Quantization of the category of coherent sheaves

on symplectic varieties

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<u>Set up:</u> X/R $\quad \{\ ,\ \} : \mathcal{O}_X \underset{R}{\otimes} \mathcal{O}_X \longrightarrow \mathcal{O}_X$ Poisson bracket

<u>Def</u> A quantization of $(X, \{,\})$ is a sheaf $\mathcal{O}_\hbar$ of $R[[\hbar]]$-algebra w/ $\mathcal{O}_\hbar \to \mathcal{O}_\hbar/\hbar \simeq \mathcal{O}_X$, $a \mapsto \bar{a}$ s.t.

1) $\mathcal{O}_\hbar$ is flat over $R[[\hbar]]$ and $\hbar$-adic complete

2) $\forall a, b \in \mathcal{O}_\hbar$, $\quad ab - ba = \hbar \{\bar{a}, \bar{b}\} \mod \hbar^2$.

<u>The basic example</u>: $X = T^*Y \xrightarrow{\pi} Y$, Y/R smooth

$\omega = d\eta \in \Omega^2_X$, $\rightsquigarrow$ $\Omega^1_X \xrightarrow{\sim} T_X$

$df \rightsquigarrow H_f$

$\{f, g\} = H_f(g)$

$\mathcal{D}_{Y,\hbar} \subset \mathcal{D}_Y[[\hbar]]$

gen. by $\mathcal{O}_Y$, $\hbar \cdot T_Y$

$\mathcal{D}_{Y,\hbar}/\hbar \simeq S^\cdot T_Y$

$\mathcal{O}_\hbar(\pi^{-1}(U)) = \mathcal{D}_{Y,\hbar}(U)$

$Y = \operatorname{Spec} k[y]$, $X = \operatorname{Spec} k[x, y]$

$\mathcal{D}_Y = k\langle y, \partial_y\rangle$, $[\partial_y, y] = 1$

$\mathcal{D}_{Y,\hbar} = k\langle y, \hbar\partial_y\rangle[[\hbar]]$

$[\hbar\partial_y, y] = \hbar$

<u>Variant</u>: replace D_Y by $\mathrm{Diff}_Y(K, K)$, $\mathrm{Diff}_Y(K^{1/2}, K^{1/2})$ $(\frac{1}{2} \in R)$

$\downarrow$

$\rightsquigarrow \mathcal{O}_\hbar^{can}$ the canonical quantization of T^*_Y.

$$\mathcal{O}_{-\hbar}^{can} \simeq (\mathcal{O}_\hbar^{can})^{op}$$

<u>Some results</u>

1. de Wilde, Lecompte, Fedosov, Bezrukavnikov - Kaledin

$R = \mathbb{C}$, X symplectic, $H^i(X, \mathcal{O}_X) = 0$, $i = 1, 2$,

Quantizations $(X, \omega) \simeq H^2_{dR}(X)[[\hbar]]$

$$\begin{array}{ccc} \Psi & & \Psi \\ \mathcal{O}_\hbar^{can} & \longleftrightarrow & 0 \\ \circlearrowleft & & \\ \mathrm{Aut}(X, \omega) & & \end{array}$$

2. X/k. $\mathrm{char}\, k = p > 2$, k perfect

$\omega = d\eta$ $H^i(X, \mathcal{O}_X) = 0$, $i = 1, 2, 3$

$\rightsquigarrow \mathcal{O}_\hbar^{can} \circlearrowleft \{\varphi \in \mathrm{Aut}(X) : \varphi^*\eta - \eta \text{ exact}\}$

<u>Basic idea</u> (of 1)

local case : $X = \mathrm{Spf}\, \underbrace{\mathbb{C}[[x_1, \dots, x_n, y_1, \dots, y_n]]}_{A_0}$

$$\omega = \sum dy_i \wedge dx_i$$

$\mathcal{O}_\hbar = \mathbb{C}\{\{x_1, \dots, x_n, y_1, \dots, y_n, \hbar\}\} = A_\hbar$, $[x_i, y_j] = \hbar \delta_{ij}$

$$1 \to H \to \mathrm{Aut}(\mathcal{A}_\hbar) \overset{*}{\twoheadrightarrow} \mathrm{Aut}(\mathcal{A}_0, \omega) \to 1$$

<u>Idea</u> (Deligne, Kontsevich, Drinfeld, ...)

Look at quantizations of $\mathrm{QCoh}(X)$ instead!

<u>Digression on abelian categories</u>

$\mathcal{A}$ abelian cat. $/R$ w/ $\oplus$

$\mathrm{Hom}_{\mathcal{A}}(V, W) \curvearrowleft R$

$$\mathrm{Hom}_{\mathcal{A}}(V \otimes_R M, W) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{A}}(V, W))$$

$V \in \mathcal{A}$, $M \in \mathrm{Mod}_R$

<u>Def</u> ⓐ $V \in \mathcal{A}$ is <u>flat over R</u> if $V \otimes_R ? : \mathrm{Mod}_R \to \mathcal{A}$ is exact.

ⓑ $\mathcal{A}$ is <u>R-flat</u> if $\mathcal{A}$ is gen. by R-flat objects.

<u>Def.</u> For $R \to R'$ and $\mathcal{A}/R$, define

$\mathcal{A} \otimes_R R'$ is the cat. of R'-mods in $\mathcal{A}$, i.e. $R' \to \underset{\mathcal{A}}{\mathrm{End}_{\mathcal{A}}(V)}$

<u>Def.</u> Let $\mathcal{A}$ be a cat. flat over R, A deformation over $R[[\hbar]]$ is

$\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n \;\underset{\text{flat}}{/}\; R[\hbar]/\hbar^{n+1}$

w/ $\mathcal{A}_{n+1}/\hbar^{n+1} \simeq \mathcal{A}_n$.

Fact (Van den Bergh, Lowen)

Deformations of Mod_A / equiv. $\simeq$ def. of A / equiv.

Rmk inner auto. are different

Idea.

$$\begin{array}{c} \mathcal{A}_0 = \mathrm{Mod}_A \\ \uparrow \\ \mathcal{A}_n \end{array}$$

$A \in \mathcal{A}_0$ projective

$\Rightarrow$ admits a lift M to $\mathcal{A}_n$

$A_n = \mathrm{End}(M)^{op}$

Main result. $\frac{1}{2} \in R$, $(X/R, \quad \omega = d\eta)$

Then $QCoh(X)$ has a canonical quantization. $QCoh(X) \underset{\hbar}{\overset{can}{\circlearrowleft}} \{(\underset{\underset{Aut(X)}{\uparrow}}{\varphi}, s \in \mathcal{O}(X)) : \varphi^*\eta - \eta = ds\}$

Idea $X = \mathrm{Spf}\, R[[x_1, \dots, x_n, y_1, \dots, y_n]]$

$\downarrow$ $\qquad \eta = \sum x_i dy_i$

$Y = \mathrm{Spf}\, R[[y_1, \dots, y_n]]$

Enough to construct an action of

$$G = \{(\varphi \in \mathrm{Aut}\, X, s \in \mathcal{O}(X)) : \varphi^*\eta - \eta = ds\}$$
$$\circlearrowright$$
$$\mathrm{Mod}(A_\hbar)$$

Def. $\varphi \curvearrowright X$ is transversal if

$$\begin{tikzcd} \Gamma_\varphi \arrow[r] \arrow[dr, "\sim"'] & X \times X \arrow[d] \\ & Y \times Y \end{tikzcd}$$

Given (φ, S), define its action on $\mathrm{Mod}(A_\hbar)$

transversal

Want a bimodule $M_{(\varphi,S)}$ over $A_\hbar$

$$M_{(\varphi,S)}/\hbar = \mathcal{O}_{\Gamma_\varphi} \qquad\qquad dS = \varphi^*\eta - \eta.$$

$$M_{(\varphi,S)} = \mathcal{O}_{Y\times Y} \cdot e^{S/\hbar} [\![\hbar]\!]$$

$$A_\hbar \oplus A_\hbar^{op}$$

$$x_i f e^{S/\hbar} = \left(\hbar \frac{\partial f}{\partial y_i} + \frac{\partial S}{\partial y_i} f\right) e^{S/\hbar}$$