Around the Thom–Sebastiani theorem

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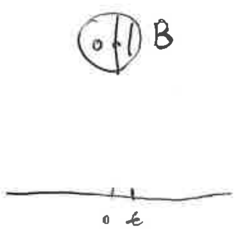
1. $/\mathbb{C}$. $\quad f : (\mathbb{C}^{m+1}, 0) \longrightarrow (\mathbb{C}, 0)$ $\quad \{0\}$ isolated critical point

$$x = (x_0, x_1, \dots, x_m) \mapsto f(x), \qquad f(0) = 0.$$



$$M_f = f^{-1}(t) \cap B \sim \underbrace{S^m \vee \cdots \vee S^m}_{\mu}$$

$$\mu_f = \dim_{\mathbb{C}} \mathbb{C}\langle x_0, \dots, x_m\rangle / \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_m}\right)$$

$$\tilde{H}^q(M_f, \mathbb{Z}) = \begin{cases} 0 & , q \neq m \\ \mathbb{Z}^{\mu}, & q = m \end{cases}$$

$$\shortparallel$$

$$R^q \phi_f(\mathbb{Z}) \circlearrowleft T \qquad \text{quasi-unipotent}$$

$$g : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0), \qquad \{0\} \text{ isolated critical pt}$$

$$y \longmapsto g(y)$$

$$f \oplus g : (\mathbb{C}^{m+n+2}, 0) \longrightarrow (\mathbb{C}, 0)$$

$$(f \oplus g)(x, y) = f(x) + g(y) \qquad \mu_{f \oplus g} = \mu_f \cdot \mu_g$$

<u>Th. (Thom–Sebastiani)</u> Construct an isom.

$$R^m \phi_f(\mathbb{Z}) \otimes R^n \phi_g(\mathbb{Z}) \xrightarrow{\sim} R^{m+n+1} \phi_{f \oplus g}(\mathbb{Z})$$

$$T_f \otimes T_g = T_{f \oplus g}$$

2. $/k$ $\quad$ $\mathrm{char}(k) = p$, $\quad$ k alg. closed

$i = 1, 2$, $\quad f_i : \mathbb{A}_k^{n_i+1} \longrightarrow \mathbb{A}_k^1$,

f_i $\quad \{0\}$ isolated pt of non smoothness

$n = n_1 + n_2$, $\quad \mathbb{A}_k^{n_1+1} \underset{k}{\times} \mathbb{A}_k^{n_2+1} \xrightarrow[]{\overset{f}{\overset{\shortparallel}{f_1 \times_k f_2}}} \mathbb{A}_k^1 \underset{k}{\times} \mathbb{A}_k^1 \xrightarrow{a} \mathbb{A}_k^1$

$(x, y) \longmapsto a(x,y) = x + y$

$f_1 \oplus f_2 = a \circ (f_1 \times_k f_2)$

$\Lambda = \mathbb{Z}/\ell^\nu, \ \mathbb{Z}_\ell, \ \mathbb{Q}_\ell, \ \overline{\mathbb{Q}_\ell}$

$R\Phi_{f_i}(\Lambda)[n_i]$ $\quad$ perverse

$$R^q\Phi_{f_i}(\Lambda) = \begin{cases} 0, & q \neq n_i \\ \Lambda^{r_i}, & q = n_i \end{cases}$$

$$R^q\Phi_{af}(\Lambda) = \begin{cases} 0, & q \neq n+1 \\ \Lambda^r, & q = n+1 \end{cases}$$

<u>Question</u>. Can one find an isom. $R^{n_1}\Phi_{f_1}(\Lambda)_0 \otimes R^{n_2}\Phi_{f_2}(\Lambda)_0 \xrightarrow{\sim} R\Phi_{af}(\Lambda)_0$?

No! $\quad$ 2 kinds of obstructions.

(1) <u>Wild ramification</u>. $\quad \mu_{f_i} \geq r_i$

$>$ possible

$\mu_{f_i} = \mathrm{tot.\ dim}\ R^{n_i}\Phi_{f_i}(\Lambda)_0 = r_i + s_i$, $\quad s_i = \mathrm{Sw}(R^{n_i}\Phi_{f_i})(\Lambda)$

$\mu_{af} = \mu_{f_1} \cdot \mu_{f_2} \qquad (r_1 + s_1)(r_2 + s_2) = r + s$

$r_1 r_2 \neq r$ in general

$n_1 = n_2 = 0, \qquad f_1 = f_2 : \mathbb{A}^1_{\mathbb{F}_p} \longrightarrow \mathbb{A}^1_{\mathbb{F}_p}. \quad f_i(x) = x^2$

$f_1 \oplus f_2 = af : \mathbb{A}^2 \to \mathbb{A}^1, \quad (x_1, x_2) \mapsto x_1^2 + x_2^2.$

$$\left[\begin{array}{ll} R^0 \phi_{f_i} (\Lambda)_0 = \Lambda(\chi), & \chi(\sigma) = \dfrac{\sigma(\sqrt{t})}{\sqrt{t}} \\ R^1 \phi_{af} (\Lambda)_0 = \underset{\substack{\uparrow \\ \text{Tate twist}}}{\Lambda(-1)} \otimes \varepsilon & , \ \varepsilon(F_p) = \left(\frac{-1}{p}\right) \end{array}\right.$$

1981 Deligne replace $\otimes$ by local convolution product.

$$\begin{array}{c} A_h \\ \shortparallel \\ \{0\} \longrightarrow A^1_h \xhookleftarrow{j} \eta \\ \shortparallel \\ \text{hensel. of } \mathbb{A}^1 \\ \text{at } 0 \end{array}$$

$M_i \in Sh(\eta, \Lambda)$ free f.t. $(i = 1, 2)$

$$(A^1_h \underset{k}{\times} A^1_h)_h =: A^2_h \xrightarrow{a} A^1_h$$

$$R\phi_a (j_1! M_1 \boxtimes j_2! M_2)_0 \in D^b_c (A_h, \Lambda)_0$$

$$R^q \phi_a (j_! M_1 \boxtimes j_2! M_2)_0 = \left[\begin{array}{ll} 0 & , q \neq 1 \\ M_1 \underset{1}{*} M_2 & \end{array}\right.$$

$*$ Laumon

Deligne's formula: $R^{n_1}\phi_{f_1}(\Lambda)_0 \ast_1 R^{n_2}\phi_{f_2}(\Lambda)_0 \xrightarrow{\sim} R^{n+1}\phi_{af}(\Lambda)_0$

Fu Lei: $F^{(0,\infty')}$

$\ast_1 \longleftrightarrow \otimes$ + Gabber's formula for $R\psi \otimes R\psi$.

$[X_1 \underbrace{\times_S}_{\text{trait}} X_2]$

$$\begin{array}{ccc} x_i & \to & X_i \\ \downarrow & & \downarrow f_i \\ \{0\} & \to & A_h \end{array}$$

$K_i \in D_{ctf}(\Lambda)$

Deligne's conjecture

(D) Assume (f_i, K_i) ULA outside x_i, then

$$R\phi_{f_1}(K_1)_{x_1} \overset{L}{\ast} R\phi_{f_2}(K_2)_{x_2} \xrightarrow{\sim} R\phi_{af}(K_1 \overset{L}{\boxtimes} K_2)_x,$$

$x = (x_1, x_2)$

$$\begin{array}{ccccc} X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & A_h \times A_h & & \\ \uparrow & & \uparrow & & \\ \square & \xrightarrow{f} & A_h^2 & \xrightarrow{a} & A_h \end{array}$$

Th 1. (D) is true, ULA assumption superfluous

$$\begin{array}{ccc} X_1 & X_2 & X_1 \times_k X_2 \\ \downarrow & \downarrow & \downarrow \\ A^1 & A^1 & \boxed{A^1 \times_k A^1} \end{array}$$

compatibility of $R\psi$ w/ external products:

Σ

3. Review of nearby cycle over general bases

Deligne (1981)

$$\begin{array}{ccccc} & & T & & \\ & \swarrow q_1 & \downarrow h & q_2 \searrow & \\ & & X \overleftarrow{\times}_S Y & & \\ & \swarrow p_1 & & p_2 \searrow & \\ X & & \overset{\tau}{\Leftarrow} & & Y \\ & f \searrow & & \swarrow g & \\ & & S & & \end{array}$$

$\tau: g p_2 \to f p_1$

$t: g q_2 \to f p_1$

$\to \exists!\, h, \quad t = \tau \circ h$

$$\begin{array}{ccccc} U & \to & V & \leftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \to & S & \leftarrow & Y \end{array}$$

$$\mathrm{Pts}\left(X \overset{\leftarrow}{\times}_S Y\right) = \left(x \to X,\ y \to Y,\ f(x) \xleftarrow{\text{specialization}} g(y) \right)$$

$X \overset{\leftarrow}{\times}_S S$ vanishing topos

$S \overset{\leftarrow}{\times}_S Y$ co-vanishing topos (Faltings, Abbes-Gros)

$$\begin{array}{ccc} x \overset{\leftarrow}{\times}_X X & \longrightarrow & X_{(x)} \\ \downarrow & & \downarrow f_{(x),s} \\ s \overset{\leftarrow}{\times}_S S & = & S_{(s)} \end{array}$$

$$\begin{array}{ccc} x & \longrightarrow & X \\ \downarrow & & \downarrow f \\ s & \longrightarrow & S \end{array}$$

geom. pts

$$\begin{array}{ccccc} & & X & & \\ & \Vert & \downarrow \Psi_f & \searrow f & \\ X & \xleftarrow{p_1} & X \overset{\leftarrow}{\times}_S S & \xrightarrow{p_2} & S \\ & \searrow f & & \Vert & \\ & & S & & \end{array}$$

$$R\Psi_f : D^+(X, \Lambda) \longrightarrow D^+(X \overset{\leftarrow}{\times}_S S, \Lambda)$$

S trait, s, η, $\quad X_s \overset{\leftarrow}{\times}_S \eta$

$R\Psi_f(K)\big|_{X_s \overset{\leftarrow}{\times}_S \eta}$ usual $R\Psi$

$$p_1^* K \longrightarrow R\Psi_f K \longrightarrow R\Phi_f K$$

$R\Phi_f(K)\big|_{X_s \overset{\leftarrow}{\times}_S \eta}$ usual $R\Phi$

Pages

$\mathcal{F} \in Sh(X \overleftarrow{\times}_S S, \Lambda)$ constructible $\quad \mathcal{F}|_{X_\alpha \overleftarrow{\times}_S S_\beta}$ lisse constructible

$$\begin{array}{l} X \\ \downarrow f t \\ S \text{ noetherian} \end{array} \qquad D^b_c(--)$$

<u>Def</u> (f, K) <u>Ψ-good</u> if $R\Psi_f K \in D^b_c(X \overleftarrow{\times}_S S, \Lambda)$ and base change compatible

$K \in D^b_c(X, \Lambda)$

<u>Ex</u> ① $\dim S \leq 1$, $\quad (f, K)$ Ψ-good (Deligne)

② (f, K) ULA $\iff$ $R\Phi_f K = 0$ universally

③ (f, K) ULA outside $\Sigma \subset X$ $\Rightarrow$ (f, K) Ψ-good ($\Rightarrow$ flatness of Ψ)

$\Sigma \xrightarrow{\text{quasi-finite}} S$

<u>Th. (Orgogozo)</u> $\underset{\text{noeth}}{X \longrightarrow S}$ ft $\quad$. $\Lambda = \mathbb{Z}/\ell^\nu$, ℓ invertible on S

$\rightarrow$ $\exists$ modification $g: S' \to S$ s.t. (f', K') Ψ-good.

<u>Künneth formula</u>

$$\begin{tikzcd} K_1 \quad X_1 \arrow[d, "f_1"'] & & X_2 \arrow[d, "f_2"] \quad K_2 \\ Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{array}{l} X = X_1 \times_S X_2 \\ f \downarrow \\ Y = Y_1 \times_S Y_2 \\ \downarrow \\ S \end{array}$$

(*)
$$R\Psi_{f_1} K_1 \overset{L}{\boxtimes} R\Psi_{f_2} K_2 \longrightarrow R\Psi_f(K) \quad , \quad K = K_1 \overset{L}{\boxtimes} K_2$$
$K_i \in D_{ctf}(X_i, \Lambda)$

Th 2 Assume (f_i, K_i) Ψ-good, then $(*)$ = isom and (f, K) Ψ-good

$$\begin{tikzcd} X_1 \arrow[d, "\Psi_{f_1}"'] & X \arrow[l] \arrow[r] \arrow[d, "\Psi_f"] & X_2 \arrow[d, "\Psi_{f_2}"] \\ X_1 \overleftarrow{\times}_{Y_1} Y_1 & X \overleftarrow{\times}_{Y} Y \arrow[l] \arrow[r] & X_2 \overleftarrow{\times}_{Y_2} Y_2 \end{tikzcd}$$

Th 2 $\Rightarrow$ Th 1

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\Psi_f"] \arrow[dr, "\Psi_{gf}"'] & X \overleftarrow{\times}_Y Y \arrow[d, "\overleftarrow{g}"] \\ & X \overleftarrow{\times}_Z Z \end{tikzcd}$$

$$R\Psi_{gf} = R\overleftarrow{g}_* R\Psi_f$$

$g = a : A_k^2 \to A$, $\quad R\Psi K = R\Psi K_1 \overset{L}{*} R\Psi K_2$
$\quad R\Phi K = R\Phi K_1 \overset{L}{*} R\Phi K_2$] a loc. acyclic

(geom. const) $\overset{L}{*} j_! V_1 = 0$

$\Downarrow$

$R\overleftarrow{a}_*$

Tame case $/k = \bar{k}$, $\quad *_1 = \otimes$.

$g \in I_t = \hat{\mathbb{Z}}'(1)$

$V_1 \quad V_2$

$(V_1)_{\bar{\eta}} \quad (V_2)_{\bar{\eta}}$

$$g^* \big| (V_1 \underset{1}{*} V_2) = g_1^* \underset{1}{*} g_2^*$$

$$\begin{tikzcd}
(X_1 \times_k X_2)|_{A^2_k} \arrow[d, "f"] \arrow[dd, bend right=60, "af"'] \\
A^2_k \arrow[d, "a_{(0,0)}"] \\
A_k
\end{tikzcd}$$

$$R\Psi(K_1 \overset{L}{\boxtimes} K_2) = R\Psi_{f_1} K_1 \boxtimes R\Psi_{f_2} K_2$$

$$R\overleftarrow{a}_* R\Psi(K) = R\Psi_{af}(K)$$

$$\shortparallel$$

$$R\overleftarrow{a}_*(R\Psi K_1 \overset{L}{\boxtimes} R\Psi K_2)$$

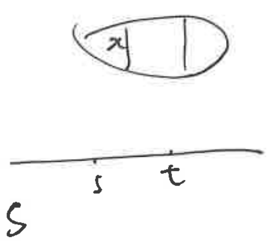
$$R\Psi_{af}(K)_0 = R a_{(0,0)*}\left((R\Psi)_0 \boxtimes (R\Psi)_0\right)$$

$$=: R\Psi \overset{L}{*} R\Psi.$$

$$\begin{tikzcd}
X_{(x)} \arrow[d] \\
S_{(s)}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{(x)} \arrow[d, "\Psi"'] \arrow[dr, "f(x,0)"] & \\
X_{(x)} \overleftarrow{\times}_{S_{(s)}} S_{(s)} \arrow[r, "p_2"] & S_{(s)} \arrow[l, bend left, "\sigma_x"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{(x)} \arrow[d] \\
S_{(s)}
\end{tikzcd}$$

$$\sigma^* R\Psi = Rf_{(x,0)*}$$

$$\sigma^* = p_{2*}$$



$$R\Psi_f(K)_{(x \to s \leftarrow t)} = R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, K)$$

$$\neq R\Gamma(X_{(x)t}, K)$$