Torsors over the Fargues-Fontaine curve

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G-bundles on FF curve, $G/\mathbb{Q}_p$

$S = \mathrm{Spa}(R, R^+)/\mathbb{F}_p$ perfectoid

$0 \neq \varpi \in R^+$ <u>top. nilpotent</u> e.g. $\mathbb{F}_p((t^{\frac{1}{p^\infty}}))$

<u>Goal</u>. $X_{FF,S} = X_{FF}$.

- $S = \mathrm{Spa}(C)$ * alg. closed perfectoid field
- family version
- Bun_G moduli v-stack $/\mathbb{F}_p$

(*) $[*/G(\mathbb{Q}_p)] \overset{\text{open}}{\hookrightarrow} \mathrm{Bun}_G$

§0. perfd S $\xrightarrow{\text{cover}}$  $/\mathbb{F}_p$ finite type.

remember <u>top. info</u>.

<u>Q</u>. What is a family of "$\mathbb{Z}_p$-HT"$/S$?

"$\mathbb{Q}_p$-HT"

finite $\mathbb{Z}_p$-local system on S

(Artin-Schreier-Witt) $\pi_1 \twoheadrightarrow \mathbb{F}_p$, $\pi_1 \twoheadrightarrow \mathbb{Z}_p$

Def. R any ring, $\varphi: R \to R$ endo.

A φ-module is finite projective R-module M + iso. $\varphi_M: \varphi^* M \xrightarrow{\sim} M$.

(adic) " v.b. on $Y/\varphi^{\mathbb{Z}}$ ".

Thm (Kedlaya – Liu) R perfect $/\mathbb{F}_p$, then

$$\{\mathbb{Z}_p\text{-local system on } R\} \simeq \{\varphi\text{-module over } W(R)\}$$

$\mathbb{Q}_p$- local system $\rightsquigarrow$ FF curve

$\boxed{\text{FF curve}}$ $S = \mathrm{Spa}(R, R^+)/\mathbb{F}_p$ perfd.

$$\varphi \circlearrowright Y = Y_{[0,+\infty]} = \mathrm{Spa}\, W(R^+) - V(p = [\varpi] = 0)$$

$$\cup$$

$$Y_{<+\infty} = Y_{[0,+\infty)} = \mathrm{Spa}\, W(R^+) - V([\varpi] = 0)$$

<u>analytic adic space</u>

$$Y_{<+\infty} = S \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p, \quad Y^{\diamond}_{<+\infty} = S \times \mathrm{Spd}\, \mathbb{Z}_p$$

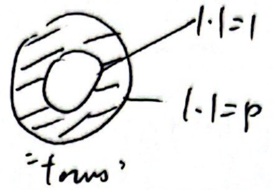
$$Y_{(0,+\infty)} = \mathrm{Spa}\, W(R^+) - V(p \cdot [\varpi] = 0)$$

$\forall\, [a,b] \subset [0,+\infty]$, $a, b \in \mathbb{Q} \cup \infty$, $a \neq b$

$$Y_{[a,b]} := \{ |p|^b \le |[\varpi]| \le |p|^a \} \subset Y$$

adic FF curve

$$X_{FF} = Y_{(0,+\infty)}/\varphi^{\mathbb{Z}} = Y_{[1,p]}/\varphi$$



⑥ Thm (KL) equiv. of cats

$\{$pro-étale $\mathbb{Q}_p$-local system on $S\} \cong \{$v.b. $\mathcal{E}$ on X_{FF} s.t. $\mathcal{E}$ is pro-étale locally trivial on $S\}$.

Q: are there other v.b. on X_{FF}?

Consider φ-module $(\mathcal{O}_{Y_{(0,+\infty)}}, p^{-1}\varphi)$ $\rightsquigarrow$ line bundle $\mathcal{O}(1)$ on X_S.

$\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, $n \in \mathbb{Z}$

☆ $(\mathcal{O}(\lambda),\ \lambda \in \mathbb{Q}$, v.b. on $X_S)$

eg. $\lambda = \frac{1}{2}$, rk 2 v.b.

Fact. $\mathcal{O}(1)$ is not trivial.

$H^0(X_S, \mathcal{O}) = \mathbb{Q}_p$ $(S = \mathrm{Spa}\, C)$

$H^0(X_S, \mathcal{O}(1))$ "huge" "ample line bundle"

§1. Thm.

- $S = \mathrm{Spa}(C)$ field

  $Y_{[0,+\infty)}$ is locally PID.

More precisely, $U \subset Y_{[0,+\infty)}$ aff. open, then $U = \mathrm{Spa}(B, B^+)$ has finitely many components and if U is connected, then B is PID.

$$|Y_{[0,+\infty)}|^{\text{classical}} \overset{\sim}{\longleftrightarrow} \text{untilt of } C/\mathbb{Z}_p$$

$$|Y_{(0,+\infty)}|^{cl} \overset{\sim}{\longleftrightarrow} \text{untilt } /\mathbb{Q}_p$$

$$|X_{FF}|^{cl} \overset{\sim}{\longleftrightarrow} \{\text{untilt } /\mathbb{Q}_p\} / \text{Frob}.$$

$$\wr\ \{\text{degree 1 Cartier divisor on } X_{FF}\}$$

$$= \mathrm{Div}^1 \longrightarrow \mathrm{Spd}\, \mathbb{F}_p$$

Pf. Idea. $C^\# \longleftrightarrow \ker(\theta : W(C^+) \to C^{+,\#})$

$= (\xi)$

- $S = \mathrm{Spa}(C)$ alg. closed field

$\exists$ functor $\mathrm{Iso}_{\overline{\mathbb{F}}_p|\mathbb{F}_p} \longrightarrow \{\text{v.b. on } X_{FF,S}\}$.

$$D_\lambda \longmapsto \mathcal{O}(\lambda)$$

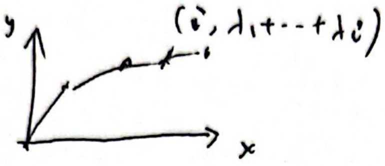
$\lambda \in \mathbb{Q}$ faithful + bijective on isom. classes.

h. ..

rk n v.b. on X_S

Newton polygon. $(\lambda_1, \cdots, \lambda_n) \in \mathbb{Q}^n$

$\lambda_1 \geq \cdots \geq \lambda_n$



$(\lambda_1, \cdots, \lambda_n) \geq (\mu_1, \cdots, \mu_n)$

if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad (1 \leq i \leq n)$

and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$.

- family version.

- $|X_S| \to |S|$ is open and closed.

- $\forall\ \mathcal{E}$ v.b. on X_S,

$\exists\, n_0$ s.t. $\forall\, n \gg n_0$, Serre twist $\mathcal{E}(n)$ is globally generated $+ H^i = 0,\ i > 0$.

ampleness of $\mathcal{O}(1) \Rightarrow$ algebraic version

alg. FF $\qquad P = \bigoplus_{n \geq 0} H^0(X_{FF}, \mathcal{O}(n))$ graded ring

$X_{FF}^{alg} = X_S^{alg} := \mathrm{Proj}(P)$ scheme

graded ring $\qquad f \in P$ positive deg d. covered by $D_+(f) = \mathrm{Spec}(P[\frac{1}{f}])_{\deg = 0}$

Natural map $X_S \to X_S^{alg}$.

Thm (CAGA) pullback induces

$$\{\text{v.b. on } X_S^{alg}\} \simeq \{\text{v.b. on } X_S\}.$$

Rmk. $B = H^0(Y_{(0,\infty)}, \mathcal{O})$ ring

$r>0$ $\tilde{R}^r = H^0(Y_{(0,r]}, \mathcal{O})$ Robba ring

$\tilde{R}^{r,int} = H^0(Y_{[0,r]}, \mathcal{O})$ integral Robba ring.

$$\tilde{R} = \lim_{r\to 0} \tilde{R}^r$$

(If $S = *$, $\tilde{R}^{Int}$ DVR p uniformizer, residue field C)

$$\tilde{R}^{Int,\wedge} = W(C)$$

Prop [KL] eq. of cats

$$\{\text{v.b. on } X_{FF}\} \simeq \{\varphi\text{-module on } Y_{(0,+\infty)}\}$$

$$\simeq \{\varphi\text{-module on } Y_{[r,+\infty)},\ r>0\}$$

$$\simeq \{\varphi^{-1}\text{-module on } Y_{(0,r]}\}$$

$$\simeq \{\varphi\text{-eq. finite projective } \tilde{R}\text{-modules}\}.$$

crys (field) give $S^\sharp \longrightarrow X_{FF,S}$

$\ker\theta = (\xi)$

$B^+ = H^0(Y_{(0,\infty]}, \mathcal{O})$

A_{crys} = divided power envelope for

$\theta : (A_{inf} = W(R^+) \longrightarrow R^{\sharp,+})$

A_{crys} <u>torsion free</u> (R^+ perfd)

$B^+_{crys} = A_{crys}[\frac{1}{p}] = A_{inf}[\frac{\xi^n}{n!}]^\wedge[\frac{1}{p}]$

<u>Thm</u> $H^0(X_{FF}, \mathcal{O}(n)) = B^{\varphi=p^n} = (B^+)^{\varphi=p^n} = (B^+_{crys})^{\varphi=p^n}$

use B^+_{crys} to define X_S^{alg}.

<u>Ex</u>. Structure of $X^{alg}_{FF,C}$

<u>Thm</u> • $X^{alg}_{FF,C}$ is a regular Noether scheme of Krull dim 1

• If $C^\sharp \in |X_{FF,C}|$ closed pt.

$X^{alg}_{FF,C} - C^\sharp = \mathrm{Spa}\ B_e$

(1) $B_e = B_{crys}^{\varphi=id}$ is a PID

(2) $\widehat{\mathcal{O}}_{Y^{alg}_{FF}, C^\sharp} = B^+_{dR} = (A_{inf}[\frac{1}{p}]^{\wedge,\xi})$ DVR, residue field $C^\sharp$

h.

$B_{dR} = B_{dR}^{+}[\frac{1}{\xi}]$

Then $\{$v.b. on $X^{alg.}_{FF,C}\}$

$\Leftrightarrow$ $\{$v.b. on Spec B_e + v.b. on Spec $B_{dR}^{+}\}$

Spec B_{dR}

(3) $t \in H^0(\mathcal{O}(1))$ w/ $V(t) = C^\flat$

then $0 \to \mathcal{O} \xrightarrow{t} \mathcal{O}(1) \to i_{*} C^\flat \to 0$ SES

Take H^0, use $H^1(\mathcal{O}) = 0$

fundamental exact seq. in p-adic Hodge theory.

$\boxed{\S 2}$ G-torsor $G/\mathbb{Q}_p$ smooth affine.

Def'n $X/\mathbb{Q}_p$ scheme / sous perf'd space

(Geometric) $P \to X$ scheme / adic space

$\circlearrowleft$

G s.t. étale locally on X

have $P \simeq G \times X$ G-eq. isom.

$\circlearrowleft$

G

(Cohomological) A <u>cohomological G-torsor</u> is an étale sheaf Q on X $\circlearrowleft G$

s.t. étale locally on X have $Q \simeq G$.

(Tannakian) exact $\otimes$-functor

$$P : \mathrm{Rep}\, G \longrightarrow \{\text{v.b. on } X\}.$$

<u>Thm</u> These three notions are equiv.

$$\begin{tikzcd} (G) \arrow[r, Leftrightarrow] \arrow[r, bend right, "\text{section to } p"'] & (C) \arrow[r, Leftrightarrow] \arrow[r, bend right, "V \in \mathrm{Rep}(G)"'] & (T) \arrow[ll, bend right, "\mathrm{Spec}\, P(\mathcal{O}(G)) \to X"'] \end{tikzcd}$$

$$Q \times_G V$$

<u>Cor.</u> $H^1_{ét}(X, G)$ classify <u>G-torsors</u>.

<u>Prop</u> $\mathrm{Bun}_G : \mathrm{Perfd}_{\mathbb{F}_p} \longrightarrow \mathrm{Grpds}$

$$S \longmapsto \{G\text{-torsors on } X_{FF,S}\}$$

is a <u>v-stack</u>.

<u>Pf.</u> <u>Tannakian</u> $G \hookrightarrow GL_N$ <u>v-descent</u>. □

<u>G-isocrystals</u>. $\rightsquigarrow B(G)$ <u>Kottwitz category</u>.

$k/\mathbb{F}_p$ alg. closed, $L = W(k)[\frac{1}{p}] . \circlearrowleft \sigma$

$B(G)$ obj. $b \in G(L)$

Hom $\quad \mathrm{Hom}(b, b') = \{ c \in G(L) : c b \sigma(c)^{-1} = b' \}$

$B(G) =$ Kottwitz set

$\mathcal{E}_1$ trivial G-bundle on X_S

$\Leftrightarrow \tilde{\mathcal{E}}_1$ on Y_S w/ $\alpha : \varphi_S^* \tilde{\mathcal{E}}_1 \xrightarrow{\sim} \tilde{\mathcal{E}}_1$

$b \rightsquigarrow (\tilde{\mathcal{E}}_1, b^{-1} \alpha^* \varphi^* b)$ on Y_S

$\big\}$ descent

$\mathcal{E}_b$ on X_S

Thm. $b \mapsto \mathcal{E}_b$ gives an equiv. of cats

$$B(G) \xrightarrow{\sim} \mathrm{Bun}_G(\mathrm{Spd}\, k) = \mathrm{Hom}_{v\text{-stack}}(\mathrm{Spd}\, k, \mathrm{Bun}_G)$$

Note. k is not perfectoid.

$\boxed{\text{Finally}}$ Newton polygon

Choose $T \subset B \subset G / \overline{\mathbb{Q}}_p$, $\quad \Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p | \mathbb{Q}_p)$

$\exists$ Newton map $\quad \nu_G : B(G) \longrightarrow (X_*(T)_{\mathbb{Q}} / W)^{\Gamma}$

Kottwitz map $\quad \kappa_G : B(G) \longrightarrow \underbrace{\pi_1(G)_{\Gamma}}_{\text{Borovoi}}$ $= X_*(T) /$ coroot lattice

Prop $\nu_G \times \kappa_G$ injective

$b_1 \le b_2$ if $\kappa_G([b_1]) = \kappa_G([b_2])$

$b_1, b_2 \in B(G)$ and $\nu([b_2]) - \nu([b_1])$

$= \sum (\geq 0)$ positive coroots in $X_*(T)/W$

$\cong X_*(T)^+$

Thm. $B(G) \simeq |\mathrm{Bun}_G|$

topological space

$b_2 \in \overline{\{b_1\}} \iff b_2 \geq b_1$

$b \in B(G)$

$\mathrm{Bun}_G^b = \mathrm{Bun}_G \times_{|\mathrm{Bun}_G|} \{b\} \longrightarrow \mathrm{Bun}_G$

$\star$ $S \longmapsto \{\tilde{G}_b\text{-torsors on } S\}$.

$\tilde{G}_b : S \longmapsto \mathrm{Aut}_{X_S}(\mathcal{E}_b)$

v-group

Classically. $G_b = \sigma\text{-central}(b)/\mathbb{Q}_p$

$\boxed{b \text{ basic}} \iff \mathcal{E}_b$ semistable G-bundle.

then $\tilde{G}_b = G_b(\mathbb{Q}_p)$.

In particular, $b = \mathrm{id}$ is basic

$$\left[*/\underline{G(\mathbb{Q}_p)}\right] \hookrightarrow \mathrm{Bun}_G$$

$$\mathbin{\rotatebox{90}{$\cong$}}$$

$$\mathrm{Bun}_G^1.$$

Example. GL_2.

$\mathcal{O}(\frac{1}{2})$	$\rightsquigarrow$	$\mathcal{O} \oplus \mathcal{O}(1)$
rk 2 v.b.		rk 2 v.b.
basic		non-basic.

Goal. ν upper semi-continuous $\rightsquigarrow$ specialization.

κ locally const.

Ext. of v.b. $\sim$ "$\mathbb{Q}_p$-HT"

p-div. on $R^{\#,+}$