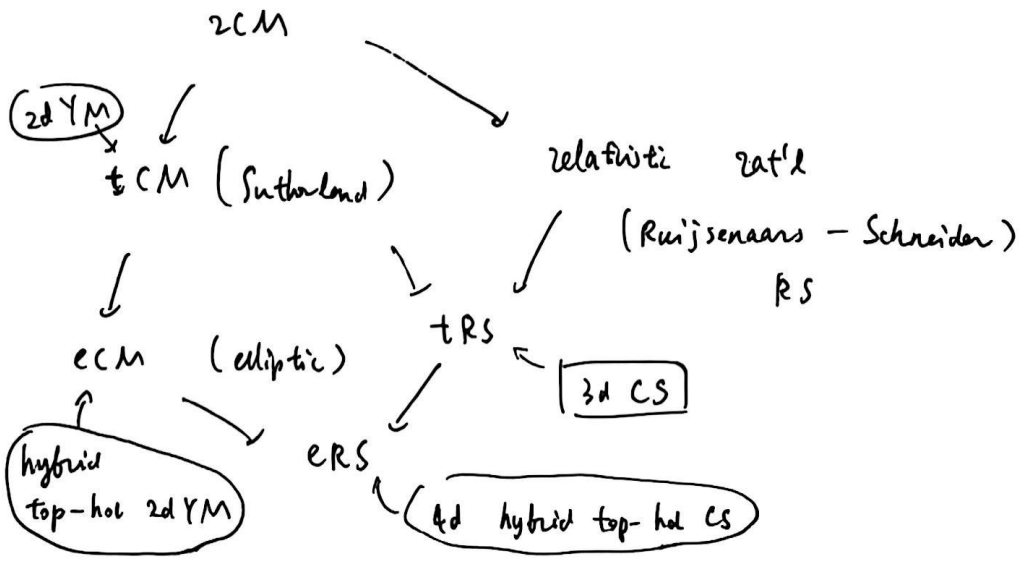
<u>Maybe</u>? P and/or X are also components of a connection



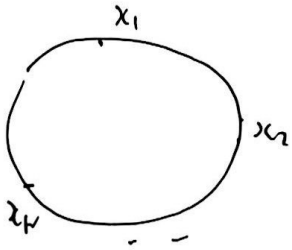
$A_x = X$

$A_y = P$? $\rightsquigarrow$ 2CM has deformations ?

4 generalizations



tCM (Sutherland)



$$H_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{8} \nu^2 \sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j))}$$

$\mathbb{R} \to \mathbb{R}/\mathbb{Z}$

Promote $u(N)$ to $\widehat{u(N)} = \{ (E(x), c) : x \in S^1, E(x) \in u(N), c \in \mathbb{R} \}$

$$[(E_1(x), c_1), (E_2(x), c_2)] = ([E_1(x), E_2(x)], \int_{S^1} \mathrm{Tr}\, E_1 \, dE_2)$$

Check Jacobi identity

$$T^* \widehat{u(N)} = \{ (E(x), c; k\partial_x + A_x) : k, c \in \mathbb{R}, E, A_x - \mathfrak{g}\text{-valued functions on } S^1 \}$$

$$kc + \int_{S^1} \mathrm{Tr}\, E A_x \, dx$$

LG-action on (E, A_x, k, c) is $(g^{-1}(x) E(x) g(x), g^{-1}(x) A_x g(x) + k g^{-1}(x) \partial_x g(x), k, c + \cdots)$

electric field $\quad P \quad X \quad$ vector potential

$k = 1$

was

$\mu = [P, X] \longrightarrow \mu = D_x E = \partial_x E + [A_x, E]$ Gauß Law

$\nu \cdot 1_N - z \cdot z^+ \longrightarrow \delta(x)(\nu \cdot 1_N - z \otimes z^+)$ analogue of $X = \mathrm{diag}(x_1, \dots, x_N)$ is now $A_x = \mathrm{diag}(x_1, \dots, x_N)$

$\delta(x) = 0$

const$_x$

anti-herm.

$N \times N$ matrix valued function on S^1

$A_x(x) \mapsto g^{-1}(x) A_x(x) g(x) + g^{-1}(x) \partial_x g(x)$ $\qquad \mathcal{H}_2 = \frac{1}{2} \int dx \, \mathrm{Tr}\, E^2$

$g : S^1 \to U(N)$

Floquet - Lyapunov theory $\quad$ (I want to kill $A_x \iff$ find g s.t. $A_x = \partial_x g g^{-1}$

obstruction to kill A_x

= monodromy $\quad P\exp \oint_0^{2\pi} A_x \, dx = G$

$g(x) = P\exp \int_0^x A_x \, dx$

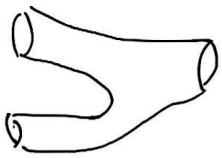
$\partial_x g = A_x g - g X$ ← constant diagonal

$g(x) = \mathrm{diag}\left(e^{in_1 x}, \dots, e^{in_N x}\right)$

$0 \le x \le 2\pi \qquad n_i \in \mathbb{Z}$

$x_i \longmapsto x_i + n_i$

$g(x) = \mathrm{diag}\left(e^{i\theta_1}, \dots, e^{i\theta_N}\right), \qquad \theta_i = \text{const.}$



$\longrightarrow$ 2d YM on forms (origami) $\longrightarrow$ lots of int. syst.

<u>Lecture 2</u> <u>Complexify</u>. $\mathbb{R} \longrightarrow \mathbb{C}$

$\mathbb{Z}/2 \longrightarrow U(1), \mathbb{C}^\times$

classical mechanics

paths in phase space $\mathcal{P}$ $\longrightarrow$ real paths in $\mathcal{P}_{\mathbb{C}}$

$\searrow$ "complex" paths in $\mathcal{P}$ (STRING THEORY!)

$\mathcal{P}_{\mathbb{C}}$ complex mfd w holomorphic symplectic str.

$\dim_{\mathbb{C}} \mathcal{P}_{\mathbb{C}} = 2n, \ \omega_{\mathbb{C}} \in \Omega^{2,0}(\mathcal{P}_{\mathbb{C}}), \quad \omega_{\mathbb{C}}^n \neq 0$

Many body Calogero–Moser
elliptic

$$H_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \nu^2 \sum_{i<j} \wp(x_i - x_j; \tau)$$

$$\wp(x;\tau) = \frac{1}{x^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(x+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right] + \text{const.}$$

$\omega_{\mathbb{C}} = \sum_{i=1}^{N} dp_i \wedge dx_i$

Weierstraß $\wp$-function on $E =$ $= \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$

$\tau_2 = \mathrm{Im}\,\tau > 0$

It is nat'l to allow $(p_i, x_i)_{i=1}^N$ to be complex $\qquad [\omega_{\mathbb{C}}]$ is non-trivial in cohomology

phase space is actually $(T^*E)^N / \mathfrak{S}_N$ – smooth hyper Kähler mfd $\qquad \boxed{\int_{S^2} \omega_{\mathbb{C}} \sim \nu}$

Dasgeb

Quantum

$p_i = -\sqrt{-1}\,\hbar \frac{\partial}{\partial x_i}$

$\hat{H}_2 = H_2\left(p = \sqrt{-1}\,\hbar \frac{\partial}{\partial x}, x\right)$ will have complex eigenvalues, as functions of (ν, τ).

$N=2$ center-of-mass frame

$p_1 + p_2 = 0$

$$P_{\mathbb{C}} /\!/ E$$

$$(x_1, \cdots, x_N) \mapsto (x_1 + z, \cdots, x_n + z)$$

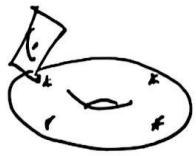
$$z \in E$$

$$\mu = p_1 + \cdots + p_N$$

$(T^* \underset{\substack{\shortparallel \\ x}}{E})/(\mathbb{Z}/2)$ this involution has fixed points.

$x = x_1 - x_2 \mapsto -x$

$p = p_1 - p_2 \mapsto -p$



$x \mapsto -x$ Fixed points in T^*E

$x = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$

$(p, x) = (0, 0), (0, \frac{1}{2}), (0, \frac{\tau}{2}), (0, \frac{1+\tau}{2})$

$X^2 = YZ$ in $\mathbb{C}^3$

model on $\mathbb{C}^2/(\mathbb{Z}/2)$

$\omega_{\mathbb{C}} = dz_1 \wedge dz_2 = \frac{dY \wedge dZ}{4X}$ $(z_1, z_2) \mapsto (-z_1, -z_2)$ $X = z_1 z_2$, $Y = z_1^2$, $Z = z_2^2$

Algebraically, $\operatorname{Spec} \mathbb{C}[X, Y, Z] / (YZ - X^2)$ now $\omega_{\mathbb{C}}$ is not exact!

blow-up $\mathcal{O}(-2)$

$\downarrow$

$\mathbb{P}^1$

parameter

complex deformation. $\operatorname{Spec} \mathbb{C}[X, Y, Z] / (YZ - X^2 - \zeta^2)$ $X^2 - YZ = \zeta^2 \neq 0$

$$\begin{cases} X = \zeta x \\ Y = \zeta(y + \sqrt{-1}\, z) \\ Z = -\zeta(y - \sqrt{-1}\, z) \end{cases} \qquad X^2 - YZ = \zeta^2 \iff x^2 + y^2 + z^2 = 1.$$

$$\omega_{\mathbb{C}} = \frac{\sqrt{-1}\,\zeta}{2} \frac{dy \wedge dz}{x} \quad , \quad \frac{dy \wedge dz}{x}\Big|_{S^2 = \{(x,y,z) \text{ are real}\}} = \mathrm{vol}_{S^2}$$

$(p, x) \sim (-p, -x)$,

$$\begin{cases} \wp(x;\tau) = \wp(-x;\tau) = X_1 \\ p\,\wp'(x;\tau) = Y \\ p^2 = X_2 \end{cases}$$

$H = p^2 + \nu^2 \wp(x;\tau)$

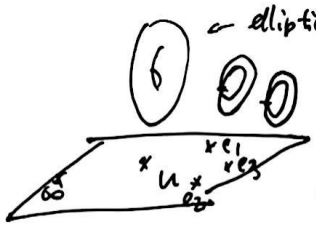
$\omega_{\mathbb{C}} = dp \wedge dx$

$$Y^2 = X_2\left(4X_1^3 - g_2(\tau) X_1 - g_3(\tau)\right)$$

phase space is a complex surface

$$(\wp')^2 = 4\wp^3 - g_2(\tau)\wp - g_3(\tau)$$
$$= 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

algebraic curve



$H = X_2 + \nu^2 X_1$

$\mathcal{B} = \mathbb{CP}^1 \quad \dim_{\mathbb{C}} \mathcal{B} = 1$

$$Y^2 = 4(H - \nu^2 X_1)(X_1 - e_1)(X_1 - e_2)(X_1 - e_3)$$

$$H = \nu^2 u \qquad = 4\nu^2 (u - X_1)(X_1 - e_1)(X_1 - e_2)(X_1 - e_3)$$

Complex str. of $\mathcal{E}_u$ depends on $[u : e_1 : e_2 : e_3]$

Lax rep'n for eCM.

Krichever 1980

"complexification of 2d YM construction from yesterday".

Yesterday.

$$P_\infty = \left\{ (E, A) : \begin{array}{l} E \in \mathrm{Maps}(S^1, \mathfrak{g}) \\ A \in \Omega^1(S^1) \otimes \mathfrak{g} \end{array} \right\} \times \mathcal{O} \quad /\!/ LG$$

$\mathfrak{g} = su(N)$, $\mathcal{O} = \mathbb{CP}^{N-1}$, $G = SU(N)$

$x \in S^1$

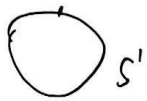
$g(x) \cdot (E, A) = \left(g^{-1} E g,\ g^{-1} A g + g^{-1} dg \right)$.

$g(x) \cdot (z_1, \cdots, z_N) = \left((g^{-1}(x) z)_1 \ \cdots\ : (g^{-1}(x) z)_N \right)$

$\uparrow$

$\mathcal{O}$

$0 = x$ base point



S^1

$\mu = D_x E + J \delta(x)$

$J : \mathcal{O} \to \mathfrak{g}^*$

$J(z_1, \cdots, z_N) = \left\| \sqrt{-1}\, \nu \left(z_i \bar{z}_j - \delta_{ij} \right) \right\|_{i,j=1}^{N}$

$\sum_{i=1}^{N} |z_i|^2 = 1$

$\uparrow$

traceless anti-hermitian $N \times N$.

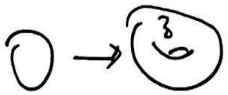
$$H_2 = \frac{1}{2} \int_{S^1} dx\, \mathrm{Tr}\, E^2 \longrightarrow t\mathcal{CM}$$

hamiltonian.

Now. $P_\infty^{\mathbb{C}} = \left\{ (E, \bar{A}) \right.$ $\to$ (0,1)- connection

$\searrow$ function of $z, \bar{z}$. (1,0)- form

on T^2, valued in $sl_N(\mathbb{C})$



$T^2 = E$

$$\mathcal{O}_{\mathbb{C}} = \left\{ (z \in \mathbb{C}^N, w \in (\mathbb{C}^N)^*) : w(z) = 1 \right\} / \mathbb{C}^\times$$

$$J(z, w) = \left\| \sqrt{-1}\, \nu \left(z_i w_j - \delta_{ij} \right) \right\|_{i,j=1}^{N}$$

$\in$ traceless $N \times N$ mat.

$$g(z, \bar{z}) \left(E, \bar{A} : (z, w) \right) = \left(g^{-1} E g,\ g^{-1} \bar{A} g + g^{-1} \bar{\partial} g,\ \left(g^{-1}(0) z,\ w g(0) \right) \right)$$

$$\mu = \bar{\partial} E + [\bar{A}, E] + J \delta^{(2)}(z, \bar{z}) = 0$$

$\uparrow$

complexified version of $D_x E$

$\exists$ gauge for generic $\bar{A}$

$\bar{A} \longrightarrow \mathrm{diag}(\alpha_1, \cdots, \alpha_N)$

$\alpha_i \in \mathbb{C}$, const $z, \bar{z}$

remaining $g(z, \bar{z}) = \mathrm{diag}\left(e^{2\pi i n_i \frac{z - \bar{z}}{\tau - \bar{\tau}} + 2\pi i m_i \frac{z \bar{\tau} - \bar{z} \tau}{\tau - \bar{\tau}}} \right)$

$$\alpha_i \longmapsto \alpha_i + \frac{2\pi i}{\tau - \bar{\tau}} (n_i + m_i \tau)$$

$$x_i = \frac{\tau_2}{\pi} d_i$$

$$x_i \sim x_i + m_i + n_i \tau$$

$$\bar\partial E_{ij} + (\alpha_i - \alpha_j) E_{ij} + J_{ij}\, \delta^{(2)}(z, \bar z) = 0$$

$$i = j : \quad \bar\partial E_{ii} + J_{ii}\, \delta^{(2)}(z, \bar z) = 0$$

For comparison: $\partial_x E_{ii} + J_{ii}\, \delta(x) = 0$

$E_{ii} = \text{const}$ outside $x = 0$

$E_{ii}(+0) - E_{ii}(-0) = J_{ii}$

$\Rightarrow J_{ii} = 0$

E_{ii} is holomorphic outside 0, $\quad E_{ii} \sim \frac{J_{ii}}{z}$, $z \to 0$

$\to J_{ii} = 0, \quad E_{ii} = p_i = \text{const}.$

$$J_{ii} = \sqrt{-1}\,\nu\,(z_i w_i - 1) = 0 \quad \to \quad z_i w_i = 1$$

$g(z, \bar z) = \operatorname{diag}(t_1, \dots, t_N), \quad \prod t_i = 1, \quad t_i \in \mathbb{C}^\times$. Use these transformations $\to z_i = w_i = 1$.

$$i \neq j, \quad E_{ij}(z, \bar z) = e^{(z - \bar z)(\alpha_i - \alpha_j)} L_{ij}(z)$$

$$L_{ij} \sim \frac{J_{ij}}{z} = \frac{\sqrt{-1}\,\nu\,(1 - \delta_{ij})}{z}, \quad z \to 0$$

$$L_{ij}(z + 1) = L_{ij}(z)$$

$$L_{ij}(z + \tau) = e^{(\bar\tau - \tau)(\alpha_i - \alpha_j)} L_{ij}(z)$$

$$H_k = \int \operatorname{tr} E^k$$

$$L_{ij}(z) = \sqrt{-1}\,\nu \frac{\theta(z + x_i - x_j)\,\theta'(0)}{\theta(z)\,\theta(x_i - x_j)}$$

$\operatorname{tr} E^2$ = meromorphic function of z w/ 2nd order pole at $z = 0$.

$$= \hbar \nu^2 \wp(z; \tau) + H_2$$

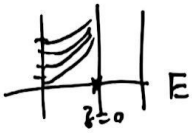
$$\text{v.p.} \int_{T^2} d^2 z \, \operatorname{tr} E^2 \qquad \lim_{\varepsilon \to 0} \int_{|z| \geq \varepsilon} \frac{dz\, d\bar z}{z^2}$$

$R(\lambda, z)$ poly. meromorphic

$\|$ spectral curve C_u

$$\det(\lambda - L(z)) = 0$$

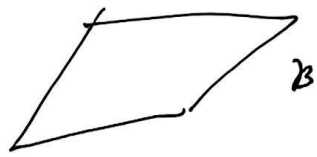
$T^*(E \setminus 0)$



$$L(z) \sim \frac{\sqrt{-1}\,\nu\,(1 - z \otimes w)}{z}$$

$g = N - 1$ curve

P_C



$B \simeq \mathbb{C}^N$

Abelian var. ("complexification" of Liouville tori of real int. systems)

Lecture3. New ~~bag of tricks~~ Cast of Characters

Supersymmetric gauge theory in 4, 5, 6 dimensions

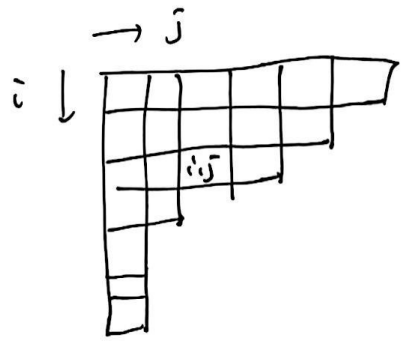
$\downarrow$

Class of statistical models

μ - measures on the set of multi-partitions

Partitions

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0)$, size $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$, length $\ell(\lambda)$.



λ_1 boxes

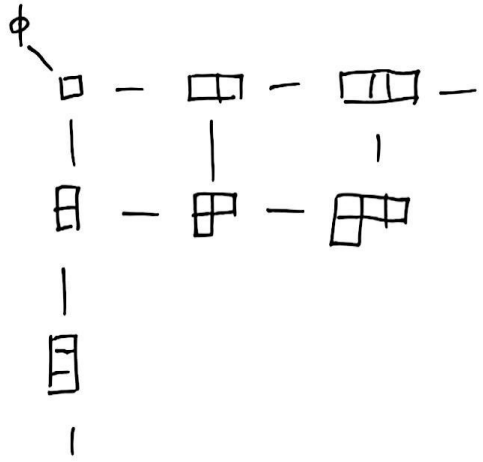
λ_2

$\vdots$

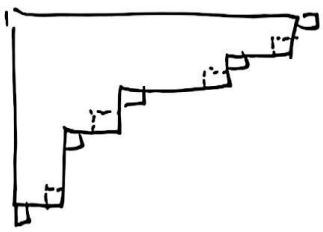
Young diagram

$(i,j) = \square \in \lambda \iff 1 \leq j \leq \lambda_i \iff 1 \leq i \leq \lambda_j^t$

λ^t



Young graph



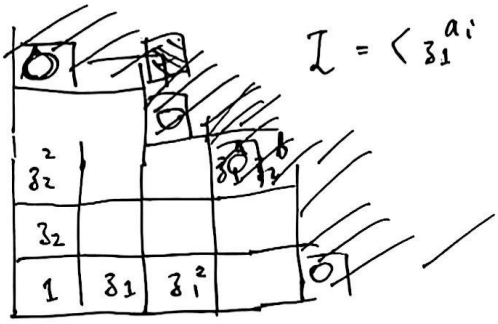
$\square \in \partial_+ \lambda$

$\square \in \partial_- \lambda$

$\# \partial_+ \lambda - \# \partial_- \lambda = 1$

$$\mathrm{Ch}\,\lambda = \chi_\lambda(q_1, q_2) = \sum_{(i,j)\in\lambda} q_1^{i-1} q_2^{j-1} = \text{character of } \mathbb{C}^\times\times\mathbb{C}^\times\text{-action on } K_\lambda.$$

$\lambda \longleftrightarrow$ monomial ideal in $\mathbb{C}[z_1, z_2]$ of codim $|\lambda|$



$$I = \langle z_1^{a_i} z_2^{b_i} \rangle_{i=1,\dots,N} \leftarrow \text{generators}$$

$$J_\lambda = \langle z_1^{i-1} z^{\lambda_i} \rangle_{i=1,\dots,\ell(\lambda)}$$

$\uparrow$ Young diagram

$\boxed{\circ} \longleftrightarrow$ generators of J_λ.

$$\langle z_1^{i-1} z_2^{j-1} \rangle_{(i,j)\in\lambda} \in \mathbb{C}[z_1, z_2]/J_\lambda = K_\lambda$$

$\partial_+\lambda$ generators of J_λ

$\partial_-\lambda$ relations of J_λ

J_λ is invariant under the $\mathbb{C}^\times\times\mathbb{C}^\times$-action $(z_1, z_2) \mapsto (t_1 z_1, t_2 z_2)$

You have seen this an hour ago

$$S_\lambda = 1 - (1-q_1)(1-q_2)\,\chi_\lambda(q_1, q_2)$$

$$= \underbrace{\sum_{\square\in\partial_+\lambda} e^{c(\square)}}_{\chi_{H^2}(q_1,q_2)} - q_1 q_2 \underbrace{\sum_{\blacksquare\in\partial_-\lambda} e^{c(\blacksquare)}}_{\chi_{H^1}(q_1,q_2)}$$

$$e^{c(\square)} = q_1^{i-1} q_2^{j-1}, \quad \square = (i,j)$$

$$\frac{S_\lambda}{(1-q_1)(1-q_2)} = \sum_{(i,j)\text{ s.t. } z_1^{i-1} z_2^{j-1}\in J_\lambda} q_1^{i-1} q_2^{j-1} = \frac{\sum_{\square\in\partial_+\lambda} e^{c(\square)}}{(1-q_1)(1-q_2)} - \frac{q_1 q_2 \sum_{\blacksquare\in\partial_-\lambda} e^{c(\blacksquare)}}{(1-q_1)(1-q_2)}$$

$$\| $$

$$\frac{1}{(1-q_1)(1-q_2)} - \chi_\lambda(q_1, q_2)$$

Complex ($\mathbb{C}^\times \times \mathbb{C}^\times$ - equivariant) (tautological) associated to λ

$$0 \to K_\lambda \xrightarrow{d_1} K_\lambda \otimes Q \oplus \mathbb{C} \xrightarrow{d_2} K_\lambda \otimes \Lambda^2 Q, \to 0 \qquad Q = \mathbb{C}^2 \text{ - rep'n of } \mathbb{C}^\times \times \mathbb{C}^\times$$

$$d_1(k) = ((z_1 k), (z_2 k); 0)$$

$$d_2(k_1, k_2; u) = u + z_2 k_1 - z_2 k_2 \qquad d_2 \circ d_1 = 0$$

Multi-partition = collection $(\lambda^{(\alpha)})$ of partitions, $\alpha \in$ Finite Set A

$$Z = \sum_{(\lambda^{(\alpha)})} \prod_{\alpha \in A} q_\alpha^{|\lambda^{(\alpha)}|} \mu_{(\lambda^{(\alpha)})}(a_\alpha, \varepsilon_1, \varepsilon_2, \cdots)$$

$$\begin{tikzcd} & & \\ H \arrow[dl] & K \arrow[l] \arrow[d] & \mathcal{E}M \arrow[l] \arrow[dl] \\ G & S/S^1 & G/S^1 \times S^1 = E \end{tikzcd}$$

There are μ's associated w/ Σ_g

$\mu(\lambda^{(\alpha)})$ originate from localization in elliptic cohomology of some moduli spaces of sol'ns to PDE on $\mathbb{R}^4$.

$$\theta(x) = \begin{cases} x & H \\ 1 - e^{-x} & K \\ \prod_{n=1}^{\infty} (1 - p^{n-1} e^{-x})(1 - p^n e^{x}) & E \\ p = \exp(2\pi \sqrt{-1}\, \sigma), & \\ \operatorname{Im} \sigma > 0 & \end{cases}$$

<u>Example</u> ($\hat{A}_0$) $A = \{1, \cdots, N\}$ - "colours"

parameters are
$$\boxed{\begin{array}{l} (a_1, \cdots, a_N) \in \mathbb{C}^N \\ (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{C}^4, \ \sum_{a=1}^{4} \varepsilon_a = 0 \\ q = \exp(2\pi \sqrt{-1}\, \tau), \qquad \operatorname{Im} \tau > 0. \end{array}}$$

RANDOM VAR's $= (\lambda^{(1)}, \cdots, \lambda^{(N)})$

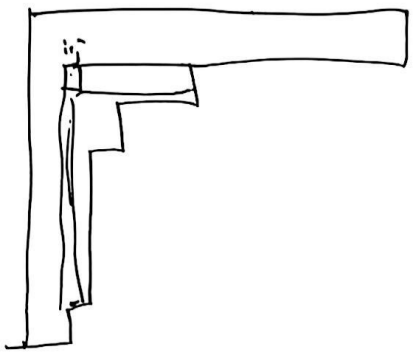
$$\mu = q^{\sum_{\alpha=1}^{N} |\lambda^{(\alpha)}|} \prod_{\alpha, \beta = 1}^{N} \prod_{(i,j) \in \lambda^{(\beta)}} \frac{\theta(\varepsilon_3 + \varepsilon_1(i - \lambda_j^{(\beta)}) + \varepsilon_2(1 + \lambda_i^{(\alpha)} - j) + a_\alpha - a_\beta)}{\theta(\varepsilon_1(i - \lambda_j^{(\beta)}) + \varepsilon_2(1 + \lambda_i^{(\alpha)} - j) + a_\alpha - a_\beta)}$$

leg

arm + 1

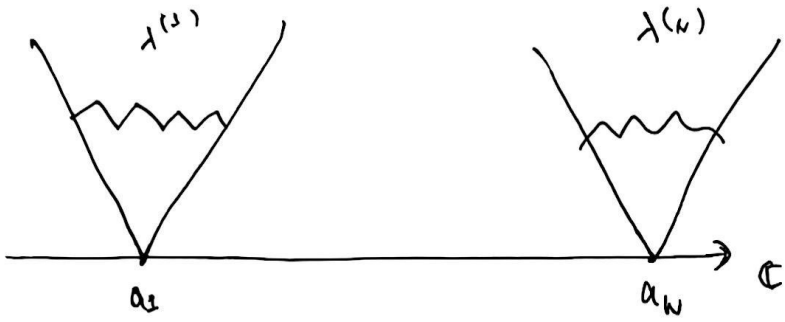
$$\times \prod_{(i,j) \in \lambda^{(\alpha)}} \frac{\theta(\varepsilon_3 + \varepsilon_1(\lambda_j^{(\alpha)t} + 1 - i) + \varepsilon_2(j - \lambda_i^{(\beta)}) + a_\alpha - a_\beta)}{\theta(\varepsilon_1(\lambda_j^{(\alpha)t} + 1 - i) + \varepsilon_2(j - \lambda_i^{(\beta)}) + a_\alpha - a_\beta)}$$

$\alpha = \beta$

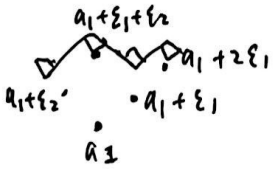


$\lambda_i - j = \mathrm{arm}(i,j)$

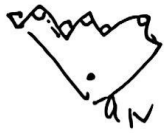
$\lambda^t_j - i = \mathrm{leg}(i,j)$



dipole interaction



$q_3 = e^{\varepsilon_3}$



$$\sum_{n=1}^{\infty} q^n \int_{\mathrm{Hilb}^{(n)}(\mathbb{C}^2)} \sum_i^{\varepsilon M} q_3^i \Lambda^i T^* \mathrm{Hilb}^{(n)}(\mathbb{C}^2)$$

$\mathbb{R}^4 = \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \curvearrowright (\mathbb{C}^*)^3$

$\cap$

$SL(4)$

$\parallel$

$\underline{N=1}$.
$$\sum_{\lambda} q^{|\lambda|} \prod_{\square \in \lambda} \frac{\theta(\varepsilon_3 + \varepsilon_1(-l_\square) + \varepsilon_2(a_\square+1))\,\theta(\varepsilon_3 + \varepsilon_1(l_\square+1) - \varepsilon_2 a_\square)}{\theta(\varepsilon_1(-l_\square) + \varepsilon_2(a_\square+1))\,\theta(\varepsilon_1(l_\square+1) - \varepsilon_2 a_\square)}$$

$\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 = 0$

in the rat'l case $\theta(x) = x$,

$$= \left(\prod_{n=1}^{\infty}(1-q^n)\right)^{-\frac{(\varepsilon_3+\varepsilon_1)(\varepsilon_3+\varepsilon_2)}{\varepsilon_1\varepsilon_2}}$$

$\varepsilon_1 \leftrightarrow \varepsilon_2$

$\varepsilon_3 \leftrightarrow \varepsilon_4$

Limits

$\varepsilon_1 = -\varepsilon_2$ $\quad \varepsilon_3 \to \infty$, $q \to 0$ s.t. $\varepsilon_3^2 q = \Lambda^2$ fixed

$\downarrow$

$\mathfrak{S}_N$ rep'n theory

$$\mu = \frac{(\dim\lambda)^2}{|\lambda|!} = \prod_{\square\in\lambda} \frac{1}{h_\square^2}, \quad h_\square = a_\square + l_\square + 1$$

limit shape $\quad \varepsilon_2 \to 0, \quad \Lambda$ fixed

For $N > 1$, NOT SO SIMPLE.

$N=2$ $\quad Z$ already describes conformal blocks of 2d CFT on

Poles at $a_\alpha - a_\beta + \varepsilon_1 i + \varepsilon_2 j = 0$ for some α, β; $i, j \geq 1$.

e.g. $\varepsilon_1 / \varepsilon_2 \in \mathbb{Q}_+$

$\downarrow$

fixed point set is not isolated

$$\frac{1}{Z} \mu(\lambda^{(\alpha)})$$

$$\langle \mathcal{O} \rangle := \frac{1}{Z} \sum_{(\lambda^{(\alpha)})} \mu(\lambda^{(\alpha)}) \, \mathcal{O}[\lambda^{(\alpha)}]$$

$\exists\, \mathcal{O}$ (observables), $\langle \mathcal{O} \rangle$ solves interesting eqs.

- Knizhnik - Zamolodchikov $\quad \widehat{sl_N}$
- Belavin - Polyakov - Zamolodchikov $\quad (\mathrm{Vir}, W_3, \dots)$

wave functions of $\underset{\hat{A}, \hat{D}, \hat{E}}{\mathrm{ADE}}$ spin chains

$\varepsilon_1, \varepsilon_2 \to 0$

$$Z \sim e^{\frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F}(a, \varepsilon_3, q, \dots)}$$

$\downarrow$

prepotential of some (special Kähler) geom. a.k.a. Seiberg - Witten geometry.

$\hookrightarrow$ family of (curves w/ Krichever data

$\varepsilon_2 \to 0$, ε_1 fixed. $\quad Z \sim e^{\frac{1}{\varepsilon_2} W(a, \varepsilon_1, \varepsilon_3, q)}$

$\searrow$ describes "quantization".

Lecture 4. $Z^{inst} = \sum_{k=0}^{\infty} q^k \, Kr\left(\mathcal{M}_k^{framed}(N), \overset{q_3 = e^{\varepsilon_3}}{\overset{\shortparallel}{z}}\right)$

$\mathcal{M}_k^{framed}(N)$: k N torsion free $GL(N)\times GL(2)$ sheaves on $\mathbb{P}^2$, (q_1, q_2) trivial over $\mathbb{P}^1_\infty$, $q_1 = e^{\varepsilon_1}$, $q_2 = e^{\varepsilon_2}$; $k = ch_2$

Democracy of (q_1, q_2, q_3, q_4) is restored by CROSSED INSTANTONS.

Z^{inst} = partition function of some stat. mech. model $\lambda^{(1)}, \dots, \lambda^{(N)}$

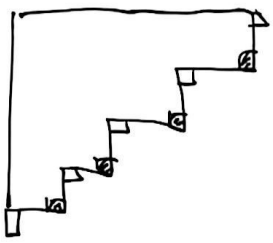
ORDER and DISORDER operators (observables)

$\downarrow$

$\mathcal{O}[\lambda^{(1)}, \dots, \lambda^{(N)}]$

Y- observable

$$Y(x)[\lambda^{(1)}, \dots, \lambda^{(N)}] = \prod_{\alpha=1}^{N} \frac{\prod_{\square \in \partial_+\lambda^{(\alpha)}} \theta\left(x - a_\alpha - c_{12}(\square)\right)}{\prod_{\blacksquare \in \partial_-\lambda^{(\alpha)}} \theta\left(x - a_\alpha - c_{12}(\blacksquare) - \varepsilon_1 - \varepsilon_2\right)}$$



$$c_{12}(\square = (i,j)) = \varepsilon_1(i-1) + \varepsilon_2(j-1)$$

valued in rat'l functions of x (H)

deg N line bundles on E ($\mathcal{M}$)

In rat'l case, $Y(x) \underset{x\to\infty}{=} x^N \exp \sum_{i=1}^{\infty} \frac{(-1)}{i} x^{-i} ch_i(\mathcal{E})$

universal sheaf localized to the pt $0 \in \mathbb{C}^2 = \mathbb{P}^2 \setminus \mathbb{P}^1_\infty$

"=" $\det(x - \Phi)$

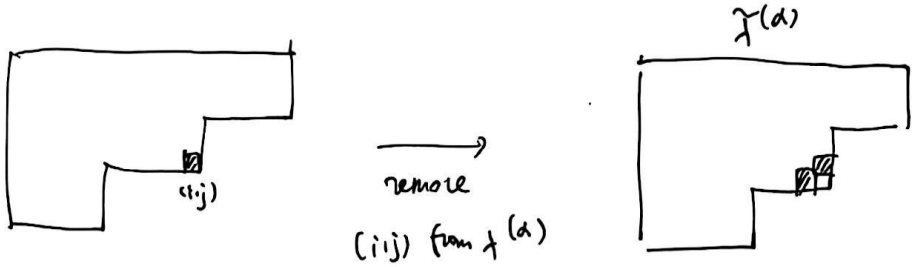
$$\langle Y(x+\varepsilon_1+\varepsilon_2)\rangle = \frac{1}{Z^{inst}} \sum_{\lambda^{(1)},\dots,\lambda^{(N)}} \mu^{(\vec{a},\vec{\varepsilon},q)}_{\lambda^{(1)},\dots,\lambda^{(N)}} Y(x+\varepsilon_1+\varepsilon_2)[\lambda^{(1)},\dots,\lambda^{(N)}]$$

$\uparrow$

has poles in x.

poles at $x = a_\alpha + \varepsilon_1(i-1) + \varepsilon_2(j-1)$ for some $(i,j) \in \mathbb{N}\times\mathbb{N}$

from $(\vec{\lambda})$, where $\partial_- \lambda^{(\alpha)} \ni (i,j)$



$$(\lambda^{(1)},\dots,\lambda^{(\alpha)},\dots,\lambda^{(N)}) \longrightarrow (\lambda^{(1)},\dots,\tilde{\lambda}^{(\alpha)},\dots,\lambda^{(N)})$$

equiv. pairs

qq-character

$$\mathcal{X}(x) = \langle Y(x+\varepsilon_1+\varepsilon_2) + q\,\frac{Y(x+\varepsilon_3)\,Y(x-\varepsilon_4)}{Y(x)} + \dots \rangle$$

$(\varepsilon_1+\varepsilon_3)/(\varepsilon_1+\varepsilon_4)/\varepsilon_3\varepsilon_4$

$\mathcal{H}$?

$$+ q^2(\dots)$$

$\uparrow$

to cancel poles at

$a_\alpha + \varepsilon_3 + \mathbb{N}\varepsilon_1 + \mathbb{N}\varepsilon_2$

ε_4

$+ \dots$

$$Y(x)\Big|_{a_\alpha+\varepsilon_1(i-1)+\varepsilon_2(j-1)} [\lambda^{(1)}\dots\tilde{\lambda}^{(\alpha)}\dots\lambda^{(N)}] = 0$$

for proper $\mathcal{H}$ the poles at $a_\alpha + \varepsilon_1(i-1) + \varepsilon_2(j-1)$ will cancel.

$k = |\lambda|$ $\quad$ $\mathbb{C}^\times\times\mathbb{C}^\times$-equiv. w/ wts q_3, q_4

$\mathrm{Kr}(\mathrm{Hilb}^{[k]}(\mathbb{C}^2), z = e^{\varepsilon_1} = q_1)$

fundamental qq-character

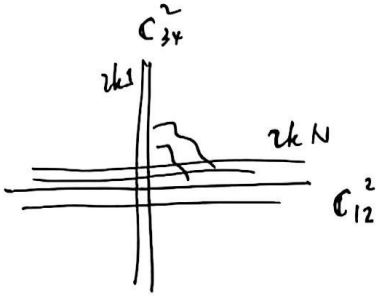
$$\mathcal{X}(x) = \sum_\lambda q^{|\lambda|} \prod_{\square\in\lambda} \frac{\theta(\varepsilon_3(a_\square+1) - \varepsilon_4 \ell_\square + \varepsilon_1)\,\theta(-\varepsilon_3 a_\square + \varepsilon_4(\ell_\square - 1) + \varepsilon_1)}{\theta(\varepsilon_3(a_\square+1) - \varepsilon_4 \ell_\square)\,\theta(-\varepsilon_3 a_\square + \varepsilon_4(\ell_\square+1))}$$

$$\times \frac{\prod_{\square\in\partial_+\lambda} Y(x+\varepsilon_1+\varepsilon_2+c_{34}(\square))}{\prod_{\blacksquare\in\partial_-\lambda} Y(x+c_{34}(\blacksquare))}$$

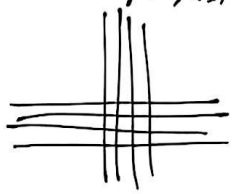
$$c_{34}(\square = (i,j)) = \varepsilon_3(i-1) + \varepsilon_4(j-1)$$

$\langle *(x) \rangle$ is holomorphic in x.

$$\prod_{\alpha=1}^{N} \theta(x - A_\alpha) \qquad A_\alpha(q) = a_\alpha + q(\cdots)$$



$\mathbb{C}^*$



$M, b_1, \dots b_M \qquad \sum_{\beta=1}^{M} b_\beta = 0$

$\times$

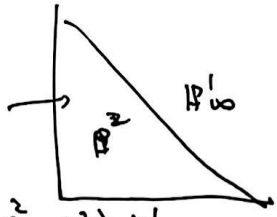
$N, a_1, \dots a_N \qquad \sum_{\alpha=1}^{N} a_\alpha = 0$

$$Z\left(\left(U(M) \times U(N)\right)/U(1)\right) = U(1) \times$$

DISORDER OPERATORS AKA SURFACE DEFECTS

– VIA ORBIFOLDING

$U(N)$ gauge theory on $\mathbb{R}^4 \sim \mathbb{C}^2 = \mathbb{P}^2 \setminus \mathbb{P}^1_\infty$

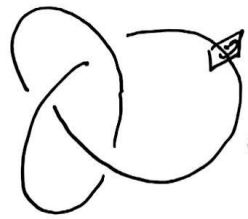


$$\int DA\, e^{-\frac{1}{g^2}\int \mathrm{Tr}\, F_A \wedge * F_A + \cdots}$$

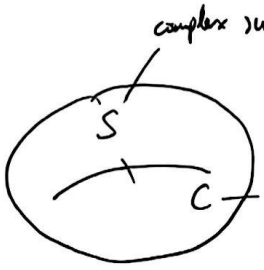
codim'n 2 defects

Observables $\left\langle \mathrm{Tr}_R\, P \exp \oint_{\mathcal{C}} A \right\rangle$

In 3d $CS(A)$



$F_A \sim J_R \delta_{\mathcal{C}}$



$z_2 = 0$ eqn of C

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ S \end{array} \quad . \quad \mathcal{E}|_C \supset \mathcal{E}_1 \supset \mathcal{E}_2 \supset \cdots \supset \mathcal{E}_n$$

$$\mathcal{E} \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_n \supset z_2 \mathcal{E}$$

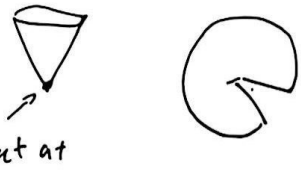
$F_A \sim J \delta^{(2)}_{\mathcal{C}}$

We do gauge theory on $(z_1, \tilde{z}_2)$, $\tilde{z}_2^N = z_2$

$$\downarrow \text{ project}$$

$$(z_1, z_2)$$

$$\mathbb{C}\times\mathbb{C}/(\mathbb{Z}/N) = \Gamma$$



defect at the origin

$\mathbb{C}^4_{1234} \circlearrowleft \Gamma$ $\quad \tilde{z}_4$

Γ-action $(z_1, \tilde{z}_2, z_3, \tilde{z}_4) \mapsto (z_1, \Omega\tilde{z}_2, z_3, \Omega^{-1}\tilde{z}_4)$

$\Omega^N = 1$

Now we define new measure on the set of N-tuples $(\lambda^{(1)}, \dots, \lambda^{(N)})$.

$\mathbb{C}^x \times \mathbb{C}^y \times \mathbb{C}^x \times \Gamma$

$q_1 \quad q_2 \quad q_3$

Space N-regular rep'n of Γ

assign $[a_\alpha] = c(\alpha) \in \mathbb{Z}/N = \mathrm{Rep}(\Gamma)$

$[\varepsilon_1] = 0$ $\quad R_0$ - trivial rep

$[\varepsilon_2] = 1$

$[\varepsilon_3] = 0$

$[\varepsilon_4] = -1$

$c : \{1, \dots, N\} \xrightarrow{1:1} \{0, \dots, N-1\}$

$\qquad\qquad\qquad\qquad\qquad\qquad \ni w$

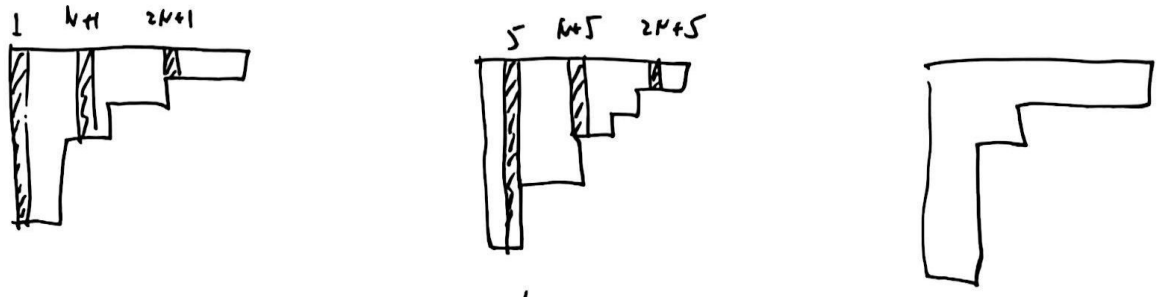
$R_w \in \mathrm{Rep}(\Gamma)$, $\quad R_w : \Omega \mapsto e^{\frac{2\pi\sqrt{-1}\,w}{N}}$

e.g. $c(\alpha) = \alpha - 1$

$$\sum n_\alpha c(\alpha) + k_2 \equiv 0 \ (N)$$

$$\mu^{new}_{\lambda^{(1)}, \dots, \lambda^{(N)}} = \prod_{w=0}^{N-1} q_w^{k_w} \prod_{\substack{\text{the same} \\ (\alpha, \beta, \square, \dots)}} \theta\left(\sum n_\alpha a_\alpha + k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3\right)$$

$$k_w = \sum_{\alpha=1}^{N} \# \left\{ (i,j) : (i,j) \in \lambda^{(\alpha)},\ c(\alpha) + j - 1 \equiv w (N) \right\}$$



$\alpha = 1$ $\qquad \alpha = 5$ $\qquad \alpha = N$

proj:

$$\pi : \left(\lambda^{(\alpha)}\right)_{\alpha=1}^{N} \longmapsto \left(\Lambda^{(\alpha)}\right)_{\alpha=1}^{N}$$

$$\boxed{\Lambda_j^{(\alpha)t} = \lambda^{(\alpha)t}_{\alpha+N(j-1)}}$$

$$\sum_{\vec{\lambda} \in \pi^{-1}(\Lambda^{(1)}, \dots, \Lambda^{(N)})} \mu^{new}_{\lambda^{(1)}, \dots \lambda^{(N)}} = q^{|\vec{\lambda}|} \mu^{old}_{\vec{\Lambda}} \times \underline{\text{observable}}$$

$$\sum \prod_w q_w^{k_w} \int_{\substack{\text{parabolic str.} \\ \text{on given } \mathcal{E}_{\vec{\Lambda}}}} 1$$

$$q = \prod_{w=0}^{N-1} q_w \qquad q_w = \frac{z_w}{z_{w-1}}, \quad z_{w+N} = q z_w$$

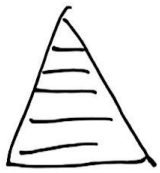
$$\Lambda_1^{(1)t} = \lambda_1^{(1)t}$$

$$\Lambda_1^{(2)t} = \lambda_2^{(2)t} \le \lambda_1^{(2)t} \qquad \text{- 1 integer freedom}$$

$$\Lambda_1^{(3)t} = \lambda_3^{(3)t} \le \lambda_2^{(3)t} \le \lambda_1^{(3)t} \qquad \text{- 2 integers}$$

$$\vdots$$

$$\Lambda_1^{(N)t} = \lambda_N^{(N)t} \le \lambda_{N-1}^{(N)t} \le \dots \le \lambda_1^{(N)t} \qquad N-1$$

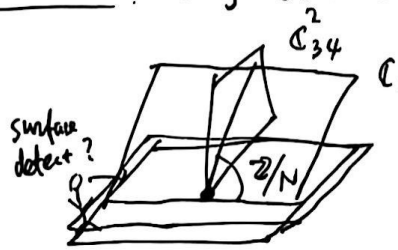


Gelfand-Zeitlin

$\mathcal{C}\mathcal{L}$ Maps $(\mathbb{C}_{z_1} \mathbb{P}^1, T^* \text{Flag})$

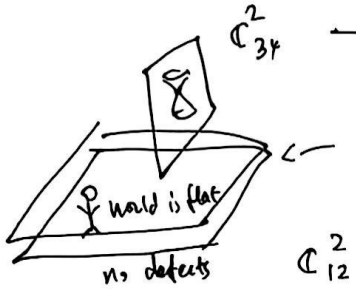
Relation to parabolic sheaves.....

Lecture 5. Why characters?



$$\mathbb{C}^2_{12} \hookrightarrow \mathbb{C}^4_{1234}$$

Let's change the orbifold action!



$\mathbb{C}^2_{34} \longrightarrow \mathbb{C}^2_{34}/\Gamma$ $\quad \Gamma \subset SU(2)$ finite subgroup, A, D, E

$\longleftarrow$ rk N sheaves ($U(N)$ instantons)

$\mathbb{C}^2_{12}$

<u>Example</u>

$\Gamma = \mathbb{Z}_{r+2}$ acts by $(z_3, z_4) \mapsto (\omega z_3, \omega^{-1} z_4)$, $\omega^{r+2} = 1$

$\mathbf{N}^{\text{framing}}$ = functions on N copies of Γ-orbits

$A = z_3^{r+2}$, $B = z_4^{r+2}$, $C = z_3 z_4$. $\quad \mathbb{C}^2_{34}/\Gamma \quad AB = C^{r+2}$

rk $N(r+2)$ sheaf w/ Γ-action

$$\mathrm{Ch}\,\mathbf{N} = \sum_{\alpha=1}^{N} \sum_{s=0}^{r+1} e^{a_{\alpha,s}} \hat{R}_s \qquad \mathrm{Rep}(\Gamma) = \{\hat{R}_0, \dots, \hat{R}_{r+1}\}$$

$\hat{R}_s$ – irrep where $\omega \mapsto \omega^s$

Framing group is now $(GL(N))^{\times(r+2)} = GL(N(r+2))^{\mathbb{Z}_{r+2}}$

moduli space of quiver instantons

$$\bigsqcup_{\vec{k}} \mathcal{M}_{\vec{k}} = \left(\begin{array}{l} \text{Derived moduli space} \\ \text{of framed rk } N(r+2) \text{ sheaves} \\ \text{on } \mathbb{P}^2 \text{ w/ trivialization} \\ \text{over } \mathbb{P}^1_\infty \quad \bigoplus_{s=0}^{r+1} N_s \otimes \hat{R}_s \end{array} \right)^{\mathbb{Z}_{r+2}} = \left\{ (\mathcal{E}_0, \dots, \mathcal{E}_{r+1}) : \begin{array}{l} \mathcal{E}_i \text{ torsion free on } \mathbb{P}^2, \\ \mathcal{E}_s|_{\mathbb{P}^1_\infty} \simeq N_s \otimes \mathcal{O}, \\ \mathrm{rk}\,\mathcal{E}_s = N \\ \mathrm{ch}_2(\mathcal{E}_s) = k_s \end{array} \right\}$$

$U(N)^{r+2}$ – instantons

$\sum n_{\alpha,s} s + m_3 - m_4 \overset{\text{mod } r+2}{=} 0$

$[\sum n_{\alpha,s} a_{\alpha,s} + m_a \varepsilon_a]$

$q \longrightarrow (q_0, \dots, q_{r+1})$

from Wednesday

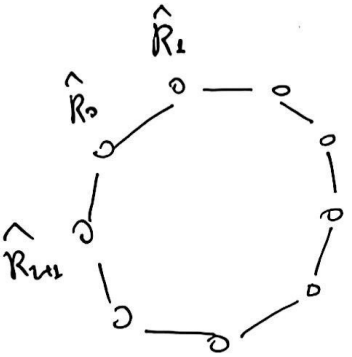
$\mathbb{Z}_{r+2}$ – charge $= 0$

$$Z^{\text{inst}}(a_{\alpha,s}; \vec{q}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \sum_{\vec{k}} \prod_{s=0}^{r+1} q_s^{k_s} \sum_{\lambda^{(\alpha,s)}} \prod \left(\frac{\quad}{\quad} \right) \qquad k_s = \sum_{\alpha=1}^{N} |\lambda^{(\alpha,s)}|$$

$N(r+2)$-tuple of Young diagrams

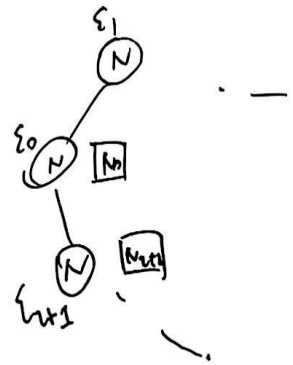
$[a_{\alpha,s}] = s \qquad [\varepsilon_3] = 1$

$[\varepsilon_1] = [\varepsilon_2] = 0. \qquad [\varepsilon_4] = -1$



$\mathbb{Z}_{r+2}$ McKay graph

$$\hat{R}_s \otimes \mathbb{C}^2_{3\psi} = \bigoplus_{s'} \mathbb{C}^{m_{ss'}} \otimes \hat{R}_{s'}$$



$$\int_{\varepsilon_s} c_1\left(\bigoplus_s R\mathrm{Hom}(\varepsilon_s, \varepsilon_{s+1})\right)$$

$m_0^{\text{trace}} \times m_1^{\text{trace}} \times \cdots \times m_{r+1}^{\text{trace}}$

$\hat{A}_{r+1}$ - theory

$$m_s = \frac{\varepsilon_3}{r+2} + \sum_{\alpha=1}^{N} \left(a_{\alpha,s+1} - a_{\alpha,s}\right)$$

mass of bi-

<u>Example</u>. $q_0 = 0 = q_{r+1}$

$\downarrow$

A_r type theory

$$\mu_{A_1}(\lambda^{(\alpha)}) = \prod_{\alpha,\alpha'} \frac{\prod_{s=1}^{N} \prod_{\square \in \lambda^{(\alpha)}} \theta\left(a_\alpha - m_{\alpha}^{-} + c_{12}(\square)\right) \theta\left(a_\alpha - m_{\alpha}^{+} + c_{12}(\square)\right)}{\prod_{\square \in \lambda^{(\alpha)}} \theta(\cdots) \prod_{\square \in \lambda^{(\alpha')}} \theta(\cdots)}$$

$\left(\lambda^{(\alpha,s)}\right)_{s=1,\dots,r}$

$r \times N$ partitions to sum over

as before w/ legs & arms

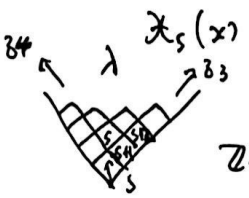
<u>Notation</u>. $a_{\alpha,1} = m_{\alpha}^{-}$

$a_{\alpha,r+2} + \varepsilon_1 + \varepsilon_2 = m_{\alpha}^{+}$

$$Y_s(x) = \prod_{\alpha=1}^{N} \frac{\prod_{\square \in \partial_+ \lambda^{(\alpha,s)}} \left(x - a_{\alpha,s} - c_{12}(\square)\right)}{\prod_{\blacksquare \in \partial_- \lambda^{(\alpha,s)}} \left(x - a_{\alpha,s} - \varepsilon_1 - \varepsilon_2 - c_{12}(\blacksquare)\right)}$$

fundamental qq-character

$$\mathcal{X}_s(x) = Y_s(x + \varepsilon_1 + \varepsilon_2) + q_s \frac{Y_{s-1}(x + \varepsilon_1 + \varepsilon_2) Y_{s+1}(x)}{Y_s(x)} + \cdots$$



$\mathbb{Z}_{r+2}$- charge $\left(\prod_{(i,j)\in\lambda} q_{s+i-j}\right)$ height only λ's for which $(i,j) \in \lambda$ $0 < s+i-j < r+1$

$$Y_0(s) = \prod_{\alpha=1}^{N} (x - m_\alpha^-) \,, \quad Y_{r+1}(s) = \prod_{\alpha=1}^{N} (x - m_\alpha^+)$$

$\varepsilon = \varepsilon_1 + \varepsilon_2$

$$\mathcal{X}_s(x) = Y_0(x + \varepsilon(1-s)) \sum_{0 \le i_1 < \cdots < i_l \le r+1} \prod_{b=1}^{l} \frac{\Lambda_{i_b}(x + \varepsilon(b-l))}{z_{b-1}}$$

$$\Lambda_i(x) = z_i \frac{Y_{i+1}(x)}{Y_i(x)} \,, \quad i = 0, \dots, r+1. \quad z_i = z_0 q_1 \cdots q_i$$

Let's take $\varepsilon_1, \varepsilon_2 \to 0$ limit.

$$\langle Y(x) \rangle \downarrow$$

$$\langle Y^{-1}(x) \rangle = \frac{1}{\langle Y(x) \rangle}$$

limit shape

$\langle \mathcal{X}_s(x) \rangle$: has no poles in x

$= T_s(x)$ – holomorphic section of $\deg = N$ $\mathcal{L}|_E$

$$T_s(x) = \frac{Y_0(x)}{z_0 \cdots z_{s-1}} e_s(\Lambda_0, \dots, \Lambda_{r+1})$$

$$\downarrow$$

$$\Lambda_i = z_i \left\langle \frac{Y_{i+1}(x)}{Y_i(x)} \right\rangle \quad \varepsilon_1, \varepsilon_2 \to 0$$

$$\sum_{s=0}^{r+1} z_0 \cdots z_{s-1} \, T_s(x) (-z^{-1})^s = Y_0(x) \prod_{i=0}^{r+1} \left(1 - z^{-1} z_i \frac{Y_{i+1}(x)}{Y_i(x)} \right)$$

spectral curve! $\quad R(x, z) \leftarrow$ algebraic curve in $\underset{x}{E} \times \underset{z}{\mathbb{P}^1}$

$$\det(\hat{R}(x, z)) = R(x, z)$$

rat'l limit

$a_{\alpha,s}$ = periods of x

$g = r \times N$

Add surface defect ($\mathbb{Z}_N$ - orbifold)

$\mathbb{Z}_N \times \mathbb{Z}_{r+2}$ - group

$$Y_s(x) \longrightarrow Y_{s,w}(x) = \theta(x - a_{s,w}) \prod_{\beta=1}^{N} \prod_{\square \in \lambda^{(\beta,s)}} \left(\frac{\theta(x - a_{s,w} - c_{12}(\square) - \varepsilon_2)}{\theta(x - a_{s,w} - c_{12}(\square) - \varepsilon_1 - \varepsilon_2)} \right)^{\delta(\beta+j-1 \equiv s\ (N))}$$

$s = 0, 1, \dots, r+1$

$w = 0, 1, \dots, N-1$

$$\prod_{\square \in \lambda^{(\beta,s)}} \left(\frac{\theta(x - a_{s,w} - c_{12}(\square) - \varepsilon_1)}{\theta(x - a_{s,w} - c_{12}(\square))} \right)$$

$\hat{Y}_s(x) = \mathrm{diag}(Y_{s,0}, \dots, Y_{s,N-1})$

$N \times N$ matrix

$\hat{D}(x,z) = 0$

$\varepsilon_1, \varepsilon_2 \to 0$

$$\hat{Y}_0(x) \prod_{i=0}^{\vec{r}} \left(1 - \hat{C}_z \hat{z}_i \frac{\hat{Y}_{i+1}(x)}{\hat{Y}_i(x)} \right) = \sum \prod_{i=0}^{s-1} (\hat{C}_z \hat{z}_i) \hat{\mathcal{X}}_s(x)$$

$$\hat{C}_z = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ z^{-1} & & & 0 \end{pmatrix}$$

$$\frac{z_{i,w+1}}{z_{i,w}} = q_{i,w}$$

$\hat{\mathcal{X}}_s(x) = \mathrm{diag}(\mathcal{X}_{s,0}, \dots, \mathcal{X}_{s,N-1})$

$\langle \mathcal{X}_{s,w}(x) \rangle$ - holomorphic section of deg 1 $\mathcal{L}|_E$.

$$\det(1 - \hat{C}_z \hat{A}) = 1 - z^{-1} \det \hat{A}$$

$\hat{D}(x,z)$ is a deg 1 poly. in x

$$x \hat{D}_0(z) + \hat{D}_1(z) = \hat{D}_0(z)(x - \hat{L}(z))$$

$$\hat{D}_0(z) = \prod_{i=0}^{r+1} (1 - \hat{C}_z \hat{z}_i) \qquad \text{Lax } \hat{L}(z) = \hat{D}_0(z)^{-1} \hat{D}_1(z)$$

$$\det \hat{D}_0(z) = 1 - z^{-1} z_i$$

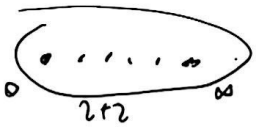
poles of $\hat{L}(z)$ are at $z = z_0, z_1, \dots, z_{r+1}$

$$\hat{L}(z) = \sum_{i=0}^{r+2} \frac{\hat{L}_i}{z - z_i}$$

, $\hat{L}_i$ depends on $(\hat{Z}_i, \hat{A}_i)$

$(N-1) \times r$ dynamical parameters

lax Gaudin-Garnier



mass)

$1 + \underbrace{N-1}_{\text{dynamical}}$ $\quad a_{r,s}$

$r+2 = r + 2$

masses

$$\phi(z) = \frac{dz}{z} \hat{L}(z)$$

$$\Psi = e^{\frac{1}{\varepsilon} S(z_{s,w};\, a_{s,w}, \dots)}$$

Ψ = surface defect expectation value

$\frac{\partial S}{\partial z_{s,w}}$ = entries of u_i, v_i

$$\varepsilon_1 q_{s,w} \frac{\partial \Psi}{\partial q_{s,w}} = \langle a_{s,w} + \varepsilon_2 k_{s,w} \rangle$$