Modular representations of affine Lie algebras

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Basic summary

- in char 0, highest weight reps of affine Lie algebras are fairly well understood

- for some particularly tricky representations, ideas from geom. Langlands play a role in computing their chars. (Feigin-Frenkel-Gaitsgory) * critical level

- What about char p?

- we'll state a series of results (in progress) and conjectures, but the basic fun new feature is that phenomena seen by FFG at critical level now appear at all levels.

0. affine Lie algebras and category $\mathcal{O}$.

G split reductive gp / field k

$F = k((t))$, G_F loop gp, $\mathfrak{g}_F = \mathfrak{g} \otimes_k k((t))$ loop Lie algebra.

$H \curvearrowright \mathbb{P}(V)$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d] & \widetilde{H} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & GL(V) \arrow[r] & PGL(V) \arrow[r] & 1 \end{tikzcd}$$

for G_F, $\mathfrak{g}_F$, most representations require actual non-trivial central ext'ns.

$$\{ 0 \to k \cdot \mathbb{1} \to \hat{\mathfrak{g}} \to \mathfrak{g}_F \to 0 \} / \text{iso}$$

centr ext'n

$\wr\!\!\!\!\!\wr$

$$(\mathrm{Sym}^2 \mathfrak{g}^*)^G$$

example. G simple, then $(\mathrm{Sym}^2 \mathfrak{g}^*)^G \simeq k \cdot$ Killing form.

Def $\hat{\mathfrak{g}}_\kappa$ is the central ext'n corresponding to κ, and

$$\hat{\mathfrak{g}}_\kappa\text{-mod} = \left\{ \begin{array}{l} \text{(smooth) rep'n} \\ \text{on which } \mathbb{1} \text{ acts by identity} \end{array} \right\}$$

Category $\mathcal{O}$:

$$\begin{tikzcd} \mathring{I} \arrow[r] \arrow[d] & I \arrow[r, "\text{Iwahori}"] \arrow[d] & G[[t]] \arrow[d, "t=0"] \\ N \arrow[r] & B \arrow[r, "\text{Borel}"'] & G \end{tikzcd}$$

def $\tilde{\mathcal{O}}_\kappa := (\hat{\mathfrak{g}}_\kappa, \mathring{I})\text{-mod}$

i.e. modules for $\hat{\mathfrak{g}}_\kappa$ for which the action of $\mathrm{Lie}(\mathring{I})$ is integrated to $\mathring{I}$.

examples

$$\begin{tikzcd} & I \arrow[dl] \arrow[dr, hook] & \\ T \simeq I/I^\circ & & G((t)) \end{tikzcd}$$

Cartan of G

any module M for $\mathrm{Lie}(T)$,

$$\mathrm{pind}(M) = \mathrm{ind}^{(\hat{\mathfrak{g}}_\kappa, \mathring{I})}_{(\mathrm{Lie}(I), \mathring{I})} \ \mathrm{res}^{(\mathrm{Lie}(I), \mathring{I})}_{\mathrm{Lie}(T)} M$$

induction at the level of Lie alg rep

$\mathfrak{t} = \mathrm{Lie}(T)$, $\lambda \in \mathfrak{t}^*$, k_λ 1-dim $\mathfrak{t}$-module $\rightsquigarrow$ $M_\lambda = \mathrm{pind}(k_\lambda)$

How do you write down elements in M_λ?

example of sl_2:

$f^i \cdot e\otimes t^{-1}|\lambda\rangle \quad f^i \cdot h\otimes t^{-1}|\lambda\rangle \quad f^i \cdot f\otimes t^{-1}|\lambda\rangle$

$\cdots \quad f^2|\lambda\rangle \quad f|\lambda\rangle \quad \cdots \quad |\lambda\rangle$

h.w. state

to discuss characters, i.e. $M_\lambda \simeq \bigoplus$ f.d. weight spaces

you should tackle:

① $\mathfrak{t}$ grading (T)

② # powers of t^{-1} $(\mathbb{G}_m)$

loop rotation $\mathbb{G}_m$

in char p, $\mathfrak{t}$ thinks

$|\lambda\rangle$ & $t^p|\lambda\rangle$ have the same weight

So use full T, not just $\mathrm{Lie}(T) = \mathfrak{t}$.

if λ lies in $\mathfrak{t}^* \setminus X^*$, really $\int$ not action of $\mathfrak{t}$ but a shift of it.

<u>def'n</u> $\mathcal{O}_\kappa = \left(\hat{\mathfrak{g}}_\kappa, (I, \chi)\right)\text{-mod}^{\mathbb{G}_m^{\text{rot}}}$, weak

$\chi \in \mathfrak{t}^* \setminus X^*$

have Verma modules M_λ in $\mathcal{O}_\kappa$ and unique simple quotients L_λ

<u>remark</u>. Concretely, integrating the $\mathring{I}$ action gives divided powers of $\mathrm{Lie}(\mathring{I})$ $\left(e, \frac{e^p}{p!}, \frac{e^{p^2}}{p^2!}, \ldots\right)$

<u>theorem</u> (Kac - Kazhdan) G simple, char $k = 0$, $\kappa \neq \kappa_c = -\frac{1}{2}\kappa_{\text{Killing}}$

Then generic $\lambda \in \mathfrak{t}^*$, M_λ irreducible.

$$\text{so } \mathrm{ch}\, L_\lambda = \mathrm{ch}\, M_\lambda = \frac{e^\lambda}{\prod_{\alpha \in \Phi^{\text{pos}}_{\text{aff}}} (1 - e^{-\alpha})}$$

$\Phi^{\text{pos}}_{\text{aff}}$ ← multi-set

$(1 - e^{-\alpha})$ ⟵ $\prod_{\substack{\alpha_f \in \Phi^-_{\text{fin}} \\ n \geq 0}} (1 - q^n e^{\alpha_f})$ ← q energy grading

$\cdot \prod_{\substack{\alpha_f \in \Phi^+_{\text{fin}} \\ n > 0}} (1 - q^n e^{\alpha_f})$

$\cdot \prod_{n > 0} (1 - q^n)^{\dim \mathfrak{t}}$ ← character of $\mathrm{Sym}\left(\mathfrak{t}((t)) / \mathfrak{t}[[t]]\right)$

<u>remark</u>. similar statement G in char. 0 and char. p.

⚠ false for $\hat{\mathfrak{g}}$ in char. p.

<u>Conjecture</u> (Kac - Kazhdan thm of Hayashi, Rocha-Caridi-Wallach, Feigin - Frenkel ...) For generic λ at $\kappa = \kappa_c$,

$$\mathrm{ch}\, L_\lambda = \frac{e^\lambda}{\prod_{\Phi_{\text{fin}}} (-) \prod_{\Phi^+_{\text{fin}}} (-)}$$ ← 3rd term is gone

Thm (Feigin-Frenkel) char. 0 $\quad Z(U(\hat{\mathfrak{g}}_{k_c})) \simeq \mathrm{Fun}(Op_{\check{G}}^{(D^\times)})$

$(k \neq k_c,\ Z \simeq k)$

Thm (in progress) (D.-Losev)

(k, λ)-generic, then

$$\mathrm{ch}\, L_\lambda = \frac{e^\lambda}{\prod_{\Phi^+_{fin}} (-) \prod_{\Phi^-_{fin}} (-) \prod_{\substack{n \geq 0 \\ (n,p)=1}} (1-q^n)^{\dim t}}$$

eat every pth element in t_F / t_O.

Thm (D.-Losev) (*) fine part about completions, k generic.

$$U(\hat{\mathfrak{g}}_k)^{G((t))} \simeq \mathrm{Fun}(Op_{\check{G}}(D^{\times(1)}))$$

Remarks

for G:

$$\begin{tikzcd} & U(\mathfrak{g}_{\mathbb{Z}})^G \arrow[dl, "-\otimes_{\mathbb{Z}} \mathbb{F}_p"'] \arrow[dr, "-\otimes_{\mathbb{Z}} \mathbb{C}"] & \\ U(\mathfrak{g}_{\mathbb{F}_p})^G & & U(\mathfrak{g}_{\mathbb{C}})^G \end{tikzcd}$$

Not the case for $\hat{\mathfrak{g}}$.

Conjecture

k level for G

$\updownarrow$

$k^\vee$ dual level for $\check{G}$

$$\hat{\mathfrak{g}}_k\text{-mod}^{G_O} \simeq D\text{-mod}_{\check{k}}(\check{N}_{F,\psi} \backslash \check{G}_F / \check{G}_O) \overset{RHS}{\approx} T^*(\check{N}^{(1)}_{F,\psi} \backslash \check{G}_F / \check{G}_O)$$

twisted cotangent bundle

$$\downarrow$$

$$Op_{\check{G},\, \check{k}^p - \check{k}}(D^{(1)})$$

Conj. k, k' are non-critical integral levels.

$\hat{\mathfrak{g}}_k\text{-mod}^{G_O} \simeq \hat{\mathfrak{g}}_{k'}\text{-mod}^{G_O}$