From geometric realization of affine Hecke algebras to character formulas

Vasya Krylov

G reductive $/F = \mathbb{F}_q((t))$

Local Langlands: $\{\text{sm. irred. rep'n } V \text{ of } G(F)\}$ $(*)$

$I = ev_0^{-1}(B(\mathbb{F}_q))$, ev_0:

$$\begin{tikzcd} G(\mathbb{F}_q[[t]]) \arrow[r, "ev_0"] & G(\mathbb{F}_q) \\ I \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & B(\mathbb{F}_q) \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

<u>Claim</u> $(*)_I \simeq \mathrm{Irrep}\Big(\underbrace{\overset{*}{C_c^\infty}(I\backslash G(F)/I)}_{\mathcal{H}_q(\hat{W}) = \mathcal{H}}\Big)$

$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}(T_w : w \in \hat{W})$

$$\begin{cases} T_{w_1} T_{w_2} = T_{w_1 w_2} & \text{if } l(w_1 w_2) = l(w_1) + l(w_2) \\ (T_s - q)(T_s + 1) = 0, & \forall s \in \hat{S} \end{cases}$$

<u>Important</u>: $\hat{W} \cong W \ltimes Q^\vee$ $\leftarrow$ coroot lattice

$\mathcal{H} \leftarrow \underbrace{\{\theta_\gamma : \gamma \in Q^\vee\}}_{\text{coroot}} \cup \{T_w : w \in W\}$ $\Rightarrow$ irred. rep of $\mathcal{H}$ are fin. dim'l

$\mathbb{C}\hat{W} \supset \underbrace{(\mathbb{C}Q^\vee)^W}_{\text{central}}$

Thm [KL]

$$\mathcal{H} \simeq \left(K^{\check{G}\times\mathbb{C}^\times}\big(\overbrace{\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}}^{St}\big), \star \right)$$

$$\begin{array}{ccc} \tilde{\mathcal{N}} = T^*(G/B) & = & \{(x,\mathfrak{b}) : x \in \operatorname{rad}\mathfrak{b}\} \\ \downarrow & & \downarrow \\ \mathcal{N} & & x \end{array}$$

$$\theta_\gamma \longmapsto \mathcal{O}_{\Delta\tilde{\mathcal{N}}}(\gamma)$$

Thm [KL, Xi] $P_W(z) := \sum_{w \in W} z^{\ell(w)}$ = "$\chi(G/B)$"

Assume $P_W(q) \neq 0$,

$$\operatorname{Irrep}(\mathcal{H}_q) \longleftrightarrow \left\{ (e,s,\rho) : \begin{array}{l} e \in \mathcal{N} \quad s \in \check{G} \text{ semisimple} \\ ses^{-1} = qe \\ \rho \in \operatorname{Irrep}\big(\underbrace{Z_{\check{G}}(e,s)/Z_{\check{G}}(e,s)^\circ}_{=A(e,s)}\big) \\ \uparrow \\ \text{adm.} \end{array} \right\} /\sim$$

$$(e,s,\rho) \longmapsto K(\mathcal{B}_e^s)_\rho \quad : \quad \mathcal{B}_e = \text{Springer fiber}, \quad \mathcal{B}_e^s = s\text{-fixed pt}$$

$K(\mathcal{B}_e^s)_\rho$ ← Standard module

$$K(\mathcal{B}_e^s) = \bigoplus_{\rho \in \operatorname{Irrep}(A(e,s))} \rho \otimes K(\mathcal{B}_e^s)_\rho$$

$$(e,s,\rho) \rightsquigarrow L(e,s,\rho), \qquad K(\mathcal{B}_e^s)_\rho \longrightarrow L(e,s,\rho)$$

$L(e,s,\rho)$ ← unique simple quotient

$\underline{q=1}$. $\hat{W} = W \ltimes Q^\vee$, $\operatorname{Irrep} \hat{W} \overset{\text{Mackey theory}}{\longleftrightarrow} \left\{ (\chi, V) : \begin{array}{l} \chi : Q^\vee \to \mathbb{Z} \\ V : \text{irrep of } W_\chi \end{array} \right\}$

$$\operatorname{Ind}_{W_\chi \ltimes Q^\vee}^{W \ltimes Q^\vee}(V \otimes \mathbb{C}_\chi) \longleftarrow\!\shortmid (\chi, V)$$

$K(\mathcal{B}_e^s)_\rho$

$$\mathcal{B}^s = \coprod_{W/W_L} \mathcal{B}_L \quad , \quad \mathcal{B}_e^s = \coprod_{W/W_L} \mathcal{B}_{L,e}$$

$L = Z(s)^\circ$

$$K(\mathcal{B}_e^s)_\rho \xleftarrow{\sim} \mathrm{Ind}_{\hat W_L}^{\hat W} K(\mathcal{B}_{L,e}^s)_\rho$$

$$\mathcal{B}_{L,e}^s = \mathcal{B}_{L,e}$$

$$\downarrow$$

$$\mathrm{Ind}_{\hat W_L}^{\hat W} H_{top}(\mathcal{B}_{L,e}^s)_\rho \curvearrowleft W_L$$

$\chi \longleftrightarrow s$

$V \longleftrightarrow (e,\rho)$

Ex. $q=1$, $e = e_{reg}$, $\mathcal{B}_e = pt \Rightarrow \hat W \curvearrowright \mathbb{C}$

$q=1$, $e = e_{subreg}$, $\mathbb{O}_{sreg} \subset \mathcal{N} \setminus \mathbb{O}_{reg}$

Thm [Slodowy]

$$\begin{array}{ccc} \tilde S_e & \longrightarrow & S_e \\ \wr\wr & & \wr\wr \\ \widetilde{\mathbb{A}^2/\Gamma} & \longrightarrow & \mathbb{A}^2/\Gamma \end{array}$$

$\tilde{\mathcal{N}} \supset \tilde S_e \leftarrow$ Slodowy variety

$\mathbb{O}_{sreg}$

$\mathcal{N}$

$S_e = (e + \mathfrak{z}_{\mathfrak{g}^\vee}(f)) \cap \mathcal{N}$ Slodowy slice

$\Gamma \subset SL_2(\mathbb{C})$

$\uparrow$

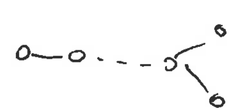
corresp. to $\tilde{\mathfrak{g}}^\vee$

$\uparrow$

unfolding of $\mathfrak{g}^\vee$

$\mathfrak{g}^\vee$ type A

$\Gamma = \mathbb{Z}/n\mathbb{Z}$

$\mathfrak{g}^\vee =$  C , $\tilde{\mathfrak{g}}^\vee =$  D

$A(e) = \mathrm{Aut}_{\mathfrak{g}^\vee}(\tilde{\mathfrak{g}}^\vee)$

$\mathfrak{g}^\vee =$  , $\tilde{\mathfrak{g}}^\vee =$ 

$\mathfrak{g} = \underline{ADE}$.

$\mathcal{B}_e =$ 

$$\begin{tikzcd} K(\mathcal{B}_e) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\parallel"] & \tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \arrow[loop below, "\hat{W} = W(\check{\mathfrak{h}})"] \\ \mathrm{Span}\left([\mathcal{O}_{\mathbb{P}_{\alpha_i}}(-1)],\ [\mathbb{C}_{pt}]\right) & \end{tikzcd}$$

$[\mathcal{O}_{\mathbb{P}_{\alpha_i}}(-1)] \leftrightarrow \alpha_i$, $\quad [\mathbb{C}_{pt}] \leftrightarrow pt$

<u>Question</u>: characters?

$$\begin{tikzcd} \mathcal{H} \ni x \arrow[r, mapsto] \arrow[d] & \mathrm{tr}_V x \in \mathbb{C} \\ \mathcal{H}/[\mathcal{H},\mathcal{H}] \arrow[ur] \arrow[d, phantom, "\parallel"] \arrow[r, dashed] & \mathcal{C}_{\mathcal{H}} \\ HH_0(\mathcal{H}) & \end{tikzcd}$$

1) How to describe $\mathcal{C}_{\mathcal{H}}$?

2) corresp. functional on $\uparrow$

<u>Filtration on $\mathcal{H}$.</u>

$$\mathcal{H} = K^{\check{\mathfrak{h}} \times \mathbb{C}^\times}(St) \qquad \begin{array}{c} St \\ \downarrow \\ \mathcal{N} = \sqcup\, \mathbb{O}_e \end{array}$$

$$\uparrow$$

$$F^{\leq \mathbb{O}} := K^{\check{\mathfrak{h}} \times \mathbb{C}^\times}(St_{\overline{\mathbb{O}}})$$

$\mathcal{H} \ni T_w$

$\Downarrow$

$C_w \leftarrow$ canonical basis

$w \in \hat{W}$, $F^{\leq w}\mathcal{H}$ = minimal "C-based" two-sided ideal in $\mathcal{H}$ which contains C_w.

$w_1 \sim w_2$ if $F^{\leq w_1}\mathcal{H} = F^{\leq w_2}\mathcal{H}$

$\hat{W} = \coprod C \leftarrow$ two-sided cells

$C_{reg} = \{1\}$

$C_{sreg} = \{w \in \hat{W} : w$ has $\underline{\text{unique}}$ minimal decomp$\}$

$\Downarrow$ s_i, $\Downarrow$ $s_i s_j$ ($i - j$), $\not\Downarrow$ $s_i s_j = s_j s_i$ ($i \ast j$)

$\underline{\text{Thm}}$ [Lusztig, Bezrukavnikov] 1) $\exists$ $\{N/\check{G}\} \longleftrightarrow \{$two-sided cells$\}$

2) $F^{\leq \mathbb{O}}\mathcal{H} = F^{\leq C_{\mathbb{O}}}\mathcal{H}$.

$\underline{1)}$ J = "$\lim_{q \to 0} \mathcal{H}_q$" asymptotic Hecke alg | $J = \mathrm{Span}_{\mathbb{C}}(t_x : x \in \hat{W})$

$$C_x C_y = \sum h_{x,y,z} C_z$$

$\wr$ $\quad h_{x,y,z} \in \hat{\mathcal{H}}$

$t_x t_y = \sum \gamma_{x,y,z} t_z$. $\gamma_{x,y,z}$ = highest term in $h_{x,y,z}$

$J = \bigoplus_{c} J_c$

$\mathcal{H}_q \overset{\phi}{\hookrightarrow} J[q^{\pm 1}]$ $\quad \phi^* L = K(B_e^s)_\rho$ $\quad (e,s,\rho) \longmapsto \{c\}$, $e \in \tilde{N}/G$

$J \curvearrowright L$

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2) $C_w \rightsquigarrow \mathcal{L}_w$, $\operatorname{supp} \mathcal{L}_w \subset St_{\overline{\mathbb{O}_w}}$

$C_w = [\mathcal{L}_w]$

Thm $\quad C_{\mathcal{H}} \simeq C_J \simeq \bigoplus_e \boxed{C_{J_e}}$

Thm [Bezrukavnikov - Karpov - Krylov] $\quad K^{Z_e \times \mathbb{C}^\times}(\mathcal{B}_e^{\mathbb{C}^\times} \times \mathcal{B}_e^{\mathbb{C}^\times}) \simeq J_e$

$\mathcal{B}_e^{\mathbb{C}^\times} = \widetilde{S}_e^{\mathbb{C}^\times}$

$$\operatorname{gr} \mathcal{H} = \bigoplus_{e \in \mathcal{N}/\check{G}} K^{\check{G} \times \mathbb{C}^\times}(St_{\mathbb{O}}) \oplus K^{Z_e \times \mathbb{C}^\times}(\mathcal{B}_e \times \mathcal{B}_e)$$

$Z_{\check{G} \times \mathbb{C}^\times}(e)$

1) depends on q

2) non-unital

Corollary. $G = GL_n$. 1) $C_{\mathcal{H}} = \bigoplus_e \mathbb{C}[Z_e /\!/ Z_e] = \bigoplus_e \mathbb{C}[T_e]^{W_e}$

2) $\operatorname{tr}_{K(\mathcal{B}_e^s)}(-) : \quad t_e \longmapsto t_e(s)$.

One example

$$\begin{array}{ccc} St & \supset & St_U \\ \downarrow & & \downarrow \\ \mathcal{N} & \underset{\text{open}}{\supset} & U \\ & & \check{G}\text{-inv.} \end{array}$$

$$K^{\check{G}}(St) \twoheadrightarrow K^{\check{G}}(St_U)$$

$$\begin{array}{c} \psi \\ C_w \longmapsto \begin{cases} \overline{C_w} \\ 0, & \mathbb{O}_w \notin U \end{cases} \end{array}$$

inverse Kazhdan-Lusztig poly (1)

$$T_w = \sum (-1)^{wv} m^w_v C_v$$

$$T_w * 1 = \sum_{\mathbb{O}_v \subset U} (-1)^{wv} m^w_v C_v$$

$$\lambda + \hat{\rho} \in \hat{\Lambda}^+, \quad \mathrm{ch}\, L(v^{-1} \cdot \lambda) = \sum_{w \in \hat{W}} (-1)^{wv} m^w_v \, \mathrm{ch}\, M(w^{-1} \cdot \lambda)$$

$$\mathrm{ch}\left(L(v^{-1} \cdot \lambda)\right) \longleftrightarrow \left[\begin{array}{l} \hat{W} \curvearrowright K^{\check{G}}(St_{U \geq v}) \\ \qquad\qquad \in \\ \quad [L_w] \end{array} \right.$$

Ex 0 $\quad v = \{1\}, \quad U = \mathbb{O}_{reg}, \quad \underset{1}{\overset{}{\mathbb{C}}} \curvearrowleft \hat{W} \longrightarrow$ Weyl-Kac char. formula.

Ex 1 $\quad v \in C_{sreg}, \quad U = \mathbb{O}_{reg} \cup \mathbb{O}_{sreg}$

$$v \in \hat{W}, \quad U = \bigcup_{c \geq C_v} \mathbb{O}_c$$

$$0 \to \underline{K^{Z_e}(B_e \times B_e)} \longrightarrow K^{\check{G}}(St_U) \longrightarrow \mathbb{C} \longrightarrow 0$$

$$K^{Z_e}(B_e)$$

$$\begin{array}{ccc} \tilde{N} & \supset & \tilde{U} \\ \downarrow & & \downarrow \\ N & \supset & U \end{array}$$

$$\alpha_i \updownarrow [\mathcal{O}_{\mathbb{P}_{\alpha_i}}(-1)[1]], \quad \alpha_0 \updownarrow [\mathcal{O}_{B_e}]$$

$$\hat{W} \curvearrowright K^G(\tilde{U}) \ni$$

$$0 \longrightarrow K^{Z_e}(B_e) \longrightarrow K^G(\tilde{U}) \longrightarrow \mathbb{C} \longrightarrow 0$$

$$\begin{array}{ccc} \wr\wr & \wr\wr & \| \\ \mathfrak{h} \oplus \mathbb{C}K & \longrightarrow \hat{\mathfrak{h}} & \longrightarrow \mathbb{C} \end{array} \quad \mathfrak{g} = D/E$$

$$\hat{\mathfrak{h}} \ni \alpha_i$$

$$\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$$

$$\| $$

$$\mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$$\mathrm{tr} \cdot d = d \sum a_i \alpha_i$$

Application

$$\{4D \ \mathcal{N}=2 \text{ SCFT}\} \longrightarrow \{\text{vertex algebra}\}$$

$$\mathcal{T} \qquad\qquad V(\mathcal{T})$$

1) $\mathrm{ch}\, V(\mathcal{T})$ = flavored Schur index of $\mathcal{T}$

2) $\mathrm{ch}\, L$ = flavored Schur index of $\mathcal{T}$ in presence of some surface defect

3) $X_{V(\mathcal{T})}$ = Higgs $(\mathcal{T})$

4) $V(\mathcal{T})$ quasi-lisse   union of finite set of symplectic leaves   implies quasi-modularity

Argyres-Douglas theory

$$L_k(\mathfrak{g}) = L_{\hat{\mathfrak{g}}}(k\Lambda_0), \quad \mathcal{O}(L_k(\mathfrak{g})) \subset \mathcal{O}(\hat{\mathfrak{g}})$$

$$V^k(\mathfrak{g})$$

Thm $k = -b$, b: ...

$b = \max(\Lambda_i(K))$

AC	1
BDG$_2$	2
FE$_6$	3
E$_7$	4
E$_8$	8

D_4, E_6, E_7, E_8

$$b = \frac{h^\vee}{6} + 1$$

$\mathfrak{g} =$ ADE, G_2, B_3,

$\mathrm{Irr}\, \mathcal{O}(L_k) \ni L(\Lambda) \rightsquigarrow w \in C_{\text{subreg}}$

$$(L(\Lambda)) \longleftrightarrow \left\{ w \in C_{\text{subreg}} : \begin{array}{l} w = s_i \cdots \\ \Lambda_i(K) = b \end{array} \right\}$$

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$$\longleftrightarrow \quad X_{L_k(\mathfrak{g})} = D(\mathbb{O}_{subreg})$$

ADE : $\mathbb{O}_{min}$

B_3 : $\mathbb{O}_{short\ root}$

G_2 : $\mathbb{O}_{subreg}$

Ex. $k=-2$, $\mathfrak{g} = D_n, G_2$.

D_n: $$\hat{R}\, ch\, L_{-2}(D_n) = \frac{1}{2} \sum_{u \in W} (-1)^u \sum_{\gamma \in Q^\vee} (\langle \theta - \alpha_2, \gamma \rangle + 1) e^{u t_\gamma (-2\Lambda_0 + \hat{\rho})}$$

G_2 $$\hat{R}\, ch\, L_{-2}(G_2) = -\sum_{u \in W} (-1)^u \sum_{\gamma^\vee = m\alpha^\vee + n\beta^\vee} \left(\left\lfloor \frac{m-1}{3} \right\rfloor + \frac{1}{4}\left(m - 1_{m \equiv n\,(2)} - 1_{n \equiv 0\,(2)}\right)\right) e^{u t_\gamma (-\Lambda + \hat{\rho})}$$