Proof of the Geometric Langlands Conjecture

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Tutorial 1.1 (Justin Campbell) Introduction to Bun_G.

char $k=0$, $k=\bar{k}$. G = conn'd reductive gp $/k$

pt/G classifying stack of G

$\mathrm{Map}(S, pt/G) = \{$étale locally trivial principal G-bundles on $S\}$

(S affine scheme)

E.g. pt/GL_n classifies rank n vector bundles.

X - smooth conn'd projective curve $/k$.

$\mathrm{Bun}_G := \underline{\mathrm{Map}}(X, pt/G)$.

$\mathrm{Map}(S, \mathrm{Bun}_G) = \{G$- bundles on $S \times X\}$.

Prop Bun_G is an algebraic stack, locally of finite type $/k$.

More precisely,

i) $\Delta: \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_G \times \mathrm{Bun}_G$ is affine.

ii) $\exists$ scheme $\mathcal{U}$ locally of finite type $/k$ and smooth surj. mor. $\mathcal{U} \longrightarrow \mathrm{Bun}_G$. (an "atlas")

References: Sorger, notes from Dennis's 2009 seminar

Prop. Bun_G is smooth.

Proof sketch: An alg stack locally of fin. type has a tangent complex which (for sufficiently nice stacks) is a perfect complex.

Such a stack is smooth $\Leftrightarrow$ its tangent complex is connective.

(i.e. concentrated in nonpositive cohomological degrees)

A standard calculation shows the fiber of tangent complex of Bun_G at a G-bundle $\mathcal{P}_G$ is

$$R\Gamma(X, \mathfrak{g} \overset{G}{\times} \mathcal{P}_G)[1].$$

$$R^i\Gamma(X, \underset{\text{coherent}}{\mathcal{F}}) = 0, \quad \forall\, i > 1 \quad \text{because } \dim X = 1.$$

$\Rightarrow \mathrm{Bun}_G$ is smooth.

Rmk. None of the above used the assumption that G is reductive.

Aside: for G reductive, Bun_G is never quasi-compact.

Eg. $\pi_0 \mathrm{Bun}_{G_m} \overset{\deg}{\cong} \mathbb{Z}$

For G nonabelian, the connected compts of Bun_G are not q.cpt.

Eg. in $\mathrm{Bun}_{GL_2}(\mathbb{P}^1)$ we have

$$\mathcal{O}^{\oplus 2} \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}(-1) \rightsquigarrow \mathcal{O}(2) \oplus \mathcal{O}(-2) \rightsquigarrow \cdots \rightsquigarrow \mathcal{O}(n) \oplus \mathcal{O}(-n) \rightsquigarrow$$

The Hecke action

$x \in X(k)$. Suppose that $k = \mathbb{C}$ and we work in the analytic topology.

Then a G-bundle on the Riemann surface X is determined by

- a G-bundle P'_G on a small disk D_x^{an};
- a G-bundle P''_G on $X \setminus \{x\}$;
- an isom. $P'_G|_{\mathring{D}_x^{an}} \overset{\alpha}{\simeq} P''_G|_{\mathring{D}_x^{an}}$

$$\mathring{D}_x^{an} := D_x^{an} - \{x\}$$

In alg geom, $\hat{\mathcal{O}}_x$ = completed local ring of X at x

$$\hat{K}_x = \mathrm{Frac}(\hat{\mathcal{O}}_x)$$

$$D_{x,S} = \mathrm{Spec}(\mathcal{O}_S \hat{\otimes} \hat{\mathcal{O}}_x), \quad \mathring{D}_{x,S} := \mathrm{Spec}(\mathcal{O}_S \hat{\otimes} \hat{K}_x) = D_{x,S} \setminus (S \times \{x\}).$$

Theorem (Beauville – Laszlo)

$$\mathrm{Map}(S, \mathrm{Bun}_G) \xrightarrow{\sim} \{(\underbrace{P'_G}_{\text{on } D_{x,S}}, \underbrace{P''_G}_{\text{on } S \times (X \setminus \{x\})}, \overbrace{\alpha}^{\text{on } \mathring{D}_{x,S}})\}$$

$\mathcal{H}_{G,x}$ = stack of Hecke modifications at x.

$$\mathrm{Map}(S, \mathcal{H}_{G,x}) = \{(P^1_G, P^2_G, \alpha) : P^1_G, P^2_G \text{ } G\text{-bundles on } D_{x,S},$$
$$P^1_G|_{\mathring{D}_{x,S}} \xrightarrow[\sim]{\alpha} P^2_G|_{\mathring{D}_{x,S}}\}$$

$\mathcal{H}_{G,x}$ is a groupoid over $\mathrm{Bun}_G(D_x)$ via composition of α's.

Beauville - Laszlo $\Rightarrow$ $\mathcal{H}_{G,x} \underset{\mathrm{Bun}_G(D_x)}{\curvearrowright} \mathrm{Bun}_G$

$$\mathcal{H}_{G,x} \underset{\mathrm{Bun}_G(D_x)}{\times} \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_G$$

<u>Uniformization</u>

$L_x G$ = loop group of G at x

$\cup$

$L_x^+ G$ = arc group of G at x

$\mathrm{Map}(S, L_x G) = \mathrm{Map}(\mathring{D}_{x,S}, G)$

$\mathrm{Map}(S, L_x^+ G) = \mathrm{Map}(D_{x,S}, G)$

Prop i) $L_x^+ G$ is an affine group scheme

ii) $L_x G$ is an ~~ind~~ - affine group ind-scheme.

Observation: locally on S, any G-bundle on $D_{x,S}$ descends to S.

$\Rightarrow \mathrm{Bun}_G(D_x) \simeq pt / L_x^+ G$

The same is true for G-bundles on $\mathring{D}_{x,S}$ which extend to $D_{x,S}$.

$$\Rightarrow \mathcal{H}_{G,x} = \mathrm{Bun}_G(D_x) \underset{\mathrm{Bun}_G(\mathring{D}_x)}{\times} \mathrm{Bun}_G(D_x) \simeq pt/L_x^+ G \underset{pt/L_x G}{\times} pt/L_x^+ G \simeq L_x^+ G \backslash L_x G / L_x^+ G$$

$Gr_{G,x}$ = affine Grassmannian $:= L_x G / L_x^+ G$

ind-proper ind-scheme

$\mathcal{H}_{G,x} = L_x^+ G \backslash Gr_{G,x}$

Tutorial 1.2 (Dima Arinkin) Local Systems.

char $\mathbb{C} = 0$, $\bar{\mathbb{C}} = \mathbb{C}$. X = smooth proj. curve, G = reductive group.

Def. $\mathrm{LocSys} = \mathrm{LocSys}_{G,X} = \{G\text{- local systems on } X\}$

de Rham local systems

1st approach. Local system:

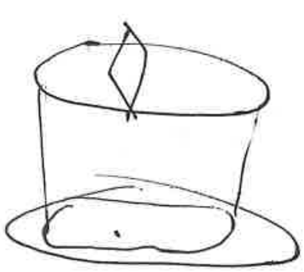
G- bundle w/ an (integrable) connection

Eg. If $G = GL(n)$,

(E, ∇): E = rk n v. bundle on X

$\nabla: E \to E \otimes \Omega_X$ $\mathbb{C}$-linear

$\nabla(fs) = f\nabla s + s \otimes df$



$(E,\nabla) \in \mathrm{LocSys}_G$

Bun_G

$\in$

E

For fixed E, ∇'s form an affine space (over $R\Gamma(X, \mathrm{End}(E) \otimes \Omega_X)$)

Properties. LocSys is a (derived) alg stack of finite type, quasi-smooth (derived l.c.i.)

Rmk Why is LocSys only quasi-smooth while Bun_G is smooth?

– because $H^i(X, \text{vec. bdle}) = 0$ for $i > 1$.

but $H^i_{dR}(X, \text{loc. sys}) = 0$ for $i > 2$.

Example $G = GL_2$, $X = \mathbb{P}^1$

Bun_G

$$\mathcal{O}^2 \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}(-1) \rightsquigarrow \mathcal{O}(2) \oplus \mathcal{O}(-2) \rightsquigarrow \cdots$$

$$\mathcal{O} \oplus \mathcal{O}(-1) \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}(-2) \rightsquigarrow \cdots$$

So connections on $\mathcal{O}^2$? $\quad d + \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} dx \quad x \rightsquigarrow \frac{1}{\xi}$

polynomials

$$a_{11}(x) = a_{11}^{(d)} x^d + a_{11}^{(d-1)} x^{d-1} + \cdots + a^{(0)}$$

$$a_{11}(x)\,dx = \left(-\frac{a_{11}^{(d)}}{\xi^{d+2}} - \frac{a_{11}^{(d-1)}}{\xi^{d+1}} - \cdots - \frac{0}{\xi} \right) d\xi$$

Answer. $\mathrm{LocSys}_{G,\mathbb{P}^1} = \left(\mathrm{pt} \times_{\mathfrak{g}} \mathrm{pt}\right)/G$

2nd approach

Local systems = G-bundles on X_{dR}.

Def $X_{dR}(S) = \mathrm{Maps}(S^{red}, X)$ [formalize parallel transport]

$\mathrm{LocSys} = \underline{\mathrm{Maps}}(X_{dR}, \mathrm{pt}/G)$, so $\mathrm{LocSys}(S) = \mathrm{Maps}(S \times X_{dR}, \mathrm{pt}/G)$

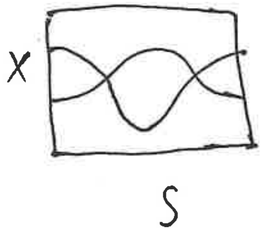
Ran space (of X)

$\mathrm{Ran}(X) = \{$ finite subsets of $X \}$.

non empty

Def. $\mathrm{Maps}(S, \mathrm{Ran}\,X) = \{$ finite nonempty subsets of $\mathrm{Maps}(S, X_{dR})\}$

$$\mathrm{Maps}(S, X_{dR}) = \mathrm{Maps}(S^{red}, X)$$



Less formally. $n > 0$,

$$X^n_{dR} \to \mathrm{Ran}\,X$$

$X^n_{dR} \to X^m_{dR}$ for any surjective $\{1, \dots, m\} \twoheadrightarrow \{1, \dots, n\}$.

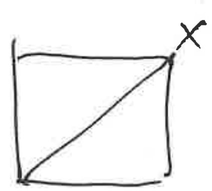
(e.g. diagonals, permutations)

Prop. $\varinjlim X^n_{dR} = \mathrm{Ran}\,X$

What is a q.coh. sheaf on Ran X?

① a sheaf on X_{dR} (a q coh. sheaf w/ a connection, i.e. D_X-module)

②



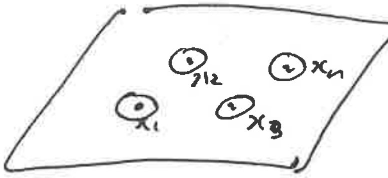
D_{X^2}-module $\quad X^2 : (x_1, x_2) \mapsto (x_2, x_1)$

$\mathfrak{S}_2$-equivariant

......

Def $\mathrm{Rep}(G)_{\mathrm{Ran}} = \mathrm{QCoh}((pt/G)_{\mathrm{Ran}})$

$(pt/G)_{\mathrm{Ran}} = \{(\underline{x} \in \mathrm{Ran}\,X,\ G\text{-local system on } D_{\underline{x}}$ - (its formal nbhd)$)\}$



X

$(pt/G)^n$

Lecture 1 (Sam Raskin) Construction of the (coarse) Langlands functor

$$\begin{tikzcd}
 & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}) \arrow[d, "quot"] \\
\text{D-mod}(Bun_G) \arrow[ur, dashed, "\exists \mathbb{L}_G"] \arrow[r, "\mathbb{L}_{G,coarse}"] & \mathrm{QCoh}(LS_{\check{G}})
\end{tikzcd}$$

$\cup\!\!\!\!\mid\!\mid$ LocSys

goal for today's lecture

Goal: $\mathbb{L}_G$ is an equiv.

Two ingredients in construction:

1) $\mathrm{Poinc}_! = \mathrm{Poinc}_!^{Vac} \in \text{D-mod}(Bun_G)$ compact

$\mathrm{Hom}(\mathrm{Poinc}_!, \mathcal{F}) =: \mathrm{coeff}(\mathcal{F})$ commutes w/ direct sums $\Leftrightarrow$ w/ all colimits.

2) The spectral action $\mathrm{QCoh}(LS_{\check{G}}) \curvearrowright \text{D-mod}(Bun_G)$ (related to Hecke action) — most of the talk is about this.

A variant on Yoneda for $\mathrm{QCoh}(\mathcal{Y})$, $\mathcal{Y}$ alg stack s.t. .....

$$\mathcal{G} \in \mathrm{QCoh}(\mathcal{Y}) \rightsquigarrow F_{\mathcal{G}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$$

$$\mathcal{H} \longmapsto \Gamma(\mathcal{Y}, \mathcal{G} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{H})$$

This functor commutes w/ colimits (assump. on $\mathcal{Y}$)

Claim. $\mathrm{QCoh}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{Funct}^{\mathrm{cts}}_{k\text{-lin}}(\mathrm{QCoh}(\mathcal{Y}), \mathrm{Vect})$ is an equiv.

Assumptions. $\mathcal{Y}$ is quasi-cpt, affine diag, $\mathrm{QCoh}(\mathcal{Y})$ is dualisable (true for quasi-smooth)

True for $LS_{\check{G}}$.

To define $\mathbb{L}_{G,\mathrm{coarse}}(\mathcal{F})$, $\mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)$, I'll use dual Yoneda,

$\forall\, \mathcal{G} \in \mathrm{QCoh}(LS_{\check{G}})$,

spectral action

$$\Gamma(LS_{\check{G}}, \mathcal{G} \underset{\mathcal{O}}{\otimes} \mathbb{L}_{G,\mathrm{coarse}}(\mathcal{F})) := \mathrm{coeff}(\mathcal{G} * \mathcal{F})$$

Axioms of GLC: $\mathrm{Poinc}_! \longmapsto \mathcal{O}_{LS_{\check{G}}}$.

Def. of $\mathbb{L}_{G,\mathrm{coarse}}$ is forced by this requirement.

Prop $\mathbb{L}_{G,\mathrm{coarse}}$ is the unique $\mathrm{QCoh}(LS_{\check{G}})$-linear functor s.t. TFDC:

$$\begin{tikzcd}
\text{D-mod}(\mathrm{Bun}_G) \arrow[rr, "\mathbb{L}_{G,\mathrm{coarse}}"] \arrow[dr, "\mathrm{coeff}"'] & & \mathrm{QCoh}(LS_{\check{G}}) \arrow[dl, "\Gamma(LS_{\check{G}}, -)"] \\
& \mathrm{Vect} &
\end{tikzcd}$$

Poincaré sheaf: $N \subset B \twoheadrightarrow T$

unip rad of B   Borel   Cartan

Think. $\quad \mathrm{Bun}_N = \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{ P_T^{\mathrm{triv}} \}$

Example $\quad G = SL_2, \quad B = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad T = \mathbb{G}_m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$

$$\mathrm{Bun}_B = \{ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \to 0 \}$$

$$\mathrm{Bun}_T = \{ \mathcal{L} \} \qquad \mathrm{Bun}_N = \{ 0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0 \}$$

$$\mathrm{Bun}_N^{\Omega} := \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{ \rho(\Omega_X) \}, \qquad 2\rho = \sum_{\substack{\alpha > 0 \\ \text{coroot}}} \alpha : \mathbb{G}_m \to T$$

Choose $\Omega_X^{1/2}$ square root of Ω_X

$$\rho(\Omega_X) := 2\rho(\Omega_X^{\frac{1}{2}})$$

Ex. $G = SL_2, \quad \mathrm{Bun}_N^{\Omega} = \{ 0 \to \Omega^{1/2} \to \mathcal{E} \to \Omega^{-1/2} \to 0 \}$

$$= \underline{R\Gamma}(\Omega_X)[1] \longrightarrow \underline{H^1(X, \Omega_X)} = \mathbb{A}^1$$

For gen'l G, each simple root $\rightsquigarrow N \to \mathbb{G}_a$

$$\mathrm{Bun}_N^{\Omega} \longrightarrow \mathrm{Bun}_{\mathbb{G}_a}^{\Omega} = \underline{R\Gamma}(X, \Omega) \longrightarrow \mathbb{A}^1$$

$$\rightsquigarrow \mathrm{Bun}_N^{\Omega} \longrightarrow \underbrace{\mathbb{A}^{\mathrm{rk}(G)} \xrightarrow{\text{sum coords}} \mathbb{A}^1}_{\Psi}$$

$$\begin{pmatrix} 1 & \Omega & \Omega^2 \\ & 1 & \Omega \\ & & 1 \end{pmatrix}$$

$\mathrm{Poinc}_! := p_{N!}(\psi^*(\exp))$ where $p_N : \mathrm{Bun}_N^{\Omega} \to \mathrm{Bun}_G$ is the projection

$\exp = (\mathcal{O}_{\mathbb{A}^1}, \nabla = d - dt) \in \text{D-mod}(\mathbb{A}^1)$

$\mathrm{Coeff}(\mathcal{F}) = C_{dR}(\mathrm{Bun}_N^{\Omega}, p_N^!(\mathcal{F}) \otimes \psi^*(\exp))$ [shift ?]

dR coh

<u>Example</u>. $G = \mathbb{G}_m$, $\mathrm{Poinc}_! = \delta_{\mathrm{triv},!} \in \text{D-mod}(\mathrm{Bun}_{\mathbb{G}_m})$

$\underbrace{pt \xrightarrow{\mathrm{triv}} \mathrm{Bun}_{\mathbb{G}_m}}_{\text{fibers are } \mathbb{G}_m}$! - pushforward of const. sheaf.

<u>Hecke action</u>:

$x \in X$, Justin said: Hecke groupoid acting on Bun_G.

Geometric Satake: $\mathrm{Rep}\,\check{G} \longrightarrow \text{D-mod}(\mathcal{H}_{G,x})$

$V^\lambda \longmapsto \mathrm{IC}_{\overline{\mathcal{H}^\lambda_{G,x}}}$ [?]

$V \in \mathrm{Rep}\,\check{G}$, $x \in X$, $H_{V,x} : \text{D-mod}(\mathrm{Bun}_G) \longrightarrow \text{D-mod}(\mathrm{Bun}_G)$

Hecke functor

pull-back to $\mathcal{H}_x^{\mathrm{global}}$, $\otimes$ w/ Satake sheaf, pushforward

$\mathcal{H}_x^{\mathrm{glob}} = \{ P_G, \tilde{P}_G \in \mathrm{Bun}_G + \alpha : P_G|_{X \setminus x} \xrightarrow{\sim} \tilde{P}_G|_{X \setminus x} \}$

$$\begin{tikzcd}
& \mathcal{H}^{glob}_{G,x} \arrow[dl, "\overleftarrow{h}"'] \arrow[d] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \\
& \mathrm{H}_{G,x} &
\end{tikzcd}$$

$P_G \qquad \tilde{P}_G$

$\mathrm{H}_{G,x}$

$\shortparallel$

$\{P_G^{loc}, \tilde{P}_G^{loc}$ on D_x,

w/ α on $\overset{\circ}{D}_x \}$

<u>Variant 1.</u> Don't fix $x \in X$

$V \in \mathrm{Rep}\,\check{G}$, $\quad H_V : \text{D-mod}(\mathrm{Bun}_G) \longrightarrow \text{D-mod}(\mathrm{Bun}_G \times X)$

$$\begin{tikzcd}
\text{D-mod}(\mathrm{Bun}_G) \arrow[r, "H_V"] \arrow[dr, "H_{V,x}"'] & \text{D-mod}(\mathrm{Bun}_G \times X) \arrow[d, "(\mathrm{id} \times x)^!"] \\
& \text{D-mod}(\mathrm{Bun}_G)
\end{tikzcd}$$

<u>Variant 2:</u> $V, W \in \mathrm{Rep}\,\check{G}$ or $\{V_i\}_{i \in I}$

$H_V \circ H_W : \text{D-mod}(\mathrm{Bun}_G) \longrightarrow \text{D-mod}(\mathrm{Bun}_G \times X^2)$

Relations from geom Satake:

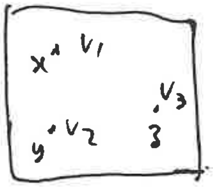
a) $\mathrm{Swap}_{X^2} \circ H_V \circ H_W = H_W \circ H_V$

b) restricting along $X \subset X \times X$, obtain $H_{V \otimes W}$

a) & b) compatible.

Variant 3 $\quad \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ acts on $D\text{-mod}(\mathrm{Bun}_G)$

V_x etc. $\in \mathrm{Rep}\,\check{G}$



X

$\rightsquigarrow$ obj. of $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ acts by $H_{V_1,x} \circ H_{V_2,y} \circ H_{V_3,z}$

$$\begin{array}{c} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \\ \Big\downarrow \ \mathrm{Loc}^{\mathrm{spec}} = \text{evaluation bundles} \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{array}$$

$x \in \mathrm{Ran}$

$\mathrm{LS}_{\check{G}} \longrightarrow (pt/\check{G})_{\mathrm{Ran}}\big|_x$ $\qquad$ $\mathrm{Loc}^{\mathrm{spec}}$ is pull back.

Thm (Gaitsgory–Rozenblyum) $\mathrm{Loc}^{\mathrm{spec}}$ is a quotient functor (Ref: GLC IV)

Thm (Drinfeld–Gaitsgory) action of $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ on $D\text{-mod}(\mathrm{Bun}_G)$ factors through action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

($\mathcal{V} \in \mathrm{Ker}(\mathrm{Loc}^{\mathrm{spec}})$, acts by zero on $D\text{-mod}(\mathrm{Bun}_G)$) "Generalized vanishing conj."

+ input from GLC 2 & GLC 4

Tutorial 1.3 (Dima Arinkin) IndCoh & Sing

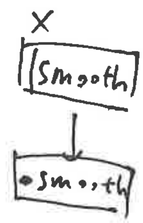
① Singular support : refinement of support to measure direction of non-vanishing.

for coherent sheaves on a quasi-smooth derived scheme (eg. l.c.i. scheme)

apply to LocSys.

q. smooth : cotangent complex is in degrees -1 and above. (Tor amplitude)

locally. $X = pt \times_{Smooth} Smooth$



$\mathcal{F} \in Coh(X) \quad (= D^b_{Coh}(X))$

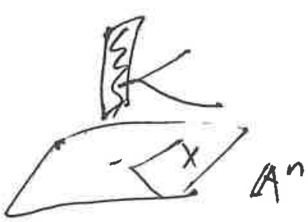
$$\begin{array}{ccc} \mathcal{F} & \rightsquigarrow & \underbrace{Sing\,Supp\,\mathcal{F} \subset Sing(X)}_{\text{Zariski closed \& conical}} \\ \cap & & \\ Coh(X) & & \end{array}$$

Zariski closed & conical

$$Sing(X) = H^{-1}T^*X = Spec_X\, Sym^{\bullet} H^1(TX)$$

Ex. $\mathbb{A}^n \supset X = \{x : f_1(x) = \cdots = f_k(x) = 0\}$

$Sing\, X = \{ (x, a) : a_1 df_1(x) + \cdots + a_k df_k(x) = 0 \}$, $x \in X$, $a \in \mathbb{A}^k$



Properties. ① $\mathcal{F}$ has bounded tor amplitude (i.e., $\mathcal{F} \in \mathrm{Perf}(X)$)

$\Leftrightarrow$ Sing Supp $\mathcal{F}$ $\subset$ zero section

② $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \quad \Rightarrow \quad$ Sing Supp $\mathcal{F}_2 \subset$ Sing Supp $\mathcal{F}_1 \cup$ Sing Supp $\mathcal{F}_3$.

③ Behavior w.r.t. maps $f: Y \to X$:

e.g., pull back f^* for smooth f : $\mathrm{Sing}\, Y = \mathrm{Sing}\, X \times_X Y$

Def (sketch) $H^1(TX)$ maps to $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$

Thm (Gulliksen) $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F})$ is f.g. as a $\mathrm{Sym}^\bullet(H^1(TX))$-module.

SingSupp $\mathcal{F}$:= support of this module.

Application to LocSys

Let $\mathcal{X}$ be a quasi-smooth derived stack, define Sing $\mathcal{X}$ and Sing Supp $\mathcal{F}$ for $\mathcal{F}$ on $\mathcal{X}$ via smooth covers (using ③)

Example. $\mathrm{Sing}(\mathrm{LocSys}_G)$

$$\parallel$$

$$\left(L \in \mathrm{LocSys}_G,\ \xi \in H^{-1}(T^*_L \mathrm{LocSys}_G)\right)$$

$$H^1 T_L(\mathrm{LocSys}_G) = H^2_{dR}(X, \mathfrak{g}_L)$$

$$\xi \in H^1 T_L(\mathrm{LocSys}_G)^* = H^2_{dR}(X, \mathfrak{g}_L)^* = H^0_{dR}(X, \mathfrak{g}_L)$$ infinitesimal symmetry of L

Def. $\overset{\text{global nilp cone}}{\mathrm{Nilp}} \subset \mathrm{Sing}(\mathrm{LocSys})$

$\quad = \{(L,\xi) : \xi \text{ is nilpotent}\}$

Ind coherent sheaves on a (derived) scheme

$\mathrm{QCoh}(X) \subset \mathrm{IndCoh}(X)$ $\qquad X$ f. type, $\mathcal{O}_X$ bounded

$= \text{"}D_{qc}(X)\text{"}$

cocomplete (admits $\varinjlim$)

$$\begin{tikzcd}
(\mathrm{QCoh}(X))^{\text{compact}} \arrow[d, equal] & \mathrm{QCoh}(X) \arrow[l, dashed, "\text{compact obj.}"'] \arrow[d, equal] \\
\underline{\text{0-section}}\ \mathrm{Perf}(X) \arrow[r, dashed, "\text{ind-completion}"] \arrow[d, phantom, "\cap"] & \mathrm{Ind}(\mathrm{Perf}(X)) \arrow[d, phantom, "\cap"] \\
\underline{\mathrm{Sing}\,X}\ \mathrm{Coh}(X) \arrow[r, dashed, "\text{ind-completion}"] & \mathrm{Ind}(\mathrm{Coh}(X))
\end{tikzcd}$$

$\mathrm{QCoh}(X)$ – compactly generated

(adding formal colimits) $=$ cat. of functors $\mathrm{Perf}(X)^{op} \to \mathrm{Vect}$

Given conical $\mathcal{N} \subset \mathrm{Sing}(X)$

define $\mathrm{IndCoh}(X)_{\mathcal{N}} = \mathrm{Ind}\left(\{\mathcal{F} \in \mathrm{Coh}(X) : \mathrm{SingSupp}(\mathcal{F}) \subset \mathcal{N}\}\right)$

<u>Rmk</u> Def'n works locally; for q-smooth stacks $\mathcal{X}$, define $\mathrm{IndCoh}(\mathcal{X})_{\mathcal{N}}$

$$= \lim_{\substack{S \to \mathcal{X} \\ \text{smooth}}} \mathrm{IndCoh}(S)_{\mathcal{N} \times_{\mathcal{X}} S}$$

$\cancel{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})} \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{Nilp}}$

Lecture 2 (Dennis Gaitsgory)

1) $\mathrm{Hom}(\mathcal{Y}, pt/G) = $ Sym. monoidal $\underbrace{\ }_{\text{right t-exact}}$ functors $\mathrm{Rep}(G) \longrightarrow \mathrm{QCoh}(\mathcal{Y})$

ii) $\mathrm{Hom}(S, LS_{\check{G}}) = $ sym. mon. right t-exact functors $\mathrm{Rep}(\check{G}) \longrightarrow \mathrm{QCoh}(S) \otimes \mathrm{Dmod}(X)$

$\mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) = \mathrm{QCoh}(S_1 \times S_2)$

2) $(pt/G)_{\mathrm{Ran}}$ ← give a def'n

$$\begin{tikzcd} (pt/G)_{\mathrm{Ran}} \arrow[d] \\ \mathrm{Ran} \end{tikzcd}$$

2') $\mathrm{Rep}(G)_{\mathrm{Ran}} = \mathrm{QCoh}((pt/G)_{\mathrm{Ran}})$

2-) $\mathrm{QCoh}(\mathcal{Y}) = \lim\limits_{\substack{\longleftarrow \\ S \to \mathcal{Y}}} \mathrm{QCoh}(S)$

3)
$$\begin{tikzcd} LS_{\check{G},X} & LS_{\check{G},X} \times \mathrm{Ran} \arrow[l, "p"'] \arrow[r, "ev"] \arrow[dr, dashed] & (pt/\check{G})_{\mathrm{Ran}} \arrow[d, dashed] \\ & & \mathrm{Ran} \end{tikzcd}$$

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow[\substack{\| \\ p_* \circ ev^*}]{\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \mathrm{QCoh}(LS_{\check{G}})$$

3') $x_1, \dots, x_n \in X$

$V_1, \dots, V_n \in \mathrm{Rep}(\check{G}) \rightsquigarrow \hat{\bigotimes}_{i=1} V_{i, x_i} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$

$$\left(\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}\left(\hat{\bigotimes}_{i=1} V_{i,x_i}\right)\right)_\sigma = \hat{\bigotimes}_{i=1} \left((V_i)_\sigma\right)_{x_i}$$

$\sigma \in LS_{\check{G}}$

3'') <u>Thm</u> $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$ is a quotient.

$$\begin{tikzcd}
 & \mathcal{U}^{\mathrm{glob}}_x \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "s"] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_G & \mathcal{U}^{\mathrm{loc}}_x & \mathrm{Bun}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{U}^{\mathrm{glob}}_{\mathrm{Ran}} \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "s"] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_G & \mathcal{U}^{\mathrm{loc}}_{\mathrm{Ran}} & \mathrm{Bun}_G
\end{tikzcd}$$

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Sat}^{\mathrm{nv}}_G} \mathrm{Dmod}(\mathcal{U}^{\mathrm{loc}}_{\mathrm{Ran}})$$

$$\begin{array}{ccc}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Dmod}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{Hecke}} & \mathrm{Dmod}(\mathrm{Bun}_G) \\
\quad\;\; \Psi \qquad\qquad\quad \Psi & & \\
\quad\;\; V \qquad\qquad\quad \mathcal{M} & &
\end{array}$$

$$\mathrm{Hecke}(V \otimes \mathcal{M}) = \overrightarrow{h}_*\big(\underbrace{s^!(V)}_{\mathrm{Sat}^{\mathrm{nv}}_G(V)} \overset{!}{\otimes} \overleftarrow{h}^!(\mathcal{M})\big)$$

<u>Thm</u>. The above binary operation factors through

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[d, "\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} \otimes \mathrm{Id}"] \arrow[dr, "\mathrm{Hecke}"] & \\
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r] & \mathrm{Dmod}(\mathrm{Bun}_G)
\end{tikzcd}$$

5) $\mathrm{Dmod}(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

$\mathrm{Poinc}^{\mathrm{Vac}}_!$

$$\begin{array}{ccc}
\mathrm{Dmod}(\mathrm{Bun}_G) & \underset{\mathbb{L}_{G,\mathrm{coarse}}}{\overset{\mathbb{L}^L_{G,\mathrm{coarse}}}{\rightleftarrows}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
 & & \Psi \\
\mathrm{Poinc}^{\mathrm{Vac}}_! & \longleftarrow\!\shortmid & \mathcal{O}_{\mathrm{LS}_{\check{G}}}
\end{array}$$

$$\begin{tikzcd}
 & & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}) \arrow[d, "\Psi"] \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[urr, dashed, "\mathbb{L}_G"] \arrow[rr, "\mathbb{L}_{G,coarse}"'] & & \mathrm{QCoh}(LS_{\check{G}})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathcal{Y})^{>-\infty} \arrow[d, "\Psi_{\mathcal{Y}}"'] \\
\mathrm{QCoh}(\mathcal{Y})^{>-\infty}
\end{tikzcd}$$

$\longleftarrow$ t-exact, equiv on eventually coconnective

$$\begin{tikzcd}
\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \arrow[d] \\
\mathrm{QCoh}(\mathcal{Y})
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}})^{>-\infty} \arrow[d, "\Psi"', "\wr"] \\
\mathrm{Dmod}(\mathrm{Bun}_G)^c \arrow[urr, dashed] \arrow[rr, "\mathbb{L}_{G,coarse}"] & & \mathrm{QCoh}_{Nilp}(LS_{\check{G}})^{>-\infty}
\end{tikzcd}$$

Want to show

__Thm 0__. $\mathbb{L}_{G,coarse}$ sends compact objects in $\mathrm{Dmod}(\mathrm{Bun}_G)$ to eventually coconnective objects in $\mathrm{QCoh}(LS_{\check{G}})$.

__Thm 1__. $\mathrm{Dmod}(\mathrm{Bun}_G)^c \subset \mathrm{Dmod}(\mathrm{Bun}_G)^{>-\infty}$

__Thm 2__ $\mathbb{L}_{G,coarse}$ is left exact $[-1000]$

__NB__ $\mathbb{L}_G$ is not left-exact. (eg. apply to constant sheaf)

__Thm 2'__ $\mathbb{L}_{G,coarse}$ is right t-exact $[1500]$. __Cor__. $\mathbb{L}_G$ is right t-exact $[1500]$

Proof of Thm 1

$\mathcal{Y}$ qc algebraic stack

$\mathrm{Dmod}(\mathcal{Y})^c \subset \mathrm{Dmod}(\mathcal{Y})^{>-\infty}$

$$\begin{tikzcd} \mathrm{IndCoh}(\mathcal{Y}) \arrow[r, shift left, "\mathrm{ind}^r"] & \mathrm{Dmod}(\mathcal{Y}) \arrow[l, shift left, "\mathrm{oblv}^r"] \end{tikzcd}$$

t-exact

$$\overset{qc}{U} \overset{j}{\hookrightarrow} \mathrm{Bun}_G$$

Every compact in $\mathrm{Dmod}(\mathrm{Bun}_G)$ is of the form $\underbrace{j_!}_{\text{not defined in general}}(\mathcal{F}_U)$, $\mathcal{F}_U \in \mathrm{Dmod}(U)^c$.

Claim. Bun_G is a union of U's for which $j_!$ is defined.

$U \hookrightarrow U'$.

Proof of Thm 2

$\mathcal{M} \in \mathrm{Dmod}(\mathrm{Bun}_G)^{>-\infty}$, $\mathbb{L}_{G,\text{coarse}}(\mathcal{M}) = \mathcal{F}$.

$\mathcal{Y}$ - algebraic stack, $\mathrm{Spec}(k) \overset{i_y}{\hookrightarrow} \mathcal{Y}$

$i_y^!(\mathcal{F})$

BBT $\exists d$ s.t. $\forall y$, $i_y^!(\mathcal{F}) \geq d \Rightarrow \mathcal{F} \geq 0$, $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$. $\leftarrow$ $\mathcal{Y}$ eventually connective?

allow all pts, not just k-pts. i.e. all base change of $\mathcal{Y}$.

Counterexample. $\mathcal{Y} = \mathrm{Spec}\, k[\xi]$, $\deg(\xi) = -2$, $\mathcal{F} = k[\xi, \xi^{-1}]$

Counterexample $\mathcal{Y} = \mathbb{A}^1$, $\mathcal{F} = \bigoplus_n k(t)[n]$

Need to show $\forall \sigma \in LS_{\check{G}}(k)$, $pt \overset{i_\sigma}{\hookrightarrow} LS_{\check{G}}$

$i_\sigma^!(\mathbb{L}_{G,\text{coarse}}(M)) \geq d$.

[AGKRRV] $LS_{\check{G}}^{\text{restr}} \overset{i^{\text{Spec}}}{\subset} LS_{\check{G}}$

$$\begin{tikzcd} pt \arrow[u, hook, "i_\sigma"] \end{tikzcd}$$

Enough to show : $(i_\sigma)^! \circ (i^{\text{Spec}})^! \mathbb{L}_{G,\text{coarse}}(M) \geq d$.

$\text{Dmod}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Dmod}(\text{Bun}_G)$

Thm [AGKRRV] $\text{Dmod}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{QCoh}(LS_{\check{G}}^{\text{restr}}) \underset{\text{QCoh}(LS_{\check{G}})}{\otimes} \text{Dmod}(\text{Bun}_G)$

$$\begin{tikzcd} \text{Dmod}_{\text{Nilp}}(\text{Bun}_G) \arrow[r, shift left, "i"] & \text{Dmod}(\text{Bun}_G) \arrow[l, shift left, "i^R"] \end{tikzcd}$$

$$\begin{tikzcd} \text{Dmod}(\text{Bun}_G) \arrow[r, "\mathbb{L}_{G,\text{coarse}}"] \arrow[d, shift left, "i^R"] & \text{QCoh}(LS_{\check{G}}) \arrow[d, shift left, "(i^{\text{Spec}})^!"] \\ \text{Dmod}_{\text{Nilp}}(\text{Bun}_G) \arrow[r, "\mathbb{L}_{G,\text{coarse}}^{\text{restr}}"'] \arrow[u, shift left, "i"] & \text{QCoh}(LS_{\check{G}}^{\text{restr}}) \arrow[u, shift left, "i_*^{\text{Spec}}"] \end{tikzcd}$$

$(i^{\text{Spec}})^! \circ \mathbb{L}_{G,\text{coarse}}$

$\quad\quad ||$

$\mathbb{L}_{G,\text{coarse}}^{\text{restr}} \circ i^R$

$(i_\sigma)^! \cdot \underbrace{\mathbb{L}_{G,\text{coarse}}^{\text{restr}}}_{\text{t-exact [FR]}} \cdot i^R(M)$

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_M) \arrow[d, "\mathrm{Eis}"'] & \mathrm{IndCoh}(\mathrm{LS}_{\check{M}}) \arrow[d, "\mathrm{Eis}^{\mathrm{Spec}}"] \\ \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"'] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

Bonus Material A (Joakim Færgeman)

Plan Introduce other versions of GLC

1) Betti 2) Restricted 3) Tempered

§1 Betti GLC (Ben-Zvi – Nadler '16)

Let X be smooth proj. curve $/\mathbb{C}$, Fix $e = \bar{e}$ char 0.

Goal: Define a version of GL replacing $\mathrm{LS}_{\check{G}}$ w/ $\mathrm{LS}_{\check{G}}^{\mathrm{Betti}} = $ "$\mathrm{Hom}(\pi_1(X), \check{G})$"$/\check{G}$

Spectral side.

$$\mathrm{LS}_{\check{G}}^{\mathrm{Betti}} := \mathrm{Maps}_{\mathrm{Spc}}(X, B\check{G})$$

$$\mathrm{LS}_{\check{G}}^{\mathrm{Betti}}(S) = \mathrm{Maps}_{\mathrm{Spc}}(X, \mathrm{Maps}_{\mathrm{Stk}}(S, B\check{G}))$$

Alternatively, $g = g(X)$ genus of X

$$\pi_1(X) = \langle a_i, b_i,\ i = 1, \cdots, g : \prod_i [a_i, b_i] = 1 \rangle$$

$$\mathrm{LS}_{\check{G}}^{\mathrm{Betti}} = \check{G}^{2g} \underset{\check{G}}{\times} \{e\} \Big/ \check{G}$$

Ex $X = \mathbb{P}^1$, $LS^{Betti}_{\check{G}} = pt \times_{\check{G}} pt / \check{G} = \Omega \check{\mathfrak{g}} / \check{G}$

Ex $G = T$, then $LS^{Betti}_{\check{T}} \simeq \check{T}^{2g} \times_{\check{T}} \{e\} / \check{T} \simeq \check{T}^{2g} \times \Omega \mathfrak{t}^* \times B\check{T}$

Automorphic side.

For $\mathcal{Y}/\mathbb{C}$ an alg stack. let $Shv^{all}(\mathcal{Y})$ to be the cat. of sheaves of

e-vector spaces on top. space underlying $\mathcal{Y}(\mathbb{C})^{an}$.

$$\text{I.e. } Shv^{all}(\mathcal{Y}) = \lim_{S \to \mathcal{Y}} Shv^{all}(S)$$

If $\Lambda \subset T^*\mathcal{Y}$ closed conical subset, we may consider

$$Shv^{all}_{\Lambda}(\mathcal{Y}) = \{ \mathcal{F} \in Shv^{all}(\mathcal{Y}) : SS(\mathcal{H}^i(\mathcal{F})) \subset \Lambda \}$$

$\mathcal{Y} = Bun_G$. Recall $T^* Bun_G = \{ (\mathcal{P}_G, \varphi \in \Gamma(X, \mathfrak{g}^*_{\mathcal{P}_G} \otimes \Omega_X)) \}$

$\Lambda = Nilp \subset T^* Bun_G$ subset of $(\mathcal{P}_G, \varphi)$: φ is nilpotent

Betti GLC: $Shv^{all}_{Nilp}(Bun_G) \rightleftharpoons IndCoh_{Nilp}(LS^{Betti}_{\check{G}})$

Motivation

1) Laumon '87 conjectured (proved in [AGKRRV'20]) that Hecke eigensheaves have nilp sing support.

2) $Eis_!$: $Shv^{all}(Bun_M) \longrightarrow Shv^{all}(Bun_G)$ preserves nilp. sing. supp.

$\underline{G=T}$ $\quad Nilp = 0$

$\underline{\text{Spectral}}$ $\quad LS^{Betti}_{\check{T}} \simeq \check{T}^{2g} \times \Omega \mathfrak{t}^* \times B\check{T}$

$$QCoh\left(LS^{Betti}_{\check{T}}\right) \simeq QCoh(\check{T}^{2g}) \otimes QCoh(\Omega \mathfrak{t}^*) \otimes QCoh(B\check{T})$$

$$\simeq k[H_1(X) \otimes \check{\Lambda}]\text{-mod} \otimes Sym(\mathfrak{t}[1])\text{-mod} \otimes Vect^{\check{\Lambda}}$$

$\underline{\text{Automorphic side}}$

$$Bun_T = Pic^{\circ}_T(X) \times BT \times \check{\Lambda}$$

$$Shv^{all}_0(Bun_T) \simeq k\left[\pi_1(Pic^{\circ}_T(X))\right]\text{-mod} \otimes H_*(T)\text{-mod} \otimes Vect^{\check{\Lambda}}$$

$$\pi_1\left(Pic^{\circ}_T(X)\right) = H_1(X) \otimes \check{\Lambda}$$

$$H_*(T) \simeq Sym(\mathfrak{t}[1])$$

$\underline{\S 2}$. Restricted GLC.

Original motivation: define stack of étale local systems on $X/\mathbb{F}_q$.

For us: (Restricted GLC)

$$\begin{tikzcd} & (\text{Restricted GLC}) \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\ (\text{Betti GLC}) & & (\text{dR GLC}) \end{tikzcd}$$

$\underline{\text{Return to dR-case}}$

$X/k = \bar{k}$ char 0. $\quad Maps(S, LS_{\check{G}}) = \left\{ \begin{array}{l} \text{right exact } \otimes\text{-functors} \\ Rep(\check{G}) \longrightarrow QCoh(S) \otimes D(X) \end{array} \right\}$

Let $\mathrm{QLisse}(X) \subset D(X)$ subcat. of $F \in D(X)$: $H^i(F)$ union of $(\text{v.b} + \nabla)$

<u>Def'n</u> $\mathrm{Maps}(S, LS^{\mathrm{restr}}_{\check{G}}) = \left\{ \begin{array}{l} \text{right exact } \otimes\text{-functors :} \\ \mathrm{Rep}(\check{G}) \longrightarrow \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \end{array} \right\}$

$\iota : LS^{\mathrm{restr}}_{\check{G}} \longrightarrow LS_{\check{G}}$

<u>Facts</u> 1) ι ~~surjective~~ (bijection) on k-pts : $\mathrm{Rep}(\check{G}) \longrightarrow D(X)$

$$\begin{tikzcd} \mathrm{Rep}(\check{G}) \arrow[r] \arrow[dr] & D(X) \\ & \mathrm{QLisse}(X) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Moreover, for $\sigma \in LS_{\check{G}}$, $LS^{\mathrm{restr}\,\wedge}_{\check{G}\ \sigma} \xrightarrow{\sim} LS^{\wedge}_{\check{G}\ \sigma}$

2) Two $\sigma_1, \sigma_2 \in LS_{\check{G}}$ lie in the same conn'd comp. of $LS^{\mathrm{restr}}_{\check{G}}$ iff they have isom. semisimplifications.

The conn'd comp. containing irred. σ is isom. to $LS^{\wedge}_{\check{G}\ \mathrm{pt}/\mathrm{Stab}_{\check{G}}(\sigma)}$. $\quad \mathrm{pt}/\mathrm{Stab}_{\check{G}}(\sigma) \to LS_{\check{G}}$

<u>Rmk</u>: σ semisimple if whenever it admits a reduction $\sigma_{\check{P}} \in LS_{\check{P}}$, it further admits reduction to $LS_{\check{M}} \to LS_{\check{P}}$.

σ local system, $\tilde{\sigma}$ semisimple local system, $\tilde{\sigma}$ is SS'n of σ if have reductions $\sigma_{\check{P}}, \tilde{\sigma}_{\check{P}}$ and they become iso. after $LS_{\check{M}}$.

<u>Restricted GLC</u>:

$$D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(LS^{\mathrm{restr}}_{\check{G}})$$

<u>Motivations</u>: 1) Any obj. of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is regular holonomic.
There is analogous Betti restricted GLC and they are equiv. by Riemann-Hilbert.

2) Given cat. $\mathcal{C}$

action $QCoh(LS^{restr}_{\check{G}}) \curvearrowright \mathcal{C}$

$\Leftrightarrow$ $Rep(\check{G})^{\otimes I} \otimes \mathcal{C} \longrightarrow \mathcal{C} \otimes QLisse(X)^{\otimes I}$

$\forall$ finite sets I + compatibilities

Thm (Nadler–Yun) $Rep(\check{G}) \otimes D(Bun_G) \longrightarrow D(Bun_G) \otimes D(X)$

$$\cup$$

$Rep(\check{G}) \otimes D_{Nilp}(Bun_G) \to D(Bun_G) \otimes QLisse(X)$

$\rightsquigarrow$ $QCoh(LS^{restr}_{\check{G}}) \curvearrowright D_{Nilp}(Bun_G)$.

In fact, $D_{Nilp}(Bun_G) \simeq D(Bun_G) \underset{QCoh(LS_{\check{G}})}{\otimes} QCoh(LS^{restr}_{\check{G}})$

dR GLC $\Rightarrow$ Restricted GLC. $IndCoh_{Nilp}(LS^{restr}_{\check{G}}) \simeq IndCoh_{Nilp}(LS_{\check{G}}) \underset{QCoh(LS_{\check{G}})}{\otimes} QCoh(LS^{restr}_{\check{G}})$

Restricted GLC $\Rightarrow$ dR GLC.

$LS^{restr}_{G_m} = \coprod_{\sigma \in LS_{G_m}} LS^{\wedge}_{G_m\, BG_m}$

$LS_{G_m} \simeq BG_m \times Loc \times \Omega k$,

Loc = coarse moduli space of rk 1 v.b. on $X + \nabla$

Tempered GLC

$$\begin{array}{ccc} D(\mathrm{Bun}_G) & \simeq & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\ \cup & & \cup \\ D(\mathrm{Bun}_G)^{\mathrm{temp}} & \simeq & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{array}$$

Q. How to char. $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$?

A. Choose $x \in X$: $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \underset{\mathrm{IndCoh}_{\mathrm{Nilp}}(\Omega\check{\mathfrak{g}}/\check{G})}{\otimes} \mathrm{QCoh}(\Omega\check{\mathfrak{g}}/\check{G})$

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}}(\Omega\check{\mathfrak{g}}/\check{G}) & \curvearrowright & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\ \uparrow & & \\ \mathrm{Rep}(\check{G}) & & \end{array}$$

Derived Satake.

$$D(L_x^+G \backslash L_xG / L_x^+G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\Omega\check{\mathfrak{g}}/\check{G})$$

So: $D(\mathrm{Bun}_G)^{\mathrm{temp}} := D(\mathrm{Bun}_G) \underset{\mathrm{IndCoh}_{\mathrm{Nilp}}(\Omega\check{\mathfrak{g}}/\check{G})}{\otimes} \mathrm{QCoh}(\Omega\check{\mathfrak{g}}/\check{G})$

Tutorial 2.1 (Andreas Hayash)

Goal: Discuss functors coeff + left adjoint Poinc!

$\rightsquigarrow$ Introduce Whittaker category on the affine Grassmannian.

Recall $\mathrm{Gr}_{G,\underline{x}}$, $\forall\, \underline{x} \in \mathrm{Ran}$, also $\mathrm{Gr}_{G,\mathrm{Ran}}$

$$\overset{\shortparallel}{(\mathcal{L}G)_{\underline{x}} / (\mathcal{L}^+G)_{\underline{x}}}$$

$(\mathcal{L}G)_{\underline{x}} \curvearrowright \mathrm{Gr}_{G,\underline{x}} \rightsquigarrow (\mathcal{L}N)_{\underline{x}} \curvearrowright \mathrm{Gr}_{G,\underline{x}}$, N unip. rad. of a Borel $B \subset G$.

The group $(\mathcal{L}N)_{\underline{x}}$ admits a character χ.

χ factors as

$$\begin{tikzcd} \mathcal{L}N \arrow[r] \arrow[rrrr, bend left, "\chi"] & \mathcal{L}(N/[N,N]) \simeq \prod_I \mathcal{L}(\mathbb{G}_a) \arrow[r, "\mathrm{Res}"] & \prod_I \mathbb{G}_a \arrow[r, "\chi_0"] & \mathbb{G}_a \end{tikzcd}$$

I: vertices of the Dynkin diagram

χ_0: character, nontrivial on each coord.

Rmk Suppressing twists,

Def. $\mathrm{Whit}^!(G)_{\underline{x}} = \mathrm{Dmod}\left(\mathrm{Gr}_{G,\underline{x}}\right)^{\mathcal{L}N,\chi}$

i.e. Objects of $\mathrm{Whit}^!(G)_{\underline{x}}$ are D-modules C on $\mathrm{Gr}_{G,\underline{x}}$ + an isom.

$\mathrm{act}: \mathcal{L}N \times \mathrm{Gr}_{G,\underline{x}} \to \mathrm{Gr}_{G,\underline{x}}$ $\quad \mathrm{act}^!(C) \simeq \chi^!(\exp) \boxtimes C$.

$m: H \times H \to H$, $m^!(\mathcal{F}) \simeq \mathcal{F} \boxtimes \mathcal{F}$ + cocycle cond.

$\mathcal{L}N = \bigcup N_\alpha$, can consider $D(\mathrm{Gr}_{G,\underline{x}})^{N_\alpha,\chi} \xrightarrow{\mathrm{oblv}_{\alpha,\beta}} D(\mathrm{Gr}_{G,\underline{x}})^{N_\beta,\chi}$

$$\begin{tikzcd} D(\mathrm{Gr}_{G,\underline{x}})^{N_\beta,\chi} \arrow[d, hook, "\mathrm{f.f.}"] \\ D(\mathrm{Gr}_{G,\underline{x}}) \end{tikzcd}$$

$$\mathrm{Whit}^!(G)_{\underline{x}} = \lim_\alpha D(\mathrm{Gr}_{G,\underline{x}})^{N_\alpha,\chi} = \bigcap_\alpha D(\mathrm{Gr}_{G,\underline{x}})^{N_\alpha,\chi} \subset D(\mathrm{Gr}_{G,\underline{x}})$$

$\exists$ oblv: $\mathrm{Whit}^!(G)_{\underline{x}} \longrightarrow D(\mathrm{Gr}_{G,\underline{x}})$ fully faithful.

It admits a (non-cts) right adjoint $\mathrm{Av}_*^{IN,x} : D(\mathrm{Gr}_{G,\underline{x}}) \to \mathrm{Whit}^!(G)_{\underline{x}}$

Rmk The cat. $\mathrm{Whit}^!(G)_{\underline{x}}$ does not depend on the choice of x.

Variant of Whit: $\mathrm{Whit}_*(G)_{\underline{x}} = D(\mathrm{Gr}_{G,\underline{x}})_{IN,x}$

which is equipped w/ a projection $D(\mathrm{Gr}_{G,\underline{x}}) \longrightarrow \mathrm{Whit}_*(G)_{\underline{x}}$.

However, $\mathrm{Whit}^!(G)_{\underline{x}} \xleftarrow[\sim]{\exists\, \Theta} \mathrm{Whit}_*(G)_{\underline{x}}$

Structure of $\mathrm{Whit}^!(G)_{\underline{x}}$

Recall. Naive geometric Satake

$$D(\mathcal{H}^{loc}_{G,\underline{x}}) = \mathrm{Sph}_{G,\underline{x}} \qquad \text{(technically should renormalize)}$$

$$\text{“} \quad D(\mathrm{Gr}_{G,\underline{x}})^{(L^+G)_{\underline{x}}} = D\left((L^+G)_{\underline{x}} \backslash (LG)_{\underline{x}} / (L^+G)_{\underline{x}}\right)$$

Naive Satake functor is a monoidal functor $\mathrm{Rep}(\check{G})_{\underline{x}} \longrightarrow \mathrm{Sph}_{G,\underline{x}}$.

e.g. V^λ h.w. $\lambda \longmapsto IC_{\overline{\mathrm{Gr}}^\lambda_G}$, $\overline{\mathrm{Gr}}^\lambda_G$ = closure of $L^+G \cdot t^\lambda$

Whenever $\mathrm{Sph}_{G,\underline{x}} \curvearrowright \mathcal{C} \rightsquigarrow \mathrm{Rep}(\check{G})_{\underline{x}} \curvearrowright \mathcal{C}$

In particular, $\mathrm{Rep}(\check{G})_{\underline{x}} \curvearrowright \mathrm{Whit}^!(G)_{\underline{x}} = \left(\mathrm{Whit}^!(\underset{\substack{\circlearrowleft \\ (LG)_{\underline{x}}}}{LG_{\underline{x}}})\right)^{(L^+G)_{\underline{x}}}$

$\mathrm{Vac}_{\mathrm{Whit}} \in \mathrm{Whit}^!(G)_{\underline{x}}$, $\quad \mathrm{Gr}_{N,\underline{x}} \subset \mathrm{Gr}_{G,\underline{x}}$

$$\mathrm{Vac}_{\mathrm{Whit}} = *\text{-extend } \omega_{\mathrm{Gr}_{N,\underline{x}}} \otimes \chi^!(\exp) \text{ to } \mathrm{Whit}^!(G)_{\underline{x}}.$$

Rmk. Extension is clean, $! \Rightarrow *$.

$$\begin{tikzcd} \mathrm{Rep}(\check{G})_{\underline{x}} \arrow[r, "\mathrm{Sat}^{nv}"] \arrow[rr, bend right] & \mathrm{Sph}_{G,\underline{x}} \arrow[r, "* \mathrm{Vac}_{\mathrm{Whit}}"] & \mathrm{Whit}^!(G)_{\underline{x}} \end{tikzcd}$$

Thm (Frenkel - Gaitsgory - Vilonen, Raskin)

Composition is an equiv., inverse is called CS_G.

Coeff: $\quad \mathrm{Gr}_{G,\underline{x}} \xrightarrow{\pi_{\underline{x}}} \mathrm{Bun}_G$

$$\rightsquigarrow \quad D(\mathrm{Bun}_G) \xrightarrow{\pi_{\underline{x}}^!} D(\mathrm{Gr}_{G,\underline{x}}) \xrightarrow{\mathrm{Av}_*} \mathrm{Whit}^!(G)_{\underline{x}}$$

composition is coeff $\underline{x}$. $\qquad$ unital factorization cat.

The functor coeff $\underline{x}$ admits a left adjoint $\mathrm{Poinc}_{!,\underline{x}}$.

$$D(\mathrm{Gr}_{G,\underline{x}}) \overset{\pi_!}{\dashrightarrow} D(\mathrm{Bun}_G)$$

partially defined, but defined on $\mathrm{Whit}^!(G)_{\underline{x}}$.

Obs 1 If $\pi_!(\mathcal{F})$ is defined, then so is $\pi_!(S \star \mathcal{F}) = S \star \pi_!(\mathcal{F})$, $S \in \mathrm{Sph}_G$

Obs 2 $\pi_!(\mathrm{Vac}_{\mathrm{Whit}})$ is defined

Tutorial 2.2 (Gurbir Dhillon) Kac – Moody modules

basic idea: to construct objects of $D(\mathrm{Bun}_G)$ there are 3 basic moves:

① Eisenstein series ② Poincaré series ③ KM localization

classical picture: $\mathbb{H} = \{ x+iy : x,y \in \mathbb{R},\ y>0 \}$ $\quad \dfrac{dx\,dy}{y^2}$

beautiful differential operator $\Delta \propto y^2(\partial_x^2 + \partial_y^2)$

Δ plays well w/ modular functions.

ex. $\underset{\substack{\uparrow \\ \text{Eisenstein series}}}{\Delta E_s} = s(1-s) E_s$ $\qquad \Delta - s(1-s)$

$\Rightarrow$ bounds microsupport of special functions (nilpotent)

$G = SL_2(\mathbb{R})$

$\cdot\ \mathbb{H}/\Gamma = SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) / \Gamma$, $\Gamma = SL_2(\mathbb{Z})$

$\mathrm{Diff}(G)^{G\times G} \simeq Z(\mathfrak{g}) = \mathbb{R}[\tfrac{1}{2}h^2 + ef + fe]$

$\Delta f = \lambda f$

$(\Delta - \lambda) f = 0$ on symbols: $\mathrm{symbol}(\Delta) = 0$

$$\begin{tikzcd}
T^*G \simeq G\times \mathfrak{g}^* \arrow[d, "\mu"'] & & G \times \mathcal{N} \arrow[ll] \arrow[d] \\
\mathfrak{g}^* = \mathfrak{g}^* \arrow[d] & & \mathcal{N} \arrow[ll] \arrow[d] \\
\mathfrak{g}^*/\!/G & & 0 \arrow[ll]
\end{tikzcd}$$

$x \in X \quad \rightsquigarrow \quad K_x \simeq \mathbb{C}((t))$

$\mathcal{O}_x \simeq \mathbb{C}[[t]]$

$$\begin{array}{ccc} SL_2(\mathbb{R}) & & G(K_x) \\ \downarrow & & \downarrow \\ \mathbb{H}/\Gamma & & \mathrm{Bun}_G = \prod G(\mathcal{O}_x) \backslash \prod' G(K_x) / G(F) \end{array}$$

F ← func. field of X

Def $\mathfrak{g}(K_x) := \mathrm{Lie}(\mathcal{I}_x G)$

$\mathfrak{g}(K_x) \simeq \mathfrak{g} \otimes \mathbb{C}((t))$ pointwise Lie bracket

Want : $Z(U(\mathfrak{g}(K_x)))$ should be BIG $\approx LA^{\mathrm{rk}\, G}$

example G f.d. red. gp

$$Z(\mathfrak{g}) \simeq \mathrm{Fun}(\mathfrak{t}^* /\!/ W)$$
$$\simeq \mathrm{Fun}(\mathfrak{t}^\vee /\!/ W)$$
$$\simeq \mathrm{Fun}(\check{\mathfrak{g}} /\!/ \check{G})$$

ex $G = T$ torus, $\mathfrak{t}(K_x)$ still commutative, so

$$U(\mathfrak{t}(K_x)) \simeq \mathrm{Sym}(\mathfrak{t}(K_x)) \simeq \mathrm{Fun}((\mathfrak{t}(K_x))^*)$$
$$\simeq \mathrm{Fun}(\check{\mathfrak{t}} \otimes \Omega^1)$$
$$\simeq \mathrm{Fun}(\mathrm{Conn}_{\check{T}}(\mathring{D}))$$

t^n dual to $t^{-n-1}dt$

$$\begin{array}{c} K_x \otimes \Omega^1 \\ \sigma \downarrow \omega \\ \mathbb{C} \end{array} \qquad \int \sigma\omega$$

G_m itself

$$Z \simeq \mathbb{C}[b]$$
$$\uparrow$$
$$t\partial_t$$

LG_m

$$Z \simeq \mathbb{C}[b_i : i \in \mathbb{Z}]$$

ex SL_2

$$Z(sl_2) = \mathbb{C}[\Delta]$$

$$Z(sl_2(K_x)) \simeq \mathbb{C}[\Delta_n : n \in \mathbb{Z}]$$

$$\Delta = \frac{1}{2}h^2 + ef + fe \qquad , \; x_n = x \otimes t^n$$

$$\text{"}\Delta_n = \sum_{i+j=n} \frac{1}{2}h_ih_j + e_if_j + f_je_i\text{"}$$

$$= \sum_{i+j=n} \frac{1}{2} :h_ih_j: + :e_if_j: + :f_je_i:$$

$$:x_ny_m: = \begin{cases} x_ny_m & n<0 \\ y_mx_n & n \geq 0 \end{cases}$$

<u>next</u>: which modules to use?

$$K \backslash SL_2(\mathbb{R}) / \Gamma \qquad \Big| \qquad G(\mathcal{O}_x) \backslash G(K_x) / \text{"}\Gamma\text{"}$$

$K = SO_2(\mathbb{R})$ max. cpt

<u>Def</u> $KL := (\mathfrak{g}(K_x), G(\mathcal{O}_x))$ - modules.

modules for $\mathfrak{g}(K_x)$ on which $\mathfrak{g}(\mathcal{O}_x)$ is integrated.

Examples $\quad V \curvearrowleft G(\mathcal{O}_x)$

$$\mathrm{ind}_{\mathfrak{g}(\mathcal{O}_x)}^{\mathfrak{g}(K_x)} V \in KL$$

V f.d., $\rightsquigarrow$ ind system of compact generators

$\lambda \in \Lambda^+$, V^λ irred. G-module

$$\begin{tikzcd} & G(\mathcal{O}_x) \arrow[dl] \arrow[dr] & \\ G & & G(K_x) \end{tikzcd} \qquad \mathbb{V}^\lambda := \mathrm{ind}_{\mathfrak{g}(\mathcal{O}_x)}^{\mathfrak{g}(K_x)} V^\lambda$$

Weyl module

$\mathbb{V}^0$ called vacuum rep'n

Example $\quad \mathbb{G}_m \curvearrowright \ell \quad$ 1-dim v.s.

$\rightsquigarrow$ diff $\quad t\partial_t = n \cdot \mathrm{id} \in \mathbb{Z}$

$t\partial_t \sim \lambda \in \mathbb{C}$ defines a rep of Lie algebra, doesn't integrate unless $\lambda \in \mathbb{Z}$.

Lemma $\quad V \in \mathfrak{g}(K_x)\text{-mod}^\heartsuit$ lifts to KL iff

① X_n, $n > 0$ act locally nilpotently

② eg. G semisimple s.c., $U(\mathfrak{g})v$ finite dim'l, $\forall v \in V$.

G torus $\longrightarrow$ int. eigenvalues from example.

Beautiful fact. $\quad V, W$ 2 f.d. $G(\mathcal{O}_x)$-modules

① $\mathrm{Hom}(V, W) \simeq \mathrm{Hom}(W^*, V^*)$

② $\mathrm{Hom}(\mathrm{ind}\, V, \mathrm{ind}\, W) \simeq \mathrm{Hom}(\mathrm{ind}(W^*), \mathrm{ind}(V^*))$

comes from a self duality $KL \simeq KL^{\vee}$.

$$\begin{array}{ccc} KL \otimes KL & \longrightarrow & Vect \\ M \boxtimes N & \longmapsto & C^{\frac{\infty}{2}+\bullet}(\mathfrak{g}(K_x),\ G(\mathcal{O}_x),\ M \otimes N) \end{array}$$

$\longleftarrow$ relative semi-infinite cohomology

$C^{\frac{\infty}{2}+\bullet}(\mathfrak{g}(K_x),\ G(\mathcal{O}_x),\ M \otimes N)$

informal def'n. ① group cohomology along $G(\mathcal{O}_x)$

② Lie alg. homology along $\mathfrak{g}(K_x)/\mathfrak{g}(\mathcal{O}_x)$.

$\approx$ Hamiltonian reduction w.r.t. $G(\mathcal{O}_x)$

3. DS reduction

idea. $Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g})$ subalgebra but also "quotient".

(Hamiltonian reduction)

ex $G = GL_n$

$$\begin{tikzcd} \mathfrak{g}^* \arrow[d] \\ \mathfrak{g}^*/\!/G \arrow[u, bend left, "S"] \end{tikzcd}$$

$$S = \begin{pmatrix} & & & & * \\ 1 & & & & * \\ & 1 & & & * \\ & & \ddots & & \vdots \\ & & & 1 & * \end{pmatrix}$$

f principal sl_2

$$\begin{tikzcd} \begin{pmatrix} & & & * \\ 1 & & & * \\ & \ddots & & \vdots \\ & & 1 & * \end{pmatrix} \arrow[r] \arrow[rr, bend right, "\sim"] & f + \mathfrak{b} \arrow[r] & f + \mathfrak{b}/N \end{tikzcd}$$

definition

$$\begin{array}{ccccc} C_*(n, - \otimes \mathbb{C}_\psi) & : & g\text{-mod} & \longrightarrow & \mathrm{Vect} \\ \wr\| & & & \searrow & \uparrow \\ \mathrm{Hom}(\mathrm{ind}_n^g |\psi\rangle, -) & & & & \mathrm{End}(\mathrm{ind}_n^g |\psi\rangle)\text{-mod} \end{array}$$

① restict to n

② $\otimes\, \psi$ a generic char ψ

③ Lie algebra (co) homology

$\mathrm{End}(\mathrm{ind}_n^g |\psi\rangle) \cong Z(g)$ ← quantum Hamiltonian picture of $Z(g)$

def'n DS reduction is the functor

$$C^{\frac{\infty}{2}+*}(n(K_x), - \otimes \mathbb{C}_\psi) : g(K_x)\text{-mod} \longrightarrow \mathrm{Vect}$$

↑ derivative of char from Andreas's talk

Lecture 3 (Dima Arinkin)

$$\begin{array}{ccc} \mathrm{Whit}(\mathrm{Gr}_{G,\underline{x}}) & & \underline{x} \in \mathrm{Ran} \\ \mathrm{Poinc}_{\underline{x}} \downarrow \uparrow \mathrm{coeff}* & & \\ \mathrm{Dmod}(\mathrm{Bun}_G) & & \end{array}$$

$$\begin{array}{c} KL(G)_{\underline{x}} \\ \mathrm{Loc}_{G,\underline{x}} \downarrow \uparrow \Gamma_{G,\underline{x}} \\ \mathrm{Dmod}(\mathrm{Bun}_G) \end{array}$$

1) $KL(G)_{\underline{x}}$ - was defined

2) $KL(G)_{\underline{x}}$ is self dual ← will be recapped

3) $C^{\frac{\infty}{2}}(Lg) : KL(G)_{\underline{x}} \longrightarrow \mathrm{Vect}$

$C^{\frac{\infty}{2}}(Ln) : KL(N)_{\underline{x}} \longrightarrow \mathrm{Vect}$

$C^{\frac{\infty}{2}}(Ln, \chi) : KL(N)_{\underline{x}} \longrightarrow \mathrm{Vect}$

G-opers

G-bundles w/ connection on a curve X, $\quad G$ reductive gp

$\cup$

B – Borel

(Beilinson-Drinfeld)

Def: $\mathrm{Oper} = \left(\mathcal{F}_B : B\text{-bundle on } X,\ \nabla\text{- connection on } \mathcal{F}_G = G \overset{B}{\times} \mathcal{F}_B\right)$

discrepancy between $\mathcal{F}_B$ and ∇ sits in the "next term".

Details

$\mathcal{F}_B \rightsquigarrow \underline{\mathrm{Conn}}(\mathcal{F}_B)$ – torsors over $(\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega$

$\downarrow$

$\mathcal{F}_G \quad \underline{\mathrm{Conn}}(\mathcal{F}_G)$ – torsor over $(\mathfrak{g})_{\mathcal{F}_B} \otimes \Omega$

$$\underline{\mathrm{Conn}}(\mathcal{F}_B) \longrightarrow \underline{\mathrm{Conn}}(\mathcal{F}_G) \xrightarrow{\delta} (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega$$

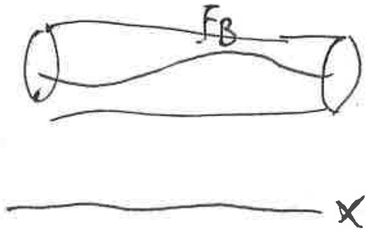
discrepancy

$\cup$

$\frac{d}{dx} +$  $\quad \bigoplus_\alpha (\mathfrak{g}_{-\alpha})_{\mathcal{F}_B} \otimes \Omega = (\mathfrak{g}_{-1}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega$

$\delta(\nabla) \in (\mathfrak{g}_{-1}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega$

and each component in $(\mathfrak{g}_{-\alpha})_{\mathcal{F}_B} \otimes \Omega$ is nowhere vanishing.



$(G/B)_{\mathcal{F}_G}$

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{g}_{-1} \supset \mathfrak{b} \supset \mathfrak{n}$$

$\mathfrak{g}_{-1} = \{x \in \mathfrak{g} : [x, \mathfrak{n}] \subset \mathfrak{b}\}$, $\quad \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$

$$\mathfrak{g}_{-1}/\mathfrak{b} = \bigoplus_{\alpha \text{ simple}} \mathfrak{g}_{-\alpha}$$

<u>Variation</u>. make $\mathcal{F}_B$ a point of Bun_N^{Ω}.

$$\mathrm{Bun}_N^{\Omega} \cong \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{\check{\rho}(\Omega)\}$$

<u>Example</u> $\mathcal{E} =$ rk 3 local system

$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3$

$\mathcal{E}_1/\mathcal{E}_0 = \Omega$, $\mathcal{E}_2/\mathcal{E}_1 = \mathcal{O}$, $\mathcal{E}_3/\mathcal{E}_2 = \Omega^{-1}$

$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega$

$\nabla(\mathcal{E}_i) \subset \mathcal{E}_{i+1} \otimes \Omega \rightsquigarrow (\mathcal{E}_i/\mathcal{E}_{i-1}) \to (\mathcal{E}_{i+1}/\mathcal{E}_i) \otimes \Omega$

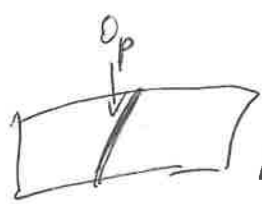
is not just nonzero, but the given iso.

<u>Informally</u>:

<u>Oper</u>. ① $\frac{d}{dt} + \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} dt$ rat'l canonical form for connections

② Opers are f. connections

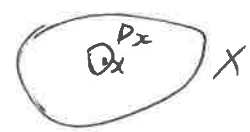
n-th order ODEs --- Systems of n 1st order ODEs.



Global opers $\rightsquigarrow$ D-modules on $\mathrm{Bun}_{\check{G}}$

local opers $\rightsquigarrow$ reps of K-M

Ideally on Ran. <u>Now</u> at a single $x \in X$.



Several spaces

$Op^{reg}(D_x)$	$\frac{d}{dx} + \begin{pmatrix} a_1(x) & a_2(x) & a_3(x) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $a_1, a_2, a_3 \in \mathbb{C}[[x]]$ $\cap$ $\left[\frac{d^3}{dx^3} \pm a_1(x)\frac{d^2}{dx^2} \pm a_2(x)\frac{d}{dx} \pm a_3(x)\right] y(x) = 0$ (extends to $\mathfrak{g}$, consider centralizer of $f \in \mathfrak{g}$)	space is $\mathbb{A}^\infty$ scheme
$Op^{mer}(D_x)$ Opers on punctured disk	$\left[\ \right)$ $a_i \in \mathbb{C}((x))$ $\bigcup_{n \to \infty} x^{-n}\mathbb{C}[[x]]$	Ind-scheme: $\bigcup_n Op^{mer, \leq n}$ is $\mathbb{A}^\infty$
$Op^{mon\text{-}free}(D_x)$ monodromy-free $\shortparallel$ $Op^{mer}(D_x) \times_{LocSys(D_x^\circ)} LocSys(D_x)$	$a_1(x), a_2(x), a_3(x)$ s.t. there are 3 linearly indep solutions in $\mathbb{C}[[x]]$.	Ex ① Formally smooth ② $\bigcup$ closed subschemes $\shortparallel$ $Y_n \times \mathbb{A}^\infty$ finite type

What kind of coherent sheaves live on such spaces $(Y \times \mathbb{A}^\infty)$

We need $\mathrm{IndCoh}^*(Y \times \mathbb{A}^\infty)$

$$= \varinjlim \mathrm{IndCoh}(Y \times \mathbb{A}^n)$$

under $*$ pullbacks

$$\begin{array}{c} Y \times \mathbb{A}^\infty \\ \downarrow \\ Y \times \mathbb{A}^n \\ \downarrow p \\ Y \times \mathbb{A}^m \end{array}$$

Tutorial 2.3 (Kevin Lin) Introduction to factorization

A factorisation space is a prestack $\mathcal{Y}_{Ran} \to Ran$ equipped equipped w/ isoms

$$\mathcal{Y}_{x \sqcup x'} \simeq \mathcal{Y}_x \times \mathcal{Y}_{x'}.$$ for $x, x' \in Ran$ disjoint.

$$[\, \mathcal{Y}_{x_1 \sqcup \cdots \sqcup x_n} \simeq \mathcal{Y}_{x_1} \times \cdots \times \mathcal{Y}_{x_n} \,]$$

A factorisation cat. is a sheaf of cat. $\mathcal{C}_{Ran}$ on Ran w/

$$\mathcal{C}_{x \sqcup x'} \simeq \mathcal{C}_x \otimes \mathcal{C}_{x'}.$$

e.g. $D(\mathcal{Y}_{Ran}) = D(\mathcal{Y})_{Ran}$ for $\mathcal{Y}$ a fact space

$IndCoh^*(\mathcal{Y}_{Ran}) = IndCoh^*(\mathcal{Y})_{Ran}$

for $\mathcal{Y}_{Ran} = Ran$, this cat is denoted $Vect_{Ran}$.

$Ran = \operatorname{colim} X^I_{dR}$

sheaf of cats on $(X_{dR})^I$ is a module cat. over $D(X^I)$.

A factorisation alg. (in a fact. cat. $\mathcal{C}_{Ran}$) is a D-module $\mathcal{A}_{Ran}$ on Ran space (global section of $\mathcal{C}_{Ran}$) w/ $\mathcal{A}_{x \sqcup x'} \simeq \mathcal{A}_x \otimes \mathcal{A}_{x'}$ (in $\mathcal{C}_{x \sqcup x'} \simeq \mathcal{C}_x \otimes \mathcal{C}_{x'}$).

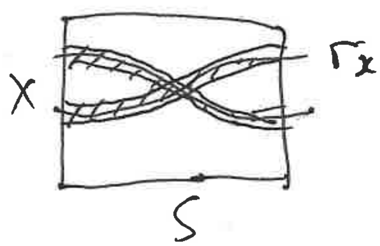
e.g. ω_{Ran} is a factorisation algebra.

A functor $F_{Ran} : \mathcal{C}_{Ran} \to \mathcal{D}_{Ran}$ is a factorization functor if

$$\begin{tikzcd} \mathcal{C}_{x \sqcup x'} \arrow[r, "\sim"] \arrow[d, "F_{x\sqcup x'}"'] & \mathcal{C}_x \otimes \mathcal{C}_{x'} \arrow[d, "F_x \otimes F_{x'}"] \\ \mathcal{D}_{x \sqcup x'} \arrow[r, "\sim"] & \mathcal{D}_x \otimes \mathcal{D}_{x'} \end{tikzcd}$$ commutes.

Prop A fact. functor sends fact. algebras to fact. algebras.

Let x be an S-point of Ran



Γ_x = union of graphs

D_x = formal completion at Γ_x (affine)

$\overset{\circ}{D}_x = D_x \setminus \Gamma_x$.

eg. Let $\mathcal{Y} \to X$ be a prestack, equipped w/ a connection (i.e. descent to X_{dR})

$$\begin{tikzcd} (\mathfrak{L}^+_\nabla \mathcal{Y})_{Ran} = \{x \in Ran,\ \text{horizontal section of } \mathcal{Y} \text{ over } D_x\} \arrow[d] \\ Ran \end{tikzcd}$$

For $\mathcal{Y} = BG \times X$, this produces $(BG)_{Ran}$.

$\mathrm{IndCoh}_*((\mathfrak{L}^+_\nabla \mathcal{Y})_{Ran})$ is a fact. cat.

$\mathcal{O}_{\mathfrak{L}^+_\nabla \mathcal{Y}, Ran}$ is a fact. alg in the fact. cat.

Rmk if $x_1, \ldots, x_n$ are k-points of X,

$$\left(\mathcal{L}^{+}_{\nabla} \mathcal{Y}\right)_{x_1 \sqcup \cdots \sqcup x_n} = \prod_{i=1}^{n} \mathcal{Y}_{x_i}.$$

Assume $\mathcal{Y}$ is affine over X

$$\begin{array}{c} \left(\mathcal{L}_{\nabla} \mathcal{Y}\right)_{\mathrm{Ran}} = \left\{ x \in \mathrm{Ran},\ \text{horizontal section of } \mathcal{Y} \text{ over } \overset{\circ}{D}_x \right\} \\ \downarrow \\ \mathrm{Ran} \end{array}$$

There is a constant map $\iota : \left(\mathcal{L}^{+}_{\nabla} \mathcal{Y}\right)_{\mathrm{Ran}} \to \left(\mathcal{L}_{\nabla} \mathcal{Y}\right)_{\mathrm{Ran}}$

So $\iota_* : \mathrm{IndCoh}_*\left(\mathcal{L}^{+}_{\nabla} \mathcal{Y}\right)_{\mathrm{Ran}} \longrightarrow \mathrm{IndCoh}_*\left(\mathcal{L}_{\nabla} \mathcal{Y}\right)_{\mathrm{Ran}}$ is a fact. functor.

Hence $\iota_* \mathcal{O}_{\mathcal{L}^{+}_{\nabla} \mathcal{Y}, \mathrm{Ran}}$ is a fact. alg. (in the latter cat.)

In general, $\left(\mathcal{L}_{\nabla} \mathcal{Y}\right)_x$ is an ind-scheme of infinite type.

Let x be a point of X.

$$\mathrm{Ran}_x = \{ y \in \mathrm{Ran} : x \in y \}$$

A <u>factorization module</u> for a fact. alg $\mathcal{A}_{\mathrm{Ran}}$ (in a fact. cat. $\mathcal{C}_{\mathrm{Ran}}$)

is a D-module (a global section of $\mathcal{C}_{\mathrm{Ran}_x}$) $\mathcal{M}_{\mathrm{Ran}_x}$ w/

$$\mathcal{M}_{y \sqcup y'} \simeq \mathcal{M}_y \otimes \mathcal{A}_{y'} \quad \text{for } y \in \mathrm{Ran}_x,\ y' \in \mathrm{Ran} \text{ disjoint}$$

$$\mathcal{M}_{x \sqcup y_1 \sqcup \cdots \sqcup y_n} \simeq \mathcal{M}_x \otimes \mathcal{A}_{y_1} \otimes \cdots \otimes \mathcal{A}_{y_n}$$

e.g. $\mathcal{A}_{\mathrm{Ran}_x}$ is a fact. module for $\mathcal{A}_{\mathrm{Ran}}$.

Claim

$$\begin{tikzcd} & \mathcal{O}_{\mathcal{L}^+_\nabla \mathcal{Y}, \mathrm{Ran}}\text{-mod}^{\mathrm{fact}}_x \arrow[d, "\text{fiber at } x"] \\ \mathrm{IndCoh}_*(\mathcal{L}_\nabla \mathcal{Y})_x \arrow[r, "\Gamma"'] \arrow[ur, dashed] & \mathrm{Vect} \end{tikzcd}$$

(conflating $\mathcal{O}_{\mathcal{L}^+_\nabla \mathcal{Y}, \mathrm{Ran}}$ w/ $\Gamma(\mathcal{L}^+_\nabla \mathcal{Y}, \mathcal{O}_{\mathcal{L}^+_\nabla \mathcal{Y}})_{\mathrm{Ran}}$)

Suffices to define a functor

$$\mathrm{ins.vac} : \mathrm{IndCoh}_*(\mathcal{L}_\nabla \mathcal{Y})_x \longrightarrow \mathrm{IndCoh}_*(\mathcal{L}_\nabla \mathcal{Y})_{\mathrm{Ran}_x}$$

which fiberwise performs $\mathcal{F} \mapsto \mathcal{F} \boxtimes \iota_* \mathcal{O}_{\mathcal{L}^+_\nabla \mathcal{Y}, y_1} \boxtimes \cdots \boxtimes \iota_* \mathcal{O}_{\mathcal{L}^+_\nabla \mathcal{Y}, y_n}$

Key input: if $y \in \mathrm{Ran}_x$, we have

$$\overset{\circ}{D}_x \hookrightarrow D_y \setminus x \hookleftarrow \overset{\circ}{D}_y$$

$$\rightsquigarrow \begin{tikzcd} & \mathrm{Sect}_\nabla(D_y \setminus x, \mathcal{Y}) \arrow[dl] \arrow[dr] & \\ (\mathcal{L}_\nabla \mathcal{Y})_x & & (\mathcal{L}_\nabla \mathcal{Y})_y \end{tikzcd}.$$

$$\rightsquigarrow \mathrm{IndCoh}_*(\mathcal{L}_\nabla \mathcal{Y})_x \longrightarrow \mathrm{IndCoh}_*(\mathcal{L}_\nabla \mathcal{Y})_y$$

then allow y to vary.

At $x \sqcup y_1 \sqcup \cdots \sqcup y_n$

$$\overset{\circ}{D}_x \hookrightarrow \overset{\circ}{D}_x \sqcup D_{y_1} \sqcup \cdots \sqcup D_{y_n} \hookleftarrow \overset{\circ}{D}_x \sqcup \overset{\circ}{D}_{y_1} \sqcup \cdots \sqcup \overset{\circ}{D}_{y_n}$$

$$\begin{tikzcd} & (\mathcal{L}_\nabla \mathcal{Y})_x \times (\mathcal{L}^+_\nabla \mathcal{Y})_{y_1} \times \cdots \times (\mathcal{L}^+_\nabla \mathcal{Y})_{y_n} \arrow[dl] \arrow[dr] & \\ (\mathcal{L}_\nabla \mathcal{Y})_x & & (\mathcal{L}_\nabla \mathcal{Y})_x \times (\mathcal{L}_\nabla \mathcal{Y})_{y_1} \times \cdots \times (\mathcal{L}_\nabla \mathcal{Y})_{y_n} \end{tikzcd}$$

Thm. the functor $\mathrm{IndCoh}_*(\mathcal{I}_{\nabla}\mathcal{Y})_x \longrightarrow \mathcal{O}_{\mathcal{L}^+_{\nabla}\mathcal{Y}}\text{-mod}^{\mathrm{fact}}_x$

is an "equivalence" of categories on $>-\infty$ part.

A unital str. on a fact. cat. $\mathcal{C}_{\mathrm{Ran}}$ is:

- a fact. alg. $\mathrm{unit}_{\mathcal{C},\mathrm{Ran}}$ (called factorization unit)
- and the ability to assemble

$$\mathcal{C}_x \longrightarrow \mathcal{C}_x \otimes \mathcal{C}_{y_1} \otimes \cdots \otimes \mathcal{C}_{y_n}, \quad \mathcal{F} \mapsto \mathcal{F} \boxtimes \mathrm{unit}_{\mathcal{C},y_1} \boxtimes \cdots \boxtimes \mathrm{unit}_{\mathcal{C},x_n}$$

into a functor $\mathrm{ins}_{\mathrm{vac}} : \mathcal{C}_x \longrightarrow \mathcal{C}_{\mathrm{Ran}_x}$.

e.g. $\hat{\mathfrak{g}}_\kappa\text{-mod}$ Kac–Moody modules

is a factorization cat, w/ unit $\mathrm{Vac}(G)_\kappa$

$\hat{\mathfrak{g}}_\kappa\text{-mod} \xrightarrow{\mathrm{oblv}} \mathrm{Vect}$ is a fact. functor

$\rightsquigarrow \mathbb{V}_{\mathfrak{g},\kappa} = \mathrm{oblv}(\mathrm{Vac}(G)_\kappa)$ is a fact. alg.

$$\begin{tikzcd} & \mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}_x \arrow[d] \\ \hat{\mathfrak{g}}_\kappa\text{-mod}_x \arrow[r, "\mathrm{oblv}"] \arrow[ur, dashed, "(*)"] & \mathrm{Vect} \end{tikzcd}$$

Thm $(*)$ is an equiv. on $>-\infty$.

Lecture 4 (Yifei Zhao)

Summary. $\exists$ $\underbrace{\text{fact. alg}}_{\text{pretty mild}}$ def'n

Pretty complex notion carries some weight

Factorization algs "$\longleftrightarrow$" Vertex algebras

$\exists$ notion of unital fact. alg.

modules (at $x \in X$)

unital modules

Local theory of fact. algs:

1) $\mathcal{Y}$ affine scheme $/X$ w/ ∇

$\rightsquigarrow$ Fact. alg $\mathcal{O}_{\mathcal{Y}}$

$\mathcal{O}_{\mathcal{Y}}\text{-mod}_x^{fact} \approx$ coherent sheaves on $\mathcal{L}_\nabla \mathcal{Y}_x$] space of horizontal sections on $\overset{\circ}{D}_x$.

corrections: work w/ ev. coconn. modules, IndCoh* on RHS.

2) $\exists$ $\mathrm{Vac}(G)_k$ $\leftarrow$ "Kac–Moody factorization algebra"

fiber of $\mathrm{Vac}(G)_k$ at $x \in X$ is $\mathbb{V}_k^\circ = \mathbb{V}_k$] vacuum rep'n of $\hat{\mathfrak{g}}_k$

$\mathrm{ind}_{\mathfrak{g}[[t]] \oplus k}^{\hat{\mathfrak{g}}_k} (\mathrm{triv})$

$\mathrm{Vac}(G)\text{-mod}_x^{fact} \approx \mathfrak{g}((t))\text{-mod}$

$\uparrow$

really on coconn. objects

w/ k: get modules for central ext'n

Goal. Prove

Thm There is a canonical equiv. of unital factorization categories

$$KL(G)_{crit} \simeq \mathrm{IndCoh}_*\left(\mathrm{Op}^{\text{mon-free}}_{\check{G}}\right) \qquad [\mathrm{FLE}_{crit}]$$

$$KL(G)_{crit} := \left(\hat{\mathfrak{g}}\text{-mod}_{crit}\right)^{L^+G}$$

level: κ is a G-inv't sym. bilinear form $\mathfrak{g}\otimes\mathfrak{g} \to k$.

$$0 \to k\mathbb{1} \to \mathfrak{g}^{\kappa} \to \mathfrak{g}(K_x) \to 0 \quad \text{defined by the cocycle}$$

$$\xi\otimes f,\ \xi'\otimes f' \longmapsto \kappa(\xi,\xi')\,\mathrm{Res}\left((df)\,f'\right)$$

$\mathrm{crit} = -\frac{1}{2}\,\mathrm{Killing}$

Rmk

$$\mathrm{Loc}: KL(G)_{crit} \to \mathrm{Dmod}_{crit}(\mathrm{Bun}_G) \underset{\text{Pfaffian} \leftarrow \text{uses } \omega^{1/2}}{\simeq} \mathrm{Dmod}(\mathrm{Bun}_G)$$

$$\mathrm{Sat}^{nv}: \mathrm{Rep}(\check{G})^{\heartsuit} \underset{\text{sym.}}{\simeq} \mathrm{Dmod}_{crit}(\mathcal{H}^{loc})^{\heartsuit} \simeq \mathrm{Dmod}(\mathcal{H}^{loc})^{\heartsuit}, \text{ modified comm.}$$

$$\begin{tikzcd}
\mathrm{Op}^{reg}_{\check{G}} \arrow[r, "\mathfrak{z}^{+,\text{mon-free}}"] \arrow[rr, bend left, "\mathfrak{z}^{+}"] & \mathrm{Op}^{\text{mon-free}}_{\check{G}} \arrow[r] \arrow[d] & \mathrm{Op}^{mer}_{\check{G}} \arrow[d] \\
& \mathrm{LS}^{reg}_{\check{G}} \arrow[r] & \mathrm{LS}^{mer}_{\check{G}}
\end{tikzcd}$$

units match under FLE_{crit}: $\mathrm{Vac}(G)_{crit} \longmapsto \left(\mathfrak{z}^{+,\text{mon-free}}\right)^{\mathrm{IndCoh}}_{*}\left(\mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}}\right)$

At $x \in X$,

$$H^0 \mathrm{End}\left(\mathrm{Vac}(G)_{crit}\right) \simeq H^0 \mathrm{End}\left(\iota_*^{+,mf}\, \mathcal{O}\right) \simeq \Gamma\left(\mathrm{Op}^{reg}_{\check{G},x}, \mathcal{O}\right)$$

$$\wr\!\!\mathrm{S}$$

$$\mathrm{ind}^{\hat{g}_{crit}}_{g(\mathcal{O}_x)}(k)^{G(\mathcal{O}_x)}$$

(Feigin – Frenkel at $x \in X$)

<u>Step 1</u> Establish FF.

$H^0\mathrm{End}(A^{(I)})$ A classical fact. alg

$\mathfrak{z}_{g,crit}$:= center of the fact. algebra $\mathbb{V}_{g,crit}$ (:= image of $\mathrm{Vac}(G)_{crit}$ under the forgetful functor)

Construct an isom. of $\mathcal{D}_X$-algebras $\mathfrak{z}_{g,crit} \simeq \mathcal{O}_{\mathrm{Op}_{\check{G}}}$

$$\mathrm{Op}_{\check{G}} \xrightarrow{\text{affine}} X_{dR}$$

$$\mathcal{L}^+_\nabla(\mathrm{Op}_{\check{G}}) \simeq \mathrm{Op}^{reg}_{\check{G}}$$

$$\mathcal{L}_\nabla(\mathrm{Op}_{\check{G}}) \simeq \mathrm{Op}^{mer}_{\check{G}}$$

<u>Step 2</u> Construct a functor (enhanced Drinfeld- Sokolov)

$$KL(G)_{crit} \longrightarrow \mathrm{IndCoh}_*\left(\mathrm{Op}^{mer}_{\check{G}}\right) \qquad (DS^{enh})$$

<u>Step 3</u>. (DS^{enh}) factors through an equiv.

$$\begin{tikzcd} KL(G)_{crit} \arrow[r, "FLE_{crit}", "\sim"'] \arrow[dr, "DS^{enh}"'] & \mathrm{IndCoh}_*\left(\mathrm{Op}^{mon\text{-}free}_{\check{G}}\right) \arrow[d, "\iota_*^+"] \\ & \mathrm{IndCoh}_*\left(\mathrm{Op}^{mer}_{\check{G}}\right) \end{tikzcd}$$

Step 1 Construction of (FF)

Reduction to the universal case

$\hat{D}_0 := \mathrm{Spf}\, k[[t]]$

group ind-scheme - $\mathrm{Aut} :=$ automorphism group of $\hat{D}_0$

group scheme - $\mathrm{Aut}^+ :=$ ——— preserving the base point

$$\mathrm{Aut}/\mathrm{Aut}^+ \simeq \hat{D}_0 .$$

$$\begin{array}{l} \hat{X}_{dR} := \{ x \in X_{dR} ;\ \hat{D}_x \simeq \hat{D}_0 \} \\ \big\downarrow \mathrm{Aut} \\ X_{dR} \end{array}$$

pullback to X is induced from an Aut^+-torsor.

$$\mathfrak{z}_{\mathfrak{g},\mathrm{crit},0} := \mathrm{ind}_{\mathfrak{g}[[t]]}^{\hat{\mathfrak{g}}^{\mathrm{crit}}}(k)^{G[[t]]} \quad \circlearrowleft \quad \mathrm{Aut}$$

$\mathrm{Op}_{\check{G},0}$:= affine scheme parametrizing $\check{G}$-opers on $\hat{D}_0$ $\circlearrowleft \mathrm{Aut}$ (actually a double cover of Aut)

Suffices to construct an Aut-equiv. isom. $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},\mathrm{crit},0}) \simeq \mathrm{Op}_{\check{G},0}$.

Birth of opers

$$\underset{\mathrm{Aut}\text{-equiv}}{\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},\mathrm{crit},0}) \longrightarrow \mathrm{Op}_{\check{G},0}} \iff \check{G}\text{-oper on } \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},\mathrm{crit},0}) \times \hat{D}_0 \text{ w/ Aut-equivariance}$$

General paradigm $\quad H^+ \subset H \curvearrowright Y$

$$\begin{tikzcd} Y \overset{H}{\times} H/H^+ \arrow[r, phantom, "\simeq"] \arrow[d] & Y/H^+ \arrow[d] \\ Y \overset{H}{\times} H/H \arrow[r, phantom, "\simeq"] & Y/H \end{tikzcd}$$

Suffices to construct

1) Aut^+-equiv. $P_{\check{B}}$ and $(P_{\check{B}})_{\check{T}} \xrightarrow{\sim} \rho(\Omega)$ over $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},\mathrm{crit},0})$

2) upgrade Aut^+-equiv. on $(P_{\check{B}})_{\check{G}}$ to an Aut-equiv.

The $\check{G}$-bundle over $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},\mathrm{crit},0})$:

$$\mathrm{Rep}\,\check{G} \longrightarrow \mathfrak{z}_{\mathfrak{g},\mathrm{crit},0}\text{-mod}$$

$$V \longmapsto \Gamma^{\mathrm{IndCoh}}(\mathrm{Gr}_G, \mathrm{Sat}^{nv}(V))^{G[[t]]} \qquad [\text{Raskin}]$$

Consequence of the construction:

$$\mathrm{Sat}^{nv}(V) * \mathrm{Vac}(G)_{\mathrm{crit}} \xrightarrow{\sim} ev^*(V) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}}{\otimes} \mathrm{Vac}(G)_{\mathrm{crit}}$$

Step 2

Affine Skryabin theorem (Raskin)

The functor $DS: \hat{\mathfrak{g}}^k\text{-mod} \longrightarrow \mathrm{Vect}$ induces an equiv

$$DS^{enh}: \mathrm{Whit}_*(\hat{\mathfrak{g}}^k\text{-mod}) \xrightarrow{\sim} \left(DS(\mathrm{Vac}(G)_k)\text{-mod}^{\mathrm{fact}}\right)_{\mathrm{ren}}$$

At $x \in X$, $\mathrm{Whit}_* \simeq \underset{n}{\mathrm{colim}}\, \mathrm{Whit}^{\leq n}$, $\mathrm{Whit}^{\leq n}$ is gen. by a single obj. W_k^n. omitted.

At $k=\mathrm{crit}$, $DS(\mathrm{Vac}(G)_{\mathrm{crit}}) \simeq \mathfrak{z}_{\mathfrak{g},\mathrm{crit},0} \overset{FF}{\simeq} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}$ (affine Kostant thm)

Have $DS^{enh}: \hat{\mathfrak{g}}^{crit}\text{-mod} \longrightarrow \mathrm{IndCoh}_*(\mathrm{Op}^{mer}_{\check{G}})$

$$\begin{tikzcd} \hat{\mathfrak{g}}^{crit}\text{-mod} \arrow[r] \arrow[dr] & \mathrm{IndCoh}_*(\mathrm{Op}^{mer}_{\check{G}}) \arrow[d] \\ & \mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}}\text{-mod}^{fact} \end{tikzcd}$$

The desired functor is

$$KL(G)_{crit} := (\hat{\mathfrak{g}}^{crit}\text{-mod})^{L^+G} \longrightarrow \hat{\mathfrak{g}}^{crit}\text{-mod} \xrightarrow{DS^{enh}} \mathrm{IndCoh}_*(\mathrm{Op}^{mer}_{\check{G}}).$$

Bonus Material B (Dennis Gaitsgory)

1) $\mathrm{Sph}_G \otimes KL \xrightarrow{*} KL$

$V \in \mathrm{Rep}(\check{G})$, $\boxed{\mathrm{Sat}^{nv}(V) * \mathbb{V}\mathrm{ac}_{crit} \simeq \tau^*(V) \underset{\mathfrak{Z}}{\otimes} \mathbb{V}\mathrm{ac}_{crit}}$

$$\mathrm{Op}_{\check{G}} \xrightarrow{\tau} LS^{reg}_{\check{G}} \simeq pt/\check{G}$$

2) $\mathbb{V}\mathrm{ac}_{crit} \in KL(G)_{crit} \longrightarrow KM(G)_{crit}$

$DS : KM(G)_{crit} \longrightarrow \mathrm{Vect}$

$$\mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}} \overset{FF}{\simeq} \mathfrak{Z} \xrightarrow{\sim} DS(\mathbb{V}\mathrm{ac}_{crit})$$

<u>Thm</u>

3) DS automatically enhances to a functor

$$\begin{tikzcd} DS^{enh}: KM(G)_{crit} \arrow[r] & DS(\mathbb{V}\mathrm{ac}_{crit})\text{-mod}^{fact} \\ KL(G)_{crit} \arrow[u] & \mathfrak{Z}\text{-mod}^{fact} \arrow[u, phantom, "\wr"] \end{tikzcd}$$

$\overset{FLE}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{mon\text{-}free}_{\check{G}}) \xrightarrow{\text{Page 50}} \mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}}\text{-mod}^{fact} \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}})$

($\mathfrak{Z}\text{-mod}^{fact} \overset{FF}{\simeq} \mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}}\text{-mod}^{fact}$)

$LG \qquad C$

$$\mathrm{Sph}(C) = C^{L^+G}$$

Ex. $C = KM$, $\mathrm{Sph}(KM) = KL$

$C = \mathrm{Dmod}(LG)$, $\mathrm{Sph}(C) = \mathrm{Dmod}(LG/L^+G)$

$$\mathrm{Sph}_G = \mathrm{Dmod}(L^+G \backslash LG / L^+G) \curvearrowright \mathrm{Sph}(C).$$

$$C^{\text{sph-gen}} := \mathrm{Dmod}(\mathrm{Gr}_G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(C) \rightleftarrows C$$

$\mathrm{Whit}^!(C)$ and $\mathrm{Whit}_*(C)$.

Ex $C = \mathrm{Dmod}(\mathrm{Gr}_G)$, $\mathrm{Whit}_*(\mathrm{Dmod}(\mathrm{Gr}_G)) = \mathrm{Whit}_*(G)$.

$\mathrm{Whit}_*(KM)$

$$\begin{tikzcd} KM \arrow[rr, "DS^{enh}"] \arrow[dr] & & \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}}) \\ & \mathrm{Whit}_*(KM) \arrow[ur, "\overline{DS}^{enh}"'] & \end{tikzcd}$$

Thm (Raskin) $\overline{DS}^{enh}$ is an equiv.

$$\mathrm{Whit}(G) \underset{\mathrm{Sph}_G}{\otimes} C^{L^+G} \simeq \mathrm{Whit}_*(C^{\text{sph-gen}}) \hookrightarrow \mathrm{Whit}_*(C)$$

$\rightleftarrows$

Derived Satake

$$\begin{tikzcd} \mathrm{Rep}(\check{G}) \arrow[r, "\mathrm{Sat}^{nv}"] \arrow[d] & \mathrm{Sph}_G \arrow[dl, "\mathrm{Sat}", "\sim"'] \\ \mathrm{Sph}^{spec}_{\check{G}} := \mathrm{IndCoh}(LS^{reg}_{\check{G}} \times_{LS^{mer}_{\check{G}}} LS^{reg}_{\check{G}}) & \end{tikzcd}$$

$$\begin{tikzcd} & pt/\check{G} \arrow[d] \\ pt/\check{G} \times_{\check{\mathfrak{g}}^{\wedge}_0/\check{G}} pt/\check{G} \simeq & pt/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} pt/\check{G} \end{tikzcd}$$

$$\mathrm{Sph}^{\mathrm{Spec}}_{\check{G}} \longrightarrow \mathrm{Sph}^{\mathrm{Spec}}_{\check{G},\mathrm{temp}}$$

$$\mathrm{IndCoh}(\quad) \xrightarrow{\Psi} \mathrm{QCoh}(\quad)$$

$$\mathrm{Sph}^{\mathrm{Spec}}_{\check{G}} \curvearrowright \tilde{C} \rightsquigarrow \tilde{C}_{\mathrm{temp}} := \mathrm{Sph}^{\mathrm{Spec}}_{\check{G},\mathrm{temp}} \underset{\mathrm{Sph}^{\mathrm{Spec}}_{\check{G}}}{\otimes} \tilde{C}$$

$$\mathrm{QCoh}((\mathrm{LS}^{\mathrm{um}}_{\check{G}})^{\wedge}_{\mathrm{reg}}) \curvearrowright \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \curvearrowleft \mathrm{Sph}^{\mathrm{Spec}}_{\check{G},\mathrm{temp}}$$

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \simeq \mathrm{Rep}(\check{G})$$

This bimodule defines a Morita equivalence.

$$\mathrm{Whit}^!(G) \overset{CS}{\simeq} \mathrm{Rep}(\check{G})$$

$$\boxed{\mathrm{Whit}_*(G) \overset{\mathrm{FLE}_\infty}{\simeq} \mathrm{Rep}(\check{G})}$$

$$\mathrm{Sph}_G \curvearrowright \qquad \mathrm{Sph}^{\mathrm{Spec}}_{\check{G}} \curvearrowright$$

$$\mathrm{Sph}_G \xrightarrow[\sim]{\mathrm{Sat}} \mathrm{Sph}^{\mathrm{Spec}}_{\check{G}}$$

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \underset{\mathrm{Sph}^{\mathrm{Spec}}_{\check{G}}}{\otimes} \mathrm{Sph}(C) \overset{①}{\simeq} \mathrm{Whit}_*(C^{\mathrm{Sph\text{-}gen}})$$

$$\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G}})^{\wedge}_{\mathrm{reg}})$$

Cor. $\mathrm{Sph}(C)_{\mathrm{temp}} \overset{②}{\simeq} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \underset{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G}})^{\wedge}_{\mathrm{reg}})}{\otimes} \mathrm{Whit}_*(C^{\mathrm{sph\text{-}gen}})$

Apply ① to $C = KM$

$$\begin{tikzcd} \mathrm{QCoh}(LS^{reg}_{\check{G}}) \otimes KL \arrow[r, "\simeq"] \arrow[dr, "\mathrm{Sph}^{spec}_{\check{G}}"', "\sim"] & \mathrm{Whit}_*(KM^{sph\text{-}gen}) \arrow[r, hook] \arrow[d, "\wr"] & \mathrm{Whit}_*(KM) \arrow[d, "\wr"] \\ & \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}})_{mf} \arrow[r, hook] & \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}}) \end{tikzcd}$$

$$\mathrm{Op}^{mf}_{\check{G}} = \coprod_{\check{\lambda} \in \Lambda^+} \mathrm{Op}^{\check{\lambda}\text{-}reg, \sim}_{\check{G}}$$

$\mathbb{V}^{\check{\lambda}} \in KL$, $DS(\mathbb{V}^{\check{\lambda}}) \simeq \mathcal{O}_{\mathrm{Op}^{\check{\lambda}\text{-}reg}_{\check{G}}}$

Cor. $KL_{temp} \simeq \mathrm{QCoh}(LS^{reg}_{\check{G}}) \underset{\mathrm{QCoh}((LS^{mer}_{\check{G}})^\wedge_{reg})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}})_{mf}$

Recall. want $KL \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{mf}_{\check{G}})$

$$\begin{tikzcd} \mathrm{Op}^{mf}_{\check{G}} \arrow[r] \arrow[d] & (\mathrm{Op}^{mer}_{\check{G}})^\wedge_{mf} \arrow[d] \\ LS^{reg}_{\check{G}} \arrow[r] & (LS^{mer}_{\check{G}})^\wedge_{reg} \end{tikzcd}$$

$$\mathrm{IndCoh}^*(\mathrm{Op}^{mf}_{\check{G}}) \simeq \mathrm{QCoh}(LS^{reg}_{\check{G}}) \underset{\mathrm{QCoh}((LS^{mer})^\wedge_{reg})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}})_{mf}$$

Prop. $KL \xrightarrow{\sim} KL_{temp}$

Tutorial 3.1 (Sam Raskin) Kac-Moody localization

Finite dim'l analogue:

H affine alg gp, $K \subset H$ subgp

$$\begin{tikzcd} H \curvearrowright \tilde{\mathcal{Y}} \arrow[d] \\ \mathcal{Y} \end{tikzcd} \leftarrow \text{stack/sm. variety}$$

$\mathcal{Y} = \tilde{\mathcal{Y}}/K$

Construction: $\mathrm{Loc}_{\mathcal{Y}} : \underbrace{\mathfrak{h}\text{-mod}^K}_{\substack{K\text{-integrated} \\ \mathfrak{h}\text{-modules}}} \longrightarrow \text{D-mod}(\mathcal{Y})$

$H^\wedge_K$ reps $\Leftrightarrow$ $\mathfrak{h}\text{-mod}^K$.

First. $\mathfrak{h}\text{-mod} \longrightarrow \text{D-mod}(\tilde{\mathcal{Y}})$

There is a map $\underset{\substack{\uparrow \\ \text{Lie alg map}}}{\mathfrak{h} \longrightarrow}$ Vect fields on $\tilde{\mathcal{Y}}$ b/c $H \curvearrowright \tilde{\mathcal{Y}}$.

$$\mathcal{U}(\mathfrak{h}) \longrightarrow \Gamma(\tilde{\mathcal{Y}}, \mathrm{Diff}_{\tilde{\mathcal{Y}}}) = \mathrm{End}(\mathrm{Diff}_{\tilde{\mathcal{Y}}})$$

Given $M \in \mathfrak{h}\text{-mod}$, $\mathrm{Loc}(M) = M \overset{\leftrightarrow}{\underset{\mathcal{U}(\mathfrak{h})}{\otimes}} \mathrm{Diff}_{\tilde{\mathcal{Y}}} \in \text{D-mod}(\tilde{\mathcal{Y}})$

$\mathrm{Loc}(\mathcal{U}(\mathfrak{h})) = \mathrm{Diff}_{\tilde{\mathcal{Y}}}$.

Correspondence-ly:

$$\begin{tikzcd} & \tilde{\mathcal{Y}}/H^\wedge \arrow[dl] \arrow[dr] & \\ \tilde{\mathcal{Y}}_{dR} & & \mathbb{B}H^\wedge \end{tikzcd}$$

] K_{dR} acts here; quotient by K_{dR} to get HC version

<u>Second</u>: Pass to K-equiv. objects

Obtain $\quad \mathfrak{h}\text{-mod}^K \xrightarrow{\text{Loc}} \text{Dmod}(\tilde{\mathcal{Y}})^K = \text{D-mod}(\tilde{\mathcal{Y}}/K) = \text{D-mod}(\mathcal{Y})$

Concretely, $\quad M \in \mathfrak{h}\text{-mod}^K, \quad$ can see $\quad \text{Loc}(M) \in \text{D-mod}(\tilde{\mathcal{Y}})$

HC str. $\rightsquigarrow$ descent datum

<u>Example</u> $\quad K = H, \quad \mathfrak{h}\text{-mod}^H = \text{Rep}\, H \longrightarrow \text{Dmod}(\mathcal{Y})$

$V \longmapsto \text{Diff}_{\mathcal{Y}} \otimes V_{\mathcal{Y}}$

$$\begin{array}{c} \tilde{\mathcal{Y}} \\ \downarrow \\ \mathcal{Y} \longrightarrow BH \end{array}$$

$V_{\mathcal{Y}} :=$ pullback of V

a.k.a. assoc. bundle

<u>Generalization</u>:

$V \in \text{Rep}\, K, \quad \text{ind}^{\mathfrak{h}}_{\text{Lie}K}(V) \in \mathfrak{h}\text{-mod}^K$

Loc sends this to D-modules induced from $V_{\mathcal{Y}}$. $\quad \mathcal{Y} \xrightarrow{\tilde{\mathcal{Y}}} BK$

$\tilde{\mathcal{Y}} = H/H_0 \quad$ for some $\quad H_0 \subset H$ subgroup, $\quad \mathcal{Y} = K\backslash H/H_0$

fiber of $\text{Loc}(M)$ at $1 \in K\backslash H/H_0$ is $C_\cdot(\text{Lie}\, H_0, M) \leftarrow$ derived coinvariants.

Loc has a right adjoint (if $\mathcal{Y}$ is quasi-cpt)

$\Gamma \simeq \Gamma(\tilde{\mathcal{Y}}, \pi^!(-))$

<u>Ex</u> $\quad \Gamma(\delta_{1 \in H/H_0}) = \text{ind}^{\mathfrak{h}}_{\mathfrak{h}_0}(\text{triv})$

<u>Formalism of Loc</u>

<u>Setup</u> $\quad \mathcal{Y} = \text{Bun}_G, \quad x \in X \;$ (or $x \in \text{Ran}$)

$$\mathcal{Y} = \mathrm{Bun}_G^{\mathrm{lvl},x} = \{P_G \text{ on } X\setminus x \text{ w/ triv. on } \mathring{D}_x \subset X\setminus x\}$$

$\curvearrowleft$

$H = L_x G$ $\qquad \mathrm{Bun}_G = \mathrm{Bun}_G^{\mathrm{lvl},x} / L_x^+ G$

Origin of Hecke action

∞-dim'l version of earlier discussion:

$$\mathrm{Loc}: KL(G) \longrightarrow D\text{-mod}(\mathrm{Bun}_G)$$

or $\quad \mathrm{Loc}: KL(G)_k \longrightarrow D\text{-mod}_k(\mathrm{Bun}_G)$

Ditto for any $\mathcal{U} \subset \mathrm{Bun}_G$ (in practice, $\mathcal{U}$ is quasi-cpt)

$$\begin{tikzcd}
\mathrm{Bun}_G(\hat{D}_x) = \mathbb{B}L_x^+G & \mathrm{Rep}(L_x^+G) \arrow[r, "\mathrm{ind}"] \arrow[d, "\text{evaluation bundles}"] & KL(G)_k \arrow[d] \\
\mathrm{Bun}_G \arrow[u] & \mathrm{QCoh}(\mathrm{Bun}_G) \arrow[r, "\mathrm{ind}"] & D\text{-mod}_k(\mathrm{Bun}_G)
\end{tikzcd}$$

tensor w/ Diff_k

commutes

Equivariant for action of $\mathrm{Sph}_k(G)_x = D\text{-mod}_k(L^+G\backslash LG/L^+G)$

<u>Example</u> $\mathrm{Loc}(\mathbb{V}_k) = \mathrm{Diff}_k$

$\qquad \mathbb{V}_k = \mathrm{ind}(\mathrm{triv}) \qquad \mathrm{Diff}_k = \mathrm{ind}(\mathcal{O})$

<u>Example</u> $\mathbb{V}^\lambda_{k,x} = \mathrm{ind}_{\mathfrak{g}[[t]]}^{\hat{\mathfrak{g}}_k}(V^\lambda)$

$\mathrm{Loc}(\mathbb{V}^\lambda) =$ (twisted) D-module induced from $\mathcal{E}^\lambda_x$ on Bun_G

$$\begin{tikzcd}
\mathcal{P}^{\mathrm{univ}} \arrow[d] & \mathcal{P}_x^{\mathrm{univ}} \arrow[l] \arrow[d] \\
X \times \mathrm{Bun}_G & \mathrm{Bun}_G \arrow[l, "x \times \mathrm{id}"']
\end{tikzcd}$$

twist V^λ by $\mathcal{P}_x^{\mathrm{univ}}$ to get $\mathcal{E}^\lambda_x$.

<u>Application (BD)</u>: $k = \text{crit}$. $\mathfrak{Z}_x \curvearrowright$ - by functoriality

$\text{Diff}_{\text{Bun}_G, \text{crit}} = \text{Loc}(\mathbb{V}^{x}_{\text{crit}})$

Choose $x \in X$

$$\text{Fun}(\text{Op}^{\text{reg}}_{\check{G}, x}) = \mathfrak{Z}_x \curvearrowright \mathbb{V}_{\text{crit}, x}$$

<u>Rmk</u> endos of Diff $\Leftrightarrow$ global diff'l ops

says: elements of $\mathfrak{Z}_x$ $\rightsquigarrow$ global diff. ops on Bun_G.

<u>Rmk</u> (Preview) : Action of $\mathfrak{Z}_x$ factors through an action of $\text{Fun}(\text{Op}_{\check{G}}(X))$.

<u>Further</u>. Birth of opers Yoga:

$$\text{Sat}^{nv}(V) * \mathbb{V}_{\text{crit}, x} \simeq \mathcal{V}_{\text{Op}^{\text{reg}}} \underset{\mathfrak{Z}_x}{\otimes} \mathbb{V}_{\text{crit}, x}$$

<u>Deduce</u>. $\text{Sat}^{nv}_{x}(V) * \text{Diff}_{\text{crit}} \simeq \mathcal{V}_{\text{Op}^{\text{reg}}_x} \underset{\mathfrak{Z}_x}{\otimes} \text{Diff}_{\text{crit}}$

Hecke eigenproperty (in simplified form)

<u>Aside</u>: If you wanted an "honest" eigensheaf, believe $\text{Fun}(\text{Op}_{\check{G}}(X))$ act on $\text{Diff}_{\text{crit}}$. Fix $\sigma \in \text{Op}_{\check{G}}(X)$, quotient $\text{Diff}_{\text{crit}}$ by corr. max'l ideal, result call $\mathcal{F}_\sigma$

<u>Fact</u>. $\mathcal{F}_\sigma$ is holonomic (in fact, regular, irred. if G is s.c.)

<u>Tutorial 3.2 (Nick Rozenblyum)</u> <u>Factorization algebras</u>

<u>Recall</u> X smooth <u>proper</u> conn'd curve

$\text{Ran}(X)$ prestack, $\text{Ran}(X)(S) = \left\{ \begin{array}{l} \text{non empty finite} \\ \text{subsets of } X(S^{\text{red}}) \end{array} \right\}$

$$\mathrm{Ran}(X) = \underset{I \in (\mathrm{fSurj})^{op}}{\mathrm{colim}} X^I_{dR}$$

$\mathrm{QCoh}(\mathrm{Ran}(X))$ has some important features

1) $\pi : \mathrm{Ran}(X) \longrightarrow pt$

$\pi^!$ has a left adjoint $\pi_!$

2) $\mathrm{Ran}(X)$ is <u>contractible</u> $\Rightarrow$ $\underbrace{\pi_! \pi^!}_{H_*(\mathrm{Ran}) \simeq k} \xrightarrow{\sim} id$

$\exists$ sub prestack $(\mathrm{Ran} \times \mathrm{Ran})_{disj} \subset \mathrm{Ran} \times \mathrm{Ran}$

Ran is a semi-group. $\quad U : \mathrm{Ran} \times \mathrm{Ran} \longrightarrow \mathrm{Ran}$

A factorization alg is (roughly) the following data:

1) $\mathcal{A} \in \mathrm{Dmod}(\mathrm{Ran})$

2) factorization isos

$$(U^! \mathcal{A})\big|_{(\mathrm{Ran} \times \mathrm{Ran})_{disj}} \simeq (\mathcal{A} \boxtimes \mathcal{A})\big|_{(\mathrm{Ran} \times \mathrm{Ran})_{disj}}$$

+ higher coherence

$$\boxed{\overset{\mathcal{A}_x}{x} \rightarrow \ \leftarrow \overset{\mathcal{A}_y}{y}}$$

Def. Let A be a factorization alg. The factorization (a.k.a. chiral) homology of A is $C_\bullet^{\mathrm{Fact}}(X, A) = \pi_!(A^{(\mathrm{Ran})})$

Examples 1) A comm. fact alg

$$A \in \mathrm{Comm\,Alg}(\mathrm{Dmod}(X)) \xrightarrow{FA} \underset{\substack{\text{cat. of fact algs} \\ \shortparallel}}{\mathrm{Fact\,Alg}(X)}$$

Fact (to be explained) A comm. fact. alg

$C_\bullet^{\mathrm{Fact}}(X, A)$ is a comm. alg and

$\mathrm{Spec}\, C_\bullet^{\mathrm{Fact}}(X, A) =$ (derived) space of horizontal sections of $\mathrm{Spec}\, A \longrightarrow X$.

Equiv $\mathrm{Maps}_{\mathrm{ComAlg}}(C_\bullet^{\mathrm{Fact}}(X, A), B) \simeq \mathrm{Maps}_{\mathrm{Comm\,Alg}(\mathrm{Dmod}(X))}(A, B \otimes \omega_X)$

Ex. $A = \mathcal{O} \otimes k[t]$ constant scheme $X \times \mathbb{A}^1 \longrightarrow X$

then $C_\bullet^{\mathrm{Fact}}(X, A) = \mathrm{Sym}(H^\bullet_{dR}(X))^\vee$

2) Kac - Moody (at level 0)

$\mathrm{Vac}(\mathfrak{g}) \leftarrow$ fact alg corresponding to Kac-Moody

$C_\bullet^{\mathrm{Fact}}(X, \mathrm{Vac}(\mathfrak{g})) \simeq C_\bullet^{\mathrm{Lie}}(H^\bullet(X) \otimes \mathfrak{g})$

Exercise deduce 2) in the abelian case from 1)

Obs $\mathcal{P} \in \mathrm{Bun}_G$, $\quad \mathrm{Pitt}_{\mathcal{P}} \simeq C_\cdot^{Lie}\big(H^1(X, \mathfrak{g}_{\mathcal{P}})\big)$

Units

What is a unital factorization algebra?

Version 0. $\exists$ a "unit" alg $\omega_{\mathrm{Ran}\,X}$.

Can ask $\omega_{\mathrm{Ran}_X} \longrightarrow \mathcal{A}$.

A unit is more!

Rough idea for every $x \in X$, have a map $k \to \mathcal{A}_x$ compatible w/ factorization

e.g. $x \neq y \in X$, $\quad \mathcal{A}_x \longrightarrow \mathcal{A}_{\{x,y\}} \simeq \mathcal{A}_x \otimes \mathcal{A}_y$

Idea. remember the inclusion relation on subsets.

Want to consider categorical prestacks (a.k.a. prestacks valued in cats)

i.e. functors $(\mathrm{Sch}^{\mathrm{aff}})^{op} \longrightarrow \mathrm{Cat}$

Main example.

$$\mathrm{Ran}^{\mathrm{unit}}(X)(S) = \left\{ \begin{matrix} \text{poset of finite} \\ \text{subsets of } X(S^{\mathrm{red}}) \end{matrix} \right\}$$

If $\mathcal{X}$ is a categorical prestack, can consider $\mathrm{QCoh}(\mathcal{X})$, $\mathrm{IndCoh}(\mathcal{X})$, $\mathrm{Dmod}(\mathcal{X})$

Concretely, $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ is the data of

- $\forall\, x: S \to \mathcal{X}$, $\quad \mathcal{F}_{S,x} \in \mathrm{QCoh}(S)$
- $\alpha: x \to x' \in \mathcal{X}(S)$, $\quad \mathcal{F}_{S,x} \longrightarrow \mathcal{F}_{S,x'}$
- Compatibility w/ composition

Def A unital factorization algebra $\mathcal{A}$ is the data of

1) $\mathcal{A}^{(Ran)} \in Dmod(Ran^{unit})$

2) $\cup^! (\mathcal{A}^{(Ran)})\Big|_{(Ran^{unit})^2_{disj}} \simeq (\mathcal{A}^{(Ran)} \boxtimes \mathcal{A}^{(Ran)})\Big|_{(Ran^{unit})^2_{disj}}$

3) $\mathcal{A}_\phi \simeq k$

...

Consider some k-points of $Ran^{unit}(X)$

$J \subset X$ finite subset

$\mathcal{A}_J \simeq \bigotimes_{j \in J} \mathcal{A}_{\{j\}}$ factorization

$J \subset J'$

$\mathcal{A}_J \to \mathcal{A}_{J'}$

e.g. $J = \phi$, $J' = \{x\}$ $\to k \to \mathcal{A}_x$

& $\mathcal{A}_{\{x\}} \longrightarrow \mathcal{A}_{\{x,y\}}$.

Factorization homology

Ran^{unit} can also be expressed in terms of X^I.

$\Rightarrow \pi^{\text{unit},!}: Vect \to Dmod(Ran^{unit})$ has a left adjoint $\pi_!^{\text{unit}}$

Def $\mathcal{A}$ is a unital fact. alg $C_\cdot^{fact}(X, \mathcal{A}) = \pi_!(\mathcal{A}^{(Ran)})$

$\exists$ nat'l map

$t : \mathrm{Ran} \to \mathrm{Ran}^{\mathrm{unit}}$

$t^! : FA^{\mathrm{untl}}(X) \longrightarrow FA(X)$

Cor of contractibility of Ran: $\forall\, \mathcal{F} \in \mathrm{Dmod}(\mathrm{Ran}^{\mathrm{untl}})$

$$\pi_! t^! \mathcal{F} \simeq \pi_!^{\mathrm{untl}} \mathcal{F}.$$

Categorical version

Have unital fact. cat. defined in the evident way.

More concretely, if $\mathcal{Y}$ is a categorical prestack, we can think of sheaves on $\mathcal{Y}$

as follows:

$$\begin{array}{c} \vdots \\ \mathrm{Maps}([1], \mathcal{Y})^{\mathrm{grpd}} \quad \text{"prestack of morphisms"} \\ s \downarrow \downarrow t \\ \mathcal{Y}^{\mathrm{grpd}} = \text{"prestack of objects"} \end{array}$$

$\mathcal{Y}^{\mathrm{grpd}}(S) = \mathcal{Y}(S)^{\mathrm{grpd}}$

$\mathrm{Maps}([1], \mathcal{Y})^{\mathrm{grpd}} = \left\{ \begin{array}{c} x, y \in \mathcal{Y}(S) \\ x \to y \end{array} \right\}$

- sheaf on $\mathcal{Y}$ is the data of a sheaf $\mathcal{F}$ on $\mathcal{Y}^{\mathrm{grpd}}$ w/ $s^! \mathcal{F} \to t^! \mathcal{F}$ + compatibilities

Apply this to $\boxed{\mathrm{Ran}^{\mathrm{untl}}}$

$(\mathrm{Ran}^{\mathrm{untl}})^{\mathrm{grpd}} = \mathrm{Ran}\ (\amalg\, \phi)$

$$\mathrm{Maps}([1], \mathrm{Ran}^{\mathrm{untl}})^{\mathrm{grpd}}(S) = \left\{ \begin{array}{c} \underline{x}, \underline{x}' \in X(S_{\mathrm{red}}) \\ \underline{x} \subset \underline{x}' \end{array} \right\} = \mathrm{Ran}_{\subseteq}$$

$$\begin{tikzcd} & \underline{\mathrm{Ran}}_{\underline{c}} \arrow[dl, "p_{small}"'] \arrow[dr, "p_{big}"] & \\ \mathrm{Ran} & & \mathrm{Ran} \end{tikzcd}$$

<u>Upshot</u>: $\mathcal{C}$ is a fact cat. The data of unitality on $\mathcal{C}$ is given by

$$p_{small}^*(\underline{\mathcal{C}}) \longrightarrow p_{big}^*(\underline{\mathcal{C}}) + \cdots$$

<u>Fact</u> p_{small}^* has a right adjoint $p_{small,*}$

Can rewrite the str. as a map $\underline{\mathcal{C}} \to p_{small\,*}\, p_{big}^*(\underline{\mathcal{C}})$

More explicitly, suppose $\underline{x} \subset \underline{x}'$,

$\Rightarrow$ ins. unit$_{\underline{x} \subset \underline{x}'}$: $\mathcal{C}_{\underline{x}} \longrightarrow \mathcal{C}_{\underline{x}'}$.

<u>Example</u> 1) KL

2) A fact alg. $\exists$ (weak) fact. cat. $A\text{-mod}^{fact}$

ins. unit$_{\phi \subset \{x\}}$ $(k) = A_x$

Let $\underline{\mathcal{C}}^{loc}$ be a sheaf of cats on Ran^{untl}

$\mathcal{C}^{glob}$ a category

<u>Def</u> A unital local-to-global functor is a map

$$\underline{\mathcal{C}}^{loc} \longrightarrow \underline{\mathrm{Dmod}}_{*}(\mathrm{Ran}^{untl}) \otimes \mathcal{C}^{glob}.$$

Note: what happens w/ units:

ins. unit$_{\underline{x} \subset \underline{x}'}$: $\mathcal{C}^{loc}_{\underline{x}} \to \mathcal{C}^{loc}_{\underline{x}'}$

unital str.

$$\begin{tikzcd} \mathcal{C}^{loc}_{\underline{x}} \arrow[r] \arrow[dr, "F_{\underline{x}}"'] & \mathcal{C}^{loc}_{\underline{x}'} \arrow[d, "F_{\underline{x}'}"] \\ & \mathcal{C}^{glob} \end{tikzcd}$$

$(*)$ $F_{\underline{x}} \xrightarrow{\sim} F_{\underline{x}'} \circ \text{ins. unit}_{\underline{x} \subset \underline{x}'}$

Can weaken this to a "lax unital str.":

where $(*)$ need not be an isom.

$$\mathrm{Fun}^{loc-glob, unit}(\underline{\mathcal{C}}^{loc}, \mathcal{C}^{glob}) \hookrightarrow \mathrm{Fun}^{loc-glob, lax-untl}(\underline{\mathcal{C}}^{loc}, \mathcal{C}^{glob}) \qquad (**)$$

Claim $(**)$ has a left adjoint given by $F \mapsto F^{\int \text{ins. unit}}$

$$F^{\int \text{ins. untl}} : \mathcal{C}^{loc}_{Z} \xrightarrow{\text{ins. untl}} \mathcal{C}^{loc}_{Z^{\leq}} \xrightarrow{F} \mathcal{C}^{glob} \otimes \mathrm{Dmod}(Z^{\leq}) \xrightarrow{(p_{small})_!} \mathcal{C}^{glob} \otimes \mathrm{Dmod}(Z)$$

for $Z \to \mathrm{Ran}$

where $Z^{\leq} = Z \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{\subseteq}$, $\mathrm{Ran}_{\subseteq} \xrightarrow{p_{big}} \mathrm{Ran}$

Key example

$\mathcal{C}^{loc} = A\text{-mod}^{fact}$ for a unital fact. alg A

$F : A\text{-mod}^{fact} \longrightarrow \mathrm{Vect} = \mathcal{C}^{glob}$

$F(M) = \mathrm{oblv}(M)$

Claim $F^{\int \text{ins. untl}} \simeq C^{Fact}_{\cdot}(X, A, -)$

for $Z \longrightarrow Ran$,

$$F^{\int \text{ins. untl}}_{Z}(M) = p_{small,!}(M_{Z^{\le}}) \in Dmod(Z)$$

$$p_{small} : Z^{\le} \longrightarrow Z \qquad\qquad f : Z \longrightarrow Ran$$

$$Ran_{f(z)} \longrightarrow \underset{z}{\overset{\in}{}}$$

M_x

$$F \longrightarrow F^{\int \text{ins. untl}}$$

$$\Rightarrow M_x \longrightarrow C^{Fact}_{\cdot}(X, A, M)$$

$M_x \quad N_y$

$$M_x \otimes N_y \longrightarrow C^{Fact}_{\cdot}(X, A, M_x \otimes N_y)$$

if $N_y \simeq A$, then $C^{Fact}_{\cdot}(X, A, M_x \otimes N_y) \simeq C^{Fact}_{\cdot}(X, A, M_x)$.

Set up

$\underline{\mathcal{C}}^{loc} \in ShvCat(Ran^{untl})$

$\mathcal{C}^{glob}$

(lax-) unital local-to-global functor $\quad F : \underline{\mathcal{C}}^{loc} \to \mathcal{C}^{glob}$

$\underline{x} \in Ran(X)$, $F_{\underline{x}} : \underline{\mathcal{C}}^{loc}_{\underline{x}} \to \mathcal{C}^{glob}$

$$\underline{x} \subset \underline{x}' : \quad \begin{tikzcd} \mathcal{C}^{loc}_{\underline{x}} \arrow[rr, "\text{ins. unit}"] \arrow[dr] & \arrow[d, phantom, "\Rightarrow"] & \mathcal{C}^{loc}_{\underline{x}'} \arrow[dl] \\ & \mathcal{C}^{glob} & \end{tikzcd}$$

$$\mathcal{Z} \xrightarrow{\text{diag}} \mathcal{Z}_{\le} = \mathcal{Z} \underset{\text{Ran}}{\times} \text{Ran}_{\le} \quad (\text{Ran}_{\le} \xrightarrow{p_{\text{small}}} \text{Ran})$$

$$\begin{tikzcd}
\mathcal{C}^{\text{loc}}_{\mathcal{Z}} \arrow[r, "\text{ins. unit}"] \arrow[drr, equal] & \mathcal{C}^{\text{loc}}_{\mathcal{Z}_{\le}} \arrow[r, "F"] \arrow[d, "\text{diag}^!"] & \mathcal{C}^{\text{glob}} \otimes \text{Dmod}(\mathcal{Z}_{\le}) \arrow[r, "(p_{\text{small}})_!"] \arrow[d, "\text{diag}^!"] & \mathcal{C}^{\text{glob}} \otimes \text{Dmod}(\mathcal{Z}) \arrow[dl, equal] \\
& \mathcal{C}^{\text{loc}}_{\mathcal{Z}} \arrow[r, "F"] & \mathcal{C}^{\text{glob}} \otimes \text{Dmod}(\mathcal{Z}) &
\end{tikzcd}$$

$$\Rightarrow F \longrightarrow F^{\int \text{ins. unital}}$$

Lecture 5 (Lin Chen)

$$\begin{tikzcd}
\text{KL}(G)_{\text{crit, Ran}} \arrow[r, "\text{FLE}_{\text{crit}}", "\sim"'] \arrow[d, "\text{Loc}"] & \text{IndCoh}_*(\text{Op}^{\text{mon-free}}_{\check{G}}) \arrow[d, "\text{Poinc}^{\text{spec}}"] \\
D(\text{Bun}_G)_{\text{crit}} \arrow[r, "\mathbb{L}"] \arrow[d, "\text{coeff}_*"] & \text{IndCoh}_{\text{nilp}}(\text{LS}_{\check{G}}) \arrow[d, "\Gamma^{\text{spec}}"] \\
\text{Whit}^!(G)_{\text{crit, Ran}} \arrow[r, "\text{CS}", "\sim"'] & \text{Rep}(\check{G})_{\text{Ran}}
\end{tikzcd}$$

Primary Goal: describe $\text{coeff}_* \circ \text{Loc}$ via local data (+ Ran magic)

Step 1. $\text{coeff}^{\text{Vac}} \circ \text{Loc}$ $\qquad \text{Coeff}^{\text{Vac}} : D_{\text{crit}}(\text{Bun}_G) \longrightarrow \text{Vect}$

Step 2. Hecke action + Ran magic

Other goals $\quad CT_x \circ \mathrm{Loc}, \quad \text{ddle} \circ \mathrm{Loc}.$

§0 Recollections

① KL $\quad KL(G) = \{ (L\mathfrak{g}, L^+G)\text{-modules} \}$

$$\mathrm{Bun}_G = \mathrm{Bun}_G^{\text{level } \infty x} / L^+G$$

$$LG \curvearrowright \mathrm{Bun}_G^{\text{level} \infty x} \quad \text{via regluing}$$

$$\rightsquigarrow \quad L_x\mathfrak{g} \longrightarrow T\left(\mathrm{Bun}_G^{\text{level} \infty x}\right)$$

$$\rightsquigarrow \quad L_x\mathfrak{g}\text{-mod} \longrightarrow D\left(\mathrm{Bun}_G^{\text{level} \infty x}\right)$$

$$M \longmapsto \mathrm{Diff} \underset{\underline{U(L\mathfrak{g})}}{\otimes} \underline{M}$$

$$\rightsquigarrow \quad L_x\mathfrak{g}\text{-mod}^{L^+G} \longrightarrow D(\mathrm{Bun}_G)$$

This works for any $x \in \mathrm{Ran}$.

$\underline{\mathrm{Loc}} : \underline{KL}(G) \longrightarrow D(\mathrm{Bun}_G) \otimes \underline{\mathrm{Vect}}$ in $\mathrm{ShvCat}(\mathrm{Ran})$

It is even in $\mathrm{ShvCat}(\mathrm{Ran}^{\mathrm{untl}})$

Ex.

$$\begin{tikzcd} KL(G)_x \arrow[r, "\mathrm{Loc}_x"] \arrow[d, "\text{ins.vac.}"'] & D(\mathrm{Bun}_G) \arrow[d, equal] \\ KL(G)_x \otimes KL(G)_{y \setminus x} = KL(G)_y \arrow[r, "\mathrm{Loc}_y"] & D(\mathrm{Bun}_G) \end{tikzcd}$$

commutes.

for $x \subset y \in \mathrm{Ran}^{\mathrm{untl}}$

$$\mathrm{Loc} : KL(G)_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_{\mathrm{Ran}}} D(\mathrm{Bun}_G \times \mathrm{Ran}) \xrightarrow{\int_{\mathrm{Ran}}} D(\mathrm{Bun}_G)$$

This is the functor Loc.

② Coeff_*

$$\mathrm{Whit}^!(G) = D(\mathrm{Gr}_{G,x})^{(LN,x)}$$

$$\mathrm{Gr}_{G,x} \xrightarrow{\pi} \mathrm{Bun}_G$$

$$\mathrm{Poinc}_{!,x} : \mathrm{Whit}^!(G)_x \underset{\mathrm{Av}_*^{\mathrm{Whit}} \circ \pi^!}{\overset{\pi_!}{\rightleftarrows}} D(\mathrm{Bun}_G) : \mathrm{Coeff}_{*,x}$$

$$\underline{\mathrm{Poinc}_!} : \underline{\mathrm{Whit}^!(G)} \rightleftarrows D(\mathrm{Bun}_G) \otimes \underline{\mathrm{Vect}} : \underline{\mathrm{Coeff}_*}$$

$$\underline{\mathrm{Vect}} \xrightarrow{\mathrm{unit}} \underline{\mathrm{Whit}(G)}$$

is given by $p_!(\exp_{\mathrm{Gr}_N})$

$$\begin{tikzcd} \mathrm{Gr}_G & \mathrm{Gr}_N \arrow[l, "p"'] \arrow[d] & \ni \exp \\ & G_a & \ni \exp \arrow[u] \end{tikzcd}$$

$$p_!(\exp)$$

Exer ① $\underline{\mathrm{unit}}^R : \underline{\mathrm{Whit}^!(G)} \xrightarrow{e^!} \underline{\mathrm{Vect}}$

②
$$\begin{tikzcd} \underline{\mathrm{Whit}^!(G)} \arrow[rr, "CS", "\sim"'] \arrow[dr, "\underline{e^!}"'] & & \underline{\mathrm{Rep}(\check{G})} \arrow[dl, "\underline{\mathrm{inv}}"] \\ & \underline{\mathrm{Vect}} & \end{tikzcd}$$

Coeff_*:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r] \arrow[ddrr, "\mathrm{coeff}^{\mathrm{Vac}}"'] & D(\mathrm{Bun}_G \times \mathrm{Ran}) \arrow[r, "\mathrm{coeff}_{\mathrm{Ran}}"] & \mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[d, "e^!_{\mathrm{Ran}}"] \\ & & D(\mathrm{Ran}) \arrow[d, "\int_{\mathrm{Ran}}"] \\ & & \mathrm{Vect} \end{tikzcd}$$

Exer. $\mathrm{coeff}^{\mathrm{Vac}}$ is given by $q_*\left(p^!(-) \overset{!}{\otimes} \chi^! \exp\right)$

$$\begin{tikzcd} \mathrm{Bun}_G & \mathrm{Bun}_N^{\Omega} \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\chi"] & pt \\ & G_a & \end{tikzcd}$$

Advanced Exer ① $\mathrm{Poinc}_!$ is unital, coeff_* is right lax unital.

② $\phi \in \mathrm{Ran}^{\mathrm{untl}}$. $\mathrm{coeff}_{*,\phi} : D(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$ is $\mathrm{coeff}^{\mathrm{Vac}}$.

§1. Statements

Main-Thm

$$\begin{tikzcd} \mathrm{KL}(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{Loc}"] \arrow[d, "\mathrm{ins.vac}"'] & D(\mathrm{Bun}_G) \arrow[d, "\mathrm{coeff}^{\mathrm{Vac}}"] \\ \mathrm{KL}(G)_{\mathrm{Ran}^c} \arrow[r, "\mathrm{DS}"] & D(\mathrm{Ran}^c) \arrow[r, "\int_{\mathrm{Ran}^c}"] & \mathrm{Vect} \end{tikzcd}$$

$\mathrm{Ran}^c = \{ x \subset y : x, y \in \mathrm{Ran} \}$

$$\begin{tikzcd} & \mathrm{Ran}^c \arrow[dl, "pr^{\mathrm{small}}"'] \arrow[dr, "pr^{\mathrm{big}}"] & \\ \mathrm{Ran} & & \mathrm{Ran} \end{tikzcd}$$

$\mathcal{C} \in \mathrm{ShvCat}(\mathrm{Ran})$

$pr_{\mathrm{big}}^*(\mathcal{C}) \in \mathrm{ShvCat}(\mathrm{Ran}^c)$

$\mathcal{C}_{\mathrm{Ran}^c} := \Gamma(\mathrm{Ran}^c, pr_{\mathrm{big}}^*(\mathcal{C}))$

Recollection on DS.

$$DS(M) = C^{\frac{\infty}{2}}(\mathcal{L}\mathfrak{n}, L^+N, M \otimes k_x)$$

For $LN = \bigcup N_i$ $\quad (N_i = \mathrm{Ad}_{t^{-i\rho}} L^+N)$

$$C^{\frac{\infty}{2}}(\mathcal{L}\mathfrak{n}, L^+N, M) = \operatorname*{colim}_i C^\cdot(\mathfrak{n}_i, M \otimes \det(\mathfrak{n}_i/\mathfrak{n}_0)[\dim N_i/N_0])$$

Exer Define the connecting maps.

$\underline{DS}: \underline{KL(G)} \longrightarrow \underline{Vect}$

is a lax unital functor.

$$DS(\mathrm{Vac}) = \mathfrak{z} = Z(\mathrm{Vac})$$

Exercise (Tautology)

$\underline{DS^{enh}}: \underline{KL(G)} \longrightarrow \underline{\mathfrak{z}\text{-mod}^{fact}}$

$\underline{C^{fact}_\cdot(X, \mathfrak{z}, -)}: \underline{\mathfrak{z}\text{-mod}}^{fact} \longrightarrow \underline{Vect}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "F^{enh}"'] & \underline{\mathcal{D}} \\ & \underline{F(\mathbb{1}_{\mathcal{C}})\text{-mod}^{fact}} \arrow[u, "oblv"'] \end{tikzcd}$$

Rewrite $\hookrightarrow$ as

$$\begin{tikzcd} KL(G)_{Ran} \arrow[d, "DS^{enh}"'] & & \\ \mathfrak{z}\text{-mod}^{fact}_{Ran} \arrow[r, "C^{fact}_\cdot(X,\mathfrak{z},-)_{Ran}"] & D(Ran) \arrow[r, "\int_{Ran}"] & Vect \end{tikzcd}$$

$\underline{\quad\quad\quad\quad} \Sigma$

1) $KL_{\underline{x}} \xrightarrow{Loc} D\mathrm{mod}(\mathrm{Bun}_G)$

$\underline{x} \in Ran$

2) $\underline{KL} \xrightarrow{Loc} Dmod(Bun_G) \otimes \underline{Vect}$ ← unital <u>local-to-global</u>

3) $KL_{Ran} \xrightarrow{Loc\,Ran} Dmod(Bun_G) \otimes Dmod(Ran) \xrightarrow{Id \otimes \int_{Ran}} Dmod(Bun_G)$

4) $\underline{KL} \xrightarrow{\underline{DS}} \underline{Vect}$ lax unital

$$\begin{tikzcd} \underline{KL} \arrow[r, "\underline{Loc}"] \arrow[d, "\underline{DS}"'] & D(Bun_G) \otimes \underline{Vect} \arrow[d, "coeff \otimes id"] \\ \underline{Vect} \arrow[r, equal, "id"'] & \underline{Vect} \end{tikzcd}$$

$\longrightarrow \mathcal{E}$

$\boxed{\text{Construction}}$

$$\begin{tikzcd} \underline{KL(G)} \arrow[r, "\underline{Loc}"] \arrow[d, "\underline{DS}"'] & D(Bun_G) \otimes \underline{Vect} \arrow[d, "coeff^{Vac} \otimes id"] \\ \underline{Vect} \arrow[r, equal] \arrow[ur, phantom, "\Nearrow"] & \underline{Vect} \end{tikzcd}$$

lax unital

$$\begin{tikzcd} \mathbb{V}ac \arrow[r, mapsto] \arrow[d, mapsto] & Diff \otimes \underline{\mathbb{1}} \arrow[d, mapsto] \\ DS(\mathbb{V}ac) & coeff^{Vac}(Diff) \otimes \underline{\mathbb{1}} \end{tikzcd}$$

$DS(\mathbb{V}ac) = \mathfrak{z} \neq coeff^{Vac}(Diff) \otimes \underline{\mathbb{1}}$

$F: \underline{\mathcal{C}} \longrightarrow D \otimes \underline{Vect}$ lax unital local-to-global

$F \longrightarrow F^{\int ins.vac} = F^{fix}$

$$\begin{tikzcd} \mathcal{C}_x \arrow[r, "F_x^{fix}"] \arrow[d, "ins.vac"'] & D \\ \mathcal{C}_{Ran_x} \arrow[r, "F_{Ran_x}"] & D \otimes D(Ran_x) \arrow[u, "\int_{Ran_x}"'] \end{tikzcd}$$

$F_x^{fix}: \mathcal{C}_x \longrightarrow D$

$\wr$

$\underline{F}^{fix}: \underline{\mathcal{C}} \longrightarrow D \otimes \underline{Vect}$

$$\begin{tikzcd} \underline{DS} \arrow[r] \arrow[dr] & \underline{coeff}^{Vac} \circ \underline{Loc} \\ & \underline{DS}^{fix} \arrow[u, dashed] \end{tikzcd}$$

Construction $\Rightarrow \underline{DS}^{fix} \longrightarrow \underline{coeff}^{Vac} \circ \underline{Loc}$

Main-Thm' $\quad \underline{DS}^{fix} \xrightarrow{\sim} \underline{coeff}^{Vac} \circ \underline{Loc}$

---

§2. Loc & oblv

$$\begin{tikzcd}
\mathrm{triv} \in \underline{\mathrm{Rep}}(L^+G) \arrow[rr, "\underline{\mathrm{Loc}}^{\mathrm{QCoh}}"] \arrow[d, shift right, "\mathrm{ind}"'] & & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \underline{\mathrm{Vect}} \ni \mathcal{O} \boxtimes \underline{\mathbb{1}} \arrow[d, shift right, "\mathrm{ind} \otimes \underline{\mathrm{id}}"'] \\
\mathrm{Vac} \in \underline{\mathrm{KL}(G)} \arrow[u, shift right, "\underline{\mathrm{oblv}}"'] \arrow[rr, "\underline{\mathrm{Loc}}"'] \arrow[urr, phantom, "\Rightarrow"] & & D(\mathrm{Bun}_G) \otimes \underline{\mathrm{Vect}} \ni \mathrm{Diff} \boxtimes \underline{\mathbb{1}} \arrow[u, shift right, "\mathrm{oblv} \otimes \underline{\mathrm{id}}"']
\end{tikzcd}$$

- is unital
- is lax unital

$$\begin{tikzcd}
\left(\underline{\mathrm{Loc}}^{\mathrm{QCoh}} \circ \underline{\mathrm{oblv}}\right)^{fix} \arrow[d] \\
\underline{\mathrm{oblv}} \circ \underline{\mathrm{Loc}}
\end{tikzcd}$$

$\underline{\text{Thm 2}}$ This is $\simeq$.

---

§3. Loc & pullback

$$\begin{tikzcd}
H_1 \arrow[d] & \curvearrowright & Y_1 \arrow[d, "p"] \\
H_2 & \curvearrowright & Y_2
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{h}_1\text{-mod} \arrow[r, "\mathrm{Loc}"] & D(Y_1) \\
\mathfrak{h}_2\text{-mod} \arrow[u, "\mathrm{res}"] \arrow[r, "\mathrm{Loc}"'] \arrow[ur, phantom, "\Rightarrow"] & D(Y_2) \arrow[u, "p^!"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathrm{KL}(H_1)} \arrow[r, "\underline{\mathrm{Loc}}_{H_1}"] & D(\mathrm{Bun}_{H_1}) \otimes \underline{\mathrm{Vect}} \\
\underline{\mathrm{KL}(H_2)} \arrow[u, "\mathrm{res}"] \arrow[r, "\underline{\mathrm{Loc}}_{H_2}"'] \arrow[ur, phantom, "\Rightarrow"] & D(\mathrm{Bun}_{H_2}) \otimes \underline{\mathrm{Vect}} \arrow[u, "\underline{p}^!"']
\end{tikzcd}$$

Thm 3 $\left(\mathrm{Loc}_{H_1} \circ \mathrm{res}\right)^{\mathrm{fix}} \xrightarrow{\sim} p^! \circ \mathrm{Loc}_{H_2}$

Ex Thm 2 $\Rightarrow$ Thm 3.

(Hint $(F \circ \phi)^{\mathrm{fix}} \simeq F^{\mathrm{fix}} \circ \phi$ if ϕ is unital

$(\psi \circ F)^{\mathrm{fix}} \simeq \psi \circ F^{\mathrm{fix}}$ if $\psi : D \otimes \mathrm{Vect} \longrightarrow D' \otimes \mathrm{Vect}$)

$$
\begin{tikzcd}
KL(H_1) \arrow[d, "\mathrm{res}^\vee"'] \arrow[r, shift right, "\mathrm{Loc}_{H_1}"'] & D(\mathrm{Bun}_{H_1}) \otimes \mathrm{Vect} \arrow[l, shift right, "\Gamma^{\mathrm{level}\,\infty}_{H_1}"'] \arrow[d, "p_*"] \\
KL(H_2) \arrow[r, shift right, "\mathrm{Loc}_{H_2}"'] \arrow[ur, "\mathrm{Loc}"] & D(\mathrm{Bun}_{H_2}) \otimes \mathrm{Vect} \arrow[l, shift right, "\Gamma^{\mathrm{level}\,\infty}_{H_2}"']
\end{tikzcd}
\quad \Longleftarrow \quad \text{Duality + Beck-Chevalley}
$$

$C^{\frac{\infty}{2}}(\mathfrak{L}n, L^+N, -) = \mathrm{BRST}_n$

Thm 4 If $H_1 \twoheadrightarrow H_2$, kernel $= N$,

$\left(\mathrm{Loc}_{H_2} \circ \mathrm{BRST}_n\right)^{\mathrm{fix}} \xrightarrow{\sim} p_* \circ \mathrm{Loc}_{H_1}$

Thm 3 + Thm 4 + fact $\Rightarrow$ Main Thm

$$
\begin{tikzcd}
KL(N) \arrow[r, "\mathrm{Loc}_N"] \arrow[d, "{C^{\frac{\infty}{2}}(\mathfrak{L}n, L^+N, -) = \mathrm{BRST}_n}"'] & D(\mathrm{Bun}_N) \otimes \mathrm{Vect} \arrow[d, "\Gamma_{dR}(\mathrm{Bun}_N)"] \\
\mathrm{Vect} \arrow[ur, Rightarrow] \arrow[r, equal] & \mathrm{Vect}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Vac}_N \in KL(N)_x \arrow[r, "\mathrm{Loc}"] \arrow[d, "\mathrm{ins.\,vac}"'] & \mathrm{Dist} \in D(\mathrm{Bun}_N) \arrow[r, "\Gamma_{dR}"] & \mathcal{O}(\mathrm{Bun}_N) \in \mathrm{Vect} \ni C^{\mathrm{fact}}_\cdot(X, \Omega_n; \Omega_n) \\
\mathrm{Vac}_{\mathrm{Ran}_x} \in KL(N)_{\mathrm{Ran}_x} \arrow[r, "\mathrm{BRST}_n"] & D(\mathrm{Ran}_x) \ni \mathcal{O}(\mathbb{B}L^+N)_{\mathrm{Ran}_x} = \Omega_n \arrow[ur, "\int_{\mathrm{Ran}_x}"] &
\end{tikzcd}
$$

$$\mathrm{BRST}_n(\mathbb{V}ac) = \mathrm{End}_{\mathrm{Rep}(L^+N)}(\mathrm{triv}, \mathrm{triv}) \simeq \mathcal{O}(\mathbb{B}L^+N)$$

<u>Thm 4'</u>: $\mathcal{O}(\mathrm{Bun}_N) = C_\cdot^{\mathrm{fact}}(X, \mathcal{O}(\mathbb{B}L^+N))$

$$\mathrm{Bun}_N = \mathrm{Maps}_X(X, \mathbb{B}N \times X)$$

$$\mathbb{B}L^+N = \mathrm{Maps}_X(D, \mathbb{B}N \times X)$$

$Y \longrightarrow X$ affine D-scheme

$$\mathcal{O}(\mathrm{Sect}_\nabla(X,Y)) = C_\cdot^{\mathrm{fact}}(X, \mathcal{O}(L^+_\nabla Y))$$

<u>Now</u>

- Y is Weil restriction of $\mathbb{B}N \times X$ along $X \longrightarrow X_{dR}$
- $\mathbb{B}N$ is as good as affine $\mathrm{QCoh}(\mathbb{B}N) \simeq \mathcal{O}(\mathbb{B}N)\text{-mod}$

<u>Tutorial 3.3 (Kevin Lin)</u>

(fix a point $x \in X$)

Recall $\mathrm{Op}^{\mathrm{mer}}_{\check{G}} = \check{G}$-opers on $\overset{\circ}{D}_x$

e.g. if $\check{G} = GL_3$

$$d + \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} dt \qquad \begin{array}{l} a_1 \in k((t)) \\ a_2 \in k((t))\,dt \\ a_3 \in k((t))\,dt^2 \end{array}$$

In general, $\mathrm{Op}^{\mathrm{mer}}_{\check{G}} = \mathrm{colim}\, V_{\leq n}$

$$V_{\leq n} = \lim V_{\leq n}/V_{\leq -m} \qquad (\text{fin dim. terms})$$

$$\begin{tikzcd} \mathrm{Op}^{m.f.}_{\check{G}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Op}^{mer}_{\check{G}} \arrow[d] \\ \mathrm{LS}^{reg}_{\check{G}} \arrow[r] & \mathrm{LS}^{mer}_{\check{G}} \end{tikzcd}$$

Want to define a functor $\mathrm{Poinc}^{Spec}_{*} : \mathrm{IndCoh}_{*}(\mathrm{Op}^{m.f.}_{\check{G}}) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$

We have:

$$\begin{tikzcd} & \mathrm{Op}^{mer,glob}_{\check{G}} \arrow[dl, "ev^{mer}"'] \arrow[dr] & \\ \mathrm{Op}^{mer}_{\check{G}} & & \mathrm{LS}^{mer,glob}_{\check{G}} \end{tikzcd}$$

$\mathrm{Op}^{mer,glob}_{\check{G}}$ = opers on $X \setminus x$

$\mathrm{LS}^{mer,glob}_{\check{G}}$ = local systems on $X \setminus x$

the diagram is over $\mathrm{LS}^{mer}_{\check{G}}$

Base change along $\mathrm{LS}^{reg}_{\check{G}} \to \mathrm{LS}^{mer}_{\check{G}}$, to get

$$\begin{tikzcd} & \mathrm{Op}^{m.f.,glob}_{\check{G}} \arrow[dl, "ev^{mf}"'] \arrow[dr, "r"] & \\ \mathrm{Op}^{m.f.}_{\check{G}} & & \mathrm{LS}_{\check{G}} \end{tikzcd}$$

"Def" $\mathrm{Poinc}^{Spec}_{*} = r_{*} \circ (ev^{m.f.})^{*}$

Because we did base change along $\mathrm{LS}^{reg}_{\check{G}} \to \mathrm{LS}^{mer}_{\check{G}}$, $\mathrm{Poinc}^{Spec}_{*}$ is linear over $\mathrm{IndCoh}(\mathrm{LS}^{reg}_{\check{G}} \times_{\mathrm{LS}^{mer}_{\check{G}}} \mathrm{LS}^{reg}_{\check{G}})$. One can show directly that $\mathrm{IndCoh}_{*}(\mathrm{Op}^{m.f.}_{\check{G}})$ is tempered.

$\Rightarrow$ $\mathrm{Poinc}^{Spec}_{*}$ lands in $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})^{temp} \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

Warning even if $f : X \to Y$ is a map of finite type schemes, the functor

$f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ need not admit a left adjoint f^*

– e.g. $pt \to pt \times_{\mathbb{A}^1} pt$ has no IndCoh $*$-pullback.

– e.g. if f is quasi-smooth, f^* exists.

(In general, need f to have finite tor amplitude.)

Why $(ev^{mer})^*$ exists?

$$\mathrm{IndCoh}_*(Op_{\check{G}}^{mer}) = \underset{*\text{-push}}{\mathrm{colim}}\ \mathrm{IndCoh}_*(V_{\leq n})$$

$$\mathrm{IndCoh}_*(V_{\leq n}) = \underset{*\text{-pull}}{\mathrm{colim}}\ \mathrm{IndCoh}(V_{\leq n}/V_{\leq -m})$$

$Op_{\check{G}}^{mer,glob} = \mathrm{colim}\ L_{\leq n}$ (each fin. dim)

for GL_3 $\quad d + \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} dt \quad a_i \in \Gamma(X \setminus x, \omega^{\otimes ?})$

$$\begin{tikzcd} L_{\leq n} \arrow[r, hook] \arrow[d] & Op_{\check{G}}^{mer,glob} \arrow[d] \\ V_{\leq n} \arrow[r] \arrow[d] & Op_{\check{G}}^{mer} \\ V_{\leq n}/V_{\leq -m} & \end{tikzcd}$$

$*$-pullback along $L_{\leq n} \to V_{\leq n}/V_{\leq -m}$ exists.

Thm

$$\begin{tikzcd}
\mathrm{IndCoh}_*(\mathrm{Op}^{m.f.}_{\check{G}})_x \arrow[r, "\mathrm{Poinc}^{\mathrm{Spec}}_*"] \arrow[d, "(2+)^*"'] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \arrow[dd, "\Gamma"] \\
\mathrm{IndCoh}_*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \arrow[d] & \\
\mathcal{O}_{\mathrm{Op}_{\check{G}}}\text{-mod}^{\mathrm{fact}}_x \arrow[r, "C^{\mathrm{Fact}}_\cdot(-)"] & \mathrm{Vect}
\end{tikzcd}$$

commutes.

Proof by example. Plug in $(2)^!_* \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}} \in \mathrm{IndCoh}_*(\mathrm{Op}^{m.f.}_{\check{G}})$

$$\begin{tikzcd}
\mathrm{Op}_{\check{G}} \arrow[d, "2^!"'] & \mathrm{Op}^{\mathrm{glob}}_{\check{G}} \arrow[l] \arrow[d] \\
\mathrm{Op}^{m.f.}_{\check{G}} & \mathrm{Op}^{m.f.,\mathrm{glob}}_{\check{G}} \arrow[l]
\end{tikzcd}$$

upper circuit is functions on $\mathrm{Op}^{\mathrm{glob}}_{\check{G}}$

lower circuit is $C^{\mathrm{Fact}}_\cdot(X, \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}})$

Nick $\Rightarrow$ $\mathrm{Op}^{\mathrm{glob}}_{\check{G}}$ is an affine scheme w/ functions

$C^{\mathrm{Fact}}_\cdot(X; \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}})$

$\mathcal{Y} \to X$ affine D-scheme

It is possible to find $\mathcal{Y} \to \mathcal{Y}_i$ modifying $\mathcal{Y}$ at x, $\mathcal{L}_\nabla \mathcal{Y} = \mathrm{colim}\, \mathcal{L}^+_\nabla \mathcal{Y}_i$.

$$\begin{tikzcd}
\mathrm{IndCoh}_*(\mathcal{L}_\nabla \mathcal{Y})_x \arrow[r] \arrow[d, "*\text{-pull}"'] & \mathcal{O}_{\mathcal{L}^+_\nabla \mathcal{Y}}\text{-mod}^{\mathrm{fact}}_x \arrow[d, "C^{\mathrm{Fact}}_\cdot"] \\
\mathrm{IndCoh}_*(\mathrm{Sect}_\nabla(X \setminus x, \mathcal{Y})) \arrow[r, "\Gamma"] & \mathrm{Vect}
\end{tikzcd}$$

$$\begin{tikzcd}
(B\check{G})_{\mathrm{Ran}} & \mathrm{LS}_{\check{G}} \times \mathrm{Ran} \arrow[l, "ev"'] \arrow[d, "p"] \\
& \mathrm{LS}_{\check{G}}
\end{tikzcd}$$

$\mathrm{Loc}^{\mathrm{Spec}} = p_! \circ ev^*$, $\Gamma^{\mathrm{Spec}} = ev_x \circ p^! = (\mathrm{Loc}^{\mathrm{Spec}})^R$

Day 3 Q & A

mon-free opers for $\mathbb{G}_m$:

$Op_{\mathbb{G}_m}$ = connections on trivial bundle = {1-forms}

Ex $\ ^{cl}\mathrm{Spec}\,\Gamma(\mathrm{Bun}_{\mathbb{G}_m}, \mathrm{Diff}) = \mathrm{AP}\,\Gamma(\Omega^1) = Op_{\mathbb{G}_m}(X)$

$$Op^{mf}(D) \subset Op^{mer}(D)$$

$$\begin{array}{cc} \| & \| \end{array}$$

$\left\{ d + f\,dt : \begin{array}{l} a_{-1} \in \mathbb{Z}, \\ a_{-n} \text{ is nilp} \\ \text{for } n > 1 \end{array} \right\} \quad \{ d + f(t)\,dt : f(t) = \sum a_i t^i \in k((t)) \}$

$$\begin{array}{c} \downarrow a_{-1} \\ \mathbb{Z} \end{array} \qquad Op^{mf} = \underbrace{\coprod Op^{mf,n}} \qquad \text{each cpt: } k[[t]] \times \left(k((t))/t^{-1}k[[t]]\right)^\wedge_0$$

$$\mathrm{QCoh}(LS_{\check{G}}) \otimes D(\mathrm{Bun}_G) \xrightarrow{\text{spec. act.}} D(\mathrm{Bun}_G) \xrightarrow{\text{coeff}} \mathrm{Vect}$$

dualize $\quad D(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_{G,\text{coarse}}} \mathrm{QCoh}(LS_{\check{G}})^\vee = \mathrm{QCoh}(LS_{\check{G}})$

$$\mathrm{Rep}\,\check{G} \otimes KL_G \xrightarrow{\mathrm{Sat}^{nv} \otimes \mathrm{id}} \mathrm{Sph}(G) \otimes KL_G \xrightarrow{\text{conv}} KL_G \xrightarrow{DS} \mathrm{Vect}$$

$$DS(M) = C^{\frac{\infty}{2}}(Ln, L^+n, M \otimes \psi)$$

$$\psi : n((t)) \to k, \quad \frac{e_i}{t} \mapsto 1$$

every other basis vector $\mapsto 0$

$$\begin{array}{ccc} \mathbb{V}_k & \overset{DS}{\longmapsto} & W_k \\ \wr & & \wr \\ L^+\mathfrak{g} & & L^+(f + b/N) \end{array} \qquad f + b/N \supset \mathfrak{g}/\!/G \simeq \mathfrak{t}/\!/W$$

$$W_{crit} = \mathcal{O}_{Op^{reg}}$$

dualize: $KL_G \xrightarrow{DS^{enh}} (\mathrm{Rep}\,\check{G})^\vee = \mathrm{Rep}\,\check{G}$

$$DS^{enh}(\mathrm{Vac}_{crit}) \simeq \tau_*\mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}} = R_{\mathfrak{z}}, \quad \tau: \mathrm{Op}^{reg}_{\check{G}} \longrightarrow LS_{\check{G}}(D) = \mathbb{B}\check{G}$$

↑ birth of opers

$$KL_G \longrightarrow \mathrm{Rep}\,\check{G}$$

$$\mathrm{Vac}_{crit} \longmapsto R_{\mathfrak{z}}$$

$$KL_G \longrightarrow R_{\mathfrak{z}}\text{-mod}^{fact}(\mathrm{Rep}\,\check{G}) \overset{\text{up to left completion}}{\approx} \mathrm{IndCoh}(\mathrm{Op}^{mf})$$

$$\mathfrak{z}\text{-mod}^{fact} \simeq \mathrm{IndCoh}(\mathrm{Op}^{mer})$$

Tutorial 4.1 (Sam Raskin)

$G = PGL_2$

Analogy w/ no. theory:

Given σ : "Gal$_{\mathbb{Q}}$" $\longrightarrow SL_2 = \check{G}$ irred (odd) rep'n, unram.

$$a_p = \mathrm{tr}(\sigma(Fr_p)), \qquad Fr_p \in \text{"Gal}_{\mathbb{Q}}\text{"}$$

Write $q = e^{2\pi i \tau}$, $\qquad f(\tau) = \sum a_n q^n$

where $a_0 = 0 \longleftarrow$ const. term $= 0$ "cuspidal" $\longleftrightarrow$ irred

$a_1 = 1$

$a_p = \mathrm{tr}(\sigma(Fr_p))$

$a_n \cdot a_m = a_{nm}$ if $(n,m) = 1$

$a_{p^{n+1}} + p\,a_{p^{n-1}} = a_p \cdot a_{p^n}$

Langlands conj.

$f(\tau)$ (defined on $\operatorname{Im}\tau>0$) is a modular form of level 1

+ "Essentially every modular form has this type".

In geometry:

$\sigma \in LS_{\check{G}}$ (ideally irred)

Geom. Langlands: $\exists$ eigensheaf $\mathcal{F}_\sigma \in Dmod(Bun_G)$

Motto: Characterize $\mathcal{F}_\sigma$ by its "q-expansion".

Analogy: $f \mapsto a_0(f) \longleftrightarrow$ $CT_* : D(Bun_G) \longrightarrow D(Bun_M)$ $\quad M$ Levi

mod forms $\longrightarrow$ $\{$funcs on $\mathbb{R}_{>0}\}$

$a_0(f)=0 \longleftrightarrow$ $\mathcal{F}_\sigma$ is "cuspidal" for σ irred.

$a_1(f) \longleftrightarrow$ $\underline{\text{coeff}_0} : D(Bun_G) \longrightarrow Vect$

"vacuum Whit. coeff."

$a_1(f) = 1 \longleftrightarrow$ $\text{coeff}(\mathcal{F}_\sigma) = k$

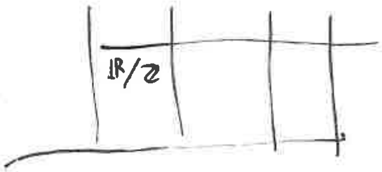
"Whittaker normalization"

Analogy harmonic analysis $/\mathbb{R}$ & adelic stuff & geometry

$$\begin{array}{ccc} \mathbb{R} & \mathbb{A}_{\mathbb{Q}} & \mathbb{A}_X \\ \cup & \cup & \cup \\ \mathbb{Z} & \mathbb{Q} & k(X) \\ K\backslash G(\mathbb{R})/G(\mathbb{Z}) & \longleftrightarrow & Bun_G \end{array}$$

$$a_1(f) = \int_{\mathbb{R}/\mathbb{Z}} f(\tau)\, e^{-2\pi i \tau}\, d\tau$$

$$\text{coeff}(F) = \int_{\text{Bun}_N^{\Omega}} F \cdot \exp$$



Analogy: $\mathbb{R}/\mathbb{Z} \rightsquigarrow {}_{\text{int adeles}}\mathbb{A}/\mathbb{G}_a \rightsquigarrow \text{Bun}^{\Omega}_{G_a} = \text{Bun}_N^{\Omega}$ ($G = PGL_2$)

Other coeffs.

$n \geq 1 \iff p_1^{\nu_1} \cdots p_s^{\nu_s} \rightsquigarrow D = \sum \nu_i [p_i]$ — eff. divisor on $\text{Spec}\,\mathbb{Z}$

Whittaker coeffs for PGL_2 indexed by $D \geq 0$ on X.

$$\text{Coeff}_D : D(\text{Bun}_{PGL_2}) \longrightarrow \text{Vect}$$

$$\{0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O} \to 0\} = \text{Bun}_N^{\Omega(-D)}$$

$$\swarrow \qquad \searrow$$

$$\text{Bun}_{PGL_2} \qquad H^1(\Omega(-D)) \longrightarrow H^1(\Omega) = \mathbb{A}^1$$

$f(q) = \sum a_n q^n$ $\quad a_n$] given by same procedure

Analogy: $\sigma \in LS_{SL_2}$

$D = \sum n_i x_i$, $\text{Coeff}_D(F_\sigma) = \bigotimes_i \text{Sym}^{n_i}(\sigma_{x_i})$

For general G

Whittaker coeffs are (naively) indexed by divisors D on X valued in Λ^+

$\uparrow$ dominant coweights

$\sum \lambda_i x_i, \quad \lambda_i \in \Lambda^+.$

$$\begin{tikzcd} & \mathrm{Bun}_N^{\Omega(-D)} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \prod_i H^1(\Omega(\alpha_i(-D))) \arrow[d] \\ & & \prod H^1(\Omega) \arrow[d, "\mathrm{sum}"] \\ & & \mathbb{A}^1 = H^1(\Omega) \end{tikzcd}$$

$$\mathrm{coeff}_D(\mathcal{F}_\sigma) = \bigotimes_i \left(V_\sigma^{\lambda_i}\right)\Big|_{x_i}, \quad \sigma \in LS_{\check{G}}$$

$D = \sum \lambda_i x_i$

Need: Smarter versions of Whittaker coefficients w/ more geometry.

Easy upgrade

$G = PGL_2$

$$\begin{tikzcd} & & \mathrm{Vect} \\ D(\mathrm{Bun}_G) \arrow[rr, "\mathrm{coeff}_d"] \arrow[urr, "\mathrm{coeff}_D"] & & D(\mathrm{Sym}^d X) \arrow[u, "!\text{-fiber at } D"'] \end{tikzcd}$$

$\forall d \geq 0$, can easily define $\mathrm{coeff}_d : D(\mathrm{Bun}_G) \to D(\mathrm{Sym}^d X)$

$\mathrm{Sym}^d X$: space of $D \geq 0$, $\deg D = d$

Can easily adapt the above to say what $\mathrm{coeff}_d(\mathcal{F}_\sigma)$ is.

$$D(\mathrm{Bun}_{PGL_2}) \longrightarrow \prod_{d \geq 0} D(\mathrm{Sym}^d)$$

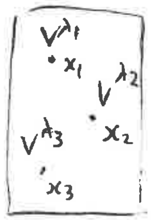
$\uparrow$

can't be anywhere near fully faithful

{mod. forms} {q-expansions}

<u>Fix</u>

coeff : $D(\mathrm{Bun}_G) \longrightarrow \underline{\mathrm{Rep}\, \check{G}_{\mathrm{Ran}}}$



Def'n of coeff is more subtle.

$$\begin{array}{ccc} D(\mathrm{Bun}_G) & \longrightarrow & \mathrm{Whit}_*(\mathrm{Gr}_{G,\mathrm{Ran}}) \\ & & \wr\!\mid \quad \leftarrow \text{Casselman - Shalika} \\ & & \mathrm{Rep}\, \check{G}_{\mathrm{Ran}} \end{array}$$

What happened secretly on Monday:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, dashed, "\mathbb{L}_{G,\mathrm{coarse}}"] \arrow[dr, "\mathrm{coeff}"'] & \mathrm{QCoh}(LS_{\check{G}}) \arrow[d, "\Gamma^{\mathrm{spec}}"] \\ & \mathrm{Rep}\, \check{G}_{\mathrm{Ran}} \end{tikzcd}$$

<u>True thms</u>:

$$\begin{array}{ccc} D(\mathrm{Bun}_G)_{\mathrm{cusp}} & & \mathrm{THM!} \\ \cap & & \\ D(\mathrm{Bun}_G)_{\mathrm{temp}} & \subset & \mathrm{Rep}\, \check{G}_{\mathrm{Ran}} \end{array}$$

What happened yesterday

BD thm: $\quad \mathrm{Fun}(\mathrm{Op}_{\check{G}}(X)) \curvearrowright \mathrm{Diff}_{crit} \in D(\mathrm{Bun}_G)$

$\sigma \in \mathrm{Op}_{\check{G}}(X)$ defined $\mathcal{F}_\sigma$ as $\mathrm{Diff}_{crit}/\mathfrak{m}_\sigma$

Big goal:

$$\begin{tikzcd}
KL(G) \arrow[r, "FLE_{crit}", "\sim"'] \arrow[d, "Loc"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{mf}) \arrow[d, "\mathrm{Poinc}_*^{Spec}"] \\
D(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] \arrow[dr, "\mathrm{coeff}"'] & \mathrm{IndCoh}_{nilp}(LS_{\check{G}}) \arrow[d] \\
& \mathrm{Rep}\,\check{G}_{Ran}
\end{tikzcd}$$

$\mathrm{Diff}_{crit} = \mathrm{Loc}(\mathbb{V})$, $\quad \mathbb{L}_G(\mathrm{Diff}_{crit}) = \mathrm{Poinc}_*^{Spec}\, FLE(\mathbb{V}_{crit})$

$$= \mathrm{Poinc}_*^{Spec}\left(\mathcal{O}_{\mathrm{Op}_{\check{G}}^{reg}}\right)$$

$$= \mathcal{O}_{\mathrm{Op}(X)} \in \mathrm{QCoh}(LS_{\check{G}})$$

Work from yesterday $\Rightarrow \mathbb{L}_G(\mathcal{F}_\sigma) =$ skyscraper at $\sigma \in \mathrm{Op}(X) \subset LS_{\check{G}}$

in other words, $\mathcal{F}_\sigma$ has the right Whit coeff.

Lecture 6 (Nick Rozenblyum)

Langlands functor.

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})_{Ran} \arrow[d] & \curvearrowright \ \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{Vac}"] & \mathrm{Vect} \\
\mathrm{QCoh}(LS_{\check{G}}) & &
\end{tikzcd}$$

$\rightsquigarrow$ $\mathbb{L}_{\text{coarse}}$: $\mathrm{Dmod}(\mathrm{Bun}_G) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

$\mathbb{L}$ is a renormalized version of $\mathbb{L}_{\text{coarse}}$

$$\begin{tikzcd} \mathrm{coeff}^{\mathrm{Vac}} : \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[rr] \arrow[dr] & & \mathrm{Vect} \\ & \mathrm{Whit}(G)_{\mathrm{Ran}} \arrow[ur] & \end{tikzcd}$$

equiv. for Satake action

$\Rightarrow$ we have a comm. diagram

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}"] \arrow[d, "\mathrm{coeff}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[d] \\ \mathrm{Whit}(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}", "\sim"'] \arrow[d] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{inv}"] \\ \mathrm{Vect} \arrow[r, equal] & \mathrm{Vect} \end{tikzcd}$$

<u>Thm</u>

$$\begin{tikzcd} \mathrm{KL}_{\mathrm{Ran}} \arrow[r, "\mathrm{FLE}", "\sim"'] \arrow[d, "\mathrm{Loc}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mf}}_{\check{G}})_{\mathrm{Ran}} \arrow[d, "\mathrm{Poinc}^{\mathrm{Spec}}"] \\ \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

canonically commutes.

<u>Recall</u> <u>temperedness</u>.

<u>Derived Satake</u>

$$\mathrm{Dmod}(L^+G\backslash LG/L^+G) = \mathrm{Sph}_G \simeq \mathrm{IndCoh}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathrm{pt}/\check{G})$$

$$\mathrm{QCoh}(\quad " \quad) = \underset{\text{Pages}}{\mathrm{Sph}_{G,\mathrm{temp}}}$$

$\mathrm{Sph}_G \curvearrowright \mathcal{C}, \quad \mathcal{C}^{temp} := \mathcal{C} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}_{G,temp} \hookrightarrow \mathcal{C}$

In particular, we have $\mathrm{Dmod}(\mathrm{Bun}_G)_{temp} \subset \mathrm{Dmod}(\mathrm{Bun}_G)$

<u>Fact</u>: $\mathbb{L}$ commutes w/ Sph_G-actions

$$\mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})_{temp} \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

<u>Upshot</u>

$$\begin{tikzcd}
\mathrm{Dmod}(\mathrm{Bun}_G)_{temp} \arrow[r, "\mathbb{L}"] \arrow[d] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d] \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})
\end{tikzcd}$$

<u>Facts</u>. 1) $(\mathrm{KL})_{temp} \simeq \mathrm{KL}$

2) $(\mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}))_{temp} = \mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}})$

<u>Upshot</u>: we have

$$\begin{tikzcd}
\mathrm{KL}(G) \arrow[r, "\overset{FLE}{\simeq}"] \arrow[d, "\mathrm{Loc}"] & \mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}) \arrow[d, "\mathrm{Poinc}"] \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{temp} \arrow[r, "\mathbb{L} = \mathbb{L}_{coarse}|_{temp}"] \arrow[d] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d] \arrow[dd, bend left=60, "\text{fully faithful}"] \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}"] \arrow[d, "\mathrm{coeff}"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}}) \arrow[d, "\Gamma^{Spec}"] \\
\mathrm{Whit}(G)_{Ran} \arrow[r, "CS", "\sim"'] & \mathrm{Rep}(\check{G})_{Ran}
\end{tikzcd}$$

Need to show commutativity of

$$\begin{tikzcd}
KL(G) \arrow[r, "FLE", "\sim"'] \arrow[d, "Loc"'] & \mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}) \arrow[d, "\mathrm{Poinc}^{Spec}"] \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{temp} \arrow[dd] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "\Gamma^{Spec}"] \\
& \\
\mathrm{Whit}(G)_{\mathrm{Ran}} \arrow[r, "CS", "\sim"'] \arrow[d, "\mathrm{coeff}^{Vac}"'] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "(\,)^{\check{G}}"] \\
\mathrm{Vect} \arrow[r, equal] & \mathrm{Vect}
\end{tikzcd}$$

Lin $KL(G) \xrightarrow{Loc} \mathrm{Dmod}(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}^{Vac}} \mathrm{Vect}$ is given by

$$KL(G) \xrightarrow{DS^{enh}} \mathfrak{z}\text{-mod}^{fact} \xrightarrow{C^{fact}_{\cdot}(X, \mathfrak{z}, -)} \mathrm{Vect}$$

Kevin $\mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma} \mathrm{Vect}$ is given by

$$\mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}) \xrightarrow{i_*} \mathrm{IndCoh}_*(\mathrm{Op}^{mer}_{\check{G}}) \longrightarrow \mathfrak{z}\text{-mod}^{fact} \xrightarrow{C^{fact}_{\cdot}(X, \mathfrak{z}, -)} \mathrm{Vect}$$

Point FLE is constructed so that

$$KL(G) \xrightarrow{FLE} \mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}) \xrightarrow{i_*} \mathrm{IndCoh}_*(\mathrm{Op}^{mer}_{\check{G}})$$

agrees w/ $KL(G) \longrightarrow KM(G) \longrightarrow \mathrm{Whit}_*(KM) \xrightarrow{DS^{enh}} \mathrm{IndCoh}^*(\mathrm{Op}^{mer}_{\check{G}})$

So.

$$\begin{tikzcd}
KL(G) \arrow[r, "FLE"] \arrow[d] & \mathrm{IndCoh}_*(\mathrm{Op}^{mf}_{\check{G}}) \arrow[d, "i_*"] & \\
\mathrm{Whit}_*(KM) \arrow[r, "DS^{enh}"] & \mathrm{IndCoh}_*(\mathrm{Op}^{mer}_{\check{G}}) \arrow[r] & \mathfrak{z}\text{-mod}^{fact}
\end{tikzcd}$$

commutes.

Lecture 7. (Justin Campbell) Eisenstein series and constant terms

$P \subset G$ parabolic, $P = M N_P$

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_M \end{tikzcd}$$

Constant term: $CT_* : \mathrm{Dmod}(\mathrm{Bun}_G) \longrightarrow \mathrm{Dmod}(\mathrm{Bun}_M)$

$CT_* = q_* p^! \, [?]$

Prop. CT_* admits a left adjoint $\mathrm{Eis}_!$.

Ref. Braverman-Gaitsgory, Drinfeld-Gaitsgory

Rmk. can modify $\mathrm{Eis}_!$, CT_* so that $\text{D-mod}_{crit}(\mathrm{Bun}_M) \underset{CT_*}{\overset{\mathrm{Eis}_!}{\rightleftarrows}} \text{D-mod}_{crit}(\mathrm{Bun}_G)$

The cuspidal part

$\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{Eis}} \subset \text{D-mod}(\mathrm{Bun}_G)$ full subcat. gen. by images of $\mathrm{Eis}_{P,!}$ for all proper parabolic subgp. $P \subsetneq G$

$\text{D-mod}(\mathrm{Bun}_G)_{cusp} = \{ \mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G) : CT_{P,*} \mathcal{F} = 0, \ \forall P \subsetneq G \}$

Theorem (Drinfeld-Gaitsgory) $\exists$ open substack $j: U \hookrightarrow \mathrm{Bun}_G$ s.t.

1) U has quasi-cpt intersection w/ each conn. comp of Bun_G.

2) $\forall \mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)_{cusp}$, $j_! j^! \mathcal{F} \xrightarrow{\sim} \mathcal{F} \xrightarrow{\sim} j_* j^* \mathcal{F}$.

The spectral side

$$\begin{tikzcd} & LS_{\check{P}} \arrow[dl, "p^{Spec}"'] \arrow[dr, "q^{Spec}"] & \\ LS_{\check{G}} & & LS_{\check{M}} \end{tikzcd}$$

q^{Spec} quasi-smooth $\Rightarrow$ $q^{Spec,*}$ exists (for IndCoh)

p^{Spec} proper $\Rightarrow$ p^{Spec}_* is left adjoint to $p^{Spec,!}$

Prop (Arinkin-Gaitsgory) $p^{Spec}_* q^{Spec,*}$ and $q^{Spec}_* p^{Spec,!}$ preserve nilpotent singular supp.

Key point: if $\alpha \in \mathfrak{g}^*$ is s.t. it's image in $\mathfrak{p}^*$ lands in $\mathfrak{m}^*$ and is nilp, then α is nilpotent.

$$\mathrm{Eis}^{Spec} := p^{Spec}_* q^{Spec,*} : \mathrm{IndCoh}_{Nilp}(LS_{\check{M}}) \rightleftarrows \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}) : q^{Spec}_* p^{Spec,!} =: \mathrm{CT}^{spec}$$

Thm (Arinkin-Gaitsgory)

1) $\mathrm{QCoh}(LS^{irred}_{\check{G}}) \xrightarrow{\sim} \mathrm{IndCoh}_{Nilp}(LS^{irred}_{\check{G}})$

2) $\mathrm{IndCoh}_{Nilp}(LS_{\check{G}})$ is gen. by $\mathrm{QCoh}(LS_{\check{G}})$ and essential image $\mathrm{Eis}^{Spec}(\mathrm{QCoh}(LS_{\check{M}}))$ for all $\check{P} \subsetneq \check{G}$.

Spectral Constant term via factorization homology

Local analogue of constant term:

$$\begin{tikzcd} & LS^{reg}_{\check{P}} \arrow[dl, "p^{Spec}_{loc}"'] \arrow[dr, "q^{Spec}_{loc}"] & \\ LS^{reg}_{\check{G}} & & LS^{reg}_{\check{M}} \end{tikzcd}$$

$$J^{\mathrm{Spec},!} := (q^{\mathrm{Spec}}_{\mathrm{loc}})_* \, (p^{\mathrm{Spec}}_{\mathrm{loc}})^* : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \longrightarrow \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{M}})$$

More concretely, $J^{\mathrm{Spec},!} : \mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{res}^{\check{G}}_{\check{P}}} \mathrm{Rep}(\check{P}) \xrightarrow{\mathrm{inv}\, N_{\check{P}}} \mathrm{Rep}(\check{M})$

lax unital factorization functor.

$$\begin{tikzcd}
\underline{\mathrm{Rep}(\check{G})} \arrow[r, "J^{\mathrm{Spec},!}"] \arrow[d, "\mathrm{Loc}^{\mathrm{Spec}}_{\check{G}}"'] \arrow[dr, phantom, "\swarrow \eta"] & \underline{\mathrm{Rep}(\check{M})} \arrow[d, "\mathrm{Loc}^{\mathrm{Spec}}_{\check{M}}"] \\
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \otimes \underline{\mathrm{Vect}} \arrow[r, "\mathrm{CT}^{\mathrm{Spec}} \otimes \mathrm{id}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \otimes \underline{\mathrm{Vect}}
\end{tikzcd}$$

Prop $\exists$ nat'l transf. η of lax unital local-to-global functors which induces

$$\left(\mathrm{Loc}^{\mathrm{Spec}}_{\check{M}} \circ J^{\mathrm{Spec},!}\right)^{\mathrm{ins.unit}} \xrightarrow{\sim} (\mathrm{CT}^{\mathrm{Spec}} \otimes \mathrm{id}) \circ \mathrm{Loc}^{\mathrm{Spec}}_{\check{G}}.$$

Proof (sketch) By def'n.

$$\begin{tikzcd}
\underline{\mathrm{Rep}(\check{G})} \arrow[d, "\mathrm{Loc}^{\mathrm{Spec}}"'] \arrow[dr, "\mathrm{Loc}^{\mathrm{Spec}}"] & \\
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \otimes \underline{\mathrm{Vect}} \arrow[r, phantom, "\overset{\sim}{\Rightarrow}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \underline{\mathrm{Vect}} \arrow[l]
\end{tikzcd}$$

Arinkin-Gaitsgory :

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[r, "\mathrm{CT}^{\mathrm{Spec}}_{\mathrm{QCoh}}"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \arrow[d] \\
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[r, "\mathrm{CT}^{\mathrm{Spec}}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}})
\end{tikzcd}$$

$$\mathrm{CT}^{\mathrm{Spec}}_{\mathrm{QCoh}} := q^{\mathrm{Spec}}_{*}\, p^{\mathrm{Spec},*}$$

$$\begin{tikzcd}
\underline{\mathrm{Rep}}(\check{G}) \arrow[r, "\mathrm{res}^{\check{G}}_{\check{P}}"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \underline{\mathrm{Rep}}(\check{P}) \arrow[d] \\
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \underline{\mathrm{Vect}} \arrow[r, "p^{\mathrm{Spec},*} \otimes \mathrm{id}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{P}}) \otimes \underline{\mathrm{Vect}}
\end{tikzcd}$$

Draw the same square, replacing $\check{P} \to \check{G}$ by $\check{P} \to \check{M}$. Then pass to right adjoints of the horizontal functors.

$$\begin{tikzcd}
\underline{\mathrm{Rep}}(\check{P}) \arrow[r, "\mathrm{inv}_{N_{\check{P}}}"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \underline{\mathrm{Rep}}(\check{M}) \arrow[d] \\
\mathrm{QCoh}(\mathrm{LS}_{\check{P}}) \otimes \underline{\mathrm{Vect}} \arrow[r, "q^{\mathrm{Spec}}_{*} \otimes \mathrm{id}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \otimes \underline{\mathrm{Vect}}
\end{tikzcd}$$

WTS $\left(\mathrm{Loc}^{\mathrm{Spec}}_{\check{M}} \circ \mathrm{inv}_{N_{\check{P}}}\right)^{\mathrm{sing.\,unit}} \xrightarrow{\sim} \left(q^{\mathrm{Spec}}_{*} \otimes \mathrm{id}\right) \circ \mathrm{Loc}^{\mathrm{Spec}}_{\check{P}}$

Enough to show after base changing along $\sigma : \mathrm{Spec}\, k \to \mathrm{LS}_{\check{M}}$.

For simplicity, take $\sigma = \sigma^{\mathrm{triv}}$.

Lecture 8 (Dennis Gaitsgory)

Goal of this talk

$$
\begin{tikzcd}
\mathrm{Dmod}(\mathrm{Bun}_M) \arrow[r, "\mathbb{L}_M"] \arrow[d, shift right, "\mathrm{Eis}_!"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check M}) \arrow[d, shift right, "\mathrm{Eis}^{\mathrm{Spec}}"'] \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] \arrow[u, shift right, "\mathrm{CT}_*"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check G}) \arrow[u, shift right, "\mathrm{CT}^{\mathrm{Spec}}"']
\end{tikzcd}
\quad + \quad \text{induction step}
$$

$$
\begin{tikzcd}
\mathrm{Whit}^!(\mathrm{Gr}_M)_{\mathrm{Ran}} \arrow[rr] \arrow[dr, "(\mathrm{coJ})^{\mathrm{fix}^\vee}"] & & \mathrm{Rep}(\check M)_{\mathrm{Ran}} \arrow[dr, "\mathrm{ind}(\mathrm{fix})^\vee"] & \\
\mathrm{Dmod}(\mathrm{Bun}_M) \arrow[u, "\mathrm{coeff}_M"] \arrow[dr, "\mathrm{Eis}_!"'] & \mathrm{Whit}^!(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\mathbb{L}_{G,\mathrm{coarse}}"] \arrow[rr, bend left] & \mathrm{QCoh}(\mathrm{LS}_{\check M}) \arrow[u] \arrow[dr, "\mathrm{Eis}^{\mathrm{Spec}}_{\mathrm{coarse}}"] & \mathrm{Rep}(\check G)_{\mathrm{Ran}} \\
& \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] \arrow[rr, "\mathbb{L}_{M,\mathrm{coarse}}"] & & \mathrm{QCoh}(\mathrm{LS}_{\check G}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check G}"']
\end{tikzcd}
$$

Need $\mathrm{coJ} : \mathrm{Whit}^!(\mathrm{Gr}_M) \longrightarrow \mathrm{Whit}^!(\mathrm{Gr}_G)$ so that

- the upper lid commutes
- the left lid commutes

Define

$$
\begin{tikzcd}
\mathrm{Whit}^!(\mathrm{Gr}_M) \arrow[r, "\mathrm{CS}_M", "\simeq"'] \arrow[d, "\mathrm{coJ}"] & \mathrm{Rep}(\check M) \arrow[d, "\mathrm{ind}"] \\
\mathrm{Whit}^!(\mathrm{Gr}_G) \arrow[r, "\mathrm{CS}_G", "\simeq"'] & \mathrm{Rep}(\check G)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Whit}^!(\mathrm{Gr}_G) \arrow[r, "\mathrm{CS}_G", "\simeq"'] \arrow[d, "\mathrm{J}"] & \mathrm{Rep}(\check G) \arrow[d, "\mathrm{inv}_{N_P}"] \\
\mathrm{Whit}^!(\mathrm{Gr}_M) \arrow[r, "\mathrm{CS}_M", "\simeq"'] & \mathrm{Rep}(\check M)
\end{tikzcd}
$$

Thm (Sith) The diagram commutes.

$$\begin{tikzcd}
\mathrm{Whit}^!(\mathrm{Gr}_G) \otimes \mathrm{Dmod}(\mathrm{Gr}_G)^{LN_{P^-}\cdot L^+M} \arrow[r] \arrow[d, "\wr"] & \mathrm{Whit}^!(\mathrm{Gr}_M) \\
\mathrm{Whit}^!(\mathrm{Gr}_G) \otimes \mathrm{Dmod}(\mathrm{Gr}_G \times \mathrm{Gr}_M)^{LP^-} \arrow[d] & \\
\mathrm{Dmod}(\mathrm{Gr}_G) \otimes \mathrm{Dmod}(\mathrm{Gr}_G) \otimes \mathrm{Dmod}(\mathrm{Gr}_M) \arrow[r, "\langle , \rangle_{\mathrm{Gr}_G} \otimes \mathrm{Id}"] \arrow[d, "\langle , \rangle"] & \mathrm{Dmod}(\mathrm{Gr}_M) \\
\mathrm{Vect} &
\end{tikzcd}$$

$\Delta^{-,\frac{\infty}{2}} = i_!^! \omega_{S^-}$, $\quad S^- \xhookrightarrow{i} \mathrm{Gr}_G$

$$S^- = LN_{P^-}\cdot 1$$

$$\begin{tikzcd}
 & & \overline{\mathrm{Bun}}_{P^-} \times_{\mathrm{Bun}_G} \mathrm{Bun}_N^{\Omega(\omega)} \arrow[dl] \arrow[dr] & & \mathrm{Zast} \arrow[ll, hook'] \\
\mathrm{Bun}_{P^-} \arrow[r, hook] \arrow[d] & \overline{\mathrm{Bun}}_{P^-} \arrow[dr, "\text{proper}"'] & & \mathrm{Bun}_N^{\Omega(\omega)} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{Bun}_M & & \mathrm{Bun}_G & \mathbb{A}^1 & \mathrm{pt}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Dmod}(\mathrm{Bun}_M) \arrow[rr, "\mathbb{L}_M"] & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[ur, "\mathrm{CT}_*"] \arrow[rr, "\mathbb{L}_G"] & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[ur, "\mathrm{CT}^{\mathrm{Spec}}"] & \\
 & \mathrm{KL}(M)_{\mathrm{Ran}} \arrow[uu, "\mathrm{Loc}_M"] \arrow[rr] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mf}}_{\check{M}}) \arrow[uu, "\mathrm{Poinc}^{\mathrm{Spec}}_{\check{M}}"'] \\
\mathrm{KL}(G)_{\mathrm{Ran}} \arrow[uu, "\mathrm{Loc}_G"] \arrow[ur, "J^{(G)}_{KM}"] \arrow[rr, "\mathrm{FLE}_G", "\sim"'] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mf}}_{\check{G}}) \arrow[uu, "\mathrm{Poinc}^{\mathrm{Spec}}_{\check{G}}"'] \arrow[ur, "J_{\mathrm{Op}}^{\mathrm{fix}}"] &
\end{tikzcd}$$

want commutation of the top lid

- $J_{KM} = \mathrm{BRST} = C^{\frac{\infty}{2}}(L\mathfrak{n}_P, -)$

- J_{Op} = pull-push

$$\begin{tikzcd} & ? \arrow[dl] \arrow[dr] & \\ Op^{mf}_{\check{G}} & & Op^{mf}_{\check{M}} \end{tikzcd}$$

$$\begin{tikzcd} KL(M) \arrow[r, "FLE_M"] & \mathrm{IndCoh}^*(Op^{mf}_{\check{M}}) \\ KL(G) \arrow[u, "J_{KM}"] \arrow[r, "FLE_G"'] & \mathrm{IndCoh}^*(Op^{mf}_{\check{G}}) \arrow[u, "J_{Op}"] \end{tikzcd}$$

Bottom lid VERY HARD

Cor. $\mathbb{L}_G$ admits a left adjoint.

Proof

$$C \xrightarrow[F]{} D$$

WTS that $(\mathbb{L}_G)^L$ is defined on a generating set of objects.

$$\begin{tikzcd} C_1 \arrow[r, "F_1"] & D_1 \\ C \arrow[u, "i_C"] \arrow[r, "F"'] & D \arrow[u, "i_D"'] \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[d, "i_C^L"'] & D_1 \arrow[l, "F_1^L"'] \arrow[d, "i_D^L"] \\ C & D \arrow[l, "F^L"] \end{tikzcd}$$

Lemma F^L is defined on the essential image of i_D^L so that commutes

$$\begin{tikzcd} \mathrm{Whit}^!(\mathrm{Gr}_G) \arrow[r, "CS", "\sim"'] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"'] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{Spec}}_{\check{G}}"'] \end{tikzcd}$$

Poinc$_!$; $\mathrm{Loc}^{\mathrm{Spec}}_{\check{G}}$

Obtain $\mathbb{L}^L_G$ is defined on $QCoh(LS_{\check{G}}) \subset IndCoh_{Nilp}(LS_{\check{G}})$

$$\begin{tikzcd}
Whit^!(Gr_G) \arrow[r, "\sim", "LS_{\check{G}}"'] \arrow[d, "Poinc_!"'] & Rep(\check{G})_{Ran} \arrow[d, "Loc^{Spec}_{\check{G}}"] \\
Dmod(Bun_G) & IndCoh_{Nilp}(LS_{\check{G}}) \arrow[l, "\mathbb{L}^L_G"]
\end{tikzcd}$$

$$\begin{tikzcd}
Dmod(Bun_M) \arrow[r, "\mathbb{L}_M"] \arrow[d, shift right, "Eis_!"'] & IndCoh_{Nilp}(LS_{\check{M}}) \arrow[d, shift right, "Eis^{Spec}"'] \\
Dmod(Bun_G) \arrow[r, "\mathbb{L}_G"] \arrow[u, shift right, "CT_*"'] & IndCoh_{Nilp}(LS_{\check{G}}) \arrow[u, shift right, "CT^{Spec}"']
\end{tikzcd}$$

Obtain $\mathbb{L}^L_G$ is defined on the essential image of Eis^{Spec} and

$$\begin{tikzcd}
Dmod(Bun_M) \arrow[d, "Eis_!"'] & IndCoh_{Nilp}(LS_{\check{M}}) \arrow[l, "\mathbb{L}^L_M"'] \arrow[d, "Eis^{Spec}"] \\
Dmod(Bun_G) & IndCoh_{Nilp}(LS_{\check{G}}) \arrow[l, "\mathbb{L}^L_G"']
\end{tikzcd}$$

Lemma ([AG]) $IndCoh_{Nilp}(LS_{\check{G}})$ is generated by $QCoh(LS_{\check{G}})$ and Eis^{Spec}.

Def $Dmod(Bun_G)_{Eis} = \langle Eis_!,$ proper parabolics$\rangle$

$$\begin{tikzcd}
Dmod(Bun_G)_{Eis} \arrow[r, shift left] & Dmod(Bun_G) \arrow[l, shift left] \arrow[r, shift left, "e^L"] & Dmod(Bun_G)_{cusp} \arrow[l, shift left, "e"]
\end{tikzcd}$$

Def $\mathrm{IndCoh}_{Nilp}(LS_{\check{G}})_{Eis} = \langle \mathrm{Eis}^{Spec}, \text{ proper parabolics} \rangle$

$$\begin{tikzcd}
\mathrm{IndCoh}_{Nilp}(LS_{\check{G}})_{Eis} \arrow[r, shift left] \arrow[d, phantom, "\parallel"] & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}) \arrow[l, shift left] \arrow[r, shift left, "(e^{Spec})^L"] & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}})_{cusp} \arrow[l, shift left, "e^{Spec}"] \arrow[d, phantom, "\parallel"] \\
\mathrm{IndCoh}_{Nilp}(LS_{\check{G}})_{red} & & \mathrm{IndCoh}_{Nilp}(LS^{irred}_{\check{G}}) \arrow[d, phantom, "\parallel"] \\
& & \mathrm{QCoh}(LS^{irred}_{\check{G}})
\end{tikzcd}$$

$e^{Spec} = (j^{Spec})_*$

$(e^{Spec})^L = (j^{Spec})^*$

$$\begin{tikzcd}
\mathrm{Dmod}(\mathrm{Bun}_G)_{Eis} \arrow[r, shift left] \arrow[d, shift left, "\mathbb{L}_{G,Eis}"] & \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[l, shift left] \arrow[r, shift left, "e^L"] \arrow[d, shift left, "\mathbb{L}_G"] & \mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \arrow[l, shift left, "e"] \arrow[d, shift left] \\
\mathrm{IndCoh}_{Nilp}(LS_{\check{G}})_{red} \arrow[r, shift left] \arrow[u, shift left, "\mathbb{L}^L_{G,Eis}"] & \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}) \arrow[l, shift left] \arrow[r, shift left, "(j^{Spec})^*"] \arrow[u, shift left, "\mathbb{L}^L_G"] & \mathrm{QCoh}(LS^{irred}_{\check{G}}) \arrow[l, shift left, "(j^{Spec})_*"] \arrow[u, shift left]
\end{tikzcd}$$

Thm $(\mathbb{L}^L_{G,Eis}, \mathbb{L}_{G,Eis})$ are mutually inverse equivalences.

$$\mathrm{Eis}^{Spec} \longrightarrow \mathbb{L}_{G,Eis} \cdot \mathbb{L}^L_{G,Eis} \cdot \mathrm{Eis}^{Spec}$$

$$\cong$$

$$\mathbb{L}_G \cdot \mathrm{Eis}_! \cdot \mathbb{L}^L_M$$

$$\cong$$

$$\mathrm{Eis}^{Spec} \longrightarrow \mathrm{Eis}^{Spec} \cdot \mathbb{L}_M \cdot \mathbb{L}^L_M$$

$$\begin{tikzcd}
\mathrm{Whit}^!(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, phantom, "\simeq"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[u] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{Spec}}_{\check{G}}"'] \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{\mathrm{cusp}} \arrow[u, "e"] \arrow[uu, bend left] \arrow[r] & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[u, "(j^{\mathrm{spec}})_*"'] \arrow[uu, bend right]
\end{tikzcd}$$

<u>For</u> $G = GL_n$

Bonus Material D (Joakim Færgeman)

<u>Recall</u> $D(\mathrm{Bun}_G)_{\mathrm{Eis}} \simeq \mathrm{IndCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{Eis}}$

leaves $D(\mathrm{Bun}_G)_{\mathrm{cusp}} \xrightarrow{\mathbb{L}_{G,\mathrm{cusp}}} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$

<u>Goal</u>. $\mathbb{L}_{G,\mathrm{cusp}}$ is conservative: $\mathbb{L}_{G,\mathrm{cusp}}(\mathcal{F}) = 0 \Rightarrow \mathcal{F} = 0$.

In fact, we'll prove that $\mathbb{L}_{G,\mathrm{temp}} : D(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ is conservative.

<u>Rmk</u>. 1) To prove GLC, it remains to show that $\mathbb{L}^L_{G,\mathrm{temp}}$ fully faithful.

2) The above proof is microlocal in nature.

<u>Reduction</u>. <u>Claim</u> It suffices to show that $\mathbb{L}_{G,\mathrm{temp}} : D_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ is conservative.

<u>Pf</u> Given $\mathcal{F} \in D(\mathrm{Bun}_G)_{\mathrm{temp}}$, $\exists\ \sigma : \mathrm{Spec}\, k \to \mathrm{LS}_{\check{G}}$, $\sigma \otimes \mathcal{F} \neq 0$,

$\sigma : \operatorname{Spec} k \to LS^{restr}_{\check{G}} \longrightarrow LS_{\check{G}}$

so $\sigma \otimes \mathcal{F} \in D(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(LS_{\check{G}})}{\otimes} \mathrm{QCoh}(LS^{restr}_{\check{G}}) \overset{[AGKRRV]}{\simeq} D_{Nilp}(\mathrm{Bun}_G)$

If $0 \neq \mathbb{L}_{G,temp}(\sigma \otimes \mathcal{F}) = \sigma \otimes \underbrace{\mathbb{L}_{G,temp}(\mathcal{F})}_{\neq 0}$

Going to explicitly describe $\mathrm{Ker}\left(\mathbb{L}_{G,coarse} : D_{Nilp}(\mathrm{Bun}_G) \longrightarrow \mathrm{QCoh}(LS^{restr}_{\check{G}})\right)$

<u>Def'n</u> $x \in \mathfrak{g}^* \simeq \mathfrak{g}$ regular if $C_{\mathfrak{g}}(x)$ minimal dim $(= \mathrm{rk}(\mathfrak{g}))$

<u>Better for us:</u> fix some non-degenerate character $\psi \in \mathfrak{n}^*$,

Consider $\mathfrak{g}^*/N = \mathfrak{g}^*/G \underset{pt/G}{\times} pt/N$

<u>Define</u> Kos_ψ as sitting in a Cartesian diagram:

$$\begin{tikzcd} \mathrm{Kos}_\psi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \psi/N \arrow[d] \\ \mathfrak{g}^*/N \arrow[r] & \mathfrak{n}^*/N \end{tikzcd}$$

Identifying $\mathfrak{g}^* \simeq \mathfrak{g}$, $\mathfrak{n}^* \simeq \mathfrak{n}^-$, $\psi = f$, then $\mathrm{Kos}_\psi \simeq f + \mathfrak{b}/N \quad (\simeq f + \mathfrak{g}^e)$

The map $f + \mathfrak{b}/N \to \mathfrak{g}_{reg}/G$ bijection on k-pts

$\mathfrak{g}_{irreg} := \mathfrak{g} \setminus \mathfrak{g}_{reg}$, $\quad \mathcal{N}_{irreg} = \mathcal{N} \cap \mathfrak{g}_{irreg}$

<u>Globally</u> Define $\mathrm{Nilp}_{irreg} \subset T^* \mathrm{Bun}_G$ to be pairs

$(\mathcal{P}_G, \varphi \in \Gamma(X, \mathfrak{g}^*_{\mathcal{P}_G} \otimes \Omega^1_X))$ s.t. φ is nilp. irreg (i.e. locally factors through $\mathcal{N}_{irreg}$)

$\ulcorner$ $T_{P_G} \mathrm{Bun}_G = H^{*}(X, \mathfrak{g}_{P_G})[1]$. Using Serre Duality,

$$H^0 T^*_{P_G} \mathrm{Bun}_G = \Gamma(X, \mathfrak{g}^*_{P_G} \otimes \omega^!_X)$$
$\lrcorner$

Thm A $\quad \mathrm{Ker}\left(\mathbb{L}_{G,\mathrm{coarse}} \middle|_{D_{\mathrm{Nilp}}(\mathrm{Bun}_G)}\right) = D_{\mathrm{Nilp\,irreg}}(\mathrm{Bun}_G)$

Thm B $\quad D_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \cap D_{\mathrm{Nilp\,irreg}}(\mathrm{Bun}_G) = 0$.

$G = SL_2$: $\mathrm{Nilp\,irreg} = \{0\}$

$$D_{\mathrm{Nilp\,irreg}}(\mathrm{Bun}_G) = \{\mathcal{F} \in D(\mathrm{Bun}_G) : H^i(\mathcal{F}) = \bigcup (\text{ext's of } \omega)\}$$

Thm A

$$\begin{tikzcd} & \mathrm{QCoh}(LS_{\check{G}}) \arrow[d, "\Gamma"] \\ \mathrm{Whit}_0 : D(\mathrm{Bun}_G) \arrow[ur, "\mathbb{L}_{G,\mathrm{coarse}}"] \arrow[r] & \mathrm{Vect} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Bun}_N^{\Omega} \arrow[r, "p"] \arrow[d, "\psi"'] & \mathrm{Bun}_G \\ \mathbb{A}^1 & \end{tikzcd} \qquad \mathrm{Whit}_0(\mathcal{F}) = C_{dR}\left(\mathrm{Bun}_N^{\Omega}, p^! \mathcal{F} \overset{!}{\otimes} \psi^!(\exp)\right)$$

Thm 1A $\quad \mathrm{Whit}_0 \middle|_{D_{\mathrm{Nilp}}(\mathrm{Bun}_G)}$ satisfies

1) t-exact

2) commutes w/ Verdier duality

3) $\mathcal{F} \in D_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\heartsuit}$, then $\dim \mathrm{Whit}_0(\mathcal{F}) = \mathrm{mult}(\mathrm{Nilp}^{\mathrm{Kos}}, CC(\mathcal{F}))$

Defining $\mathcal{N}ilp^{Kos}$:

$\psi : Bun_N^{\Omega} \longrightarrow \mathbb{A}^1$

$\rightsquigarrow d\psi : Bun_N^{\Omega} \longrightarrow T^* Bun_N^{\Omega}$

Define Kos_ψ^{glob} by

$$\begin{tikzcd} Kos_\psi^{glob} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Bun_N^{\Omega} \arrow[d, "d\psi"] \\ T^*Bun_G \times_{Bun_G} Bun_N^{\Omega} \arrow[r, "dp"] & T^* Bun_N^{\Omega} \end{tikzcd}$$

Key properties

$$Kos_\psi \times_{\mathfrak{g}^*/G} \mathcal{N}/G = \{f\}$$

$$Kos_\psi^{glob} \times_{T^*Bun_G} \mathcal{N}ilp = \{f^{glob}\}$$

Define $\mathcal{N}ilp^{Kos} \subset \mathcal{N}ilp$ the irred. comp. containing f^{glob}.

$G = SL_2$

$$\mathcal{N}ilp = \bigcup_{d \geq 1-g} \overline{\mathcal{N}ilp^d}$$

$$\mathcal{N}ilp^d = \{ 0 \to \underset{\deg \mathcal{L} = d}{\mathcal{L}} \to \mathcal{E} \to \mathcal{L}^\vee \to 0 ,\ \underset{0 \neq}{\varphi} : \mathcal{L}^\vee \to \mathcal{L} \otimes \Omega \}$$

$$\mathcal{E} \longrightarrow \mathcal{L}^\vee \xrightarrow{\varphi} \mathcal{L} \otimes \Omega^1_X \longrightarrow \mathcal{E} \otimes \Omega^1_X$$

$Nilp^{1-g} = Nilp^{kos}$

$$f^{glob} = \Big(\Omega^{-1/2} \oplus \Omega^{1/2}, \quad \varphi : \Omega^{-1/2} \oplus \Omega^{1/2} \to (\Omega^{-1/2} \oplus \Omega^{1/2}) \otimes \Omega$$
$$= \Omega^{1/2} \oplus \Omega^{3/2} \Big)$$
$$\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Thm 2A If $\mathcal{F} \in D_{Nilp}(Bun_G)$ s.t. $SS(\mathcal{F}) \not\subset Nilp_{irreg}$, then $\exists\ \check{\lambda} \in \check{\Lambda}^{+}$

$$SS(\mathcal{H}_{V_x^{\check{\lambda}}} * \mathcal{F}) \supset Nilp^{kos}.$$

$\forall x \in X$

Exer formulate and prove analogous result for $D_{\mathcal{N}}(\mathfrak{g}/G)$ using Fourier transform

Thm 1A + 2A $\Rightarrow$ Thm A.

$\uparrow$

$$D_{Nilp_{irreg}}(Bun_G) = \mathrm{Ker}\left(\mathbb{L}_{G,coarse} \big| D_{Nilp}(Bun_G)\right)$$

"$\subset$" $\mathcal{F} \in D_{Nilp_{irreg}}(Bun_G)$, we get that $\mathrm{Whit}_0(\mathcal{H}_{V_x^{\check{\lambda}}} * \mathcal{F}) = 0$

This means $\Gamma(LS_{\check{G}}^{restr}, \mathbb{L}_{G,coarse}(\mathcal{F}) \otimes E_x^{\check{\lambda}}) = 0$

$\Rightarrow \mathbb{L}_{G,coarse}(\mathcal{F}) = 0$ because $LS_{\check{G}}^{restr} \underset{pt/G}{\times} pt$ formal affine scheme.

"$\supset$" Given $\mathcal{F} \in \mathrm{Ker}(\quad)$, $\Gamma(LS_{\check{G}}^{restr}, E_x^{\check{\lambda}} \otimes \mathbb{L}_{G,coarse}(\mathcal{F})) = 0, \ \forall \check{\lambda}$

$$\| $$
$$\mathrm{Whit}_0(\mathcal{H}_{V_x^{\check{\lambda}}} * \mathcal{F})$$

$\Rightarrow SS(\mathcal{F}) \subset Nilp_{irreg}$ $\square$.

Thm A + Thm B $\Rightarrow$ $\ker\left(\mathbb{L}_{G,\text{coarse}} \big|_{D_{Nilp}(Bun_G)}\right) \cap D_{Nilp}(Bun_G)_{temp} = 0.$

Lecture 9 (Dennis Gaitsgory) Ambidexterity - I

$$\begin{tikzcd}
Dmod(Bun_G)_{Eis} \arrow[r, shift left] \arrow[d, shift left, "\mathbb{L}_G"] & Dmod(Bun_G) \arrow[l, shift left] \arrow[r, shift left, "e^L"] \arrow[d, shift left, "\mathbb{L}_G"] & Dmod(Bun_G)_{cusp} \arrow[l, shift left, "e"] \arrow[d, shift left, "\mathbb{L}_{G,cusp}"] \\
IndCoh_{Nilp}(LS_{\check{G}})_{red} \arrow[u, shift left, "\mathbb{L}^L_G"] \arrow[u, phantom, "\simeq"] \arrow[r, shift left] & IndCoh_{Nilp}(LS_{\check{G}}) \arrow[l, shift left] \arrow[u, shift left, "\mathbb{L}^L_G"] \arrow[r, shift left, "(j^{spec})^*"] & IndCoh_{Nilp}(LS^{irred}_{\check{G}}) \arrow[l, shift left, "(j^{spec})_*"] \arrow[u, shift left, "\mathbb{L}^L_{G,cusp}"]
\end{tikzcd}$$

$$IndCoh_{Nilp}(LS^{irred}_{\check{G}}) = QCoh(LS^{irred}_{\check{G}})$$

$\mathbb{L}_{G,temp} : Dmod(Bun_G)_{temp} \longrightarrow QCoh(LS_{\check{G}})$ is conservative

$$\Downarrow$$

$\mathbb{L}_{G,cusp} : Dmod(Bun_G)_{cusp} \longrightarrow QCoh(LS^{irred}_{\check{G}})$ is conservative

Goal to show $\mathbb{L}^L_{G,cusp}$ is fully faithful.

$$\mathbb{L}_{G,temp} \circ \mathbb{L}^L_{G,temp} : QCoh(LS_{\check{G}}) \longrightarrow QCoh(LS_{\check{G}})$$

$$\simeq \quad \mathcal{A}_G \otimes - , \qquad \mathcal{A}_G \in AssocAlg(QCoh(LS_{\check{G}}))$$

$$\mathbb{L}_{G,cusp} \circ \mathbb{L}^L_{G,cusp}$$

$$= \quad \mathcal{A}_{G,cusp} \otimes - , \qquad \mathcal{A}_{G,cusp} = (j^{spec})^*(\mathcal{A}_G)$$

Fully-faithfulness $\Leftrightarrow$ $\mathcal{O}_{LS_{\check{G}}} \xrightarrow{\sim} \mathcal{A}_G$ is an isom.

on cusp part $\Leftrightarrow$ $\mathcal{O}_{LS^{irred}_{\check{G}}} \xrightarrow{\sim} \mathcal{A}_{G,cusp}$ is an isom.

Know already: $(\iota^{spec})^{\wedge !}(\mathcal{O}_{LS_{\check{G}}}) \xrightarrow{\sim} ((\iota^{spec})^{\wedge})^{!}(\mathcal{A}_G)$

<u>Thm</u> (Ambidexterity) $\mathbb{L}_{G,cusp}$ is ambidexterous $\Leftrightarrow$ $\mathbb{L}^L_{G,cusp} \simeq \mathbb{L}^R_{G,cusp}$

eg. $\mathrm{Vect} \xrightarrow[\otimes V]{} \mathrm{Vect}$

<u>Cor</u> $\mathcal{A}_{G,cusp}$ is perfect $\in \mathrm{QCoh}(LS^{irred}_{\check{G}})$ and self-dual.

$\mathcal{A}_{G,cusp} \otimes - = \mathbb{L}_{G,cusp}\mathbb{L}^L_{G,\cancel{cusp}}$

$(\mathbb{L}_{G,cusp} \cdot \mathbb{L}^L_{G,cusp})^R = (\mathbb{L}^L_{G,cusp})^R \circ \mathbb{L}^R_{G,cusp} = \mathbb{L}_{G,cusp} \cdot \mathbb{L}^L_{G,cusp}$

$C^\vee = \mathrm{Funct}_{cts}(C, \mathrm{Vect})$

$C = \mathrm{Ind}(C^c)$ $\qquad$ $C \otimes C^\vee \xrightarrow{ev} \mathrm{Vect}$

$C^\vee = \mathrm{Ind}((C^c)^{op})$ $\qquad$ $\mathrm{unit}_C \in C \otimes C^\vee$.

$C \xrightarrow{F} D$

$C^\vee \xleftarrow{F^\vee} D^\vee$

$$C \underset{F^R}{\overset{F}{\rightleftarrows}} D \qquad F \text{ preserves compactness}$$

$$C^c \xrightarrow{F} D^c$$

$$(C^c)^{op} \xrightarrow{F^{op}} (D^c)^{op}$$

$$\wr$$

$$C^\vee \xrightarrow{F^{op}} D^\vee$$

<u>Lemma</u> $(F^{op}, F^\vee)$ is an adjoint pair.

$$\mathbb{D}_C : C^c \to (C^\vee)^c$$

$$F^{op} = \mathbb{D}_D \circ F \circ \mathbb{D}_C$$

<u>Example</u> $C = \mathrm{Dmod}(\mathcal{Y})$, $\mathcal{Y}$ - qc alg stack

$$\mathrm{Dmod}(\mathcal{Y}) \otimes \mathrm{Dmod}(\mathcal{Y}) \xrightarrow{\langle , \rangle} \mathrm{Vect}$$

$$\mathcal{F}_1 \quad \mathcal{F}_2 \longmapsto C_{dR}(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

$$\Delta_*(\omega_{\mathcal{Y}}) \in \mathrm{Dmod}(\mathcal{Y} \times \mathcal{Y}) \simeq \mathrm{Dmod}(\mathcal{Y}) \otimes \mathrm{Dmod}(\mathcal{Y})$$

$$(\mathrm{Dmod}(\mathcal{Y})^c)^{op} = \mathrm{Dmod}(\mathcal{Y})^c, \quad \mathbb{D}_{\mathcal{Y}}$$

$$\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$$

$$f_* : \mathrm{Dmod}(\mathcal{Y}_1) \longrightarrow \mathrm{Dmod}(\mathcal{Y}_2)$$

$$(f_*)^\vee : \mathrm{Dmod}(\mathcal{Y}_2) \longrightarrow \mathrm{Dmod}(\mathcal{Y}_1)$$

$$\parallel$$

$$f^!$$

$$(f_*)^{op} = f_! \simeq f_*, \quad f \text{ proper}$$

f smooth, $(f^!)^{op} = f^*$

$\mathcal{U} \subset \mathrm{Bun}_G$

We obtain $\mathrm{Dmod}(\mathcal{U})$ is self-dual.

Take $\mathcal{U}$ to be large enough so that $\mathcal{F} \in \mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \Rightarrow \mathcal{F} \xrightarrow{\sim} j_* j^* \mathcal{F}$.

$$\begin{tikzcd} C_1 \arrow[r, shift right, "e"'] & C \arrow[l, shift right, "e^L"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \arrow[r, shift right, "e_{\mathcal{U}}"'] \arrow[rr, bend right, "e"'] & \mathrm{Dmod}(\mathcal{U}) \arrow[l, shift right, "(e_{\mathcal{U}})^L"'] \arrow[r, "j_*"] & \mathrm{Dmod}(\mathrm{Bun}_G) \end{tikzcd}$$

$$\begin{tikzcd} C_1^\vee \arrow[r, shift right, "(e^L)^\vee"'] & C^\vee \arrow[l, shift right, "e^\vee"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G)^\vee_{cusp} \arrow[r, shift right, "(e_{\mathcal{U}}^L)^\vee"'] & \mathrm{Dmod}(\mathcal{U}) \arrow[l, shift right, "(e_{\mathcal{U}})^\vee"'] \end{tikzcd}$$

<u>Lemma</u> $\mathrm{Dmod}(\mathrm{Bun}_G)^\vee_{cusp} = \mathrm{Dmod}(\mathrm{Bun}_G)_{cusp}$

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G)^\vee_{cusp} \arrow[dr, hook] & & \mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \arrow[dl, hook'] \\ & \mathrm{Dmod}(\mathcal{U}) & \end{tikzcd}$$

$\mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \subset \mathrm{Dmod}(\mathcal{U})$ can be realized as $(j^* \circ \mathrm{Eis}_!)^\perp$

$(\mathrm{Dmod}(\mathrm{Bun}_G)_{cusp})^\vee \subset \mathrm{Dmod}(\mathcal{U})$ can be realized as $(\mathbb{D}_{\mathcal{U}} \cdot (j^* \cdot \mathrm{Eis}_!) \cdot \mathbb{D}_{\mathrm{Bun}_M})^\perp$

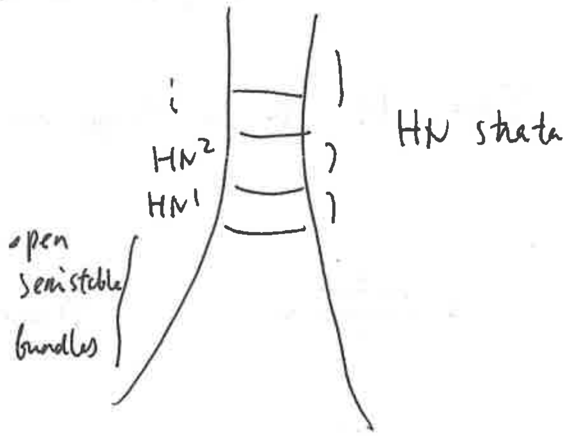
$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_M \end{tikzcd}$$

$\mathrm{Eis}_! = p_! \circ q^*$, $\quad \mathrm{Eis}_* = p_* \circ q^!$

<u>Claim</u> $\mathrm{Eis}_!$ and Eis_* after composing w/ j^* are finite extensions of copies of each other

$$\mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \xrightarrow{\mathbb{L}_{G,cusp}} \mathrm{QCoh}(\mathrm{LS}^{irred}_{\check{G}})$$

$$\mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \xleftarrow{\mathbb{L}^{\vee}_{G,cusp}} \mathrm{QCoh}(\mathrm{LS}^{irred}_{\check{G}})$$



__Thm 1__ $\quad \mathbb{L}^{\vee}_{G,cusp} = (\mathbb{L}_{G,cusp})^L$

__Thm 2__ $\quad \mathbb{L}^{\vee}_{G,cusp} = (\mathbb{L}_{G,cusp})^R$

$$HN^n = \{ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{\vee} \to 0, \ \deg \mathcal{L} = n > 0 \}$$

$\mathbb{L}^{L}_{G,cusp} \simeq (\mathbb{L}^{L}_{G,cusp})^{op} \Leftrightarrow$ __Thm 1__

$n \gg 0$, $H^1(\mathcal{L}^2) = 0$.

__Ex__ $\quad HN^n$ unip gerbe over $\mathrm{Bun}^n_{G_m}$, $n \gg 0$

__Exer__ $\quad CT^?_* =$!-restr. to HN^n for $n \gg 0$

$$\begin{tikzcd}
\mathrm{Whit}^!(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "CS_G", "\sim"'] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{spec}_{\check{G}}"'] \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \arrow[u, "e"] \arrow[r, "\mathbb{L}_{G,cusp}"'] & \mathrm{QCoh}(\mathrm{LS}^{irred}_{\check{G}}). \arrow[u, "(\iota^{spec})_*"']
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{cusp} \arrow[ur, "CS \circ \mathrm{coeff} \circ e"] \arrow[rr, "\mathbb{L}_{G,cusp}"'] & & \mathrm{QCoh}(\mathrm{LS}^{irred}_{\check{G}}) \arrow[ul, "\Gamma^{\check{G}}_{spec} \circ (\iota^{spec})_*"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Vect} \arrow[dl, "((\mathrm{coeff}^{\mathrm{Vac}} \circ e)^L)^{op}"'] \arrow[dr, "((\Gamma(LS_{\check{G}}, -) \circ j_*^{\mathrm{Spec}})^L)^{op}"] & \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{\mathrm{cusp}} & & \mathrm{QCoh}(LS^{\mathrm{irred}}_{\check{G}}) \arrow[ll, "((\mathbb{L}_{G,\mathrm{cusp}})^L)^{op}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Vect} \arrow[dl, "(\mathrm{coeff}^{\mathrm{Vac}} \circ e)^L"'] \arrow[dr, "(\Gamma(LS_{\check{G}}, -) \circ j_*^{\mathrm{Spec}})^L"] & \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{\mathrm{cusp}} & & \mathrm{QCoh}(LS^{\mathrm{irred}}_{\check{G}}) \arrow[ll, "(\mathbb{L}_{G,\mathrm{cusp}})^L"]
\end{tikzcd}$$

$$\underset{\mathcal{O}_{LS^{\mathrm{irred}}_{\check{G}}}}{\underset{\parallel}{(\Gamma(LS_{\check{G}}, -) \circ j_*^{\mathrm{Spec}})^L}} = (j_*^{\mathrm{Spec}})^L \circ \Gamma(LS_G, -)^L = (j^{\mathrm{Spec}})^* \circ \mathcal{O}_{LS_{\check{G}}}$$

$$\underset{\mathcal{O}_{LS^{\mathrm{irred}}_{\check{G}}}}{\underset{\parallel}{(\Gamma(LS_{\check{G}}, -) \circ j_*^{\mathrm{Spec}})^\vee}} = (j_*^{\mathrm{Spec}})^\vee \circ \Gamma(LS_{\check{G}}, -)^\vee = (j^{\mathrm{Spec}})^* \circ \mathcal{O}_{LS_{\check{G}}}$$

$$(\mathrm{coeff}^{\mathrm{Vac}} \circ e)^L = e^L \cdot (\mathrm{coeff}^{\mathrm{Vac}})^L = e^L(\mathrm{Poinc}^{\mathrm{Vac}}_!) \in \mathrm{Dmod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

$$((\mathrm{coeff}^{\mathrm{Vac}} \circ e)^L)^{op} \simeq \underset{e^L(\mathrm{Poinc}^{\mathrm{Vac}}_!)}{\underset{\parallel}{(\mathrm{coeff}^{\mathrm{Vac}} \circ e)^L}}$$

$$D(\mathrm{Bun}_G) \xrightarrow{j^*} \mathrm{Dmod}(U)$$

$$e^L(\mathrm{Poinc}^{\mathrm{Vac}}_!) = e^L\Big(\underset{\mathrm{Poinc}^{\mathrm{Vac}}_*}{\underset{\parallel}{\mathbb{D}(\mathrm{Poinc}^{\mathrm{Vac}}_!)}}\Big)$$

$$e^L(\mathrm{Poinc}^{\mathrm{Vac}}_!) = e^L(\mathrm{Poinc}^{\mathrm{Vac}}_*)$$

$$\begin{tikzcd}
 & \mathrm{Bun}_N^{\Omega} \arrow[dl, "p"'] \arrow[dr, "ev"] & \\
\mathrm{Bun}_G & & \mathbb{A}^1
\end{tikzcd}$$

Lecture 10 (Lin Chen)

Goal 1 : $KL_{Ran} \xrightarrow{Loc} D(Bun_G) \xrightarrow{j^*} D(U)$ is a quotient, For $U \xrightarrow{j} Bun_G$

Goal 2. $\mathbb{L}^{\vee}_{G,cusp} \xrightarrow{\sim} \mathbb{L}^{R}_{G,cusp}$

Goal 3 $\mathcal{A}_{G,cusp} \otimes - \overset{def}{:=} \mathbb{L}_{G,cusp} \circ \mathbb{L}^{L}_{G,cusp} : QCoh(LS^{irred}_{\check{G}}) \to QCoh(LS^{irred}_{\check{G}})$

$\sigma \in LS^{irred}_{\check{G}}$, $\mathcal{A}_{G,cusp}|_\sigma = C_\cdot(\qquad)$

Goal 1 ; General paradigm $F : \mathcal{C} \to \mathcal{D}$

$F^R : \mathcal{D} \to \mathcal{C}$

$$(F^\vee)^L = (F^R)^\vee = F^{conj} : \mathcal{C}^\vee \to \mathcal{D}^\vee$$

Exer $unit_1 \in \mathcal{C} \otimes \mathcal{C}^\vee$

$unit_2 \in \mathcal{D} \otimes \mathcal{D}^\vee$

Construct $F \otimes F^{conj}(unit_1)$

$\downarrow$

$unit_2$

Exer. TFAE

- F is a Verdier quot
- F^{conj} is a Verdier quot
- $F \otimes F^{conj}(unit_1) \xrightarrow{\sim} unit_2$
- $F \circ F^R \xrightarrow{\sim} id$

Prop 1 $\quad j^* \circ \mathrm{Loc}$ is a quotient.

Proof.

$$\begin{tikzcd}
KL_{Ran} \arrow[r, "Loc"] & D(Bun_G) \arrow[r, "j^*"] & D(U) \\
Rep(L^+G)_{Ran} \arrow[u, "ind"] \arrow[r, "Loc^{QCoh!}"] & QCoh(Bun_G) \arrow[u, "ind"] \arrow[r, "j^*"] & QCoh(U) \arrow[u, "ind"]
\end{tikzcd}$$

$j^* \circ Loc^{QCoh}$ preserves cpt's.

$$\begin{tikzcd}
& U \times Ran \arrow[dl, "*\text{-pull}"'] \arrow[dr, "\int_{Ran}"] & \\
\mathbb{B}L^+G_{Ran} & & U
\end{tikzcd}$$

$\Rightarrow j^* \circ Loc$ preserves cpt's

$KL \simeq KL^\vee$

$D(U) \simeq D(U)^\vee$

Claim (Exer.) $\quad j^* \circ Loc$ is self-conjugate via the above self-dualities

$$\mathfrak{g}\text{-mod}^K \xrightarrow{Loc^{l/r}} D^{l/r}(\tilde{Y}/K)$$

$$Loc^l \xleftrightarrow{conj.} Loc^r$$

Apply paradigm, only need $(j^* \otimes j^*)(Loc \otimes Loc)(unit_{KL}) \xrightarrow{\sim} \Delta_{*,dR}\, \omega_U$

$\Delta : U \to U \times U$

Stronger claim: $\boxed{\text{Key Cal 1}}$

$(Loc \otimes Loc)(unit_{KL}) \xrightarrow{\sim} \Delta_{*,dR}\, \omega_{Bun_G}$, $\Delta : Bun_G \to Bun_G \times Bun_G$.

Rmk. ① $unit_{KL}$ is CADO $\quad$ chiral alg. diff. operator

$$D(LG) \xrightarrow{\Gamma^{IndCoh}} Lg\text{-mod} \otimes Lg\text{-mod}$$

$LG \curvearrowright LG \curvearrowleft LG$ $\quad \delta_{L^+G} \longmapsto$ CADO $\quad (L^+G \times L^+G\text{-integrable})$

$(L^+G \times L^+G\text{-inv})$ $\quad$ (fact. unit) $\quad$ Pagelog

② $\mathrm{Rep}(L^+G) \longrightarrow \mathrm{QCoh}(U)$ ✓

$\boxed{\text{Key Cal 2}}$ $\mathrm{Loc}^{\mathrm{QCoh}} \otimes \mathrm{Loc}^{\mathrm{QCoh}}(\mathrm{unit}_{\mathrm{Rep}(L^+G)})$ is $\Delta_* \mathcal{O}_{\mathrm{Bun}_G}$ $\Leftrightarrow$ $j^*, \mathrm{Loc}^{\mathrm{QCoh}}$ is a quotient $\forall j : U \to \mathrm{Bun}_G$

③ Ex. Key 2 $\Rightarrow$ Key 1.

$(\mathrm{ind} \otimes \mathrm{id})(\mathrm{unit}_{\mathrm{Rep}(L^+G)}) \simeq (\mathrm{id} \otimes \mathrm{oblv})(\mathrm{unit}_{KL})$ $\left[F \otimes \mathrm{id}(\mathrm{unit}_1) = \mathrm{id} \otimes F^\vee(\mathrm{unit}_2) \right]$

④ $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}^{\mathrm{Spec}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

$\boxed{\text{Key Cal 3}}$ $\mathrm{Loc}^{\mathrm{Spec}} \otimes \mathrm{Loc}^{\mathrm{Spec}}(\mathrm{unit}_{\mathrm{Rep}(\check{G})}) \longrightarrow \Delta_* \mathcal{O}_{\mathrm{LS}_{\check{G}}}$

$\Leftrightarrow$ $\mathrm{Loc}^{\mathrm{Spec}}$ is a quotient

⑤ relative Nick

$$\mathrm{Fun}(\mathrm{Sect}_\nabla(X, Z)) = C^{\mathrm{fact}}_{\cdot}(X, \mathrm{Fun}(L^+_\nabla Z)) \qquad \begin{matrix} Z \\ \downarrow \\ pt \end{matrix}$$

$Z \to Y$ $\quad (\mathbb{B}\check{G} \to \mathbb{B}\check{G} \times \mathbb{B}\check{G})$

for Key 3

$W(\mathbb{B}G) \to W(\mathbb{B}G \times \mathbb{B}G)$ for Key 2

Weil restr. along $X \to X_{dR}$

⑥ $\mathrm{Poinc}_! \otimes \mathrm{Poinc}_*(\mathrm{unit}_{\mathrm{Whit}}) \overset{?}{\longrightarrow} \Delta_* \omega_{\mathbb{B}\check{G}}$ can't prove.

Goal 2

$$\begin{tikzcd} \mathrm{KL}_{\mathrm{Ran}} \arrow[rr, shift left, "\mathrm{FLE}", "\sim"'] \arrow[d, shift right, "\mathrm{Loc}^{\mathrm{cusp}}"'] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mf}}_{\check{G}})_{\mathrm{Ran}} \arrow[ll, shift left, "\mathrm{FLE}^{-1}"] \arrow[d, shift right, "\mathrm{Poinc}^{\mathrm{irred}}"'] \\ D(\mathrm{Bun}_G)_{\mathrm{cusp}} \arrow[u, shift right, "\Gamma^{\mathrm{cusp}}"'] \arrow[rr, shift left, "\mathbb{L}_{\mathrm{cusp}}"] & & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[ll, shift left, "\mathbb{L}^R_{\mathrm{cusp}}"] \arrow[u, shift right, "\mathrm{coeff}^{\mathrm{irred}}"'] \end{tikzcd}$$

Claim Every cat above is self-dual.

Every functor is self-conjugate.

① $\mathrm{KL} \simeq \mathrm{KL}^\vee$

② $D_{\mathrm{cusp}} \simeq D_{\mathrm{cusp}}^\vee$

③ $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})^\vee$

④* $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mf}}_{\check{G}}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mf}}_{\check{G}})^\vee$

Cheat: FLE is an equivalence ① ⇒ ④

$\mathrm{IndCoh}_* \simeq (\mathrm{IndCoh}^!)^\vee$

$\mathrm{IndCoh}^!(Y) \sim \mathrm{IndCoh}_*(Y)$

Need ω take in $\mathrm{IndCoh}_*(Y)$

⑤ FLE ✓

⑥ $\mathrm{Loc}^{\mathrm{cusp}}$ ✓

⑦ $\mathrm{Poinc}^{\mathrm{irred}}$ is self-conjugate

⑧ $\mathbb{L}_{\mathrm{cusp}}$ is ...

⑦ ⇒ ⑧ ✓

Goal 2 $\mathbb{L}^{\mathrm{conj}}_{\mathrm{cusp}} = \mathbb{L}_{\mathrm{cusp}}$

$\overset{\shortparallel}{(\mathbb{L}^R_{\mathrm{cusp}})^\vee} \rightsquigarrow \mathbb{L}^R_{\mathrm{cusp}} \simeq \mathbb{L}^\vee_{\mathrm{cusp}}$

Recall

$$\begin{tikzcd} & \mathrm{Op}^{\mathrm{mf,glob,irred}}_{\check{G},\mathrm{Ran}} \arrow[dl, "\pi'"'] \arrow[r] & \mathrm{Op}^{\mathrm{mf,\,glob}}_{\check{G},\mathrm{Ran}} \arrow[dl, "\pi"] \arrow[dr, "s"] \arrow[dr, bend left, "s'"] & \\ \mathrm{LS}^{\mathrm{irred}}_{\check{G}} \arrow[r] & \mathrm{LS}_{\check{G}} & & \mathrm{Op}^{\mathrm{mf}}_{\check{G},\mathrm{Ran}} \end{tikzcd}$$

By def'n,
$$\mathrm{Poinc}^{\mathrm{irred}} = \pi'_* \circ \boxed{s'^*}$$
$$(\mathrm{Poinc}^{\mathrm{irred}})^{\mathrm{conj}} = \pi'_! \cdot \boxed{(s')^*}$$

<u>Lemma</u>. π' is proper.

$$\mathcal{A}_{G,\mathrm{cusp}} \otimes - = \mathbb{L}_{\mathrm{cusp}} \cdot \mathbb{L}^{L}_{\mathrm{cusp}}$$
$$= \mathbb{L}_{\mathrm{cusp}} \cdot \mathbb{L}^{R}_{\mathrm{cusp}} \quad (\text{as endomorphisms})$$
$$\wr\!\wr \quad (\text{b/c } \Gamma_{\mathrm{cusp}} \text{ is f.f.})$$
$$\mathbb{L}_{\mathrm{cusp}} \cdot \Gamma^{\mathrm{cusp}} \cdot \mathrm{Loc}^{\mathrm{cusp}} \cdot \mathbb{L}^{R}_{\mathrm{cusp}}$$
$$\wr\!\wr$$
$$\mathrm{Poinc}^{\mathrm{irred}} \cdot \mathrm{FLE}^{-1} \cdot \mathrm{FLE} \cdot \mathrm{coeff}^{\mathrm{irred}}$$
$$\wr\!\wr$$
$$\mathrm{Poinc}^{\mathrm{irred}} \cdot \mathrm{coeff}^{\mathrm{irred}}$$

<u>Want</u>. $\mathrm{Poinc}^{\mathrm{irred}} \cdot \mathrm{coeff}^{\mathrm{irred}}\left(\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}\right) = \mathcal{A}_{G,\mathrm{cusp}}$

$\pi_! \cdot s^* \cdot s_* \cdot \pi^!$

Prop 1. $\pi_! \cdot s^* \cdot s_* \cdot \pi^! \xrightarrow{\sim} \pi_! \cdot \pi^!$

Prop 2

$$\begin{tikzcd}
Op^{mf, glob, irred}_{\check{G}, Ran} \arrow[d, hook] \arrow[dd, bend right, "\pi"'] & Op_{\sigma, Ran} \arrow[d] \\
(LS^{irred}_{\check{G}})^{\wedge} \arrow[d, "\omega"] & (Op_{\sigma, Ran})_{dR} \arrow[d] \\
LS^{irred}_{\check{G}} & \sigma \arrow[l, phantom, "\ni"]
\end{tikzcd}$$

Prop 2. $\pi_! \pi^! \xrightarrow{\sim} \omega_! \omega^!$

Prop 2' $End_{IndCoh}(\omega_{Op_{\sigma, Ran}}) = End_{Dmod}(\omega_{Op_{\sigma, Ran}}) = C\cdot(Op_{\sigma, Ran})$

$Op_{\check{G}} \longrightarrow \mathbb{B}\check{G}$

$Z \longrightarrow X$ affine D-scheme

$$\begin{tikzcd}
& Sect_{\nabla}(X - \underline{x}, Z)_{Ran} \arrow[dl, "\pi"'] \arrow[d] \arrow[dr, "s"] & \\
* & Sect_{\nabla}(X - \underline{x}, Z)_{Ran, dR} \arrow[l, "\pi_{dR}"] & L_{\nabla}(Z)_{Ran}
\end{tikzcd}$$

Prop 1. $\pi_! \cdot s^* \cdot s_* \cdot \pi^! \xrightarrow{\sim} \pi_! \pi^!$

Prop 2 $\pi_! \circ \pi^! = \pi_{!, dR} \cdot \pi^!_{dR}$

Lecture 11 (Kevin Lin)

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] \arrow[d, "\mathrm{coeff}^{\mathrm{Vac}}_*"'] & \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}}) \arrow[d, "\Gamma"] \\ \mathrm{Vect} \arrow[r, equal] & \mathrm{Vect} \end{tikzcd}$$

$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ - linear

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, equal] \arrow[d, "\mathrm{Poinc}^{\mathrm{Vac}}_!"'] & \mathrm{Vect} \arrow[d, "\mathcal{O}_{\mathrm{LS}_{\check{G}}}"] \\ D(\mathrm{Bun}_G) & \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}}) \arrow[l, "\mathbb{L}_G^L"'] \end{tikzcd}$$

$\mathcal{A}_G$ algebra in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ producing the monad $\mathbb{L}_G \circ \mathbb{L}_G^L$.

We need to show $\mathcal{O}_{\mathrm{LS}_{\check{G}}} \to \mathcal{A}_G$ is an isom.

Its restriction along $(\iota^{\mathrm{red}})^\wedge : (\mathrm{LS}_{\check{G}})^\wedge_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}} \to \mathrm{LS}_{\check{G}}$ is an isom

It remains to study $\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}} \longrightarrow \mathcal{A}_G^{\mathrm{cusp}}$

Facts about $\mathcal{A}_G^{\mathrm{cusp}}$

(i) it's a classical vector bundle

(ii) it admits a connection w/ finite monodromy.

Proof of (i) Dennis explained $\mathcal{A}_G^{\mathrm{cusp}}$ is a self-dual perfect complex.

Lin explained $(\iota_\sigma)^* \mathcal{A}_G^{\mathrm{cusp}} \simeq C_\cdot(\text{generic oper str. on } \sigma)$ $\square$

Dima showed this space is non empty.

[BKS] proved contractibility for classical groups

$\check{G}$ simply conn'd, $g \geq 2$.

More facts: $\check{G}$ simply conn'd, $g \geq 2$,

I. $LS_{\check{G}}$ is a classical lci stack [BD1]

II. $LS^{irred}_{\check{G}} \subset LS_{\check{G}}$ has codim ≥ 2

III. $LS^{irred}_{\check{G}}$ is simply conn'd.

IV. $\Gamma(LS_{\check{G}}, A_G) = k$.

Why does this imply $\mathcal{O}_{LS^{irred}_{\check{G}}} \Rightarrow A^{cusp}_G$?

$$\text{III} \Rightarrow A^{cusp}_G \simeq \left(\mathcal{O}_{LS^{irred}_{\check{G}}}\right)^{\oplus 2k}$$

$$H^0(LS^{irred}_{\check{G}}, A^{cusp}_G) \simeq H^0(LS^{irred}_{\check{G}}, \mathcal{O}_{LS^{irred}_{\check{G}}})^{\oplus 2k}$$

$$(\text{I}+\text{II}) \Uparrow$$

$$H^0(LS_{\check{G}}, A_G) \underset{\text{IV}}{\simeq} k$$

Proof of II

$$\begin{tikzcd} LS^{stable}_{\check{G}} \arrow[r, phantom, "\subset"] \arrow[d, "\text{affine fibration}"'] & LS^{irred}_{\check{G}} \ (\text{smooth}) \arrow[r, phantom, "\subset"] & LS_{\check{G}} \arrow[d] \\ Bun^{stable}_{\check{G}} \arrow[rr, "\text{codim} \geq 2 \text{ complement}"'] & & Bun_{\check{G}} \end{tikzcd}$$

if $\check{G} = SL_n$, $(\mathcal{E}, \Lambda^n \mathcal{E} \simeq \mathcal{O})$ is stable iff every subbundle $\mathcal{E}' \subset \mathcal{E}$ has $\deg(\mathcal{E}') < 0$.

The obstruction to $P_{\check G}$ admitting a connection is the Atiyah class

$$\alpha_{P_G} \in H^1(X, \Omega^1 \otimes P_G \overset{\check G}{\times} \check{\mathfrak g}) \simeq H^0(X, P_{\check G} \overset{\check G}{\times} \check{\mathfrak g}^\vee)$$

$\mathrm{Bun}_{\check G}$ is simply conn'd

Proof sketch of IV.
$$\Gamma(LS_{\check G}, \mathcal A_G) = \mathrm{Maps}_{\mathrm{IndCoh}_N(LS_{\check G})}(\mathcal O_{LS_{\check G}}, \mathbb L_G \circ \mathbb L_G^L \circ \mathcal O_{LS_{\check G}})$$

$$\simeq \mathrm{End}_{D(\mathrm{Bun}_G)}(\underbrace{\mathbb L_G^L \cdot \mathcal O_{LS_{\check G}}}_{\mathrm{Poinc}_!^{\mathrm{Vac}}})$$

$$\begin{tikzcd}
\mathrm{Bun}_N^\Omega \arrow[r] \arrow[d, "\text{contractible fibers}"'] & \mathrm{Bun}_N^\Omega/T \arrow[r, hook, "\text{locally closed}"] \arrow[d] & \mathrm{Bun}_G \\
\mathbb A^2 \arrow[r, "\pi"] \arrow[d, "\mathrm{sum}"'] & \mathbb A^2/T & \\
\mathbb A^1 & &
\end{tikzcd}$$

this End group is $\mathrm{End}_{D(\mathbb A^2/T)}(\pi_! \cdot \mathrm{sum}^*(\exp))$

for $G = SL_2$, $\mathrm{Bun}_N^\Omega = \{\Omega^{1/2} \to \mathcal E \to \Omega^{-1/2}\} \simeq H^1(X, \Omega^1) \simeq \mathbb A^1$

We are looking at $\pi_! \cdot \exp \simeq j_* k$ [shift] $\quad \mathbb A^1/\mathbb G_m \overset{j}{\hookleftarrow} \mathbb G_m/\mathbb G_m = pt$