Topics in number theory : Bruhat - Tits theory

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Lecture 1. k nonarch. $\underbrace{\text{local field}}_{\text{complete}} \supset \mathcal{O} = $ val. ring $\twoheadrightarrow f = $ (finite) residue field

$G = $ conn'd reductive gp $/k$ (e.g. $SL_n(D)$, Sp_{2g}, $SO(q)$, $U(h)$, G_2, $\cdots$)

Lemma. Let $H = $ smooth affine k-gp, then $H(k)$ has nbhd basis of 1 consisting of compact open subgroups.

Pf. Choose closed k-subgp inclusion $H \hookrightarrow GL_{N,k}$, so $H(k) \subset GL_N(k)$ is closed subgroup.

$\therefore$ WLOG $H = GL_N$. $GL_N(k) \underset{\substack{\text{open}\\\text{compact}}}{\supset} GL_N(\mathcal{O}) \supset \underbrace{1 + m^i \mathrm{Mat}_N(\mathcal{O})}_{\text{Nbhd basis of } 1}$ $(i \geq 1)$

Rmk. $X = $ affine scheme of f. type $/\mathcal{O}$ (e.g. $GL_{N,\mathcal{O}}$), then $X(\mathcal{O}) \subset X(k)$ is compact open subset : $X \underset{\text{closed}}{\hookrightarrow} \mathbb{A}^N_{\mathcal{O}}$, so WLOG $X = \mathbb{A}^N_{\mathcal{O}}$. ✓.

There is much interest in (usually ∞-dim'l) $\mathbb{C}$-reps V of $G(k)$ s.t.

1) <u>smooth</u> : each $v \in V$ has $\mathrm{Stab}_{G(k)}(v)$ open $\underset{\text{Lemma}}{\supset} K = $ compact open subgp.

   so $\Leftrightarrow V = \bigcup_K V^K$ (directed union : $V^K, V^{K'} \subset V^{\overset{K \cap K'}{\underset{\text{compact open}}{\uparrow}}}$)

2) <u>admissible</u> : $\dim_{\mathbb{C}} V^K < \infty$, $\forall K$.

When classify such <u>irred.</u> V, focus on "biggest" K s.t. $V^K \neq 0$.

How to analyze possibilities for "large" K in $G(k)$?

Prop (Langlands) Each compact subgp of $G(k)$ lies in a max'l one, and all max'l compact subgps are open.

- Pf next time in wider generality, w/ k = "henselian".
- Over $\mathbb{R}$, max'l compact subgps in $G(\mathbb{R})$ are all $G(\mathbb{R})$-conjugate. (See Ch XV of Hochschild's book: Str. of Lie gps)

but for non-arch. local k, often multiple $G(k)$-conj. classes of max'l compact open subgps.

Bruhat-Tits theory gives a way to analyze this.

Ex 1. $GL_n(k)$ has all K conjugate to $GL_n(\mathcal{O}) = GL_{n,\mathcal{O}}(\mathcal{O})$

- classical pf uses $K \cap GL_n(\mathcal{O}) \underset{\text{open}}{\subset} K$ = compact has finite index in K
- Pf next time for "henselian" k

Consider $K = g\, GL_n(\mathcal{O}) g^{-1}$ for some $g \in G(k)$. (usually $\neq GL_n(\mathcal{O})$)

$$(*) \quad G = GL_{n,k} \xrightarrow[c_{g^{-1}}]{\simeq} GL_{n,k} \underset{\text{open}}{\supset} GL_{n,\mathcal{O}}$$

defines an $\mathcal{O}$-structure on G: smooth affine $\mathcal{O}$-gp $\mathcal{G}$ equipped with isom $\alpha: \mathcal{G}_k \simeq G$ $\left(\Rightarrow \mathcal{G}(\mathcal{O}) \underset{\substack{\text{cpt}\\\text{open}}}{\subset} \mathcal{G}(k) \overset{\alpha}{\simeq} G(k)\right)$

Observe $(*)$ yields $\mathcal{O}$-structure $\mathcal{G}$ for G with $\mathcal{G} \underset{\mathcal{O}\text{-gp}}{\simeq} GL_{n,\mathcal{O}}$ but $\alpha: \mathcal{G}_k \simeq G_k = GL_{n,k}$ is NOT id_{G_k}. Unraveling this, $\mathcal{G}(\mathcal{O}) \subset G(k) = GL_n(k)$ is exactly $g\,GL_n(\mathcal{O})g^{-1} = K$.

Rmk. If $(\mathcal{G},\alpha)$, $(\mathcal{G}',\alpha')$ are two $\mathcal{O}$-structures on G, there's at most one

isom $\quad \mathcal{G} \overset{\mathcal{O}}{\simeq} \mathcal{G}'$

s.t. $\quad \mathcal{G}_k \overset{\mathcal{O}_k}{\simeq} \mathcal{G}'_k$, $\quad \alpha: \mathcal{G}_k \simeq G$, $\quad \alpha': \mathcal{G}'_k \simeq G$

If $\mathcal{G}(\mathcal{O}) \neq \mathcal{G}'(\mathcal{O})$ as subgroups of $G(k)$, then $\mathcal{O}$-str's are NOT "same"!

Later see for analogue w/ $f = \bar{f}$, converse holds: $\mathcal{G}(\mathcal{O}) = \mathcal{G}'(\mathcal{O})$ inside $G(k)$,

then $\mathcal{G}' \simeq \mathcal{G}$ as $\mathcal{O}$-structures. (Smoothness $/_\mathcal{O}$ crucial)

a) $\begin{pmatrix} 1 & 0 \\ 0 & \pi^r \end{pmatrix} GL_2(\mathcal{O}) \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-r} \end{pmatrix} = \left\{ \begin{pmatrix} \mathcal{O} & \mathfrak{m}^{-r} \\ \mathfrak{m}^r & \mathcal{O} \end{pmatrix} \subset GL_2(k) \right\}$ $\quad (r \in \mathbb{Z})$

π = uniformizer of $\mathcal{O}$

$$\left\{ \begin{pmatrix} x & \pi^{-r} y \\ \pi^r z & w \end{pmatrix} : x,y,z,w \in \mathcal{O} \right\}$$

$\leadsto$ write down $\mathcal{G}$

Corresponds different $\mathcal{O}$-subalg. of $k[GL_2]$ than $\mathcal{O}[GL_{2,\mathcal{O}}]$.

b) $0 \le m \le n-1$, $\gamma_m = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \pi & \\ & & & & \ddots \\ & & & & & \pi \end{pmatrix}$ (m entries π) $\Rightarrow \gamma_m GL_n(\mathcal{O}) \gamma_m^{-1} = \left\{ \begin{array}{c|cc} & n-m & m \\ n-m & \mathcal{O} & \mathfrak{m}^{-1} \\ m & \mathfrak{m} & \mathcal{O} \end{array} \right\} \subset GL_n(k)$

depends on m.

When $K = \mathcal{G}(\mathcal{O})$ for an $\mathcal{O}$-str. $\mathcal{G}$ of G, get interesting compact open subgps of K

from subgp structure of $\mathcal{G}_f$:

$$
\begin{tikzcd}
GL_n(k) \supset K_\theta = GL_n(\theta) \arrow[r, two heads] \arrow[d, equal] & GL_n(f) \arrow[d, equal] \\
GL_{n,\theta}(\theta) \arrow[r, two heads] & GL_{n,\theta}(f) \\
1 + m\,\mathrm{Mat}_n(\theta) \subset \underbrace{\mathcal{H}}_{} \arrow[u, phantom, "\cup" description] \arrow[r] \arrow[ur, phantom, "\llcorner", very near start] & P(f) \arrow[u, phantom, "\cup" description]
\end{tikzcd}
$$

Surjectivity manifestation of θ-smoothness of $GL_{n,\theta}$.

for parabolic f-subgps $P \subset GL_{n,f}$.

these will be examples of parahoric subgps of $G(k)$.

(Rmk. For $P \neq Q$ parabolic f-subgps of $GL_{n,f}$, $P(f) \neq Q(f)$)

Rmk. For probing structure of $\mathcal{G}_f$, often need to work over $\bar{f}$ in pts. This is one reason not to limit to finite ~~fs~~.

Ex2. $G = SL_{n,k}$ has n-conj. classes of max'l compact subgps in $G(k)$: rep'td by $K_m = \gamma_m G(\theta) \gamma_m^{-1} \subset SL_n(k)$ for $\gamma_m = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \pi & & \\ & & & \ddots & \\ & & & & \pi \end{pmatrix}$ (m entries π), $\gamma_m \notin G(k)$ (not obvious)

Lecture 2. Action on Building

Bruhat - Tits theory provides a metric space $\mathcal{B}(G)$ (building of G), w/ an isometric action of $G(k)$ having open stabilizers $G(k)_x$, and satisfies "negative curvature" ensuring fixed pt thm $\Rightarrow$ every compact subgroup $K \subset G(k)_x$ for some x.

If $S \subset G$ is max. split torus, and $S' = (S \cap \underset{\substack{\uparrow \\ \text{derived group}}}{DG})^\circ_{red}$ be associated max'l split torus in DG.

then $\mathcal{B}(G) \supset \mathcal{A}(S)$ = apartment attached to S

= affine space for $X_*(S')_{\mathbb{R}}$

Compatibly w/ action of $N_G(S)(k)$, and $g.\mathcal{A}(S) = \mathcal{A}(gSg^{-1})$, and these cover $\mathcal{B}(G)$.

Rmk. For $G = ss$, $G(k)_x$ are compact, so max'l compact subgps are among these $G(k)_x$'s.

When G is also simply conn'd (e.g. SL_n, Sp_{2g}, not $SO(g)$, not PGL_n) then max'l compacts are exactly $G(k)_x$'s for x those are vertices in the apartments.

$G = \prod_i G_i$, G_i simply conn'd, k-simple ($G_i = R_{k_i'/k}(G_i')$)

finite separable; abs. simple $/k_i'$ s. connected

$S = \prod_i S_i$ $\Leftrightarrow$ $\Phi(G_i, S_i)$ is irred (or empty)

max'l split torus in G_i

$r_i = k$-rank of $G_i = \dim(S_i) = \dim \Phi(G_i, S_i)$

Max. compact $K \subset G(k)$ $= \prod K_i$

max'l compact in $G_i(k)$ $(= G_i'(k_i'))$

BT theory will yield # of conj. classes of $K = \prod(r_i + 1)$ (G = s. conn'd)

For $G = SL_n$, $r = n-1 \Rightarrow r+1 = n$, So the n examples in $SL_n(k)$ at the end of Lecture 1 have no $SL_n(k)$- conj. among them.

<u>Visual meaning</u> via building

Consider (G, S) split (S max'l k-torus in G) (e.g. $SL_n, PGL_n, Sp_{2g}, \dots$)

$$S' = (S \cap \underset{\substack{\| \\ G'}}{\mathcal{D}G})^{0}_{red}$$

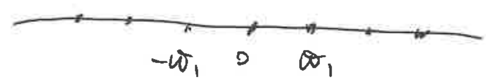
$$\mathcal{A} = X_*(S')_{\mathbb{R}} \curvearrowleft W_{aff} = \langle \text{reflections in the hyperplanes} \rangle, \quad \Phi(G,S) = \Phi(G', S') \subset X^*(S')_{\mathbb{R}}$$

$$\cup$$

$$\{\text{hyperplanes } a(x) = k \text{ for } k \in \mathbb{Z}, a \in \Phi\}$$

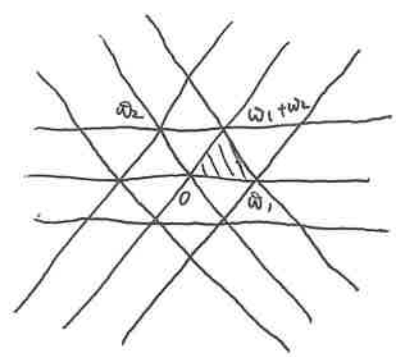
$\mathcal{C} \subset \mathcal{A}$ to be (open) fundamental domain for W_{aff}.

SL_2:



The "corners" of chamber closures in apartments are called vertices.

SL_3



For G <u>s.conn'd</u>, max'l compacts are exactly $G(k)_x$ for $x \in B(G)$ a vertex.

If $\mathcal{C} \subset \mathcal{A}(S)$ is a chamber, then

$\{G(k)_x\}_{x \in vert(\bar{\mathcal{C}})}$ are representatives of $G(k)$-conj. classes of max'l compact subgps w/o repetition.

The examples $K_m = \gamma_m SL_n(\mathcal{O}) \gamma_m^{-1}$ are exactly these for a specific chamber in $\mathcal{B}(SL_n)$

<u>Rmk</u>. $G' = DG = k$-anisotropic $:= (S' = 1)$

$$\begin{array}{c} \Updownarrow \text{ (later)} \\ G'(k) \text{ is } \underline{\text{compact}}. \end{array} \qquad \begin{array}{l} \Leftrightarrow \ \mathcal{A}(S) = pt \\ \Leftrightarrow \ \mathcal{B}(G) = pt \end{array}$$

Naturally. $\mathcal{B}(G) = \mathcal{B}(G^{ad} = G/Z_G)$

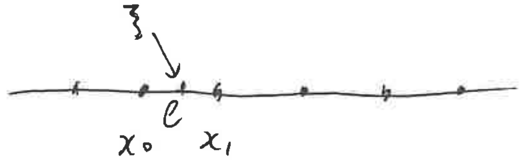
$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ G(k) & \longrightarrow & G^{ad}(k) \end{array} \ \Big] \text{ enrichment of } G(k)\text{-action}$$

$$\begin{array}{ccc} \mathcal{B}(SL_n) & = & \mathcal{B}(PGL_n) \\ \circlearrowleft & & \circlearrowleft \\ SL_n(k) & \longrightarrow & PGL_n(k) \end{array} \ \Big] \text{ coker } \overset{\det}{=} k^\times/(k^\times)^n$$

All $K_m \subset SL_n(k)$ land in subgps of conjugates of $PGL_n(\mathcal{O})$.

$\left(SL_n(\mathcal{O}) \longrightarrow PGL_n(\mathcal{O}) \text{ has coker } \mathcal{O}^\times/(\mathcal{O}^\times)^n \right)$

<u>Case $n=2$</u>.



$$\begin{array}{c} PGL_2(k)_{x_0} \\ \parallel \\ PGL_2(\mathcal{O}) \end{array}$$

$$PGL_2(k)_{x_1} = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} PGL_2(\mathcal{O}) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}^{-1}$$

$\begin{pmatrix} 0 & \pi^{-1} \\ -1 & 0 \end{pmatrix}$ flips e: swaps x_0, x_1, $\widetilde{K} = PGL_2(k)_\xi$ is a "new" max'l compact.

$= \left\langle \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{m} & \mathcal{O} \end{pmatrix}, \begin{pmatrix} 0 & \pi^{-1} \\ -1 & 0 \end{pmatrix} \right\rangle$

BT theory provides $\mathcal{O}$-structure $\tilde{\mathcal{G}}$ for $\tilde{K}$, w/ $\tilde{\mathcal{G}}_F$ is disconnected (order 2)

$$\text{w/ } \tilde{\mathcal{G}}_f^0 = \mathbb{G}_m \ltimes (\mathbb{G}_a \times \mathbb{G}_a)$$

$$t \cdot (x, y) = (tx, \bar{t}y)$$

$$\left(\text{so } \tilde{K} = \tilde{\mathcal{G}}(\mathcal{O}) \text{ is pro-solvable} \right)$$

Warning. $PGL_3(k)$ also has 2 conj. classes!

Lecture 3. Henselian fields and Boundness

Need to go beyond local fields.

Ex. Later on, having $f = \bar{f}$ will be convenient.

For $K = \mathbb{Q}_p$, have unramified extensions

$$\begin{array}{ccc} K = \mathbb{Q}_p & \mathbb{Q}_p(\zeta_n) \text{ for } p \nmid n. & \text{equivalently, } \mathbb{Q}_p(\zeta_{p^r-1}) \\ \cup & \cup & \cup \\ \mathcal{O} = \mathbb{Z}_p & \mathbb{Z}_p[\zeta_n] & \mathbb{Z}_p[\zeta_{p^r-1}] \end{array}$$

to get

$$\begin{array}{l} \mathbb{Q}_p^{un} = \bigcup_{p \nmid n} \mathbb{Q}_p(\zeta_n) = \bigcup_{r > 0} \mathbb{Q}_p(\zeta_{p^r-1}) \Big\} \text{ directed unions} \\ \cup \\ \mathbb{Z}_p^{un} = \bigcup_{p \nmid n} \mathbb{Z}_p[\zeta_n] = \bigcup_{r > 0} \mathbb{Z}_p[\zeta_{p^r-1}] \Big\} \text{ dvr w/ residue field } \bar{\mathbb{F}}_p \text{ and uniformizer } p. \end{array}$$

Which is not complete, $\because$ $\sum_{n \geq 0} \zeta_{p^n-1} \cdot p^n$ $\circlearrowleft$ not stable by any open subgp of $\mathrm{Gal}(\mathbb{Q}_p^{un} | \mathbb{Q}_p)$

But $\mathbb{Z}_p^{un}$ satisfies Hensel's Lemma in strong form (as for any complete DVR).

$f \in R[X]$ monic w/ reduction $f_0 = g_0 h_0$ for monic coprime g_0, h_0 over residue field. Then uniquely lifts to $f = gh$ w/ monic $g, h \in R[X]$.

Idea. $A = R[X]/f$ has $A/pA = k[X]/(f_0) \simeq k[X]/(g_0) \times k[X]/(h_0)$

and lift $e = (1,0)$ from A/p to idempotent in A by $\underbrace{\text{successive approx.}}$

when R is complete DVR

$\Downarrow$

A also p-adically separated + complete

Since f arises "at finite level" over $\mathbb{Z}_p$ (i.e. $f \in \mathbb{Z}_p[\zeta_n][X]$ for some $p \nmid n$)

can run complete case there; and uniqueness at all higher levels $\Rightarrow$!ness over $\mathbb{Z}_p^{un}$.

But $\hat{\mathbb{Z}}_p^{un} = W(\overline{\mathbb{F}}_p)$ has fraction field not algebraic over $\mathbb{Q}_p$.

All above works idem for any complete DVR $\mathcal{O}$ in place of $\mathbb{Z}_p$ w/ tr. field k

due to: $\left\{ \begin{array}{l} \text{finite unram. field} \\ \text{ext } k'/k \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \text{finite separable} \\ f'/f \text{ of res. field} \end{array} \right\}$

Key pt: f'/f finite separable, $f' = f[X]/(g_0)$ for monic g_0.

Pick $g \in \mathcal{O}[X]$ monic lift of g_0, so $\mathcal{O}' := \mathcal{O}[X]/(g)$ is DVR $\left(\mathcal{O}'/\pi\mathcal{O}' = f[X]/(g_0) = f'\right)$

= finite local (free) $\mathcal{O}$-alg.

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}', \mathcal{O}'') \xrightarrow[\sim]{\text{Hensel}} \mathrm{Hom}_{f}(f', f'' = \mathcal{O}''/\pi\mathcal{O}'')$$

root of g in $\mathcal{O}''$; $\mathcal{O}''$: finite unram. DVR over $\mathcal{O}$; root of g_0 in $\mathcal{O}''$

Using above, if fix separable closure $f_s | f$ and for each $f' \subset f_s$ of finite f-degree, build a corresponding $\mathcal{O}'/\mathcal{O}$ ($!$ up to $\mathcal{O}$-alg. isom).

to get $\mathcal{O}^{un} := \varinjlim \mathcal{O}'$ (such $\mathcal{O}'$ form directed system lifting that of $f' \subset f_s$)

This $\mathcal{O}^{un}$ is called max'l unramified extn, has same unif. as $\mathcal{O}$, res field f_s, <u>not</u> complete (usually), but satisfies strong form of Hensel's Lemma.

Check for inertia subgroup $I_k \subset \mathrm{Gal}(k_s|k)$,

$$(k_s)^{I_k} = \mathrm{Frac}(\mathcal{O}^{un}) =: k^{un} \quad (k \text{ any complete discretely valued field})$$

<u>Later</u>. Deep Thm of Steinberg on non-abelian Galois cohom. of such k^{un}.

Advantage of unramifiedness is "étale descent $/\mathcal{O}$" is Galois descent unlike finite flat descent w/ ramified k'/k.

<u>Prop</u>. Let R be DVR, fr. field K, TFAE:

① valuation on K extends $!$ly to all algebraic extns $k'|K$. ($\Leftrightarrow$ all finite separable)

② R satisfies strong form of Hensel's Lemma.

③ R is henselian in sense of EGA IV_4, 18.5.11.

Pf. ① ⟺ ② is Prop 2.4.3 in Ber IHES (1993)

② ⟺ ③ is EGA IV_4, Rem 18.5.13

Def. Say K is <u>henselian</u> when these hold.

If K is henselian, so is any alg. extn $K'|K$ on which lifted valuation is <u>discrete</u> (or: remove "discreteness" in Prop).

Via ② in prop, any henselian K has K^{un} w/ residue field k_s

(R^{un} is also "strict henselian")

Have $I_K \subset G_K$ associated to K^{un}.

Prop. For K henselian, $L \longmapsto L \otimes_K \hat{K}$ is an equivalence of categories

$$\left\{ \begin{matrix} \text{finite separable} \\ \text{fields } |K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{finite separable} \\ \text{fields } |\hat{K} \end{matrix} \right\}.$$

In particular, $\mathrm{Gal}(k_s|k) \xrightarrow{\sim} \mathrm{Gal}((\hat{k})_s|\hat{k})$.

Pf. Prop 2.4.1 in Ber IHES.

In the henselian setting, what replace compactness for subgroups of $G(K)$ for finite type affine K-gps G?

Lemma. Let K be discretely valued field, Let X be affine f.type $/K$.

$j : X \hookrightarrow \mathbb{A}^n_K$ closed imm.

For $B \subset X(k)$, TFAE

① $j(B) \subset k^n$ is bounded (for the valuation)

② $\forall h \in k[X]$, $h(B) \subset X$ is bounded.

Pf. $k[X]$ is generated as k-alg. by component functions of j.

Def Such B are called bounded.

Ex. If k = local field (complete, f finite)

then $B \subset X(k)$ bounded $\Leftrightarrow$ $\bar{B}$ = compact.

Ex. If $X = GL_n$, $B \subset GL_n(k)$ is bounded in the sense of defn.

$\underset{\text{exercise}}{\Leftrightarrow}$ $B, B^{-1} \subset \mathrm{Mat}_n(k)$ are bounded.
(use Cramer)

For subgp $B \subset GL_n(k)$, boundedness in GL_n $\Leftrightarrow$ boundedness in Mat_n.

Lecture 4. More on bounded subgroups

Now k to be henselian (discretely-valued) field, general residue field f.

Here are some basic properties of boundedness.

Prop. ① If $f: X \to Y$ is a k-map between affine k-schemes of finite type, and $B \subset X(k)$ is bounded (in X), then $f(B) \subset Y(k)$ is bounded.

② For G an affine k-group of finite type, and $B, B' \subset G(k)$ are bounded, then $BB' = \{bb' : b \in B, b' \in B'\} \subset G(k)$ is bounded.

③ If $X = R_{k'/k}(X')$ for finite separable k'/k, X' = affine of finite type over k' (= henselian), then the bijection $X(k) = X'(k')$ preserves boundedness both ways.

Rmk. (i) Importance of ③ is for G = s. conn'd or adjoint type, always being

$$G = \prod_i R_{k_i'/k}(G_i') \text{ for } G_i' \text{ that are abs. simple } / k_i'$$

(and s. conn'd, resp. adjoint type)

(ii) Statement of ③ in [KP, Lemma 2.2.5] is missing henselian conditions on k.

Pf. ① For $h \in k[Y]$, want $h(f(B)) \subset k$ to be bounded.

But $h(f(B)) = \underbrace{(f^*(h))}_{\in k[X]}(B)$ and B is bounded in X.

② Applying ① to $B \times B' \subset (G \times G)(k)$ and $f : G \times G \to G$ the mult map.

③ Pick closed immersion $X' \hookrightarrow \mathbb{A}^n_{k'}$, so get

$X = R_{k'/k}(X') \longrightarrow \mathrm{Res}_{k'/k}(\mathbb{A}^n_{k'})$ also closed immersion,

$\simeq (\mathbb{A}_k)^{nd}$ $\quad d = [k' : k]$

use a k-basis of k'

Reduces us to the case $X' = \mathbb{A}^n_{k'}$.

A subset $B \subset X'(k') = (k')^n$ is bounded $\Leftrightarrow$ image in each factor is bdded.

More generally, a subset of $(Y_1 \times Y_2)(k')$ is bounded in $(Y_1 \times Y_2)$

$\Leftrightarrow$ images in $Y_1(k')$, $Y_2(k')$ are bounded

$\left(k'[Y_1 \times Y_2] = k'[Y_1] \otimes_{k'} k'[Y_2] \right)$. Similarly over k.

$\therefore$ WLOG $X' = \mathbb{A}^1_{k'}$. Pick k-basis of k', so get $X \cong \mathbb{A}^d_k$.

So task reduces to showing a subset of k' is bounded for valuation topology.

$\Leftrightarrow$? it is bounded for k-vec. sp. top. (using valuation of k)

Want valuation norm on k' is "equivalent" (bdd both ways w/ constant multiples)

For <u>complete</u> k this is true $\because$ only <u>one</u> complete norm on any f. dim'l k-vector space.

To reduce to the complete case, look at $\hat{k} \otimes_k k' \xrightarrow{\sim} \widehat{k'}$ (as $\hat{k}$-algebras)

$$\begin{tikzcd} \hat{k} \otimes_k k' \arrow[r, "\sim", "(k'|k \text{ separable})"'] & \widehat{k'} \\ k' \arrow[u, "\text{topologies arising from } k\text{-basis's of } k' \text{ are compatible}"] \arrow[r, equal] & k' \arrow[u, "k \text{ henselian}"', "\text{val. top. are compatible}"] \end{tikzcd}$$

Apply "uniqueness up to equivalence" for complete norms on f. dim'l $\hat{k}$-vector spaces.

Let's now turn to bounded subgps of $G(k)$ for affine k-gp G of f. type.

<u>Thm</u> (Bruhat–Tits, Rousseau) If G is conn'd reductive $/k$ = henselian,

$G(k)$ is bounded (in G) $\Leftrightarrow$ G is k-anisotropic

(no nontrivial k-split tori in G)

<u>Pf</u> Elegant proof by Prasad in [KP, Thm 2.2.9]

Key case is $\underline{\underline{ss}}$ G, since

1) $G \xrightarrow[\text{isogeny}]{} \underbrace{G^{ad}}_{ss} \times \underbrace{(G/\mathcal{D}G)}_{\text{torus}}$, $\underbrace{G^{sc}}_{ss} \times \underbrace{(Z^{\circ}_{G})_{red}}_{\text{torus}} \xrightarrow[\text{isogeny}]{} G$

2) tori can be treated directly via torus being an "isogeny factor" of $\prod_i R_{k_i/k}(\mathbb{G}_m)$ for finite separable $k_i | k$.

3) For $\underline{\underline{\text{finite}}}$ $\varphi: X \longrightarrow Y$ between affine k-schemes of f. type and $B \subset Y(k)$ is bounded, then $\varphi^{-1}(B) \subset X(k)$ is bounded.

<u>Pf.</u> $k[Y] \xrightarrow{\varphi^*} k[X]$ is module-finite, so each of a finite set of k-alg. generators of $k[X]$ satisfies a <u>monic</u> poly. relation over $k[Y]$. Then use bound on $|\text{roots}|$ of a poly. of given degree d with given bounds on $|\text{coeffs}|$.

Rmk. In item (3), finiteness is crucial, consider

$$GL_n \underset{\text{open}}{\hookrightarrow} \mathrm{Mat}_n .$$

From now on k is henselian.

Prop. (Langlands) Let G be conn'd reductive k-gp.

Then every bounded subgp $\mathcal{G} \subset G(k)$ lies in a maximal one, and max'l ones are open.

Pf: Let's show $\mathcal{G}$ lies in a bounded <u>open</u> subgp, so then can focus on <u>open</u> $\mathcal{G}$.

Pick closed immersion of k-gps $G \xrightarrow{j} GL_n$

$$\text{So } \mathcal{G} \subset G(k) \subset GL_n(k)$$

is bounded in $GL_n(k)$.

If $\mathcal{G} \subset GL_n(k)$ lies in <u>open</u> bounded subgp $\mathcal{U} \subset GL_n(k)$,

then $G(k) \cap \mathcal{U} = j^{-1}(\mathcal{U})$ is <u>bounded</u> open $\supset \mathcal{G}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \hookrightarrow j$ finite.

We'll show any bounded subgp $\mathcal{G} \subset GL_n$ preserves an $\mathcal{O}$-lattice Λ. $\mathcal{G} \subset \underbrace{GL(\Lambda)}_{\text{bdd, open}} \subset GL_n(k)$

Lecture 5. Maximal bounded subgroups

k = henselian ($\Rightarrow$ discrete valued)

Let $\mathcal{G} \subset GL_n(k)$ be bounded, seek an $\mathcal{O}$-lattice $\Lambda \subset k^n$ stable under $\mathcal{G}$

(so $\mathcal{G} \subset \underline{GL(\Lambda)} \subset GL_n(k)$)

$\uparrow$ bdd open

conjugate to $GL_n(\mathcal{O})$

since $\Lambda = \gamma \cdot (\mathcal{O}^n)$ for

some $\gamma \in GL_n(k)$

Once done, saw last time that for any conn'd reductive k-gp G and bounded subgp $\mathcal{G} \subset G(k)$, $\exists$ $\underline{\text{bounded}}$ $\underline{\text{open}}$ subgp $\mathcal{U} \subset G(k)$ s.t. $K \subset \mathcal{U}$.

Consider action $\quad GL_n \times \mathbb{A}^n_k \xrightarrow{\alpha} \mathbb{A}^n_k$.

$$\begin{array}{c} GL_n \times \mathbb{A}^n_k \\ \cup \\ GL_n(k) \times k^n \\ \cup \\ \mathcal{G} \times \Lambda_0 \;\;] \text{ bounded} \end{array}$$

Pick $\Lambda_0 = \mathcal{O}^n \subset k^n$

We conclude $\alpha(\mathcal{G} \times \Lambda_0) \subset k^n$ is $\underline{\text{bounded}}$.

$$\begin{array}{c} \alpha(\mathcal{G}\times\Lambda_0) \\ \shortparallel \\ \Lambda := \mathcal{G}\cdot\Lambda_0 \end{array}$$

$\Lambda_0 \overset{1\in\mathcal{G}}{\subset} \Lambda \subset \frac{1}{\pi^N}\Lambda_0$ since $\Lambda \subset k^n$ is bounded.

and Λ is $\mathcal{O}$-submodule of k^n, so Λ is an $\mathcal{O}$-lattice in k^n, $\mathcal{G}$-stable by design.

Back to conn'd reductive G over k and bounded open subgp $\mathcal{G} \subset G(k)$.

Seek maxim'l bounded subgp $\tilde{\mathcal{G}} \subset G(k)$ containing $\mathcal{G}$ ($\Rightarrow \tilde{\mathcal{G}}$ open)

By Zorn's lemma, suffices to show for a chain $\{\mathcal{G}_\alpha\}$ of bounded subgps of $G(k)$ containing $\mathcal{G}$, $\bigcup_\alpha \mathcal{G}_\alpha =: \mathcal{H}$ (= subgp!) is bounded.

Note $\mathcal{H}$ is open $\because \mathcal{H} \supset \mathcal{G}$. This ensures $\mathcal{H} \subset G$ is Zariski-dense:

Lemma. For X a smooth conn'd affine k-schemes, any non-empty open $\mathcal{U} \subset X(k)$ is Zar-dense in X.

Pf. (k-complete) X = smooth, conn'd $\Rightarrow$ irred., reduced $\rightarrow k[X]$ is domain.
Enough to show if $f \in k[X]$ with $f(\mathcal{U}) = \{0\}$, then $f = 0$.

Pick $u_0 \in \mathcal{U}$, so $k[X] \hookrightarrow k[X]_{m_{u_0}} = \mathcal{O}_{X,u_0} \hookrightarrow \hat{\mathcal{O}}_{X,u_0}$

Assume k complete, so $X(k)$ has structure of naive k-analytic manifold X^{an},

$\mathcal{O}_{X,u_0} \longrightarrow \mathcal{O}_{X^{an},u_0} = \mathcal{O}_{\mathcal{U}^{an},u_0}$ inducing isomorphisms $\hat{\mathcal{O}}_{X,u_0} = \hat{\mathcal{O}}_{X^{an},u_0} = \hat{\mathcal{O}}_{\mathcal{U}^{an},u_0}$

But $f|_{\mathcal{U}} = 0 \Rightarrow f$ has vanishing image in $\mathcal{O}_{X^{an},u_0} \subset \hat{\mathcal{O}}_{X^{an},u_0} = \hat{\mathcal{O}}_{X,u_0}$,
$f = 0$. $\square$

Assume $\mathcal{H}$ is unbounded, seek contradiction. In Prasad's pf of B-T-R Thm, uses reductivity to deduce [KP, Lemma 2.2.11] via Jacobson Density Thm that an unbounded $\mathbb{Z}$-dense subgp of $G(k)$ contains $\gamma \in G(k)$ s.t.

γ has non-integral eigenvalue in $\bar{k}$ wrt. chosen $G \hookrightarrow GL_{N,k}$.

Conversely, any subgp $\mathcal{G}' \subset G(k)$ containing such a γ <u>must</u> be unbdd:

if $\mathcal{G}'$ were bounded in G, hence in GL_N, we'd have uniform bounds

on $|\text{coeffs}|$ for char. polys on $\mathcal{G}'$, so get uniform bounds on all

$|\text{roots in } \bar{k}| = |\text{eigenvalues in } \bar{k}|$.

But γ^m has λ^m as eigenvalue, and $|\lambda^m| = |\lambda|^m \to \infty$, as $m \to \infty$ $(|\lambda| > 1)$

But $\mathcal{H} = \bigcup_\alpha \mathcal{G}_\alpha$, so $\gamma \in \mathcal{H}$ lies in some $\mathcal{G}_\alpha$, so that $\mathcal{G}_\alpha$ also unbdd. $\square$

<u>Method of pf yields</u>:

<u>Cor</u>: max'l bounded subgps of $GL_{n,k}$ are exactly conjugates of $GL_n(\mathcal{O})$.

<u>Pf</u>. Know max'l bdd subgp $K \subset GL_n(k)$ exists. But $K \subset \underbrace{GL(\Lambda)}_{\text{bdd}}$ for

some $\mathcal{O}$-lattice $\Lambda \subset k^n$, so $K = GL(\Lambda)$ by maximality.

$\therefore$ $GL(\Lambda)$ is max'l for some Λ. $\square$

For $\mathcal{X}$ an affine (flat) $\mathcal{O}$-scheme of f. type w/ $X = \mathcal{X}_k$,

$\mathcal{X}(\mathcal{O}) \subset X(k)$ is bounded (and open): use $\mathcal{X} \hookrightarrow \mathbb{A}^N_{\mathcal{O}}$ closed immersion

to define choice of $X \hookrightarrow \mathbb{A}^N_k$ over k, and $\mathcal{X}(\mathcal{O}) = X(k) \cap \mathbb{A}^N_{\mathcal{O}}(\mathcal{O})$

$= X(k) \cap \mathcal{O}^N \subset k^N$] bdd, open

The preceding Corollary can be used to prove:

Prop. If G is conn'd reductive $/k$, $K \subset G(k)$ is max'l bounded, then $K = \mathcal{G}(\mathcal{O})$ for some (smooth) $\mathcal{O}$-structure $\mathcal{G}$ of G.

Rmk ① For any $\mathcal{O}$-structure, $\mathcal{G}(\mathcal{O}) \subset G(k)$ is bounded open subgp. These often not max'l: given such $\mathcal{G}$, consider

$$K_1 = \ker(\mathcal{G}(\mathcal{O}) \to \mathcal{G}(f)) = \mathcal{G}^+(\mathcal{O}),$$

for $\mathcal{G}^+ =$ (smooth) dilatation of $\mathcal{G}$ along $e_0 \in \mathcal{G}_f$

$$= \mathrm{Spec}\left(\mathcal{O}[\mathcal{G}]\left[\frac{I_{e_0}}{\pi}\right]\right).$$

(affine open in $\mathrm{Bl}_{e_0}(\mathcal{G})$)

($\Leftarrow$ $\mathcal{G}_f$ is conn'd reductive)

② BT theory will yield that for $\mathcal{G}$ a reductive $\mathcal{O}$-gp, $\mathcal{G}(\mathcal{O}) \subset G(k)$ is max'l bounded.

Lecture 6. Boundedness and $\mathcal{O}$-structures

Prop $G =$ conn'd reductive $/k$, $K \subset G(k)$ max'l bounded. Then $K = \mathcal{G}(\mathcal{O})$ for some (smooth!) $\mathcal{O}$-structure $\mathcal{G}$ of G.

Rem: 1) For typical $\mathcal{O}$-structure $\mathcal{G}$ of G, $\mathcal{G}(\mathcal{O}) \subset G(k)$ is NOT max'l, but it is for $\mathcal{G}$ reductive (i.e. $\mathcal{G}_f^0$ reductive): called hyperspecial.

2) If $\mathcal{G}, \mathcal{G}'$ are reductive models of G, then $G^{ad}(k) \curvearrowright G$ does carry one to the other (at least for $k = \hat{k}$, f finite), yet $G(k) \to G^{ad}(k)$ usually NOT surjective. (e.g. $G = SL_2$, $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ - conjugation)

Pf: Pick closed immersion of k-grps $\rho: G \to GL_{n,k}$, so

$K \subset G(k) \overset{\rho}{\subset} GL_n(k)$ is bounded subgp, so by replacing ρ w/ $GL_n(k)$-conjugate, can rearrange $K \subset GL_n(\mathcal{O})$.

Consider scheme-closure $\mathcal{G}'$ of $G \hookrightarrow GL_{n,k} \xhookrightarrow{\text{open}} GL_{n,\mathcal{O}}$:

$\mathcal{O}[\mathcal{G}'] = \mathrm{im}(\mathcal{O}[GL_n] \to k[G])$; since $\mathcal{O}$ is Dedekind, $\mathcal{G}'$ is $\mathcal{O}$-flat, and then an $\mathcal{O}$-subgp of $GL_{n,\mathcal{O}}$ (check!)

By design, $\mathcal{G}'_k = G$ inside $GL_{n,k}$.

We have $\mathcal{G}'(\mathcal{O}) = GL_n(\mathcal{O}) \cap G(k)$ inside $GL_n(k)$

$\supset K$ inside $G(k)$, where K is max'l bounded

yet $\mathcal{G}'(\mathcal{O})$ is certainly bounded (in $G(k)$).

So $K = \mathcal{G}'(\mathcal{O})$ inside $G(k)$. By [BLR, §7.1, Lemma 4 - Thm 5], $\exists$ affine f. type $\mathcal{O}$-gp map $\mathcal{G} \to \mathcal{G}'$ s.t. • $\mathcal{G}$ is $\mathcal{O}$-smooth, • $\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}'_k = G$, w/ $\mathcal{G}(\mathcal{O}) \simeq \mathcal{G}'(\mathcal{O})$ (even $\mathcal{O}^{un}$-pts)

Built via finitely many "dilatations" along closed subschemes of special fiber.

("gp smoothening"). $\square$

Because: this $\mathcal{G}$ may have $\mathcal{G}_f$ disconn'd. (so lucky accident that $\mathcal{G}_f$

conn'd when $\mathcal{G}_f^\circ$ is reductive: [C, Prop 3.1.12]

$G \to S$ smooth affine and all G_s° are reductive, then

$\{s \in S : G_s \text{ conn'd}\} \subset S$ is closed, so if S irred., then

G_η conn'd $\Rightarrow$ all G_s are conn'd.

If $\mathcal{G} \to \mathrm{Spec}(\mathcal{O})$ is flat gp of f. type, and $\mathcal{G}_k$ is conn'd, then

$\mathcal{G}$ is conn'd as top. space, even if $\mathcal{G}_f$ is disconn'd.

Consider $\mathcal{G} \to \mathrm{Spec}(\mathcal{O})$ smooth affine gp w/ $\mathcal{G}_k$ conn'd. Then union

of fibral identity components: in closed $\mathcal{G}_f \subset \mathcal{G}$, removing closed locus of

non-id (conn'd) components.

This union is open in $\mathcal{G}$, called $\mathcal{G}^\circ \subset \mathcal{G}$.

(w/ evident open subscheme str. an $\mathcal{O}$-subgp). This is affine:

(Raynaud)

Prop: If R = dvr, $K = \mathrm{Frac}(R)$, $\mathcal{H} \to \mathrm{Spec}(R)$ is sep'td flat gp of

finite type, then $\mathcal{H}_K$ = affine $\Rightarrow$ $\mathcal{H}$ is affine.

Pf. See [Prasad–Yu, Prop 3.1] – overkill for $\mathcal{G}$ smooth $\because$

in smooth case, hard Appendix ref in pf is not needed.

Direct pf that $\mathcal{G}^\circ$ is affine for smooth affine R-gp $\mathcal{G}$

(w/ $\mathcal{G}_K$ conn'd)

Observe: for R-algebra A,

$$\mathcal{G}^\circ(A) = \{ g \in \mathcal{G}(A) : g \bmod \pi \in \mathcal{G}(A/\pi A) \text{ is in } \mathcal{G}_f^\circ(A/\pi A) \}$$

R-flat A $\curvearrowright$ $= \mathcal{Z}(A)$

for $\mathcal{Z} = \mathrm{Spec}\left(R[\mathcal{G}]\left[\frac{I_{\mathcal{G}_f^\circ}}{\pi}\right] \right)$ = affine.

$\nearrow$

R-flat

Q: Given smooth affine k-scheme X of f. type, and smooth affine $\mathcal{O}$-schemes $\mathfrak{X}, \mathfrak{X}'$ of f. type w/ $\mathfrak{X}_k \simeq X = \mathfrak{X}'_k$ and $\mathfrak{X}(\mathcal{O}) = \mathfrak{X}'(\mathcal{O})$.

then can $\mathfrak{X} \neq \mathfrak{X}'$? (i.e. $\mathcal{O}[\mathfrak{X}] \neq \mathcal{O}[\mathfrak{X}']$ inside $k[X]$).

Yes: if $\mathfrak{X}_f$ has a conn'd comp. $\overline{\mathcal{C}}$ w/ no f-pts, use $\mathfrak{X}' = \mathfrak{X} - \mathcal{C}$

(= affine)

Prop: $\mathfrak{X}, \mathcal{Y}$ smooth affine $\mathcal{O}$-schemes w/ k-fibers X, Y resp.

Assume $\mathfrak{X}(f) \subset \mathfrak{X}_f$, $\mathcal{Y}(f) \subset \mathcal{Y}_f$ are Z-dense (e.g. $f = f_s$ by smoothness)

Then a k-map $\varphi: X \to Y$ extends to $\mathcal{O}$-map $\varphi': \mathfrak{X} \to \mathcal{Y} \iff \varphi(\mathfrak{X}(\mathcal{O})) \subset \mathcal{Y}(\mathcal{O})$.

Rmk. For $f = f_s$, say a bounded subset in $X(k)$
$\quad\hookleftarrow$ smooth affine k-scheme

is <u>schematic</u>, if $B = \mathfrak{X}(\mathcal{O})$ for an $\mathcal{O}$-str. $\mathfrak{X}$ of X (then unique!)

Pf. $(\Rightarrow)$ ✓

$(\Leftarrow)$ Given $\varphi(\mathfrak{X}(\mathcal{O})) \subset \mathcal{Y}(\mathcal{O})$ inside $Y(k)$ and want

$\varphi^* : k[Y] \to k[X]$ carries $\mathcal{O}[\mathcal{Y}]$ into $\mathcal{O}[\mathfrak{X}]$.

For $h \in \mathcal{O}[\mathcal{Y}]$, consider $\varphi^*(h) \in k[X]$.

For $\tilde{x} \in \underset{\subset X(k)}{\mathfrak{X}(\mathcal{O})}$, $(\varphi^* h)(\tilde{x}) = h_k(\underbrace{\varphi(\tilde{x})}_{\in \mathcal{Y}(\mathcal{O})}) \in k$ is inside $\mathcal{O}$

Remains to show if $\alpha \in k[X]$ has $\alpha(\mathfrak{X}(\mathcal{O})) \subset \mathcal{O}$, want to deduce $\alpha \in \mathcal{O}[\mathfrak{X}]$. Can write $\alpha = \frac{A}{\pi^n}$ for $A \in \mathcal{O}[\mathfrak{X}]$, $n \geq 0$

$n = 0$, done. $\qquad (k[X] = \mathcal{O}[\mathfrak{X}][\frac{1}{\pi}])$

<u>$n \geq 1$</u>. $\pi^n \alpha = A \in \mathcal{O}[X]$

$\mathfrak{X}$-smooth, $\mathcal{O}$ = hens $\Rightarrow$ $\mathfrak{X}(\mathcal{O}) \twoheadrightarrow \mathfrak{X}(f)$.

Consider $A_0 \in f[\mathfrak{X}_f]$: A_0 on $\mathfrak{X}(f)$ is reductions of values of $\underbrace{\pi^n \alpha \text{ on } \mathfrak{X}(\mathcal{O})}_{\mathcal{O}\text{-valued}}$. $\Big\}$ $\pi^n \mathcal{O}$-valued

$\therefore$ A_0 vanishes on $\mathfrak{X}(f) \underset{Z\text{-dense}}{\subset} \mathfrak{X}_f$, so $A_0 = 0$, so $A = \pi A'$ for $A' \in \mathcal{O}[\mathfrak{X}]$. $\pi^n \alpha = \pi A'$, so $\alpha = \frac{A'}{\pi^{n-1}}$ ....

Lecture 7 $G(k)^1$ and Néron models of tori

Let G be conn'd reductive $/k$. The valuation homomorphism of G is

$$\omega_G : G(k) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X_k^*(G), \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Q}}(X_k^*(G^{ab})_{\mathbb{Q}}, \mathbb{Q})$$

$$X_k^*(G) = \mathrm{Hom}_k(G, G_m) = X_{k_s}^*(G)^{\Gamma_k} = X_k^*(G^{ab})$$

$$G^{ab} = G/\mathcal{D}G = k\text{-torus}$$

by $\omega_G(g) = \big(\chi \mapsto \mathrm{ord}(\chi(g))\big)$, where $\chi(g) \in k^\times$.

The kernel is denoted $G(k)^1 = \{ g \in G(k) : |\chi(g)| = 1,\ \forall \chi \in X_k^*(G) \}$.

- $G(k)^1$ is normal in G
- $G(k)^1$ is functorial in G, $\because$ ω_G is so.

Note $G(k)^1 = G(k)$ when $X_k^*(G)$ (i.e. G has no nontrivial central G_m: $(Z_G)^\circ_{red} \longrightarrow G^{ab}$ is isogeny.)

e.g. $G = GL_n$, $G_m \xrightarrow{\mathfrak{z}^n} G_m$

All bounded subgroups $K \subset G(k)$ lie in $G(k)^1$: $\chi : G \longrightarrow G_m$

$\Rightarrow$ $\chi(K) \subset k^\times$ is bounded subgp of GL_1, so $\chi(K) \subset \mathcal{O}^\times$.

Note for quotient map $q: G \to \underset{\substack{\shortparallel \\ k\text{-torus}}}{G^{ab}}$, have $G(k)^1$ is preimage of

$(G^{ab})(k)^1$ under q as k-pts.

Rmk. For G ss, $G(k)^1 = G(k)$ is usually NOT bounded. Loosely speaking,

$G(k)^1$ removes "unbounded part" from $Z(G)(k)$. $[\, Z(G) := (Z_G)^\circ_{red} \,]$

Q: Is $G(k)^1$ generated by the bounded subgroups?

Ex. Suppose $G = T$ is k-torus, $w_T : T(k) \to \mathrm{Hom}_{\mathbb{Z}}(X^*_k(T), \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Q}}(X^*_k(T)_{\mathbb{Q}}, \mathbb{Q})$

$X_{*,k}(T)_{\mathbb{Q}}$

$X_*(T_{\bar{k}})^{\Gamma_k}_{\mathbb{Q}}$

$X^*(T_{\bar{k}})^{\Gamma_k}_{\mathbb{Q}}$

& $X_{*,k}(T) \times X^*_k(T) \xrightarrow{\langle , \rangle} \mathbb{Z}$ is usually NOT perfect pairing.

Ex. $T = R_{k'|k}(\mathbb{G}_m)$ for $k'|k$ finite separable of degree d.

$\mathbb{G}_m \underset{can}{\overset{j}{\hookrightarrow}} R_{k'|k}(\mathbb{G}_m) = T, \quad A^\times \to (k' \otimes_k A)^\times$

$N_{k'|k} : T = R_{k'|k}(\mathbb{G}_m) \to \mathbb{G}_m, \quad (k' \otimes_k A)^\times \xrightarrow[\text{norm}]{} A^\times$

Check $X^*_k(T) = N_{k'|k}^{\mathbb{Z}}$, $X_{*,k}(T) = j^{\mathbb{Z}}$, and $\langle N_{k'|k}, j \rangle : z \mapsto z^d$

Bit of work, [KP, Lemma 2.5.7] $\Rightarrow$ image of w_T contains $X_{*,k}(T)$ (w/ finite index)

Special case: $T = (\mathbb{G}_m)^N$, $w_T : (k^\times)^N \to \mathbb{Z}^N$ is ord. so surjective.
and $(\mathbb{G}_m^N)^1 = (\mathcal{O}^\times)^N \subset (k^\times)^N = T(k)$.

Prop Let T be a torus.

① $T(k)^1$ is bounded, hence is max'l bounded subgp.

② $T(k)^1 = \{t \in T(k) : |\chi(t)| = 1,\ \forall \chi \in X^*(T_{\bar{k}}) = X^*(T_{k'})$

for finite Galois $k'|k$, splitting $T\}$

Pf For ②, observe for finite separable $\ell | k$, $T(k) \hookrightarrow T(\ell)$ is "same" as

$T \hookrightarrow R_{\ell|k}(T_\ell)$ on k-pts.

So $T(k) \hookrightarrow T(\ell)$ carries bounded subgps onto bounded subgps.

and $T(k) \cap (\text{bounded in } T(\ell))$ is bounded in $T(k)$.

But $T(k) \hookrightarrow T(\ell)$ carries $T(k)^1$ into $T(\ell)^1$, so to show $T(k)^1$ is bounded, it suffices to do for $T(\ell)^1$, reducing us to split torus case which is clear. Settles ①.

For ②, need to show for $t \in T(k)^1$ and $\chi : T_{k'} \to G_m$ (for $T_{k'}$ split) that $\chi(t) \in (\mathcal{O}')^\times$. But $T(k)^1 \subset T(k')^1$ is bounded in $T(k')$, so $\chi(t^{\mathbb{Z}}) \subset (k')^\times$ is bounded in G_m: $\chi(t^{\mathbb{Z}}) \subset (\mathcal{O}')^\times$. $\square$

Cor. For conn'd reductive G/k, $G(k)^1 = \{g \in G(k) : |\chi(g)| = 1,\ \forall \chi \in X^*(G_{\bar{k}})\}$

$X^*(G_{\bar{k}}) = X^*(G^{ab}_{\bar{k}})$

In particular, for finite separable $k'|k$, $G(k')^1 \cap G(k) = G(k)^1$.

Enough: for separable algebraic $k'|k$ w/ $|k'^\times| \subset \mathbb{R}_{>0}$ discrete. (eg. $k' = k^{un}$)

Rmk. For f perfect, $|k'|^\times$ discrete $\iff [(k')^{un} : k^{un}] < \infty$ since $f_s = \bar{f}$.

Why is $G(k)^1$ of interest? We'll see (for f perfect) that all max'l bounded subgps of $G(k)$ have form $G(k)^1_x$ for certain pts $x \in \mathcal{B}(G)$.

used in BT theory

(and all $G(k)^1_x$ are bounded)

We'll build canonical $\mathcal{O}$-structure $\mathcal{G}^1_x$ for G w/ $\mathcal{G}^1_x(\mathcal{O}) = G(k)^1_x$

(and likewise over k^{un}),

But often, $(\mathcal{G}^1_x)_f$ is disconn'd. and want to get a handle on $(\mathcal{G}^1_x)^\circ$.

Using Néron models of k-tori, we'll define finite index subgp

$G(k)^\circ \subset G(k)^1$ (open normal in $G(k)$) s.t. $G(k)^\circ_x = (\mathcal{G}^1_x)^\circ(\mathcal{O})$.

For simply conn'd G, will have $G(k)^\circ = G(k)^1$, but usually not otherwise.

Lecture 8. The subgroup $G(k)^\circ$.

Loose end on $G(k)^1$. A_G = max'l split central k-torus in G

= max split torus in $Z(G) = (Z_G)^\circ_{red}$.

$$w_G : G(k) \to \mathrm{Hom}_{\mathbb{Z}}(X^*_k(G), \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Q}}(X^*_k(G^{ab})_{\mathbb{Q}}, \mathbb{Q}) = X_{*,k}(G^{ab})_{\mathbb{Q}}$$
$$\Leftarrow X_{*,k}(Z(G))_{\mathbb{Q}}$$

$Z(G) \to G^{ab} = G/\mathcal{D}G$ is an isogeny. $\qquad = X_{*,k}(A_G)_{\mathbb{Q}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = X^*(A_G)^\vee_{\mathbb{Q}}$

and $G(k)^1 = \ker(w_G)$

For parabolic k-subgp $P \subset G$, say $P = M \ltimes U$ for Levi M

$(\exists\, \lambda : \mathbb{G}_m \to G$ s.t. $M = Z_G(\lambda)$, $P = P_G(\lambda))$

Can be convenient to relate $M(k)^1$ to $G(k)^1$.

$$A_G \subset Z_G(\lambda) = M \Rightarrow A_G \subset A_M \quad (\to X_*(A_G)_{\mathbb{Q}} \subset X_*(A_M)_{\mathbb{Q}})$$

For any k-split torus $S \subset G$ w/ $A_G \subset S$ (e.g. $S = A_M$),

have isogeny via mult. $A_G \times S' \to S$ for $S' = (S \cap \mathcal{D}G)^0_{red}$

$= \max.$ k-torus of S inside $\mathcal{D}G$.

mult: $Z(G) \times \mathcal{D}G \to G$ isogeny, so

$$Z(G) \times (Z(G)\cdot S)' \to Z(G)\cdot S \quad \text{isogeny.}$$

max. split subtori: $A_G \times S' \to A_G \cdot S = S$ an isogeny

$\therefore A_M = A_G \cdot A_M'$ as almost product.

$\therefore X_*(A_M)_{\mathbb{Q}} = X_*(A_G)_{\mathbb{Q}} \oplus X_*(A_M')_{\mathbb{Q}}$

<u>Lemma</u>. ①
$$\begin{tikzcd} M(k) \arrow[r, "w_M"] \arrow[d] & X_*(A_M)_{\mathbb{Q}} \arrow[d, "pr"] \\ G(k) \arrow[r, "w_G"] & X_*(A_G)_{\mathbb{Q}} \end{tikzcd}$$
coincides.

② $\left(M(k) \cap G(k)^1\right) / M(k)^1 \hookrightarrow \ker(\mathrm{pr}) = X_*(A'_M)_{\mathfrak{a}}$.

Pf. ① is defn chasing, ① ⇒ ② : see [KP, Lemma 2.6.19].

Ex. Consider P_0 a minimal parabolic, then $M_0 = Z_G(S)$ for max. split k-tori $S \subset G$. Then $\left(Z(k) \cap G(k)^1\right) / Z(k)^1 \hookrightarrow X_*(S')_{\mathfrak{a}}$, where $Z_G(S) =: Z$

$A_{M_0} = S$, $S' = (S \cap \mathcal{D}G)^\circ_{\mathrm{red}}$

underlies why $G(k)^\circ \subset G(k)^1$.

To define $G(k)^0$, we first treat the case of k-tori T.

For this, we Néron models of tori [BLR, §10.1, Prop 4 – Prop 6].

For any k-torus T, ∃ smooth separated (comm) $\mathcal{O}$-gp $\mathcal{J}$ w/ $\mathcal{J}_k = T$, s.t.

∀ smooth $\mathcal{O}$-schemes $\mathcal{Y}$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{Y}, \mathcal{J}) \xrightarrow{\sim} \mathrm{Hom}_k(\mathcal{Y}_k, T)$.

(so for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}')$ for finite unram. $k'|k$, $\mathcal{J}(\mathcal{O}') = T(k')$)

($\Leftrightarrow$ $\mathcal{O} \to \mathcal{O}'$ étale)

Typical special fiber : $\mathcal{J}_{\bar{f}} \simeq$ (conn'd comm affine) × (f.g. $\mathbb{Z}$-module)

$\mathcal{J}^\circ_{\bar{f}} =$ affine, $\pi_0(\mathcal{J}_{\bar{f}}) =$ f.g. $\mathbb{Z}$-module.

Properties : ⓪ $\mathcal{J}^\circ \overset{\text{open}}{\subset} \mathcal{J}$ affine.

① $\mathcal{J} \otimes_{\mathcal{O}} \mathcal{O}^{un}$ is Néron model of $T_{k^{un}}$.

② $\pi_0(\mathcal{J}_{\bar{f}}) = \mathcal{J}_{\bar{f}} / \mathcal{J}^\circ_{\bar{f}}$ is f.g., hence has finite torsion subgp.

$\quad \| $

$\pi_0(\mathcal{J}_{f_s})$

So get $\mathcal{J}^{ft} \subset \mathcal{J}$ open subgp (so $\mathcal{J}_f^\circ \subset \mathcal{J}_f^{ft}$)

w/ $\pi_0(\mathcal{J}_{\bar{f}}^{ft}) = \pi_0(\mathcal{J}_{\bar{f}})_{tor}$.

<u>Def</u>: $T(k)^\circ := \mathcal{J}^\circ(\mathcal{O})$. $\left(\subset (\mathcal{J}^\circ)_k(k) = T(k) \text{ is open}\right)$

<u>Note</u>: $T(k)^\circ = T(k) \cap T(k^{un})^\circ$, since $T(k^{un})^\circ = \mathcal{J}^\circ(\mathcal{O}^{un})$ ($k \cap \mathcal{O}^{un} = \mathcal{O}$)

$\uparrow$

(use ⓪, ① above)

The Iwahori subgroup of T.

<u>NOT</u> the defn in [KP, § 2.5].

The above 3 properties will emerge from <u>construction</u> of $\mathcal{J}$ (next time)

<u>Prop</u>. ① $\mathcal{J}^{ft}(\mathcal{O}) = T(k)^1$ (so $T(k)^1 / T(k)^\circ \subset \pi_0(\mathcal{J}_{\bar{f}}^{ft}) =$ finite) and $T(k)^1 = T(k)^\circ$ if $\pi_0(\mathcal{J}_{\bar{f}})$ is torsion free

* ② If $f = \bar{f}$ and $k'|k$ finite separable splitting T, then

$N: R_{k'|k}(T_{k'}) \longrightarrow T$ (as for comm. f. type k-gps in general).

induces $T(k') \longrightarrow T(k)$ on k-pts

$$\begin{array}{ccc} T(k') & \longrightarrow & T(k) \\ \cup & & \cup \\ (T(k')^\circ =) \, T(k')^1 & \longrightarrow & T(k)^\circ \end{array}$$

③ If $T_{k^{un}}$ is induced tori, then $T(k)^\circ = T(k)^1$.

$\nearrow$

assume f perfect

<u>Rmk</u>. For general k w/ f perfect, $T(k)^\circ = T(k) \cap \underbrace{T(k^{un})^\circ}_{\text{has Galois theoretic description}}$ via ② over k^{un}.

Lecture 9. Néron models of tori and $G(k)^\circ$

Let's see how Néron model $\mathcal{T}$ of a k-torus T is built.

From defn of Néron mapping property, if T_1, T_2 have Néron models $\mathcal{T}_1, \mathcal{T}_2$, then $\mathcal{T}_1 \times \mathcal{T}_2$ is a Néron model of $T_1 \times T_2$.

We'll first focus on construction for split T, so in effect $\mathbb{G}_m$.

<u>Ex</u>. $T = \mathbb{G}_{m,k}$. We build $\mathcal{T}$ as gluing of $\mathcal{O}$-schemes $\{U_n\}_{n\in\mathbb{Z}}$,

for $U_n = \mathbb{G}_{m,\mathcal{O}}$, using

$$\begin{array}{ccc} \mathbb{G}_{m,k} & & \mathbb{G}_{m,k} \\ \cup\!\!\shortparallel & & \shortparallel \\ (U_n)_k & \underset{\cdot\pi}{\simeq} & (U_{n+1})_k \\ \text{open}\cap & & \cap\text{open} \\ U_n & & U_{n+1} \end{array}$$

Call this $\mathcal{T}$. $\left((U_n)_k \underset{\cdot\pi^m}{\simeq} (U_{n+m})_k \right)$. using a choice of $\pi \in \mathcal{O}$.

$$\mathcal{T}_f = \mathbb{G}_{m,f} \times \mathbb{Z}, \quad \mathcal{T}(\mathcal{O}') = \bigcup_{n\in\mathbb{Z}} \underbrace{(\mathcal{O}')^\times}_{\substack{\shortparallel \\ U_n(\mathcal{O}')}} \pi^n \underset{\substack{\uparrow \\ \pi \in \mathcal{O}' \text{ uniformizer}}}{=} k'^\times \text{ for finite } \underline{\underline{\text{unram.}}}\ k'|k.$$

$$U_n(\mathcal{O}') \subset (U_n)_k(k') \underset{\pi^{-n}}{\simeq} (U_0)_k(k') = \mathbb{G}_m(k') = k'^\times.$$

<u>Check $\mathcal{T}$ separated</u>:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Delta_{\mathcal{T}/\mathcal{O}}} & \mathcal{T} \times \mathcal{T} \\ \cup & & \cup \\ \Delta^{-1}(U_n \times U_m) & \xrightarrow[\text{closed immersion}]{?} & U_n \times U_m \quad \text{open cover} \end{array}$$

graph of gluing $= \Delta^{-1}(U_n \times U_m)$

along $(U_n)_k \underset{\cdot\pi^{m-n}}{\simeq} (U_m)_k$

$n = m$, $\Delta_{U_n, \mathcal{O}}$ is closed immersion.

$n \neq m$, $\Delta^{-1}(U_n \times U_m) = \{(u, u') \in \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} : u' = \pi^{m-n} u\}$

$\overset{?}{\underset{\text{closed}}{\subset}} \; \mathbb{G}_{m,\mathcal{O}} \times \mathbb{G}_{m,\mathcal{O}}$

$\Updownarrow$ $n > m$ or $n < m$

$\pi^{n-m} u' = u$

Same eqn $/\mathcal{O}$ (w/ pos. exponent on π) cuts out <u>same</u> subscheme, hence closed.

Condition on $\mathcal{O}$-algebra A, u, u' units forces A to be a k-alg.!

Informally, J is a union of $\pi^{\mathbb{Z}}$-translates of $J^\circ = \mathbb{G}_{m,\mathcal{O}}$.

$\pi \in k^\times = T(k) = J(\mathcal{O})$. Here $\pi_0(J_{\mathbb{F}}) = \mathbb{Z}$

See [BLR, §10.1, Ex 5] for Néron mapping property.

<u>General case</u>. Pick finite separable ext'n $k'|k$ s.t. $T_{k'}$ is split: $T_{k'} \simeq \mathbb{G}_{m,k'}^d$

Have Néron model J' over $\mathcal{O}'$ for $T_{k'}$: $(J')^\circ = \mathbb{G}_{m,\mathcal{O}'}^d$, $\pi_0(J'_{\mathbb{F}'}) = \mathbb{Z}$, so $T_{k'} \underset{\text{open}}{\subset} J'$.

$$\begin{array}{ccc} T & \xrightarrow{\text{closed}} & R_{k'|k}(T_{k'}) \\ {\scriptstyle \text{open}}\downarrow & & \cap \; \text{open} \\ J^{\text{ft}} & \xrightarrow{\text{closed}} & R_{\mathcal{O}'|\mathcal{O}}(J') \underset{\text{open}}{\supset} R_{\mathcal{O}'|\mathcal{O}}((J')^\circ) \end{array}$$

(see [BLR, §7.6] for Weil restriction)

J^{ft} = scheme closure of T.

$\mathcal{O}$-flat $\mathcal{O}$-subgp

<u>Existence of $R_{\mathcal{O}'|\mathcal{O}}(\mathcal{J}')$:</u> Since $\mathcal{J}' \otimes_{\mathcal{O}} k = \mathcal{J}'_{k'} = T_{k'}$ is affine open in $\mathcal{J}'$, we need to certify any finite subset of $\mathcal{J}' \otimes_{\mathcal{O}} f$ $(= \mathcal{J}'_{f'})$ topologically is in an affine open of $\mathcal{J}'$.

Want to check for $I = \{-N, \cdots, N\} \subset \mathbb{Z}$, gluing $\mathcal{U}_I$ of $\mathcal{U}_n$'s for $n \in I$ is affine. $\bigcup_{n\in I} \mathcal{U}_n \underset{\text{open}}{\subset} \mathcal{J}$.

Pf. $I \underset{\text{bijection}}{\simeq} \mathbb{Z}/m\mathbb{Z}$ for $m = |I|$, so $\mathcal{U}_I$ as $\mathcal{O}$-scheme is contained in $\mathbb{Z}/(m+1)\mathbb{Z}$- analogue of $\mathcal{U}$. This analogue $\mathcal{G}$ is an $\mathcal{O}$-gp, smooth separated and f. type w/ generic fiber T_k = affine, so $\mathcal{G}$ = affine.

$\mathcal{U}_I$ = open subscheme of $\mathcal{G}$ complementary to <u>one</u> conn'd comp. of special fiber

so $\mathcal{U}_I$ = dilatation of $\mathcal{G}$ along clopen $Y \subset \mathcal{G}_f$

= affine.

$$\begin{array}{ccc} T & \hookrightarrow & R_{k'|k}(T_{k'}) \\ {\scriptstyle\text{open}}\big\downarrow & & \wedge\,{\scriptstyle\text{open}} \\ \mathcal{J}^{ft} & \underset{\text{closed}}{\hookrightarrow} & R_{\mathcal{O}'|\mathcal{O}}(\mathcal{J}') \\ & & \cup\,{\scriptstyle\text{open}} \\ \mathcal{J}^{ft} \cap R_{\mathcal{O}'|\mathcal{O}}((\mathcal{J}')^0) & & R_{\mathcal{O}'|\mathcal{O}}((\mathcal{J}')^0) \end{array}$$

$\mathcal{J}^{ft} \cap R_{\mathcal{O}'|\mathcal{O}}((\mathcal{J}')^0)$ is open in $\mathcal{J}^{ft}$, closed subscheme, hence affine. so has finite geometric π_0

$R_{\mathcal{O}'|\mathcal{O}}(\mathcal{J}')$, $R_{\mathcal{O}'|\mathcal{O}}((\mathcal{J}')^0)$ is relative identity component, so

$R_{(\mathcal{O}'/\pi)/f}$ applied to $(\mathcal{J}')^0_{\mathcal{O}'/\pi}$ - torsor $\mathcal{J}'_{\mathcal{O}'/\pi} \to \mathbb{Z}^d$

$\pi_0(R_{\mathcal{O}'|\mathcal{O}}(\mathcal{J}')_f) = \mathbb{Z}^d$

$R_{\mathcal{O}'|\mathcal{O}}((\mathcal{J}')^0)$ "$(\mathbb{G}_m, \mathcal{O}')^d$" $= \underbrace{R_{\mathcal{O}'|\mathcal{O}}(\mathbb{G}_m)^d}_{\text{geom. conn'd fibers}}$ ($=$ "$\mathcal{O}'$ as $\mathcal{O}$-scheme") $\delta = [k' : k]$

(open complement in $\mathbb{A}^\delta_{\mathcal{O}}$ of zero locus of $N_{\mathcal{O}'|\mathcal{O}} : \mathbb{A}^\delta_{\mathcal{O}} \to \mathbb{A}^1_{\mathcal{O}}$)

$\pi_0(J^{ft}_{\bar f})$ = f.g. $\mathbb{Z}$-module, so $J^{ft}(\mathcal{O}^{un}) / (J^{ft})^\circ(\mathcal{O}^{un})$ is f.g.

$(\hookrightarrow \pi_0((J^{ft})_{\bar f}))$

Note $J^{ft}(\mathcal{O}) = T(k)$, similarly over $\mathcal{O}^{un}$ (use Néron property of J')

By [BLR, §10.1, Prop 4], the gp smoothening $J \xrightarrow[\text{affine}]{q} J^{ft}$ is a Néron model of T.

$J \underset{\text{open}}{\supset} \underbrace{q^{-1}((J^{ft})^\circ) \supset J^\circ}_{\text{finite index on special fiber}}$ $\qquad$ $\pi_0(J_{\bar f})$ is f.g.

Recall: $\pi_0(J_{\bar f})_{tor}$ = finite.

Lecture 10. The subgroup $G(k)^e$

Thm (Steinberg) $f = \bar f \Rightarrow H^1(k, G) = 1$ for any conn'd reductive k-gp. G

In general, if f perfect, then $H^1(k^{un}, G) = 1$ for any conn'd red. k-gp G.

Ex. Str. theory of conn'd red. gps $\Rightarrow$ G over k has ! (up to $\cong$) q-split inner form. The set of inner forms is image of $H^1(k, G^{ad})$ in $H^1(k, \mathrm{Aut}_{G/k})$, $1 \to G^{ad} \to \mathrm{Aut}_{G/k} \to \underbrace{\Gamma}_{\text{étale}} \to 1$

Thus, G is q-split over k^{un}.

Rmk. Equivalence of (Néron model def. of $T(k)^\circ$) and (Galois theoretic defn in [KP, Def 2.3.15]) use perfectness of f.

Still allow general f.

Def. $G(k)^\natural := \operatorname{im}(G^{sc}(k) \to G(k))$ for s. conn'd central cover $G^{sc} \to \mathcal{D}G$.

Have $(G^{sc})^{ad} = G^{ad}$, so G^{ad} acts on "$G^{sc} \to G$", so G does to $(G \twoheadrightarrow G^{ad})$, so $G(k)$-action on itself via conjugation lifts to $G(k)$-action on $G^{sc}(k)$, so $G(k)^\natural \lhd G(k)$, and $G(k)/G(k)^\natural$ is comm. and $G(k)^\natural$ is <u>functorial</u> in G ($\because G^{sc} \to G$ is)

Let $M_0 = Z_G(S)$ be Levi k-subgp of a minimal parabolic k-subgp $P_0 = M_0 \ltimes U_0$.
("minimal Levi")

Have Bruhat decomp. $G(k) = \bigcup_{w \in W(G,S)} P_0(k)\,\dot{w}\,P_0(k)$

where $W(G,S) = N_G(S)(k)/Z_G(S)(k)$, $\dot{w} \in N_G(S)(k)$ any rep. of w.

$$= W(G^{sc}, S^{sc})$$

Also, U_0 is the unip. radical of corresponding preimage $P_0^{sc} \subset G^{sc}$.

$\therefore$ in the commutative quotient $G(k)/G(k)^\natural$, image of $M_0(k)$ is full.

This proves [KP, Fact 2.6.22]:

<u>Prop</u> $G(k) = G(k)^{\natural} \cdot M_0(k)$ for minimal Levi $M_0 = Z_G(S)$

(hence for any Levi)

<u>From now on f is perfect</u>.

<u>Def.</u> Define $G(k)^{\circ} \subset G(k)$ as follows:

① If G is q-split (as happens over k^{un}),

$G(k)^{\circ} := G(k)^{\natural} \cdot T(k)^{\circ}$ for a max. k-torus T in a Borel subgp (T is minimal Levi of G, all $G(k)$-conjugate.)

② If G^{ad} is anisotropic (e.g. $G = M_0$ = minimal Levi $Z_H(S)$ for conn'd reductive H/k), then $G(k)^{\circ} := G(k) \cap G(k^{un})^{\circ}$ (via ① over k^{un}).

③ <u>General case</u>: $G(k)^{\circ} := G(k)^{\natural} M_0(k)^{\circ}$ for minimal Levi M_0 (via ② for M_0).

Why are ①–③ indep. of T, M_0? Such choices are $G^{ad}(k)$-conjugate.

Also, $G(k)^{\circ} \supset G(k)^{\natural}$: by hand for ①, ③, (by structure theory over fields).

for ②, we see $G(k)^{\natural} \subset G(k) \cap G(k^{un})^{\natural} \subset G(k) \cap G(k^{un})^{\circ} = G(k)^{\circ}$

Since $G(k)/G(k)^{\natural}$ is comm. w/ <u>trivial</u> $G^{ad}(k)$-actions (check), the above " $G^{ad}(k)$-conjugacy" becomes invisible in this qt.

Likewise, comm. of $G(k)/G(k)^{\natural} \Rightarrow G(k)^{\circ} \trianglelefteq G(k)$

<u>Lemma</u>: $G(k)^{\circ} \subset G(k)$ is open.

See [KP, Rem 2.6.25] for consistency among overlaps of ①, ②, ③ in Def.

$\rightsquigarrow$ Functoriality of $G(k)^{\circ}$ in G is <u>NOT</u> obvious.

<u>Lemma</u>: $G(k)^{\circ} \subset G(k)^{1}$.

<u>Rmk</u>. $G(k)^{1}/G(k)^{\circ}$ is comm $\because G(k)/G(k)^{\circ} \twoheadleftarrow G(k)/G(k)^{\natural}$ comm.

<u>Pf</u>: We check cases ①, ②, ③ using $G(k)^{\natural} \subset G(k)^{1}$ $\because G^{sc}(k) = G^{sc}(k)^{1}$ and $H(k)^{1}$ is functorial in H.

① $G(k)^{\circ} = G(k)^{\natural}\, T(k)^{\circ} \subset G(k)^{1}\, T(k)^{1} = G(k)^{1}$.

② $G(k)^{\circ} = G(k) \cap G(k^{un})^{\circ} \underset{\substack{\text{① for} \\ G_{k^{un}}}}{\subset} G(k) \cap G(k^{un})^{1} = G(k)^{1}$.

③ $G(k)^{\circ} = G(k)^{\natural}\, M_{0}(k)^{\circ} \underset{\substack{\text{② for} \\ M_{0}}}{\subset} G(k)^{1}\, M_{0}(k)^{1} = G(k)^{1}$.

<u>Lecture 11</u> <u>Refined aspects of $G(k)^{\circ}$</u>

We defined $G(k)^{\circ} \subset G(k)$ an open normal subgroup, $\underset{\substack{\| \\ \mathrm{im}(G^{sc}(k))}}{G(k)^{\natural}} \subset \underbrace{G(k)^{\circ} \subset G(k)^{1}}_{\text{comm. quotient}}$

From def'n, NOT evident that $G(k)^\circ$ is functorial in G.

However, over k^{un}, we do have functoriality by direct argument (notes from last time). Hence, $G(k^{un})^\circ$ is functorial in G. $\leftarrow$ Try it!

Prop. $G(k)^\circ \subset G(k) \cap G(k^{un})^\circ$. ($\supset$ is later, yield functoriality)

Pf. Check $G(k)^\circ \subset G(k^{un})^\circ$ casewise from def'n of $G(k)^\circ$:

① G = q-split, $G(k)^\circ = G(k)^{\natural} \cdot T(k)^\circ$ for T = max'l torus in Borel k-subgp $B \subset G$.

$\subset G(k^{un})^{\natural} \cdot T(k^{un})^\circ$ $\because \mathcal{T}_{\mathcal{O}^{un}}$ is Néron model of $T_{k^{un}}$ ($\Rightarrow T(k)^\circ \subset T(k^{un})^\circ$)

$= G(k^{un})^\circ$ $\because T_{k^{un}}$ is max'l in Borel $B_{k^{un}} \subset G_{k^{un}}$.

② G^{ad} is k-anisotropic: $G(k)^\circ = G(k) \cap G(k^{un})^\circ$ by def'n.

③ (general) For minimal Levi $M_0 \subset G$, $G(k)^\circ = G(k)^{\natural} \cdot M_0(k)^\circ$

$\subset G(k^{un})^{\natural} M_0(k^{un})^\circ$ (② for M_0)

$G(k^{un})^{\natural} \subset G(k^{un})^\circ$ by def'n. $\overset{?}{\subset} G(k^{un})^\circ$

$M_0(k^{un})^\circ \subset G(k^{un})^\circ$ by functoriality $/k^{un}$ applied to $M_0 \hookrightarrow G$.

Main result:

Thm $G(k)^\circ \subset G(k)^1$ has finite index.

For $G = T$ a torus, this is known because $T(k)^1 = \mathcal{J}^{ft}(\mathcal{O})$, $T(k)^\circ = \mathcal{J}^\circ(\mathcal{O})$,

so $T(k)^1 / T(k)^\circ \hookrightarrow \pi_0(\mathcal{J}_{\bar{f}})_{tor} \overset{=}{} \text{finite}$.

To prove thm, we need two lemmas on relation of $M_0(k)^\circ$ and

$G(k)^\circ$ for minimal Levi $M_0 \subset G$.

Lemma 1. If $G = G^{sc}$, $M_0(k^{un})^\circ = M_0(k^{un})^1$. $\left(\xRightarrow{M_0(k)\cap\cdot} M_0(k)^\circ = M_0(k)^1 \right)$.

Rem. If G is q-split, then $M_0 = T$ is <u>induced</u> torus (uses $G = G^{sc}$).

hence $T(k)^\circ = T(k)^1$ (Prop 8.5 (iii)).

Pf. Let $K = k^{un}$. Want $M_0(K)^\circ \subset M_0(K)^1$ is equality. By def'n,

$M_0(K)^\circ = \underbrace{M_0(K)^q}_{\substack{\text{image of } K\text{-pts} \\ \text{of } \underbrace{M_0^{sc} \to \mathcal{D}M_0}_{\substack{\text{equality, see Cor 9.5.11} \\ \text{in Alg Gps II (use } G = G^{sc})}}}} T(K)^\circ$ for max. K-torus $T \subset (M_0)_K$ of a Borel K-subgp.

so $M_0(K)^\circ = (\mathcal{D}M_0)(K)\, T(K)^\circ$.

M_0 = Levi of $G \Rightarrow (M_0)_K \subset G_K$ is Levi, so max. tori of Borels

of $(M_0)_K$ are <u>also</u> max. tori of Borel of G_K:

Since max. torus in a Borel = max. torus that is max'lly split, and

$(M_0)_K$ and G_K have max. <u>split</u> tori of <u>same</u> dim.

$\therefore$ T is max. torus in Borel of q-split $G_K = G_K^{sc}$, so T is induced.

$T' := T \cap \mathcal{D}(M_0)_K$ is max. K-torus of $(\mathcal{D}M_0)_K$ w/ max'l split rk.

So T' is a max'l torus of a Borel of $\mathcal{D}M_0)_K = (\text{q-split}/K)$
$= \underline{\underline{sc}}$

So T' is also induced.

$$1 \to T' \to T \to \overset{=:\bar{T}}{T/T'} = \left(M_0/\mathcal{D}(M_0)\right)_K$$

Also, from how we <u>build</u> $((M_0)_K, T)$ by Galois-twisting, T/T' also induced.

$$\begin{array}{ccc} M_0(K) & \supset M_0(K)^1 \searrow \bar{T}(K)^1 & \overset{\bar{T}\text{ induced}}{=} \bar{T}(K)^0 \\ \cup & & \\ T(K)^0 & \xrightarrow{\text{surjective}} & \bar{T}(K)^0 \end{array}$$

defn of $M_0(K)^0$ for q split $(M_0)_K$.

surjective $\because$ $1 \to T' \to T \to \bar{T} \to 1$ exact seq. of tori$/K$.

$\therefore$ $M_0(K)^1/M_0(K)^0$ comes from $\ker\left(M_0(K)^1 \to \bar{T}(K)^1\right)$

$$\subset \underbrace{\mathcal{D}(M_0)(K)}_{\text{s.conn'd}} = M_0(K)^q \subset M_0(K)^0.$$

Lemma 2. For conn'd reductive group G,

$$G(k)^0 \cap M_0(k)^1 = M_0(k)^0.$$

Pf. $M_0(k)^0 := M_0(k) \cap M_0(K)^0$.

$M_0(k)^1 = M_0(k) \cap M_0(K)^1$.

$\therefore$ enough to show $G(K)^0 \cap M_0(K)^1 = M_0(K)^0$.

(since $M_0(k) \cap (\cdot)$ yields result)

By functoriality over K, have $M_0(K)^0 \overset{(*)}{\subset} \underbrace{G(K)^0}_{G(K)^q T(K)^0 \text{ for max. } K\text{-torus } T \text{ in Borel of } G_K} \cap M_0(K)^1$

(applied to $M_0 \hookrightarrow G$)

Such T can be chosen to be in Borel of $(M_0)_K$.

For such T, have $T(K)^0 \subset M_0(K)^0$

(by def'n of $M_0(K)^0$ for q. split $(M_0)_K$).

Want $(*)$ to be equality.

Lecture 12. Finiteness result and Steinberg's Thm

Last time we were proving

Lemma. For minimal Levi M_0 in G, $G(k)^0 \cap M_0(k)^1 = M_0(k)^0$.

We reduced to showing the containment $M_0(K)^0 \subset G(K)^0 \cap M_0(K)^1$

is an equality. $G(k)^0 = G(k)^q T(k)^0$ for $T \subset (M_0)_K$ max torus in Borel K-subgp ($\Rightarrow$ same for T in G_K).

so $G(k)^0 \cap M_0(k)^1 = \left(G(k)^q \cap M_0(k)^1\right) T(k)^0$.

$\therefore$ suffices to show $G(k)^q \cap M_0(k)^1 \overset{(+)}{\subset} M_0(k)^0$.

$$\begin{tikzcd} M_0' \arrow[r, phantom, "\subset"] \arrow[d, "\substack{\text{central isogeny}\\\text{onto smooth}}"'] \arrow[dr, phantom, "\square"] & G^{sc} \arrow[d] \\ \underbrace{M_0}_{\text{torus centralizer}} \arrow[r, phantom, "\subset"] & G \end{tikzcd}$$

conn'd normal subgp

} central isogeny onto <u>normal</u> $\mathcal{D}G \triangleleft G$.

($M_0 \cap \mathcal{D}G$ <u>is</u> a minimal Levi of $\mathcal{D}G$).

Minimal Levi's of G^{sc} and $\mathcal{D}G$ "correspond" via image & pre image.

So M_0' is minimal Levi of G^{sc}: by Cor 9.5.11 of [C2],

$\mathcal{D}M_0'$ is also <u>s. conn'd</u>, so $\mathcal{D}M_0'$ <u>is</u> M_0^{sc}.

$M_0^{sc} = M_0' \longrightarrow M_0$ is central isogeny onto normal subgp,

$A_{M_0'} \to A_{M_0}$ is isogeny onto subtorus.

$\therefore M_0'(k)^1$ is preimage of $M_0(k)^1$ ($w_{M_0} : M_0(k) \to X_*(M_0^{ab})_{G} = X_*(A_{M_0})_{G}$)

$\therefore G(k)^q \cap M_0(k)^1$ is image of $M_0'(k)^1$ in $G(k)$.

Previous Lemma (over k): $M_0' \subset G^{sc}$ is min. Levi has $M_0'(k)^1 = M_0'(k)^0$

But $H(k)^\circ$ is functorial in conn'd red. H/k.

So $M_0' \to \underset{\underset{G}{\cap}}{M_0}$ takes $M_0'(k)^\circ$ into $M_0(k)^\circ$. $\square$

Pf that $G(k)^\circ \subset G(k)^1$ has finite index.

① Reduce task for G to same for min. Levi $Z = Z_G(S)$

($S \subset G$ max. split torus $/k$)

② Reduce task for Z to case of $\underbrace{\text{tori over } K}$.

known: $T(k)^\circ \subset T(k)^1$ for all k

via Néron model defn ($\pi_0(\mathcal{J}_{\bar{f}})_{\text{for}}$ finite)

Grant finiteness for H w/ H^{ad} anisotropic $/k$ (e.g. $H = Z = Z_G(S)$)

$G(k)^\circ = G(k)^q \underbrace{Z(k)^\circ}_{\text{finite index in } Z(k)^1}$

$\underbrace{G(k)/G(k)^q}_{\text{comm.}} \twoheadrightarrow \underbrace{G(k)/G(k)^\circ}$ $\supset$ $G(k)^1/G(k)^\circ$

Bruhat. $G(k) = \coprod_{\substack{w \in W(G,S) \\ \| \\ W(G^{sc}, S^{sc})}} P_0(k)\dot{w}\underbrace{P_0(k)}_{Z(k) \ltimes U_0(k)}$

So can choose $\dot{w}$ to come from $N_{G^{sc}(k)}(S^{sc}) \subset G^{sc}(k)$.

and corresponding min. parabolic $P_0^{sc} \subset G^{sc}$ has $R_u(P_0^{sc}) \xrightarrow{\simeq} U_0$, So $U_0(k) \subset G(k)^{g}$. So $G(k)/G(k)^0$ is entirely rep'td by $Z(k)$, so

$$G(k)^1/G(k)^0 = \big(Z(k)\cap G(k)^1\big) \Big/ \underbrace{\big(Z(k)\cap G(k)^0\big)}_{(G(k)^g \cap Z(k))Z(k)^0}.$$

$\wr\wr$

finite ?

Since assuming $Z(k)^0 \subset Z(k)^1$ has finite index, and $Z(k)^1 \subset Z(k)\cap G(k)^1$ suffices to show

$$\big(Z(k)\cap G(k)^1\big)\Big/\big((G(k)^g \cap Z(k))Z(k)^1\big) \quad \text{is finite.}$$

Let's look $\dfrac{Z(k)\cap G(k)^1}{Z(k)^1}$. In study of relation of W_G and W_Z ("W_M")

$\left(\subset Z(k)/Z(k)^1 \subset \mathrm{Hom}(Z, \mathbb{G}_m)^*_{\mathbb{Q}}\right)$ we saw $\dfrac{Z(k)\cap G(k)^1}{Z(k)^1} \hookrightarrow X_*(S')_{\mathbb{Q}}$

$S' = (S \cap \mathcal{D}G)^0_{red}$, $Z = Z_G(S)$.

is a lattice, contains $X_*(S')$

$$\begin{array}{ccc}
Z(k)\cap G(k)^1 & \longrightarrow & X_*(S')_{\mathbb{Q}} \\
\cup & & \bigcup \\
Z(k)\cap (\mathcal{D}G)(k) & & \\
\cup & & \\
S'(k) & \longrightarrow\!\!\!\!\rightarrow & X_*(S')
\end{array}
\qquad \left(k^\times \underset{ord}{\longrightarrow\!\!\!\!\rightarrow} \mathbb{Z}\right).$$

Run same consideration for $G^{sc} \supset Z_{G^{sc}}(S^{sc}) = Z'$, $G^{sc}(k) = G^{sc}(k)^1$

$$\begin{array}{ccc}
G^{sc} & & Z' \\
\downarrow & & \downarrow \\
G & \supset & Z
\end{array}$$

to see $\underset{\text{image of } S^{sc}(k)}{\overset{}{G(k)^g \cap Z(k)}} \longrightarrow X_*(S')_{\mathbb{Q}} = X_*(S^{sc})_{\mathbb{Q}}$ has image containing

(image of $S^{sc}(k)$ $\subset$ $G(k)^g \cap Z(k)$)

$X_*(S^{sc}) \subset X_*(S)$
(finite index)

$\therefore \ \left(Z(k) \cap G(k)^1\right) / \left(G(k)^0 \cap Z(k)\right) Z(k)' = \text{lattice} / \text{lattice} = \text{finite}$

By $\underline{\underline{\text{defn}}}$, $Z(k)^0 = Z(k) \cap Z(K)^0$,
$Z(k)^1 = Z(k) \cap Z(K)^1$,

so $Z(k)^1 / Z(k)^0 \hookrightarrow Z(K)^1 / Z(K)^0$.

But Z_K is q-split, so its minimal Levi are tori.

Rerun argument "$G \rightsquigarrow Z$ over k" as "$Z \rightsquigarrow$ torus over K". $\square$

Back to Steinberg Theorem: $H^1(K, G) = 1$ for conn'd red. G.

Actual thm (more general) requires $\hat{k} | K$ be separable ($\Leftrightarrow \mathcal{O}_K$ is excellent dvr)

Lemma. $H^1(k, G) \hookrightarrow H^1(\hat{k}; G)$ for Henselian k, smooth affine G.

Say E, E' are G torsors over k, become isomorphic $/ \hat{k}$,

Want $E \simeq E'$. $\mathcal{I} = \mathrm{Isom}_G(E, E')$ is H-torsor w/ $\hat{k}$-pt
pre compose
w/ $H = \mathrm{Aut}_G(E) = $ form of G

$H^1(k, H) \to H^1(\hat{k}, H)$

$\xi \longmapsto$ trivial $\qquad \mathcal{I}(k) \overset{\text{dense}}{\subset} \mathcal{I}(\hat{k}) \neq \phi$

Want $\xi \overset{?}{=}$ trivial $\qquad$ smooth.

Lecture 13. Fields of dim ≤ 1 & non-positively curved metric spaces

For a field F, the followings are equivalent [Serre, Galois cohomology, Ch. II, § 3.1, Prop 5]

① $\underbrace{\mathrm{cd}}_{\text{torsion discrete modules}}(\mathrm{Gal}(F_s | F)) \le 1$, and when $\mathrm{char}(F) = p > 0$, also $\mathrm{Br}(E)[p] = 0$, $\forall$ finite separable $E|F$.

② $\mathrm{Br}(E) = 0$, $\forall$ finite separable $E|F$

③ $\forall$ finite sep. extn $E|F$ and finite Galois $L|E$, $N_{L|E} : L^\times \twoheadrightarrow E^\times$.

Such F are say to satisfy "$\dim(F) \le 1$".

Ex. (Lang) For discretely-valued henselian K w/ $f = \bar{f}$ and $\hat{K}|K$ separable, ($\Leftrightarrow$ $\mathcal{O}_K$ is excellent) [separability automatic when $K = \hat{K}$], then $\dim(K) \le 1$.

Thm (Steinberg) If F is a field w/ $\dim(F) \le 1$, then $H^1(F, G) = 1$, $\forall$ conn'd reductive F-gps G.

- Steinberg's pf assumed F is perfect (then allowed any smooth conn'd affine F-gp G)

See Lotner's write-up on course website to avoid perfectness of F.

Rmk. For F of $\dim \le 1$ and F-torus T, direct pf that $H^i(F, T) = 1$, $\forall i \ge 1$ is [KP, Lemma 2.5.4].

Upshot: we saw for K henselian, and smooth affine k-gp G, we have

$H^1(K,G) \hookrightarrow H^1(\hat{K},G)$ as sets.

$\therefore$ if $f=\bar{f}$, then $H^1(\hat{K},G)=\{*\}$ for conn'd reductive G,

So $H^1(k,G)=\{*\}$, too.

When $\underset{\curvearrowright}{\overset{G(k)}{}} \mathcal{B}(G)$ is constructed, it will be covered by affine spaces $\underset{}{\overset{N_G(S)(k)}{\curvearrowright}} \mathcal{A}(S)$

for $\underset{\underset{W(G,S)}{\circlearrowleft}}{X_*(S')_{\mathbb{R}}}$, where $S'=(S\cap \mathcal{D}G)^0_{red}$, $S=$ max. split torus.

Need a criterion for ① bounded subgp $K\subset G(k)$ to have a fixed pt.

$x\in \mathcal{B}(G) \; (\Rightarrow K\subset G(k)^1_x \underset{\text{later}}{=\!=} \text{bounded})$

② $\Gamma=\mathrm{Gal}(K|k) \curvearrowright \mathcal{B}(G_K)$ has fixed pts.

(define $\mathcal{B}(G)=\mathcal{B}(G_K)^{\Gamma}\neq\phi$, and make the desired structures on this).
e.g. to make $A(S)\subset \mathcal{B}(G)$

Rmk. $\mathcal{B}(G)$ is pt for G that is k-anisotropic.

Rmk Metric on $\mathcal{B}(G)$ restricted to each $A(S)$ "comes from" a $W(G,S)$-invariant inner product on $X_*(S')_{\mathbb{R}}$, and any two $x,y\in\mathcal{B}(G)$ lie in some common $\mathcal{A}(S)$.

Def. A <u>curve</u> in a metric space (X, ρ) is cont. $c:[0,1]\to X$,

and say c is <u>rectifiable</u> if $\ell(c) = \sup\limits_{0\le t_0<\dots<t_n\le 1} \sum_{i=0}^{n-1} \rho(c(t_i), c(t_{i+1})) < \infty$

Easy: if c is rectifiable, then so is $c|_{[a,b]}$ for $0\le a<b\le 1$.

Say c is a <u>geodesic</u> if rectifiable and "parameterized by arc length":

$$\ell(c|_{[t_0,t_1]}) = \rho(c(t_0), c(t_1)) = \rho(c(0), c(1))\,|t_0-t_1|.$$

Write $[x,y]$ for a geodesic from x to y (if one exists).

Call (X,ρ) a <u>geodesic space</u> if all $x,y\in X$ are joined by geodesic.

and say (X,ρ) is <u>uniquely geodesic</u> if $\forall\, x,y\in X$, $\exists!\,[x,y]$.

Ex. $\mathbb{R}^n$, affine spaces over $\mathbb{R}^n$, $\mathcal{B}(G)$.

Rmk. If (X,ρ) is geodesic, then $\rho(x,y) = \inf\limits_{c:x\to y} \ell(c)$.

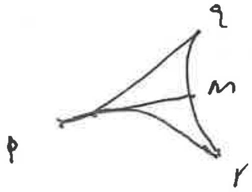
Def. Say (X,ρ) is <u>non-positively curved</u> if $\forall\, x,y\in X$, $\exists\, m\in X$ s.t.

$\forall z\in X$, $(\star)\ \rho(x,z)^2+\rho(y,z)^2 \ge 2\rho(m,z)^2 + \frac{1}{2}\rho(x,y)^2$ (Equality for $\mathbb{R}^n$).

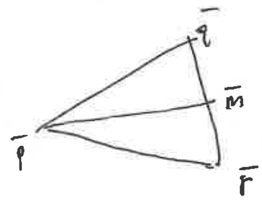
Lemma. A non-positively curved geodesic space is !ly geodesic, and m in $(\star)$ is unique: $m = c(\frac{1}{2})$ for $c=[x,y]$.

Pf: [KP, Lemma 1.1.11]

Rmk. In [BH, Part II, Exer 1.9]: for geodesic space X, $(\star)$ is equivalent to CAT(0), and to



and

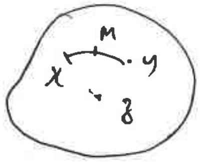


satisfies $\rho(p,m) \le d(\bar p, \bar m)$.

Lemma. For non-positively curved geodesic X, all $\bar B(z,r)$ are convex

$$\left(x,y \in \bar B(z,r) \Rightarrow [x,y] \subset \bar B(z,r) \right).$$

Pf



$$2r^2 \geq \rho(x,z)^2 + \rho(y,z)^2 \geq 2\rho(m,z)^2 + \frac{1}{2}\rho(x,y)^2$$
$$\geq 2\rho(m,z)^2 \qquad \square$$

Iterate using $[x,m]$, $[m,y]$!

For any bounded M in such X, have $M \subset$ some $\bar B(z,r)$ = convex, closed, bounded,

$$M \subset \bigcap_{\substack{C \supset M \\ \text{convex, closed, bounded}}} C =: \text{convex hull of } M,$$

convex, closed, bounded.

Next: Bruhat–Tits fixed pt lemma.

For $K \subset \mathrm{Isom}(X)$, note K preserves orbits $K.x_0 = M$.

Lecture 14 Affine Spaces and affine root systems

(G, T) $\qquad Ad: T \to GL(\mathfrak{g})$

$\uparrow$ ss-gp $\qquad$ $\uparrow$ max. split torus. $\qquad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_a \mathfrak{g}_a$

$0 \neq a \in X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$

$\Phi = \{a : \mathfrak{g}_a \neq 0\} \subset X^*(T)_{\mathbb{R}} = \mathfrak{t}^* =: V^*, \quad V = \mathfrak{t}$

(V, Φ) forms a <u>root system</u>:

<u>Def</u>. (root system) A root system consists of (V, Φ) where V f.d. $\mathbb{R}$-v.s., $\Phi \subset V^* \setminus \{0\}$ finite subset satisfying

(1) $\mathbb{R}\Phi = V^*$

(2) $\forall a \in \Phi, \exists a^\vee \in V$ s.t. • $\langle b, a^\vee \rangle \in \mathbb{Z}$, $\langle a, a^\vee \rangle = 2$

• $r_a : V^* \to V^*$, $\phi \mapsto \phi - \langle \phi, a^\vee \rangle a$, leaves Φ invariant

<u>Rmk</u>.
(0) Let $a \in \Phi$, Suppose $\lambda a \in \Phi \Rightarrow \lambda \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$

(1) $a^\vee$ is unique.

(2) $(V^*, \Phi^\vee = \{a^\vee\})$ is also a root system, called the dual root system.

(3) $W_\Phi = \langle r_a \rangle \subset GL(V^*)$ is a finite group, $W_\Phi = W_{\Phi^\vee}$.

(4) $\exists \Delta \subset \Phi$ s.t. every $a \in \Phi$ can be uniquely written as

$a = \sum_{a_i \in \Delta} \lambda_i a_i$ w/ either all $\lambda_i \in \mathbb{Z}_{\geq 0}$ or all $\lambda_i \in \mathbb{Z}_{\leq 0}$.

Δ is called a set of simple roots.

- $w(\Delta)$, $w \in W_\Phi$ is also a set of simple roots.
- $w(\Delta) = \Delta \Rightarrow w = 1$.
- For every two sets of simple roots Δ_1, Δ_2, $\exists!\ w$ s.t. $w(\Delta_1) = \Delta_2$.

(5) (V_1, Φ_1), (V_2, Φ_2) are two root systems, then

$(V_1 \oplus V_2, (\Phi_1, 0) \sqcup (0, \Phi_2))$ is a root system.

<u>Def</u>. A root system is called <u>irreducible</u> if it is <u>not</u> the product of two root systems.

<u>Def</u>. A root system is called <u>reduced</u> if $a \in \Phi \Rightarrow 2a \notin \Phi$.

<u>Rmk</u>. $\exists$ a complete classification of (irr.) reduced root systems.

A, B, C, D, E, F, G

<u>Affine situation</u>.

$G / K \supset \mathcal{O}$ ← local field, T max. split torus.

$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{a \in \Phi} \mathfrak{g}_a$, $\mathfrak{g}_a$ is a K- vector space.

↑ filtration of $\mathfrak{g}_a$ by $\mathcal{O}$-lattices satisfying certain properties.

filtrations of $\{\mathfrak{g}_a\}$ will be parametrized by affine roots.

Def. Let V be a v.s. $/(k=\mathbb{R})$. An affine space is a V-torsor.

for some f.d. v.s. V, i.e. a set equipped w/ a simply transitive

action of V.

Def. $A^* = \{\varphi : A \to k : \exists\, \dot{\varphi} \in V^*$ s.t. $\forall x \in A, v \in V, \ \varphi(x+v) = \varphi(x) + \dot{\varphi}(v)\}$

s.t. $\dot{\varphi}$ if exists, is unique, & called the derivative (vector part) of φ.

& is also denoted by $\nabla\varphi$.

$$0 \to k \to A^* \to V^* \to 0.$$
$$\varphi \mapsto \dot{\varphi}$$

Splitting $V^* \to A^*$ $\Leftrightarrow$ choosing a point $x \in A$.

Let (V_i, A_i), $i = 1,2$ be two affine spaces, an affine transformation

$F : A_1 \to A_2$ is a map s.t. $\exists\, \dot{F} : V_1 \to V_2$ linear,

s.t. $\forall x \in A_1, v \in V_1, \quad F(x+v) = F(x) + \dot{F}(v)$.

$(V_1, A_1) = (V_2, A_2) \rightsquigarrow \mathrm{Aff\,Aut}(A)$

$$1 \to V \to \mathrm{Aff\,Aut}(A) \to GL(V) \to 1.$$

$\alpha \in A^*$, $H_\alpha = \{x \in A : \alpha(x) = 0\}$ hyperplane, is an affine space

under the action of $\ker \dot{\alpha}$.

Again once choosing $x \in A$, $\begin{array}{c} V \xrightarrow{\sim} A \\ v \mapsto v + x \end{array}$, can write $\alpha = \dot{\alpha} + c$.

Def. An affine root system consists of (V, A, Φ_{aff}) where $V/\mathbb{R}$ f.d. v.s., A an affine space over V, $\Phi_{aff} \subset A^* - \{0\}$

s.t. (1) Φ_{aff} spans A^*

(2) $\forall \alpha \in \Phi_{aff}$, $\exists \alpha^\vee \in V$ s.t.

$\langle \dot{\alpha}, \alpha^\vee \rangle = 2$. $\langle \dot{\beta}, \alpha^\vee \rangle \in \mathbb{Z}$, $\forall \beta \in \Phi_{aff}$

$r_\alpha : A \to A$, $x \mapsto x - \alpha(x)\alpha^\vee$ leaves Φ_{aff} invariant

(3) $\forall a$, $\Phi_{aff, a} := \{\alpha : \dot{\alpha} = a\}$ has no accumulation points.

Rmk. (1) $\alpha^\vee$ is unique. In fact, $(V, \Phi = \{\dot{\alpha}\})$ is a root system.

(2) $W_{aff} = \langle r_\alpha \rangle \in \mathrm{Aff\,Aut}(A)$

$1 \to X_* \to W_{aff} \to W_\Phi \to 1$

lattice in V.

Lecture 15 Fixed pt thms and more on affine spaces

(☆) Given $x, y \in X$, $\exists m \in X$ s.t. $\forall z \in X$,

$\rho(x,z)^2 + \rho(y,z)^2 \geq 2\rho(m,z)^2 + \frac{1}{2}\rho(x,y)^2$] non-positively curved

Thm. Let X be complete non-positively curved geodesic space.

① If $M \subset X$ is nonempty closed bounded convex subset, and define

$$f: M \longrightarrow \mathbb{R}$$

$$y \longmapsto \operatorname{diam}(y, M) = \sup_{y' \in M} \rho(y, y')$$

and "radius" $r(M) = \inf_{y \in M} f(y) \geq 0$

then $\exists !$ barycenter $m_0 \in M$ (means $\operatorname{diam}(m_0, M) = r(M)$).

Rmk. NOT the usual notion of barycenter when $M \subset \mathbb{R}^n$ is n-simplex.

In particular, m_0 is a fixed pt of $\operatorname{Stab}_{\underbrace{\operatorname{Isom}(X)}_{\text{isometry}}}(M)$

$$\operatorname{Stab}_{\operatorname{Isom}(X)}(M) = \{\varphi \in \operatorname{Isom}(X) : \varphi(M) = M\}$$

② (Bruhat–Tits fixed pt lemma) For nonempty bounded $M \subset X$,

$\operatorname{Stab}_{\operatorname{Isom}(X)}(M)$ has a fixed pt $x_0 \in X$.

Ex. $X = \mathcal{B}(G)$, $M =$ some "facet" in $\mathcal{A}(S)$ w.r.t. an affine root system. $G(k) \circlearrowright$ via isometries

Pf. ① $\Rightarrow$ ②: $\operatorname{Stab}_{\operatorname{Isom}(X)}(M) \subset \operatorname{Stab}_{\operatorname{Isom}(X)}(\operatorname{Conv}(M))$, apply ① to $\operatorname{conv}(M)$.

Pf of ①. Consider pts $y \in M$ where $f(y) = \mathrm{diam}(y, M) \approx r(M)$

and show such y's "get collectively close" and build m_0 as Cauchy seq. limit.

Suppose $r(M) = 0$, then forces M to be a pt:

$$\forall x, x' \in M, \ y \in M, \quad \rho(x,x') \le \rho(y,x) + \rho(y,x') \le 2\,\mathrm{diam}(y, M)$$

$\Rightarrow \rho(x,x') = 0, \ x = x'$.

$\therefore$ Can assume $\underset{r}{r(M)} > 0$. Pick $0 < \varepsilon < r$. Consider $y, y' \in M$

s.t. $f(y), f(y') \le r + \varepsilon$. We claim $\rho(y,y')^2 \le 16 r \varepsilon$.

Pf: M convex $\Rightarrow [y,y'] \subset M \Rightarrow$ Midpt m of $[y,y']$ is in M too.

$\therefore f(m) = \mathrm{diam}(m, M) \ge r$. $\therefore \exists z \in M$ s.t. $\rho(m,z) \ge r - \varepsilon$.

$$2(r+\varepsilon)^2 \ge f(y)^2 + f(y')^2 \ge \rho(y,z)^2 + \rho(y',z)^2$$

$$\ge 2\rho(m,z)^2 + \frac{1}{2}\rho(y,y')^2$$

$$\ge 2(r-\varepsilon)^2 + \frac{1}{2}\rho(y,y')^2$$

$\Rightarrow \rho(y,y')^2 \le 16 r\varepsilon$.

Pick $\varepsilon_n \to 0^+$, and $y_n \in M$ w/ $f(y_n) \le r + \varepsilon_n$. For $m \ge n$,

$\rho(y_m, y_n)^2 \le 16 r \varepsilon_n \Rightarrow \{y_n\}$ is a Cauchy sequence, so $\exists\, m_0 = \lim_{n\to\infty} y_n \in M$ (closed)

and $f: y \mapsto \mathrm{diam}(y, M)$ is continuous, so $f(m_0) = r$.

Uniqueness: If $m_1 \in M$ also satisfies $f(m_1) = r$, then

$$f(m_0), f(m_1) \leq r + \varepsilon, \ \forall\, 0 < \varepsilon < r, \text{ so}$$

$$\rho(m_0, m_1)^2 \leq 16 r \varepsilon, \ \forall\, 0 < \varepsilon < r, \text{ so } m_0 = m_1. \quad \square$$

Ex. For $X = B(G)$, $K \subset G(k)$ is <u>bounded</u> subgp, then

$M = K \cdot x_0 \subset B(G)$ is bounded, and $K \subset \mathrm{Stab}_{\mathrm{Isom}(X)}(M)$,

so K has fixed pt in $B(G)$.

$\therefore$ as x varies through $B(G)$, $\underbrace{G(k)_x^1}_{\text{will be bounded}}$ will capture all bounded K.

Likewise, $\Gamma = \mathrm{Gal}(K|k) \curvearrowright B(G_K)$ via isometries w/ bounded orbits,

so $B(G_K)^\Gamma \neq \phi$ (will be the <u>construction</u> of $B(G)$).

Prop. Let (X, ρ) be a non-positively curved complete geodesic space,

$Y \subset X$ closed convex, $\neq \phi$,

1. $\forall\, x \in X$, $\exists !\, y = \pi(x) \in Y$ at which $\rho(x, y)$ is minimized.
2. $\pi : X \longrightarrow Y$ is continuous.
3. π is equivariant for $\mathrm{Stab}_{\mathrm{Isom}(X)}(Y)$.

Rmk. This is often applied to $Y = \mathcal{A}(S) \subset B(G) = X$

Pf. [KP, Prop 1.1.20] ① + ② $\Rightarrow$ ③ clear

Pf of ①: Similar to the preceding Cauchy argument

Use $r(x) = \inf_{y \in Y} \rho(x,y) \geq 0$.

If $r(x) = 0$, then $x \in Y$ by closedness, so $\pi(x) = x$ works.

If $r(x) > 0$, for $0 < \varepsilon < r(x)$, $\rho(x,y_1), \rho(x,y_2) < r + \varepsilon$

$\Rightarrow \rho(y,y') < 16 r(x) \varepsilon$.

Lecture 16. Surprises w/ affine root systems:

Loose ends on affine spaces Let A be an affine space for a f. dim'l vec. sp. V over a field L (e.g. $L = \mathbb{R}$)

Rmk ($L=\mathbb{R}$). Picking an inner product on V yields translation-invariant metric on V, hence on A, i.e. $d(a,a') = \|\underbrace{a-a'}_{\in V}\|$

Let $A^* = \{A \xrightarrow{\text{affine}} L\}$ = L-vec. sp. of affine L-valued functions on A

of $\dim = \underbrace{1}_{\text{constant functions}} + \dim V = 1 + \dim A$.

SES $0 \to L \to A^* \xrightarrow{\nabla} V^* \to 0$

$\psi \longmapsto \dot{\psi}$

affine hypeplane $\{\psi = 0\} \subset A$

① $\forall \psi \in A^* - L$ ($\Leftrightarrow \dot{\psi} \in V^* - \{0\}$), $H_{\psi+s} \cap \overbrace{H_\psi}^{\{\psi=0\} \subset A} = \phi$ for $s \in L^\times$.

② For $\psi, \eta \in A^* - L$, $H_\psi = H_\eta \iff \eta = s\psi$ for some $s \in L^\times$.

Also, $\nabla(s\psi) = s\nabla\psi$, so $s\psi$ is characterized by (i) $H_{s\psi} = H_\psi$

(ii) $\nabla(s\psi) = s\nabla\psi$

$\uparrow$

determines $s\psi$ up to adding elt. of L.

③ $\underbrace{(A^*)^*}_{\text{vector space}}$ is NOT A'^* for any naturally associated affine space A' for $V^{**} = V$.

$\therefore$ to define $\Psi^\vee$ for affine root system (A, Ψ), we'll have to make a choice that realizes $\Psi^\vee \subset A^*$.

④ For (V, A) an affine space and $V = V_1 \oplus V_2$ (e.g. (A, Ψ) affine rootsyst and decompose $(V^*, \nabla\Psi = \Phi)$ into irred. constituents),

then $A_1 = A/V_2 \circlearrowleft V/V_2 = V_1$
$A_2 = A/V_1 \circlearrowleft V/V_1 = V_2$ are affine spaces

and
$$\begin{array}{ccc} A & \xrightarrow{\sim} & A_1 \times A_2 \\ \circlearrowleft & & \circlearrowleft \\ V & \xrightarrow{\sim} & V_1 \oplus V_2 \end{array}$$
Same for $V = \bigoplus_{i=1}^{n} V_i$

$\rightsquigarrow$
$$\begin{array}{ccccccc} 0 \to L \to & A_1^* \oplus A_2^* & \xrightarrow{q} & A^* & \to 0 \\ \| & \cup & & \cup & \\ \{(c,-c) : c \in L\} & L \oplus L & \xrightarrow{+} & L & \end{array}$$
, Similarly $\bigoplus A_j^* \twoheadrightarrow (\prod A_j)^*$ has kernel $\{\sum x_j = 0\} \subset L^n$.

Surjectivity of q ensures spanning property for $(A_1 \times A_2, \Psi_1 \amalg \Psi_2)$.

Now $L = \mathbb{R}$

<u>Def</u>. An affine root system $\underset{\substack{(A, \Psi) \\ \hat{A}^* - \mathbb{R}}}{}$ is <u>reducible</u> if $\Psi = \Psi_1 \sqcup \Psi_2$, w/ $\Psi_j \neq \phi$

and $\forall\, \psi_j \in \Psi_j$, $\langle \dot{\psi}_1, \dot{\psi}_2^\vee \rangle = 0$ (with $\dot{\psi}_1 \in V^*$, $\dot{\psi}_2^\vee \in V$)

(axioms for affine root system include $r_{\psi, \dot{\psi}^\vee}$)

and $\langle \dot{\psi}_2, \dot{\psi}_1^\vee \rangle = 0$

i.e. $r_{\psi_2}\big|_{\Psi_1} = \mathrm{id}$.

Want $A = A_1 \times A_2$ wrt. some $V = V_1 \oplus V_2$, s.t. $A_1^*, A_2^* \hookrightarrow A^*$ have

$\Psi_j \subset A_j^*$ (inside A^*) and (A_j, Ψ_j) is affine root system.

Look at $\Phi = \nabla \Psi \supset \Phi_j = \nabla(\Psi_j) \subset (V^* - \{0\})$ where $\Phi = \Phi_1 \sqcup \Phi_2$.

Satisfying defn of reducibility for (V^*, Φ).

$\therefore V_j^* = \mathrm{span}(\Phi_j)$ yields $V^* = V_1^* \oplus V_2^*$, making

$(V^*, \Phi) = (V_1^*, \Phi_1) \times (V_2^*, \Phi_2)$

This gives $V = V_1 \oplus V_2$, so $\underset{\substack{\circlearrowleft \\ V_1}}{A} = A_1 \times A_2$ ... more precisely $A = \underset{\substack{\curvearrowleft \\ V_1}}{A_1} \times \underset{\substack{\curvearrowleft \\ V_2}}{A_2}$ w/ $\Psi_j \subset A_j^*$ $(\subset A^*)$

Want (A_j, Ψ_j) is affine root system. Non-obvious part is that for

$\psi \in \Psi_1$, $r_{\psi, \dot{\psi}^\vee} \curvearrowright A$ respects $A_1 \times A_2$ via reflection on A_1, and id on A_2.

This uses notion of "basis" and "chamber" for affine root systems.

Rmk. (A, Ψ) is irreducible if not reducible.

($\sim (0, \phi)$ is NOT affine root system, $\because$ affine dual 0^* is $\mathbb{R}$, not spanned by ϕ)

To speak usefully of "ired. decomp." of (A, Ψ), need notion of isom for affine root systems.

(i) Natural to try this defn: $f : (A, \Psi) \xrightarrow{\sim} (A', \Psi')$ means isom. of affine spaces $f : A \xrightarrow{\sim} A'$ s.t. $\underset{\text{lof}}{A^*} \xleftarrow{\sim} \underset{\text{l}}{A'^*}$: f^* carries Ψ' onto Ψ.

Problem. Effect of f^* on lines $\mathbb{R} \subset A^*$, $\mathbb{R} \subset A'^*$ of constant funcs is identity map!

(Above notion of isom. is too restrictive)

Consider $(A, s\Psi)$ for $s \in \mathbb{R}^\times - \{\pm 1\}$.

$(A, s\Psi) \not\simeq (A, \Psi)$ in ired. cases when $s \neq \pm 1$.

Suppose are, then apply $\nabla : V^* \xrightarrow[\sim]{\text{linear}} V^*$
$s\Phi \xrightarrow{\sim} \Phi$ — ~~two~~ such are mult. by $\pm s$, and would have to be one of these.

But effect of isom. on A^* has to be id on $\mathbb{R}$. Need broader notion.

(ii) Next time, given (V, Φ), we'll build $(\underset{\text{for } V}{A}, \Psi_\Phi)$, where $\nabla(\Psi_\Phi) = \Phi$,

but $(\Psi_{\Phi^\vee})^\vee \not\simeq \Psi_\Phi$ for Φ not simply-laced.

Lecture 17. Construction and Duality of Affine Root Systems

We defined the notion of reducibility for (A, Ψ), and from that defn, we see Ψ is reducible $\Leftrightarrow$ $\Phi = \nabla\Psi$ is reducible; and so Ψ is irred. $\Leftrightarrow$ Φ is irred.

$\therefore$ to find all irred. Ψ (possibly non-reduced), we should pick irred. Φ (known via conn'd Dynkin diagram in reduced case, and BC_n $\forall n \geq 1$ in non-reduced case), and seek Ψ w/ $\nabla\Psi = \Phi$.

First, let's review reflections in axioms for (A, Ψ) for A an affine space for V: $\forall \psi \in \Psi$, $\exists \overset{\circ}{\psi}{}^\vee \in V = (V^*)^*$, s.t. the linear reflection

$$r_{\psi,\overset{\circ}{\psi}{}^\vee} : A^* \longrightarrow A^*$$
$$y \longmapsto y - \overset{\circ}{y}(\overset{\circ}{\psi}{}^\vee)\psi \qquad (\text{reflection} \Leftrightarrow \langle \overset{\circ}{\psi}, \overset{\circ}{\psi}{}^\vee \rangle = 2)$$

preserves Ψ.

Say $r: A \longrightarrow A$ affine transformation is an (affine) reflection if $r \neq 1$, $r^2 = 1$, and $\exists$ affine hyperplane $H \subset A$ st. $r|_H = \mathrm{id}$.

Check: $r_\psi : A \longrightarrow A$, is an affine reflection, and
$$x \longmapsto x - \underbrace{\psi(x)\overset{\circ}{\psi}{}^\vee}_{\in V}$$

$$A^* \longrightarrow A^*$$
$$y \longmapsto y \circ r_\psi$$
is $r_{\psi,\overset{\circ}{\psi}{}^\vee}$.

Inspired by [Bourbaki, Ch VI, §2, no. 1]

<u>Construction</u>: Let (V, Φ^*) be a root system, not $(0, \phi)$, possibly non-reduced,

Let $A = V = (V^*)^*$. Define $\Psi = \Psi_\Phi \subset A^* = V^* \oplus \mathbb{R}$ as follows:

$\Psi = \{a+n : a \in \Phi, n \in I_a\}$. where $I_a = \begin{cases} \mathbb{Z}, & a \notin 2\Phi \ (\Leftrightarrow a \in \Phi^{nd}) \\ 2\mathbb{Z}+1, & a \in 2\Phi (\Rightarrow \frac{a}{2} \in \Phi^{nd}) \end{cases}$

Using $(\nabla(a+n))^\vee = a^\vee \in \Phi^\vee$, $\forall n \in I_a$
$\subset V \setminus \{0\}$.

This <u>is</u> affine root system, where $\overset{\text{affine reflection}}{r_{a+n}} : A \to A$

$$x \mapsto \underbrace{(x - a(x)a^\vee)}_{r_a(x)} - na^\vee$$

Also, $\nabla \Psi = \Phi$.

(really $r_{a^\vee} : V \simeq V$ from $\Phi^\vee$)

<u>Rmk</u>. ① $I_a = 2\mathbb{Z}+1$ for $a \in 2\Phi$ makes Ψ <u>reduced</u>:

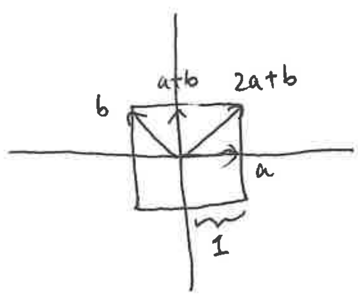
if $a = 2b$ for $b \in \Phi$, then $b \in \Phi^{nd}$ and $a + 2k = 2\underbrace{(b+k)}_{\uparrow\, I_b} \in 2\Psi$

If took $I_a = \mathbb{Z}$ $\forall a \in \Phi$, then would get affine root system, but non-reduced when Φ is. Thus, $\forall$ irred. Φ (perhaps non-reduced), get reduced <u>irred</u> Ψ w/ $\nabla\Psi = \Phi$.

② If replace $I_a \subset \mathbb{R}$ w/ $s \cdot I_a$ for $s \in \mathbb{R}^\times$, then get $s \cdot (\Psi_{\frac{1}{s}\Phi})$ since $a + ns = s(\frac{1}{s}a + n)$, where $\frac{1}{s}\Phi \simeq \Phi$, but $s\Psi, \Psi$ not isom. in the sense of last time for $s \neq \pm 1$, Ψ irred.
$H_{s\psi} = H_\psi$, $r_{s\psi} = r_\psi$, $(s\psi)^\vee = \frac{1}{s}\psi^\vee$

Come back to "right" notion of isom. next time.

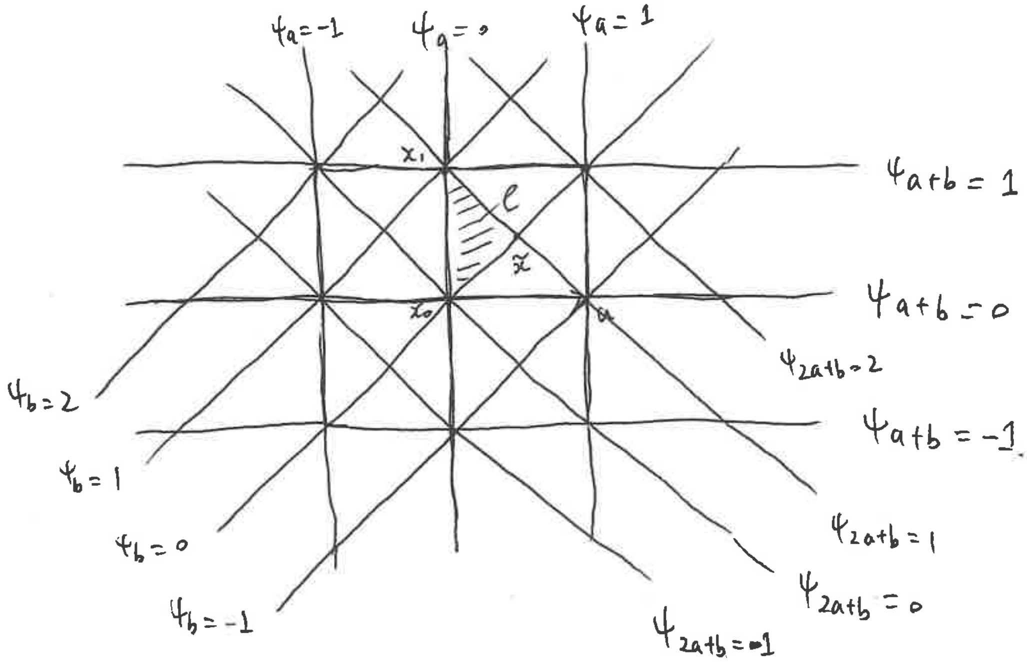
$\underline{\text{Ex}}$. $\Phi = B_2 = C_2$, choice of Φ^+ ($\Leftrightarrow$ basis $\Delta \subset \Phi$)



$V = \mathbb{R}^2 \simeq \mathbb{R}^2 = V^*$
dot product

affine roots $\psi_{c+n} = \langle c, - \rangle + n$

for $c \in \Phi$, $n \in \mathbb{Z}$.



Conn'd components of $A - \left(\bigcup_{\psi \in \Psi} H_\psi \right)$ are called $\underline{\text{chambers}}$.

e is cut out by 3 inequalities $\psi_a > 0$, $\psi_b > 0$, $\psi_{2a+b} < 1$

($1 - \psi_{\underbrace{2a+b}_{\text{highest root in } \Phi^+}} > 0$)

$\bar{e}$ is not equilateral, unlike Φ_{A_2}.

x_0, x_1 are "hyperspecial" vertices ($K_j = G(k)^1_{x_j}$ is $\mathcal{O}$-pts of reductive $\mathcal{O}$-model, $\mathcal{G}_{x_j}$, where K_j's are $\underline{\text{not}}$ $G(k)$-conjugate.)

and $\tilde{x}$ is not hyperspecial, and $\tilde{K} = G(k)^1_{\tilde{x}}$ is $\mathcal{O}$-pts of $\mathcal{O}$-model $\tilde{\mathcal{G}}$

w/ $(\tilde{\mathcal{G}}_t)/R_u \simeq SL_2 \times SL_2$.

$\tilde{\mathcal{G}}_t$ conn'd of NOT reductive

Next time: $(\Psi_\Phi)^\vee$ not always $\Psi_{\Phi^\vee}$.

Lecture 18 Duality, Isomorphisms, and Classification

Ex. $(\overset{\curvearrowleft V}{A}, \Psi)$, (W^*, Φ')

$$\underset{\substack{\circlearrowleft \\ V \oplus W}}{(A \times W)^*} = \underset{\text{affine dual}}{A^*} \oplus \underset{\text{linear dual}}{W^*}$$

$\Psi \times \Phi' \subset A^* \oplus W^* = (A \times W)^*$ w/ $\psi^\vee, (a')^\vee$: satisfies all axioms, including spanning.

If $\Phi = \nabla \Psi$ is reducible: $\Phi = \prod \Phi_i$, $V^* = \prod V_i^*$, then $\rightsquigarrow A = \prod A_i$

$$\Psi_i = \nabla^{-1}(\Phi_i) \subset \Psi \subset A^* = \left(\prod A_i\right)^* \twoheadleftarrow \prod A_i^* \leftarrow A_i^*$$

has $\Psi_i \subset A_i^*$ $(\subset A^*)$, and $\psi_i^\vee \in V^*$ live in V_i^*, and

(A_i, Ψ_i) satisfies all affine root system axioms, except possibly spanning:

$$\begin{array}{ccc} \mathrm{span}(\Psi_i) & \subset & A_i^* \\ \downarrow & & \downarrow{\scriptstyle \nabla} \\ \mathrm{span}(\Phi_i) & = & V_i^* \end{array}$$

Q: Is $\mathrm{span}(\Psi_i) \cap \mathbb{R} \neq \phi$?

<u>Fix</u>: Should add in axiom that $\forall a \in \Phi = \nabla \Psi$, $\exists \geq$ two $\psi \in \Psi$ s.t. $\dot{\psi} = a$, i.e. $\forall H_\psi$, $\exists \psi' \neq \psi$ in Ψ, s.t. $H_{\psi'} \cap H_\psi = \phi$.

Let $(\overset{\vee}{\tilde{A}}, \Psi)$ be an affine root system. To define dual $\Psi^\vee$, we work inside A^* because linear dual $(A^*)^*$ does not seem to be $(A')^*$ for some "natural" affine space A' for V^*. Let (V^*, Φ) be $\nabla\Psi$.

Pick inner product $\langle \cdot, \cdot \rangle$ on V that is invariant under $W(\Phi^\vee) = W(\Phi)^*$.

<u>Rmk</u>. If Φ is irred., then $W(\Phi) \curvearrowright V^*$ is <u>abs. irred.</u> so $\langle \cdot, \cdot \rangle$ is unique up to $\mathbb{R}_{>0}$-scaling. In general, $\Phi = \coprod \Phi_i$ $(= \oplus \Phi_i)$ is irred. decomp., so $V^* = \oplus V_i^*$ $(\Rightarrow V = \oplus V_i)$. $W(\Phi) = \prod W(\Phi_i) \curvearrowright \prod V_i^*$, abs. irred. on each factor. So $\langle \cdot, \cdot \rangle$ must be $\perp \langle \cdot, \cdot \rangle_i$, unique up to $\mathbb{R}_{>0}$ on each $\langle \cdot, \cdot \rangle_i$. We already have $(A, \Psi) = \prod \underset{\text{irred}}{(A_i, \Psi_i)}$

What we've about to do can be viewed as $\Psi^\vee := \prod (\Psi_i^\vee)$.

<u>Recall</u>. For usual root systems (W, Φ), upon choosing Weyl-inv. $\langle \cdot, \cdot \rangle$ to identify $W^* \simeq W$, get $a^\vee = \dfrac{2a}{\langle a, a \rangle}$ $\forall a \in \Phi$.

<u>Construction</u>. Define $\psi^\vee = \dfrac{2\psi}{\langle \dot{\psi}, \dot{\psi} \rangle} \in A^*$ $(\underset{\text{nonzero}}{\dot{\psi} \in V^* \simeq V})$

$$\nabla(\psi^\vee) = \frac{2\dot{\psi}}{\langle \dot{\psi}, \dot{\psi} \rangle} = (\dot{\psi})^\vee \quad \text{usual coroot for } \Phi \quad (\text{in axioms for } \Psi!)$$

and $\langle \dot{\psi}, \nabla(\psi^\vee) \rangle = 2$ (want $\dot{\psi}$ to be "coroot" for $\psi^\vee$)

So have reflection $r_{\psi^\vee, \dot\psi} : A^* \longrightarrow A^*$

$$y \longmapsto y - \dot y(\underbrace{\dot\psi)\psi^\vee}_{\in V^* \simeq V}.$$

$$\uparrow_{\in V^*}$$

$$= y - \langle \dot y, \nabla(\psi^\vee)\rangle \psi$$

$$= r_{\psi, \dot\psi^\vee}(y)$$

$r_{\psi^\vee, \dot\psi} = r_{\psi, \dot\psi^\vee}$. Let $\Phi^\vee = \{\psi^\vee : \psi \in \Phi\}$

<u>Claim</u>: $r_{\psi^\vee, \dot\psi} : A^* \xrightarrow{\sim} A^*$ carries Φ^* into itself.

<u>Pf</u>. For $\psi, \eta \in \Phi$, $\underset{\underset{\frac{2\psi}{\langle\dot\psi,\dot\psi\rangle}}{\parallel}}{r_{\eta^\vee}(\psi^\vee)} = \frac{2}{\langle \dot\psi, \dot\psi\rangle} \underbrace{r_{\eta^\vee}}_{r_\eta}(\psi) = \frac{2}{\langle \dot\psi, \dot\psi\rangle} r_\eta(\psi)$

$$\overset{?}{=} r_\eta(\psi)^\vee.$$

So need $\langle \dot\psi, \dot\psi\rangle \overset{?}{=} \langle \nabla(r_\eta(\psi)), \nabla(r_\eta(\psi))\rangle$. But $\nabla(r_\eta(\psi)) = \underbrace{r_{\dot\eta}}_{\in W(\Phi)}(\dot\psi)$

and $r_{\dot\eta}$ leaves $\langle \cdot, \cdot \rangle$ unchanged. ▨

This verifies all axioms for $(A, \Phi^\vee)$ since $H_{\psi^\vee} = H_\psi$.

<u>Def</u>. The preceding is called <u>dual</u> affine root system.

By construction, $\nabla(\Phi^\vee) = \Phi^\vee$ ($\subset V^* \simeq V$).

<u>Puzzle</u>: What happens if change $\langle \cdot, \cdot \rangle$? Rescales each $\Phi_i^\vee$ for irred. components Φ_i of Φ.

Prop. Using same $\langle\cdot,\cdot\rangle$ to identify $(V^*)^*$ w/ V^*, get $(\Psi^\vee)^\vee = \Psi$ inside A^*.

Pf. $\nabla(\psi^\vee) = \dfrac{2\dot\psi}{\langle\dot\psi,\dot\psi\rangle}$, so $(\psi^\vee)^\vee = \dfrac{2\psi^\vee}{\left(\frac{4\langle\dot\psi,\dot\psi\rangle}{\langle\dot\psi,\dot\psi\rangle^2}\right)} = \dfrac{\langle\dot\psi,\dot\psi\rangle}{2}\psi^\vee = \psi$. $\square$

Ex. Consider $\Psi = \Psi_\Phi$ for irred. reduced (V^*, Φ), using $\langle\cdot,\cdot\rangle$ so longer root length is $\sqrt{2}$.

Check, $(\Psi_{\Phi^\vee})^\vee = \{a+n : a \in \Phi,\ n \in J_a\}$. $J_a = \begin{cases} \mathbb{Z}, & a \text{ of longest length} \\ \frac{1}{\ell}\mathbb{Z}, & a \text{ of shortest length.} \end{cases}$

$\nabla = \Phi^{\vee\vee} = \Phi$

$\ell = \left(\frac{\text{long}}{\text{short}}\right)^2 \in \{1,2,3\}$

$\therefore$ for $\ell = 1$ (simply laced), have $(\Psi_{\Phi^\vee})^\vee = \Psi_\Phi$

$\left(\Leftrightarrow \Psi_{\Phi^\vee} = (\Psi_\Phi)^\vee\right)$.

but for $\ell = 2,3$ (B_n, C_n $(n \geq 2)$, F_4, G_2), get new reduced irred. root systems!

Put in notes good def'n of "isom" clarifies naive notion + scaling

Thm [KP, Thm 1.3.63] The irred. reduced Ψ are exactly (up to isom) are exactly Ψ_Φ's or their duals.

Lecture 19   Bases and special points

Loose end: let's discuss classification of non-reduced irred. affine root systems Ψ.

Necessarily $\Phi = \nabla\Psi$ is irred. and non-reduced ($\psi = 2\eta \Rightarrow \dot{\psi} = 2\dot{\eta}$)

so $\Phi = BC_n$, $n = \dim A = \dim V^*$. $\Psi = \Psi^{nd} \cup \Psi^{nm}$,

where $\underbrace{\Psi^{nd} = \{\psi \in \Psi : \psi \notin 2\Psi\}, \quad \Psi^{nm} = \{\psi \in \Psi : 2\psi \notin \Psi\}}$

are both affine root systems w/ same hyperplanes

and reflections as Ψ, and these are reduced and irred.

(so same Weyl gp)

Possibilities for "irred. + reduced" are known, organized by derivative root system.

$$\underset{B_n}{\Phi^{nd}} \subset \nabla(\Psi^{nd}) \subset \nabla(\Psi) = \underset{BC_n}{\Phi}$$

intermediate, stable under common Weyl gp. of rk n

$a = \dot{\psi} \in \Phi^{nd} \Rightarrow \psi \notin 2\Psi$

$$\underset{C_n}{\Phi^{nm}} \subset \nabla(\Psi)^{nm} \subset \nabla(\Psi) = \underset{BC_n}{\Phi}$$

$a = \dot{\psi} \in \Phi^{nm} \Rightarrow 2\psi \notin \Psi$

$$\Rightarrow \nabla(\Psi^{nd}) = \underset{BC_n}{\Phi} \text{ or } \underset{B_n}{\Phi^{nd}}, \qquad \nabla(\Psi^{nm}) = \underset{BC_n}{\Phi} \text{ or } \underset{C_n}{\Phi^{nm}}$$

$$\therefore \Psi^{nd} \in \{\Psi_{BC_n}, \Psi_{B_n}, \underbrace{\Psi_{C_n}^{\vee}}_{(n>1)}\}, \quad \Psi^{nm} \in \{\Psi_{BC_n}, \Psi_{C_n}, \underbrace{\Psi_{B_n}^{\vee}}_{(n>1)}\}$$

In errata for [KP, Thm 1.3.69].

Thm Nonreduced irred. Φ is determined up to isom by pair of isom. classes of Φ^{nd}, Φ^{nm}, and possibilities for (Φ^{nd}, Φ^{nm}) are $(\Phi_{B_n}, \Phi_{B_n}^\vee)$, $(\Phi_{C_n}^\vee, \Phi_{C_n})$ } same for $n=1$.

Pf. Pf first gives explicit construction of all 4, and then pf. of exhaustiveness. (via more refined analysis)

$(\Phi_{BC_n}, \Phi_{C_n})$, $(\Phi_{C_n}^\vee, \Phi_{BC_n})$ } dual

Let (A, Φ) be affine root system, let $\mathcal{C} \subset A$ be chamber (= conn'd component of $A - \left(\bigcup_{\psi \in \Phi} H_\psi\right)$)

For $\psi \in \Phi$, $H_\psi \cap \underset{\text{conn'd}}{\underline{\mathcal{C}}} = \phi$.

so $\mathcal{C} \subset A^{\psi > 0}$ or $A^{\psi < 0}$.

Say ψ is <u>positive</u> (resp. negative) w.r.t. $\mathcal{C}$ if $\psi(\mathcal{C}) \subset \mathbb{R}_{>0}$ or $\subset \mathbb{R}_{<0}$.

This gives $\Phi = \Phi(\mathcal{C})^+ \sqcup \Phi(\mathcal{C})^-$.

The <u>basis</u> $\Delta_{\mathcal{C}}$ is $\{\psi \in \Phi^{nd} : \underbrace{\psi \in \Phi(\mathcal{C})^+,\ H_\psi \text{ is a wall of } \mathcal{C}}_{\mathcal{C} \text{ is on the "positive side" of the wall}}\}$

and call elts of $\Delta_{\mathcal{C}}$ <u>simple</u> w.r.t. $\mathcal{C}$.

<u>Rmk</u>. $\mathcal{C} = \bigcap_{\psi \in \Delta_{\mathcal{C}}} A^{\psi > 0}$.

If $\Psi = \oplus \Psi_i$ is irred. decomp. $(A = \prod A_i)$, then $\mathcal{C} = \prod \mathcal{C}_i$.

where $W(\Psi_i) \curvearrowright V_i$ is (abs) irred., and $W(\Psi_i)$ is infinite.

$\left(\psi \in \Psi_{i_0} \Rightarrow H_\psi = \left(\prod_{i \neq i_0} A_i\right) \times \left(\psi = 0 \text{ in } A_{i_0}\right)\right)$

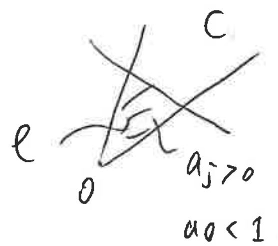
$(r_{H+v} \circ r_H =$ translation by $2v)$

By [Bourbaki, Ch IV, §3, no.9, Prop 8], each $\mathcal{C}_i$ is an open simplex.

So $\mathcal{C}$ is "polysimplex", and $\bar{\mathcal{C}} \underset{\text{bijection}}{\cong} A/W(\Psi)$. ([Bourbaki, Ch V, §3, no.3, Thm 2])

Prop. If Ψ irred, then $\Delta_{\mathcal{C}} = \{\psi_1, \dots, \psi_n\}$ is a basis of A^*, and $\{$vertices of $\bar{\mathcal{C}}\} = \{x_0, \dots, x_n\}$, where $\psi_i(x_j) \in \delta_{ij} \mathbb{R}_{>0}$.

Ex. Let $\Psi = \Psi_\Phi$ for Φ reduced and irred. Pick basis Δ_0 of Φ, to get Φ^+, cone chamber C of V^* for Φ^+



$\exists!$ chamber $\mathcal{C}$ of Ψ w/ $\mathcal{C} \subset C$ and $0 \in \bar{\mathcal{C}}$

this $\mathcal{C}$ has $\Delta_{\mathcal{C}} = \{a_1, \dots, a_n, 1 - a_0\}$,

where $a_0 =$ highest root in Φ^+. [Bourbaki, Ch VI, §2, no.2, Prop. 10]

$-\Delta$ is never a basis for affine root systems!

Thm [KP, Props 1.3.20/22]. Let (A, Φ) be affine root system,

$W(\Phi) \overset{\text{def}}{=} \langle r_H \rangle_{H = H_\psi}$ for $\psi \in \Phi$.

① $W(\Phi)$ acts simply transitively on {chambers} ($\Leftrightarrow$ {bases})

② $W(\Phi) = \langle r_H \rangle_{H \in \text{wall}(e)}$ for any chamber e.

Cor 1. $\Phi^{nd} = W(\Phi) . \Delta$ for $\Delta = \Delta_e$ for choices of e.

Pf. Pick $\psi \in \Phi^{nd}$.

Pick one such e' in $A^{\psi > 0}$ w/ $H_\psi \in \text{wall}(e')$, so $\psi \in \Delta_{e'}$.

But $W(\Phi)$ moves e' to e, hence $\Delta_{e'}$ onto Δ_e.

Cor 2. Each $\psi \in \Phi(e)^+$ is in $\underline{\mathbb{Z}_{\geq 0} \cdot \Delta}$ for $\Delta = \Delta_e$ for choice of e

unique if Φ ired ($\Rightarrow \Delta$ is basis of A^*).

Pf. WLOG Φ is ired, so $\Delta \subset A^*$ is basis.

Let $L = \mathbb{Z} . \Delta \subset A^*$ (lattice). WLOG $\psi \in \Phi^{nd}$, so

$\psi \in W(\Phi) . \Delta \subset W(\Phi)(L) = \langle r_H \rangle_{H \in \text{wall}(e)} (\mathbb{Z} . \Delta)$

For $\psi_i \in \Delta$, $r_{\psi_i}(\underset{\in L}{\ell}) = \ell - \underbrace{\langle \psi_i, \ell \rangle}_{\in \mathbb{Z}} \underbrace{\psi_i}_{\in \Delta}$, so $W(\Phi)$ preserves L

so $\psi \in L = \mathbb{Z} . \Delta$.

$\psi = \sum n_i \psi_i$, for $n_i \in \mathbb{Z}$.

Evaluate on vertices x_j of $\bar{e}$:

$$0 \le \psi(x_j) = \sum n_i \psi_i(x_j) = n_j \underbrace{\psi_j(x_j)}_{>0}. \qquad \square$$

Lecture 20 Special points and Extra special pts.

Let (A, Ψ) be affine root system, A is affine space for V, $\Phi = \nabla\Psi \subset V^*$.

Def. For $x \in A$, let $A_x^* = \{\psi \in A^* : \psi(x) = 0\}$ (so $\mathbb{R} \oplus A_x^* = A^*$)

so $\nabla : A_x^* \xrightarrow{\sim} V^*$

$$\begin{array}{ccc} \Psi_x = \{\psi \in \Psi : \psi(x) = 0\} & \xhookrightarrow{\nabla} & \Phi \\ \cap & & \cap \\ A_x^* & \xrightarrow{\sim} & V^* \end{array}$$

So Ψ_x is finite, stable under r_η for $H_\eta \ni x$.

and $r_\eta \curvearrowright \Psi_x$ lies "over" $r_\eta \curvearrowright \Phi$.

So Ψ_x is a "subroot system" of Φ (root system in its span inside A_x^*),

and $W(\Psi_x) \subset W(\Phi)$ is $\langle r_H \rangle_{H = H_\psi \ni x}$.

Rmk. Need to justify restriction to span(Ψ_x) is injective on $W(\Psi_x) \subset W(\Phi)$.

Can check $\{\psi^\vee : \psi \in \Psi_x\} \xrightarrow{\sim} (\Psi_x)^\vee$.

Since $\{H_\psi\}_{\psi \in \Psi}$ is locally finite in A, so A is a disjoint union of facets

(based on looking at hyperplanes passing through each $x \in A$)

Each facet $\mathcal{F}$ is in the closure of a chamber, so using $W(\Psi)$-action,

describes all $\mathcal{F}$'s via looking at $\bar{e}$ for one e. (see [Bourbaki] for

Coxeter gps acting on vector spaces.)

The subset $\Psi_x \subset \Psi$ depends only on facet $\mathcal{F} \ni x$, so denote as $\Psi_{\mathcal{F}}$.

<u>Lemma</u> For $x \in \mathcal{F} \subset A$, $W(\Psi)_x = \{w \in W(\Psi) : w|_{\mathcal{F}} = \mathrm{id}\}$

$$= \langle r_H \rangle_{H = H_\psi \supset \mathcal{F}}.$$

<u>Pf.</u> [KP, Lemma 1.3.17]

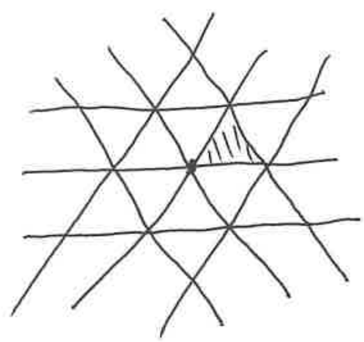
Moreover,

$$\nabla : W(\Psi)_x \xrightarrow{\sim} W(\Psi_x = \Psi_{\mathcal{F}})$$

<u>Rmk.</u> Later $\Psi_{\mathcal{F}}$ will be root system of max. reductive quotient of special fiber of $\mathcal{G}^\circ_{\mathcal{F}}$ = Bruhat-Tits $\mathcal{O}$-gp

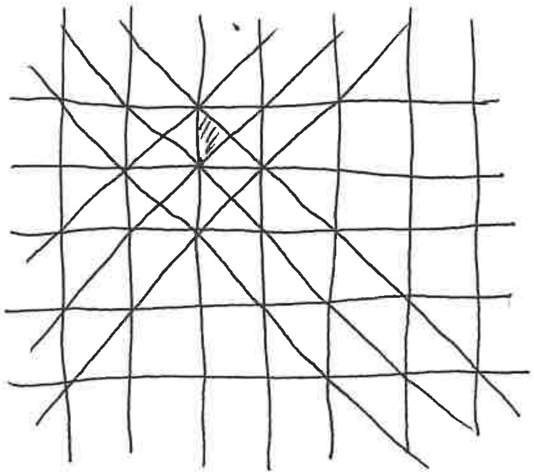
w/ $\mathcal{G}^\circ_{\mathcal{F}}(\mathcal{O}) = G(k)^\circ_{\mathcal{F}}$ (ptwise $\mathcal{F}$ stabilizer) for $A = A(S) \subset \mathcal{B}(G)$.

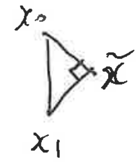
<u>Ex.</u> $\Psi = \Psi_{A_2}$



All vertices x have $\Psi_x \xrightarrow[\sim]{\nabla} \Phi = A_2$.

Ex. $\Psi = \Psi_{B_2 (= C_2)}$





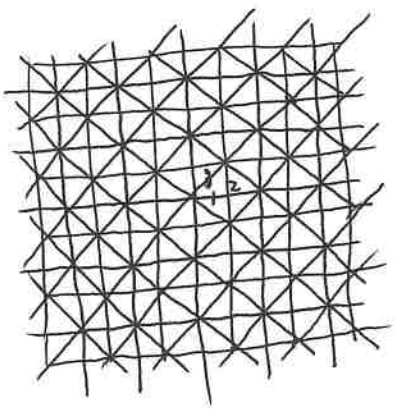
$\Psi_{x_0}, \Psi_{x_1} \xrightarrow[\sim]{\nabla} \Phi = B_2$

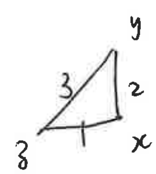
$\Psi_{\tilde{x}} \xrightarrow[\sim]{\nabla} A_1 \times A_1 \subset \Phi.$

(later, $((\mathcal{G}^{\circ}_{\tilde{x}})_f)^{red} \simeq SL_2 \times SL_2$ for $G = Sp_4$)

Ex. $\Psi = \Psi_{BC_2} = \{ \pm e_i^* + \mathbb{Z},\ \pm e_1^* \pm e_2^* + \mathbb{Z},\ \underbrace{2e_i^* + (2\mathbb{Z}+1)}_{\psi \text{ w/ } \dot{\psi} \in 2\Phi} \}$

$2e_i^* + (2\mathbb{Z}+1) = 2(e_i^* + (\mathbb{Z} + \tfrac{1}{2}))$





$\nabla: \Psi_x, \Psi_y, \Psi_z \hookrightarrow \Phi = BC_2$

$\Psi_x \xrightarrow{\sim} A_1 \times A_1$

$\Psi_y = B_2 = \Phi^{nd} \subset \Phi.$

$\Psi_z = C_2 = \Phi^{nm} \subset \Phi.$

Rmk. When $F \subset \bar{e}$, $\Delta_e \cap \Psi_F$ ($\subset \Psi$) is a basis of Ψ_F.

Pf. [KP, Prop. 1.3.35(6)]

Def. Say $x \in A$ is special if each H_ψ is parallel to $H_{\psi'} \ni x$ (i.e. $\psi' \in \Psi_x$)

($\Leftrightarrow W(\Psi_x) = W(\Phi)$)

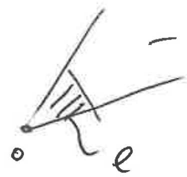
[KP, Lemma 1.3.42]

Say x is <u>extra special</u>, if Φ has <u>basis</u> of form $\{\psi_1^\circ, \dots, \psi_\ell^\circ\}$

for $\psi_i \in \Psi_x$ $\quad$ ($\Leftrightarrow$ $\Psi_x \overset{\nabla}{\hookrightarrow} \Phi$ contains a basis).

Ex. $\Psi = \Psi_\Phi$ for Φ irred. and reduced.

This has $0 \in \mathrm{Vert}(\bar{\mathcal{C}})$ as extra special.



— positive Weyl chamber cone.

Ex. A_2: all vertices are special and extra special.

B_2: x_0, x_1 both special and extra special, $\tilde{x}$ is neither.

BC_2: x is neither special nor extra special,

y is both special and extra special

z is special but not extra special.

<u>Prop</u> ① Extra special pts always exist.

② x special $\Leftrightarrow$ $W(\Psi)_x \hookrightarrow W(\Phi)$ is equality

③ Extra special $\Rightarrow$ special, converse holds if Φ reduced.

<u>Pf.</u> see notes.

① Pick a basis of Φ: $\{\psi_1^\circ, \dots, \psi_\ell^\circ\}$, $\ell = \dim A = \dim V^*$ so $H_{\psi_i} \subset A$ are defined by affine

linear eqns w/ <u>linear indep.</u> derivative, so $\cap H_{\psi_i} = \{pt\}$.

extra special pt.

Lecture 21 Affine diagrams

Last time we introduced special & very special points for an affine root system (A, Ψ), and saw these are always vertices.

Prop. Let (A, Ψ) be an irred. affine root system, $\Phi = \nabla\Psi$, $\Delta \subset \Psi$ a basis, at most one $\psi \in \Delta$ can have derivative $\dot{\psi} \in \Phi$ that is divisible.

Pf. Let $\mathcal{C} \subset A$ be a chamber s.t. $\Delta = \Delta_{\mathcal{C}}$. We know there is an extra special pt $x_0 \in \bar{\mathcal{C}}$, so Ψ_{x_0} contains affine roots $\psi_1, \cdots, \psi_\ell$ s.t.

$\{\dot{\psi}_1, \cdots, \dot{\psi}_\ell\}$ is a basis of Φ. This latter basis defines positive systems of roots for Φ and its dual, so in particular a positive Weyl chamber cone $C \subset V$.

The canonical isomorphism of pointed spaces $(A, x_0) \simeq (V, 0)$ carries each H_{ψ_i} over to the vanishing linear hyperplane $H_{\dot{\psi}_i}$ that varies through the walls of C.

The action of $W(\Psi)_{x_0} = W(\Psi_{x_0}) = W(\Phi) = W(\Phi^\vee)^*$ is transitive on the set of Weyl chamber cones of Φ, and so can be used to bring us to the case that the affine chamber $\mathcal{C}$ is contained in C.

But $x_0 \mapsto 0$, so the open $\mathcal{C}$ inside C has the origin in its closure and hence (check by reasoning w/ locally finite collections of hyperplanes) the walls H_{ψ_i} of C are among the walls of $\mathcal{C}$.

Since $\psi_i(e) \subset \psi_i(c) \subset \mathbb{R}_{>0}$ and $\psi_i \in \Psi^{nd}$ (as its derivative $\dot{\psi}_i$ is in Φ^{nd} due to being part of a basis of Φ), we conclude that each ψ_i belongs to

$$\{\psi \in \Psi(e)^+ \cap \Psi^{nd} : H_\psi \in \text{wall}(e)\} = \Delta_e = \Delta.$$

We have thereby built affine roots $\psi_1, \dots, \psi_\ell \in \Delta$ for which $\dot{\psi}_i \in \Phi^{nd}$ for all i. Here $\ell = \dim \Phi = \dim A = \#\Delta - 1$, so there is <u>exactly one</u> affine root $\psi \in \Delta$ not among the ψ_i's. This is the only one which could possibly have derivative divisible in Φ, so we are done. $\square$

The Dynkin diagram $\mathrm{Dyn}(\Psi)$ will be a weighted ("multiplicity on edges") graph whose edges w/ multiplicity > 1 are assigned a direction. (except for some edges w/ mult. 4, which never arise for Dynkin diagrams of usual root systems)

The set of <u>vertices</u> in $\mathrm{Dyn}(\Psi)$ is a choice of basis Δ, where $\#\Delta = 1 + \dim A$
$= 1 + \dim \Phi$
(note that when Φ is irred., $\mathrm{Dyn}(\Phi)$ is connected).

For distinct $\psi, \eta \in \Delta$, no edges join them if $\dot{\psi}$ and $\dot{\eta}$ are orthogonal in Φ (equivalently $\langle \dot{\psi}, \dot{\eta}^\vee \rangle = 0$, and equivalently $\langle \dot{\eta}, \dot{\psi}^\vee \rangle = 0$).

For the irred. components Ψ_i of Ψ, we have $\Delta = \coprod \Delta_i$ for bases Δ_i of Ψ_i (since $e = \prod e_i$ for chambers e_i of Ψ_i and $\text{wall}(e) = \coprod \text{wall}(e_i)$).
In particular, since the irred. decomp. $\Psi = \oplus \Psi_i$ has the irred. Ψ_i's as the preimages

of the irreducible components Φ_i of Φ, w/ roots in Φ_i orthogonal to roots in Φ_j when $i \neq j$, we have no edges in $\mathrm{Dyn}(\Phi)$ joining a vertex in $\mathrm{Dyn}(\Phi_i)$ to a vertex in $\mathrm{Dyn}(\Phi_j)$ when $i \neq j$. Thus, we can focus the rest of the definition on the case <u>when Φ is irreducible</u>.

By irred., a $W(\Phi)$- inv. inner product on V (or V^*) is unique up to scaling, so the ratio of lengths $\frac{\|\dot{\psi}\|}{\|\dot{\eta}\|}$ for $\psi, \eta \in \Delta$ is intrinsic (as is the angle $\angle(\dot{\psi}, \dot{\eta})$).

Furthermore, by irreducibility and Prop., at most one of ψ, η has derivative divisible in Φ, so we can arrange $\dot{\psi} \in \Phi^{nd}$.

Suppose ψ and η are <u>not orthogonal</u>, so

$f(\psi, \eta) = \langle \dot{\psi}, \dot{\eta}^\vee \rangle \langle \dot{\eta}, \dot{\psi}^\vee \rangle = 4\cos^2(\angle(\dot{\psi}, \dot{\eta})) \in \{1, 2, 3, 4\}$, where the last equality comes from the formula (using a $W(\Phi)$-invariant inner product)

$$\langle a, b^\vee \rangle = \frac{2(a \cdot b)}{(b \cdot b)} = 2\frac{\|a\|}{\|b\|}\cos(\angle(a, b)) \quad \text{for } a, b \in \Phi.$$

Note that the value $f(\psi, \eta)$ is equal to 4 if and only if the angle between $\dot{\psi}$ and $\dot{\eta}$ is 0 or π, which is to say these derivatives are linearly dependent. We have arranged that $\dot{\psi} \in \Phi^{nd}$, so the possibilities for linear dependence are $\dot{\eta} = \pm\dot{\psi}$ or $\dot{\eta} = \pm 2\dot{\psi}$. (The phenomenon of edges w/ mult. 4 never arises for Dynkin diagrams of usual root systems, since for those there is never linear dependence among distinct roots in a basis.)

By staring at possibilities for pairs of non-orthogonal roots in a root system, when $\|\eta\| \neq \|\psi\|$, we have $f(\psi,\eta) = \frac{\|\text{long}\|^2}{\|\text{short}\|^2}$.

We assign an edge between ψ and η w/ multiplicity $f(\psi,\eta)$, and when $f(\psi,\eta) > 1$ w/ $\|\dot\eta\| \neq \|\dot\psi\|$, we put an arrow from long to short.

Rmk. From the list of possibilities early in [Bour. Ch. VI, §1, no. 3], we have $\|\dot\eta\| = \|\dot\psi\|$ w/ $f(\psi,\eta) > 1$ exactly when $\dot\eta = \pm\dot\psi$ (in which case $f(\psi,\eta) = 4$).

But this says H_η and H_ψ are parallel, and a simplex (such as $\bar{c}$, by irred. of Ψ) has parallel distinct walls exactly in the 1-dim'l case.

Hence, this can only possibly occur for Ψ_{A_1}, Ψ_{BC_1}, and the three non-reducible irred. Ψ's w/ dimension 1. We'll see those for which it happens in our tabulation of Dynkin diagrams below.

A typical example:

Ex. Let $\Psi = \Psi_\Phi$ for Φ a reduced and irred. root system. Pick a basis $\Delta_0 = \{a_1, \cdots, a_n\}$ of Φ, and let Φ^+ be the associated positive system of roots.

One basis of Ψ is $\Delta = \{1-a_0, a_1, \cdots, a_n\}$ for a_0 the highest root in Φ^+.

Then $\mathrm{Dyn}(\Psi)$ as a bare graph is obtained from $\mathrm{Dyn}(\Phi)$ (w/ vertex set Δ_0) by joining $1-a_0$ to a_i for $1 \le i \le n$ exactly when $\langle a_0, a_i^\vee\rangle \neq 0$.

Some such non-vanishing must occur since otherwise the nonzero a_0 would be orthogonal to $a_1, \dots, a_n$ that span the entire space, an absurdity.

<u>Lemma</u>. If Ψ is irred. w/ deviatile Φ, then $Dyn(\Psi)$ consists of $Dyn(\Phi)$ joined along some weighted (possibly directed) edges to an additional vertex. In particular, $Dyn(\Psi)$ inherits connectedness from $Dyn(\Phi)$.

<u>Proof</u>. Since $W(\Phi)$ acts transitively on the set of bases, we can pick an extra special vertex x_0 s.t. Δ contains $\{\psi_1, \dots, \psi_\ell\}$ w/ $\psi_i \in \Psi_{x_0}$ and w/ $\{\dot{\psi}_1, \dots, \dot{\psi}_\ell\}$ a basis of Φ. But $\# \Delta = 1 + \dim A = 1 + \dim \Phi$, so $\ell = \dim A$ and hence

$$\Delta = \{\psi_0, \psi_1, \dots, \psi_n\} \quad \text{w/} \quad \psi_0(x_0) > 0 \qquad (\text{as } \psi_i(x_0) = 0 \text{ for all } i > 0).$$

Thus, $Dyn(\Psi) = \{\psi_0\} \amalg Dyn(\Phi)$ in which $Dyn(\Phi)$ is a subgraph w/ the same weighted edges and arrows, so we just have to make sure there is some edge joining ψ_0 to $Dyn(\Phi)$. But if there is no such edge, then $\dot{\psi}_0$ would be orthogonal to $\dot{\psi}_1, \dots, \dot{\psi}_\ell$ that span V^*, forcing $\dot{\psi}_0 = 0$, contradicting that ψ_0 is non-constant. $\square$

<u>Affine Dynkin diagrams for reduced irred. cases</u>.

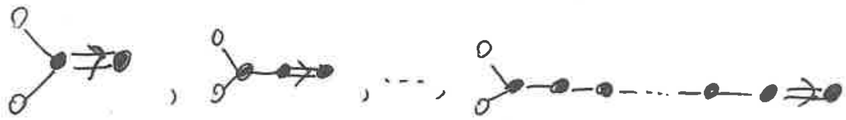
Non-special vertices : filled-in circles

extra special vertices : empty circles

Special but not extra special vertices : circles having an X inside

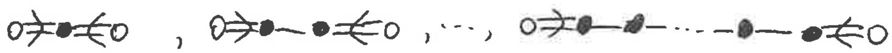
A_n $(n \geq 1)$

B_n $(n \geq 3)$



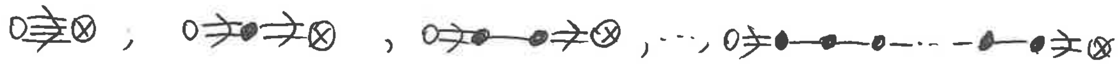
$B_n^\vee$ $(n \geq 3)$

C_n $(n \geq 2)$



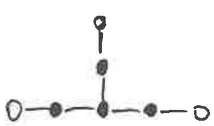
$C_n^\vee$ $(n \geq 2)$

BC_n $(n \geq 1)$



D_n $(n \geq 4)$

E_6



E_7

E_8

F_4

G_2

$F_4^\vee$

G_2

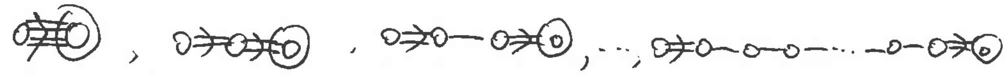
Rmk. An observation comes out of the preceding exhaustive tabulation is that the only connected affine diagrams which contain a loop are Φ_{A_n} for $n > 1$. This may seem like a niche fact of no significance, but it is a key ingredient of a striking fact to come later via Bruhat-Tits theory: if an absolutely simple and connected semisimple k-group is anisotropic, then it must be of type A and split over k^{un} (more can be said when the residue field is finite: inner type A, which is to say G^{sc} is the algebraic group of units of reduced norm 1 in a finite-dimensional central division algebra over k).

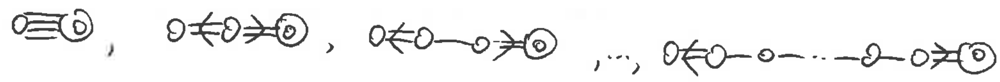
To appreciate how striking this is, recall that over $\mathbb{R}$, there are anisotropic absolutely simple and connected semisimple groups of every Dynkin type ("compact forms").

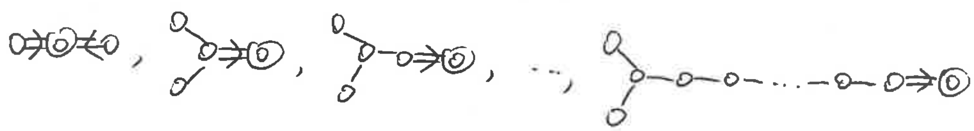
Over non-archimedean local fields k w/ finite residue field, this badly fails due to local class field theory. For example, consider adjoint type B_n, which is to say $SO(q)$ for a non-degenerate quadratic space (V, q) w/ dimension $2n+1$.

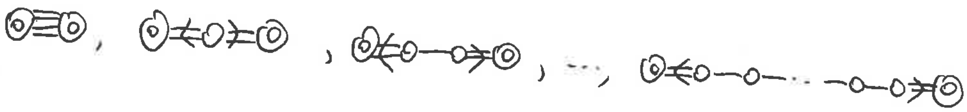
Taking $n \geq 2$, this quadratic space has dim. at least 5, so the quadratic form has a non-trivial zero by local class field theory and hence $SO(q)$ is k-isotropic.

Affine diagrams for non-reduced irreducible Φ. Rather than keeping track of special and extra special pts, we only keep track of which vertices are multipliable (by drawing an extra circle around such vertices).

(BC_n, C_n) $(n \geq 1)$ 

$(C_n^\vee, BC_n)$ $(n \geq 1)$ 

$(B_n, B_n^\vee)$ $(n \geq 2)$ 

$(C_n^\vee, C_n)$, $(n \geq 1)$ 

Lemma. For any irred. Φ w/ basis $\Delta = \Delta_e$ and facet F of $\bar{e}$, the usual root system Φ_F has Dynkin diagram obtained from $\mathrm{Dyn}(\Phi)$ by removing the vertices of $\bar{F}$ from $\Delta = \mathrm{Vert}(\mathrm{Dyn}(\Phi))$ and removing the edges of $\mathrm{Dyn}(\Phi)$ touching such removed vertices.

Pf. This is seen by combining two facts: upon writing $\Delta = \{\psi_0, \dots, \psi_n\}$, we can label $\mathrm{vert}(e)$ as $\{x_0, \dots, x_n\}$ where $\psi_i(x_j) \in \mathbb{R}_{>0}\, \delta_{ij}$ and Φ_F has $\Delta \cap \Phi_F$ as a basis.

Prop. The isom. class of an irred. and reduced Φ is determined by the weighted direct graph $\mathrm{Dyn}(\Phi)$.

Since $\mathrm{Dyn}(\Phi) = \mathrm{Dyn}(\Phi^{\mathrm{red}})$ for non-reduced irred. Φ (disregarding the decorations we add to the diagram record additional information), we cannot remove "reduced" from this result.

Proof. Use the table.

Lecture 22 Tits systems (or BN pairs)

The facets of the apartments of the building of a connected reductive group over a henselian (discretely-valued) field k w/ perfect residue field will correspond to bounded open subgroups called parahoric subgroups. To define such subgroups in general require having the building in hand (though for complete k w/ finite residue field there is a more direct characterization), but some aspects of parahoric subgroups rest on the purely group-theoretic notion of a "Tits system". We now discuss this concept, and next time we relate it to the notion of "abstract building".

Let G be an abstract gp. A Tits system for G is a triple (B, N, S) consisting of subgroups $B, N \subset G$ and a subset $S \subset N/(B \cap N)$ s.t.

(TS1) $T := B \cap N$ is normal in N, and $\langle B, N \rangle = G$

(TS2) the subset $S \subset W := N/T$ (the Weyl gp of the Tits system) consists of elements of order 2 generates W;

(TS3) for all $s \in S$ and $w \in W$, $sBw \subset BwB \cup BswB$ (so in particular, taking $w = s$, we have $sBs \subset BsB \cup B$, so $BsB \cup B$ is a subgp of G);

(TS4) for all $s \in S$, we have $sBs \neq B$ (equivalently, $sBs \not\subset B$, since if $sBs \subset B$, then $B \subset s^{-1}Bs^{-1} = sBs$ because $s^2 \in B \cap N \subset B$)

If moreover $\bigcap_{w \in W} wBw^{-1} = T$, then we say the Tits system is saturated.

If S is empty, then $W = 1$ and so $N \subset B$, hence $G = \langle B, N \rangle = B$. Conversely if $B = G$ then clearly $W = 1$ and so S is empty. When the Tits system is saturated, then $B = T$ (equivalently, $B \subset N$), so S is empty if and only if $N = B = G$. Of course, it is the case of non-empty S that is of interest (much as the Borel-Tits structure theory of connected reductive gps over general fields has nothing to say when the derived group is anisotropic, and Bruhat-Tits theory has nothing to say when the derived group is anisotropic.)

Expanding on the parenthetical observation we made for (TS3), by [Bour. Ch IV, §2, no.5] the subset $S \subset W$ is uniquely determined as the set of those $w \in W$ for which $BwB \cup B$ is a subgp of G. For this reason, the data of the Tits system is uniquely determined by the pair (B, N) and hence the Tits systems are often called <u>BN-pairs</u>.

We are interested in Tits systems for which the associated Weyl gp is infinite (in fact it will be the Weyl gp of an affine root system). but let's first recall the classical source of examples w/ finite W (giving rise to "spherical buildings").

<u>Ex</u>. Let $G = \underline{G}(F)$ for an arbitrary field F and connected reductive F-gp $\underline{G}$. Let $B = \underline{P}(F)$ for a minimal parabolic F-subgp $\underline{P}$ of $\underline{G}$, and $N = N_{\underline{G}}(\underline{S})(F)$ for a max. split F-torus $\underline{S} \subset \underline{P}$.

(so $T = B \cap N = Z_{\underline{G}}(\underline{S})(F)$ by the structure theory of connected reductive gps over fields, $W = W(\underline{G}, \underline{S}) \simeq W(\Phi(\underline{G}, \underline{S}))$).

For the basis Δ of $\Phi(\underline{G}, \underline{S})$ corresponding to $\underline{P}$ under the bijection between the set of minimal parabolic F-subgroups $\underline{Q} \supset \underline{S}$ and the set of positive systems of roots in $\Phi = \Phi(\underline{G}, \underline{S})$ via $\underline{Q} \mapsto \{a \in \Phi : a \text{ is an } \underline{S}\text{-weight on } \mathrm{Lie}(R_{u,F}(\underline{Q}))\}$, if we let $S = \{r_a\}_{a \in \Delta} \subset W(\Phi) = W$, then (G, B, N, S) is a Tits system.

In such cases, S is empty precisely when $D\underline{G}$ is F-anisotropic

By [C2, Rem. V.2.8], such Tits systems not only have finite W but are saturated and clearly satisfy $B = T \ltimes U$ w/ $U = \underline{U}(F)$ for $\underline{U}$ the F-split smooth conn'd unipotent $R_{u,F}(\underline{P})$. This U is a nilpotent group (since $\underline{U}$ is nilpotent as an alg. gp) and one says the BN-pair is weakly split when such a semi-direct product str exists for B using a nilpotent subgp $U \subset B$.

Rmk 22.2. In the setting of previous Ex, for $s = r_a$ the root group $\underline{U}_a$ has trivial schematic intersection w/ $(s\underline{P}s^{-1})$ (since conjugation by r_a on $X^*(\underline{S})_{\mathbb{Q}}$ negates a and carries $\Phi^+ - \mathbb{Q}\{a\}$ into itself), so $\underline{U}_a(F) \subset B/(B \cap s^{-1}Bs)$. It follows that when F is infinite, so $\underline{U}_a(F)$ is infinite for all $a \in \Phi$, then $B/(B \cap s^{-1}Bs)$ is infinite for all $s \in S$.

Remarkably there is a converse when F is infinite and $D\underline{G}$ is F-simple; in such situations, any saturated and weakly split BN-pair (B,N) for $\underline{G}(F)$ w/ finite Weyl gp and $B/(B\cap sBs^{-1})$ infinite for all $s\in S$ must arise from a (uniquely determined) $(\underline{P},\underline{S})$ by [Pra, Thm. B].

Def For a Tits system (G,B,N,S), a parabolic subgp $P\subset G$ is a subgp containing conjugate of B.

Ex. For $X\subset S$ and $W_X := \langle X\rangle \subset W$, $P_X := BW_XB$ is a parabolic subgp (by (TS 3), since $Bww'B\subset (BwB)(Bw'B)$ and W is generated by S). These parabolic subgps clearly contain B. The parabolic subgps containing B are called standard.

Thm 22.5. The following hold for any Tits system (G,B,N,S).

(1) (Bruhat decomposition) $G=\coprod_{w\in W} BwB$

(2) Each parabolic subgp P is conj. to P_X for a unique $X\subset S$ (called the "type" of P)

(3) Each P is its own normalizer in G.

Proof. See Bourbaki. $\square$

We emphasize how remarkable Thm 22.5 (2) is: it says in particular that the only subgroups of G containing $B=P_\emptyset$ are the subgroups P_X for subsets $X\subset S$.

Even for $G = GL_n(F)$ for a field F and B its subgroup of upper triangular matrices, this is not at all obvious!

That fact is crucial for encoding group-theoretic information in terms of a geometric object (a "polysimplicial" building). Tits was motivated to discover buildings in the search for unified approach to geometrically interpreting groups through an action on a naturally associated non-positively curved space much as some aspects of connected Lie groups G are understood through their action on the "symmetric space" G/K of non-compact type (for a max'l cpt subgp K in G).

For BT theory for a conn'd red. gp $\underline{G}$ over a henselian field k, the relevant BN-pair will be $G = \underline{G}(k)^{\circ}$, $B = \underline{G}(k)^{\circ}_{e}$ for a chamber e in the apartment $\mathcal{A}(\underline{S}) \subset \mathcal{B}(\underline{G})$ associated to a max. split k-torus $\underline{S} \subset \underline{G}$, $N = N_{\underline{G}}(\underline{S})(k) \cap G = N_G(\underline{S})$,

W will be naturally identified w/ $W(\Psi)$ for an affine root system $\Psi(\underline{G}, \underline{S})$ (to be built!) on the affine space $A(\underline{S})$ giving rise to a "polysimplicial structure" encoded in the data of the building, and $S \subset W = W(\Psi)$ will correspond to the set of reflections in walls of the chamber e.

The bounded parabolic subgps associated to this Tits system will be those of the form $P = G(k)^{\circ}_{F}$ for F facets of the "polysimplicial" $\mathcal{B}(\underline{G})$, and the "type" of P will be encoded by using the $G(k)^\circ$-action on the building to move $\mathcal{F}$ over to a uniquely determined facet of the choice of $\mathcal{C}$. The boundedness of a parabolic subgp will have a characterization in purely group-theoretic terms as we shall see (but bear in mind that the ambient group for the Tits system in $G(k)^\circ$, which is generally not $G(k)$).

Lecture 23. Admissible types and abstract buildings

Loose ends on general Tits systems:

① For a parabolic subgp $P \subset G$ for Tits system (G, B, N, S),

we define its type $X \subset S$ where P is conjugate to a unique parabolic subgp $\underbrace{P_0 \supset B}_{\text{"standard"}}$: $P_0 = P_X = BW_X B$ for $! X \subset S$.

By [Bourbaki, Ch IV, §2, no. 5, Thm 3] : $P_X \subset P_Y \iff X \subset Y$,

so $P_X \cap P_{X'} = P_{X \cap X'}$.

In particular, max'l proper parabolics are $P_{S - \{x\}} =: G_x$ for $x \in S$.

So standard parabolic $P_X = \bigcap_{y \notin X} \underbrace{P_y}_{\substack{\text{exactly max'l proper} \\ \text{parabolic subgps} \supset P_X}}$

For S finite, each proper parabolic P is contained in <u>finitely many</u> max'l proper parabolics. $P_1, \ldots, P_m$ and $P = \bigcap_i P_i$ and $\text{type}(P_i) =$ elts of S <u>not</u> in $\text{type}(P)$.

Later: $P =$ proper parabolic $\overset{\text{inclusion reversing}}{\Leftrightarrow}$ facets $\mathcal{F}$ of an abstract building for (G, B, N, S), and $\text{vert}(\mathcal{F}) \Leftrightarrow$ max'l proper parabolics $Q \supset P$.

<u>Rmk</u>. For conn'd reductive G/k, Tits system to be made will have $G = G(k)^0$, <u>not</u> $G(k)$.

G is <u>split</u> $\underset{\text{defn}}{\Rightarrow}$ $G(k)^0 = G(k)^q S(k)^0$ for $G(k)^q =$ image of $G^{sc}(k)$

$S =$ max. split k-torus in G

$\Rightarrow S(k)^0 = \mathcal{S}(\mathcal{O})$ for ! $\mathcal{O}$-torus model.

For $G = PGL_2$, then $G(k)^0 = (\text{image of } SL_2(k)) \cdot \mathcal{S}(\mathcal{O})$

for $\mathcal{S} = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset PGL_{2,\mathcal{O}}$.

But $PGL_2(k) \xrightarrow{\det} k^\times/(k^\times)^2$ has kernel image of $SL_2(k)$, and

$\mathcal{S}(\mathcal{O}) \hookrightarrow \mathcal{O}^\times/(\mathcal{O}^\times)^2 \underset{2}{\subsetneq} k^\times/(k^\times)^2$ $\quad \because k^\times/\mathcal{O}^\times \xrightarrow{\sim} \mathbb{Z}$

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \cup \\ (k^\times)^2 & \hookrightarrow & 2\mathbb{Z} \end{array}$$

So $PGL_2(k)^0 \underset{2}{\subsetneq} PGL_2(k)$.

Later: $\quad B(PGL_2) = B(SL_2)$.

$$\begin{array}{ccc} B(PGL_2) & = & B(SL_2) \\ \circlearrowleft & & \circlearrowleft \\ PGL_2(k)^\circ \underset{2}{\subsetneq} PGL_2(k) & \longleftarrow & SL_2(k) = SL_2(k)^\circ \end{array}$$

general for G s.conn'd

When we relate Tits systems to (abstract) buildings for a facet F,

$$\mathrm{Stab}_F(G) = \{ g \in G : g(x) = x, \ \forall x \in F \}$$

For $PGL_2(k)$ acting on tree $B(SL_2)$, the "extra" elt in

$\underbrace{PGL_2(k) - PGL_2(k)^\circ}_{PGL_2(k)^1}$ will flip an edge: $\bullet\!\!-\!\!\circ \overset{\text{flip}}{\longleftrightarrow} \circ\!\!-\!\!\bullet$

Eventually, for a central qt $G \to G'$ of conn'd red. gps, a natural isometry

$$\begin{array}{ccc} B(G) & \xrightarrow{\sim} & B(G') \\ \circlearrowleft & & \circlearrowleft \\ G(k) & \longrightarrow & G'(k) \end{array}$$

, but beware usually

$$\begin{array}{ccc} G(k) & \longrightarrow & G'(k) \\ \cup & & \cup \\ G(k)^\circ & \longrightarrow & G'(k)^\circ \end{array}$$

usually not surj.

Some hard work [KP, Lemma 7.5.2] will verify hypotheses of the following for such $G(k)^\circ \longrightarrow G'(k)^\circ$ when $k = K$:

Prop Let (G, B, N, S), (G', B', N', S') be two Tits systems,

$f : G \to G'$ s.t. $f(B) \subset B'$, $f(N) \subset N'$, and

(i) $\ker(f) \subset T := B \cap N$

(ii) $f(G) \trianglelefteq G'$, $G'/f(G)$ abelian

(iii) $T' := B' \cap N'$ normalizes $f(B)$, $B' = f(B)T'$, $N' = f(N)T'$.

Then • $f: W \xrightarrow{\sim} W'$, $S \xrightarrow{\sim} S'$.

• $\mathrm{Par}(G) \xrightarrow{\sim} \mathrm{Par}(G')$ via $f^{-1}(P) \longleftarrow P'$, also type preserving.

$P \longmapsto N_{G'}(f(P))$ inclusion preserving

For conn'd reductive G/k, w/ $\mathcal{D}G$ <u>not</u> k-anisotropic, $\left(\underset{\text{later}}{\Rightarrow} B(G) \neq pt\right)$

then $B(G) = B(G^{ad})$ where $G^{ad} = \prod G_i^{ad}$ for k-simple "factors"

G_i of $\mathcal{D}G$.

some are k-isotropic.

$= \prod_j B(G_j^{ad})$, for $\{G_j^{ad}\}$ set of k-<u>isotropic</u> simple

factors of G^{ad}.

Inside here, for chambers we'll have $C = \prod C_i$ for chambers $C_j \subset B(G_j^{ad})$

that <u>are simplices</u> w/ wall $(C_j) \longleftrightarrow S_i$ w/ $S = \coprod S_j$

where S comes from Tits system of G,

S_j — — — — G_j^{ad}

where Tits system for G will have Weyl gp

$W = W(\Psi)$ for affine root system $(A(\mathcal{S}), \Psi)$ for max. split $\mathcal{S} \subset G$ w/

irred. decomp is $(A(\mathcal{S}_j), \Psi_j)$ for max. split $\mathcal{S}_j \subset G_j^{ad}$ that's image of $\mathcal{S}$.

Moreover, $W(\Psi) = \prod W(\Psi_j)$ will induce $\underset{(*)}{S = \coprod S_j}$ corresponding to bases.

Now state purely gp-theoretic mechanism to get $(*)$:

For Tits system (G, B, N, S), w/ S finite $\neq \phi$, by [Bourbaki, Ch IV, §1]

(W, S) is a Coxeter system. and $\exists!$ decomp $S = \coprod_j S_j$ for non-empty

pairwise commuting $S_j \subset S$ that are themselves irred. ($S_j \neq S_j' \amalg S_j''$ for

nonempty commuting $S_j', S_j'' \subset S_j$).

(Concretely, $S_j \longleftrightarrow$ vertices of conn'd comp of Coxeter graph of (W, S)).

For what we need, S_j will be vertices of conn'd affine Dynkin diagrams,

so $\# S_j \geq 2$.

We call $X \subset S = \coprod S_j$ admissible if $X \cap S_j \subsetneq S_j$ for all j: we'll care

about parabolics w/ admissible type.

Lecture 24. Abstract buildings.

Today: combinatorial "geometric" notion of building ({vertices} is a proxy for a simplex.) will encode gp-theoretic data for a given Tits system. (*)

Can go in reverse (next time): given a building w/ "sufficiently rich" gp-action, we can construct from that a Tits system which recovers the given building via (*). (**)

One uses (*) to make $B(G_K)$ via Tits system made "by hand" in $G(K)^\circ$.

Rmk. Roll of SL_2 for split gps $/F$ is replaced with SL_2 and $SU_3(F'|F)$ quadratic Gal. ext.

for q-split gps $/F$

rank 1 abs simple s. connected q-split $/F$

For $\Gamma = \mathrm{Gal}(K|k)$, show $B(G_K)^\Gamma \circlearrowleft G(k)$ is a building (real work).

whose $G(k)^0$- action fulfills $(**)$. $\longrightarrow$ "unramified descent"

Def. An <u>(abstract) simplicial complex</u> is a pair $(V, \mathcal{B})$ where V is a non-empty set ("vertices") and $\mathcal{B}$ is a nonempty set of finite nonempty subsets $\mathcal{F} = \{x_1, \dots, x_n\} \subset V$ (called <u>facets</u>), s.t.

- $\{x\} \in \mathcal{B}$
- $\forall \mathcal{F} \in \mathcal{B}$, and $\mathcal{F}' \subset \mathcal{F}$ is nonempty subset then $\mathcal{F}' \in \mathcal{B}$.

Rmk. For actual n-simplex $\Delta \subset \mathbb{R}^{n+1}$, facets (of dim $\in \{0, \dots, n\}$) are open convex hulls of nonempty sets of vertices. So $\Delta = \coprod_{\substack{\mathcal{F} \subset \Delta \\ \text{facets}}} \mathcal{F}$.

Define $\dim \mathcal{F} = \# \mathcal{F} - 1$. $\ulcorner \exists$ evident notions of isom. for simplicial complex. $\lrcorner$

Ex. Say $\mathcal{F} \neq \mathcal{F}'$ of same dimension share a common codim 1 facet $\mathcal{F}_0$ if $\mathcal{F}_0 \subset \mathcal{F}, \mathcal{F}'$. For $\mathcal{F}, \mathcal{F}' \in \mathcal{B}$, $\mathcal{F} \cap \mathcal{F}'$ is empty $(\notin \mathcal{B})$ or a facet.

Def. An (abstract) <u>polysimplicial complex</u> $\ulcorner$ Recall for (A, Φ), chambers are simplices $\Leftrightarrow \Phi$ irred., and are products of chambers of irred. components Φ_i in general.] is an ordered n-tuple $\mathcal{B} = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$

where $n \geq 1$ and $\mathcal{B}_i$ are abstract simplicial complex.

$V = \{$vertices of $\mathcal{B}\} := \prod V_i$, facets of $\mathcal{B}$ are $F = \prod F_i$ for $F_i \in \mathcal{B}_i$

$\dim(F) := \sum_i (\# F_i - 1)$.

Note F is a non-empty finite subset of V,

A facet that is max'l w.r.t. inclusion is called a chamber.

Easy to see a chamber of $\mathcal{B}$ is exactly $C = \prod C_i$ for chambers $C_i \in \mathcal{B}_i$

An isom. $\varphi : \mathcal{B} \xrightarrow{\sim} \mathcal{B}'$ of polysimplicial complexes is a collection of

isoms $\varphi_i : \mathcal{B}_i \xrightarrow{\sim} \mathcal{B}'_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_n$ (req'd that $\mathcal{B}, \mathcal{B}'$ have same # of "factors")

Def. A polysimplicial $\mathcal{B}$ is called chamber complex if

(i) Each facet F is contained in a chamber, and all chambers have same dim $d \geq 0$.

(ii) For two chambers C, C', $\exists$ sequence of chambers $\overbrace{C = C_0, C_1, \dots, C_N = C'}^{\text{"gallery"}}$

s.t. $\forall\, 0 \leq i < N$, $\underset{\in \mathcal{B}}{C_i \cap C_{i+1}}$ is of codim. 1 ("common wall") = "panel"

Rmk. ① $\mathcal{B}$ is a chamber complex $\Leftrightarrow$ all $\mathcal{B}_i$ are so.

② A 0-dim'l chamber complex is a point. ($\phi \notin \mathcal{B}_i$)

? Tits allows $\phi \in \mathcal{B}_i$.

Def. Say chamber complex $\mathcal{B}$ is thick if each codim 1 facet $F \in \mathcal{B}$ is in ≥ 3 chambers. ($\Leftrightarrow$ all $\mathcal{B}_i$ [that are not pts] are thick).

Say chamber complex is thin if every codim 1 $F \in \mathcal{B}$ is contained in exactly two chambers ($\Leftrightarrow$ all $\mathcal{B}_i$ [that are not pts] are thin).

Def. An (abstract) building is a thick chamber complex $\mathcal{B}$ equipped w/ a collection of thin chamber subcomplexes $\mathcal{A} \subset \mathcal{B}$ (called apartments)

$F \in \mathcal{A}$, $F' \in \mathcal{B}$, w/
$F' \subset F$, then $F' \in \mathcal{A}$.

s.t. ① Any two chambers $C, C' \in \mathcal{B}$ lie in a common apartment $\mathcal{A}$.

② For any two facets F_1, F_2 and apartments $\mathcal{A}, \mathcal{A}' \ni F_1, F_2$, $\exists$ isom. $\mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ of polysimplicial complexes that is identity in F_1, F_2.

Thm. Let (G, B, N, S) be a Tits system, w/ S finite, $\# S \geq 2$ ($S \neq \phi \Rightarrow B \neq G$)

Let $V = \{$ max'l <u>proper</u> parabolic $P \subset G\}$,

$$\mathcal{B} = \{ F = \{ \underset{\in V}{P_1}, \dots, \underset{\in V}{P_n} \} : P_F = \bigcap_i P_i \text{ is parabolic} \}$$

$\circlearrowleft$

G via $g.F = \{ g P_1 g^{-1}, \dots, g P_n g^{-1} \}$.

- $\mathcal{C}_0 = \{ Q \in V : Q \supset B \}$ $\quad (P_{\mathcal{C}_0} = B)$.
- $A_0 = \{ n. \mathcal{C}_0 : n \in N \}$.

(1) $\mathcal{B}$ is a simplicial building using set of apartments $\{ g. A_0 : g \in G \}$ and chambers are exactly $g. \mathcal{C}_0$ for $g \in G$ (so $\{$minimal parabolics$\}$ $\leftrightarrow$ $\{$chambers$\}$).

(2) $P_F = \{ g \in G : g(F) = F \}$, and $V^{P_F} = F$.

(3) G acts transitively on $\{ (A, C) : C \subset A \text{ is chamber} \}$

P_F acts transitively on $\{ A \supset F \}$.

(4) Each facet $F \in \mathcal{B}$ is G-conjugate to a <u>unique</u> facet $F_0 \subset \mathcal{C}_0$ ("type" of F or of P_F)

and if $F, F' \subset A_0$ that are G-conjugate, then they are N-conjugate.

Pf. [KP, Prop 1.5.6, 1.5.13]. $\quad \mathcal{B} =$ Tits building of (G, N, B, S).

Lecture 25 — From buildings to Tits systems

Last time: $(G, B, N, S) \rightsquigarrow$ simplicial bldg $\mathcal{B}$ (imposed $P \subsetneq G$,

no condition related to possibility that (W, S) is reducible).

For ex: $G = G_1 \times G_2$, $P = P_1 \times P_2 \subsetneq G$ does not force $P_i \subsetneq G_i$, for both $i = 1, 2$.

This is relevant to $G = G(k)^0$ when $G^{ad} = \prod_i G_i$ is not k-simple.

Idea: Want to focus on those P for which $\overset{X}{\overset{\shortparallel}{\text{type}}}(P) \subset S = \coprod S_i$, has

$X_i = X \cap S_i \subsetneq S_i$ for all i.

Prop. Let (G, B, N, S) be a Tits system w/ finite $S \neq \phi$, assume irred. components $S_i \subset S$ have $\# S_i \geq 2$, $\forall i$.

Let $\mathcal{B}$ be associated (simplicial) Tits bldg.

[Rmk: $V^{P_F} = F$ as subsets of V, also $\mathcal{B}^{P_F}$ = set of facets $F' \subset F$, rests on fact that parabolic subgps in Tits systems are own normalizer:

$P_F \subset Q \iff Q = P_{F'}$ for $F' \subset F$.]

Define $\mathcal{B}' = \{ F \in \mathcal{B} : \text{type}(P_F) \subset S = \coprod S_i$ is admissible $\}$

This is a polysimplicial complex using $\text{Vert}(\mathcal{B}') = \{ P \subset G$: max'l among parabolics w/ adm. type $\}$

$= \{ P \subset G : \text{type}(P) = \coprod_i (S_i - \{s_i\}) \}$

Observe if $X = \coprod X_i \subset \coprod S_i = S$ is admissible then $X_i = \bigcap_{s \in S_i - X_i} (S_i - \{s\})$

these encode "vertices" of B' for "facet" associated to P of type X.

[Previously: $P = \bigcap_{P_i \supset P} P_i$ ← vertices of facet for P.

P_i max'l proper parabolic

Now: $P = \bigcap Q_j$ for Q_j the max'l adm parabolics $\supset P$

P admissible

$\text{type}(P) = \coprod X_i$, $X_i \subsetneq S_i$, then $\text{type}(Q_j) = \coprod (S_i - \{s_i\})$ for $s_i \in S_i - X_i$

s_i vary!]

B' is a bldg w/ same chambers as B and apartments $A \cap B'$ for apts A of B.

Explicitly, $B' \simeq \prod B_i$ for B_i are Tits bldg for $(G, \underset{P_{S-S_i}}{\overset{\shortparallel}{B_i}}, N, S_i)$

Pf [KP, Prop 1.5.18].

Call B' the <u>restricted Tits bldg</u> for (G, B, N, S). From construction, it inherits properties (1) – (4) of B (except just polysimplicial).

By design, B' inherits G-action. Both B and B' satisfy

<u>Cor</u>. For two pairs (A_1, e_1), (A_2, e_2), $\exists\, g \in G$ s.t. $g.(A_1, e_1) = (A_2, e_2)$ and g is <u>id</u> on $e_1 \cap e_2$.

Pf. Pick a $g \in G$ s.t. $g.(A_1, C_1) = (A_2, C_2)$.

If $C_1 \cap C_2 = \phi$, nothing to do.

Suppose $C_1 \cap C_2 \neq \phi$, so $F = C_1 \cap C_2$ is a facet. Then g carries $F \subset C_1$ to $g.F \subset C_2$, but $F \subset C_2$. But G carries each facet to only one facet of a chosen chamber! $\therefore g.F = F$, so $g \in P_F$, so $g|_F = \mathrm{id}$. $\square$

Reverse construction. Let $\mathcal{B} = \{B_1, \dots, B_n\}$ be abstract polysimplicial bldg, equipped w/ action by a group G s.t.

(i) $\forall (A, C), (A', C')$, $\exists g \in G$ s.t. $g.(A, C) = (A', C')$ w/ $g|_{C \cap C'} = \mathrm{id}$.

(ii) $\mathrm{Stab}_G(C)$ fixes vert(C) ptwise.

Choose apt A_0 and chamber $C_0 \in A_0$. Let $B = \mathrm{Stab}_G(C_0)$, $N = \mathrm{Stab}_G(A_0)$.

Then: ① (G, B, N) is a saturated Tits system, where $S \subset W = N/(B \cap N)$ has elts in bijection w/ walls F of C_0. where $r_F \in S$ acts by carrying C_0 to ! chamber $C \neq C_0$ in A sharing same wall F.

② If irred. decomp. $(W, S) = (\prod W_i, \coprod S_i)$ has all $\# S_i \geq 2$, then (e.g. if (W_i, S_i) comes from irred. affine root system.)

$F \longmapsto P_F = \mathrm{Stab}_G(F) \subset G$ is an G-equiv. isom. of the given $\mathcal{B}$ onto the restricted bldg of (G, B, N, S).
$\mathcal{B} \ni F$

Pf [KP, Thm 1.5.28, Thm 1.5.30]

Axioms for Bruhat - Tits theory: G be conn'd red. gp over k = henselian disc-valued w/ perfect res. field f.

(Axioms 4.1.x, $x = 1, 2, 4, 6, 8, 9, 16, 17, 20, 22, 27$)

Axiom 1. $\exists$ polysimplicial bldg $B(G) \circlearrowleft G(k)$ having apartments

$\{A(S)\}_{\substack{S \subset G \\ \text{max. split tori}}}$ and • $\mathrm{Stab}_{G(k)}(A(S)) = N_G(S)(k)$.

• for any central qt $G \twoheadrightarrow \bar{G}$ have equiv. isom. $B(G) \xrightarrow{\sim} B(\bar{G})$

Axiom 2. $\forall$ facets $F \in B(G)$, $G(k)^\circ_F = \mathrm{Stab}_{G(k)^\circ}(F)$ is bounded open subgp.

Call these parahoric subgp, F = chamber: Iwahori subgps.

Lecture 26. More axioms

Axiom 1. $B(G) \cdots$ w/ apartments $\underbrace{A(S)}$ + indexing of apts by {all S}

bijective labeling by set of max. split k-tori S

is $G(k)$-equivariant: $g \cdot A(S) = A(gSg^{-1})$

⌈In defn of bldg, require isoms to respect apts⌋

Axiom 2. bounded open for $G(k)^\circ_F = \mathrm{Stab}_{G(k)^\circ}(F) \ldots$

Prop. $B(G) = pt \Leftrightarrow \mathcal{D}G$ is k-anisotropic $(\Leftrightarrow \Phi(G,S) = \phi \Leftrightarrow S' = (S \cap \mathcal{D}G)^\circ_{red} = 1 \ldots)$

Pf. In notes (using just Axiom 1.)

Upcoming: Axiom 4: enhancing $\mathcal{A}(S)$ to an affine space over $V' = X_*(S')_{\mathbb{R}}$.

Axiom 6: affine root system $\Psi = \Psi(G,S)$ in A^* (w/ $\nabla\Psi = \Phi$ in $V'^* \overset{\shortparallel}{=} X^*(S')_{\mathbb{R}}$)

we'll require $\dim \mathcal{B} > 0$. [$\Leftrightarrow S' \neq 1$

$\Leftrightarrow$ DG is k-isotropic]

Fix $S \subset G$ max'l split torus, $N = N_G(S)$, $Z = Z_G(S)$.

Let $S' = (S \cap DG)^0_{red} \subset DG$ be corresponding max'l split k-torus in DG.

Since set of apts is $G(k)$-equivariantly by bijectively labeled by set of S's,

$\mathrm{Stab}_{G(k)}(\mathcal{A}(S)) = N(k)$. In particular, $\mathcal{A}(S)$ as poly simplicial complex

has $N(k)$-action. (pass to G^{ad} w/ same bldg and $S^{ad} \subset G^{ad}$, get product

decomp. of $\mathcal{A}(S) \circlearrowleft N(k)$).

<u>Lemma</u>: Up to ! isom, $\exists$ affine space $A = A(S)$ for $V' = X_*(S')_{\mathbb{R}}$.

(so $\dim A = \dim S'$)

equipped w/ $N(k) \xrightarrow{f} \underbrace{\mathrm{Aff}(A)}_{\text{affine automorphisms}}$

s.t. ① $\forall n \in N(k)$, $f(n): A \xrightarrow{\sim} A$ w/ linear $\nabla(f(n)): V' \xrightarrow{\sim} V'$ is exactly

natural $N(k)$-action on $X_*(S')_{\mathbb{R}}$ ($N(k) \curvearrowright G$ preserves S, hence S')

(dual $W(G,S)$-action) $\overset{\shortparallel}{N(k)/Z(k)}$

② $\forall z \in Z(k)$, $f(z): A \to A$ is translation by $v(z) \in V' = X_*(S')_{\mathbb{R}}$

that is image of $w_G(z)$ under pr_1 for.

$$w_Z : Z(k) \longrightarrow X^*(Z)_{\mathbb{R}} \overset{\hookrightarrow = X^*(S)_{\mathbb{R}}}{} \quad X_*(S)_{\mathbb{R}} = X_*(S')_{\mathbb{R}} \oplus X_*(A_G)_{\mathbb{R}}$$

$$g \longmapsto [\chi \mapsto \mathrm{ord}(\chi(g))]$$

$$\begin{tikzcd}
S \subset Z \arrow[r] \arrow[rr, bend left] & Z/DZ \arrow[r] & \mathbb{G}_m \\
C \text{ (max. central)} \arrow[ur, "\text{isogeny}"'] & & \\
S \arrow[r, "\text{isogeny}"'] & (\text{max. split qt of } Z/DZ) \arrow[uur] &
\end{tikzcd}$$

via isogeny

$$\begin{tikzcd}
S' \times A_G \arrow[r, "\text{isogeny}", "\text{mult}"'] & S \\
DG \times Z_G \arrow[u, phantom, "\cap"] \arrow[r, "\text{mult}", "\text{isogeny}"'] & G \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

Pf [KP, Prop 4.4.3]

Rmk (i) Construction of A is "too abstract" to be useful in most pfs later.

(ii) $Z(k)^1 = \ker(w_Z)$ acts trivially on A, hence all bounded subgps of $Z(k)$ do so, too.

(iii) For $z = s \in S(k) \subset Z(k)$ and $a \in \Phi(G,S) \subset X^*(S')_{\mathbb{R}} = V'^*$,

Δ is an $\mathbb{R}$-basis.

(*) $a(\nu(s)) = -\mathrm{ord}(\underset{k^\times}{a(s)}) \in \mathbb{Z}$

Pf. $S(k) \supset \underbrace{S'(k)}_{\text{defn.}} \underbrace{A_G(k)}_{\text{killed by } \nu}$ OK for $s \in S'(k) A_G(k)$

qt is killed by some $n \in \mathbb{Z}^+$ ($\because S' \times A_G \to S$ isogeny), and to prove (*), can first multiply both sides by n to replace s w/ $s^n \in S'(k) A_G(k)$. $\square$

Rmk. Such A has an interpretation in terms of "valuation on root datum"

$\uparrow$ = way of put "norms" on $U_a(k)$'s to cut out bounded subgps.

only practical for G q-split (e.g. split)

Reflections $r_a \in N(k)/Z(k)$ for $a \in \Phi$ come from:

Prop (Tits) For $a \in \Phi = \Phi(G,S)$, $u \in U_a(k) - \{1\}$, $\exists!\ u', u'' \in U_{-a}(k)$

s.t. $m(u) = u'uu'' \in N(k)$

These $u', u'' \neq 1$, and $m(u) \curvearrowright X^*(S)_{\mathbb{R}}$ is r_a.

If $a \in \Phi^{nm}$, then $u' = u'' = m(u)^{-1}u\,m(u)$ and $m(u)^2 \in S(k)$.

Pf [CGP, Prop C.2.24]

We'll use $m(u)$'s to cut out hyperplanes in $A = A(S)$ (next time)

(fixed pt locus of $m(u) \curvearrowright A$)

Ex. $G = SL_2$, $a\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} = t^2$, $U_a = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$

$u = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \Rightarrow u' = u'' = \begin{bmatrix} 1 & 0 \\ -\frac{1}{x} & 1 \end{bmatrix}$, $m(u) = \begin{bmatrix} 0 & x \\ -\frac{1}{x} & 0 \end{bmatrix}$

$x \neq 0$

$G = SU_3(k'|k)$ - in notes ($u' \neq u''$).

Lecture 27 Apartment axioms and metric

Let G, $S \subset G$, $N = N_G(S)$, $Z = Z_G(S)$ be as before, $\Phi = \Phi(G,S)$.

Last time: $\exists!$ (up to $\underline{!}$ isom) affine space $A = A(S)$ for $V' = X_*(S')_{\mathbb{R}}$ equipped w/ $f: N(k) \longrightarrow \mathrm{Aff}(A)$ satisfying certain properties. (added constr. of A: set of splitting of certain exact seq. of gps $1 \to V' \to E \to W \to 1$)

$$\underset{\substack{\circlearrowright \\ W \text{ natural}}}{V'} \qquad \underset{\substack{\| \\ W(\Phi)}}{W}$$

"up to ! isom" includes A has no nontrivial $N(k)$-equiv. affine auts.

Variant: Assuming $S' \neq 1$ ($\Leftrightarrow A \neq pt$) put a (thin) chamber complex str. on A (using an affine root system)

$N(k)$-equiv. $\qquad\qquad\qquad N(k)$-equiv.

and ask if the underlying polysimplicial complex w/ $N(k)$-action has no nontrivial auts? (ignoring the affine space str.)

<u>Axiom 6</u>: Assume $S' \neq 1$ ($\Leftrightarrow$ $\mathcal{D}G$ is k-isotropic $\Leftrightarrow \Phi \neq \phi \Leftrightarrow$ $\overset{\dim A > 0}{A \neq pt}$)

(AS1) $\forall a \in \Phi$, $\forall u \in U_a(k) - \{1\}$, action on A by $m(u) = u'uu'' \in N(k)$ $(m(u)^2 \in Z(k))$ is a reflection in a hyperplane $H_u = A^{m(u)=1}$ (i.e. $\dim H_u = \dim A - 1$) that is an affine space for $\ker(\underline{a}) \subset V'$ ($a \in X^*(S')_{\mathbb{R}} = V'^*$)

In particular, H_u is zero locus of a unique $\psi \in A^* - \mathbb{R}$ w/ $\dot{\psi} = \underline{a}$, call it ψ_a^u.

(AS2) $\Psi' = \{\psi_a^u \in A^* - \mathbb{R} : \overset{a \in \Phi,}{u \in \mathcal{U}_a(k) - \{1\}}\}$ is an affine root system

in A^* w/ $\nabla\Psi' = \Phi$ in $V'^* = X^*(S')_{\mathbb{R}}$.

(AS3) For $\psi \in A^*$ w/ $\dot\psi = a \in \Phi$, the subset

$$\mathcal{U}_\psi = \{u \in \mathcal{U}_a(k) : u = 1 \text{ or } \psi_a^u \geq \psi \ (\Leftrightarrow \psi(H_u) \leq 0 \Leftrightarrow H_u \subset A^{\psi \leq 0})\}$$

$\left[\ell, \ell' \in A^* - \mathbb{R},\ \ell' \geq \ell \Leftrightarrow \underbrace{\ell' - \ell \geq 0}_{\text{affine linear}} \Leftrightarrow \ell' - \ell \in \mathbb{R}_{\geq 0},\ \text{so } \dot\ell' = \dot\ell \right]$

is a subgp of $\mathcal{U}_a(k)$. (later: these $\mathcal{U}_\psi$ will be <u>bounded</u> in $\mathcal{U}_a(k)$)

For

$$\mathcal{U}_{\psi+} = \{u \in \mathcal{U}_a(k) : \cdots \psi_a^u > \psi\} = \bigcup_{\underbrace{\psi' > \psi}_{\text{directed as } \psi' \downarrow \psi}} \mathcal{U}_{\psi'}. \quad (= \text{subgp of } \mathcal{U}_\psi),$$

$$\Psi = \{\psi \in \Psi' : \mathcal{U}_\psi \not\subset \mathcal{U}_{\psi+}\underbrace{\mathcal{U}_{2a}(k)}_{=1 \text{ when } 2a \notin \Phi}\}$$ is an affine root system on A.

w/ $\nabla\Psi' = \Phi$.

(AS4) $(\Psi \subset)\Psi' \subset \Psi \cup \frac{1}{2}\Psi$ (i.e. if $\psi \in \Psi'$ <u>not</u> in Ψ, then $2\psi \in \Psi$).

In particular, Ψ and Ψ' define same collection of (vanishing) hyperplanes in A,

So <u>same</u> thin chamber complex str. on A.

<u>Rmk</u>. For $n \in N(k)$, $u \in \mathcal{U}_a(k) - 1$, then $nun^{-1} \in \mathcal{U}_{n.a}(k) - \{1\}$

and $m(nun^{-1}) = n\,m(u)\,n^{-1}$, so $N(k)$ acting on A preserves Ψ, Ψ',

$\{ \psi_a^u : u \in U_a(k) - \{1\},\ a \in \Phi \}$.

(A5) $N(k) \xrightarrow{f} \mathrm{Aff}(A)$ satisfies $\underset{\text{(automatic)}}{W(\Phi) \subset \mathrm{im}(f)} \subset W(\Phi)^{\mathrm{ext}}$

where <u>extended Weyl gp</u> is $W(\Phi)^{\mathrm{ext}} = \{ \alpha \in \mathrm{Aff}(A) : \alpha(\Phi) = \Phi$ and $\dot{\alpha} \in W(\bar{\Phi}) \}$

notes $\rightarrow$ $\underset{\text{finite}}{\triangleright} W(\Phi)$ $\quad$ $W(\bar{\Phi}) \cong W(\Phi^\vee) \subset GL(V')$

$(\subset \mathrm{Aut}(e))$

and

- $G = G^{sc} \Rightarrow \mathrm{im}(f) = W(\Phi)$
- G q-split adjoint type $\Rightarrow \mathrm{im}(f) = W(\Phi)^{\mathrm{ext}}$ $\quad (SL_2$ vs $PGL_2)$

$\uparrow$

$\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} \in N(k)$

flips an edge in tree

** (A6) The $N(k)$-equiv. polysimplicial complex A_0 arising from (A, Φ) (or $(A, \Phi'))$ is $\mathcal{A}(S) \circlearrowleft N(k)$.

This needs:

<u>Lemma</u>. A_0 has no nontrivial $\overset{N(k)}{\text{equiv.}}$ automorphisms.

<u>Pf.</u> Fix vertex $v \in A_0$, want to reconstruct v from elts of $N(k)$ acting on A_0: show $\bigcap_j (A_0)^{m(u_j)=1} = \{v\}$

A  $H_{u_j} = A^{m(u_j)=1}$

$\square$

Axiom 4 - Via $A(S) \simeq A(S)_0$ from Axiom 6, for $g \in G(k)$, the isom

$g : A(S) \simeq A(gSg^{-1})$ arising from a (necessarily unique)

$\circlearrowright$
$B(G)$

<u>affine</u> aut $g : A(S) \simeq A(gSg^{-1})$ and

$\nabla g : X_*(S')_{\mathbb{R}} \simeq X_*(gS'g^{-1})_{\mathbb{R}}$ is exactly

natural from g conj. b/w the tori.

Lecture 28 <u>Metric and Enrichment of Bldg</u> Fix $S_0 \subset G$ max'l split k-torus

We've had Axioms 1, 2, 6, 4. Here is another: $N = N_G(S)$, $\Phi_0 = \Phi(G, S)$.

Axiom 9 (Tits system) Assume $\mathcal{D}G$ is k-isotropic ($\Leftrightarrow$ S_0 not central

Choose chamber $\mathcal{C}_0 \subset A_0 = A(S_0)$ $\left(\begin{array}{l}\text{recall}\\ N(k) = \mathrm{Stab}_{G(k)}(A(S_0))\end{array}\right)$ $\Leftrightarrow S_0' \neq 1$ $\Leftrightarrow B(G)$ not pt).

Let $\mathcal{G} = G(k)^0$, $\mathcal{I} = G(k)^0_{\mathcal{C}}$, $\mathcal{N} = N(k) \cap \mathcal{G}$
$= \mathrm{Stab}_{\mathcal{G}}(A_0)$

Then $\mathcal{G} \curvearrowright B(G)$ satisfies hypotheses for the "reverse construction", so

- $(\mathcal{I}, \mathcal{N})$ is a saturated BN-pair for $\mathcal{G}$, and the finite set $\mathcal{S}$ of reflections in $\mathcal{W} = W/(\mathcal{N} \cap W)$ has all irred. comps $\mathcal{S}_i$ w/ $\# \mathcal{S}_i \geq 2$.

- and $B(G) \circlearrowleft \mathcal{G}$ $\underline{\underline{\text{is}}}$ the restricted bldg of this gp theoretic data.
(unambiguous!)
since $\mathrm{Aff}_{N(k)}(A) = 1$

when $\mathcal{D}G$ is k-isotropic.

Two consequences : ① for any facet $\mathcal{F} \subset \mathcal{B}(G)$, $G(k)^{\circ}_{\mathcal{F}} \curvearrowright$ vertices $\mathrm{vert}(\mathcal{F})$ is trivial

② $G(k)^{\circ}_{\mathcal{F}}$ are exactly the admissible parabolic subgps of (I, N).

When $\mathcal{D}G$ is k-anisotropic, then $\mathcal{B}(G) = pt$, and ①, ② still OK.

Rmk. The subgps $G(k)^{\circ}_{\mathcal{F}}$ are open and bounded in $G(k)$ (by Axiom [–]).

Now we enrich combinatorial $\mathcal{B}(G)$ to interesting "space" $B(G)$.

$\mathcal{B}(G)$ is gluing of $A(S)$ along polysimplicial subcomplexes

$A(S) \cap A(S')$ for max. split k-tori $S, S' \subset G$.

Define $B(G)$ to be gluing of $A(S)$'s along subsets corresponding to union of facets in each $A(S) \cap A(S')$. (check triple overlap).

$$\mathrm{Facets}(B(G)) \longleftrightarrow \mathrm{Facets}(\mathcal{B}(G)) \qquad (*)$$

$$F \longmapsto \mathcal{F} = \mathrm{vert}(F) = F_0.$$

and $\mathcal{F}' \subset \mathcal{F}$ in $\mathcal{B}(G)$ is exactly $\underbrace{F' \subset \overline{F}}_{\text{denoted } F' \preceq F}$ in $B(G)$, where

$\overline{F}$ = union of subfacets of F (using subsets of $\mathrm{vert}(F)$)

(computed in any apt $A(S) \subset F$)

? In $B(G)$, distinct facets are disjoint.

For any $g \in G(k)$, have affine isom. $g: A(S) \xrightarrow{\sim} A(gSg^{-1})$

inducing $g: \mathcal{A}(S) \simeq \mathcal{A}(gSg^{-1})$

These define $G(k)$-action on $B(G)$, and it makes $(*)$ be $G(k)$-equivariant.

For facet F, we have $G(k)^0_F = G(k)^0_{F_0}$, and this acts trivially on

$F_0 = \mathrm{vert}(F)$, but also acts as affine isoms $\begin{array}{ccc} A(S) & \simeq & A(gSg^{-1}) \\ \cup & & \cup \\ F & \Rightarrow & F \end{array}$

via id on vert (F), so such g is id on F

and even on $\overline{F}$.

Thus $G(k)^0_F = G(k)^0_x$ for each $x \in F$. (use $B(G) = \coprod$ facets)

In particular, (i) $G(k)^0_x = G(k)^0_y$ for $x, y \in B(G) \Leftrightarrow x, y$ in same facet

Moreover, (ii) $B(G)^{G(k)^0_F} = \overline{F}$ (since $\mathcal{B}(G)^{G(k)^0_F} = \{\text{subfacets of } F\}$)

? $G(k)^1_x$ is sensitive to where $x \in F \subset B(G)$ lies: $G = PGL_2$, $x \in F =$ o—o

midpt or not midpt

Now it's safe to rename $B(G), A(S)$ as $\mathcal{B}(G), \mathcal{A}(S)$.

(if ever need to pass to underlying polysimplicial complex, we may write

$\mathcal{B}_0, \mathcal{A}_0$).

<u>Metric construction</u> For any two $x, y \in \mathcal{B}$, have $x \in F \subset \overline{C}$, $y \in F' \subset \overline{C'}$,

and $\exists$ apt $A \supset \overline{C}, \overline{C'}$, so $A \ni x, y$.

Pick $S_0 \subset G$, choose $W(\Phi_0)$ - inv't $\langle \cdot, \cdot \rangle$ on $X_*(S'_0)_{\mathbb{R}}$.

This defines metric on $A(S_0)$: $d(a, a') = \|a - a'\|$, where $a - a' \in X_*(S'_0)_{\mathbb{R}}$.

For any $S \subset G$ max split k-torus, pick $g \in G(k)$, so $g S_0 g^{-1} = S$.

So have $g : A(S_0) \xrightarrow{\simeq} A(S)$ as affine spaces over $X_*(S'_0)_{\mathbb{R}} \simeq X_*(S')_{\mathbb{R}}$

Transport the metric to $d_{A(S)}$ on $A(S)$

Claim: This is indep. of g.

Pf: Must check $N_G(S_0)(k) \curvearrowright A(S_0)$ is isometry.

Action is affine, so reduces to derivative action on $X(S'_0)_{\mathbb{R}}$ being an isometry.

- ok $\because$ that goes through $W(\Phi_0)$-action on $X(S'_0)_{\mathbb{R}}$. $\square$

Define $d : \mathcal{B}(G) \times \mathcal{B}(G) \longrightarrow \mathbb{R}_{\geq 0}$

$d(x, y) = d_A(x, y)$ for any $A \ni x, y$.

Well-defined: Suppose $A, A' \ni x, y$, can pick facets $F_x, F_y \subset A$ with

$x \in F_x$, $y \in F_y$. $A' \ni x, y \Rightarrow A' \supset F_x, F_y$.

One of bldg axioms (in refined form)

$\exists$ isom. of apts $\alpha : A \simeq A'$ that is the identity on F_x, F_y.

- this α comes from $g \in G(k)^\circ \subset G(k)$ $\therefore$ $A' = gA$, but $g : A \xrightarrow{\simeq} A'$

carries d_A to $d_{A'}$, get $g(x) = x$, $g(y) = y$. $\square$

$\mathcal{B}(G) \simeq \mathcal{B}(G^{ad}) = \prod \mathcal{B}(G_i^{ad})$

Lecture 29. Boundedness and Applications

$\Phi = \Phi(G,S)$

Let $S \subset G$ be max split k-torus, $N = N_G(S)$, $Z = Z_G(S)$, $A = A(S) \subset \mathcal{B}(G)$,

$\Psi \subset \Psi' \subset A^*$ from Axiom 6.

Given $\psi \in \Psi'$ w/ $\dot{\psi} = a \in \Phi$, get $U_\psi \subset U_a(k)$ certain subgp (bounded)

Axiom 8: (i) $\forall \psi \in \Psi$, $A^{\psi \geq 0}$ is fixed ptwise by $U_\psi \subset U_a(k) \subset G(k)$ $(a = \dot{\psi} \in \Phi)$

(ii) For $\Omega \subset A$ a nonempty bounded subset and $a \in \Phi$, let

$\psi_a^\Omega \in \Psi$ be minimal among all $\psi \in \Psi$ w/ $\dot{\psi} = a$ s.t. $\Omega \subset A^{\psi \geq 0}$.

Then $G(k)^\circ_\Omega$ is gen'td by subgps $Z(k)^\circ$ $\left(\subset Z(k)^1 \text{ acts trivially on } A\right)$.

and $U_{\psi_a^\Omega}$. $\left[Z(k)^\circ := Z(K)^\circ \cap Z(k) \right]$

(iii) $G(k)^\circ_\Omega \cap U_a(k) = U_{\psi_a'^\Omega}$ for $\psi_a'^\Omega \in \Psi'$ is minimal $\psi' \in \Psi'$ s.t. $\dot{\psi}' = a$, $\Omega \subset A^{\psi' \geq 0}$.

Rmk. Using full force of BT integral models + "unram. descent"

$\Rightarrow G(k)^\circ_\Omega \cap Z(k) = Z(k)^\circ$ [KP, §9.4].

Axiom 16 (Iwahori decomp.) Let $\mathcal{C} \subset \mathcal{A}$ be chamber, $I := G(k)^{\circ}_{\mathcal{C}}$, let $\underline{\Phi}^{+} \subset \underline{\Phi}$ be pos. system of roots.

Let $U^{\pm} = R_{u,k}(P^{\pm})$, for $P^{\pm} \supset S$ min. parabolic $/k$ corresponding to $\underline{\Phi}^{+}$, $\underline{\Phi}^{-} = -\underline{\Phi}^{+}$. Then

- $(U^{+}(k) \cap I) \times Z(k)^{\circ} \times (U^{-}(k) \cap I) \xrightarrow{\text{mult}} I$ is bijective.
- for fixed $\pm$, $\underbrace{\prod_{a \in \underline{\Phi}^{\pm, nd}}}_{\text{in any enumeration of } \underline{\Phi}^{\pm, nd}} U_{f_a^{\mathcal{C}}} \xrightarrow{\text{mult}} U^{\pm}(k) \cap I$ is bijective.

Rmk. $\exists$ Axiom 17 on behavior of $\mathcal{B}(G)$ w.r.t. isoms in (k, G).

This is used to make $\mathrm{Gal}(K/k)$ acts on $\mathcal{B}(G_K)$ when BT theory is set up $/K$.

Content: for $\gamma \in \mathrm{Gal}(K/k)$, $T \subset G_K$ max. split K-torus,

get $[\gamma] : \mathcal{A}(T) \xrightarrow{\sim} \mathcal{A}(\gamma^{*}(T))$ (as affine spaces).

s.t. $[\gamma'\gamma] = [\gamma'][\gamma]$.

In "unram. descent" (next time), construct $\mathcal{B}(G)$ as $\mathcal{B}(G_K)^{\Gamma}$.

Prop. $G(k) = I\, N(k)\, I$.

Rmk. With more work using Tits system bldgs, can show $InI = In'I$ $\Leftrightarrow n^{-1}n' \in Z(k)^{\circ}$ ($\subset I$). This is a special case of Cartan decomp. [KP, Thm 5.2.1]

$N(k)/Z(k)^\circ$ is "nearly" $W(\Phi)$ (v.s. $W(\Phi)^{ext}\ldots$)

Pf. Let $\mathcal{G} = G(k)^\circ$, $\mathcal{N} = N(k) \cap \mathcal{G} = \mathrm{Stab}_{\mathcal{G}}(\mathcal{A})$ $\because N(k) = \mathrm{Stab}_{G(k)}(\mathcal{A} = \mathcal{A}(S))$.

We have $(\mathcal{I}, \mathcal{N})$ is a (saturated) BN-pair for $\mathcal{G}$, so $\mathcal{G} = \mathcal{I}\mathcal{N}\mathcal{I}$

(Bruhat decomp. for Tits system)

Using $\mathcal{G}$ is transitive on $\underset{\underset{G(k)}{\circlearrowleft}}{\{\text{apts}\}}$, get $G(k) = N(k)\,\mathcal{G}$.

But $\mathcal{G}$ acts transitively on set of pairs $(\mathcal{A}', e')$ – in particular, using $\mathcal{A}' = \mathcal{A}$,

get $\mathcal{G} \cap N(k) = \mathcal{N}$ is transitive on set of chambers in $\mathcal{A}$.

$\therefore$ $N(k) = N(k)_e \mathcal{N}$.

$G(k) = N(k)\,\mathcal{G} = N(k)_e \underbrace{(\mathcal{N}\mathcal{G})}_{= \mathcal{G}} = N(k)_e\, \mathcal{I}\mathcal{N}\mathcal{I}$

But $N(k)_e$ normalizes $G(k)^\circ_e = \mathcal{I}$. $\quad = \mathcal{I}\, N(k)_e\, \mathcal{N}\mathcal{I} = \mathcal{I}\, N(k)\, \mathcal{I}$. $\square$

$\underset{\text{finite}}{N(k) \supset Z(k)} \supset \underbrace{Z(k)^1 \supset Z(k)^\circ}_{\text{finite}}$ — This motivates the idea that a subset $\Sigma \subset G(k)$ is bounded

$\Leftrightarrow$ Its Iwahori reps in $N(k)$ have f. many images mod $Z(k)^\circ$.

$Z(k)/Z(k)^1 \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(X_k^*(Z), \mathbb{Z}) = \mathbb{Z}$-lattice.

Thm For $\Sigma \subset G(k)^1$ (e.g. any bounded subgp), TFAE:

① Σ is bounded in $G(k)$

② $\Sigma \subset \underbrace{I N(k) I}_{G(k)}$ meets finitely many double cosets.

③ $\Sigma \cdot x \subset B(G)$ is bounded for all $x \in B(G)$

③' $\Sigma \cdot x$ bounded for one $x \in B(G)$.

Pf. See notes.

Cor ① For $x \in B(G)$ vertex, then $G(k)^1_x$ is max. bounded subgp.

② Each max'l bounded (open) subgp $K \subset G(k)$ is $G(k)^1_x$ for some $x \in B(G)$

(2 not nec. vertex. PGL_2, x = midpt of edge)

If $G(k)^1 = G(k)^0$ (e.g. $G = ss, sc$) then such x must be vertex.

③ Max bounded subgps of $G(k)^0$ are exactly $\underbrace{G(k)^0_x}_{\text{max. parahorics!}}$ for vertices x.

Pf. Suppose $K \subset G(k)$ is max bounded, so $K \subset G(k)^1$, then $K \cdot x \subset B(G)$ is bounded non-empty. presented by K.

$\therefore \exists y \in B(G)$ s.t. $K \subset G(k)^1_y$ (BT fixed pt thm) $\therefore K = G(k)^1_y$ if K is max'l.

Suppose x is vertex, so $G(k)^1_x \subset G(k)^1_y$ for some y.

$$\subset G(k)^1_{\mathcal{F}} \quad \text{for facet } \mathcal{F} \ni y.$$

Suppose $G(k)^1 = G(k)^0$, so $\underset{\substack{\| \\ \text{vertex}}}{G(k)^0_x} \subset G(k)^0_{\mathcal{F}} \subset \underset{= \text{max'l}}{G(k)^0_v}$ for vertex v of $B(G)$.

$\therefore x = v$.

Lecture 30 Bruhat-Tits gp schemes and applications

The axioms so far uniquely determine $B(G)$ up to ! isom:

Prop. If $\underset{\substack{\circlearrowleft \\ G(k)}}{B_1, B_2}$ are covered by the canonical $\underset{\substack{\circlearrowleft \\ N_G(S)(k)}}{A(S)'s}$, and satisfy Axiom 1, 4, 8, 9,

then $\exists!$ $G(k)$-equiv. isom. $B_1 \simeq B_2$ inducing $A_1(S) \underset{!}{\simeq} A_2(S)$. for some

($\Rightarrow$ all) S.

Pf. See [KP, §4.4] - esp. Prop 4.4.6, Cor 4.4.7.

Axiom 27 (unramified descent) Let $\underset{(\text{isometry})}{\Gamma = \mathrm{Gal}(K|k) \curvearrowright B(G_K)}$ commuting w/ $G(k) \subset G(K)$.

(i) $\forall$ max split k-torus $S \subset G$, $\exists$ $\underbrace{k\text{-torus } T \supset S}_{\text{"special" } k\text{-torus}}$ s.t. $T_K \subset G_K$ is $\underset{K}{\text{max'l split}}$.

and for such T, $A(T_K)^\Gamma \subset B(G_K)$ is indep of such T, hence preserved by $N_G(S)(k)$. (ii) Moreover, it is an affine space for $X_*(S')_{\mathbb{R}} \subset X_*(T_K)_{\mathbb{R}}$, and as such is isomorphic (nec. uniquely !) to $\mathcal{A}(S)$.

$N_G(S)(k)$-equiv.

(gives $\mathcal{A}(S) \hookrightarrow \mathcal{B}(G_K)$).

(iii) The injections $\mathcal{A}(S) \hookrightarrow \mathcal{B}(G_K)$ glue nec. uniquely to $G(k)$-equiv.

$\mathcal{B}(G) \hookrightarrow \mathcal{B}(G_K)$. onto $\mathcal{B}(G_K)^\Gamma$

? vertex$(\mathcal{B}(G)) \in$ chamber$(\mathcal{B}(G_K))$ can happen.

(iv) $\mathcal{A}(T_K)^* \longrightarrow \mathcal{A}(S)^*$ carries $\Phi(G_K, T_K)$ into $\Phi(G,S) \cup \mathbb{R}$

when $S' \neq 1$ hitting all of $\Phi(G,S)$.

($\Leftarrow$) $\mathcal{D}G$ is isotropic $\Leftarrow$) $\mathcal{B}(G) \neq pt$)

Rmk. ① For nonempty bounded $\Omega \subset \mathcal{A}(S)$, can view Ω as nonempty bounded in $\mathcal{A}(T_K) \subset \mathcal{B}(G_K)$, so $G(K)^*_\Omega$, $G(k)^*_\Omega$ for $* = 0, 1$ make sense,

and $G(K)^*_\Omega \cap G(k)^* = G(k)^*_\Omega$. ($G(K)^\circ \cap G(k) = G(k)^\circ$ is hard...)

② Pf of existence of special $T \supset S$ needs full force of BT/K including BT gp schemes. [KP, Prop 9.3.4]

Axiom 20. For nonempty bounded $\Omega \subset \mathcal{A}(S)$, $\exists$ $\mathcal{O}$-str. $\mathcal{G}^1_\Omega$ of G s.t. (nec. !)

$\mathcal{G}^1_\Omega(\mathcal{O}) = G(K)^1_\Omega$, so $\mathcal{G}^1_\Omega(\mathfrak{o}) = \mathcal{G}^1_\Omega(\mathcal{O}) \cap G(k) = G(K)^1_\Omega \cap G(k) = G(k)^1_\Omega$

($G(K)^1 \cap G(k) = G(k)^1$)

(K/k unram. $\Rightarrow$ existence of $\mathcal{G}^1_\Omega/\mathcal{O}$ implies existence over $\mathfrak{o}$).

It's (affine!) rel. id. component $\mathcal{G}^\circ_\Omega$ satisfies

- $\mathcal{G}^\circ_\Omega(\mathcal{O}) = G(k)^\circ_\Omega$ $\left(K = k : \mathcal{G}^\circ_\Omega(\mathcal{O}) = G(K)^\circ_\Omega \right)$

** • If $S \hookrightarrow G = (\mathcal{G}^\circ_\Omega)_k = (\mathcal{G}^1_\Omega)_k$ is $\mathcal{S}_k$ for $\underbrace{\text{split}}_{\simeq \mathbb{G}^N_{m,\mathcal{O}}}$ closed $\mathcal{O}$-torus $\mathcal{S} \hookrightarrow \mathcal{G}^\circ_\Omega$.

- If $\Omega \subset \mathcal{F}$ $(\subset \mathcal{A}(S))$, then $\mathcal{G}^\circ_\Omega = \mathcal{G}^\circ_{\mathcal{F}}$ ← parahoric gpschemes.
  (recall $G(k)^\circ_\Omega = G(k)^\circ_{\mathcal{F}}$)

Rmk $\overline{\mathcal{S}} \hookrightarrow (\mathcal{G}^\circ_{\mathcal{F}})_f$ is max'l f-split: use deformation theory of tori in smooth affine gps + $S \subset G$ is max. split.

$\mathcal{G}^1_{\overline{\Omega}} = \mathcal{G}^1_\Omega$ (look at $\mathcal{O}$-pts), so if $\Omega \prec \Omega'$ $(\Omega \subset \overline{\Omega'})$,

$$\mathcal{G}^1_{\Omega'}(\mathcal{O}) = G(K)^1_{\overline{\Omega'}} \subset G(K)^1_\Omega = \mathcal{G}^1_\Omega(\mathcal{O}),$$

So get $\rho_{\Omega,\Omega'} : \mathcal{G}^1_{\Omega'} \to \mathcal{G}^1_\Omega$ as $\mathcal{O}$-models of G_K, hence as $\mathcal{O}$-models of G

$\therefore$ get $\mathcal{G}^\circ_{\Omega'} \to \mathcal{G}^\circ_\Omega$ (Gal descent = f'et descent for finite unram. $\mathcal{O} \to \mathcal{O}'$)

So get $\overline{\rho}_{\Omega,\Omega'} : \overline{\mathcal{G}}^\circ_{\Omega'} \to \overline{\mathcal{G}}^\circ_\Omega$

Axiom 22. $\ker(\overline{\rho}_{\Omega,\Omega'})$ is smooth conn'd unipotent $/f$. For $\mathcal{F} \prec \mathcal{F}'$, then $P_{\mathcal{F},\mathcal{F}'} = \overline{\rho}_{\mathcal{F},\mathcal{F}'}(\overline{\mathcal{G}}^\circ_{\mathcal{F}'}) \subset \overline{\mathcal{G}}^\circ_{\mathcal{F}}$ is a parabolic f-subgp, and as vary through all $\mathcal{F}' \succ$ given $\mathcal{F}$, these $P_{\mathcal{F},\mathcal{F}'}$ vary w/o repetition through parabolic

$\boxed{\text{inclusion reversing biject.}}$

f-subgps of $\overline{\mathcal{G}}^\circ_{\mathcal{F}}$.

For $A(S) \supset \mathcal{F}$, can describe root datum of $\left((\overline{\mathcal{G}}^\circ_{\mathcal{F}})_{red}, \overline{S}\right)$ in terms of

$\mathcal{F}$ and $\nabla: \Psi(G,S) \twoheadrightarrow \Phi(G,S)$.

Moreover, $\mathcal{G}^\circ_{\mathcal{F}'}(\mathcal{O})$ is preimage of $P_{\mathcal{F},\mathcal{F}'}(f)$ under $\mathcal{G}^\circ_{\mathcal{F}}(\mathcal{O}) \to \mathcal{G}^\circ_{\mathcal{F}}(f)$.

For $\mathcal{G}^\circ_{\mathcal{F}}$ reductive ($\Leftrightarrow$ $\mathcal{F}$ = "hyperspecial" vertex),

Iwahori $G(k)^\circ_{e}$ for $e \supset \mathcal{F}$ is "preimage of Borel" when $\dim f \leq 1$.

Thm. If $\dim f \leq 1$, G is k-anisotropic, and abs. simple (Φ irred.), then G is type A.

(more work: f finite $\Rightarrow$ inner type A: $SL_1(D)$).

Pf. $B(G) \subset B(G_K) \circlearrowleft \Gamma$

$B(G) = \{x\}$

$x \in \mathcal{F}' \subset A(T_K)$

$\mathcal{F}' = \mathcal{F}'^{\Gamma}$

$(\mathcal{G}^\circ_{\mathcal{F}})_{\mathcal{O}}$ is "Γ-stable".

So $\mathcal{G}^\circ_{\mathcal{F}}$ descends to $\mathcal{O}$-model $\mathcal{H}$ of G. But $\mathcal{H}_f$ has no proper parabolic (∵ no nontrivial split tori)

deformation theory of tori

$\therefore \mathcal{H}_f$ soluble, so $\overline{\mathcal{G}}^\circ_{\mathcal{F}}$ is soluble. Δ transitive

so $\mathcal{F}$ is chamber: $x \in e \circlearrowleft \Gamma$. $\text{vert}(e) \circlearrowleft \Gamma$. If ≥ 2 orbits, bary centers of their convex hulls would be distinct in $B(G_K)^\Gamma = B(G) = \{\cdot\}$. Look at t-shirt $\Rightarrow$ type A.