Quiver varieties & Symplectic resolutions

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(1) A quiver is a directed graph. $Q = (Q_0, Q_1, s,t : Q_1 \to Q_0)$

A rep'n of a quiver : $-$ V_i vec. sp. $\forall i \in Q_0$ $/\mathbb{C}$

$-$ $\varphi_a : V_{s(a)} \longrightarrow V_{t(a)}$ $\mathbb{C}$-linear, $\forall a \in Q_1$

$(d_i) = d \in \mathbb{N}^{Q_0}$ dimension vector, $\mathrm{Rep}_d(Q) = \{$ rep'ns of Q, $\dim V_i = d_i \}$

$$\begin{array}{ccl} & \circlearrowright & \\ G_d & & = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}}) \\ \parallel & & \\ \prod GL_{d_i}(\mathbb{C}) & & = \mathbb{A}^N \end{array}$$

$$(g_i) \cdot (\varphi_a) = g_{t(a)} \circ \varphi_a \circ g_{s(a)}^{-1}$$

Goal: understand the orbit space $\mathrm{Rep}_d(Q)/G_d$ $-$ quiver variety

$$\mathrm{Spec}\left(\mathbb{C}[\mathrm{Rep}_d(Q)]^{G_d}\right)$$

Rmk $\mathbb{C}^\times \subset G_d$ acts trivially.

Ex 1 $\underset{d=2}{\cdot} \circlearrowleft A$ $\mathrm{Mat}_{2\times 2}(\mathbb{C})/\mathrm{Conj}.$

Ex 1' $\rho_A : \mathbb{C}^2 \xrightarrow{A} \mathbb{C}^2$ if A, B invertible, $(A,B)\,\rho_A = \mathbb{C}^2 \xrightarrow{B} \mathbb{C}^2$

$B = tI$, $t \to 0$ not closed orbit

Thm [Le Bruyn – Procesi] $\mathbb{C}[\mathrm{Rep}_d(Q)]^{G_d} = \mathbb{C}[\mathrm{tr}(-\gamma)]$ γ directed cycle. $\mathrm{len}(\gamma) \leq \left(\sum d_i\right)^2$

Rmk $A \longrightarrow \det(A)$ G_d-inv. $\det(A) = \frac{1}{2}\left(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\right)$.

Enhancements (1) $Q \rightsquigarrow \overline{Q}$ double

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "a^*"] \arrow[loop above, "b"] \arrow[loop below, "b^*"] \end{tikzcd}$$

$$\mathrm{Rep}_d(\overline{Q}) = \mathrm{Rep}_d(Q) \oplus \mathrm{Rep}_d(Q^{op})$$

$$= T^* \mathrm{Rep}_d(Q) \circlearrowleft G_d$$

$$\mu_d : \mathrm{Rep}_d(\overline{Q}) \to \mathfrak{g}_d^* \simeq \mathfrak{g}_d$$

$$\rho \longmapsto \sum_{a \in Q_1} [\rho(a), \rho(a^*)]$$

(2) $\mathrm{Rep}(Q) \longleftrightarrow \mathbb{C}Q$-mod

$\mathrm{Rep}(Q, R) \longleftrightarrow \mathbb{C}Q/(R)$-mod

$\mathrm{Rep}\left(\overline{Q}, R = \sum_{a \in Q} [a, a^*]\right) \longleftrightarrow \Pi(Q)$-mod

(3) [King] Let $\theta \in \mathbb{Z}^{Q_0}$, $\theta \cdot d = 0$.

$V \in \mathrm{Rep}_d(Q)$ is θ-semistable if $\forall\, W \subset V$, $\dim(W) \cdot \theta \leq 0$.

Ex. $\mathbb{C} \overset{a}{\underset{b}{\rightrightarrows}} \mathbb{C} \setminus (a,b) = (0,0) \,/\, \mathbb{C}^\times \longrightarrow \mathbb{P}^1_{[a:b]}$

$\theta_0 = (0,0)$ all reps

$\theta_1 = (-1,1)$
$$\begin{tikzcd} V = \mathbb{C} \arrow[r, "a"] & \mathbb{C} \\ W = 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$
no reps

$\theta_2 = (1,-1)$,
$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[u] \end{tikzcd}$$

Commutes: $(a,b) = (0,0)$

Def. [中岛] $\mathcal{M}(Q,d,\theta) = \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{N}} \mathbb{C}[\mu_d^{-1}(0)]^{n\theta, G_d}\right)$ x_θ^n

$$\downarrow$$

$$\mathcal{M}(Q,d,0) = \mathrm{Spec}\left(\mathbb{C}[\mu_d^{-1}(0)]\right)^{G_d}$$

$\theta \rightsquigarrow$

$$\chi_\theta \in \mathrm{Hom}(G_d, \mathbb{C}^\times), \quad (g_i) \mapsto \prod \det(g_i)^{\theta_i}$$

$$\begin{array}{c} \downarrow \ \uparrow \\ PG_d \end{array} \qquad \alpha I \mapsto \prod \alpha^{d_i \theta_i} = \alpha^{\theta \cdot d} = 1$$

Crawley - Boevey :
$$\mathcal{M}(Q, d, 0) \simeq \prod_{d = n_1 d^1 + \dots + n_m d^m} \mathrm{Sym}^{n_i}(\mathcal{M}(Q, d^i, 0))$$

$d = n_1 d^1 + \dots + n_m d^m$ — canonical decomposition

$$p(d) > \sum n_i \, p(d^i)$$

Ex. $Q = \bullet \rightrightarrows \bullet$, $d = (1,1)$, $\mathcal{M}(Q, d, 0) = \mathbb{V}(z^2 - xy) = \mathbb{C}^2/\mathbb{Z}/2$ A_1- sing.

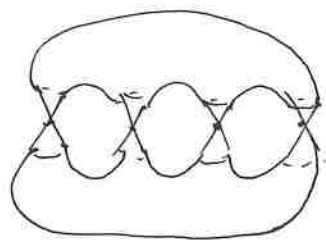
$\mathcal{M}(Q, d, (1,-1)) = T^*\mathbb{P}^1$

$d = (2,2)$ $\quad \mathrm{Hilb}^2(T^*\mathbb{P}^1) \to \mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}/2) \leftarrow ?$

(3) Ex. $X = (\mathbb{C}^\times)^2 / \mathbb{Z}/2$, $\quad (x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$, $\quad \omega = \frac{dx}{x} \wedge \frac{dy}{y}$

$(\pm 1, \pm 1)$ each A_1- sing.

$T_{(1,1)} X \qquad (1+\varepsilon_1, 1+\varepsilon_2) \mapsto (1-\varepsilon_1, 1-\varepsilon_2)$



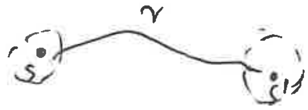
Thm [K- Schedler] X sympl. singularity w/ S sympl. leaves

$$\underset{\text{open}}{U \subset X} \mapsto SR(U) = \{\text{sympl. resol. } \pi : \tilde{U} \to U\} \Big/ \begin{array}{c} U_1 \xrightarrow{\varphi} U_2 \\ {\scriptstyle \pi_1} \searrow_{U} \swarrow {\scriptstyle \pi_2} \end{array}$$

is an S- constructible sheaf of sets.

Procedure : (0) Define SR at singular points

(1) Extend along any simple exit



$\gamma_*(\pi_s) = \pi_{s'}$

$\pi_1(S, s) \curvearrowright SR(U_s)$

(2) If monodromy-free, then extends from U_s to U_S.

(3) If compatible, then extend to X.

$X = (\mathbb{C}^\times)^2 / \mathbb{Z}/2$, $Y = \mathrm{Sym}^2(X)$

4 dim smooth

2 dim $\Delta = \{(x, x) : x \in X\}$,

$\{(s, x) : s \in X^{\mathrm{sing}}\}$

0-dim $\{(s, s') : s, s' \in X^{\mathrm{sing}}\}$

$\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_2)$

$\{(s, s) : s \in X^{\mathrm{sing}}\}$ - 4 most singular pts

$\# SR = 2^4$