Murphy's law for deformation spaces

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Hilbert scheme

Hilb_n parametrizes closed subschemes of $\mathbb{P}^n$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\text{flat}"'] & \mathbb{P}^n_T \arrow[d] \\ & T \end{tikzcd}$$

Hilbert scheme $\mathrm{Hilb}_n = \coprod_{p(t)} \mathrm{Hilb}_{n,p(t)}$

↑ Hilbert poly

proj. schemes

Murphy's law (Harris – Morrison, p18)

There is no geometric possibility so horrible that it cannot be found on some Hilbert scheme.

$= \coprod_n \mathrm{Hilb}_n$

Pathologies II (Mumford)

$\mathbb{P}^3$ $d=14$ $g=24$

$C \hookrightarrow \mathbb{P}^3$

$X = \mathrm{Bl}\ \mathbb{P}^3$

aut X

def X

ob X

Hilb non-reduced here!

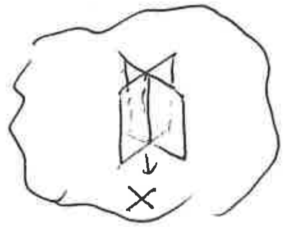
Def X

$0 \to \mathrm{Aut}\ \mathbb{P}^3 \to \mathrm{Def}(C \hookrightarrow \mathbb{P}^3) \twoheadrightarrow \mathrm{Def}\ X \to 0$

Define Singularity = pted scheme (X,p)

C

$\sim$ equiv. rel'n generated by $(X,p) \xrightarrow{\text{smooth}} (Y,q)$, then $(X,p) \sim (Y,q)$



$f.t./\mathbb{Z}$

Def. We say Murphy's law holds for a scheme (or stack) X if every singularity type appears on it.

Thm Murphy's law holds for the Hilbert scheme of smooth curves in proj. spaces.

- of smooth surfaces in $\mathbb{P}^5$
- of surfaces in $\mathbb{P}^4$.
- Kontsevich's space of stable maps
- Chow varieties
- moduli space of surfaces (smooth, very ample, ...)

Mod space of smooth surfaces w/ v. ample can. bundle.

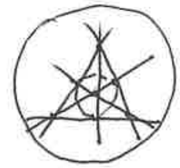
Maybe

S general type, $h^1(\mathcal{O}_S) = 0$, K_S ~~very~~ ample,

then unobstructed.

$5^7 = 0$

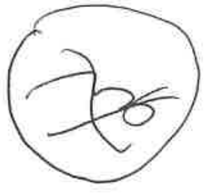
Serre, Raynaud



$\mathbb{P}^2$

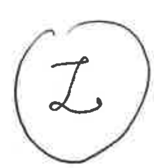
$\mathbb{F}_2$

Plane curves



equisingular deformation

Severi def



~~smooth~~

nodes & cusps

- stable coherent sheaves
- def of isolated CM 3-fold singularity.

Philosophy.

Good spaces	Bad spaces
curves	surfaces
branched covers of $\mathbb{P}^1$	branched cover of $\mathbb{P}^2$
surfaces in $\mathbb{P}^3$	surfaces in $\mathbb{P}^4$
def ☺	def ☹

Informal.

$$S \xrightarrow{|2K|} \mathbb{P}^5$$

$$\begin{array}{ccc} \mathbb{C}^\times \swarrow & \mathrm{Def}(S, L, 6 \text{ sections}) & \\ \mathrm{Def}(S \hookrightarrow \mathbb{P}^5) & \downarrow & \\ & \mathrm{Def}(S, L = K) & \\ & \downarrow & h^1(\mathcal{O}) = 0 \\ & \mathrm{Def}(S) & \end{array}$$

$$S \overset{\pi}{\hookrightarrow} \mathbb{P}^5$$

$$0 \to T_S \to \pi^* T_{\mathbb{P}^5} \to N \to 0$$

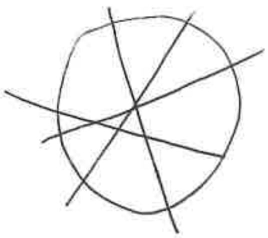
$$H^0(N) \to H^1(T_S) \to H^1(\pi^* T_{\mathbb{P}^5})$$

$$\mathrm{Def}(S \hookrightarrow \mathbb{P}^5) \to \mathrm{Def}(S)$$

$$H^1(N) \to H^2(T_S)$$

$$\to \mathrm{Ob}(S \hookrightarrow \mathbb{P}^5) \to \mathrm{Ob}(S)$$

Mnëv's Thm

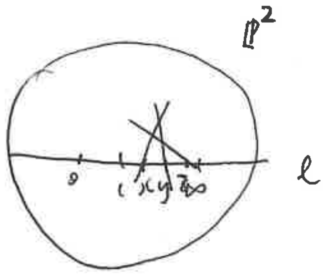


Fix m, n, require $p_1 \in l_2$, $p_2 \in l_3$, ...

incidence scheme $\subset (\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n$

$\bigcup$ incidence schemes satisfies Murphy's law.

Fact



i) force $x+y=z$

ii) force $xy=z$

iii) $x=-y$

$a^2+4b=0$

$b^7+c^7=0$.

"$\mathbb{C}$ cares about $\mathbb{Q}$"

Mod. space of $\mathbb{C}$-surfaces

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{C}} & \rightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} \mathbb{Z} \end{array}$$

$yx(y-x)(y-\pi x)=0$

Suppose you have a "nice" object X over $\mathbb{C}$.

Def X is "defined over $\mathbb{Z}$"

TRUE: great!

FALSE: a simple counterexample, great!