<u>Jiahao's correction</u>: $T/\mathbb{C}$ torus, acting on $X/\mathbb{C}$ scheme of finite type,

$t \in T$, $i : X^t \hookrightarrow X$, $K^T(X^t) \xrightarrow{i_*} K^T(X)$ becomes an isomorphism

after localizing $\underset{\{f : f(t) \neq 0\}}{\underset{\shortparallel}{S}} \subset K^T(pt) = R(T) = \mathbb{Z}[X^*(T)]$.

Anti-spherical module

Daniel Kim

$\mathcal{O} \twoheadrightarrow k$
$\cap$
F non-arch. local field, G/F split reductive.

B Borel, T max. split torus, Iwahori $I = \ker(G(\mathcal{O}) \to G(k)/B(k))$

<u>Extended affine Weyl group</u>. $\tilde{W} = N_{G(F)}(T(F))/T(\mathcal{O})$.

$$1 \to X_*(T) \to \tilde{W} \to W \to 1$$

finite Weyl group.

$\mathcal{H}_I = C_c^\infty(I \backslash G(F)/I ; \mathbb{C})$

$G(F) = \coprod_{w \in \tilde{W}} I \dot{w} I$ ($\dot{w}$ is a lift) $\rightsquigarrow T_w = \mathbb{1}_{I\dot{w}I} \in \mathcal{H}_I$.

① $(T_s + 1)(T_s - q) = 0$ for $s \in \tilde{W}$ simple reflection.

② $T_{vw} = T_v \cdot T_w$, when $\ell(vw) = \ell(v) + \ell(w)$.

"Iwahori–Matsumoto presentation"

<u>Rmk</u>. We can treat q as a formal variable, work over $\mathbb{Z}[q]$.

If $\lambda, \mu \in X_*(T)^+$ are dominant, $\Rightarrow T_{\lambda+\mu} = T_\lambda T_\mu = T_\mu T_\lambda$.

& if we work over $\mathbb{Z}[q^{\pm 1}]$, then all T_i's are invertible.

$\forall \lambda \in X_*(T)$, write $\lambda = \lambda_1 - \lambda_2$ for $\lambda_1, \lambda_2 \in X_*(T)^+$,

$\rightsquigarrow$ define $e^\lambda = q^{-\langle \lambda, \rho \rangle} T_{\lambda_1} \cdot T_{\lambda_2}^{-1}$ well-defined & $e^{\lambda+\mu} = e^\lambda \cdot e^\mu$

(we may want $\rho \in X^*(T)$, otherwise need to introduce $q^{\frac{1}{2}}$).

<u>Thm (Bernstein)</u> $e^\lambda T_w$ for $\lambda \in X_*(T)$, $w \in W$ form a $\mathbb{Z}[q^{\pm 1}]$-basis for $\mathcal{H}_I$, $T_s e^{s(\lambda)} - e^\lambda \cdot T_s = (1-q) \dfrac{e^\lambda - e^{s(\lambda)}}{1 - e^{\alpha^\vee}}$ where $s = s_\alpha \in W$ is a simple reflection

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<u>Steinberg variety</u> $\check{G}, \check{B}, \check{T} / \mathbb{C}$ dual groups

<u>Def</u>. An element $x \in \check{\mathfrak{g}}$ is nilpotent when its image under $\check{\mathfrak{g}} \hookrightarrow \mathfrak{gl}_n$ is

<u>Fact</u>: $\check{\mathfrak{n}} = \mathrm{Lie}\, \check{N}$ covers all nilpotent elements under the adjoint action.

<u>Def</u>. $\mathcal{N} \subset \check{\mathfrak{g}}$ the <u>nilpotent cone</u> is the set of nilp. elts regarded as a var.

Def. Springer resolution:

$$\tilde{\mathcal{N}} = \{(x, g\check{B}) \in \mathcal{N} \times \check{G}/\check{B} : x \in g\check{\mathfrak{n}}g^{-1}\}$$

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d, "\text{projective}"'] & (x, g\check{B}) \arrow[d, mapsto] \\ \mathcal{N} & x \end{tikzcd}$$

Given $g\check{B}$, $(\mathrm{ad}_g \check{\mathfrak{n}})^{\perp} = \mathrm{ad}_g \check{\mathfrak{b}}$ under the Killing form

$$0 \to \mathrm{ad}_g \check{\mathfrak{b}} \to \mathfrak{g} \to T_{g\check{B}}(\check{G}/\check{B}) \to 0$$

$$\Rightarrow \mathrm{ad}_g \check{\mathfrak{n}} \cong T^*_{g\check{B}}(\check{G}/\check{B}) \rightsquigarrow \tilde{\mathcal{N}} = T^*\check{\mathcal{F}}, \ (\check{\mathcal{F}} = \check{G}/\check{B}).$$

Def. $Z = \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} \quad \left(\neq \tilde{\mathcal{N}} \overset{L}{\underset{\mathcal{N}}{\times}} \tilde{\mathcal{N}}\right)$ the Steinberg variety.

Want to understand K-theory of Z.

Ring str. comes from

$$\begin{tikzcd} & \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} \arrow[dl, "\mathrm{id}\times\pi\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\Delta\times\mathrm{id}"] & \\ Z = \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} & & \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} \underset{\mathbb{C}}{\times} \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}} = Z \underset{\mathbb{C}}{\times} Z \end{tikzcd}$$

pullback $\tilde{\mathcal{N}}$ smooth.

$$\check G \times \mathbb{G}_m \curvearrowright \begin{array}{c} \tilde{\mathcal N} \\ \downarrow \\ \check{\mathfrak g} \end{array} \qquad \begin{array}{l} (g,c).(x, h\check B) = (\mathrm{ad}_g x \cdot c,\ gh\check B). \\ \\ (\mathrm{ad},\ \text{scalar}) \end{array}$$

$\rightsquigarrow$ Z also has a $\check G \times \mathbb{G}_m$-action $\rightsquigarrow$ $K^{\check G \times \mathbb{G}_m}(Z)$ is defined!

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \wr\wr$ spoiler.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathcal{H}_I$

<u>Subalgebras</u> $\quad \Delta : \tilde{\mathcal N} \longrightarrow \tilde{\mathcal N} \overset{L}{\underset{\check{\mathfrak g}}{\times}} \tilde{\mathcal N} = Z \qquad \check G \times \mathbb{G}_m$-equiv

$$\rightsquigarrow \Delta_* : \underset{\substack{\text{algebra } (\because\ \tilde{\mathcal N} \text{ is smooth}).}}{K^{\check G \times \mathbb{G}_m}(\tilde{\mathcal N})} \longrightarrow K^{\check G \times \mathbb{G}_m}(Z)$$

<u>Claim</u>. This is an algebra homomorphism.

$$\begin{tikzcd}
\tilde{\mathcal N} \arrow[r, equal] \arrow[d, "\Delta"'] & \tilde{\mathcal N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \tilde{\mathcal N} \times_{\mathbb C} \tilde{\mathcal N} \arrow[d, "\Delta \times \Delta"] \\
\tilde{\mathcal N} \overset{L}{\underset{\check{\mathfrak g}}{\times}} \tilde{\mathcal N} & \tilde{\mathcal N} \overset{L}{\underset{\check{\mathfrak g}}{\times}} \tilde{\mathcal N} \overset{L}{\underset{\check{\mathfrak g}}{\times}} \tilde{\mathcal N} \arrow[l] \arrow[r, "\mathrm{id} \times \Delta \times \mathrm{id}"'] & \tilde{\mathcal N} \overset{L}{\underset{\check{\mathfrak g}}{\times}} \tilde{\mathcal N} \times_{\mathbb C} \tilde{\mathcal N} \overset{L}{\underset{\check{\mathfrak g}}{\times}} \tilde{\mathcal N}
\end{tikzcd}$$

$$K^{\check G \times \mathbb{G}_m}(\tilde{\mathcal N} = T^*\check{\mathcal F}) = K^{\check G \times \mathbb{G}_m}(\check{\mathcal F} = \check G/\check B) = K^{\check B \times \mathbb{G}_m}(*)$$

$$= R(\check B \times \mathbb{G}_m) = \mathbb{Z}[X^*(\check T) \oplus \mathbb{Z}] = \mathbb{Z}[X_*(T)][q^{\pm 1}]$$

Thm. There is a diagram of algebras

$$\begin{tikzcd}
K^{\check{G}\times \mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[r, "\Delta_*"] & K^{\check{G}\times \mathbb{G}_m}(Z) \\
\mathbb{Z}[X_*(T)][q^{\pm 1}] \arrow[u, "\cong"] \arrow[r] & \mathcal{H}_I \arrow[u, "\cong"'] \\
\lambda \arrow[r, mapsto] & e^{\lambda}
\end{tikzcd}$$

$[\mathcal{O}_\lambda] \mapsto -\lambda$

$\|$

Equiv. l.b. on $\check{\mathcal{F}}$ corresponding to $\check{B} \xrightarrow{\lambda} \mathbb{G}_m$.

$$\begin{tikzcd}
Z \arrow[r, "i"] & \check{\mathcal{F}} \times \check{\mathcal{F}}
\end{tikzcd} \qquad \check{G} \text{ acts } \mathrm{diag}$$

$(x, g_1\check{B}, g_2\check{B}) \mapsto (g_1\check{B}, g_2\check{B})$

$$\rightsquigarrow \quad \check{\mathcal{F}} \times \check{\mathcal{F}} = \coprod_{w\in W} \underbrace{(\check{\mathcal{F}} \times \check{\mathcal{F}})_w}_{=:\, Y_w}$$

$$\begin{tikzcd}
\check{\mathcal{F}} \times \check{\mathcal{F}} & Y_w \\
Z \arrow[u] & \pi^{-1}(Y_w) \arrow[u]
\end{tikzcd}$$

$$Z = \coprod_{w\in W} \pi^{-1}(Y_w)$$

each $\pi^{-1}(Y_w) \to Y_w$ is a vec. bundle.

In fact, $\pi^{-1}(Y_w) = T^*_{Y_w}(\check{\mathcal{F}} \times \check{\mathcal{F}})$

$\left(\text{ker. of } T^*(\check{\mathcal{F}} \times \check{\mathcal{F}})|_{Y_w} \to T^*_{Y_w}\right)$.

$\dim \pi^{-1}(Y_w) = \dim Y_w + \dim \text{fiber} = \left(\dim \check{\mathcal{F}} + \ell(w)\right) + \left(\dim \check{\mathcal{F}} - \ell(w)\right)$

$= 2 \dim \check{\mathcal{F}}$

$\overline{\pi^{-1}(Y_w)}$ are the irred. components of Z.

Fact. For $s \in W$ a simple refl, $\check{Y}_s \subset \check{\mathcal{F}} \times \check{\mathcal{F}}$ is smooth

$$\& \quad \begin{tikzcd} & \check{Y}_s \arrow[dl, "pr_1"'] \arrow[dr] & \\ \check{\mathcal{F}} & & \check{\mathcal{F}} \end{tikzcd} \quad \text{are } \mathbb{P}^1\text{- bundles.}$$

$\Omega^1_{\check{Y}_s/\check{\mathcal{F}}}$ (first proj.) $\longrightarrow$ pull back along $T^*_{\check{Y}_s}(\check{\mathcal{F}} \times \check{\mathcal{F}}) \longrightarrow \check{Y}_s$

$\overset{\text{def}}{=} Q_s$ $\qquad T^*_{\check{Y}_s}(\check{\mathcal{F}} \times \check{\mathcal{F}}) \subset Z$

Claim. $T_s \longleftrightarrow -[q Q_s] - [\mathcal{O}_0]$

$T_s \in \mathcal{H}_I$, $\quad -[q Q_s] - [\mathcal{O}_0] \in K^{\check{G} \times \mathbb{G}_m}(Z)$.

Modules.

Note, Whenever $\check{G} \times \mathbb{G}_m$ - equivariant $Y \to \check{\mathfrak{g}}$, $K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} Y)$ has a convolution

action by $K^{\check{G} \times \mathbb{G}_m}(Z = \tilde{\mathcal{N}} \overset{L}{\underset{\check{\mathfrak{g}}}{\times}} \tilde{\mathcal{N}})$

Take $Y = \check{\mathfrak{g}}$ $\rightsquigarrow$ $K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ left module over $K^{\check{G} \times \mathbb{G}_m}(Z)$

$$K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \cong \mathbb{Z}[X_*(T)][q^{\pm 1}]$$

$$\begin{tikzcd} R(\check{G} \times \mathbb{G}_m) \arrow[r, "\pi^*"] \arrow[rr, bend right] & K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[r, "\Delta_*", "\text{ring hom.}"'] & K^{\check{G} \times \mathbb{G}_m}(Z) \end{tikzcd} \curvearrowright K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}})$$

Claim ① this composition is central; ② The action of $K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ on itself is mult.

① Given $V \in \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$, $\check{G} \times \mathbb{G}_m$-equiv. $\mathcal{F}$ on $\mathcal{Z}$,

$$V * \mathcal{F} = \mathcal{F} * V = \left(\mathcal{F} \otimes V \text{ w/ diag action} \right)$$

② Use base change isom.

<u>Hecke algebra side:</u>

$$\begin{tikzcd} \mathbb{Z}[X_*(T)]^W[q^{\pm 1}] \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z}[X_*(T)][q^{\pm 1}] \arrow[r] & \mathcal{H}_I \curvearrowright_{\text{left}} \mathcal{H}_I \otimes_{\mathcal{H}_{fin,\varepsilon}} \mathbb{Z}[q^{\pm 1}]. \\ Z(\mathcal{H}_I) & & \end{tikzcd}$$

<u>Def</u>. $\mathcal{H}_{fin} = \mathbb{Z}[q^{\pm 1}][T_w : w \in W]$.

$\varepsilon : \mathcal{H}_{fin} \to \mathbb{Z}[q^{\pm 1}]$, $T_w \mapsto q^{\ell(w)}$ is a ring hom.

$\left[(T_s + 1)(T_s - q) = 0 \text{ for } T_s = q \right]$

<u>Lemma</u>. Identify $\mathcal{H}_I \otimes_{\mathcal{H}_{fin,\varepsilon}} \mathbb{Z}[q^{\pm 1}] \cong K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}})$, $\mathbb{Z}[q^{\pm 1}]$-linear.

$$e^{-\lambda} \otimes 1 \longmapsto [\mathcal{O}_\lambda]$$

Then

$$\begin{tikzcd} R(\check{G} \times \mathbb{G}_m) \arrow[r, hook] \arrow[d, phantom, "\wr\!\shortparallel"] & K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[r, phantom, "\curvearrowleft"] \arrow[d, phantom, "\wr\!\shortparallel"] & K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[d, phantom, "\wr\!\shortparallel"] \\ \mathbb{Z}[X_*(T)]^W[q^{\pm 1}] \arrow[r, hook] & \mathbb{Z}[X_*(T)][q^{\pm 1}] \arrow[r, phantom, "\curvearrowleft"] & \mathcal{H}_I \otimes_{\mathcal{H}_{fin,\varepsilon}} \mathbb{Z}[q^{\pm 1}] \end{tikzcd}$$

$[\mathcal{O}_\lambda] \leftrightarrow e^{-\lambda}$ — commutative diagram.

$$\begin{tikzcd} \rightsquigarrow K^{\check{G} \times \mathbb{G}_m}(\mathcal{Z}) \arrow[r, "\alpha"] \arrow[d, dashed] & \mathrm{End}_{R(\check{G} \times \mathbb{G}_m)}\left(K^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \right) \arrow[d, phantom, "\wr\!\shortparallel"] \\ \mathcal{H}_I \arrow[r, hook] & \mathrm{End}_{\mathbb{Z}[X_*(T)]^W[q^{\pm 1}]}\left(\mathcal{H}_I \otimes_{\mathcal{H}_{fin,\varepsilon}} \mathbb{Z}[q^{\pm 1}] \right) \end{tikzcd}$$

① Will prove α is injective

② The actions of T_s and $-[q\mathcal{O}_s]-[\mathcal{O}_0]$ agree $\rightsquigarrow \mathcal{H}_I \hookrightarrow K^{\check{G}\times\mathbb{G}_m}(Z)$

③ Check that this is everything.

Recall. $\check{\mathcal{F}}\times\check{\mathcal{F}} = \coprod_{w\in W} Y_w, \rightsquigarrow Z = \coprod_{w\in W} T^*_{Y_w}(\check{\mathcal{F}}\times\check{\mathcal{F}})$

Prop. Write $Z_{\leq w} = \coprod_{y\leq w} T^*_{Y_y}(\check{\mathcal{F}}\times\check{\mathcal{F}})$ and $Z_{<w}$ similarly.

Then there is a split SES

$$0 \to K^{\check{G}\times\mathbb{G}_m}(Z_{<w}) \to K^{\check{G}\times\mathbb{G}_m}(Z_{\leq w}) \to K^{\check{G}\times\mathbb{G}_m}(Y_w) \to 0.$$

Pf. Cell decomposition for
$$\begin{array}{c} Z \\ \downarrow \scriptstyle{pr_1} \\ \check{\mathcal{F}} \end{array}.$$

Claim. ① $K^{\check{G}\times\mathbb{G}_m}(Z_{\leq w})$ are left $K^{\check{G}\times\mathbb{G}_m}(\tilde{\mathcal{N}})$- submodules.

② $K^{\check{G}\times\mathbb{G}_m}(Y_w) \cong \mathbb{Z}[X_*(T)][q^{\pm 1}]$

because $Y_w = \check{G}/\underbrace{(\check{B}\cap w\check{B}w^{-1})}$ is solvable

$\Rightarrow R(\check{B}\cap w\check{B}w^{-1}) = R(\check{T})$.