Étale cohomology of v-stacks

Tony Feng

Overview: there is a well-behaved theory of étale sheaves on perfectoid spaces.

Want to extend it to a more general class of spaces.

$$\begin{array}{ccccc} \text{perfectoid spaces} & \hookrightarrow & \text{(loc. spatial) diamonds} & \hookrightarrow & \text{(Artin) v-stacks} \\ & & \therefore & & \\ \text{schemes} & \hookrightarrow & \text{algebraic spaces} & \hookrightarrow & \text{(Artin) stacks} \end{array}$$

1st step: diamonds = quot. of perf. $/\mathbb{F}_p$ by proétale equiv. rel'ns.

$$\cap$$

proétale sheaf $(\mathrm{Perf}/\mathbb{F}_p)$

$$Y = \mathrm{Coeq}(R \overset{\text{proétale}}{\rightrightarrows} X)$$

perfectoid

Rmk. While algebraic spaces are arguably esoteric, diamonds are ubiquitous.

Ex. $\mathrm{Spd}\, \mathbb{Q}_p$ is a diamond.

$$\begin{array}{ccccc} \mathbb{Q}_p^{cyc} & \overset{\flat}{\rightsquigarrow} & \mathbb{F}_p((t^{1/p^\infty})) & & \\ \mathbb{Z}_p^\times \ \big| & \zeta_{p-1} \longleftrightarrow t & \big\downarrow & \circlearrowleft \mathbb{Z}_p^\times & \gamma \cdot t = (1+t)^\gamma - 1 \\ \mathbb{Q}_p & & \mathrm{Spd}\, \mathbb{Q}_p & & \end{array}$$

$\underline{\text{Ex}}$. $\mathrm{Spd}\,\mathbb{Q}_p \times_{\mathbb{F}_p} \mathrm{Spd}\,\mathbb{Q}_p$

$\underline{\text{Ex}}$ There is a functor

$$\left\{\begin{array}{l}\text{analytic} \\ \text{adic } /\mathbb{Q}_p\end{array}\right\} \longrightarrow \text{diamonds}$$

$$X \longmapsto X^\diamond \qquad \text{generalizes tilting}$$

$$\begin{tikzcd}
\text{perf'd} \quad \tilde{X} \times_X \tilde{X} \arrow[r, squiggly] \arrow[d, shift left] \arrow[d, shift right] & (\tilde{X} \times_X \tilde{X})^\flat \arrow[d, shift left] \arrow[d, shift right] \\
\text{perf'd} \quad \tilde{X} \arrow[r, squiggly] \arrow[d, "\text{proét}"'] & \tilde{X}^\flat \arrow[d, dashed] \\
X & X^\diamond
\end{tikzcd}$$

$\Rightarrow \mathbb{A}^1_{\mathbb{Q}_p}, \; G^\diamond_{\mathbb{Q}_p}$, etc.

$\underline{\text{Ex}}$. Artin v-stacks will include $pt/\underline{G(\mathbb{Q}_p)}$, Bun_G.

Let Λ = fin. coeff. ring w/ $p \in \Lambda^\times$

$\underline{\text{Goal}}$: define $D_{\acute{e}t}(X, \Lambda)$ together w/ attendant 6-functor formalism.

$\underline{\text{Ex}}$. X perfectoid $\Rightarrow D_{\acute{e}t}(X, \Lambda) \overset{\text{Want}}{=}$ usual*.

$\underline{\text{Ex}}$. G = alg. gp $/\mathbb{Q}_p$, $\quad D_{\acute{e}t}\left([pt/\underline{G(\mathbb{Q}_p)}], \Lambda\right) \overset{\text{Want}}{=} D(\mathrm{Rep}^{sm}_\Lambda \, G(\mathbb{Q}_p))$

$\underline{\text{Ex}}$. G reductive $/\mathbb{Q}_p$, Want $D_{\acute{e}t}(\mathrm{Bun}_G)$ to have semiortho. decomp. into

$D(\mathrm{Rep}^{sm}_\Lambda(G(\mathbb{Q}_p)))$

Diamonds

Proétale topology $f : X \to Y$ map of perf'd spaces.

Def (i) f is <u>finite étale</u> $\forall$ opens $\mathrm{Spa}(S, S^+) \subset Y$,

$$f^{-1}(V) = \mathrm{Spa}(R, R^+), \quad R \text{ fin. étale}/S, \quad R^+ = (S^+)^{\mathrm{int}}.$$

(ii) f is <u>étale</u> if locally on X, Y,

$$\begin{tikzcd} X \arrow[r, hook, "\text{open}"] & X' \arrow[d, "\text{fin étale}"] \\ & Y' \arrow[r, hook, "\text{open}"] & Y \end{tikzcd}$$

(iii) f is <u>affinoid proétale</u> if

$$\begin{tikzcd} X = \mathrm{Spa}(R, R^+) = \varprojlim \underbrace{\mathrm{Spa}(R_i, R_i^+)}_{X_i} \arrow[d, "f"] \\ Y = \mathrm{Spa}(S, S^+) \end{tikzcd}$$

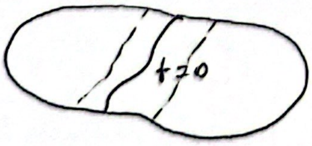
w/ $f_i : X_i \to Y$ étale

+ f is proétale if it is affinoid proétale locally on X, Y.

Rmk. There are many more proétale maps than one might guess.

Ex. Any Zariski closed immersion is affinoid proétale.

$X = V(f) \hookrightarrow Y = \mathrm{Spa}(R, R^+)$



$X_n = \{ |f| \le 1/p^n \} \overset{\text{open}}{\subset} Y$

$\Rightarrow \varprojlim_n X_n = X$

$\{f = 0\} \cup \{f \neq 0\} \twoheadrightarrow Y$ not a proétale cover b/c covers have quasicompactness condition.

Ex. $\mathrm{Spa}\, C\langle T^{1/p^\infty}\rangle \to \mathrm{Spa}\, C$ isn't proétale.

Ex. $\mathrm{Spa}\, C\langle T^{1/2p^\infty}\rangle \xrightarrow{\text{not proétale}} \mathrm{Spa}\, C\langle T^{1/p^\infty}\rangle$ (quasi pro-étale)

Recall. $\mathrm{Perf} \xhookrightarrow{\text{Yoneda}} \left\{\begin{matrix}\text{proétale} \\ \text{sheaves}\end{matrix}\right\}$

Def. A diamond is a proétale sheaf on Perf of the form $Y = \mathrm{Coeq}(R \overset{s}{\underset{t}{\rightrightarrows}} X)$

$R, X \in \mathrm{Perf}$, s, t proétale.

- spatial = qcqs + $|Y|$ $(= |X|/|R|)$ has a basis coming from open subdiamonds.
- locally spatial

Recall: v-topology. A map $f: X \to Y$ is a v-cover if $\forall$ qc open $V \subset Y$, $\exists$ qc open $U \subset X$ s.t. $f(U) = V$.

perfectoid : proétale : v

$\vdots$

scheme : étale : fpqc

Fact: Diamonds are automatically v-sheaves

(analogous to Gabber's result: alg. spaces are fpqc sheaves)

Diamonds $\hookrightarrow$ v-sheaves $\subset$ v-stacks.

Derived cat. of étale sheaves

X = (small) v-stacks

$\updownarrow$

admits surj. from perfectoid

Want: $D_{\text{ét}}(X, \Lambda)$ and 6 functors

- (f^*, f_*), • $\otimes$, RHom.
- $(f_!, f^!)$ under hyp. on f.

projection formula, base change, etc...

(1) $D_{ét}(X;\Lambda)$ agrees* w/ "known" $D_{ét}(X,\Lambda)$ if X perfectoid

(2) $D_{ét}(X;\Lambda)$ satisfies v-descent. i.e.

$X \to Y$ is a v-cover, then

$$(*) \quad D_{ét}(Y) = \varprojlim \left(D_{ét}(X) \rightrightarrows D_{ét}(X \times_Y X) \rightrightarrows \cdots \right)$$

<u>Philosophy</u>: (1), (2) <u>should</u> already determine $D_{ét}(-)$ since perfectoid X form a basis for v-topology.

Not literally true since $(*)$ not well-defined. (no limits of triangulated cats)

$\rightsquigarrow$ work with stable ∞-cat. $\mathcal{D}_{ét}(-)$

$$D_{ét} \xleftarrow{\text{homotopy cat.}} \mathcal{D}_{ét}$$

- obj. = complexes of injectives.
- mor = homotopy equiv. classes of chain maps.

- obj. = C_0
- 1-morphisms = $C_0 \to C_1$
- 2-morphisms

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C_1 \arrow[dl] \arrow[dl, phantom, shift left=2, "\Rightarrow"] \\ C_2 & \end{tikzcd}$$

- 3-morphisms

$\vdots$

<u>Technical implementation</u>.

Perf $\supset$ Strictly totally discon'd (qcqs and every etale cover splits)

<u>Ex</u>. Coproduct of points $\quad \mathrm{Spa}\left(\prod_i (C_i, C_i^+)\right)$ $\quad C_i$ = compl. alg. closed

Structure

$$\begin{tikzcd} \mathrm{Spa}(C,C^+) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{pt} \arrow[r, phantom, "\in"] & \pi_0(X) \end{tikzcd}$$

C = alg. closed field

profinite set

Slogan. "profinite union of geom. points".

$$|\mathrm{Spa}(C,C^+)| = |\mathrm{Spec}\, \underbrace{C^+/m_{\mathcal{O}_C}}_{\text{val'n ring}}|$$

$\bullet \rightsquigarrow \bullet \rightsquigarrow \bullet$.

Prop For any qc $X \in \mathrm{Perf}$, $\exists$ proetale cover $\underset{\text{str. tot. disc.}}{\tilde{X}} \xrightarrow{\text{proét}} X$

Will: Give a characterization of $\boxed{D_{\text{ét}}(X)}$ for str. tot. disc. X that manifestly descends in v-topology.

X (locally spatial) diamond $\Rightarrow$ étale site $X_{\text{ét}}$

$V \to X$ is étale if étale after base change to perfectoid space $\to X$.

$$X_v \to X_{\text{ét}}$$

$$\to \quad D(X_{\text{ét}}) \underset{\lambda_*}{\overset{\lambda^*}{\rightleftarrows}} D(X_v)$$

Prop. X locally spatial diamond, (str. tot. disc.) then $D^{(+)}(X_{\text{ét}}) \xrightarrow{\lambda^*} D^{(+)}(X_v)$ is fully faithful.

Prop 2 If $X' \to X$ is a $\underline{v\text{-cover}}$, and $\mathcal{F} \in D(X_v)$, then $\mathcal{F} \in D^+(X_{\acute{e}t})$

$\Leftrightarrow$ $\mathcal{F}|_{X'} \in D^+(X_{\acute{e}t})$

Def'n. $D_{\acute{e}t}(X) \subset \overset{D_v(X)}{\overset{/\!/}{D(X_v)}}$ spanned by $\mathcal{F}$ s.t. $\mathcal{F}|_{X'} \in D(X'_{\acute{e}t})$; $\forall$ str. tot. disc. $\begin{array}{c} X' \\ \downarrow \\ X \end{array}$.

This descends in v-topology

$\Rightarrow$ defined for v-stacks.

Proof sketch (Prop 1).

$$\begin{tikzcd} X_v \arrow[d] \arrow[dd, bend right, "\lambda"'] \\ X_{pro\acute{e}t} \arrow[d, "\nu"] \\ X_{\acute{e}t} \end{tikzcd}$$

First show $\nu^* : D(X_{\acute{e}t}) \to D(X_{pro\acute{e}t})$ fully faithful.

Want $\mathcal{F} \simeq R\nu_* \nu^* \mathcal{F}$.

Key points: (1) $\nu^{-1}\mathcal{F}(\varprojlim U_j) = \varinjlim_J \mathcal{F}(U_j)$

- true on presheaf pullback by def'n.
- presheaf pullback is already sheafy.

need exactness of Čech complexes

both by filtered colim from Čech cpxes for étale covers (already exact), but filtered colim is exact.

$\to H^i_{pro\acute{e}t}(\varprojlim_J U_j, \nu^{-1}\mathcal{F}) = \varinjlim_J H^i_{\acute{e}t}(U_j, \mathcal{F})$

$R^i\nu_* \nu^* \mathcal{F}$ = étale sheafification of $U \mapsto H^i_{pro\acute{e}t}(U, \nu^{-1}\mathcal{F}) = H^i_{\acute{e}t}(U, \nu^*\mathcal{F})$

$\Rightarrow$ vanishes locally in étale top.

<u>Upshot</u>: want $\mathcal{F} \Rightarrow R\lambda_* \lambda^* \mathcal{F}$

have proétale descent. $\Rightarrow$ can base change to assume X str tot disc.

Want: $H^i(X_v, \lambda^{-1}\mathcal{F}) = 0$, $i > 0$.

$$\alpha \in H^i(X_v, \lambda^{-1}\mathcal{F}) \Rightarrow \exists \begin{array}{c} Y = \mathrm{Spa}(S, S^+) \\ \downarrow f \quad v\text{-cover} \\ X = \mathrm{Spa}(R, R^+) \end{array} \quad \text{s.t. } f^*\alpha = 0$$

Want $\alpha = 0$

$$S = R\langle T_i \rangle_{i \in I} / \langle (f_j) \rangle_{j \in J} \qquad I, J \text{ possibly infinite}$$

$$Y \hookrightarrow \{f_j = 0\}_{j \in J} \subset \mathbb{B}^I_X$$

$$\parallel$$

$$\varprojlim \underbrace{\left(\{|f_j| \le \varepsilon\}_{j \in J' \text{ finite} \subset J} \subset \mathbb{B}^{I'} \text{ finite.} \right)}_{Y_\delta}$$

"smooth": open in fin. dim. ball $/X$

$\mathcal{F}$ étale $\quad H^i(\varprojlim_\delta Y_\delta, \mathcal{F}) = \varinjlim_\delta H^i(Y_\delta, \mathcal{F})$

$\alpha \mapsto 0$

$\Rightarrow \alpha \mapsto 0$ in some $H^i(Y_\delta, \mathcal{F})$

<u>Punchline</u>:
$$\begin{tikzcd} & Y_\delta \arrow[d, "\text{``smooth''}"] \\ X' \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$
$\Rightarrow \exists$ section after étale cover of X.

$\Rightarrow \alpha = 0$ in $H^i(X_v, \nu^{-1}\mathcal{F})$

Ex. $D(pt;\Lambda) = D(\Lambda)$

$\overset{\shortparallel}{\text{“}\mathrm{Spd}\,\overline{\mathbb{F}}_p\text{”}}$

not tautology, b/c pt far from being a diamond.

Need to compute $D(pt;\Lambda)$ by v-descent.

Let C = complete alg. closed.

$$\mathrm{Spa}\, C \longrightarrow pt$$

$$D_{\acute{e}t}(\mathrm{Spa}\, C) = D((\mathrm{Spa}\, C)_{\acute{e}t}) = D(\Lambda)$$

$$\begin{tikzcd} D(pt,\Lambda) \arrow[r, hook] & D_{\acute{e}t}(\mathrm{Spa}\, C,\Lambda) \arrow[d, phantom, "\wr\!\!\shortparallel"] \\ & D(\Lambda) \arrow[ul] \end{tikzcd}$$

Prop. For any small v-stack $X/\overline{\mathbb{F}}_p$, any complete alg. closed C,

$$D_{\acute{e}t}(X) \hookrightarrow D_{\acute{e}t}(X \times_{\overline{\mathbb{F}}_p} \mathrm{Spa}\, C) \quad \text{fully faithful.}$$

Ex. Want $D_{\acute{e}t}([pt/\underline{G(\mathbb{Q}_p)}]) = D(\mathrm{Rep}^{sm}\, G(\mathbb{Q}_p))$

$pt/\underline{G}$

locally pro-p $\qquad G$

ad hoc def'n of $X_{\acute{e}t}$ for v-stack X

$X = pt/\underline{G}$, get $X_{\acute{e}t} \longleftrightarrow \{G\text{-sets}\}$

Claim $D(X_{\bar{e}t}) \simeq D_{\bar{e}t}(X)$.

Similarly for $\underbrace{X \times \mathrm{Spa}\, C}_{X_C} = [\mathrm{Spa}\, C / \underline{G}]$

Want. $\mathcal{F} \Rightarrow R\lambda_* \lambda^* \mathcal{F}$, $\mathcal{F} \in D(X_{\bar{e}t})$

$\lambda: X_{C,v} \longrightarrow X_{C,\acute{e}t}$

This is an étale local statement $\Rightarrow$ assume $\underline{G}$ pro-p.

Key: $H^i_v([\mathrm{Spa}\, C / \underline{G}] ; \mathcal{F}) = \begin{cases} \mathcal{F}^G & i=0 \\ 0, & i>0 \end{cases}$

$$\mathrm{Spa}\, C \longrightarrow [\mathrm{Spa} / \underline{G}]$$

Its Čech complex coincides w/ continuous cohomology cochain complex.

$\rightarrow H^i_v([\mathrm{Spa}\, C / \underline{G}], \mathcal{F}) = H^i_{cts}(\underline{G}, \mathcal{F}|_{\mathrm{Spa}\, C})$