Integral representation of classical L-function: Rankin-Selberg case

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volume | Rankin-Selberg (1940) 1983, J-PSS | Mirabolic $GL_n/P_n \approx \mathbb{A}^n - 0$ | basic affine space G/U "Fourier transform"

F no. field, $D_F = |d_{F/\mathbb{Q}}|$. Tamagawa $dg = D_F^{-\frac{\dim G}{2}} \prod_v' |\omega|_v$

G/F reductive gp. (gauge form ω) on $[G] = G(F)\backslash G(\mathbb{A})$

Ex $G = GL_n$, $\omega = \frac{1}{(\det g)^n} \bigwedge_{i,j} dg_{ij}$

$$\mathrm{vol}_\omega(GL_n(\mathcal{O}_v)) = \frac{1}{q_v^{n^2}} \# GL_n(\mathbb{F}_{q_v})$$
$$= (1-q_v^{-1}) \cdots (1-q_v^{-n})$$
$$= \zeta_v(1)^{-1} \cdots \zeta_v(n)^{-1} = \Delta_{GL_n,v}^{-1}$$
$$= L_v(0, M_{GL_n})^{-1}$$

$\mathrm{Res}_{s=0} L(s, M_G)$

$M_{GL_n} = \mathbb{Q}(1) \oplus \cdots \oplus \mathbb{Q}(n)$

✪ $dg = (\Delta^*)^{-1} \prod_v' \Delta_v |\omega|_v$

$\tau(G) := \mathrm{vol}(G(F)\backslash G(\mathbb{A}))$

$$\tau(\mathbb{G}_{m,F}) = 1 \iff \zeta_F^*(1) = \frac{2^{r_1}(2\pi)^{r_2} h \cdot R}{w_F D_F^{1/2}}$$

$$\tau(SL_n) = 1 \overset{n\geq 2}{\iff} \mathrm{vol}\left(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})\right) \in \mathbb{Q}\,\zeta(2)\cdots\zeta(n)$$

$\tau(\mathbb{G}_a) = 1 \iff$ Tate thesis, $\mathrm{vol}(\mathbb{A}_F/F) = 1$

Metric $\underset{\pi_1}{\langle \phi_1}, \underset{\pi_2}{\phi_2} \rangle_{Pet} = \int_{[G]^1} \phi_1(g)\,\phi_2(g)\,dg = \prod_v{}' \langle \phi_{1,v}, \phi_{2,v} \rangle_{Pet,v}$

$\mathrm{Hom}_G(\pi_1 \otimes \pi_2, \mathbb{C})$

$\boxed{\text{Thm A}}$ (Jacquet-Shalika) $\pi, \pi' \in \mathcal{A}_{cusp}(GL_n)$, $\omega_\pi \overline{\omega_{\pi'}} = 1$

then ① $L(s, \pi \times \pi')$ has analytic continuation to $s \in \mathbb{C}$

② (S large set of places of F)

$$\langle \phi, \phi' \rangle_{Pet} = n \cdot \frac{L^{S,*}(1, \pi \times \pi')}{\Delta^{S,*}_{GL_n}} \prod_{v \in S} \alpha_v(\phi_v, \phi_v').$$

Whittaker model $W_{\phi_v}(g_v) = \int_{N_n(F_v)} \phi_v(u_v g_v)\,\psi_v^{-1}(u_v)\,du_v$ | $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}$

$g_v \in GL_n(F_v)$ $\qquad N_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \subset GL_n$

$$\alpha_v(\phi_v, \phi_v') = \int_{N_{n-1}(F_v)\backslash GL_{n-1}(F_v)} W_{\phi_v}\begin{pmatrix} h & \\ & 1 \end{pmatrix} \overline{W_{\phi_v'}\begin{pmatrix} h & \\ & 1 \end{pmatrix}}\,dh_v$$

Thm B (Flicker - Rallis, twisted case) $F' | F$ quad ext'n,

$\pi \in \mathcal{A}_{cusp}(GL_{n,F'})$, $\omega_\pi |_{\mathbb{A}_F^\times} = 1$

$${}^L\left(\mathrm{Res}_{F'|F}\, GL_{n,F'}\right) = \left(GL_n(\mathbb{C}) \times GL_n(\mathbb{C})\right) \rtimes \langle \underset{\text{involution}}{\sigma} \rangle$$

① $L(s,\pi, As)$ analytic cont. to $s \in \mathbb{C}$.

has a pole at $s=1$ iff π comes from base change

②
$$\int_{[GL_{n,F}]^1} \phi(h)\, dh = \frac{n \cdot L^{S,*}(1, \pi, As)}{\Delta^{S,*}_{GL_n}} \prod_v \alpha_v^1(\phi)$$

$\boxed{\text{General}}$ $H \hookrightarrow G$, $\pi \in \mathcal{A}_{cusp}(G)$, $\omega =$ "Special function on $H(\mathbb{A})$".

$$P_{H,\varphi_\omega}(\phi) = \int_{[H]^1} \phi(h)\, \varphi_\omega(h)\, dh \qquad \varphi_\omega \in \omega$$

- When $P_{H,\omega} \neq 0$ on π ?

- $P_{H,\omega} \in \mathbb{C}\, L^S_{M^v}(\pi) \prod_{v \in S} \alpha_v(\phi_v)$

☆ local branching law $\alpha_v \neq 0$

Ex $H = GL_n \longrightarrow G = GL_n \times GL_{n+1}$ std ⊗ std , $\frac{1}{2}$ Unfolding

JPPS 1983 $H = GL_n \longrightarrow G = GL_n \times GL_n$ std ⊗ std , $\frac{1}{2}$

$GL_n \times GL_m$ $\omega =$ mirabolic Eis series Φ on $\mathbb{A}^n$

Why study L-functions (using $P_{H,W}$)?

degree formula. Deligne conjecture. converse thm, class number, BSD

volume formula.

$G/\mathbb{Q}$ split

$d_i = e_i + 1$ exponent of G

$$\mathrm{vol}(G(\mathbb{R})/\Gamma) \in \mathbb{Q} \prod_{i=1}^{r} \zeta(d_i)$$

$$\mathrm{Vol}(E_6(\mathbb{R})/\Gamma) \in \mathbb{Q}\, \zeta(2)\zeta(5)\zeta(6)\zeta(8)\zeta(9)\zeta(12)$$

Rankin-Selberg method (p-adic L-func.) f, g cusp form $\Gamma = SL_2(\mathbb{Z})$ wt $k \geq 2$

$$f = \sum_{n=1}^{\infty} a_n q^n, \quad g = \sum_{n=1}^{\infty} b_n q^n$$

$$L(s, f \times g) = \sum_{n=1}^{\infty} \frac{a_n \bar{b}_n}{n^s} \longleftarrow I(s) = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} E(z,s) \frac{dx\,dy}{y^2}$$

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}(\gamma z))^s \quad = \int_{\Gamma_\infty \backslash \mathbb{H}} f(z)\overline{g(z)}\, y^{k+s-2}\, dx\,dy$$

func. eq.

$$= \int_0^\infty \frac{a_n \bar{b}_n}{n^{k-2}} \underbrace{\int_0^\infty e^{-4\pi n y} y^{k+s-2}\, dy}_{\Gamma(s)}$$

$\Gamma_\infty \ni \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$ $z \mapsto z + n$

$GL_n \times GL_1$ Godement-Jacquet

$GL_n \longrightarrow GL_n \times GL_{n+1}$

$\Phi \in \mathcal{A}_{cusp}(GL_{n+1})$

$\phi \in \mathcal{A}_{cusp}(GL_n)$

$$I(s,\Phi,\varphi) = \int_{N_n(\mathbb{A})\backslash GL_n(\mathbb{A}_F)} W_\phi(g) \int_{N_n(F)\backslash N_n(\mathbb{A})} \Phi\begin{pmatrix} g & \\ & 1\end{pmatrix} \overline{\psi_N(n)}\, dn\, |g|^{s-\frac{1}{2}} dg$$

full Whittaker: $\displaystyle \phi(g) = \sum_{\gamma \in N_n(F)\backslash GL_n(F)} W_\phi(\gamma g)$

$$= \int_{N_n(\mathbb{A})\backslash GL_n(\mathbb{A})} |g|^{s-\frac{1}{2}} W_\phi(g) W_\Phi\begin{pmatrix} g & \\ & 1\end{pmatrix} dg$$

$$= \prod_v I_v(s,\Phi_v,\varphi_v)$$

use $N_n \hookrightarrow N_{n+1}$ $\qquad W_\Phi\begin{pmatrix} g & \\ & 1\end{pmatrix}$

$$\left(\begin{array}{c|c} N_n & u \\ \hline & 1\end{array}\right)$$

$P_n = \left(\begin{array}{c|c} * & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline * & 1\end{array}\right) \subset GL_n$ $\qquad W_\Phi\begin{pmatrix} g & \\ & 1\end{pmatrix} = \displaystyle\int_u \int_{N_n} \Phi(nug)\, \overline{\psi_{N_n}(n)\, \psi_U(u)}\, dn\, du$

<u>Mirabolic expansion</u> $\qquad \displaystyle \Phi(h) = \sum_{\chi \in \widehat{U(F)\backslash U(\mathbb{A})} \simeq F^n} \Phi_\chi(h)$

Φ cuspidal $\Rightarrow \Phi_0 = 0$

$$GL_n(F) \overset{\text{conj}}{\curvearrowright} U(F) = F^n$$

$$F^n - 0 = P_n(F)\backslash GL_n(F)$$

Check. ① $v \gg 0$, $I_v = L(s, \pi_v \times \pi_v')$

② any v, (GCD of I_v) $= L(s, \pi_v \times \pi_v')$
(ramified)

Jacquet 2003 ③ explicit test vector $(v | \infty)$ $\quad I_v(\phi_v^{test}, \phi_v^{test}) = L(s, \pi_v \times \pi_v')$
$\neq 0$

④ local func. eq of $I_v(s, \phi_v, \Phi)$

Baruch (2003) local proof of Kirillov's conjecture

π_1 generic

$$\mathrm{Hom}_{GL_n}(\pi_1 \otimes \pi_2, \mathbb{C}) = \mathrm{Hom}_{P_n}(\pi_1 \otimes \pi_2, \mathbb{C})$$

$/\mathbb{Q}_p$
$/\mathbb{R}$