Nearby cycles on bases of dimension greater than 1

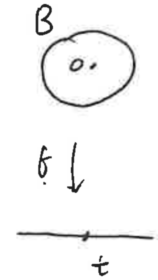
Luc Illusie

Orgogozo, Gabber.

1. Historical sketch

Milnor ~ 1966

$$\begin{array}{c}(\mathbb{C}^{n+1}, 0)\\ f\downarrow\\ (\mathbb{C}, 0)\end{array}$$



$B \cap f^{-1}(t) = V_t$

$V_t \sim \underbrace{S^n \sqcup \cdots \sqcup S^n}_{\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}[[\frac{\partial}{\partial t_i}]]}{(\frac{\partial f_i}{\partial t_i})}}$

Grothendieck $R\Psi$, $R\Phi$.

étale variants SGA 7

S strictly local trait

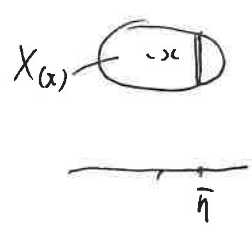
$$\begin{tikzcd} x \arrow[r] & X_s \arrow[r, "i"] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, "j"'] \arrow[d] & X_{\bar\eta} \arrow[l] \arrow[d] \arrow[ll, bend right, "\bar j"'] \\ & s \arrow[r] & S & \eta \arrow[l] & \bar\eta \arrow[l] \end{tikzcd}$$

$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, ℓ invertible in S.

$F \in D^+(X_\eta, \Lambda)$

$R\Psi F := i^* R\bar j_* (F|X_{\bar\eta}) \in D^+(X_s, \Lambda)$

$(R\Psi F)_x = R\Gamma(\underbrace{X_{(x)\bar\eta}}_{\text{Milnor fiber}}, F)$



$R\Psi$ nearby cycles

$F \in D^+(X, \Lambda)$

$F|X_s \to R\Psi F \to \underset{\text{vanishing cycles}}{R\Phi F} \xrightarrow{+1}$

$\mathrm{Gal}(\bar\eta|\eta)$-action

Basic properties

(1) $$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

If f is proper,

$\mathcal{F} \in D^+(X_\eta, \Lambda)$,

$$R\Psi_Y Rf_{\eta *} \mathcal{F} \xrightarrow{\sim} Rf_{s *} R\Psi_X \mathcal{F}$$

In particular, $Y = S$, $R\Gamma(X_s, R\Psi \mathcal{F}) \simeq R\Gamma(X_\eta, \mathcal{F})$

(2) X/S finite type. $\mathcal{F} \in D^b_c(X_\eta, \Lambda)$, $\Rightarrow R\Psi(\mathcal{F}) \in D^b_c(X_s, \Lambda)$

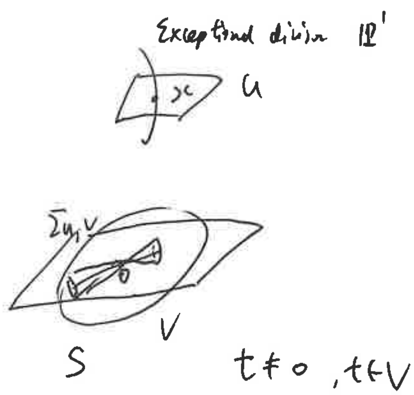
Compatible w/ base change of traits (Deligne SGA4 $\frac{1}{2}$)

How to generalize to $\dim S > 1$?

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l, hook'] \arrow[d, "f_{U,V}"] & \ni x \\ S & V \arrow[l, hook'] & \ni f(x) \end{tikzcd}$$

$Rf_{U,V *} \mathbb{Z}$ usually bad

e.g. $X = Bl_{\{0\}} S$
$f \downarrow$
$S = \mathbb{A}^2$



$t \neq 0$, $t \in V$

$$\left(Rf_{U,V *} \mathbb{Z}\right)_t = \begin{cases} \mathbb{Z} & \text{if } t \in \Sigma_{U,V} \\ 0, & t \notin \Sigma_{U,V} \end{cases}$$

Sabbah

1983, product formula $\varepsilon = \prod \varepsilon_v$

Deligne constructed $R\Psi$ in the étale context, written up by Laumon

Laumon : alternative proof $\left[\begin{array}{l}\text{Fourier transform}\\ \text{stationary phase}\end{array}\right.$

2. Oriented products and vanishing toposes (topoi)

$$\begin{tikzcd}
 & X \overleftarrow{\times}_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & S \arrow[u, phantom, "\leftarrow"] &
\end{tikzcd}$$

$g p_2 \to f p_1$

2-universal

points $x \to X$, $y \to Y$

$g(y) \to f(x)$

$f(x) \leftarrow g(y)$

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] & W \arrow[l] \arrow[d] \\
X \arrow[r] & S & Y \arrow[l]
\end{tikzcd}$$

coverings $\left\{ \begin{array}{l}
\begin{tikzcd}
(U_i) \arrow[d] \arrow[dr] & & \\
U \arrow[r] & V & W \arrow[l]
\end{tikzcd} \\[2ex]
\begin{tikzcd}
 & & (W_i) \arrow[dl] \arrow[d] \\
U \arrow[r] & V & W \arrow[l]
\end{tikzcd} \\[2ex]
\begin{tikzcd}
 & V' \arrow[d] & W' \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\
U \arrow[r] \arrow[ur] & V & W \arrow[l]
\end{tikzcd}
\end{array} \right.$

e.g. (1) $X = Y = S$, $\overleftarrow{S} = X \overleftarrow{\times}_S Y$

$x \leftarrow y$

(2) $s \in S$ $\quad s \overleftarrow{\times}_S S = S_{(s)}$.

$X \overleftarrow{\times}_S S$ vanishing topos of f.

$$\begin{tikzcd}
 & X \overleftarrow{\times}_S S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & S \arrow[dl, equal] \\
 & S &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \times_S Y \arrow[d, "\Psi"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & X \overleftarrow{\times}_S Y \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & S &
\end{tikzcd}$$

$\Psi^{-1}(U \to V \leftarrow W)$

$= U \times_V W$

$\mathcal{F} \in D^+(X, \Lambda)$,

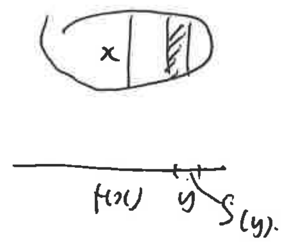
$R\Psi \mathcal{F} \ (= R\Psi_* \mathcal{F}) \in D^+(X \overleftarrow{\times}_S S, \Lambda)$

$$X \overleftarrow{\times}_S S \supset X_s \overleftarrow{\times}_S \eta = \left\{ \begin{array}{l} \text{sheaves on } X_{\bar\eta} \\ + \text{ action of } \mathrm{Gal}(\bar\eta / \eta) \end{array} \right\}$$

"$X_s \times_S \eta$".

$(R\Psi \mathcal{F})|_{X_s \overleftarrow{\times}_S \eta}$ classical $R\Psi \mathcal{F}$.

$(R\Psi \mathcal{F})_{x,\, f(x) \leftarrow y} = R\Gamma(X_{(x)} \times_{S_{(f(x))}} S_{(y)}, \mathcal{F})$



"Milnor tube".

3. Main results

How to generalize (1) and (2).

(1) $$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
 & S &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Psi_X"'] & Y \arrow[d, "\Psi_Y"] \\
X \overleftarrow{\times}_S S \arrow[r, "\overleftarrow{f}"] & Y \overleftarrow{\times}_S S
\end{tikzcd}$$

$R\overleftarrow{f}_* R\Psi_X \simeq R\Psi_Y Rf_*$

If f is proper, $R\overleftarrow{f}_*$ commutes w/ base changes $S' \to S$, recover classical (1).

$X \overleftarrow{\times}_S Y$, $X, Y/S$ of finite type. so noetherian.

G Λ-module on $X \overleftarrow{\times}_S Y$ is constructible

$\iff$ $X = \cup X_\alpha$, $Y = \cup Y_\beta$ s.t. $G|_{X_\alpha \overleftarrow{\times}_S Y_\beta}$ locally constant, fibers of finite type.

In general, $F \in D^b_c(X, \Lambda) \not\Rightarrow R\Psi F \in D^b_c(X \overleftarrow{\times}_S S, \Lambda)$

<u>Thm 1</u> (Deligne, 1983) Assume X/S of f.t., Fix $F \in D^b_c(X; \Lambda)$. $\exists$ open $U \subset X$ s.t. $X - U = \Sigma$ is quasi-finite $/S$, and $F|_U$ universally locally acyclic

$$\left(\begin{array}{l} x \to U, \quad F_x \xrightarrow{\sim} R\Gamma(X_{(x)_t}, F) \\ t \to x \end{array}\right)$$

Then $R\Psi F \in D^b_c(X \overleftarrow{\times}_S S, \Lambda)$ and commutes w/ any base change $S' \to S$.

Moreover, $R\Phi F$ is concentrated on $\Sigma \overleftarrow{\times}_S S$.

(local to global argument)

<u>Eg.</u>
$$\begin{tikzcd} \Sigma \arrow[r] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$
semistable curve

$F = \Lambda_X$

Thm 2 (Orgogozo) $f: X \to S$ f. type, $\mathcal{F} \in D^b_c(X, \Lambda)$, then there exists a modification $g: S' \to S$ (surjective birational) s.t.

if
$$\begin{tikzcd} X \arrow[d] & X' = X_{S'} \arrow[l] \arrow[d] & \mathcal{F}' \\ S & S' \arrow[l, "g"'] \end{tikzcd}$$

then $R\Psi_{X'}(\mathcal{F}') \in D^b_c(X' \times_{S'} S', \Lambda)$ and commutes w/ base change.

Strategy of proof.

Reduce thm 2 to thm 1, using proper cohomological descent and de Jong's alterations.

$$\begin{tikzcd} S_1 \arrow[d, "\text{alteration}"'] & S' \arrow[l] \arrow[d, "\text{flat, finite, surj.}"] \\ S & S_2 \arrow[l, "\text{modification}"'] \end{tikzcd}$$

↑
Raynaud

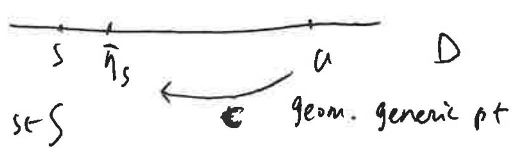
4. Lefschetz pencils

$k = \bar{k}$, $X \hookrightarrow P = \mathbb{P}^n_k$, X projective smooth

$\check{P} \supset D$ Lefschetz pencil, $X_D = X \cap H_D$

axis of D transversal to X

$\exists$ finite $S \subset D$ s.t. $t \in D$, $X_t = \begin{cases} \text{smooth} & \text{if } t \notin S \\ \exists!\ x_s \text{ ordinary quadratic} & \text{if } t \in S \end{cases}$



$n = \begin{cases} 2n' \\ 2n'+1 \end{cases}$ $\quad \Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$

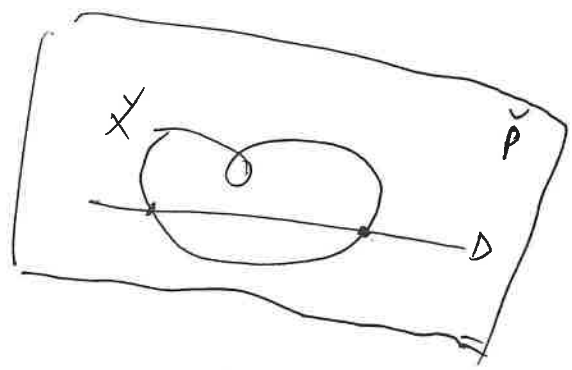
$\ell \neq$ char

$\pm\delta_s \in H^n(X_{\bar\eta_s}, \Lambda(n')) \longmapsto \pm\delta_c \in H^*(X_u, \Lambda(n'))$ $\quad$ vanishing cycle

<u>Thm3</u> (Gabber - Orgogozo)

(1) $\{\pm\delta_c\}$ conjugate under $G = \mathrm{Im}\Big(\pi_1(D - S, u) \to GL\big(H^*(X_u, \Lambda(n'))\big)\Big)$

(2) $G = G_0$, where $G_0 = \mathrm{Im}\big(\pi_1(\check{P} - \check{X}, u) \to GL(-)\big)$



$\check{X} = \{t : H_t$ not transversal to $X\}$

<u>Rmk</u> (1) $p \neq 2$ or $n = 2n'+1$, SGA 7 <u>XVIII</u>

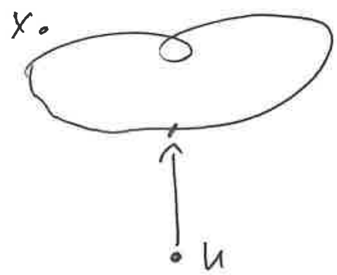
$$\pi_1^{tame}(D - S, u) \twoheadrightarrow \pi_1^{tame}(\check{P} - \check{X}, u)$$

(2) $p = 2$, $n = 2n'$, Weil II, D general, $\Lambda = \mathbb{Q}_\ell$.

$Z = \{x \in X,\ t \in \check{P},\ x \in H_t\}$.
$$\begin{array}{ccc} Z & \longleftarrow & Z_U \\ \downarrow & & \downarrow f_U \\ \check{P} & \longleftarrow & U = \check{P} - \check{X} \end{array}$$

$R^* f_{U*} \Lambda(n')$ lisse (on U)

$\check{X}_0$ good locus



$\check{X}_0 \overset{\leftarrow}{\times}_{\check{P}} U \xrightarrow{p_2} U$

$\pm\delta \subset p_2^* R^* f_{U*}(-)$

local system of rank 1 or 2

$\check{X}_0 \overset{\leftarrow}{\times}_{\check{P}} U$ connected

$$\begin{tikzcd} & u \arrow[dl, "c"'] \arrow[dr, "c'"] & \\ s \arrow[rr, "d"] & & s' \end{tikzcd}$$

d path on $\check{X}_0 \overset{\leftarrow}{\times}_{\check{P}} U$

$\pm\delta_c \longmapsto \pm\delta_{c'}$

$p_2(d) \in \pi_1(\check{P} - \check{X}, u)$