The Plancherel algebra

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[BZSV, §8] Recall

$\mathbb{F} = \mathbb{C}, \overline{\mathbb{F}_q}$, $\quad k$ alg. closed char 0

$(G, M) \qquad (\check{G}, \check{M})$

focus on (G, T^*X) untwisted polarized

Saw
$$SHV(X_F/G_O) \xrightarrow[\sim]{} QC^{\natural}(\check{M}/\check{G})$$

(1) $\delta_X \longleftrightarrow \mathcal{O}^{\natural}_{\check{M}/\check{G}}$

(2) $Shv(G_O \backslash G_F) =: \mathcal{H}_G$ - action $\longleftrightarrow QC^{\natural}(\check{\mathfrak{g}}^{\vee *}/\check{G}^{\vee})$

$Perv_{G_O}(G_F) =:$ Sat - action $\longleftrightarrow Rep^{\natural}(\check{G})$

Want: decategorify this by taking End of δ_X

Note:

$\delta_X \longleftrightarrow \mathcal{O}^{\natural}_{\check{M}/\check{G}}$

$T_V * \delta_X \longleftrightarrow V \otimes \mathcal{O}^{\natural}_{\check{M}/\check{G}} \in QC^{\natural}(\check{M}/\check{G})$, $\forall V \in Rep(\check{G})$

they generate know hom spaces. $\Leftarrow$ they generate the cat.

$Hom(V \otimes \mathcal{O}^{\natural}_{\check{M}/\check{G}}, W \otimes \mathcal{O}^{\natural}_{\check{M}/\check{G}}) = Hom_{\check{G}}(V \otimes \check{W}, \mathcal{O}^{\natural}_{\check{M}/\check{G}})$

explicit

Construction

$\mathcal{C} = \mathrm{Rep}(\check{G})$ rigid tensor cat. $\curvearrowright$ cat. $\mathcal{M}$

$F \in \mathcal{M}$ ~~cpt.~~ $\rightsquigarrow$ $\mathcal{C} \underset{\underline{\mathrm{Hom}}(F,-)}{\overset{-\otimes F}{\rightleftarrows}} \mathcal{M}$

in particular, • $\underset{\underline{\mathrm{End}}(F,F)}{\overset{\underline{\mathrm{End}}(F)}{\shortparallel}} \in \mathcal{C}$

• characterized by $\mathcal{H}om_{\mathcal{C}}(V, W \otimes \underline{\mathrm{End}}(F)) \simeq \mathrm{Hom}_{\mathcal{M}}(V \otimes F, W \otimes F)$

• unital algebra in $\mathcal{C}$.

Def. (Plancherel algebra / Coulomb branch)

$$\mathbb{PL}_X := \underline{\mathrm{End}}_{\overline{\mathcal{H}}_G}(\delta_X)$$

inner endomorphism in $\mathrm{Alg}(\overline{\mathcal{H}}_G) \simeq \mathrm{Alg}(QC^{\sslash}(\check{\mathfrak{g}}^*/\check{G}))$ ($\mathbb{E}_1$)

Remarks. • $\mathbb{PL}_X$ is dga over $\check{\mathfrak{g}}^{*\sslash}$ w $\check{G}$-action.

Conj. (Plancherel algebra conjecture)

$$\mathbb{PL}_X \simeq \mathcal{O}^{\sslash}_{\check{M}/\check{G}} \quad \text{as } \check{G}\text{-equiv. dga over } \check{\mathfrak{g}}^{*\sslash}$$

Note. consequence of local conjecture.

$$\mathbb{PL}_X = \underline{\mathrm{End}}_{\overline{\mathcal{H}}_G}(\delta_X) \underset{\text{local duality}}{=} \underline{\mathrm{End}}_{\overline{\mathcal{H}}_G}(\mathcal{O}^{\sslash}_{\check{M}/\check{G}}) = \mathcal{O}^{\sslash}_{\check{M}/\check{G}}$$

Consequences

- $\mathbb{PL}_X$ formal
- $\mathbb{PL}_X$ commutative. (doesn't mean $\mathbb{E}_\infty$)
- indep. of sheaf theory
- $\mathbb{PL}_X$ encodes whole $SHV(X_F/G_O)$

Recap.

$$\underset{\mathbb{F}}{(G,M) = (G,T^*X)} \longleftrightarrow \underset{k}{(\check{G},\check{M})}, \quad k=\bar{k}, \text{ char } 0$$

$$SHV(X_F/G_O) \overset{\text{conj.}}{\simeq} QC^!(\check{M}/\check{G})$$

Plancherel algebra $\quad \mathbb{PL}_X = \underline{End}_{\mathcal{H}_G}(\delta_X)$

alg. in $\quad \mathcal{H}_G = SHV(G_O \backslash Gr) = QC^!(\check{\mathfrak{g}}^*/\check{G})$

Def. (relative Grassmannian / variety of triples)

relative Grassmannian of X is $\{(y,x,g) : y = xg\}$

$$\begin{tikzcd} X_O \times Gr & Gr^X \arrow[l, hook', "i"'] \arrow[r] \arrow[d] & X_O \overset{G_O}{\times} G_F \circlearrowleft G_O \arrow[d] \\ & X_O \arrow[r] & X_F \end{tikzcd}$$

$(y, g^{-1}) \longleftarrow\!\shortmid$; (x,g) ; y ; $X_O \circlearrowleft G_O$

Note $Gr^X \circlearrowleft G_O$

have correspondence

$$\begin{tikzcd}
 & Gr^X/G_O \arrow[dl, "p_1"'] \arrow[d, "r"] \arrow[dr, "p_2"] & \\
X_O/G_O & G_O\backslash Gr_G & X_O/G_O
\end{tikzcd}$$

$$\begin{tikzcd}
 & & G_O\backslash Gr^X \arrow[dll, bend right] \arrow[drr, bend left] & & \\
 & & X_F \times^{G_O} G_F/G_O \arrow[dl] \arrow[dr] & & \\
X_O/G_O \arrow[r, hook] & X_F/G_O \arrow[dr] & & Y_F/G_O \arrow[dl] & X_O/G_O \arrow[l] \\
 & & X_F/G_F & &
\end{tikzcd}$$

Facts (1) the functor of points, R

$$X_O \times Gr_G = \{ \mathcal{G} : G\text{-bundle on } R[[t]],\ \sigma \text{ section over } R((t)),\ x \in X(R[[t]]) \}$$

$$\uparrow$$

$$Gr^X = \{ (\mathcal{G}, \sigma, x) : x^{\mathcal{G}} \text{ extends to } R[[t]] \}$$

denote : $\mathcal{X} :=$ assoc. X-bundle on $R[[t]]$

$x^{\mathcal{G}} =$ trivialization over $R((t))$.

(2) X affine $\rightsquigarrow$ i^{Gr} is closed immersion

(3) $Gr^X/G_O = \left\{ \begin{array}{l} \text{pairs of } \mathcal{G}\text{-bundles on } R[[t]] \\ \text{pairs of sections of } X\text{-bundles on } R[[t]] \\ \text{isom. over } R((t)) \end{array} \right\}$

(4) restrict $Gr_{\le m} \hookrightarrow Gr$.

$$\begin{tikzcd}
X_O \times Gr_{\le m} \arrow[d] & Gr^X_{\le m} \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\
X_N \times Gr_{\le m} & Gr^X_{\le m, N} \arrow[l]
\end{tikzcd}$$

Example $\quad X = H\backslash G$

then $\quad Gr^X / G_O \simeq H_O \backslash H_F / H_O$

Rmk $\quad \Psi \to X \quad$ twisted case

$\rightsquigarrow Gr^X \to \mathbb{A}^1.$

Now

$\mathbb{PL}_X$ algebra in $Rep(\check{G})$. Want to describe isotypic parts

$$\mathbb{PL}_X = \bigoplus_{V \in Irr(\check{G})} V \otimes \mathbb{PL}_X^{(V)} \qquad \mathbb{PL}_X^{(V)} = Hom_{\check{G}}(V, \mathbb{PL}_X)$$

Notation. $\quad T_V^X = \Gamma^* T_V \quad$ pull back along $\Gamma : Gr^X/G_O \to G_O \backslash Gr_G$

$Gr_{\leq m}$ = sufficiently big stratum supporting T_V.

Prop (Multiplicity spaces in $\mathbb{PL}_X$)

$$\mathbb{PL}_X^{(V)} \langle -\deg T_V \rangle \simeq Hom_G(T_V^X, \pi^! k_{X_N})$$

$Gr^{*}_{\leq m, N}$

$$\simeq H^{\cdot}_G(Gr^X_{\leq m, N}, \mathbb{D} T_V^{*}) \langle -2\dim X_N \rangle$$

$$\boxed{\begin{array}{l} \eta : G \to G_m \\ \deg : G_F \xrightarrow{\eta} G_{m,F} \xrightarrow{\deg} \mathbb{Z} \\ g : \mathcal{F} \simeq g^* \mathcal{F} \langle \deg g \rangle \end{array}}$$

Remark $\quad \mathbb{PL}_X^{(V)} \langle -\deg T_V \rangle = Hom_{Gr^X/G_O}(T_V^X, \omega^{ren}_{Gr^X})$

$$= Hom_{G_O \backslash Gr}(T_V, \Gamma_* \omega^{ren}_{Gr^X})$$

$\mathcal{A} = \Gamma_* \omega^{ren}_{Gr^X} \quad$ algebra

Proof sketch

$$
\begin{tikzcd}
X_O/G_O \arrow[d] & Gr^{\times}/G_O \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "i^{Gr}"] & X_O/G_O \arrow[d] \\
& X_O \times^{G_O} G_F/G_O \arrow[ul] \arrow[ur] \arrow[d] \arrow[dd, bend right, "\tilde{\Gamma}"'] & \\
X_F/G_O & X_F \times^{G_O} G_F/G_O \arrow[l, "p_2^F"] \arrow[r, "p_1^F"'] \arrow[d, "\Gamma_F"] & X_F/G_O \\
& G_O \backslash Gr &
\end{tikzcd}
$$

$$T * \delta_X = p^F_{2,*}\left(\Gamma_F^! T \overset{!}{\otimes} p_1^{F,!}\delta_X\right)$$

[need smoothness of X]

$$\delta_X = i_* \omega_{X_O}$$

$$\mathcal{H}om_{X_F/G_O}(T * \delta_X, \delta_X) = \mathcal{H}om_{X_O \times^{G_O} G_F/G_O}\left(\tilde{\Gamma}^! T, i_*^{Gr}\, \omega_{Gr^{\times}/G_O}\right)$$

$$= \mathcal{H}om_{X_N \times^{G_O} Gr_{\leq m}}\Big(\underbrace{\Gamma_N^! T}_{= \Gamma_N^* T\langle -2\dim X_N\rangle}, i_*\, \omega_{Gr^{\times}_{\leq m, N}}\Big)$$

$$- \Sigma$$

Noncommutative deformation

$X \circlearrowleft G$

$X_F \circlearrowleft G_O \times \mathbb{G}_m^{rot}$

$SHV(X_F / G_O \times \mathbb{G}_m^{rot})$

Can take $\mathbb{G}_m$-equiv. coh.

$\rightsquigarrow$ lands in $H^*(B\mathbb{G}_m) = k[u]$

Def. $\mathbb{PL}_X^{\hbar} := H^{\bullet}_{\mathbb{G}_m}(\underline{\mathrm{End}}(\delta_X))$

Fact exist deformation quantization of $\check{M}$

Conj: $\mathbb{PL}_X^{\hbar} \simeq$ Rees construction of this quantization of $\check{M}$.

Conj. Poisson bracket on $\mathbb{PL}_X$ $\longleftrightarrow$ Poisson bracket on $\check{M}$ coming from the symplectic str.

Example (Iwasawa-Tate case) $X = \mathbb{A}^1$, $G = \mathbb{G}_m$

$$(G, M) = (\mathbb{G}_m, T^*\mathbb{A}^1) \longleftrightarrow (\check{G}, \check{M}) = (\mathbb{G}_m, T^*\mathbb{A}^1)$$

$$X_m^{\ell} = t^{-\ell} R[t]/t^m$$

$$G_O = R[\![t]\!]^{\times}$$

$$\delta_{\ell} = \underline{k}_{X^{\ell}}\langle \ell \rangle$$

$$\mathcal{H}om(\delta_{\ell_1}, \delta_{\ell_2}) = H^{\bullet}(B\mathbb{G}_m)\langle -d \rangle$$

$$d = |\ell_1 - \ell_2|$$

$$\mathcal{H}om(\underline{k}_{X^{\ell_1}}, \underline{k}_{X^{\ell_2}}) = \begin{cases} H^{\bullet}(B\mathbb{G}_m), & \ell_1 \geq \ell_2 \\ H^{\bullet}(B\mathbb{G}_m)\langle -2d \rangle, & \ell_2 \geq \ell_1 \end{cases}$$

V_n = wt n rep'n, TV_n = perverse sheaf.

$T_{V_n} * \delta_x = \delta_n$

$\mathcal{H}om(T_{V_n} * \delta_x, T_{V_m} * \delta_x) = \mathcal{H}om(\delta_n, \delta_m) = H^\bullet(BG_m)\langle -|m-n|\rangle$

$\mathcal{H}om(T_{V_i} * \delta_x, \delta_x) =$ coh. deg $|i|, |i|+2, |i|+4, \dots$

$\mathcal{O}_{\check{M}} = \mathcal{O}[T^* \mathbb{A}^1] = k[x, \xi]$

G_m-act $\quad \lambda \cdot x = \lambda^{-1} x$

$\lambda \cdot \xi = \lambda \xi$

G_{gr}-act $\quad \alpha \cdot x = \alpha x$

$\alpha \cdot \xi = \alpha \xi$

$\mathcal{H}om(V_n \otimes \mathcal{O}_{\check{M}/\check{G}}, V_m \otimes \mathcal{O}_{\check{M}/\check{G}}) = \mathcal{H}om(V_{n-m}, \mathcal{O}_{\check{M}/\check{G}})$

$= \langle x^a \xi^b ; \ a, b \geq 0, \quad b-a = n-m \rangle$

$$= \begin{cases} \langle \xi^d, x^1 \xi^{d+1}, x^2 \xi^{d+2}, \dots & n \geq m \\ \langle x^d, x^{d+1} \xi^1, x^{d+2} \xi^2, \dots & m \geq n \end{cases}$$

$d \quad d+2 \quad d+4$

$\mathbb{PL}_x^{(v_i)} = \langle x^a \xi^b : \ a, b \geq 0, \ b-a = i \rangle$