Elliptic Gamma function

Dan Bump

Felder - Varchenko : Elliptic gamma function

FV show the EGF is an automorphic form of "degree 1" for $SL(3;\mathbb{Z}) \ltimes \mathbb{Z}^3$.

Jacobi triple product identity:

$$\sum_{-\infty}^{\infty} (-x)^n q^{n^2/2} = \prod_{j=0}^{\infty} (1-q^j)(1-xq^{j-1/2})(1-x^{-1}q^{j-1/2})$$

$|q|<1$, $x \neq 0$, $x = e^{2\pi i z}$

$q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$.

This is quasiperiodic in z.

$$\theta(z,\tau) = \sum e^{2\pi i (n+\frac{1}{2})^2 + 2\pi i (n+\frac{1}{2})(z+\frac{1}{2})}$$

$z \to z+\lambda$, $\lambda \in \mathbb{Z}+\mathbb{Z}\tau \simeq \mathbb{Z}^2$

multiplies θ by a nonzero multiplier. Odd Jacobi theta function

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q-Pochhammer symbol $\quad (x;q) = \prod_{j=0}^{\infty} (1-xq^j)$

$$e^{\frac{2\pi i}{24}} (q;q) = \eta(\tau), \quad q = e^{2\pi i \tau}$$

Using Jacobi triple product.

$$(z, \tau) \longrightarrow \left(\frac{z}{c\tau+d}, \frac{a\tau+d}{c\tau+d} \right)$$

$SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$

Relevant automorphy group for θ

"Jacobi group"

For example, $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in SL(2;\mathbb{Z})$

$$\theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\frac{i\pi z^2}{\tau}}\, \theta(z,\tau)$$

can be proved by Poisson summation.

$$\theta_0(z,\tau) = (x;q)\,(q/x;q)$$

$$= \prod_{j=0}^{\infty} (1-xq^j)\,(1-x^{-1}q^{1+j})$$

$$\theta(z,\tau) = ie^{i\pi(\tau/4 - x)}\, \underbrace{(x,q)\left(\frac{q}{x};q\right)}_{\theta_0(z,\tau)}(q,q)$$

$x = e^{2\pi i z}$

$q = e^{2\pi i\tau}$ $\qquad \tau, \sigma \in \mathcal{H}$

$r = e^{2\pi i\sigma}$

$$(x;q,r) = \prod_{j=0}^{\infty}\prod_{k=0}^{\infty} \left(1 - xq^j r^k\right)$$

$$\Gamma(z;\tau,\sigma) = \frac{\left(\frac{qr}{x};q,r\right)}{(x;q,r)}$$

Theorem 3.1 $\qquad \Gamma(z;\tau,\sigma) = \Gamma(z,\sigma,\tau)$

$$\Gamma(z+1;\tau,\sigma) = \Gamma(z;\tau,\sigma)$$

$$(13) \qquad \Gamma(z+\sigma,\tau,\sigma) = \Gamma(z,\tau,\sigma)\,\theta_0(z,\tau)$$

$$(14) \qquad \Gamma(z+\tau,\tau,\sigma) = \Gamma(z,\tau,\sigma)\,\theta_0(z,\sigma)$$

If we check (13), (14) follows..

$$(qx,q,r) = \frac{(x;q,r)}{(x,r)}$$

Want to extend to σ,τ having values in $\mathbb{C}$

First extend to lower 1/2 plane, then consider continuity on $\mathbb{R}$.

$$(x;q^{-1}) := \frac{1}{(xq;q)}$$

Why is this natural.

$$-\log(x;q) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} \frac{x^j q^{nj}}{j}$$

$$= \sum_{j=1}^{\infty} \frac{x^j}{j(1-q^j)}$$

$$(4) \qquad (x;q) = \exp\left(-\sum_{j=1}^{\infty} \frac{x^j}{j(1-q^j)}\right)$$

$$= \exp\left(-\sum \frac{(xq)^j}{j(1-q^j)}\right)^{-1}$$

If replace q by q^{-1}: $\quad (x;q^{-1}) = \exp\left(-\sum \frac{x^j}{j(1-q^{-j})}\right)$

FV substitute this to get a def'n of $\Gamma(z;\tau,\sigma)$ for τ in lower half plane

In section 3.5, they consider limit as $\tau \to$ real axis.

Section 7 they consider modularity $G = SL(3;\mathbb{Z}) \ltimes \mathbb{Z}^3$

has generators & rel'ns due to Steinberg (Chevalley). Milnor's book Alg. K-theory.

presentation of $SL(3;\mathbb{Z})$ is in terms of elementary matrices:
$SL(n;\mathbb{Z})$ $(n \geq 3)$

$$e_{ij} = 1 + X_{ij} = \mathrm{Exp}(X_{ij})$$

X_{ij} = matrix w/ 1 in ij pos.

$$[e_{ij}, e_{k,\ell}] = 1 \quad \text{if } i \neq k,\ j \neq \ell$$

$$[X_{ij}, X_{k,\ell}] = 0 \quad \text{in } \mathrm{Lie}(GL(3)).$$

because $(e_i - e_j) + (e_k - e_\ell)$ is not a root.

$$e_{ij}\, e_{jk} = e_{ik}\, e_{jk}\, e_{ij}$$

$$[X_{ij}, X_{jk}] = X_{ik}$$

$$(e_{13}\, e_{31}^{-1}\, e_{13})^4 = 1$$

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

Claim. These rel'ns are presentation of $SL(3;\mathbb{Z})$

Adds more generators for $\mathbb{Z}^3$

Consider a manifold X and a group $G \curvearrowright X$

Examples: $G = SL(2;\mathbb{Z}) \ltimes \mathbb{Z}^2$, $X = \underset{z}{\mathbb{C}} \times \underset{\tau}{\mathcal{H}}$

$SL(3,\mathbb{Z}) \times \mathbb{Z}^3$, $X = \mathbb{C} \times \mathbb{C}^3$

Consider N = meromorphic func. on X

M = nonvanishing hol. func.

Theta multiplier (multiplier system) is a 1-cocycle in $H^1(G, M)$

$$G \curvearrowright M, \qquad (g.m)(x) = m(g^{-1}x)$$

$$\phi(g_1 g_2) = \phi(g_1) \cdot {}^{g_1}\phi(g_2)$$

$$\phi(g_1 g_2, x) = \phi(g_1, x)\, \phi(g_2, g_1^{-1} x) \qquad \text{1-cocycle cond.}$$

An automorphic form of class $[\phi]$ is $\mu \in C^0(G, N)$

$(\delta\mu)(g) =$

coboundary in $C^1(G,N)$

This is supposed to agree w/ the multiplier $\phi \in C^1(G, M)$

$$\phi(g, x) = \frac{\mu(x)}{g\,\mu(x)} = \frac{\mu(x)}{\mu(g^{-1}x)} \qquad \text{Automorphic condition}$$

$\mu = \theta_0$ is a rep. of the automorphic class of type $[\phi] \in H^1(G, M)$

ϕ = theta multiplier

The framework extends to $SL(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$ except

$\mu \in C^1(G, N)$

$\delta\mu \in C^2(G, M)$