Gaitsgory's central functor

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June 12, 2024

Abstract

This is the note for a seminar talk in Tsinghua. My task is to introduce Gaitsgory's central functor.

1 Introduction

Let G be a connected reductive group, defined over $\mathbb{Z}$. For simplicity, let us assume that G is split. We fix standard notations B, T, N etc.

Temporarily, set $\mathcal{K} = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$. Let I be the Iwahori in $G(\mathcal{O})$. We have the affine Hecke algebra

$$\mathcal{H}^{\mathrm{aff}} = (C_c(I\backslash G(\mathcal{K})/I), *) = C_{I,c}(\mathbf{Fl})$$

and the spherical affine Hecke algebra

$$\mathcal{H}^{\mathrm{sph}} = (C_c(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})), *) = C_{G(\mathcal{O}),c}(\mathbf{Gr}).$$

Here $\mathbf{Fl} = G(\mathcal{K})/I$ and $\mathbf{Gr} = G(\mathcal{K})/G(\mathcal{O})$.

By integration along $G(\mathcal{O})/I$, I get a map

$$\mathcal{H}^{\mathrm{aff}} = C_{I,c}(\mathbf{Fl}) \to C_{G(\mathcal{O}),c}(\mathbf{Fl}).$$

I also have a map

$$\mathcal{H}^{\mathrm{sph}} = C_{G(\mathcal{O}),c}(\mathbf{Gr}) \to C_{G(\mathcal{O}),c}(\mathbf{Fl})$$

via pull-back.

Theorem 1.1 (Bernstein)**.** The image of $Z(\mathcal{H}^{\mathrm{aff}})$ and $\mathcal{H}^{\mathrm{sph}}$ in $C_{G(\mathcal{O}),c}(\mathbf{Fl})$ agree, and we have an isomorphism

$$Z(\mathcal{H}^{\mathrm{aff}}) \simeq \mathcal{H}^{\mathrm{sph}}.$$

From now on, let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. By the fonctions-faisceaux correspondence, a natural categorification of $\mathcal{H}^{\mathrm{sph}}$ is the Satake category $\mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, while the affine Hecke category $\mathsf{Perv}_I(\mathbf{Fl})$ is a categorification of $\mathcal{H}^{\mathrm{aff}}$. Gaitsgory's central functor is a categorification of Bernstein's theorem above.

Theorem 1.2 (Gaitsgory)**.** There is a functor

$$Z\colon \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \to \mathsf{Perv}_I(\mathbf{Fl})$$

such that

1. For any $\mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$ and $\mathcal{F} \in \mathsf{Perv}(\mathbf{Fl})$, the convolution $\mathcal{F} *_{\mathbf{Fl}} Z(\mathcal{G})$ is a perverse sheaf.
2. For any $\mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$ and $\mathcal{F} \in \mathsf{Perv}_I(\mathbf{Fl})$, there is a canonical isomorphism
$$Z(\mathcal{G}) *_{\mathbf{Fl}} \mathcal{F} \simeq \mathcal{F} *_{\mathbf{Fl}} Z(\mathcal{G}).$$
3. $Z(\delta_{1_{\mathbf{Gr}}}) = \delta_{1_{\mathbf{Fl}}}$.
4. For any $\mathcal{G}^1, \mathcal{G}^2 \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, there is a canonical isomorphism
$$Z(\mathcal{G}^1) *_{\mathbf{Fl}} Z(\mathcal{G}^2) \simeq Z(\mathcal{G}^1 *_{\mathbf{Gr}} \mathcal{G}^2).$$
5. For any $\mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, we have $\pi_!(Z(\mathcal{G})) \simeq \mathcal{G}$. Here π is the projection $\pi\colon \mathbf{Fl} \to \mathbf{Gr}$.

Here canonicity or naturality means certain higher compatibility isomorphisms.

2 Principal bundles

Moduli problems of principal G-bundles are ubiquitous in the study of affine grassmannians and affine flag varieties, so I feel like it's part of my duty to clarify what do we mean by principal bundles.

2.1 Grothendieck topologies

Let k be a commutative ring, $k\text{-}\mathsf{Alg}$ the category of k-algebras. We know that $k\text{-}\mathsf{Alg}^{\mathrm{op}}$ is equivalent to the category of affine k-schemes, so a presheaf of set on the category affine k-schemes is the same as a functor $k\text{-}\mathsf{Alg} \to \mathsf{Set}$. Similarly, I have the category of presheaves of groups $\mathsf{Fun}(k\text{-}\mathsf{Alg}, \mathsf{Grp})$, the category of presheaves of abelian groups $\mathsf{Fun}(k\text{-}\mathsf{Alg}, \mathsf{Ab})$, etc.

Recall that for a topological space X, a sheaf on X is a presheaf on X satisfying the sheaf axiom, namely certain gluing property. More precisely, suppose $\bigcup_{i\in I} U_i$ is an open covering of some open subset U, we require the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

to be an equalizer. By specifying a collection of "open coverings" of objects $\mathrm{Spec}(R) \in k\text{-}\mathsf{Alg}^{\mathrm{op}}$, I can define certain Grothendieck topology on $k\text{-}\mathsf{Alg}^{\mathrm{op}}$.

Definition 2.1 (fpqc topology)**.** A collection

$$\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\}_{i\in I}$$

is an open covering of $\mathrm{Spec}(R)$ in the *fpqc topology*, if

1. I is finite;
2. each $R \to S_i$ is flat;
3. $R \to \prod_{i\in I} S_i$ is faithfully flat.

Definition 2.2 (fppf topology)**.** A collection

$$\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\}_{i\in I}$$

is an open covering of $\mathrm{Spec}(R)$ in the *fppf topology*, if it is an open covering in the fpqc topology and each S_i is finitely presented over R.

Definition 2.3 (étale topology)**.** A collection

$$\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\}_{i\in I}$$

is an open covering of $\mathrm{Spec}(R)$ in the *étale topology*, if it is an open covering in the fppf topology and each S_i is étale over R.

Definition 2.4 (Zariski topology)**.** A collection

$$\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\}_{i\in I}$$

is an open covering of $\mathrm{Spec}(R)$ in the *Zariski topology*, if it is an open covering in the étale topology and each S_i is of the form R_f for some $f \in R$.

Now let $\tau \in \{\mathrm{fpqc}, \mathrm{fppf}, \text{ét}, \mathrm{Zar}\}$ be one of these Grothendieck topologies.

Definition 2.5. A presheaf $F \in \mathsf{Fun}(k\text{-}\mathsf{Alg}, \mathsf{Set})$ is a τ-sheaf, if for any τ-cover

$$\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\}_{i\in I},$$

the diagram

$$F(R) \to \prod_{i\in I} F(S_i) \to \prod_{i,j\in I} F(S_i \otimes_R S_j)$$

is an equalizer.

Let $\mathsf{Shv}_\tau(k\text{-}\mathsf{Alg}^{\mathrm{op}})$ be the category of τ-sheaves on $k\text{-}\mathsf{Alg}^{\mathrm{op}}$.

By construction, we have

$$\mathsf{Shv}_{\text{fpqc}}(k\text{-}\mathsf{Alg}^{\text{op}}) \subset \mathsf{Shv}_{\text{fppf}}(k\text{-}\mathsf{Alg}^{\text{op}}) \subset \mathsf{Shv}_{\text{ét}}(k\text{-}\mathsf{Alg}^{\text{op}}) \subset \mathsf{Shv}_{\text{Zar}}(k\text{-}\mathsf{Alg}^{\text{op}}).$$

Remark 2.1. From this point of view, Grothendieck's faithfully flat descent theorem tells us the presheaf $R \mapsto R$ is a fpqc-sheaf.

Remark 2.2. For any $\tau \in \{\text{fppf}, \text{ét}, \text{Zar}\}$, the inclusion $\mathsf{Shv}_\tau(k\text{-}\mathsf{Alg}^{\text{op}}) \hookrightarrow \mathsf{Fun}(k\text{-}\mathsf{Alg}, \mathsf{Set})$ has a left adjoint, the τ-sheafification. The fpqc-sheafification is more subtle for some set-theoretic issues. I will ignore these obstacles by just avoiding talking about fpqc-sheafification.

Remark 2.3. I can perform the same constructions over any base scheme S.

2.2 Yoneda embedding

By the Yoneda embedding, we have a faithful embedding

$$k\text{-}\mathsf{Alg}^{\text{op}} \hookrightarrow \mathsf{Fun}(k\text{-}\mathsf{Alg}, \mathsf{Set}), \operatorname{Spec}(R) \mapsto [S \mapsto \operatorname{Hom}_{k\text{-}\mathsf{Alg}}(R, S)].$$

For any $\tau \in \{\text{fpqc}, \text{fppf}, \text{ét}, \text{Zar}\}$, the image of this embedding lies in $\mathsf{Shv}_\tau(k\text{-}\mathsf{Alg}^{\text{op}})$, the category of τ-sheaves, by Grothendieck's faithfully flat descent. Moreover, we have

Proposition 2.1. There is a faithful embedding

$$h\colon k\text{-}\mathsf{Sch} \hookrightarrow \mathsf{Shv}_\tau(k\text{-}\mathsf{Alg}^{\text{op}}), X \mapsto [h_X\colon \operatorname{Spec}(S) \mapsto \operatorname{Hom}_{k\text{-}\mathsf{Sch}}(\operatorname{Spec}(S), X) = X(S)]$$

from the category of k-schemes to the category of τ-sheaves, for any $\tau \in \{\text{fpqc}, \text{fppf}, \text{ét}, \text{Zar}\}$.

Intuitively, this is an embedding because k-schemes are glued from Zariski open affine k-schemes.

A τ-sheaf in the essential image of the inclusion h is said to be representable in schemes. I do not distinguish a τ-sheaf representable in schemes with the representing scheme.

Since Set is cocomplete, I can talk about presheaves representable in ind-schemes. I also don't distinguish an ind-scheme with the presheaf it represents.

2.3 Principal bundles

Choose a Grothendieck topology $\tau \in \{\text{fpqc}, \text{fppf}, \text{ét}, \text{Zar}\}$ for $k\text{-}\mathsf{Alg}^{\text{op}}$, and let $\mathcal{G}$ be a τ-sheaf of groups.

Definition 2.6 (torsor)**.** By a *$\mathcal{G}$-torsor*, I mean a τ-sheaf $\mathcal{P}$, endowed with a right action of $\mathcal{G}$ (i.e. a morphism $\mathcal{P} \times \mathcal{G} \to \mathcal{P}$ satisfying the usual axioms), such that

1. for any $\operatorname{Spec}(R) \in k\text{-}\mathsf{Alg}^{\text{op}}$, there exists a τ-cover $\{\operatorname{Spec}(S_i) \to \operatorname{Spec}(R)\}_{i \in I}$ such that $\mathcal{G}(S_i) \neq \emptyset$ for any $i \in I$;

2. the map
$$\mathcal{P} \times \mathcal{G} \to \mathcal{P} \times \mathcal{P}, (x, g) \mapsto (x, xg)$$
is an isomorphism of τ-sheaves.

Suppose now $\mathcal{G} = h_G$ is represented by a k-group scheme G.

Definition 2.7. By a *principal G-bundle*, I mean a k-scheme X endowed with a right action of G, such that

1. the morphism of schemes
$$X \times_k G \to X \times_k X, (x, g) \mapsto (x, xg)$$
is an isomorphism;

2. there exists a fpqc covering $\{\operatorname{Spec}(S_i) \to \operatorname{Spec}(k)\}_{i \in I}$ such that each $X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(S_i)$ is isomorphic, as a G-scheme, to $G \times_{\operatorname{Spec}(k)} \operatorname{Spec}(S_i)$.

By definition, any principal G-bundle is fpqc locally trivial. I say that this principal G-bundle is τ-locally trivial, if the covering can be chosen to be a τ-covering.

Let X be a k-scheme. By the fully faithfulness of the Yoneda embedding, the datum of a right h_G-action on h_X is equivalent to the datum of a right G-action on X. Moreover, X is a τ-locally trivial principal G-bundle if and only if the τ-sheaf h_X is an h_G-torsor.

Suppose G is smooth, then any principal G-bundle X is smooth since the property of being smooth is fpqc on the base. Surjective smooth morphisms admit sections étale locally, so now X is automatically étale locally trivial.

Suppose G is affine, then any h_G-torsor in the fpqc, fppf or étale topology is representable by a principal G-bundle. The basic idea is to use affine descent. Let $\mathcal{P}$ be an h_G-torsor, $\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(k)\}_{i\in I}$ be a covering (for the corresponding Grothendieck topology) over which $\mathcal{P}$ is trivial. The restriction of $\mathcal{P}$ to each $\mathrm{Spec}(S_i)$ is representable by a scheme P_i (noncanonically isomorphic to $G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(S_i)$). Each P_i is affine over $\mathrm{Spec}(S_i)$ because the property of being affine is fpqc on the base. These schemes P_i are naturally endowed with a descent datum relative to the covering $\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(k)\}_{i\in I}$, which is effective by affine descent. So we can "glue" these P_i's to obtain a scheme P representing $\mathcal{P}$.

From above discussions, we know that the following notions coincide when G is smooth and affine:

- h_G-torsors for the fpqc topology;
- h_G-torsors for the fppf topology;
- h_G-torsors for the étale topology;
- principal G-bundles;
- fppf locally trivial principal G-bundles;
- étale locally trivial principal G-bundles.

From now on, I only consider the case in which G is smooth and affine, and I only use the term principal G-bundles.

Due to some mental block, I will set $k = \mathbb{C}$ to be the field of complex numbers.

Recall that $G_{\mathcal{K}}$ is the presheaf

$$G_{\mathcal{K}} \colon R \mapsto G(R((t)))$$

and $G_{\mathcal{O}}$ is the presheaf

$$G_{\mathcal{O}} \colon R \mapsto G(R[[t]]).$$

The affine grassmannian $\mathbf{Gr}$ is defined to be the fppf sheafification of $G_{\mathcal{K}}/G_{\mathcal{O}}$. We know that $G_{\mathcal{O}}$ is representable in schemes, and $G_{\mathcal{K}}, \mathbf{Gr}$ are representable in ind-schemes.

Similarly, I define I to be the presheaf

$$I \colon R \mapsto \mathrm{ev}^{-1}(B(R)), \mathrm{ev} \colon G(R[[t]]) \to G(R),$$

$\mathbf{Fl}$ to be the fppf sheafification of $G_{\mathcal{K}}/I$. It is known that $\mathbf{Fl}$ is represented by an ind-scheme, called the *affine flag variety.*

3 Convolution product

Last time, Liangshi draw the fundamental convolution diagram

$$\mathbf{Gr} \times \mathbf{Gr} \xleftarrow{p} G_{\mathcal{K}} \times \mathbf{Gr} \xrightarrow{q} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathbf{Gr} \xrightarrow{m} \mathbf{Gr}$$

For $\mathcal{G}^1 \in \mathsf{Perv}(\mathbf{Gr}), \mathcal{G}^2 \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, $p^*(\mathcal{G}^1 \boxtimes \mathcal{G}^2)$ is a $G_{\mathcal{O}}$-equivariant perverse sheaf on $G_{\mathcal{K}} \times \mathbf{Gr}$, and hence descends to a perverse sheaf $\mathcal{G}^1 \widetilde{\boxtimes} \mathcal{G}^2 \in \mathsf{Perv}(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathbf{Gr})$.

The convolution product is defined by

$$- *_{\mathbf{Gr}} - \colon \mathsf{Perv}(\mathbf{Gr}) \times \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \to \mathsf{D}^b_c(\mathbf{Gr}), (\mathcal{G}^1, \mathcal{G}^2) \mapsto \mathcal{G}^1 *_{\mathbf{Gr}} \mathcal{G}^2 = m_!(\mathcal{G}^1 \widetilde{\boxtimes} \mathcal{G}^2).$$

Miraculously, the convolution of two $G_{\mathcal{O}}$-equivariant perverse sheaves is also perverse (and obviously $G_{\mathcal{O}}$-equivariant). Moreover, the convolution product $*_{\mathbf{Gr}}$ on $\mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$ comes with a natural commutativity constraint, making $(\mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}), *_{\mathbf{Gr}})$ a symmetric monoidal category.

Similarly, I can draw the fundamental convolution diagram for the affine flag variety

$$\mathbf{Fl} \times \mathbf{Fl} \xleftarrow{p} G_{\mathcal{K}} \times \mathbf{Fl} \xrightarrow{q} G_{\mathcal{K}} \times^{I} \mathbf{Fl} \xrightarrow{m} \mathbf{Fl}$$

For $\mathcal{F}^1 \in \mathsf{Perv}(\mathbf{Fl}), \mathcal{F}^2 \in \mathsf{Perv}_I(\mathbf{Fl})$, $p^*(\mathcal{F}^1 \boxtimes \mathcal{F}^2)$ is I-equivariant, and hence descends to a perverse sheaf $\mathcal{F}^1 \widetilde{\boxtimes} \mathcal{F}^2 \in \mathsf{Perv}(G_{\mathcal{K}} \times^I \mathbf{Fl})$.

The convolution product is defined by

$$- *_{\mathbf{Fl}} - \colon \mathsf{Perv}(\mathbf{Fl}) \times \mathsf{Perv}_I(\mathbf{Fl}) \to \mathsf{D}^b_c(\mathbf{Fl}), (\mathcal{F}^1, \mathcal{F}^2) \mapsto \mathcal{F}^1 *_{\mathbf{Fl}} \mathcal{F}^2 = m_!(\mathcal{F}^1 \widetilde{\boxtimes} \mathcal{F}^2).$$

It restricts to a map

$$- *_{\mathbf{Fl}} - \colon \mathsf{Perv}_I(\mathbf{Fl}) \times \mathsf{Perv}_I(\mathbf{Fl}) \to \mathsf{D}^b_I(\mathbf{Fl}),$$

but the image does not lie in the heart $\mathsf{Perv}_I(\mathbf{Fl})$ in general.

4 Constructions

By finding a moduli interpretation of (ind) schemes arising before, I can study the global/factorisation analogue of these objects, with the aid of Beauville–Laszlo's theorem.

4.1 Moduli interpretation

Let $D = \operatorname{Spec}(\mathcal{O}), D^* = \operatorname{Spec}(\mathcal{K})$. For a $\mathbb{C}$-algebra R, let $D_R = \operatorname{Spec}(R[[t]]), D^*_R = \operatorname{Spec}(R((t)))$. CAVEAT: $R[[t]] \neq R \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ and $R((t)) \neq R \otimes_{\mathbb{C}} \mathbb{C}((t))$ in general.

For a scheme X, let $\mathcal{E}^0_X = X \times_{\mathbb{C}} G$ be the trivial principal G-bundle on X. Very often, I omit the subscript X for the sake of brevity.

Recall the moduli interpretation of (the presheaf represented by) $\mathbf{Gr}$:

$$\mathbf{Gr}(R) = \{(\mathcal{E}, \beta) : \mathcal{E} \text{ a principal } G\text{-bundle on } D_R, \beta \colon \mathcal{E}|_{D^*_R} \simeq \mathcal{E}^0_{D^*_R} \text{ a trivialisation}\}.$$

Similarly, I have a moduli description of $\mathbf{Fl}$:

$$\mathbf{Fl}(R) = \{(\mathcal{E}, \beta, \epsilon) : (\mathcal{E}, \beta) \in \mathbf{Gr}(R), \epsilon \text{ a reduction of } \mathcal{E} \text{ to } B \text{ over } \operatorname{Spec}(R)\}.$$

Clearly, I have a natural projection $\pi \colon \mathbf{Fl} \to \mathbf{Gr}$ by forgetting ϵ. The fiber at 1 is G/B.

Let X be a pointed smooth geometrically connected curve. I have the global version of the affine grassmannian:

$$\mathbf{Gr}_X(R) = \{(y, \mathcal{E}, \beta) : y \in X(R), \mathcal{E} \text{ a principal } G\text{-bundle on } X(R), \beta \colon \mathcal{E}|_{(X\setminus y)(R)} \simeq \mathcal{E}^0_{(X\setminus y)(R)} \text{ a trivialisation}\}.$$

I have a natural projection $\mathbf{Gr}_X \to X$. By Beauville–Laszlo's theorem, the fiber $\mathbf{Gr}_{X,y} \simeq \mathbf{Gr}$ for any $y \in X$.

Now fix a closed point $x \in X$, I have the global version of the affine flag variety:

$$\mathbf{Fl}_{(X,x)}(R) = \{(y, \mathcal{E}, \beta, \epsilon) : (y, \mathcal{E}, \beta) \in \mathbf{Gr}_X(R), \epsilon \text{ a reduction of } \mathcal{E}_{x(R)} \text{ to } B\}.$$

I have a natural projection $\mathbf{Fl}_{(X,x)} \to X$. By Beauville–Laszlo's theorem,

$$\mathbf{Fl}_{(X,x)}|_{X\setminus x} \simeq \mathbf{Gr}_X|_{X\setminus x} \times G/B, \mathbf{Fl}_{(X,x),x} \simeq \mathbf{Fl}.$$

4.2 Construction of the functor

Set $(X,x) = (\mathbb{A}^1, 0)$. Now we view the projection $\mathbf{Fl}_{(\mathbb{A}^1,0)} \to \mathbb{A}^1$ as a regular function on the global affine flag variety $\mathbf{Fl}_{(\mathbb{A}^1,0)}$. Associated is the nearby cycle functor

$$\Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}} \colon \mathsf{D}^b_c(\mathbf{Fl}_{(\mathbb{A}^1,0)}|_{\mathbb{G}_m}) \to \mathsf{D}^b_c(\mathbf{Fl}_{(\mathbb{A}^1,0),0})$$

who has the virtue that

$$\Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathsf{Perv}(\mathbf{Fl}_{(\mathbb{A}^1,0)}|_{\mathbb{G}_m})) \subset \mathsf{Perv}(\mathbf{Fl}_{(\mathbb{A}^1,0),0}).$$

Last time, Liangshi explained that any $G_{\mathcal{O}}$-equivariant perverse sheaf $\mathcal{G}$ on $\mathbf{Gr}$ can be spread out to a $G_{\mathbb{A}^1,\mathcal{O}}$-equivariant perverse sheaf $\mathcal{G}_{\mathbb{A}^1}$ on $\mathbf{Gr}_{\mathbb{A}^1}$. He used the global coordinate on $\mathbb{A}^1$ which enables him to make an identification $\mathbf{Gr}_{\mathbb{A}^1} \simeq \mathbf{Gr} \times \mathbb{A}^1$.

Remark 4.1. The spreading out procedure can be performed over any smooth algebraic curve X. Let's consider the pro-algebraic group $\mathrm{Aut}(\mathcal{O})$. We have a canonical $\mathrm{Aut}(\mathcal{O})$-principal bundle $\mathrm{Aut}(X)$ over X. As Liangshi briefly explained, $\mathbf{Gr}_X \simeq \mathrm{Aut}(X) \times^{\mathrm{Aut}(\mathcal{O})} \mathbf{Gr}$, so any $\mathrm{Aut}(\mathcal{O})$-equivariant perverse sheaf on $\mathbf{Gr}$ can be spread out. We know that any $G_{\mathcal{O}}$-equivariant perverse sheaf on $\mathbf{Gr}$ is automatically $\mathrm{Aut}(\mathcal{O})$-equivariant. This can be seen, for example, from the classification of simple objects in $\mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$ and the semisimplicity of $\mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$.

Now we can state Gaitsgory's construction of the central functor Z.

$$Z \colon \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \to \mathsf{Perv}(\mathbf{Fl}), \mathcal{G} \mapsto \Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}).$$

Proposition 4.1. We have $Z(\delta_{1_{\mathbf{Gr}}}) \simeq \delta_{1_{\mathbf{Fl}}}$.

Proof. I have a canonical section $1_{\mathbf{Fl}_{(\mathbb{A}^1,0)}} \colon \mathbb{A}^1 \to \mathbf{Fl}_{(\mathbb{A}^1,0)}$ sending y to the quadruple $(y, \mathcal{E}^0, \beta^0, \epsilon^0)$ such that $1_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}|_{\mathbb{G}_m} = 1_{\mathbf{Gr}_{\mathbb{A}^1}}|_{\mathbb{G}_m} \times 1_{G/B}, 1_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}|_0 = 1_{\mathbf{Fl}}$. □

Proposition 4.2. For any $\mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, $\pi_!(Z(\mathcal{G})) \simeq \mathcal{G}$. Here π is the projection $\pi \colon \mathbf{Fl} \to \mathbf{Gr}$.

Proof. This follows from the fact that nearby cycle commutes with proper pushforward. Consider the diagram

$$\begin{tikzcd}
\mathbf{Fl}_{(\mathbb{A}^1,0),0} \arrow[r] \arrow[d, "\pi_0"] & \mathbf{Fl}_{(\mathbb{A}^1,0)} \arrow[d, "\pi_{(\mathbb{A}^1,0)}"] & \mathbf{Fl}_{(\mathbb{A}^1,0)}|_{\mathbb{G}_m} \arrow[l] \arrow[d, "\pi_{\mathbb{G}_m}"] \\
\mathbf{Gr}_{\mathbb{A}^1,0} \arrow[r] \arrow[d] & \mathbf{Gr}_{\mathbb{A}^1} \arrow[d] & \mathbf{Gr}_{\mathbb{A}^1}|_{\mathbb{G}_m} \arrow[l] \arrow[d] \\
0 \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

We have

$$\pi_!(Z(\mathcal{G})) = \pi_{0!}(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})) \simeq \Psi_{\mathbf{Gr}_{\mathbb{A}^1}}(\pi_{\mathbb{G}_m!}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})) = \Psi_{\mathbf{Gr}_{\mathbb{A}^1}}(\mathcal{G}_{\mathbb{G}_m}).$$

We know that the vanishing cycle $\Phi_{\mathbf{Gr}_{\mathbb{A}^1}}(\mathcal{G}_{\mathbb{A}^1}) = 0$ (the support of the vanishing cycle lies in the singular locus), so $\Psi_{\mathbf{Gr}_{\mathbb{A}^1}}(\mathcal{G}_{\mathbb{G}_m}) = \mathcal{G}_{\mathbb{A}^1}|_0 = \mathcal{G}$, we are done. □

4.3 Factorisation

Like the construction of the commutativity constraint in the geometric Satake equivalence explained by Liangshi last time, I need to construct a factorisation (Beilinson–Drinfeld) version of the affine grassmannian and the affine flag variety.

Let (X,x) be a pointed smooth geometrically connected curve. The following definition should be understood properly (using the functor of points point of view)

$$\mathbf{Gr}^{\mathrm{BD}}_{(X,x)} = \{(y, \mathcal{E}, \beta') : y \in X, \mathcal{E} \text{ a principal } G\text{-bundle on } X, \beta' \text{ a trivialisation of } \mathcal{E} \text{ away from } x \cup y\},$$

$$\mathbf{Fl}^{\mathrm{BD}}_{(X,x)} = \{(y, \mathcal{E}, \beta', \epsilon) : (y, \mathcal{E}, \beta') \in \mathbf{Gr}^{\mathrm{BD}}_{(X,x)}, \epsilon \text{ a reduction of } \mathcal{E} \text{ to } B \text{ at } x\}.$$

The factorisation affine grassmannian and the factorisation affine flag variety are representable in ind-schemes.

Using Beaville–Laszlo (type) theorem, I have

$$\mathbf{Gr}^{\mathrm{BD}}_{(X,x)}|_{X\backslash x} \simeq \mathbf{Gr}_X|_{X\backslash x} \times \mathbf{Gr}, \mathbf{Gr}^{\mathrm{BD}}_{(X,x),x} \simeq \mathbf{Gr},$$

$$\mathbf{Fl}^{\mathrm{BD}}_{(X,x)}|_{X\backslash x} \simeq \mathbf{Gr}_X|_{X\backslash x} \times \mathbf{Fl}, \mathbf{Fl}^{\mathrm{BD}}_{(X,x),x} \simeq \mathbf{Fl}.$$

4.4 Construction of the fusion

Now set $(X, x) = (\mathbb{A}^1, 0)$. Last time, Liangshi explained to us the construction of the commutativity constraint of $\mathsf{Perv}_{G_\mathcal{O}}(\mathbf{Gr})$:

$$C_{\mathbf{Gr}}(\cdot,\cdot)\colon \mathsf{Perv}_{G_\mathcal{O}}(\mathbf{Gr}) \times \mathsf{Perv}(\mathbf{Gr}) \to \mathsf{Perv}(\mathbf{Gr}), (\mathcal{G}^1, \mathcal{G}^2) \mapsto \Psi_{\mathbf{Gr}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(\mathcal{G}^1_{\mathbb{G}_m} \boxtimes \mathcal{G}^2).$$

Similarly, I can construct a fusion

$$C_{\mathbf{Fl}}(\cdot,\cdot)\colon \mathsf{Perv}_{G_\mathcal{O}}(\mathbf{Gr}) \times \mathsf{Perv}(\mathbf{Fl}) \to \mathsf{Perv}(\mathbf{Fl}), (\mathcal{G}, \mathcal{F}) \mapsto \Psi_{\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \mathcal{F}).$$

Proposition 4.3. Let $\mathcal{G}$ be an object of $\mathsf{Perv}_{G_\mathcal{O}}(\mathbf{Gr})$, then

- For any $\mathcal{F} \in \mathsf{Perv}_I(\mathbf{Fl})$, there is a canonical isomorphism $C_{\mathbf{Fl}}(\mathcal{G}, \mathcal{F}) \simeq Z(\mathcal{G}) *_{\mathbf{Fl}} \mathcal{F}$.
- For any $\mathcal{F} \in \mathsf{Perv}(\mathbf{Fl})$, there is a canonical isomorphism $C_{\mathbf{Fl}}(\mathcal{G}, \mathcal{F}) \simeq \mathcal{F} *_{\mathbf{Fl}} Z(\mathcal{G})$.

Proof. I prove the first statement, the proof of the second one is similar. To do so, I need a global/factorisation version of the fundamental convolution diagram

$$\mathbf{Fl} \times \mathbf{Fl} \xleftarrow{p} G_\mathcal{K} \times \mathbf{Fl} \xrightarrow{q} G_\mathcal{K} \times^I \mathbf{Fl} \xrightarrow{m} \mathbf{Fl}$$

More precisely, I consider the diagram

$$\mathbf{Fl}_{(\mathbb{A}^1,0)} \times_{\mathbb{A}^1} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)} \xleftarrow{p_{(\mathbb{A}^1,0)}} G_{\mathcal{K},\mathbb{A}^1} \times_{\mathbb{A}^1} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)} \xrightarrow{q_{(\mathbb{A}^1,0)}} G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)} \xrightarrow{m_{(\mathbb{A}^1,0)}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}$$

Notice that $m_{(\mathbb{A}^1,0)}$ is ind-proper. From the diagram

$$\begin{tikzcd}
G_\mathcal{K} \times^I \mathbf{Fl} \arrow[r, equal] \arrow[d, "m"] & G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}|_0 \arrow[r] \arrow[d, "m_{(\mathbb{A}^1,0)}|_0"] & G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)} \arrow[d, "m_{(\mathbb{A}^1,0)}"] & G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}|_{\mathbb{G}_m} \arrow[l] \arrow[d, "m_{(\mathbb{A}^1,0)}|_{\mathbb{G}_m}"] \\
\mathbf{Fl} \arrow[r, equal] & \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}|_0 \arrow[r] \arrow[d] & \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)} \arrow[d] & \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}|_{\mathbb{G}_m} \arrow[l] \arrow[d] \\
 & 0 \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

I have that

$$C_{\mathbf{Fl}}(\mathcal{G}, \mathcal{F}) = \Psi_{\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \mathcal{F}) = \Psi_{\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(m_{(\mathbb{A}^1,0)!}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}(\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})))$$

$$= m_! \Psi_{G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}(\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})).$$

By construction,

$$Z(\mathcal{G}) *_{\mathbf{Fl}} \mathcal{F} = m_!(Z(\mathcal{G})\widetilde{\boxtimes}\mathcal{F}) = m_!(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}\mathcal{F}),$$

so it suffices to show that

$$\Psi_{G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}(\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) = \Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}\mathcal{F}.$$

Noticing that $q_{(\mathbb{A}^1,0)}$ is smooth and that smooth pullback is conservative, it suffices to show that

$$q^*_{(\mathbb{A}^1,0)}\Psi_{G_{\mathcal{K},\mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}(\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) = q^*_{(\mathbb{A}^1,0)}(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})\widetilde{\boxtimes}\mathcal{F}).$$

Let $n_{(\mathbb{A}^1,0)}$ be the projection $n_{(\mathbb{A}^1,0)}\colon \mathcal{G}_{\mathcal{K},\mathbb{A}^1} \to \mathbf{Fl}_{(\mathbb{A}^1,0)}$, $n_{(\mathbb{A}^1,0)}$ is smooth. Smooth pullback commutes with nearby cycle, so

$$
\begin{aligned}
&q^*_{(\mathbb{A}^1,0)}\Psi_{G_{\mathcal{K},\mathbb{A}^1}\times^{I_{\mathbb{A}^1}}\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}((\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}})\widetilde{\boxtimes}(\mathcal{F}\boxtimes\delta_{1_{\mathbf{Gr}}}))\\
=&\Psi_{G_{\mathcal{K},\mathbb{A}^1}\times_{\mathbb{A}^1}\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(q^*_{(\mathbb{A}^1,0)}((\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}})\widetilde{\boxtimes}(\mathcal{F}\boxtimes\delta_{1_{\mathbf{Gr}}})))\\
=&\Psi_{G_{\mathcal{K},\mathbb{A}^1}\times_{\mathbb{A}^1}\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(n^*_{(\mathbb{A}^1,0)}(\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}})\boxtimes(\mathcal{F}\boxtimes\delta_{1_{\mathbf{Gr}}}))\\
=&\Psi_{G_{\mathcal{K},\mathbb{A}^1}}(n^*_{(\mathbb{A}^1,0)}(\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}}))\boxtimes\Psi_{\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(\mathcal{F}\boxtimes\delta_{1_{\mathbf{Gr}}})\\
=&n^*_{(\mathbb{A}^1,0)}\Psi_{G_{\mathcal{K},\mathbb{A}^1}}(\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}})\boxtimes\Psi_{\mathbf{Fl}^{\mathrm{BD}}_{(\mathbb{A}^1,0)}}(\mathcal{F}\boxtimes\delta_{1_{\mathbf{Gr}}})\\
=&n^*_{(\mathbb{A}^1,0)}\Psi_{G_{\mathcal{K},\mathbb{A}^1}}(\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}})\boxtimes\mathcal{F}\\
=&q^*_{(\mathbb{A}^1,0)}(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1,0)}}(\mathcal{G}_{\mathbb{G}_m}\boxtimes\delta_{1_{G/B}})\widetilde{\boxtimes}\mathcal{F}).
\end{aligned}
$$

We are done. □

Proposition 4.4. For any $\mathcal{G}^1,\mathcal{G}^2 \in \mathsf{Perv}_{G_\mathcal{O}}(\mathbf{Gr})$, there is a canonical isomorphism

$$Z(\mathcal{G}^1) *_{\mathbf{Fl}} Z(\mathcal{G}^2) \simeq Z(\mathcal{G}^1 *_{\mathbf{Gr}} \mathcal{G}^2).$$

The proof is analogous, using the fact that nearby cycle commutes with proper pushforward, external tensor product, and smooth pullback.

5 An example for SL(2)

Let $G = \mathrm{SL}(2)$. Let's see what does Z do for $G_\mathcal{O}$-equivariant perverse sheaves on $\mathbf{Gr}$ supported on $\mathbb{P}^1 = \overline{\mathbf{Gr}_{\alpha^\vee/2}}$. To do so, it suffices to consider the following degeneration

$$\mathcal{Y} = \{([x:y:z],\lambda) \in \mathbb{P}^2\times\mathbb{A}^1 : xy = \lambda z^2\} \to \mathbb{A}^1, ([x:y:z],\lambda)\mapsto \lambda.$$

We see that $\mathcal{Y}_\lambda \simeq \mathbb{P}^1$ for $\lambda \neq 0$ and $\mathcal{Y}_0$ is the transversal intersection of two $\mathbb{P}^1$'s.

I want to understand perverse sheaves on $\mathcal{Y}_0$ lisse along certain stratification. Let $Y = \{xy = 0\} \subset \mathbb{C}^2$, then $\mathcal{Y}_0 = \overline{Y}$. Let Λ be the following stratification of Y:

$$Y = \{0\} \sqcup \mathbb{C}^\times_{x\text{-axis}} \sqcup \mathbb{C}^\times_{y\text{-axis}}.$$

I have a description of $\mathsf{Perv}_\Lambda(Y)$ using Beilinson's gluing. Namely, consider the regular function

$$f\colon \mathbb{C}^2 \to \mathbb{C}, (x,y)\mapsto x-y.$$

The zero locus of f restricts to $\{0\}$ on Y. Now using Beilinson's gluing, we see that

$$
\mathsf{Perv}_\Lambda(Y) = \left\{\begin{array}{rr}
\mathcal{F}\in\mathsf{Perv}(\mathbb{C}^\times_{x\text{-axis}}\sqcup\mathbb{C}^\times_{y\text{-axis}}), & \mu \text{ monodromy of } \Psi_f(\mathcal{F}),\\
\mathcal{F} \text{ lisse along } \mathbb{C}^\times_{x\text{-axis}}, \mathbb{C}^\times_{y\text{-axis}}, & c\colon V_0\to\Psi_f(\mathcal{F}),\\
& v\colon \Psi_f(\mathcal{F})\to V_0,\\
V_0 \in \mathsf{Perv}(\{0\}), & c\circ v = 1-\mu.
\end{array}\right\}
$$

$$
= \left\{
\begin{tikzcd}
V_y \arrow[loop above, "\mu_y"] \arrow[dr, shift left, "v_y"] & & V_x \arrow[loop above, "\mu_x"] \arrow[dl, shift left, "v_x"] \\
& V_0 \arrow[ul, shift left, "c_y"] \arrow[ur, shift left, "c_x"] &
\end{tikzcd}
:
\begin{array}{r}
c_x\circ v_x = 1-\mu_x,\\
c_y\circ v_y = 1-\mu_y,\\
c_x\circ v_y = 0,\\
c_y\circ v_x = 0.
\end{array}
\right\}
$$

By requiring the lisse condition at ∞_x and ∞_y, i.e. requiring that $\mu_x = \mathrm{id}, \mu_y = \mathrm{id}$, I have

$$
\mathsf{Perv}_{\overline{\Lambda}}(\mathcal{Y}_0) = \left\{
\begin{tikzcd}
V_y \arrow[dr, shift left, "v_y"] & & V_x \arrow[dl, shift left, "v_x"] \\
& V_0 \arrow[ul, shift left, "c_y"] \arrow[ur, shift left, "c_x"] &
\end{tikzcd}
:
\begin{array}{r}
c_x\circ v_x = 0,\\
c_y\circ v_y = 0,\\
c_x\circ v_y = 0,\\
c_y\circ v_x = 0.
\end{array}
\right\}
$$

Let us compute $Z(\mathbb{C}_{\mathbb{P}^1}[1]) = \Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ explicitly. I have a short exact sequence of perverse sheaves

$$0 \to \mathbb{C}_{\mathcal{Y}_0}[1] \to \Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2]) \to \Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2]) \to 0.$$

Noticing that $\bullet = [0:0:1]$ is the only singular point of the function $\mathcal{Y} \to \mathbb{A}^1$, $\Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ is supported on this point. Let us compute the stalk $\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_\bullet$, which is the same as the cohomology of the Milnor fiber at $\bullet$ (up to some cohomological shift). Here the Milnor fiber is just

$$\{x, y \in \mathbb{C} : xy = \lambda\} \simeq S^1 \text{ (for sufficiently small } \lambda),$$

so

$$\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_\bullet = R\Gamma(S^1, \mathbb{C})[1] = \mathbb{C} \oplus \mathbb{C}[1].$$

Therefore I get $\Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_\bullet = \mathbb{C}$ and hence $\Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_\bullet = \mathbb{C}_\bullet = \mathrm{IC}_\bullet$. One now knows that $\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ is an extension of $\mathrm{IC}_\bullet$ by $\mathbb{C}_{\mathcal{Y}_0}[1]$. Moreover, there is a short exact sequence of perverse sheaves

$$0 \to \mathrm{IC}_\bullet \to \mathbb{C}_{\mathcal{Y}_0}[1] \to \mathrm{IC}_{\mathbb{C}_x}(\mathbb{C}_x^\times) \oplus \mathrm{IC}_{\mathbb{C}_y}(\mathbb{C}_y^\times) \to 0,$$

so the Loewy diagram of $\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ is

$\mathrm{IC}_\bullet$
$\mathrm{IC}_{\mathbb{C}_x}(\mathbb{C}_x^\times) \oplus \mathrm{IC}_{\mathbb{C}_y}(\mathbb{C}_y^\times)$
$\mathrm{IC}_\bullet$

Using the quiver description, I have

$$\mathrm{IC}_\bullet = \begin{tikzcd} 0 \arrow[dr, shift left] & & 0 \arrow[dl, shift left] \\ & \mathbb{C} \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd},$$

$$\mathrm{IC}_{\mathbb{C}_x}(\mathbb{C}_x^\times) = \begin{tikzcd} 0 \arrow[dr, shift left] & & \mathbb{C} \arrow[dl, shift left] \\ & 0 \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd},$$

$$\mathrm{IC}_{\mathbb{C}_y}(\mathbb{C}_y^\times) = \begin{tikzcd} \mathbb{C} \arrow[dr, shift left] & & 0 \arrow[dl, shift left] \\ & 0 \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd},$$

$$\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2]) = \begin{tikzcd} \mathbb{C} \arrow[dr, bend left] & & \mathbb{C} \arrow[dl, bend right] \\ & \begin{matrix} \mathbb{C} \\ \oplus \\ \mathbb{C} \end{matrix} \arrow[ul, bend left] \arrow[ur, bend right] & \end{tikzcd}.$$

6 Confession

I didn't explain the following.

6.1 I-equivariance

I want that for any $\mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, $Z(\mathcal{G})$ is automatically I-equivariant. This is not obvious from Gaitsgory's original construction (and I cannot understand the argument in Gaitsgory's paper). In the book by Achar–Riche, they proposed another resolution of the problem. Instead of the constant group scheme $G_{\mathcal{O}} \times \mathbb{A}^1$ over $\mathbb{A}^1$, they used a nonconstant group scheme $\mathcal{G} \to \mathbb{A}^1$ such that $\mathcal{G}|_{\mathbb{G}_m} \simeq G_{\mathcal{O}} \times \mathbb{G}_m$ and $\mathcal{G}_0 \simeq I$. Now from the construction of $\mathcal{G}$-equivariant nearby cycles, the image is automatically $\mathcal{G}_0 \simeq I$-equivariant. Their construction of the nonconstant group scheme $\mathcal{G}$ used Bruhat–Tits theory. Hope somebody can explain this to us.

6.2 Higher nearby cycles

To check higher compatibilities between isomorphisms constructed above, I need a theory of nearby cycles over $\mathbb{A}^2$. This is explained in detail in a paper by Achar–Riche (also in their book). Hope somebody can explain it to us.

6.3 Monodromy

Constructed using nearby cycle, there is a natural monodromy action on Z. One can show that the monodromy action on Z is unipotent, hence inducing a monodromy weight filtration on Z as explained in Weil II.

6.4 Epitaph

> `Did you know: In one's afterlife, one is condemned to finding counterexamples to all false statements made in life?`
>
> `Hence the advice: Start early!`

I am still confused by the following issues:

1. I don't understand Gaitsgory's proof of I-equivariance. Can somebody help?

> `Confusion will be my epitaph.`