Proof of Arkhipov - Bezrukavnikov's equivalence

Calder Morton - Ferguson

1. Finish the proof of [AB]

2. Discuss t-structures on $D^b_{IW}(\mathcal{F}\ell)$

   $\hookrightarrow$ any conn'd exact perverse sheaf in $Perv_I(\mathcal{F}\ell)$ <u>averages</u> to a tilting one.

$$F_{IW} : D^b Coh^{\check{G}}(\tilde{\mathcal{N}}) \longrightarrow D^b_{IW}(\mathcal{F}\ell_G)$$

$$\left\{ \begin{array}{ccc} V \otimes \mathcal{O}_{\tilde{\mathcal{N}}} & \longmapsto & Z(V) \\ \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) & \longmapsto & J_\lambda \end{array} \right\}$$

<u>Theorem</u>. F_{IW} is an equiv. of categories.

<u>Cor.</u> $P_I^{asph} \longrightarrow P_{IW}$ is an equiv. of categories.

<u>Lemma</u>. $Av_{IW}(J_\lambda)$ w/ $\lambda \in X^\vee$ generate $D_{IW}(\mathcal{F}\ell)$.

<u>Proof</u>. $[J_\lambda] = [\Delta_\lambda]$ in K_0

$\Rightarrow$ $Av_{IW}(J_\lambda)$ is supported on $\overline{\mathcal{F}\ell_{G,\lambda}}$ w/ restriction to $\mathcal{F}\ell_{G,\lambda}$ of rank 1.

$\Rightarrow$ The objects generate $D^b_{IW}(\mathcal{F}\ell_G)$.

Lemma. For any $V \in Rep(G^\vee)$, the morphism

$$Hom_{D^b Coh}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}) \longrightarrow Hom_{D^b_{IW}}(F_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}), F_{IW}(V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}))$$

is injective.

Proof. $F_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}) = \Delta_0^{IW}$, $F_{IW}(V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}) = Z^{IW}(V)$. ETS

$$Hom_{Coh^{G^\vee}(\tilde{\mathcal{N}})}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}) \longrightarrow Hom_{P_I^{asph}}(\Pi_{asph}(\Delta_e^I), \Pi_{asph}(Z(V)))$$

injective.

$$\searrow$$

$$Hom_{P_I^0}(\delta^0, Z^0(V))$$

$\hookrightarrow$ quotient of P_I by $\langle IC_w \rangle$ $w \neq 1$.

$$Hom_{Coh^{G^\vee}(\tilde{\mathcal{N}})}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}) \hookrightarrow Hom_{Rep(H)}(\bar{a}_e, V)$$

$$LHS = (V \otimes \mathcal{O}(\tilde{\mathcal{N}}))^{G^\vee} \xrightarrow{\sim} (V \otimes \mathcal{O}(\mathcal{N}))^{G^\vee} \xrightarrow{\sim} (V \otimes \mathcal{O}(\mathcal{O}_2))^{G^\vee}$$

complement of $\mathcal{O}_2$ has codim 2

regular nilpotent orbit

$$\mathcal{O}(\mathcal{O}_2) = Ind_{Z_{G^\vee}(n_0)}^{G^\vee}(\bar{a}_e)$$

so that $(V \otimes \mathcal{O}(\mathcal{O}_2))^{G^\vee} = (V \otimes Ind_{Z_{G^\vee}(n_0)}^{G^\vee}(\bar{a}_e))^{G^\vee} = V^{Z_{G^\vee}(n_0)}$.

So the map is the inclusion $V^{Z_{G^\vee}(n_0)} \hookrightarrow V^H$.

Done?

Cor. For any $V \in \mathrm{Rep}(G^\vee)$, any $\lambda \in X_+^\vee$, and any $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{D^b\mathrm{Coh}}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)[n]) \longrightarrow \mathrm{Hom}_{D^b_{IW}}(F_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}), F_{IW}(V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))[n])$$

is injective.

Proof. $\mathrm{LHS} = (V \otimes H^n(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)))^{G^\vee}$, $H^n(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = 0$ unless $n = 0$.

$\Rightarrow \mathrm{LHS} = 0$ unless $n = 0$.

There always exists $V' \in \mathrm{Rep}(G^\vee)$ s.t. $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \hookrightarrow V' \otimes \mathcal{O}_{\tilde{\mathcal{N}}}$ in $\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & \mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes V' \otimes \mathcal{O}_{\tilde{\mathcal{N}}}) \arrow[d, "\text{just had}"] \\
\mathrm{Hom}_{D^b_{IW}}(F_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}), F_{IW}(V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))) \arrow[r] & \mathrm{Hom}_{D^b_{IW}}(F_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}), F_{IW}(V \otimes V' \otimes \mathcal{O}_{\tilde{\mathcal{N}}}))
\end{tikzcd}$$

Let's show that F_{IW} is fully faithful.

$$\mathrm{Hom}_{D^b\mathrm{Coh}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\eta_{\mathcal{F},\mathcal{G}}} \mathrm{Hom}_{D^b_{IW}}(F_{IW}(\mathcal{F}), F_{IW}(\mathcal{G}))$$

In the case $\mathcal{F} = \mathcal{O}_{\tilde{\mathcal{N}}}$, and $\mathcal{G} = V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)[n]$, it's an isom.

Why? Injectivity ✓. Surjectivity: we claim both sides have the same dimension.

$$\mathrm{RHS} = \mathrm{Hom}_{D^b_{IW}}(\Delta_0^{IW}, \mathcal{Z}^{IW}(V) * J_\lambda[n]) \simeq \mathrm{Hom}_{D^b_{IW}}(\Delta_0^{IW} * J_{-\lambda}, \mathcal{Z}^{IW}(V)[n])$$
$$\simeq \mathrm{Hom}_{D^b_{IW}}(\Delta_{-\lambda}^{IW}, \mathcal{Z}^{IW}(V)[n]) \overset{\text{TILTING}}{=\!=} \begin{cases} 0, & n \neq 0 \\ \dim(V_{-\lambda}), & n = 0 \end{cases}$$

LHS identifies w/

$$\left(V \otimes \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))\right)^{G^\vee}$$

$N(\lambda)$ co-Weyl module, $\mathrm{Ind}_{B^\vee}^{G^\vee}(k_\lambda)$

so its dimension is

$$\sum_{\nu \in X^\vee_+} [V^* : N(\nu)]\,[\Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) : N(\nu)]$$

we know $[\Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) : N(\nu)] = \dim(N(\nu)_\lambda)$

$\Rightarrow$ this sum is $= \dim((V^*)_\lambda) = \dim(V_{-\lambda})$.

$\Rightarrow$ $\eta_{\mathcal{F}, \mathcal{G}}$ is an isom. when $\mathcal{F} = \mathcal{O}_{\tilde{\mathcal{N}}}$, $\mathcal{G}$ is <u>anything</u>.

So if $\mathcal{F} = \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ for some $\lambda \in X^\vee$,

$$\mathrm{Hom}_{D^b\mathrm{Coh}}(\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda), \mathcal{G}) \simeq \mathrm{Hom}_{D^b\mathrm{Coh}}(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{G} \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(-\lambda))$$

and $\mathrm{Hom}_{D^b_{IW}}(F_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)), F_{IW}(\mathcal{G})) = \mathrm{Hom}_{D^b_{IW}}(\Delta_0^{IW} * J_\lambda, F_{IW}(\mathcal{G}))$

$$= \mathrm{Hom}(\Delta_0^{IW}, F_{IW}(\mathcal{G}) * J_{-\lambda}) \simeq \mathrm{Hom}(\Delta_0^{IW}, F_{IW}(\mathcal{G} \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(-\lambda)))$$

$\Rightarrow$ F_{IW} is fully faithful.

$\Rightarrow$ The essential image of F_{IW} is a triangulated subcat. of $D^b_{IW}(\mathcal{F}\ell)$

It contains $\mathrm{Av}_{IW}(J_\lambda)$ which we showed generate all of $D^b_{IW}(\mathcal{F}\ell)$.

We have $X = \mathcal{F}\ell$

$$D^b_{IW}(X) \simeq D^b Coh^{G^\vee}(\tilde{\mathcal{N}}) \simeq D^b(A\text{-mod}^{G^\vee}_{fg})$$

noncommutative Springer resolution.

Arinkin – Gaitsgory,
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Is there a constructible description of the heart of "the NCS t-structure".

$$D^b_B(G/B) \hookrightarrow D^b_I(\mathcal{F}\ell)$$
$$\mathrm{IS}$$
$$\theta_0$$

$\Delta_w \quad w \in W_{fin}$

$\nabla_w \quad w \in W_{fin}$

$T_w \quad w \in W_{fin}$

$P_w \quad w \in W_{fin}$

$I_w \quad w \in W_{fin}$

Ringel duality $D^b_B(G/B) \circlearrowleft$

$\mathcal{F} \longmapsto \mathcal{F} * \Delta_{w_0}$

$\nabla_w \longmapsto \Delta_{ww_0}$

$T_w \longmapsto P_{ww_0}$

$I_w \longmapsto T_{ww_0}$

"thin affine flag var." $\quad LG/I =: \mathcal{F}\ell \quad G(\mathcal{O}) \xrightarrow{\pi} G(k), \quad I = \pi^{-1}(B)$

"thick affine flag var." $\quad LG/I^- =: Th \quad k[t^{-1}] \xrightarrow{\pi} k, \ t^{-1} \mapsto 0, \quad I^- = \pi^{-1}(B^-)$

$I \backslash LG / I \qquad I \backslash LG / I^-$

$$D^b_I(Fl) \xrightarrow[R]{X} D^b_I(Th)$$

Yun

$\nabla_w \longmapsto \Delta_w$

$T_w \longmapsto P_w$

$R^{-1}(\mathrm{Perv}_{IW}(Th)) \simeq A\text{-mod}^{G^\vee}_{f.g.}$.

Thm If $\mathcal{F} \in D^b_I(Fl)$ is convolution exact, then $Av_{IW}(\mathcal{F})$ is tilting

Lemma. If $\mathcal{F}$ is convolution exact, it's an A-module.

Proof of thm. (Mirković) If $\mathcal{F} \in \mathrm{Perv}_B(G/B)$ is convolution exact, then it's tilting.

$$\left.\mathcal{F} \longmapsto \underbrace{\mathcal{F} * \Delta_{w_0}}_{\Delta_w[-d],\ d\geq 0}\right] \quad \text{affine analogue : } R.$$

$$\underset{d=0}{\nabla_y[-d]} \xleftarrow{*\nabla_{w_0}}$$