Iwahori – Whittaker category and averaging functor

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$/\overline{\mathbb{F}_p}$, coeff $\overline{\mathbb{Q}_\ell}$

$\boxed{\text{Goal}}$.

$$Av_{IW} : D^b_I(Fl_G) \longrightarrow \text{"} D^b_{IW}(Fl_G) \text{"}$$

$$\mathcal{F} \longmapsto \Delta_0^{IW} * \mathcal{F}$$

$$m : LG \overset{I}{\times} Fl_G \longrightarrow Fl_G$$

Ⓐ t-exact

Ⓑ $P_I \longrightarrow P_{IW}$ fully faithful after quotient $P_I \twoheadrightarrow {}^f P_I$

Simple $(P_I) \longleftrightarrow W$

$IC^I_w \longleftrightarrow w$

Serre quotient.

$${}^f P_I = P_I / \langle IC^I_w \rangle ,\ w \notin {}^f W$$

$W = W_f \ltimes X^\vee$ extended affine Weyl gp

${}^f W$ = minimal length elt in $W_f \backslash W$.

Ⓒ (next time) ${}^f P_I \cong P_{IW}$.

Toy model [B-R, Top. approach to Soergel] $/\overline{\mathbb{F}_p}$. $G \supset B \supset T$, B^+ opposite of B, $U^+ \subset B^+$

$X = G/U$, $Y = G/B$. $Av_\chi : D^b_{U^+}(G/B) \longrightarrow D^b_{U^+, \underline{AS}}(G/B)$

$$\mathcal{F} \longmapsto (a_+)_* (\underline{AS} \boxtimes \mathcal{F}) [\dim U^+]$$

$$a_+ : U^+ \times G/B \longrightarrow G/B$$

Ex $G = PGL_2$, $U^+ = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$

$U^+ \curvearrowright \mathbb{P}^1$, $U \curvearrowright \mathbb{P}^1$. $Av_\chi : D^b_U(\mathbb{P}^1) \longrightarrow D^b_{U^+, AS}(\mathbb{P}^1)$

$\pi : \mathbb{A}^1 \to \mathbb{A}^1$
$x^p - x \leftarrow\!\shortmid x$

$\pi_* \overline{\mathbb{Q}}_\ell \leftarrow$ nontrivial direct summand $\underline{AS}$.

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§1. $k \supset \mathbb{F}_q$, $\mathcal{O} = k[[t]]$, $F = k((t))$, G/k

$W = W_f \ltimes X^\vee \curvearrowright X^\vee$

$w = s \cdot \lambda$

$W_{cox} = W_f \ltimes \mathbb{Z}R^\vee$

$\Omega = \{w \in W : \ell(w) = 0\}$

$\cong X^\vee / \mathbb{Z}R^\vee$ ab gp.

$t(\lambda) = 1 \cdot \lambda$ translation. $\lambda : \mathbb{G}_m \to T$ $\ell(t(\lambda)) = \langle \lambda, 2\rho \rangle$, $\forall \lambda \in X^\vee_+$.

$Fl_G = \bigcup_{w \in W} \underline{\underline{Fl_{G,w}}}$ length $(w) = \dim Fl_{G,w}$

$\qquad\qquad \shortparallel$

$\qquad I\dot{w}I/I$ $\qquad j_w : Fl_{G,w} \hookrightarrow Fl_G$

$\underline{\text{Bruhat order}}$ $w \leq_{Bru} w'$

$\Updownarrow$

$Fl_{G,w} \subset \overline{Fl_{G,w'}}$

Standard Δ^I_w, costandard ∇^I_w

$$\underset{\text{std}}{\mathrm{Im}(\Delta^I_w} \longrightarrow \underset{\text{costd}}{\nabla^I_w)} = \underset{\text{Simple perverse sheaf}}{IC^I_w} \in P_I$$

$\boxed{\text{Facts}}$. $P_I = Perv_I(Fl_G) \hookrightarrow D^b_I(Fl_G)$

① simple obj. in $P_I \longleftrightarrow w \in W$

$IC^I_w \longleftrightarrow w$

② Duality $\Delta_w^I \longleftrightarrow \nabla_w^I$.

③ (<u>tilting exercises</u>) $\forall w \in W$, write $w = \underset{\Omega}{\varpi} \cdot W_{Cox}$.

Then $\mathrm{Soc}(\Delta_w^I) = IC_{\varpi}^I$.

IC_{ϖ}^I mult. 1 in Δ_w^I.

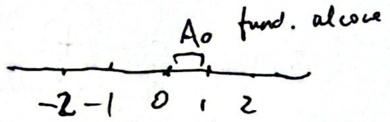
Ex. $w \in W_f$. Δ_w^I, $\varpi = e$, $\mathrm{Soc}(\Delta_w^I) = IC_e^I = \delta_e = \Delta_e = \nabla_e$.

④ $\Delta_w^I * \nabla_{w^{-1}}^I \simeq \delta_e$

If $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, then $\Delta_{w_1}^I * \Delta_{w_2}^I \Rightarrow \Delta_{w_1 w_2}^I$.

Ex $G = PGL_2$, $\mathbb{Z}R^\vee = 2\mathbb{Z} \subset X^\vee = \mathbb{Z}$

$W_f = \langle 1, s \rangle$



$S = \langle s, s_0 \rangle$ for W_{Cox} $\qquad s : x \longmapsto -x$

$s_0 : x \longmapsto 2 - x$

$\lambda \in X^\vee = \mathbb{Z}$ $\qquad t(\lambda) : x \longmapsto x + \lambda$

$\Omega = \{e, \varpi\}$ $\qquad \varpi : x \longmapsto 1 - x$

$\ell(w) = \#$ aff. hyp. separating A_0 & $w(A_0)$

${}^f W =$ minimal length representatives in $W_f \backslash W$.

$$\begin{bmatrix} x \longmapsto x + n, \ n \geq 0 \\ x \longmapsto n - x, \ n \geq 1 \end{bmatrix}$$

Upshot: ${}^f W \cong W_f \backslash W \cong X^\vee$

$w_\lambda \longleftarrow\!\!\!\longmapsto \lambda$

§2 $f: \mathbb{G}_a \to \mathbb{G}_a$, $x \mapsto x^p - x$.

$AS = \oplus(f_* \overline{\mathbb{Q}}_p)^{\sim}$ where $\mathbb{Z}/p$ acts by $\mu_p \hookrightarrow \overline{\mathbb{Q}}_\ell$

character sheaf

* Key: $H^\bullet(\underline{AS}) = H^\bullet_c(\underline{AS}) = 0$

Affine. $G \curvearrowright X$ $\mathcal{L}$ on G character sheaf.

$\mathcal{F}$ sheaf on X is $(G, \mathcal{L})$-equiv. if $a^* \mathcal{F} \simeq \mathcal{L} \boxtimes \mathcal{F}$ + compatibility

In fact, $U(F)$ not cpt. use pro-p Iwahori $I^+_u = pr^{-1}(u^+) \subset I^+$

I^+ opposite Iwahori, $pr: I^+ \twoheadrightarrow B^+$

pro-p unipotent.

$$\chi: I^+_u \to u^+ \to u^+/[u^+, u^+] \cong \prod_{\alpha \text{ simple}} \mathbb{G}_a \xrightarrow{\Sigma} \mathbb{G}_a$$

AS

$\chi^* \underline{AS}$.

$D^b_{IW}(Fl_G) = (I^+_u, \chi^* \underline{AS})$ - eq. sheaves on Fl_G.

triangulated cat., perverse t-str., $P_{IW} \hookrightarrow D^b_{IW}$.

Obs. ① right convolution. $*: D^b_{IW} \times D^b_I \to D^b_{IW}$

② $\boxed{I^+_u\text{-orbits on } Fl_G}$ $(= I^+\text{-orbits})$ $w \in W$, $Fl^{IW}_w := I^+_u \dot{w} I / I$

$\boxed{\text{Fact}}$: $\exists\ \chi_w: Fl^{IW}_w \to \mathbb{G}_a$ defined by $\chi_w(g w I) = \chi(g)$, $g \in I^+_u$

iff $w \in {}^f W$. $(\chi|_{\mathrm{stab}_{I^+_u}(wI)} = 0)$.

Cor Only Fl_w^{Iw} $(w \in {}^f W)$ supp. rk 1 $(I_u^+, x^*\underline{AS})$-eq. local system

$\lambda \in X^\vee \rightsquigarrow w_\lambda \in {}^f W$ $\quad \mathcal{L}_\lambda := x_{w_\lambda}^*(\underline{AS})$

Define $\mathrm{Im}\left(\Delta_\lambda^{Iw} \longrightarrow \nabla_\lambda^{Iw}\right) = IC_w^{Iw}$

$$\begin{matrix} \Delta_\lambda^{Iw} \\ \| \\ \hat{j}_{!,w_\lambda}^{Iw} \longrightarrow \hat{j}_{*,w_\lambda}^{Iw} \end{matrix}$$

Prop. $\mathrm{Im} = IC_\lambda^{Iw} \in P_{Iw}$ simple.

$$\mathrm{Hom}(\Delta_\lambda^{Iw}, \nabla_\mu^{Iw}[n]) = \begin{cases} \bar{\mathbb{Q}}_\ell, & \lambda = \mu, n = 0 \\ 0, & \text{else} \end{cases}$$

Cor. P_{Iw} is a highest weight cat. ($\Rightarrow$ tilting obj.)

weight poset $= X^\vee$, std ∇_λ^{Iw}, costd Δ_λ^{Iw}.

Define a new order

Def. $\lambda \leq \mu \iff Fl_{w_\lambda}^{Iw} \subset \overline{Fl_{w_\mu}^{Iw}}$

NOT the same as $\lambda \leq_{Bru} \mu$. ① (same if $\lambda, \mu \in X_+^\vee$)

② $\mu \in X_+^\vee$, then $\lambda \leq \mu$ iff $V(\mu)_\lambda \neq 0$

③ $\lambda \leq \mu$ then $\mu - \lambda \in \mathbb{Z}R^\vee$

Δ^{I}_{w} - - $\overset{?}{-}$ - - Δ^{IW}_{λ}

$w \in W$ $\qquad w_\lambda \longleftrightarrow \lambda$ $\qquad P_{IW} \hookrightarrow D^b_{IW}$

$\qquad \in {}^f W \quad \in X^\vee$

Note $\overline{Fl^{IW}_{0}} = \overline{I_u^+ e I/I} = L^+G/I \cong G/B$

has no other $Fl^{IW}_{w_\lambda}$ $(\lambda \neq 0)$

$\Rightarrow \Delta^{IW}_{0} = \nabla^{IW}_{0} = IC^{IW}_{0} = \delta^{IW}_{0} \in P_{IW}$

kernel sheaf.

Def. $Av_{IW} : D^b_I(Fl_G) \to D^b_{IW}(Fl_G)$

$\mathcal{F} \longmapsto \delta^{IW}_{0} * \mathcal{F}$.

Goal (A) t-exact

(B) ${}^f P_I \xrightarrow{Av_{IW}} P_{IW}$ fully faithful. (left inverse or section)

$\quad ||$

$P_I / \langle IC^I_w, w \notin {}^f W \rangle$

(A) Lemma. $Av(IC^I_w) = 0$ if $w \notin {}^f W$

$\delta^{IW}_{0} * IC^I_w$

Pf. $\exists s \in W_f$ simple s.t. $sw < w$. $\quad IsI \overset{I}{\times} IwI/I \to \overline{IwI/I}$.

IC^I_w is J_s-equivariant. $I \subset J_s$ parahoric. $J_s/I \cong \mathbb{P}^1$

$\exists\, \mathcal{F} \in D^b_{J_S}$, $\quad \mathrm{For}^{J_S}_{I} \mathcal{F} = IC^{\pm}_{w}$

$\pi_0 : {}_I\backslash Fl_G \to {}_{J_S}\backslash Fl_G$

$$\delta_0^{Iw} * IC^I_w = \left((\pi_0)_* \Delta_0^{Iw} \right) \overset{J_S}{*} \mathcal{F} = 0.$$

$\underset{0}{\shortparallel}$ $\leftarrow$ check for stalks.

② $w \in W$, $w = x\, w_\lambda$, $x \in W_f$, $w_\lambda \in {}^f W$,

Then $Av_{Iw}(\Delta^I_w) = \Delta^{Iw}_{w_\lambda}$ $\quad (\nabla \to \nabla)$

$\underline{\text{Pf}}$. $\boxed{x=1}$ We $\quad Fl_0^{Iw} \tilde{\times} Fl_{G,w} \cong Fl^{Iw}_{G,w}$

$I_u^+ e I \overset{I}{\times} IwI/I \to I_u^+ wI/I$.

$w_0 \in W_f$ longest elt.

$I^+ = w_0 I w_0$.

$\boxed{x \neq 1}$ $\underline{\text{Tilting exercises}}$.

$\exists\, IC^I_e \hookrightarrow \Delta^I_x \to \text{coker}$

$\uparrow$

$- * \Delta^I_{w_\lambda}$ $\Big\}$ $\quad$ composition factor IC^I_y, $y \in W_f$, $y \neq e$

$\Rightarrow y \notin {}^f W$

$\Delta^I_{w_\lambda} \to \Delta^I_w \to \underline{\text{cone}}$

gen. by $IC^I_y * \Delta^I_{w_\lambda}$

$Av.(IC^I_y) = \delta.$ $\quad \square$

Ⓐ $\underline{t\text{-exactness}}$.

$$\left[\begin{array}{ll} D_I^{b,p\geq 0} = \langle \Delta_w^I[n] \rangle_{\substack{n\in\mathbb{Z}_{\geq 0} \\ w\in W}} & \\ & \Rightarrow t\text{-exact.} \\ D_I^{b,p\leq 0} = \langle \nabla_w^I[n] \rangle_{\substack{n\leq 0, \\ w\in W}} & \end{array}\right.$$

Ⓑ $w \in {}^f W, \quad Av(IC_w^I) = IC_{w_\lambda}^{IW} \neq 0$

Ⓑ $Av : P_I \longrightarrow P_{IW}$

$\hookleftarrow$ h.w. cat.

${}^f P_I = P_I / \langle IC_w^I \rangle \quad w \notin {}^f W$

antispherical cat.

$\underline{\text{Goal}}$: $\delta_e^I \xrightarrow{Av} \delta_e^{IW}$

?

$\boxed{\text{Claim}}$: $\exists\ \Xi : P_{IW} \longrightarrow P_I$ s.t. ${}^f\Xi \circ Av_{IW} = id$ in ${}^f P_I$.

$\Rightarrow$ Ⓑ : Av_{IW} is injective on Ext^1 (by splitting)

$\uparrow$

$Av_{IW}: {}^f P_I \to P_{IW}$

Av_{IW} (simple) = simple

induction on length of A, B to show Av_{IW} preserves hom.

Obs δ_e^{Iw} is $(I_u^+, x^*\underline{AS})$-equiv. $I_0 = I \cap I_u^+ = \ker(I \to B)$

I_0 - equiv.

Obs left induction = left av. along constant local system.

$${}^*\mathrm{Ind}_{I_0}^{I} : D^b_{I_0} \longrightarrow D^b_I$$

$$\mathcal{F} \longmapsto a_*(\overline{\mathcal{G}_e} \tilde{\boxtimes} \mathcal{F})[\dim B]$$

Thm. [BR18] Lemma 12.1

$\exists$ morphism in D^b_I.

$$A_e^I \longrightarrow {}^*\mathrm{Ind}_{I_0}^{I}(\Delta_0^{Iw})[-\dim T] \longrightarrow \text{cone}$$

Cone is gen. by $IC_w^I[n]$, $w \in W_f$, $n \in \mathbb{Z}_{\leq 0}$

$n = 0$, then $w \neq 0$.

$${}^p\mathcal{H}^0({}^*\mathrm{Ind}_{I_0}^{I}(-)[\dim T]) \in P_I$$