Adic spaces

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Def. An <u>adic ring</u> is a topological ring A, s.t. the topology on A is the I-adic topology for an ideal $I \trianglelefteq A$. Such an ideal is called an <u>ideal of definition</u> for A.

Def. A top. ring A is <u>Huber</u> if there is an open subring $A_0 \subset A$ s.t. A_0 is adic w/ a finitely generated ideal of definition.

A_0 is called <u>ring of definition</u>.

<u>Examples</u>
- Discrete rings are adic, $I = 0$
- $\mathbb{Z}_p$, $F[[t]]$ adic rings
- K - discretely valued field $\Rightarrow$ $\mathcal{O}_K \subset K$ open
  $\mathcal{O}_K$ m-adic topology.
- Tate algebras $K\langle T_1, \dots, T_n\rangle$ are Huber, because $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$ is open, and carries the m-adic topology.

Def. A <u>Tate ring</u> is a Huber ring, which contains a topologically nilpotent unit.

K and $K\langle T_1, \dots, T_n\rangle$ are Tate, because a pseudo uniformizer ϖ is a top. nilpotent unit.

Def. A subset S of a top. ring A is <u>bounded</u>, if for any nbhd $\mathcal{U}$ of 0, there is a nbhd V of 0 s.t. $V\cdot S \subset \mathcal{U}$.

$$V\cdot S = \{v\cdot s : v\in V,\ s\in S\}$$

S bounded in norm, $\forall s\in S$, $|s|\le c$.

$\mathcal{U}\supset B(0,r)$. $V = B(0,\frac{r}{c})$, $V\cdot S\subset B(0,r)\subset \mathcal{U}$

<u>Conversely</u> $\mathcal{U} = B(0,1)$. $V\cdot S\subset B(0,1)$. $V\supset B(0,r)$

$B(0,r)\cdot S\subset V\cdot S\subset B(0,1)$, $\forall s\in S$, $|s|\le \frac{1}{r}$.

Def. An element $a\in A$ is power bounded, if $\{a^n\}$ is bounded, and <u>topologically nilpotent</u>, if $a^n\to 0$.

A° is the set of power bounded elts, $A^{\circ\circ}$ is the set of top. nilp. elements.

If A is a Huber ring, A_0 is a ring of def. for A, I an ideal of def for A_0.

A_0 is <u>bounded</u>: $\mathcal{U}$ nbhd of 0, $I^n\subset \mathcal{U}$, $I^n A_0\subset A_0 \Rightarrow A_0\subset A^\circ$

Ex. $A = \mathbb{Q}_p[\varepsilon]/\varepsilon^2$, $a+b\varepsilon$ is power bounded iff a is power bounded

$(a+b\varepsilon)^n = a^n + a^{n-1}\cdot nb\varepsilon$. $A^\circ = \mathbb{Z}_p\oplus \mathbb{Q}_p\,\varepsilon$

Def. A Huber ring A is <u>uniform</u> if A° is bounded in A.

Def. Let A be a Huber ring. A subring $A^+ \subset A$ is called a <u>ring of integral elements</u> if A^+ is open and integrally closed in A.

A <u>Huber pair</u> is a pair (A, A^+) where A is a Huber ring and A^+ is a ring of integral elements.

Ex. A° is an example of a ring of integral elts.
A is an example of a ring of integral elts.

Def. A <u>valuation</u> on a ring A is a map $v : A \to \Gamma \cup \{0\}$, where Γ is an ordered abelian group, written multiplicatively, s.t.

- $v(ab) = v(a)v(b)$, • $v(1) = 1$
- $v(a+b) \le \max(v(a), v(b))$ • $v(0) = 0$

Two valuations v and v' are equiv.
if $\forall a, b \in A$, $v(a) \ge v(b) \iff v'(a) \ge v'(b)$

$a |_v b \iff v(a) \ge v(b)$.

Def. If A is a top. ring, then a valuation v is called <u>continuous</u>, if for any $\gamma \in \Gamma$, the set $\{a \in A : v(a) < \gamma\}$ is open.

Def. $v^{-1}(0)$ is called the kernel of v.

Def. If (A, A^+) is a Huber pair, then define the adic spectrum $\mathrm{Spa}(A, A^+)$ to be the set of equiv. classes of cts valuations on A s.t. $\forall a \in A^+$, $v(a) \leq 1$. We write for $x \in \mathrm{Spa}(A, A^+)$ and $f \in A$, instead of $x(f)$, $|f(x)|$

$\mathrm{Spa}(A, A^+)$ has a topology w/ a basis consisting of rat'l domains

$$U\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{Spa}(A, A^+) : |f_1(x)| \leq |g(x)|, \dots, |f_n(x)| \leq |g(x)|, |g(x)| \neq 0\}$$

for $f_1, \dots f_n, g \in A$ s.t. $(f_1, \dots, f_n)$ is open.

[f_i have no common zero.]

$$U\left(\frac{f_1, \dots f_n}{g}\right) \cap U\left(\frac{h_1, \dots h_m}{k}\right) = U\left(\frac{f_1 k, \dots, f_n k, h_1 g, \dots, h_m g}{gk}\right)$$

Any cover of a rat'l domain by other rat'l domains is refined by a std cover:

$$\mathrm{Spa}(A, A^+) = U\left(\frac{f_1, \dots, f_n}{f_1}\right) \cup \dots \cup U\left(\frac{f_1, \dots, f_n}{f_n}\right).$$

We want to define a structure sheaf $\mathcal{O}_{\mathrm{Spa}(A, A^+)}$ on $\mathrm{Spa}(A, A^+)$ by using the equivalence of sheaves w/ sheaves on the base.

Define $\mathcal{O}_{\mathrm{Spa}(A,A^+)}\left(U\left(\frac{f_1,\dots,f_n}{g}\right)\right)$ to be the universal complete Huber ring over A in which g is invertible and $\frac{f_1}{g},\dots,\frac{f_n}{g}$ are power bounded.

It is denoted $\hat{A}\langle\frac{f_1}{g},\dots,\frac{f_n}{g}\rangle$.

Thm $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a presheaf on the category of rat'l domains and inclusions.

But $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is not always a sheaf

Def (A,A^+) is sheafy, if $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a sheaf of topological rings.

Thm (Huber) (A,A^+) is sheafy, if

- A is discrete
- A is finitely generated over a noetherian ring of definition
- A is Tate and strongly noetherian, that is, $A\langle T_1,\dots,T_n\rangle$ is noetherian for all n.

Thm (Buzzard-Verberkmoes) (A,A^+) is sheafy, if A is Tate and stably uniform, that is, if $A\langle\frac{f_1}{g},\dots,\frac{f_n}{g}\rangle$ is uniform for any $f_1,\dots,f_n,g\in A$ s.t. $(f_1,\dots,f_n)=A$

Def. A valued locally ringed space is a triple $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on X, s.t. $\mathcal{O}_{X,x}$ is a local ring for any $x \in X$, and v_x is an equiv. class of cts valuations on $\mathcal{O}_{X,x}$.

A morphism $f: (X, \mathcal{O}_X, \{v_x\}) \to (Y, \mathcal{O}_Y, \{v_y\})$

consists of a cts map $f: X \to Y$, a morphism of sheaves of top. rings $\mathcal{O}_Y \to f_* \mathcal{O}_X$, s.t.

$$\begin{array}{ccc} \mathcal{O}_{Y, f(x)} & \xrightarrow{f^*} & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

commutes for some choice of v_x and $v_{f(x)}$, equivalently,

$$a \,|_{v_{f(x)}}\, b \iff f^*(a) \,|_{v_x}\, f^*(b).$$

and $\mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a local map.

Def An <u>adic space</u> is a valued locally ringed space which is locally isom. to a space of the form $(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, \{x\}_x)$ for a sheafy Huber pair (A, A^+).

Thm (Γ - Spa adjunction) There is a natural isom.

$$\mathrm{Hom}_{\substack{\text{adic}\\ \text{spaces}}}(X, \mathrm{Spa}(A, A^+)) \cong \mathrm{Hom}_{\substack{\text{Huber}\\ \text{pairs}}}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))$$

Def For an adic space X, define a subsheaf $\mathcal{O}_X^+ \subset \mathcal{O}_X$ via

$f \in \mathcal{O}_X(U)$ is in $\mathcal{O}_X^+(U)$ if for any $x \in U$, the stalk of f at x has valuation at most 1.

Thm
$$\mathcal{O}_{\mathrm{Spa}(A, A^+)}(\mathrm{Spa}(A, A^+)) \cong \hat{A}$$
$$\cup$$
$$\mathcal{O}^+_{\mathrm{Spa}(A, A^+)}(\mathrm{Spa}(A, A^+)) \cong \hat{A^+}.$$

Def. A complete Tate ring R is <u>perfectoid</u>, if

$\boxed{p \text{ is a fixed prime.}}$

- R is uniform
- $\exists$ pu ϖ s.t. $\varpi^p | p$ in R°
- $F: R^\circ/p \twoheadrightarrow R^\circ/p$ is surjective.

($\Leftrightarrow$ $F: R^\circ/\varpi \to R^\circ/\varpi^p$ is an isom.)

Def. K is a perfectoid field, if K is complete.

- K is not discretely valued,
- $|p| < 1$
- $F: K^\circ/p \to K^\circ/p$ is surjective.

Ex. $\mathbb{Q}_p$ is <u>not</u> perfectoid, $p^p \nmid p$.

- $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ is perfectoid.

  $F : \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}/p \longrightarrow \widehat{\mathbb{Z}[\zeta_{p^\infty}]}/p$

- $\mathbb{F}_p((t^{\frac{1}{p^\infty}})) = \widehat{\mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})}$

- K perfectoid field, $\Rightarrow$ $K\langle T^{\frac{1}{p^\infty}}\rangle$ is a perfectoid ring.

$$\overset{\shortparallel}{\widehat{\varinjlim_n K\langle T^{\frac{1}{p^n}}\rangle}}$$

- $\mathbb{Z}_p^{cyc}((T^{\frac{1}{p^\infty}}))\langle \frac{p}{T}\rangle$ - perfectoid ring not containing a field.

Prop. If R is a <u>perfect</u> ($F : R \xrightarrow{\sim} R$) complete Tate ring of char. p, then R is perfectoid.

<u>Pf (sketch)</u> How to show R is uniform:

start w/ a ring of definition $R_0 \subset R$, prove that $\bigcap_n F^{-n}(R_0) =: S$ also a ring of definition.

$R^{\circ\circ} \subset S$, $R^\circ \subset \varpi^{-1}R^{\circ\circ} \subset \varpi^{-1}S$, bounded

<u>Cor</u> A perfect complete Tate ring is sheafy.

<u>Thm</u> A perfectoid ring is sheafy.

$$\begin{tikzcd} R \arrow[d] & R\langle \frac{f}{g} \rangle \arrow[d, squiggly, dash] \\ R^\flat \arrow[r] & R^\flat\langle \frac{f}{g} \rangle \end{tikzcd}$$

<u>Def</u>. A <u>perfectoid space</u> is an adic space locally isom. to $\mathrm{Spa}(R, R^+)$,

R is perfectoid.