

p-adic integration, buildings and BPS-invariants

Michael Brückening

p-adic integration = integration theory of \mathbb{R} -valued functions (on \mathbb{A}_p) / $\mathbb{F}_q, \mathbb{A}_p$
 motivic integration = $\widehat{k(\text{Var})}[\mathbb{L}^{-1}]$ -valued integration / \mathbb{C}

$$\mathbb{A}_p \longleftrightarrow \mathbb{C}((t)) \quad \mathbb{Z}_p \longleftrightarrow \mathbb{C}[[t]]$$

Why?

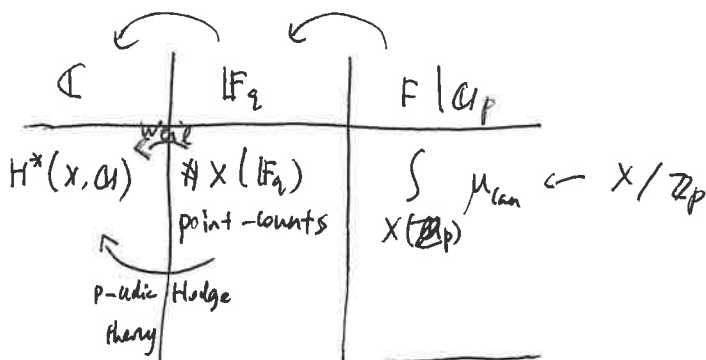
X smooth proj.

$$X = [Y/h],$$

h finite gp

$$\alpha \in Br(X)$$

$$\alpha \in H_{\text{ét}}^2(X, \mathbb{Z})$$



$$H_{\text{ét}}^*(X, \alpha) = \sum_{g \in G/h} \gamma^g(F_q) \quad \int K_{\text{can}} h \quad h: X(\mathbb{A}_p) \rightarrow \mathbb{A}^x$$

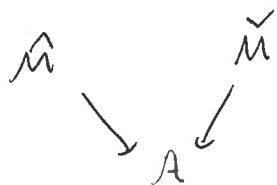
$$\bigoplus_{g \in G/h} H^*(X^g/C(g), \alpha) \quad \text{twist } w(g)$$

$$x \mapsto e^{2\pi i \text{Hasse}(\alpha)_x}$$

Batyrev

$$X, Y \text{ birat'l } CY \Rightarrow h^{\text{ét}}(X) = h^{\text{ét}}(Y)$$

$$\text{Hasse: } Br(F) = \mathbb{A}/\mathbb{Z} \quad \bigwedge \quad \mathbb{Z}^1[\text{tors}]$$



Assumption:
 - Smooth DM stack
 - tameness

$$\text{Thm (MWZ)} \quad \int_{\hat{M}(\mathcal{O}_F)} h_\alpha = \int_{\check{M}(\mathcal{O}_F)} h_{\check{\alpha}}$$

$$\Rightarrow \#_{\text{orb}}^{\check{\alpha}} \hat{M} = \#_{\text{orb}}^{\alpha} \check{M}$$

$$\text{If } \exists \text{ Hitchin-Kostant section} \\ \text{vol}(\hat{M}(\mathcal{O}_F)) = \text{vol}(\check{M}(\mathcal{O}_F))$$

Prop ($G \curvearrowright \mathbb{Z}$)

$$\mathrm{Irr} \hat{M}_G = \bigsqcup_{\substack{L_H \\ \in \mathrm{End}(L_G)}} \hat{M}_H$$

What happens if \mathcal{X} is not a smooth DM stack?

\mathcal{X} smooth alg stack / \mathcal{O}_F
 \downarrow
 X coarse moduli space
 \uparrow
 singular

$$\mathrm{vol}(X(\mathcal{O}_F)) = ?$$

point-count / \mathbb{F}_q
 cohomology / \mathbb{C}

$$[b] \quad \mathcal{X} = \mathcal{M}_{2,d}(\mathbb{C}) = \text{semi-stable Higgs bundles on } \mathbb{C}$$

$$\downarrow$$

$$X = M_{2,d}(\mathbb{C})$$

$$\int_{X/\mathcal{O}_F} h_d \text{ is degree-independent } \begin{cases} n, d \text{ coprime} \\ \text{general} \end{cases}$$

\leadsto BPS cohomology is degree (n, d) independent

Kinjo - Koike proved χ -indep. & describe BPS cohom. using IC.

Setting \mathcal{M} = smooth moduli stack of semistable (same slope) objects
in an abelian cat. \mathcal{O}_F / \mathbb{F}_q

$$(+): \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$$

$$CF(\mathcal{M}) = \left\{ f: \mathcal{M}(\mathbb{F}_q) \rightarrow \bigvee_{n \in \mathbb{Z}} \mathbb{T} \mathbb{C} \right\}$$

$$\psi_n(f)(x) = \begin{cases} \psi_n(f(x)) & , x = nx' \\ 0 & \end{cases}$$

$$\psi_m(cn) = (c_{mn})$$

$$\text{Exp: } CT_0 \rightarrow CT_0$$

$$f \mapsto \exp\left(\sum \frac{\psi_n(f)}{n}\right)$$

$$\text{Log}(1+f) = \sum_{n \geq 1} \frac{\mu(n)}{n} \psi_n(\log(1+f))$$

$$\text{Thm (GWZ)} \quad \exists \text{ specialization map } sp: \mathcal{M}(\mathcal{O}_F) \rightarrow \mathcal{M}(\mathbb{F}_q)^{is}$$

$$\forall E \in \mathcal{M}(\mathbb{F}_q), \quad F(E) = \left(\int_{sp^{-1}(E)(\mathcal{O}_{\overline{F}})} h_\alpha \right), \text{ then}$$

$$F(E) = \text{Log} \left(\frac{\mathbb{L}^{\dim}}{\mu} \right)$$

Compare this w/

$$\text{Exp}_{\text{Syn}} (H^*(B\mathcal{G}_m) \otimes BPS_m) = \bigoplus_{\frac{m}{p} = n} H^{BM}(M(\dots)) \mathbb{L}^{\dots(q, \dots)}$$