Enriques Surfaces

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$$S \quad, \quad b_2(S) = 10, \quad k(S) = 0, \quad 2K_S = 0, \quad \chi(\mathcal{O}_S) = 1$$

Def (1) If $h^{0,1} = 0$, we say S is classical

(2) If $h^{0,1} = 1$, & $F : H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S)$ bijective, S is ordinary

(3) $-\,''\,-$ $F = 0$, we say S is supersingular

Prove ① all Enriques have $h^{0,1} = P_g = 0$ unless char $k = 2$.

char $k = 0$,
$$0 = \Delta_S = 2h^{0,1} - \underbrace{b_1}_{0} \qquad \Rightarrow h^{0,1} = 0$$

char $k = p > 0$: assume $h^{0,1} = 1$. take basis $e \in H^1(\mathcal{O}_S)$

$F e = \lambda e$. $F + \lambda : G_{a,S} \to G_{a,S}$, $\alpha_\lambda = \ker(F - \lambda)$

$$0 \longrightarrow \alpha_\lambda \longrightarrow G_a \xrightarrow{F - \lambda} G_a \longrightarrow 0$$

$$H^1(S, \alpha_\lambda) \longrightarrow \ker(F - \lambda) \longrightarrow 0$$

$$\exists \text{ non trivial } \alpha_\lambda \text{ torsor } X \xrightarrow{\pi} S$$

Claim (i) X is Gorenstein & $\omega_X \simeq \mathcal{O}_X$.

(ii) $\pi_* \mathcal{O}_X$ has a compo series & p factors $\mathcal{O}_S$.

$$\chi(\mathcal{O}_X) \leq h^0(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 2$$

$$\chi(\mathcal{O}_X) = \chi(\pi_* \mathcal{O}_X) = p\chi(\mathcal{O}_S) = p \qquad \Rightarrow p \leq 2.$$

② $\left[\text{Thm 3} \quad \text{in} \quad \text{Mumford III} \right] \qquad \exists \ S \longrightarrow \mathbb{P}^1 \ \text{w} \ P_a = 1 \ \text{fibers}$

Def   A curve $D = \sum n_i E_i$ is of canonical type if $K \cdot E_i = 0 = D \cdot E_i$.

D is indecomposable if $\gcd(n_i) = 1$.

Pf ②: ETS $\exists \ D$ of canonical type.

Let $P_0 = \min\limits_{\substack{C \text{ irred}}} P_g(C) \qquad \text{s.t. } C^2 > 0$

$$= \min \left(\tfrac{1}{2} c^2 + 1 \right)$$

Take $D \in |C|$ s.t. D is reducible but not $\overset{\underset{\text{non singular rat'l}}{\swarrow \ \searrow}}{E + E'}$

Step 1:   Let E be a comp of D, $D = E + E'$

$0 < c^2$, $\qquad D \cdot E' = C \cdot E' \geq 0$

$\qquad \qquad D \cdot E = D^2 - D \cdot E' \leq c^2$

$\qquad \qquad \quad \underset{\underset{C^2}{\parallel}}{} \quad \underset{\geq 0}{\underbrace{\qquad}}$

But $D \cdot E = E^2 + E \cdot E'$ So either $E \cdot E' > 0$ or $E^2 < 0$ in either case,

$E^2 < c^2$ so $E^2 \leq 0$.   So $P_g(E) \in \{0, 1\}$. $\iff E = 0$ or -2.

<u>Step 2.</u> $D = \sum m_i E_i$ $E_i^2 = 0 \Rightarrow E_i$ is of canonical type

So wLoG $E_i^2 = -2$.

- $\sum m_i \geq 3$

- $E_i \cdot E_j \geq \overset{\text{Assume}}{2}$, $(E_i + E_j)^2 = E_i^2 + E_j^2 + 2 E_i E_j \geq 2$

$1 \leq \dim |E_i + E_j|$ By R-R,

$$\chi(E_i + E_j) = \chi(\mathcal{O}_s) + \frac{1}{2}(E_i + E_j)^2 \geq 2$$

But $|E_i + E_j| + |D - E_i - E_j| \leq |c|$

<u>Step 3.</u>   If $E_i E_j = 2$, then $E_i + E_j$ canonical

So wLoG $E_i \cdot E_j \in \{0, 1\}$.

Graph: vertices E_i,   Edge if $E_i \cdot E_j = 1$

$\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$

$\underset{\text{or}}{}$ is Dynkin diagram $D^2 < 0 < c^2 = D^2$ $\Rightarrow \Leftarrow$.

③ $p = b_2 = 10$. $\forall S$ Enriques.

Pf [Mum I, Thm 4]   <u>Thm.</u> $p = b_2$ $\forall$ quasi-elliptic surfaces.

Pf $\exists$ $Y \xrightarrow{\pi} S$ Y smooth ruled surface
        π proper of degree p.

$$\text{Pic}(Y) \otimes \mathbb{Z}_\ell \longrightarrow H^2(Y, \mathbb{Z}_\ell)$$

$$\pi^* \uparrow \downarrow \pi_* \qquad\qquad \pi^* \uparrow \downarrow \pi_*$$

$$\text{Pic}(S) \otimes \mathbb{Z}_\ell \longrightarrow H^2(S, \mathbb{Z}_\ell)$$

Prop If S Enriques surface is elliptic $S \xrightarrow{\phi} \mathbb{P}^1$, it suffices to show

$$P = b_2 \quad \text{on} \quad J(S) \xrightarrow{t} \mathbb{P}^1.$$

Fact :
$$b_i(J(S)) = b_i(S) \qquad [\text{Prop 4.3.84 in Dolg I}]$$
$$P(S) = P(J(S))$$

$[\text{Thm 2 of Mum I}]$ gives $\omega_{J(S)} = t^*\left((R^1 t_* \mathcal{O}_S)^{-1} \otimes \omega_{\mathbb{P}^1}\right)$

So $\omega_{J(S)} = t^* \mathcal{O}(-1)$, so all plurigenera of $J(S)$ vanish

by Castelnuovo's rat'lity crit, $J(S)$ is rat'l

so $P(J(S)) = b_2(J(S)).$ $\qquad$ □

④ $\text{Pic}^{\tau}_{S/k} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & S \text{ classical} \\ \mu_2 & \text{ordinary} \\ \alpha_2 & \text{super singular} \end{cases}$

Pf S classical $\quad H^1(S, \mathcal{O}_S) = 0 \quad \Rightarrow \text{Pic}_S$ smooth

$\text{Pic}^{\tau}_S = \text{Tors}(\text{Pic}(S))$

$\not{F}$ — torsion

$$1 = \chi(F)$$

so $F \to F^\vee \otimes \omega_S$ has a section

s. $F = \mathcal{O}_S$ or ω_S

So Pic_S^τ supported at a point if S non-classical.

Thm [1.3.1, Dolg I] Any Enriques surface has a degree 2 cover $\pi: X \to S$ which is a torsor for $(\text{Pic}_S^\tau)^\vee$.

Classical:
$$1 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$
$$H^1(S, \mu_2) \to H^1(S, \mathbb{G}_m)[2] \to 0$$

non-classical. □

So we call $X \to S$ the "K3" cover of S.

Thm. X is Gorenstein & $H^1(\mathcal{O}_X) = 0$, $\omega_X \cong \mathcal{O}_X$.

(1) If $p \neq 2$ or S ordinary, X is smooth

(2) Else X is $\underline{not}$ smooth.

In case II, $\pi: X \to S$ is purely inseparable & π is a homeomorphism $X_{\text{ét}} \to S_{\text{ét}}$

so $c_2(X) = c_2(S) = 12$

$\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_S) = 2$

But if X smooth, $12 \chi(\mathcal{O}_X) = K_X^2 + c_2(X) = 12$
$\underset{2\chi}{\underset{||}{}}$