Topics in algebraic geometry

Ravi Vakil

Lecture 1 : Hilbert scheme

Moduli Spaces.

parametrize objects of interest by some sort of geometric spaces

$G(k,n)$ k-dim subspace of n-space, eg $\mathbb{C}^n$.

$$\begin{array}{ccc} \int\int\{\,\} & & \\ \downarrow & & \\ B & \longrightarrow & M \end{array}$$

"families over all B" "maps all $B \to M$"

Yoneda's Lemma.

Any X is $\underline{Sch}$ gives contravariant functor $h_X : \underline{Sch} \to Sets$

$$B \longmapsto Mor(B,X)$$

$$h_X \simeq h_Y \Leftrightarrow X \simeq Y.$$

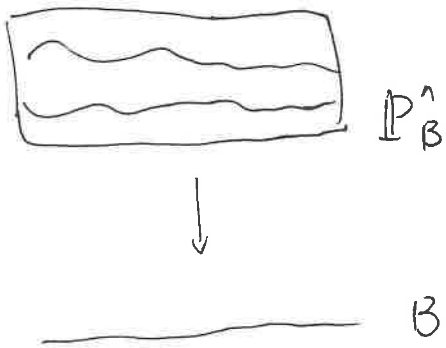
Restate question:

I tell you the family, You tell me $\underline{\text{if}}$ the moduli space exists (the describe it).

Hilbert schemes: (made up history)

Space of smooth subvarieties in projective space. $Hilb^{sm} \subset Hilb$ proper e.g. Degree d hypersurfaces in $\mathbb{P}^n$.

$? x_0^d + \cdots + ? x_n^d$ $\qquad\qquad$ $\mathbb{P}^{\binom{n+d}{d}-1}$



$x_0^d + x_1^d + \cdots + x_n^d = 0$

$\varepsilon = 0$

$\bullet \to \qquad B = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$

↳ fibers degree d hypersurface

effective Cartier divisor

$$\boxed{\text{Ex. If } B \text{ reduced, } X \hookrightarrow \mathbb{P}^n_B \text{ closed subvar, all fibers are degree } d \text{ hypersurface. Flat??}}$$

Theorem. Fix n, d, the following is a bijection between

$$\left\{\begin{array}{c}\text{families of degree } d \\ \text{hypersurfaces in } \mathbb{P}^n \\ \text{over } B\end{array}\right\} \quad \text{and} \quad \left\{ B \longrightarrow \mathbb{P}_B^{\binom{n+d}{d}-1} \right\}$$

which plays well with $B \to B'$.

One direction: $B \longrightarrow \mathbb{P}_B^{\binom{n+d}{d}-1}$

$$\mathbb{P}^{\binom{n+d}{d}-1} \times \mathbb{P}^n \supset a_{x_0^d} x_0^d + a_{x_0^{d-1}x_1} x_0^{d-1} x_1 + \cdots + a_{x_n^d} x_n^d = 0$$

$$\begin{array}{c} \downarrow \\ B \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \end{array}$$

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n \times B \\ & & \downarrow \pi \\ & & B \xrightarrow{\exists !} \mathbb{P}_B^{\binom{n+d}{d}-1} \end{array}$$

satisfying our definition

$$0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}^n_B}(d) \to \mathcal{O}_X(d) \to 0$$

rank 1 line bundle

rank $\binom{n+d}{d}-1$

$$0 \to \pi_* \mathcal{I}_X(d) \to \pi_* \mathcal{O}_{\mathbb{P}^n_B}(d) \to \pi_* \mathcal{O}_X(d) \to R^1\pi_* \mathcal{I}_X(d)$$

Vector bundles of deg d polys
rank $\binom{n+d}{d}$

Rank

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

loc. free
finite rank

$\mathcal{F}', \mathcal{F}''$ lffr $\Rightarrow$ $\mathcal{F}$ lffr

$\mathcal{F}, \mathcal{F}''$ lffr $\Rightarrow$ $\mathcal{F}'$ lffr

$\mathcal{F}', \mathcal{F}$ lffr $\not\Rightarrow$ $\mathcal{F}''$ lffr

$$0 \to \mathcal{O}(-p) \to \mathcal{O} \to \mathcal{O}|_p \to 0$$

Map to $\mathbb{P}^n$ from B, $0 \to \mathcal{L} \to \mathcal{O}^{\oplus(n+1)} \to Q \to 0$

$\mathcal{L}$: rank 1; Q: loc. free

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus(n+1)} \to Q \to 0$$

on $\mathbb{P}^n$

$$0 \to S \to \mathcal{O}^{\oplus n} \to Q \to 0$$

S: rank k; Q: rank $n-k$

Corresponds to $B \to G(k,n)$
projective

Cohomology + Base Change Thm.

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

$X \curvearrowleft \mathcal{F}$ coherent sheaf on X, flat over Y

π proper

Y (loc. Noe)

$$\Phi : \psi^* R^i \pi_* \mathcal{F} \to R^i \pi'_* ((\psi')^* \mathcal{F})$$

Suppose Φ^p surjective for $q \to Y$
$$\uparrow$$
$$X$$

Then

(i) $\exists \; q \in U \overset{\text{open}}{\subset} Y$ so Φ^p_U is iso.

(ii) Φ^{p-1}_q is surj iff $(R^p \pi_* \mathcal{F})$ is loc. free near p.

Lecture 2 Quot scheme

Where we are: - moduli spaces

- Intuition: category $\mathcal{G}$ (eg. $\underline{Sch}$)

- Contr. functor $\mathcal{G} \to Sets$
FUNCTOR

e.g. $\underset{\in \mathcal{G}}{B} \longmapsto \{\text{families on } B\}$ moduli functor

$\cong \text{Maps}(-, \mathcal{M})$
representable functors

Examples.

(i) Grassmannian $G(k,n)$

$$B \longmapsto \left\{ \begin{array}{c} 0 \to \underset{\substack{\text{rk.} \\ n-k}}{S} \to \mathcal{O}^{\oplus n} \to \underset{\substack{\text{rk } k \\ \text{v.b.}}}{Q} \to 0 \\ \text{over } B \end{array} \right\}$$

isom:

$$\begin{array}{ccccccccc} 0 & \to & S & \to & \mathcal{O}^{\oplus n} & \to & Q & \to & 0 \\ & & \downarrow{\wr} & & \| & & \downarrow{\wr} & & \\ 0 & \to & S' & \to & \mathcal{O}^{\oplus n} & \to & Q' & \to & 0 \end{array}$$

$$G(k,n) \rightleftarrows G(n-k,n)$$

(ii) "Stacks"

(iii) exists ??? by hand? connect to other / Artin's criteria

Hall's thesis

Hilbert FUNCTOR. Fix n

families of subschemes of $\mathbb{P}^n$

$$B \longmapsto \left\{ \begin{array}{ccc} X & \xrightarrow{\text{closed}} & \mathbb{P}^n \times B \\ \text{flat}\downarrow & \swarrow & \\ B & & \end{array} \right\} \Big/ \sim \text{isom.}$$

isom.

$$\begin{array}{ccc} & X' & \\ \swarrow{\sim} & & \searrow{\text{closed}} \\ X & \xrightarrow{\text{closed}} & \mathbb{P}^n \times B \\ \text{flat}\downarrow & \swarrow & \\ B & & \end{array}$$

Need either B loc. Noetherian

or $X \to B$ loc. finitely presented

Thm (Grothendieck, FGA) Hilbert FUNCTOR is representable.

$$\mathrm{Hilb}\,\mathbb{P}^n = \coprod \boxed{\mathrm{Hilb}_{p(t)}\,\mathbb{P}^n}$$

projective

Hilbert polynomial

smooth

$$\mathrm{Hilb}^{sm}_{p(t)}\,\mathbb{P}^n \underset{\text{open}}{\subset} \mathrm{Hilb}_{p(t)}\,\mathbb{P}^n$$

Quot FUNCTOR for $\mathbb{P}^n$ $\mathcal{F}$

fix coherent sheaf on $\mathbb{P}^n$

$\boxed{\text{fin. pr.}}$
$\mathrm{Quot}_{\mathcal{F}/\mathbb{P}^n}$

$$\begin{tikzcd} p^*\mathcal{F} \arrow[r, two heads] & \mathcal{Q} & \mathcal{F} \arrow[d, dash] \\ \mathbb{P}^n \times B \arrow[rr, "p"] \arrow[d] & & \mathbb{P}^n \\ B \end{tikzcd}$$

flat over B

$\boxed{n=0}$.

$$0 \to \underset{\text{rk } N-k}{\mathcal{S}} \to \underset{\mathcal{F}}{\mathcal{O}^{\oplus N}} \to \underset{\text{rk } k}{\mathcal{Q}} \to 0.$$

$\rightsquigarrow$ Grassmannian

$$\mathcal{O}_{\mathbb{P}^n \times B} \to \underset{\text{flat}}{\mathcal{O}_X} \to 0$$

$\rightsquigarrow$ Hilbert FUNCTOR

Example

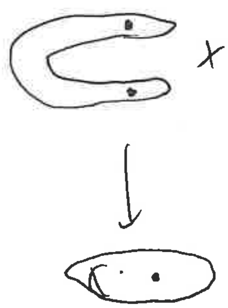
X ~~proj.~~ proper variety $/ \mathbb{C}$

$\mathbb{Z}/2$ acts on X ($X \circlearrowleft \mathbb{Z}/2$)

Quotient doesn't exist as a variety (alg. space)

Hilbert scheme of 2 points

$\mathrm{Conf}^2(X)$



$Y \subset \mathrm{Conf}^2(X)$

Hilbert schemes of m pts on $\mathbb{A}^n$

$\mathrm{Hilb}_m Y$ Def (FUNCTOR)

$$X \xhookrightarrow{\text{closed}} Y \times B$$

$\downarrow$ flat, loc. fin. pre.

B

$\mathrm{Hilb}_m(\mathbb{A}^n)$ new def. — coord. $x_1, \dots, x_n$

$$\begin{tikzcd} X \arrow[rr, hook, "\text{closed}"] \arrow[dr, "\pi"'] & & \mathbb{A}^n \times B \arrow[dl] \\ & B & \end{tikzcd}$$

$\pi_* \mathcal{O}_X$ locally free of rank m on B

$\mathcal{O}_B \to \pi_* \mathcal{O}_X$

$\pi_* \mathcal{O}_X$ is an $\mathcal{O}_B$-algebra

Special case $n=2$

$x_1 = x$, $x_2 = y$.

Aside. $n = 1$ or 2, smooth

($n=3$. lots of components

$n = \dim Y$, Y smooth

$\underline{m=3}$. $\mathrm{Hilb}_3 \mathbb{A}^2$

$\pi_* \mathcal{O}_X$ rank 3 vec. bdle

$1, x, x^2$

$\mathrm{Hilb}_3^{1,x,x^2} \mathbb{A}^2$: $1, x, x^2$ bases for $\boxed{\pi_* \mathcal{O}_X}$ rank 3 vector bundle.

(basis)

$\cong$

$\mathbb{A}^6$

free!

$$\pi_* \mathcal{O}_X = \mathcal{O}_B \oplus x\mathcal{O}_B \oplus x^2 \mathcal{O}_B$$

$$x^3 = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$$

$$y \cdot 1 = b_0 \cdot 1 + b_1 \cdot x + b_2 x^2$$

From subscheme, get a_0, a_1, a_2, b_0, b_1, b_2.

From $a_0, a_1, a_2, b_0, b_1, b_2$

$$\mathrm{Spec}\ \mathbb{C}[x,y] \big/ (x^3 - a_0 - a_1 x - a_2 x^2,\ y - b_0 - b_1 x - b_2 x^2)$$

We have missed $\mathbb{C}[x,y]/(x^2, xy, y^2)$, basis $1, x, y$.

$\mathrm{Hilb}_3^{1,x,y} \mathbb{A}^2$

Algebra.

	1	x	y
1	1	x	y
x	x		
y	y		

$$x^2 = a + bx + cy$$

$$y^2 = d + ex + fy$$

$$xy = g + hx + iy$$

$$xy\, xy = x^2 y^2$$

$\mathrm{Hilb}_3^{1,x,y} \mathbb{A}^2 \subset \mathrm{Hilb}_3 \mathbb{A}^2$

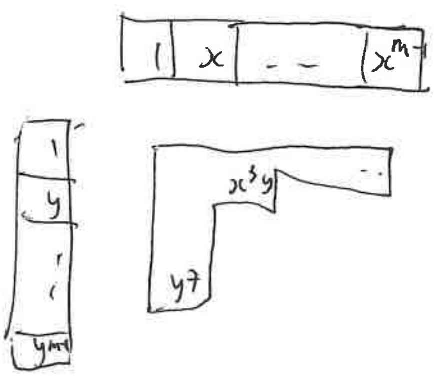
$1 .\ x .\ x^2$

$1,\ y,\ y^2$

$1,\ x,\ y$

$\mathcal{O}_{\mathbb{A}^3}/I$ spanned by $1, x, y, x^2, xy, y^2, \dots$

$\mathrm{Hilb}_m \mathbb{A}^2$



Lecture 3

open subFUNCTOR

$\mathrm{Hilb}_m^{\boxed{\dots}} \mathbb{A}^n \subset \mathrm{Hilb}_m \mathbb{A}^n$

Recall. FUNCTOR ; rep FUNCTOR $\supseteq$ Schemes

Defn. subFUNCTOR $F' \to F$ is a subFUNCTOR if

$\forall B \in Sch$, $F'(B) \xrightarrow{\text{subset}} F(B)$.

Def. representable morphism of functors

$F \to F'$ a morphism of FUNCTORs is a representable morphism

if $\forall B \in Sch$,

$$\begin{tikzcd} \boxed{A} \arrow[r] \arrow[d] & B = h_B \arrow[d, "\text{i.e. elt of } F'(B)"] \\ F \arrow[r] & F' \end{tikzcd}$$

also a scheme

Def (same notation)

$F \to F'$ is an open SUBFUNCTOR, if

$A \to B$ is an open embedding of schemes

closed Idem.

Three statements? Fix F FUNCTOR.

Suppose $F_i \to F$ bunch of open subFUNCTORs + they cover F,

then if F_i are representable, so is F^{sheetify}.

Thm If F is representable, and $F' \to F$ (a representable morphism of FUNCTORs) an open subfunctor,

then F' is representable.

$X \subset \mathbb{A}^n$ closed subscheme,

$\mathrm{Hilb}_m X \subseteq \mathrm{Hilb}_m \mathbb{A}^n$ closed subscheme.

$\mathrm{Hilb}_m \mathbb{P}^n$ cover it with $\mathrm{Hilb}_m X$, $X \subset \mathbb{P}^n$ affine

locally of finite type (FACT: projective)

$\to$ val. criteria for properness

X projective, $X \hookrightarrow \mathbb{P}^n$

$\mathrm{Hilb}_m X \hookrightarrow \mathrm{Hilb}_m \mathbb{P}^n$

Biggest version.

Suppose $\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd}$ projective morphism.

$\mathrm{Hilb}_m(X/Y) \longrightarrow Y$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hilb}_{X'/Y'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Hilb}_{X/Y} \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

$$B \to \mathrm{Hilb}_m(X/Y) \iff \begin{tikzcd} W \arrow[r, "\text{closed}"] \arrow[dr, "\substack{\text{flat} \\ \text{finitely presented}}"'] & X \times_Y B \arrow[r] \arrow[d] & X \arrow[d] \\ & B \arrow[r] & Y \end{tikzcd}$$

FUNCTOR: Schemes $/Y \longrightarrow$ Sets

$$Y = \bigcup_{\substack{\text{affine} \\ \text{open covers}}} Y_i$$

(SHEAF)

Theorem. Suppose $\begin{tikzcd} F \arrow[d] \\ X \text{ scheme} \end{tikzcd}$ is a FUNCTOR, and open cover $\{X_i \subset X\}_{i \in J}$

and $F|_{X_i}$ $\left(\begin{tikzcd} F|X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_i \arrow[d] \\ F \arrow[r] & X \end{tikzcd} \right)$ is rep., then F is rep.

Push pull map

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[d, "\pi'"', "\text{proj}"] & X \arrow[d, "\pi"', "\text{proj}"] \\ Y' \arrow[r, "p"'] & Y \end{tikzcd}$$

$\mathcal{F}$ on X

Schemes

$$p^* R^i \pi_* \mathcal{F} \longrightarrow R^i \pi'_* (p')^* \mathcal{F}.$$

Moduli space of hypersurfaces in $\mathbb{P}^n$

Setting notation, $\mathbb{P}^n$, $x_0, \ldots, x_n$

universal family

$$\begin{tikzcd} \sum_I a_I x^I \arrow[r] \arrow[d, "\text{flat, f.pr}"'] \arrow[dr, phantom, "\ulcorner"] & \{\sum a_I x_I = 0\} \arrow[r, hook] \arrow[d, "\text{flat, f. presented}"] & \mathbb{P}^N \times \mathbb{P}^n \arrow[dl] \\ B \arrow[r] & \in \mathbb{P}^N & \end{tikzcd}$$

$$N = \binom{n+d}{n} - 1$$

Lecture 4 When is a FUNCTOR a SHEAF?

FUNCTOR $\downarrow$ contr. functor

(Schemes) $\longrightarrow$ (Sets)

A FUNCTOR is a SHEAF:

identity: $U_i \overset{\text{open}}{\hookrightarrow} B$ cover,

glueability:

$$F(B) \longrightarrow \prod F(U_i) \text{ injective},$$

$$F(B) \to \prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j).$$

Exercise. If F is a SHEAF, and $\{U_i \to F\}$ open cover by representable FUNCTORs, then F is representable.

$$\left\{\begin{matrix}\text{rep.}\\ \text{FUNCTORs}\end{matrix}\right\} \subset \{\text{SHEAF}\} \subset \{\text{FUNCTOR}\}$$
$$\begin{matrix}\text{\rotatebox{90}{$\cong$}} \\ \text{Sch}\end{matrix}$$

Degree d hypersurfaces in $\mathbb{P}^n$ $\leftarrow$ $\mathbb{P}^n_{\mathbb{Z}}$, $\mathbb{P}^n_{\mathbb{C}}$, $\mathbb{P}^n_B$

FUNCTOR.

$$B \quad \begin{tikzcd} X \arrow[r, hook, "\text{closed}", "\text{sub}"'] \arrow[d, "\text{flat}"', "\text{f.pr.}"] & \mathbb{P}^n_B \arrow[dl, "\pi"] \\ B & \end{tikzcd}$$

for any $q \in B$, $\begin{tikzcd} X_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ q \arrow[r] & B \end{tikzcd}$

X_q is a degree d hypersurface.

Thm. represented by $\mathbb{P}^N$. $N = \binom{n+d}{n} - 1$

$$\begin{tikzcd} \{\sum a_I x^I = 0\} \arrow[r, hook] \arrow[d] & \mathbb{P}^n \times \mathbb{P}^N \arrow[dl, "\pi"] \\ \mathbb{P}^N & \end{tikzcd}$$

$\mathbb{P}^n$ — coord. $x_0, \dots, x_n$

$\mathbb{P}^N$ — coord. a_I

Pf. On $\mathbb{P}^n \times B$ $\sim$ $\quad 0 \to \mathcal{I}_X^{(d)} \to \mathcal{O}_{\mathbb{P}^n_B}^{(d)} \to \mathcal{O}_X^{(d)} \to 0$

flat $\quad$ flat. $\quad$ flat

$\downarrow$ want to get the map $B \to \mathbb{P}^N$

Then

$$0 \to \pi_* I_X(d) \to \pi_* \mathcal{O}_{\mathbb{P}^n_B}(d) \to \pi_* \mathcal{O}_X(d) \to R^1 \pi_* I_X(d) \to \cdots$$

$\leftarrow$ locally free

all zero.

If over a field, $B = \operatorname{Spec} k$,

$$0 \to H^0(\mathbb{P}^n, I_X(d)) \to H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_X(d))$$

$$\to H^1(\quad) \to H^1(\quad) \to H^1(\quad)$$

$\underset{0}{\parallel} \quad \underset{0}{\parallel} \quad \underset{0}{\parallel}$

Grassmannian (geometric version)

$$\begin{array}{c} \mathbb{P}^{n-1} \\ \downarrow \\ B \end{array}$$

"$\mathbb{P}^{k-1}$'s in $\mathbb{P}^{n-1}$"

Claim FUNCTOR $\quad G(k-1, n-1) \longleftrightarrow G(k,n)$

fibers are all pts are $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{n-1}$

$$\begin{tikzcd} X \arrow[r, hook, "d"] \arrow[dr, "\text{flat, f.pr.}"'] & \mathbb{P}^{n-1} \arrow[d] \\ & B \end{tikzcd}$$

$B \to G(k,n)$: over B, $0 \to S_{n-k} \to \mathcal{O}^{\oplus n} \to Q_k \to 0$

$\rightsquigarrow \operatorname{Sym}^\bullet \mathcal{O}^{\oplus n} \to \operatorname{Sym}^\bullet Q_k \to 0$

$\mathbb{P}^n = \mathrm{Proj}\ \mathrm{Sym}^\bullet \mathcal{O}^{\oplus n} \xleftarrow{\text{closed}} \mathrm{Proj}\ \mathrm{Sym}^\bullet Q_k$

Conversely, $0 \to I_X(1) \to \mathcal{O}_{\mathbb{P}^n_B}(1) \to \mathcal{O}_X(1) \to 0$

Input:

Over a field, $B = \mathrm{Spec}\ k$.

can see $\to$

$$0 \to H^0(I(1)) \to H^0(\mathbb{P}^{n-1}, \mathcal{O}(1)) \to H^0(\mathbb{P}^{k-1}, \mathcal{O}(1))$$
$$\to H^1(I(1)) \to H^1(\mathbb{P}^{n-1}, \mathcal{O}(1)) \to H^1(\mathbb{P}^{k-1}, \mathcal{O}(1))$$
$$\to \ldots$$

$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

<u>Question</u>:

$$\begin{tikzcd} X \arrow[rr, hook, leftarrow] \arrow[dr, "\text{flat, f.pr.}"'] & & X' \arrow[dl, "\text{flat, f.pr.}"] \\ & B & \end{tikzcd}$$

iso. over B^{red}, show $X = X'$.

(Unmotivated) The Flatening Stratification (pf. later)

Situation

$$\begin{tikzcd} & \mathcal{F} \text{ coherent (f.pr.??)} \arrow[dl, dash] \\ \mathbb{P}^n_B \arrow[d] & \\ B & \end{tikzcd}$$

Motivation

$$\begin{tikzcd} & \mathcal{F} \text{ flat} \arrow[dl, dash] \\ \mathbb{P}^n_{B'} \arrow[d] & \\ B' & \end{tikzcd}$$

$B \longleftarrow B'$

Which B-schemes "flatten" $\mathcal{F}$?

Thm $\exists$ "minimal" B-scheme. $Fl_{\mathcal{F}} \overset{\text{obvious}}{\twoheadrightarrow} B$, flattening $\mathcal{F}$.

The cat. of B-schemes flattening $\mathcal{F}$ has a final object.

Better.

Thm (cont'd). In fact, $Fl_{\mathcal{F}} = \coprod B_i$, $B_i \to B$ are locally closed embeddings,

where these form a stratification,

indexed by Hilbert polynomial.

$$Fl_{\mathcal{F}} = Fl_{\mathcal{F}, p(m)}, \qquad P_{\mathcal{F}}(m) = \chi(\mathbb{P}^n, \mathcal{F}(m))$$
$$= \sum_i (-1)^i h^i(\mathbb{P}^n, \mathcal{F}(m))$$

Ordering in hilb. poly. $p > q$ if $p(m) > q(m)$ for $m \gg 0$

Hilb FUNCTOR of $\mathbb{P}^n$

$$\begin{array}{ccc} X & \overset{cl}{\hookrightarrow} & \mathbb{P}^n \\ \downarrow {\scriptstyle \text{flat f.pr.}} & & \\ B & & \end{array}$$

Thm Representable. Ex. $\mathrm{Hilb}\,\mathbb{P}^n = \coprod \underbrace{\mathrm{Hilb}_{p(m)}\,\mathbb{P}^n}_{\text{Instead, representable} \to \text{projective.}}$

Quot FUNCTOR Fix n, r,

$$\begin{array}{c} \mathcal{O}^{\oplus r} \\ / \\ \mathbb{P}^n \\ \downarrow \\ B \end{array}$$

$$\begin{array}{ccccc} \mathcal{O}^{\oplus r} & \to & Q & \to & 0 \\ | & & & & \\ \mathbb{P}^n & & & & \\ | & & & & \\ B & & & & \end{array}$$

coh. f.pr.
(has Hilb. polynomial)

$$\mathrm{Quot}_{\mathcal{O}^{\oplus r}} = \coprod \mathrm{Quot}_{P(m)}$$

Lecture 5.

Recall Goal: show two FUNCTORs are representable.

Hilb Fix $n \in \mathbb{Z}_{\geq 0}$, $p(t) \in \mathbb{Q}[t]$, $\mathrm{Hilb}_{p(t)}(\mathbb{P}^n)$

$\mathrm{Hilb}_{p(t)}(\mathbb{P}^n)$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\text{flat, f.pr.}"'] & \mathbb{P}^n_B = \mathbb{P}^n \times B \arrow[dl] \\ B & \end{tikzcd}$$

For every pt $p \in B$, X_p has Hilb. poly. $p(t)$.

Quot Fix $n \in \mathbb{Z}_{\geq 0}$, $p(t) \in \mathbb{Q}[t]$, $r \in \mathbb{Z}_{>0}$ $0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n_B}^{\oplus r} \twoheadrightarrow \mathcal{G} \to 0$ s.t.

$$\begin{array}{c} \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

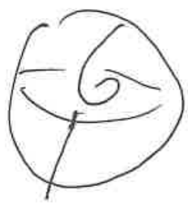
$\mathcal{G}$ has Hilb. poly. $p(t)$, flat, coherent$/B$

hypersurface : write it down

Grassmannians : $G(k-1, n-1) = G(k,n)$

Hilb pts :

<u>Example</u>. twisted cubic in $\mathbb{P}^3_{w,x,y,z}$



$$\operatorname{rk}\begin{bmatrix} w & x & y \\ x & y & z \end{bmatrix} = 1$$

$$wy - x^2 = wz - xy = xz - y^2 = 0$$

$p(t) = 3t+1$. $\mathrm{Hilb}_{3t+1}\, \mathbb{P}^3$.

maybe : choose 3 quadrics, $G(3, 10)$ space of quadrics

$\mathrm{Fl}_{3t+1}\, G(3,10)$

I — 3 quadric equations

"S_2" — all quadric

$$7 = p(2) = \chi(X, \mathcal{O}(2)) = h^0(X, \mathcal{O}(2)) - \underbrace{h^1(X, \mathcal{O}(2))}_{=0}$$

Let's make this work. Fix n, $p(t)$, r. Pick $d_0 \gg 0$ to make this work.

<u>Goal</u>:

$$G(p(d_0), \underbrace{\dim S_{d_0}}_{\binom{d_0+n}{n}})$$

i.e. $\binom{d_0+n}{n} - p(d_0)$ eqns

$$\mathcal{O}(-d_0)^{\oplus \binom{d+n}{n} - p(d_0)} \xrightarrow{\alpha} \mathcal{O}^{\oplus r} \longrightarrow \operatorname{coker} \alpha \longrightarrow 0$$

$$\begin{array}{c} \downarrow \\ \mathbb{P}^n \times G(p(d_0), S_{d_0}) \\ \Big\downarrow \\ G(p(d_0), \dim S_{d_0}) \end{array}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\text{flat f.pr.}"'] & \mathbb{P}^n_B \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

$0 \to \mathcal{I}_X \to \mathcal{O} \to \mathcal{O}_X \to 0.$

$0 \to \mathcal{I}_X(d) \to \mathcal{O}(d) \to \mathcal{O}_X(d) \to 0.$

$$0 \to \pi_* \mathcal{I}_X(d) \to \pi_* \mathcal{O}(d) \to \pi_* \mathcal{O}_X(d)$$
$$\to R^1 \pi_* \mathcal{I}_X(d) \to \cdots$$

$\boxed{\text{Iou}}$ As $d \gg 0$, for any k field, any X w/ this Hilb. Poly.,

$$h^{>0}(\sim) = 0$$

Also for any such X (w/ this $p(t)$, n, r),

$$I_{d'} \otimes S_1 \to I_{d'+1} \quad \text{is surjective, } d' \geq d.$$

Whole argument: . —

<u>Castelnuovo - Mumford Regularity</u>

<u>Situation</u>

/ field K

$$\begin{array}{ccc} \mathcal{O}(1) & & \mathcal{F} \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

$\mathcal{F}$ coherent sheaf

X proj. scheme $/K$

<u>Def</u>. We say $\mathcal{F}$ is m-regular, if $H^i(X, \mathcal{F}(m-i)) = 0$ for all $i > 0$.

<u>Thm</u>. If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m')$ is generated by global sections and has no higher cohomology for $m' \geq m$.

(Iou, if $\mathcal{O}^{\oplus r} \to \mathcal{F}$ and $\mathcal{F}$ has Hilb. poly. $p(t)$

$\mathbb{P}^n$ Then $\mathcal{F}$ is m-regular,

where $m = m(r, p(t), n)$.)

<u>Observe</u>. $\mathcal{F}$ m-regular $\Leftrightarrow$ $\mathcal{F}(1)$ is $(m-1)$-regular.

<u>Observe</u>. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ on $\mathbb{P}^n$

- $\mathcal{F}'$ and $\mathcal{F}''$ m-regular $\Rightarrow$ $\mathcal{F}$ m-regular
- $\mathcal{F}'$ $(m+1)$-regular, $\mathcal{F}$ m-regular $\Rightarrow$ $\mathcal{F}''$ m-regular
- $\mathcal{F}''$ $(m-1)$-regular, $\mathcal{F}$ m-regular $\Rightarrow$ $\mathcal{F}'$ m-regular.

<u>Prop</u>. $\mathcal{F}$ m-regular $\Rightarrow$ $\mathcal{F}$ is $(m+1)$-regular.

<u>Pf</u>. Induction on n, $n = 0$, $H^i(\mathbb{P}^n, \text{any}) = 0$, $i > 0$

Can assume $k = \bar{k}$

Let's prove it for $n > 0$ (assuming "smaller")

Choose hyperplane $H \subset \mathbb{P}^n$, miss associated pts of F

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

$\otimes F$, $\quad 0 \to F(-1) \to F \to F|_H \to 0.$

$F|_H$ m-regular $\Rightarrow$ $F|_H$ $(m+1)$-regular

$\Rightarrow F$ $(m+1)$-regular $\quad (F(-1)$ $(m+1)$-reg.$)$

Prop. Suppose F on $\mathbb{P}^n$ is m-regular, for $r \geq m$,

$$H^0(\mathcal{O}(1)) \otimes H^0(F(r)) \longrightarrow H^0(F(r+1)) \text{ is surjective.}$$

Pf. Assume $k = \bar{k}$. Induct on n. $n=0$, ✓

Inductive Step:

Lecture 6 Castelnuovo - Mumford regularity.

Situation. $\mathbb{P}^n_k$, F coherent $\qquad F$ is m-regular $\Leftrightarrow H^i(\mathbb{P}^n, F(m-i)) = 0$, $i > 0$

- F is m-regular $\Leftrightarrow F(1)$ is $(m-1)$-regular.

  $\Rightarrow F$ is $(m+1)$-regular.

- $0 \to F' \to F \to F'' \to 0$ SES

F'	F	F''
m	$\therefore m$	m
$(m+1)$	m	$\therefore m$
$\therefore m$	m	$(m-1)$

• If H hyperplane missing assoc. pts of $\mathcal{F}$, then m-regular $\Rightarrow \mathcal{F}|_H$ m-regular.

Prop: Suppose $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular, then

$\mu: H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{F}(r)) \longrightarrow H^0(\mathcal{F}(r+1))$ is surjective for $r \geq m$.

Pf. $k = \bar{k}$. By induction on n. $n=0$ ✓.

$$\begin{tikzcd}
& 0 \arrow[d] \\
& H^0(\mathcal{F}(r)) \arrow[d, "\times \ell"] \\
H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{F}(r)) \arrow[r, "\mu_r"] \arrow[d, two heads] & H^0(\mathcal{F}(r+1)) \arrow[d, two heads] \\
H^0(\mathbb{P}^{n-1}, \mathcal{O}(1)) \otimes H^0(\mathcal{F}|_H(r)) \arrow[r, two heads] & H^0(\mathcal{F}|_H(r+1)) \arrow[d] \\
& H^1(\mathcal{F}(r)) = 0
\end{tikzcd}$$

$$\operatorname{im}\mu + \underset{\underset{\operatorname{im}\mu}{\cap}}{\ell \times H^0(\mathcal{F}(r))} = H^0(\mathcal{F}(r+1))$$

$$\Rightarrow \operatorname{im}\mu = H^0(\mathcal{F}(r+1)).$$

Prop. Suppose $\mathcal{F}$ m-regular, then $\mathcal{F}(m)$ is generated by global sections.

Side Claim. $\alpha: \mathcal{G} \to \mathcal{H}$ coh. sheaves on $\mathbb{P}^n$ is surjective if $H^0(\mathcal{G}(s)) \to H^0(\mathcal{H}(s))$ is surjective for $s \gg 0$.

$\mathcal{O}_{\mathbb{P}^n}^{\oplus N} \twoheadrightarrow \mathcal{F}(m)$

$\mathcal{O}_{\mathbb{P}^n}^{\oplus N}(-m) \twoheadrightarrow \mathcal{F}$

Better. $\mathcal{G} \to \mathcal{H} \to \mathcal{I}$ coh. on $\mathbb{P}^n_k$

exact iff $H^0(\mathcal{G}(s)) \to H^0(\mathcal{H}(s)) \to H^0(\mathcal{I}(s))$ exact for $s \gg 0$.

$\Rightarrow$: Serre vanishing.

$\Leftarrow$: ...

Goal: $H^0(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{F}(m)$ is surjective.

$H^0\left(H^0(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n}(s)\right) \longrightarrow H^0(\mathcal{F}(m+s))$ is surjective for $s \gg 0$.

$$\begin{array}{c} H^0\left(H^0(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n}(s)\right) \\ \| \\ H^0(\mathcal{F}(m)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(s)) \end{array} \qquad \square$$

Theorem. If $\mathcal{F}$ is m-regular, then for $r \geq m$, $\mathcal{F}(r)$ is gen. by global sections, has no higher coh.

$$H^0(\mathcal{O}(r-m)) \otimes H^0(\mathcal{F}(m)) \twoheadrightarrow H^0(\mathcal{F}(r))$$

Theorem. Fix k field, $r \in \mathbb{Z}^{>0}$, $n \in \mathbb{Z}^{\geq 0}$, $p(t) \in \mathbb{Q}[t]$, then there is some $m = m(n, r, p(t))$ s.t. for any coh. sheaf $\mathcal{F} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ w/ Hilb. poly. $p(t)$, $\mathcal{F}$ is m-regular

Pf $k = \bar{k}$, induction on n. $n = 0$ ✓

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \to \mathcal{G} \to 0.$$

Inductive step: pick H missing ass. pts of $\mathcal{F}$ and $\mathcal{G}$.

$$0 \to \mathcal{F}|_H \to \mathcal{O}_H \to \mathcal{G}|_H \to 0$$

$$0 \to \mathcal{F}\overset{t}{(-1)} \to \mathcal{F}^{(t)} \to \mathcal{F}|_H^{(t)} \to 0.$$

Hilb. poly. of $\mathcal{F}|_H$

$= P(t) - P(t-1)$.

$\mathcal{F}|_H$ is m'-regular for $m'(n, r, p(t))$

$$H^{p-1}(\mathcal{F}|_H(t)) \to H^p(\mathcal{F}(t-1)) \to H^p(\mathcal{F}(t)) \to H^p(\mathcal{F}(t)|_H)$$

If $t \geq m'$, $p \geq 2$, first term & last term $= 0$.

$$H^p(\mathcal{F}(t-1)) \xrightarrow{\simeq} H^p(\mathcal{F}(t))$$

Take $t \to \infty$, $H^p(\mathcal{F}(t)) = 0$, so $H^p(\mathcal{F}(t)) = 0$ for $t \geq m'-1$, $p \geq 2$.

$t \geq m'-1$

$$H^0(\mathcal{F}(t)) \xrightarrow{\alpha_t} H^0(\mathcal{F}|_H(t)) \to H^1(\mathcal{F}(t-1)) \to H^1(\mathcal{F}(t)) \to \underset{0}{\underset{\shortparallel}{\cancel{H^1(\mathcal{F}|_H(t))}}}$$

Is α_t surjective? Want it to be!

$$\begin{array}{ccc} H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{F}(t-1)) & \xrightarrow{\alpha_{t-1}} & H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{F}|_H(t-1)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{F}(t)) & \xrightarrow{\alpha_t} & H^0(\mathcal{F}|_H(t)) \end{array}$$

so $\twoheadrightarrow$

[α_{t-1} surj. $\Rightarrow$ α_t surj.]

$H^1(\mathcal{F}(t-1)) = H^1(\mathcal{F}(t))$ for some t

$\Rightarrow$ equal for all large t.

But eventually $0 \Rightarrow$ all 0.

I will bound $h^1(\mathcal{F}(m'))$, $m = m' +$ this bound

(+1)

M.

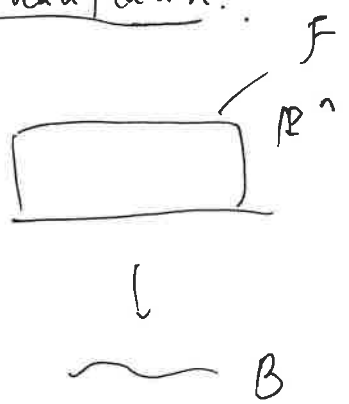
$$\chi(\mathcal{F}(M)) = p(M) = h^0(\mathcal{F}(M)) - h^1(\mathcal{F}(M)) + \underbrace{\cancel{h^2(\mathcal{F}(M)) \cdots}}_{0}$$

$$h^1(\mathcal{F}(M)) = h^0(\mathcal{F}(M)) - p(M)$$

$$0 \to \mathcal{F}(M) \to \mathcal{O}^{\oplus r}(M)$$

$$\Rightarrow h^0(\mathcal{F}(M)) \le h^0(\mathcal{O}^{\oplus r}(M)) = r\binom{M+n}{n}.$$

Flattening stratification.



f.pr. coh, $s \gg 0$.

$\pi_* \mathcal{F}(s)$

A ring (noetherian?) M f.pr. A-module,

Spec A

Lecture 7. Today: More criteria for cohomology commuting w/ base change. Toward the flattening stratification.

Reminder: Goal.

$$\begin{tikzcd} X \subset \mathbb{P}^n \arrow[d] & \mathcal{F} \text{ f. pr. } / A \\ S \overset{\text{suffices}}{=} \operatorname{Spec} A. & \end{tikzcd}$$

"fl. stratification"

First today: $n=0$.

$\mathcal{F}$ $\qquad \tilde{M}$

$/ \ \mathbb{P}^0_S$

$S = \operatorname{Spec} A$

Plan
(local)
- top stratification
- scheme structure.
- universal property.

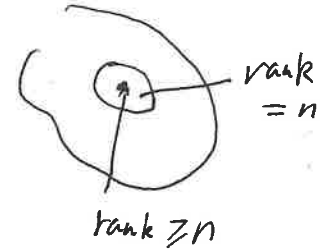
Step 1. Define $S_n \subset S$ subset. where $\mathcal{F}$ has rank n

$$= \{ p \in S : \mathcal{F}|_p = \mathcal{O}_p^{\oplus n} \}.$$

Claim. $S = \coprod_{n \geq 0} S_n$.

$\longrightarrow \overline{S_n} \subset \bigcup_{i \geq n} S_i$ (rank only jumps up)

Trans. rank is upper semicts.

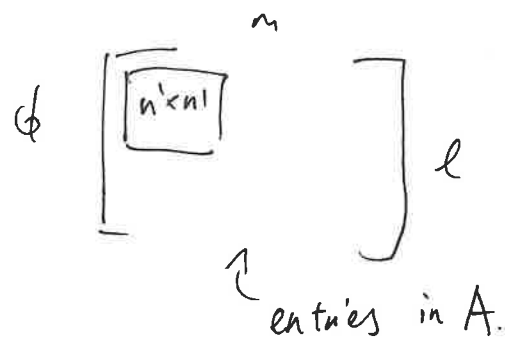


Pf. $A^{\oplus \ell} \xrightarrow{[\phi]} A^{\oplus m} \to M \to 0$. at p rank n.

All $(n'+1)$ minors of $\phi = 0$
Closed condition
$\Leftrightarrow$ rank $\phi \leq n'$
$\Leftrightarrow$ rank $M \geq m - n'$

Scheme structure: Pick $n' \times n'$ minor, $n' + n = m$.

localize @ det(Minor)



$B \leftarrow A$

$\operatorname{Spec} B \to \operatorname{Spec} A$

Def. I_{St} = ideal gen. by $(n'+1) \times (n'+1)$ minors.
↑
Stepan

$B \leftarrow A$, $\quad \operatorname{Spec} B \xrightarrow{\pi} \operatorname{Spec} A$

Then $M \otimes_A B$, i.e. $\pi^* \widetilde{M}$ is free if $B \leftarrow A$ locally free, factors through

$B \leftarrow A/I_{St} \leftarrow A$.

Another proof. $A^{\oplus n} \twoheadrightarrow M$

$\tilde{M}$
|
Spec A

$\cdot p$

Choose bases for $\mathcal{F}|_p$.
lift elements of $\mathcal{F}_p$

$$A^{\oplus n} \to M \to K \to 0.$$

$$\otimes k(p): \quad k(p)^{\oplus n} \xrightarrow{\simeq} M \otimes k(p) \to \underbrace{K \otimes k(p)}_{=0} \to 0$$

$$K_p \overset{\text{Nakayama}}{=} 0$$

Shrink Spec A, $\widetilde{A^{\oplus n} \to M \to 0}$

$$A^{\oplus m} \xrightarrow{\boxed{\phi}} A^{\oplus n} \to M \to 0$$

For which $B \leftarrow A$, is $M \underset{A}{\otimes} B$ free?

$$B^{\oplus m} \xrightarrow{[-]} B^{\oplus n} \to M \underset{A}{\otimes} B \to 0$$

Ans: Iff entries in $[\phi_B]$

M f.pr. /A. M is loc. free iff M flat.

Given $\underbrace{0 \to N \to A^{\oplus n} \xrightarrow{[\,]} A^{\oplus n} \to 0}$, want $N = 0$?

More: when cohomology commutes w/ base change

$$\begin{tikzcd} \mathbb{P}^n_T \arrow[r, "\mu"] \arrow[d, "\beta"'] & \mathbb{P}^n_S \arrow[d, "\alpha"] \\ T \arrow[r, "\rho"] & S = \operatorname{Spec} A \end{tikzcd}$$

$T = \operatorname{Spec} B$, $\mathcal{F}$ on $\mathbb{P}^n_S$

$$\varphi_i : \rho^* R^i\alpha_* \mathcal{F} \to R^i\beta_*(\mu^*\mathcal{F})$$

Today (i) All flat base change ρ

(ii) $\mathcal{F}$ flat, yes if twist enough (any base change)

(iii) No flatness, one base change, twist enough

(o) CBC Thm

Thm. ρ flat $\Rightarrow$ φ_i is iso.

Pf. To compute $R^i\alpha_*\mathcal{F} = H^i(\alpha_*\mathcal{F})$

We Čech cpx $\check{C}_A$

$$0 \to \prod_{i=0}^{n} \mathcal{F}(U_i) \to \prod \mathcal{F}(U_{ij}) \to \cdots \to 0$$

LHS $= H^i(\check{C}_A) \otimes_A B \to$ RHS $H^i(\check{C}_B) = H^i(\check{C}_A \otimes_A B)$, H^* Commutes w/ $-\otimes_A B$ exact functor

Thm. $\mathcal{F}$ flat, choose any $m \gg 0$ so that $H^i(\mathbb{P}^n_A, \mathcal{F}(m)) = 0$, $i > 0$

i.e. $R^i\alpha_*(\mathcal{F}(m)) = 0$ (Serre vanishing)

Then φ_i is an iso. for $\mathcal{F}(m)$ for all i

$$0 \to H^0(\mathcal{F}(m)) \to \prod \mathcal{F}(m)(U_i) \to \prod \mathcal{F}(m)(U_{ij}) \to \cdots \to 0 \quad \text{exact}$$

$\therefore$ flat

$\therefore$ flat $\quad$ flat

$0 \to \mathcal{F}' \to \cdots - \cdot \mathcal{F} \to 0$

Aside. $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

$\therefore$ flat $\quad$ flat $\quad$ flat

$\boxed{\text{BONUS}}$ $\pi_* \mathcal{F}(m)$ is flat. is a finite rank vector bundle

$C_B^\vee = C_A^\vee \otimes B$. Let's apply $\otimes_A B$.

$\therefore$ $0 \to H^0(\mathcal{F}(m) \otimes_A B \to C_B^\vee \to 0$

is exact

Aside:

$0 \to \mathcal{F} \to \cdots \to 0$

flat

$- \otimes_A B$ preserves exactness

$\therefore$ $H^{i>0}(\mathcal{F}(m) \otimes B) = 0$, $H^0(\mathcal{F}(m) \otimes B) = H^0(\mathcal{F}(m)) \otimes_A B$

Lecture 8

Goal: Flattening stratification.

$$\begin{tikzcd} \mathcal{F}_Y & & & \mathcal{F} \\ & \mathbb{P}^n_Y \arrow[r] \arrow[d] & \mathbb{P}^n_X \arrow[d] & \\ & Y \arrow[r] & X & \end{tikzcd}$$

$X = \coprod_{\text{set}} X_{p(t)}$. loc. closed subscheme

set-theoretic:

$X_{q(t)} \overset{\text{open}}{\subset} \coprod_{p(t) \geq q(t)} X_{p(t)}$

an poly,

Recall: Cohomology commutes w/ base change.

$$\begin{tikzcd} \mathcal{F}_X & & \mathcal{F} \\ \mathbb{P}^n_X \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & \mathbb{P}^n_A \arrow[d, "\pi"] & \\ X \arrow[r, "\rho"'] & \operatorname{Spec} A & \end{tikzcd}$$

(i) Yes. All flat $X \to \operatorname{Spec} A$.

(ii) If $\mathcal{F}$ flat, then $\mathcal{F}(m)$, $m \gg 0$, $\forall$ base change.

(iii) Given $X \to \operatorname{Spec} A$, $\mathcal{F}(m)$, $m \gg 0$.
(X quasi-cpt)
noetherian

Pf for (iii) Do it for $X = \operatorname{Spec} B$, B noetherian.

(Serre vanishing)

$m \gg 0$, $R^{i>0} \pi_* \mathcal{F}(m) = 0$, i.e. $H^{>0}(\mathbb{P}^n_A, \mathcal{F}(m)) = 0$

$R^{i>0} \pi'_* (p'^* \mathcal{F}(m)) = 0$, i.e. $H^{>0}(\mathbb{P}^n_B, (p')^* \mathcal{F}(m)) = 0$

For $i > 0$, $m \gg 0$. $p^* R^i \pi_* \mathcal{F}(m) \xrightarrow{\sim} (R^i \pi'_*)(p')^* \mathcal{F}(m)$ is iso!

$i = 0$. $a \gg 0$, $\mathcal{F}(a)$ is generated by global sections.

$$\mathcal{O}^{\oplus \text{finite}} \to \mathcal{F}(a) \to 0$$

$$0 \to \underset{\text{coherent}}{\overset{\text{def}}{\mathcal{G}}} \to \mathcal{O}^{\oplus \text{finite}}(-a) \to \mathcal{F} \to 0$$

Same game for $\mathcal{G}$. $\overset{\text{finite}}{\bigoplus} \mathcal{O}_{\mathbb{P}^n_B}(-b) \to \mathcal{O}_{\mathbb{P}^n_B}^{\oplus \text{finite}}(-a) \to \mathcal{F}_B \to 0$.

$$0 \to \mathcal{G}(m) \to \oplus \mathcal{O}(m-a) \to \mathcal{F}(m) \to 0 \qquad \text{on } \mathbb{P}^n_A$$

$$0 \to \pi_* \mathcal{G}(m) \to \oplus \pi_* \mathcal{O}(m-a) \to \pi_* \mathcal{F}(m) \to \underset{\underset{0}{\|}}{R^1 \pi_* \mathcal{G}(m)} \quad \text{on } \operatorname{Spec} A$$

p^*:

$$p^* \pi_* \mathcal{G}(m) \to \oplus p^* \pi_* \mathcal{O}(m-a) \to p^* \pi_* \mathcal{F}(m) \to 0$$

$\pi_* \oplus \mathcal{O}(m-b) \twoheadrightarrow \pi_* \mathcal{G}(m)$, $m \gg 0$.

$\oplus p^* \pi_* \mathcal{O}(m-b) \to \oplus p^* \pi_* \mathcal{O}(m-a) \to p^* \pi_* \mathcal{F}(m) \to 0$ for $m \gg 0$.

Exercise. Suppose $\overbrace{\mathcal{G} \to \mathcal{H} \to \mathcal{F}}^{\text{coherent}} \to 0$ is exact on $\mathbb{P}^n_B$

then for $m >> 0$, $\pi'_* \mathcal{G}(m) \to \pi'_* \mathcal{H}(m) \to \pi'_* \mathcal{F}(m) \to 0$

More generally, given $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ exact on $\mathbb{P}^n_B$, coherent.

Show $\pi_* \mathcal{G}'(m) \to \pi_* \mathcal{G}(m) \to \pi_* \mathcal{G}'(m)$ is exact for $m >> 0$

Pf of flattening stratification:

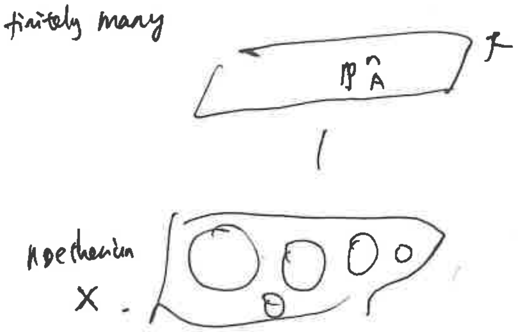
step 1

Black box thm: generic flatness. $\mathcal{F}$ over $\mathbb{P}^n_A$, A noeth. + integral.
(reduced is enough)

$\exists$ dense open of Spec A over which it is flat.

Step 2. (toward the topological "stratification")

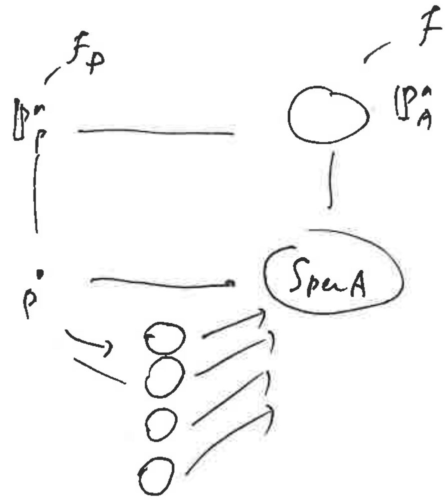
bunch of affine locally closed subsets where Hilb. pol. is constant.
finitely many



generic flatness + Noetherian induction.

Conclusion: # possible Hilb. poly. is finite.

Step 3. $\exists\, m \gg 0$ for which coho. + base change commutes for every pt of Spec A.



affine open cover (finite) s.t. $\mathcal{F}|_{U_i}$ flat over U_i
U_i

Step 4. The (candidate) top. stratification.

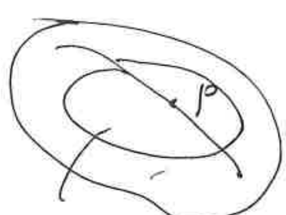
$$\operatorname{rank}\left(\pi_* \mathcal{F}(10^6)\right)_{\text{at } p}$$

$$= \text{hil. pol.}\,(10^6) \text{ at } p.$$

Next: Let's get scheme structure on these strata

— Later: satisfies universal property.

looking for a closed subscheme structure.



replace A by A_f

$M \gg 0$, Flattening str. for this $\pi_* \mathcal{F}(M)$
$\pi_* \mathcal{F}(M+1)$, $\pi_* \mathcal{F}(M+2)$

$\pi_* \mathcal{F}(M+i)$ is locally free of rank $p(M+i)$, $\operatorname{Spec} A / I_{M+i}$

$I_M + I_{M+1} + I_{M+2} + \cdots = I.$

Lecture 9 Flattening stratification

$$\begin{tikzcd} \mathcal{F}_T \arrow[d, dash] & & \\ \mathbb{P}^n_T \arrow[r] \arrow[d] & \mathbb{P}^n_A \arrow[d, "\pi"] & \mathcal{F} \text{ coherent} \arrow[l, dash] \\ T \arrow[r] \arrow[ur, dashed, "Z_{f_i}"] & \operatorname{Spec} A & \text{Noetherian} \end{tikzcd}$$

$T = \operatorname{Spec} B$

Goal. Flatening "stratification"

Know: Finitely many Hilb. polys appearing $f_1(m) < \cdots < f_3(m)$ "rank $\pi_* \mathcal{F}(m)$"

$Z_{f_i} \subset \operatorname{Spec} A$, $\bigcup_{i \geq i_0} Z_{f_i}$ is closed.

Goal. loc. closed subscheme structure on Z_{f_i}, $i \geq i_0$, satisfying univ. property.

Question: How can I tell if $\mathcal{F}_B - \mathbb{P}^n_B$ is flat.

$$\begin{array}{c} \mathbb{P}^n_B \\ \downarrow \pi \\ \operatorname{Spec} B = T \end{array}$$

Criterion $\mathcal{F}_B$ is flat iff If for $m \gg 0$, say $m \geq m_0$, $R^{i>0} \pi_* \mathcal{F}_B(m) = 0$, $\pi_* \mathcal{F}_B(m)$ locally free. (i.e. vector bundle)

Pf. $\Gamma_\cdot(\mathcal{F}) = \bigoplus_{m \geq m_0} H^0(\mathcal{F}(m))$, $\widetilde{\Gamma_\cdot(\mathcal{F})} = \mathcal{F}$

flat.

Pf. Suppose $\mathcal{F}$ flat $\Rightarrow$ Serre vanishing.

Čech complex for $\mathcal{F}(m)$, $m \geq m_0$, $\quad 0 \to \check{C}_\bullet(\mathcal{F}(m)) \to 0$

Exact. $\quad 0 \to H^0(\mathcal{F}(m)) \to \underset{\text{flat}}{\check{C}_\bullet}(\mathcal{F}(m)) \to 0$

Suppose
$$\begin{tikzcd} \mathbb{P}^n_B \ni \mathcal{F}_B \arrow[r, "\rho_B"] \arrow[d, "\pi_B"'] & \mathbb{P}^n_A \ni \mathcal{F} \arrow[d, "\pi"] \\ \operatorname{Spec} B \arrow[r, "\rho"'] & \operatorname{Spec} A \end{tikzcd}$$

$\mathcal{F}_B$ is flat over B, w/ Hilb. poly. f.

then what does this mean for $\operatorname{Spec} B \to \operatorname{Spec} A$?

$\Leftrightarrow$ $R^0 \pi_{B*} \rho_B^* \mathcal{F}(m)$ is a v.b. rank $f(m)$.

$R^{>0} \pi_{B*} \rho_B^* \mathcal{F}(m) = 0$

$\Leftrightarrow$ $\rho^* \pi_* \mathcal{F}(m)$ is a v.b. of rank $f(m)$ for $m \gg 0$

$\Leftrightarrow$ ρ maps $\operatorname{Spec} B$ into the right flattening stratum for $\pi_* \mathcal{F}(m)$ for $m \gg 0$.

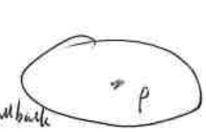
$$\begin{tikzcd} \mathbb{P}^n_A \ni \mathcal{F} \arrow[d] & \\ \operatorname{Spec} A & Z_f \arrow[l, hook'] \end{tikzcd}$$
set

look at all the flattening strata of

$\pi_* \mathcal{F}$, $\pi_* \mathcal{F}(1)$, $\cdots$ $\pi_* \mathcal{F}(m)$, $\cdots$

I_1 $\cdots$ I_m $\cdots$

Go to the local ring,


Spec A

p

$\operatorname{Spec} A_p \xrightarrow{\text{flat pullback}} \operatorname{Spec} A$

Sequence of ideals:

$I_1 + I_2 + \cdots$

$\quad I_2 + \cdots$

$\quad\quad I_3 + \cdots$

What I am looking for is the ideal that is $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A/I_m$

i.e. $B \longleftarrow A$ for $m >> 0$, sends I_m to 0 for $m >> 0$

New topic, Kontsevich's Thm on rational curves in projective spaces.

(? Graber - Harris - Starr on rational connectedness)

Old question: curves in $\mathbb{P}^2$

How many lines $(d=1)$
$(g=0)$ through 2 pts

How many conics through 5 "general" pts? $\longrightarrow$ $\mathbb{P}^5$ of conics.

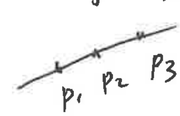
$p_1, \ldots, p_5$.

$\mathbb{P}^4$ of conics through p_1

$\mathbb{P}^3$ of conics through p_1, p_2

$\mathbb{P}^2$ of conics through p_1, p_2, p_3

$\mathbb{P}^1$ of conics through p_1, p_2, p_3, p_4



How many cubics in $\mathbb{P}^2$, $g=0$

How many degree d, $g=0$ curves $\boxed{N_d}$? in plane through $3d-1$ pts?

Where does "$3d-1$" come from?

Consider $\mathbb{P}^1_{uv} \xrightarrow{\deg d} \mathbb{P}^2_{x,y,z}$ $[f_0(u,v) : f_1(u,v) : f_2(u,v)]$ have degree d

$3(d+1) = 3d+3$

+ base point free

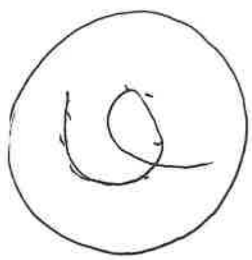
-1 scalar $\boxed{3d+2}$

$-\dim \mathrm{Aut}\, \mathbb{P}^1$ $\boxed{3d-1}$

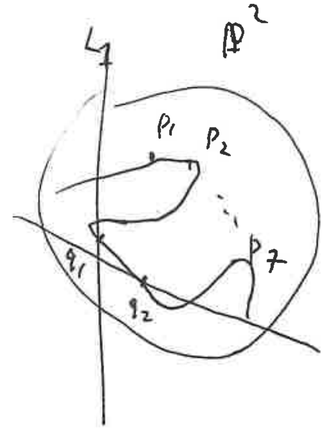
$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = ???$

<u>Question</u>

$\mathbb{P}^2$



$\mathbb{P}^2$



rat'l curves

S-dim'l family

through $p_1, \dots, p_7$

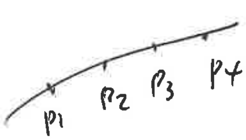
1-dim'l family.

$$M \xrightarrow[p_1 p_2 q_1 q_2]{\text{Cross ratio}} \mathbb{P}^1$$

What is the degree of this map ?

Answer, count preimage of $0 \in \mathbb{P}^1$, $\infty \in \mathbb{P}^1$

Cross ratio map :



$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$

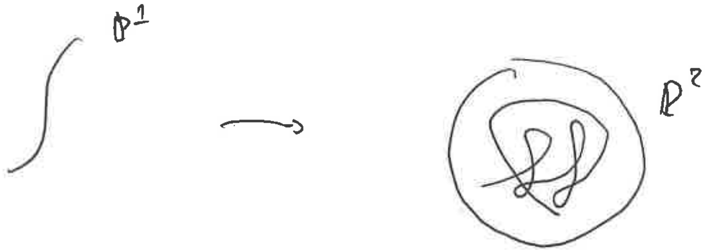
$M_{0,4}$

$z_1, z_2, z_3, z_4 \in \mathbb{C}, \quad \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$

Lecture 10   Today: Kontsevich's enumeration of rational plane curves

/$\mathbb{C}$. consider # rational curves in $\mathbb{P}^2$, degree d through $3d-1$ gen. chosen pts $p_1, \dots, p_{3d-1}$. Call it N_d.

$N_1 = N_2 = 1$.



moduli space of $\quad Mor_d(\mathbb{P}^1, \mathbb{P}^2) / Aut(\mathbb{P}^1)$.

Def. $\mathcal{M}_0(\mathbb{P}^2, d)$ from genus 0 curve to $\mathbb{P}^2$ "of degree d".

Roughly, degree d plane curves of geometric genus 0.

is it a FUNCTOR? No!!

Want: a (smooth) DM stack / or orbifold

Rmk. $Mor_d(\mathbb{P}^1, \mathbb{P}^2)$ is a variety.

Exercise: Prove that what I wrote down represents that FUNCTOR.

Rmk. More generally, given projective $X, Y / \mathbb{C}$,

$\mathrm{Mor}(X, Y)$ is representable [Thm (Grothendieck)]

$$\mathrm{Mor}(X, Y) \subset \underbrace{\mathrm{Hilb}(X \times Y)}_{\text{projective}}$$

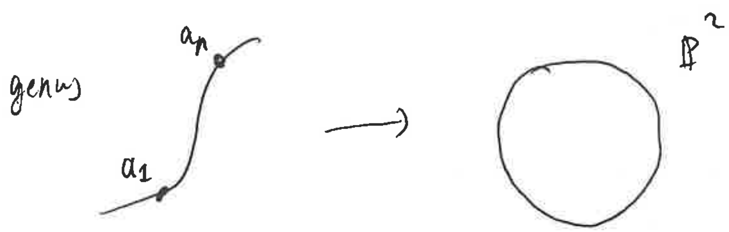
Rmk. Consider $f \in \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^2)$

$$[x : y] \longmapsto [x^d : y^d : 0]$$

$$\mathbb{P}^1 \circlearrowleft^{\times 3d} \longrightarrow \mathbb{P}^2$$

Revisit Konsevich's question:

$$\underbrace{M_{0,n}}_{\#\text{ marked pts}}(\mathbb{P}^2, d) \xrightarrow{ev_1, \cdots, ev_n} \mathbb{P}^2$$



$a_1, \cdots, a_n$ distinct.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\text{universal curve}"'] & \mathbb{P}^2 \\ M_{0,n}(\mathbb{P}^2) \arrow[u, bend right, "ev_i"'] \arrow[ur, "ev_i"'] & \end{tikzcd}$$

Choose general $p_1, \cdots, p_{3d-1}$, $M_{0,3d-1}(\mathbb{P}^2, d)$.

consider $\# \, ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1}) = N_d$.

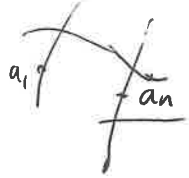
First.

$\overline{M_{0,n}}(\mathbb{P}^2, d)$

Stability condition.

parametrize

every irr. components contracted by p have ≥ 3 "special" pts on it.



$\xrightarrow{\ \ p\ \ } \mathbb{P}^2$

nodal genus 0, $a_1, \ldots, a_n$ distinct

Translation $(C, p_1, \ldots, p_n) \longrightarrow \mathbb{P}^2$

has finite # of automorphism

Miracle 1: This is a proper DM stack

Miracle 2: not hard.

$$\underset{\text{dense open}}{M_{0,n}(\mathbb{P}^2, d)} \subset \underset{\text{proper smooth}}{\overline{M_{0,n}}(\mathbb{P}^2, d)}$$

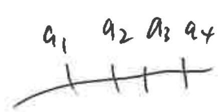
(boundary: normal crossings)

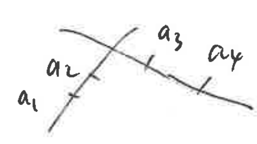
N_d is # $ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1})$ on $\overline{M_{0,n}}(\mathbb{P}^2, d)$

$$= ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1}) \cap [\overline{M_{0,n}}(\mathbb{P}^2, d)]$$

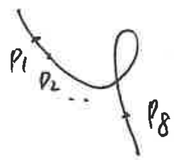
Example. $\overline{M_{0,n}}(\mathbb{P}^2, d)$

a_1, a_2, a_3, a_4 $\xrightarrow{\text{cross ratio of 4 pts}}$ $\overline{M_{0,4}} = \overline{M_{0,4}}(pt)$

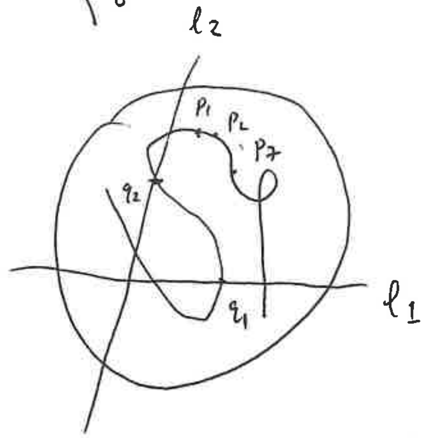




Let's compute $N_3 = 12$. à la Kontsevich



Consider instead: pick general $p_1, \dots, p_7 \in \mathbb{P}^2$, line $\ell_1, \ell_2 \subset \mathbb{P}^2$
rat'l curves through (later $p_8 = \ell_1 \cap \ell_2$)

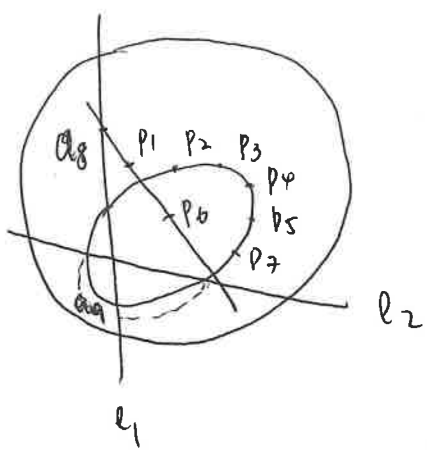


$$ev_1^{-1}(p_1) \cap \dots \cap ev_7^{-1}(p_7) \cap ev_8^{-1}(\ell_1) \cap ev_9^{-1}(\ell_2)$$

$$\cap\ \overline{M}_{0,9}(\mathbb{P}^2, 3)$$

$$\downarrow CR \quad p_1, p_2, q_1, q_2$$

$$\overline{M}_{0,4}$$



Case p_1 on degree 1, p_2 on degree 2.

$\longrightarrow$ 5 choices for splitting $(p_3, \dots, p_7)$

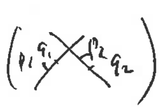
$\longrightarrow$ 2 choices for q_2 on ℓ_2

$\longrightarrow$ 1 choice for q_1 on ℓ_1

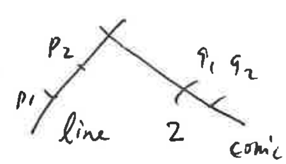
$\longrightarrow$ 2 choices for node

Total: 20.

Case p_1 on degree 2, p_2 on degree 1. $\boxed{20}$

Conclusion: # $CR^{-1}\Big($  $\Big) = 40$.

Case

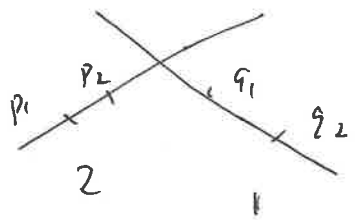


$\longrightarrow$ 2 choices for q_1

$\longrightarrow$ 2 choices for q_2

$\longrightarrow$ 2 choices for nodes

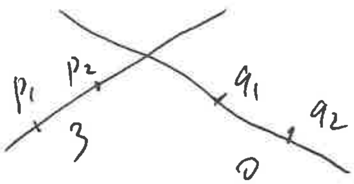
total 8.

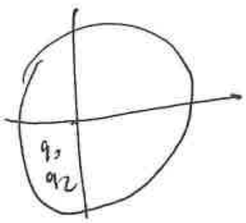


$-\ 10 = \binom{5}{2}$ split of $p_3, \dots, p_7$.

$-\ 2$ choices for the node.

<u>total 20.</u>





$\boxed{N_3}$

$N_3 + 28 = 40$

$\Rightarrow \underline{N_3 = 12}$

Ex. $\boxed{N_4 = 620}$

<u>Lecture 11</u>. Enumerative geometry

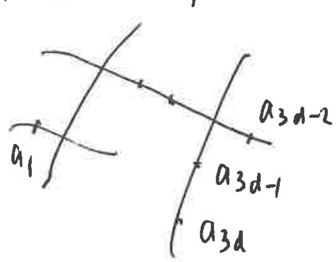
12 rational plane cubics through 8 general pts.

620 rational plane quartics through 11 general pts.

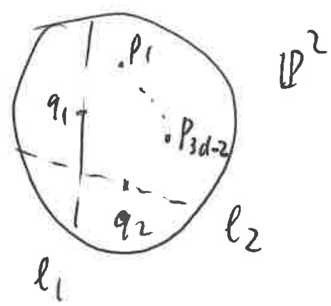
<u>moduli spaces</u>:

smooth 1-dim'l proper variety

≤ 1 nodes



$\xrightarrow[d]{\text{degree}}$

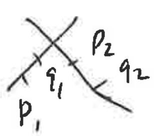


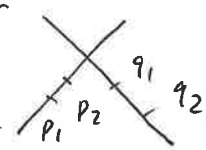
(with) genus 0 curves nodal.

$$\mathcal{M} \xrightarrow{CR} \overline{\mathcal{M}_{0,4}}$$

$p_1 \ p_2$

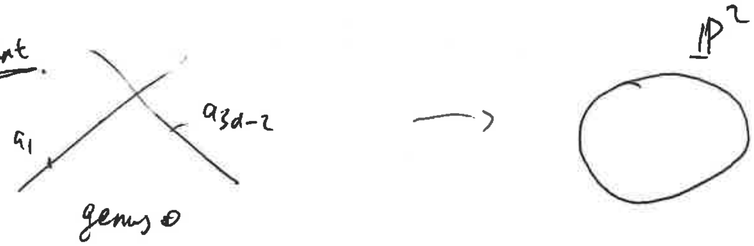
$q_1 \ q_2$

$\#\, CR^{-1}\Big($ 

$\Big) = \#\, CR^{-1}\Big($ 

$\Big)$

Space.

Want. 

"stable"

✓ 

$\mathcal{M}_{0,3d}(\mathbb{P}^2, d)$,

know dimension


"stable"

$$\overline{\mathcal{M}_{0,3d}}\,(\mathbb{P}^2, d)$$

target; degree d.

genus 0; # marked pts

: smooth, proper, known dim.

$\mathcal{M}_{0,3d}(\mathbb{P}^2, d)$ dense.

Bertini's Thm (Kleiman–Bertini Thm, 1973)

$\alpha : X \longrightarrow \mathbb{P}^2$ / field k char. 0

X smooth, ℓ general line, then $\alpha^{-1}(\ell)$ is smooth codim. 1.

$ev_{3d-1}^{-1}(\ell_2) \cap \overline{\mathcal{M}_{0,3d}}(\mathbb{P}^2, d)$

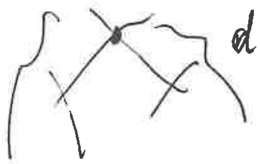
Smooth.

"$q_1 \in \ell_1$"

$q_2 \in \ell_2$.

Let's understand the locus in $\overline{\mathcal{M}_{0,n}}(\mathbb{P}^2, d)$ where source has at least one node, i.e. $\Delta\overline{\mathcal{M}_{0,n}}(\mathbb{P}^2,d) = \overline{\mathcal{M}_{0,n}}(\mathbb{P}^2,d) \setminus \mathcal{M}_{0,n}(\mathbb{P}^2,d)$.

Δ_{d_1,d_2}:

d_1  d_2 $\quad [n_1]\amalg[n_2]=[n]$ $\overline{\mathcal{M}_{0,n_1+1}}(\mathbb{P}^2, d_1) \times_{\mathbb{P}^2} \overline{\mathcal{M}_{0,n_2+1}}(\mathbb{P}^2, d_2) \subset \overline{\mathcal{M}_{0,n}}(\mathbb{P}^2, d)$

an irred. component of $\Delta\overline{\mathcal{M}_{0,n}}(\mathbb{P}^2,d)$

(everyone twice)

$n = n_1 + n_2$.

$[n] = [n_1] \amalg [n_2]$

CR: $\overline{\mathcal{M}_{0,3d}}(\mathbb{P}^2,d) \dashrightarrow \overline{\mathcal{M}_{0,4}}$

<u>Fact</u>. ✓

<u>Issue</u>, $\exists\ \overline{\mathcal{M}_{g,n+1}} \longrightarrow \overline{\mathcal{M}_{g,n}}$.

More generally,  B flat family genus g curve (connected) at most nodal w/ n marked sections, pairwise disjoint

(hard). $\hookrightarrow \overline{\mathcal{M}_{g,n}}$

<u>Fortunately</u>. For 620,

$X \supseteq \underset{F}{\cup} X \supseteq X_E^D$ $\boxed{\text{Smooth surface!!!}}$

$\downarrow$

M

Smooth curve

$E \cdot D = 1$

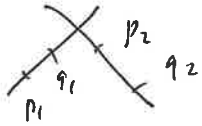
$E \cdot F = 0$

$D + E = F$

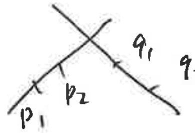
$M \dashrightarrow \overline{M_{0,4}}$

$\Rightarrow E \cdot E = -1$

$M \longrightarrow \overline{M_{0,4}}$ $\Rightarrow$ blow down E!

$CR^{-1}\Big($ 

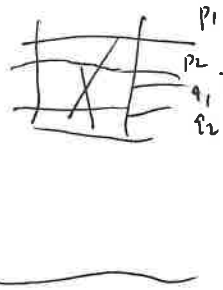
$\Big)$ is what we thought.

$CR^{-1}\Big($ 

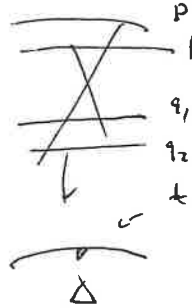
$\Big)$

<u>Next question:</u>

A_{m-1} surface singularity

surface 

$xy = s^m$

Smooth curve $\xrightarrow{CR}$ 

(x, y)

surface

"$xy = t$"

$\overline{\mathcal{M}}_{0,4}$

$t = s^m$

$m =$ mult $CR^{-1}(\Delta)$?

$\boxed{\overline{\mathcal{M}}_{g,n}(X, \beta)}$

virtual fund. class

<u>Lecture 12</u>. <u>Today</u>: Why $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ smooth (Deformation theory).

Moduli of curves $\mathcal{M}_g$. later: "turn on" points, singularities, maps.

translation: If M is a f.t./k moduli space,

Smooth: Show (Zar) tangent space at every closed pt has $\dim = \dim \mathcal{M}$ [for DM stacks, alg. spaces, f.t. varieties]

$T_p \mathcal{M}_g = H^1(C, T_C) =: \operatorname{def}(C)$ $\qquad H^0(C, T_C) = \operatorname{aut} C$

$p = [C] \xrightarrow{k = \bar{k}} \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \longrightarrow \mathcal{M}_g$

$\operatorname{Spec} k \longrightarrow p$

If X is a smooth variety,

$$H^1(X, T_X) = \operatorname{def} X \qquad \underline{\text{Exercise}}.$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\cdot} \cdot \arrow[r] & \mathcal{X} \arrow[d, "\text{flat}"] \\ \cdot \arrow[rr] & & \cdot = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \end{tikzcd}$$

<u>Rmk</u>. If X is affine, $\operatorname{def} X = 0$.

<u>Ex</u>.

$$\begin{tikzcd} X \times \operatorname{Spec} k[\varepsilon]/\varepsilon^2 = \tilde{X} \arrow[d] \\ \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \end{tikzcd}$$

auts of $\tilde{X}$ preserving X.

obstruction space

X smooth

	$H^0(X, T_X)$	$H^1(X, T_X)$	$H^2(X, T_X)$, ...
	3	0-dim	0
	0	dim 6	0
	1-dim	1-dim	0
	0	3	0
$g \geq 2$	0	$3g-3$	0

Artin stacks smooth dim $3g-3$

DM stack. Smooth dim $3g-3$.

Formally locally.

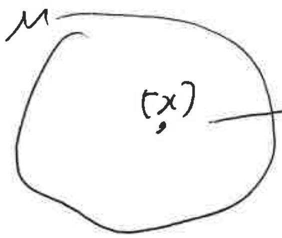
cut out by equations $\Leftrightarrow H^2(T_X)$.

$ob = 0 \Rightarrow M$ is smooth.

Example   Mod. space f.t.

$h^0(X, T_X) = 0$

$h^1(X, T_X) = 7, \quad h^2(X, T_X) = 1.$



$\longrightarrow$ tangent space has dim 7.

If dim is 7, $\mathcal{M}$ smooth at X.

If dim is 6, it has hypersurface singularity. It is Cohen-Macaulay, Gorenstein.

$\boxed{\mathcal{M}_{g,1}}$

$$\mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g$$

$$\operatorname{def}(C, p) \longrightarrow \operatorname{def}(C)$$

$$H^1(C, T_C)$$

$$\operatorname{aut}(C, p) \longrightarrow \operatorname{aut}(C)$$

$$H^0(C, T_C(-p)) \longrightarrow H^0(C, T_C)$$

$$0 \to T_C(-p) \to T_C \to T_C|_p \to 0$$

$$0 \to \underset{\operatorname{aut}(C,p)}{H^0(T_C(-p))} \to \underset{\operatorname{aut} C}{H^0(T_C)} \to \underset{\operatorname{def}_C P}{H^0(T_C|_p)} \to \underset{\operatorname{def}(C,p)}{H^1(T_C(-p))} \to \underset{\operatorname{def} C}{H^1(T_C)} \to 0$$

DM stack

$\boxed{\dim \mathcal{M}_{1,1} = 1}$

$\mathcal{M}_{1,0}$ Artin stack

<u>Next</u>. X smooth. Consider $M_g(X)$

$$\underset{\text{smooth}}{C} \underset{\substack{\text{closed}\\ \text{embedding}}}{\overset{\pi}{\hookrightarrow}} X$$

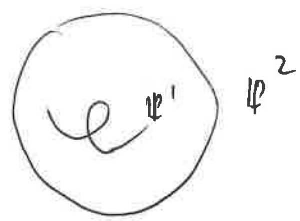
$$0 \to T_C \to \pi^* T_X \to N_{C|X} \to 0 \quad \text{on } C$$

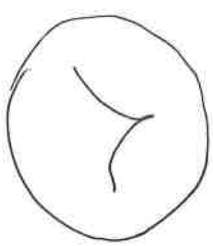
$$0 \to \underset{\text{aut } C}{H^0(T_C)} \to \underset{\text{def } \pi}{H^0(\pi^* T_X)} \to \underset{\text{def}(C,\pi)}{H^0(N_{C|X})}$$

$$\to \underset{\text{def } C}{H^1(T_C)} \to \underset{\text{Ob } \pi}{H^1(\pi^* T_X)} \to \underset{\text{Ob}(C,\pi)}{H^1(N_{C|X})}$$

$$\to \underset{\text{Ob } C}{H^2(\cancel{T_C})}^{\,0} \to \cdots \qquad \cancel{}\ 0 \qquad \cancel{}\ 0$$

Slightly more generally, $\underset{\mathbb{P}^1}{\overset{\|}{C}} \to \mathbb{P}^2$ <u>nonconstant</u> ok!





$$0 \to \underset{\substack{\|\\ 0}}{\ker} \to T_C \to \pi^* T_X \to \operatorname{coker} \to 0$$

<u>Claim</u>. $H^1(C, \pi^* T_{\mathbb{P}^2}) = 0$ (hence moduli space is smooth)

for $C = \mathbb{P}^1$, nonconstant

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \to 0$$

Suffices to show $H^1(C, \mathcal{O}(1)) = 0$.

Constant case: $0 \to K_{C/X} \to T_C \to \pi^* T_X \to N_{C/X} \to 0$

complex. $\left[T_C \to \pi^* T_X \right]$ instead of a sheaf.

Now allow C to be singular. $H^1(C, T_C)$?

$\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$.

(works for singular C.

Idea for stable maps in general.

aut / def / ob C $\quad \mathrm{Ext}^i(\Omega_C, \mathcal{O})$.

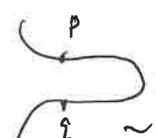
aut / def / ob π $\quad H^i(\pi^* T_X)$

have $\boxed{\pi^* \Omega_X \to \Omega_C}$ $\quad$ Apply $R\mathrm{Hom}(\cdot, \mathcal{O}_C)$ to Ω_C

$\pi^* \Omega_X$

$\pi^* \Omega_X \to \Omega_C$.

Last calculation of the day.

Def.  C compared to 

$\tilde{C}$

$\mathrm{Def}(\tilde{C}, p, q) \to \mathrm{Def}(C) \to$ 1-dim'l (smoothing of the node)

$H^0(T_{\tilde{C}}(-p-q)) \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$

Lecture 13. Today. Irreducibility of the moduli space of curves

rational connection — Deligne – Mumford '69

Kollár, Mori, Graber – Harris – Starr

moduli space of curves $\mathcal{M}_g$ is irreducible.

$\dim_{\mathbb{C}} \mathcal{M}_g = 3g-3$.

$\exists$ irreducible

$$\begin{array}{c} \int\!\!\int\!\!\int \\ \downarrow \\ N_g \xrightarrow{\text{surj.}} \mathcal{M}_g \end{array}$$

every curve appear.

Goal v1. Find an irred. family N_g in which every genus g curve sits.

genus g curve

$\downarrow$ deg d cover

$\mathbb{P}^1$

Riemann

Riemann - Hurwitz formula:

$$2-2g = d\cdot(2-2\cdot 0) - r$$

$$\boxed{r = 2d+2g-2}$$

$N_g = N_{d,g}$

$d \gg 0 \sim 2g$

moduli space of degree d branched cover of $\mathbb{P}^1$

Goal: irreducible.

d copies of 

Condition: $\sigma_1 \cdots \sigma_d = e$ in $\mathfrak{S}_d$

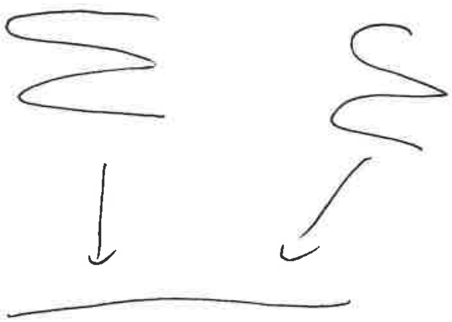
Conversely, this determines the cover complex-analytically

Riemann existence theorem.

<u>Fact 1</u> (soon): $N_{d,g}$ is smooth

$\boxed{= \mathcal{M}_g(\mathbb{P}^1, d)}$ of dim $2d + 2g - 2$.

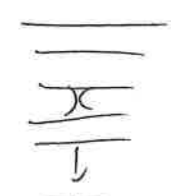
Goal: connected.



same g, d.

Most such covers are of this form:

$\sigma_1, \cdots, \sigma_d$ are transpositions $\sigma_j = (a\,b)$

i.e.  i.e. analytically locally, for each branch pt 1 ram. pt,

and analytically $y^2 = x$.

$\sigma_1 \cdots \sigma_b = e$ b pts   reduces the problem to a

transportation

$\sigma_1' \cdots \sigma_b' = e$ another b pts   Combinatorial game.

If you don't have simple branching.

nonsimple branching.

$N_{d,g} = M_g(\mathbb{P}^1, d)$

Back to Deligne–Mumford (1969)

M_g (black box) finite type $/\mathbb{C}$, smooth dim $3g-3$.

$\bar{M}_g$ tangent space at C is ~~$H^1(C, T_C)$~~

$\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$

Goal: show connected.

M_g connected | $\bar{M}_g$ corresponds to stable curves

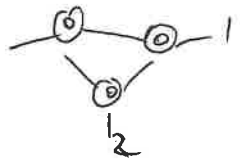
genus 0 components have ≥ 3 "special pts"

For convenience, describe it as a "stable graph", encoding the topological type of a stable curve.

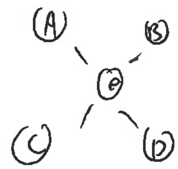
Q. What is the dimension of the space of curves w/ a given topological type?

0-dim strata

$\overline{M}_{1,3}$



1-dim stratum

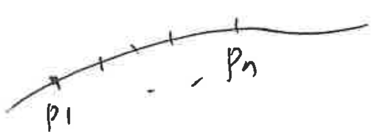


Exercise. Any 2 trivalent graphs in $\overline{M}_{g,n}$ are connected by such 1-strata.

Needed. Any curve in $M_{g,n}$ can be connected to a curve in $\overline{M}_{g,n} \setminus M_{g,n}$.

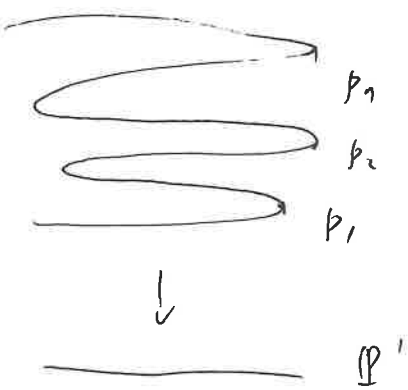
That suffices!

Now let us finish.



C

genus g.



$\mathcal{O}(?p_1 + ?p_2 + \dots + ?p_n)$

branch pts that deform "independently" of each other.

fixed branching at ∞ $\curvearrowright$ $\overline{M}_g^1(\mathbb{P}^1, d) \xrightarrow{br} \mathrm{Sym}^b\, \mathbb{P}^1 \simeq \mathbb{P}^d$.

$M_g^1(\mathbb{P}^1, d)$

$(C, p_1, \dots, p_d,$

<u>Lecture 14</u>. Graber - Harris - Starr Thm

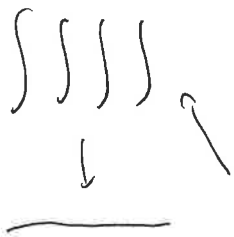
$\nwarrow$ rational connectedness

birational geometry / diophantine geometry

mod. spaces
- $\hookrightarrow$ Hilbert — enum. geom.
- Kontsevich
  - $\to$ GW theory
  - $\hookrightarrow$ deformation theory of maps
  - $\searrow$ irreducibility of M_g, Fulton
  - Hurwitz space

trichotomy

g curve	$g=0$	$g=1$	$g \geq 2$
X			
T_X	pos.	zero.	neg.
X arbitrary dim, $\det \Omega_X = K_X$	neg: Fano $\boxed{\text{rational}}$	zero	pos: general type



proj. smooth family

cubic 4 fold

$\mathbb{P}^{\binom{3+5}{3}-1}$

$\boxed{\text{Rationality ???}}$

neither open nor closed

Hard meta-question: given X, is it rat'l ?

Def. Given X projective cpx smooth irred. var., we say X is <u>rationally connected</u> if one of these equiv. definitions holds:

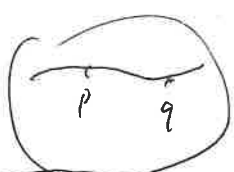
[Kollár, Miyaoka, Mori. — Campana]

For any two general pts $p, q \in X$,

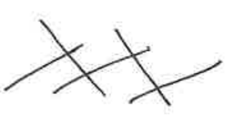
$\exists$ 

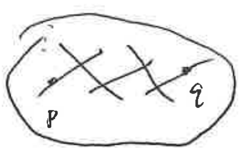
any 2 pts p, q

any finite set of pts

$\mathbb{P}^1 \longrightarrow$ 

$\boxed{\text{maybe immersed}}$

$\exists$ 

chain of $\mathbb{P}^1$'s $\longrightarrow$ 

$\boxed{\begin{array}{l}\exists\ \mathbb{P}^1 \xrightarrow{\text{rat'l}} X \text{ immersed} \\ \text{w/ positive normal bundle}\end{array}}$

normal bundle:

$$0 \longrightarrow T_{\mathbb{P}^1} \longrightarrow \pi^* T_X \longrightarrow \underset{\text{positive}}{N} \longrightarrow 0$$

$N = \bigoplus \mathcal{O}(n_i)$, $n_i > 0$

Observations RC birat'l invt

rat'l $\Rightarrow$ RC

$\Downarrow$ $\nearrow$

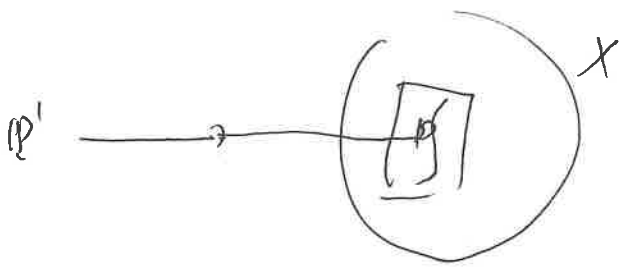
unirational

Def X unirational $\mathbb{P}^n \dashrightarrow X$ dominant

$\dim X = 1$ or 2, rat'l $\Leftrightarrow$ RC (classification)

Prop Suppose $D \subset X$ smooth ample, D rat'l conn'd $\Rightarrow$ X rat'l conn'd.

Pf.



$$0 \to N_{C,D} \to N_{C,X} \to N_{D,X}|_C \to 0$$

N positive $\Leftrightarrow$ $H^1(N(-1)) = 0$.

Cor. smooth Cubic hypersurfaces dim ≥ 2 are all RC.

Cubic 3 folds are never rat'l

Clemens – Griffiths – '69

Rmk. Thm Kollár – Miyaoka – Mori., X Fano (i.e. $-K_X$ ample),

$\Rightarrow X$ rationally connected.

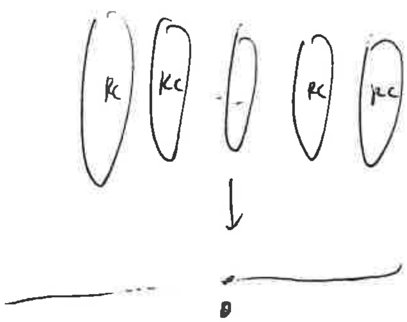
Prop Rational conn'dness is an open + closed condition in families

Sketch Pf:

open

$$\begin{array}{c} M_0(X,\beta) \\ \downarrow \\ B \end{array} \qquad \begin{array}{l} H^1(N(-1))=0 \text{ at } \pi: C \to X_0. \\ \text{smooth at } \pi: \mathbb{P}^1 \to X_0. \end{array}$$

closed.



curve

$$\begin{array}{c} X \\ \downarrow \\ B \end{array}$$

$$\begin{array}{c} \overline{M}_{0,2}(X,d) \\ \downarrow \\ X \times_B X. \end{array}$$

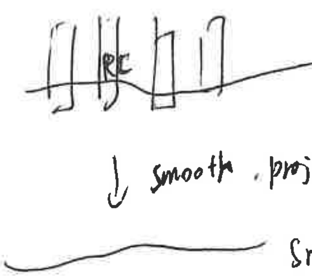
RC $X_1 \subset \mathbb{P}^1$ proj

$\exists\, d > 0.$

$\overline{M}_{0,2}(X_1,d) \to X_1 \times X_1$

Graber - Harris - Starr Thm

$$\begin{array}{c} X \\ \downarrow \\ B \end{array}$$



smooth, proj.

Smooth curve

Then has a section

Cor. $\begin{array}{c} \overset{\text{smooth}}{X} \\ \downarrow \\ Y \end{array}$ has one RC fiber, Y RC (proj. smooth), then X is RC

(gen. of Tsen's thm)

Thm. Let K be a field of tr. deg $1/\mathbb{C}$, then any RC variety X $/K$ has a rat'l pt.

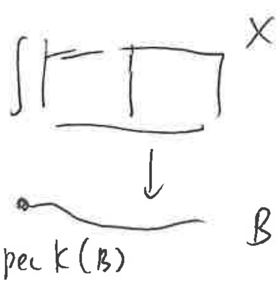
pol. in $n+1$ variables degree $d < n+1$

coeff. in K has a solution.

$t x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ has a sol'n in $\mathbb{C}(t)$.

$K = FF(B)$

$\quad \uparrow$ smooth curve



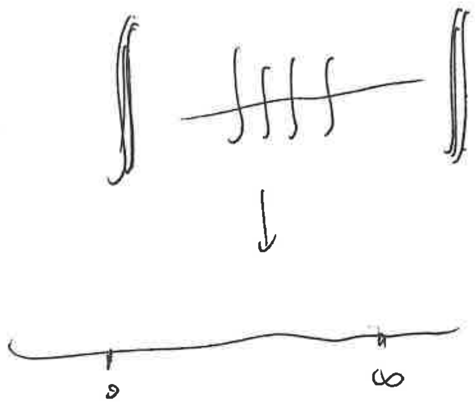
Example of a RC variety of degree $d \geq n+1$ in $\mathbb{P}^n / K$ w/o a rat'l pt.

$X_0 = \{ z_0^d + \cdots + z_n^d = 0 \text{ in } \mathbb{P}^n \}$

ξ primitive dth root of unity

$\mu_d \curvearrowright X_0$, $\xi \cdot [z_0 : \cdots : z_n] = [z_0 : \xi z_1 : \cdots : \xi^n z_n]$

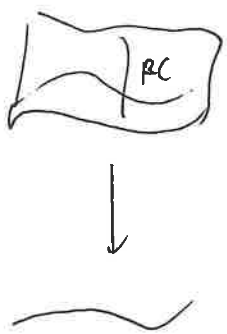
$$\begin{array}{ccc} X_0 \times \mathbb{P}^1 & \xrightarrow{/\mu_d} & X \\ \downarrow & & \downarrow \\ \mathbb{P}^1/\mu_d & \longrightarrow & \mathbb{P}^1 \end{array}$$



mult d.

X smooth when $\boxed{d \geq n+1}$

Lecture 15 Graber – Harris – Starr and weak approximation



X smooth $/\mathbb{C}$

$\Rightarrow \exists$ section

B smooth proper curve

$/\mathbb{Q}$

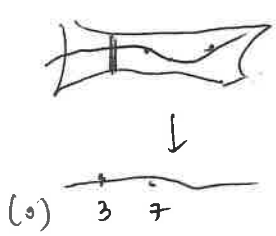
deg $d<n+1$, hypersurfaces in $\mathbb{P}^n$

i.e. Fano

i.e. $-K_X$ very ample.

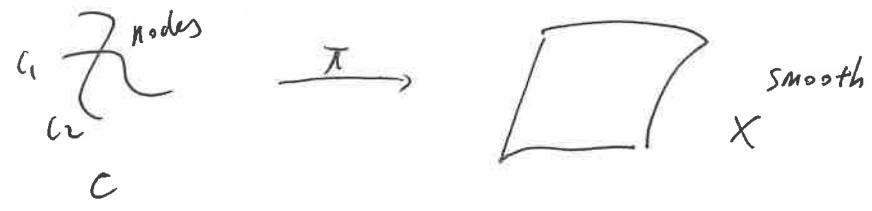
$x^3+y^3+z^3+w^3=0$ in $\mathbb{P}^3$.

With constraints mod $p_i^{a_i}$



Spec $\mathbb{Z}$

Deformation theory facts



N: $0 \to T_C \to \pi^* T_X \to N_{C|X} \to 0$

conormal sheaf of cl. embedding $\mathcal{I}/\mathcal{I}^2$. vector bdle for C nodal.

normal bundle ... ok

$H^0(N)$ 1st order deformation of map π

$H^1(N)$ obstruction

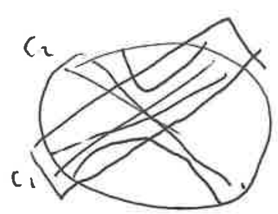
$$N_{C|X}|_{C_1} \cong N_{C_1|X}(p), \quad N_{C|X}|_{C_2} \cong N_{C_2|X}(p)$$

$(X = \mathbb{P}^2)$

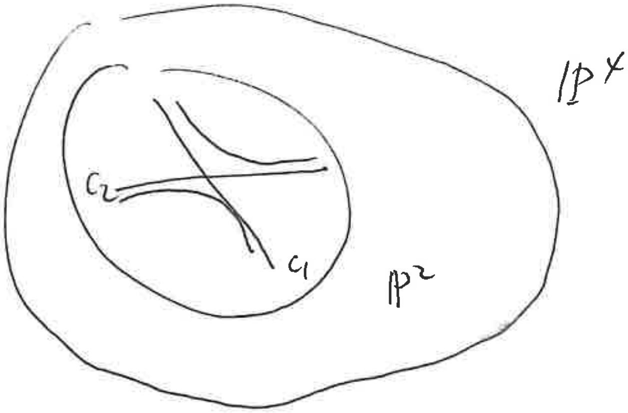
(C_1, p), (C_2, p) near π

$$\mathcal{M}((C_1,p)\to X) \times_X \mathcal{M}((C_2,p)\to X) \xrightarrow{\text{divisor}} \mathcal{M}(C \to X)$$

$$\square \to \underset{\text{}}{\mathrm{def}_{C\to X}} \to \overset{\text{1-dim'l}}{\mathrm{def}_{\text{node}}} \to 0$$



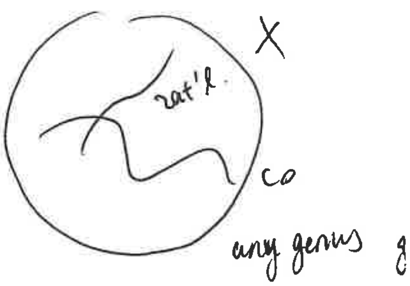
consider def. to conics in $\mathbb{P}^2$.



$N_{C|X}|_{C_1} \cong N_{C_1|X}[p]$ ← only along one direction

$N_{C|X}|_{C_2} \cong N_{C_2|X}[p]$

<u>Suppose</u> X is Rationally connected

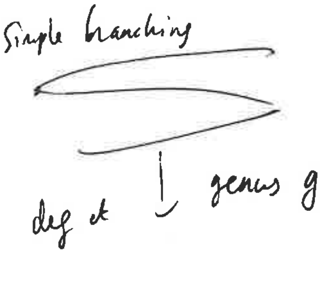


$H^0(N)$

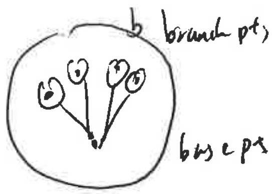
$H^1(N)$

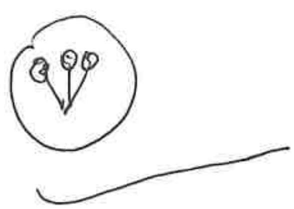
Fulton's pf of irreducibility of M_g

Simple branching

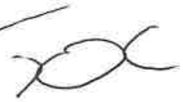


b transposition mult. identity

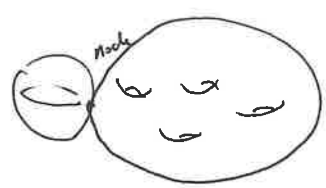




d copies



$y^2 = x(x-t), \quad t \to 0$



Graber - Harris - Starr