Schobers for Grothendieck resolutions

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C curve $g > 1$, $\quad D(C) \hookrightarrow D(X)$, X Fano ?

Kapranov - Schechtman : <u>Schobers</u>.

<u>Spherical functors</u> $\quad$ $D_1 \xrightarrow{\psi} D_2$ is spherical. if $\bullet$ $\exists\, \psi^!, \psi^*$ right & left adjoint

$$\begin{array}{ccc} D_1 & \xrightarrow{\psi} & D_2 \\ \circlearrowright & & \circlearrowleft \\ C_1 & & T_2 \end{array}$$

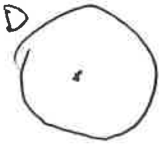
$\bullet$ C_1, T_2 are equivalences

$C_1 \to id \to \psi^! \psi$

$\psi \psi^! \to id \to T_2$

Seidel - Thomas, Horja, Logvinenko - Anno

D



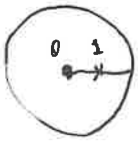
$D^b_{c,(D,0)}(Sh(D)) \neq D(Sh_c(D,0))$

$\uparrow$ t-structure (middle perversity)

the heart of the t-str. $=$ $Perv_{(D,0)}(D)$ abelian cat

$$\begin{tikzcd} 1 \arrow[r, bend right, "\alpha"'] & 2 \arrow[l, bend right, "\beta"'] \end{tikzcd}$$

- $\alpha\beta - id_2$ is invertible
- $\beta\alpha - id_1$ is invertible.



K radius $\quad$ $F \in Perv(D,0)$,

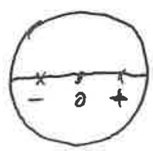
$\mathbb{H}^1_K(F)$ $\qquad$ $\mathbb{H}^{\neq 1}_K(F) = 0$

$\uparrow$ a sheaf locally const in strata

$$V_0 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V_1 \qquad VD(\mathcal{F}) \in \mathrm{Perv}(D,0)$$

$$\rightsquigarrow \quad V_0^\vee \longrightarrow V_1^\vee$$

D



$K =$ diameter

$$\left\{ \begin{array}{c} V_- \underset{\delta_-}{\overset{\gamma_-}{\rightleftarrows}} V_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} V_+ \\ \text{fibers of } \mathcal{H}^1_K(\mathcal{F}) \\ \\ \gamma_+\delta_- \text{ is isom.}, \quad \gamma_-\delta_+ \text{ is isom} \\ \gamma_+\delta_+ = id_{V_+} \\ \gamma_-\delta_- = id_{V_-} \end{array} \right\}$$

$$\simeq \mathrm{Perv}(D,0)$$

$$R\Gamma(D,\mathcal{F}) = \{ \overset{0}{V_- \oplus V_+} \xrightarrow{(\delta_-,\delta_+)} \overset{1}{V_0} \}$$

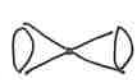
$$R\Gamma_c(D,\mathcal{F}) = \{ \underset{1}{V} \xrightarrow{(\gamma_-,\gamma_+)} \underset{2}{V_- \oplus V_+} \}$$

Atiyah flop $\quad X_0 = X_- \underset{Z}{\times} X_+$

$$\begin{tikzcd}[column sep=small] & & X_0 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 \cong C_- \arrow[r, hook] \arrow[dr] & X_- \arrow[dr] & \mathbb{P}^1 \cong C_+ \arrow[r, hook] \arrow[dl] & X_+ \arrow[dl] \\ & pt \in & Z & \end{tikzcd}$$

X_-, X_+ smooth

Z singular

Atiyah flop: $\quad Z =$  $\quad \dim Z = 3 \quad , X_-, X_+$ small resol'n.

(fibers are curves, not divisors)

$$D(X_-) \underset{Rf_{-,*}}{\overset{Lf_-^*}{\rightleftarrows}} D(X_0) \underset{Lf_+^*}{\overset{Rf_{+,*}}{\rightleftarrows}} D(X_+)$$

$$\mathcal{E}_1 \rightleftarrows \mathcal{E}_0 \leftrightarrows \mathcal{E}_+$$ Schober

$T_{+-} : \mathcal{E}_- \to \mathcal{E}_+$ equiv

$T_{-+} : \mathcal{E}_+ \to \mathcal{E}_-$ equiv.

$D(X_0)/K$, $K = \ker Rf_{-,*} \cap \ker Rf_{+,*}$

$D(X_-) \leftrightarrows D(X_0)/K \rightleftarrows D(X_+)$ "intermediate extension"

$j_{!*}$

Spherical pair.

$K_0(\mathcal{F})$

$K_0(\mathcal{E}_-) \rightleftarrows K_0(\mathcal{E}_0) \leftrightarrows K_0(\mathcal{E}_+)$ $\otimes \mathbb{Q}$

Prop $j : D \setminus 0 \hookrightarrow D$, $K_0(\mathcal{F}) = Rj_!(\mathcal{L})$

$\mathbb{Q}^2 \rightleftarrows \mathbb{Q}^4 \leftrightarrows \mathbb{Q}^2$

$\mathcal{L}$ is a local system on $D - 0$ w/ monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$X_- \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$

$X_0 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1)$

$X_+ \simeq$

Thm (B-K-S) $H^0(D, \mathcal{F}) = D^b(X_-) \times^h_{D^b(X_0)} D^b(X_+) = \underleftarrow{\mathrm{holim}}$ Mct $\{ D^b(X_-) \xrightarrow{Lp_-^*} D^b(X_0) \xleftarrow{Lp_+^*} D^b(X_+) \}$

$H^0(D, \mathcal{F}) = \mathrm{Perf}(\mathcal{Z})$

$$\begin{tikzcd} & X_0 \arrow[dl] \arrow[dr] & \\ X_- \arrow[dr, "f_-"'] & & X_+ \arrow[dl, "f_+"] \\ & Z & \end{tikzcd}$$

VdB set up

$\dim f_+, \dim f_- \leq 1$

$\operatorname{codim} \operatorname{Ex}(f_-), \operatorname{Ex}(f_+) \geq 2$

Z Gorenstein, mult. 2 sing

X_-, X_+ smooth

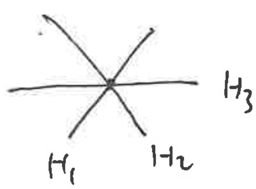
$$\mathbb{H}^2_c(D, \mathcal{F}) := D^b(X_-) \coprod_{D^b(X_0)} D^b(X_+)$$

$$= \underrightarrow{\operatorname{holim}} \left\{ D^b(X_-) \xleftarrow{Rp_{-,*}} D^b(X) \xrightarrow{Rp_{+,*}} D^b(X_+) \right\}$$

$$\mathbb{H}^2_c = D^b(Z)$$

$\mathcal{H} = \{H_i \subset \mathbb{C}^n\}$

given by real eqns



A_2 $\mathbb{C}^2$

(K.-S.) $\operatorname{Perv}(\mathbb{C}^n, \{H_i\}) \ni \mathcal{F}$

$K \xrightarrow{\text{replace}} \mathbb{R}^n \subset \mathbb{C}^n$

13 cells $\ni C$ E_C, $C' \subset C$

$$E_C \rightleftarrows E_{C'} \qquad C' \leq C$$

$$\forall (C, C'), \quad \begin{tikzcd} E_C \arrow[rr, "\psi_{CC'}"] \arrow[dr] & & E_{C'} \\ & E_0 \arrow[ur] & \end{tikzcd}$$

- $\dim C = \dim C' \Rightarrow \psi_{CC'}$ isom.

- C, C', C'' colinear $\Rightarrow \psi_{CC''} = \psi_{C'C''} \circ \psi_{CC'}$

H-schober

$\tilde{g} \to g$ Grothendieck resol'n