Singular support of coherent sheaves

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Lecture 1

A sort of introduction. (w/ D. Arinkin) k alg. closed, char $k = 0$

X curve $/k$, G reductive gp $/k$

$$\mathrm{Dmod}(\mathrm{Bun}_G) \underset{?}{\overset{\mathbb{L}_G}{\simeq}} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

$$G \hookleftarrow P \longrightarrow M \qquad \check{G} \hookleftarrow \check{P} \longrightarrow \check{M}$$

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_M \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{LocSys}_{\check{P}} \arrow[dl, "p_{spec}"'] \arrow[dr, "q_{spec}"] & \\ \mathrm{LocSys}_{\check{G}} & & \mathrm{LocSys}_{\check{M}} \end{tikzcd}$$

$$\mathrm{Eis} = p_! \circ q^* \qquad \mathrm{Eis}_{spec} = p_{spec,*} \circ q^!_{spec}$$

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G", "\simeq"'] & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \\ \mathrm{Dmod}(\mathrm{Bun}_M) \arrow[u, "\mathrm{Eis}"] \arrow[r, "\mathbb{L}_M", "\simeq"'] & \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \arrow[u, "\mathrm{Eis}_{spec}"'] \end{tikzcd}$$

IMPOSSIBLE!

Def. A triangulated cat. $\mathcal{C}$ is cocomplete if it contains all direct sums.

Def $c \in C$ is compact if $\mathrm{Hom}(c, -)$ commutes w/ direct sums.

$A\text{-mod}^{\heartsuit}$

Q: When is an object $M \in A\text{-mod}^{\heartsuit}$ s.t. $\mathrm{Hom}(M, -)$ commutes w/ direct sums ?

A: Iff M is finitely genrated.

$A\text{-mod}$

$M \in A\text{-mod}$

Q: When is M compact ?

A: It's perfect := equiv. to a finite cpx of f.g. projective modules.

A is a f.g. $\overset{\text{comm.}}{}$ k-alg.

$S = \mathrm{Spec}\, A$. $\quad QCoh(S) := A\text{-mod}$

Lemma, $\mathcal{F} \in QCoh(S)$ is compact iff $\mathcal{F}$ satisfies

- $\mathcal{F} \in Coh(S)$
- $\forall s \in S$, $\quad k_s \overset{L}{\underset{\mathcal{O}_S}{\otimes}} \mathcal{F}$ has finitely many cohomologies.

$Perf(S) \subset Coh(S)$

The inclusion is an equality iff S is smooth.

$$\mathcal{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_2 \quad (F \dashv G)$$

Terminology: a functor is continuous if it takes direct sums to direct sums.

Lemma (a) A left adjoint is always continuous.

(b) If G is continuous, then F takes compacts to compacts.

Eis is the left adjoint to $CT = q_* \circ p^!$

Hence Eis takes compacts to compacts.

The functor $\mathrm{Eis}_{\mathrm{spec}}$

- Does send $\mathrm{Coh}(\mathrm{LocSys}_{\check{M}})$ to $\mathrm{Coh}(\mathrm{LocSys}_{\check{G}})$
- Doesn't send $\mathrm{Perf}(\mathrm{LocSys}_{\check{M}})$ to $\underline{\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})}$

$A \rightsquigarrow \mathrm{Spec}(A)$ $\qquad f \in A_{red} \rightsquigarrow A_f$

$A \in \mathrm{CDGA}$ $\leftarrow$ non-positively graded

$A \in \mathrm{CDGA} \rightsquigarrow \mathrm{Spec}(A)$

The topological space of $\mathrm{Spec}\, A = \mathrm{Spec}\, H^0(A)$ $\qquad f \in (H^0(A))_{red} \rightsquigarrow A_f$

$\mathrm{Hom}(S_1, S_2)$

$\mathrm{Maps}(S_1, S_2)$ s.t. $\mathrm{Hom}(S_1, S_2) = \pi_0(\mathrm{Maps}(S_1, S_2))$

$$\begin{tikzcd} S_1 \arrow[dr] & & S_2 \arrow[dl] \\ & S_3 & \end{tikzcd}$$

$\mathrm{Hom}(T, S_1 \underset{S_3}{\times} S_2) = \mathrm{Hom}(T, S_1) \underset{\mathrm{Hom}(T, S_3)}{\times} \mathrm{Hom}(T, S_2)$

$S_1 \underset{S_3}{\times} S_2 = \mathrm{Spec}\left(A_1 \overset{L}{\underset{A_3}{\otimes}} A_2\right)$

$\mathrm{Maps}(T, S_1 \underset{S_3}{\times} S_2) = \mathrm{Maps}(T, S_1) \underset{\mathrm{Maps}(T, S_3)}{\times} \mathrm{Maps}(T, S_2)$

$$\underset{\mathrm{Spec}\, A}{\overset{\shortparallel}{S}} \rightsquigarrow \underset{\mathrm{Spec}\, H^0(A)}{\overset{\shortparallel}{{}^{cl}S}}$$

$S = \mathrm{Spec}\, A$ is almost of finite type if

- $H^0(A)$ is of finite type
- $H^i(A)$'s are f.g. as modules over $H^0(A)$.

$TS \in \mathrm{QCoh}(S)$

$s \in S \rightsquigarrow T_s S \in \mathrm{Vect}^{\geq 0}$

$$H^0(T_s S) = \mathrm{Hom}(\mathrm{Spec}(k \oplus \varepsilon k), S)$$

$$H^i(T_s S) = \mathrm{Hom}(\mathrm{Spec}(k \oplus \varepsilon k), S)$$

$\deg(\varepsilon) = -i$

<u>Ex.</u> Recover $T_s S$.

$\mathrm{Perf}(S) \subset \mathrm{Coh}(S)$

<u>Lemma.</u> S is a smooth classical scheme, iff $H^i(T_s S) = 0$, $\forall s \in S$, $\forall i \geq 1$.

<u>Def.</u> S is quasi-smooth if $H^i(T_s S) = 0$, $\forall s \in S$, $\forall \underline{i \geq 2}$.

This is tautologically equiv. to $\underset{\text{locally}}{T^*S} \simeq \underset{\text{free}}{E_{-1}} \to \underset{\text{free}}{E_0}$

<u>Lemma</u> S is quasi-smooth if locally it can be written as a fiber product

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^n \arrow[d] \\ pt \arrow[r] & \mathbb{A}^m \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Y_1 \arrow[d, "f"] \\ pt \arrow[r] & Y_2 \end{tikzcd}$$

$$TS = \text{fiber}\left(TY_1|_S \to TY_2|_S\right)$$

$$TS = \text{fiber}\left(\mathcal{O}_S^{\oplus n} \to \mathcal{O}_S^{\oplus m}\right)$$

$$0 \qquad\qquad 1$$

<u>NB</u>: classical l.c.i $\iff$ quasi-smooth + classical

$T_s S[-1]$

$$pt \times_S pt = \text{Inertia}(S; s)$$

$$\text{Lie}(\text{Inertia}(S, s)) = T_s S[-1]$$

$$R\text{Hom}_{\mathcal{O}_S}(k_s, k_s) = U(\text{Lie}(\text{Inertia}(S, s)))$$

S - almost of finite type

$\text{Coh}(S) = \{ \mathcal{F} \in \text{QCoh}(S) : \mathcal{H}^i(\mathcal{F})$ are f.g. over $H^0(S)$, and zero for all but finitely many $i \}$

$\mathcal{F} \in \text{Coh}(S)$, $s \in S$: $\quad R\text{Hom}(k_s, k_s) \curvearrowright R\text{Hom}(k_s, \mathcal{F})$

The DGLA $T_s S[-1]$ acts on $R\text{Hom}(k_s; \mathcal{F})$

$\bigoplus_i H^i(T_s S[-1])$ acts on $\bigoplus_i \text{Ext}^i(k_s; \mathcal{F})$.

Assume S is quasi-smooth

$H^1(T_s S)$ acts on $\bigoplus_i \operatorname{Ext}^i(k_s, \mathcal{F})$.
(placed in deg 2)

$$\operatorname{Sym}(H^1(T_s S)) \curvearrowright \bigoplus_i \operatorname{Ext}^i(k_s, \mathcal{F})$$

$\operatorname{Sing\,Supp}_s(\mathcal{F})$ = the support $\subset (H^1(T_s S))^*$
of $\bigoplus_i \operatorname{Ext}^i(k_s; \mathcal{F})$
as a module over $\operatorname{Sym}(H^1(T_s S))$

$$\operatorname{Sing}(S) = \bigcup_{s \in S} (H^1(T_s S))^*$$

$$\operatorname{Sing}(S) \simeq \operatorname{Spec}_{\mathcal{O}_S} \operatorname{Sym}_{\mathcal{O}_S}(H^0(TS[1]))$$

<u>Def</u>. $\mathcal{F} \in \operatorname{Coh}(S)$

$$\operatorname{Sing\,Supp}(\mathcal{F}) = \bigcup_{s \in S} \operatorname{Sing\,Supp}_s(\mathcal{F}) \subset \operatorname{Sing}(S)$$

<u>Prop</u>. $\operatorname{Sing\,Supp}(\mathcal{F})$ is Zariski-closed in $\operatorname{Sing}(S)$.

<u>Thm 1</u> $\operatorname{Sing\,supp}(\mathcal{F}) \subset \{0\}$
$\Updownarrow$
$\mathcal{F} \in \operatorname{Perf}(S)$.

$$S = pt \times_V pt = \operatorname{Spec} \operatorname{Sym}(V^*[1])$$

$$\operatorname{Sing}(S) = V^*$$

$$\operatorname{Coh}(S) \simeq \operatorname{Sym}(V[-2])\text{-mod}^{f.g.} \qquad (\text{Koszul duality})$$

$$\bigcup \quad \mathcal{F} \longmapsto R\operatorname{Hom}(k_S, \mathcal{F}) \quad \bigcup$$

$$\operatorname{Perf}(S) \simeq \operatorname{Sym}(V[-2])\text{-mod}^{f.g.}_{\{0\}}$$

Lecture 2

S/k is quasi-smooth if

$H^i(T_S S) = 0, \ \forall i \geq 2.$

$\mathcal{F} \in \operatorname{Coh}(S)$

$$\operatorname{SingSupp}_S(\mathcal{F}) = \operatorname{supp}_{\operatorname{Sym}(H^1(T_S))} \left(\bigoplus_i \operatorname{Ext}^i(k_S, \mathcal{F}) \right) \subset \left(H^1(T_S S)\right)^* = H^{-1}(T_S^* S)$$

$$\operatorname{Sym}(H^1(T_S)) \curvearrowright \bigoplus_i \operatorname{Ext}^i(k_S, \mathcal{F})$$

$$\operatorname{Sing}(S) = \operatorname{Spec}_{S_{cl}} \left(\operatorname{Sym}(H^1(T_S)) \right)$$

$$= \bigcup_S H^{-1}(T_S^* S)$$

$$\mathcal{F} \rightsquigarrow \operatorname{SingSupp}(\mathcal{F}) \subset \operatorname{Sing}(S)$$

<u>Prop</u> $\operatorname{SingSupp}(\mathcal{F})$ is Zariski closed.

<u>Thm</u> $\operatorname{SingSupp}(\mathcal{F}) \subset \{0\} \iff \mathcal{F} \in \operatorname{Perf}(S)$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U \arrow[d, "f"] \\ pt \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$$\mathrm{Sing}(S) = \left(T^*\mathbb{A}^1|_S \xrightarrow{df^*} T^*U|_S \right)$$

$$\| \qquad \mathcal{O}_S \longrightarrow T^*U_S$$

at $s \in S$

$$\begin{cases} 0 & \text{if } df_s \neq 0 \\ k & \text{if } df_s = 0 \end{cases}$$

$\mathrm{Sing\,Supp}_S(\mathcal{F})$ will be nonzero iff $\mathcal{F}$ is not perfect on a Zariski nbhd of s.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U \arrow[d, "h"] \\ pt \arrow[r] & \mathbb{V} \end{tikzcd}$$

$$\mathrm{Sing}(S) = \ker\left(T^*\mathbb{V}|_S \xrightarrow{dh^*} T^*U|_S\right)$$

$$V = T_{pt}\mathbb{V} \qquad \mathbb{V}^* \times S$$

$\xi \in V^*$, want to say when $(s, \xi) \in \mathrm{Sing\,Supp}(\mathcal{F})$.

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d] & S' \arrow[r] \arrow[d] & U \arrow[d, "h"] \\ pt \arrow[r] & \mathbb{V}' \arrow[r] \arrow[d] & \mathbb{V} \arrow[d, "f"] \\ & pt \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$$(df)_{pt} = \xi$$

<u>Thm.</u> $(s, \xi) \in \mathrm{SingSupp}(\mathcal{F}) \Leftrightarrow i_*\mathcal{F}$ is not perfect on S' on a Zariski nbhd of s.

Koszul definition

$$G_V = pt \underset{V}{\times} pt \quad , \quad Coh(G_V) \overset{KD}{\simeq} Sym(V[-2])\text{-mod}^{perf}$$

$$\begin{tikzcd}
 & pt \underset{V}{\times} pt \arrow[dl] \arrow[dr] & \\
pt \arrow[dr] & & pt \arrow[dl] \\
 & V &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & S \underset{U}{\times} S \arrow[dll] \arrow[dd] \arrow[drr] & & \\
S \arrow[dd] & & & & S \arrow[dd] \\
 & & G_V \arrow[dll] \arrow[drr] & & \\
pt & & & & pt
\end{tikzcd}$$

The group G_V acts on S.

$$act : G_V \times S \longrightarrow S$$

$$\mathcal{F} \in Coh(S), \quad act^*(\mathcal{F}) \in Coh(G_V \times S)$$

$$\wr\, KD$$

$$\mathcal{O}_S \otimes Sym(V[-2])\text{-mod}$$

<u>Thm</u> $\quad SingSupp(\mathcal{F}) = Supp(KD \circ act^*(\mathcal{F})) \subset S \times V^*$

C cocomplete triangulated cat.

$C^c \subset C$

$\uparrow$

cpt objects

$$Funct_{cts}(Ind(\overset{\circ}{C}), C) \simeq Funct(\overset{\circ}{C}, C),$$

functorially in C.

$\overset{\circ}{C} \rightsquigarrow Ind(\overset{\circ}{C})$

Def C is compactly generated if

$$(C^c)^{\perp} = 0$$

Ex. 1. $\mathrm{Ind}(\overset{\circ}{C}) = \mathrm{Funct}((\overset{\circ}{C})^{op}, \mathrm{Vect})$

Thm. (a) $\mathrm{Ind}(C^c) \to C$ is fully faithful

(a') Is an equiv. iff C is cptly gen.

(b) $\overset{\circ}{C} \to (\mathrm{Ind}(\overset{\circ}{C}))^c$ realizes $(\mathrm{Ind}(\overset{\circ}{C}))^c$ as $(\overset{\circ}{C})^{kar}$.

$\mathrm{Coh}(S) \sim \overset{\circ}{C}$

$\mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S))$

$C = A\text{-mod}$

C is compactly generated $\Rightarrow$ $\mathrm{Ind}(\mathrm{Perf}(S)) \xrightarrow{\sim} \mathrm{QCoh}(S)$.

$\mathrm{Hom}(A, -) = A\text{-mod} \to \mathrm{Vect}$

$\mathrm{Ind}(\mathrm{Coh}(S)) =: \mathrm{IndCoh}(S)$

$$\begin{array}{c} \mathrm{IndCoh}(S) \\ \Xi_S \uparrow \;\; \downarrow \Psi_S \\ \mathrm{QCoh}(S) \end{array}$$

$\mathrm{IndCoh}(S)$ has a t-structure s.t. Ψ_S is t-exact, and it defines an equiv. $\mathrm{IndCoh}(S)^{\geq -n} \xrightarrow{\sim} \mathrm{QCoh}(S)^{\geq -n}$.

$$\begin{array}{c} \mathrm{Coh}(S) \\ \Xi_S \uparrow \\ \mathrm{Perf}(S) \end{array}$$

$S = \mathrm{Spec}\, k[\xi]$, $\deg \xi = -2$, $\mathcal{O}_S$ NOT coherent!

Def S is eventually coconnective if $H^{-i}(\mathcal{O}_S) = 0$ for $i >> 0$

Example (a) Classical schemes are eventually coconnective

(b) Quasi-smooth schemes are eventually coconnective.

$\uparrow$

Exer.

Lemma $\begin{array}{c} S_1 \\ f \downarrow \\ S_2 \end{array}$ Then f is of finite Tor dim iff its fibers are eventually coconnective

S eventually coconnective

Claim. (a) (Ξ_S, Ψ_S) form an adjoint pair

(b) Ξ_S is fully faithful

] "colocalization"

$$\ker(\Psi_S) = \{ \mathcal{F} \in \mathrm{IndCoh}(S) : H^i(\mathcal{F}) = 0, \ \forall i \}$$

$\uparrow$

w.r.t. t-structure

$$S = pt \underset{V}{\times} pt = \mathrm{Spec}\ \mathrm{Sym}\ V^*[1]$$

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & \overset{KD}{\simeq} & \mathrm{Sym}(V[-2])\text{-mod} \\ \Xi_S \uparrow \ \downarrow \Psi_S & & \uparrow \Xi_S \ \downarrow \Psi_S \leftarrow \text{cohomology w/ support} \\ \mathrm{QCoh}(S) & \simeq & \mathrm{Sym}(V[-2])\text{-mod}_{\{0\}} \end{array}$$

$KD(\mathcal{F}) = R\mathrm{Hom}(k_S, \mathcal{F})$.

$\dim V = 1$, $\mathrm{Sym}(V[-2]) = k[\eta]$, $\deg \eta = 2$,

$M = k[\eta, \eta^{-1}]$

$$\mathrm{IndCoh}(S)^{\mathbb{G}_m} / \mathrm{QCoh}(S)^{\mathbb{G}_m} \simeq \mathrm{QCoh}(\mathbb{P}(V^*))$$

$$\mathrm{ker}(\psi)^{\mathbb{G}_m}$$

<u>E.g.</u> if $\dim V = 1$, $\mathrm{ker}(\psi)^{\mathbb{G}_m} \simeq \mathrm{Vect}$.

$\mathcal{N} \subset \mathrm{Sing}(S)$ conical subset

$$\begin{array}{c} \mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S)) \\ \uparrow\downarrow \\ \mathrm{IndCoh}_{\mathcal{N}}(S) := \mathrm{Ind}(\mathrm{Coh}_{\mathcal{N}}(S)) \\ \uparrow\downarrow \\ \mathrm{QCoh}(S) = \mathrm{Ind}(\mathrm{Perf}(S)) \end{array}$$

$\mathrm{IndCoh}(\mathrm{LocSys}_G)$

- IndCoh (algebraic stack)
- How is LocSys_G an algebraic stack.

$\mathrm{QCoh}(\mathcal{Y})$

$\mathcal{Y}$ an arbitrary prestack

$(\mathrm{Sch}^{\mathrm{aff}})^{op} \longrightarrow \mathrm{Spaces}$

To specify $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$

$S \xrightarrow{y} \mathcal{Y} \rightsquigarrow \mathcal{F}_{S,y} \in \mathrm{QCoh}(S)$

$S_1 \xrightarrow{f} S_2 \xrightarrow{y_2} \mathcal{Y} \rightsquigarrow f^*(\mathcal{F}_{S_2,y_2}) \simeq \mathcal{F}_{S_1,y_1}$

$y_1 = y_2 \circ f$

$$S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3 \xrightarrow{y_3} \mathcal{Y}$$

$$\begin{tikzcd}
f^* \circ g^* (\mathcal{F}_{S_3, y_3}) \arrow[r, "\simeq"] \arrow[d, "\wr"] \arrow[dr, phantom, "\Downarrow"] & (g\circ f)^* \mathcal{F}_{S_3,y_3} \arrow[d, "\wr"] \\
f^*(\mathcal{F}_{S_2, y_2}) \arrow[r, "\simeq"] & \mathcal{F}_{S_1, y_1}
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[r, mapsto] & \mathrm{QCoh}(S) \\
(\mathrm{Sch}^{\mathrm{aff}})^{op} \arrow[r, "\mathrm{QCoh}"] & \mathrm{DGCat} \\
(\mathrm{Sch}^{\mathrm{aff}}/\mathcal{Y})^{op} \arrow[u]
\end{tikzcd}$$

$$\mathrm{QCoh}(\mathcal{Y}) = \lim_{\substack{\longleftarrow \\ (\mathrm{Sch}^{\mathrm{aff}}/\mathcal{Y})^{op}}} \mathrm{QCoh} \in \mathrm{DGCat}$$

Rmk. If $\mathcal{Y}$ is algebraic, $\mathrm{QCoh}(\mathcal{Y})^{+} \simeq$ LB-definition

Def. An algebraic derived stack is a prestack s.t.

1) satisfies étale-descent

2) $\exists$ derived scheme $S \xrightarrow{f} \mathcal{Y}$ equipped w/ a schematic smooth surjective map.

étale descent:

$$\begin{tikzcd}
\tilde{S} \arrow[d] \\
S
\end{tikzcd}$$

$$\mathcal{Y}(S) \xrightarrow{\sim} \mathrm{Tot}\left(\mathcal{Y}(\tilde{S}/S)\right)$$

Observation Let $\mathcal{Y}$ be algebraic, one can replace the limit

$$\begin{array}{c} \lim\limits_{\longleftarrow} \\ (Sch^{aff}/\mathcal{Y})^{op} \end{array} QCoh$$

$$\downarrow \quad S$$

$$(Sch^{aff}/_{\text{smooth over } \mathcal{Y}})^{op}$$

$$\downarrow \quad S$$

$$((Sch^{aff}/_{\text{smooth over } \mathcal{Y}})_{\text{smooth}})^{op}$$

$$S_1 \xrightarrow{f} S_2$$

Def f is smooth if its fibers are classical smooth schemes.

$$IndCoh(\mathcal{Y}) = \lim_{\substack{\longleftarrow \\ (Sch^{aff}_{\text{smooth over } \mathcal{Y}})_{\text{smooth}}}} IndCoh$$

$$S_1 \xrightarrow{f} S_2$$

$f^*: IndCoh(S_2) \longrightarrow IndCoh(S_1)$ $\rightsquigarrow f^!$

$Coh(S_2) \xrightarrow{f^*} Coh(S_1)$ (finite tor dim)

$$Perf(\mathcal{Y}) = \lim_{\longleftarrow} Perf(S)$$

$$Coh(\mathcal{Y}) = \lim_{\longleftarrow} Coh(S) \quad \leftarrow \mathcal{Y} \text{ algebraic}$$

Prop. (a) $IndCoh(\mathcal{Y}) \simeq Ind(Coh(\mathcal{Y}))$ $\leftarrow$ $\mathcal{Y}$ is quasi-compact, affine stabilizer

(b) $QCoh(\mathcal{Y}) \not\simeq Ind(Perf(\mathcal{Y}))$

Let $\mathcal{Y}$ be quasi-compact w/ affine stabilizers

(b') $\mathrm{Perf}(\mathcal{Y}) = (\mathrm{QCoh}(\mathcal{Y}))^c$

(b'') The functor $\mathrm{Ind}(\mathrm{Perf}(\mathcal{Y})) \to \mathrm{QCoh}(\mathcal{Y})$ is an equiv. ??

$\mathcal{Y}$ - algebraic stack (locally almost of finite type)

Def. $\mathcal{Y}$ is quasi-smooth if $\forall\ y \in \mathcal{Y}(k)$, $H^i(T_y \mathcal{Y}) = 0$ for $i \geq 2$

(if $\mathcal{Y}$ n-Artin stack, $[-n, \infty)$)

<0 : automorphisms

>0 : singularities

Equivalently, $\mathcal{Y}$ is quasi-smooth if for every $S \xrightarrow{f} \mathcal{Y}$ s.t. f is smooth, S is quasi-smooth.

$\mathrm{Sing}(\mathcal{Y}) = \mathrm{Spec}_{\mathcal{Y}_{cl}}(\mathrm{Sym}(H^1(T\mathcal{Y})))$

$S \xrightarrow{f} \mathcal{Y}$ f is smooth

$\mathrm{Sing}(\mathcal{Y}) \underset{\mathcal{Y}}{\times} S = \mathrm{Sing}(S)$

$S_1 \xrightarrow{f} S_2$ smooth $\mathrm{Sing}(S_2) \underset{S_2}{\times} S_1 \simeq \mathrm{Sing}(S_1)$

$\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$, $\mathcal{N}_S \subset \mathrm{Sing}(S)$, $\forall S \xrightarrow{f} \mathcal{Y}$

$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{IndCoh}(\mathcal{Y})$

$\quad \| \ \varprojlim \mathrm{IndCoh}_{\mathcal{N}_S}(S)$

Lecture 3.

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & = & \mathrm{Ind}(\mathrm{Coh}(S)) \\ \Xi_S \uparrow \downarrow \Psi_S & & \\ \mathrm{QCoh}(S) & = & \mathrm{Ind}(\mathrm{Perf}(S)) \end{array}$$

If S is eventually coconnective, Ψ_S has a left adjoint Ξ_S.

If S is quasi-smooth, for every $N \subset \mathrm{Sing}(S)$,

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & & \\ \uparrow\downarrow & & \\ \mathrm{IndCoh}_N(S) & = & \mathrm{Ind}(\mathrm{Coh}_N(S)) \\ \uparrow\downarrow & & \\ \mathrm{QCoh}(S) & & \end{array}$$

of bounded cohomological dimension

$\Downarrow$

$\Gamma(\mathcal{Y}, -)$ is cts

$\mathcal{Y} \rightsquigarrow \mathrm{QCoh}(\mathcal{Y})$

If $\mathcal{Y}$ is algebraic, $\mathcal{Y} \rightsquigarrow \mathrm{IndCoh}(\mathcal{Y})$ ($\simeq \mathrm{Ind}(\mathrm{Coh}(\mathcal{Y}))$ if $\mathcal{Y}$ is quasi-cpt w/ affine stabilizers)

If $\mathcal{Y}$ is quasi-smooth for every $N \subset \mathrm{Sing}(\mathcal{Y})$

$\mathcal{Y} \rightsquigarrow \mathrm{IndCoh}_N(\mathcal{Y})$. ($\mathrm{Coh}_N(\mathcal{Y})$ is the cpts in $\mathrm{IndCoh}_N(\mathcal{Y})$, but not known to generate)

For $\mathcal{Y} = \mathrm{LocSys}$, it will be true that $\forall N$, $\mathrm{IndCoh}_N(\mathcal{Y})$ is cptly generated by $\mathrm{Coh}_N(\mathcal{Y})$.

LocSys_G

Bun_G

$$\mathrm{Hom}(S, \underline{\mathrm{Maps}}(X, \mathcal{Y})) = \mathrm{Hom}(S \times X, \mathcal{Y})$$

$\mathrm{Bun}_G = \underline{\mathrm{Maps}}(X, pt/G)$

$X \rightsquigarrow X_{dR}$

$\mathrm{Hom}(S, X_{dR}) = \mathrm{Hom}(S_{red}, X)$

$\mathrm{QCoh}(X_{dR}) = \mathrm{Dmod}(X)$

If X is affine, $\mathrm{Dmod}(X)$ is the full subcat. of $\mathrm{Dmod}(\tilde{X})$ w/ supp on X.

$X \hookrightarrow \tilde{X}$
$\uparrow$ smooth.

$\mathrm{LocSys}_G := \underline{\mathrm{Maps}}(X_{dR}, pt/G)$

- If $\mathcal{Y}$ a stack, $\mathrm{Dmod}(\mathcal{Y}) = \mathrm{QCoh}(\mathcal{Y}_{dR})$

or $\mathrm{Dmod}(\mathcal{Y}) = \lim\limits_{\leftarrow S} \mathrm{Dmod}(S)$

One proves that if X is proper, then LocSys_G is <u>an algebraic stack</u>.

$\sigma : X \to \mathcal{Y}$

$$T_\sigma(\underline{\mathrm{Maps}}(X, \mathcal{Y})) \underset{\substack{\uparrow \\ \text{Exe.}}}{=} \Gamma(X, \sigma^*(T\mathcal{Y}))$$

- $\mathcal{Y} = pt/G$, $\quad T_\sigma(\mathrm{Bun}_G) = \Gamma(X, \mathfrak{g}_\sigma[1])$

- $\mathcal{Y} = pt/G$, $\quad X_{dR}$. $\quad T_\sigma(\mathrm{LocSys}_G) = \Gamma_{dR}(X, \mathfrak{g}_\sigma[1])$

<u>Proof of prop</u>. Just the fact that $\Gamma_{dR}(X, \text{local system})$ lives in degrees $[0, 2]$.

$$\mathrm{Sing}(\mathrm{LocSys}_G) = (\sigma, A)$$

$$H^1(T_\sigma(\mathrm{LocSys}_G)) = H^2_{dR}(X, \mathfrak{g}_\sigma)$$

$$H^{-1}(T^*_\sigma(\mathrm{LocSys}_G)) = H^0_{dR}(X, \mathfrak{g}^*_\sigma) \overset{\text{reductive}}{\simeq} H^0_{dR}(X, \mathfrak{g}_\sigma)$$

$$\mathrm{Nilp} \subset \mathrm{Sing}(\mathrm{LocSys}_G)$$

where we require A to be nilpotent.

Geometric Langlands

$$\begin{array}{ccc} \mathrm{Dmod}(\mathrm{Bun}_G) & \overset{\mathbb{L}_G}{\simeq} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \\ \cup & & \cup \\ \mathrm{Dmod}(\mathrm{Bun}_G)_{\mathrm{temp}} & \simeq & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \end{array}$$

$$\begin{array}{ccc} (\mathrm{Dmod}(\mathrm{Bun}_G)^c)^{op} & \simeq & (\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}))^{op} \\ \Big\downarrow \cancel{\boxed{\mathbb{D}^{\mathrm{Verdier}}}} & & \Big\downarrow \mathbb{D}^{\mathrm{Serre}} \\ \mathrm{Dmod}(\mathrm{Bun}_G)^c & \simeq & \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \end{array}$$

$$\mathrm{Hom}_{\mathrm{Dmod}(Y)}(\mathbb{D}\mathcal{F}_1, \mathcal{F}_2) = \Gamma_{dR}(Y, \mathcal{F}_1 \otimes \mathcal{F}_2)$$

Cpt objects in $\mathrm{Dmod}(\mathrm{Bun}_G)$ all look as follows:

$$U \overset{j}{\hookrightarrow} \mathrm{Bun}_G$$

$$\mathcal{F}_U \qquad j_!(\mathcal{F}_U)$$

If $\mathcal{F}_U \in \mathrm{Dmod}(U)$ was compact, $j_*(\mathcal{F}_U)$ may no longer be compact.

$$\mathbb{D}(j_!(\mathcal{F}_U)) = j_*(\mathbb{D}(\mathcal{F}_U))$$

Drinfeld: miraculous duality. $\mathrm{Mir}: \mathrm{Dmod}(\mathrm{Bun}_G)^c \to \mathrm{Dmod}(\mathrm{Bun}_G)$

$$\mathrm{Mir} \circ \mathrm{Eis}_* = \mathrm{Eis}_!^-$$

$$\mathrm{Mir} \circ \mathbb{D}^{\mathrm{Verdier}} \longleftrightarrow \mathbb{D}^{\mathrm{Serre}}$$

$$\begin{tikzcd} & \mathrm{LocSys}_P \arrow[dl, "p_{spec}"'] \arrow[dr, "q_{spec}"] & \\ \mathrm{LocSys}_G & & \mathrm{LocSys}_M \end{tikzcd}$$

$$\mathrm{Eis}_{spec} = (p_{spec})_* \circ (q_{spec})^! : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_M) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$$

$$\mathrm{Sing}(\mathrm{LocSys}_P) = (\sigma_P, A \in (\mathfrak{g}/\mathfrak{n}(\mathfrak{p}))_\sigma) \supset \mathrm{Nilp}$$

$$p^* = \mathfrak{g}/\mathfrak{n}(\mathfrak{p})$$

$$\begin{array}{c} (\mathfrak{g}/\mathfrak{n}(\mathfrak{p}))_\sigma \\ \cup \\ (\mathfrak{p}/\mathfrak{n}(\mathfrak{p}))_\sigma \\ \| \\ \mathfrak{m}_\sigma \\ \cup \\ \mathrm{Nilp} \end{array}$$

<u>Prop</u> (a) $(q_{spec})^! : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_M) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_P)$ and preserves compactness

(b) $(p_{spec})_* : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_P) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$ and preserves coherence.

Thm

$$\left\langle \mathrm{Eis}_{\mathrm{Spec}}\left(\mathrm{QCoh}(\mathrm{LocSys}_M)\right)\right\rangle = \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$$

all P

$$\mathrm{LocSys}_G(\mathbb{P}^1) = \left(pt \underset{\mathfrak{g}}{\times} pt\right)/G$$

$$\begin{tikzcd}
 & \mathrm{IndCoh}(pt \times_{\check{\mathfrak{g}}} pt/\check{G}) \arrow[r, phantom, "\simeq"] \arrow[d, shift left] & \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check{G}} \arrow[d, shift left] \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G", "\sim"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(pt \times_{\check{\mathfrak{g}}} pt/\check{G}) \arrow[r, phantom, "\simeq"] \arrow[u, shift left] \arrow[d, shift left] & \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check{G}}_{\mathrm{Nilp}} \arrow[u, shift left] \arrow[d, shift left] \\
 & \mathrm{QCoh}(pt \times_{\check{\mathfrak{g}}} pt/\check{G}) \arrow[r, phantom, "\simeq"] \arrow[u, shift left] & \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check{G}}_{0} \arrow[u, shift left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Bun}_G(\mathbb{P}^1) \\
pt/G \arrow[u, hook, "j"] \\
pt \arrow[u, "\pi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{IndCoh}_{\mathrm{Nilp}}(pt \times_{\check{\mathfrak{g}}} pt/\check{G}) \arrow[d, phantom, "\ni"] \\
j_!(k) \arrow[r, leftrightarrow, "\mathbb{L}_G"] & i_*(k)
\end{tikzcd}$$

diagonal

Constant sheaf is NOT cpt on pt/G !

$$pt/\check{G} \xrightarrow{i} pt \times_{\check{\mathfrak{g}}} pt/\check{G}$$

$j_! \circ \pi_!(k) \overset{\mathbb{L}_G}{\longleftrightarrow}$ Structure sheaf of Nilp in $\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check{G}}_{\mathrm{Nilp}}$.

$$\begin{tikzcd} S_1 \arrow[d, "f"] \\ S_2 \end{tikzcd}$$

$$f^{\mathrm{IndCoh}}_* : \begin{tikzcd}
\mathrm{IndCoh}(S_1) \arrow[r] \arrow[d, shift left, "\Psi_{S_1}"] & \mathrm{IndCoh}(S_2) \arrow[d, shift left, "\Psi_{S_2}"] \\
\mathrm{QCoh}(S_1) \arrow[r, "f_*"] \arrow[u, shift left, "\Xi_{S_1}"] & \mathrm{QCoh}(S_2) \arrow[u, shift left, "\Xi_{S_2}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Coh}(S_1) \arrow[r] \arrow[d] & \mathrm{QCoh}(S_1)^+ \arrow[r, "f_*"] & \mathrm{QCoh}(S_2)^+ & \mathrm{IndCoh}^+(S_2) \arrow[l, "\Psi_{S_2}"', "\sim"] \arrow[d] \\
\mathrm{IndCoh}(S_1) \arrow[rrr, "f_*^{\mathrm{IndCoh}}"] & & & \mathrm{IndCoh}(S_2)
\end{tikzcd}$$

$$\begin{tikzcd}
S_1 \underset{S_2}{\times} T^*S_2 \arrow[r] \arrow[d] & T^*S_1 \\
T^*S_2
\end{tikzcd}$$

$$\begin{tikzcd}
S_1 \underset{S_2}{\times} \mathrm{Sing}(S_2) \arrow[r, "\mathrm{Sing}(f)"] \arrow[d] & \mathrm{Sing}(S_1) \\
\mathrm{Sing}(S_2)
\end{tikzcd}$$

<u>Thm</u>. $N_1 \subset \mathrm{Sing}(S_1)$, $\quad N_2 \subset \mathrm{Sing}(S_2)$

Assume $\quad S_1 \underset{S_2}{\times} N_2 \supset \mathrm{Sing}(f)^{-1}(N_1)$

$$\Rightarrow \quad f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}_{N_1}(S_1) \longrightarrow \mathrm{IndCoh}_{N_2}(S_2)$$

<u>Exe</u>. Deduce Prop (b) from theorem

A positively graded com. alg that maps to the graded center of T

T

$\forall a \in A_n$,
$$\begin{tikzcd}
t' \arrow[r, "a"] & t'[2n] \\
t \arrow[r, "a"] \arrow[u] & t[2n] \arrow[u]
\end{tikzcd}$$
and they commute.

Spec A

$a \rightsquigarrow Y_a \subset \operatorname{Spec} A$, the zero locus of a

$$\mathcal{T}_{Y_a} \rightleftarrows \mathcal{T} \underset{\hookleftarrow}{\xrightarrow{Loc_a}} \mathcal{T}_{\operatorname{Spec} A - Y_a}$$

$$\left\{ t \in \mathcal{T}_{\operatorname{Spec} A - Y_a} \text{ if } t \xrightarrow{a} t[2n] \text{ is an isom.} \right\}$$

$$Loc_a(t) = \operatorname*{hocolim}_k t[2nk]$$

$$t \to t[2n] \to t[2 \cdot 2n] \to t[3 \cdot 2n]$$

$N \subset \operatorname{Spec} A$

$$\mathcal{T}_N = \bigcap_{a,\, N \subset Y_a} \mathcal{T}_{Y_a}$$

<u>Prop.</u> $\mathcal{T}_{N_1 \cap N_2} \subset \mathcal{T}_{N_1} \cap \mathcal{T}_{N_2}$ is <u>an equality</u>.

S quasi-smooth, $\mathcal{T} = H_0(\operatorname{IndCoh}(S))$

$B = HH^*(S)$ Hochschild cohomology

$$H^0(\Gamma(S, \mathcal{O}_S)) \to HH^0(S)$$

$$\Gamma(X, T_X[-1]) \to HC(S)$$

$$H^1(X, T_X) \to HH^2(S)$$

$$A = \Gamma(\operatorname{Sing} S, \mathcal{O}) \to HH^*(S)$$

Lecture 4

$S \rightsquigarrow \mathrm{IndCoh}(S)$

$X \xrightarrow{f} Y$

$$\begin{tikzcd} \mathrm{IndCoh}(X) \arrow[r, "f_*^{\mathrm{IndCoh}}"] \arrow[d, shift left, "\Psi_X"] & \mathrm{IndCoh}(Y) \arrow[d, shift left, "\Psi_Y"] \\ \mathrm{QCoh}(X) \arrow[r, "f_*"] \arrow[u, shift left, "\Xi_X"] & \mathrm{QCoh}(Y) \arrow[u, shift left, "\Xi_Y"] \end{tikzcd}$$

$f : X \xrightarrow{\text{proper}} Y$

$(f_*, f^!)$

$$\mathrm{QCoh}(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} \mathrm{QCoh}(Y)$$

However, $f^!$ is not <u>continuous</u>. (does not commute w/ infinite direct sums)

$$C \underset{G}{\overset{F}{\rightleftarrows}} D$$

(a) If G is cts, then F sends cpt to cpt.

(b) If C is cptly gen., (a) is iff.

$$\mathrm{IndCoh}(X) \underset{f^!}{\overset{f_*^{\mathrm{IndCoh}}}{\rightleftarrows}} \mathrm{IndCoh}(Y)$$

Why continuous functors?

1) $C \xrightarrow{F} D$ Have more control

2) $\mathrm{DGCat}_{\mathrm{cont}}$, $C, D \rightsquigarrow C \otimes D$

Ex. C - A-mod

D - B-mod

$C \otimes D \simeq (A \otimes B)$-mod

$C \underset{\mathcal{A}}{\otimes} D$, $\mathcal{A} \in Alg(DGCat_{cont})$, a.k.a. monoidal DG-category

$X \xrightarrow{f} Y$

$f^!: IndCoh(Y) \to IndCoh(X)$

$\left[\begin{array}{l} \text{If } f \text{ is an open embedding, } f^! = (f_*^{IndCoh})^L \\ \text{If } f \text{ is a proper map, } f^! = (f_*^{IndCoh})^R \end{array}\right.$

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[rr, bend right, "f"'] & \bar{X} \arrow[r, "\bar{f}"] & Y \end{tikzcd}$$

Nagata ?

$f^! = j^! \circ \bar{f}^!$

Yifeng Liu & Weizhe Zheng (constructible sheaves)

$GR: (Sch_{aft})^{op} \xrightarrow{IndCoh^!} DGCat$

$$\begin{array}{l} PreStk_{laft} := ({}^{<\infty}Sch^{aft}_{aft})^{op} \longrightarrow \text{Groupoids} \\ \downarrow \\ PreStk \end{array}$$

$(PreStk_{laft})^{op} \xrightarrow{IndCoh^!} DGCat$

$$IndCoh(\mathcal{Y}) = \lim_{\overleftarrow{(S \xrightarrow{y} \mathcal{Y})}} IndCoh(S)$$

$\mathrm{Dmod}(X) = \mathrm{IndCoh}(X_{dR})$ Today

$\mathrm{Dmod}(X) = \mathrm{QCoh}(X_{dR})$ last week

$\mathrm{Perf}(S) \curvearrowright \mathrm{Coh}(S)$

$\mathrm{QCoh}(S) \curvearrowright \mathrm{IndCoh}(S)$

$\wr$

$\mathrm{QCoh}(\mathcal{Y}) \curvearrowright \mathrm{IndCoh}(\mathcal{Y})$

$\mathcal{Y} \xrightarrow{p_{\mathcal{Y}}} pt \quad , \omega_{\mathcal{Y}} = (p_{\mathcal{Y}})^{!}(k)$

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \longrightarrow \mathrm{IndCoh}(\mathcal{Y})$$
$$\mathcal{F} \longmapsto \mathcal{F} \otimes \omega_{\mathcal{Y}}$$

Thm (a) If $\mathcal{Y} = \mathcal{X}_{dR}$ for $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}} \Rightarrow \Upsilon_{\mathcal{X}_{dR}}$ is an equiv.

(b) $\mathcal{Y} = Y$ derived scheme, Υ_Y is an equiv. $\Leftrightarrow$ Y is smooth.

$X \xrightarrow{f} Y$

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{\Upsilon_X} & \mathrm{IndCoh}(X) \\ f^* \uparrow & & \uparrow f^! \\ \mathrm{QCoh}(Y) & \xrightarrow{\Upsilon_Y} & \mathrm{IndCoh}(Y) \end{array}$$

X

$$\begin{array}{c} \mathrm{IndCoh}(X) \\ \uparrow \Upsilon_X \\ \mathrm{QCoh}(X) \end{array}$$

$$\begin{array}{c} \mathrm{IndCoh}(X) \\ \Xi_X \uparrow \ \downarrow \Psi_X \\ \mathrm{QCoh}(X) \end{array}$$

Example X is smooth

$$\Xi_X = \Psi_X = \mathrm{Id}_{QCoh(X)},$$

Then Υ_X is tensoring by the canonical bundle.

$$\begin{tikzcd} & \mathrm{LocSys}_P \arrow[dl, "p_{spec}"'] \arrow[dr, "q_{spec}"] & \\ \mathrm{LocSys}_G & & \mathrm{LocSys}_M \end{tikzcd}$$

$$\mathrm{Eis}_{spec} = (p_{spec})_*^{\mathrm{IndCoh}} \circ (q_{spec})^!$$

Want $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_M) \longrightarrow \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_G)$

Last time: introduced $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_P)$ and explained that

$(p_{spec})_*$: $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_P)$ to $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_G)$

Today. $(q_{spec})^!$ maps $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_M)$ to $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_P)$

Reminder: $X \xrightarrow{f} Y$

$$\mathrm{IndCoh}(X) \xrightarrow{f_*^{\mathrm{IndCoh}}} \mathrm{IndCoh}(Y)$$

$$\begin{tikzcd} \mathrm{Sing}(Y) \underset{Y}{\times} X \arrow[r, "\mathrm{Sing}(f)"] \arrow[d] & \mathrm{Sing}(X) \\ \mathrm{Sing}(Y) & \end{tikzcd}$$

Thm. $N_X \subset \mathrm{Sing}(X)$, $N_Y \subset \mathrm{Sing}(Y)$.

Assume that $\mathrm{Sing}(f)^{-1}(N_X) \subset N_Y \underset{Y}{\times} X$

$\Rightarrow f_*^{\mathrm{IndCoh}}$ maps $\mathrm{IndCoh}_{N_X}(X)$ to $\mathrm{IndCoh}_{N_Y}(Y)$.

When does $f^!$ send $\mathrm{IndCoh}_{N_Y}(Y)$ to $\mathrm{IndCoh}_{N_X}(X)$?

Thm. This happens provided that

$$N_Y \underset{Y}{\times} X \subset (\mathrm{Sing}(f))^{-1}(N_X)$$

$$\Updownarrow$$

$$(\mathrm{Sing}(f))(N_Y \underset{Y}{\times} X) \subset N_X$$

Exe. deduce from thm.

When is $f^!(\mathcal{F})$ coherent if $\mathcal{F} \in \mathrm{Coh}(Y)$?

Thm X quasi smooth, $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Coh}(X)$, TFAE.

(a) $\mathcal{F}_1 \otimes \mathcal{F}_2$ is coherent

(b) $\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2$ is coherent $\quad \Leftrightarrow \quad \mathrm{SingSupp}(\mathcal{F}_1) \cap \mathrm{Sing\,Supp}(\mathcal{F}_2) \subset \{0\}$.

(c) $\underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is coherent

(d) $\underline{\mathrm{Hom}}(\mathcal{F}_2, \mathcal{F}_1)$ is coherent

$\sigma \in \mathrm{LocSys}_T$

$k_\sigma \in \mathrm{QCoh}(\mathrm{LocSys}_T) \qquad \mathrm{Eis}_{spa}(k_\sigma) \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$

$\qquad\qquad \shortparallel$

$\qquad \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_T)$

$\mathcal{F} \in \mathrm{IndCoh}(X) \rightsquigarrow \mathrm{Singsupp}(\mathcal{F}) \subset \mathrm{Sing}(X)$

$\mathcal{F} \in \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(X) = \mathrm{IndCoh}(X)/\mathrm{QCoh}(X) \rightsquigarrow \mathbb{P}(\mathrm{Sing}(X)) \supset \overset{\circ}{\mathrm{Singsupp}}(\mathcal{F})$

Thm. $\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(X))) \curvearrowright \mathrm{IndCoh}^{\circ}(X)$

$\{$ categories equipped w/ an action of $G((t))\}$ $\quad$ $\{$sheaves of categories over $\mathrm{LocSys}_{\check{G}}(\mathring{D})\}$

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \rightsquigarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

$$2\text{-}\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \rightsquigarrow 2\text{-}\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

Thm.

$$\langle \mathrm{Eis}_{\mathrm{spec}}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})) \rangle = \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

$$C \underset{G}{\overset{F}{\rightleftarrows}} D \qquad (D = \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}))$$

Lemma. The image of F generates D iff G is conservative.

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad X \longrightarrow X_{dR} \underset{Y_{dR}}{\times} Y$$

$$\begin{tikzcd} \mathrm{QCoh}(X_{dR} \underset{Y_{dR}}{\times} Y) \arrow[r] \arrow[rr, bend left, "\Upsilon_{X_{dR} \underset{Y_{dR}}{\times} Y}"] & \boxed{\mathrm{IndCoh}_{\{0\}}(X_{dR} \underset{Y_{dR}}{\times} Y)} \arrow[r] \arrow[d] & \mathrm{IndCoh}(X_{dR} \underset{Y_{dR}}{\times} Y) \arrow[d] \\ & \mathrm{QCoh}(X) = \mathrm{IndCoh}_{\{0\}}(X) \arrow[r, hook] & \mathrm{IndCoh}(X) \end{tikzcd}$$

$I(X/Y)$

$$\begin{tikzcd} \mathrm{LocSys}_P \arrow[d] \\ \mathrm{LocSys}_G \end{tikzcd} \qquad \text{We're interested in } I(\mathrm{LocSys}_P / \mathrm{LocSys}_G).$$

(By induction on semi-simple rank, this category fully faithfully embeds into the P-degenerate Whittaker category.)

<u>Thm</u>. $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_G) \longrightarrow \mathrm{Glue}(\mathcal{I}(\mathrm{LocSys}_P/\mathrm{LocSys}_G), P \in \mathrm{Par}(G))$

is fully faithful.

$\ulcorner$ $D(\mathrm{Bun}_G) \hookrightarrow \mathrm{Glue}(\mathrm{Whit}^P, \quad P \in \mathrm{Par}(G))$ $\lrcorner$

$$\begin{array}{c} \mathrm{QCoh}(\check{G}) \\ \downarrow \\ \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \simeq \mathrm{Whit}^G \end{array}$$

$$A \longrightarrow \mathrm{DGCat}$$
$$a \longmapsto C_a$$
$$[a \to b] \longmapsto [C_a \xrightarrow{F_{a\to b}} C_b]$$

$$\underset{\underset{a\in A}{\longleftarrow}}{\lim} C_a = \left\{ \underset{a\in A}{c_a \in C_a}, \quad F_{a\to b}(c_a) \simeq c_b \right\}$$

$$\underset{\underset{a\in A}{\longleftarrow}}{\mathrm{laxlim}}\, C_a = \left\{ \underset{a\in A}{c_a \in C_a}, \quad F_{a\to b}(c_a) \to c_b \right\}$$
$$\parallel$$
$$\mathrm{Glue}(C_a, a\in A)$$

$A = (0 \to 1)$ $\qquad Y_0 \overset{j}{\hookrightarrow} Y \overset{i}{\hookleftarrow} Y_1$

$C_0 = \mathrm{Shv}(Y_0)$

$C_1 = \mathrm{Shv}(Y_1)$, $\qquad F_{0\to 1} = i^! \cdot j_!$

$\mathrm{Glue}(\mathrm{Shv}(Y_i)) \simeq \mathrm{Shv}(Y)$

$$\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_G) \longrightarrow I(\mathrm{LocSys}_P/\mathrm{LocSys}_G)$$

$$\begin{tikzcd}
\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_G) \arrow[r] & \mathrm{IndCoh}(\mathrm{LocSys}_G) \arrow[d, "!"] \\
& \mathrm{IndCoh}\left((\mathrm{LocSys}_P)_{dR} \underset{(\mathrm{LocSys}_G)_{dR}}{\times} \mathrm{LocSys}_G\right) \arrow[d, "\text{right adjoint}"] \\
& I(\mathrm{LocSys}_P/\mathrm{LocSys}_G)
\end{tikzcd}$$

Thm $\exists$ action of $\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(Y))) \curvearrowright \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) = \mathrm{IndCoh}(Y)/\mathrm{QCoh}(Y)$ s.t.

(a) For $\mathcal{M} \subset \mathbb{P}(\mathrm{Sing}(Y))$

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{M}}(Y) = \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \underset{\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(Y)))}{\otimes} \mathrm{Dmod}(\mathcal{M})$$

$$\overset{\circ}{I}(X,Y) = I(X/Y)/\mathrm{QCoh}(X_{dR} \underset{Y_{dR}}{\times} Y)$$

(b) $$\overset{\circ}{I}(X/Y) = \mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(X/Y))) \underset{\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(Y)))}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$$

$$\begin{tikzcd}
\mathrm{Sing}(X/Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathrm{Sing}(Y) \underset{Y}{\times} X \arrow[r] \arrow[d] & \mathrm{Sing}(X) \\
\mathrm{Sing}(Y) &
\end{tikzcd}$$

Proof of thm. reduce to prove $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{Nilp}(\mathrm{LocSys}_G) \hookrightarrow \mathrm{Glue}(\overset{\circ}{I}(\mathrm{LocSys}_P/\mathrm{LocSys}_G),\ P \in \mathrm{Par}(G))$

Now, both sides are obtained from $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{LocSys}_G)$ by tensoring over $\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(\mathrm{LocSys}_G)))$ w/ something.

$$\mathrm{Dmod}(\mathbb{P}\,\mathrm{Nilp}) \hookrightarrow \mathrm{Glue}\left(\mathrm{Dmod}\left(\mathbb{P}\,\mathrm{Sing}(\mathrm{LocSys}_P/\mathrm{LocSys}_G)\right),\ P \in \mathrm{Par}(G)\right)$$

The latter statement is that certain homotopy types have trivial homology.

$(A, \sigma_G) \in \mathrm{Nilp}$

$\sigma_G \in \mathrm{LocSys}_G$, $\quad A \in \Gamma_{dR}(X, \mathfrak{g}_{\sigma_G})$

Let P be a parabolic $\rightsquigarrow$ $\{$reductions σ_P of σ_G to P s.t. $A \in \mathfrak{n}(P)_{\sigma_P}\}$

$$H_*(\mathrm{Glue}(\text{reductions}_P)) \xrightarrow{\sim} k$$