Towards Bezrukavnikov via p-adic central sheaves

João Lourenço

$\ell \neq p$ prime, G' split conn. red gp $/ \bar{\mathbb{F}}_q((t))$. I' Iwahori model of $G' / \bar{\mathbb{F}}_q[[t]]$.
(w/ pinning B', T', w')

$\Lambda = \bar{\mathbb{Q}}_\ell$, $\hat{G}$ dual gp of G', $\hat{\mathcal{N}} \subset \hat{\mathfrak{g}}$ nilpotent cone. $\hat{\mathcal{N}}_{Spr} = \hat{G} \overset{\hat{B}}{\times} \hat{\mathfrak{n}}$ Springer res.

$\hat{St} = \hat{\mathcal{N}}_{Spr} \underset{\hat{\mathfrak{g}}}{\times} \hat{\mathcal{N}}_{Spr}$

Kazhdan-Lusztig : $K_0(Coh(\hat{G} \backslash \hat{St})) \simeq \mathbb{Z}[\tilde{W}] \simeq K_0(Shv_c(L^+I' \backslash Fl_{I'}))$

Iwahori-Weyl gp

Bezrukavnikov: $\Phi : D^b_{coh}(\hat{G} \backslash \hat{St}) \xrightarrow{\sim} D^b_{ét,c}(L^+I' \backslash Fl_{I'})$

Now $F | \breve{\mathbb{Q}}_p$ finite, G split conn. red gp $/F$, I Iwahori $\mathcal{O}$-model of G,
$\mathcal{O}$ ring of integers, (w/ pinning B, T, w)
$k = \bar{\mathbb{F}}_p$

$Gr_G = B^+_{dR}$- Grassmannian $/ Spd\, F$

$\|$

$\operatorname{colim}_{\mu \in X_*(T)/W} Gr_{G, \leq \mu}$ ← Schubert diamond (Bruhat stratification)

$Fl_I = LG/L^+I$ Witt flag variety / Spec k

$\|$

$\operatorname{colim}_{w \in \tilde{W}} Fl_{I, \leq w}$ ← Schubert perfect varieties

Beilinson - Drinfeld Grassmannian: Gr_I v-sheaf / $\mathrm{Spd}\, \mathcal{O}$

w/ gen fiber Gr_G, special fiber $\mathcal{F}\ell_I^\diamond$. $= \operatorname{colim}_{\mu \in X_*(T)/W} M_{I,\mu}$ ← local models

closures of $\mathrm{Gr}_{G, \le \mu}$.

Thm (AGLR) ① μ minuscule $\Rightarrow$ $M_{I,\mu}$ is represented by a unique flat normal $\mathcal{O}$-scheme

② The special fiber of $M_{I,\mu}$ is $A_{I,\mu}^\diamond$ <— Kottwitz - Rapoport's μ-admissible locus

$$A_{I,\mu}^\diamond = \bigcup_{\substack{\nu \in X_*(T) \\ \mathrm{I} \\ \mu \in X_*(T)/W}} \mathcal{F}\ell_{I, \le \nu}$$

② goes via supp. of nearby cycles of Satake sheaves.

<u>Fargues - Scholze</u> Sat: $\operatorname{Rep} \hat{G} \xrightarrow{\sim} \operatorname{Perv}^{ULA}(\mathrm{Hk}_{G,C})$, $C = \hat{\bar{F}}$

$$\mathrm{Hk}_{G,C} = L^+G \backslash \mathrm{Gr}_G$$

$$\mathrm{Hk}_{I,k} \xrightarrow{i} \mathrm{Hk}_{I,\mathcal{O}_C} \xleftarrow{j} \mathrm{Hk}_{G,C}$$

$$R\psi = i^* Rj_* : D_{\acute{e}t}(\mathrm{Hk}_{G,C}) \longrightarrow D_{\acute{e}t}(\mathrm{Hk}_{I,k})$$

Thm (AGLR) ① $c : D^b_{\acute{e}t,c}(\mathrm{Hk}_I^{\mathrm{Witt}}) \xrightarrow{\sim} D^{ULA}_{\acute{e}t}(\mathrm{Hk}_{I,k})$

② $R\psi$ respects ULA sheaves.

Get $Z = R\psi \circ \mathrm{Sat} : \operatorname{Rep}(\hat{G}) \longrightarrow D^b_{\acute{e}t,c}(\mathrm{Hk}_I)$

Thm (ALWY): Z is a central monoidal functor.

<u>Idea</u>. Classically due to Gaitsgory.

Let points collide to get $Z(V) * A \simeq A * Z(V)$.

Also need compatibility w/ symmetry constraint : use nearby cycles $/ (\mathrm{Spd}\, O_C)^2$

For perversity, need Wakimoto functor $J_\nu : D^b_{\acute{e}t,c}(*) \longrightarrow D^b_{\acute{e}t,c}(Hk_I)$
$\nu \in X_*(T)$

uniquely determined by

i) $J_{\nu_1}(A_1) * J_{\nu_2}(A_2) \simeq J_{\nu_1+\nu_2}(A_1 \otimes A_2)$

ii) $J_\nu(A) = \nabla_{t_\nu} \otimes^L A$, $\nu \in X_*(T)_+$, ∇_{t_ν} costandard perv. sheaf on $Fl_{I,\, t_\nu}$.

$J = \bigoplus_{\nu \in X_*(T)} J_\nu$ lift to $\mathrm{Rep}(\hat{T})$

<u>Thm</u> (ALWY): $Z(V)$ is perverse and admits a filtr. w/ graded equal to $J(V|_{\mathrm{Rep}\,\hat{T}})$

<u>Idea</u> Classically due to AB, using perversity.

Instead, use centrality to write $Z(V)$ as an ext'n of complexes $\in$ ess im J

Identify the pieces by taking CTs.

Have

$$\begin{array}{ccccc} (V,\nu) & & \mathrm{Rep}(\hat{G} \times \hat{T}) & \xrightarrow{Z \times J} & \mathrm{Perv}(Hk_I) \\ \downarrow & & \downarrow & & \downarrow \\ V \otimes \mathcal{O}_{\hat{\mathcal{N}}_{Spr}}(\nu) & & D^b_{coh}(\hat{G} \backslash \hat{\mathcal{N}}_{Spr}) & \xrightarrow{F} & D^b_{\acute{e}t,c}(Hk_I) \end{array}$$

want to extend

AB functor

Consider $\hat{\mathcal{N}}_{Spr}^{aff} \subset (\hat{G}/\hat{U})^{aff} \times \hat{\mathfrak{g}}$ stabilizer for $\hat{\mathfrak{g}}$-action

Nilpotent monodromy endo. + Plücker relation $\Rightarrow$ $Coh_{fr}(\hat{G} \backslash \hat{\mathcal{N}}^{aff}_{Spr}) \xrightarrow{\tilde{F}} \mathrm{Perv}(Hk_I)$

$\uparrow$
direct sum $V \boxtimes \mathcal{O}(\nu)$

$D^b_{Coh}(\hat{G}\backslash\hat{\mathcal{N}}_{Spr})$ is a Verdier quotient of $Ch^b\left(Coh_{fr}(\hat{G}\backslash\hat{\mathcal{N}}^{aff}_{Spr})\right)$

$\longrightarrow$ Verity complexes get killed by $Ch^b(\tilde{F})$.

$\mathcal{L}_{AS}$ Artin-Schreier local system

$\rightsquigarrow D^b_{\acute{e}t,c}(Hk_{IW})$ = cat. of sheaves on Fl_I w/ a RL^+I^{op}-action governed by $\mathcal{L}_{AS}$.

$\nearrow Av_{IW}$ from $D^b_{\acute{e}t,c}(Hk_I)$; $\uparrow$ Iwahori-Whittaker

<u>Advantage</u>: $Perv(Hk_{IW})$ is h.w. cat. w/ tilting objs indexed by $X_*(T)$

(Assume for now on G is of type A)

Thm (ALWY) The composition $av_{IW}\circ F: D^b_{Coh}(\hat{G}\backslash\hat{\mathcal{N}}_{Spr}) \xrightarrow{\sim} D^b_{\acute{e}t,c}(Hk_{IW})$

$\exists$ 2 main ingredients

- $Z_{IW}(V) = av_{IW}(Z(V))$ is tilting (extend by monoidality from minuscule μ).
- regular quotient.

$\exists$ quot. $Coh_{fr}(\hat{G}\backslash\hat{\mathcal{N}}_{Spr}) \twoheadrightarrow Coh(\hat{G}\backslash\hat{\mathcal{O}}_r)$

$\hookleftarrow$ regular orbit $\subset \hat{\mathcal{N}}$

Similarly, $\exists$ $Perv(Hk_I) \twoheadrightarrow Perv^0(Hk_I) = Perv(Hk_I)/\langle IC_w,\ \ell(w)>0\rangle$

$$\begin{array}{c} \uparrow \qquad \cup\ \ Rep\ H \\ Rep(Z_{\hat{G}}(m)) \xrightarrow{res} \end{array}$$

$\hookleftarrow$ nilpotent elt in $\hat{\mathfrak{g}}$ comes from monodromy.

<u>Prop</u> (ALWY) m is regular

<u>Idea</u> AB argue via weights

Gabber: $R\Psi$(filt) = monodromy-filt, i.e. $\mathrm{Fil}_{wt} = \mathrm{Fil}_{mon}$

$\longrightarrow$ The corresponding image of Fil_{wt} $Z(V)$ is $Z^{\circ}(V)$.

Hoven - Zavyalov. Gabber still works for formal schemes w/ smooth gen. fiber

+ some disk open covering condition.

$\rightsquigarrow$ apply to $M_{I,\mu}$ for μ minuscule, then extend by monoidality.