The ε-connection and algebraic K-theory

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1) The classical story [Deligne, Beilinson - Bloch - Esnault]

2) 2 constructions

3) Patel's work

4) ε-connection

5) Alg K-theory viewpoint on the ε-connections

6) A further higher-dim'l gen. of ε-lines (if time permits)

1) k field, char. 0 $\qquad X/k$ curve, smooth proper

$$U \subset X \quad \text{Zariski open } (\neq \emptyset)$$

$\varepsilon = (E, \nabla)$ flat connection on U

<u>Goal</u>: $\det^{\mathbb{Z}} H^{\bullet}_{dR}(E,\nabla) \simeq \bigotimes_{x \in X \setminus U} \underbrace{\varepsilon_x(E,\nabla)}_{\text{local}}$

Need $\omega \in (\Omega^1_U)^{\times}$ (no poles & zeroes on U) in order to define

$\underbrace{\varepsilon_{x,\omega}(E,\nabla)}_{\text{graded lines}}$

Important: the local factors $\varepsilon_{x,\omega}(\varepsilon)$ only depends on ω & ε "<u>near</u>" x

$(\mathcal{E},\omega)\big|_{D_x^\circ}$, where $D_x^\circ = \operatorname{Spec} \operatorname{Frac} \hat{\mathcal{O}}_x$, should be sufficient to compute

$\varepsilon_{x,\omega}(\mathcal{E})$.

2 constructions

Idea: $\mathcal{E} = (E, \nabla)$ defined on all of X

$$H^*_{dR}(\mathcal{E}) = \mathbb{H}^*(E \xrightarrow{\nabla} E \otimes \underbrace{\Omega^1_X}_{l.b.})$$

$$\Rightarrow \det{}^{\mathbb{Z}}(H^*_{dR}(\mathcal{E})) \cong \det{}^{\mathbb{Z}}(\mathbb{H}^*(X, E \oplus E \otimes \Omega^1_X[-1]))$$

$$\simeq \det{}^{\mathbb{Z}}(\mathbb{H}^*(X, E \xrightarrow{\omega} E \otimes \Omega^1_X))$$

where ω is a regular 1-form on X, nonzero

$$\Rightarrow \bigotimes_{\substack{x \in X \\ \omega(x)=0}} \det{}^{\mathbb{Z}}(F_x)$$

cpx supp. at x

Deligne's construction of de Rham ε-lines

$\mathcal{E}/U$

Deligne showed [LNM 163] that there exist vec. bundles $\underbrace{M, N}_{\text{called good lattices}} \in VB(X)$ s.t.

- $M, N|_U = E$
- $\nabla(M) \subset N \otimes \Omega^1_{X,\log}$
- $[M \xrightarrow{\nabla} N \otimes \Omega^1_{X,\log}] \overset{\text{qisom.}}{\simeq} j_*^{dR}\mathcal{E}$, where $j: U \hookrightarrow X$

Consequence: $H^*_{dR}(U, \mathcal{E}) \simeq \mathbb{H}^*(X, M \xrightarrow{\nabla} N \otimes \Omega^1_{X,\log})$

$\rightsquigarrow \det^{\mathbb{Z}}(H_{dR}(U,\mathcal{E})) \simeq \det^{\mathbb{Z}}(\mathbb{H}^*(X, M \xrightarrow{0} N \otimes \Omega^1_{\log}))$

Restricting to U, we can use ω as a differential

$\rightsquigarrow$ K-theory point $M \oplus N \otimes \Omega^1_{\log}[-1]$ is sent to 0 in $K(U)$.

$\xrightarrow{\text{localization}}$ lies in $K(X, X \setminus U)$

$[M \oplus N \otimes \Omega^1_{\log}[-1]] \simeq \bigoplus_{x \in X \setminus U} [F_x]$. we define

$\mathcal{E}_{x,\omega} = \det^{\mathbb{Z}}[F_x]$.

For reg. singularities, $M = N$, $M \xrightarrow{\nabla} M \otimes \Omega^1_{\log}$

Second construction (BBE)
reformulated slightly

$X \rightsquigarrow \mathbb{D} = \operatorname{Spec} k[[t]]$, $U \rightsquigarrow \mathbb{D}^0 = \operatorname{Spec} k((t))$

$\mathcal{E} = (E, \nabla)/\mathbb{D}^0$ & $\omega \in (\Omega^1_U)^\times$

$\rightsquigarrow \left[E \overset{\nabla}{\underset{\omega}{\rightrightarrows}} E \otimes \Omega^1\right]$

binary cpx (Grayson)

In fact: acyclic binary cpx in the quotient of the cat. of lin. loc. cpt k-vec. sps (a.k.a. Tate vec. spaces)

$$\{\text{lin. loc. cpt vec. sps}\} / \{\text{lin. cpt vec. sps}\}$$

In here, the binary cpx above is acyclic

$$\Omega K\left(\{\text{loc. cpt.}\}/\{\text{cpt}\}\right) \overset{\text{S. Saito}}{\simeq} \Omega\Sigma K(k) \simeq K(k) \xrightarrow{\det^{\mathbb{Z}}} \mathrm{Pic}^{\mathbb{Z}}$$

$$\left[E \underset{\omega}{\overset{\nabla}{\rightrightarrows}} E\otimes\Omega^1_X\right] \in K(k)$$

3) Patel's construction

X smooth proper variety, U open, $S \subset T^*X$ (sing. supp)

$\omega \in \Omega^1_U$ s.t. $\omega(U)\cap S = \phi$

Thm (Patel) There is a map of spectra

$$\begin{tikzcd}
\mathcal{E}_\omega : \mathbb{K}(D_X, S) \arrow[r] \arrow[dr, bend right, "R\Gamma_{dR}"'] & \mathbb{K}(X, X\setminus U) \arrow[d, "R\Gamma"] \\
& \mathbb{K}(k)
\end{tikzcd}$$

$\mathbb{K}(D_X,S)$: D-module on X w/ sing. supp in S

$\left(\mathbb{K}(X,Z) = \mathbb{K}(\mathrm{Perf}_Z(X))\right)$

$$\begin{tikzcd}
\mathbb{K}(\mathcal{D}_X) \arrow[r, "\text{Quillen}", "\sim"'] & \mathbb{K}(T^*X) \arrow[r, "\sim"] & \mathbb{K}(X) \\
\mathbb{K}(\mathcal{D}_{X,S}) \arrow[r] \arrow[u] & \mathbb{K}(T^*X, S) \arrow[r] \arrow[u] & \mathbb{K}(X, X\setminus U) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K}(\mathcal{D}_{X,S}) \arrow[d] \arrow[r, bend left] \arrow[rr, bend left, "0"] & & \\
\mathbb{K}(T^*X;S) \arrow[r] \arrow[d, dashed] & \mathbb{K}(T^*X) \arrow[r] \arrow[d, "\int"] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & \mathbb{K}(T^*X\setminus S) \arrow[d, "\omega^*"] \\
\mathbb{K}(X, X\setminus U) \arrow[r] & \mathbb{K}(X) \arrow[r] & \mathbb{K}(U)
\end{tikzcd}$$

$\omega(U) \cap S = \emptyset$

composition: $\mathcal{E}_\omega : \mathbb{K}(\mathcal{D}_{X,S}) \to \mathbb{K}(X, X\setminus U)$

2nd part: $\mathcal{E}$- connection.

4) BBE actually define $\mathcal{E}$-lines for families of flat connections:

A/k comm. k-alg A,

$\mathcal{E}_A = (E_A, \nabla_A)$ on $U_A = U \times_k A$, $\omega_A \in (\Omega^1_{U_A/A})^\times$

BBE: If $\mathcal{E}_A = (E_A, \nabla_A)$ is $\mathcal{E}$-nice (a.k.a. blissful), one can define an $\mathcal{E}$-line $\mathcal{E}_{X,\omega_A}(\mathcal{E}_A) \in \mathrm{Pic}^{\mathbb{Z}}(A)$.

Furthermore, there is a nat'l flat connection (or crystal str.) on this line bdle, called $\mathcal{E}$- connection.

A K- theoretic approach

A naive picture (wrong!!!)

Let $\mathcal{E}_A = (E, \nabla)_A$ be a constant family, but allow the 1- form $\omega_A \in (\Omega^1_{\cancel{X}A/A})^\times$ to be nonconstant.

$$\left[E_A \underset{\omega_A}{\overset{\nabla_A}{\rightrightarrows}} E_A \otimes \Omega^1_{X_A/A} \right] \in \mathbb{K}(A) \xrightarrow{\det ?} \mathrm{Pic}^{\mathbb{Z}}(A)$$

or use any other construction of ε-lines.

Use the following (wrong!) idea to constr. a connection on $\varepsilon_{x,\omega}$:

We want a crystal str. on $\varepsilon_{x,\omega}$: i.e. if $\omega_0, \omega_1 \in \Omega^{\times}_{U_A}$ s.t.

$\omega_0 |_{A^{red}} = \omega_1 |_{A^{red}}$, then we want an isom.

$$\varepsilon_{x,\omega_1} \simeq \varepsilon_{x,\omega_2}.$$

Furthermore, the cocycle cond. should be satisfied for triples $\omega_0, \omega_1, \omega_2, \cdots$

<u>Consider</u>: $(1-t)\omega_0 + t\omega_1 = \omega_t$ $\quad \mathbb{A}^1$-homotopy between ω_0 & ω_1

$$\omega_t |_{A^{red}} = \omega_0 |_{A^{red}} = \omega_1 |_{A^{red}}$$

$$\rightsquigarrow \varepsilon_{x,\omega_t}(\varepsilon) \in \mathbb{K}(A[t])$$

$$\begin{array}{ccc} \| & & \\ \mathbb{K}(\mathbb{A}^1_A) & \xrightarrow{ev_0} & \mathbb{K}(A) \\ \downarrow{\scriptstyle ev_1} & & \\ \mathbb{K}(A) & & \end{array}$$

(wrong!) Using $\underbrace{\mathbb{A}^1\text{-invariance of } \mathbb{K}}_{\text{only for regular rings}}$, we get $\varepsilon_{x,\omega_0} \simeq \varepsilon_{x,\omega_1}$.

<u>Problem</u>: $\mathbb{A}^1$-inv. doesn't hold in the required generality.

Solution: We use $\mathbb{K}(\mathbb{P}^1_A) \simeq \mathbb{K}(A) \oplus \mathbb{K}(A)$ (Thomason / Quillen)

Furthermore, we have a Cartesian diagram

$$\begin{tikzcd} \mathbb{K}(A) \arrow[r, "i_*"] \arrow[d] & \mathbb{K}(\mathbb{P}_A) \arrow[d, "z^*"] \\ 0 \arrow[r] & \mathbb{K}(A) \end{tikzcd}$$

$i : \mathrm{Spec}\, A \to \mathbb{P}^1_A$, $z : \mathrm{Spec}\, A \to \mathbb{P}^1_A$

s.t. $z(\mathrm{Spec}\, A) \cap i(\mathrm{Spec}\, A) = \phi$

Define. $\mathbb{K}_\infty(\mathbb{P}^1_A) = \mathrm{cofib}\left(\mathbb{K}(A) \xrightarrow{i_*} \mathbb{K}(\mathbb{P}^1_A)\right)$

The result above amounts to $\mathbb{K}_\infty(\mathbb{P}^1_A) \simeq \mathbb{K}(A)$

This is our replacement for $\mathbb{A}^1$- invariance of K-theory in the construction

of the ε-conn.

$$\begin{tikzcd} & & \mathbb{K}(A) \arrow[d, phantom, "\wr"] \\ \mathbb{K}(VB^{\nabla}_{/U}) \arrow[r, "\varepsilon_{x,w_t}"] \arrow[urr, bend left, "\varepsilon_{x,w_0}"] \arrow[drr, bend right, "\varepsilon_{x,w_1}"'] & \mathbb{K}_\infty(\mathbb{P}^1_A) \arrow[ur, "ev_0", "\sim"'] \arrow[dr, "ev_1"', "\sim"] \arrow[r, phantom, "\simeq"] & \mathbb{K}(A) \arrow[d, phantom, "\wr"] \\ & & \mathbb{K}(A) \end{tikzcd}$$

Rmk 1. MG expects the ε-conn. constr. above to agree w/ BBE's ε- connection.

Rmk 2 There is now a candidate def'n of blissful families of flat connections in arb. dimensions. This is based on recent work by Esnault - Sabbah

<u>Thm</u> (ES) Good lattices also exist in higher dimensions:

$$\mathcal{E} = (E, \nabla)/U, \quad U \subset X$$

$$E_0 \xrightarrow{\nabla} E_1 \otimes \Omega^1_{\log} \xrightarrow{\nabla} E_2 \otimes \Omega^2_{\log} \to \cdots \to E_n \otimes \Omega^n_{\log}$$

<u>Expectation</u>: Blissful families $\longleftrightarrow$ families of flat connections where a family of good lattices exists.