Fargues' categorical conjecture for elliptic parameters for $SL(n)$

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$F/\mathbb{Q}_p$ p-adic field, G/F reductive, $\ell \neq p$

local Langlands

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} G(F) \xrightarrow{\sim} \{\text{"enhanced" } L\text{-parameters}\}$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel$$

$$\varphi : W_F \longrightarrow \check{G}(\overline{\mathbb{Q}}_\ell) + \text{extra data}$$

Many constructions:

for GL_n, SL_n, classical groups, classical constructions

Fargues - Scholze: produce a map $\longrightarrow$.

Question: When do the classical constructions & Fargues - Scholze's fancy construction match?

Known for: GL_n (Fargues - Scholze)

GSp_4 (Hamann)

U_{2n+1}? (Peng)

⋮

[S., Hur - S.]

Thm. Fargues - Scholze construction and Gelbart - Knapp's construction match for $G = SL_n$ for "elliptic parameters"

§1. Classical picture

[Henniart, Harris - Taylor, Scholze] give a ~~bijection~~ surjection:

$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} GL_n(F) \longrightarrow \{n\text{-dim'l reps of } W_F\}/\mathrm{iso}$, bijection over $\underbrace{\text{irred. rep. of } W_F}_{\text{"elliptic"}}$

More generally, G/F split reductive.

naively LLC: $\operatorname{Irr} G(F) \longrightarrow \{W_F \longrightarrow \check{G}(\overline{\mathbb{Q}_\ell})\}/\text{conj}$. (semisimple)

(when $G = GL_n$, $\check{G} = GL_n$)

bijection over "elliptic parameters"

<u>Example</u>. When $G = SL_n$, LLC:

$$\operatorname{Irr} SL_n(F) \longrightarrow \{W_F \longrightarrow PGL_n(\overline{\mathbb{Q}_\ell})\}/\text{conj}.$$

π — elliptic = irred. as projective rep'n.

find Π of $GL_n(F)$ s.t. $\pi \overset{\oplus}{\hookrightarrow} \Pi|_{SL_n(F)}$

expected compatibility of LLC w/ isogenies for $SL_n \hookrightarrow GL_n$:

$$\begin{tikzcd} \Pi \in \operatorname{Irr} GL_n(F) \arrow[r, squiggly] & \varphi_\Pi : W_F \arrow[r] \arrow[dr, "\overline{\varphi_\Pi}"'] & GL_n \arrow[d] \\ & & PGL_n \end{tikzcd}$$

$$\varphi_\pi = \overline{\varphi_\Pi}$$

this will <u>NOT</u> be a bijection.

$$\Pi|_{SL_n(F)} = \underbrace{\pi_1 \oplus \cdots \oplus \pi_r}_{\text{distinct}} \rightsquigarrow \varphi_{\pi_1} = \cdots = \varphi_{\pi_r} \quad \text{not a bijection}$$

~~$GL_n(F)$~~: $GL_n(F) \curvearrowright SL_n(F)$

conjugation (outer automorphisms)

$$\begin{pmatrix} p & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{\det = p}$$

outer automorphism $GL_n(F)/F^\times \cdot \overbrace{SL_n(F)}^{\text{inner}} \curvearrowright \{\pi_1, \ldots, \pi_r\}$

$$GL_n(F) / F^\times \cdot SL_n(F) \xrightarrow[\sim]{\det} F^\times/(F^\times)^n$$

$$F^\times/(F^\times)^n \underset{\text{transitive}}{\curvearrowright} \{\pi_1, \dots, \pi_2\}$$

$$\underbrace{\left(F^\times/(F^\times)^n\right) / \mathrm{Stab}(\pi_1)}_{} \xrightarrow{\sim} \underbrace{\{\pi_1, \dots, \pi_2\}}_{\text{fiber of the LLC}}$$

is there a L-parameter description?

In terms of L-parameters:

$S_\varphi = Z_{PGL_n}(\bar{\varphi})$ ← analogous to S_σ in Sam's talk.

$= \{ g \in PGL_n : g \bar{\varphi} g^{-1} = \bar{\varphi} \}$

$= \{ g \in PGL_n : g \varphi(w) \cdot g^{-1} = \gamma_g(w) \cdot \varphi(w)$ for $\gamma_g : W_F \to \mu_n(\bar{\mathbb{Q}}_\ell) \}$

$$\begin{tikzcd} W_F \arrow[d] \arrow[r] & \mu_n(\bar{\mathbb{Q}}_\ell) \\ F^\times \arrow[ur] \arrow[r] & F^\times/(F^\times)^n \arrow[u] \end{tikzcd}$$

$$F^\times/(F^\times)^n \twoheadrightarrow \mathrm{Hom}(S_\varphi, \bar{\mathbb{Q}}_\ell)$$

$$t \longmapsto [g \mapsto \gamma_g(t)]$$

kernel is exactly $\mathrm{Stab}(\pi_1)$ [Gelbart-Knapp]

⇝ if you keep track of $W_F \to PGL_n(\bar{\mathbb{Q}}_\ell)$ + $\chi \in \mathrm{Hom}(S_\varphi, \bar{\mathbb{Q}}_\ell^\times) \simeq \underline{\mathrm{Irr}(S_\varphi)}$

"enhancement" — appeared in Sam's talk

Thm [GK] surjection: $\mathrm{Irr}\, SL_n(F) \to \left\{ \begin{array}{l} W_F \xrightarrow{\varphi} PGL_n(\bar{\mathbb{Q}}_\ell) \leftarrow \text{irreducible} \\ + \chi \in \mathrm{Irr}(S_\varphi) \end{array} \right\} / \mathrm{conj}.$

bijection over $\underline{\text{elliptic parameters}}$

More generally, LLC should state: Surjection

$$\mathrm{Irr}\, G(F) \longrightarrow \{ \underset{\text{semisimple}}{W_F \to \check{G}(\overline{\mathbb{Q}}_\ell)} \} / \mathrm{conj}.$$

whose fiber over φ elliptic is bijection w/ $\mathrm{Irr}(\underline{S_\varphi})$
↑
Some finite gp.

§2. Categorical picture

Fargues-Scholze constructed an action: "Geometric Langlands" for X_{FF}.

$$\underbrace{\mathrm{Perf}(\mathrm{Par}_{\check{G}})}_{\text{stack of L-parameters}} \curvearrowright \underbrace{D_{lis}(\mathrm{Bun}_G)}_{\text{one part is isom. to } \mathrm{Rep}\, G(F)}$$

define a map $\mathrm{Irr}(G(F)) \longrightarrow \{\varphi : W_F \to \check{G}\}/\mathrm{conj}.$

predict: $D_{coh}(\mathrm{Par}_{\check{G}}) \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G)$ (on elliptic locus)

$$M \longmapsto M * \underbrace{W_\varphi}_{\text{Whittaker sheaf}}$$

S_φ automorphism gp, $*/S_\varphi \hookrightarrow \mathrm{Par}_{\check{G}}$

Conj. (Fargues) G split, φ elliptic parameter, $\exists!$ generic s.c. rep π of $G(F)$ s.t. $\varphi_\pi = \varphi$, and the functor

$$\mathrm{Perf}(*/S_\varphi) \xrightarrow{\sim} D_{lis}^{L_\varphi}(\mathrm{Bun}_G) \quad \leftarrow \text{L-par } \varphi$$

$$W \longmapsto W * \pi$$

t-exact equiv.

Thm. Yes for SL_n.

Why can't you naively do this?

Eventually, use geometric Satake.

$V \in Rep(\check{G}) \rightsquigarrow T_V$ Hecke operator

$G = GL_n$, $\check{G} = GL_n$, $std \in Rep(GL_n)$

$G = SL_n$, $\check{G} = PGL_n \rightsquigarrow ad \in Rep(PGL_n)$

<u>Categorical way</u>.

On spectral side, $Perf(Par_{PGL_n}) \curvearrowright D_{lis}(Bun_{SL_n})$

$$\{\varphi : W_F \longrightarrow PGL_n\} / {}^{ad} PGL_n.$$

$$\uparrow \mu_n\text{-torsor}$$

$$\{\varphi : W_F \to PGL_n + \widetilde{\varphi(w)} \in SL_n\} / PGL_n, \ w \in W_F$$

$\rightsquigarrow I_w \overset{\text{line bundle}}{\in} Perf(Par_{PGL_n})$ only depends on $[w] \in F^\times/(F^\times)^n$

$I_w * - : D_{lis}(Bun_{SL_n}) \to D_{lis}(Bun_{SL_n})$

<u>Suffices to show</u>: $D(Bun_{SL_n}) \approx Rep\, SL_n(F)$

$$\circlearrowleft F^\times/(F^\times)^n$$

$\mu_n \curvearrowright SL_n$, $Bun_{\mu_n} \curvearrowright Bun\, SL_n$ $\qquad (L, \psi).(V, \varphi) = (L \otimes V. \psi \otimes \varphi)$

$\{L \text{ l.b.} : L^{\otimes n} \overset{\psi}{\simeq} \mathcal{O}\}$ $= Bun_{\mu_n}$, $\{V \text{ v.b.} : \det(V) \overset{\varphi}{\simeq} \mathcal{O}\} = Bun\, SL_n$

$(\mathcal{O}, t)$ $t: \mathcal{O}^{\otimes n} \xrightarrow{\sim} \mathcal{O}$

<u>Claim</u>. action of $(\mathcal{O}, t) \in \mathrm{Bun}_{\mu_n}$ on $D(\mathrm{Bun}_{SL_n})$ matches outer automorphism action.