# Modules over the loop group

## Dennis Gaitsgory

**Talk 1** Gurbir Dhillon, Yakov Varshavsky, David Yang

$K$ local field, $G$ reductive group$/K$, $LG$ group indscheme over $k$.

**Twisted Levi**: $M \subset G$ that becomes a Levi over $\bar{K}$.

- $LG(k)$-mod – classical theory
- $LG$-Cat $\Leftrightarrow$ $D(LG)$-mod
- $LG$-Cat $\Leftrightarrow$ $D(LG)$-mod ($AG$ Cat)
- $G_{\sigma,r} \supset G_{\sigma,r+}$ Moy–Prasad subgroups
- $G_\sigma := G_{\sigma,0}$ – parahoric, $G_{\sigma,0+}$ – unipotent radical. $L_\sigma = G_\sigma / G_{\sigma,0+}$

Lemma. f.f.

Let $C \in LG$-Cat,

$$\begin{tikzcd}
D(LG/G_{\sigma,r+}) \underset{D(G_{\sigma,r+}\backslash LG/G_{\sigma,r+})}{\otimes} C^{G_{\sigma,r+}} \arrow[r, shift left] \arrow[d, shift left, dashed] & C \arrow[l, shift left, dashed] \\
D(LG/G_{\sigma,r+}) \underset{D(G_\sigma/G_{\sigma,r+})}{\otimes} C^{G_{\sigma,r+}} \arrow[u, shift left] &
\end{tikzcd}$$

**Def** $C^{\leq r}$ to be the full $LG$-invt subcat. in $C$ generated by the essential images of these functors for all $\sigma$.

**Def** $C$ has depth $\leq r$ if $C^{\leq r} \to C$ is an equality.

Def. $LG\text{-}Cat^{=r} \subset LG\text{-}Cat^{\leq r}$

$$\{C \in LG\text{-}Cat^{\leq r} : C^{\leq r'} = 0, \ \forall r' < r\}$$

$LG\text{-}Cat^{=0}$ is "understood".

- $LG\text{-}Cat^{=0}$ can be expressed in terms of $L_\sigma\text{-}Cat$, $\forall \sigma$.
- $LG\text{-}Cat^{=0} = D(\overset{\circ}{I}\backslash LG/\overset{\circ}{I})\text{-}\underline{mod}$

connection: filtration

Thm. $LG\text{-}Cat^{=r} = \prod_M LG\text{-}Cat^{=r, M}$, $r>0$

$$LG\text{-}Cat^{=r,M} \simeq \left(LM\text{-}Cat^{=r,M,\,\text{non-deg}}\right)^{W_M}, \quad W_M = \mathrm{Norm}(M)/M$$

Pf | W |.

$$LG\text{-}Cat^{=r,G} = LG\text{-}Cat^{<r} \underset{L(coZ_G^\circ)\text{-}Cat^{<r}}{\otimes} L(coZ_G^\circ)\text{-}Cat^{\leq r}$$

$coZ_G^\circ$ - max'l quotient $\underbrace{\text{torus}}$

$G$ split, $A = \Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ the apartment, $x \in A$, $r \in \mathbb{Q} \rightsquigarrow G_{x,r}$, $G_{x,r+}$.

$$\mathfrak{g}_{x,r} = \bigoplus_\alpha \mathfrak{g}_\alpha t^i, \quad \langle \alpha, x\rangle + i \geq r$$

$$\mathfrak{g}_{x,r+} = \bigoplus_\alpha \mathfrak{g}_\alpha t^i, \quad \langle \alpha, x\rangle + i > r$$

Examples $x=0$. $G_{x,0} = L^+G$, $G_{x,0+} = G_1$, $G_{x,n} = G_n$, $G_{x,n+} = G_{n+1}$.

$x = \varepsilon\rho$, $0<|\varepsilon|\ll 1$, $G_{x,0} = I$, $G_{x,0+} = \overset{\circ}{I}$.

Hack $A$ by hyperplanes with 2 conditions

- Must contain $H_{\alpha,\frac{m}{n}} = \{x : \langle x, \alpha \rangle = \frac{m}{n}\}$, $n =$ denominator of $r$.

- Invariant w.r.t. the extended affine Weyl group $W^{\mathrm{aff,ext}}$
$$\cup$$
$$\Lambda$$

$\sigma \subset \bar{\sigma}$

$\uparrow$ open $\quad \uparrow$ closed facet

Ex. torus

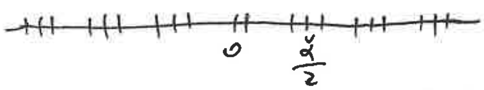
$x_1, x_2 \in \sigma$

$G_{x_1,r} = G_{x_2,r}$

$G_{x_1,r+} = G_{x_2,r+} \quad \Rightarrow \quad G_{\sigma,r+} \subset G_{\sigma,r} \subset G_\sigma$

normal subgroups

Ex $G = SL_2$, $r =$ integer.




Basic inclusion. $\quad \tau < \sigma \quad \tau \subset \bar{\sigma}$

$$G_{\tau,r+} \subset G_{\sigma,r+} \subset G_{\sigma,0} \subset G_{\tau,0}$$

$$\left( \text{e.g. } G_1 \subset \mathring{I} \subset I \subset L^+G \right)$$

$BG_{\leq r}: \Sigma \longrightarrow$ indschemes w/ action of loop group

$\Sigma =$ facets$/W^{\mathrm{aff,ext}}$, $\quad BG_{\leq r}(\sigma) \longmapsto LG/G_{\sigma,r+}$.

Facets $\longrightarrow$ ind-schemes w/ $LG$-action

$\sigma \longmapsto LG/G_{\sigma,r+}$

$\tau \subset \bar{\sigma}$ $\qquad LG/G_{\sigma,r+} \longleftarrow LG/G_{\tau,r+}$

---

Talk 2. $\quad A = \mathbb{R} \underset{\mathbb{Z}}{\otimes} \Lambda$

$x \in A \rightsquigarrow G_{x,r} \supset G_{x,r+}$

$$\begin{array}{ccc} \mathfrak{g}_{x,r} & \supset & \mathfrak{g}_{x,r+} \\ \| & & \| \\ \bigoplus_{\alpha,n} \mathfrak{g}_\alpha t^n & & \bigoplus_{\alpha,n} \mathfrak{g}_\alpha t^n \\ \langle \alpha, x\rangle + n \geq r & & \langle \alpha, x\rangle + n > r \end{array}$$

Exer. $\quad G_{x,r} = G_{x,(r-\varepsilon)+}$

We hack $A$ w/ hyperplanes
- $W^{\text{aff, ext}}$ - invariant
- Must contain $H_{\alpha,\frac{m}{n}} = \{x : \langle \alpha, x\rangle = \frac{m}{n}\}$

for $m \in \mathbb{Z}$, $n$ is the denominator of $r$

$\Sigma$ = set of facets

$x_1, x_2 \in \sigma$, $\quad G_{x_1,r+} = G_{x_2,r+} \qquad \sigma \rightsquigarrow G_{\sigma,r+}$

$\tau \leq \sigma \iff \tau \in \bar{\sigma}$

$G_{\tau,r+} \subset G_{\sigma,r+} \subset G_\sigma \subset G_\tau$

$\Sigma / W^{\mathrm{aff,ext}}$

objects: facets

$\sigma_1 \to \sigma_2 \iff w \in W^{\mathrm{aff,ext}}, \; w(\sigma_1) \subset \overline{\sigma_2}$

$$\Sigma / W^{\mathrm{aff,ext}} \xrightarrow[B(LG)_{\mathrm{Sr}}]{} \mathrm{Corr}(\mathrm{PreStk}) / (pt/LG)$$

$$\sigma \longmapsto pt / (G_\sigma / G_{\sigma,r+})$$

$$\begin{tikzcd} & pt/G_\sigma \arrow[dl] \arrow[dr] & \\ pt/(G_\sigma/G_{\sigma,r+}) & & pt/LG \end{tikzcd}$$

$\sigma \xrightarrow[\sim]{w} w(\sigma)$, $w \in W^{\mathrm{ext,aff}}$.

Pick $g \in N(LT)/L^+T$ lifting $w$

$$\begin{array}{cccc} \mathrm{Ad}_g : & G_\sigma & \xrightarrow{\sim} & G_{w(\sigma)} \\ & \cup & & \cup \\ & G_{\sigma,r+} & \xrightarrow{\sim} & G_{w(\sigma),r+} \end{array}$$

$g'' = g' \cdot k, \quad k \in G_\sigma$

Important. $\mathrm{Ad}_k$ is canonically trivial as an automorphism of $pt/(G_\sigma/G_{\sigma,r+})$

$\tau \subset \bar{\sigma}$

$$\begin{tikzcd}
& pt/(G_\sigma/G_{\tau,2+}) \arrow[dl] \arrow[dr] & \\
pt/(G_\tau/G_{\tau,2+}) & & pt/(G_\sigma/G_{\sigma,2+})
\end{tikzcd}$$

$$\mathrm{ShvCat}(B(G)_{\leq 2}) = \lim_{\substack{\longleftarrow \\ \sigma \in \Sigma/W^{\mathrm{aff,ext}}}} \mathrm{ShvCat}(B(G)_{\leq 2}(\sigma))$$

$$\| $$

$$(G_\sigma/G_{\sigma,2+})\text{-Cat}$$

I.e. $\forall \sigma \in \Sigma \quad \rightsquigarrow \quad C_\sigma \in (G_\sigma/G_{\sigma,2+})\text{-Cat}$

$\sigma \xrightarrow[\sim]{w} w(\sigma)$

Choose $g \in N(LT)/L^+T$ ,

$$\begin{tikzcd}
C_\sigma \arrow[r, "\varphi_g", "\sim"'] & C_{w(\sigma)} \\
G_\sigma/G_{\sigma,2+} \arrow[u, bend left] \arrow[r, "Ad_g"] & G_{w(\sigma)}/G_{w(\sigma),2+} \arrow[u, bend right]
\end{tikzcd}$$

$g'' = g' \cdot k \qquad \varphi_{g''} = \varphi_{g'} \cdot k$

$$\begin{array}{c} k \\ \uparrow \\ G_\sigma \end{array}$$

$\tau \leq \sigma$ , $C_\sigma \simeq (C_\tau)^{(G_{\sigma,2+}/G_{\tau,2+})}$

<u>NB</u> It's canonically independent of the hacking.

$$\begin{array}{ccc} LG\text{-}Cat & \xrightarrow{Loc_{\leq r}} & ShvCat(B(G)_{\leq r}) \\ \Psi & & \\ C & \longmapsto & \left[\sigma \mapsto C^{G_{\sigma,r+}}\right] \end{array}$$

$$C^{G_{\sigma,r+}} \in (G_\sigma/G_{\sigma,r+})\text{-}Cat$$

**Thm** $Loc_{\leq r}\big|_{LG\text{-}Cat^{\leq r}}$ is an equiv.

**Example** $G = T$

$\Sigma = \{*\}, \quad W^{aff,ext} = \Lambda$

$$LT/T_{r+}\text{-}Cat \simeq \left(L^+T/T_{r+}\text{-}Cat\right)^{\Lambda}$$

$$1 \to L^+T/T_{r+} \to LT/T_{r+} \to \Lambda \to 1$$

**Example** $r=0$

$$LG\text{-}Cat^{\leq 0} \simeq \varprojlim_{\sigma \in \Sigma/W^{aff,ext}} L_\sigma\text{-}Cat$$

**Example** $r=\infty$

$$LG\text{-}Cat \simeq \varprojlim_{\sigma \in \Sigma/W^{aff,ext}} G_\sigma\text{-}Cat$$

$$\Updownarrow$$

$\operatorname{colim}_{\sigma \in \Sigma/W^{aff,ext}} pt/G_\sigma \xrightarrow{\sim} pt/LG$ as prestacks localized in the h-topology.

$$\begin{tikzcd} \operatorname{colim} LG/I \arrow[r] \arrow[dr] & LG/L^+G \\ & LG/\widetilde{L^+G} \simeq pt \end{tikzcd}$$

$G = SL_2$

after h- localization

<u>Exer</u> Prove at the classical level (follows from contractibility of building)

$$\begin{tikzcd} \tilde{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X} \arrow[d] & \tilde{U} \arrow[l] \arrow[d, "\wr"] \\ Z \arrow[r] & X & U \arrow[l] \end{tikzcd}$$

$$\operatorname*{colim}_{\sigma \in \Sigma/W^{\mathrm{aff},\mathrm{ext}}} LG/G_\sigma \simeq pt \qquad h\text{-topology}$$

$$\begin{tikzcd} \mathrm{ShvCat}(B(G)_{\leq r}) \arrow[r, bend left, "\Gamma^{\leq r}"] & LG\text{-}\mathrm{Cat} \arrow[l, bend left, "\mathrm{Loc}^{\leq r}"] \end{tikzcd}$$

$$\Gamma^{\leq r}([\sigma \mapsto C_\sigma]) = \operatorname*{colim}_{\sigma \in \Sigma/W^{\mathrm{aff},\mathrm{ext}}} \operatorname{ind}_{G_\sigma}^{LG}(C_\sigma)$$

<u>Cor.</u> $\begin{tikzcd} LG\text{-}\mathrm{Cat}^{\leq r} \arrow[r, bend left, "i^{\leq r}"] & LG\text{-}\mathrm{Cat} \arrow[l, bend left, "(i^{\leq r})^R"] \end{tikzcd}$

$$i^{\leq r}(i^{\leq r})^R \simeq \operatorname*{colim}_{\sigma} \operatorname{ind}_{G_\sigma}^{LG}\left(C^{G_{\sigma, r+}}\right)$$

$$\mathcal{C}^{\leq r} \underset{(\text{counit})^R}{\overset{\text{counit}}{\rightleftarrows}} \mathcal{C}$$

$$\text{counit} \circ (\text{counit})^R = P_{\leq r}$$

projector on the $\leq r$ subcat.

Question: can we describe it explicitly?

Theorem

$$P_{\leq r} \cong \operatorname*{colim}_{\sigma \in \Sigma/W^{\text{aff,ext}}} \mathrm{Av}_!^{LG/G_\sigma}\left(e_{G_{\sigma,r+}}\right)$$

---

Talk 3. $A = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$

$\Sigma$ - poset of facets

$\Sigma^\sim$

$\sigma \in \Sigma \rightsquigarrow G_{\sigma,r+} \subset G_{\sigma,r}$

$\tau \subset \bar{\sigma}$

$$G_{\tau,r+} \subset G_{\sigma,r+} \subset G_{\sigma,r} \subset G_{\tau,r}$$

$$\begin{tikzcd} & G_\sigma/G_{\tau,r+} \arrow[dl] \arrow[dr] & \\ G_\tau/G_{\tau,r+} & & G_\sigma/G_{\sigma,r+} \end{tikzcd}$$

$$B(G)_r : \Sigma^\sim \longrightarrow \mathrm{Corr}(\mathrm{PreStk})$$
$$\sigma \longmapsto pt/(G_\sigma/G_{\sigma,r+})$$

$$\mathrm{ShvCat}(B(G)_2) = \left\{ \sigma \in \Sigma \rightsquigarrow C_\sigma \in (G_\sigma / G_{\sigma,2+})\text{-Cat}, \quad C_\sigma = (C_\tau)^{G_{\sigma,2+}} \right\}$$

$$\mathrm{ShvCat}(B(G)_2) \xleftarrow{\mathrm{Loc}^2} LG\text{-Cat}$$

$$\{\sigma \mapsto C^{G_{\sigma,2+}}\} \longleftarrow\!\shortmid\; C$$

**Thm.** (DVY)

$$\begin{tikzcd}
\mathrm{ShvCat}(B(G)_2) \arrow[rr, bend left=40, "\Gamma^2"] & LG\text{-Cat}^{\leq 2} \arrow[l, "\sim"'] \arrow[r, shift left, "i^{\leq 2}"] & LG\text{-Cat} \arrow[l, shift left, "(i^{\leq 2})^R"] \arrow[ll, bend right=30, "\mathrm{Loc}^2"']
\end{tikzcd}$$

$I$ - index cat

$i \in I \longmapsto C_i$

$i \to j, \quad \varphi_{ij} : C_i \to C_j$

$\lim_{i \in I} C_i$

$I^{op}$

$i \in I^{op} \longmapsto C_i \qquad C_i \xleftarrow{\varphi_{ij}^L} C_j$

$$\operatorname{colim}_{i \in I^{op}} C_i \simeq \lim_{i \in I} C_i$$

Example $\quad \operatorname{IndCoh}(\mathbb{A}^\infty)$

$$\|$$

$$\operatorname{colim} \operatorname{IndCoh}(\mathbb{A}^n) = \lim \operatorname{IndCoh}(\mathbb{A}^n)$$

$$\operatorname{ShvCat}(B(G)_r) = \operatorname*{colim}_{\sigma \in \Sigma^\sim} (G_\sigma / G_{\sigma,r+})\text{-Cat}$$

$$\tau \subset \bar{\sigma}$$

$$(G_\tau / G_{\tau,r+})\text{-Cat} \longleftarrow (G_\sigma / G_{\sigma,r+})\text{-Cat}$$

$$\operatorname{Ind}^{G_\tau / G_{\tau,r+}}_{G_\sigma / G_{\tau,r+}} \left( C_\sigma |_{G_\sigma / G_{\tau,r+}} \right) \longleftarrow\!\!\shortmid \; C_\sigma$$

The functor $\Gamma^r$ is given by

$$(G_\sigma / G_{\sigma,r+})\text{-Cat} \longrightarrow LG\text{-Cat}$$

$$C_\sigma \longmapsto \operatorname{Ind}^{LG}_{G_\sigma} \left( C_\sigma |_{G_\sigma} \right)$$

$$C_i \xleftarrow{F_i} D$$

$$\wr$$

$$\lim_{i \in I} C_i \xleftarrow{F} D$$

$$\operatorname*{colim}_{i \in I^{op}} C_i \xrightarrow{F^L} D$$

$$\Updownarrow$$

$$C_i \xrightarrow{F_i^L} D$$

$$\Gamma^2(\{C_\sigma\}) = \underset{\sigma \in (\Sigma^\sim)^{op}}{\mathrm{colim}} \ \mathrm{Ind}^{LG}_{G_\sigma}(C_\sigma)$$

$$\tau \subset \sigma$$

$$\mathrm{Ind}^{LG}_{G_\sigma}(C_\sigma) \longrightarrow \mathrm{Ind}^{LG}_{G_\tau}(C_\tau)$$

$$LG\text{-}\mathrm{Cat}^{\leq 2} \underset{(i^{\leq 2})^R}{\overset{i^{\leq 2}}{\rightleftarrows}} LG\text{-}\mathrm{Cat}$$

$$C^{\leq 2} \longleftarrow\!\shortmid \ C$$

$$i^{\leq 2} \circ (i^{\leq 2})^R \longrightarrow \mathrm{Id}$$

$$C^{\leq 2} \underset{\mathrm{counit}^R}{\overset{\mathrm{counit}}{\rightleftarrows}} C$$

$$P^{\leq 2} = \mathrm{counit} \circ \mathrm{counit}^R.$$

<u>Thm</u> $$P^{\leq 2} = \underset{\sigma \in (\Sigma^\sim)^{op}}{\mathrm{colim}} \ \mathrm{Av}_*^{LG/G_\sigma}(e_{G_{\sigma,2+}})$$

$$\uparrow$$

$$D(LG)$$

<u>Proof</u> $$\underset{i \in I^{op}}{\mathrm{colim}} \ C_i \xrightarrow{F^L} D \qquad F^L \circ F = \underset{i \in I^{op}}{\mathrm{colim}} \ F_i^L \circ F_i$$

$$C_i \xrightarrow{F_i^L} D$$

$$\int \qquad C_i \xleftarrow{F_i} D \qquad \underset{\sigma}{\mathrm{colim}} \ \mathrm{Ind}^{LG}_{G_\sigma}(C^{G_{\sigma,2+}}) \longrightarrow C$$

$$\underset{i \in I}{\lim} \ C_i \xleftarrow{F} D \qquad \qquad \mathrel{\rotatebox{90}{$\subseteq$}}\ C^{\leq 2}$$

Want to show: $\forall \sigma$,

$$\mathrm{Ind}_{G_\sigma}^{LG}\left(\mathcal{C}^{G_{\sigma,r+}}\right) \underset{(i_\sigma)^R}{\overset{i_\sigma}{\rightleftarrows}} \mathcal{C}$$

$i_\sigma \circ (i_\sigma)^R$ is given by $\mathrm{Av}_*^{LG/G_\sigma}\left(\underline{e}_{G_{\sigma,r+}}\right)$

$$\begin{array}{ccccc} H_1 & \triangleleft & H_0 & \subset & H \\ G_{\sigma,r+} & & G_\sigma & & LG \end{array} \quad , \quad \underline{H/H_0 \text{ proper}}$$

$$D(H/H_1) \underset{D(H_0/H_1)}{\otimes} D(H_1 \backslash H) \underset{i^R}{\overset{i}{\rightleftarrows}} D(H)$$

$$i^R = \mathrm{Av}^{H/H_0}\left(\underline{e}_{H_1}\right)$$

$$\begin{array}{c} LG\text{-}\mathrm{Cat}^{\leq r} \\ \cup \\ LG\text{-}\mathrm{Cat}^{=r} \end{array}$$

$$\mathcal{C}^{<r} \rightleftarrows \mathcal{C}^{\leq r} \rightleftarrows \mathcal{C}^{=r}$$

Want to describe $\mathcal{C}^{=r}$ in terms of the localization picture.

$$\mathcal{C}^{G_{\sigma,r+}} \curvearrowleft G_{\sigma,r}/G_{\sigma,r+} \quad r>0$$
$$\cong$$
$$\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+}$$

Define $\left(\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+}\right)^{*,\text{unst}} \subset \left(\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+}\right)^*$

$\uparrow$ all points whose $L_\sigma$-orbit contains 0 in its closure

$\left(\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+}\right)^* \curvearrowleft L_\sigma$

For $G = SL_2$, here are the possibilities:

if $\sigma$ is a vertex, $\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+} \simeq \mathfrak{g} \curvearrowleft G$, in this case unstable = nilpotent.

$\sigma$ is not a vertex, there are two scenarios:

(1) $(\mathfrak{g}_\sigma/\mathfrak{g}_{\sigma,r+})^* = \operatorname{span}\{e,f\}$, $L_\sigma = \mathbb{G}_m$ acting hyperbolically

unstable locus is $\operatorname{span}(e) \cup \operatorname{span}(f)$

(2) $(\mathfrak{g}_\sigma/\mathfrak{g}_{\sigma,r+})^* = \operatorname{span}(e)$ or $\operatorname{span}(f)$. in this case, the whole thing unstable.

We say that a rat'l $r$ is relevant if $\exists\, x \in A$ s.t. not the whole $(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+})^*$ is unstable.

Example for $G = SL_2$, the only relevant rat'ls are $\frac{1}{2}\mathbb{Z}$.

Thm. $r>0$

$$\left(C^{=r}\right)^{G_{\sigma,r+}} = C^{G_{\sigma,r+}} \otimes \frac{D\left((\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+})^{*,0}\right)}{D\left((\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+})^{*}\right)}$$

where $(\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+})^{*,0} = (\mathfrak{g}_{\sigma,r}/\mathfrak{g}_{\sigma,r+})^* -$ unstable.

Cor If $z$ is irrelevant, $LG\text{-}Cat^{=z} = 0$.

Thm
$$\begin{array}{ccc} \{C_\sigma \in G_\sigma/G_{\sigma,z+}\text{-}Cat\} & \simeq & LG\text{-}Cat^{\leq z} \\ \cup & & \cup \\ \{C_\sigma \in (G_\sigma/G_{\sigma,z+})\text{-}Cat^{\circ}\} & & LG\text{-}Cat^{=z} \end{array}$$

$$(G_\sigma/G_{\sigma,z+})\text{-}Cat^{\circ} = G_\sigma/G_{\sigma,z+}\text{-}Cat \underset{G_{\sigma,z}/G_{\sigma,z+}\text{-}Cat}{\times} (G_{\sigma,z}/G_{\sigma,z+}\text{-}Cat)^{\circ}$$

where
$$\begin{array}{ccc} G_{\sigma,z}/G_{\sigma,z+}\text{-}Cat^{\circ} & \subset & G_{\sigma,z}/G_{\sigma,z+}\text{-}Cat \\ \shortparallel & & \wr\shortparallel \\ Shv\,Cat((\mathfrak{g}_{\sigma,z}/\mathfrak{g}_{\sigma,z+})^{*,\circ}) & \subset & Shv((\mathfrak{g}_{\sigma,z}/\mathfrak{g}_{\sigma,z+})^{*}) \end{array}$$

Lemma The following subsets of $(\mathfrak{g}_{\sigma,z}/\mathfrak{g}_{\sigma,z+})^*$ coincide:

(1) the unstable locus

(2) The union $_{\tau \subset \bar\sigma}$ of $L_\tau$- orbits of kernels of the above maps for all $\tau$

$$G_{\tau,z+} \subset G_{\sigma,z+} \subset G_{\sigma,z} \subset G_{\tau,z}$$

$$\begin{array}{c} G_{\sigma,z} \\ \downarrow \\ G_\tau/G_{\tau,z+} \end{array}$$

(3) The union of $L_\tau$ - orbits of the characters that vanish on $G_{\sigma,z} \cap G_{\tau,z}$, $\forall\ \sigma \in \Sigma$.

Talk 4 (Yasha Varshavsky)

Goal: Thm $\mathrm{Loc}^{2} : L(G)-\mathrm{Cat}^{\leq 2} \xrightarrow{\Phi} \lim_{\sigma \in \Sigma} G_\sigma / G_{\sigma,2+} - \mathrm{Cat}$

$$\Phi(C) = \{ C^{G_{\sigma,2+}} \}_\sigma$$

$\Sigma = \tilde{\Sigma}_{\mathrm{fund}}$ - Subdivided fund. alcove.

Reduction 1 $\mathrm{Loc}^2$ has left adjoint $\Gamma^2\{(C_\sigma)\} = \mathrm{colim}_\sigma\, D(LG) \otimes_{D(G_\sigma)} C_\sigma$

want to show $\Gamma^2$ is f.f.

Reduction 2 $\forall\, Y \subset LG$ closed $I \times I$-inv subscheme

$$C_Y := \mathrm{colim}_\sigma\, D(Y G_\sigma) \otimes_{D(G_\sigma)} C_\sigma$$

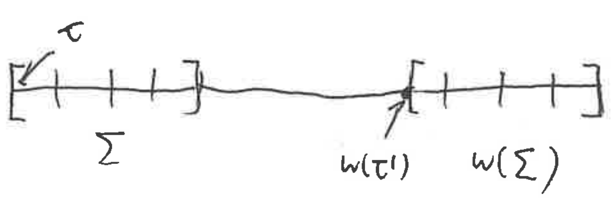
Enough to show:

Prop $\forall\, Y_1 \subset Y_2$ and every vertex $\tau \in \Sigma$, the functor

$$(C_{Y_1})^{G_{\tau,2+}} \xrightarrow{\sim} (C_{Y_2})^{G_{\tau,2+}} \quad \text{is equiv.}$$

Goal of today: to show it for $G = SL_2$.

By induction, can assume $Y_2 - Y_1 = IwI$.



Let $\tau' \in \Sigma$ vertex s.t. $w(\tau')$ is the closest vertex to $\tau$.

**Notation:** $\forall\ \Delta \subset \Sigma$ denote

$$C_{Y,\Delta} := \operatorname{colim}_{\sigma \in \Delta} D(YG_\sigma) \otimes_{D(G_\sigma)} C_\sigma$$

**Claim** if $\tau' \in \Delta$, then the functor

$$\left(C_{Y_1,\Delta}\right)^{C_{\tau,2+}} \xrightarrow{\sim} \left(C_{Y_2,\Delta}\right)^{C_{\tau,2+}} \quad \text{is an equiv.}$$

**Rmk** Claim for $\Delta = \Sigma \iff$ Proposition.

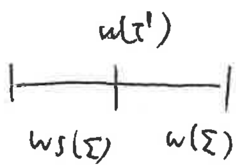
**Basis of induction:** $\Delta = \tau'$. In this case, $Y_1 G_{\tau'} = Y_2 G_{\tau'}$

$$\Updownarrow$$

$$w \in Y_1 G_{\tau'}$$

$\Rightarrow \quad C_{Y_1,\tau'} \xrightarrow{\sim} C_{Y_2,\tau'}$.

By construction:

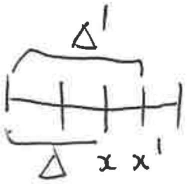

$s$ - reflection assoc. to $\tau'$

$ws < w \Rightarrow \boxed{ws \in Y_1}$

$\Rightarrow Y_1 \subset Y_2$ closed

$w = wss \in wsG_{\tau'} \subset Y_2 G_{\tau'}$.

**Induction step.** Assume claim for $\Delta$




$$\Delta' = \Delta \coprod_x [x, x']$$

$$C_{Y,\Delta'} = C_{Y,\Delta} \coprod_{C_{Y,[x,x']}} C_{Y,x'}$$

So enough to show the diagram

$$\begin{tikzcd}
\left( D(Y_1 G_\sigma) \otimes_{D(G_\sigma)} C_\sigma \right)^{G_{\tau,r+}} \arrow[r] \arrow[d] & \left( D(Y_2 G_\sigma) \otimes_{D(G_\sigma)} C_\sigma \right)^{G_{\tau,r+}} \arrow[d] \\
\left( D(Y_1 G_{\sigma'}) \otimes_{D(G_{\sigma'})} C_{\sigma'} \right)^{G_{\tau,r+}} \arrow[r] & \left( D(Y_2 G_{\sigma'}) \otimes_{D(G_{\sigma'})} C_{\sigma'} \right)^{G_{\tau,r+}}
\end{tikzcd}$$

is pushout diagram.

<u>Criterion</u> Let

$$\begin{tikzcd}
C_1 \arrow[r, "F"] \arrow[d, "G_1"'] \arrow[dr, phantom, "\circledast"] & C_2 \arrow[d, "G_2"] \\
C_1' \arrow[r, "F'"] & C_2'
\end{tikzcd}$$

be comm. diag s.t.

(1) $F$ & $F'$ are fully faithful and have cont. right adjoints

(2) $G_1$, $G_2$ have right adjoints

(3) Beck-Chevalley condition holds

$$F \circ G_1^R \xrightarrow{\sim} G_2^R \circ F'.$$

(4) $G_2$ induces equiv $\ker(F^R) \xrightarrow{\sim} \ker(F'^R)$

Then it is a pushout diagram.

It suffices to show $\circledast$ satisfies conditions (1) - (4)

<u>Rmk</u> (1)–(3) hold before $G_{\tau,2+}$ – invariants.

(1) $Y_1 G_\sigma \subset Y_2 G_\sigma$ is closed, so $i_*: D(Y_1 G_\sigma) \to D(Y_2 G_\sigma)$ is f.f. w/ right adjoint $i^!$.

(2) Recall $C_\sigma = (C_{\sigma'})^{G_{\sigma,2+}}$, so WTS

$$G: D(YG_\sigma) \underset{D(G_\sigma)}{\otimes} C_\sigma \to D(YG_{\sigma'}) \underset{D(G_{\sigma'})}{\otimes} C_{\sigma'}$$

has a right adjoint.

$G$ is obtained by $- \underset{D(G_{\sigma'})}{\otimes} C_{\sigma'}$ from corresp. functor for $C_\sigma = D(G_\sigma)$.

$G: D(YG_\sigma \overset{G_\sigma}{\times} (G_{\sigma,2+} \backslash G_{\sigma'})) \to D(YG_{\sigma'})$ corresponding to the diagram

$$\begin{tikzcd} YG_\sigma \overset{G_\sigma}{\times} (G_{\sigma,2+}\backslash G_{\sigma'}) & YG_\sigma \overset{G_\sigma}{\times} G_{\sigma'} \arrow[l, "\text{(pro) smooth}"'] \arrow[r, "\text{proper}"] & YG_{\sigma'} \end{tikzcd}$$

So $G$ has its right adjoint.

⋮