Modular functoriality in the local Langlands correspondence

Tony Feng

F = local field of res. char. p

G reductive $/F$ (pretend split)

K = coeff. field, char $\neq p$, $W_F \subset \mathrm{Gal}(F^s|F)$, $\check{G}$ dual gp of G

<u>LLC</u>

$$\{\mathrm{irrep}_K\, G(F)\}/\sim \;\longrightarrow\; \{W_F \to \check{G}(k)\}/\sim$$

$$\pi \longmapsto \rho_\pi$$

Fargues - Scholze

<u>Ex</u>

G	$\check{G}$
GL_n	GL_n
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n}
$SO_{2a+1} \times SO_{2b+1} \not\subset SO_{2a+2b+1}$	$Sp_{2a} \times Sp_{2b} \subset Sp_{2a+2b}$

NOT functorial

(Harris - Taylor)

<u>Langlands functoriality</u>

$\check{H} \to \check{G}$

$\Rightarrow \mathrm{Rep}_K H(F) \dashrightarrow \mathrm{rep}_K G(F)$?

$$\begin{tikzcd}
\{\mathrm{irrep}_K G(F)\}/\sim \arrow[r] & \{W_F \to \check{G}(k)\}/\sim \\
\{\mathrm{irrep}_K H(F)\}/\sim \arrow[r] \arrow[u, dashed, "\text{transfer}"] & \{W_F \to \check{H}(k)\}/\sim \arrow[u]
\end{tikzcd}$$

Ex $\check{H} = \check{M} \subset \check{G}$

$P = MN \subset G$

$\mathrm{Rep}_k M(F) \longrightarrow \mathrm{Rep}_k G(F)$ parabolic induction

$\sigma \longmapsto \left\{ C^\infty \ f : G(F) \xrightarrow{P(F)} V_\sigma \right\}$

Adjoint

$$\mathrm{Rep}_K M(F) \xleftarrow{\text{Jacquet module}} \mathrm{Rep}_K G(F)$$

$$\pi \big/ \langle x - n.x,\ n \in N(F) \rangle \longleftarrow\!\shortmid \pi$$

Goal: analogue of Jacquet module for more situations.

"Modular functoriality" (Treumann – Venkatesh)

- char $k = \boxed{\ell} \neq p$
- $\sigma \in \mathrm{Aut}(G)$ order $\boxed{\ell}$
- $H = G^\sigma$ connected reductive

Ex. (Cyclic base change)

Start w/ H, $\mathrm{Gal}(E/F) = \mathbb{Z}/\ell$

$\Downarrow$ generator σ

$G = \mathrm{Res}_{E/F}(H_E)$, $G^\sigma = H$

Ex. (Triality)  σ $G = \mathrm{Spin}(8)$, $H = G_2$

Claim $\exists$ "nat'l" $\check{H}_k \to \check{G}_k$ in this situation.

Ex $G = Sp_{2a+2b}$, char $F \neq 2$

$\sigma =$ conj. by $\begin{pmatrix} Id_{2a} & \\ & -Id_{2b} \end{pmatrix}$ $\quad \ell = 2$

$H = Sp_{2a} \times Sp_{2b}$

Looking for $SO_{2a+1} \times SO_{2b+1} \longrightarrow SO_{2a+2b+1}$.

In char 2, it does exist!

$SO_{2a(+1)} (x_0^2 +) x_1 x_2 + \cdots + x_{2a-1} x_{2a}$

$SO_{2b(+1)} (y_0^2 +) y_1 y_2 + \cdots + y_{2b-1} y_{2b}$ $\quad z_0^2 \quad z_0 = (x_0 + y_0)$

Why suspect claim?

$\hookrightarrow$ Hecke algebras

$[G] := G(F)/G(\mathcal{O}_F)$

$\mathcal{H}(G,k) = \mathrm{Fun}^c_{G(F)}([G]^2, k)$

$\quad \wr\!\text{S}$ Satake

$\mathcal{O}(\check{G}_k /\!/ \check{G}_k)$

Treumann–Venkatesh There is an "exceptional" $\mathcal{H}(G,k) \xrightarrow{br} \mathcal{H}(H,k)$

Brauer homomorphism

Assume $[G]^\sigma = [H]$

Key point: $\mathcal{H}(G,k)^\sigma \xrightarrow{\text{restriction}} \mathcal{H}(H,k)$ is an algebra homomorphism

Pf Additivity ✓

$$f_1 \underset{G}{*} f_2 (x,z) = \sum_{y \in [G]} f_1(x,y)\, f_2(y,z)$$

$$f_1|_H \underset{H}{*} f_2|_H \,(x,z) = \sum_{y \in [H]} f_1(x,y)\, f_2(y,z)$$

$$\text{diff} = \sum_{\substack{y \in [G] \setminus [H] \\ \circlearrowleft \\ \sigma \text{ acts freely}}} \underset{\sigma_y}{f_1(x,y)}\ \underset{\sigma_y}{f_2(y,z)}$$

$$f_i(x, \sigma y) = f_i(\sigma x, \sigma y) = f_i(x,y)$$

Everything cancels in orbits! □

... more work ...

$$\begin{array}{ccc} \mathcal{H}(G,k) & \xrightarrow{br} & \mathcal{H}(H,k) \\ \wr\!\! & & \wr\!\!\ \text{Satake} \\ \mathcal{O}(\check{G}_k /\!/ \check{G}_k) & \longrightarrow & \mathcal{O}(\check{H}_k /\!/ \check{H}_k) \end{array}$$

Q. Does this come from $\check{H}_k \to \check{G}_k$?

Thm (TV) Yes (in ~~most cases~~ omit E_6) if G simply conn'd, H semisimple.

Pf is via classification + case exhaustion.

Thm (F) Yes if ℓ not $\underbrace{\text{too small}}_{\text{(i.e. good)}}$ via uniform proof.

Ex. σ = conj. by strongly regular s.s. $\in G(F)$

$\Rightarrow$ H = (not. nec. split) torus T , ${}^LT \longrightarrow {}^LG = \check{G} \rtimes W_F$

Proof of idea

$$\mathcal{H}(G, k) \rightsquigarrow \mathrm{Sat}(\mathrm{Gr}_G ; k)$$

$$\begin{tikzcd}
\mathrm{Sat}(\mathrm{Gr}_G^{B_{dR}^+}, k) \arrow[r, equal, "\text{Geom. Sat}"] \arrow[d, "\text{Brauer functor}"'] & \mathrm{Rep}(\check{G}) \arrow[d, dashed] \\
\mathrm{Sat}(\mathrm{Gr}_H, k) \arrow[r, "\simeq"] & \mathrm{Rep}(\check{H})
\end{tikzcd}$$

Applications to LLC.

$$FS : \{\mathrm{irrep}_k\, G(F)\} \to \{\rho_\pi : W_F \to \check{G}(k)\}$$

- surjective ?
- finite ?

Kaletha explicit param.

$$\begin{tikzcd}
\mathrm{irrep}_k^{r.s.c.}\, G(F) & & \text{L-param}^{s.c.} \\
& & \\
\left\{ \begin{array}{l} (T \text{ elliptic} \subset G \\ \text{reg } \chi : T(F) \to \mathbb{C}^\times) \end{array} \right\} \arrow[uu] & W_F \arrow[r, "{}^L\chi \cdot \varepsilon"] \arrow[dr, dashed] & {}^LT \arrow[uu] \arrow[d, hook] \\
& & {}^LG
\end{tikzcd}$$

<u>Thm (F).</u> ${}^L\psi : \check{H} \to \check{G}$

If $\pi \in \mathrm{Rep}\, H(F)$ has param. ρ_π, then ${}^L\psi \circ \rho_\pi$ arises from some $\Pi \in \mathrm{Rep}_k\, G(F)$.

$\sigma \circlearrowright \Pi \in \mathrm{Irrep}_k(G(F)) \circlearrowleft \sigma$

Suppose Π fixed under $\sigma \Rightarrow \sigma \circlearrowright \Pi$, $\quad T^0(\Pi) = \dfrac{\Pi^\sigma}{(1+\sigma+\dots+\sigma^{\ell-1})\Pi} \in \mathrm{Rep}_k\, H(F)$

Thm (Conj. of TV)

$$\Pi \underset{\text{Tr}}{\otimes} k$$

$$\text{" } T^{0 \text{ or } 1}\,\Pi \text{ transfers to } \Pi^{(1)} \text{ "}$$

in subqt of $\bar{\pi}$

$$\begin{tikzcd}
W_F \arrow[rr, "\rho_\pi"] \arrow[dr, "\rho_{\pi(1)}"'] & & \check{H} \arrow[dl, "L\psi"] \\
& \check{G} &
\end{tikzcd}$$

<u>Thm-in-progress</u>. $k = \mathbb{C}$

$(T, \chi) \rightsquigarrow \pi_{(T,\chi)}$

If T unram., then $\rho_{\pi_{(T,\chi)}}$ = Kaletha's.

$Gr_G \circlearrowleft \mathbb{Z}/\ell$ from action on G

$Gr_G \circlearrowleft \mathbb{Z}/\ell$ through loop rotation.

$$\begin{tikzcd}
& \mathcal{H}(G,k) \arrow[dl, "Fr"'] & \\
\mathcal{H}(G,k)^\sigma & \cdots \arrow[r] & \mathcal{H}(H,k)
\end{tikzcd}$$