Moduli spaces: functors, algebraic spaces, stacks, algebraic stacks

Max Lieblich

Lecture 1. What is a moduli space?

(0) varieties

(1) curves of genus g

(2) line bundles on X

(3) maps between X & Y

(4) closed subschemes of X (e.g. $X = \mathbb{P}^n$)

(5) subspaces of a fixed vector space

$\{$pts of $M_i\} \longleftrightarrow \{$objects of flavor $i\}$

Q: How should a scheme be described?

(1) "Absolute": affine chart

(2) "Relative"

Analogy: functions

Q. How should an element $f \in C^\infty([0,1])$ be described?

(1) $f(x)$

(2) Relative: pairing $g \in C^\infty([0,1]) \rightsquigarrow \int_0^1 fg \in \mathbb{R}$

$$L_f : C^\infty([0,1]) \longrightarrow \mathbb{R}$$
$$g \longmapsto \int_0^1 fg$$

"Thm" $L : C^\infty([0,1]) \longrightarrow \mathrm{Hom}(C^\infty([0,1]), \mathbb{R})$ is an injective linear transf.

Same procedure in a cat.: pairing $X, Y \longmapsto \mathrm{Hom}(X,Y)$, a set

X object, $\quad Y \longmapsto \mathrm{Hom}(Y, X) = h_X(Y)$

$\mathcal{C} \longrightarrow \mathrm{Func}(\mathcal{C}^o, \underline{Set})$, functor, fully faithful

$$Z \to Y \rightsquigarrow \begin{array}{ccc} h_X(Y) & \to & h_X(Z) \\ \| & & \| \\ \mathrm{Hom}(Y, X) & \to & \mathrm{Hom}(Z, X) \end{array}$$

composition

h_X is a contravariant functor from $\mathcal{C}$ to $\underline{Set}$.

$\underline{Ex}$. $\mathcal{C} = Sch_{\mathbb{Z}}$, $h_{\mathbb{A}^1}(Y) = \Gamma(Y, \mathcal{O}_Y)$

$Z \to Y$, $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\text{pullback}} \Gamma(Z, \mathcal{O}_Z)$

$h_{\mathbb{G}_m}(Y) = \Gamma(Y, \mathcal{O}_Y^\times)$

$h_{\mathbb{P}^n}(Y) = \{ \mathcal{O}_Y^{n+1} \twoheadrightarrow \mathcal{L}, \ \mathcal{L} \text{ invertible sheaf on } Y \} / \sim$

$\mathrm{Func}(\mathcal{C}^o, \underline{Set})$ is a cat. obj. = functors, mor = nat'l transf.

e.g. $\Gamma(Y, \mathcal{O}_Y^\times) \hookrightarrow \Gamma(Y, \mathcal{O}_Y)$, $\quad h_{\mathbb{G}_m}(Y) \to h_{\mathbb{A}^1}(Y)$

$h_{\mathbb{G}_m} \to h_{\mathbb{A}^1}$.

$X \to X' \overset{\text{comp.}}{\rightsquigarrow} h_X \to h_{X'}$.

<u>Analogue of bumps</u>

<u>Yoneda Lemma</u>. $h : \mathcal{C} \to \mathrm{Func}(\mathcal{C}^o, \underline{Set})$ is fully faithful.

e.g. $h_{\mathbb{A}^1} \to h_{\mathbb{P}^n}$ comes from a unique map of scheme $\mathbb{A}^1 \to \mathbb{P}^n$.

<u>Notation</u>. h_X = "the functor of pts of X"

$\mathcal{C} = Sch_{\mathbb{C}}$, $h_X(\operatorname{Spec}\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(\operatorname{Spec}\mathbb{C}, X) = X(\mathbb{C})$

<u>Ex</u>. $\left.\begin{array}{l} \mathbb{G}_m \longrightarrow \mathbb{G}_m \\ x \longmapsto x^2 \end{array}\right\}$ work on all pts, def a nat'l transf $h_{\mathbb{G}_m} \to h_{\mathbb{G}_m}$. $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$

<u>Philosophy</u>: Functor = "generalized space"

Schemes = distinguished class of spaces

Instead of $F: \mathcal{C}^\circ \longrightarrow \underline{Set}$

think: F

Internal structure: $F(T)$

<u>Exer</u>: $\exists$ a nat'l bijection $\operatorname{Hom}(h_T, F) \xrightarrow{\sim} F(T)$

<u>Def</u> A functor F is <u>representable</u> if $\exists X \in \mathcal{C}$ s.t. $F \simeq h_X$.

Anything we can do w/ sets, we can do w/ functors of sets.

e.g. $\begin{array}{ccc} F & & G \\ & \searrow \ \swarrow & \\ & H & \end{array} \rightsquigarrow (F \times_H G)(T) = F(T) \times_{H(T)} G(T)$ This works.

$h_{X \times_Z Y} = h_X \times_{h_Z} h_Y$.

<u>Q</u>. What is the functor of points of M_i ?

$\mathcal{C} = Sch_{\mathbb{Z}}$, $M_0(T) = \left\{ X \to T, \begin{array}{l} \text{finite presentation,} \\ \text{flat,} \\ \text{geom. integral fiber} \end{array} \right\} / \simeq$

$M_g(T) = \left\{ C \to T, \begin{array}{l} \text{proper smooth [finite presentation]} \\ \text{fibers are curves of genus } g \end{array} \right\} / \simeq$

$M_2(T) = \{ \mathcal{L} \text{ invertible sheaf on } X \times T \} / \simeq$

$M_3(T) = \mathrm{Hom}_T(X_T, Y_T)$

$$M_4(T) = \left\{ \begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & X \times T \arrow[d] \\ & T \end{tikzcd} , \quad \begin{array}{l} Z \text{ is } T\text{-flat} \\ \text{closed subscheme of fin. present. of } X \times T \end{array} \right\} / \simeq$$

(base is $\mathbb{C}$)

$M_5(T) = \{ W \subset V \otimes \mathcal{O}_T \text{ s.t. cokernel is loc. free} \}$

Lecture 2. X scheme, h_X has a nice property:

Fix Y, $\quad U \subset Y \mapsto \mathrm{Hom}(U, X) = h_X(U)$ is a sheaf in Zar. top.

$\{U_i \subset Y\}$ open cover. $\quad h_X(Y) \xrightarrow{a} \prod_i h_X(U_i) \underset{c}{\overset{b}{\rightrightarrows}} \prod_{i,j} h_X(U_i \cap U_j)$

is exact. $\quad$ [a injective, & $\mathrm{im}(a) = \{ \alpha : b(\alpha) = c(\alpha) \}$]

Problem: Zar. topology is not "geometric".

Serre: FAC

Serre: other things better.

Grothendieck: abstract categorical topology.

Obs. X top. space, get a cat: $\quad$ obj. $U \subset X$ open

mor. $\mathrm{Hom}(U, V) = \{ \text{inclusions } U \subset V \}$.

$|\mathrm{Hom}(U, V)| \leq 1$.

Presheaf. contravariant functor to Set.

$U \to V$ $\quad$ $F(V) \to F(U)$.
$(U \subset V)$

For sheaves, need to remember more: open coverings.

Retain $\underbrace{\{V_i \subset U\}}_{\text{set of arrows } V_i \to U \text{ in cat.}}$

<u>Silly properties</u>. (i) $\{U \subset U\}$ is a covering

(ii) If $\{V_i \subset U\}$ is a covering & $W \subset U$,

$\{V_i \cap W \subset W\}$ is a covering

(iii) If $\{W_{ij} \subset V_i\}$ are coverings

$\{V_i \subset U\}$ is a covering, then $\{W_{ij} \subset U\}$ is a covering.

<u>Def</u>. Given a cat. $\mathcal{C}$, a <u>Grothendieck top.</u> is a collection of sets of arrows $\{V_i \to U\}$ for each $U \in \mathcal{C}$ ["coverings"] s.t.

(i) Any isom. is a covering.

(ii) If $\{V_i \to U\}$ is a covering, & $W \to U$, then $V_i \times_U W$ exists for each i & $\{V_i \times_U W \to W\}$ is a covering.

(iii) If $\{W_{ij} \to V_i\}$ cov's, $\{V_i \to U\}$ cov., then $\{W_{ij} \to U\}$ covering.

<u>Site</u> is a cat. w/ a Grothendieck top.

<u>Examples</u>. X a scheme

X_{Zar} = small Zariski site

obj. = $U \hookrightarrow X$ open immersions

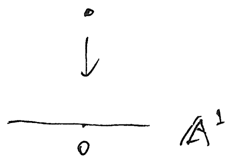
arrows =
$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X \\ U \arrow[ur] & \end{tikzcd}$$

coverings = $\{V_i \xrightarrow{\varphi_i} U : \bigcup \varphi_i(V_i) = U\}$.

X_{ZAR} = big Zariski site

$\mathcal{C} = Sch_X \ni Z$

coverings $\left\{ \begin{tikzcd} Y_i \arrow[r, "\varphi_i"] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd} : \text{each } \varphi_i \text{ is an open immer.} \ \& \ \bigcup_i \varphi_i(Y_i) = Z \right\}$.

ex. 

$X = \mathbb{A}^1$, $Y = x$

h_Y

$X_{\acute{e}t}$ = small étale site

$\mathcal{C} = \{ Z \to X \text{ étale} \} \subset Sch_X$

coverings : $\left\{ \begin{tikzcd} Y_i \arrow[r, "\varphi_i"] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd} : \bigcup \varphi_i(Y_i) = Z \right\}$ NOTE: each φ_i is étale.

$X_{\acute{E}T}$ = big étale site

$\mathcal{C} = Sch_X$, coverings of Z = $\left\{ \begin{tikzcd} Y_i \arrow[r, "\varphi_i"] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd} : \begin{array}{l} \varphi_i \text{ is étale} \\ \bigcup \varphi_i(Y_i) = Z \end{array} \right\}$

X_{fppf} = fppf site

$\mathcal{C} = Sch_X$, coverings = $\left\{ \begin{tikzcd} Y_i \arrow[r, "\varphi_i"] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd} : \begin{array}{l} \varphi_i \text{ is flat, loc. of fin. pres.,} \\ \bigcup \varphi_i(Y_i) = Z \end{array} \right\}$

Def. Given a site $\mathcal{C}$, a sheaf on $\mathcal{C}$ is a functor $F : \mathcal{C}^\circ \to \underline{Set}$ s.t. $\forall$ covering $\{Y_i \to Z\}$ in $\mathcal{C}$, the diagram $F(Z) \to \prod F(Y_i) \rightrightarrows \prod F(Y_i \times_Z Y_j)$ is exact.

[Need to consider $i = j$, eg. $\operatorname{Spec} \mathbb{Q}(\sqrt{2}) \to \operatorname{Spec} \mathbb{Q}$]

Thm (Grothendieck) For any X-scheme S, the functor $h_S : Sch_X^\circ \to \underline{Set}$ is an fppf sheaf.

Baby $\overset{\text{most}}{}$ case: Fix $\{Y_i \to Z\}$ covering, show

$h_S(Z) \to \prod h_S(Y_i) \rightrightarrows \prod h_S(Y_i \times_Z Y_j)$ is exact.

$\{Y_i \to Z\}$ is $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \to B$ faithfully flat

& $S = \operatorname{Spec} C$

Diagram becomes $\operatorname{Hom}(C, A) \to \operatorname{Hom}(C, B) \rightrightarrows \operatorname{Hom}(C, B \otimes_A B)$

$\operatorname{Hom}(C, A \to B \rightrightarrows B \otimes_A B)$

Lemma. $A \to B \rightrightarrows B \otimes_A B$ is exact.

$b \mapsto b \otimes 1$, $b \mapsto 1 \otimes b$

[equiv. $0 \to A \to B \to B \otimes_A B$ exact seq. of A-modules
$b \mapsto b \otimes 1 - 1 \otimes b$]

Pf. special case: $\exists\, B \xrightarrow{\sigma} A$ augmentation

$$A \to B \xrightarrow{\sigma} A, \quad \text{composite } = \mathrm{id}$$

Get $B \otimes_A B \to B$

$b \otimes c \mapsto \sigma(b) c$

Show: if $b \otimes 1 = 1 \otimes b$, then $b \in A$.

$\Rightarrow \underset{A}{\underset{\uparrow}{\sigma(b)}} = \underset{A}{\underset{\uparrow}{b}}$ ✓.

Observe: to prove $0 \to A \to B \to B \otimes_A B$ is exact, it's enough to prove it after a faithfully flat base change $A \to B$:

$$B \to B \otimes_A B \xrightarrow{\text{mult}} B, \quad \text{composite } = \mathrm{id}$$

□

Lemma. $F : Sch_X^{\circ} \longrightarrow Set$ is an fppf sheaf iff

(1) F is a Zariski sheaf

(2) $\forall$ $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \to B$ faithfully flat & of fin. presentation
$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \| & & \| \\ U & \longrightarrow & V \end{array}$$

$$F(V) \to F(U) \rightrightarrows F(U \times_V U) \quad \text{is exact.}$$

Cor. If S is affine, then h_S is an fppf sheaf.

Sketch of general case (S arb.) $\quad h_S$ Zariski sheaf : no problem.

Let $S_i \subset S$ be an affine covering.

$U \to V$ fppf covering.

$$h_S(V) \to h_S(U) \rightrightarrows h_S(U \times_V U)$$

$U \to S$ s.t. the two maps
$$\begin{array}{c} U \times_V U \\ \downarrow\downarrow \\ U \\ \downarrow \\ S \end{array} \quad \text{agree}$$

$\exists\, |V| \xrightarrow{f} |S|$ $\quad$ magic fppf

s.t. $|U| \to |V| \to |S|$ corresp. to $U \to S$.

Pull back $S_i \subset S$, $V_i := f^{-1}(S_i)$, $U_i = U \times_V V_i$

$$\begin{array}{ccccc} U_i \times_{V_i} U_i & \rightrightarrows & U_i & \longrightarrow & V_i \\ & & \searrow & \swarrow & \\ & & & S_i = \text{affine} & \end{array}$$

$\leftarrow$ exists by affine case; $\underbrace{}$ these glue to give $V \to S$ as desired.

Lecture 3. Descent theory = Gluing

Zariski-land: X scheme, $\{U_i \subset X\}$ open covering, F_i on U_i, q-coh.

$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$

$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, $\forall i, j, k$ on $U_i \cap U_j \cap U_k$

<u>Def</u>. $X' \xrightarrow{f} X$ an fpqc morphism.

$$\begin{array}{ccc} & X'' := X' \times_X X' & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ X' & & X' \end{array}$$

$\mathcal{F}'$ qcoh. sheaf on X'.

A <u>descent datum</u> is an isom. $\varphi : p_1^* \mathcal{F}' \xrightarrow{\sim} p_2^* \mathcal{F}'$ s.t. $p_{23}^* \varphi \circ p_{12}^* \varphi = p_{13}^* \varphi$.

$$\begin{tikzcd}
p_{12}^* p_1^* \mathcal{F}' \arrow[r, "p_{12}^* \varphi"] \arrow[dd, equal] & p_{12}^* p_2^* \mathcal{F}' \arrow[d, equal] & \\
& p_{23}^* p_1^* \mathcal{F}' \arrow[r, "p_{23}^* \varphi"] & p_{23}^* p_2^* \mathcal{F}' \arrow[d, equal] \\
p_{13}^* p_1^* \mathcal{F}' \arrow[rr, "p_{13}^* \varphi"] & & p_{13}^* p_2^* \mathcal{F}'
\end{tikzcd}$$

commutes.

$$\begin{array}{c} X' \times_X X' \times_X X' \\ \downarrow {\scriptstyle p_{ij}} \\ X' \times_X X' . \end{array}$$

<u>Reinterpretation</u>,

$\varphi_{t,t} = id$.

<u>Def'</u>. A descent datum on $\mathcal{F}'$ consists an isom. $\varphi_{t_1, t_2} : t_1^* \mathcal{F}' \xrightarrow{\sim} t_2^* \mathcal{F}'$

for all $t_1, t_2 \in X'(T)$, fixed $T \in Sch_X$ $(T \to X)$ s.t. $\forall t_1, t_2, t_3 \in X'(T)$,

$\varphi_{t_2, t_3} \circ \varphi_{t_1, t_2} = \varphi_{t_1, t_3}$. & this is functorial in T, t_i.

<u>Note</u>. If $\mathcal{F}' = f^* \mathcal{F}$, there is a nat'l descent datum. (can)

$\varphi_{t_1, t_2} : t_1^* f^* \mathcal{F} \xrightarrow{\sim} t_2^* f^* \mathcal{F}$. $f t_1 = f t_2$.

$$\begin{array}{ccc} (f t_1)^* & = & (f t_2)^* \\ \wr\downarrow & & \wr\downarrow \\ t_1^* f^* & \xrightarrow{\sim} & t_2^* f^* \end{array}$$

Def. The cat. of <u>descent data for f</u>, $\mathcal{D}_f$, is the cat. of pairs $(\mathcal{F}', \varphi)$, where $\mathcal{F}'$ is a q-coh. sheaf on X' & φ is a descent datum.

Maps: $\psi : \mathcal{F}_1' \to \mathcal{F}_2'$, s.t.

$$\begin{tikzcd} p_1^* \mathcal{F}_1' \arrow[r, "\varphi_1"] \arrow[d, "p_1^*\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & p_2^* \mathcal{F}_1' \arrow[d, "p_2^*\psi"] \\ p_1^* \mathcal{F}_2' \arrow[r, "\varphi_2"] & p_2^* \mathcal{F}_2' \end{tikzcd}$$

$\Rightarrow$ pullback defines a functor

$$\widetilde{f^*} : \mathrm{QCoh}(X) \longrightarrow \mathcal{D}_f$$
$$\mathcal{F} \longmapsto (f^*\mathcal{F}, \mathrm{can}).$$

Def. f is a <u>descent morphism</u> if $\widetilde{f^*}$ is fully faithful.

f is an <u>effective descent morphism</u> if $\widetilde{f^*}$ is an equiv.

<u>Thm (Grothendieck)</u>. If $f : X' \to X$ is fpqc, then f is an effective descent morphism for q-coh. sheaves. [WE CAN GLUE!].

<u>Thm (Giraud / Grothendieck)</u>. If f has a section, then f is an effective descent morphism.

<u>Pf</u>.
$$\begin{tikzcd} X' \arrow[d, "f"] \\ X \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

$\widetilde{f^*}$ equiv: $\widetilde{f^*}$ fully faithful; essentially surj.

$\widetilde{f^*}$ is clearly faithful: $\sigma^* f^* = \mathrm{id}$. $\alpha : \mathcal{F} \to \mathcal{G}$ s.t. $f^*\alpha = 0$

$\Rightarrow \sigma^* f^* \alpha = \alpha = 0$.

<u>Full</u>: a map $(\mathcal{F}, \varphi) \xrightarrow{\psi} (\mathcal{F}', \varphi')$ $\equiv$

$$\begin{tikzcd} T \arrow[r, "t"] \arrow[dr] & X' \arrow[d] \\ & X \end{tikzcd}, \quad t^*\mathcal{F} \xrightarrow{\psi_t} t^*\mathcal{F}'$$

s.t. $\forall\, t_1, t_2$,

$$\begin{tikzcd} t_1^* \mathcal{F} \arrow[r, "\psi_{t_1}"] \arrow[d, "\varphi_{t_1,t_2}"'] \arrow[dr, phantom, "\circlearrowleft"] & t_1^* \mathcal{F}' \arrow[d, "\varphi'_{t_1,t_2}"] \\ t_2^* \mathcal{F} \arrow[r, "\psi_{t_2}"] & t_2^* \mathcal{F}' \end{tikzcd}$$

Ref. pt $= \sigma_T$: $\left(T \to X \xrightarrow[\sigma]{} X'\right)$

$$\begin{tikzcd}
\sigma^* \mathcal{F} \arrow[r, "\varphi_\sigma = \sigma^*\varphi"] \arrow[d, "\simeq"', "\varphi_{\sigma,t}"] & \sigma^* \mathcal{F}' \arrow[d, "\simeq \; \varphi'_{\sigma,t}"] \\
t^* \mathcal{F} \arrow[r, "\varphi_t"] & t^* \mathcal{F}'
\end{tikzcd}$$

$\to$ can propagate φ_σ.

Essential surjectivity. $(\mathcal{F}, \varphi) \in \mathcal{D}_f$, $t \in X'(T)$, $T \in Sch_X$.

Hope: $(\mathcal{F}, \varphi) \simeq \tilde{f}^*(\sigma^* \mathcal{F}) = (f^* \sigma^* \mathcal{F}, \text{can})$

$\varphi_{t, \sigma f t}: t^* \mathcal{F} \to t^* f^* \sigma^* \mathcal{F}$

Given t_1, t_2,

$$\begin{tikzcd}
t_1^* \mathcal{F} \arrow[r, "\varphi_{t_1, \sigma f t_1}"] \arrow[d, "\varphi_{t_1,t_2}"'] \arrow[dr, phantom, "\circlearrowleft"] & t_1^* f^* \sigma^* \mathcal{F} \arrow[d, "\varphi_{\sigma f t_1, \sigma f t_2} = \text{can.}"] \\
t_2^* \mathcal{F} \arrow[r, "\varphi_{t_2, \sigma f t_2}"'] & t_2^* f^* \sigma^* \mathcal{F}
\end{tikzcd}$$

[think] $\quad \sigma f t_1 = \sigma f t_2$

Comes from gluing for $\mathcal{F}$.

Pf of Thm. Special case: X, X' affine. $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \to B$ faithfully flat

a) $\tilde{f}^*$ fully faithful $\Leftrightarrow$ M, N are A-modules, show that the following is exact.

$$\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B) \rightrightarrows \operatorname{Hom}_{B \otimes_A B}(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B)$$

$$\operatorname{Hom}_B(M \otimes_A B, N \otimes_A B) = \operatorname{Hom}_A(M, N \otimes_A B), \qquad \operatorname{Hom}_{B \otimes_A B}(\ldots) = \operatorname{Hom}_A(M, N \otimes_A B \otimes_A B)$$

$$M \otimes_A B \otimes_A B \cong M \otimes_A B \otimes_B (B \otimes_A B) \cong M \otimes_A B \otimes_A B$$

X	X'	$X'' = X' \times_X X'$
M	$M \otimes B$	$b \mapsto b \otimes 1$, $B \to B \otimes_A B$, $b \mapsto 1 \otimes b$
$\mathcal{F}$	$f^* \mathcal{F}$	

$$\operatorname{Hom}_A(M, \underbrace{N \to N \otimes_A B \rightrightarrows N \otimes_A B \otimes_A B}_{\text{enough to show this is exact.}})$$

reduce to the case where aug. $B \to A$. then follow nose.

b) $\widetilde{f^*}$ essentially surj.

$$\begin{array}{ll} (\mathcal{F}, \varphi) & \\ \wr \quad \wr & \\ M \quad & \varphi: B \otimes_A M \xrightarrow{\sim} M \otimes_A B \end{array}$$ as $B \otimes_A B$- modules.

Guess what $\mathcal{G}$ on X should be s.t. $\widetilde{f^*}(\mathcal{G}) \simeq (\mathcal{F}, \varphi)$

$$\begin{array}{l} \wr \\ N = \{ m \in M : m \otimes 1 = \varphi(1 \otimes m) \} \end{array}$$

Obs. $\exists$ a map $\nu : N \otimes_A B \to M$ which is compatible w/ descent datum.

Goal: show this is an isom.

Can do this after faithfully flat base change.

$\Rightarrow$ may assume $\exists$ aug. $B \to A$, i.e. a section $X' \rightleftarrows X$.

Now: KNOW descent is effective. The proof in this case shows that ν is an isom. in D_f. $\square$.

Lecture 8 Return to moduli : do we get sheaves?

(3) $\mathrm{Hom}(X, Y)$

(4) Closed subschemes of X

(5) Subspaces of V

$$h_{M(3)}(T) = \mathrm{Hom}_T(X_T, Y_T) = \mathrm{Hom}(X \times T, Y)$$

Proved: Y is a sheaf. $\Rightarrow$ $h_{M(3)}$ is a fppf sheaf.

$$h_{M(4)}(T) = \{ \underbrace{Z \overset{\text{closed}}{\hookrightarrow} X \times T}_{} : Z \ T \text{ flat} \} / \simeq$$

$\Leftrightarrow I_Z \subset \mathcal{O}_{X \times T}$ isoms are unique when they exist.

sheaf cond'n $\rightsquigarrow$ descent data on the inclusion $I_Z \subset \mathcal{O}_{X \times T}$.

fppf descent is effective for q-coh. $\Rightarrow$ these things glue $\Rightarrow$ $h_{M(4)}$ is a sheaf.

$$\{T_i \to T\}$$

$$h_{M(4)}(T) \to \prod h_{M(4)}(T_i) \rightrightarrows \prod h_{M(4)}(T_i \times_T T_j)$$

isom. classes

uniqueness $\Rightarrow$ harmless to choose rep's:

Know: $I_Z|_{T_i}$ is T_i-flat, $\forall i$.

Conclude: I_Z is T-flat.

<u>Lemma</u> $f: X' \to X$ faithfully flat. A q-coh. sheaf $\mathcal{F}$ on X is X-flat, resp. finite pres.... iff $f^*\mathcal{F}$ is.

$$h_{M(5)}(T) = \{W \subset \mathcal{O}_T \otimes V : \text{cokernel is loc. free}\}$$

Again: isoms are unique if they exist. Same descent argument applies.

(0) Varieties

(1) Curves of genus g

(2) line bundles on X

$$h_{M(2)}(T) = \{\mathcal{L} \text{ on } X \times T\}/\simeq \; = \operatorname{Pic}(X \times T)$$

Sheaf cond. $\{T_i \to T\}$

$$\operatorname{Pic}(X \times T) \to \prod \operatorname{Pic}(X \times T_i) \rightrightarrows \prod \operatorname{Pic}(X \times T_i \times_T T_j)$$

<u>Note</u>. this completely un-exact.

Claim. Exactness always fails on the left.

Pf. Choose T s.t. $\mathrm{Pic}(T) \neq \{0\}$.

Let $\mathcal{M}$ be a non-trivial inv. sheaf on T

$\rightsquigarrow p_2^* \mathcal{M} \in \mathrm{Pic}(X \times T)$. Choose an open covering $\{T_i \subset T\}$ s.t.

$X \times T \xrightarrow[p_2]{} T$ $\qquad \mathcal{M}|_{T_i} \simeq \mathcal{O}_{T_i}$.

$$\mathrm{Pic}(X \times T) \to \prod \mathrm{Pic}(X \times T_i)$$

They are not isom. $\left[\begin{array}{ccc} p_2^* \mathcal{M} & \longmapsto & (\mathcal{O}_{X \times T_i}) \\ \mathcal{O}_{X \times T} & \longmapsto & \end{array}\right.$

Claim. Exactness fails at the middle (in general)

Pf. $X/\mathbb{R}$: $(x^2+y^2+z^2=0) \subset \mathbb{P}^2_{\mathbb{R}}$.

Know $X \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{P}^2_{\mathbb{C}}$. but $X \not\simeq \mathbb{P}^2_{\mathbb{R}}$.

$\Rightarrow$ there are no divisors of deg 1 (use R-R!)

Consider the covering $\mathrm{Spec}\, \mathbb{C} \to \mathrm{Spec}\, \mathbb{R}$

$$\begin{array}{ccccc} \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X \otimes \mathbb{C}) & \rightrightarrows & \mathrm{Pic}(X \otimes \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \\ \| & & \| & & \wr\| \ \ \mathbb{C} \times \mathbb{C} \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \rightrightarrows & \mathbb{Z} \times \mathbb{Z} \\ & & 1 & \longmapsto & (1,1) \end{array}$$

this is the proof.

Descent fails because

- local l.b. $\mathcal{L}$ on $X \times T'$
- $p_1^* \mathcal{L} \xrightarrow{\sim} p_2^* \mathcal{L}$ on $X \times T''$
- $\varphi_{jk} \circ \varphi_{ij} \neq \varphi_{ik}$.

Recall

Fix our problem. think about categories instead of sets.

Def. A <u>groupoid</u> is a cat. where all morphisms are invertible.

Def A groupoid $\mathcal{C}$ is <u>discrete</u> if $\forall x \in \mathcal{C}$, $\mathrm{Aut}(x) = \mathrm{id}$.

Def. A gpoid is <u>connected</u> if any two objects are isom.

$\chi :$ Set $\to$ <u>Grpoid</u> $\leftarrow$ a cat. arrows are functors

$S \longmapsto \left(\begin{array}{l} \text{obj} = S \\ \text{arrows just id arrows} \end{array} \right)$

Lem. Essential image of χ is the discrete grpoids.

More good things:

$M_{(i)}(T)$ groupoid, e.g. $M_2(T)$ $\{$groupoid of $\mathcal{L}$ on $X \times T\}$

$$S \xrightarrow{f} T, \quad M_{(2)}(T) \xrightarrow{(X \times f)^*} M_{(2)}(S) \qquad \text{functor}$$

$$\mathcal{L} \text{ on } X \times T \longmapsto (\mathrm{id} \times f)^* \mathcal{L} \text{ on } X \times S$$

<u>Guess</u>: $M_{(2)} : \mathrm{Sch}^{\circ} \to$ <u>Grpoid</u>.

$$T'' \xrightarrow{g} T' \xrightarrow{f} T$$

$\exists$ isom. $g^* f^* \xrightarrow{\sim} (fg)^*$ universal property of pullback.

⌜ Pullbacks are unique up to unique isom. ⌟

$$T''' \xrightarrow{h} T'' \xrightarrow{g} T' \xrightarrow{f} T$$

$$\begin{tikzcd}
 & h^*(fg)^* \arrow[dr] \arrow[d, phantom, "|||"] & \\
h^* g^* f^* \arrow[ur] \arrow[r] & (gh)^* f^* \arrow[r] & (fgh)^*
\end{tikzcd}$$

commutes.

Def A <u>fibered cat. w/ clivage</u> (a pseudo-functor) over a cat. $\mathcal{C}$ is

(1) for each $c \in \mathcal{C}$, a groupoid $F(c)$

(2) $\forall$ arrow $f: c \to d$ in $\mathcal{C}$, a functor $f^*: F(d) \to F(c)$

(3) for each pair of arrows $c \xrightarrow{f} d \xrightarrow{g} e$, an isom. $\nu_{f,g}: f^* g^* \xRightarrow{\sim} (gf)^*$

s.t. the diagram $c \xrightarrow{f} d \xrightarrow{g} e \xrightarrow{i} h$ ... commutes.

<u>Lecture 5</u>

Def. A functor $F: \mathcal{D} \to \mathcal{C}$ is a cat. <u>fibered in groupoids</u> if

(i) $\forall \beta: c_1 \to c_2 \in \mathcal{C}$, $\forall d_2 \in \mathcal{D}$ s.t. $F(d_2) = c_2$, $\exists\ \alpha: d_1 \to d_2$ s.t. $F(\alpha) = \beta$.

(ii) $\forall$

$$\begin{tikzcd} d_1 \arrow[dr, "\alpha_1"] & \\ & d_2 \\ d_3 \arrow[uu, dashed, "\alpha_3"] \arrow[ur, "\alpha_2"'] \arrow[u, phantom, "\cong"] & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} c_1 \arrow[dr, "\beta_1"] & \\ & c_2 \\ c_3 \arrow[uu, dashed, "\beta_3"] \arrow[ur, "\beta_2"'] \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

Given β_3, $\exists!$ α_3 s.t. $F(\alpha_3) = \beta_3$.

Def Given $c \in \mathcal{C}$, the fiber category $\mathcal{D}_c$ (F_c) has objects

$d \in \mathcal{D}$ s.t. $F(d) = c$, & arrows $d_1 \xrightarrow{\alpha} d_2$ s.t. $F(\alpha) = id$.

Def A 1-morphism of cat. fibered in groupoid $F_1: \mathcal{D}_1 \to \mathcal{C}$, $F_2: \mathcal{D}_2 \to \mathcal{C}$ is

a functor $F: \mathcal{D}_1 \to \mathcal{D}_2$

$$\begin{tikzcd} \mathcal{D}_1 \arrow[rr, "F"] \arrow[dr, "F_1"'] & \arrow[d, phantom, "\cong"] & \mathcal{D}_2 \arrow[dl, "F_2"] \\ & \mathcal{C} & \end{tikzcd}$$

F is an <u>equiv</u>. ("isom") if $\forall c \in \mathcal{C}$, the induced $F_c: (\mathcal{D}_1)_c \to (\mathcal{D}_2)_c$ is an equiv.

Remark

Note: $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}_1, \mathcal{D}_2)$ is a groupoid. (arrows are nat'l isom-s) between functors

$\mathcal{C} = \mathrm{Sch}_S$

Old Friend: $\mathrm{Func}(\mathcal{C}^o, \mathrm{Set})$

Older friend: Schemes $/S$

Set $\subset$ Grpoid $\rightsquigarrow$ our old(er) friends naturally define cat's fibered in grpoid.

Ex. $\mathcal{D}_1 = h_X$, $X \in \mathrm{Sch}_S$

$$\mathrm{Hom}_{\mathcal{C}}(h_X, \mathcal{D}_2) \underset{\text{equiv. of cats}}{\xrightarrow{\sim}} (\mathcal{D}_2)_X$$

$\mathcal{M}_{(0)} =$ moduli of varieties

$$\{X \to \mathcal{M}_{(0)}\} \longleftrightarrow \left(\begin{array}{c}\text{flat families}\\ \mathcal{V}\\ \downarrow\\ X\\ \text{of varieties}\end{array}\right)$$

Ex. $X \mapsto \underline{\mathrm{QCoh}(X)}$ defines a cat. fibered in grpoids.

cat. of q. coh. sh on X w/ isoms as arrows

Bonus: descent theory = glueing = "sheafiness"

Gluing in general. fix $\mathcal{D} \to \underline{\mathcal{C} = \mathrm{Sch}_S}$

think of this as the big étale site.

Def Given a covering $\{Y_i \to X\}$,

<u>Cat. of descent data</u> (w.r.t. this covering)

$\mathcal{D}_{\{Y_i \to X\}}$: obj. : (d_i, φ_{ij}) where $d_i \in \mathcal{D}_{Y_i}$,

$$\varphi_{ij} : \underbrace{d_i|_{Y_i \times_X Y_j}}_{pr_1^* d_i} \xrightarrow{\sim} \underbrace{d_j|_{Y_i \times_X Y_j}}_{pr_2^* d_j}$$

w/ arrows $(d_i, \varphi_{ij}) \to (d_i', \varphi_{ij}')$

$d_i \to d_i'$

compatible w/ the $\varphi_{ij}, \varphi_{ij}'$.

s.t. $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, $\forall i,j,k$, on $Y_i \times_X Y_j \times_X Y_k$.

Any object d of $\mathcal{D}_X$ gives rise to an obj. of $\mathcal{D}_{\{Y_i \xrightarrow{\psi_i} X\}}$:

$d_i = d|_{Y_i} = \psi_i^*(d)$

$$\begin{tikzcd} & Y_i \arrow[dr, "\psi_i"] & \\ Y_i \times_X Y_j \arrow[ur, "pr_1"] \arrow[dr, "pr_2"'] & & X \\ & Y_j \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

$\psi_i \circ pr_1 = \psi_j \circ pr_2$

$pr_1^* \psi_i^* \xrightarrow{\sim} pr_2^* \psi_j^* \quad \rightsquigarrow \quad pr_1^* d_i \xrightarrow{\sim} pr_2^* d_j$.

Cocycle: built in to pseudo-functors.

<u>Upshot</u> get a <u>functor</u> $\nu_{\{Y_i \to X\}} : \mathcal{D}_X \to \mathcal{D}_{\{Y_i \to X\}}$.

<u>Def</u>. $\mathcal{D}$ is a <u>prestack</u> on $\mathcal{C}$ if $\nu_{\{Y_i \to X\}}$ is <u>fully faithful</u> for all $\{Y_i \to X\}$

("descent morphism")

$\mathcal{D}$ is a <u>stack</u> if $\nu_{\{Y_i \to X\}}$ is an <u>equiv.</u> of cats for all $\{Y_i \to X\}$ ("effective descent morphism").

Prestacks: a reinterpretation

Given $a, b \in \mathcal{D}_X$, define a presheaf $I(a,b)$ on Sch_X as follows:

Given $f: Y \to X$, assign $I(a,b)(f) = \mathrm{Isom}_{\mathcal{D}_Y}(f^*a, f^*b)$

<u>Lemma (exer)</u> $\mathcal{D}$ is a prestack iff $\forall X, a, b$, $I(a,b)$ is a sheaf on X_{ET}.

"isomorphisms form a sheaf".

Just as one can sheafify a prestack, one can stackify a prestack (in fact, any fibered cat.)

<u>Thm</u> Given a $\overbrace{\text{fibered cat.}}^{\text{prestack}}$ $\mathcal{D} \to \underset{\text{site}}{\mathcal{C}}$, $\exists$ a stack $\mathcal{D}^s$ & a 1-morphism
$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[dr] & \mathcal{D}^s \arrow[d] \\ & \mathcal{C} \end{tikzcd}$$

s.t. $\forall$ stack $\mathcal{S} \to \mathcal{C}$, the map

$\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}^s, \mathcal{S}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{S})$ is an equiv. of groupoids.

<u>Prop</u> $QCoh$ is a stack on $(\mathrm{Spec}\,\mathbb{Z})_{ET} = Sch_{\mathbb{Z}}$ (in fact, $(\mathrm{Spec}\,\mathbb{Z})_{fppf}$)

<u>Prop</u> Sheaves on $(\mathrm{Spec}\,\mathbb{Z})_{ET}$ form a stack.

$Sh_T = \{\text{sheaves on } T_{ET}\}$.

<u>Our problems</u> "Is it a stack?"

(5) Subspaces of V: STACK - SHEAF!

(4) Closed subschemes of X: STACK - SHEAF!

(3) $\mathrm{Hom}(X, Y)$: STACK - SHEAF!

(2) Line bdle on X. : $\boxed{\text{STACK - BUT NOT A SHEAF}}$

(1) Curves of genus $g \neq 1$: STACK – NOT A SHEAF.

(0) Varieties PRESTACK ($\mathrm{Isom}(X,Y)$ is a sheaf)

BUT NOT A STACK!

Ex. $\exists\, X/\mathbb{C}$, smooth 3-fold (not q. proj.), w/ a descent datum relative to $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,\mathbb{R}$, which does NOT descend.

Funny: a scheme X is a sheaf $\rightsquigarrow$ a family $\begin{array}{c} X \\ \downarrow \\ T \end{array}$ is a sheaf on T_{ET}.

$\{\text{Schemes}\} \subset \{\text{Sheaves}\}$ Why not take the stacky closure of $(Sch) \subset Sh$?!

Lecture 6.

Picard category is a groupoid P together w/ the following extra structure

(a) A functor $+ : P \times P \to P$

(b) An isom. of functors

$$\begin{tikzcd} & P \times P \times P \arrow[dl, "+\times 1"'] \arrow[dr, "1\times +"] & \\ P \times P \arrow[dr, "+"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & P \times P \arrow[dl, "+"] \\ & P & \end{tikzcd}$$

$\sigma_{x,y,z} : (x+y)+z \xrightarrow{\sim} x+(y+z)$

(c) a nat'l transf.

$$\begin{tikzcd} P \times P \arrow[rr, "\mathrm{flip}"] \arrow[dr, "+"'] & \overset{\tau}{\Rightarrow} & P \times P \arrow[dl, "+"] \\ & P & \end{tikzcd}$$

$\tau_{x,y} : x+y \xrightarrow{\sim} y+x$.

Lecture 6

S = scheme, $\mathcal{C} = Sch_S$ (big étale site)

$\mathbb{P}^n$: two competing descriptions

(1) $h_{\mathbb{P}^n}(T) = \{ \mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{L},\ \mathcal{L} \text{ invertible on } T \} / \simeq$

(2) "$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \mathbb{G}_m$" $\qquad \mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T^\times)$

If (2) make sense, $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ "is" a $\mathbb{G}_m$-torsor.

$$\begin{tikzcd} T \arrow[r, "\mathbb{G}_m\text{-equiv.}"] \arrow[d, "\mathbb{G}_m\text{-torsor}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^{n+1} \setminus \{0\} \arrow[d] \\ X \arrow[r] & \mathbb{P}^n \end{tikzcd}$$

$\forall\, t \in T,\ \alpha \in \mathbb{G}_m,\quad f(\alpha t) = \alpha f(t)$.

Prop/exercise. There is a nat'l equiv. of cats

$$\begin{pmatrix} \text{Rel. Affine } X\text{-schemes} \\ \text{w/ } \mathbb{G}_m\text{-action} \\ \text{\& } \mathbb{G}_m\text{-equiv. maps} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \mathbb{Z}\text{-graded} \\ \text{q-coh. } \mathcal{O}_X\text{-algs} \\ \text{w/ graded maps} \end{pmatrix}^{\circ}$$

Idea: given

$$\begin{tikzcd} Y \arrow[d, "f"] \\ X \end{tikzcd} \to \mathbb{G}_m\text{-action on } f_*\mathcal{O}_Y \text{ over } X$$

$\rightsquigarrow$ breaks up as a sum of eigenspaces indexed by the characters $= \mathbb{Z}$

Ex. Action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}_X$

$\qquad\qquad \shortparallel$

$\underline{\mathrm{Spec}}_X\, \mathcal{O}_X[x_1, \dots, x_{n+1}]$ $\qquad$ grading by total degree.

$t \in \mathbb{G}_m,\ x_i \mapsto t x_i$

$$\begin{array}{c} T \\ \downarrow \\ X \end{array} \quad \mathbb{G}_m\text{-torsor} \longrightarrow \text{rel. affine by descent theory}$$

($\mathbb{G}_m$ is affine)

Prop. Given a $\mathbb{G}_m$-torsor $T \to X$, $\exists$ an inv. sheaf $\mathcal{L}$ on X s.t.

$$T \cong \underline{\mathrm{Spec}}_X \left(\bigoplus_{i \in \mathbb{Z}} \mathcal{L}^i \right) \quad \text{nat'l grading by "}i\text{"} \longleftrightarrow \text{action}$$

$$\subset \underline{\mathrm{Spec}}_X \left(\bigoplus_{i \geq 0} \mathcal{L}^i \right) =: \overline{T}$$

Pf. fppf locally on X, $T \cong \underline{\mathrm{Spec}}_X \mathcal{O}_X[x, x^{-1}]$. note: each graded piece has the form $x^i \mathcal{O}$.

Descent datum: graded isom. $\mathcal{O}[x, x^{-1}] \xrightarrow{\sim} \mathcal{O}[x, x^{-1}]$

ETC --- $\square$

A $\mathbb{G}_m$-equiv. map

$$\begin{tikzcd} \underset{\underline{\mathrm{Spec}}_X \bigoplus_{i \in \mathbb{Z}} \mathcal{L}^i}{T} \arrow[r] \arrow[d, hook] \arrow[dr] & \mathbb{A}^{n+1} \setminus \{0\} \arrow[d, hook] \\ \overline{T} \arrow[r] & \mathbb{A}^{n+1} \end{tikzcd}$$

$$\underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathcal{L}^i \to \underline{\mathrm{Spec}}_X \mathcal{O}_X[x_1, \dots, x_{n+1}]$$

$$\bigoplus_{i \geq 0} \mathcal{L}^i \xleftarrow[\text{graded}]{} \mathcal{O}_X[x_1, \dots, x_{n+1}]$$

$$\Big\updownarrow$$

$$\mathcal{L} \twoheadleftarrow \mathcal{O}_X^{n+1} \qquad (T \to \mathbb{A}^{n+1} \setminus \{0\})$$

Conclusion. The functor of pts tells us that in fact, $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor.

G group scheme

X scheme. $\quad X \circlearrowleft G$. Love: make a quotient X/G s.t. $X \to X/G$ G-torsor.

Def. The quotient stack $[X/G]$ has as fiber cat. over Y the following

Obj: pairs $\left(\begin{smallmatrix} T \\ \downarrow \\ Y \end{smallmatrix}, \varphi\right)$

$\uparrow$ G-torsor, $\varphi : T \to X$ G-equiv. morphism

Arrows: $\left(\begin{smallmatrix} T \\ \downarrow \\ Y \end{smallmatrix}, \varphi\right) \to \left(\begin{smallmatrix} T' \\ \downarrow \\ Y \end{smallmatrix}, \varphi'\right)$.

$$\begin{tikzcd} T \arrow[rr, "\psi"] \arrow[dr] & & T' \arrow[dl] \\ & Y & \end{tikzcd}$$

s.t. ψ is G-equiv. isom. & $\varphi'\psi = \varphi$.

Note: $\exists$ nat'l map $\nu : X \to [X/G]$.

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d] & X \\ X & \end{tikzcd}$$

trivial G-torsor

$$\begin{tikzcd} G \times X \arrow[r, "a"] \arrow[d] & X \\ X & \end{tikzcd}$$

a = action

$h\,(g,x) \mapsto h.gx$

$\quad\mathrm{I} \qquad\qquad \|$

$(hg, x) \mapsto hgx$

Claim. ν makes X a G-torsor over $[X/G]$.

Pf.

show this a G-torsor

$$\begin{tikzcd} Y \times_{[X/G]} X \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & {[X/G]} \end{tikzcd}$$

Def. Given morphisms of stacks

$\mathcal{X} \xrightarrow{\alpha} \mathcal{Z}$, $\mathcal{Y} \xrightarrow{\beta} \mathcal{Z}$

the fiber product has fiber cats

$$\left(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}\right)_T = \left\{ (x, y, \varphi) : x \in \mathcal{X}_T,\ y \in \mathcal{Y}_T,\ \varphi : \alpha(x) \underset{\sim}{\overset{\text{arrow in } \mathcal{Z}_T}{\longrightarrow}} \beta(y) \right\}$$

$(x, y, \varphi) \xrightarrow{\sim} (x', y', \varphi')$: $\begin{matrix} x \xrightarrow{\gamma} x' \\ y \xrightarrow{\delta} y' \end{matrix}$ s.t.

$$\begin{tikzcd} \alpha(x) \arrow[r, "\alpha(\gamma)"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\mathrm{III}"] & \alpha(x') \arrow[d, "\varphi'"] \\ \beta(y) \arrow[r, "\beta(\delta)"'] & \beta(y') \end{tikzcd}$$

Ex $\left(X \times_{[X/G]} Y \right)_T$

$$\begin{tikzcd} G \times X \arrow[r, "\nu"] \arrow[d] & X \\ X & \end{tikzcd} \quad , \quad \begin{tikzcd} U \arrow[r, "\beta"] \arrow[d] & X \\ Y & \end{tikzcd}$$

1) $x \in X(T)$, $y \in Y(T)$, $y : T \to Y$

2) $\nu(x) \in [X/G]$ $\iff$ pullback of $\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \\ X & \end{tikzcd}$ to $\begin{tikzcd} G \times T \arrow[r] \arrow[d] & X \\ T & \end{tikzcd}$

$\beta(y) \in [X/G]$ $\quad \begin{tikzcd} y^* U \arrow[r] \arrow[d] & U \arrow[r] & X \\ T & & \end{tikzcd}$

$$\varphi : \begin{tikzcd} G \times T \arrow[r] \arrow[d] & y^* U \arrow[r] \arrow[dl] & X \\ T & & \end{tikzcd} = \text{a choice of point } U(T)$$

Isoms : $(x, y, \varphi) \xrightarrow{\sim} (x', y', \varphi')$

$$\begin{matrix} x \xrightarrow{\sim} x' \\ y \xrightarrow{\sim} y' \end{matrix} \Rightarrow \begin{matrix} x = x' \\ y = y' \end{matrix} \quad (X, Y \text{ are schemes}) \Rightarrow \varphi = \varphi'$$

$\Rightarrow$ only isoms are id maps

$$\Rightarrow \begin{tikzcd} Y \times_{[X/G]} X \arrow[r] \arrow[d, phantom, "\wr"] & Y \\ U \arrow[ur, "G\text{-torsor}"'] & \end{tikzcd}$$

Ex. $\begin{tikzcd} & T \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$ $\quad (X \times_Z Y)_T$: (α, β, φ)

$\alpha \in X(T)$

$\beta \in Y(T)$

$\varphi : f \circ \alpha \xrightarrow{\sim} g \circ \beta$

$\Updownarrow$

$(f \circ pr_1) \circ (\alpha \times \beta) \xrightarrow{\sim} (g \circ pr_2) \circ (\alpha \times \beta)$

$$\begin{tikzcd} T \arrow[r, "\alpha \times \beta"] & X \times Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

$$X \times_Z Y \to X \times Y$$
$$\shortparallel$$
$$\underline{\mathrm{Isom}} (pr_1^* f, pr_2^* g)$$

Def $\mathcal{X} \to \mathcal{Y}$ is <u>representable</u> (by schemes) if $\forall\, T \to \mathcal{Y}$,

$\mathcal{X} \times_{\mathcal{Y}} T \to T$ is equiv. to a fiber cat. assoc. to a scheme.

$[*/G]_T$ = cat. of G-torsors

$$\begin{array}{c} G \\ \downarrow \\ T \end{array} \qquad [*/G] = BG$$

$$* \longrightarrow [*/(\mathbb{Z}/2)]$$

finite étale of deg 2

Lecture 7 <u>Algebraic stacks.</u>

<u>Q.</u> What is geometry?

<u>A.</u> Local str. (on top of topology).

<u>Ex.</u> F: a sheaf on S_{fppf}.

<u>Claim.</u> F is a scheme iff $\exists$ a scheme U & a map $U \xrightarrow{a} F$ which is Zariski-locally an isom. i.e. $\exists$ a covering $\{G_i \subset F\}$ open subfunctor s.t. for each i, $\exists\, U_i \subset U$ open w/

$$\begin{tikzcd} U_i \arrow[r] \arrow[d, "\wr"'] & F \\ G_i \arrow[ur] & \end{tikzcd}$$

a: <u>uniformization</u>.

Def (temp) An <u>étale algebraic space</u> over S is a sheaf F on S_{ET} s.t. $\exists$ a scheme U & a <u>surjective étale representable</u> morphism $U \to F$.

An fppf alg. space: [same, but $U \to F$ is only fppf].

<u>Hypotheses</u>

i) $F \to F \times F$ quasi-affine

...

Thm (Artin) Any fppf alg. space is an étale alg. space. [same hypotheses]

Hypotheses: – F locally of finite presentation $/ S$.

– $F \to F \times F$ representable & finite type ($\Rightarrow$ q-affine)

Ex. 1) $\exists$ smooth 3-fold $/\mathbb{C}$ w/ descent datum w.r.t. $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ which is not effective. BUT: $\exists$ a sheaf $\bar{T}/\mathbb{R}$ s.t. $\bar{T} \otimes \mathbb{C} \cong T$

$$\begin{tikzcd} \bar{T} \\ T \arrow[u, "\text{finite étale}"'] \end{tikzcd}$$

2) group quotients ...

3) contractions ...

Def A stack $\mathcal{X}$ on $S_{\acute{E}T}$ is <u>Deligne-Mumford stack</u> (DM stack) if

(i) $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by schemes, q-cpt, separated.

(ii) $\exists$ an <u>étale</u> surjection $X \to \mathcal{X}$

$$\begin{tikzcd} X \arrow[r] & \mathcal{X} \\ \text{scheme.} \arrow[u] & \end{tikzcd}$$

⌜ (i) $\Rightarrow$ any map from scheme to $\mathcal{X}$ is representable ⌟

(i) says $\forall f : T \to \mathcal{X}$

$g : T' \to \mathcal{X}$, $\Rightarrow$ $\underline{\mathrm{Isom}}(pr_1^* f, pr_2^* g)$ "is" a q-cpt sep'd map of schemes.

$$\begin{tikzcd} \underline{\mathrm{Isom}}(pr_1^* f, pr_2^* g) \arrow[d] \\ T \times T' \end{tikzcd}$$

$$\begin{tikzcd} X \times_{\mathcal{X}} p \arrow[r] & X \arrow[d, "\text{étale}"] \\ p \arrow[u, dashed, bend left, "\text{étale}"] \arrow[r] \arrow[ur, dashed] & \mathcal{X} \\ \operatorname{Spec} k \arrow[u, "\shortparallel"] & \end{tikzcd}$$

$$\begin{tikzcd} X \times_{\mathcal{X}} p \arrow[d, "\text{ét}"'] & \supset \underline{\mathrm{Aut}}(p) \arrow[dl, "\text{ét}"] \\ p & \end{tikzcd}$$

$\Leftrightarrow$ there is no non-trivial infinitesimal auto. of the obj. par. by p.

DIAGONALS ARE OF FINITE TYPE

What about $B\mathbb{G}_m$?

$$\begin{tikzcd} B\mathbb{G}_m \\ * \arrow[u, "\mathbb{G}_m\text{-torsor}"'] \end{tikzcd}$$

<u>Def.</u> An <u>Artin stack</u> on S_{ET} is a stack $\mathcal{X}$ s.t.

(i) $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is rep'ble by alg. spaces, q-cpt & separated

(ii) $\exists$ a scheme X & a smooth surjection $X \to \mathcal{X}$.

<u>Thm (Artin)</u>. If $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ rep'ble by alg. spaces, q-cpt, sep'd, then $\exists\ X \to \mathcal{X}$ fppf surj.

iff $\mathcal{X}$ is an Artin stack.

<u>Prop</u> Suppose M is a moduli stack (loc. of finite pres.) s.t. isoms are rep'ble (by alg spaces), q-cpt, sept'd.

then M is Artin iff $\exists\ X \to M$ which is formally smooth.
(loc. of finite pres)

$$\begin{tikzcd} X \arrow[r] & M \\ \bar{Y} \arrow[u] \arrow[r, hook] & Y \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$
lift up to isom.

<u>Thm (Artin)</u> An Artin stack $\mathcal{X}$ is DM iff $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified

iff no object has non-trivial infinitesimal aut's.

<u>Ex</u>. $S = \operatorname{Spec} \mathbb{C}$, or $\operatorname{Spec} \mathbb{Z}$ $\qquad \mathcal{M}_{1,1}$ = stack of elliptic curves

$$(\mathcal{M}_{1,1})_T = \left\{ \mathcal{E} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} T \ : \ \pi \text{ is proper \& smooth, } \forall\ \bar{t} \to T \text{ (geom. pt)}, \ \mathcal{E}_{\bar{t}} \text{ conn'd, } g(\mathcal{E}_{\bar{t}}) = 1 \right\}$$

<u>Claim</u>. $\mathcal{M}_{1,1}$ is a DM stack.

- Condition on Isoms : not so bad.

  saving grace: no non-trivial inf. auts.

  Inf. auts of

$$\begin{array}{c}(E,p)\\ \downarrow\\ \operatorname{Spec}\bar{k}\end{array} \qquad \begin{array}{c}H^0(E,T_E)\\ \wr\!\wr\\ H^0(E,\mathcal{O}_E)\cong\bar{k}\\ \cup\\ H^0(E,\mathcal{O}_E(-p))=0\end{array}$$

$\Rightarrow$ enough to show $\mathcal{M}_{1,1}$ is an Artin stack.

To prove this: find a formally smooth family $B \longrightarrow \mathcal{M}_{1,1}$.

$$\begin{array}{c}\mathcal{E}\\ \downarrow \circlearrowleft \sigma\\ B\end{array}$$

<u>Idea</u>. uniformize by the family of plane cubics

(1) There is a scheme $\mathcal{U}$ representing the functor $T \mapsto$

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & \mathbb{P}^2_T \arrow[d] \\ & T \end{tikzcd}$$

smooth family of cubic curves

étale locally on T, $T' \to T$.

Pf of (1), take the universal cubic

$$\Big(\sum_{i+j+k=3} \alpha_{ijk} x^i y^j z^k\Big) \subset \mathbb{A}^{10}\times\mathbb{P}^2$$
$$\downarrow$$
$$\mathbb{A}^{10}$$

s.t.

$$\begin{tikzcd} \mathcal{C}_{T'} \arrow[r] \arrow[dr] & \mathbb{P}^2_{T'} \arrow[d] \\ & T' \end{tikzcd}$$

: $\mathcal{C}$ is the vanishing locus of a section of $\mathcal{O}_{\mathbb{P}^2_{T'}}(3)$.

$\exists\ \tilde{u} \subset \mathbb{A}^{10}$ param. smooth cubics.

$\mathcal{U}$ = image of $\tilde{u}$ in $\mathbb{P}^9 \leftarrow \mathbb{A}^{10}\setminus\{0\}$

(2) $\exists$ a scheme $P\ (\to \mathcal{U})$ rep. the functor $T \mapsto \left(\begin{array}{c}\mathcal{E}\subset\mathbb{P}^2_T\\ \sigma\big(\downarrow \swarrow\\ T\end{array}\right)$ pt'd smooth cubics

(3) force $\mathcal{O}(1)|_{\mathcal{E}} \simeq \mathcal{O}(3\sigma)|_{\mathcal{E}} \longrightarrow P'$

(4) Action of PGL_3 on P' coming from choosing coord. of $\mathbb{P}^2$. (5) $[P'/PGL_3] \simeq \mathcal{M}_{1,1}$.

Lecture 8

S scheme locally of finite type / excellent Dedekind scheme

$\mathcal{F}$: stack on $S_{\acute{E}T}$.

loc. of finite pres: $A = \varinjlim A_i$

$\varinjlim \mathcal{F}_{\operatorname{Spec} A_i} \xrightarrow{\sim} \mathcal{F}_{\operatorname{Spec} A}$ is an equiv. of cat

Brian C.: $\operatorname{Spec} k \xrightarrow{x} \mathcal{F}$ If x admits an <u>effective</u> <u>versal formal deformation</u>,

then $\exists$

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[dr] & \mathcal{F} \\ & X \arrow[u, "f"'] \end{tikzcd}$$

s.t. f is "formally smooth at x".

X of finite type $/S$.

$$\begin{tikzcd} \bar{Y} \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & \mathcal{F} \end{tikzcd}$$

local Artin schemes

pt of $\bar{Y}$ maps to x

Content:

1) Schlessinger $\Rightarrow$ $\exists$ versal formal deform. [hull]

$\boxed{\text{Infinitesimal}}$

2) Formal $\to$ effective

Grothendieck Existence Thm.

$\approx$ étale - local existence.

Given $X \to S$, $a \in \mathcal{F}_X$, let $\mathcal{F}_a$ be the groupoid, $(X \xrightarrow{f} Y)$,

$(\mathcal{F}_a)_Y = \{ \alpha : a \to b \text{ s.t. } \operatorname{im}(\alpha) \text{ in } S_{\acute{E}T} \text{ is } f \}$

$= \{ b \in \mathcal{F}_Y,\ \varphi : a \xrightarrow{\sim} f^* b \}$

$\overline{\mathcal{F}_a}(Y)$ = isom. classes of $(\mathcal{F}_a)_Y$.

(S1')

$$\begin{tikzcd} & B \arrow[d] \arrow[dr] & \\ A' \arrow[r, two heads] & A \arrow[r, two heads] & A_0 \end{tikzcd}$$

$\uparrow$ reduced

infinitesimal thickenings

at $\mathcal{F}_{\operatorname{Spec} A}$ $\ker(A' \to A)$ is an A_0-mod.

$\quad\ \ \shortparallel$

$\ \ \mathcal{F}(A)$

$(\mathcal{F}a)$

$$\begin{array}{c} (\mathcal{F}a)(A' \times_A B) \\ \downarrow \\ \mathcal{F}a(A') \times \mathcal{F}a(B) \end{array}$$

is an equiv. of cats.

(S2) $a_0 = a|\operatorname{Spec} A_0$ $\qquad D_{a_0}(M)$ is a finite A_0-module

Martin: $\mathcal{F}a(A_0[M]) = D_{a_0}(M)$ an A_0-module

M: finite A_0-module.

Suppose given an obstruction (à la Martin)

$\underset{\text{inf. ext.}}{A \twoheadrightarrow A_0}$, $a \in \mathcal{F}(A)$, $\mathcal{O}_a : (A_0\text{-Mod}_{ft}) \to (A_0\text{-Mod}_{ft})$

s.t. $\forall$ $\underbrace{A' \to A}$ $\to A_0$ deformation situation

$\ker(A' \to A) = M$ is an A_0-module, then

$o_a(A') \in \mathcal{O}_a(M)$ s.t. $o_a(A') = 0$ iff a lifts to A'.

In addition, assume: $A \twoheadrightarrow A_0$ inf. ext. A_0 of f-type $/S$

(4.1) (i) Étale localization: then $D_{a_0}(M_0 \otimes B_0) \xleftarrow{\sim} D_{a_0}(M_0) \otimes B_0$

$B_0 = A_0 \otimes_A B$, $M_0 \in A_0\text{-mod}_{ft}$, $\mathcal{O}_{b_0}(M_0 \otimes B_0) \xleftarrow{\sim} \mathcal{O}_{a_0}(M_0) \otimes B_0$

$b_0 = a_0|_{B_0}$.

(4.1) (ii) Completion. If $m \subset A_0$ max'l, then

$$D_{a_0}(M) \otimes \hat{A}_0 \xrightarrow{\sim} \varprojlim D_{a_0}(M/m^n M)$$

(4.1) (iii) Constructibility: $\exists$ a dense set of closed pts $p \in \operatorname{Spec} A_0$ s.t.

$D_{a_0}(M) \otimes k(p) \xrightarrow{\sim} D_{(a_0)_p}(M \otimes k(p))$

$O_{a_0}(M) \otimes k(p) \xrightarrow{\sim} O_{(a_0)_p}(M \otimes k(p))$

<u>Thm (Artin)</u>. Given F, O satisfying (S1'), (S2), & (4.1), if $X \xrightarrow{f} F$, $X \to S$ finite type, $x \in X$, f is formally smooth at x, then $\exists\, U \subset X$, $x \in U$, s.t. $f|_U : U \to F$ is formally smooth.

<u>Prop</u> (Artin) F is an Artin stack, loc. of finite type over S if

(1) $F \to F \times F$ is rep'ble by alg. spaces, q-cpt, sep'td,

(2) (S1'), (S2) hold

(3) If $(\hat{A}, m)$ is a complete loc. noeth. ring $/S$, then $F(\hat{A}) \xrightarrow{\sim} \varprojlim F(\hat{A}/m^n)$ is an equiv.

(4) D, O satisfy (4.1).

<u>Ex</u>. $\mathcal{M}_g =$ stack of curves of genus g $\boxed{g>1}$: $\begin{matrix} \mathcal{C} \\ \downarrow \\ T \end{matrix}$ proper smooth geom. conn'd fibers of genus g.

(1) $\mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$: Grothendieck or [one second]

(2) Schlessinger : no problem.

(3) Grothendieck existence thm:

(4) $\begin{matrix} \mathcal{C} \\ \downarrow \\ \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A' \end{matrix}$: $\left.\begin{matrix} O_{\mathcal{C}}(M) = H^2(\mathcal{C}_{A_0}, T_{\mathcal{C}_{A_0}/A_0} \otimes M) \\ D_{\mathcal{C}}(M) = H^1(\mathcal{C}_{A_0}, T_{\mathcal{C}_{A_0}/A_0} \otimes M) \end{matrix}\right|$ A_0-modules

$M = \ker(A' \to A)$

(i) compatible w/ étale base change $A_0 \to B_0$ (Hartshorne)

(ii) Completion [less obvious, but OK] (iii) Constructibility: cohomological base change.

$$\begin{array}{c} \mathcal{C}_0 \\ \downarrow f \\ \operatorname{Spec} A_0 \end{array}$$

want: $R^i f_* \left(T_{\mathcal{C}_{A_0}/A_0} \otimes M \right) \otimes k(p) \underset{\sim}{\rightarrow} H^i(\mathcal{C}_p, T_{\mathcal{C}_p/p} \otimes M)$

isom.

Note, no non-triv. inf. auts $\left(H^0(\mathcal{C}, T)\right)$

$\rightsquigarrow$ DM stack.

Thm (Artin). $\mathcal{F}$ is an Artin stack loc. of fin. type $/S$ if

(1) $(S1')$, $(S2)$ hold, & if $a_0 \in \mathcal{F}(A_0)$ & M is a finite A_0-module, then $\mathrm{Aut}^{\inf}_{a_0}(A_0[M])$ is a finite A_0-module.

(2) $\mathcal{F}(\hat{A}) \rightarrow \varprojlim \mathcal{F}(\hat{A}/m^n)$ equi.

(3) D, $\mathcal{O}$, $\mathrm{Aut}^{\inf}_{a_0}(A_0[M])$ satisfy (4.1)

(4) If φ is an aut. of a_0 s.t. $\varphi = \mathrm{id}$ at a dense set of pts of $\operatorname{Spec} A_0$, then $\varphi = \mathrm{id}$.

(5) (1)-(4) $\Rightarrow$ $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is rep'ble & sep'td.

Check that it is q-cpt.