Automorphic lifting (II)

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Lecture 1

Thm (Wiles) If $a, b, c \in \mathbb{Z}_{\neq 0}$ and $n \in \mathbb{Z}_{>2}$, then $a^n + b^n \neq c^n$.

ℓ-adic rep'ns / modular forms.

WLOG, $n = \ell$ prime, $\ell > 3$, a, b, c are coprime, b even, $a \equiv -1 \ (4)$

Frey: $E: \quad y^2 = x(x - a^\ell)(x + b^\ell)$

$$E[\ell](\overline{\mathbb{Q}}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^2 \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$$

$\bar{r}_{E,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow GL_2(\mathbb{F}_\ell)$ dual of

- $\det \overline{r_{E,\ell}} = \bar{\varepsilon}_\ell^{-1}$, ε_ℓ = ℓ-adic cyclotomic char.

$$\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathbb{Z}_\ell^\times$$

$\bar{\varepsilon}_\ell = \varepsilon_\ell \bmod \ell$. gives action on ℓ-power roots of 1.

- $\bar{r}_\ell$ ramified only at 2 & ℓ.
- $\bar{r}_\ell|_{G_{\mathbb{Q}_2}} \sim \begin{pmatrix} \chi & * \\ 0 & \varepsilon_\ell^{-1}\chi^{-1} \end{pmatrix}$, χ unram. char.
- $\bar{r}_\ell|_{G_{\mathbb{Q}_\ell}}$ is Fontaine-Laffaille w/ HT no's $\{0, 1\}$.
- $\bar{r}_\ell$ is irred. (Mazur)

$L | \mathbb{Q}_\ell$ finite ext'n, $\quad \mathcal{O} = \mathcal{O}_L, \quad \mathbb{F} = \mathcal{O}_L / \lambda$

Complete local noeth. $\mathcal{O}$-algebras (R, m) : $\mathcal{O} \longrightarrow R$

residue field $\mathbb{F}$ $\quad \lambda \longrightarrow m \qquad \mathcal{O}/\lambda \xrightarrow{\sim} R/m$

R complete in m-adic top.

$\Rightarrow R \simeq \varprojlim R_i$ ← artinian local $\mathcal{O}$-algs

$\varrho : G_{\mathbb{Q}} \longrightarrow GL_2(R)$ cts

non-standard, ϱ is hardly ramified if

- $\det \varrho = \varepsilon_\ell^{-1}$
- ϱ unram. outside 2 & ℓ
- $\varrho|_{G_{\mathbb{Q}_2}} \simeq \begin{pmatrix} \chi & * \\ 0 & \varepsilon_\ell^{-1}\chi^{-1} \end{pmatrix}$. χ unram.
- $\varrho|_{G_{\mathbb{Q}_\ell}}$ is FL w/ HT no's $\{0, 1\}$.

Want to prove : if $\varrho : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_\ell)$ is hardly ramified, then ϱ is reducible.

$$\left.\begin{array}{l} \overline{\varrho_{E,\ell}} \text{ hardly ramified} \\ \text{Mazur: irred.} \end{array}\right] \text{contradiction}$$

Thm A If $\bar{\varrho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_{3^m})$ is hardly ramified, then $\bar{\varrho} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_3^{-1} \end{pmatrix}$

(∴ reducible)

- uses discriminant bounds: Minkowski, Odlyzko, ...

$\overline{\mathbb{Q}}^{\ker \bar{\varrho}}$ small discriminant $\Rightarrow [\overline{\mathbb{Q}}^{\ker \bar{\varrho}} : \mathbb{Q}]$ - small.

Cor. If $L | \mathbb{Q}_3$ finite, and $\varrho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O}_L)$ is hardly ramified, then

$$\varrho^{ss} \simeq 1 \oplus \varepsilon_3^{-1}.$$

Thm B. Suppose $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_\ell)$ is hardly ramified, and irred.. then

$\exists\ M | \mathbb{Q}$ a finite ext'n + for each prime $\lambda^{(\nmid 2)}$ of M, a cts rep'n

$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O}_{M,\lambda})$ s.t. 1) ρ_λ is hardly ramified

compatible system

2) $\forall\ \lambda_1, \lambda_2$ above ℓ_1, ℓ_2, if $p \nmid 2\ell_1\ell_2$,

$$\underset{\underset{M_{\lambda_1}}{\cap}}{\operatorname{tr} \rho_{\lambda_1}(\mathrm{Frob}_p)} = \underset{\underset{M_{\lambda_2}}{\cap}}{\operatorname{tr} \rho_{\lambda_2}(\mathrm{Frob}_p)} \in M$$

3) $\exists\ \lambda_0 | \ell$ w/ $\rho_{\lambda_0} \bmod \lambda_0 = \bar{\rho}$

Thms A, B $\Rightarrow$ FLT

$\overline{\rho_{E,\ell}} \rightsquigarrow \exists\ M,\ \{\rho_\lambda\},\ \overline{\rho_{\lambda_0}} \simeq \bar{\rho}$

$\lambda_1 | 3$, $\rho_{\lambda_1} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O}_{M,\lambda_1})$

$$\rho_{\lambda_1}^{ss} \simeq 1 \oplus \varepsilon_3^{-1}$$

$$\therefore \operatorname{tr} \rho_{\lambda_0}(\mathrm{Frob}_p) = 1 + p, \quad \forall\ p \nmid 6\ell$$

$$\qquad \| $$

$$\operatorname{tr}(1 \oplus \varepsilon_\ell^{-1})(\mathrm{Frob}_p)$$

Chebotarev density $\Rightarrow$

$$\left.\begin{array}{l} \operatorname{tr} \overline{\rho_{\lambda_0}} = \operatorname{tr}(1 \oplus \bar{\varepsilon}_\ell^{-1}) \\ \det \overline{\rho_{\lambda_0}} = \det(1 \oplus \bar{\varepsilon}_\ell^{-1}) \end{array}\right] \Rightarrow \overline{\rho_{\lambda_0}}^{ss} \simeq 1 \oplus \bar{\varepsilon}_\ell^{-1}$$

$\Rightarrow \overline{\rho_{\lambda_0}} = \overline{\rho_{E,\ell}}$ is reducible.

Pf of Thm B. Choose $M|\mathbb{Q}$ an imaginary quadratic field, unramified above 6ℓ.
(ramified somewhere)

$p>3$, p split in M. $p \nmid 6\ell$.

$$x=(x_v), \quad \|x\| = \prod_v |x_v|_v$$

Find: $N|M$ finite, $\theta : \mathbb{A}_M^\times / M_\infty^\times \longrightarrow N^\times$ s.t.

1) $\theta|_{M^\times} = \mathrm{Id}$

2) $\theta|_{\mathbb{A}_\mathbb{Q}^\times} : x \longmapsto \|x\|^{-1} x_\infty \delta_{M|\mathbb{Q}}(x)$ ⊛

$$\begin{tikzcd}
\mathrm{Gal}(M|\mathbb{Q}) \arrow[d, phantom, "\wr"] & \mathbb{A}_\mathbb{Q}^\times / (N_{M/\mathbb{Q}} \mathbb{A}_M^\times)\mathbb{Q}^\times \arrow[l, "\sim"'] \arrow[dl, "\delta_{M|\mathbb{Q}}"] \\
\{\pm 1\} &
\end{tikzcd}$$

3) θ unramified at all v except $v|p$, or v ramified in M.

4) $\theta|_{\mathcal{O}_{M_v}^\times}$ has order $p-1$, $\forall\, v|p$. (**)

Choose ℓ', $\ell' \neq \ell$, ℓ' splits in N, $\underset{(***)}{\ell' \nmid (p-1)}$, $\bar{\rho}, \theta$ unram. @ ℓ'.

$\lambda'|\ell'$ prime of N, $\mathcal{O}_{N,\lambda'} \simeq \mathbb{Z}_{\ell'}$.

automorphic by CFT

$$\begin{tikzcd}
\theta_{\lambda'} : \mathbb{A}_M^\times / M^\times M_\infty^\times \arrow[r] \arrow[d, phantom, "\simeq"] & \mathcal{O}_{N,\lambda'}^\times \simeq \mathbb{Z}_{\ell'}^\times \\
\mathrm{Gal}(M^{ab}|M) & x \longmapsto \theta(x)\, x_{\lambda'|M}^{-1} \\
G_M \arrow[u] &
\end{tikzcd}$$

$\boxed{\text{automorphic}}$

$\uparrow$

$\mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} : G_\mathbb{Q} \longrightarrow GL_2(\mathbb{Z}_{\ell'})$, $\det \mathrm{Ind}_{G_M}^{G_\mathbb{Q}} \theta_{\lambda'} = \varepsilon_{\ell'}^{-1}$ by ⊛

$\overline{\theta_{\lambda'}} = \theta_{\lambda'} \bmod \lambda'$. $\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta_{\lambda'}}$ is irred. $\Leftarrow$ $(**)$, $(***)$

Moret-Bailly. $\Rightarrow \exists\, F|\mathbb{Q}$ totally real

$\ell, \ell', 2$ unram. in F

$[F:\mathbb{Q}]$ even

$F|\mathbb{Q}$ linearly disjoint from

$\overline{\mathbb{Q}}^{\ker \bar{r} \,\cap\, \ker \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta_{\lambda'}}}(\zeta_{\ell\ell'})$

X $(X(\overline{\mathbb{Q}_v}))$

$P_v \in X(\mathbb{Q}_v)$, $v \in S$, $\# S < \infty$

$\exists ?\ P \in X(\mathbb{Q})$ s.t. P close to P_v for $v \in S$?

$(X(\overline{\mathbb{Q}}))$

$\exists\, E|F$ w/ elliptic curve

- $E[\ell]^\vee \simeq \bar{r}|_{G_F}$.
- $E[\ell']^\vee \simeq \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta_{\lambda'}}$

$\Longrightarrow$ E has semistable reduction everywhere

- E has good reduction at ℓ and ℓ'.

and if E has bad reduction at v, then $\bar{r}|_{G_{F_v}}$, $\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta_{\lambda'}}\big|_{G_{F_v}}$ are trivial.

$\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \theta_{\lambda'}\big|_{G_F}$ is automorphic$/F$ $\Rightarrow$ $\mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}} \overline{\theta_{\lambda'}}\big|_{G_F} \simeq \bar{r}_{E,\ell'}$ automorphic$/F$

$\underset{\text{ALT}}{\Rightarrow}$ $r_{E,\ell'}$ auto.$/F$

r is part of compatible system.

$\Uparrow \leftarrow$ (use base change) $\Rightarrow$ $r_{E,\ell}$ auto. $/F$

$r|_{G_F}$ is part of a compatible sys.

$\Uparrow$ $\Rightarrow$ $\overline{r_{E,\ell}}$ auto $/F$ $\overset{\text{ALT}}{\Longrightarrow}$ $R^{\text{univ}}_{\bar{r}}|_{G_F}$ finite $/\mathcal{O}$.

$r|_{G_F}$ auto.$/F$ $\overset{\text{ALT}}{\Leftarrow}$ $\overline{r_{E,\ell}}$ is $\bar{r}$

$\Rightarrow$ $R^{\text{univ}}_{\bar{r}}$ finite $/\mathcal{O}$ $\Rightarrow$ $\bar{r}$ has handly ramified ℓ-adic lift.

$r: G_{\mathbb{Q}} \to GL_2(\mathcal{O}')$

$$\rho^{univ} : G_{\mathbb{Q}} \longrightarrow GL_2(R^{univ}_{\bar{\rho}})$$

↑ classifies all handly ramified lifts

Galois cohomology ⇒

$$\mathcal{O}' \overset{?}{\dashleftarrow} R^{univ}_{\bar{\rho}} \simeq \frac{\mathcal{O}[[T_1, \dots, T_s]]}{(f_1, \dots, f_s)}$$

⇒ Krull dim $R^{univ}_{\bar{\rho}} \geq 1$.

$$\left(R^{univ}_{\bar{\rho}} \overset{??}{\simeq} \mathbb{F}[[T]] \right)^{\sim}$$

automorphic?

Statement of ALT.

Lecture 2. automorphy lifting thm: $\bar{\rho}$ mod l Galois rep'n, auto

↑ concrete def'n

some Galois def. ring is finite / $\mathbb{Z}_l$ / ⇓ a suitable l-adic lift is auto

$F|\mathbb{Q}$ totally real field, $[F:\mathbb{Q}]$ even. l prime, $l > 3$, unramified in F.

$D|F$ quaternion alg,

$$D \otimes_F F_v = \begin{cases} \mathbb{H}, & v | \infty \\ M_{2\times 2}(F_v), & v \nmid \infty \end{cases}$$

$$D \otimes_F \mathbb{A}_F^{\infty} \simeq M_{2\times 2}(\mathbb{A}_F^{\infty})$$

$D^{\times} \subset (D \otimes \mathbb{A}_F^{\infty})^{\times}$ discrete, cocompact.

$L|\mathbb{Q}_l$ finite

$\mathcal{O} = \mathcal{O}_L$, $\mathcal{O}/\lambda = \mathbb{F}$

S a finite set of finite places of F, $S \not\ni$ primes above ℓ.

$v \in S$, $\quad \underset{\substack{\uparrow \\ \text{residue field}}}{k(v)^\times} \supset \Delta_v$ subgroup, $\quad \underset{\text{homomorphism}}{\chi_v : \Delta_v \to \mathcal{O}^\times}$ $\qquad \Delta = \prod_{v \in S} \Delta_v$

$\qquad \chi = \prod_{v \in S} \chi_v$

$$U_\Delta(S) = \prod_{v \notin S} GL_2(\mathcal{O}_{F,v}) \times \prod_{v \in S} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,v}) : \begin{array}{l} v(c) > 0 \\ a/d \bmod v \in \Delta_v \end{array} \right\}$$

$$\begin{tikzcd} U_\Delta(S) \arrow[d, "\chi"] & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[d, mapsto] \\ \mathcal{O}^\times & \chi_v(a/d) \end{tikzcd}$$

(e.g. $\Gamma_1(v), \Gamma_0(v) \ldots$) $\qquad \subset GL_2(\mathbb{A}_F^\infty)$

open compact subgp

A : $\mathcal{O}$-alg.

$$S(U_\Delta(S), A)_\chi$$

$$= \{ \varphi : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / (\mathbb{A}_F^\infty)^\times \to A \; : \; \varphi(gu) = \chi(u)\varphi(g), \; \forall u \in U_\Delta(S) \}$$

$$\begin{tikzcd} \varphi \arrow[d, mapsto] \\ (\varphi(g))_{g \in D^\times \backslash (D \otimes \mathbb{A}_F^\infty)^\times / (\mathbb{A}_F^\infty)^\times U_\Delta(S)} \end{tikzcd}$$

$$= \bigoplus A(\chi)^{((\mathbb{A}_F^\infty)^\times U_\Delta(S) \cap g^{-1} D^\times g)/F^\times}$$

finite group

finite set.

ℓ nr in F, $\ell > 3$ $\Rightarrow$ order prime to ℓ

$\therefore$ finite free $/A$.

$\mathbb{T}(U_\Delta(S), A)_\chi$ = A-subalg. of $\mathrm{End}_A(S(U_\Delta(S), A)_\chi)$ generated by T_v for $v \notin S$, $v \nmid \ell$

$$(T_v \varphi)(h) = \int_{\{g \in M_{2\times 2}(\mathcal{O}_{F,v}) : v(\det g) = 1\}} \varphi(hg) \, dg \leftarrow \text{Haar measure } dg(GL_2(\mathcal{O}_{F,v})) = 1$$

$$= \sum_{\alpha \in \mathcal{O}_{F,v}/v} \varphi\left(h\begin{pmatrix} \pi_v & \alpha \\ 0 & 1 \end{pmatrix}\right) + \varphi\left(h\begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix}\right)$$

$\mathbb{T}(U_\Delta(S), A)_x$ – commutative, contained in a finite free A-module

If $A \to B$, then $\mathbb{T}(U_\Delta(S), A)_x \otimes_A B \twoheadrightarrow \mathbb{T}(U_\Delta(S), B)_x$ has nilpotent kernel.

isom. if B is ℓ-torsion free.

$$\mathbb{T}(U_\Delta(S), \mathcal{O})_x = \bigoplus_{\substack{\mathfrak{m} \text{ max'l} \\ \text{ideal}}} \mathbb{T}(U_\Delta(S), \mathcal{O})_{x,\mathfrak{m}}$$

If $\mathfrak{m}$ is a max'l ideal of $\mathbb{T}(U_\Delta(S), \mathcal{O})_x$, then $\exists!$

$\bar{r}_{\mathfrak{m}} : G_F \longrightarrow GL_2(\overline{k(\mathfrak{m})})$ residue field for $\mathfrak{m}$

cts, semisimple rep'n, s.t.

1) $\det \bar{r}_{\mathfrak{m}} = \bar{\varepsilon}_\ell^{-1}$

2) if $v \notin S$, $v \nmid \ell$, then $\bar{r}_{\mathfrak{m}}$ is unramified @ v

and $\operatorname{tr} \bar{r}_{\mathfrak{m}}(\mathrm{Frob}_v) = T_v$

3) if $v \mid \ell$, then $\bar{r}_{\mathfrak{m}}|_{G_{F_v}}$ is FL w/ HT nos $\{0, 1\}$

4) If $v \in S$, $\exists\ \psi_{v,1}, \psi_{v,2} : G_{F_v}^{ab} \longrightarrow \overline{k(\mathfrak{m})}^\times$ tamely ramified

w/ $\bar{r}_{\mathfrak{m}}|_{G_{F_v}}^{ss} = \psi_{v,1} \oplus \psi_{v,2}$

$$\begin{tikzcd}
I_{F_v^{ab}/F_v} \arrow[d, "\psi_{v,1}"] & \mathcal{O}_{F,v}^\times \arrow[l, "\mathrm{Art}", "\sim"'] \arrow[d] \\
 & k(v)^\times \\
\overline{k(\mathfrak{m})}^\times & \Delta_v \arrow[l, "\chi_v"'] \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

__Def.__ If $\overline{r}_m$ is irreducible, we call m __non-Eisenstein__.

We now assume m always __non-Eisenstein__.

$\uparrow$

In this case, $\exists\ r_m : G_F \longrightarrow GL_2\left(\mathbb{T}\left(U_\Delta(S), \mathcal{O}\right)_{\chi, m}\right)$ s.t.

1) $\det r_m = \varepsilon_\ell^{-1}$

2) $v \notin S, v \nmid \ell \Rightarrow$ r_m is unram. @ v and $\operatorname{tr} r_m(\mathrm{Frob}_v) = T_v$

3) if $v \mid \ell$, then $r_m|_{G_{F_v}}$ is FL w/ HT no's $\{0, 1\}$.

4) if $v \in S$, then r_m is tamely ramified @ v.

__ad hoc def'n of automorphic__.

if $L' \mid L$ finite ext'n, $\quad$ / or $\mathbb{F}' \mid \mathbb{F}$ finite

$v \nmid \ell$ and r ram. @ v, $r|_{G_{F_v}}^{ss} \simeq \psi_1 \oplus \psi_2$

ψ_i tamely ram.

$r : G_F \to GL_2(L')$ cts s.s. rep'n.

tamely ramified (away from ℓ)

$\overline{r} : G_F \to GL_2(\mathbb{F}')$

(of wt 0, level prime to ℓ) for some S, Δ, χ

We call r (resp. $\overline{r}$) automorphic if $\exists\ \theta : \mathbb{T}\left(U_\Delta(S), \mathcal{O}\right)_\chi \longrightarrow L'$ or $\mathbb{F}'$

s.t. $\theta(T_v) = \operatorname{tr} r(\mathrm{Frob}_v) \sim \operatorname{tr} \overline{r}(\mathrm{Frob}_v)$, $\forall\ v \notin S$, $v \nmid \ell$.

$+\ r / \overline{r}$ unram. outside $S \cup \{v \mid \ell\}$.

If $\overline{r}$ irred. $\Rightarrow$ $\overline{r} = \theta \circ (\overline{r}_m)$ for some m.

r $\qquad r \qquad r_m$

__Specialize__. $S = Q \amalg R$ $\quad$ $v \in R$. $\quad \# k(v) \equiv 1\ (\ell)$

$\overline{r}_m|_{G_{F_v}}$ is trivial

$\Delta_v = k(v)^\times$, χ_v has ℓ power order

$v \in Q$: $\quad \# k(v) \equiv 1 \ (\ell)$

Δ_v max'l subgp of $k(v)^\times$ of order prime to ℓ

$H_v = k(v)^\times / \Delta_v$ $\quad \ell$- power order

$\chi_v = 1$

$\bar{r}_m$ unram. @ v, $\quad \bar{r}_m(\mathrm{Frob}_v)$ has distinct eigvals α_v, β_v.

$v \in Q$ $\quad$, $\sigma \in G_{F_v} \longmapsto \mathrm{Frob}_v \in G_{k(v)}$

$$\mathrm{char}_{r_m(\sigma)}(X) = X^2 - \mathrm{tr}\, r_m(\sigma) X + \varepsilon_\ell(\sigma)^{-1} \equiv (X-\alpha_v)(X-\beta_v) \quad (m)$$

$$\overset{\text{Hensel}}{\underline{=}} (X - A_{v,\sigma})(X - B_{v,\sigma}) \qquad \begin{matrix} A_{v,\sigma} \equiv \alpha_v \\ B_{v,\sigma} \equiv \beta_v \end{matrix} \ (\text{mod } m)$$

$\Rightarrow \exists\, \chi_{\alpha_v}, \chi_{\beta_v} : G_{F_v} \longrightarrow \mathbb{T}(U_\Delta(S), \mathcal{O})_{\chi,m}^\times$ s.t.

$\forall \sigma \in G_{F_v}$, $\quad \mathrm{char}_{r_m(\sigma)}(X) = (X - \chi_{\alpha_v}(\sigma))(X - \chi_{\beta_v}(\sigma))$

if $\sigma \mapsto \mathrm{Frob}_v \Rightarrow \chi_{\alpha_v}(\sigma) \bmod m = \alpha_v$

$\chi_{\beta_v}(\sigma) \bmod m = \beta_v$

$S(U_\Delta(S), \mathcal{O})_{\chi,m}$

$\uparrow$

U_a satisfies

$(X - \chi_{\alpha_v}(\mathrm{Art}_{F_v} a))(X - \chi_{\beta_v}(\mathrm{Art}_{F_v} a))$

$v \in Q$

$a \in \mathcal{O}_{F,v} \setminus \{0\}$

$$(U_a \varphi)(h) = \int_{U_\Delta(S)_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} U_\Delta(S)_v} \varphi(hg)\, dg \qquad dg\,(U_\Delta(S)_v) = 1$$

U_a commutes w/ each other

$U_{ab} = U_a U_b$. commute w/ T_v

fix σ_0 above Frob_v

$\cap$

$// \; G_{F_v}^{ab}$

$\mathrm{Art}\, a_0$, $a_0 \in \mathcal{O}_{F,v}$, $v(a_0) = 1$ $\quad e_{\alpha v} = \dfrac{U_{a_0} - B_{v,\sigma_0}}{A_{v,\sigma_0} - B_{v,\sigma_0}}$, $e_{\beta v} = \dfrac{U_{a_0} - A_{v,\sigma_0}}{B_{v,\sigma_0} - A_{v,\sigma_0}}$

$e_{\alpha v}^2 = e_{\alpha v}$ $\quad e_{\alpha v} e_{\beta v} = 0$

$e_{\beta v}^2 = e_{\beta v}$ $\quad e_{\alpha v} + e_{\beta v} = 1$

$$S(\chi, Q, \mathcal{O}) := \prod_{v \in Q} e_{\alpha v} S(U_\Delta(S), \mathcal{O})_{\chi, m}$$

$$\cup$$

$$\mathbb{T}(\chi, Q) := \mathbb{T}(U_\Delta(S), \mathcal{O})_{\chi, m} \hookrightarrow \mathrm{End}_{\mathcal{O}}(\quad)$$

$\rho^{mod}_{\chi, Q} : G_F \longrightarrow GL_2(\mathbb{T}(\chi, Q))$, $v \in Q$, $\rho^{mod}_{\chi, Q}\big|_{G_{F_v}} \simeq \chi_{\alpha v} \oplus \chi_{\beta v}$

U_a acts on $S(\chi, Q, \mathcal{O})$

$a \in \mathcal{O}_{F_v} \setminus \{0\}$, $v | a$. $\chi_{\alpha v}(\mathrm{Art}\, a) \in \mathbb{T}(\chi, Q)$

$H_v = k(v)^\times / \Delta_v$. $\quad v \in Q$

↑ cyclic order the largest power of ℓ dividing # $k(v)^\times$

$$H_Q = \prod_{v \in Q} H_v \longrightarrow \mathbb{T}(\chi, Q)$$

$a \in H_v \longmapsto \chi_{\alpha v}(\mathrm{Art}\, a)$ = action of U_a on $S(\chi, Q, \mathcal{O})$!

$=$ action of $U_{\Delta_0}(S) / U_\Delta(S) \simeq \Delta_0/\Delta \simeq H_Q$

$\Delta_{0,v} = k(v)^\times, \forall v$

$\mathcal{O}[H_Q] \longrightarrow \mathbb{T}(\chi, Q)$

$\triangledown$
$\mathfrak{a}_Q \leftarrow$ augmentation ideal $= \langle h-1 : h \in H_Q \rangle$

Lemma. 1) $S(\chi, Q, \mathcal{O})$ is a free $\mathcal{O}[H_Q]$-module

2) $$\begin{tikzcd} S(\chi, Q, \mathcal{O})_{H_Q} \arrow[r, "tr", "\sim"'] \arrow[d, equal] & S(\chi, \phi, \mathcal{O}) \\ S(\chi, Q, \mathcal{O}) / \mathfrak{a}_Q S(\chi, Q, \mathcal{O}) & \end{tikzcd}$$

Lecture 3.

$F|\mathbb{Q}$, $R =$ finite set of primes of F away from ℓ.

$v \in R$, $\chi_v : k(v)^\times \to \mathcal{O}^\times$
ℓ-power order

$q_v = \# k(v) \equiv 1 \pmod{\ell}$

$L | \mathbb{Q}_\ell$ finite
$\mathcal{O} = \mathcal{O}_L$
$\mathcal{O}/\lambda = \mathbb{F}$.

Space of modular forms

Hecke alg

m non-Eisenstein max'l ideal $\quad \bar{r}_m |_{G_{F_v}} = 1$

Q finite sets of primes, $Q \cap (R \cup \{v | \ell\}) = \phi$

$v \in Q$, $\quad q_v \equiv 1 \pmod{\ell}$

$v \in Q$, H_v max'l ℓ power order quot. of $k(v)^\times$.

$H_Q = \prod_{v \in Q} H_v$

$\mathbb{T}(Q, \chi) \longleftarrow \mathcal{O}[H_Q] \triangleright \mathfrak{a}_Q$ aug. ideal

$\bar{r}_m(\mathrm{Frob}\, v)$ has distinct eigen vals $\alpha_v \neq \beta_v \in \mathbb{F}$.

$\circlearrowleft$
$S(Q, \chi)$ = modular forms localized at m_Q.
level Q
$\circlearrowright$
H_Q

$\boxed{r^{mod}_{Q,\chi} : G_F \to GL_2(\mathbb{T}(Q,\chi))}$

Some conditions on the U_a operators, $a \in \mathcal{O}_{F_v} - \{0\}$, $v \in Q$

two characterizations:
- Hecke operators
- action of I_{F_v}, $v \in Q$

$S(Q,\chi)$ finite free $/\ \mathcal{O}[H_Q]$

$S(Q,\chi)/\mathfrak{a}_Q S(Q,\chi) \xrightarrow{\sim} S(\phi,\chi)$.

$R^{univ}_{Q,\chi}$ = universal deformation ring for lifts r of $\bar{r}_m$ such that

- $\det r = \varepsilon_\ell^{-1}$
- r unramified away from $R \cup Q \cup \{v|\ell\}$
- $v|\ell$, then $r|_{G_{F_v}}$ is FL w/ HT no's $\{0,1\}$
- $v \in R$ and $\sigma \in I_{F_v}$, then $r(\sigma)$ has char. poly.

$$(X - \chi_v(\sigma))(X - \chi_v(\sigma)^{-1}).$$

$$I_{F_v} \twoheadrightarrow I_{F_v^{ab}/F_v} \cong \mathcal{O}_{F_v}^\times \twoheadrightarrow k(v)^\times \xrightarrow{\chi_v} \mathcal{O}^\times$$

$r^{univ}_{Q,\chi}$;

$\mathrm{tr}\, r^{univ}_{Q,\chi}(\mathrm{Frob}_v) \to T_v$

$R^{univ}_{Q,\chi} \twoheadrightarrow \mathbb{T}(Q,\chi)$

$r^{univ}_{Q,\chi} \rightsquigarrow r^{mod}_{Q,\chi}$

$v \in Q \Rightarrow r^{univ}_{Q,\chi}|_{G_{F_v}} \sim \chi_{\alpha_v} \oplus \chi_{\beta_v}$

$$\chi_{\alpha_v/\beta_v} : G_{F_v} \to (R^{univ}_{Q,\chi})^\times$$

mod $\mathfrak{m}$ unramified, $\chi_{\alpha_v/\beta_v}(\mathrm{Frob}_v) = \alpha_v/\beta_v$

$\chi_{\alpha_v}|_{I_{F_v}} : H_v \to (R^{univ}_{Q,\chi})^\times$

$R^{univ}_{\chi,Q}/\mathfrak{a}_Q \xrightarrow{\sim} R^{univ}_{\chi,\phi}$

Thm. $R^{univ}_{\phi,\chi} \longrightarrow \mathbb{T}(\phi,\chi)$ has nilpotent kernel

(only need case $\chi = 1$)

$$\begin{tikzcd} & R^{univ}_{\phi,\chi} \arrow[r] \arrow[d] & \mathbb{T}(\phi,\chi) \arrow[dl, dashed] \\ \Rightarrow & \mathcal{O}' & \end{tikzcd}$$

$\Big|$ $R^{univ}_{\phi,\chi}$ finite $/\mathcal{O}$.

$R^{univ}_{\phi,\chi} \leftarrow$ local def. ring at $v \in R$

$\exists$ local lifting ring $R_v^{\square} \leftarrow$ parametrizes lifts w/ no equivalence.

<u>framed deformations</u>

$(r, \{\alpha_v\}_{v\in R})$ $\quad r: G_F \longrightarrow GL_2(T)$

satisfies all condition listed before

$/\sim$

$\alpha_v \in \ker\left(GL_2(T) \longrightarrow GL_2(\mathbb{F})\right)$

$(r, \{\alpha_v\}_{v\in R}) \sim (\beta r \beta^{-1}, \{\beta\alpha_v\})$, $\quad \forall \beta \in \ker\left(GL_2(T) \longrightarrow GL_2(\mathbb{F})\right)$.

representable $/ R^{\square}_{Q,\chi}$

$(\alpha_v^{univ})^{-1} r^{univ} (\alpha_v^{univ}) \Big|_{G_{F_v}}$ well defined $v \in R$

$\therefore R_v^{\square} \longrightarrow R^{\square}_{Q,\chi} \rightsquigarrow R^{loc}_{Q,\chi} := \left(\hat{\bigotimes}_{v\in R} R_v^{\square}\right) \longrightarrow R^{\square}_{Q,\chi}$

Fix $r^{univ}_{Q,\chi} : G_F \longrightarrow GL_2(R^{univ}_{Q,\chi})$ in its equiv. class.

non-canonical

$R^{\square}_{Q,\chi} \overset{\downarrow}{\cong} \underset{\substack{\uparrow \\ \text{power series in } 4|R|-1 \text{ vars.}}}{R^{univ}_{Q,\chi}[[A_{ijv} : v \in R,\ i,j = 1,2]]/(A_{v_0 11})}$, $\quad \left(r^{univ}_{Q,\chi}, \left\{\begin{pmatrix} 1 + A_{11v} & A_{12v} \\ A_{21v} & 1 + A_{22v} \end{pmatrix}\right\}_{v\in R}\right)$.

$$R^{univ}_{Q,\chi} \longrightarrow \mathbb{T}(Q,\chi) \curvearrowright S(Q,\chi)$$

$$\Lambda_Q = \mathcal{O}[H_Q][[A_{ij,v}]] / (A_{11 v_0})$$

$$R^{\square}_{Q,\chi} = R^{univ}_{Q,\chi} \hat{\otimes}_{\mathcal{O}[H_Q]} \Lambda_Q$$

$$S^{\square}(Q,\chi) = S(Q,\chi) \otimes_{\mathcal{O}[H_Q]} \Lambda_Q$$

$$\mathfrak{a}_Q \triangleleft \Lambda_Q$$
$$\rotatebox{90}{=}$$
$$\langle h-1,\ A_{ijv} : h \in H_Q \rangle$$

$$\begin{tikzcd}
 & R^{loc}_\chi[[x_1,\dots,x_{d_Q}]] \arrow[dr] & & \Lambda_Q \arrow[dl] & \\
\forall Q \text{ as above,} & & R^{\square}_{Q,\chi} \arrow[dl] \arrow[r, phantom, "\curvearrowright"] & S^{\square}(Q,\chi) \arrow[dl] & \text{finite free } /\Lambda_Q \arrow[l] \\
\text{mod } \mathfrak{a}_Q, & R^{univ}_{\phi,\chi} \arrow[r, phantom, "\curvearrowright"] & S(\phi,\chi) & &
\end{tikzcd}$$

Q. What is the smallest value of d_Q we can take?

Ans: $d_Q = \dim H^1_{L_Q^\perp}(G_F, ad^0 \bar{r}(1)) + |R| + |Q| - 1$

$$d_Q = \dim \left(m^{\square}_{Q,\chi} \Big/ \left((m^{\square}_{Q,\chi})^2 + \underset{\uparrow \text{ max'l ideal in } R^{loc}_\chi}{m^{loc}_\chi} \right) \right)^{\vee} \leftarrow Hom(-,\mathbb{F})$$

$$\updownarrow$$

$$\begin{tikzcd}
R^{\square}_{\chi,Q} \arrow[r] & \mathbb{F}[\varepsilon]/(\varepsilon^2) \\
R^{loc}_\chi \arrow[r] \arrow[u] & \mathbb{F} \arrow[u]
\end{tikzcd}
\quad \longleftrightarrow$$

liftings $(1+\varepsilon\phi)\bar{r}_m$ of $\bar{r}_m$ / $\mathbb{F}[\varepsilon]$

plus elts $(1+\varepsilon\alpha_v)$, $\alpha_v \in M_{2\times 2}(\mathbb{F})$

s.t. $\phi \in Z^1(G_F, ad\bar{r}) \leftarrow$ unram. away from $Q \cup R \cup \{v|\ell\}$

$tr\phi = 0$

$v|\ell$, $[\phi|_{G_{F_v}}] \in H^1_f(G_{F_v}, ad^0\bar{r})$

$$v \in R, \quad (1-\alpha_v \varepsilon)(1+\varepsilon\phi|_{G_{F_v}})\,\bar{\rho}_m|_{G_{F_v}}(1+\alpha_v\varepsilon) = \bar{\rho}_m|_{G_{F_v}}$$

$$\Updownarrow$$

$$\phi|_{G_{F_v}} = \partial\alpha_v$$

$L_v = H^1_f \qquad v | \ell \qquad\qquad L_v^{\perp} = H^1_f$

$L_v = (0), \qquad v \in R \qquad\qquad L_v^{\perp} = H^1$

$L_v = H^1. \qquad v \in \mathcal{Q} \qquad \mathcal{L}_{\mathcal{Q}} = \{L_v\} \qquad L_v^{\perp} = (0)$

$L_v = H^1_f = H^1(G_{k(v)}, ad^0\bar{\rho}) \qquad\qquad L_v^{\perp} = H^1_f$

$$H^1_{\mathcal{L}_{\mathcal{Q}}}(G_F, ad^0\bar{\rho}) = \{[\phi] \in H^1(G_F, ad^0\bar{\rho}) : res_v[\phi] \in L_v, \forall v\}$$

$Z^1_{\mathcal{L}_{\mathcal{Q}}}$ preimage in Z^1 of $H^1_{\mathcal{L}_{\mathcal{Q}}}$

$\longleftrightarrow \phi \in Z^1_{\mathcal{L}_{\mathcal{Q}}}, \quad \alpha_v \in ad\bar{\rho}, \quad v \in R$

$\phi|_{G_{F_v}} = \partial\alpha_v \qquad /\sim \qquad \beta \in ad\bar{\rho},$

$(\phi, \{\alpha_v\}) \sim (\phi + \partial\beta, \{\alpha_v + \beta\})$

$$d_{\mathcal{Q}} = \dim H^1_{\mathcal{L}_{\mathcal{Q}}}(G_F, ad^0\bar{\rho}) + 3 - \dim H^0(G_F, ad^0\bar{\rho})$$

$$+ \underbrace{\sum_{v \in R} H^0(G_{F_v}, ad\bar{\rho}) - 4}_{H^2_{\mathcal{L}} + H^0_{\mathcal{L}} - H^3_{\mathcal{L}}}$$

Tate duality

$$\dim H^1_{\mathcal{L}_{\mathcal{Q}}^{\perp}}(G_F, (ad^0\bar{\rho})(1)) - \dim H^0(G_F, (ad^0\bar{\rho})(1))$$

$$+ \sum_{v|\infty}(3 - \dim H^0(G_{F_v}, (ad^0\bar{\rho}))) - \sum_{v|\ell} 3[F_v : \mathbb{Q}_\ell]$$

$$+ \sum_{v|\ell}(\dim L_v - \dim H^0(G_{F_v}, ad^0\bar{\rho})) + \sum_{v \in R}(\dim L_v - \dim H^0(G_{F_v}, ad^0\bar{\rho}))$$

$$+ \sum_{v \in \mathcal{Q}}(\dim L_v - \dim H^0(G_{F_v}, ad^0\bar{\rho})) \qquad + \sum_{v \in R} H^0(G_{F_v}, ad\bar{\rho}) - 1$$

$$d_{Q} = \dim H^1_{\mathcal{L}_Q^\perp}(G_{F}, (ad^\circ \bar r)(1)) + |R| - 1 + |Q|.$$

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathcal{L}_Q^\perp} \arrow[r] & H^1_{\mathcal{L}_\phi^\perp} \arrow[r] & \bigoplus_{v\in Q} H^1(G_{k(v)}, (ad^\circ \bar r)(1)) \arrow[d, phantom, "\wr" description] \\
 & & & (ad^\circ \bar r)(1) / (\mathrm{Frob}_v - 1)(ad^\circ \bar r)(1)
\end{tikzcd}$$

$r = \dim H^1_{\mathcal{L}_\phi^\perp}$

Prop $\forall N \in \mathbb{Z}_{>0}$, $\exists$ a finite set Q_N of places of F s.t.

1) $Q_N \cap (R \cup \{v \mid \ell\}) = \phi$

2) $|Q_N| = r$

3) $v \in Q_N$, then $q_v \equiv 1 \ (\ell^N)$

4) $v \in Q_N \Rightarrow \bar r_m(\mathrm{Frob}_v)$ has eig. vals $\alpha_v \neq \beta_v$.

5) $H^1_{\mathcal{L}_{Q_N}^\perp}(G_F, ad^\circ \bar r) = (0)$.

$(\therefore d_{Q_N} = |R| - 1 + r)$

STP. $\forall\, 0 \neq \phi \in H^1_{\mathcal{L}_\phi^\perp}$, $\exists\, v \notin R \cup \{v \mid \ell\}$

s.t. - v splits completely in $F(\zeta_{\ell^N})$

- $\bar r_m(\mathrm{Frob}_v)$ does not have ℓ-power order

- $\phi(\mathrm{Frob}_v) \notin (\mathrm{Frob}_v - 1)(ad^\circ \bar r)(1)$

Lecture 4

$r := \dim H^1_{\mathcal{L}^\perp_\emptyset}(G_F, (\mathrm{ad}^0 \bar{r})(1))$ $\qquad R$

Prop. $\forall N \in \mathbb{Z}_{>0}$, $\exists$ $Q_N \subset$ primes of F s.t.

1) $Q_N \cap (R \cup \{v | \ell\}) = \emptyset$

2) $|Q_N| = r$

3) $v \in Q_N$, $\overset{|k_v|}{\overset{\shortparallel}{q_v}} \equiv 1 \pmod{\ell^N}$

4) $v \in Q_N$, then $\bar{r}(\mathrm{Frob}_v)$ has distinct eigenvalues.

5) $H^1_{\mathcal{L}^\perp_{Q_N}}(G_F, (\mathrm{ad}^0 \bar{r})(1)) = (0)$.

STP. $0 \neq \phi \in H^1_{\mathcal{L}^\perp_\emptyset}(G_F, (\mathrm{ad}^0 \bar{r})(1)) \to \exists\, v \notin R$, $v \nmid \ell$, $q_v \equiv 1\ (\ell^N)$

$\bar{r}(\mathrm{Frob}_v)$ distinct eig. vals

$\mathrm{res}_{G_{F_v}} \phi \neq 0$. $\qquad 0 \neq \phi(\mathrm{Frob}_v) \in (\mathrm{ad}^0 \bar{r})(1) / (\mathrm{Frob}_v - 1)\, \mathrm{ad}^0 \bar{r}(1)$

$\Uparrow$ Cebotarev

STP. $\forall\, 0 \neq \phi \in H^1_{\mathcal{L}^\perp_\emptyset}(G_F, (\mathrm{ad}^0 \bar{r})(1))$

$\exists\, \sigma \in G_{F(\zeta_{\ell^N})}$ s.t. 1) $\bar{r}(\sigma)$ has distinct eig. vals

2) $\phi(\sigma) \notin (\sigma - 1)\, \mathrm{ad}^0 \bar{r}(1)$

$E_N = \bar{F}^{\ker \bar{r}}(\zeta_{\ell^N})$ $\qquad \sigma \in \mathrm{Gal}(E_N | F(\zeta_{\ell^N})) \xleftarrow{\text{Want}} \tilde{\sigma} \in G_{F(\zeta_{\ell^N})}$

$\tau \in G_{E_N}$

$\sigma_1 = \tau\tilde{\sigma}$ $\qquad$ s.t. $\bar{r}(\sigma)$ has distinct eig vals. i.e. ~~and~~ $\bar{r}(\sigma)$ Order prime to ℓ

$\phi(\tau) + \phi(\tilde{\sigma}) \notin (\sigma - 1)(\mathrm{ad}^0 \bar{r})(1)$

o.k. if

σ has eig. val 1 on $\langle \phi(G_{E_N}) \rangle_{\mathbb{F}}$

STP 1) $\underset{\overset{\#}{0}}{\mathrm{res}\,\phi} \in H^1(G_{E_N}, (\mathrm{ad}^0 \bar{r})(1)) = \mathrm{Hom}(G_{E_N}, (\mathrm{ad}^0 \bar{r})(1))$

$\therefore \phi(G_{E_N}) \neq (0)$.

2) $\forall\ 0 \neq W \subset (\mathrm{ad}^0 \bar{r})$ G_F - invariant.

$\exists\ \sigma \in \mathrm{Gal}(E_N \mid F(\zeta_{\ell N}))$ s.t. σ has an eig. val. 1 on W, and $\mathrm{ad}\,\sigma$ not ℓ- power order.

2).

$$\begin{tikzcd} & E_N \arrow[dl, dash] \arrow[dr, dash] & \\ E_1 & & F(\zeta_{\ell N}) \end{tikzcd}$$

$\mathrm{Gal}(E_N \mid F(\zeta_{\ell N})) \hookrightarrow \mathrm{Gal}(E_1 \mid F(\zeta_\ell))$

w/ ℓ- power order index.

reduce to case $N = 1$.

a) ad^0 $\bar{r}$ irreducible

$\bar{r}(\sigma)$ α, β

α/β, β/α, 1

So just need $\sigma \in G_{F(\zeta_\ell)}$ s.t. $\mathrm{ad}\,\bar{r}(\sigma)$ not ℓ-power order

$(\mathrm{ad}^0 \bar{r})^{G_{F(\zeta_\ell)}} = 0$ (as else $\mathrm{ad}^0 \bar{r}$ would have a line inv't by G_F as $\mathrm{Gal}(F(\zeta_\ell) \mid F)$ abelian. $\lightning$)

$\therefore \mathrm{ad}^0 \bar{r}(G_{F(\zeta_\ell)})$ does not ℓ- power order

b) ad^0 $\bar{r}$ reducible. $\Rightarrow \exists$ char δ s.t. $\bar{r} \simeq \bar{r} \otimes \delta$, $\bar{r}$ irred. $\Rightarrow \delta \neq 1$

$\Rightarrow \delta^2 = 1$ (take det)

$L = \bar{F}^{\ker \delta}$, $\bar{r} \simeq \mathrm{Ind}_{G_F}^{G_L} \theta$. $\bar{r}|_{G_L} = \theta \oplus \theta'$.

$\mathrm{ad}^0 \bar{r} = (\mathrm{Ind}_{G_F}^{G_L} \theta/\theta') \oplus \delta$

b) i) $\underline{\mathrm{Ind}_{G_F}^{G_L} \theta/\theta' \text{ irred.}}$

$W = \delta$, need σ s.t. $\delta(\sigma) = 1$, i.e. $\sigma \in G_L$. and $(\theta/\theta')(\sigma) \neq 1$.

$W = \mathrm{Ind}_{G_F}^{G_L} \theta/\theta'$ can look for σ w/ $\delta(\sigma) = -\underline{1}$.

when $\left(\mathrm{Ind}_{G_F}^{G_L} \theta/\theta'\right)(\sigma) \sim \begin{pmatrix} 0 & 1 \\ (\theta/\theta')(\sigma^2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma \in G_{F(\zeta_\ell)}$

$$(\theta/\theta')(\sigma^2) = \frac{\theta(\sigma^2)}{\theta(\sigma\sigma^2\sigma^{-1})} = \underline{1}.$$

b) ii) $\mathrm{ad}^0\bar{\tau} \simeq \underbrace{\delta \oplus \chi \oplus \chi\delta}$.

all distinct. (as $\bar{\tau}$ irred.)

all quadratic

$\bar{F}^{\ker\delta} \not\subset F(\zeta_\ell)$

$\bar{F}^{\ker\chi} \not\subset F(\zeta_\ell)$

$\bar{F}^{\ker\chi\delta} \not\subset F(\zeta_\ell)$.

want σ s.t. $\delta(\sigma) = -1$, $\chi(\sigma) = 1$

$\delta(\sigma) = 1$ $\chi(\sigma) = -1$.

$\chi\delta(\sigma) = 1$, $\delta(\sigma) = -1$.

1) $H^1(\mathrm{Gal}(E_N|F), (\mathrm{ad}^0\bar{\tau})(1)) = 0$.

$$\begin{tikzcd} H^1(\mathrm{Gal}(E_N|F), (\mathrm{ad}^0\bar{\tau})(1)) \arrow[d, phantom, "\wr\shortparallel"] \\ \ker\left(H^1(G_F, (\mathrm{ad}^0\bar{\tau})(1)) \rightarrow H^1(G_{E_N}, (\mathrm{ad}^0\bar{\tau})(1))\right) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathrm{Gal}(E_1|F), (\mathrm{ad}^0\bar{\tau})(1)) \arrow[r] \arrow[d] & H^1(\mathrm{Gal}(E_N|F), (\mathrm{ad}^0\bar{\tau})(1)) \\ & \Gamma := \mathrm{Im}\,\bar{\tau} \subset GL_2(\mathbb{F}) & \end{tikzcd}$$

$$\rightarrow H^1(\mathrm{Gal}(E_N|E_1), (\mathrm{ad}^0\bar{\tau})(1))^{\mathrm{Gal}(E_1|F)}$$

$$\shortparallel$$

$$\mathrm{Hom}(\mathrm{Gal}(E_N|E_1), (\mathrm{ad}^0\bar{\tau})(1))^{\mathrm{Gal}(E_1|F)}$$

$$\shortparallel$$

$$\mathrm{Hom}(\mathrm{Gal}(E_N|E_1), \underset{\overset{\shortparallel}{0}}{(\mathrm{ad}^0\bar{\tau})(1)^{G_F}})$$

$\Gamma = \mathrm{Im}\,\bar{\iota} \subset GL_2(\mathbb{F})$.

Suppose $\mathbb{F} = \mathbb{F}_\ell$.

Need $H^1(\Gamma, (\mathrm{ad}^\circ) \otimes (\det)^{-1}) = 0$.

$\ell \nmid |\Gamma|$, ✓

$\ell \mid |\Gamma|$, up to conj. $\Gamma \supset \left\{\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right\} = S$

$$\begin{tikzcd}
H^1(\Gamma, (\mathrm{ad}^\circ\bar{\iota})\otimes\det^{-1}) \arrow[r, hook] & H^1(S, (\mathrm{ad}^\circ\bar{\iota})\otimes\det^{-1})^N \arrow[d, phantom, "\cap"] \\
 & \left(\left((\mathrm{ad}^\circ\bar{\iota})\otimes\det^{-1}\right)_S\right)^N \\
0 \arrow[uu] & \mathbb{F}\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix} \arrow[u] \arrow[loop below]
\end{tikzcd}$$

0 if $\exists$ $\begin{pmatrix}\alpha & * \\ 0 & \beta\end{pmatrix} \in \Gamma$ w/ $\beta^2 \neq 1$

$\begin{pmatrix}\alpha & * \\ 0 & \beta\end{pmatrix}$ acts as β^2

Γ acts on $\mathbb{P}^1(\mathbb{F}_\ell)$, has no fixed pt

, contains an ℓ-cycle

$\therefore$ action transitive.

$L \subset \mathbb{F}^2$ is any line.

$\delta \in \Gamma$, $\delta L = \langle \begin{pmatrix}1 \\ 0\end{pmatrix} \rangle$ $\delta S \delta^{-1}$ fixes every elt of L

$\begin{pmatrix}1 & 0 \\ c & 1\end{pmatrix}\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ b & 1\end{pmatrix} = \begin{pmatrix}1+ab & a \\ b+c+cab & 1+ac\end{pmatrix}$ $\therefore N = \left\{\begin{pmatrix}* & * \\ 0 & *\end{pmatrix}\right\}$

$ab \neq -1$, $c = \dfrac{-b}{1+ab} \longrightarrow \begin{pmatrix}1+ab & * \\ 0 & (1+ab)^{-1}\end{pmatrix}$

$v \in R$, $R^{\square}_{\chi_v}$ parametrizes $r : G_{F_v} \longrightarrow GL_2(\overset{\swarrow \text{ test ring}}{T})$

$$r \text{ tamely ramified} \Leftarrow \begin{cases} - \ r \bmod m_T = 1. \\ - \ \det r = \varepsilon_\ell^{-1} \\ - \ \sigma \in I_{F_v}, \ \mathrm{char}_{r(\sigma)}(X) = (X - \chi_v(\sigma))(X - \chi_v(\sigma)^{-1}) \end{cases}$$

$$\boxed{R^{loc}_{\chi}} := \hat{\bigotimes_{v \in R}} R^{\square}_{\chi_v}$$

$\uparrow$ need to know a bit about this

① If $\mathrm{Spec}\, R^{loc}_1$ has irred. comp. $C_1, \dots, C_s$, $/\mathcal{O}$

then $C_i \otimes_{\mathcal{O}} \mathbb{F}$ are irred., and distinct, and exhaust the irred. components of $\mathrm{Spec}\left(R^{loc}_1 \otimes_{\mathcal{O}} \mathbb{F}\right)$.

② If $\chi_v \neq 1$, $\forall v$, then $\mathrm{Spec}\, R^{loc}_{\chi}$ is irred. and ℓ is not nilpotent in R^{loc}_{χ}.

Choose $\tau_v \in I_{F_v}$ lifting a generator of tame inertia.

$\phi_v \in G_{F_v}$ —— Frob_v

| gen. $G_{F_v}/$wild inertia.

$$\underset{\Sigma_v}{\overset{\parallel}{r(\tau_v)}} \ \& \ \underset{\Phi_v}{\overset{\parallel}{r(\phi_v)}} \text{ determine } r.$$

$R^{\square}_{\chi_v}$ parametizes pairs $(\Phi_v, \Sigma_v) \in \ker\left(GL_2(T) \longrightarrow GL_2(\mathbb{F})\right)^2$

w/ $\det \Phi_v = q_v$, $\Phi_v^{-1} \Sigma_v \Phi_v = \Sigma_v^{q_v}$, $\mathrm{char}_{\Sigma_v}(X) = (X - \chi_v(\sigma_v))(X - \chi_v(\sigma_v)^{-1})$

$\mathbb{Z}/\ell^N\mathbb{Z} \times \mathbb{Z}/\ell^N\mathbb{Z} \to GL_2(T/m^2)$

$(1,0) \longmapsto \Sigma_v$

$(0,1) \longmapsto \Phi_v$

$\varprojlim \mathbb{Z}_\ell \times \mathbb{Z}_\ell \to GL_2(T)$ (+)

Lecture 5 R set of bad primes (of F)

$L / \mathbb{Q}_\ell$

$\mathcal{O}, \mathcal{O}/\lambda = \mathbb{F}$

$v \in R$, $q_v \equiv 1 \pmod{\ell}$

$\chi_v : k(v)^\times \to \mathcal{O}^\times$ ℓ power order

$\chi = \prod \chi_v$

$R^\square_{\chi_v}$ universal lifting ring $/\mathcal{O}$

for $r : G_{F_v} \to GL_2(\underset{\nwarrow}{T})$ test CNL $\mathcal{O}$-alg.

$r \bmod m_T = 1$.

$\det r = \varepsilon_\ell^{-1}$

$v \in R$

$\phi_v \in G_{F_v}$ a lift of Frob.

$\tau_v \in I_{F_v}$ generates I_{F_v}/P_{F_v}

$\zeta_v = \chi_v(\tau_v)$

$\Phi_v = r^{univ}(\phi_v)$

$\Sigma_v = r^{univ}(\tau_v)$

If $\sigma \in I_{F_v}$, then $\text{char}_{r(\sigma)}(X) = (X - \chi_v(\sigma))(X - \chi_v(\sigma)^{-1})$

$\Rightarrow$ trivial on P_{F_v}

$\Phi_v, \Sigma_v \equiv 1 \ (m_T)$

$R^{loc}_\chi := \widehat{\bigotimes_{v \in R}} R^\square_{\chi_v} \longleftarrow$ parametizes matrices $(\Phi_v, \Sigma_v) \in M_{2\times 2}(T)$ $\det \Phi_v = q_v$

$\text{char}_{\Sigma_v}(X) = (X - \zeta_v)(X - \zeta_v^{-1})$

$\Phi_v^{-1}\Sigma_v\Phi_v = \Sigma_v^{q_v}$

Prop. 1) If $\chi_v \neq 1$ $\forall v \in R$, then $\text{Spec}\, R^{loc}_\chi$ is irreducible, its generic pt has char. 0, and Krull dim $R^{loc}_\chi = 3|R| + 1$.

2) Let $C_1, \dots, C_n$ denote the irred. components of $\text{Spec}\, R^{loc}_1$. then C_i has generic pt in char 0, $\dim C_i = 3|R| + 1$, and $C_i \otimes_{\mathcal{O}} \mathbb{F}$ are irreducible and distinct.

$\mathcal{M}_\chi | \mathcal{O}$ moduli space of matices $(\Phi_v, \Sigma_v)_{v \in R}$ s.t.

$\shortparallel$

$\times_{v \in R} \mathcal{M}_{\chi_v}$ det $\Phi_v = q_v$, $\mathrm{char}_{\Sigma_v}(X) = (X - \zeta_v)(X - \zeta_v^{-1})$

$$(*) \quad \Phi_v^{-1} \Sigma_v \Phi_v = \Sigma_v^{q_v}.$$

$\uparrow$

parametizes (Φ_v, Σ_v) as above.

$R_\chi^{loc} \simeq \hat{\mathcal{O}}_{\mathcal{M}_\chi, \underline{1}}$

$\underline{1} = ((1,1), (1,1), \dots)$

$\in \mathcal{M}_\chi(\mathbb{F})$

if $\zeta_v \not\equiv 1$, $\forall v$

then $\mathrm{char}_{\Sigma_v}(x) \mid (x^{q_v} - x)$

$\therefore \Sigma_v^{q_v} = \Sigma_v$

$\therefore (*) \iff \Sigma_v \Phi_v = \Phi_v \Sigma_v$

If $\chi_v = 1$, $\forall v$,

$\mathcal{M}_1 \simeq \mathcal{N} \leftarrow$ parametizes (Φ_v, N_v)

det $\Phi_v = q_v$

$\mathrm{char}_{N_v}(X) = X^2$

$\Phi_v^{-1} N_v \Phi_v = q_v N_v$ $\quad (N_v = \Sigma_v - 1)$

$$\mathcal{N} \simeq \times_{v \in R} \mathcal{N}_v$$

L Suppose $X/\mathcal{O}$ is a scheme of f.type, and $x \in X(\mathbb{F})$.

Let $X_1, \dots, X_r$ denote the irred. comp. of X. Suppose the X_i all have generic pt in char. 0, are all irred. of dim d, and the $X_i \times_{\mathcal{O}} \mathbb{F}$ are irred. distinct

Let $C_1, \dots, C_s$ denote the irred. comp. of Spec $\hat{\mathcal{O}}_{X,x}$, then the C_i have gen. pt in char. 0, have dim d, and the $C_i \times_{\mathcal{O}} \mathbb{F}$ are irred. + distinct.

[$\tilde{X} \longrightarrow X$ normalisation. ...]

∴ 2) ⟺ 2') same true for $\mathcal{N}$. ⟺ same true for $\mathcal{N}_v$, $v \in R$

& the irred. comp. of $\mathcal{N}_v$ and $\mathcal{N}_v \times \mathbb{F}$

are geom. irred.

L. Suppose R is a complete noeth. local $\mathcal{O}$-alg, then $R[\frac{1}{\ell}]$ is Jacobson

(i.e. any prime ideal is intersection of max. ideals).

∴ max'l ideals dense in Spec.

Moreover, if $\mathfrak{p}$ is a max'l ideal of $R[\frac{1}{\ell}]$, then $k(\mathfrak{p})$ is a finite ext'n of L.

and the image of R in $k(\mathfrak{p})$ is a finite $\mathcal{O}$-module.

('pf' WLOG $\mathfrak{p} \cap R = (0)$, then $R[\frac{1}{\ell}]$ is a field.

R flat $/\mathcal{O}$

$$\Downarrow$$

Krull dim $(R/\ell) = 0$

$$\Downarrow$$

R/ℓ Artinian. ∴ finite $/\mathbb{F}$)

∴ R finite $/\mathcal{O}$.

1) reduces to 1')

a) Spec $R_\chi[\frac{1}{\ell}]$ is conn'd

Spec $R_\chi[\frac{1}{\ell}]$ ⟸ is irred.

b) If $\mathfrak{p}$ is a max'l ideal of $R_\chi[\frac{1}{\ell}]$, then $R_\chi[\frac{1}{\ell}]^\wedge_{\mathfrak{p}}$ is a power series ring over $k(\mathfrak{p})$

c) $\mathcal{M}_\chi^{red}$ is flat $/\mathcal{O}$.

d) $\mathcal{M}_\chi$ has $3|R|+1$) ↘ $\boxed{\mathcal{M}_{\chi v}}$.

$\mathcal{M}_\chi \times \mathbb{F} \simeq \mathcal{M}_1 \times \mathbb{F} \simeq \mathcal{N} \times \mathbb{F}$

2') $\mathcal{N}_v \qquad (\Phi, N) \qquad \det \Phi = q_v$

$\cup \qquad \Phi^{-1} N \Phi = q_v N$

$\mathcal{N}_0$ locus where $N = 0 \qquad \mathrm{char}_N(X) = X^2$

$$\begin{tikzcd} g\begin{pmatrix} q_v & 0 \\ 0 & 1 \end{pmatrix} & \mathcal{N}_0 & \\ & & SL_2 \arrow[ul, "\sim"'] \\ & & g \arrow[uull, mapsto] \end{tikzcd}$$

$\mathcal{N}_v \supset \mathcal{N}_\pm$ locus where $\mathrm{tr}\, \Phi = \pm(1 + q_v)$

$\mathcal{N}(L) \ni (\Phi, N)$ either $N = 0$ and $(\Phi, N) \in \mathcal{N}_0(L)$

$\uparrow$

any field

or $N = g\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} g^{-1}$, $\alpha \neq 0$

$\Rightarrow \Phi = g \begin{pmatrix} \beta & \gamma \\ 0 & \beta q_v \end{pmatrix} g^{-1}$ w/ $\beta^2 = 1$, i.e. $\beta = \pm 1$.

$\therefore (\Phi, \mathcal{N}) \in \mathcal{N}_\pm(L)$

$$(PGL_2 \times \mathbb{A}^1)/\mathbb{G}_a \xrightarrow{\sim} (\mathcal{N}_\pm - \mathcal{N}_0)^{\mathrm{red}}$$

$$(g, a) \longmapsto \left(\pm g \begin{pmatrix} 1 & a \\ 0 & q_v \end{pmatrix} g^{-1}, \; g\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g^{-1} \right)$$

$\mathbb{G}_a$ acts $\quad x : (g, a) \longmapsto \left(g\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \; a + (q_v - 1)x \right)$

$\mathcal{N}_\pm^{\mathrm{red}}$ is the closure of $(\mathcal{N}_\pm - \mathcal{N}_0)^{\mathrm{red}}$ in $\mathcal{N}_v$.

$$PGL_2 \times \mathbb{A}^1 \longrightarrow (\mathcal{N}_\pm \cap \mathcal{N}_0)^{\mathrm{red}} \qquad \text{surj. on field pts}$$

$$(g, a) \longmapsto \left(\pm g \begin{pmatrix} 1 & a \\ 0 & q-1 \end{pmatrix} g^{-1}, \; 0 \right)$$

$\hookrightarrow$ all generalize to $\mathcal{N}^\pm - \mathcal{N}_0$.

$\therefore$ N_0, $N_\pm$ are the irred. comp. of N_v. all dim ~~4~~. all generically char. 0

all geom. irred. $N_0 \times \mathbb{F}$ and $N_\pm \times \mathbb{F}$ are geom. irred. and dist.

2') $M_{X_v} \underset{\mathcal{O}}{\times} L \longleftarrow PGL_2/T \times \mathbb{G}_m$

$$\begin{tikzcd} PGL_2/T \\ \text{diagonal torus} \arrow[u] \end{tikzcd}$$

$$\left(g \begin{pmatrix} a & 0 \\ 0 & q_v a^{-1} \end{pmatrix} g^{-1}, \; g \begin{pmatrix} \zeta_v & 0 \\ 0 & \zeta_v^{-1} \end{pmatrix} g \right) \longleftarrow\!\shortmid (g, a)$$

over a field of char. 0 $\supset L$

any $(\Phi, \Sigma) \in M_{X_v}(K)$ is in the image of this map.

$\therefore$ $(M_{X_v} \underset{\mathcal{O}}{\times} L)^{red} \xleftarrow{\sim} PGL_2/T \times \mathbb{G}_m$ $\therefore$ geom. irred. dim 3.

c) Sufficient to show that we can find a pt on each irred. compt of $N_v \times \mathbb{F}$

$(M^{red}_{X_v} \times L)^{Zc} \subset M^{red}_{X_v}$ on no other comp. which lifts to char. 0.

$(M_{X_v} \times \mathbb{F})^{red} \simeq (N_v \times \mathbb{F})^{red}$

$\uparrow$

every irred comp of special fiber has dim ≥ 3

$\therefore$ union of compts of $N_v \times \mathbb{F}$

$$\begin{tikzcd} N_0^{\times \mathbb{F}} \ni \begin{pmatrix} \alpha & 0 \\ 0 & q_v/\alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad \alpha \neq \pm 1 \arrow[d, dash] \\ N_\pm \end{tikzcd}$$

lifts to $\begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & q_v/\tilde{\alpha} \end{pmatrix}, 0$

$\tilde{\alpha}$ lifts α

$N_\pm \times \mathbb{F} - N_0 \supset \left(\pm \begin{pmatrix} 1 & 0 \\ 0 & q_v \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$

lifts to $\left(\begin{pmatrix} 1 & \frac{1-q_v}{\zeta_v-\zeta_v^{-1}} \\ 0 & q_v \end{pmatrix}, \begin{pmatrix} \zeta_v & 1 \\ 0 & \zeta_v^{-1} \end{pmatrix}\right)$

1') b)

Usually

$R_{\bar r}$ $\quad r^{univ}$

Rmk. $k(\mathfrak{p}) \hookrightarrow R^{loc}_{\chi}[\frac{1}{\ell}]^{\wedge}_{\mathfrak{p}}$ $\qquad R_{\bar r}[\frac{1}{\ell}]^{\wedge}_{\mathfrak{p}}$ = universal def. ring for r^{univ} mod $\mathfrak{p}$

$\uparrow$

look mod $\mathfrak{p}^2$, $\forall r$

$r = 2$ | ^ deformations unobstructed.

Lecture 6. $L | \mathbb{Q}_\ell$ finite, $\mathcal{O} = \mathcal{O}_L$, $\mathcal{O}/\lambda = \mathbb{F}$.

F, v, $q_v \equiv 1 \pmod{\ell}$

R set of primes of F : $v \in R \Rightarrow q_v \equiv 1 \pmod{\ell}$

$\downarrow$

$v \in R$, $\qquad R^{\square}_{\chi_v}$ universal lifting ring for $1 : G_{F_v} \to GL_2(\mathbb{F})$

$\chi_v : k(v)^\times \longrightarrow \mathcal{O}^\times$

ℓ-power order

s.t. 1) for $\sigma \in I_{F_v}$,

$\mathrm{char}_{r(\sigma)}(X) = (X - \chi_v(\sigma))(X - \chi_v(\sigma)^{-1})$

2) $\det r = \varepsilon_\ell^{-1}$

$$R^{loc}_{\chi} = \hat{\bigotimes_{v \in R}} R^{\square}_{\chi_v}$$

Prop 1) If $\chi_v \neq 1$, $\forall v \in R$, then $\mathrm{Spec}\, R^{loc}_{\chi}$ is irred., of dim $1 + 3|R|$, w/ char. 0 generic points.

✓ 2) If $\chi_v = 1$, $\forall v \in R$ and if $C_1, \cdots, C_r$ denote the irred. comp. of Spec R_1^{loc}, then C_i has dim $1+3|R|$ has generic pt of char. 0. and the $C_i \underset{\mathcal{O}}{\times} \mathbb{F}$ are irred. & distinct.

Still need to prove 2 things: in case $\chi_v \neq 1$, $\forall v$

c) If $\mathfrak{p}$ is a max'l ideal of $R_\chi^{loc}[\frac{1}{\ell}]$, then $R_\chi^{loc}[\frac{1}{\ell}]^\wedge_{\mathfrak{p}}$ is a power series ring over $k(\mathfrak{p})$.

d) Spec $R_\chi^{loc}[\frac{1}{\ell}]$ is conn'd.

$\mathcal{M}_\chi/\mathcal{O}$, $R_\chi^{loc} = \hat{\mathcal{O}}_{\mathcal{M}_\chi, 1}$

finite type, $\mathcal{M}_\chi \underset{\mathcal{O}}{\times} L$ is smooth.

c) $R_\chi^{loc}[\frac{1}{\ell}]/\mathfrak{p} \simeq k(\mathfrak{p})$

$R_\chi^{loc}[\frac{1}{\ell}]/\mathfrak{p}^2 \simeq k(\mathfrak{p})[[X_1, \cdots, X_r]]/(X_1, \cdots, X_r)^2$

$\cdots$

Show successively $\exists$ $R_\chi^{loc}[\frac{1}{\ell}]/\mathfrak{p}^s \xrightarrow{f_s} k(\mathfrak{p})[[X_1, \cdots, X_r]]/(X_1, \dots X_r)^s$

$\Rightarrow$ f_s surjective.

dim $\mathfrak{p}^a/\mathfrak{p}^{a+1} \leq$ dim $(X_1, \cdots X_r)^a/(X_1, \cdots X_r)^{a+1}$

$\Rightarrow$ f_s isomorphism.

$B = k(\mathfrak{p})[[X_1, \cdots, X_r]]/(X_1, \cdots X_r)^{s+1}$

$I = (X_1, \cdots, X_r)^s$

$$\begin{tikzcd} A/\mathfrak{p}^s \arrow[r, "\sim"'] & B/I \\ A/\mathfrak{p}^{s+1} \arrow[u, two heads] \arrow[r, dashed] & B \arrow[u] \end{tikzcd}$$

$\mathfrak{p} I = (0)$

$r_v(\tau_v)\, e_{1,v} = \zeta_v\, e_{1,v}$ $\qquad$ $\tau_v \in I_{F_v}$ gen. tame inertia

$r_v(\tau_v)\, e_{2,v} = \zeta_v^{-1}\, e_{2,v}$ $\qquad$ $\phi_v \in G_{F_v}$ lifts Frob.

$\zeta_v = \chi_v(\tau_v)$

over $R^{loc}_{\chi}[\frac{1}{\ell}]$ $\Rightarrow$ $r_v(\phi_v)\, e_{1,v} = \alpha_{1,v}\, e_{1,v}$

$r_v(\phi_v)\, e_{2,v} = \alpha_{2,v}\, e_{2,v}$

Choose $\tilde{\alpha}_{i,v} \in B$ lifting $\alpha_{i,v}$ mod p^s

$$\mathcal{O}'_{k(p)} = \{\beta \in \mathcal{O}_{k(p)} : \beta \bmod \overset{\text{max}}{\text{ideal}} \in \mathbb{F}\}$$

$$B^0 = \mathcal{O}'_{k(p)}[\underbrace{\tilde{\alpha}_{i,v} - \bar{\alpha}_{i,v}}_{\text{nilpotent}}] \subset B$$

B^0 is a CNL $\overset{\mathcal{O}\text{-alg.}}{\text{ring}}$, res. field $\mathbb{F}$

$$r'_v : G_{F_v} \longrightarrow GL_2(B^0)$$

$r'_v(\tau_v)\, e_{1,v} = \zeta_v\, e_{1,v}$ $\qquad$ $r'_v(\phi_v)\, e_{1,v} = \tilde{\alpha}_{1,v}\, e_{1,v}$

$r'_v(\tau_v)\, e_{2,v} = \zeta_v^{-1}\, e_{2,v}$ $\qquad$ $r'_v(\phi_v)\, e_{2,v} = \tilde{\alpha}_{2,v}\, e_{2,v}$

a) $\chi_v \neq 1, \forall v$, Spec $R^{loc}_{\chi}[\frac{1}{\ell}]$ conn'd

Step 1 Suppose p is a max'l ideal of $R^{loc}_{\chi}[\frac{1}{\ell}]$,

then $\exists\, p'$ a max'l ideal in the same irred. comp. of Spec $R^{loc}_{\chi}[\frac{1}{\ell}]$

s.t. $\forall v \in R$, r_v mod p' is upper triangular, $\tau_v \mapsto \begin{pmatrix} \zeta_v & * \\ 0 & \zeta_v^{-1} \end{pmatrix}$

Pf. $0 \neq e_{1,v}$ of $\mathcal{O}^2_{k(p)}$ such that $r_v(\tau_v)\, e_{1,v} = \zeta_v\, e_{1,v}$

primitive $\qquad$ $r_v(\phi_v)\, e_{1,v} = \alpha_{1,v}\, e_{1,v}$

extend to basis $\{e_{1,v}, e_{2,v}\}$ of $\mathcal{O}^2_{k(p)}$. $\quad A_v = (e_{1,v}, e_{2,v})$

$$A_v^{-1} r_v(\tau_v) A_v = \begin{pmatrix} \zeta_v & * \\ 0 & \zeta_v^{-1} \end{pmatrix}$$

$$A_v^{-1} r_v(\phi_v) A_v = \begin{pmatrix} \alpha_{1,v} & * \\ 0 & * \end{pmatrix}$$

λ' max. ideal of $\mathcal{O}_{K(p)}$

$$S = \mathcal{O}_{K(p)} \left\langle X_{v,ij}, Y_v ; \begin{matrix} v \in R \\ ij = 1 \text{ or } 2 \end{matrix} \right\rangle \Big/ \langle \det(X_{vij}) Y_v - 1 \rangle$$

$\langle\ \rangle$ ⟵ power series whose terms ⟶ 0 ℓ-adically

S ⟵ ired.

$$\left((X_{v,ij})^{-1} r_v(\phi_v) (X_{v,ij}),\ (X_{v,ij})^{-1} r_v(\tau_v) (X_{v,ij}) \right) \in M_{2\times 2}(S)^{|R|}$$

$\mathrm{Spec}\, S$ ⟵ λ'-adically complete

gives

$$\begin{tikzcd}
\mathrm{Spec}\, S \arrow[r] \arrow[ddr, dashed] & M_x \\
\mathrm{Spec}\, S/\lambda' \arrow[u] \arrow[r] & \mathrm{Spec}\, \mathbb{F} \arrow[u] \\
 & \mathrm{Spec}\, \hat{\mathcal{O}}_{\mathcal{M}_x,(1,1)} \arrow[uu, bend right]
\end{tikzcd}$$

point $((1,1), \dots, (1,1))$

$r_v \bmod \lambda'$ trivial

/ get $S \overset{f}{\longleftrightarrow} R_x^{loc}$.

$$f \bmod \langle X_{v,ij} - \delta_{ij},\ Y_v - 1 \rangle$$

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \hat{B}_m \arrow[l] \\
\mathcal{O}_{K(p)} & & \mathcal{O}_{K(p)}
\end{tikzcd}$$

$X_{v,ij} \mapsto \delta_{ij}$, $Y_v \mapsto 1$

$\left(" \; R_x^{loc} \right)$

$(X_{v,ij}) \mapsto A_v$

$Y_v \mapsto \det(A_v)^{-1}$

Step 2. $\mathfrak{p}$ as in Step 1, then $\exists\, \mathfrak{p}'$ in the same conn'd comp. of $\operatorname{Spec} R_\chi^{\text{loc}}[\frac{1}{\ell}]$ w/ each $\rho_v \bmod \mathfrak{p}'$ diagonal

Pf. Can assume $\rho_v \bmod \mathfrak{p}$ upper triangular, $\forall v$

$S = \operatorname{Spec} \mathcal{O}_{k(\mathfrak{p})}\langle Y_v : v \in R \rangle$

$$\in M_{2\times 2}(S)^{2|R|}.$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & Y_v^{-1} \end{pmatrix} (\rho_v \bmod \mathfrak{p})(\phi_v) \begin{pmatrix} 1 & 0 \\ 0 & Y_v \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & Y_v^{-1} \end{pmatrix} (\rho_v \bmod \mathfrak{p})(\tau_v) \begin{pmatrix} 1 & 0 \\ 0 & Y_v \end{pmatrix} \right)$$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} a & Y_v b \\ 0 & c \end{pmatrix} \quad \text{entries in } S$$

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r] & M_\chi \\ \operatorname{Spec} S/\mathfrak{p}' \arrow[u] \arrow[r] & \operatorname{Spec} \mathbb{F} \arrow[u] \end{tikzcd} \quad - (1,1), (1,1), \cdots$$

$$\begin{tikzcd} R_\chi^{\text{loc}} \arrow[r] \arrow[d, "\ker \mathfrak{p}"'] & S \arrow[ld] \arrow[d] & \\ \mathcal{O}_{k(\mathfrak{p})} & \mathcal{O}_{k(\mathfrak{p})} & \ker \mathfrak{p}' \end{tikzcd}$$

$Y_v \mapsto 1$, $\quad Y_v \mapsto 0$

$\rho_v \bmod \mathfrak{p}'$ is diagonal, $\forall v$

Step 3 $\mathfrak{p}$ max'l in $R_\chi^{\text{loc}}[\frac{1}{\ell}] \Rightarrow \mathfrak{p}$ is in the same conn'd comp. as $\mathfrak{p}_0$,

where $\rho_{\mathfrak{p}_0}(\tau_v) = \begin{pmatrix} \zeta_v & 0 \\ 0 & \zeta_v^{-1} \end{pmatrix}$

$\rho_{\mathfrak{p}_0}(\phi_v) = \begin{pmatrix} 1 & 0 \\ 0 & q_v \end{pmatrix}$

Pf WLOG $\tau_v \bmod p$ is diagonal, $\forall v$

$$(r_v \bmod p)(\tau_v) = \begin{pmatrix} \zeta_v & 0 \\ 0 & \zeta_v^{-1} \end{pmatrix}$$

$$(r_v \bmod p)(\phi_v) = \begin{pmatrix} \alpha_v & 0 \\ 0 & q_v/\alpha_v \end{pmatrix}$$

$$S = \operatorname{Spec} \mathcal{O}_{k(p)} \langle Y_v, Z_v \rangle / (Y_v Z_v - 1 \rangle$$

$$\left(\begin{pmatrix} Y_v & 0 \\ 0 & q_v Z_v \end{pmatrix}, \begin{pmatrix} \zeta_v & 0 \\ 0 & \zeta_v^{-1} \end{pmatrix} \right) \in M_{2\times 2}(S)^{2|R|}$$

~~Spec~~ $S = \mathcal{O}'_{k(p)} [\![Y_v]\!]$, $\mathcal{O}'_{k(p)} = \{ \alpha \in \mathcal{O}_{k(p)} : \alpha \bmod \text{max'l ideal} \in \mathbb{F} \}$

$$\left(\begin{pmatrix} (1+Y_v) & 0 \\ 0 & q_v/(1+Y_v) \end{pmatrix}, \begin{pmatrix} \zeta_v & 0 \\ 0 & \zeta_v^{-1} \end{pmatrix} \right) \in M_{2\times 2}(S)^{2|R|}$$

$$\begin{tikzcd} R^{loc}_x \arrow[r] & S \arrow[dl, "Y_v = 0"'] \arrow[d, "Y_v = \alpha_v - 1"] \\ \mathcal{O}'_{k(p)} & \mathcal{O}'_{k(p)} \end{tikzcd}$$

Lecture 7 A noeth. local ring, M a f.g. A-module

$$\operatorname{Supp}_A(M) = \{ \mathfrak{p} \in \operatorname{Spec} A : M_\mathfrak{p} \neq 0 \} \subset \operatorname{Spec} A$$

$$= V(\operatorname{Ann}_A(M))$$ closed

Lemma 1) If $I \triangleleft A$, then $\operatorname{supp}_{A/I}(M/IA) = \operatorname{Supp}(M) \cap \operatorname{Spec}(A/I)$

$\overset{\downarrow}{\mathfrak{p}}$

$A_\mathfrak{p} > \mathfrak{p} > I$

If $(M/IM)_\mathfrak{p} = 0$
$\overset{\shortparallel}{M_\mathfrak{p}/I_\mathfrak{p} M_\mathfrak{p}}$ by Nakayama $\Rightarrow M_\mathfrak{p} = 0$.

2) If A is a local $\mathcal{O}$-alg., ~~and~~ if the irred. compts $C_1, \dots, C_r$ of $\operatorname{Spec} A$ — C_i gen char. 0 satisfy the $C_i \times_{\mathcal{O}} \mathbb{F}$ are irred, distinct, and exhaust the irred. compts of $A/\lambda A$.

and if M is θ- torsion free and if $\mathrm{Supp}_{A/\lambda A}(M/\lambda M) = \mathrm{Spec}(A/\lambda A)$

then $\mathrm{Supp}_A(M) = \mathrm{Spec}(A)$.

Pf. $\mathfrak{p}$ minimal prime of A, $\lambda \notin \mathfrak{p}$

$\wp \supset (\mathfrak{p}, \lambda)$ minimal prime. $\wp \in \mathrm{Supp}_{A/\lambda}(M/\lambda M)$

$\mathfrak{p}$ is the unique prime ideal properly contained in $\wp$.

$M_\wp \neq (0)$. $\lambda^2 : M \hookrightarrow M$

$\lambda^2 : M_\wp \hookrightarrow M_\wp$

$\therefore \lambda^2 \notin \mathrm{Ann}_{A_\wp}(M_\wp)$ $\therefore \lambda \notin \sqrt{\mathrm{Ann}_{A_\wp}(M_\wp)}$

$\exists\ q$ prime of $A_\wp$. $q \in \mathrm{Supp}_A(M)$

$\wp$ or $\mathfrak{p}$

$\lambda \notin q$ $\therefore q = \mathfrak{p}$.

Lemma Same assumptions, $\mathrm{depth}(m_A, M) \le \dim$ any irred. comp. of $\mathrm{Supp}_A(M)$.

length of the longest sequence

$x_1, \cdots x_d \in m_A$ s.t.

x_i is not a zero divisor of $M/(x_1, \cdots x_{i-1})M$, $\forall i$

Lemma If $\mathrm{depth}(m_A, M) \geq \dim A$, then $\mathrm{Supp}_A(M)$ is a union of irred. comp. of $\mathrm{Spec}\, A$.

$\dim_{\mathbb{F}} H^1_{\mathcal{L}_\phi^\perp}\left(G_F, (\mathrm{ad}^\circ \bar{r})(1)\right) = r$

$H_\infty = \mathbb{Z}_\ell^r$.

Krull dim $4|R| + r$

power series ring $/ \mathcal{O}$

$$\Lambda_\infty = \Lambda_\phi [\![H_\infty]\!] \simeq \Lambda_\phi [\![T_1, \dots, T_r]\!]$$

$$\Lambda_Q = \mathcal{O} [\![A_{v,i,j} : \underset{i,j = 1,2}{v \in R}]\!] \Big/ (A_{v_0,1,1}) [H_Q]$$

$\triangledown$

$\mathfrak{a}_{Q_N}$

$= \langle A_{v,i,j},\ h-1 : h \in H_Q \rangle$

$H_Q = \prod_{v \in Q} (\text{max'l } \ell\text{-power order quot } k(v)^\times)$.

$\exists\ H_\infty \twoheadrightarrow H_Q,\ \Lambda_\infty \twoheadrightarrow \Lambda_Q$ not canonical.

$N \in \mathbb{Z}_{>0}$, $\quad Q_N$, $\quad |Q_N| = r$

$\chi = \prod_{v \in R} \chi_v$.

$\chi_v : k(v)^\times \longrightarrow \mathcal{O}^\times$

ℓ-power order

$v \in Q_N$, then $q_v \equiv 1\ (\ell^N)$

$$\begin{tikzcd}
R^{\mathrm{loc}}_\chi \arrow[d] \arrow[dr, bend left] & \Lambda_{Q_N} \arrow[d] & \Lambda_\infty \arrow[l, two heads] & & \\
R^{\mathrm{loc}}_\chi [\![x_1, \dots, x_{|R|+r-1}]\!] \arrow[r, two heads, "\circledast", "\exists"'] & R^\square_{Q_N,\chi} \arrow[d] & \circlearrowright & S^\square_{Q_N,\chi} \arrow[d] & \text{finite free} / \Lambda_{Q_N} \arrow[l] \\
\dim 4|R| + r \arrow[u] & R^{\mathrm{univ}}_{\phi,\chi} & \circlearrowright & S_{\phi,\chi} \simeq S^\square_{Q_N,\chi} / \mathfrak{a}_{Q_N} &
\end{tikzcd}$$

$$R^{\mathrm{loc}}_\chi = \widehat{\bigotimes_{v \in R}} R^\square_{\chi_v}$$

(diagram for χ) mod λ $=$ (diagram for 1) mod λ

Choose $\circledast$ for $\chi = 1$. and then choose $(*)$ for each χ which mod λ equals choice for $\chi = 1$.

$R^{\mathrm{loc}}_{\chi,\infty} := R^{\mathrm{loc}}_\chi [\![x_1, \dots, x_{|R|+r-1}]\!]$

$\mathfrak{a}_\infty = (A_{v,i,j},\ h-1 : h \in H_\infty)$

$\bigcap_N C_N = 0$, $\quad C_N = \ker(\Lambda_\infty \longrightarrow \Lambda_{Q_N})$

$$\begin{tikzcd}
 & \mathfrak{a}_\infty \arrow[dd, mapsto] & \triangleleft & \Lambda_\infty \arrow[d] & & & \\
R^{\mathrm{loc}}_{1,\infty} \arrow[rrr, two heads] & & & R^\square_{Q_N,1} \arrow[d] & \circlearrowright & S^\square_{Q_N,1} \arrow[d] & \text{finite free} / \Lambda_\infty / C_N \arrow[l] \\
 & (0) & \triangleleft & R^{\mathrm{univ}}_{\phi,1} & \circlearrowright & S^{\mathrm{univ}}_{\phi,1} &
\end{tikzcd}$$

Choose χ s.t. $\chi_v \neq 1$, $\forall v \in R$.

get same diagram for χ. isom. to mod λ

$\mathfrak{b}_N \triangleleft \Lambda_\infty$ open ideal, $\bigcap_N \mathfrak{b}_N = (0)$

$$\mathfrak{b}_N \supset c_M, \ \forall M \geq N$$

$$\begin{tikzcd} \mathfrak{b}_N \arrow[r, phantom, "\supset"] & c_M \\ ((1+T_i)^{\ell^N}-1) \arrow[u, phantom, "\supset"] \arrow[ur, phantom, "\cap"] & \end{tikzcd}$$

$\mathfrak{b}_N \supset \mathfrak{b}_{N+1}$. $\quad \Lambda_\infty \simeq \mathcal{O}[[A_{v,i,j}]] / (A_{v_0,1,1}) [[T_1, \dots, T_r]]$

$$c_N = \left((1+T_i)^{\ell^{N_i}} - 1 \right)$$

$$N_i = \mathrm{ord}_\ell (q_v - 1) \geqslant N$$

Choose $d_{N,1} \triangleleft R^{\mathrm{univ}}_{\phi,1}$ open ideal

$d_{N,1} \supset d_{N+1,1}$, $\bigcap_N d_{N,1} = (0)$.

$$\mathrm{Ann}_{R^{\mathrm{univ}}_{\phi,1}} \left(S_{\phi,1} / \mathfrak{b}_N \right) \supset d_{N,1} \supset \mathfrak{b}_N R^{\mathrm{univ}}_{\phi,1} \quad (*)$$

also choose $d_{N,\chi} \triangleleft R^{\mathrm{univ}}_{\phi,\chi}$ same properties.

Let $e_{N,1} = d_{N,1} \cap$ (preimage of $d_{N,\chi}$ mod λ)

$e_{N,\chi} = d_{N,\chi} \cap$ (preimage of $d_{N,1}$ mod λ).

Still satisfy $(*)$

$e_{N,1}$ mod λ = $e_{N,\chi}$ mod λ.

Set $R_{M,N,\chi} = \mathrm{Im} \left(R_{Q_M,\chi}^{\square} \longrightarrow R^{\mathrm{univ}}_{\phi,\chi} / e_{N,\chi} \oplus \mathrm{End}_{\Lambda_\infty} \left(S^{\square}_{Q_M,\chi} / \mathfrak{b}_N \right) \right)$

for $M \geqslant N$

If we fix N, then as M varies over $\mathbb{Z}_{\geqslant N}$,

$\# R^{\mathrm{univ}}_{\phi,\chi} / e_{N,\chi}$, $\quad \# S^{\square}_{Q_N,\chi} / \mathfrak{b}_N$, $\quad \# S_{\phi,\chi} / \mathfrak{b}_N$, $\quad \# R_{M,N,\chi}$

are bounded indep. of M.

$\forall M \geq N$

$$\begin{tikzcd}
 & \Lambda_\infty \rhd \mathfrak{a}_\infty \arrow[d] & & \\
R^{loc}_{x,\infty} \arrow[r, two heads] & R_{M,N,x} \arrow[r, phantom, "\curvearrowright"] \arrow[d] & S^{\square}_{\mathfrak{a}_{M},x}/\mathfrak{b}_N \arrow[d] & - \text{ finite free over } \Lambda_\infty/\mathfrak{b}_N \\
 & R^{univ}_{\phi,x}/e_{N,x} \arrow[r, phantom, "\curvearrowright"] & S_{\phi,x}/\mathfrak{b}_N & \Big\} \text{ mod out by } \mathfrak{a}_\infty + \mathfrak{b}_N
\end{tikzcd}$$

for $x = 1$ and other x, isomorphic mod λ.

for fixed N, only finitely many choices of 2 diagrams for 1 and x + comparisons mod λ.

look at diagrams for $(1, M)$

one diagram arises for ∞'ly many M, say diag for $(1, M_1)$

look at diagrams for $(2, M)$ s.t. $\text{diag}(1, M) \simeq \text{diag}(1, M_1)$

one diagram $\text{diag}(2, M_2)$ arises for ∞'ly many other M's.

etc.

$\forall N \in \mathbb{Z}_{\geq 1}$, we have a diagram s.t. $\text{diag}(N, M_N)$ reduces mod $\mathfrak{b}_{N-1}$ and $e_{N-1,x}$ to the diagram for $(N-1, M_{N-1})$

Take $\varprojlim_N$

$$\begin{tikzcd}
 & {} \arrow[dd] & \mathfrak{a}_\infty \lhd \Lambda_\infty \arrow[d] & & \\
R^{loc}_{1,\infty} \arrow[rr, two heads] & & R_{\infty,1} \arrow[r, phantom, "\curvearrowright"] \arrow[d] & S_{\infty,1} \arrow[d] & \text{finite free}/\Lambda_\infty \arrow[l] \\
 & (0) \arrow[r, phantom, "\lhd"] & R^{univ}_{\phi,1} \arrow[r, phantom, "\curvearrowright"] & S_{\phi,1} \cong S_{\infty,1}/\mathfrak{a}_\infty &
\end{tikzcd}$$

$\nearrow$ dim $4|R| + n$

Same for x + isomorphisms of 2 diagrams mod λ

$$\begin{tikzcd} & \Lambda_\infty \arrow[dl, dashed] \arrow[d] & \text{- powerseries } /\mathcal{O} \text{ of dim } 4|R|+r \\ R^{loc}_{\chi,\infty} \arrow[r, two heads] & R_{\infty,\chi} \arrow[r, phantom, "\curvearrowright"] & S_{\infty,\chi} \end{tikzcd}$$

$$\operatorname{depth}\left(m_{\Lambda_\infty}, S_{\infty,\chi}\right) \geq 4|R|+r$$

$m_{R^{loc}_{\chi,\infty}}$

$\operatorname{Spec} R^{loc}_{\chi,\infty}$ irred. (as $\chi_v \neq 1, \forall v$)

$\operatorname{Supp}_{R^{loc}_{\chi,\infty}}\left(S_{\infty,\chi}\right) = \operatorname{Spec} R^{loc}_{\chi,\infty} \leftarrow \dim = 4|R|+r$.

$\Rightarrow \operatorname{Supp}_{R^{loc}_{\chi,\infty}/\lambda}\left(S_{\infty,\chi}/\lambda\right) = \operatorname{Spec}\left(R^{loc}_{\chi,\infty}/\lambda\right)$

$\therefore \operatorname{Supp}_{R^{loc}_{1,\infty}/\lambda}\left(S_{\infty,1}/\lambda\right) = \operatorname{Spec}\left(R^{loc}_{1,\infty}/\lambda\right)$.

$R^{loc}_{1,\infty}$ has property that compts in char. 0 + char l are in bijection under reduction.

$\therefore \operatorname{Supp}_{R^{loc}_{1,\infty}}\left(S_{\infty,1}\right) = \operatorname{Spec}\left(R^{loc}_{1,\infty}\right)$

$\therefore \operatorname{Supp}_{R^{loc}_{1,\infty}/a_\infty}\left(S_{\phi,1}\right) = \operatorname{Spec}\left(R^{loc}_{1,\infty}/a_\infty\right)$.

$R^{loc}_{1,\infty}/a_\infty \twoheadrightarrow R^{univ}_{\phi,1} \curvearrowright S_{\phi,1} \quad \Rightarrow \quad \operatorname{Supp}_{R^{univ}_{\phi,1}}\left(S_{\phi,1}\right) = \operatorname{Spec} R^{univ}_{\phi,1} \; \left(= \operatorname{Spec} R^{loc}_{1,\infty}/a_\infty\right)$

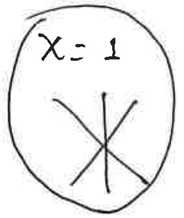
$\therefore R^{loc}_{1,\infty}/a_\infty \twoheadrightarrow R^{univ}_{\phi,1} \twoheadrightarrow \mathbb{T}_{\phi,1}$ have nilpotent kernels.

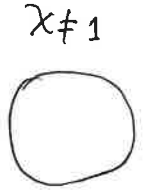
Lecture 8. $r_1, r_2 : G_F \longrightarrow GL_2(\mathcal{O})$

$\bar{r}_1 \simeq \bar{r}_2$, r_1 automorphic

$r_1|_{G_{F_v}}$ & $r_2|_{G_{F_v}}$ lie on the same irred. comp. of $R^{\square}_{\bar{r}_2|G_{F_v}}$, then r_2 also automorphic.

$v \nmid \ell$, we can switch components of $R^{\square}_{\bar{r}_1}|_{G_{F_v}}$





$v \mid \ell$ serious problem : avoided because

Fontaine - Laffaille deformations $\downarrow$ local lifting ring smooth

- $\ell \nmid$ level
- wt - small $\lesssim \ell$
- ℓ unram. in F

$v \mid \ell$, If $F_v = \mathbb{Q}_\ell$, $G = GL_2$

- Kisin : Breuil - Mezard
- Emerton : completed cohomology

$\uparrow$ Pan

depend on ℓ-adic local Langlands

What else can go wrong :

1) $\dim R^{loc}_{\chi,\infty} < \dim \Lambda_\infty$: Calegari - Geraghty

automorphic forms occur in > 1 degree in cohomology.

derived approach

[2) Cebotarev argument to find Q_N can fail.

- Skinner - Wiles / Pan $\leftarrow$ rarely works

$\pm\ell \equiv 1 \pmod 4$, $\mathbb{Q}(\sqrt{\pm\ell}) \subset \mathbb{Q}(\zeta_\ell)$

$\bar{\chi} : G_{\mathbb{Q}(\sqrt{\pm\ell})} \longrightarrow \mathbb{F}^\times$

$\bar{\chi}' = \bar{\chi} \circ$ conjugation by some $\tau \in G_{\mathbb{Q}} - G_{\mathbb{Q}(\sqrt{\pm\ell})}$

$\bar{\chi} \neq \bar{\chi}'$, $\bar{r} = \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{\pm\ell})}}^{G_{\mathbb{Q}}} \bar{\chi}$

F totally real, even degree

$\bar{r} : G_F \longrightarrow GL_2(\mathbb{F})$ modular, irreducible

$r : G_F \longrightarrow GL_2(\mathcal{O})$ crystalline lift (any weight)

$\Rightarrow r$ is modular

- any wt - allow induced from quadratic subfield of $\mathbb{Q}(\zeta_\ell)$

wanted to find $\sigma \in G_{\mathbb{Q}}$ s.t. 1) $\sigma \in G_{\mathbb{Q}(\zeta_\ell)} \Rightarrow \delta(\sigma) = 1$. eig. val of $(\mathrm{ad}^\circ \bar{r})(\sigma)$ are 1, $\bar{x}/\bar{x}'(\sigma)$, $\bar{x}'/\bar{x}(\sigma)$

2) $\bar{r}(\sigma)$ has distinct eigenvalues

$$\mathrm{ad}^\circ \bar{r} = \left(\mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{\pm \ell})}}^{G_{\mathbb{Q}}} \bar{x}/\bar{x}'\right) \oplus \delta$$

3) σ w/ eig. val. 1 on $\mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{\pm\ell})}}^{G_{\mathbb{Q}}} (\bar{x}/\bar{x}')$

$$\delta : \mathrm{Gal}(\mathbb{Q}(\sqrt{\pm \ell})|\mathbb{Q}) \twoheadrightarrow \{\pm 1\}$$

$$\Downarrow$$

$$\underbrace{\bar{x}/\bar{x}'(\sigma) = 1, \qquad \bar{x}(\sigma) = \bar{x}'(\sigma)}_{\text{impossible.}}$$

$$\longrightarrow \varepsilon$$

1) Pseudo-representations

R a top. ring, $2 \in R^\times$ Γ a top. f.g. profinite gp

Def. A (cts) pseudo-rep of Γ valued in R is a cts func. $T : \Gamma \to R$ s.t.

1) $T(1) = 2$, 2) $T(\sigma\tau) = T(\tau\sigma)$

3) $T(\sigma\tau\rho) + T(\sigma\rho\tau) - T(\sigma)T(\tau\rho) - T(\tau)T(\sigma\rho) - T(\rho)T(\sigma\tau) + T(\sigma)T(\tau)T(\rho) = 0$

$\forall\ \sigma, \tau, \rho \in \Gamma$

Lemma. If $r : \Gamma \to GL_2(R)$ is a cts rep, then $\mathrm{tr}\, r : \Gamma \to R$ is a pseudo-rep.

Pf. $A, B, C \in M_{2\times 2}(R)$. $\det A = \frac{1}{2}\left[(\mathrm{tr} A)^2 - \mathrm{tr}(A^2)\right]$

$\lambda A + \mu B$. $(\lambda A + \mu B)^2 - \mathrm{tr}(\lambda A + \mu B)(\lambda A + \mu B) + \frac{1}{2}\left[(\mathrm{tr}(\lambda A + \mu B))^2 - \mathrm{tr}((\lambda A + \mu B)^2)\right] = 0$

$R[\lambda, \mu]$

Coeff. of $\lambda\mu$: $AB + BA - \mathrm{tr}(A)B - \mathrm{tr}(B)A + \frac{1}{2}\left(2(\mathrm{tr} A)(\mathrm{tr} B) - \mathrm{tr}\, AB - \mathrm{tr}\, BA\right) = 0$

$= AB + BA - (\mathrm{tr} A)B - (\mathrm{tr} B)A - \mathrm{tr}(AB) + (\mathrm{tr} A)(\mathrm{tr} B)$

$\Rightarrow ABC + BAC - (\mathrm{tr} A)(BC) - (\mathrm{tr} B)(AC) - \mathrm{tr}(AB)C + (\mathrm{tr} A)(\mathrm{tr} B)C = 0$

$\Rightarrow$ take trace.

Facts (harder)

1) If R is an alg. closed field and if $T : \Gamma \to R$ is a (2-dim'l) pseudo-rep, then $\exists$ a semisimple rep $r : \Gamma \to GL_2(R)$ w/ $T = \mathrm{tr}\, r$.

2) Given Γ, $\exists S \subset \Gamma$ a finite subset s.t. any pseudo-rep $T : \Gamma \to R$ is determined by its values on S.

top. f.g.

$L|\mathbb{Q}_\ell$ finite, $\mathcal{O}/\lambda \simeq \mathbb{F}$

$\mathcal{F} : \begin{pmatrix} \text{CArt. } L\ \mathcal{O}\text{-algs} \\ \text{res. field } \mathbb{F} \end{pmatrix} \longrightarrow \underline{\text{Sets}}$

$\mathcal{F}$ is pro-representable by a complete noeth local $\mathcal{O}$-alg R iff

1) $\mathcal{F}(\mathbb{F}) = \{*\}$

2) $\forall$

$$\begin{tikzcd} & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

$\mathcal{F}(A \times_B C) \xrightarrow{\sim} \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C)$] Sufficient to treat certain special cases

3) $\mathcal{F}(\mathbb{F}[\varepsilon]/\varepsilon^2)$ is finite. (for noetherian)

$\overline{T} : \Gamma \to \mathbb{F}$ pseudo-rep

$\mathcal{F} : R \longrightarrow$ lifts of $\overline{T}$ to a pseudo rep $T : \Gamma \to R$

1) + 2) clear. 3) also true: $S \subset \Gamma$, $|S| < \infty$ s.t. any pseudo-rep is determined by its values on S.

$$\# \mathcal{F}\left(\mathbb{F}[\varepsilon]/(\varepsilon^2)\right) \leq (\# \mathbb{F})^{\# S} < \infty$$

rep't'd by $R^{ps}_{\overline{T}}$.

$(\det T)(\sigma) = \frac{1}{2}\left(T(\sigma)^2 - T(\sigma^2)\right)$, $\det T : \Gamma \to R^\times$ homomorphism

$\chi : \Gamma \to \mathcal{O}^\times$ cts char. $\quad R^{ps}_{\overline{T},\chi} \leftarrow$ parametrizes pseudo-reps w/ determinant χ.

1) $\bar{r} : \Gamma \to GL_2(\mathbb{F})$

universal lifting ring

$$\begin{tikzcd} R^{ps}_{tr\,\bar{r}} \arrow[r] & R^{univ}_{\bar{r}} \arrow[r] & R^{\square}_{\bar{r}} \end{tikzcd}$$

image is the subring top. generated by $tr\, r^{univ}(\sigma)$ $\forall \sigma \in \Gamma$

If the centralizer of $\bar{r}$ in $GL_2(\mathbb{F})$ is $\mathbb{F}^\times$ $\Rightarrow$ $R^{\square}_{\bar{r}} \simeq R^{univ}_{\bar{r}}[[A_1, A_2, A_3]]$

$r_1 \sim r_2 \Leftrightarrow r_1 = A r_2 A^{-1}$

some $A \in \ker(GL_2(R) \to GL_2(\mathbb{F}))$.

2) If $\bar{r}$ is absolutely irred., then $R^{ps}_{tr\,\bar{r}} \xrightarrow{\sim} R^{univ}_{\bar{r}}$ (Nyssen)

eg. (Bellaiche) $F | \mathbb{Q}_\ell$ finite, $\zeta_\ell \notin F$.

$\overline{\chi_1}, \overline{\chi_2} : G_F \to \mathbb{F}^\times$

$\overline{\chi_1}/\overline{\chi_2} \neq 1, \bar{\varepsilon}_\ell^{\pm 1}$.

$\overline{T} = \overline{\chi_1} + \overline{\chi_2} = tr\,\bar{r}$

$\bar{r} = \begin{pmatrix} \overline{\chi_1} & * \\ 0 & \overline{\chi_2} \end{pmatrix}$

* non-trivial

$\dim m_R/(\lambda, m_R^2\rangle$ $\quad$ for $R = R^{ps}_{\overline{T}}, R^{univ}_{\bar{r}}, R^{\square}_{\bar{r}}$

$R^{ps}_{\bar{T}}$ dim = $\left(1+[F:\mathbb{Q}_\ell]\right)^2+1$

$R^{univ}_{\bar{r}}$ dim = $1+4[F:\mathbb{Q}_\ell]$

$R^{\square}_{\bar{r}}$ dim = $4\left(1+[F:\mathbb{Q}_\ell]\right)$

* * *

mod ℓ rep'n of $GL_2(\mathbb{Q}_\ell)$

$(A, \mathfrak{m})$ CNL $\mathcal{O}$-alg., res. field $\mathbb{F}$.

$G = GL_2(\mathbb{Q}_\ell)$ $\quad [G = GL_2(\mathbb{Q}_\ell)^n]$

$Z = \mathbb{Q}_\ell^\times$ $\quad [Z = (\mathbb{Q}_\ell^\times)^n]$

$G_0 = GL_2(\mathbb{Z}_\ell)$ $\quad [G_0 = GL_2(\mathbb{Z}_\ell)^n]$

$B = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{Q}_\ell)\right\}$ $\quad [B = (\ )^n]$

$\mathrm{Mod}^{sm}_G(A)$ = cat. of smooth reps of G on A-modules

i.e. $\forall m \in M$, $\exists r$ s.t. $\mathfrak{m}^r m = (0)$.

$\mathrm{Stab}_G(m)$ open

$\mathrm{Mod}^{\ell fin}_G(A)$ full subcat. of those M s.t. $\forall m \in M$, $\exists m \in N \subset M$

$\uparrow$ finite length subobject $\rightarrow$ in $\mathrm{Mod}^{sm}_G(A)$.

$\cap$

$\mathrm{Mod}^{sm}_G(A)$

$\mathrm{Mod}^{sm}_{G,1}(A) \subset \mathrm{Mod}^{sm}_G(A)$ full subcat. where Z acts trivially

(or Z acts via η for any cts char $\eta: Z \to A^\times$, $\mathrm{Mod}^{sm}_{G,\eta}(A)$).

$\mathrm{Mod}^{\ell fin}_{G,1}(A) = \cdots$

FACTS 1) all abelian cats

2) admit exact small inductive limits

3) have a single generator X

i.e. $M \overset{f}{\underset{g}{\rightrightarrows}} N$ $\quad f \neq g$.

$\exists\, h : X \to M$, $\quad f \circ h \neq g \circ h$.

$\bigoplus_{r, U} \text{Sm-Ind}_U^G (A/m^r)$ or Sm-Ind^G_{UZ}

$\uparrow$ open subgp of G

Lecture 9 $GL_2(\mathbb{Q}_\ell)$, $\quad GL_2(\mathbb{Q}_\ell)^n$

$L | \mathbb{Q}_\ell$ finite, $\mathcal{O}$, $\mathcal{O}/\lambda = \mathbb{F}$

A CNL $\mathcal{O}$-alg res. field $\mathbb{F}$

m = max. ideal of A

$G = GL_2(\mathbb{Q}_\ell)$ or $GL_2(\mathbb{Q}_\ell)^n$

$Z = \mathbb{Q}_\ell^\times$ or $(\mathbb{Q}_\ell^\times)^n$

$G_0 = GL_2(\mathbb{Z}_\ell)$ or $GL_2(\mathbb{Z}_\ell)^n$

$\mathrm{Mod}^{sm}_G(A) \supset \mathrm{Mod}^{sm}_{G,1}(A)$ $\quad G \curvearrowright M / A$ $\quad \forall m \in M$, $\exists\, U \subset G$ open, $Um = m$.

$\cup \qquad \cup$ $\qquad$ and $\exists\, r$ s.t. $m^r m = (0)$.

$\mathrm{Mod}^{\ell fin}_G(A) \supset \mathrm{Mod}^{\ell fin}_{G,1}(A)$ $\quad \forall m \in M$, $\exists\, m \in N \subset M$, N finite length

- abelian cats
- exact-inductive limits
- $\exists$ single generator
- $\exists$ a set of finite length generators $\leftarrow$ for ℓ fin cases

$\therefore$ has injective envelopes

$\forall M$, $\exists\, I$ injective, $M \hookrightarrow I$

and if $\underset{\neq 0}{N} \subset I$, then $N \cap M \neq 0$.

loc. finite length cases

π, σ are irreds

$\pi \sim \sigma$ iff $\pi = \pi_0, \pi_1, \dots, \pi_r \simeq \sigma$ irreds

s.t. $\forall i$, either $\pi_{i+1} \simeq \pi_i$ or $\mathrm{Ext}^1(\pi_i, \pi_{i+1}) \neq 0$ or $\mathrm{Ext}^1(\pi_{i+1}, \pi_i) \neq 0$.

$\sim$ class is called a **block**. $\mathcal{B}$

$\mathrm{Mod}^{l\,fin}_{G,(1)}(A)_{\mathcal{B}}$: objects all whose irred subqts in $\mathcal{B}$.

$\uparrow$ cent. char.

$$\mathrm{Mod}^{l\,fin}_{G,(1)}(A) \simeq \bigoplus_{\mathcal{B}} \mathrm{Mod}^{l\,fin}_{G,(1)}(A)_{\mathcal{B}}.$$

$\mathrm{Mod}^{pro\,aug}_{G}(A)$ = cat. of profinite A-mods M together w/ an action of $A[G]$ s.t. for some (hence every) open cpt subgp $U \subset G$, the $A[U]$ action comes w/ an ext'n to a cts $A[[U]]$-action.

$\exists$ anti equiv.

$$\begin{array}{ccc} \mathrm{Mod}^{sm}_G(A) & \longleftrightarrow & \mathrm{Mod}^{pro\text{-}aug}_G(A) \\ M & \longmapsto & \mathrm{Hom}_{\mathcal{O}}(M, L/\mathcal{O}) \\ \mathrm{Hom}^{cts}_{\mathcal{O}}(N, L/\mathcal{O}) & \longleftarrow\!\shortmid & N \end{array}$$

$$\mathrm{Mod}^{l\,fin}_G(A) \longleftrightarrow \mathcal{C} \qquad \mathcal{C} = \prod_{\mathcal{B}} \mathcal{C}_{\mathcal{B}}$$

$$\mathrm{Mod}^{l\,fin}_{G,1}(A) \longleftrightarrow \mathcal{C}_1 \qquad \mathcal{C}_1 = \prod_{\mathcal{B}} \mathcal{C}_{\mathcal{B},1}$$

Look now at $\mathrm{Mod}^{\mathrm{lfin}}_{G,1}(A)$ or $\mathcal{C}_1$.

FACTS 1) π irred., $\mathrm{End}(\pi)$ are finite / $\mathbb{F}$.

2) blocks are finite

3) If $N \in \mathrm{Mod}^{\text{pro-aug}}_{G,1}(A)$ is finitely generated over $A[[U]]$ for one (hence every) open cpt subgp $U \subset G$, then $N \in \mathcal{C}_1$.

$\mathcal{B} = \{\pi_1, \cdots, \pi_n\}$ a block.

$P_i \twoheadrightarrow \pi_i^\vee$ projective envelope, $P_{\mathcal{B}} = \bigoplus P_i$, $E_{\mathcal{B}} = \mathrm{End}(P_{\mathcal{B}})$

$E_{\mathcal{B}}$ has topology: if $P_{\mathcal{B}} \xrightarrow{\pi_M} M$ ← finite length, $I_M = \{\alpha \in E_{\mathcal{B}} : \pi_M \circ \alpha = 0\}$

↑ basis of open nbhds of 0.

$\mathfrak{m}_{\mathcal{B}} = I_{(P_{\mathcal{B}} \twoheadrightarrow \bigoplus_i \pi_i)}$

$E_{\mathcal{B}}$ is compact,

$\mathfrak{m}_{\mathcal{B}}$ = Jacobson radical of $E_{\mathcal{B}}$

$$E_{\mathcal{B}}/\mathfrak{m}_{\mathcal{B}} \xrightarrow{\sim} \mathrm{End}\left(\bigoplus_i \pi_i\right)$$

$\mathcal{C}_{1,\mathcal{B}} \simeq$ (?) f.g. left $E_{\mathcal{B}}$-mods.

$M \longmapsto \mathrm{Hom}_{\mathcal{C}_1}(P_{\mathcal{B}}, M)$

$P_{\mathcal{B}} \otimes_{E_{\mathcal{B}}} N \longleftarrow\!\shortmid\ N$

Paškunas : $G = GL_2(\mathbb{Q}_\ell) \supset B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Cases for $\mathcal{B}$

$\mathrm{End} = \mathbb{F}[T]$

1) $\mathcal{B} = \{\pi\}$, $\pi = \pi_{r,\eta} = \left(\text{c-ind}_{ZG_0}^{G} \left(\mathrm{Sym}^r(\mathbb{F}^2) \right) / T \right) \otimes (\eta \circ \det)$

ℓ acts trivially

$0 \leq r \leq \ell - 1$

r even

$\eta : \mathbb{Q}_\ell^\times \longrightarrow \mathbb{F}^\times$ w/ $\eta^2 = \omega^{-r}$

$$\begin{aligned} \omega : \mathbb{Q}_\ell^\times &\longrightarrow \mathbb{F}^\times \\ \ell &\longmapsto 1 \\ \mathbb{Z}_\ell^\times \ni a &\longmapsto (a \bmod \ell) \end{aligned}$$

$\chi_1 \times \chi_2 : B \longrightarrow \mathbb{F}^\times$, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$

2) $\mathcal{B} = \{ \mathrm{Ind}_B^G(\chi \times \chi^{-1}), \mathrm{Ind}_B^G(\omega\chi \times \chi\omega^{-1}) \}$ $\chi_i : \mathbb{Q}_\ell^\times \to \mathbb{F}^\times$

nat'l induction

$\chi : \mathbb{Q}_\ell^\times \to \mathbb{F}^\times$ w/ $\chi^2 \neq 1$ and $(\chi\omega)^2 \neq 1$. (similar w/ χ valued in $\mathbb{F}' | \mathbb{F}$ finite)

3) $\mathcal{B} = \{1, \mathrm{Sp}, \mathrm{Ind}_B^G\, \omega \otimes \omega^{-1}\} \otimes \chi$, $\chi^2 = 1$.

Def. $\bar{\rho}_B : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\mathbb{F})$ in cases 1), 2), 3).

semisimple cts

1) $\left(\mathrm{Ind}_{G_{\mathbb{Q}_{\ell^2}}}^{G_{\mathbb{Q}_\ell}} \omega_2^{r+1} \right) \otimes \eta$

ω_2 = 2nd fundamental char.

$$\begin{aligned} \omega_2 : \mathbb{Q}_{\ell^2}^\times &\longrightarrow \mathbb{F}^\times \\ \ell &\longmapsto 1 \\ \mathbb{Z}_{\ell^2}^\times \ni a &\longmapsto a \end{aligned}$$

2) $\bar{r}_B^\vee = \chi \oplus \omega \chi^{-1}$

3) $\bar{r}_B^\vee = \chi \oplus \omega \chi^{-1}$

$\bar{r}_B$ runs over all ss mod ℓ reps of $G_{\mathbb{Q}_\ell}$

w/ simple factors defined over $\mathbb{F}$ and w/ $\det \bar{r}_B = \bar{\varepsilon}_\ell^{-1}$.

Thm (Paškūnas) 1) E_B is finite over $Z(E_B)$

2) $Z(E_B) \simeq R^{ps}_{\mathrm{tr}\, r_B,\, \underset{\substack{\uparrow \\ \det = \varepsilon_\ell^{-1}}}{\varepsilon_\ell^{-1}}}$

extends to $G = GL_2(\mathbb{Q}_\ell)^n$

irreds
if abs irred $= \bigotimes$ irred.

blocks $= \prod$ blocks.

finite. (often not always known to be an isom.)

Thm basically still holds. $\hat{\bigotimes}_i R^{ps}_{\mathrm{tr}\, r_{B_i},\, \varepsilon_\ell^{-1}} \longrightarrow Z(E_B)$

$B = \hat{\prod}_{i=1} B_i$

$R^{ps}_{\mathrm{tr}\, r_B,\, \varepsilon_\ell^{-1}}[\frac{1}{\ell}] \rhd \mathfrak{p}$ max'l ideal $\qquad k(\mathfrak{p}) \,|\, L$ finite.

$r : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\mathcal{O}_{k(\mathfrak{p})})$ crystalline, HT nos $\{w, -1-w\}$

w/ $\mathrm{tr}\, r = T^{univ}_{mod\, \mathfrak{p}}$

$(r \otimes B_{cris})^{G_{\mathbb{Q}_\ell}} \circlearrowleft \phi$ eig. values α, β

Suppose $\alpha/\beta \neq \ell^{\pm 1}$

$$\tilde{V}_p = \left(\mathrm{Sym}^{2w}(k(p)^{\oplus 2}) \otimes \det{}^{w}\right)^{\vee} \otimes \text{sm-}\mathrm{Ind}_B^G(\mu_{\alpha/\ell} \times \mu_\beta)$$

$$\begin{matrix} \cup \\ G \end{matrix}$$

$\mu_\gamma : \mathbb{Q}_\ell^\times/\mathbb{Z}_\ell^\times \longrightarrow \mathbb{F}^\times$

$\ell \longmapsto \gamma$

V_p = universal completion of $\tilde{V}_p$ $\longleftarrow$ $\varprojlim \tilde{V}_p / \text{f.g. } \mathcal{O}_{k(p)}[G]\text{-submods}$

unitary Banach rep. W of G, $\underset{\text{dense}}{W \supset \tilde{V}_p^{\oplus n}}$ $\Rightarrow$ $\exists\ V_p^{\oplus n} \twoheadrightarrow W$.

Thm. V_p is top. irred. + non-trivial + admissible unitary Banach rep of G.

$V_p \supset$ unit ball V_p°

$(V_p^\circ/\lambda')^U$ fin. dim'l / $\mathbb{F}'$

$\forall\ U \subset G$ open cpt subgp.

$M_p = \mathrm{Hom}_{\mathcal{O}}(V_p^\circ, \mathcal{O}) \in \mathcal{C}_1$

$V_p^\circ \otimes_{\mathcal{O}} L/\mathcal{O}$

Thm (Paškunas) The $R^{ps}_{\mathrm{tr}_B, \epsilon_\ell^{-1}}$ action on $\mathrm{Hom}(P_B, M_p)[\frac{1}{\ell}]$ factors through

$$R^{ps}_{\mathrm{tr}_B, \epsilon_\ell^{-1}}[\tfrac{1}{\ell}]/p$$

$\mathrm{Spec}\left(R^{\mathrm{cr}, \{w, -1-w\}, \square}_{\bar{r}|_{G_{\mathbb{Q}_v}}, \epsilon_\ell^{-1}}\right)$ has lots of irred. compts if $w \gg 0$

$v | \ell$

"solution" use $\underline{R^{\square}_{\bar{r}|_{G_{\mathbb{Q}_\ell}}, \epsilon_\ell^{-1}}}$ $\longleftarrow$ $R^{\mathrm{univ}}_{\bar{r}, \epsilon_\ell^{-1}}$ is not expected to correspond to a usual Hecke algo

$\uparrow$ no cond. @ ℓ – need to package all Hecke alg. together

Spec is often irred.
In fact, often a power series ring.

$H^2(G_{\mathbb{Q}_\ell}, (ad^0\bar{\tau})) \simeq H^0(G_{\mathbb{Q}_\ell}, (ad^0\bar{\tau})(1))^\vee$ often zero.

Lecture 10

$$\begin{tikzcd} \mathcal{C}_{1,\mathcal{B}} \arrow[d, phantom, "\wr"] & M \arrow[d, mapsto] & P_{\mathcal{B}} \\ \text{profinite} \\ \text{left } E_{\mathcal{B}}\text{-modules} & \mathrm{Hom}_{\mathcal{C}_{\mathcal{B}}}(P_{\mathcal{B}}, M) & E_{\mathcal{B}} \end{tikzcd}$$

$E_{\mathcal{B}}$ finite $/ Z(E_{\mathcal{B}})$, $G = GL_2(\mathbb{Q}_\ell)$

$$R^{ps}_{tr\,\tau_{\mathcal{B}},\, \varepsilon_\ell^{-1}} \xrightarrow{\sim} Z(E_{\mathcal{B}})$$

$G = GL_2(\mathbb{Q}_\ell)^n$: $\hat{\bigotimes} R^{ps}_{tr\,\tau_{\mathcal{B},v},\, \varepsilon_\ell^{-1}} \xrightarrow[\text{finite}]{} Z(E_{\mathcal{B}})$

$\pi = \otimes \pi_i$

$\mathcal{B} = \prod \mathcal{B}_i$

$P_{\mathcal{B}} = \hat{\otimes} P_{\mathcal{B}_i}$

$E_{\mathcal{B}} = \hat{\otimes} E_{\mathcal{B}_i}$

A, B $\underset{\text{comm.}}{\underset{\uparrow}{R}}$-algebras.

$Z(A) \otimes_R Z(B) \longrightarrow Z(A \otimes_R B)$ not in general an isom.

iso. if A, B free R.

$R^{cr,H}_{\bar{\tau}}$ $\qquad$ $\underline{\ell > 3}$. $\qquad$ $\bar{\tau} : G_{\mathbb{Q}_\ell} \to GL_2(\mathbb{F})$

$R^{\square}_{\bar{\tau}}$ $\qquad$ 1) $\bar{\tau}$ irred. $\qquad$ $R^{\square}_{\bar{\tau}, \varepsilon_\ell^{-1}}$ formally smooth $\qquad$ $H^2(G_{\mathbb{Q}_\ell}, ad^0\bar{\tau}) = (0)$

$$\begin{tikzcd} R^{\square}_{\bar{\tau}, \varepsilon_\ell^{-1}} \arrow[d, phantom, "\wr"] & H^2(G_{\mathbb{Q}_\ell}, ad^0\bar{\tau}) \arrow[d, phantom, "\wr"] \\ \mathcal{O}[[X_1, \dots, X_6]] & H^0(G_{\mathbb{Q}_\ell}, (ad^0\bar{\tau})(1))^\vee \end{tikzcd}$$

2) $\bar{r} = \begin{pmatrix} \chi & * \\ 0 & \chi^{-1}\bar{\varepsilon}_\ell^{-1} \end{pmatrix}$ non-split : $\chi^2 \neq \bar{\varepsilon}_\ell^{-2}$

formally smooth, $\mathcal{O}[[X_1, \dots, X_6]]$.

3) $\bar{r} = \chi \oplus \chi^{-1}\bar{\varepsilon}_\ell^{-1}$, $\chi^2 \neq \bar{\varepsilon}_\ell^{-2}$ nor 1 : $\mathcal{O}[[X_1, \dots, X_6]]$.

2') $\bar{r} = \begin{pmatrix} \chi\bar{\varepsilon}_\ell^{-1} & * \\ 0 & \chi^{-1} \end{pmatrix}$ non-split, $\chi^2 = 1$. : $\mathcal{O}[[A_1, A_2, A_3, x, y, z, w]]/(xy - zw)$

3') $\bar{r} = \chi \oplus \chi^{-1}\bar{\varepsilon}_\ell^{-1}$, $\chi^2 = 1$, $\longrightarrow$ $\mathcal{O}[[A_1, A_2, A_3, x, y, z, w]]/(xy - zw)$

$F|\mathbb{Q}$ totally real, $[F:\mathbb{Q}]$ even, ℓ split completely in F

$L|\mathbb{Q}_\ell$ finite, $\mathcal{O}/\lambda = \mathbb{F}$, $L \supset \operatorname{Im}\tau$, $\forall\, \tau : F \hookrightarrow \bar{L}$.
$\zeta_\ell \in L$.

$\bar{r} : G_F \longrightarrow GL_2(\mathbb{F})$. Assume $\forall\, \sigma \in G_F$, the eig vals of $\bar{r}(\sigma)$ lie in $\mathbb{F}$.

- unramified away from ℓ.
- R a finite set of primes of F s.t. if $v \in R$, then $q_v = |k(v)| \equiv 1 \ (\ell)$

  $\bar{r}|_{G_{F_v}} = 1$

$\chi = \prod_{v \in R} \chi_v$, $\chi_v : k(v)^\times \longrightarrow \mathcal{O}^\times$
ℓ power order

$v \in R$, $R^\square_{\chi_v}$ universal lifting ring of $\bar{r}|_{G_{F_v}}$ for lifts r w/ $\det r = \varepsilon_\ell^{-1}$

and for $\sigma \in I_{F_v}$, $\operatorname{tr} r(\sigma) = \chi_v(\sigma) + \chi_v(\sigma)^{-1}$.

also think $\chi_v : I_{F_v} \twoheadrightarrow k(v)^\times \longrightarrow \mathcal{O}^\times$

$v | \ell$. $R_v^\square$ ———— for lifts of $\det = \varepsilon_\ell^{-1}$

$R_\chi^{loc} = \hat{\bigotimes}_{v|\ell} R_v^\square \ \hat{\otimes} \ \hat{\bigotimes}_{v \in R} R_{\chi_v}^\square$ has dim. $6 \# \{v|\ell\} + 3 \# R + 1$.

Q: finite set of primes of F: $q_v = |k(v)| \equiv 1 \ (\ell)$

H = max ℓ-power order qt of $k(v)^\times$

$\bar{r}(\mathrm{Frob}_v)$ has distinct eig vals $\alpha_v \neq \beta_v \in \mathbb{F}$

$H_Q = \prod_{v \in Q} H_v$

$\left(\therefore Q \cap (R \cup \{v|\ell\}) = \phi \right)$.

$R^{univ}_{Q,\chi}$ ← universal deformation ring for lifts r of $\bar{r}$ s.t.

- $\det r = \varepsilon_\ell^{-1}$
- $v \in R$ and $\sigma \in I_{F_v}$, then $\mathrm{tr}\,\sigma = \chi_v(\sigma) + \chi_v(\sigma)^{-1}$.

$R^\square_{Q,\chi}$ "framed" @ $R \cup \{v|\ell\}$

↖ power series ring over $R^{univ}_{Q,\chi}$ in $4 \left| R \cup \{v|\ell\} \right| - 1$.

$v \in Q$, $r^{univ}|_{G_{F_v}} \sim \Psi_{\alpha_v} \oplus \Psi_{\beta_v}$, $(\Psi_{\alpha_v} \bmod \mathfrak{m})(\mathrm{Frob}_v) = \alpha_v$

$\Psi_{\alpha_v}|_{I_{F_v}} : H_v \longrightarrow R^{univ}_{Q,\chi}$

$\mathfrak{a}_Q \triangleleft \mathcal{O}[H_Q] \longrightarrow R^{univ}_{Q,\chi}$ $\qquad R^{univ}_{Q,\chi} / \mathfrak{a}_Q = R^{univ}_{\phi,\chi}$

augmentation

$$\Lambda_Q = \mathcal{O}[H_Q][[A_{v,i,j} : \begin{smallmatrix} v \in R \cup \{v|\ell\} \\ i,j = 1,2 \end{smallmatrix}]] / (A_{v_0 11})$$

$$\begin{tikzcd}
\Lambda_Q \\
\tilde{\mathfrak{a}}_Q = (A_{v,i,j},\ h-1 : h \in H_Q) \arrow[u, phantom, "\triangleleft"] \arrow[d] \\
R^\square_{Q,x}
\end{tikzcd}$$

$$R^\square_{Q,x} \underset{\substack{\uparrow \\ \text{Non-can}}}{\simeq} R^{univ}_{Q,x} \otimes_{\mathcal{O}[H_Q]} \Lambda_Q$$

$$R^\square_{Q,x} / \tilde{\mathfrak{a}}_Q \xrightarrow{\sim} R_{\phi,x}$$

$$\begin{tikzcd}
R^{loc}_x \arrow[r] & R^\square_{Q,x} & \\
 & \Lambda_Q \arrow[d] & \dim 4|\{v|\ell\}| + 4|R| \arrow[l] \\
R^{loc}_x[[x_1, \dots, x_s]] \arrow[r, two heads] & R^\square_{Q,x} & \\
\dim 6|\{v|\ell\}| + 4|R| + |Q| + \dim H^1_{L_Q^\perp} \arrow[u] & &
\end{tikzcd}$$

$$s = |R| + |Q| - 1 + \dim H^1_{L_Q^\perp}(G_{F,\{v|\ell\} \cup R}\ ;\ (ad^0 \bar{r})(1))$$

↑ no condition if $v|\ell$ or $v \in R$

trivial if $v \in Q$.

D/F quat. alg. ram. @ exactly ∞ places $\quad D \otimes_{\mathbb{Q}} \mathbb{A}^\infty \simeq M_{2\times 2}(\mathbb{A}^\infty_F)$

$$U^\ell_Q \subset GL_2(\mathbb{A}^{\infty,\ell}_F)$$

$$\parallel$$

$$\prod_{v \nmid \ell} U_{Q,v} \qquad U_{Q,v} = \begin{cases} GL_2(\mathcal{O}_{F_v}) & \text{if } v \notin Q \cup R \cup \{v|\ell\} \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F_v}) : v | c \right\} & \text{if } v \in R \end{cases}$$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F_v}) : \begin{array}{l} v \mid c, \\ a/d \bmod v \text{ has order prime to } \ell \end{array} \right\} \text{ for } v \in Q$$

$$\chi : U_Q^\ell \longrightarrow \mathcal{O}^\times$$

$$\prod_v \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \mapsto \prod_{v \in R} \chi_v(\underset{\bmod v}{a_v/d_v})$$

$$U_\ell \subset GL_2(\mathcal{O}_{F,\ell}) \text{ any open compact}$$

$$\begin{tikzcd} GL_2(\mathcal{O}_{F,\ell}) \arrow[d, phantom, "\shortparallel"] \\ \prod_{v \mid \ell} GL_2(\mathcal{O}_{F_v}) \end{tikzcd}$$

For A an $\mathcal{O}$-mod. $S(U_Q^\ell U_\ell, A)_\chi = \{ \varphi : D^\times \backslash GL_2(\mathbb{A}_F^\infty) / (\mathbb{A}_F^\infty)^\times \longrightarrow A :$

$\varphi(gu) = \chi(u)\varphi(g), \ \forall u \in U_Q^\ell U_\ell \}$

$$\simeq \bigoplus_{\underbrace{g \in D^\times \backslash GL_2(\mathbb{A}_F^\infty)/(\mathbb{A}_F^\times) U_Q^\ell U_\ell}_{\text{finite}}} A \qquad \begin{array}{c} \varphi \\ \rotatebox{-90}{$\mapsto$} \\ (\varphi(g)) \end{array}$$

$$\left(g^{-1} D^\times g (\mathbb{A}_F^\infty)^\times \cap U_Q^\ell U_\ell \right) \Big/ (\mathbb{A}_F^\infty)^\times \cap U_Q^\ell U_\ell$$

$$\uparrow$$

finite group order prime to ℓ $\Big|$ $\Leftarrow$ $\ell > 3$, ℓ unr. in F.

$S^{sm}(U_Q^\ell, A)_\chi = \{ \varphi : D^\times \backslash GL_2(\mathbb{A}_F^\infty)/(\mathbb{A}_F^\infty)^\times \longrightarrow A :$

φ locally const, $\varphi(gu) = \chi(u)\varphi(g)$, $\forall u \in U_Q^\ell \}$

e.g. $A = \mathcal{O}/\lambda^n$, $L/\mathcal{O}$. $= \varinjlim_{U_\ell} S(U_Q^\ell U_\ell, A)_\chi$

$$S^{cts}(U^\ell_Q, \underset{\substack{\cap \\ L}}{\mathcal{O}})_x = \{ \text{——}, \ \varphi \text{ cts}, \ \text{——} \}$$

$\uparrow$ instead of locally constant.

"completed cohomology"

N.B. $S^{cts}(U^\ell_Q, L)_x = S^{cts}(U^\ell_Q, \mathcal{O})_x[\frac{1}{\ell}]$

$$(*) \quad \overset{\curvearrowleft GL_2(F_\ell)}{S^{cts}(U^\ell_Q, \mathcal{O})_x} \simeq \overset{\curvearrowleft GL_2(F_\ell)}{\mathrm{Hom}_{\mathcal{O}}(L/\mathcal{O}, S^{sm}(U^\ell_Q, L/\mathcal{O})_x)} \in \mathrm{Mod}^{sm}_{GL_2(F_\ell),1}(\mathcal{O})$$

$$S^{sm}(U^\ell_Q, L/\mathcal{O})_x = \bigoplus_{g \in D^\times \backslash GL_2(\mathbb{A}_F^\infty) / (\mathbb{A}_F^\infty)^\times U^\ell_Q GL_2(\mathcal{O}_{F,\ell})} C^{sm}(PGL_2(\mathcal{O}_{F,\ell}), L/\mathcal{O})(x)^{\Sigma_g}$$

$$\varphi \longmapsto [u_\ell \mapsto \varphi(g u_\ell)]$$

$$\Sigma_g = g D^\times g^{-1} (\mathbb{A}_F^\infty)^\times \cap U^\ell_Q GL_2(\mathcal{O}_{F,\ell}) \Big/ (\mathbb{A}_F^\infty)^\times \cap U^\ell_Q GL_2(\mathcal{O}_{F,\ell}).$$

order prime to ℓ.

$$S^{cts}(U^\ell_Q, \mathcal{O})_x = \bigoplus_{g \text{ as before}} C^{cts}(PGL_2(\mathcal{O}_{F,\ell}), \mathcal{O})(x)^{\Sigma_g}$$

$$(*) \Leftarrow C^{cts}(\underbrace{PGL_2(\mathcal{O}_{F,\ell})}_{\text{true for any profinite group } \Gamma}, \mathcal{O}) \simeq \mathrm{Hom}_{\mathcal{O}}(L/\mathcal{O}, C^{sm}(PGL_2(\mathcal{O}_{F,\ell}), L/\mathcal{O})).$$

$(\pi) = \lambda$

$$[\gamma \mapsto f(\pi^{-n})(\gamma)_{\pi^n}] \longleftarrow\!\shortmid f$$

$$C^{cts}(\Gamma, \mathcal{O}) = \varprojlim_n \overset{sm}{C^{cts}}(\Gamma, \mathcal{O}/\lambda^n) = \varprojlim_n \mathrm{Hom}(\lambda^{-n}/\mathcal{O}, C^{sm}(\Gamma, L/\mathcal{O}))$$

$$= \mathrm{Hom}(L/\mathcal{O}, C^{sm}(\Gamma, L/\mathcal{O}))$$

$$M(U^\ell_Q, \mathcal{O})_\chi \overset{\in \mathrm{Mod}^{\text{pro-aug}}_{GL_2(F_\ell),1}(\mathcal{O})}{=} \mathrm{Hom}(S^{sm}(U^\ell_Q, L/\mathcal{O})_\chi, L/\mathcal{O})$$

$\circlearrowleft$

$GL_2(F_\ell)$

completed homology.

s.t. $\forall\, U \subset GL_2(F_\ell)$ the action extends continuously from $\mathcal{O}[U]$ to $\mathcal{O}[[U]]$.

$$= \bigoplus_{g \text{ as before}} \mathrm{Hom}(C^{sm}(PGL_2(\mathcal{O}_{F,\ell}), L/\mathcal{O})(\chi)^{\Sigma_g}, L/\mathcal{O})$$

$$= \bigoplus_{g \text{ as before}} \mathcal{O}[[PGL_2(\mathcal{O}_{F,\ell})]](\chi^{-1})^{\Sigma_g}$$

$$M(U^\ell_Q, \mathcal{O})_\chi \simeq \mathrm{Hom}_\mathcal{O}(S^{cts}(U^\ell_Q, \mathcal{O})_\chi, \mathcal{O})$$

$$S^{cts}(U^\ell_Q, \mathcal{O})_\chi \simeq \mathrm{Hom}^{cts}_\mathcal{O}(M(U^\ell_Q, \mathcal{O})_\chi, \mathcal{O}).$$

$\Uparrow$

Γ profinite.

$$\underset{\wr}{\mathcal{O}[[\Gamma]]} \simeq \mathrm{Hom}_\mathcal{O}(C^{cts}(\Gamma, \mathcal{O}), \mathcal{O})$$

$$\mathrm{Hom}_\mathcal{O}(C^{sm}(\Gamma, L/\mathcal{O}), L/\mathcal{O}) \simeq \mathrm{Hom}_\mathcal{O}(\mathrm{Hom}_\mathcal{O}(L/\mathcal{O}, C^{sm}(\Gamma, L/\mathcal{O})), L/\mathcal{O})$$

$$C^{cts}(\Gamma, \mathcal{O}) \simeq \mathrm{Hom}^{cts}_\mathcal{O}(\mathcal{O}[[\Gamma]], \mathcal{O})$$

Lecture 11. F, D, $D^\times_\infty \backslash F^\times_\infty$ compact. $(D \otimes_\mathbb{Q} \mathbb{A}^\infty)^\times \simeq GL_2(\mathbb{A}^\infty_F)$

$\ell > 3$, $L \mid \mathbb{Q}_\ell$, $\mathcal{O}/\lambda = \mathbb{F}$. $U^\ell_Q \subset GL_2(\mathbb{A}^{\infty,\ell}_F)$, $U^\ell_Q = \prod_v U^\ell_{Q,v}$

$U_\ell \subset GL_2(F_\ell)$ open cpt subgp, sufficiently small. $v \notin R \cup Q$, $U^\ell_{Q,v} = GL_2(\mathcal{O}_{F,v})$.

$$S^{sm}(U^\ell_Q, L/\mathcal{O})_\chi = \{\varphi : D^\times \backslash GL_2(\mathbb{A}^\infty_F)/\mathbb{A}^{\infty,\times}_F \to L/\mathcal{O} :$$

$$\varphi \text{ locally constant}, \varphi(gu) = \chi(u)\varphi(g), \forall u \in U^\ell_Q\}.$$

(*)

$$S^{sm}(U^\ell_Q, L/O)_x = \bigoplus_{g\in D^\times\backslash GL_2(\mathbb{A}_F^\infty)/\mathbb{A}_F^{\infty,\times}U^\ell_Q U_\ell} C^{sm}(U_\ell/U_\ell\cap F_\ell^\times, L/O)$$

$$S^{cts}(U^\ell_Q, O)_x = \{\varphi : D^\times\backslash GL_2(\mathbb{A}_F^\infty)/\mathbb{A}_F^{\infty,\times} \to O : \varphi \text{ cts}, (*)\}$$

$$S^{cts}(U^\ell_Q, O)_x = \bigoplus_{g\in\cdots} \underset{\overset{\uparrow}{\text{cts}}}{C^{cts}}(U_\ell/U_\ell\cap F_\ell^\times ; O)$$

$$\simeq \mathrm{Hom}_O(L/O, S^{sm}(U^\ell_Q, L/O)_x)$$

$$\mathrm{Hom}_O(L/O, C^{sm}(\Gamma, L/O)) \simeq C^{cts}(\Gamma, O) \quad \text{for } \Gamma \text{ profinite.}$$

$$M(U^\ell_Q, O)_x = \mathrm{Hom}_O(S^{sm}(U^\ell_Q, L/O)_x ; L/O)$$

$$M(U^\ell_Q, O)_x \cong \bigoplus_g O[[U_\ell/U_\ell\cap F_\ell^\times]] \simeq \mathrm{Hom}_O(S^{cts}(U^\ell_Q, O)_x, O)$$

$\mathrm{Hom}_O(C^{sm}(\Gamma, L/O), L/O) \simeq O[[\Gamma]]$. " finitely generated $O[[U_\ell/U_\ell\cap F_\ell^\times]]$-module

$\mathrm{Hom}_O(C^{cts}(\Gamma, O), O) \simeq O[[\Gamma]]$ all cts action $PGL_2(\mathbb{F}_\ell)$

also a module $O[[U_\ell]]$ for any open cpt $U_\ell \subset GL_2(F_\ell)$.

$\mathrm{Hom}^{cts}_O(M(U^\ell_Q, O)_x, O) = S(U^\ell_Q, O)_x$. $\mathrm{Hom}^{cts}_O(O[[\Gamma]], O) \simeq C^{cts}(\Gamma, O)$

$\Gamma = \varprojlim \Gamma_i$, Γ_i finite

$$\mathrm{Hom}^{cts}_O(O[[\Gamma]], O) \simeq \varprojlim_n \mathrm{Hom}^{cts}_O(O/\lambda^n[[\Gamma]], O/\lambda^n)$$

$$\simeq \varprojlim_n \varinjlim_i \mathrm{Hom}_O(O/\lambda^n[\Gamma_i], O/\lambda^n)$$

$$\simeq \varprojlim_n \varinjlim_i \mathrm{Map}(\Gamma_i, O/\lambda^n) = \varprojlim_n C^{cts}(\Gamma, O/\lambda^n) \simeq C^{cts}(\Gamma, O)$$

$$\mathrm{Hom}_O(C^{cts}(\Gamma, O), O) = \varprojlim_n \mathrm{Hom}_O(C^{cts}(\Gamma, O/\lambda^n), O/\lambda^n)$$

$$= \varprojlim_n \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_i \mathrm{Map}(\Gamma_i, \mathcal{O}/\lambda^n), \mathcal{O}/\lambda^n \right)$$

$$= \varprojlim_n \varprojlim_i \mathcal{O}/\lambda^n [\Gamma_i] = \mathcal{O}[[\Gamma]].$$

direct def'n

$$S^{cts}(U^\ell_{\mathbb{Q}}, L)_x \simeq S^{cts}(U^\ell_{\mathbb{Q}}, \mathcal{O})_x [1/\ell]$$

$\uparrow$

Banach space (sup norm) $\quad GL_2(\mathbb{F}_\ell)$ is unitary — preserves norm or unit ball.

$$k = (k_\sigma) \in \mathbb{Z}_{\geq 1}^{\mathrm{Hom}(F,L)}$$

$$\text{Set } W_k = \underbrace{\bigotimes_{\sigma \in \mathrm{Hom}(F,L)} \mathrm{Sym}^{2k_\sigma - 2}(L^2) \otimes \det{}^{1-k_\sigma} \overset{\circlearrowleft}{\ } PGL_2(L)}_{PGL_2(F_\ell)}$$

on σ comp, use $\sigma : PGL_2(F_\ell) \to PGL_2(L)$

$$\begin{tikzcd}
\bigoplus_k \mathrm{Hom}_{\substack{GL_2(\mathcal{O}_{F,\ell}) \\ \text{or} \\ U_\ell}}\left(W_k^\vee, S^{cts}(U^\ell_{\mathbb{Q}}, L)_x\right) \otimes W_k^\vee \arrow[d] \\
S^{cts}(U^\ell_{\mathbb{Q}}, L)_x
\end{tikzcd}$$

Lemma This map has dense image.

Lemma 2. X finite type affine scheme $/\mathbb{Z}_\ell \implies L[X] \longrightarrow C^{cts}(X(\mathbb{Z}_\ell), L)$

algebraic reg. func. $\quad$ dense image

e.g. $L[(RS^F_{\mathbb{Q}} PGL_2)_L] \xrightarrow{\text{dense}} C^{cts}(PGL_2(\mathcal{O}_{F,\ell}), L)$ on X defined $/L$

is

$\bigoplus W_k \otimes W_k^\vee$

Pf of L2. prove it Aff^n $\qquad C^{cts}(\mathbb{Z}_\ell^n, L)$

$$\varphi(t) = \sum_{\underline{m} \in \mathbb{Z}^n_{\geq 0}} \overset{\substack{L\\ \Downarrow}}{C_{\underline{m}}} \binom{t_1}{m_1} \cdots \binom{t_n}{m_n}, \quad |c_{\underline{m}}| \to 0$$

$$|\varphi|_\infty = \max |c_{\underline{m}}|$$

$$X \subset \mathrm{Aff}^n, \quad \begin{tikzcd} C^{cts}(\mathbb{Z}_\ell^n, L) \arrow[r, two heads] & C^{cts}(X(\mathbb{Z}_\ell), L) \arrow[l, bend left] \end{tikzcd}$$

$S^{sm}(U^\ell_Q U_\ell, \lambda^{-n}/\mathcal{O})_x$

$\circlearrowleft$

$T_v, \quad v \in R \cup Q \cup \{v | \ell\}$

$\updownarrow \quad GL_2(\mathcal{O}_{F,v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathcal{O}_{F,v})$

$\mathbb{T}(U^\ell_Q U_\ell, \mathcal{O}/\lambda^n)_x \leftarrow$ finite $/\mathcal{O}/\lambda^n$ commutative

$=$ $\mathcal{O}$-subalg. of $\mathrm{End}(S^{sm}(U^\ell_Q U_\ell, \lambda^{-n}/\mathcal{O})_x)$ gen. by T_v for $v \notin R \cup Q \cup \{v|\ell\}$

$n' > n$, $S^{sm}(U^\ell_Q U_\ell, \lambda^{-n}/\mathcal{O})_x \hookrightarrow S^{sm}(U^\ell_Q U_\ell, \lambda^{-n'}/\mathcal{O})_x$

$$\mathbb{T}(U^\ell_Q U_\ell, \mathcal{O}/\lambda^n)_x \twoheadleftarrow \mathbb{T}(U^\ell_Q U_\ell, \mathcal{O}/\lambda^{n'})_x$$

$U'_\ell \subset U_\ell$, $S^{sm}(U^\ell_Q U_\ell, \lambda^{-n}/\mathcal{O})_x \hookrightarrow S^{sm}(U^\ell_Q U'_\ell, \lambda^{-n}/\mathcal{O})_x$

$$\mathbb{T}(U^\ell_Q U_\ell, \mathcal{O}/\lambda^n)_x \twoheadleftarrow \mathbb{T}(U^\ell_Q U'_\ell, \mathcal{O}/\lambda^n)$$

$\mathbb{T}(U^\ell_Q, \mathcal{O})_x := \varprojlim_{n, U_\ell} \mathbb{T}(U^\ell_Q U_\ell, \mathcal{O}/\lambda^n)_x$ profinite

$\circlearrowright$

$S^{sm}(U^\ell_Q, L/\mathcal{O})_x$ $\therefore$ also acts on $S^{cts}(U^\ell_Q, \mathcal{O})_x$, $M(U^\ell_Q, \mathcal{O})_x$

commutes w/ $GL_2(F_\ell)$-actions.

$n' \geq n$

$U'_\ell \triangleleft U_\ell$ $\quad \mathbb{T}(U^\ell_{\mathbb{Q}} U'_\ell, \mathcal{O}/\lambda^{n'})_x \twoheadrightarrow \mathbb{T}(U^\ell_{\mathbb{Q}} U_\ell, \mathcal{O}/\lambda^{n})_x$

↑ Artinian

If U_ℓ is a pro-ℓ-group, this induces a bijection on prime = max'l ideals

Pf $\quad \mathfrak{m}$ a max'l ideal of $S(U^\ell_{\mathbb{Q}} U'_\ell, \lambda^{-n'}/\mathcal{O})_{x,\mathfrak{m}} \neq 0$ ← faithful $\mathbb{T}$

$$\Downarrow$$

$$U_\ell/U'_\ell \curvearrowright S(U^\ell_{\mathbb{Q}} U'_\ell, \lambda^{-n'}/\mathcal{O})_x[\mathfrak{m}] \neq 0$$

ℓ power order

$$\therefore S(U^\ell_{\mathbb{Q}} U'_\ell, \lambda^{-n'}/\mathcal{O})_x[\mathfrak{m}]^{U_\ell/U_{\ell'}} \neq 0$$

$$\parallel$$

$$S(U^\ell_{\mathbb{Q}} U_\ell, \lambda^{-n}/\mathcal{O})_x[\mathfrak{m}] \qquad \therefore \mathfrak{m}\,\mathbb{T}(U^\ell_{\mathbb{Q}} U_\ell, \mathcal{O}/\lambda^n)_x \not\ni 1.$$

Cor. $\mathbb{T}(U^\ell_{\mathbb{Q}}, \mathcal{O})_x$ has finitely many max'l ideals, and $\simeq \prod_{\mathfrak{m} \in \mathrm{Max}(\mathbb{T}(U^\ell_{\mathbb{Q}}, \mathcal{O})_x)} \mathbb{T}(U^\ell_{\mathbb{Q}}, \mathcal{O})_{x,\mathfrak{m}}$ ↑ complete local ring

$$\Uparrow$$

Lemma If A_i are artinian rings, $A_i \twoheadrightarrow A_{i+1}$, $\mathrm{Spec}(A_{i+1}) \xrightarrow{\sim} \mathrm{Spec}(A_i)$

and if $A_\infty = \varprojlim A_i$, then $\mathrm{Max}(A_\infty) \xleftarrow{\sim} \mathrm{Max}(A_i)$, $\forall i$, $A_\infty \simeq \prod_{\mathfrak{m} \in \mathrm{Max}(A_\infty)} A_{\infty,\mathfrak{m}}$ ← complete local ring

↑ may not be noetherian

Pf $\quad A_i = \prod_{\mathfrak{m}} A_{i,\mathfrak{m}} \rightsquigarrow$ reduce to case A_i local, $\forall i$. max. ideal $\mathfrak{m}_i$.

$A_i \twoheadrightarrow A_j$

$\mathfrak{m}_i =$ preimage of $\mathfrak{m}_j$

$A_i^\times =$ preimage of $A_j^\times$

$A_\infty = \varprojlim A_i$

$\mathfrak{m}_\infty = \varprojlim \mathfrak{m}_i$

$A_\infty - \mathfrak{m}_\infty = \varprojlim A_i^\times \subset A_\infty^\times$

A_∞ is local, max'l ideal $\mathfrak{m}_\infty$.

$$\begin{tikzcd}
 & \mathbb{T}(U^\ell_Q U_\ell, \overline{L})_x = \overset{\text{finite}}{\prod} \overline{L} & \\
T^{\mathrm{mod}} : \arrow[d, equal] & G_{F, Q \cup R \cup \{v \mid \ell\}} \arrow[u, "\text{cts pseudorep}"] \arrow[dr, dashed, "\text{by Čebotarev}"'] & \\
\mathrm{tr} \circ \rho^{\mathrm{mod}} & & \mathbb{T}(U^\ell_Q U_\ell, \mathcal{O})_x \arrow[uul, hook, "\text{closed}"'] \arrow[d] \\
 & & \mathbb{T}(U^\ell_Q U_\ell, \mathcal{O}/\lambda^n)
\end{tikzcd}$$

some auts forms

$v \notin R \cup Q \cup \{v \mid \ell\}$,

$T^{\mathrm{mod}}(\mathrm{Frob}_v) = T_v$

$\det T^{\mathrm{mod}} = \varepsilon_\ell^{-1}$

compatible as vary U_ℓ, n

$$T^{\mathrm{mod}} : G_{F, Q \cup R \cup \{v \mid \ell\}} \longrightarrow \mathbb{T}(U^\ell_Q\ , \mathcal{O})_x = \prod_{m} \mathbb{T}(U^\ell_Q, \mathcal{O})_{x,m}$$

$T^{\mathrm{mod}}(\mathrm{Frob}_v) = T_v, \qquad v \notin R \cup Q \cup \{v \mid \ell\}$

$\det T^{\mathrm{mod}} = \varepsilon_\ell^{-1}$.

$$T^{\mathrm{mod}}_m : G_{F, Q \cup R \cup \{v \mid \ell\}} \longrightarrow \mathbb{T}(U^\ell_Q, \mathcal{O})_{x,m}$$

$$\begin{tikzcd}
R^{ps}_{T^{\mathrm{mod}} \bmod m} \arrow[r, two heads] & \mathbb{T}(U^\ell_Q, \mathcal{O})_{x,m} \\
T^{\mathrm{univ}}(\mathrm{Frob}_v) \arrow[r, mapsto] \arrow[u] & T_v
\end{tikzcd}$$

Noeth.

$\therefore \mathbb{T}(U^\ell_Q, \mathcal{O})_x$ noetherian

Rmk If $S \supset R \cup Q \cup \{v \mid \ell\}$ is any finite set of places of F, then $\mathbb{T}(U^\ell_Q, \mathcal{O})_x$ is top. generated by the T_v for $v \notin S$.

Pf. Frob_v for $v \notin S$ dense in $G_{F, R \cup Q \cup \{v \mid \ell\}}$

$\therefore$ if $v \in S \setminus R \cup Q \cup \{v \mid \ell\}$, then T_v is a limit of $T_{v'}$, $v' \notin S$.

$$T^{mod} \bmod \mathfrak{m} = \mathrm{tr}\, \underset{\substack{\uparrow \\ \text{semisimple}}}{\bar\rho_{\mathfrak{m}}}$$

If $\bar\rho_{\mathfrak m}$ is abs. irred., then $\exists\ \rho_{\mathfrak m} : G_{F, Q \cup R \cup \{v | \ell\}} \longrightarrow GL_2(\mathbb{T}(U^{\ell}_{Q}, \mathcal{O})_{\chi, \mathfrak{m}})$

$$\mathrm{tr}\, \rho_{\mathfrak m} = T^{mod}_{\mathfrak m}.$$

Lecture 12 Local theory.

$L | \mathbb{Q}_\ell$ finite, $\mathcal{O}/\lambda \simeq \mathbb{F}$

$G = GL_2(\mathbb{Q}_\ell)^n$, $Z = (\mathbb{Q}_\ell^\times)^n$

$$\begin{tikzcd}
 & & & \mathcal{C}_G \arrow[dl, leftrightarrow, "dual"'] & & & \\
\mathrm{Mod}^{sm}_G(\mathcal{O}) \arrow[dddd, leftrightarrow, bend right, "\text{anti-equiv}"'] & \supset & \mathrm{Mod}^{lfin}_G(\mathcal{O}) & = & \bigoplus_{\mathcal{B}} \mathrm{Mod}^{lfin}_G(\mathcal{O})_{\mathcal{B}} & \mathrm{Hom}^{cts}_{\mathcal{O}}(N, L/\mathcal{O}) & M \arrow[ddd, mapsto] \\
\cup & & \cup & & & & \\
\mathrm{Mod}^{sm}_{G,1}(\mathcal{O}) & \supset & \mathrm{Mod}^{lfin}_{G,1}(\mathcal{O}) \arrow[dr, leftrightarrow] & = & \bigoplus_{\mathcal{B}} \mathrm{Mod}^{lfin}_{G,1}(\mathcal{O})_{\mathcal{B}} & & \\
 & & & \mathcal{C}_{G,1} & & N \arrow[uuu, mapsto] & \mathrm{Hom}_{\mathcal{O}}(M, L/\mathcal{O}) \\
\mathrm{Mod}^{pro-aug}_G(\mathcal{O}) & & & & & &
\end{tikzcd}$$

$\mathrm{Mod}^{pro-aug}_G(\mathcal{O})$ = profinite $\mathcal{O}$-mod w/ an action of G s.t. $\forall\ U \subset G$ open cpt (for one such U) the $\mathcal{O}[U]$-action extends to a cts $\mathcal{O}[\![U]\!]$-action

N is admissible if it is finitely generated over $\mathcal{O}[\![U]\!]$ for one (hence all) U

$N \in \mathrm{Mod}^{pro-aug}_{G,1}(\mathcal{O})$, N admissible $\Rightarrow N \in \mathcal{C}_{G,1}$

$M(U^{\ell}_{Q}, \mathcal{O})_{\chi} \in \mathcal{C}_{G,1}$, and admissible, i.e. $\in \mathcal{C}^{adm}_{G,1}$

$S(U^{\ell}_{Q}, L/\mathcal{O})_{\chi} \in \mathrm{Mod}^{lfin}_{G,1}(\mathcal{O})$

$$\mathcal{C}_G = \prod_{\mathcal{B}} \mathcal{C}_{G,\mathcal{B}}$$
$$\cup$$
$$\mathcal{C}_{G,1} = \prod_{\mathcal{B}} \mathcal{C}_{G,1,\mathcal{B}}$$

$P_{\mathcal{B}} \twoheadrightarrow \bigoplus_{\pi \in \mathcal{B}} \pi^\vee$ projective envelope , $E_{\mathcal{B}} = \mathrm{End}_{\mathcal{C}_{G,1}}(P_{\mathcal{B}})$ finite $/ Z(E_{\mathcal{B}})$

$\mathcal{C}_{G,1,\mathcal{B}} \simeq$ profinite $E_{\mathcal{B}}$ - modules

$N \longmapsto \mathrm{Hom}(P_{\mathcal{B}}, N)$

$P_{\mathcal{B}} \otimes_{E_{\mathcal{B}}} X \longleftarrow\!\shortmid X$

<u>Lemma</u>. $N \in \mathcal{C}^{adm.}_{G,1,\mathcal{B}} \Rightarrow \mathrm{Hom}(P_{\mathcal{B}}, N)$ finitely gen. $/E_{\mathcal{B}}$

$\mathcal{B} = \prod B_i \leftarrow$ block for $\mathrm{Mod}^{lfin}_{GL_2(\mathbb{Q}_\ell), 1}(\mathcal{O})$.

$\updownarrow$

$\bar{\tau}_{B_i} : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\mathbb{F})$ or a finite extn , but won't discuss that case.

$\det \bar{\tau}_{B_i} = \bar{\varepsilon}_\ell^{-1}$ semisimple

$$\hat{\bigotimes}_{i=1,\dots,n} R^{ps}_{\mathrm{tr}\,\bar{\tau}_{B_i},\, \varepsilon_\ell^{-1}} = R^{ps}_{\mathcal{B}} \xrightarrow[\text{finite}]{} Z(E_{\mathcal{B}})$$

(usually an isom.)

$\mathcal{U}_m = \ker\left(GL_2(\mathbb{Z}_\ell)^n \longrightarrow GL_2(\mathbb{Z}/\ell^m\mathbb{Z})^n\right)$ $m \geq 1$

$\Lambda_m = \mathcal{O}[\![\mathcal{U}_m]\!]$ local, left right noetherian. $J_m \triangleleft \Lambda_m$, $\Lambda_m / J_m \simeq \mathbb{F}$

$J_m = \langle \lambda, u-1 : u \in \mathcal{U}_m \rangle$

If $\underset{\substack{\uparrow \\ \text{f.g.} /\Lambda_1}}{N} \in \mathcal{C}^{\mathrm{adm}}_{G,1}(\mathcal{O})$, then $\exists\ \varphi_N \in \mathbb{Q}[t]$ s.t. for $j \gg 0$

$$\dim_{\mathbb{F}} \left(J_1^{j} N / J_1^{j+1} N \right) = \varphi_N(j)$$

Def $\dim_{\Lambda_1} N = 1 + \deg \varphi_N$ (Gelfand-Kirillov dim)

eg $\dim_{\Lambda_1} \Lambda_1 = 1+3n$, $\dim_{\Lambda_1} M(U^{\ell}_{Q}, \mathcal{O})_{\mathfrak{X}} = 1+3n$

Prop (Pan) If X is a f.g. E_B-module, then

$$\dim_{\Lambda_1} \left(P_B \underset{E_B}{\otimes} X \right) \le \underset{\substack{\uparrow \\ \dim \operatorname{Supp}_{R_B^{ps}}(X)}}{\dim_{R_B^{ps}}(X)} + n$$

eg $n=1$, X finite length $/R_B^{ps}$. Want $\dim_{\Lambda_1}\left(P_B \underset{E_B}{\otimes} X\right) \le 1$

WLOG X irred. $\rightsquigarrow$ π irred. rep'n of $PGL_2(\mathbb{Q}_\ell)$ over $\mathbb{F}$

$$\Rightarrow \dim_{\mathcal{O}[[U_1]]} \pi^\vee \le 1$$

i.e. $\dim_{\mathbb{F}} \pi^\vee / J^r \pi^\vee \le$ linear function of r

$$J_{(m)} = \ker\left(\mathcal{O}[[U_1]] \longrightarrow \mathcal{O}[U_1/U_m]\right) \subset J^{\ell^{m-1}}$$

Sufficient to prove $\underset{\substack{\| \\ \dim \pi^{U_m}}}{\dim_{\mathbb{F}} \pi^\vee / J_{(m)} \pi^\vee} \le$ linear fun of ℓ^{m-1}

eg. π principal series

$$\dim \pi^{U_m} \leq \#\left(B(\mathbb{Q}_\ell)\backslash GL_2(\mathbb{Q}_\ell)/U_m\right)$$

$$\| $$

$$\# \; B(\mathbb{Z}_\ell)\backslash GL_2(\mathbb{Z}_\ell)/U_m$$

$$\|$$

$$\# \; B(\mathbb{Z}/\ell^m\mathbb{Z})\backslash GL_2(\mathbb{Z}/\ell^m\mathbb{Z}) = \ell^m + \ell^{m-1}$$

Concrete classification $\mathrm{Irr}\left(\mathrm{Mod}^{sm}_{GL_2(\mathbb{Q}_\ell),1}(\mathbb{F})\right)$

$$M(U^1_{\mathbb{Q}}, \mathcal{O})_{x,m} \quad \dim\ 1+3n$$

$$\updownarrow$$

$$M(U^1_{\mathbb{Q}}, \mathcal{O})_{x,m} / E_{B_m}$$

expect dim $1+2n$

1) $B_i = \{\pi\}$, $\pi = \left(c\text{-}\mathrm{Ind}^{GL_2(\mathbb{Q}_\ell)}_{\mathbb{Q}_\ell^\times GL_2(\mathbb{Z}_\ell)} \underset{\uparrow}{\mathrm{Sym}^r(\mathbb{F}^2)}\right)/T \otimes \eta\circ\det$

where $\eta^2 \omega^r = 1$ $\qquad$ $\ell \in \mathbb{Q}_\ell^\times$ acts as 1

$\omega : \mathbb{Q}_\ell^\times \longrightarrow \mathbb{F}_\ell^\times$

$\ell \longmapsto 1$

$x \in \mathbb{Z}_\ell^\times \longmapsto x \bmod \ell$

$$\overline{\tau}_{B_i} = \left(\mathrm{Ind}^{G_{\mathbb{Q}_\ell}}_{G_{\mathbb{Q}_{\ell^2}}} \omega_2^{r-1}\right) \otimes \eta$$

2) $B_i = \left\{\mathrm{Ind}^{GL_2(\mathbb{Q}_\ell)}_{B(\mathbb{Q}_\ell)}(\chi \otimes \chi^{-1}),\ \mathrm{Ind}^{GL_2(\mathbb{Q}_\ell)}_{B(\mathbb{Q}_\ell)}(\omega\chi^{-1} \otimes \chi\omega^{-1})\right\}$ $\qquad \chi^2 \neq 1$

$\uparrow$ if rat'l over $\mathbb{F}$ $\qquad (\chi\omega)^2 \neq 1$

$\chi : \mathbb{Q}_\ell^\times \longrightarrow \mathbb{F}^\times$

$$\overline{\tau}_{B_i} = \chi\omega^{-1} \oplus \chi^{-1}$$

3) $B_i = \left\{1, Sp, \mathrm{Ind}^{GL_2(\mathbb{Q}_\ell)}_{B(\mathbb{Q}_\ell)}(\omega \otimes \omega^{-1})\right\} \otimes \chi$, $\quad \chi^2 = 1$, $\quad \overline{\tau}_{B_i} = \chi\omega^{-1} \oplus \chi^{-1}$

$\mathrm{Ban}^{\mathrm{adm},\mathrm{fl}\ \leftarrow \text{finite length}}_{G,1}(L)$ = cat. of adm. unitary cts Banach space rep of G/Z

$G/Z \curvearrowright V$ = Banach space

$\cup$

V^0 unit ball, invariant by G

$V^0/\lambda \in \mathrm{Mod}^{\mathrm{adm}}_{G,1}(\mathcal{O})$

$B = \prod B_i$, $R^{ps}_B = \hat{\bigotimes}_i R^{ps}_{\mathrm{tr}\,\bar{r}_{B_i},\,\varepsilon_\ell^{-1}}$

$\mathrm{Ban}^{\mathrm{adm},\mathrm{fl}}_{G,1}(L)$ splits up as $\mathrm{Ban}^{\mathrm{adm},\mathrm{fl}}_{G,1}(L)_{\mathcal{B}}$

$M(U^\ell_G, \mathcal{O})_\chi \in \mathcal{C}^{\mathrm{adm}}_{G,1}$

$S^{sm}(U^\ell_G, L/\mathcal{O})_\chi \in \mathrm{Mod}^{\mathrm{adm}}_{G,1}(\mathcal{O})$

$S^{cts}(U^\ell_G, \mathcal{O})_\chi = S^{cts}(U^\ell_G, L)^0_\chi$

$\wedge$

$\mathrm{Ban}^{\mathrm{adm}}_{G,1}(L) \ni S^{cts}(U^\ell_G, L)_\chi$

$$\mathrm{Ban}^{\mathrm{adm},\mathrm{fl}}_{G,1}(L)_{\mathcal{B}} = \bigoplus_{\mathfrak{p} \in \mathrm{Max}(R^{ps}_{\mathcal{B}}[\frac{1}{\ell}])} \mathrm{Ban}^{\mathrm{adm},\mathrm{fl}}_{G,1}(L)_{\mathcal{B},\mathfrak{p}}$$

$\wr\!\!\downarrow$ f. length modules $E_{\mathcal{B}}[1/\ell]^\wedge_{\mathfrak{p}}$

$n=1$, $G = GL_2(\mathbb{Q}_\ell)$

$V \in \mathrm{Ban}^{\mathrm{adm},\mathrm{fl}}_{G,1}(L)_{\mathcal{B},\mathfrak{p}}$

$\cup$

V^0 $\quad \mathrm{Hom}_{\mathcal{O}}(V^0, \mathcal{O}) \in \mathcal{C}^{\mathrm{adm}}_{G,1,\mathcal{B}}$ $\quad \mathrm{Hom}_{\mathcal{C}}(P_{\mathcal{B}}, \mathrm{Hom}_{\mathcal{O}}(V^0,\mathcal{O}))$

$\circlearrowleft$

$E_{\mathcal{B}}$ action factors through $E_{\mathcal{B}}/\mathfrak{p}^m$ some power

2) $\mathfrak{p} \in \mathrm{Max}(R^{ps}_{\mathcal{B}}[\frac{1}{\ell}])$

Suppose $T \bmod \mathfrak{p} = \mathrm{tr}\, r$, $r : G_{\mathbb{Q}_\ell} \to GL_2(k(\mathfrak{p}))$

Suppose r is absolutely irred.

Then $\mathrm{Ban}^{\mathrm{adm.fl}}_{G,1}(L)_{B,\mathfrak{p}}$ has a unique irred. $\Pi_{\mathfrak{p}}$

If ρ is de Rham w/ distinct HT wts $k, 1-k$, $k \in \mathbb{Z}_{>0}$, then $\mathrm{Hom}_U(W_k^\vee, \Pi_{\mathfrak{p}}) \neq 0$

for some $U \subset GL_2(\mathbb{Q}_\ell)$ open cpt, And $E_B/\mathfrak{p}$ is a division algebra

If further ρ is crystalline, and $\left(\rho \otimes_{\mathbb{Q}_\ell} B_{cr}\right)^{G_{\mathbb{Q}_\ell}} \simeq k(\mathfrak{p})^{\oplus 2} \circlearrowleft$ Frob has eigenvalues α, β w/ $\alpha/\beta \neq \ell^{\pm 1}$.

$$\Pi_{\mathfrak{p}} = \overset{\text{universal}}{\text{completion}} \quad W_k^\vee \otimes \mathrm{Ind}_{B(\mathbb{Q}_\ell)}^{GL_2(\mathbb{Q}_\ell)}\left(\mu_{\alpha/\ell} \times \mu_\beta\right)$$

$$\mu_\gamma : \mathbb{Q}_\ell^\times / \mathbb{Z}_\ell^\times \longrightarrow k(\mathfrak{p})^\times$$
$$\ell \longmapsto \gamma$$

$$R^{ps}_{m} = \widehat{\bigotimes_{v\mid\ell}} R^{ps}_{\mathrm{tr}\,\bar\rho_m|_{G_{F_v}},\, \varepsilon_\ell^{-1}} \longrightarrow \prod \mathbb{T}(U^\ell_Q, \mathcal{O})_{x,m}$$

$T_m|_{G_{F_v}}$ gives

$$M(U^\ell_Q, \mathcal{O})_x = \prod_m M(U^\ell_Q, \mathcal{O})_{x,m}$$

$$\mathbb{T}(U^\ell_Q, \mathcal{O})_x = \prod_m \mathbb{T}(U^\ell_Q, \mathcal{O})_{x,m}$$

$$\bar\rho_m : G_F \longrightarrow GL_2(k(m))$$

$$T^{\mathrm{mod}} : G_F \longrightarrow \mathbb{T}(U^\ell_Q, \mathcal{O})_x$$

$\bar\rho_m|^{ss}_{G_{F_v}} \longleftrightarrow B_{m,v}$ block of $\mathrm{Mod}^{\ell\text{ fin}}_{GL_2(\mathbb{Q}_\ell),1}(\mathcal{O})$

$B_m = \prod B_{m,v}$ block of $\mathrm{Mod}^{\ell\text{ fin}}_{GL_2(F_\ell),1}(\mathcal{O})$

<u>Thm</u> (Pan) 1) $M(U^\ell_Q, \mathcal{O})_{x,m} \in \mathcal{C}^{\ell\text{ fin}}_{GL_2(F_\ell),1}(\mathcal{O})_{B_m}$

2) The 2-actions of R^{ps}_m on $\mathrm{Hom}(P_{B_m}, M(U^\ell_Q, \mathcal{O})_{x,m})$ agree

$$R^{ps}_m \to \mathbb{T}(U^\ell_Q, \mathcal{O})_{x,m}, \quad R^{ps}_m \longrightarrow E_{B_m}$$

Lecture 13

Thm (Pan) 1) $M(\overset{\curvearrowleft GL_2(F_\ell)}{U_Q^\ell, \mathcal{O}})_{\chi,m} \in \mathcal{C}^{adm}_{GL_2(F_\ell),\ell}(\mathcal{O})_{B_m}$

$$B_m = \prod_{v|\ell} B_{\bar{r}_m|^{ss}_{G_{F_v}}}$$

$$R^{ps}_{B_m} \longrightarrow \underset{\circlearrowleft}{\mathbb{T}(U_Q^\ell, \mathcal{O})_{\chi,m}}$$

2) Two actions of $R^{ps}_{B_m}$ on $\mathrm{Hom}(P_m, M(U_Q^\ell, \mathcal{O})_{\chi,m})$ agree.

$$\overset{\circlearrowleft}{E_{B_m}} \longleftarrow R^{ps}_{B_m} = \hat{\bigotimes_{v|\ell}} R^{ps}_{\mathrm{tr}\,\bar{r}_m|_{G_{F_v}}, \varepsilon_\ell^{-1}}$$

$$GL_2(F_\ell)$$

STP $\forall\, v|\ell$, and all open cpt $U_\ell^v \subset GL_2(F_\ell^v)$

1) $M(U_Q^\ell U_\ell^v, \mathcal{O})_{\chi,m} \in \mathcal{C}^{adm}_{GL_2(F_v),1}(\mathcal{O})_{B_{\bar{r}_m|^{ss}_{G_{F_v}}}}$

2) Two actions of $R^{ps}_{\mathrm{tr}\,\bar{r}_m|_{G_{F_v}}, \varepsilon_\ell^{-1}}$ on $\mathrm{Hom}(P_{B_{\bar{r}_m}|^{ss}_{G_{F_v}}}, M(U_Q^\ell U_\ell^v, \mathcal{O})_{\chi,m})$ agree.

$$W_k = \mathrm{Sym}^{2k-2}(L^2) \otimes \det{}^{1-k} \circlearrowleft GL_2(F_v) \quad , k \in \mathbb{Z}_{>0}.$$

$$\bigoplus_k \mathrm{Hom}_{GL_2(\mathcal{O}_{F,v})}(W_k^\vee, S^{cts}(U_Q^\ell U_\ell^v, \mathcal{O})_{\chi,m}) \otimes W_k^\vee \longrightarrow S^{cts}(U_Q^\ell U_\ell^v, \mathcal{O})_{\chi,m}$$

has dense image.

$$\mathbb{T}(U_Q^\ell, \mathcal{O})_\chi[\tfrac{1}{\ell}]$$

$$\circlearrowleft$$

$U_v \subset GL_2(F_v)$ open cpt subgp

$$\mathrm{Hom}_{U_v}(W_k^\vee, S^{cts}(U_Q^\ell U_\ell^v, L)_\chi) \qquad D^\times \backslash GL_2(\mathbb{A}_F^\infty) \longrightarrow \mathcal{O}$$

$$\wr\wr \qquad\qquad [f \mapsto f \circ \varphi]$$

$$\{\varphi : D^\times \backslash GL_2(\mathbb{A}_F^\infty)/(\mathbb{A}_F^\infty)^\times \to W_k : \varphi(gu) = \chi(u^\ell) u_v^{-1} \varphi(g)\} \qquad \varphi \longmapsto$$

$$\prod_{\mathfrak{p} \in \mathrm{Spec}(\circlearrowleft)} k(\mathfrak{p}) \hookleftarrow \mathbb{T}_k(\overset{\circlearrowleft}{U_Q^\ell} U_\ell^v U_v, L) = L\text{-subalg. of } \mathrm{End}_L(\downarrow) \quad \forall\, u \in U_Q^\ell U_\ell^v U_v$$

gen. by T_w for $w \notin R \cup Q \cup \{w|\ell\}$.

If $i: L \hookrightarrow \mathbb{C}$, $\mathrm{Hom}_{U_v}(W_k^\vee, S^{cts}(U^\ell_\mathbb{Q} U^\vee_\ell, L)_\chi) \otimes_{L,i} \mathbb{C}$

$\wr$

$$\left\{ \varphi : D^\times \backslash GL_2(\mathbb{A}_F^\infty) / (\mathbb{A}_F^\infty)^\times \longrightarrow W_k \otimes_{L,i} \mathbb{C} : \varphi(gu) = (i \circ \chi)(u^\ell)\, i(u_v)^{-1} \varphi(g) \quad \forall u \in U^\ell_\mathbb{Q} U^\vee_\ell U_v \right\}$$

φ

$\big\downarrow$

$g \mapsto g_{\infty_i}^{-1}(g_v \varphi(g^\infty))$

$\wr$

$F \to F_v \to L \xrightarrow{i} \mathbb{C}$ gives ∞-place ∞_i of F

$$\left\{ \psi : D^\times \backslash GL_2(\mathbb{A}_F) / \mathbb{A}_F^\times \longrightarrow W_k \otimes_{L,i} \mathbb{C} : \psi(gu) = (i \circ \chi)(u^\ell)\, u_{\infty_i}^{-1} \psi(g) \right\}$$

$\forall u \in U^\ell_\mathbb{Q} U^\vee_\ell U_v D_\infty^\times$

$\circlearrowright$

$\mathbb{T}_k(U^\ell_\mathbb{Q} U^\vee_\ell U_v, L)_\chi \otimes_{L,i} \mathbb{C} \simeq \mathbb{C}^?$

We call a max'l ideal $\mathfrak{p}$ of $\mathbb{T}(U^\ell_\mathbb{Q} U^\vee_\ell, \mathcal{O})_\chi[\frac{1}{\ell}]$ <u>algebraic</u> if it arises for some k, U_v from $\mathbb{T}_k(U^\ell_\mathbb{Q} U^\vee_\ell U_v, L)_\chi$.

$$\varinjlim_{U_v} \mathrm{Hom}_{U_v}(W_k^\vee, S^{cts}(U^\ell_\mathbb{Q} U^\vee_\ell, L)_\chi)[\mathfrak{p}] \simeq \pi_\mathfrak{p}^{\oplus d_\mathfrak{p}}$$

$\circlearrowleft$ $GL_2(F_v)$ smooth

for some irred. smooth rep. of $GL_2(F_v)$

If $\mathfrak{p}$ is unramified, so is $\pi_\mathfrak{p}$.

$$\bigoplus_k \bigoplus_{\substack{\mathfrak{p}\ \text{unram} \\ \text{algebraic}}} \Big(\underbrace{\varinjlim_{U_v} \mathrm{Hom}_{U_v}(W_k^\vee, S^{cts}(U^\ell_\mathbb{Q} U^\vee_\ell, L)_\chi[\mathfrak{p}])}_{\substack{\wr \\ \pi_\mathfrak{p}^{\oplus d_\mathfrak{p}}}} \otimes W_k^\vee \Big) \xrightarrow{\text{equiv}} S^{cts}(U^\ell_\mathbb{Q} U^\vee_\ell, L)_\chi$$

$GL_2(F_v) \curvearrowright$ on each of the three terms

has dense image.

$S^{cts}(U^\ell_\mathbb{Q} U^\vee_\ell, L)_\chi \supset \bigoplus_{\mathfrak{p}\ m, alg} \Pi(\mathfrak{p})$ dense

$\Pi(\mathfrak{p})$ = closure of the image of term $\leftrightarrow \mathfrak{p}$

$$M(U^{\ell}_{\alpha} U^{v}_{\ell}, \mathcal{O})_{x,m} \hookrightarrow \prod \Pi(\mathfrak{p})^{\vee}$$

it will suffice to show that for $\mathfrak{p}$ unram. algebraic w/ $\mathfrak{p}^{c} \subset m$ that we have

$$\Pi(\mathfrak{p}) \in \mathrm{Ban}^{adm,\, fl}_{GL_2(F_v),\,1}(L)_{B_{\bar{r}_m}|^{ss}_{G_{F_v}},\, \tilde{\mathfrak{p}}}$$

$$\begin{tikzcd} R^{PS}_{\mathrm{tr}\,\bar{r}_m|_{G_{F_v}},\, \varepsilon_\ell^{-1}}[\tfrac{1}{\ell}] \arrow[r] & \mathbb{T}(U^{\ell}_{\alpha} U^{v}_{\ell}, \mathcal{O})_{x,m}[\tfrac{1}{\ell}] \\ \tilde{\mathfrak{p}} & \mathfrak{p} \end{tikzcd}$$

$$M(U^{\ell}_{\alpha} U^{v}_{\ell}, \mathcal{O})_{x} \overset{def}{=} \mathrm{Hom}(S^{sm}(U^{\ell}_{\alpha} U^{v}_{\ell}, L/\mathcal{O}), L/\mathcal{O}) \simeq \mathrm{Hom}(S^{cts}(U^{\ell}_{\alpha} U^{v}_{\ell}, \mathcal{O})_{x}, \mathcal{O})$$

$$S^{cts}(U^{\ell}_{\alpha} U^{v}_{\ell}, \mathcal{O})_{x} \simeq \mathrm{Hom}(L/\mathcal{O}, S^{sm}(U^{\ell}_{\alpha} U^{v}_{\ell}, L/\mathcal{O}))$$

$$S^{cts}(U^{\ell}_{\alpha} U^{\ell}_{\ell}, L)_{x} = S^{cts}(U^{\ell}_{\alpha} U^{v}_{\ell}, \mathcal{O})[\tfrac{1}{\ell}]$$

$r_{\mathfrak{p}} : G_F \longrightarrow GL_2(k(\mathfrak{p}))$, $r_{\mathfrak{p}}|_{G_{F_v}}$ is crystalline, HT wts k & $1-k$.

$(r_{\mathfrak{p}} \otimes B_{cris})^{G_{F_v}}$ 2-dim'l over $k(\mathfrak{p})$ (μ_γ is unram. char, $\mathrm{Frob}_v \mapsto \gamma$)

$\circlearrowright$ Frob_v has evals α, β. $\alpha/\beta \neq \ell^{\pm 1}$ $\Pi_{\mathfrak{p}} \simeq \mathrm{Ind}^{GL_2(F_v)}_{B(F_v)}(\mu_{\alpha/\ell} \times \mu_\beta)$

$$\Pi(\mathfrak{p}) \underset{dense}{\supset} \left(\mathrm{Ind}^{GL_2(F_v)}_{B(F_v)}(\mu_{\alpha/\ell} \times \mu_\beta) \otimes W_k^{\vee}\right)^{\oplus d_{\mathfrak{p}}}$$

$V_{\mathfrak{p}}$ = universal unitary completion of

$$\Rightarrow V_{\mathfrak{p}}^{\oplus d_{\mathfrak{p}}} \twoheadrightarrow \Pi(\mathfrak{p}). \quad V_{\mathfrak{p}} \in \mathrm{Ban}^{adm,\, fl}_{GL_2(F_v),\,1}(L)_{B_{\bar{r}_m}|^{ss}_{G_{F_v}},\, \tilde{\mathfrak{p}}}$$

Thm Suppose $\mathfrak{p} \triangleleft \mathbb{T}(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}}[\frac{1}{\ell}]$ a max'l ideal

$r : G_F \longrightarrow GL_2(k(\mathfrak{p}))$ w/ $\mathrm{tr}\, r = T_{\mathfrak{m}} \mod \mathfrak{p}$

$\forall v | \ell$, $r|_{G_{F_v}}$ is de Rham + $\underline{\text{abs irred}}.^{(*)}$ w/ HT wts k_v, $1-k_v$, $k_v \in \mathbb{Z}_{>0}$

Then $\mathfrak{p}$ pulls back from a max'l ideal of $\mathbb{T}_k(U^{\ell}_{\mathbb{Q}} U_\ell, L)_\chi$, $k = (k_v)$

Pf Suppose 1 prime v above ℓ.

STP $\mathrm{Hom}_{U_\ell}(W_k^\vee, \underbrace{S^{cts}(U^{\ell}_{\mathbb{Q}}, L)_{\chi,\mathfrak{m}}[\mathfrak{p}]}) \neq 0$ for some U_v.

$\in \mathrm{Ban}^{adm, fl}_{GL_2(F_v),1}(L)_{\mathcal{B}_{\mathfrak{m}}, \tilde{\mathfrak{p}}}$

by $(*)$, this cat has unique irred. object $\Pi_{\tilde{\mathfrak{p}}}$ - and

$\mathrm{Hom}_{U_\ell}(W_k^\vee, \Pi_{\tilde{\mathfrak{p}}}) \neq 0$ for some U_ℓ.

Thm. $\mathbb{T}(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}}$ is finite over $R^{ps}_{\mathcal{B}_{\mathfrak{m}}} = \hat{\bigotimes}_{v|\ell} R^{ps}_{\mathrm{tr}\,\bar{r}_{\mathfrak{m}}|_{G_{F_v}}, \varepsilon_\ell^{-1}}$

and has $\dim \geq 1 + 2[F:\mathbb{Q}]$.

Pf $M(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}} = \mathrm{Hom}(P_{\mathfrak{m}}, \underbrace{M(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}}})$

$\circlearrowleft$ $\quad \mathfrak{C}^{adm}_{GL_2(F_\ell),1}(\mathcal{O})_{\mathcal{B}_{\mathfrak{m}}}$ $\quad$ f.g. $\mathcal{O}[[U_\ell]]$

$\mathbb{T}(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}}$ faithful

f.g. $E_{\mathcal{B}_{\mathfrak{m}}}$

$\therefore$ f.g. $/ R^{ps}_{\mathcal{B}_{\mathfrak{m}}} \longrightarrow \mathbb{T}(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}} \subset \mathrm{End}_{R^{ps}_{\mathcal{B}_{\mathfrak{m}}}}(M(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}})$

$\therefore$ $\mathbb{T}(U^{\ell}_{\mathbb{Q}}, \mathcal{O})_{\chi,\mathfrak{m}}$ finite $/ R^{ps}_{\mathcal{B}_{\mathfrak{m}}}$

$$\dim_{\mathcal{O}[[U_\ell]]} M(U^\ell_Q, \mathcal{O})_{\chi,m} - [F:\mathbb{Q}] \leq \dim R^{ps}_{B_m}\left(M(U^\ell_Q,\mathcal{O})_{\chi,m}\right)$$

$$\shortparallel$$

$$1+3[F:\mathbb{Q}] - [F:\mathbb{Q}]$$

$$= \dim_{\mathbb{T}(U^1_Q,\mathcal{O})_{\chi,m}}\left(M(U^\ell_Q,\mathcal{O})_{\chi,m}\right)$$

$$\leq \dim \mathbb{T}(U^\ell_Q,\mathcal{O})_{\chi,m}$$

Lecture 14. $(R^{big})^{red} = \mathbb{T}^{big}$, $\bar{r}|_{G_{F(\zeta_\ell)}}$ abs irred / then $\bar{r}$ irred, $\bar{r}|_{G_{F(\zeta_\ell)}}$ red.

R, Q, $v \in R$, $q_v \equiv 1$ (ℓ).

R: bad primes; Q: auxilliary primes

$U^\ell_Q \subset GL_2(\mathbb{A}_F^{\infty,\ell})$

χ_v, $v\in R$, char. of $k(v)^\times$ of ℓ-power order

$\chi_v = 1$, $\forall v$ [$\chi_v \neq 1$, $\forall v$

finite free over $\mathcal{O}[[U^1_\ell]]$

$$M(U^\ell_Q,\mathcal{O})_{\chi,m} = \mathrm{Hom}\left(S^{sm}(U^\ell_Q, L/\mathcal{O})_{\chi,m}, L/\mathcal{O}\right)$$

$\circlearrowleft$

$\mathbb{T}(U^\ell_Q,\mathcal{O})_m[GL_2(F_\ell)]$, $U^n_\ell = \ker\left(GL_2(\mathcal{O}_{F,\ell}) \to GL_2(\mathcal{O}_F/\ell^n)\right)$

$T_m : G_F \to \mathbb{T}(U^\ell_Q, \mathcal{O})_{\chi,m}$ pseudo-rep. $\det T_m = \varepsilon_\ell^{-1}$.

unram. away from $R \cup \{v|\ell\} \cup Q$

$w \notin$

$T_m(\mathrm{Frob}_w) = T_w$

$\mathrm{tr}\, \bar{r}_m = T_m \bmod m$

assume. $\bar{r}_m$ only ramified above ℓ.

B_m: block.

$$\shortparallel$$

$$\prod_{v|\ell} B_{T_m|_{G_{F_v}} \bmod m}$$

$$R^{ps}_{B_m} = \hat{\bigotimes_{v|\ell}} R^{ps}_{\bar{T}_m|_{G_{F_v}},\, \varepsilon_\ell^{-1}} \longrightarrow \mathbb{T}(U^\ell_Q,\mathcal{O})_{\chi,m}$$

$$M(U_Q^\ell, \mathcal{O})_{\chi, m} \in \mathcal{C}_{GL_2(F_\ell),1}(\mathcal{O})_{B_m} \ni P_m$$

$R^{ps}_{B_m}$ $\nearrow$ two actions which agree

$v \in R$, $T_m|_{I_{F_v}} = \chi_v + \chi_v^{-1}$ and assume $\overline{\tau}_m|_{G_{F_v}} = 1$

$v \in Q$: assume choose v s.t. $q_v \equiv 1 \ (\ell)$, $Q \cap (R \cup \{v | \ell\}) = \phi$.

$$(T_m \bmod m)(\mathrm{Frob}_v) = \bar{\alpha}_v + \bar{\alpha}_v^{-1}, \quad \bar{\alpha}_v^2 \neq 1$$

$\Rightarrow T_m = \psi_v + \varepsilon_\ell^{-1}\psi_v^{-1}$, $\psi_v : G_{F_v} \to \mathbb{T}(U_Q^\ell, \mathcal{O})^\times_{\chi, m}$

$(\psi_v \bmod m)(\mathrm{Frob}_v) = \bar{\alpha}_v$

max'l quot of $k(v)^\times$ of ℓ-power order

$$\psi_v|_{I_{F_v}} \text{ factors through } I_{F_v} \twoheadrightarrow k(v)^\times \twoheadrightarrow H_v$$

$$H_Q = \prod_{v \in Q} H_v$$

$$\prod \psi_v : H_Q \to \mathbb{T}(U_Q^\ell, \mathcal{O})^\times_{\chi, m}$$

$v \in Q$, $a \in \mathcal{O}_{F,v} \setminus \{0\}$ $\quad U_{v,a} \longleftrightarrow U_{Q,v}^\ell \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} U_{Q,v}^\ell$

$\uparrow$ commutes

$U_{v,ab} = U_{v,a} U_{v,b}$ $\quad T_w \quad w \notin Q \cup R \cup \{w' | \ell\}$

$U_{v',b}$, $v' \in Q$, $b \in \mathcal{O}_{F,v'} \setminus \{0\}$

$GL_2(F_\ell)$

Choose $\phi_v \longmapsto \mathrm{Frob}_v$

$\uparrow$ G_{F_v} $\quad \phi_v|_{F_v^{ab}} = \mathrm{Art}(\pi_v)$ uniformizer $\quad A_v = \psi_v(\phi)$, $B_v = \varepsilon_\ell^{-1}\psi_v^{-1}(\phi)$

$A_v \bmod m \neq B_v \bmod m$

$$(U_{\pi_v} - A_v)(U_{\pi_v} - B_v) = 0 \text{ on } M(U_Q^\ell, \mathcal{O})_{\chi, m}$$

$$e_v = \frac{U_{\pi_v} - B_v}{A_v - B_v}, \quad e_v^2 = e_v, \quad (U_{\pi_v} - A_v)e_v = 0$$

$$(U_{\pi_v} - B_v)(1 - e_v) = 0 \qquad e = \prod_{v \in Q} e_v$$

$$M(U^\ell_Q, \mathcal{O})^+_{x,m} = eM(U^\ell_Q, \mathcal{O})_{x,m}$$

$$\cup$$

$$\Pi(U^\ell_Q, \mathcal{O})_{x,m}[GL_2(F_\ell)] \quad \nwarrow \text{ finite free } \mathcal{O}[[U'_\ell]][H_Q]$$

$$S^{sm}(U^\ell_Q, L/\mathcal{O})_x = \bigoplus_{\text{finite}} C^{sm}(U'_\ell \times H_Q, L/\mathcal{O})$$

$$M(U^\ell_Q, \mathcal{O})^+_{x,m}/\sigma_Q \xrightarrow[\sim]{tr} M(U^\ell_\phi, \mathcal{O})_{x,m}$$

$$S^{sm}(U^\ell_Q, L/\mathcal{O})^{+, H_Q}_{x,m} \xleftarrow[m]{} S^{sm}(U^\ell_\phi, L/\mathcal{O})_{x,m}$$

Now assume $\bar{r}_m|_{G_{F(\zeta_\ell)}}$ is abs. irred.

Cebotarev $\forall N, \exists Q_N$ s.t.

1) $v \in Q_N \Rightarrow q_v \equiv 1 \ (\ell^N)$

2) $v \in Q_N \Rightarrow \bar{r}_m$ unram. @ v

and $\bar{r}_m(\mathrm{Frob}_v)$ has distinct evals.

3) $\# Q_N \equiv r = \dim_{\mathbb{F}} H^1_{\mathcal{L}^\perp_\phi}(G_F, (ad^0 \bar{r})(1))$

$\uparrow$ no condition at $\{v|\ell\} \cup R$

unramified elsewhere

4) $H^1_{\mathcal{L}^\perp_{Q_N}}(G_F, (ad^0 \bar{r})(1)) = 0$

$\uparrow$

no condition $v \in R \cup \{v|\ell\}$

trivial $v \in Q_N$

unramified elsewhere

$R^{univ}_{Q_N, x}$ universal def ring for $\bar{r}_m$. for lifts

- no condition above $\{v|\ell\} \cup Q_N$
- unramified elsewhere
- $tr\, r|_{I_{F_v}} = \chi_v + \chi_v^{-1}$ for $v \in R$
- $\det r = \varepsilon_\ell^{-1}$

$\downarrow$

$R^{\square}_{Q_N, x}$ $\leftarrow$ frame at primes in $R \cup \{v|\ell\}$

If we fix $r^{univ} : G_F \to GL_2(R^{univ}_{Q_N,\chi})$, then $R^{\square}_{Q_N,\chi} \simeq R^{univ}_{Q_N,\chi} \hat{\otimes} \mathcal{O}_\infty$

$\dim = 4(|R| + [F:\mathbb{Q}]) \to \mathcal{O}_\infty = \mathcal{O}[[A_{v,i,j} : \begin{smallmatrix} v\in R\cup\{v|\ell\} \\ i,j=1,2 \end{smallmatrix}]]/(A_{v_0,1,1})$

$v\in Q_N$, $r^{univ}|_{G_{F_v}} = \psi_v \oplus \varepsilon_\ell^{-1}\psi_v^{-1}$ $\qquad (\psi_v \bmod m)(\mathrm{Frob}_v) = \bar{\alpha}_v$

$\psi_v|_{I_{F_v}} : I_{F_v} \to k(v)^\times \to H_v \to (R^{univ}_{Q_N,\chi})^\times$

$\mathcal{O}[H_{Q_N}] \to R^{univ}_{Q_N,\chi}$, $\quad R^{univ}_{Q_N,\chi}/\mathfrak{a}_{Q_N} \Rightarrow R^{univ}_{\phi,\chi}$

$$M(U^\ell_{Q_N},\mathcal{O})^{+}_{\chi,m} \underset{R^{univ}_{Q_N,\chi}}{\otimes} R^{\square}_{Q_N,\chi} = M(U^\ell_{Q_N},\mathcal{O})^{+,\square}_{\chi,m}$$

$(\text{or } \hat{\otimes}_{\mathcal{O}} \mathcal{O}_\infty)$ $\qquad \uparrow$ finite free over $\mathcal{O}_\infty[H_{Q_N}][[U'_\ell]]$

mod $\tilde{\mathfrak{a}}_{Q_N}$, get $M(U^\ell_\phi,\mathcal{O})_{\chi,m}$

$\tilde{\mathfrak{a}}_{Q_N} \triangleleft \mathcal{O}_\infty[H_{Q_N}]$

$\parallel$

$\langle A_{v,i,j} : h-1 : h\in Q_N\rangle$

$R^{loc}_\chi[[X_1,\dots,X_t]] \twoheadrightarrow R^{\square}_{Q_N,\chi}$

$\mathbb{F}[\varepsilon]/(\varepsilon^2)$

$(1+\phi\varepsilon)\bar{r}_m$

$M_{2\times2}(\mathbb{F})$ $\quad (1+\alpha_v\varepsilon)$

action of $I_2 + M_{2\times2}(\mathbb{F})\varepsilon$ $\backslash$ $\quad \phi|_{G_{F_v}} + \partial\alpha_v = 0$ $\quad v\in R\cup\{v|\ell\}$

$\phi \in Z^1_{\mathcal{L}_{Q_N}}(G_F, \mathrm{ad}^0\bar{r})$

$\uparrow$

loc. trivial at $R\cup\{v|\ell\}$

anything at Q_N, unram. elsewhere

$R^{loc}_\chi \to R^{\square}_{Q_N,\chi}$

$\parallel$

$\hat{\bigotimes}_{v\in R\cup\{v|\ell\}} R^{\square}_{\bar{r}_m|_{G_{F_v}},\varepsilon_\ell^{-1}}$

$\uparrow$

$v|\ell$ $\quad$ all liftings

$v\in R$ $\quad$ look at liftings w/ $\mathrm{tr}\, r|_{G_{F_v}} = \chi_v + \chi_v^{-1}$

$$t = \dim H^1_{\mathcal{L}_{Q_N}}(G_F, ad^0\bar\rho) + 3 - \dim H^0(G_F, ad^0\bar\rho) + \sum_{v\in R\cup\{v|\ell\}} \left(1+\dim H^0(G_{F_v}, ad^0\bar\rho)\right) - 4$$

$$= \dim H^1_{\mathcal{L}^\perp_{Q}}(G_F, ad^0\bar\rho(1)) - \dim H^0(G_F, ad^0\bar\rho(1)) - 1 + \sum_{v|\infty}\left(-\dim H^0(G_{F_v}, ad^0\bar\rho)\right)$$

$$+ \sum_{v\in R\cup\{v|\ell\}}\left(-\dim H^0(G_{F_v}, ad^0\bar\rho)\right) + \sum_{v\in Q_N}\left(\dim H^1(G_{F_v}, ad^0\bar\rho) - \dim H^0(G_{F_v}, ad^0\bar\rho)\right)$$

$$+ \sum_{v\in R\cup\{v|\ell\}}\left(1+\dim H^0(G_{F_v}, ad^0\bar\rho)\right)$$

$$= -1 - [F:\mathbb{Q}] + |R\cup\{v|\ell\}| + \sum_{v\in Q_N} \dim H^0(G_{F_v}, ad^0\bar\rho(1))$$

$$\alpha_v^2, \alpha_v^{-2}, 1$$

$$= -1 + |R| + \underbrace{|Q_N|}_{r} = t$$

$$\begin{tikzcd}
H_\infty = \mathbb{Z}_\ell^2 \arrow[d] & \Lambda_\infty = \mathcal{O}_\infty[[H_\infty]] \arrow[d] \arrow[r, phantom, "\rhd"] & \tilde{\mathfrak{a}}_\infty \arrow[d, phantom, sloped, "="] \\
H_{Q_N} & R^\square_{Q_N,\chi} & \langle A_{v,i,j},\ h-1 : h\in H_\infty\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Lambda_\infty \arrow[d] & \dim\ 4|R| + 4[F:\mathbb{Q}] + r\text{, power series ring}/\mathcal{O} \arrow[l] & \\
 & & & \text{fin. free}/\Lambda_\infty/\mathfrak{f}_N[[U_\ell']] \arrow[dl] \\
R_\chi^{loc}[[X_1,\dots,X_t]] \arrow[r, two heads] & R^\square_{Q_N,\chi} \arrow[r, phantom, "\cong"] \arrow[d] & M(U^\ell_{Q_N}, \mathcal{O})^{t,\square}_{\chi,m} \arrow[d, "\text{mod out by } \tilde{\mathfrak{a}}_\infty"] & \\
 & R^{univ}_{\phi,\chi} \arrow[r, phantom, "\cong"] & M(U_\phi, \mathcal{O})_{\chi,m} & \\
\dim \arrow[uu] & & & \\
1+6[F:\mathbb{Q}]+3|R|+t \arrow[d, phantom, sloped, "="] & & & \\
6[F:\mathbb{Q}]+4|R|+r & & &
\end{tikzcd}$$

$$\mathfrak{f}_N = \ker\left(\mathcal{O}_\infty[[H_\infty]] \longrightarrow \mathcal{O}_\infty[H_{Q_N}]\right) \subset \tilde{\mathfrak{a}}_\infty$$

$$c_N = \ker\left(\mathcal{O}[[U_\ell^1]] \longrightarrow \mathcal{O}[\mathrm{PGL}_2(\mathcal{O}_{F,\ell}/\ell^N)]\right)$$

$b_N \underset{\text{open}}{\triangleleft} \Lambda_\infty$ s.t. $b_N \supset b_{N+1}$

$b_N \supset \langle A_{v,i,j},\ h-1 : h \in \ell^N \mathbb{Z}_\ell^2 \rangle$

$\cap\, b_N = 0$

$b_N R^{\mathrm{univ}}_{\phi,\chi} \subset d_{N,\chi} \underset{\text{open}}{\triangleleft} R^{\mathrm{univ}}_{\phi,\chi}$ for $\chi = 1$

$\sim \chi = \chi_0$ w/ $\chi_{0,v} \neq 1,\ \forall v$

$d_{N,\chi} \subset d_{N-1,\chi}$

$\cap$

$\mathrm{Ann}_{R^{\mathrm{univ}}_{\phi,\chi}}\left(M(U^\ell_\phi, \mathcal{O})_{\chi,m} / b_N\right)$

$\cap\, d_{N,\chi} = 0.$

$e_{N,1} = d_{N,1} \cap \mathrm{preim}\left((d_{N,1} \bmod \lambda) \cap (d_{N,\chi_0} \bmod \lambda)\right)$

$-\chi_0 \quad -\chi_0$ ———

Same properties but $e_{N,1} \bmod \lambda = e_{N,\chi_0} \bmod \lambda$.

$$M \geq N,\quad R_{M,N,\chi} = \mathrm{Im}\left(R^\square_{Q_M,\chi} \longrightarrow \mathrm{End}\left(M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m} / b_N + c_{3N}\right) \oplus R^{\mathrm{univ}}_{\phi,\chi}/e_{N,\chi}\right)$$

$\uparrow$

finite card.,
bounded only in terms of N.

Lecture 15. $M \geq N$, $\Lambda_\infty \triangleright \mathfrak{a}_\infty$, $\Lambda_\infty/\mathfrak{a}_\infty = \mathcal{O}$ $r = \dim H^1_{L_\phi^\perp}(G_F, \mathrm{ad}^0 \bar{r}(1))$

$\Lambda_\infty = \mathcal{O}[[4|R| + 4[F:\mathbb{Q}] + r - 1 \text{ variables}]]$

$$\begin{tikzcd} & \Lambda_\infty \arrow[d] & & \\ R^{\mathrm{loc}}_\chi[[x_1,\dots,x_t]] \arrow[r, two heads] & R_{M,N,\chi} \arrow[r, phantom, "\curvearrowright"] \arrow[dr] & M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(b_N + c_{3N}) \arrow[d] & \\ & & R^{\mathrm{univ}}_{\phi,\chi}/e_{N,\chi} \curvearrowright M(U_\phi, \mathcal{O})_{\chi,m}/(b_N + c_{3N}) & \end{tikzcd}$$

rank indep. of N, M $\longleftarrow$ finite free / $\Lambda_\infty / b_N \otimes_{\mathcal{O}} \mathcal{O}[[U_\ell^1]]/c_{3N}$

$\nearrow$ dim $3|R| + 6[F:\mathbb{Q}] + 1$ $\quad t = |R| + r - 1$

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$\mathfrak{b}_N \underset{\text{open}}{\triangleleft} \Lambda_\infty$

$\mathfrak{c}_N \underset{\text{open, 2-sided ideal}}{\triangleleft} \mathcal{O}[[U_\ell^1]]$

$U_\ell^n = \ker\left(GL_2(\mathcal{O}_{F,\ell}) \to GL_2(\mathcal{O}_F/\ell^n)\right)$

$\mathfrak{c}_N = \ker\left(\mathcal{O}[[U_\ell^1]] \to \mathcal{O}[U_\ell^1/U_\ell^N]\right)$

$R^{\mathrm{univ}}_{\phi,\chi}/\mathfrak{e}_{N,\chi}$ $\qquad$ $\mathfrak{e}_{N,\chi} \underset{\text{open}}{\triangleleft} R^{\mathrm{univ}}_{\phi,\chi}$

-mod λ indep. of χ

$\chi = \begin{cases} \chi_0 \\ 1 \end{cases}, \ \chi_{0,v} \neq 1, \ \forall v \in R$

$$GL_2(F_\ell) \supset G_N = \bigcup_{\substack{a,d \in F_\ell^\times \\ |v(a/d)| \leq N \\ \forall v | L}} GL_2(\mathcal{O}_{F,\ell}) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} GL_2(\mathcal{O}_{F,\ell})$$

$GL_2(F_\ell) = \bigcup G_N$

$N \leq i \leq 3N$

$g \in G_N,$

$$\begin{tikzcd} M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(\mathfrak{b}_N + \mathfrak{c}_{3N}) \arrow[r, two heads] \arrow[d, "g"] & M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(\mathfrak{b}_N + \mathfrak{c}_i) \arrow[d, "g"] \\ M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(\mathfrak{b}_N + \mathfrak{c}_{2N}) \arrow[r, two heads] & M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(\mathfrak{b}_N + \mathfrak{c}_{i-N}) \end{tikzcd}$$

$g \circ h = gh : M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(\mathfrak{b}_N + \mathfrak{c}_{3N}) \longrightarrow M(U^\ell_{Q_M}, \mathcal{O})^{+,\square}_{\chi,m}/(\mathfrak{b}_N + \mathfrak{c}_N)$

and if g or $h \in U_\ell^1$ $\longrightarrow$ replace by $/\mathfrak{b}_N + \mathfrak{c}_{3N}$

$g \in F_\ell^\times \Rightarrow g$ trivial

$\# U_\ell^1 \backslash G_N / U_\ell^1 F_\ell^\times < \infty$

diagram (M, N) of level N

choose $(M_1, N_1), (M_2, N_2), \cdots$ $\quad N_1 < N_2 < N_3 < \cdots$

$$\mathrm{diag}(M_{i+1}, N_i) \simeq \mathrm{diag}(M_i, N_i)$$

~~N_1~~. finitely many isom. classes of diag of level N_1. and so one occurs as $\mathrm{diag}(M, N_1)$ for ∞ ly many M. choose one such M_1

finitely many iso. classes of diags of level N_2 such that "mod N_1" we get $\mathrm{diag}(M_1, N_1)$, one occurs as $\mathrm{diag}(M, N_2)$ for ∞'ly many M. choose one $M_2 \cdots$

$\underset{i}{\varprojlim}$:

power series ring $/\mathcal{O}$ in $4|R| + 4[F:\mathbb{Q}] + r - 1$ variables

$\Lambda_\infty / \mathfrak{a}_\infty = \mathcal{O}$

dim $6[F:\mathbb{Q}] + 3|R| + 1$

$$\begin{tikzcd}
 & \Lambda_\infty \rhd \mathfrak{a}_\infty \arrow[d] & GL_2(F_\ell) \arrow[d, phantom, "\curvearrowright"] & \mathcal{U}_\ell^1 \text{ actions agree} \arrow[l] \arrow[d] \\
\boxed{R_\chi^{loc}}[\![X_1, \ldots, X_t]\!] \arrow[r] & R_{\chi,\infty} \arrow[r, phantom, "\curvearrowright"] \arrow[d] & M_{\chi,\infty} \arrow[d, "\text{mod out by } \mathfrak{a}_\infty"] \arrow[r, phantom, "\text{finite free}"] & \Lambda_\infty \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\mathcal{U}_\ell^1]\!] \\
 & R_{\phi,\chi}^{univ} \arrow[r, phantom, "\curvearrowright"] & M(\mathcal{U}_\phi^\ell, \mathcal{O})_{\chi,m} &
\end{tikzcd}$$

apply $\mathrm{Hom}(P_{B_m}, -)$

for $\chi = \chi_0$, $\quad \chi_{0,v} \neq 1, \forall v \in R$

and 1

mod λ, independent of χ.

$$\begin{tikzcd}
 & \Lambda_\infty \arrow[d] & & & \\
R_\chi^{loc}[\![X_1, \ldots, X_t]\!] \arrow[r, two heads] & R_{\chi,\infty} \arrow[d] & \mathfrak{a}_\infty \arrow[d, maps to] \arrow[r, phantom, "\curvearrowright"] & \mathcal{M}_{\chi,\infty} \arrow[d, "\text{mod out by } \mathfrak{a}_\infty"] & \Lambda_\infty \underset{\mathcal{O}}{\hat{\otimes}} \underset{v|\ell}{\hat{\otimes}} R^{ps}_{\bar{r}_m}|_{G_{F_v}, \varepsilon_\ell^{-1}} \arrow[l, hook'] \arrow[d] \\
 & R_{\phi,\chi}^{univ} & \mathcal{O} \arrow[r, phantom, "\curvearrowright"] & \mathcal{M}(\mathcal{U}_Q^\ell, \mathcal{O})_{\chi,m} & \underset{v|\ell}{\hat{\otimes}} R^{ps}_{\bar{r}_m}|_{G_{F_v}, \varepsilon_\ell^{-1}} \arrow[l, hook']
\end{tikzcd}$$

ps $\leftarrow$ action from $GL_2(F_\ell)$

ps / comes both from $GL_2(F_\ell)$ action + from Galois theory

finite $/ R_{\chi,\infty}$

$\Rightarrow \mathcal{M}_{x,\infty}$ finite / $R_{x,\infty}$

$\Downarrow$

$\mathcal{M}_{x,\infty}$ flat / Λ_∞

A / Λ_∞ f.g.

$X \in \mathcal{C}_{GL_2(F_\ell),1}(\mathcal{O})_{B_m}$

~~flat~~ w/ Λ_∞-action, flat / Λ_∞

$F^\bullet \longrightarrow A \longrightarrow 0$ fin. free resolution as Λ_∞-modules

$X \otimes_{\Lambda_\infty} F^\bullet \longrightarrow X \otimes_{\Lambda_\infty} A \to 0$ exact

$$\begin{tikzcd} \mathrm{Hom}(P_{B_m}, X \otimes_{\Lambda_\infty} F^\bullet) \arrow[r] \arrow[d, phantom, "\wr\wr"] & \mathrm{Hom}(P_{B_m}, X \otimes_{\Lambda_\infty} A) \arrow[r] & 0 \\ \mathrm{Hom}(P_{B_m}, X) \otimes_{\Lambda_\infty} F^\bullet & & \end{tikzcd}$$

exact

$$\mathrm{Tor}_i^{\Lambda_\infty}(A, \mathrm{Hom}(P_{B_m}, X)) = \begin{cases} 0 & \text{if } i>0 \\ \mathrm{Hom}(P_{B_m}, X \otimes_{\Lambda_\infty} A) & \text{if } i=0 \end{cases}$$

$\forall$ f.g. A. $\mathrm{Hom}(P_{B_m}, X)$ flat / Λ_∞.

$$\dim_{R_x^{loc}[\![\underline{x}]\!]}(\mathcal{M}_{x,\infty}) = 2 + 4|R| + 4[F:\mathbb{Q}] + \overbrace{\dim_{R_x^{loc}[\![\underline{x}]\!]} \mathcal{M}(U_{\mathbb{Q}}^{\ell}, \mathcal{O})_{x,m}}^{\geq 1 + 2[F:\mathbb{Q}]} - 1$$

$$\geq 2 + 4|R| + 6[F:\mathbb{Q}]$$

$\mathrm{Supp}_{R_x^{loc}[\![\underline{x}]\!]}(\mathcal{M}_{x,\infty}) = \bigcup$ irred compts of $\mathrm{Spec}\, R_x^{loc}[\![\underline{x}]\!]$.

$x = x_0$: $R_{x_0}^{loc}$ is irred $\therefore \mathrm{supp}_{R_{x_0}^{loc}[\![\underline{x}]\!]} \mathcal{M}_{x_0,\infty} = \mathrm{Spec}\, R_{x_0}^{loc}[\![\underline{x}]\!]$

$\therefore \mathrm{supp}_{R_{x_0}^{loc}/\lambda[\![\underline{x}]\!]}(\mathcal{M}_{x_0,\infty}/\lambda) = \mathrm{Spec}\, R_{x_0}^{loc}/\lambda[\![\underline{x}]\!]$.

$\mathrm{Supp}_{R_1^{loc}/\lambda[[\underline{X}]]}(\mathcal{M}_{1,\infty}/\lambda) \quad = \mathrm{Spec}\ R_1^{loc}/\lambda[[\underline{X}]]$

reduction gives a bijection

$$\begin{tikzcd} \mathrm{Irr}(\mathrm{Spec}\ R_1^{loc}[[\underline{X}]]) \arrow[d] \\ \mathrm{Irr}(\mathrm{Spec}\ R_1^{loc}/\lambda[[\underline{X}]]) \end{tikzcd}$$

$\rightarrow \quad \mathrm{Supp}_{R_1^{loc}[[\underline{X}]]}(\mathcal{M}_{1,\infty}) = \mathrm{Spec}\ R_1^{loc}[[\underline{X}]]$

$\mathrm{Supp}_{R^{univ}_{\phi,1}}(M(U^1_\phi, \mathcal{O})_{1,m}) = \mathrm{Spec}\ R^{univ}_{\phi,1}$

and $\mathcal{M}_{1,\infty}$ is λ-torsion free

$\ker(R^{univ}_{\phi,1} \twoheadrightarrow \mathbb{T}(U^1_Q, \mathcal{O})_{1,m})$ nilpotent

Connectedness dimension

R complete local noetherian ring $\Rightarrow$ Spec R conn'd.

$c(R) = \min(\dim W :\ W \subset \mathrm{Spec}\ R$ closed, $\mathrm{Spec}\ R - W$ is disconnected$)$

conn'd dim

(call ϕ disconn'd)

$\leq \dim R$

R domain : $c(R) = \dim R$

Prop. R complete local noeth domain, $f_1, \ldots, f_r \in m_R$ $\Big] \Rightarrow c(R/(f_1, \ldots, f_r)) \geq \dim R - r - 1.$

$I \triangleleft R$, M an R-mod, $\Gamma_I(M) = \{m \in M : I^n m = 0$ for some $n\}$

$= \varinjlim_n \mathrm{Hom}(R/I^n, M)$

$\uparrow$ right derived functors

$H^i_I(M) = \varinjlim_n \mathrm{Ext}^i_R(R/I^n, M)$

$\Gamma_I(M)$ and hence $H^i_I(M)$ only depend on $V(I) \subset \mathrm{Spec}\ R$

$\shortparallel$

Spec(R/I) as set

FACT 1) $H^i_{(f_1,\cdots,f_r)}(M)$ is the homology of

$$M \to \bigoplus_{i_0} M_{f_{i_0}} \to \bigoplus_{i_0<i_1} M_{f_{i_0}f_{i_1}} \to \cdots \to M_{f_1\cdots f_r}$$

$$M_{f_{i_0}\cdots \widehat{f_{i_j}} \cdots f_{i_s}} \to M_{f_{i_0}\cdots f_{i_s}}$$

is $(-1)^j$ x obvious map

If $V(I) = V(f_1,\cdots,f_r)$, then $H^i_I(M) = 0$ for $i > r$.

2) If R is a domain, $I \lhd R$, $\sqrt{I} \neq m$, then $H^i_I(M) = 0$ for $i \geq \dim R$.

3) If $M \neq 0$ f.g. $/R$, then $H^{\dim R}_{m_R}(M) \neq 0$. $m_R \lhd R$ max'l ideal

4) $J_1, J_2 \lhd R$, then LES

$$\cdots \to H^i_{J_1+J_2}(M) \to H^i_{J_1}(M) \oplus H^i_{J_2}(M) \to H^i_{J_1 \cap J_2}(M) \to \cdots$$

Lecture 16 connectedness dimension

R CLN ring, $c(R) = \min\left\{ \dim W : \begin{array}{l} W \subset \operatorname{spec} R \text{ closed} \\ \operatorname{Spec} R - W \text{ disconn'd} \end{array} \right\} \leq \dim R$

ϕ disconn'd

$= \min\left\{ \dim C_1 \cap C_2 : \begin{array}{l} C_1, C_2 \text{ are unions of irred compts of } \operatorname{Spec} R, \\ C_1 \cup C_2 = \operatorname{Spec} R \end{array} \right\}$

$I \lhd R$

M R-module $\quad T_I(M) = I^\infty$ torsion in M, $H^i_I(M)$ derived functors

R domain

1) $I_1, I_2 \trianglelefteq R$, $\sqrt{I_1 + I_2} = m$, $\sqrt{I_i} \neq m$. then $V(I_1 \cap I_2)$ cannot be cut out by less than $\dim R - 1$ elements.

<u>Pf</u> $d = \dim R$.

$\sqrt{I_i} \neq m$, R domain

$$\begin{tikzcd}
 & & 0 \arrow[d, phantom, sloped, "\neq"] & 0 \qquad 0 \arrow[d, phantom, sloped, "="] \\
\cdots \arrow[r] & H^{d-1}_{I_1 \cap I_2}(R) \arrow[r] \arrow[d, dash] & H^d_m(R) \arrow[r] \arrow[d, phantom, sloped, "="] & H^d_{I_1}(R) \oplus H^d_{I_2}(R) \\
 & \therefore \neq 0 & H^d_{I_1+I_2}(R) &
\end{tikzcd}$$

$\therefore$ $V(I_1 \cap I_2)$ cannot be cut out by less than $\dim R - 1$ equations.

2) R domain, $J_1, J_2 \trianglelefteq R$, $\dim R/J_1 > \dim R/J_1 + J_2$

$\dim R/J_2 > \dim R/J_1 + J_2$

then $V(J_1 \cap J_2)$ cannot be cut out by less than $\dim R - \dim R/J_1+J_2 - 1$ elts of R.

<u>Pf</u> induction on $d = \dim R/J_1 + J_2$ $\quad d = 0$ ✓

$d > 0$. let $p_1, \ldots, p_n$ be min primes above $J_1 + J_2$

$p_i \neq m$, $\therefore$ $m \not\subset \bigcup_i p_i$. Choose $a \in m - \bigcup_i p_i$

$J_i' = (J_i, a)$, $\dim R/J_1' + J_2' = \dim R/J_1 + J_2 - 1$

$$\wedge$$

$\dim R/J_i' \geq \dim R/J_i - 1$

$\therefore$ $V(J_1' \cap J_2')$ cannot be cut out by fewer than $\dim R - \dim R/J_1 + J_2$ elts

$\therefore$ $V(J_1 \cap J_2)$ cannot be cut out by fewer than $\dim R - \dim R/J_1 + J_2 - 1$ elts

Prop. If R is a complete noeth local domain, and $f_1, \dots, f_m \in R$, then

$$c(R/(f_1, \dots, f_m)) \geq \dim R - m - 1.$$

Pf. If $V(f_1, \dots, f_m) \subset \operatorname{Spec} R$ is irred, $c \geq \dim R - m$.

If not, then $\exists\, C_1, C_2 \subset \operatorname{Spec} R/_{(f_1, \dots, f_m)}$ unions of irred components, $C_1 \cup C_2 = \operatorname{Spec} R$.

$c = \dim C_1 \cap C_2$, $C_i = V(J_i)$, $c < \dim C_i$. (wlog C_1, C_2 have no irred compts in common).

$$(f_1, \dots, f_m) \subset J_i \triangleleft R$$

$$V(J_1 \cap J_2) = V(f_1, \dots, f_m)$$

$$m \geq \dim R - \underbrace{\dim R/J_1 + J_2}_{c} - 1, \quad c \geq \dim R - m - 1.$$

$-\!\!\xi$

$F|\mathbb{Q}$ totally real, even degree, ℓ splits completely, $\bar{r} : G_F \to GL_2(\mathbb{F})$ abs. irred.

$L|\mathbb{Q}_\ell$, $\mathcal{O} = \mathcal{O}_L$, $\mathcal{O}/\lambda = \mathbb{F}$. $\det \bar{r} = \bar{\varepsilon}_\ell^{-1}$, $\bar{r}$ ramified only above ℓ

Case of interest. $\bar{r} = \operatorname{Ind}_{G_E}^{G_F} \bar{\chi}$, $F(\zeta_\ell) \supset E \underset{2}{\supsetneq} F$.

R finite set of places of F, $v \in R \Rightarrow q_v \equiv 1\ (\ell)$, $\bar{r}|_{G_{F_v}} = 1$

Consider lifts of $\bar{r}$ which unramified away from ℓ and R w/ $\det = \varepsilon_\ell^{-1}$.

$\exists$ universal such lift $r^{univ} : G_F \to GL_2(R_\phi^{univ})$

Define Hecke algebra $\mathbb{T}_\phi$ acting on completed cohomology : $R_\phi^{univ} \twoheadrightarrow \mathbb{T}_\phi$

Def'n If $\mathfrak{p} \in \operatorname{Spec} R_\phi^{univ}$ we call it *promodular* if in image of $\operatorname{Spec} \mathbb{T}_\phi$.

Thm. A Suppose $\mathfrak{p} \in \operatorname{Spec} R_\phi^{univ}$ is promodular, and $\dim R_\phi^{univ}/\mathfrak{p} \geq [F:\mathbb{Q}] + 3|R| + 2$, then any prime contained in $\mathfrak{p}$ is also promodular.

$\Downarrow$

Thm. B If $[F:\mathbb{Q}] \geq 4|R| + 2$, then every prime of $\operatorname{Spec} R_\phi^{univ}$ is promodular.

Pf. $\dim \mathbb{T}_\phi \geq 1 + 2[F:\mathbb{Q}]$ $\therefore$ R_ϕ^{univ} has some promodular prime of dim $1 + 2[F:\mathbb{Q}]$

$\geq [F:\mathbb{Q}] + 3|R| + 2$

$\therefore$ $\operatorname{Spec} R_\phi^{univ}$ has some promodular irred comp by Thm A.

$C_1 =$ union of promodular irred compts of $\operatorname{Spec} R_\phi^{univ} \neq \phi$

$C_2 =$ union of irred comp of $\operatorname{Spec} R_\phi^{univ}$ which are not promodular

We want to prove that $C_2 = \phi$. Suppose not,

If $\dim C_1 \cap C_2 \geq [F:\mathbb{Q}] + 3|R| + 2$, then THM A shows some irred compt of C_2 is promodular. $\Rightarrow \Leftarrow$.

$c(R_\phi^{univ}) = ?$ $\quad R_\phi^{univ} = \widetilde{R}_\phi^{univ} / (|R| \text{ equations})$

$= \mathcal{O}[[x_1, \dots, x_{h_1}]] / (f_1, \dots, f_{h_2} + |R|))$

$h_i = \dim H^i(G_F, R \cup \{v|\ell\}, \operatorname{ad}^0 \bar{r})$

$$c(R_\phi^{univ}) \geq 1 + h_1 - h_2 - |R| - 1$$

$$= h_0 + \sum_{v|\infty} (3-1) - |R|$$

$$= 2[F:\mathbb{Q}] - |R|$$

$$\geq [F:\mathbb{Q}] + 3|R| + 2$$

Look for $F'|F$ Galois soluble s.t. ℓ splits completely in F'

$v \in R$, then v unram. in F' w/ res. deg. f.

and $[F':F][F:\mathbb{Q}] \geq 4 \underbrace{|R| \frac{[F':F]}{f}} + 2$ F' linearly disjoint from $\bar{F}^{\ker \bar{r}}$

i.e. $[F':F]\left([F:\mathbb{Q}] - \frac{4|R|}{f}\right) \geq 2$ $\#R_{F'} = \{w \text{ place of } F' : w|_F \in R\}$

Choose f s.t. $[F:\mathbb{Q}]f > 4|R|$

and $[F':F]$ suff. large

then $\operatorname{Spec} R^{univ}_{\phi, F'} = \operatorname{Spec} \mathbb{T}_{\phi, F'}$.

Cor If $[F:\mathbb{Q}] \geq 4|R| + 2$, and r is a regular de Rham lift of $\bar{r}$, $\det r = \varepsilon_\ell^{-1}$

and r unram. outside $\{v|\ell\} \cup R$ and for $v \in R$, $\sigma \in I_{F_v}$, $\operatorname{tr} r(\sigma) = 2$,

then r is modular.

Thm A $\mathfrak{p} \in \operatorname{Spec} R_\phi^{univ}$ promodular, $\dim R_\phi^{univ}/\mathfrak{p} \geq [F:\mathbb{Q}] + 3|R| + 2$.

$\Rightarrow$ any prime contained in $\mathfrak{p}$ is also promodular.

$$R_E^{dih} = R_\phi^{univ} \big/ \langle \mathrm{tr}\, \rho^{univ}(\sigma) = 0, \quad \forall \sigma \in G_F - G_E \rangle.$$

if $\bar\rho$ was also induced from $E'|F$ quad, also introduce $R_{E'}^{dih}$.

$\underset{\text{prime}}{p} \triangleleft R_\phi^{univ}$, then $\rho^{univ} \otimes k(p)$ is induced from G_E

$\iff p \in \mathrm{Spec}\, R_E^{dih}$.

<u>Lemma</u> $\dim R_E^{dih}/\lambda \leq [F:\mathbb{Q}]$

<u>Pf</u> q prime of R_E^{dih}/λ, $\rho^{univ} \otimes k(q) = \mathrm{Ind}_{G_E}^{G_F} \chi$.

Let $\sigma \in G_F - G_E$. $\exists\ \tau \in G_E$ s.t. $\bar\chi(\sigma\tau\sigma^{-1}) \neq \bar\chi(\tau)$

$\bar\rho = \mathrm{Ind}_{G_E}^{G_F} \bar\chi$. choose basis of R_E^{dih}/q, $\rho^{univ}(\tau) = \begin{pmatrix} \chi(\tau) & 0 \\ 0 & \underset{\nearrow \text{ different mod } m}{\chi(\sigma\tau\sigma^{-1})} \end{pmatrix}$

$\therefore \forall \tau' \in G_E$, $(\rho^{univ} \bmod q)(\tau') = \begin{pmatrix} \chi(\tau') & 0 \\ 0 & \chi(\sigma\tau'\sigma^{-1}) \end{pmatrix}$

$\rho^{univ} \bmod q' \simeq \mathrm{Ind}_{G_E}^{G_F} \chi$.

$\chi : G^{ab}_{E,\, ~~\mathrm{Ra}~~\{v|\ell\}} \longrightarrow (R^{dih}/q)^\times$

$\quad\quad\quad\wr\wr$

$1 + \mathcal{O}_{E,\ell} \xrightarrow{\text{finite index}} \mathbb{A}_E^\times \big/ \overline{(\widehat{\mathcal{O}}_E^{\ell})^\times E^\times (E_\infty^\times)^\circ}$

$\wr\wr$

$\mathbb{Z}_\ell^{2[F:\mathbb{Q}]}$ $\qquad \chi\chi^\sigma = \varepsilon_\ell^{-1}$.

$\smile$ choose generators $\tau_1, \dots, \tau_{[F:\mathbb{Q}]}, \sigma_{\tau_1}, \dots, \sigma_{\tau_{[F:\mathbb{Q}]}}$

$$\begin{tikzcd} \mathbb{F}[[T_1, \dots, T_{[F:\mathbb{Q}]}]] \arrow[d] \\ \mathbb{F}[[\chi(\tau_i') - 1]]_{i=1,\dots,[F:\mathbb{Q}]} \arrow[d] \\ R_E^{dih}/q \arrow[u, "\text{finite}", bend left] \end{tikzcd}$$

${}^{\psi}\chi(p)^N$

Lecture 17 $\quad R_\phi^{univ} \twoheadrightarrow \mathbb{T}_\phi$

Thm. Suppose $\mathfrak{p}$ is a promodular prime of R_ϕ^{univ} (i.e. $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_\phi$) and $\dim R_\phi^{univ}/\mathfrak{p} \geq [F:\mathbb{Q}] + 3|R| + 2$, then any prime $\mathfrak{q} \subset \mathfrak{p}$ is also promodular.

Lemma Suppose $\mathfrak{p}$ is a prime of R_ϕ^{univ} w/ $\dim R_\phi^{univ}/\mathfrak{p} \geq [F:\mathbb{Q}] + 3|R| + 2$, then $\exists\, \mathfrak{q} \supset \mathfrak{p}$ a prime w/ $\dim R_\phi^{univ}/\mathfrak{q} = 1$ s.t.

1) $\ell \in \mathfrak{q}$
2) $(r^{univ} \bmod \mathfrak{q})|_{G_{F_v}} = 1, \ \forall v \in R$
3) $ad^\circ r^{univ} \otimes k(\mathfrak{q})$ is irred $\longrightarrow$ $r^{univ} \otimes k(\mathfrak{q})$ is not dihedral.

Pf $v \in R$

$$\begin{tikzcd} R_v^\square \arrow[d] \\ R_\phi^{univ} \end{tikzcd}$$

$\exists\, a_{v1}, a_{v2}\ a_{v3} \in m_{R_v^\square}$ s.t. $R_v^\square/(a_{v1}, a_{v2}, a_{v3})$ is 0-dim'l.

$$\dim R_\phi^{univ}/(\mathfrak{p}, \lambda, a_{vi} : v \in R, i=1,2,3) \geq [F:\mathbb{Q}]+1$$

$\ker(R_\phi^{univ} \twoheadrightarrow R^{dih}) = J$

$\uparrow$ $\dim \leq [F:\mathbb{Q}]$

Lemma A noetherian local ring, $I \triangleleft A$, $V(I) \neq \operatorname{Spec} A$.

$\Rightarrow \exists\, \mathfrak{q} \in \operatorname{Spec} A - V(I)$ w/ $\dim A/\mathfrak{q} = 1$

[Apply with
$A = R_\phi^{univ}/(\mathfrak{p}, \lambda, a_{vi})$

$I = J + (\mathfrak{p}, \lambda, a_{vi})/(\mathfrak{p}, \lambda, a_{vi})$

$\dim A/I \leq [F:\mathbb{Q}]$

Pf $I \subset$ nilradical $\rightarrow$ ✓

else $\exists\, a \in I$, a not nilpotent.

$A_a \neq (0)$, $\quad \mathfrak{p}$ max ideal of A_a, $\mathfrak{p}^c =$ preimage of $\mathfrak{p}$ in A

$$(A/\mathfrak{p}^c)_a = A_a/\mathfrak{p} \text{ field} \Rightarrow \dim A/\mathfrak{p}^c \leq 1$$

Suffices to prove

Thm If q is a promodular prime of R_ϕ^{univ} w/

1) $l \in q$
2) $\dim R_\phi^{univ}/q = 1$
3) $\rho^{univ} \bmod q \,|_{I_{F_v}} = 1, \ \forall v \in R$
4) $\mathrm{ad}^0 \rho^{univ} \otimes k(q)$ abs irred.

then $R_{\phi,q}^{univ} \twoheadrightarrow \mathbb{T}_{\phi,q}$ has nilpotent kernel.

Pf. $B = R_\phi^{univ}/q$, $\quad A =$ normalization of B, $A \simeq \mathbb{F}'[[T]]$, $\mathbb{F}'|\mathbb{F}$ finite

a) WLOG $\mathbb{F}' = \mathbb{F}$:

$$\begin{tikzcd} L' \arrow[r, phantom, "|"] \arrow[d] & L \arrow[d] \\ \mathcal{O}'/\lambda' = \mathbb{F}' & \mathbb{F} \end{tikzcd}$$

$\beta' \subset q'$

$$\begin{tikzcd} R_\phi^{univ} \otimes_{\mathcal{O}} \mathcal{O}' \arrow[r, two heads] & \mathbb{T}_\phi \otimes \mathcal{O}' & q' \arrow[d, dash] \\ R_\phi^{univ} \arrow[r, two heads] \arrow[u] & \mathbb{T}_\phi \arrow[u] & q \end{tikzcd}$$

$\beta \subset q$

b) $\exists\, v_0 \notin R \cup \{v | l\}$ s.t. $\mathrm{tr}\,(\overbrace{\rho^{univ} \bmod q}^{\rho_q})\,(\mathrm{Frob}_{v_0}) \notin \mathbb{F}$

$B \,\Big|\, \overline{\mathbb{F}[\mathrm{tr}\, \rho_q(\mathrm{Frob}_{v_0})]}$ finite

$\therefore \exists\, v_2, v_2, \dots, v \notin R \cup \{v | l\}$ s.t.

B is top. gen. $/\mathbb{F}$ by $\mathrm{tr}\, \rho_q(\mathrm{Frob}_{v_i})$, $i = 0, \dots, 2$

$\underline{P} = \{v_0, \dots, v_2\}$

$$\begin{tikzcd}
 & \text{framed at all } v \in R \cup \{v|\ell\} \cup P \arrow[d, bend right] & & \\
R^{loc}_x \arrow[r] \arrow[d, equal] & R^{\square}_{Q,x} \arrow[r, two heads] & R^{univ}_{\phi,x} \arrow[r, two heads] & R^{univ}_{\phi,x}/\lambda \\
\widehat{\bigotimes}_{v\in R} R^{\square}_{\chi_v} \hat{\otimes} \widehat{\bigotimes}_{v|\ell} R^{\square}_v \hat{\otimes} \widehat{\bigotimes}_{v\in P} R^{\square,ur}_v & \text{deformations only ramified at } R \cup \{v|\ell\} \cup Q \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

primes $q_{x,loc}$ $\underbrace{q_{Q,x} \quad q_x \quad q}_{\text{quots all } B}$

$$R^{loc}_x / q_{x,loc} \simeq B.$$

c) $B \supset T^c A$ for some c: A is f.g. as a B-module (B excellent)

$\therefore \exists\, b \in B$ w/ $bA \subset B$ $\Big)$ $T^c A \subset B$

$b = T^c$ unit in A

L_ϕ = trivial @ all $v \in R \cup P \cup \{v|\ell\}$, unram. elsewhere

Prop. $r = \dim_{k(q)} H^1_{L_\phi^\perp}(G_F, \underbrace{ad^0 r^{univ} \otimes k(q)}_{r_q}(1)) = H^1(G_F, P\cup R\cup\{v|\ell\}, ad^0 r^{univ} \otimes k(q))$

$\exists\, c$, $\forall N$, $\exists\, Q_N$ a set of primes of F s.t.

1) $v \in Q_N \Rightarrow v \equiv 1\ (\ell^N)$

2) $|Q_N| = r$

3) $v \in Q_N \Rightarrow r_q(\mathrm{Frob}_v)$ has eigenvalues α_q, β_q,

4) $q_{Q_N,x}/(q^2_{Q_N,x}, q_{x,loc}) \otimes_B A \simeq A^g \oplus M_N$ $\ell(A/\underbrace{(\alpha_q-\beta_q)^2}_{(\mathrm{tr}\, r_q(\mathrm{Frob}_q))^2 - 4\det r_q(\mathrm{Frob}_q)}) < c$.

$g = |R| + |P| + r - 1$, $\ell_A(M_N) < c$.

5) $\exists\ R_x^{loc}[[x_1, \dots, x_g]] \longrightarrow R^{\square}_{Q_N, x}$ s.t.

$$x_i \longmapsto \text{elt of } q_{Q_N,x}$$

$$\ell\left(q_{Q_N,x} / (q_{Q_N,x}^2\ q_{x,loc},\ x_1, \dots x_g)\right) < c$$

Pf **Step 1.** condition 5 follows from 1-4.

e_i be standard basis of A^g. Suppose $B \supset T^{c_1}A$

$$T^{c_1} e_i \in q_{Q_N,x} / (q_{Q_N,x}^2,\ q_{x,loc}) \qquad T^{c_2} M_N = 0$$

$$\overset{\rotatebox{90}{$\in$}}{m}$$

$$T^{c_2} m = \sum a_i e_i \in A^g$$

$$T^{2c_1+c_2} m = \sum \underset{\overset{\uparrow}{B}}{(T^{c_1} a_i)}\ (T^{c_1} e_i)$$

then $T^{2c_1+c_2}$ will kill (*)

Step 2 We can replace 4) by 4').

4') $\forall n,\ \left| \ell_A\left(q_{Q_N,x} / (q_{Q_N,x}^2,\ q_{x,loc}) \otimes_B A/T^n\right) - ng \right| < c$

Pf $q_{Q_N,x} / (q_{Q_N,x}^2,\ q_{x,loc}) \otimes A \simeq A^{g'} \oplus M_N'$ ← finite length

$$\ell\left(q_{Q_N,x} / (q_{Q_N,x}^2,\ q_{x,loc}) \otimes_B A/T^n\right) = ng' + \overbrace{\ell\left(M_N' / T^n M_N'\right)}^{\text{bounded length indep. of } n}$$

$\therefore g = g',\quad \ell(M_N') < c.$

Step 3 can replace 4') by 4'')

$$\ell\left(H^1_{\mathcal{L}_{Q_N}^\perp}\left(G_F, (\mathrm{ad}^\circ \bar{r})(1) \otimes A/T^n\right)\right) < C, \ \forall n$$

Pf $\ell\left(\mathfrak{q}_{Q_N,\chi} / (\mathfrak{q}^2_{Q_N,\chi}, \mathfrak{q}_{\chi,\mathrm{loc}}) \otimes_B A/T^n\right)$

$$= -4n + \sum_{v \in P \cup R \cup \{v \mid \ell\}} \ell\left(H^0(G_{F_v}, \mathrm{ad}^\circ r_\mathfrak{q} \otimes A/T^n)\right)$$

$$+ \ell\left(H^1_{\mathcal{L}_{Q_N}}(G_F, \mathrm{ad}^\circ r_\mathfrak{q} \otimes A/T^n)\right) + 3n - \ell\left(H^0_{\cancel{\mathcal{L}}}(G_F, \mathrm{ad}^\circ r_\mathfrak{q} \otimes A/T^n)\right)$$

$=$

Lecture 18 Prop Suppose $\mathfrak{q}$ is a 1-dim'l prime of $\mathbb{T}_{1,\phi}/\lambda$, s.t. $r_\mathfrak{q} = r^{\mathrm{univ}} \bmod \mathfrak{q}$:
is trivial on G_{F_v}, $\forall v \in R$ and $\mathrm{ad}^\circ r_\mathfrak{q}$ is absolutely irred. $G_F \to GL_2(k(\mathfrak{q}))$

Then $R^{\mathrm{univ}}_{1,\phi,\mathfrak{q}} \longrightarrow \mathbb{T}_{1,\phi,\mathfrak{q}}$ has nilp. kernel.

normalization of $\mathbb{T}_{1,\phi}/\mathfrak{q} = \mathbb{F}[[T]]$.

L. $r = \dim_{k(\mathfrak{q})} H^1_{\mathcal{L}_\phi^\perp}(G_F, \mathrm{ad}^\circ r_\mathfrak{q}(1))$, then $\exists C$ s.t. $\forall N$, $\exists Q_N$ a set of primes of F s.t. 1) $\# Q_N = r$

2) $v \in Q_N \Rightarrow Nv \equiv 1\ (\ell^N)$

(3) $Q_N \cap (R \cup \{v \mid \ell\} \cup P) = \phi$. and $\forall v \in Q_N$, $r_\mathfrak{q}(\mathrm{Frob}_v)$ has distinct eigenvalues α_v, β_v and $v_T((\alpha_v - \beta_v)^2) < C$.

4) $\mathfrak{q}_{x,Q_N} / (\mathfrak{q}_{x,Q_N}^2, \mathfrak{q}_{x,loc}) \otimes A \simeq A^g \oplus M_N$

4'') $\ell(H^1_{\mathcal{L}^{\perp}_{Q_N}}(G_F, ad^\circ \bar{r})(1) \otimes A/T^n)) < C, \forall n$

where $g = |R| + |P| + 2 - 1$ and $\ell_A(M_N) < C$.

5) $\exists\ R_x^{loc}[[X_1, \dots X_g]] \longrightarrow R^{\square}_{x,Q_N}$ s.t. $\ell(\mathfrak{q}_{x,Q_N} / (\mathfrak{q}_{x,Q_N}^2, \mathfrak{q}_{x,loc}, X_1, \dots X_g)) < C$

$X_i \longmapsto \mathfrak{q}_{x\,Q_N}$

have shown $\ell(\mathfrak{q}_{Q,x} / (\mathfrak{q}_{Q_N,x}^2, \mathfrak{q}_{x,loc}) \otimes A/T^n) = -n$

$$+ \sum_{v \in \underbrace{P \cup R \cup \{v|\ell\}}_{=S}} \ell(H^0(G_{F_v}, ad\, r_q \otimes A/T^n)) + \ell(H^1_{\mathcal{L}_{Q_N}}(G_F, ad^\circ r_q \otimes A/T^n))$$

$$- \ell(H^0(G_F, ad^\circ r_q \otimes A/T^n))$$

$$= n(-1 + \underbrace{|P| + |R| + 2}_{g}) + \text{bdd by } \underset{\text{indep of } N}{C_2}$$

(can replace 4'') by 4''')

$$\ell\left(\begin{matrix}\ker \\ \operatorname{coker}\end{matrix}\left(H^1(G_{F,S}, ad^\circ r_q(1) \otimes A) \xrightarrow{\otimes} \bigoplus_{v \in Q_N} \left(ad^\circ r_q(1) \otimes A / (\mathrm{Frob}_v - 1)\right)\right)\right) < C$$

<u>Step 5</u> Sufficient to find $\sigma_1, \dots, \sigma_2 \in G_F(\zeta_{\ell^\infty})$ s.t. $r_q(\sigma_i)$ have distinct eigenvalues and $H^1(G_{F,S}, ad^\circ r_q(1) \otimes k(q)) \xrightarrow{\sim} \bigoplus_{i=1}^{2} ad^\circ r_q(1) \otimes k(q) / (\sigma_i - 1)$

Lt let α_i, β_i = evals of σ_i

choose C s.t. $C \geq \mathrm{val}_T((\alpha_i-\beta_i)^2)$

(**) $C \geq \ell(H^1(G_{F,S}, \mathrm{ad}^0 r_q(1) \otimes A)^{\mathrm{tor}})$

$C \geq \ell\left(\begin{matrix}\ker\\ \mathrm{coker}\end{matrix}\left(H^1(G_{F,S}, \mathrm{ad}^0 r_q(1) \otimes A) \longrightarrow \bigoplus_{i=1}^{2} \mathrm{ad}^0 r_q(1) \otimes A \big/ (\sigma_i - 1)\right)\right)$

given N, choose v_i s.t. $\mathrm{Frob}_{v_i} \in G_{F(\zeta_{\ell^N})}$ $(\Rightarrow q_{v_i} \equiv 1 \ (\ell^N))$

$\phi_j(\mathrm{Frob}_{v_i}) \equiv \phi_j(\sigma_i) \quad (T^{C+1})$

$r_q(\mathrm{Frob}_{v_i}) \equiv r_q(\sigma_i) \quad (T^{C+1})$

then 1) if $\alpha_{v_i}, \beta_{v_i}$ are eigenvalues of $r_q(\mathrm{Frob}_{v_i})$

then $\mathrm{val}_T((\alpha_{v_i}-\beta_{v_i})^2) \leq C$

2) $\ell(\ker(\underline{\circledast})) \leq C$ by (**)

3) $\ell(\mathrm{coker}(\circledast)) \leq 3C$

Step 6 STP: $\forall\ 0 \neq [\phi] \in H^1(G_{F,S}, \mathrm{ad}^0 r_q(1) \otimes k(q))$ $\exists\ \sigma \in G_{F(\zeta_{\ell^\infty})}$ s.t.

$r_q(\sigma)$ has distinct evals and $\phi(\sigma) \notin (\sigma-1)\,\mathrm{ad}^0 r_q(1) \otimes k(q)$.

Step 7 Let $F_\infty = \overline{F}^{\ker r_q}(\zeta_{\ell^\infty})$ and set

$$\begin{tikzcd} \Gamma = \mathrm{Gal}(F_\infty | F) \arrow[d, "r_q"', shift right] \arrow[d, "\text{cycle}", shift left] \\ PGL_2(A) \times \mathbb{Z}_\ell^\times \end{tikzcd}$$

STP $H^1(\Gamma, \mathrm{ad}^0 r_q(1) \otimes k(q)) = 0$.

$\Gamma_n = \ker(\Gamma \to PGL_2(A/T^n) \times (\mathbb{Z}/\ell^n\mathbb{Z})^\times)$

Pf $0 \neq [\phi] \in H^1(G_{F,S}, \mathrm{ad}^0 2q(1) \otimes k(q))$, then $\phi|_{G_{F_\infty}} \neq 0$

$0 \neq \phi \in \mathrm{Hom}(G_{F_\infty}, \mathrm{ad}^0 2q(1) \otimes k(q))$

$\mathrm{Im}\,\phi$ is G_F-invariant as $\phi(\sigma\tau\sigma^{-1}) = \sigma\phi(\tau)$, $\tau \in G_{F_\infty}$.

$\langle \mathrm{Im}\,\phi \rangle_{k(q)} = \mathrm{ad}^0 2q(1)$

Choose $\tau_0 \in G_{F(\zeta_{\ell^\infty})}$ s.t. $2q(\tau_0)$ has distinct eigenvalues.

Otherwise $\exists$ non-triv subspace where $G_{F(\zeta_{\ell^\infty})}$ acts by scalars.

$\Rightarrow \exists$ line preserved by G_F. $\Rightarrow\Leftarrow$ □

$\sigma = \tau\tau_0$, $\tau \in G_{F_\infty}$. $\phi(\sigma) = \phi(\tau) + \phi(\tau_0) \notin (\tau_0 - 1)(\mathrm{ad}^0 2q(1) \otimes k(q))$

Step 8. $H^1(\Gamma, \mathrm{ad}^0 2q(1) \otimes k(q)) = 0$.

Pf STP as $k(q)$ replaced by A. or even by A/T

STP $H^1(\Gamma/\Gamma_n, \mathrm{ad}^0 2q(1) \otimes A/T) = 0$.

induction on n: $n=1$, Γ/Γ_1 has order prime to ℓ ✓

$n>1$: $0 \to \overset{=0 \text{ by ind. hyp.}}{H^1(\Gamma/\Gamma_{n-1}, \mathrm{ad}^0 2q(1) \otimes A/T)} \to H^1(\Gamma/\Gamma_n, \mathrm{ad}^0 2q(1) \otimes A/T)$

$\to \mathrm{Hom}_\Gamma(\underset{\substack{\uparrow \\ \text{involves} \\ \mathrm{ad}^0 2q \otimes A/T,\ A/T}}{\Gamma_{n-1}/\Gamma_n}, \mathrm{ad}^0 2q(1) \otimes A/T) \quad = 0$

Lecture 19.

Prop Suppose q is a prime of $\mathbb{T}_{1,\phi/\lambda}$ of dim 1, s.t.

$r_q : G_F \longrightarrow GL_2(k(q))$ is trivial on G_{F_v}, $\forall v \in R$. and

$ad^0 r_q$ is absolutely irred. Then $R^{univ}_{1,\phi,q} \twoheadrightarrow \mathbb{T}_{1,\phi,q}$ has nilp. kernel

Choose auxiliary sets of primes Q_N s.t. ...

$$U_l^1 = \ker(PGL_2(O_{F,l}) \longrightarrow PGL_2(O_F/l))$$

$$\Lambda_\infty = O[[A_{v,ij}]]_{\substack{v \in R \cup P \cup \{v \mid l\} \\ i,j = 1,2}} \Big/ (A_{v_0,1,1}) \quad [[H_\infty]] \quad , \quad H_\infty = \mathbb{Z}_l^2$$

$$\triangledown$$

$$\mathfrak{a}_\infty = \langle A_{v,ij} , h-1 : h \in H_\infty \rangle$$

$$c_N = \ker\left(O[[U_l^1]] \longrightarrow O[PGL_2(O_{F,l}/l^N)]\right) \underset{open}{\triangleleft} O[[U_l^1]]$$

$$b_N \underset{open}{\triangleleft} \Lambda_\infty , \quad b_N \supset b_{N+1}, \quad \bigcap_N b_N = (0)$$

$$b_N \supset \langle A_{v,ij}, h-1 : h \in l^N \mathbb{Z}_l^2 \rangle$$

$e_{N,x} \underset{open}{\triangleleft} R^{univ}_{x,\phi}$

$$e_{N,x} \supset e_{N+1,x}, \quad \bigcap_N e_{N,x} = (0)$$

$e_{N,x}$ mod λ indep of x

$$e_{N,x} \subset \mathrm{Ann}_{R^{univ}_{x,\phi}}\left(M(U^l_\phi, O)_{x,m} \big/ (b_N + c_{3N})\right)$$

$$\begin{tikzcd}
\Lambda_\infty \arrow[d, "\square"'] & \\
R_{Q_N,x} \arrow[d] & \circlearrowright \quad M(U^l_{Q_N}, O)_{x,m} \otimes_{O[[H_\infty]]} \Lambda_\infty \text{ — fin. free } O[[A_{v,ij}]]/(A_{v_0,1,1})[H_{Q_N}][[U_l^1]] \\
R^{univ}_{\phi,x} & \circlearrowright \quad M(U^l_\phi, O)_{x,m} \big/ (b_N + c_{3N})
\end{tikzcd}$$

$$\begin{tikzcd}
 & R^{univ}_{\phi,x}/e_{N,x} \arrow[r, phantom, "\simeq"] & M(U^{\ell}_{\phi}, \mathcal{O})_{x,m}/(b_N + e_{3N}) \\
R^{loc}_x[[X_1,\dots,X_g]] \arrow[r] & \tilde{R}_{x,N} \arrow[u, two heads] \arrow[r, phantom, "\simeq"] & M_{x,N} \arrow[u, two heads, "\text{mod out by } \mathfrak{a}_\infty"'] \quad \text{fin. free over } \Lambda_\infty[[U^1_\ell]]/(b_N + e_{3N}) \\
 & \Lambda_\infty \arrow[u] &
\end{tikzcd}$$

- $\tilde{R}_{x,N} \hookrightarrow R^{univ}_{\phi,x}/e_{N,x} \oplus \mathrm{End}(M_{x,N})$
- indep of x mod λ
- $X_i \longmapsto \tilde{q}_{N,x}$
- $\tilde{q}_{N,x} / (q^2_{N,x}, q_{x,loc}, X_1, \dots, X_g)$ killed by $\overset{T^c \text{ in previous notation}}{f} \in R^{univ}_{x,\phi}/q \setminus \{0\}$

  indep of N

$\mathcal{D}$ diagram of level N

- we have a diagram of level N, $\forall N$
- up to isom, there are only finitely many diagrams of level N
- $N > N'$, D diagram of level $N \rightsquigarrow D^{(N')}$ a diagram of level N'.

$\exists\ D_1, D_2, \dots$ s.t. if $N > N'$, $D_N^{(N')} \simeq D_{N'}$

$\underleftarrow{\lim}$: $g = |R| + |P| + r - 1$

$$\begin{tikzcd}
 & \Lambda_\infty \arrow[d] & & & \\
R^{loc}_x[[x_1, \dots, x_g]] \arrow[r] & R_{x,\infty} \arrow[d] & q_\infty \arrow[d, mapsto] & \circlearrowright & M_{x,\infty} \arrow[d, "\text{mod } a_\infty"] \\
q \triangleleft & R^{univ}_{\phi, x} & 0 & \circlearrowright & M(U^\ell_\phi, \mathcal{O})_{x,m}
\end{tikzcd}$$

Λ_∞ ← power series ring over $\mathcal{O}$ in $4|R| + 4|P| + 4[F:\mathbb{Q}] + r - 1$ variables

$M_{x,\infty}$ ← finite free $/\Lambda_\infty[[U^1_\ell]]$

S.t. 1) mod λ indep of x

2) $R^{loc}_x \longrightarrow R^{univ}_{x,\phi}$ is the natural one

3) $\tilde{q}_{x,loc} \longrightarrow q_{x,\infty}$

4) $q_{x,\infty} / (q^2_{x,\infty}, \tilde{q}_{x,\infty})$ is killed by $b \in (R_{\phi,x}/q) - \{0\}$

5) $M_{x,\infty} \in \mathcal{C}_{PGL_2(F_\ell),1}(\mathcal{O})_{Bm}$

Prop. $\dim \mathbb{T}(U^\ell_\phi, \mathcal{O})_{x,m} \geq 1 + 2[F:\mathbb{Q}]$

Lemma. A noeth local ring, M a f.g. A-module, flat $/A$

$a_1, \dots, a_r$ an M-regular sequence in m_A,

$M/(a_1, \dots, a_r)M$ flat over $A/(a_1, \dots, a_r)A$ + minimal dim of an irred comp of $\mathrm{Supp}_A(M) \geq r +$ min dim of an irred comp of $\mathrm{Supp}_{A/(a_1 \dots a_r)}(M/(a_1 \dots a_r)M)$

$\Rightarrow$ 6) every irred comp of $\mathrm{Supp}_{R_{x,\infty}}(M_{x,\infty})$ has dim $\geq 4|P| + 4|R| + 6[F:\mathbb{Q}] + 2$

$$\begin{tikzcd}
 & R^{loc}_{x}[[\underline{X}]]^{\wedge}_{\tilde{q}_{x,loc}} \arrow[r, two heads] & \hat{R}_{x,\infty,q_{x,\infty}} \arrow[r, phantom, "\circlearrowleft"] & \hat{M}_{x,\infty,q_{x,\infty}} \\
\text{killed by } f \notin q_{x,\infty} \arrow[rr] & & \left(q_{x,\infty}/(\hat{q}_{x,\infty}, \tilde{q}_{x,loc})\right)_{q_{x,\infty}} = 0 \arrow[u] &
\end{tikzcd}$$

$\mathcal{M}_{x,\infty} = \mathrm{Hom}(P_{B_m}, M_{x,\infty})$

$\Lambda_\infty \hat{\otimes} R^{ps}_{B_m} \Rightarrow \mathcal{M}_{x,\infty}$

flat $/\Lambda_\infty$ finite $/R_{x,\infty}$

<u>Lemma</u> A an equi dim'l complete noeth. ring

$p \triangleleft A$ a prime, M a f.g. A-module

$\Rightarrow$ $\hat{A_p}$ equi dim'l of dim $\dim A - \dim A/p$

and min dim of a compt of $\mathrm{supp}_{\hat{A_p}}(\hat{M_p}) \geq$ dim of a min compt of $\mathrm{supp}_A(M)$ $- \dim A/p$

min dim of an irred comp of $\mathrm{Supp}_{R^{loc}_{x}[[\underline{X}]]^{\wedge}_{\tilde{q}_{x,loc}}}\left(\hat{\mathcal{M}}_{\infty,x,q_{x,\infty}}\right)$

$$\geq 4|P| + 4|R| + 6[F:\mathbb{Q}] + 2 - 1$$

$$\rightarrow \boxed{R^{loc}_{x}[[\underline{X}]]^{\wedge}_{\tilde{q}_{x,loc}}} \twoheadrightarrow \hat{R}_{x,\infty,q_{x,\infty}}$$

dim $6[F:\mathbb{Q}] + 4|R| + 4|P| + 2 - 1 \cancel{+1-1}$

Supp $_{R_{\chi}^{loc}[\![\underline{X}]\!]^{\wedge}_{\tilde{q}_{\chi,loc}}} \left(\mathcal{M}_{\chi,\infty}{}^{\wedge}_{q_{\chi,\infty}} \right)$ is a union of irred compts of Spec $R_{\chi}^{loc}[\![\underline{X}]\!]^{\wedge}_{\tilde{q}_{\chi,loc}}$

Prop 1) If $\chi_v \neq 1, \forall v \in R$, then Spec $R_{\chi}^{loc}[\![\underline{X}]\!]^{\wedge}_{\tilde{q}_{\chi,loc}}$ is irred.

2) If p_1, p_2 are minimal primes of $R_1^{loc}[\![\underline{X}]\!]^{\wedge}_{\tilde{q}_{\chi,loc}}$ and if p is a minimal prime of $R_1^{loc}[\![\underline{X}]\!]^{\wedge}_{\tilde{q}_{\chi,loc}}/\lambda$ w $p \supset p_1$ and p_2, then $p_1 = p_2$.

Choose $\chi_v \neq 1$, $\forall v \in R$. supp $_{R^{\wedge}_{\chi,\infty,q_{\chi,\infty}}} \left(\mathcal{M}^{\wedge}_{\chi,\infty,q_{\chi,\infty}} \right)$

$= \operatorname{Spec} R^{\wedge}_{\chi,\infty,q_{\chi,\infty}}$

$\Rightarrow$ same true mod λ $\Rightarrow$ Supp $_{R^{\wedge}_{1,\infty,q_{1,\infty}}/\lambda} \left(\mathcal{M}^{\wedge}_{1,\infty,q_{1,\infty}}/\lambda \right)$

$= \operatorname{Spec} R^{\wedge}_{1,\infty,q_{1,\infty}}/\lambda$

2) $\mathcal{M}^{\wedge}_{1,\infty,q_{1,\infty}}$ flat $/\mathcal{O}$

Supp $_{R^{\wedge}_{1,\infty,q_{1,\infty}}} \left(\mathcal{M}^{\wedge}_{1,\infty,q_{1,\infty}} \right) = \operatorname{Spec} R^{\wedge}_{1,\infty,q_{1,\infty}} \Rightarrow$ supp $_{R_{1,\infty}} (\mathcal{M}_{1,\infty})$

$\supset \operatorname{Spec} R_{1,\infty,q_{1,\infty}}$

$\Rightarrow$ Supp $_{R^{univ}_{\phi,1}} \left(\mathcal{M}(U^{\ell}_{\phi}, \mathcal{O})_{1,m} \right) \supset \operatorname{Spec} R_{\phi,1,q} \Rightarrow \ker \left(R^{univ}_{\phi_1,q} \longrightarrow \mathbb{T}_{\phi,1,q} \right)$ nilpotent

$$R^{loc}_X [\![\underline{X}]\!] = S \hat{\otimes} R_x$$

$\triangledown \qquad\qquad // \qquad\qquad \backslash\backslash$

$\tilde{q}_{x,loc} \qquad \bigotimes_{v \in P \cup \{v \mid \ell\}} R_v^\square \qquad \left(\hat{\bigotimes}_{v \in R} R_{v,\bar{x}_v}^\square \right) [\![\underline{X}]\!]$

$\backslash\backslash$

(q_S, m_{R_x})

$$R^{loc}_X [\![\underline{X}]\!]^\wedge_{\tilde{q}_{x,loc}} \simeq \hat{S}_{q_S} \hat{\otimes} R_x$$

1) Lemma $A/\mathcal{O}$ is a local noeth ring

Spec $A[\frac{1}{\ell}]$ conn'd, $A[\frac{1}{\ell}]$ normal

$A^{red}/\mathcal{O}$ flat

$\Big] \Rightarrow A^{red}$ a domain

a) Spec $(\hat{S}_{q_1} \hat{\otimes}_{\mathcal{O}} R_x)[\frac{1}{\ell}]$ conn'd : same argument as for Spec $R_x[\frac{1}{\ell}]$ works

b) $\overbrace{R^{loc}_X [\![\underline{X}]\!]^\wedge_{\tilde{q}_{x,loc}}}^{\circledast} [\frac{1}{\ell}]$ normal

$\parallel$

$R^\wedge \qquad R/\mathcal{O}$ finite type explicit $\qquad \hat{R}_m = R^{loc}_X [\![\underline{X}]\!]$

$R[\frac{1}{\ell}]$ normal

$R \to \hat{R}_m \to R^{loc}_X [\![\underline{X}]\!]^\wedge_{\tilde{q}_{x,loc}}$

regular f.f. — also regular $[\frac{1}{\ell}]$

$R[\frac{1}{\ell}] \to \circledast$ regular f.f. $\Rightarrow$ (*) normal

2) $\hat{S}_{q_s} \hat{\otimes} R_1$ is

if $\hat{S}_{q_s}$ geom irred and the irred compts of R_1 are geom. irred

$$\begin{tikzcd}
\mathrm{Irr}(\hat{S}_{q_s} \hat{\otimes} R_1) \arrow[d, leftrightarrow, "\text{bij}"] \\
\mathrm{Irr}(R_1)
\end{tikzcd}$$

+ same mod d.