Representations of affine Hecke algebras

Misha Finkelberg

Lecture 3.

1. Bernstein – Lusztig presentation of the (extended) affine Hecke algebra $\mathcal{H} \simeq \mathcal{H}_{fin} \ltimes \mathbb{Z}[X]$

Also, $\mathcal{H} \ltimes \mathbb{Z}[X_{aff}]$ = the double affine Cherednik – Hecke algebra DACHA

not a Hecke – Coxeter algebra.

2. Steinberg variety of triples $Z = \tilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \tilde{\mathcal{N}} = T^*\mathcal{B} \underset{\mathcal{N}}{\times} T^*\mathcal{B}$

$K^{\mathcal{G}}(Z) \xleftarrow{\simeq} \mathcal{H}$, $\quad \mathcal{G} = G \times \mathbb{C}^*$.

Z = the zero level of the moment map $T^*(\mathcal{B} \times \mathcal{B}) \to (\mathrm{Lie}\, G)^* = \mathrm{Lie}\, G$

$$T^*\mathcal{B} = \{(\mathfrak{b}, x) : x \in \mathrm{rad}\, \mathfrak{b}\} \qquad T^*(\mathcal{B} \times \mathcal{B}) = \{(\mathfrak{b}_1, x_1, \mathfrak{b}_2, x_2)\}$$

$$\begin{array}{cccc} T^*\mathcal{B} & (\mathfrak{b},x) & T^*(\mathcal{B}\times\mathcal{B}) & (\mathfrak{b}_1, x_1, \mathfrak{b}_2, x_2) \\ \downarrow \mu & \downarrow & \downarrow \mu & \downarrow \\ \mathcal{N} \subset \mathrm{Lie}\, G & x \in \mathcal{N} & \mathrm{Lie}\, G & x_1 - x_2 \end{array}$$

$Z = \sqcup T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$, $\quad \mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} Y_w$ $\quad G$- orbits

$Y_e = \Delta_{\mathcal{B}}$, $\; T^*_{\Delta_{\mathcal{B}}}(\mathcal{B} \times \mathcal{B})$ closed in Z, isom. to $T^*\mathcal{B}$

Y_{w_0} = the open orbit, $\; T^*_{Y_{w_0}}(\mathcal{B} \times \mathcal{B})$ = the zero section over Y_{w_0}

open in Z

$G = SL_2$, $\quad \mathcal{B} \times \mathcal{B} = \mathbb{P}^1 \times \mathbb{P}^1$, $\quad Z = T^*_\Delta \mathbb{P}^1 \sqcup (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$

Polynomial representation of $K^G(Z)$

$$\begin{array}{lll} H \subset G & \text{a reductive subgroup,} & Y \subset \mathcal{N} \quad H\text{-inv. subvar.} \\ \| & & \| \\ G & & O \qquad\qquad \tilde{Y} \subset \tilde{\mathcal{N}} \text{ the preimage of } Y \\ & & \tilde{Y} = \mathcal{B} \end{array}$$

$$Z \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \supset \tilde{\mathcal{N}} \times \tilde{Y}$$

$$Z \cap (\tilde{\mathcal{N}} \times \tilde{Y}) = Z \cap (\tilde{Y} \times \tilde{Y})$$

$$\begin{tikzcd} \tilde{\mathcal{N}} \times \tilde{Y} \arrow[d, "p'"', "\text{smooth}"] \\ \tilde{Y} \end{tikzcd} \quad , \quad Z \cap (\tilde{Y} \times \tilde{Y}) \xrightarrow[\text{proper}]{p''} \tilde{Y} \quad \text{the first projection}$$

Given $\mathcal{F} \in Coh_G(Z)$, $\mathcal{F}' \in Coh_H(\tilde{Y})$, set

$$\mathcal{F} * \mathcal{F}' := p''_*\left(\mathcal{F} \overset{L}{\underset{\mathcal{O}_{\tilde{\mathcal{N}}^2}}{\otimes}} p'^* \mathcal{F}'\right) \in K^H(\tilde{Y})$$

a structure of $K^G(Z)$-module on $K^H(\tilde{Y})$.

In case $Y = 0$, $\tilde{Y} = \mathcal{B}$, $H = G$, $K^H(\tilde{Y}) = K^G(\mathcal{B}) = K^{B \times \mathbb{C}^*}(pt) = K^{T \times \mathbb{C}^*}(pt)$

$$\| \\ R(T)[q^{\pm 1}]$$

$$K^G(Z) \supset K^G(Z_\Delta) = K^G(T^*\mathcal{B}) = K^G(\mathcal{B}) = R(T)[q^{\pm 1}]$$

The action of on $R(T)[q^{\pm 1}]$ is the multiplication action.

The generators T_s act by the Demazure - Lusztig operators.

$$\exp(\lambda) \mapsto \frac{\exp(\lambda) - \exp(s\lambda)}{\exp(\alpha) - 1} - q \frac{\exp(\lambda) - \exp(s\lambda + \alpha)}{\exp(\alpha) - 1}, \quad s = s_\alpha$$

Nil-Hecke algebra. $K^G(Z)$ $\qquad Z \subset T^*\mathcal{B} \times T^*\mathcal{B} \supset \mathcal{B}\times\mathcal{B}$

$\qquad\qquad \| \qquad\qquad\qquad\qquad \pi \downarrow$

$\qquad K^G(\mathcal{B}\times\mathcal{B}) \qquad\qquad \mathcal{B}\times T^*\mathcal{B} \xleftarrow{i}$ zero section.

injective.

Fact: $i^*\pi_* : K^G(Z) \to K^G(\mathcal{B}\times\mathcal{B})$ is an homomorphism

The presentation of $\mathcal{H}_{nil}$ is obtained from the B-L presentation of $\mathcal{H}$

via $q \mapsto 0$. $\qquad (T_s - q)(T_s+1) = 0 \rightsquigarrow T_s(T_s - 1) = 0$

$G = SL_2$, generators of $\mathcal{H}$, T, $\theta^{\pm 1}$, relations $(T+1)(T-q) = 0$

$$T\theta^{-1} - \theta T = (1-q)\theta$$

$$c = -T-1 \Rightarrow c^2 = -(q+1)c,$$

$$c\theta^{-1} - \theta c = q\theta - \theta^{-1}.$$

Set $\Xi : \mathcal{H} \longrightarrow K^G(Z)$, $\quad \theta \mapsto \mathcal{O}_{\mathbb{P}^1}(-1)$, $\theta^{-1} \mapsto \mathcal{O}_{\mathbb{P}^1}(1)$

$c \mapsto qQ$, where $Q = \mathcal{O}_{\mathbb{P}^1_1} \boxtimes \Omega^1_{\mathbb{P}^1_2} = \Omega^1_{\mathbb{P}^1_1\times\mathbb{P}^1_2/\mathbb{P}$

Lusztig: $(T+v^{-1})(T-v) = 0$,

$v = \sqrt{q}$

$Q' = \mathcal{O}(-1)\boxtimes\mathcal{O}(-1$

Claim: Ξ is a homomorphism. $\qquad \mathbb{P}^1 \underset{i}{\overset{\pi}{\leftrightarrows}} T^*\mathbb{P}^1$

Koszul complex on $T^*\mathbb{P}^1$:

$$0 \longrightarrow \mathcal{O}_{T^*\mathbb{P}^1} \longrightarrow \pi^*\Omega^1_{\mathbb{P}^1} \longrightarrow i_*\Omega^1_{\mathbb{P}^1} \longrightarrow 0.$$

$$\Rightarrow 0 \to \mathcal{O}_{\mathbb{P}^1}\boxtimes\mathcal{O}_{T^*\mathbb{P}^1} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}^1}\boxtimes\pi^*\Omega^1_{\mathbb{P}^1} \to Q \to 0,$$

δ is linear in the fibers of $T^*\mathbb{P}^1$

To make it equivariant, have to $\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1} \rightsquigarrow \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1} \cdot q^{-1}$

$$\Rightarrow qQ = q\left(\mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega^1_{\mathbb{P}^1}\right) - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1}$$

$$qQ * qQ = \left(q\, \mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega^1_{\mathbb{P}^1} - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1}\right) * \left(q\mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1}\right)$$

$$= q \underset{\substack{\parallel \\ -1}}{\langle \pi^*\Omega^1_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1} \rangle} q\mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1} - \underset{\substack{\parallel \\ 1}}{\langle \mathcal{O}_{T^*\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1} \rangle} q\mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1}$$

$$\left(\langle \underset{T^*\mathbb{P}^1}{\mathcal{F}}, \underset{\mathbb{P}^1}{\mathcal{F}'} \rangle := \chi R\Gamma\left(\mathbb{P}^1, \mathcal{F} \overset{L}{\otimes}_{\mathcal{O}_{T^*\mathbb{P}^1}} \mathcal{F}'\right) \right)$$

$$= -(q+1)\, qQ. \qquad ✓$$

$$Z \overset{\bar{\pi}}{\hookrightarrow} \mathbb{P}^1 \times T^*\mathbb{P}^1 \overset{\bar{\iota}}{\hookleftarrow} \mathbb{P}^1 \times \mathbb{P}^1, \quad \varphi = \bar{\iota}^* \bar{\pi}_* : K^G(Z) \longrightarrow K^G(\mathbb{P}^1 \times \mathbb{P}^1)$$

an injective homomorphism.

have to check $\overbrace{\varphi(qQ) + \varphi(\mathcal{O}(1)) - \varphi(\mathcal{O}(-1)) + \varphi(qQ)}^{LHS} = \overbrace{q\,\varphi(\mathcal{O}(-1))}^{RHS}$

$- \varphi(\mathcal{O}(1))$.

<u>Künneth formula</u>: $K^G(\mathbb{P}^1 \times \mathbb{P}^1) = K^G(\mathbb{P}^1) \otimes_{K^G(pt)} K^G(\mathbb{P}^1) = R(T)[q^{\pm 1}] \otimes_{R(G)[q^{\pm 1}]} R(T)[q^{\pm 1}]$

$$\varphi(qQ) = \bar{\iota}^* \bar{\pi}_* (qQ) = q\,\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2) - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}$$

$$\bar{\iota}^* \bar{\pi}_* \mathcal{O}(n) = \mathcal{O}_\Delta(n), \quad \forall n \in \mathbb{Z}$$

If a line bdle $\mathcal{L}_\Delta$ is supported on $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$, then $\forall\, \mathcal{F} \in K^G(\mathbb{P}^1 \times \mathbb{P}^1)$

$$\mathcal{F} * \mathcal{L}_\Delta = pr_2^* \mathcal{L}_\Delta \otimes \mathcal{F}, \quad \mathcal{L}_\Delta * \mathcal{F} = pr_1^* \mathcal{L}_\Delta \otimes \mathcal{F}.$$

$$\mathrm{LHS} = \tau^* \bar{\pi}_* (q\mathcal{Q}) + \mathcal{O}_\Delta(1) - \mathcal{O}_\Delta(-1) + \tau^* \bar{\pi}_* (q\mathcal{Q})$$

$$= q\,\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$$

$$- q\,\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2) + \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}.$$

To compute the RHS, replace $\mathcal{O}_\Delta(\pm 1)$ by resolutions: (Beilinson resolution)

$$0 \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_\Delta \to 0$$

$\square$

<u>Claim</u>: $\Xi : \mathcal{H} \to K^G(Z)$ is an isom.

$Z = T^*_\Delta(\mathbb{P}^1 \times \mathbb{P}^1) \sqcup (Y_{w_0} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$

closed open

<u>Proof</u>: $\mathcal{H}_0 \subset \mathcal{H}$ is generated by $\theta^{\pm 1}$.

Then $\Xi(\mathcal{H}_0) \subset K^G(Z_\Delta)$, and $\Xi|_{\mathcal{H}_0}$ is a tautological isomorphism

$R(T)[q^{\pm 1}] \xrightarrow{\sim} K^G(Z_\Delta)$

The cellular fibration lemma

dévissage

$$0 \to K^G(Z_\Delta) \to K^G(Z) \to K^G(T^*_{Y_{w_0}}(\mathbb{P}^1 \times \mathbb{P}^1)) \to 0$$

Thom isomorphism: $K^G(Y_{w_0}) \simeq K^G(\mathbb{P}^1) = R(T)[q^{\pm 1}]$

$Y_{w_0} \subset \mathbb{P}^1 \times \mathbb{P}^1$ $\Rightarrow K^G(T^*_{Y_{w_0}}(\mathbb{P}^1 \times \mathbb{P}^1))$ is a free $R(T)[q^{\pm 1}]$-mod

$\downarrow pr_1$ affine fibration w/ generator $\mathcal{O}_{T^*_{Y_{w_0}}(\mathbb{P}^1 \times \mathbb{P}^1)}$.

$\mathbb{P}^1_1$ w/ fiber $\mathbb{A}^1$

$$K^G(Z)/K^G(Z_\Delta) \simeq R(T)[q^{\pm 1}] \Rightarrow \bar{\Xi} : \mathcal{H}/\mathcal{H}_0 \to K^G(Z)/K^G(Z_\Delta) \simeq K^G(Y_{w_0}$$

sends T to $\mathcal{O}_{Y_{w_0}}$ up to an invertible element of $R(T)[q^{\pm 1}]$

$\Rightarrow \bar{\Xi}$ is an iso. $\Rightarrow \Xi$ is an isom, $\square$.

General G: $\Xi : \mathcal{H} \longrightarrow K^{\mathcal{G}}(Z)$, $\Xi(\theta_\lambda) := [\mathcal{O}(-\lambda)]$ on $Z_\Delta = T^* \mathcal{B}_\Delta$.

lifted from $T^*\mathcal{B}_\Delta \to \mathcal{B}$

$\Xi(T_s) = -q\, \mathcal{Q}_s - q$, $Y_s \subset \mathcal{B} \times \mathcal{B}$ subminimal orbit $\rightsquigarrow \overline{Y_s} \xrightarrow[\text{fiber } \mathbb{P}^1]{pr_1} \mathcal{B}$

$\rightsquigarrow \Omega^1_{\overline{Y_s}/\mathcal{B}}$ line bundle on $\overline{Y_s}$. $T^*_{\overline{Y_s}}(\mathcal{B} \times \mathcal{B})$ = closed irreducible comp. of Z.

$$T^*_{\overline{Y_s}}(\mathcal{B}\times\mathcal{B}) \xrightarrow{p} \overline{Y_s} \qquad Z = \bigsqcup_{w \in W} T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$$

$\mathcal{Q}_s = p^* \Omega^1_{\overline{Y_s}/\mathcal{B}}$: line bdle on $T^*_{\overline{Y_s}}(\mathcal{B}\times\mathcal{B})$

Lecture 4

$\Xi : \mathcal{H} \longrightarrow K^{\mathcal{G}}(Z)$ for arbitrary simply connected G

on generators : $\Xi(\theta_\lambda) = \mathcal{O}(-\lambda)$: the line bundle on $\mathcal{B}$ lifted to $T^*\mathcal{B} = Z_\Delta$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad T^*_\Delta(\mathcal{B}\times\mathcal{B}) \cap Z$

$\mathcal{H}_{fin} \ni T_s \longmapsto -q\,\mathcal{Q}_s - q$, where $Y_s \subset \mathcal{B} \times \mathcal{B}$ is a submininal orbit

$\overline{Y_s} \xrightarrow{pr_1} \mathcal{B}$: $\mathbb{P}^1$-fibration, $\Omega^1_{\overline{Y_s}/\mathcal{B}}$ relative canonical class,

$\mathcal{Q}_s = p^* \Omega^1_{\overline{Y_s}/\mathcal{B}}$, $T^*_{\overline{Y_s}}(\mathcal{B}\times\mathcal{B}) \xrightarrow{p} \overline{Y_s}$.

To check the relations, we use the polynomial rep. M.

$\mathcal{H} \hookrightarrow \operatorname{End} M \hookleftarrow K^{\mathcal{G}}(Z)$. Algebraically, almost idempotent $e = \sum_{w \in W} T_w$.

Lemma : a) $\exists$ a homomorphism $\varepsilon : \mathcal{H}_{fin} \to \mathbb{Z}[q^{\pm 1}]$, $T_w \mapsto q^{\ell(w)}$

b) $\forall\, a \in \mathcal{H}_{fin}$, $a \cdot e = e \cdot a = \varepsilon(a) e$.

c) $\mathcal{H} \cdot e$ = a free $R(T)[q^{\pm 1}]$-module of rank 1 w/ generator e.

__Proof.__ a) evident.

b) for $s \in S$, $W = W' \sqcup W''$,

$$W' = \{w \in W : \ell(w) = \ell(s) + \ell(sw)\}, \quad W'' = \{w \in W : \ell(sw) = \ell(s) + \ell(w)\}.$$

$w \in W' \iff w = sy$ for some $y \in W''$. $T_w = T_s T_y \Rightarrow T_s T_w = (q-1)T_w + q T_y$

If $y \in W'' \Rightarrow T_s T_y = T_{sy} \Rightarrow T_s e = \sum_{w \in W'} T_s T_w + \sum_{y \in W''} T_s T_y$

$$= \sum_{w \in W'} \left[(q-1) T_w + q T_{sw} \right] + \sum_{y \in W''} T_{sy} = q e$$

c) $\{\theta_\lambda T_w\}$ form a $\mathbb{Z}[q^{\pm 1}]$-basis in $\mathcal{H} \Rightarrow \{\theta_\lambda e\}$ are linearly indep. over $\mathbb{Z}[q^{\pm 1}]$.

from b) $\Rightarrow$ $\{\theta_\lambda e\}$ generate $\mathcal{H} \cdot e$ over $\mathbb{Z}[q^{\pm 1}]$. $\square$

__Corollary__. $\mathrm{Ind}_{\mathcal{H}_{fin}}^{\mathcal{H}} \varepsilon \xrightarrow{\sim} \mathcal{H} \cdot e$, $u \otimes 1 \mapsto u \cdot e$. This is $M = R(T)[q^{\pm 1}]$.

Geometrically, $K^G(\mathcal{B}) = R(T)[q^{\pm 1}] \xrightarrow{\sim} \mathcal{H} \cdot e$, $\exp(\lambda) \mapsto \theta_\lambda \cdot e$

I think should be $T^*\mathcal{B}$. Anti-spherical v.s. spherical.

__Prop 1__. The polynomial rep. is effective. $K^G(Z) \overset{\rho_Z}{\hookrightarrow} \mathrm{End}_{R(G)} K^G(\mathcal{B})$

__Prop 2__. The diagram commutes:

$$\begin{tikzcd}
\{\theta_\lambda, T_w\} \arrow[r] \arrow[dr, "\cong"'] & \mathcal{H} \arrow[r, "\rho_{\mathcal{H}}"] \arrow[d, dashed, "\wr"] & \mathrm{End}_{\mathbb{Z}[q^{\pm 1}]} M \arrow[d, equal] \\
& K^G(Z) \arrow[r, "\rho_Z"] & \mathrm{End}_{\mathbb{Z}[q^{\pm 1}]} K^G(\mathcal{B})
\end{tikzcd}$$

__Cor__. one may add the dotted arrow.

Prop: The dotted arrow is an isom.

Proof: Extend the Bruhat order on W to a total order $\leq$.

$$Z_{\leq w} := \bigsqcup_{y \leq w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B}) \quad \text{closed in } Z.$$

The cellular fibration lemma $\Rightarrow K^G(Z_{\leq w}) \hookrightarrow K^G(Z)$, and the image of these embeddings define a filtration in $K^G(Z)$ w/ ass. gr.

$$K^G(Z_{\leq w}) / K^G(Z_{<w}) = K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B})) \simeq K^G(Y_w) = \text{a free } R(T \times \mathbb{C}^*)$$

-module w/ generator $\left[\mathcal{O}_{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})}\right]$

Similarly for $\mathcal{H}$, $\mathcal{H}_{\leq w} = \langle \theta_\lambda T_y, y \leq w \rangle$

$\Rightarrow \mathcal{H}_{\leq w} / \mathcal{H}_{<w}$ free $R(T \times \mathbb{C}^*)$-module w/ generator T_w.

We have to check that $\operatorname{gr} \Xi$ is an iso.

Lemma: a) $\Xi(\mathcal{H}_{\leq w}) \subset K^G(Z_{\leq w})$

b) $\Xi(T_w) = c_w\left[\mathcal{O}_{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})}\right]$ for some $c_w \in R(T \times \mathbb{C}^*)^\times$ invertible

Proof. Demazure resolution. $\overline{Y_s}$ is smooth, $\mathbb{P}^1$-fibration over $\mathcal{B}$.

$\overline{Y_w}$ not smooth in general. the simplest example for $Gr(2,4) \twoheadleftarrow \mathcal{B}(SL_4)$

Schubert varieties in $Gr(2,4)$: $\quad \mathbb{C}^4 \supset \mathbb{C}^3 \supset \mathbb{C}^2 \supset \mathbb{C}^1 \supset 0$

$\cup$

V

B-orbits (closures) in $Gr(2,4)$

$V = \mathbb{C}^2$	$\bullet$
$\mathbb{C}^1 \subset V \subset \mathbb{C}^3$	$\mathbb{P}^1$
$V \subset \mathbb{C}^3$	$\mathbb{P}^2$
$\mathbb{C}^1 \subset V$	$\mathbb{P}^2$
$\dim(V \cap \mathbb{C}^2) \geq 1$	Y

$\overline{Y}$ is singular, and has two resolutions (Demazure – Zelevinsky)

$$\begin{array}{ccll} \overline{Y} & \leftarrow & \widetilde{Y}_1 & = \{V \supset \ell \subset \mathbb{C}^2\} \\ \cup & & \cup & \\ e & \leftarrow & \mathbb{P}^1 & \end{array}$$

$$\widetilde{Y}_1 \underset{\overline{Y}}{\times} \widetilde{Y}_2 = Bl_e \overline{Y}$$

the fiber over e is $\mathbb{P}^1 \times \check{\mathbb{P}}^1$ = the projective quadratic

$$\begin{array}{ccll} \overline{Y} & \leftarrow & \overset{\dim 3}{\widetilde{Y}_2} & = \{V \subset L \supset \mathbb{C}^2\} \\ \cup & & \cup & \\ e & \leftarrow & \check{\mathbb{P}}^1 & \end{array}$$

Zelevinsky constructed similar resolutions for arbitrary Schubert varieties on $Gr(k,n)$.

<u>Demazure resolutions</u> $\overline{Y_w} \subset \mathcal{B} \times \mathcal{B}$ choose a reduced decomposition

$$w = s_{i_1} \cdots s_{i_r}, \quad r = \ell(w)$$

$$\widetilde{Y}_{s_{i_1} \cdots s_{i_r}} = \overline{Y}_{s_{i_1}} \underset{\mathcal{B}}{\times} \overline{Y}_{s_{i_2}} \underset{\mathcal{B}}{\times} \cdots \underset{\mathcal{B}}{\times} \overline{Y}_{s_{i_r}} \xrightarrow{p} \overline{Y_w} \subset \mathcal{B} \times \mathcal{B}.$$

p is an isom. over $Y_w \subset \overline{Y_w}$. (For SL_n, it looks as

$$\begin{array}{ccccccccccccc} \mathbb{C}^1 & \subset & \mathbb{C}^2 & \subset & \cdots & \subset & \mathbb{C}^{i_1-1} & \subset & \mathbb{C}^{i_1} & \subset & \mathbb{C}^{i_1+1} & \subset \cdots \subset & \mathbb{C}^n \\ \| & & \| & & & & \| & & & & & & \\ V_1^1 & & V_1^2 & & & & V_1^{i_1-1} & \subset & V_1^{i_1} & \subset & V_1^{i_1+1} & & \\ & & & & & & \cdots & & & & & & \\ V_{r-1}^1 & \subset & V_{r-1}^2 & \subset & \cdots & \subset & V_{r-1}^{i_r-1} & \subset & V_{r-1}^{i_r} & \subset & V_{r-1}^{i_r+1} & \subset \cdots & \\ \| & & \| & & & & \| & & & & \| & & \\ V_r & & V_r^2 & & & & V_r^{i_r-1} & \subset & V_r^{i_r} & \subset & V_r^{i_r+1} & \subset \cdots & \end{array}$$

If we intersect w/ $\{B_0\} \times \mathcal{B}$, we get a tower of $\mathbb{P}^1$-fibrations

$$\downarrow$$

$$\overline{Y_w} \cap (\{B_0\} \times \mathcal{B}).$$

The preimage of $\partial \overline{Y_w} = \overline{Y_w} \setminus Y_w$ in $\widetilde{Y_w}$ is a normal crossing divisor.

The support of convolution of sheaves on $T^*_{\overline{Y_s}}(\mathcal{B} \times \mathcal{B})$ lies in

$T^*_{\overline{Y}_{s_{i_1}}}(\mathcal{B} \times \mathcal{B}) \circ \cdots \circ T^*_{\overline{Y}_{s_{i_r}}}(\mathcal{B} \times \mathcal{B})$, a closed subvariety of Z living over

$1 \dots n$

$\underbrace{\overline{Y_{s_{i_1}}} \circ \cdots \circ \overline{Y_{s_{i_r}}}}_{\subset \mathcal{B}\times\mathcal{B}} = \overline{Y_w}$ of the form $\bigcup_{y \le w} T^*_{Y_y}(\mathcal{B}\times\mathcal{B})$

$$/\!/$$

$$T^*_{Y_w}(\mathcal{B}\times\mathcal{B}) \cup \boxed{V \quad \text{closed in } Z_{<w}}.$$

$\Rightarrow$ $\mathcal{Q}_{s_{i_1}} * \cdots * \mathcal{Q}_{s_{i_r}}$ has support in $Z_{\le w}$.

$\Rightarrow$ $\Xi(T_{s_{i_1}}) * \cdots * \Xi(T_{s_{i_r}}) \subset K^G(Z_{\le w})$, proved a).

b) $\Xi(T_s)\big|_{T^*_{Y_s}(\mathcal{B}\times\mathcal{B})}$ = a line bundle $\mathcal{Q}_s$

The intersection of $Z_1 \cap \cdots \cap Z_r$ in $T^*\mathcal{B}^{r+1}$ is transversal and equal to $T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ under $pr_{1,r+1}$. Here $Z_\ell = pr_{\ell,\ell+1}^{-1}\left(T^*_{Y_{s_{i_\ell}}}(\mathcal{B}\times\mathcal{B})\right)$

Finally, $\widetilde{\mathcal{Q}}_{s_{i_\ell}}$ = the direct image of $pr^{\wedge}_{\ell,\ell+1}\mathcal{Q}_{s_{i_\ell}}$ under $Z_\ell \hookrightarrow T^*\mathcal{B}^{r+1}$

$\Xi(T_{s_{i_1}}) * \cdots * \Xi(T_{s_{i_r}})\Big|_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})} = \underbrace{\widetilde{\mathcal{Q}}_{s_{i_1}} \otimes \cdots \otimes \widetilde{\mathcal{Q}}_{s_{i_r}}}_{\text{cw}}$ the product of line bundles $\Rightarrow$ is a line bundle $\Rightarrow$ invertible.

Economical presentation of $K^G(Z) = K^{B\times\mathbb{C}^*}(Z_B) = K^{T\times\mathbb{C}^*}(Z_B)$

$Z_B = \bigsqcup_{w\in W} T^*_{\mathcal{B}_w}\mathcal{B}$, $\mathcal{B} = \bigsqcup_{w\in W}\mathcal{B}_w$, B-orbits for a fixed B

$$\begin{tikzcd} Z \arrow[r, hook, "\bar{j}"] & (T^*\mathcal{B})\times\mathcal{B} \arrow[r, "pr_2"] & \mathcal{B} \\ Z_B\times\{B\} \arrow[u, hook] \arrow[r, hook, "\bar{j}"] & (T^*\mathcal{B})\times\{B\} \arrow[u, hook] & \end{tikzcd}$$

, Z_B is the fiber of this composition over $\{B\}\in\mathcal{B}$.

$$\begin{tikzcd} Z \arrow[r, hook, "\bar{j}"] & (T^*\mathcal{B})\times\mathcal{B} & \mathcal{B}\times\mathcal{B} \arrow[l, hook', "\bar{i}"'] \arrow[r, "pr_2"] & \mathcal{B} \\ Z_B\times\{B\} \arrow[u, hook] \arrow[r, hook, "j"] & (T^*\mathcal{B})\times\{B\} \arrow[u, hook] & \mathcal{B}\times\{B\} \arrow[u, hook] \arrow[l, hook', "i"'] \arrow[r, "pr_2"] & \{B\} \arrow[u, hook] \end{tikzcd}$$

Pas 1.

Lecture 5.

$$Z \xhookrightarrow{\bar{j}} (T^*\mathcal{B}) \times \mathcal{B} \xhookleftarrow{\bar{i}} \mathcal{B} \times \mathcal{B}$$

$$\begin{tikzcd}
 & \mathcal{H} \arrow[r] & \mathrm{End}(\mathcal{H}\cdot e) \arrow[d, phantom, "\wr"] \arrow[dr, equal] & \\
\{\theta_\lambda, T_s\} \arrow[ur] \arrow[r, "\Xi"] \arrow[dr, "\bar{i}^*\bar{j}_*\Xi"'] & K^G(Z) \arrow[r, "\rho_Z"] \arrow[d, "\bar{i}^*\bar{j}_*"] & \mathrm{End}\, K^G(\mathcal{B}) \arrow[r, equal] \arrow[d, phantom, "\wr"] & \mathrm{End}(R(T)[q^{\pm 1}]) \\
 & K^G(\mathcal{B}\times\mathcal{B}) \arrow[r, "\rho_{\mathcal{B}}"] & \mathrm{End}\, K^G(\mathcal{B}) \arrow[ur, equal] &
\end{tikzcd}$$

Upper comp. Ψ_1

middle Ψ_2

lower Ψ_3.

$\Psi_1 \overset{?}{=} \Psi_2$

For general reasons, $\Psi_2 = \Psi_3$. Have to check $\Psi_1 = \Psi_3$.

Want to prove injectivity of ρ_Z.

The economical construction of ρ_Z and $\rho_{\mathcal{B}}$.

Fix $\{B\} \in \mathcal{B}$, $K^G(\mathcal{B}\times\mathcal{B})$ restrict to the fiber $\mathcal{B}\times\{b\}$

$$\begin{array}{c} K^G(\mathcal{B}\times\mathcal{B}) \\ \downarrow \wr \\ K^B(\mathcal{B}\times\{B\}) = K^B(\mathcal{B}) = K^T(\mathcal{B}) \end{array}$$

$Z_B = \bigsqcup_{w\in W} T^*_{\mathcal{B}_w}\mathcal{B}$, $\mathcal{B}_w$ is a B-orbit in $\mathcal{B}$.

$$\begin{tikzcd}
Z \arrow[r, hook, "\bar{j}"] & (T^*\mathcal{B})\times\mathcal{B} & \mathcal{B}\times\mathcal{B} \arrow[l, hook', "\bar{i}"'] \arrow[r, "pr_2"] & \mathcal{B} \\
Z_B\times\{B\} \arrow[u, hook] \arrow[r, hook] & (T^*\mathcal{B})\times\{B\} \arrow[u, hook] & \mathcal{B}\times\{B\} \arrow[l, hook', "i"'] \arrow[r, "pr_2"] \arrow[u, hook] & \{B\} \arrow[u, hook]
\end{tikzcd}$$

fibration over $\mathcal{B}$ w.r.t. pr_2

$\cup$

fibers over $\{B\}$

$$\begin{tikzcd}
K^G(Z) \arrow[r, "\bar{i}^*\bar{j}_*"] \arrow[d, phantom, "\wr\ \text{res}"] \arrow[dr, phantom, "\circlearrowleft"] & K^G(\mathcal{B}\times\mathcal{B}) \arrow[d, phantom, "\wr"] \arrow[r, equal] & K^G(\mathcal{B}\times\mathcal{B})[q^{\pm1}] \arrow[d, phantom, "\parallel"] \\
K^{T\times\mathbb{C}^*}(Z_B) \arrow[r, "i^*j_*"] & K^{T\times\mathbb{C}^*}(\mathcal{B}) \arrow[r, equal] & K^T(\mathcal{B})[q^{\pm1}]
\end{tikzcd}$$

$\rho_Z = \rho_B \circ \bar{i}^* \bar{j}_*$. To prove ρ_Z injective, need ρ_B injective and

$\bar{i}^* \bar{j}_*$ injective.

A) $\bar{i}^* \bar{j}_*$ inj. by the Localization (or Concentration) Thm (Thomason).

A torus $\mathbb{T}$ acts on X, $X^{\mathbb{T}} \xhookrightarrow[i]{} X$ fixed points

$K^{\mathbb{T}}(X^{\mathbb{T}}) \xhookrightarrow[i_*]{} K^{\mathbb{T}}(X)$. $K^{\mathbb{T}}(?)$ is a $\underset{\mathbb{Z}[\mathbb{T}]}{\underset{\shortparallel}{K^{\mathbb{T}}(pt)}}$-module.

$$K^{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{\mathbb{Z}[\mathbb{T}]} \mathrm{Frac}\,\mathbb{Z}[\mathbb{T}] \xrightarrow[i_*]{\sim} K^{\mathbb{T}}(X) \otimes_{\mathbb{Z}[\mathbb{T}]} \mathrm{Frac}\,\mathbb{Z}[\mathbb{T}].$$

$$K^{\mathbb{T}}(X^{\mathbb{T}}) = K(X^{\mathbb{T}}) \otimes_{\mathbb{Z}} K^{\mathbb{T}}(pt).$$

Take a point $a \in \mathbb{T}(\mathbb{C})$, $K^{\mathbb{T}}_{\mathbb{C}}(pt) = \mathbb{C}[\mathbb{T}]$. $m_a \subset \mathbb{C}[\mathbb{T}]$ the maximal ideal of a.

$\underset{\text{localization in } m_a}{K^{\mathbb{T}}(X^a)_a} \xrightarrow{i} K^{\mathbb{T}}(X)_a \Rightarrow$ the fibers at a are isomorphic.

In our case, $\mathbb{T} = T \times \mathbb{C}^\times$, $a = (1, c)$, $c \neq 1$.

$K^{T\times\mathbb{C}^\times}(Z_B, \cdots)$ — Z_B a union of cells. By the cellular fibration Lemma, $K^{T\times\mathbb{C}^\times}(?)$ is a free $\mathbb{C}[T\times\mathbb{C}^\times] = R(T)[q^{\pm 1}]$-module

$\Rightarrow$ enough to check injectivity after localization.

$$\begin{tikzcd} Z_B \arrow[r, hook, "j"] & T^*B \\ Z_B \cap B \arrow[u, hook, "i_Z"] \arrow[r] & B \arrow[u, hook, "i"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{matrix} Z_B^a & = & (T^*B)^a \\ \shortparallel & & \shortparallel \\ (Z_B \cap B)^a & = B = & B^a \end{matrix}$$

By the Loc. Thm.,
$K^{T\times\mathbb{C}^*}(Z_B)_a \xrightarrow{\bar{j}_*} K^{T\times\mathbb{C}^*}(T^*B)_a \xrightarrow{\bar{i}^*} K^{T\times\mathbb{C}^*}(B)_a$
equals $i^*: K^{T\times\mathbb{C}^*}(Z_B^a)_a \to K^{T\times\mathbb{C}^*}(Z^a_{B\cap B})_a$, with $Z_B^a = B = Z^a_{B\cap B}$.

$\Rightarrow \bar{i}^* \bar{j}_*$ is injective.

To prove ρ_Z injective, need $\rho_{\mathcal{B}}$ inj. and $\bar{i}^* \bar{j}_*$ inj.

B) We have reduced the injectivity of ρ_Z to the inj. of $\rho_{\mathcal{B}}$.

Direct computation: By Künneth formula, $K^G(\mathcal{B}\times\mathcal{B}) = K^G(\mathcal{B}) \otimes_{K^G(pt)} K^G(\mathcal{B})$

$= R(T)[q^{\pm1}] \otimes R(T)[q^{\pm1}]$

$R(G)[q^{\pm1}] = R(T)^W[q^{\pm1}]$

$\rho_{\mathcal{B}} : K^G(\mathcal{B}\times\mathcal{B}) \longrightarrow \mathrm{End}\, K^G(\mathcal{B}) = \mathrm{End}\, R(T)[q^{\pm1}]$

If $R(T) \simeq R(T)^\vee$ over $R(G)$, then RHS $= \mathrm{End}_{R(G)[q^{\pm1}]} R(T)[q^{\pm1}]$.

The desired duality is $\langle F_1, F_2\rangle := R\Gamma(\mathcal{B}, F_1 \otimes F_2) \in R(G)$

Non-degeneracy follows from the Pittie–Steinberg thm (Kazhdan–Lusztig)

$K^G(\mathcal{B}) = R(T) \Rightarrow \langle\, ,\rangle$ is computed by the Weyl formula

$\langle P, Q\rangle = \Delta^{-1} \sum_{w\in W} (-1)^{\ell(w)} w(PQ) \exp(w\rho)$.

The Steinberg basis e_w of $R(T)$ over $R(G)$, $\det \langle e_w, e_{w'}\rangle = \pm 1$.

$\langle e_y, e_{y'}\rangle = \Delta^{-1} \sum_w (-1)^{\ell(w)} w e_y\, w e_{y'} \exp(w\rho)$: this is a matrix element of

$A \cdot D \cdot A^t$ where D is the diagonal matrix w/ elts $\Delta^{-1}(-1)^w w \exp\rho$.

$\exp\left(\sum_{w\in W} w\rho\right) = 1$

$\det D = \pm\, \Delta^{-|W|}$

$\det(A\cdot D\cdot A^t) = \pm 1 \;\rightarrow\; \langle\, ,\rangle$ is nondegenerate, and

$R(T) \otimes_{R(G)} R(T) = \mathrm{End}_{R(G)} R(T) \curvearrowright R(T)$. $\Rightarrow K^G(\mathcal{B}\times\mathcal{B}) \xhookrightarrow{\rho_{\mathcal{B}}} \mathrm{End}\, K^G(\mathcal{B})$

$\Rightarrow \rho_Z$ injective.

To calculate Ψ_3, define $tr: K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \hookrightarrow K^T(\mathcal{B}) \otimes K^T(\mathcal{B}) \xrightarrow{\langle , \rangle} R(T)$

<u>Lemma</u>. diagram commutes

$$\begin{tikzcd} K^G(\mathcal{B}\times\mathcal{B}) \otimes K^G(\mathcal{B}) \arrow[r, "*"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & K^G(\mathcal{B}) \arrow[d, equal] \\ K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \arrow[r, "tr"] & R(T) \end{tikzcd}$$

$$K^G(\mathcal{B}\times\mathcal{B}) = K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B})$$

$$K^T(\mathcal{B}) = R(T) \otimes_{R(G)} K^G(\mathcal{B})$$

Have to check

$$\begin{tikzcd} \left(K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B})\right) \otimes K^G(\mathcal{B}) \arrow[r, "*"] \arrow[d, "\wr\ \varphi\otimes id"] \arrow[dr, phantom, "\circlearrowright"] & K^G(\mathcal{B}) \arrow[d, "\wr\ \varphi"] \\ \left(R(T) \otimes_{R(G)} K^G(\mathcal{B})\right) \otimes K^G(\mathcal{B}) \arrow[r, "tr"] & R(T) \end{tikzcd}.$$

$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F} \in K^G(\mathcal{B})$

$$\varphi((\mathcal{F}_1 \boxtimes \mathcal{F}_2) * \mathcal{F}) = \varphi(\mathcal{F}_1 \cdot \langle \mathcal{F}_2, \mathcal{F}\rangle) = \varphi(\mathcal{F}_1) \cdot \langle \mathcal{F}_2, \mathcal{F}\rangle$$

$$tr \circ (\varphi \otimes id)(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}) = tr(\varphi(\mathcal{F}_1) \cdot (\mathcal{F}_2 \boxtimes \mathcal{F})) = \varphi(\mathcal{F}_1) \cdot tr(\mathcal{F}_2 \boxtimes \mathcal{F})$$
$$= \varphi(\mathcal{F}_1) \langle \mathcal{F}_2, \mathcal{F}\rangle.$$

<u>Computation of Ψ_3</u>: a) $\Psi_3(\theta_\mu)(\exp(\lambda)) = \exp(\lambda - \mu)$

b) $\underset{s=s_\alpha}{\Psi_3(T_s)}(\exp(\lambda)) = \dfrac{\exp(\lambda) - \exp(s\lambda)}{\exp(\alpha) - 1} - q\, \dfrac{\exp(\lambda) - \exp(s\lambda + \alpha)}{\exp(\alpha) - 1}$ Demazure-Lusztig operator

This will be reduced to rank $= 1$ computation for $G' =$ subminimal Levi
w/ the derived group
$[G', G'] = SL_2$

$$\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_w \supset \mathcal{B}_s \subset \overline{\mathcal{B}_s} = \mathcal{B}_s \sqcup \mathcal{B}_e \simeq \mathbb{P}^1 \overset{\varepsilon}{\hookrightarrow} \mathcal{B}$$

$\overline{Y_s} \subset \mathcal{B} \times \mathcal{B} \xrightarrow{pr_2} \mathcal{B} \ni \{B\}$, the fiber of $\overline{Y_s}$ over $\{B\}$ is $\overline{\mathcal{B}_s}$.

Prop. a) $res \circ \bar{\imath}^* \bar{\jmath}_* [\mathcal{O}(\lambda)] = \mathbb{C}_{\{B\}, \lambda} \in B\text{-mod}$

b) $res \circ \bar{\imath}^* \bar{\jmath}_* [Q_s] = \varepsilon_* \left(q[\Omega^1_{\overline{\mathcal{B}_s}}] - [\mathcal{O}_{\overline{\mathcal{B}_s}}] \right)$

Recall that $\Xi(T_s) = qQ_s - 1$, $Q_s =$ the class of a line bundle on $T^*_{\overline{Y_s}} \mathcal{B}^2 \subset Z$,

$Q_s = \pi_s^* \Omega^1_{pr_2}$. $\quad T^*_{\overline{Y_s}} \mathcal{B}^2 \overset{\bar{\jmath}}{\hookrightarrow} (T^* \mathcal{B}) \times \mathcal{B} \overset{\bar{\imath}}{\hookleftarrow} \mathcal{B} \times \mathcal{B}$.

$$T^*_{\overline{\mathcal{B}_s}} \mathcal{B} \overset{j}{\hookrightarrow} T^* \mathcal{B} \overset{i}{\hookleftarrow} \mathcal{B}$$

$$T^* \mathcal{B}_s \overset{\bar{\imath}}{\hookleftarrow} \overline{\mathcal{B}_s}$$

General fact: $X \circlearrowleft T$, V a T-equivariant vector bundle, $z : X \hookrightarrow V$ zero section

$\mathbb{T} = T \times \mathbb{C}^\times$ acts on V, $\quad K^{\mathbb{T}}(X) \xrightarrow{z_*} K^{\mathbb{T}}(V)$ iso. after localization

$z^* z_* : K^{\mathbb{T}}(X) \circlearrowleft$ is the multiplication of the Euler class of V,

$$eul(V) = \sum_{k=1}^{rk V} (-q)^k (\Lambda^k V^*)$$

$$res \circ \bar{\imath}^* \bar{\jmath}_* q[Q_s] = \varepsilon_* \, q \cdot \left[eul(T^* \overline{\mathcal{B}_s}) \otimes \Omega^1_{\overline{\mathcal{B}_s}} \right] = \varepsilon_* \left[(q \mathcal{O}_{\overline{\mathcal{B}_s}} - T\overline{\mathcal{B}_s}) \otimes \Omega^1_{\overline{\mathcal{B}_s}} \right]$$
$$= \varepsilon_* \left(q[\Omega^1_{\overline{\mathcal{B}_s}}] - [\mathcal{O}_{\overline{\mathcal{B}_s}}] \right).$$

a) is clear.

Lecture 6.

$$\begin{tikzcd}
 & \mathcal{H} \arrow[r, "\rho_{\mathcal{H}}"] & \mathrm{End}(\mathcal{H}\cdot e) \arrow[dr, "\Upsilon"] \arrow[d, equal] & \\
\{\theta_\lambda, T_s\} \arrow[ur] \arrow[r, "\Xi"] \arrow[dr, "\bar{\imath}_* \bar{\jmath}^* \Xi"'] & K^G(Z) \arrow[r, "\rho_Z"] \arrow[d, "\bar{\imath}^* \bar{\jmath}_*"] & \mathrm{End}(K^G(\mathcal{B})) \arrow[r, equal] \arrow[d, equal] & \mathrm{End}(R(T)[q^{\pm 1}]) \\
 & K^G(\mathcal{B}\times\mathcal{B}) \arrow[r, "\rho_{\mathcal{B}}"] & \mathrm{End}(K^G(\mathcal{B})) \arrow[ur, equal] &
\end{tikzcd}$$

$\Upsilon(\theta_{-\lambda}\cdot e) = \exp(\lambda)$

The composition from $\{\theta_\lambda, T_s\}$ to End $R(T)[q^{\pm 1}]$ are Ψ_1, Ψ_2, Ψ_3.

<u>Main goal</u>: $\Psi_1 = \Psi_2$.

General fact: $\Psi_2 = \Psi_3$.

We need: $\Psi_1 = \Psi_3$.

<u>Recall</u>: $K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \hookrightarrow K^T(\mathcal{B}) \otimes K^T(\mathcal{B}) \xrightarrow{\langle , \rangle} R(T)$

composition denoted tr.

<u>Lemma 1</u>.

$$\begin{tikzcd}
K^G(\mathcal{B}\times\mathcal{B}) \otimes K^G(\mathcal{B}) \arrow[r, "*"] \arrow[d, "\mathrm{res}"] \arrow[dr, phantom, "\circlearrowright"] & K^G(\mathcal{B}) \arrow[d, equal] \\
K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \arrow[r, "tr"] & R(T)
\end{tikzcd}$$

Simple root α, simple reflection $s = s_\alpha$. $\varepsilon : \overline{\mathcal{B}_s} \simeq \mathbb{P}^1 \hookrightarrow \mathcal{B}$

<u>Lemma 2</u>. a) $\mathrm{res}\, \bar{\imath}^* \bar{\jmath}_* \mathcal{Q}_s = \varepsilon_* \left(q[\Omega^1_{\bar{\mathcal{B}}_s}] - [\mathcal{O}_{\bar{\mathcal{B}}_s}] \right)$

b) $\mathrm{res}\, \bar{\imath}^* \bar{\jmath}_* \mathcal{O}(\lambda) = \mathbb{C}_{\{B\},\lambda}$

Compute $\Phi_3(T_s) = \rho_B \bar{\imath}^* \bar{\jmath}_* \Xi(T_s)$

$$= \rho_B \bar{\imath}^* \bar{\jmath}_* (-q Q_s - 1)$$

Set $\mathcal{F} = \bar{\imath}^* \bar{\jmath}_* Q_s$. For $\mathcal{L} \in K^G(\mathcal{B})$,

$$\rho_B(\mathcal{F})(\mathcal{L}) = \mathrm{tr}\,(\mathrm{res}(\mathcal{F}) \otimes \mathcal{L}) \Rightarrow$$

$$\rho_B(\bar{\imath}^* \bar{\jmath}_* \Xi(T_s)) : \mathcal{O}(\lambda) \mapsto \mathrm{tr}\left(\varepsilon_*(-q\Omega^1_{\bar{B}_s} + \mathcal{O}_{\bar{B}_s} - \mathbb{C}_{\{B\}}) \otimes \mathcal{O}(\lambda)\right)$$

$$= -q\left[R\Gamma(\bar{B}_s, \Omega^1_{\bar{B}_s} \otimes \varepsilon^* \mathcal{O}(\lambda))\right]$$

$$+ [R\Gamma(\bar{B}_s, \varepsilon^* \mathcal{O}(\lambda))] - \exp(\lambda) \in R(T)[q^{\pm 1}]$$

This is a computation in $K^{L \times \mathbb{C}^\times}(L/B_L)$, L = subminimal Levi subgroup corresponding to $\{\alpha\}$.
$\quad\quad \shortparallel$
$\quad\quad \bar{B}_s$

$[L,L] = SL_2$, $\quad s(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$

$$\mathcal{O}(\lambda)\big|_{\bar{B}_s} = \mathcal{O}(\langle \lambda, \check{\alpha} \rangle)$$

$$[R\Gamma(\mathbb{P}^1, \mathcal{O}(\lambda))] = \exp(\lambda) \frac{1 - \exp((1 - \langle \lambda, \check{\alpha} \rangle)\alpha)}{1 - \exp(-\alpha)} \in \underset{\underset{R(T)^s}{\shortparallel}}{K^L(pt)}$$

$$\Omega^1_{\bar{B}_s} = \mathcal{O}(-\alpha), \quad \varepsilon^* \mathcal{O}(\lambda) = \exp(\lambda) \in R(T)$$

$$-q \exp(\lambda - \alpha) \frac{1 - \exp((1 - \langle \lambda, \check{\alpha} \rangle)\alpha)}{1 - \exp(-\alpha)} + \exp(\lambda) \frac{1 - \exp((-\langle \lambda, \check{\alpha} \rangle - 1)\alpha)}{1 - \exp(-\alpha)} - \exp(\lambda)$$

$$= \frac{\exp(\lambda) - \exp(s\lambda)}{\exp(\alpha) - 1} - q \frac{\exp(\lambda) - \exp(s\lambda + \alpha)}{\exp(\alpha) - 1}$$ (The Demazure - Lusztig operator)

Compare w/ Ψ_1 algebraic.

Compute $\rho_{\mathcal{H}}(T_s)(\theta_{-\lambda}\cdot e)$. By the Bernstein - Lusztig relations,

$$T_s\theta_{-\lambda} = \theta_{-s\lambda}T_s - (q-1)\frac{\theta_{-s\lambda}-\theta_{-\lambda}}{1-\theta_{-\alpha}}$$

$$\Rightarrow \rho_{\mathcal{H}}(T_s)(\theta_{-\lambda}\cdot e) = T_s\theta_{-\lambda}e = \left(\theta_{-s\lambda}T_s - (q-1)\frac{\theta_{-s\lambda}-\theta_{-\lambda}}{1-\theta_{-\alpha}}\right)e$$

$$= \left(q\theta_{-s\lambda} - (q-1)\frac{\theta_{-s\lambda}-\theta_{-\lambda}}{1-\theta_{-\alpha}}\right)e$$

Remains to apply Υ to RHS.

$$\Upsilon(\mathrm{RHS}) = q\exp(s\lambda) - (q-1)\frac{\exp(s\lambda)-\exp(\lambda)}{1-\exp(\alpha)}$$

$$= \frac{\exp(\lambda)-\exp(s\lambda)}{\exp(\alpha)-1} - q\frac{\exp(\lambda)-\exp(s\lambda+\alpha)}{\exp(\alpha)-1} \qquad \square$$

The same Demazure - Lusztig operator. $\square$

$\mathcal{H} \xrightarrow{\sim} K^G(Z)$.

The list of basic properties of equivariant K - theory.

1. The cellular fibration lemma, computation of K(union of cells)
2. Künneth formula
3. Thomason Localization Thm (Concentration)
4. $K^G(Z) \underset{i^* j_*}{\hookrightarrow} K^G(B\times B)$ is a convolution algebra homomorphism.

Applications: representations of $\mathcal{H}$

$\mathcal{H}$ over $\mathbb{Z}[q^{\pm 1}]$. For $\mathbb{C}$-repn of $\mathcal{H}$, we will use the complexification $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{C}$. From now on, $\mathcal{H}$ = complexified

<u>Recall</u>: $Z(\mathcal{H}) = R(T)^W \otimes \mathbb{C}[q^{\pm 1}] = R(G) \otimes \mathbb{C} = \mathbb{C}[G]^G$,

conjugation-inv. functions.

$\operatorname{Spec} Z(\mathcal{H}) = \{\text{semisimple conj. classes in } G\} \ni a = (s, t)$, $s \in G^{ss}$, $t \in \mathbb{C}^\times$

$\mathbb{C}_a$: 1-dim module over $Z(\mathcal{H})$.

$\mathcal{H}_a = \mathcal{H} \otimes_{Z(\mathcal{H})} \mathbb{C}_a$. Recall that $\mathcal{H}$ is a free module over $R(T)[q^{\pm 1}]$ of $rk = |W|$, and $R(T)[q^{\pm 1}]$ is a free module over $Z(\mathcal{H})$ of $rk = |W|$

$\Rightarrow \dim \mathcal{H}_a = |W|^2$.

<u>Schur Lemma</u>: If M is an irred. $\mathcal{H}$-module, then $Z(\mathcal{H})$ acts on M by a character a $\Rightarrow$ the action of $\mathcal{H}$ factors through $\mathcal{H}_a$.

<u>Proof</u>: Amitsur trick:

$\operatorname{End}_{\mathcal{H}}(M)$ is a skew-field. $\dim_{\mathbb{C}} \mathcal{H}$ is countable $\Rightarrow \dim_{\mathbb{C}} M$ is countable.

$\Rightarrow \dim_{\mathbb{C}} \operatorname{End}_{\mathcal{H}} M$ is countable. $h \mapsto h(m)$: $\operatorname{End}_{\mathcal{H}}(M) \hookrightarrow M$ for a generator $m \in M$.

Any skew field over $\mathbb{C}$ of countable dimension is $\mathbb{C}$.

Otherwise for $t \notin \mathbb{C}$, $\{(t-c)^{-1}\}_{c \in \mathbb{C}}$ is linearly indep..

From now on, we will study irred. rep. of $\mathcal{H}_a$. In $(\mathcal{H}_a)$

$Z^a \subset Z$ the fixed point subvariety.

$$Z^a = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a = (T^*\mathcal{B})^a \times_{\mathcal{N}^a} (T^*\mathcal{B})^a \subset (T^*\mathcal{B})^a \times (T^*\mathcal{B})^a$$

Thm. $\mathcal{H}_a = K(Z^a) \otimes \mathbb{C} \simeq \underbrace{H_\bullet(Z^a;\mathbb{C})}_{\text{Borel-Moore homology}}$

"rational (additive) analogue of K-theory"

= "trigonometric (multiplicative) analogue of $H_\bullet$"

$H_\bullet^G(Z)$ = the rational analogue of $\mathcal{H}$ = the degenerate or graded affine Hecke algebra. $\mathcal{H}_{deg} \neq \mathcal{H}$.

$\mathcal{H}_{deg} = gr\,\mathcal{H}$.

$(\mathcal{H}_{deg})_{\log a} \Rightarrow \mathcal{H}_a$

Definition of Borel-Moore homology

Def: $H_\bullet(X)$ = the reduced homology of the 1-pt compactification $\hat{X}$

$= H_\bullet^{ord}(\bar{X}, \bar{X} \setminus X)$ $\forall$ compactification $\bar{X}$ $= (H_c^\bullet(X,\mathbb{C}))^*$

$= H^{-\bullet}(X, \mathbb{D})$, $\mathbb{D}$ = dualizing constructible complex on X

Properties 1. $\pi : X \to Y$ proper $\Rightarrow$ $\pi_* : H_\cdot(X) \to H_\cdot(Y)$ since

$\bar{\pi} : \hat{X} \to \hat{Y}$ is continuous.

2. $\underset{\text{open}}{U} \subset X \supset \underset{\text{closed}}{Z} = X \setminus U$ $\Rightarrow$ LES

$$\cdots \to H_i(Z) \to H_i(X) \to H_i(U) \to H_{i-1}(Z) \to \cdots$$

$$H_\cdot(X) = H_\cdot^{\text{ord}}(\bar{X}, \bar{X} \setminus X) \to H_\cdot^{\text{ord}}(\bar{X}, \bar{X} \setminus U) = H_\cdot(U)$$

Embed X into a smooth compact manifold M

Poincaré duality $\quad H_i(X) = H^{m-i}(M, M \setminus X)$

$$H_i(Z) = H^{m-i}(M, M \setminus Z), \qquad m = \dim M$$

The above LES $=$

$$\cdots \to H^k(M, M \setminus Z) \to H^k(M, M \setminus X) \to H^k(M, M \setminus U) \to H^{k+1}(M, M \setminus Z) \to \cdots$$

3. A smooth oriented X has the fundamental class $[X] \in H_{\dim_{\mathbb{R}} X}(X)$.

If X is an irred. complex variety, $X^{\text{smooth}} \subset X$, $\operatorname{codim}_{\mathbb{R}}(X^{\text{sing}}) \geq 2$

$\Rightarrow$ $H_i(X^{\text{sing}}) = 0$, $\forall\, i > 2\dim_{\mathbb{C}} X - 2$ & from the LES

$$H_{2\dim_{\mathbb{C}} X}(X) \xrightarrow{\sim} H_{2\dim_{\mathbb{C}} X}(X^{\text{smooth}})$$

$\Rightarrow$ any irred. comp. of top dimension n has $[X_i] \in H_{2n}(X)$.

They form a basis in $H_{2n}(X)$.

4. M a smooth oriented real manifold, $Z, Z' \subset M$ closed $\rightsquigarrow$

$$H_i(Z) \times H_j(Z') \to H_{i+j-m}(Z \cap Z'), \quad m = \dim M$$

is Poincaré dual to the cup product $H^{m-i}(M, M \setminus Z) \times H^{m-j}(M, M \setminus Z') \to H^{2m-i-j}(M, M \setminus (Z \cap Z'))$

5. Künneth formula : $\boxtimes$

$$H_\cdot(X_1) \otimes H_\cdot(X_2) \xrightarrow{\sim} H_\cdot(X_1 \times X_2)$$

6. Restriction w/ support

$\iota: X \hookrightarrow Y$ smooth embedding of oriented manifolds, $d = \dim Y - \dim X$,

$Z \subset Y$ closed subspace, $\Rightarrow$ $\iota^*: H_\cdot(Z) \to H_{\cdot-d}(Z \cap X)$

$$c \longmapsto c \cap [X]$$

Poincaré dual to $H^\cdot(Y, Y\setminus Z) \to H^\cdot(X, X\setminus(X\cap Z))$

If $Z, Z' \subset M$, $c, c' \in H_\cdot(Z)$, $H_\cdot(Z')$, then $c \cap c' = \Delta^*(c \boxtimes c')$

7. Smooth pullback: $X \xrightarrow{p} Y$ oriented bundle w/ a fiber F smooth

$(X, Y$ not necessarily smooth$)$ $\Rightarrow$ $p^*: H_i(Y) \to H_{i+d}(X)$, $d = \dim X - \dim Y$

due to $p^* \mathbb{D}_Y = \mathbb{D}_X[-d]$

<u>Lecture 7</u>

7. $X \xrightarrow{p} Y$ an oriented fibration w/ a smooth fiber F, $\dim F = d$

$p^*: H_i(Y) \to H_{i+d}(X)$, due to the fact that $p^* \mathbb{D}_Y = \mathbb{D}_X[-d]$

If $i: Y \to X$ is a section, then $i^*: H_i(X) \to H_{i-d}(Y)$, due to

the fact that $i^* \mathbb{D}_X = \mathbb{D}_Y[d]$. $i^* p^* = Id_Y$.

If both X, Y are smooth, $Z \subset Y$, $Z' \subset X$ closed subvarieties,

$p|_{Z'}$ is proper $\Rightarrow$ $p^{-1}(Z) \cap Z' \xrightarrow{p} Y$ is also proper, and its image is denoted

$Z \circ Z'$. Then $c \in H_\cdot(Z)$, $c' \in H_\cdot(Z')$.

projection formula $p_*(p^*c \cap c') = c \cap p_* c'$.

8. The action of cohomology:

$H^i(X) \otimes H_j(X) \longrightarrow H_{j-i}(X)$, $a \otimes c \mapsto a.c$

9. Thom isomorphism:

Oriented vector bundle $V \underset{i}{\overset{\pi}{\rightleftarrows}} X$ zero section

$\Rightarrow$ the Euler class $e(V) \in H^r(X)$, $r = \operatorname{rk} V$.

$H_\cdot(X) \underset{\pi^*}{\overset{i^*}{\rightleftarrows}} H_\cdot(V)$. for $c \in H_\cdot(X)$, $i^* i_*(c) = e(V) \cdot c$

If X is a smooth closed subvariety in a smooth variety Y, $X \overset{i}{\hookrightarrow} Y$,

then $\forall\, c \in H_\cdot(X)$, $i^* i_*(c) = e(T_X Y) \cdot c$

10. <u>Excess intersection</u>:

Z_1, Z_2 smooth oriented subvarieties in an oriented manifold M, $Z = Z_1 \cap Z_2$

is smooth, and the intersection is "clean" but not transversal.

$$T_{1,2} = T_Z M / (T_Z Z_1 + T_Z Z_2).$$

Clean: $T_z Z_1 \cap T_z Z_2 = T_z Z$ at any $z \in Z$.

then $[Z_1] \cap [Z_2] = e(T_{1,2}) \cdot [Z]$.

Construction of convolution in the BM homology:

M_1, M_2, M_3 : connected smooth oriented manifolds.

closed subvarieties $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$

$$\rightsquigarrow Z_{12} \circ Z_{23} \subset M_1 \times M_3 = \{ (m_1, m_3) : \exists\, m_2,\ (m_1, m_2) \in Z_{12},\ (m_2, m_3) \in Z_{23} \}$$

Let $d = \dim_{\mathbb{R}} M_2$. Assume that $(Z_{12} \times Z_{23}) \cap \Delta M_2$ is proper.

$$\downarrow$$
$$M_1 \times M_3$$

$$H_i(Z_{12}) \times H_j(Z_{23}) \longrightarrow H_{i+j-d}(Z_{12} \circ Z_{23})$$

$$c_{12} * c_{23} = p_{13*}\underbrace{\left[(c_{12} \otimes [M_3]) \cap ([M_1] \otimes c_{23}) \right]}$$

Thm. $a = (s, t)$ semisimple element of G

$\rightsquigarrow \mathcal{H}_a$: $|W|^2$-dimensional central reduction of $\mathcal{H}$

$$\mathcal{H}_a \simeq H_\bullet(Z^a; \mathbb{C}).$$

"Proof": a chain of algebra isomorphisms

$$\mathcal{H}_a \simeq \mathbb{C}_a \underset{R(G)}{\otimes} K^G(Z) \qquad \text{the main geometric theorem}$$

$$\| $$

$$\mathbb{C}_a \underset{R(A)}{\otimes} K^A(Z)$$, where $A \subset G$ is the Zariski closure of $\{a^n\}_{n \in \mathbb{Z}}$

the neutral component $A^\circ \subset A$ is a torus

$A \subset \mathbb{T} \subset G$, $\mathbb{T} = T \times \mathbb{C}^\times$

$K^{\mathbb{T}}(X) = R(\mathbb{T}) \underset{R(G)}{\otimes} K^G(X)$, and $K^A(X) = R(A) \underset{R(\mathbb{T})}{\otimes} K^{\mathbb{T}}(X)$, due to the fact that $K^G(X)$ is a free $R(G)$-module, due to the cellular fibration lemma

Next we define a morphism $r_a : \mathbb{C}_a \otimes_{R(A)} K^A(Z) \longrightarrow \mathbb{C}_a \otimes_{R(A)} K^A(Z^A = Z^a)$

$$K^A(Z^a) = K^A(pt) \otimes K(Z^a)$$

$$\Rightarrow \mathbb{C}_a \otimes_{R(A)} K^A(Z^a) = K_{\mathbb{C}}(Z^a)$$

$r_a : \mathbb{C}_a \otimes_{R(A)} K^A(Z) \longrightarrow K_{\mathbb{C}}(Z^a)$ is the composition of

$$\mathbb{C}_a \otimes_{R(A)} K^A(Z) \xrightarrow[i^*]{\sim} \mathbb{C}_a \otimes_{R(A)} K^A(Z^a) \xrightarrow{\sim} K_{\mathbb{C}}(Z^a) \xrightarrow[1 \otimes \mathrm{eul}_a^{-1}]{\sim} K_{\mathbb{C}}(Z^a)$$

"correction factor"

Explanation: 1) i^* is the restriction w/ supports in $\tilde{N}^a \times \tilde{N}^a$.

$$\begin{tikzcd} Z^a \arrow[r, hook, "i"] \arrow[d, hook] & Z \arrow[d, hook] \\ \tilde{N}^a \times \tilde{N}^a \arrow[r, hook] & \tilde{N} \times \tilde{N} = (T^*B)^2 \end{tikzcd}$$

smooth $\qquad$ smooth

2) $\mathrm{eul}_A = \sum (-1)^i [\Lambda^i T^*_{\tilde{N}^a} \tilde{N}] \in K^A(\tilde{N}^a) = R(A) \otimes K(\tilde{N}^a)$

if $\iota : \tilde{N}^a \hookrightarrow \tilde{N}$, then $\iota^* \iota_*$ = the multiplication by $\mathrm{eul}_A : K^A(\tilde{N}^a) \circlearrowleft$ automorphism after localization in the max. ideal $\mathfrak{m}_a \in R(A)$.

3) eul_a = the image of eul_A under composition

$$K^A(\tilde{N}^a) = R(A) \otimes K(\tilde{N}^a) \xrightarrow{ev_a \otimes Id} \mathbb{C} \otimes K(\tilde{N}^a) = K(\tilde{N}^a)_{\mathbb{C}}.$$

Explicitly, if we split the conormal bundle $T_{\tilde{N}^a}\tilde{N} = \bigoplus_\nu (T^*_{\tilde{N}^a}\tilde{N})_\nu$ over the eigen characters of A, then $\mathrm{eul}_a = \bigoplus_\nu \left(\sum (-\nu(a))^i \Lambda^i (T^*_{\tilde{N}^a}\tilde{N})_\nu\right)$

Lemma 1. r_a commutes w/ the convolution, i.e. r_a is a homomorphism of convolution algebras.

$$\mathcal{H}_a \to \cdots \to K_a(Z^a) \to H_\bullet(Z^a;\mathbb{C})$$

We define the latter morphism as follows:

The Chern character $ch_\bullet : K(Z^a) \to H_\bullet(Z^a)$

$$Z^a \subset \tilde{\mathcal{N}}^a \times \tilde{\mathcal{N}}^a$$

$$\begin{array}{ccl} K(Z^a) & = K(\tilde{\mathcal{N}}^a \times \tilde{\mathcal{N}}^a) & \text{supported on } Z^a \\ \downarrow & \downarrow & \\ H_\bullet(Z^a) & = H^\bullet(\tilde{\mathcal{N}}^a \times \tilde{\mathcal{N}}^a) & \text{supported on } Z^a \end{array}$$

b) The Todd class $Td_{\tilde{\mathcal{N}}^a} \in H^\bullet(\tilde{\mathcal{N}}^a)$

c) Bivariant Riemann-Roch morphism. $RR: K(Z^a) \to H_\bullet(Z^a)$

$$\mathcal{F} \mapsto (1 \boxtimes Td_{\tilde{\mathcal{N}}^a}) \cup ch_\bullet(\mathcal{F})$$

Lemma 2. RR commutes w/ the convolutions.

From now on we will classify the irr. rep.'s of $H_\bullet(Z^a;\mathbb{C})$

Construction of $H_\bullet(Z^a)$-modules. Springer resolution $\mu: \tilde{\mathcal{N}} \to \mathcal{N}$

$$\begin{array}{ccc} \tilde{\mathcal{N}} & \to & \mathcal{N} \\ \cup & & \cup \\ \tilde{\mathcal{N}}^a & \to & \mathcal{N}^a \end{array}$$

$a = (s,t) \in G \times \mathbb{C}^\times$,

$\mathcal{N}^a = \{x \in \mathcal{N} : Ad_s x = tx\}$, $\tilde{\mathcal{N}}^a = \{(b,x) \in \mathcal{B}^a \times \mathcal{N}^a,\ x \in b\}$

The projection to $\mathcal{B}$ identifies $\mu^{-1}(x)$ w/ $\mathcal{B}^s_x$ = the fixed points of s & x ($\mathcal{B}_x \subset \mathcal{B}$ is the zero set of the vector field $x \in \mathrm{Lie}\, G$ on $\mathcal{B}$)

$\mathcal{B}_x$ is the usual Springer fiber, typically very singular and reducible.

Remark. $\mathcal{B}^s_x \neq \phi$ since $\mathcal{B}^s_x$ = fixed pt set of a group generated by s and $\exp(\mathbb{C}\cdot x)$. $\mathrm{Ad}_s x = tx \Rightarrow$ this group is solvable and contained in a Borel B $\Rightarrow \{B\} \in \mathcal{B}^s_x$.

Now $H_\cdot(Z^a)$ acts by convolution on $H_\cdot(\mu^{-1}(x)) = H_\cdot(\mathcal{B}^s_x)$

Set $G(s,x) = Z_G(s) \cap Z_G(x) \subset G$.

$C(s,x) = G(s,x)/G^\circ(s,x)$ the component group

NB: $G = GL(n)$, then $C(s,x) = \{e\}$.

For $GL(n)$, the centralizer of anything is connected: (the centralizer in $\mathrm{Mat}_{n\times n}$ = a vector space) $\cap$ ($\det \neq 0$).

The action of $C(s,x) \curvearrowright H_\cdot(\mathcal{B}^s_x)$ commutes w/ $H_\cdot(Z^a) \curvearrowright H_\cdot(\mathcal{B}^s_x)$.

Def. If $\chi \in \mathrm{Irr\,Rep}\, C(s,x)$ occurs in $H_\cdot(\mathcal{B}^s_x)$, then we write $\chi \in C(s,x)^\wedge$ (χ is relevant), we set $K_{a,x,\chi} = \mathrm{Hom}_{C(s,x)}(\chi, H_\cdot(\mathcal{B}^s_x))$ a standard $H_\cdot(Z^a)$-module.

NB: the BM homology we encounter live only in even degrees.

Costandard modules

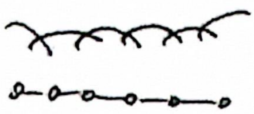
$Z_G(s)\cdot x = \mathbb{O} \subset \mathcal{N}^a$, we take a local analytic slice S to $\mathbb{O}$

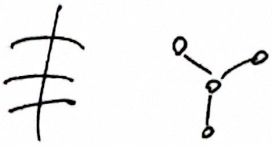
$\tilde{S} = \mu^{-1}(S)$ is a smooth tubular nbhd of $B^s_x = \mu^{-1}(x)$, and B^s_x is a homotopy retract of $\tilde{S}$.

NB. If $s = 1$, x = a subregular nilp. element. (a unique submaximal nilp. orbit)

$\mathfrak{gl}_n$, $x = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix} \oplus (0)$, type $(n-1, 1)$

$\mu^{-1}(x)$ = a union of projective lines intersecting according to some Dynkin graph:

A_{n-1} 

$\mathfrak{so}_8 = D_4$ 

G_2 ---

$\mathfrak{sp}_{2n}$ 

equal to the special fiber of the resolution of the corresp. Kleinian singularity $\mathbb{A}^2/\Gamma$, $\Gamma \subset SL(2;\mathbb{C})$, $\tilde{S}$ = the resolution itself

$\mu^{-1}(x) \subset \tilde{S}$

 $\subset \widetilde{\mathbb{A}^2/\Gamma}$

Lecture 8

Reminder: component group $C(s,x) = G(s,x)/G^\circ(s,x)$

$G(s,x) = Z_G(s) \cap Z_G(x)$, s semisimple, x nilpotent.

$\mathcal{B}_x^s \subset \mathcal{B}$: fixed point of s, x in $\mathcal{B}$. $\mathcal{B}^s = \sqcup Y_i$, $Y_i = Z_G(s)/\text{Borel subgroup}$.

For $s \in T$ regular, $\mathcal{B}^s = \mathcal{B}^T = \bigsqcup_W$ points

$\mathcal{B}_x$ = the Springer fiber, equidimensional of dim $\frac{1}{2}(\mathrm{codim}_{\mathcal{N}}\, G\cdot x)$

There is a Weyl group representation in $H_{top}(\mathcal{B}_x) \supset C(x)$

$C(s,x) \curvearrowright H_\bullet(\mathcal{B}_x^s)$. We set $C(s,x)^\wedge$ = the irr. reps appearing in $H_\bullet(\mathcal{B}_x^s)$

Springer: all the irr. reps of W appear exactly once as the multiplicity space

$\mathrm{Hom}_{C(x)}(\chi, H_\bullet(\mathcal{B}_x))$, $\chi \in C(x)^\wedge$.

Central reduction $\mathcal{H}_a$ of $\mathcal{H}$, a = semisimple element in $\mathcal{G} = G \times \mathbb{C}^\times$.

$a = (s,t)$, $t \in \mathbb{C}^\times$, $x \in \mathcal{N}^a \Leftrightarrow \mathrm{Ad}_s x = tx$.

Geometrically, $\mathcal{H}_a \curvearrowright H_\bullet(\mathcal{B}_x^s) \circlearrowleft C(s,x)$, $\chi \in C(s,x)^\wedge$.

The standard module: $K_{a,x,\chi} = \mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\mathcal{B}_x^s))$.

Costandard modules: $Z_G(s) \curvearrowright \mathcal{N}^a$, $\mathbb{O}_x$ the $Z_G(s)$-orbit of x.

S a local slice to $\mathbb{O}_x$ through x in classical topology. $\tilde{S} = \mu^{-1}(S) \supset \mu^{-1}(x)$

$\tilde{S}$ is a tubular nbhd of $\mathcal{B}_x^s$: a homotopy retract of $\tilde{S} \hookleftarrow \mathcal{B}_x^s$

BM homology not homotopy invariant. $H_\bullet(\tilde{S}) \leftarrow H_\bullet(\mathcal{B}_x^s)$

If we choose S invariant w.r.t. $K \subset Z_G(s)$, $K(s,x) \curvearrowright S, \tilde{S}$

$K(s,x)/K^\circ(s,x) = C(s,x)$. $C(s,x) \curvearrowright H_\bullet(\tilde{S})$.

Costandard module $\check{K}_{a,x,\chi} := \mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\tilde{S}))$.

$$\psi : K_{a,x,\chi} \longrightarrow \check{K}_{a,x,\chi}, \quad L_{a,x,\chi} := \mathrm{Im}\,\psi.$$

$Z_G(s)$ acts by conjugation on (x, χ), and the conjugate pairs give rise to the isomorphic $K, K^\vee, L$.

Thm (Ginzburg, Kazhdan–Lusztig) ~ 1985

a) $L_{a,x,\chi}$ is an irred. $\mathcal{H}_a$-module if $L \neq 0$

$L_{a,x,\chi} \simeq L_{a,x',\chi'}$ iff $(x, \chi) \sim (x', \chi')$ are conjugate.

b) If $t \neq \sqrt[\circ]{1}$, then $L_{a,x,\chi} \neq 0$, $\forall x \in \mathcal{N}^a$, and $\chi \in C(s,x)^\wedge$

[for $t = \sqrt[\circ]{1}$, I. Grojnowski]

c) $\forall$ simple $\mathcal{H}_a$-module arises this way.

In type A, combinatorial classification by J. Bernstein, A. Zelevinsky, ~ 1980.
$G = PGL_n$, $C(s,x) = \{e\}$.

Stupid corollary: Since $\dim \mathcal{H}_a = |W|^2$, $\#\,\mathrm{Irr}(\mathcal{H}_a)$ is finite

$\Rightarrow \#\, Z_G(s)$-orbits in $\mathcal{N}^a$ is finite ($G \backslash \mathcal{N}$ is finite)

Can be proved directly.

One can compute the characters of $K, K^\vee, L$.

$\mathcal{H} \supset R(T)[q^{\pm 1}] \supset Z(\mathcal{H})$ of $k|_{R(T)[q^{\pm 1}]}$

$\downarrow \qquad\qquad\qquad a$

$\mathcal{H}_a$

category $\mathcal{O}$ of $\mathfrak{g}$-modules

Standard Verma modules $M(\chi)$

Dual Verma modules $M^\vee(\chi)$

Irred. $L(\chi)$

$$\begin{tikzcd} M(\chi) \arrow[rr, "\psi"] \arrow[dr] & & M^\vee(\chi) \\ & L(\chi) \arrow[ur] & \end{tikzcd}$$

Computation of $\operatorname{char} K_{a,s,x,\chi}$:

$\mathcal{B}_x^s = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_r$ connected components

$C(s,x)$ acts on $\{1, \cdots, r\}$.

If $g \in C(s,x)$ stabilizes $\mathcal{B}_j$, then the Lefschetz number

$$\ell(g, \mathcal{B}_j) = \sum_p (-1)^p \operatorname{Tr}(g^* : H_p(\mathcal{B}_j)\mathbb{Z})$$

$\theta_\lambda \in R(T)[q^{\pm 1}] \subset \mathcal{H}$.

Set $\langle \lambda, s \rangle_j := \exp(\lambda)\,(s \bmod [B,B])$

$\{B\} \in \mathcal{B}_j \Rightarrow s \in B \twoheadrightarrow B/[B,B] \cong T$

$\uparrow$ abstract Cartan

The bundle over $\mathcal{B}_j$ w/ fiber $B/[B,B]$ is trivial.

$B = \mathrm{Ad}_h B'$, $\mathrm{Ad}_h : B'/[B',B'] \xrightarrow{\sim} B/[B,B]$

s regular, $\mathcal{B}^s = \mathcal{B}^{T = Z_G(s)} = \bigsqcup_W$ points

$\langle \lambda, s \rangle_j = \langle w_j \lambda, s \rangle$

$Z(U\mathfrak{g}) = \operatorname{Sym}(\mathfrak{h})^W$, $\xi \in \operatorname{Spec} Z(U\mathfrak{g})$

For a fixed (regular ξ), there are $|W|$-many χ s.t. $M(\chi), M^\vee(\chi), L(\chi)$ have the central character ξ.

$\operatorname{char}(M(\chi)) = \operatorname{char}(M^\vee(\chi))$

$= \exp\chi \cdot \operatorname{char} \operatorname{Sym}(\mathfrak{n}_-)$

$\operatorname{char}(L(\chi)) = \sum (-1)^{\ell(w)} \operatorname{char} M(w \cdot \chi)$

if χ is dominant

χ nondominant 1981 (Kazhdan–Lusztig conjecture 1979)

Beilinson–Bernstein, Brylinski–Kashiwara

$$\operatorname{char}(L(\chi)) = \sum_{w \in W} P_w(1)\, M(w \cdot \chi)$$

KL polynomials, related to singularities of Schubert varieties $\overline{\mathcal{B}_y} \subset \mathcal{B}$.

Prop. $\operatorname{Tr}(\Theta_\lambda, K_{a,x,\chi}) = \dfrac{1}{|C(s,x)|} \displaystyle\sum_{j=1}^{r} \sum_{\substack{g \in \operatorname{Stab} B_j \\ \cap \\ C(s,x)}} \operatorname{Tr}_\chi(g)\, \ell(g, B_j) \cdot \langle \lambda, s \rangle_j$

Proof. $H_\bullet(\mathcal{B}^s_x) = \displaystyle\bigoplus_{\substack{\text{orbits of } C(s,x) \\ \text{in } \{1,\dots,r\}}} \operatorname{Ind}^{C(s,x)}_{C(s,x)_j} H_\bullet(\mathcal{B}_j)$

$$\underset{\text{Stabilizer of a component}}{\overset{\shortparallel}{C(s,x)_j}}$$

By the Frobenius reciprocity, $\forall\, \chi \in \operatorname{Irr} C(s,x)$,

$$\operatorname{Hom}_{C(s,x)}(\chi, \operatorname{Ind}^{C(s,x)}_{C(s,x)_j} H_\bullet(\mathcal{B}_j)) = \operatorname{Hom}_{C(s,x)_j}(\chi|_{C(s,x)_j}, H_\bullet(\mathcal{B}_j))$$

The action of Θ_λ is $\otimes\, \mathcal{O}(\lambda)\big|_{\mathcal{B}^s_x}$

$\mathcal{B}^s = \sqcup Y_i$, $\quad Y_i$ = flag varieties of $Z_G(s)$.

$A \subset G$ generated by a $\quad$ (A° = torus)

A-action on $\mathcal{B}^s$ is trivial

A-action on $\mathcal{O}(\lambda)|_{\mathcal{B}^s}$ factors through the dilation fiberwise action via some character $A \xrightarrow{\lambda_i} \mathbb{C}^*$ on $Y_i \subset \mathcal{B}^s$.

$\lambda_i = \langle \lambda, s \rangle_j$ for some $B_j \subset Y_i$. $\qquad \begin{matrix} \mathcal{B}^s & \supset & \mathcal{B}^s_x \\ \shortparallel & & \shortparallel \\ \sqcup Y_i & & \sqcup \mathcal{B}_j \end{matrix}$

$K^A(Y_i) = R(A) \otimes K(Y_i)$

$\mathcal{O}(\lambda)|_{Y_i} = \exp(\lambda_i) \otimes \mathcal{O}(\lambda)_i$ $\qquad \mathbb{C}_a \underset{R(A)}{\otimes} K^A(Y_i) \simeq \mathbb{C}_a \underset{R(A)}{\otimes} R(A) \otimes K(Y_i) = K(Y_i) \xrightarrow{\ ch\ } H_\bullet(Y_i)$

$\langle \lambda, s \rangle_j \cdot ch(\Theta_\lambda|_{Y_i})$, $\quad ch(\Theta_\lambda|_{Y_i}) = 1 + \cdots$ unipotent

For any θ_λ - equivariant subspace

$V \subset H_\bullet(\mathcal{B}_j)$, $tr(\theta_\lambda, V) = \langle \lambda, s \rangle_i \cdot \dim V$.

$$Tr(\theta_\lambda, K_{a,x,\chi}) = \sum_{\text{orbits}} \langle \lambda, s \rangle_j \cdot \dim \mathrm{Hom}_{C(s,x)_j}\left(\chi|_{C(s,x)_j}, H_\bullet(\mathcal{B}_j)\right)$$

$$= \sum_j \frac{1}{|C(s,x)_j|} \left(\langle \lambda, s \rangle_j \sum_{g \in C(s,x)_j} Tr(\chi(g))\, Tr(g, H_\bullet(\mathcal{B}_j)) \right)$$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \parallel$ Since $H_{odd}(\mathcal{B}_j) = 0$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \ell(g, H_\bullet(\mathcal{B}_j))$

K-L - de Concini - Procesi

Shriekology (Ed Shpiz): the standard operations on constructible sheaves are f_*, f^*, $f_!$, $f^!$, $\boxtimes$, $\mathcal{H}om$ ...

<u>Constructible sheaves</u>, local systems = locally constant sheaves

= rep. of fundamental group $\pi_1(X, x)$

A sheaf $\mathcal{F}$ is <u>constructible</u> if there is a stratification $X = \bigsqcup_{s \in S} X_s$ s.t.

$\forall s$, $\mathcal{F}|_{X_s}$ is a local system. This class of sheaves is closed w.r.t.

f_*, f^*, $R^i f_*$.

<u>Artin gluing</u>: local systems $\left(= \rho_s \text{ rep. of } \pi_1(X_s, x_s) \right)$

$X_r \subset \overline{X_s}$ tubular nbhd U_r of X_r in $\overline{X_s}$, $\mathring{U}_r = U_r \cap X_s$ $\quad \rho_s \circ p_s$

$= \text{cosp} \circ \rho_r \circ p_r$

$$\begin{array}{ccccc} U_r & \hookleftarrow & \mathring{U}_r & \hookrightarrow & X_s \\ \downarrow & & & & \\ X_r & & & & \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{p_s} \pi_1(X_s, x_s) \curvearrowright_{\rho_s} V_s & \\ \pi_1(\mathring{U}_r, x_s) & & \nearrow \text{Cospecialization morphism} \\ p_r \downarrow & & \\ \pi_1(X_r, x_r) \curvearrowright_{\rho_r} V_r & & \end{array}$$

Lecture 9.

Last time defined constructible sheaves.

$X \rightsquigarrow \mathrm{Const}(X)$ abelian category

$$\cap$$

$D^b_c(X)$ derived category of complexes of sheaves w/ constructible cohomologies

Unfortunately, $D^b_c(X) \underset{\text{Nori}}{\simeq} D^b(\mathrm{Constr}(X))$

$\mathcal{B} = \bigsqcup \mathcal{B}_w \rightsquigarrow D^b\,\mathrm{Constr}_{\mathrm{Schubert}}(\mathcal{B}) \not\simeq D^b_{c,\mathrm{Schubert}}(x)$

$$\wr\wr$$

$D^b\,\mathrm{Perv}_{\mathrm{Schubert}}(\mathcal{B})$

From now on, $f: X \to Y$, $f_*: D^b_c(X) \to D^b_c(Y)$ full derived functor Rf_*.

Shriekology : list of basic relations between $f_*, f^*, f_!, f^!, \boxtimes, \mathcal{H}om$.

1. f^* is left adjoint to f_*

2. $f_! : X \to Y$, relative compactification $X \overset{j}{\hookrightarrow} \bar{X} \xrightarrow{\bar{f}} Y$

$f_! := \bar{f}_* j_!$, there is the right adjoint functor $f^!: D^b_c(Y) \to D^b_c(X)$.

Eg. if $f: X \hookrightarrow Y$ is a closed embedding, then $f^! = Rf^!$ the derived functor of sections w/ support in X.

$\underline{\mathbb{C}}$ on $\mathbb{A}^1 = Y$, $X = \{0\}$. $f^! \underline{\mathbb{C}} = \mathbb{C}[-2]$.

$\underline{\mathbb{C}} \to j_* \underline{\mathbb{C}} \to \delta_0[-1] \xrightarrow{[1]} \cdots$

3. External product. $\mathcal{F}$ on X, $\mathcal{G}$ on Y, $\mathcal{F} \boxtimes \mathcal{G}$ on $X \times Y$.

If $X = Y$, $\Delta: X \hookrightarrow X \times X$, $\mathcal{F} \otimes \mathcal{G} = \Delta^*(\mathcal{F} \boxtimes \mathcal{G})$, $\mathcal{F} \overset{!}{\otimes} \mathcal{G} = \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$.

4. Internal $\underline{Hom} = \mathcal{H}om$.

$$Hom(\mathcal{F} \otimes \mathcal{G}, \mathcal{E}) = Hom(\mathcal{F}, \underline{\mathcal{H}om}(\mathcal{G}, \mathcal{E})).$$

5. $pr : X \longrightarrow pt$, $pr^! \mathbb{C}_{pt} = \mathbb{D}_X$ the dualizing complex on X.

If X is smooth complex variety, then $\mathbb{D}_X = \mathbb{C}_X[2\dim_{\mathbb{C}} X]$.

$\mathbb{D}_X$ = sheaf of orientations of a smooth real manifold $[\dim_{\mathbb{R}} M]$.

Grothendieck - Verdier duality $D^b_c(X) \circlearrowleft$ antiinvolution

$\mathcal{F} \mapsto D\mathcal{F} := \underline{Hom}(\mathcal{F}, \mathbb{D}_X)$, $D \circ D = Id$.

6. $X \xrightarrow{f} Y$, $\underline{\mathcal{H}om}(f_! \mathcal{F}, \mathcal{G}) = f_* \underline{\mathcal{H}om}(\mathcal{F}, f^! \mathcal{G})$.

7. $f_* \underline{\mathcal{H}om}(f^* \mathcal{F}, \mathcal{G}) = \underline{\mathcal{H}om}(\mathcal{F}, f_* \mathcal{G})$.

8. $\underline{\mathcal{H}om}(\mathcal{F} \otimes \mathcal{G}, \mathcal{E}) = \underline{\mathcal{H}om}(\mathcal{F}, \underline{\mathcal{H}om}(\mathcal{G}, \mathcal{E}))$.

9. $\mathbb{D}(\mathcal{F} \otimes \mathcal{G}) = \underline{\mathcal{H}om}(\mathcal{F}, \mathbb{D}\mathcal{G}) \iff \underline{\mathcal{H}om}(\mathcal{F}, \mathcal{G}) = \mathbb{D}\mathcal{F} \overset{!}{\otimes} \mathcal{G}$.

10. $\mathbb{D} f_* \mathbb{D} = f_!$, $\mathbb{D} f^! \mathbb{D} = f^*$.

11. Base change:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g"] \arrow[dl, bend left=60] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

$f^! g_* = g_* f^!$

$f^* g_! = g_! f^*$

If g is proper, $g_* = g_!$. Otherwise, $f^* g_* \neq g_* f^*$

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

12. Monodromic sheaves. $X \underset{i}{\overset{f}{\rightleftarrows}} Y$, $\mathbb{C}^\times \curvearrowright X$, $X^{\mathbb{C}^\times} = Y$. $f \circ i = \mathrm{id}_Y$

$$\begin{tikzcd} \mathbb{C}^\times \times X \arrow[r, "act"] \arrow[d, hook] & X \\ \mathbb{C} \times X \arrow[ur, dashed, "\overline{act}"'] & \end{tikzcd}$$

X is contracted to Y by the action of $\mathbb{C}^\times$.

$\forall x$, $\exists \lim\limits_{t \to 0} (t \cdot x) = f(x)$.

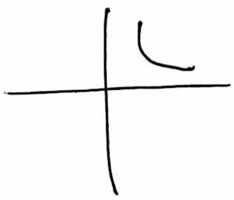
Monodromic $\mathcal{F} \in D^b_c(X)$.

$\forall t \in \mathbb{C}^*$, $t^* \mathcal{F} \simeq \mathcal{F}$.

If $\mathcal{F}$ is monodromic, then $i^! \mathcal{F} \xrightarrow{\sim} f_! \mathcal{F}$

$f_* \mathcal{F} \xrightarrow{\sim} i^* \mathcal{F}$.

13. The hyperbolic stalks



$X = \mathbb{A}^2$ $\quad t(x,y) = (tx, t^{-1}y)$

$Y = \bullet$

$\mathbb{C}^\times \curvearrowright X$, $Y = X^{\mathbb{C}^\times}$, $\mathcal{F}$ monodromic.

$\quad\quad\quad\quad \rotatebox{90}{$\subseteq$}$

$\quad\quad\quad\ \bigsqcup_i F_i$ connected components

$A_i = \{ x \in X : \lim\limits_{t \to 0} t \cdot x \in F_i \}$

$R_i = \{ x \in X : \lim\limits_{t \to \infty} t \cdot x \in F_i \}$

$\mathbb{P}^1$, $0, \infty$

$A_0 = \mathbb{A}^1$, $A_\infty = \{\infty\}$.

$R_0 = \{0\}$, $R_\infty = \mathbb{P}^1 \setminus 0$.

$F = \bigsqcup F_i$, $A = \bigsqcup A_i$, $R = \bigsqcup R_i$

$$\begin{tikzcd} F \arrow[r, shift left, "r"'] \arrow[d, shift left, "a"] & R \arrow[l, shift left, "p"'] \arrow[d, "\alpha"] \\ A \arrow[u, shift left, "q"] \arrow[r, "p"'] & X \end{tikzcd}$$

Thm (I. Mirković, T. Braden, V. Drinfeld – D. Gaitsgory)

There is an isomorphism $r^* \alpha^! \mathcal{F} \simeq a^! p^* \mathcal{F}$ (dually, $r^! \alpha^* \mathcal{F} = a^* p^! \mathcal{F}$)

the hyperbolic restriction

$\Phi \mathcal{F}$

14. Cousin spectral sequence: $X = \sqcup X_s$ stratification, $\mathcal{F}$ is smooth w.r.t. $\sqcup X_s$

$$H_c^{p+q}(X;\mathcal{F}) \Leftarrow E_2^{pq} = \bigoplus_{\mathrm{codim}\, X_s = q} H_c^{p+q}(X_s, i_s^* \mathcal{F}), \quad i_s : X_s \to X.$$

If X_s = the attractor of $\mathbb{C}^\times$-action to $s \in X_s$, then the Cousin SS is composed of the hyperbolic stalks. There is the dual co-Cousin SS.

Perverse sheaves. $X = \sqcup X_s$, smooth along this stratification.

a full subcategory $\mathrm{Perv}(\sqcup X_s) \subset D_c^b(X)$.

$\mathcal{F} \in \mathrm{Perv}$ if the cohomology of $\mathcal{F}$ are loc. constant along X_s.

$i_s^* \mathcal{F}$ on X_s, $\quad H^k(i_s^* \mathcal{F}) = 0$ for $k > -\dim X_s$

$H^k(i_s^! \mathcal{F}) = 0$ for $k < -\dim X_s$.

all X_s are smooth. If X is smooth, then $\underbrace{\mathbb{C}_X[\dim X]}_{=: C_X}$ is perverse

The two conditions are Verdier-dual, $\mathrm{Perv}(\sqcup X_s)$ is Verdier self-dual.

Thm (Beilinson - Bernstein - Deligne) $\mathrm{Perv}(\sqcup X_s)$ is abelian.

SES $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0 \iff \mathcal{F} \to \mathcal{G} \to \mathcal{E} \xrightarrow{+1}$ a distinguished triangle.

$$\mathrm{Perv}(X) = \bigcup_{\text{Stratifications}} \mathrm{Perv}(\sqcup X_s).$$

Basic example: $X = \mathbb{P}^1$, $X_0 = \mathbb{A}^1$, $X_\infty = \infty$

$$\mathbb{A}^1 \xrightarrow{j} X \xleftarrow{i} \infty$$

$C_{\mathbb{A}^1} = \mathbb{C}_{\mathbb{A}^1}[1]$, $j_! C_{\mathbb{A}^1}$, $j_* C_{\mathbb{A}^1}$ are perverse.

$i_* \mathbb{C}_0$ perverse. $\mathbb{C}_{\mathbb{P}^1}[1] = j_{!*} \mathbb{C}_{\mathbb{A}^1}$ = the image of the canonical morphism
$j_! \mathbb{C}_{\mathbb{A}^1} \to j_* \mathbb{C}_{\mathbb{A}^1}$.

Classification of irreducible perverse sheaves [BBD]

$\underline{\mathbb{C}} \hookleftarrow j_! \underline{\mathbb{C}}_U \hookleftarrow j_! \underline{\mathbb{C}}_V \hookleftarrow \cdots$

Perv is Noetherian and self-dual

$\Rightarrow$ artinian.

Constr. is not artinian.

There are no irr. Constr. sheaves except for δ_x.

$X_S \xhookrightarrow{i_S} X$ a smooth stratum,

$\mathcal{F}_S$: an irr. local system on X_S

$$\mathrm{Im}\left(i_{S!} \mathcal{F}_S [\dim X_S] \xrightarrow{\varphi} i_{S*} \mathcal{F}_S [\dim X_S] \right) =: i_{S!*} \mathcal{F}_S [\dim X_S]$$

the minimal, or intermediate, or Deligne, or Goresky - MacPherson.

Thm (M. Artin 1965) If $f : X \to Y$ is affine, then $f_* : D^b_c(X) \to D^b_c(Y)$ is right t-exact, $f_!$ is left t-exact.

$${}^pH^{<0}(f_! P) = 0$$
$${}^pH^{>0}(f_* P) = 0.$$

$$\mathcal{F} \in D^b_c(Y)$$
$$\cup \qquad \cup$$
$$\text{Constr.} \qquad \text{Perv}$$

$\underline{H}^i(\mathcal{F}) \in$ Constr, ${}^p\underline{H}^i(\mathcal{F}) \in$ Perv

In particular, if $j : X \hookrightarrow Y$ is an open affine embedding, then both $j_* P$, $j_! P$ are perverse for $P \in \mathrm{Perv}(X)$.

For X affine, $P = \mathbb{C}_X \Rightarrow$ Morse - Lefschetz thm.
$\downarrow$
$pt = Y$

Lecture 10.

Goresky – MacPherson irreducible perverse sheaves

$IC(X_s, \mathcal{L}) := j_{!*}\mathcal{L}[\dim X_s]$, $j: X_s \hookrightarrow X$

Cohomological characterization of irr. perv. sheaves.

$$\left[\begin{array}{ll} P|_{X_s} = j_s^* P & \text{lives in coh. degrees } \leq -\dim X_s \\ j_s^! P & \text{- - - - - - } \geq -\dim X_s \end{array}\right]$$

$IC(X_s, \mathcal{L})\big|_{X_s}$ lives in degree $= -\dim X_s$, irred. local system.

$\forall\, t<s$, $IC(X_s, \mathcal{L})\big|_{X_t}$ lives in degree $< -\dim X_t$

$j_t^! IC(X_s, \mathcal{L})$ lives in deg $> -\dim X_t$

$\boxed{\begin{array}{l} \mathbb{C}^\times \overset{j}{\hookrightarrow} \mathbb{C}, \\ 0 \to \underline{\mathbb{C}} \to \mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \underline{\mathbb{C}} \to 0 \\ j_{!*}\mathcal{L}[1] \\ \underline{\mathbb{C}} - \delta_0 - \underline{\mathbb{C}} \end{array}}$

In particular, from the base change, and $H^{>2\dim X}(X) = 0$.

if $\tilde{X} \xrightarrow{\pi} X$ is a small resolution of singularities,

$\pi_* \underline{\mathbb{C}}[\dim \tilde{X}]$ is an irred. perverse sheaves on $X = j^{sm}_{!*}\, \underline{\mathbb{C}}[\dim X]$, $X^{sm} \overset{j^{sm}}{\hookrightarrow} X$.

Eg. $X = Mat_{2\times 2}^{\det = 0} \subset Mat_{2\times 2}$, $\tilde{X} = \{(A, \ell \text{ line}) : A\ell = 0\}$
$$\| $$
$$Tot(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$$

Small: it is stratified, and $\dim \pi^{-1}(x) < \frac{1}{2} \operatorname{codim}_X X_s$, $x \in X_s$, unless $X_s = X^{sm}$.

Semismall: $<$ ⇝ $\leq$. $\pi_* \underline{\mathbb{C}}[\dim \tilde{X}]$ is perverse.

$\dim X = 2$,

$\pi_* = \underline{\mathbb{C}}_X[1] \oplus \delta_0$

Decomposition Theorem [BBD]: $X \xrightarrow{\pi} Y$ proper, P an irred. p.s. on X,

$\pi_* P = \bigoplus$ of its perverse cohomologies ${}^p\underline{H}^i(\pi_* P)[-i]$, and each ${}^p\underline{H}^i(\pi_* P)$ is $\oplus$ of irr. p.s.

$(\dagger)$ $\underline{H}^i(\pi_* P)$ makes no sense to talk of semisimplicity.
$\#$
$\pi_* p$

$\underline{\mathbb{C}}[\dim X] =: C_X$

Lemma. $X \overset{i}{\hookleftarrow} x$, P irr. p.s. on X,

$i^! P \longrightarrow i^* P$ is 0 unless $P = \delta_x$.

Proof: If $P \neq \delta_x$, $\dim \operatorname{supp} P \geq 1 \Rightarrow$ $i^* P$ lives in coh. deg < 0
$i^! P$ deg > 0.

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \supset & \widetilde{\mathcal{N}}^a \\ \downarrow & & \downarrow \\ \mathcal{N} & \supset & \mathcal{N}^a \\ & & \| \\ & & X \ni 0 \end{array} \qquad , a = (s,t) \in G \times \mathbb{C}^\times$$

Lemma: smooth variety $M \xrightarrow{\mu} N$, μ is proper, $M_x = \mu^{-1}(x)$, $m = \dim M$
$\qquad\qquad\qquad\qquad\qquad\qquad\uparrow i$
$\qquad\qquad\qquad\qquad\qquad\qquad x$

Then a) $H_\bullet(M_x) = H^{m-\bullet}(i^! \mu_* C_M)$

b) $H^\bullet(M_x) = H^{\bullet - m}(i^* \mu_* C_M)$

Proof. a) $\mathbb{D}_M = \underline{\mathbb{C}}_M[2\dim M] = C_M[m]$ $\Rightarrow$ $H_\bullet(M_x) = H^{-\bullet}(M_x; \mathbb{D}_{M_x})$
$= H^{-\bullet}(\mu_* \mathbb{D}_{M_x})$
$= H^{-\bullet}(\mu_* i^!_{M_x} \mathbb{D}_M)$
$= H^{-\bullet}(i^! \mu_* C_M[m])$
$= H^{m-\bullet}(i^! \mu_* C_M)$

b) Similarly.

Corollary.

$$\begin{tikzcd}
H^\bullet(i^! \mu_* C_M) \arrow[d] \arrow[r, phantom, "="] & H^\bullet(\mathbb{D}_{M_x}[-m]) \arrow[r, phantom, "="] \arrow[d, phantom, "\curvearrowright"] & H_{m-\bullet}(M_x) \arrow[d, "\nu"] \arrow[r, phantom, "\overset{\text{Poincaré}}{=}"] & H^{m+\bullet}(\tilde{U}, \tilde{U}\setminus M_x) \arrow[d] \arrow[r, phantom, "="] & H^{ord}_{m-\bullet}(\tilde{U} \arrow[d] \\
H^\bullet(i^* \mu_* C_M) \arrow[r, phantom, "="] & H^\bullet(\mathbb{C}_{M_x}[m]) \arrow[r, phantom, "="] & H^{m+\bullet}(M_x) \arrow[r, phantom, "\overset{(i^*_{M_x})^{-1}}{=}"] & H^{m+\bullet}(\tilde{U}) \arrow[r, phantom, "="] & H_{m-\bullet}(\tilde{U})
\end{tikzcd}$$

$$\nu : H_{m-\bullet}(M_x) \xrightarrow{(i_{M_x})_*} H_{m-\bullet}(M) \overset{\text{Poincaré}}{=} H^{m+\bullet}(M) \xrightarrow{i^*_{M_x}} H^{m+\bullet}(M_x)$$

If $x \in U \subset N$ contractible nbhd, $\tilde{U} = \mu^{-1}(U) \subset M$ tubular nbhd of M_x

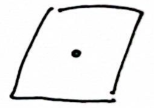
$$H^\bullet(i^* \mu_* C_M) = H^\bullet(U, \mu_* C_M) = H^\bullet(\tilde{U}, \mathbb{C}_M[m]) = H^{m+\bullet}(\tilde{U}), \text{ also } H^\bullet(\tilde{U}) \overset{\text{is}}{=} H^\bullet(M_x)$$

Finally, $H_\bullet(M_x) = H^{ord}_\bullet(\tilde{U})$ since $H_\bullet(M_x)$ is dual of $H^\bullet_c(M_x) = H^\bullet(M_x) = H^\bullet(\tilde{U})$

and $H^{ord}_\bullet(\tilde{U})$ is also dual of $H^\bullet(\tilde{U})$

By the decomposition thm, $\mu_* C_M = \bigoplus_{\varphi = (N_\beta, \mathcal{L}_\beta)} \underbrace{L_\varphi}_{\text{multiplicity space}} \otimes IC_\varphi$ $\qquad L_\varphi = \bigoplus_i L_\varphi(i)$.

$$H^{m+\bullet}(M_x) = \bigoplus_\varphi L_\varphi \otimes H^{m+\bullet}(i^* IC_\varphi)$$

$Bl_0 \xrightarrow{\pi}$  $\mathbb{A}^3$

If $IC_\varphi = \delta_x$, then $L_x := L_{\delta x}$ enters w/ multiplicity

$\pi_* C_{Bl_0} = C_{\mathbb{A}^3} \oplus \delta_0 \otimes L_0$.

$H^\bullet(i^* \delta_x) = \mathbb{C} \Rightarrow L_x \subset H_\bullet(\tilde{U})$.

$L_0 = \mathbb{C}[2] \oplus \mathbb{C}[-2]$

Prop.: a) the image of $(i_{M_x})_* : H_\bullet(M_x) \to H_\bullet(\tilde{U})$ is exactly L_x

$\operatorname{Ker}((i_{M_x})_*)$ = the kernel of the bilinear pairing

$$\begin{tikzcd}
H_{m+\bullet}(M_x) \times H_{m-\bullet}(M_x) \arrow[r] \arrow[d] & \mathbb{C}, \\
H_{2m}(\tilde{U}) = \mathbb{C} &
\end{tikzcd}$$

$\langle \xi, \eta \rangle^{\tilde{U}} = \xi \cap \eta$ in the ambient space $\tilde{U}$.

Proof. a) identify $(i_{M_x})_*$ with $H^\bullet(i^! \mu_* C_M) \to H^\bullet(i^* \mu_* C_M)$, and decompose it into $\bigoplus_\varphi L_\varphi \otimes \left[H^\bullet(i^! IC_\varphi) \to H^\bullet(i^* IC_\varphi)\right]$, and by Lemma, the image is nonzero only for $IC_\varphi = \delta_x$.

b) identify the pairing with

$$\begin{array}{ccc} H^{ord}_{m+\bullet}(\tilde{u}) \times H^{ord}_{m-\bullet}(\tilde{u}) & \xrightarrow{\cap} & H^{ord}_\bullet(\tilde{u}) = \mathbb{C} \\ \big\downarrow {\scriptstyle id \times can} & & \\ H^{ord}_{m+\bullet}(\tilde{u}) \times H_{m-\bullet}(\tilde{u}) & \xrightarrow{\cap} & H^{ord}_\bullet(\tilde{u}) = \mathbb{C} \end{array}$$

the bottom is nondegenerate by Poincaré

$\Rightarrow$ the kernel of the upper pairing $= \ker(can)$.

Corollary: $(i_{M_x})_*$ gives rise to an iso.: $H_\bullet(M_x)/\mathrm{Rad}\langle\,,\,\rangle_{\tilde{u}} \xrightarrow{\sim} L_x$.

$x \in N$ was the smallest stratum. In general, we use that all the strata are orbits $\mathbb{O}$ of some group action.

$x \in \mathbb{O}$, $\mathrm{Stab}_x Z_G(a) = G_x$ w/ component group $C_x = G_x/G_x^\circ$

$\varphi = (\mathbb{O}, \chi)$, $\chi \in \mathrm{Irr}\, C_x$

$L_\varphi = \mathrm{Im}\left(H_\bullet(M_x)_\chi \to H_\bullet(\tilde{S})_\chi\right) = H_\bullet(M_x)_\chi / \mathrm{Rad}\langle\,,\,\rangle^{\tilde{S}}_\chi$ — preimage of a slice to $\mathbb{O}$ through x.

We can reduce to the above situation of a point stratum, since there is a global Slodowy slice S (algebraic)

We apply this machinery to $M = \tilde{\mathcal{N}}^a$, $N = \mathcal{N}^a$, $\mu : \tilde{\mathcal{N}}^a \to \mathcal{N}^a$

We get $L_{a,x,x} = \mathrm{Im}\left(H_\bullet(B^s_x)_x \to H_\bullet(\mathfrak{Z})_x\right) = L_\varphi = H_\bullet(B^s_x)_x / \mathrm{Rad}\langle\, , \,\rangle^{\mathfrak{Z}}_x$

$a = (s,t)$, $x \in \mathcal{N}^a$ $\qquad$ $\cong K_{a,x,x}$ $\qquad$ $\| \; K_{a,}{}^\vee{}_{x,x}$ $\qquad$ $\|$ multiplicity of IC_φ in $\mu_* C_M$

<u>Next time</u>: $H_\bullet(Z^a = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a) = \mathcal{H}_a \simeq \mathrm{Ext}_{D^b_c(\mathcal{N}^a)}(\mu_* C_{\tilde{\mathcal{N}}^a}, \mu_* C_{\tilde{\mathcal{N}}^a})$

Yoneda algebra

Lecture 11

$$\begin{array}{ccl} \tilde{\mathcal{N}}^a & = & M, \; C_M \\ \downarrow \mu & & \downarrow \\ \mathcal{N}^a & = & N, \; \mu_* C_M = \bigoplus IC_\varphi \otimes L_\varphi, \quad L_\varphi = \bigoplus L_\varphi(i) \end{array}$$

<u>Thm.</u> M smooth, $\mu : M \to N$ proper.

$Z = M \times_N M$

$H_\bullet(Z)$ convolution algebra

is

$\mathrm{Ext}^\bullet_{D^b(N)}(\mu_* C_M, \mu_* C_M)$ <u>not</u> respecting the natural gradings.

<u>Proof</u>: $M_1 \xrightarrow{\mu_1} N \xleftarrow{\mu_2} M_2$ $\qquad$ $Z_{12} = M_1 \times_N M_2$, $H_\bullet(Z_{12}) \simeq \mathrm{Ext}^{m_1+m_2-\bullet}_{D^b(N)}(\mu_{1*} C_{M_1}, \mu_{2*} C_{M_2})$

$\dim M_{1,2} = m_{1,2}$

$$\mathbb{D}_{M_1\times M_2} = \mathbb{C}_{M_1\times M_2}[m_1+m_2] \Rightarrow \mathbb{D}_{Z_{12}} = i^! \mathbb{C}_{M_1\times M_2}[m_1+m_2],$$

where $i : Z_{12} \hookrightarrow M_1\times M_2$.

$$H_{-j}(Z_{12}) = H^j(\mathbb{D}_{Z_{12}}) = H^{j+m_1+m_2}(i^! \mathbb{C}_{M_1\times M_2})$$

$$\begin{tikzcd} Z_{12} \arrow[r, hook, "i"] \arrow[d, "\mu_{12}"'] & M_1\times M_2 \arrow[d, "\mu_1\times\mu_2"] \\ N \arrow[r, hook, "\Delta"] & N\times N \end{tikzcd}$$

$$H^\bullet(Z_{12}, i^!(\mathbb{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2)) = H^\bullet(N, \mu_{12*} i^!(\mathbb{D}\mathcal{F}_1\boxtimes\mathcal{F}_2))$$

$\underline{\underline{\text{base change}}}$ $H^\bullet(N, \Delta^!(\mu_1\times\mu_2)_*(\mathbb{D}\mathcal{F}_1\boxtimes\mathcal{F}_2))$

$$= H^\bullet(N, \Delta^!(\mu_{1*}\mathbb{D}\mathcal{F}_1 \boxtimes \mu_{2*}\mathcal{F}_2))$$

$$= H^\bullet(N, \Delta^!(\mathbb{D}\mu_{1*}\mathcal{F}_1 \boxtimes \mu_{2*}\mathcal{F}_2))$$

$$= H^\bullet(N, \mathbb{D}\mu_{1*}\mathcal{F}_1 \overset{!}{\otimes} \mu_{2*}\mathcal{F}_2)$$

$$= H^\bullet(N, \underline{\mathcal{H}om}(\mu_{1*}\mathcal{F}_1, \mu_{2*}\mathcal{F}_2)) = \mathrm{Ext}^\bullet(\mu_{1*}\mathcal{F}_1, \mu_{2*}\mathcal{F}_2)$$

$\mathcal{F}_1 = \mathbb{C}_{M_1}$, $\mathcal{F}_2 = \mathbb{C}_{M_2}$.

Let us skip the composition = convolution check.

$\mu_* \mathbb{C}_M = \bigoplus_{\varphi,k} L_\varphi(k)\otimes IC_\varphi$, $\varphi = (\mathbb{O}, \mathcal{L})$, $\mathbb{O}$ stratum of $N = \mathcal{N}^a \circlearrowleft G(s,x)$

$\mathcal{L}$ irr. local system

$$L_\varphi = \bigoplus_k L_\varphi(k)$$

$$H_\bullet(Z) = \bigoplus_k \mathrm{Ext}^k(\mu_*\mathbb{C}_M, \mu_*\mathbb{C}_M) = \bigoplus_{\varphi,\psi,i,j,k} \mathrm{Hom}(L_\varphi(i), L_\psi(j))\otimes \mathrm{Ext}^k(IC_\varphi[i], IC_\psi[j])$$

$$= \bigoplus \mathrm{Hom}(L_\varphi(i), L_\psi(j))\otimes \mathrm{Ext}^{k+j-i}(IC_\varphi, IC_\psi)$$

$$= \bigoplus_{k,\varphi,\psi} \mathrm{Hom}(L_\varphi, L_\psi)\otimes \mathrm{Ext}^k(IC_\varphi, IC_\psi)$$

$\mathrm{Ext}^{<0} = 0$, $\mathrm{Ext}^0(IC_\varphi, IC_\psi) = \delta_{\varphi\psi}$

$$= \bigoplus_{\varphi} \operatorname{End} L_\varphi \oplus \underbrace{\bigoplus_{k>0,\varphi,\psi} \operatorname{Hom}(L_\varphi, L_\psi) \otimes \operatorname{Ext}^k(IC_\varphi, IC_\psi)}_{\text{Radical}}$$

positively graded algebra w/ semisimple deg = 0 part.

Corollary. $\underbrace{\{\operatorname{Irr} H_\bullet(Z)\} = \{L_\varphi\}}$.

Kazhdan - Lusztig formulas for characters of L_φ in terms of K.

Thm. a) $[H_\bullet(M_x)_\psi : L_\varphi] = \sum \dim H^k(i_x^! IC_\varphi)_\psi$

b) $[H^\bullet(M_x)_\psi : L_\varphi] = \sum \dim H^k(i_x^* IC_\varphi)_\psi$

$$\begin{array}{ccc} M & = & \tilde{\mathcal{N}}^a \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ N & = & \mathcal{N}^a \end{array}$$

NB. For Verma modules $[M_\lambda : L_\nu]$ irreducibles in $\mathcal{O}$ over $\mathfrak{g}$

Conversely, $\operatorname{ch} L_\nu = \sum P_{\lambda\nu} \operatorname{ch} M_\lambda$. λ, ν must be in the same W-orbit.

two basis in K-group $= \mathbb{Z}[W]$, $\lambda = w \cdot \Lambda$, $\nu = y \cdot \Lambda$, Λ regular dominant

KL - BB & BK. $P_{\lambda\nu} = P_{wy} = P_{w,y}(1)$ ← KL polynomial

KL matrix is self-inverse up to something not very essential.

Proof: a) multiplicity of L_φ as $\operatorname{Ext}^\bullet(\mu_* C_M, \mu_* C_M)$-module in

$$(H^\bullet i_x^! \mu_* C_M)_\psi = H_\bullet(M_x)_\psi$$
$$\|$$
$$\bigoplus_\varphi L_\varphi \otimes H^\bullet(i_x^! IC_\varphi)_\psi \curvearrowleft \operatorname{Ext}^\bullet(\mu_* C_M, \mu_* C_M) \text{ Yoneda composition}$$

$$\mathrm{Ext}^k(\mu_* C_M, \mu_*(C_M)) \otimes \bigoplus_\varphi L_\varphi \otimes H^j(i_x^! \mu_* C_M) \to \bigoplus_\varphi L_\varphi \otimes H^{j+k}(i_x^! \mu_* C_M)$$

$\Rightarrow$ there is a descending filtration $F^p H^\cdot(i_x^! \mu_* C_M) = \bigoplus H^{\geq p}(i_x^! \mu_* C_M)$

$$\shortparallel \quad \bigoplus_{j \geq p} L_\varphi \otimes H^j(i_x^! IC_\varphi)$$

F^p is invariant under the action of $\mathrm{Ext}^\cdot(\mu_* C_M, \mu_* C_M)$.

$gr_F H^\cdot(i_x^! \mu_* C_M)$ has trivial action of $\mathrm{Ext}^{>0}$, and the action factors through $\bigoplus \mathrm{End}(L_\varphi)$

$gr_F H^\cdot(i_x^! \mu_* C_M) = \bigoplus L_\varphi \otimes H^\cdot(i_x^! IC_\varphi)$ w/ semisimple action $\Rightarrow$

$gr_F H^\cdot(i_x^! \mu_* C_M) = \bigoplus L_\varphi \otimes H^\cdot(i_x^! IC_\varphi)$ as the $\bigoplus \mathrm{End}\, L_\varphi$-module.

a) is obviously true for $gr_F H_\cdot(M_x)$, but the multiplicities do not change.

Projective modules over $H_\cdot(Z) = \mathrm{Ext}^\cdot(\mu_* C_M, \mu_* C_M)$

$$\shortparallel \quad A = \bigoplus_\varphi \mathrm{End}\, L_\varphi \oplus Rad.$$

Fix a projector e_φ of $rk = 1$ in $\mathrm{End}\, L_\varphi$.

$\mathrm{End}\, L_\varphi \cdot e_\varphi = L_\varphi$, $\{e_\varphi\}$ orthogonal projectors in $A \subset H_\cdot(Z)$

<u>Def</u>. $P_\varphi = H_\cdot(Z) e_\varphi \subset H_\cdot(Z)$ is a direct summand: $H_\cdot(Z) e_\varphi = H_\cdot(Z) \otimes_A \underbrace{(A \cdot e_\varphi)}_{\text{a direct summand}}$

$\Downarrow$ a projective module.

$P_\varphi = L_\varphi \oplus \bigoplus_{\psi, k > 0} L_\psi \otimes \mathrm{Ext}^k(IC_\psi, IC_\varphi)$ since $\mathrm{Hom}(L_\psi, L_\psi) \cdot e_\varphi = L_\varphi$

A descending filtration $F^m P_\varphi = \bigoplus \mathrm{Ext}^{\geq m} \otimes \cdots$ submodule of P_φ.

$P_\varphi / F^1 P_\varphi = L_\varphi$

$\Rightarrow$ P_φ is generated by L_φ as an $H_\cdot(Z)$-module

$\Rightarrow$ P_φ is an idecomposable projectile cover of L_φ.

Cartan matrix. $C_{\varphi\psi} := [P_\varphi : L_\psi] = \sum_k \dim \mathrm{Ext}^k(IC_\psi, IC_\varphi)$

All the indecomposable projecties are of the form P_φ.

Kazhdan – Lusztig matrix $KL_{\varphi\psi} = \sum_k [\underline{H}^k \, i_\varphi^* IC_\psi : \mathcal{L}_\varphi]$

IC_ψ on $\overline{\mathbb{O}}_\psi \subset N$ $= i_{\psi!*}\mathcal{L}_\psi$

$\cup$

$\mathbb{O}_\varphi, \mathcal{L}_\varphi$

$IC_\varphi = i_{\varphi!*}\mathcal{L}_\varphi$

Almost diagonal matrix

$$D_{\varphi\psi} = \sum_k (-1)^k \dim H^k(\mathbb{O}, \check{\mathcal{L}}_\psi \otimes \mathcal{L}_\varphi)$$

$$\varphi = (\mathbb{O}_\varphi, \mathcal{L}_\varphi)$$

$$\|$$

$$\psi = (\mathbb{O}_\psi, \mathcal{L}_\psi)$$

Thm. Let the following parity vanishing conditions hold:

1) $H_{odd}(Z) = 0$

2) $\forall x \in N$, $H_{odd}(M_x) = 0$

Then $C = KL \cdot D \cdot KL^t$.

NB: a) $N = \mathcal{N}^a$, $M = \tilde{\mathcal{N}}^a$, 1), 2) hold. deep result of Lusztig, DeConcini, Procesi.

b) in cat. $\mathcal{O}$ over $\mathfrak{g}$, $D = \mathrm{Id}$, $C = KL \cdot KL^t$ by BGG reciprocity.

$M : L = KL$, $P : M = KL^t$.

c) $\mathcal{C}$ is symmetric $\Leftrightarrow$ $\mathcal{D}$ is symmetric.

true often but not always.

If $\mathcal{L}_\varphi^\vee = \mathcal{L}_\varphi$, $\forall \varphi$, then $\mathcal{D}$ is symmetric.

e.g. $a = (1,1)$, $N = \mathcal{N}$, $\mathcal{L}_\varphi^\vee = \mathcal{L}_\varphi$,

since $\pi_1^{\text{equiv.}} = \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5$, or product thereof.

If $G = SL_N$, then $\pi_1^{\text{equiv.}}(\mathbb{O}_{reg}) = \mathbb{Z}/N\mathbb{Z} \Rightarrow \mathcal{L}_\varphi \neq \mathcal{L}_\varphi^\vee$ for $N > 2$.

d) Usually, $\mathcal{D}$ is <u>not</u> invertible over $\mathbb{Z}$. (KL is invertible) $\Rightarrow$ $\mathcal{C}$ is not invertible

$\rightarrow$ homological dimension of $\mathcal{H}_a$-mod is infinite.

and $[P_\varphi]$ do <u>not</u> generate $K(\mathcal{H}_a\text{-mod})$

Lecture 12.

<u>Remark</u>: there is a particular case when the grading on the Ext-algebra is the natural one: $M \xrightarrow{\pi} N$ when π is semismall.

$N = \bigsqcup \mathbb{O}_s$, π is stratified w/ $N = \bigsqcup \mathbb{O}_s$, $\forall s$, $\operatorname{codim}_N \mathbb{O}_s \geq 2 \dim \pi^{-1}(x)$
$x \in \mathbb{O}_s$.

$\pi|_{\mathbb{O}_s}$ is a locally trivial fibration.

Our main example, $a \in \mathfrak{g} = (1,2)$.

$N = \mathcal{N}$, $M = T^*\mathcal{B} = \tilde{\mathcal{N}}$.

Grothendieck – Springer : π is semismall

(in fact, strictly semismall. $\operatorname{codim}_{\mathcal{N}} \mathbb{O}_S = 2 \dim \pi^{-1}(x), \ \forall x \in \mathbb{O}_S$)

$\Rightarrow \ \pi_* C_M = \bigoplus IC_\varphi \otimes L_\varphi$, L_φ lives in $\deg = 0$. no grading.

$\pi_* C_M$ is a semisimple perverse sheaf on M.

$H_\bullet(Z) = \operatorname{Ext}^\bullet(\pi_* C_M, \pi_* C_M)$,

$\operatorname{Ext}^0 = \bigoplus \operatorname{End}(L_\varphi)$, $\operatorname{Ext}^{>0}$ is the radical of $H^\bullet(Z)$.

Thm Under the two parity vanishing conditions, $C = KL \cdot D \cdot KL^t$.

a) $\forall x \in \mathcal{N}$, $H_{odd}(M_x) = 0$

b) $H_{odd}(Z) = 0$.

Recall. $KL_{\varphi\psi} = \sum_k \left[\underline{H}^k (i_\varphi^* IC_\psi) : \chi_\varphi \right]$

$\varphi = (\mathbb{O}_\varphi, \chi_\varphi)$, χ_φ is an irred. rep. of the equivariant $\pi_1(\mathbb{O}_\varphi, x)$

$\equiv$ irr. local system $\mathcal{L}_\varphi$ on $\mathbb{O}_\varphi$.

$D_{\varphi\psi} = \sum_k (-1)^k \dim H^k(\mathbb{O}, \check{\mathcal{L}_\varphi} \otimes \mathcal{L}_\psi)$

almost diagonal : both $\mathcal{L}_\varphi, \mathcal{L}_\psi$ live on the same orbit $\mathbb{O}$.

Lemma 1. If $H_{odd}(M_x) = 0$, $\forall x$, and if $IC_\psi \subset \mu_* C_M$, and $x \in \overline{\mathbb{O}_\varphi}$,

then $H^{d_\varphi + k} i_x^! IC_\psi = 0$ for odd k, $d_\varphi = \dim \mathbb{O}_\varphi$.

<u>Lemma 2</u>. Under the assumptions of the theorem, for φ & ψ s.t. $IC_\varphi, IC_\psi \,\Theta\, \mu_* C$

$$\operatorname{Ext}^{d_\varphi + d_\psi + k}(IC_\varphi, IC_\psi) = 0 \quad \text{for odd } k.$$

<u>Proof of the theorem modulo Lemma</u>:

Slogan: parity vanishing $\Rightarrow$ no cancellation in the Euler char. χ.

$$\mathcal{F}_1, \mathcal{F}_2 \in D^b(N) \Rightarrow \sum (-1)^k \dim \operatorname{Ext}^k(\mathcal{F}_1, \mathcal{F}_2)$$

$$= \sum_k (-1)^k \dim H^k(N, D\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2) = \chi(N, D\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2)$$

$$= \sum_{\mathbb{O}} \chi(\mathbb{O}, i_{\mathbb{O}}^! (D\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2)) = \sum_{\mathbb{O}} \chi(\mathbb{O}, i_{\mathbb{O}}^! D\mathcal{F}_1 \overset{!}{\otimes} i_{\mathbb{O}}^! \mathcal{F}_2)$$

Take $\mathcal{F}_1 = IC_\psi$, $\mathcal{F}_2 = IC_\varphi$.

$$[P_\varphi : L_\psi] = \sum_k \dim \operatorname{Ext}^k(IC_\psi, IC_\varphi)$$

$$= \sum_k \dim \operatorname{Ext}^{d_\varphi + d_\psi + k}(IC_\psi, IC_\varphi)$$

$$\underset{\text{Lemma 2}}{\overset{\text{parity vanishing}}{=\!=\!=}} \sum_k (-1)^k \dim \operatorname{Ext}^{d_\varphi + d_\psi + k}(IC_\psi, IC_\varphi)$$

$$= (-1)^{d_\varphi + d_\psi} \sum_{\mathbb{O}} \chi(\mathbb{O}, i_{\mathbb{O}}^! DIC_\psi \overset{!}{\otimes} i_{\mathbb{O}}^! IC_\varphi)$$

Now in the K-group of $D^b(\mathbb{O})$, the class of $i_{\mathbb{O}}^! IC_\varphi = \sum_k (-1)^k \underline{H}^k i_{\mathbb{O}}^! IC_\varphi$

$$= \sum_k (-1)^{d_\varphi + k} \underline{H}^{d_\varphi + k} i_{\mathbb{O}}^! IC_\varphi = (-1)^{d_\varphi} \sum_k (-1)^k \underline{H}^{d_\varphi + k} i_{\mathbb{O}}^! IC_\varphi$$

$$= (-1)^{d_\varphi} \sum_k \underline{H}^{d_\varphi + k} i_{\mathbb{O}}^! IC_\varphi$$ by the parity vanishing of Lemma 1.

Conclusion. $i_{\mathbb{O}}^! IC_\varphi = (-1)^{d_\varphi} \sum_{k,\mathcal{L}} [\underline{H}^k i_{\mathbb{O}}^! IC_\varphi : \mathcal{L}] \cdot \mathcal{L} \in K D^b(\mathbb{O})$.

the sum runs over the irr. local systems on $\mathbb{O}$.

Similarly, $i_{\mathbb{O}}^! DIC_\psi = (-1)^{d_\psi} \sum_{k,\mathcal{L}} [\underline{H}^k i_{\mathbb{O}}^! IC_\psi ; \mathcal{L}] \mathcal{L}^\vee$.

Hence $i_{\mathbb{O}}^! DIC_\psi \overset{!}{\otimes} IC_\varphi = (-1)^{d_\psi} \sum_{k,\mathcal{L}} \left[\underline{H}^{d_\psi + k} i_{\mathbb{O}}^! IC_\psi : \mathcal{L}\right]$.

$$\cdot (-1)^{d_\varphi} \sum_{l,\mathcal{L}'} \left[\underline{H}^{d_\varphi + l} i_{\mathbb{O}}^! IC_\varphi : \mathcal{L}'\right] \cdot \mathcal{L}^\vee \overset{!}{\otimes} \mathcal{L}'.$$

$$= (-1)^{d_\varphi + d_\psi} \sum_{\mathcal{L},\mathcal{L}'} KL_{\psi\,\mathcal{L}} \cdot KL_{\varphi\,\mathcal{L}'}\ \mathcal{L}^\vee \overset{!}{\otimes} \mathcal{L}'.$$

But for local systems, there is no difference between $\mathcal{L}^\vee \overset{!}{\otimes} \mathcal{L}'$ and $\mathcal{L}^\vee \otimes \mathcal{L}'$ (up to even shift). It remains to compute the Euler characteristic and multiply by $(-1)^{d_\varphi + d_\psi}$.

$$\Rightarrow (-1)^{d_\varphi + d_\psi} \chi(\mathbb{O}, i_{\mathbb{O}}^! DIC_\psi \overset{!}{\otimes} i_{\mathbb{O}}^! IC_\varphi) = \sum_{\mathcal{L},\mathcal{L}'} KL_{\psi,\mathcal{L}} \cdot KL_{\varphi,\mathcal{L}'}\ \underbrace{\chi(\mathbb{O}, \mathcal{L}^\vee \otimes \mathcal{L}')}_{D_{\mathcal{L},\mathcal{L}'}}. \qquad \square$$

Proof of Lemma 1. Set $m = \dim M$.

$$i : x \hookrightarrow N, \quad H_\bullet(M_x) = H^{-\bullet}(i^! \mu_* C_M[m])$$

$$= \bigoplus L_\varphi(k) \otimes H^{m+k-\bullet}(i^! IC_\varphi)$$

$$\varphi = (\mathbb{O}_\varphi, \mathcal{L}_\varphi), \quad \mu_* C_M = \bigoplus_\varphi IC_\varphi[k] \otimes L_\varphi(k). \quad L_\varphi = \bigoplus_k L_\varphi(k)$$

If $x \in \mathbb{O}_\varphi$, then $i^! IC_\varphi = \mathcal{L}_\varphi[-d_\varphi]_x$

$\Rightarrow H^{d_\varphi}(i^! IC_\varphi) = H^0(\mathcal{L}_\varphi|_x) \neq 0$

$\Rightarrow H^{-j}(i^! IC_\varphi[m+k]) \neq 0$ for $-j+m+k = d_\varphi$.

Hence for $k = d_\varphi - m + j$, the summand $L_\varphi(k) \otimes H^{-j}(i^! IC_\varphi)[m+k]$ gives a

<u>nonzero</u> contribution to $H_j(M_x)$ if $L_\varphi(k) \neq 0$.

The parity vanishing condition $\Rightarrow L_\varphi(d_\varphi - m + j) = 0$ for $2 \nmid j$. $\quad (*)$

Now let $x \in \mathbb{O}$ (arbitrary stratum in the closure of $\mathbb{O}_\varphi$). Then similarly,

$L_\varphi(k) \otimes H^{-j+m+k}(i^! IC_\varphi) = 0$ for j odd. $\quad (!)$

If IC_φ enters $\mu_* C_M$, from $(*)$, we conclude for some $p \in \mathbb{Z}$,

$L_\varphi(d_\varphi - m - 2p) \neq 0$. Take $k = d_\varphi - m - 2p$, substitute into $(!)$

$H^{d_\varphi - j - 2p}(i^! IC_\varphi) = 0$ for odd j.

All in all, $H^{d_\varphi + j}(i^! IC_\varphi) = 0$ for odd j. $\square$

<u>Proof of Lemma 2</u>.

$\mathrm{Ext}^k(IC_\varphi, IC_\psi) = H^k(N, DIC_\varphi \overset{!}{\otimes} IC_\psi)$

$DIC_\varphi = IC(\mathbb{O}, \check{\mathcal{L}}_\varphi)$, since $\mu_* C_M$ is self-dual, we see that

$IC_\varphi \ominus \mu_* C_M$ iff $DIC_\varphi \ominus \mu_* C_M$, so we have to check

$H^{d_\varphi + d_\psi + k}(N, IC_\varphi \overset{!}{\otimes} IC_\psi) = 0$ for $2 \nmid k$. By the conclusion $(*)$. $\exists\, p, q \in \mathbb{Z}$

s.t. $L_\varphi(d_\varphi - m - 2p) \neq 0 \neq L_\psi(d_\psi - m - 2q) \Rightarrow L_\varphi(d_\varphi - m - 2p) \otimes L_\psi(d_\psi - m - 2q) \neq 0$.

Now $H_j(\mathcal{Z}) = \mathrm{Ext}^{-j+2m}_{D^b(N)}(\mu_* \mathbb{C}_M, \mu_* \mathbb{C}_M)$.

$$= H^{-j}(\mu_* \mathbb{C}_M[m] \overset{!}{\otimes} \mu_* \mathbb{C}_M[m])$$

$$= \bigoplus_{k,l,\varphi,\psi} L_\varphi(k) \otimes L_\psi(l) \otimes H^{-j+k+l+2m}(IC_\varphi \overset{!}{\otimes} IC_\psi)$$

By our assumption, LHS $= 0$ for $2 \nmid j$

$\Rightarrow$ all the summands in RHS vanish for $k = d_\varphi - m - 2p$ and $l = d_\psi - m - 2q$

$\Rightarrow 0 = H^{-j+k+l+2m}(IC_\varphi \overset{!}{\otimes} IC_\psi) = H^{-j+d_\varphi+d_\psi-2p-2q}(IC_\varphi \overset{!}{\otimes} IC_\psi)$

for $2 \nmid j$. Hence $H^{d_\varphi+d_\psi+k}(N, IC_\varphi \overset{!}{\otimes} IC_\psi) = 0$ for odd k.

Application of this Ginzburg - Kazhdan - Lusztig ideology to the Kazhdan - Lusztig conjecture about $[M_w : L_y]$ in the category $\mathcal{O}$ over $\mathfrak{g}$. $w \le y \in W$

M_w has h.w. $w\rho - \rho$, L_y has h.w. $y\rho - \rho$.

(regular weights in the W-orbit of 0 w.r.t. dot-action centered in $-\rho$)

they all have the same trivial central character.

We will need the notion of equivariant Borel-Moore homology. More generally, equivariant cohomology w/ coeffs. in equivariant constructible complexes.

$H^\bullet_G(X, \mathbb{C}) \overset{\text{Borel}}{=} H^\bullet(BX, \mathbb{C}) \quad BX := EG \overset{G}{\times} X$

$\Rightarrow H^\bullet_G(X, \mathbb{C})$ is a module over $H^\bullet(BG, \mathbb{C}) = H^\bullet_G(pt)$.

If $G = T$ a torus, then $H^\bullet_T(pt) = \mathbb{C}[\mathfrak{t}]$.

If $G \supset T$ = Cartan torus, $W = W(G,T)$, $H^\bullet_G(pt) = \mathbb{C}[\mathfrak{t}]^W$.

Any G is an algebraic subgroups of $GL(N)$ $\Rightarrow$ $BG \xrightarrow{GL(N)/G} BGL(N)$

$BGL(N)$ is the union of Grassmannians $Gr(N,M)$, $M \to \infty$.

Spaces $St(N,M)$, $M \to \infty$. $BG = \bigcup St(N,M)/G$, $M \to \infty$.

By definition, an equivariant constructible complex on X w.r.t. G is a compatible collection of constructible complexes on $St(N,M)/G$ ---> BG

Then $H^\bullet_G(X, \mathcal{F}) = \lim H^\bullet(St(N,M)/G, \mathcal{F}_M)$

This is the approach of J. Bernstein, V. Lunts.

$T \curvearrowright N$, $\mathcal{F}$ a semisimple T-equivariant complex $= \bigoplus IC_\varphi$.

We will study $\mathcal{E} = \mathrm{Ext}^\bullet_T(\mathcal{F}, \mathcal{F})$ contains $H^\bullet_T(pt) = \mathbb{C}[\mathfrak{t}]$. in the center.

$\Rightarrow$ for $x \in \mathfrak{t}$, we can specialize $\mathcal{E}$ at x $\Rightarrow$ $\mathcal{E}_x$ a finite dimensional algebra

It usually happens that $\mathcal{E}_x$ has some geometric meaning.

Eg. when we have a resolution $M \to N$, $x \in \mathfrak{t}$ $\rightsquigarrow$ $\{\exp cx,\ c \in \mathbb{C}\}$

$\rightsquigarrow$ take the closure. $T_x \subset T$. a subtorus, not necessarily 1-dim.

$M^{T_x} \to N^{T_x}$ fixed points. If $Z = M^{T_x} \times_{N^{T_x}} M^{T_x}$ has a cellular decomp.

then $\mathcal{E}_x = H_\bullet(Z) = \mathrm{Ext}^\bullet(\mu_* \mathbb{C}_{M^{T_x}}, \mu_* \mathbb{C}_{M^{T_x}})$.