Connections in characteristic 0 and p

Hélène Esnault

Lecture 1.

Def. Let X be a manifold (C^∞, cpx analytic, algebraic mfd)

Ω^1 sheaf of 1-forms (e.g. X Riemann surface, z local parameter, $\mathcal{O}\,dz$)

$\mathcal{E}$ coherent sheaf (locally free)

$$\underset{\text{connection}}{\nabla} : \mathcal{E} \to \Omega^1 \otimes_{\mathcal{O}} \mathcal{E}$$

abelian sheaves

Leibniz relation: defect to $\mathcal{O}$ linearity

$$\nabla(\underset{\mathcal{O}}{f}\,\underset{\mathcal{E}}{e}) = df\, e + f \nabla e$$

Curvature:

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} \mathcal{E} \xrightarrow{\nabla} \Omega^2 \otimes_{\mathcal{O}} \mathcal{E}$$

$$\nabla(\underset{\Omega^1}{w} \otimes \underset{\mathcal{E}}{e}) = dw \otimes e - w \nabla e$$

$\nabla \circ \nabla = \nabla^2 =$ curvature , $\mathcal{O}$ - linear.

Flatness of ∇ (integrality): $\nabla^2 = 0$

<u>Category of flat connections</u>

$$\text{complex}\left[\, 0 \to \mathcal{E} \to \Omega^1 \otimes \mathcal{E} \to \Omega^2 \otimes \mathcal{E} \to \Omega^3 \otimes \mathcal{E} \to \cdots \,\right]$$

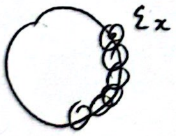
de Rham cpx

$$\nabla(\underset{\Omega^n}{w} \otimes e) = dw \otimes e + (-1)^n w \nabla e$$

<u>Poincaré Lemma</u> (cpx mfd) dR cpx exact everywhere except in degree 0. // X alg. mfd. $/\mathbb{C}$
$X^{an} =: X(\mathbb{C})$

$\mathcal{E}^{\nabla} := \ker(\nabla : \mathcal{E} \to \Omega^1 \otimes \mathcal{E})$ local system, i.e. locally trivial in the analytic topology.

$\bullet \in X^{an}$, action of $\pi_1^{top}(X^{an}, \bullet)$ on $\mathcal{E}_\bullet$ = fiber of sheaf at this pt.



$\gamma \in \pi_1^{top}(X, \bullet)$, $e \in \mathcal{E}_\bullet$ $\rightsquigarrow$ $\gamma e \in \mathcal{E}_x$

$\mathbb{C}$-vec. sp. of dim = rank $\mathcal{E}$

$\rightsquigarrow$ rep'n $\pi_1^{top}(X, \bullet) \to GL(\mathcal{E}_x)$ monodromy representation.

<u>Riemann-Hilbert correspondence</u>: $\{$category of flat connections on X analytic mfd$\}$

$\xleftrightarrow{\text{equiv.}}$ $(\pi_1^{top}(X, \bullet) \to GL(\bullet, \mathbb{C}))/$conjugacy

X alg. mfd $/\mathbb{C}$ $\underset{\text{Deligne}}{\longleftrightarrow}$ $\{\pi_1^{top}(X^{an}, \cdot) \to GL(\cdot, \mathbb{C})\}/\text{conj.}$

(regular singular)

<u>Examples.</u> X algebraic mfd $/\mathbb{C}$

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \\ x \in X^{an} \end{tikzcd}$$

abstract group

π_1^{top}: unramified cover

$\downarrow$

$\widehat{\pi_1^{top}}$ = profinite completion

$$\pi \overset{\Psi}{\longmapsto} \pi^{-1}(x)$$

$$(\text{covers}) \longrightarrow (\text{Sets})$$

$$\mathrm{Aut}(\Psi) = \pi_1^{top}(X, \cdot)$$

$$\begin{tikzcd} Y \arrow[d] \\ X^{an} \end{tikzcd}$$

$\leftarrow$ unramified <u>finite</u> cover.

<u>Riemann Existence Thm</u>: X alg. $\Rightarrow$ Y alg.

$\rightsquigarrow$ X scheme connected of f-type, $x \to X$ geom. point

Grothendieck: $\pi_1^{ét}(X, x) = \widehat{\pi_1^{top}}(X^{an}, x)$

if X mfd $/\mathbb{C}$

$$\begin{tikzcd} Y \arrow[d, "h"'] \\ X \end{tikzcd}$$

finite étale $\rightsquigarrow$ yields flat connection on X:

· $(\mathcal{O}, d)$ trivial connection

$$\mathcal{O}_Y \xrightarrow{d} \Omega^1_Y = h^* \Omega^1_X$$

sheaf of regular functions

$$h_* \mathcal{O}_Y \xrightarrow{h_* d} h_* h^* \Omega^1_X = \Omega^1_X \otimes_{\mathcal{O}_X} h_* \mathcal{O}_Y$$

flat connection.

any summand of it $(\mathcal{E}, \nabla)$ $h^*(\mathcal{E}, \nabla)$ trivial.

$\Leftrightarrow$ monodromy of $(\mathcal{E}, \nabla)$ is finite.

$\qquad\qquad ||$

image of $\pi_1^{top}(X^{an}, \cdot) \to GL(\mathcal{E}_\cdot)$

(if over $\mathbb{C}$)

particular case of a so-called Gauß - Manin connection.

$= \{ (\mathcal{E}, \nabla)$ w/ finite monodromy $\}$

RH correspondence : $\otimes$ correspondence

[linear / $\mathbb{C}$, abelian, rigid, monoidal]

$\longrightarrow$ Tannakian

So can also talk on "monodromy" on any X smooth scheme of f.type over a field k, over which

category of integrable connection is tannakian : precisely the char. 0 fields.

Yet another formulation:

$(\mathcal{E}, \nabla)$ has finite monodromy

$\Leftrightarrow \exists\ Y \xrightarrow[h]{} X$ finite étale s.t. $h^*(\mathcal{E}, \nabla)$ is trivial.

i.e. $h^*(\mathcal{E}, \nabla) \underset{\uparrow}{\simeq} \bigoplus_1^2 (\mathcal{O}_Y, d)$.

as a connection

<u>Note</u>: this last definition is meaningful in char. $p > 0$.

$h: Y \longrightarrow X$ smooth projective

connecting homomorphism

$n \in \mathbb{Z}_{\geq 0}$, $\quad R^n h_* \Omega^\bullet_{Y/X} \xrightarrow[\text{Katz}]{GM} \Omega^1_X \otimes_{\mathcal{O}_X} R^n h_* \Omega^\bullet_{Y/X}$

<u>Recall</u>.
$$\begin{array}{ccccccccc}
\Omega^\bullet_Y & = & \mathcal{O}_Y & \xrightarrow{d} & \Omega^1_Y & \xrightarrow{d} & \Omega^2_Y & \xrightarrow{d} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega^\bullet_{Y/X} & = & \mathcal{O}_Y & \xrightarrow{d/X} & \Omega^1_{Y/X} & \xrightarrow{d/X} & \Omega^2_{Y/X} & \longrightarrow & \cdots
\end{array}$$

assume X has dim 1

<u>eg</u>. $n=0$, $\quad h_* \Omega^\bullet_{Y/X} \longrightarrow \Omega^1_X \otimes h_* \Omega^\bullet_{Y/X}$

$s \in h_* \mathcal{O}_Y$, $d/X(s) = 0$

$$\begin{array}{ccccc}
 & & \mathcal{O}_Y & \xrightarrow{=} & \mathcal{O}_Y \\
 & & \downarrow & & \downarrow d/X \\
h^*\Omega^1_X & \longrightarrow & \Omega^1_Y & \longrightarrow & \Omega^1_{Y/X} \\
 & & \downarrow & & \downarrow \\
h^*\Omega^1_X \otimes \Omega^1_{Y/X} & \longrightarrow & \Omega^2_Y & \longrightarrow & \Omega^2_{Y/X} \\
 & & \downarrow & & \downarrow \\
h^*\Omega^1_X \otimes \Omega^2_{Y/X} & \longrightarrow & \vdots & & \vdots
\end{array}$$

Take h finite : did before.

Def'n Let X be a smooth scheme of finite type over a char. 0 field k. Then a connection $(\mathcal{E}, \nabla)$ is said to be of "geometric origin" if there is $\mathcal{U} \underset{\text{dense open}}{\hookrightarrow} X$ s.t. $(\mathcal{E}, \nabla)|_{\mathcal{U}}$ is a subquotient of a Gauß-Manin connection.

Deep question: how to characterize them?

Kronecker: $a \in \mathbb{C}$

(monodromy)

$$a \in \mu_\infty(\mathbb{C}) \iff ? \left[\begin{array}{l} a \in \overline{\mathbb{Z}} \text{ algebraic integer} \\ |a^\sigma|_{\mathbb{C}} = 1 \end{array}\right. \qquad \text{analytic}$$

root of unity

$$a = \exp(2\pi\sqrt{-1}\, \underset{\mathbb{C}}{\underset{\Uparrow}{b}})$$

$$a \in \mu_\infty(\mathbb{C}) \iff b \in \mathbb{Q}$$

(connection)

arithmetic

$$b \in \mathbb{Q} \iff \left[\begin{array}{l} b \in \overline{\mathbb{Q}} \text{ algebraic number, i.e. } b \in K,\ K \text{ some number field} \\ \qquad \exists\ b \in \mathcal{O}_{K,\Sigma} \leftarrow \text{finite \# of primes} \\ \forall\, \mathfrak{p} \notin \Sigma,\ b \bmod \mathfrak{p} \in \mathcal{O}_{K,\Sigma}/\mathfrak{p} = k(\mathfrak{p}) \\ \qquad\qquad \in \mathbb{F}_p \subset \end{array}\right.$$

Little challenge. arithm. (i) (ii) $\iff$ analytic (i) (ii)

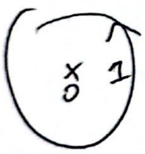
Relation to deep problem. (for finite monodromy)

$X = \mathbb{G}_m$, $X(\mathbb{C}) = \mathbb{C}\setminus\{0\}$, t parameter

$\mathcal{E} = \mathcal{O}$, $\Gamma(X,\mathcal{O}) = \mathbb{C}[t,t^{-1}]$

$$\nabla(1) = b\frac{dt}{t}\otimes 1 \quad , \quad a = \exp(2\pi\sqrt{-1}\,b)$$

$$\pi_1^{top}(\mathbb{C}\setminus\{0\},1) = \mathbb{Z}$$



$$0 = \nabla(f\cdot 1) = df + bt\frac{dt}{t}$$

$$\frac{df}{f} = -b\frac{dt}{t}$$

$$f = \text{const}\ \exp(2\pi\sqrt{-1}\,bt)$$

$$\gamma \longmapsto \exp(2\pi\sqrt{-1}\,b) = a$$

Now.

$$\begin{array}{c}(\mathbb{G}_m, z)\\ \downarrow \\ (\mathbb{G}_m, t = z^n)\end{array} \qquad f\in\overline{\mathbb{C}(t)} \iff b\in\mathbb{Q}$$

arithm (i) $b\in K$ ~~#~~ field (i.e. $(\mathcal{E},\nabla)$ is defined over K)

(ii) $b^p - b = 0 \mod \mathfrak{p}$ for almost all $\mathfrak{p}$.
p prime in $\mathbb{Z}$.

Defn $b^p - b \mod p$ is the p-curvature of $(\mathcal{E}, \nabla) \mod p$.

Obtained (Kronecker)

Thm $(X, (\mathcal{E}, \nabla))$ has finite monodromy
$$\underset{\mathbb{G}_m}{\overset{\shortparallel}{X}}$$

$\Leftrightarrow$ p-curvature $= 0$ for almost all p.

Particular case of Grothendieck's p-curvature conjecture.

Lecture 2. X smooth var. $/\mathbb{C}$
projective

Simpson: 3 moduli spaces

character variety:

$M_B(X)(r)$ ← rank $\quad$ $\mathbb{C}$-pts $= \{\pi_1^{top}(X(\mathbb{C}), \cdot) \to GL(r, \mathbb{C})\}/\text{conj}.$

Betti $\quad$ $\mathbb{C}$-variety $\quad$ irred

$M_{dR}(X)(r)(\mathbb{C}) = \{(\mathcal{E}, \nabla) :$ alg integrable connection. $\}/\text{isom}.$
irred.

de Rham

RH: $\mathbb{C}$-analytically, $M_B \simeq M_{dR}$.

$$M_{Dol}(X)(r)(\mathbb{C}) = \{(V,\theta) : \theta : V \xrightarrow{c_i = 0} \Omega^1_X \otimes V, \ \theta \wedge \theta = 0\}$$

$\uparrow$ Dolbeault $\qquad$ $\mathcal{O}$-linear $\qquad$ $*$-stable (in the sense of Mumford)

loose way: $\forall\ W \subset V$.

$$\begin{array}{ccc} W & \subset & V \\ \theta\downarrow & & \downarrow\theta \\ \Omega^1\otimes W & \subset & \Omega^1\otimes V \end{array}$$

"slope$(W) <$ slope(V)"

$$M_{Dol} \underset{\mathbb{R}-\text{analytically}}{\simeq} M_{dR}$$

e.g. $r=1$, X curve.

$$M_B(\mathbb{C}) = \{\pi_1^{top} \to \mathbb{C}^\times\} = H^1(X, \mathbb{C}^\times)$$

$$M_{Dol} = Pic^0(C) \times H^0(X, \Omega^1)$$

$$\begin{array}{ccc} M_{dR} & \xleftarrow{H^0(X,\Omega^1)} & \{(\mathcal{E},\nabla)\} \\ \downarrow & & \downarrow \\ Pic^0 & & [\mathcal{E}] \end{array}$$

$r \geq 2$.

Rmk. Rigid objects are isolated points of either M_B, or M_{dR}, or M_{Dol}.

Conjecture (Simpson) rigid objects are of geometric origin.

[i.e. $\exists\ Y \xrightarrow{f} U \overset{open}{\hookrightarrow} X$ s.t. $(\mathcal{E},\nabla)$ is a subquotient of $(R^n f_*(\Omega^\bullet_{Y/U}), \nabla^{GM})$

Y proj. smooth, $U \neq \emptyset$

$(R^n f^{an}_* \mathbb{Z}) \otimes \mathbb{C} \downarrow$ local system know:

① X proj. curve, no rigid objects

X affine curve, Katz: all coming from hypergeometric diff eqns.

X open in $\mathbb{P}^1$ (appearance of Shimura varieties)

② X higher dim'l,

Corlette - Simpson: SL_2 - connections

i.e. $(\mathcal{E}, \nabla)$ $rk = 2$, $\underset{\text{rk 1 conn.}}{\det(\mathcal{E}, \nabla)} = (\mathcal{O}, d)$.

Coming from automorphic bundles on Shimura varieties.

Rmk. If Simpson conj. is correct, then

if $\rho : \pi_1^{top}(X(\mathbb{C}), \cdot) \to GL(r, \mathbb{C})$ is rigid, then $\exists$ $L|K$ finite field ext'n

$\searrow GL(r, K)$ s.t. $\rho(\pi_1^{top}(X(\mathbb{C}), \cdot)) \subset GL(r, \mathcal{O}_L)$.

number field

mod. conj.

Simpson's geometricity conjecture $\Rightarrow$ the integrality conjecture:

a rigid $\rho : \pi_1^{top}(X(\mathbb{C}), \cdot) \to GL(r, \mathbb{C})$

has values in $GL(r, \mathcal{O}_L)$ (up to conj.)

$[L : \mathbb{Q}] < \infty$.

③ $(\mathcal{E}, \nabla)$ irred. rigid & integral.

SL_3 - connection. $\xRightarrow[\text{Langer - Simpson}]{}$ geometric.

<u>Note</u>: intersection of yesterday's lecture w/ today's lecture lies in the word "geometric". ① + ② + ③ geometric method.

<u>Q</u>: $\underbrace{①②③}$ where does $Y \to U \hookrightarrow X$ come from?

SL_2, SL_3 $(2,0), (1,1), (0,2)$
$(1,0), (0,1)$

<u>Def'n</u> X smooth $/k$, char $k = p > 0$

$X' =$ Frobenius twist

relative Frobenius

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$$\lambda^p \longleftarrow\!\shortmid \lambda$$

$$\begin{tikzcd} & k[X] \otimes_k k \arrow[d, equal] & \\ k[X] & k[X'] \arrow[l] & k[X] \arrow[l] \arrow[ll, bend right, "f^p \,\leftarrow\!\shortmid\, f"'] \\ & k \arrow[ul] \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

$x^p \leftarrow\!\shortmid 1 \otimes x$

$\lambda^p \longleftarrow\!\shortmid \lambda$

$$\sum \lambda_I^p x^{Ip} \longleftarrow \sum \lambda_I^p x^I \longleftarrow f = \sum \lambda_I x^I$$

$\mathcal{E}'$ coh. sheaf on X', $\left(\mathcal{E} := F^*\mathcal{E}', \nabla_{can}\right)$ $\quad \nabla_{can}(m' \otimes f) = m' \otimes df$

$M' \otimes_{k[X']} k[X] = M$ over $k[X]$, M' over $k[X']$ $\quad$ Well-defined as $d(\lambda^p f) = \lambda^p df$

$\mathcal{E}'$ not necessarily locally free $\Rightarrow$ char $p>0$, cat. of int. conn. is not tannakian.

<u>Def'n</u>. $(\mathcal{E},\nabla)$ integrable conn. has <u>p-curvature</u> 0 iff $\exists\ \mathcal{E}'$ on X' s.t.

$$(\mathcal{E},\nabla) \simeq (F^*\mathcal{E}', \nabla_{can}).$$

$(\mathcal{E},\nabla)$ has <u>nilpotent</u> p-curvature if $(\mathcal{E},\nabla)$ is filtered s.t.

$(gr\,\mathcal{E}, gr\,\nabla)$ has p-curvature 0.

<u>See together</u>:

<u>Prop</u>. $(\mathcal{E},\nabla)$ flat connection on X/k ← char. 0

$\exists\ (X_S, (\mathcal{E}_S, \nabla_S))$ S scheme of f.t. $/\mathbb{Z}$

$\operatorname{Spec} \mathbb{C} \to S$

s.t. $(X,(\mathcal{E},\nabla)) = (X_S,(\mathcal{E}_S,\nabla_S)) \otimes_S k$.

then if flat connection has finite monodromy, the p-curvatures of its mod p reductions are 0. — for almost all p. $\left(\psi(\nabla_s)=0,\ \forall\ s\to S' \hookrightarrow S\right)$ open $\neq\phi$

$\mathcal{E}_S \xrightarrow{\nabla_S} \Omega^1_{X_S/S} \otimes_{\mathcal{O}_{X_S}} \mathcal{E}_S$

$s \longrightarrow S$ closed pt $\qquad \mathcal{E}_s \xrightarrow{\nabla_s} \Omega^1_{X_s} \otimes_{\mathcal{O}_{X_s}} \mathcal{E}_s$

mod p reduction = restriction at s

<u>Why</u> :

$$\begin{array}{ccc} Y & Y_s & \\ \text{finite étale} \downarrow h & \downarrow h_s & \\ X & X_s & \end{array}$$

$$\begin{tikzcd} Y_s \arrow[r, "F"] \arrow[d, "h_s"] & Y'_s \arrow[d, "h'_s"] \\ X_s \arrow[r, "F"'] & X'_s \end{tikzcd}$$

h_s étale

$h^*(\mathcal{E}, \nabla)$ trivial

$\mathcal{E}'$ subqt of $h'_{s*} \mathcal{O}_{Y'_s}$

Prop. (Deligne) k char 0, $(\mathcal{E}, \nabla)$ GM connection ($Y \xrightarrow{f} X$ proj. smooth)

(likewise for subqt)

$\to (X_S, (\mathcal{E}_S, \nabla_S))$

$s \in S$ cl. pt. $(X_s, (\mathcal{E}_s, \nabla_s))$ nilpotent p-curvature.

Optimistic generalization of the Grothendieck – Katz p-curvature <u>conj.</u>:

if $(\mathcal{E}, \nabla)/k$ char. 0 has nilp. p-curvatures for almost all p, ($s \to S' \overset{\text{open}}{\hookrightarrow} S$, $\neq \emptyset$)

then $(\mathcal{E}, \nabla)$ is of geometric origin.

<u>Summarize the discussion</u>:

- Grothendieck – Katz conj. | Deligne

  finite in char. 0 $\Rightarrow$ p-curvature 0

  geometric $\Rightarrow$ nilp.

- Simpson: rigid $\Rightarrow$ geometric

  rigid $\Rightarrow$ integral $\boxed{\text{topological side}}$

Simpson geometric conj. $\Rightarrow$

<u>Conj</u>. X sm proj. $/ k$ char 0, $(\mathcal{E}, \nabla)$ rigid, $\Rightarrow$ p-curvature should be nilpotent.

<u>Thm</u> ($\mathcal{E}$.- Groechenig) Yes!

Another way of phrasing the property that p-curvature of $(\mathcal{E}_s, \nabla_s)$ is nilp., is to say:

$k(s)$ = residue field of s

$$\begin{tikzcd} X_S \arrow[d] & X_W \arrow[l] \arrow[d] \\ S & W \arrow[l] \end{tikzcd}$$

$W = W(k(s))$ - point of S, $K := \mathrm{Frac}\, W$ $\Longleftrightarrow$ $(\mathcal{E}_K, \nabla_K)$ is an isocrystal.

Witt vector

p-adic field

$(\mathcal{E}_W, \nabla_W)$

$$\begin{tikzcd} X_S \arrow[d] & X_W \arrow[l] \arrow[d] & X_K \arrow[l] \arrow[d] \\ S & \mathrm{Spec}\, W \arrow[l] \arrow[r, phantom, "\urcorner"] & \mathrm{Spec}\, K \arrow[l] \\ s \arrow[u] \arrow[ur] & & \end{tikzcd}$$

X_s

<u>Thm</u> ($\mathcal{E}$.- Groechenig) Isocrystal has a Frobenius structure.

X_s curve, s = finite field

L. Lafforgue - Abe

<u>know</u>: F-isoc. are of geom. origin. / X curve

cannot use (so far) — isoc. $\to$ char. 0?

L. Lafforgue constructed on curve $/\mathbb{F}_q$ ℓ-adic companions,

(positive answer for curves to Deligne's conjecture in Weil II)

Tomoyuki Abe : $\exists$ companions

ℓ-adic sh. $\longleftrightarrow$ F overconvergent isocrystals

curve.

Drinfeld : X smooth in any dim $/\mathbb{F}_q$,

$\Rightarrow$ ℓ-adic companions exist. (L functs are the same)

E.- Abe : F overconvergent isocrystals have ℓ-adic companions.

(L - functs are the same)

using this :

int. conj. : true if in addition to be isolated the moduli point is smooth.

cohomological condition seen by the L-funct.

Lecture 3. Thm (E.- Groechenig) : X smooth proj. $/k$ char. 0

(integrality conj.) $(\mathcal{E}, \nabla)$ rigid flat connection

$\leftrightarrow H^1_{dR}(X, (\mathcal{E}nd^\circ \mathcal{E}, \mathcal{E}nd^\circ \nabla)) = 0$ cohomologically (i.e. $[(\mathcal{E}, \nabla)] \in M_{dR}(X)(k)$ smooth) isolated

then it is integral (i.e. $k \underset{\text{cpx}}{\hookrightarrow} \mathbb{C}$, RH:
embedding

$$\begin{array}{ccc} \rho: \pi_1^{top}(X, \cdot) & \longrightarrow & GL(2, \mathbb{C}) \\ & \searrow & \cup \\ & & GL(2, \mathcal{O}_L) \end{array}$$)

L number field
$\cap$
$\mathbb{C}$

2 ingredients

Recall. from previous lecture

① p-curvature of $(\mathcal{E}, \nabla)$ are nilpotent.

② $(\mathcal{E}, \nabla)$ on $X_K \leftarrow K$ p-adic field

(X_S, ∇_S) S ft$/\mathbb{Z}$ $\operatorname{Spec} K \longrightarrow S$

isocrystal

is endowed w/ a Frobenius structure.

③ X_s , $k(s) = \mathbb{F}_q$

existence of ℓ-adic companions

(Abe - E.)

① ② ③ only need rigidity

④ companion ℓ-adic sheaves viewed on X_K (Grothendieck's specialization homo.)

Coh. rigid

Ad ①.

p-curvature:

$$\begin{tikzcd} & \mathcal{E} \\ \mathcal{E} \arrow[ur, "\nabla_\theta"] \arrow[r, "\nabla"] & \Omega^1_X \otimes \mathcal{E} \arrow[u, "\theta \in T_X"'] \end{tikzcd}$$

X/k perfect field of char. $p>0$

$\psi(\theta)(\nabla) = (\nabla_\theta)^p - \nabla_{\theta^p}$ ← only in char. $p>0$

$\psi(\nabla) : \mathcal{E} \longrightarrow F^*\Omega^1_{X'} \otimes \mathcal{E}$

$\psi(\nabla) \in H^0(X, F^*\Omega^1_{X'} \otimes \mathcal{E}nd(\mathcal{E}))$

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr] \arrow[rr, bend left, "="] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ & \operatorname{Spec} k \arrow[r, "\mathrm{Frob}_k"'] & \operatorname{Spec} k \end{tikzcd}$$

relative Frobenius

$\lambda^p \leftarrow\!\shortmid \lambda$

More is true (Katz)

$\underset{\sim}{F^*\Omega^1_{X'}} \otimes \underset{\sim}{\mathcal{E}nd(\mathcal{E})}$

$\nabla_{can} \otimes \mathcal{E}nd(\nabla) =: D$

= unique connection s.t. its flat connections $= \Omega^1_{X'}$

$k[X] \leftarrow k[X'] \leftarrow k[X]$

$x^p \leftarrow\!\shortmid x$

$\varphi^p \longleftarrow\!\shortmid \varphi$

$\psi(\nabla) \in H^0(F^*\Omega^1_{X'} \otimes \underset{\sim}{\mathcal{E}nd(\mathcal{E})})^D$ ← algebra

Define $\forall n \in \mathbb{Z}_{>0}$,

$$\operatorname{Sym}^n \psi(\nabla) \in H^0(X, F^* \operatorname{Sym}^n \Omega^1_{X'})^{D = \nabla_{can}}$$

$$= H^0(X', \operatorname{Sym}^n \Omega^1_{X'})$$

projective

Defines for us: $\mathcal{X}_{dR} = M_{dR}(X)(r) \longrightarrow \mathbb{A}'_k$

$\searrow \operatorname{Spec} k \swarrow$

$\mathbb{A}'_{/k}$ affine space

$$\mathbb{A}'(k) = \bigoplus_{1}^{r} H^0(X', \mathrm{Sym}^n \Omega^1_{X'}) \quad \text{char-}p \text{ Hitchin base}$$

Point: rigidity $^{(\mathcal{E},\nabla)}$ /char 0 $\Rightarrow$ $\chi_{dR}((\mathcal{E},\nabla)) = 0$

$$\Updownarrow$$

$\psi((\mathcal{E},\nabla))$ nilpotent

if not, then $\chi_{dR}((\mathcal{E},\nabla)) \neq 0$.

Bezrukavnikov – Braverman, Ogus – Vologodsky, Groechenig,

$$\{(\mathcal{E},\nabla)\} \longleftrightarrow \left(\begin{array}{l} \mathcal{D} \leftarrow \text{Azumaya algebra on } T^*X' \\ \text{-module} \\ \text{coherent on spectral varieties} \end{array} \right)$$

$a \to \mathbb{A}'_k \rightsquigarrow$ spectral variety: $X'_a \subset T^*X'$.

$\uparrow$

$\mathcal{O}$-coh. sheaf endowed w/ an action of an Azumaya alg.

transforms non-linear objs (connections) into $\boxed{\text{linear objs}}$ $\leftarrow$ deforms if $a \neq 0$

$\Uparrow$

[OV]

Companions (Deligne's conjecture , Weil II)

X normal variety, geom. conn'd $/\mathbb{F}_q$, $\ell \neq$ char $\mathbb{F}_q = p$

Class field theory $\Rightarrow$: V irred. $\overline{\mathbb{Q}_\ell}$ ℓ-adic sheaf

$$\det V : \pi_1^{et}(X, \cdot) \longrightarrow \overline{\mathbb{Q}_\ell}^\times$$

finite or not.

$\rightarrow \exists\ \chi : \pi_1^{et}(X) \longrightarrow \overline{\mathbb{Q}_\ell}^\times$ s.t.

$\det (V \otimes \chi)$ is finite

Conj. V irred. / $\det V$ finite

① weight $V = 0$ (i.e. $\forall\ x \in |X|$, geom. Frob $\mathrm{Frob}_x \in \pi_1^{et}(X)$

acts on $V_{\bar{x}}$ $\bar{x} \longrightarrow x$ geom. point.

well-defined up to conj.)

$\rightsquigarrow \det(1 - t\,\mathrm{Frob}_x \mid V_{\bar{x}}) \in \overline{\mathbb{Q}_\ell}[t]$

$\quad\quad = f_x(V)(t) \quad\quad E_x \subset$ # field

$\forall\ E_x \xrightarrow{\iota} \mathbb{C}$ cpx embedding , $|\iota(\text{eigenvalue})|_{\mathbb{C}} = 1$.

Curve : Lafforgue $\rightarrow$ ✓

Consequence of Langlands correspondence .

$\rightarrow$ OK for X dim > 1.

[suggest $\exists\ U \hookrightarrow X$, $Y \rightarrow U$ smooth proj. s.t. V subqt of $R^n f_* \overline{\mathbb{Q}_\ell}$ some $n \in \mathbb{N}$]

② $\exists E$ # field $\supset E_x$, $\forall x \hookrightarrow X$ closed pt

X smooth curve, OK Lafforgue

$\dim X > 1$, Deligne (2011)

[V entirely determined by all $f_x(V)(t)$; Deligne - bound the deg. of necessary x]

$\Uparrow$

Čebotarev

③ V irred. finite det, X normal / $\mathbb{F}_q$

$\sigma : \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \overline{\mathbb{Q}_{\ell'}}$ $(\forall \ell' \neq p)$

$\uparrow$

abstract isomorphism

$\wedge$

field

companion.

$\underset{?}{\Rightarrow} \exists\, V'$ irred. finite det, $f_{V'}(x) = \sigma f_V(x)$?

X normal curve OK, Lafforgue

Drinfeld 2012 OK, X smooth

(Wiesend)

$$\boxed{\text{more substance to the prediction that the } V\text{'s should be of geom. origin}}$$

④ " crystalline side to the story "

$\nearrow$

Crew

$\sigma : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, $\Rightarrow M$ F-overconvergent isocrystal,

$\sigma(f_M(x)) = f_V(x)$. and vice versa. and also $\sigma : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$

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X smooth curve. Lafforgue + T. Abe

$p \to \ell$ $\quad X$ smooth, ℓ-adic companions exist (Abe-E.)

One comment on how we use existence of ℓ-adic companions:

(Lan-Sheng-Zuo $\longleftrightarrow$ Faltings p-adic Simpson correspondence).

Why does one restrict the problem to <u>cohomologically</u> rigid connections.

k char. 0, $\quad H^1_{dR}(X, (\mathcal{E}nd^\circ \mathcal{E}, \mathcal{E}nd^\circ \nabla)) = 0. \quad (*)/k$

generic base change thm for dR cohomology:

$\rightsquigarrow (*)/k(s) \quad \forall s \to S/\mathbb{Z}$ closed pt

<u>Pb becomes</u> $\quad \ell$-adic companion. on X_s

to our $(\mathcal{E}, \nabla)$ on $X_K \leftarrow$ p-adic field

<u>Want to show.</u>

X_s/s smooth

$$\begin{tikzcd}
\pi_1^{\acute{e}t}(X_K) \arrow[r, two heads] \arrow[rr, bend right] & \pi_1^{\acute{e}t}(X_s) \arrow[r] & GL(2, \bar{\mathbb{Q}}_\ell)
\end{tikzcd}$$

Grothendieck specialization homomorphism

So can view V as an irred. $\bar{\mathbb{Q}}_\ell$-adic sheaf on X_K.

$(X_s$ proper $/s)$ $\quad H^1_{\acute{e}t}(X_{\bar{s}}, \mathcal{E}nd^\circ V)$

$\| \text{SGA}$

$H^1_{\acute{e}t}(X_{\bar{K}}, \mathcal{E}nd^\circ V_K) = 0$?

def'n of companions: by product formula, L func. of $\mathcal{E}nd^\circ(\mathcal{E}, \nabla)$ and of $\mathcal{E}nd^\circ(V)$ are the same.

L-funcs recognise Euler cha. + Frob. action.

purity (conj. ¥ 1)

$\Rightarrow$ $\mathcal{E}nd^\circ V$ weight 0

$\rightarrow$ wts of each $H^n(X_{\bar{s}}, \mathcal{E}nd^\circ V)$ are different.

$\mathcal{E}nd^\circ(\mathcal{E}, V)$.

$\Rightarrow$ If $H^1(X_{S/k}\ \mathcal{E}nd^\circ(\mathcal{E}, \nabla)) = 0$ $\Rightarrow$ $H^1_{\text{ét}}(X_s, \mathcal{E}nd^\circ V) = 0$.

behind Drinfeld's construction of ℓ-adic companions (and Abe-E). once we know on curves is the existence of a Lefschetz thm.

X sm. ft $/\mathbb{C}$, Lefschetz:

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ \uparrow_{\iota} & & \uparrow \\ C & \longrightarrow & \bar{C} \end{array}$$

good compactification

proj-, $\bar{X} \setminus X$ NCD

complete intersection curve

good position w.r.t. $\bar{X} \setminus X$

$$\pi_1^{top}(C) \twoheadrightarrow \pi_1^{top}(X)$$

$$\Rightarrow \pi_1^{\text{ét}}(C) \twoheadrightarrow \pi_1^{\text{ét}}(X)$$

ok, $\forall$ base field k char. = 0

char $= p > 0$, no longer true.

$\mathbb{A}^2 = X$

Claim: $\nexists$ sm. curve $C \subset \mathbb{A}^2$ s.t.

$$\pi_1^{\acute{e}t}(C) \twoheadrightarrow \pi_1^{\acute{e}t}(\mathbb{A}^2)$$

$C = V(f) \subset \mathbb{A}^2$, $\quad \mathbb{A}^2 \leftrightarrow k[x,y] \ni f \notin k$

Artin-Schreier

$t^p - t = f$ conn. $/\mathbb{A}^2$

on C disconn'd.

? (Wiesend): how could one prove ... using the information on curves?

Idea: no Lefschetz,

but fix one representation of $\pi_1^{\acute{e}t}(X) \xrightarrow{\rho} GL(2, \bar{\mathbb{Q}}_\ell)$ irreducible,

find a curve $C \to X$ s.t. $\rho|_C$ has the same monodromy gp.

for ℓ-adic rep's

$$\text{2 steps: } \rho: \pi_1^{\acute{e}t}(X) \longrightarrow GL(2, \mathcal{O}_{E_\ell}) \downarrow GL(2, \mathcal{O}_{E_\ell}/\ell)$$

pro-ℓ-gp

pure-ℓ-quotient

f.d. $\mathbb{F}_\ell$ v.s.

E number field, E_ℓ completion at ℓ. $\quad E \supset \mathbb{Q}$

$$X_\ell' \xrightarrow{\mathbb{F}_\ell \text{ v.s.}} X_\ell \xrightarrow{\text{f. ét cover}} X$$

Π top. gp $\to \pi_1^{\acute{e}t}(X)$

thm on profinite gp

- Hilbert irreducibility thm.