Super Geometry and Super Moduli

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Lecture 1. Theta-null divisor in the moduli space of SUSY curves.
(supercurves)

$$X \xrightarrow{1|1} S$$
superscheme

SUSY structure: $\underset{\text{rank } 0|1}{D} \subset T_{X/S}$

$$D^2 = D \otimes_{\mathcal{O}_X} D \xrightarrow[\sim]{[\,,\,]} T_{X/S}/D$$

$\Rightarrow$ exact sequence $\quad 0 \to \underset{0|1}{D} \to \underset{1|1}{T_{X/S}} \to \underset{1|0}{D^2} \to 0$

Dually, $\quad 0 \to D^{-2} \to \underbrace{\Omega^1_{X/S} \to D^{-1}}_{} \to 0$

$$\text{Ber}: \quad \underset{0|1}{\omega_{X/S}} \simeq D^{-1}$$

$$\delta: \underset{1|1}{\Omega^1_{X/S}} \longrightarrow \underset{0|1}{\omega_{X/S}} \qquad \mathcal{O}\text{-linear}$$

$\leftrightarrow$ derivation $\delta: \mathcal{O}_X \to \omega_{X/S}$

Locally, $\exists$ rel. coord. on X/S (z, θ) s.t. $D = \langle \overbrace{\partial_\theta + \theta \partial_z}^{D \,\in} \rangle$

$\delta : f \longmapsto D(f)\,[dz|d\theta]$.

Over even base S

$$\underset{\substack{\text{smooth}\\\text{curve}}}{C} \longrightarrow S, \quad \mathcal{O}_X = \mathcal{O}_C \oplus L, \quad L^2 \xrightarrow{\sim} \omega_{C/S}.$$

$$\delta \downarrow \quad \omega_{X/S} = \omega_C \oplus L$$

generating

$$\delta(\underset{\mathcal{O}_C}{f}, \underset{L}{s}) = (\underset{\omega_C}{df}, \underset{L}{s})$$

Now over $\mathbb{C}$, classical topology $\qquad \pi : X \to S$

$\underline{\mathbb{C}} = \underline{\mathbb{C}}_{X/S} = \pi^{-1}\mathcal{O}_S$ (functions constant along fibers)

<u>Ex. seq</u> on X

$$0 \to \underline{\mathbb{C}}_{X/S} \to \mathcal{O}_X \xrightarrow{\delta} \omega_{X/S} \to 0$$

$\rightsquigarrow$ LES on S:

$$0 \to \mathcal{O}_S \to \pi_*\mathcal{O}_X \to \pi_*\omega_{X/S} \to R^1\pi_*\underline{\mathbb{C}}_{X/S} \to R^1\pi_*\mathcal{O}_X$$
$$\to R^1\pi_*\omega_{X/S} \to R^2\pi_*\underline{\mathbb{C}}_{X/S} \to 0.$$

$R^1\pi_*\underline{\mathbb{C}}_{X/S} \leftarrow$ local system of $H^1(C_s, \mathbb{C})$

$R^2\pi_*\underline{\mathbb{C}}_{X/S} \simeq \mathcal{O}_S$

Thm Assume $\forall s \in S$, $H^0(C_s, L_s) = 0$, then have exact seq.

$$0 \to \underset{g}{\overset{\text{bundle}}{\pi_*\omega_{X/S}}} \to \underset{2g}{R^1\pi_*\underline{\mathbb{C}}} \to \underset{g}{\overset{\text{bundle}}{R^1\pi_*\mathcal{O}_X}} \to 0$$

Can trivialize $R^1\pi_*\mathbb{C}$ over some covering $\tilde{S} \to S$

$\rightsquigarrow \tilde{S} \to LGr(g, 2g)$ ← superperiod map

Lagrangian grassmannian

<u>Want to study</u>: behavior of periods near $s: H^0(C_s, L_s) \neq 0$.

Classical picture for theta-char. (even) (Mumford)

C smooth proj. curve, $L^2 \simeq \omega_C$, how to understand $H^0(C, L)$?

Pick a point $p \in C$. $n \gg 0$,

$$V = H^0(C, L(np)/L(-np)) \quad \leftarrow 2n\text{-dim'l}$$

$$V \otimes V \longrightarrow \mathbb{C}$$

$$s, t \longmapsto \mathrm{Res}_p(\underbrace{st}_{\in \omega_C(2np)/\omega_C})$$

symmetric nondeg. form on V

Two isotropic subspaces: max. isotropic

$$\begin{cases} L_1 = H^0(C, L(np)) \subset V \\ L_2 = H^0(C, L/L(-np)) \subset V \end{cases}$$

$$L_1 \cap L_2 = H^0(C, L)$$

$$[L_1 \to V/L_2] \quad \text{computes } H^*(C, L).$$

Next, $C \xrightarrow{\pi} S$, $\underset{p}{\cup}$ $\nearrow$, $\begin{array}{c} L \\ | \\ C \end{array}$ family of spin-curves

$L_1 = \pi_*(L(np))$, ... $\quad [L_1 \to V/L_2]$ complex of bundles on S

$\simeq R\pi_* L$

Let $\Lambda \subset V$ max. isotropic, transversal to both L_1 & L_2

$$V \simeq \Lambda \oplus L_2$$

$$L_1 = \text{graph}\ (\phi : L_2 \to \underset{\overset{\wr\wr}{L_2^\vee}}{\Lambda})$$

$$\phi^* = -\phi$$

$$[L_1 \to V/L_2] \underset{q\text{ isom}}{\simeq} [L_2 \xrightarrow{\phi} L_2^\vee]$$

$\text{pf}(\phi) \leftarrow$ local eq'n

for the theta-null divisor $H^0(L_s) \neq 0$

(generically $H^0(L_s) = 0$)

Super isotropic intersection setup

$V, (\cdot,\cdot)$ sympl. vector bundle $rk = (2m|2n)$ over S superscheme

$L_1 \subset V \supset L_2$ isotropic subbundle. of dim $(m|n)$

define a cplx on S

$$C(V, L_1, L_2) = [L_1 \xrightarrow{\overset{d_{L_1,L_2}}{\overset{\shortparallel}{d}}} V/L_2] \underset{q\text{ isom.}}{\simeq} [L_1 \oplus L_2 \to V] \simeq [L_1 \to L_2^\vee]$$

Assume L_1 and L_2 are generically transversal. d is generic. isom.

$$\rightsquigarrow \underset{\Theta(L_1, L_2)}{\overset{\shortparallel}{\text{ber}\,(d)}} \in \text{Ber}\,(L_1)^{-1} \otimes \text{Ber}\,(L_2)^{-1}.$$

rat'l section

Thm 1. Assume L_1 and L_2 are generically transversal

& at some point $s \in S$,

$$L_{1s}^{+} \cap L_{2s}^{+} = 0.$$

even part

Then locally near s, $\exists$ trivialization of $\mathrm{Ber}(L_1)^{-1} \otimes \mathrm{Ber}(L_2)^{-1}$ s.t.

$\theta(L_1, L_2)^{-1} = f^2$, f regular function

Furthermore, $f \cdot d_{L_1, L_2}^{-1}$ is regular.

Proof. reduce to purely odd rank case, use Pfaffian.

Now back to $X \xrightarrow{\pi} \underset{\cup}{S} \supset U$ supercurve. generically $H^0(C_s, L_s) = 0$.

fix $\underset{\text{Lagrangian}}{\Lambda} \subset \underbrace{R^1 \pi_* \mathbb{C}_{X/S}}_{\text{Sympl.}}$ s.t. $\Lambda|_{S_{bos}}$ transversal to $\pi_* \omega_{C/S_{bos}}$

$\pi_* \omega_{X/S}$

$\Rightarrow$ can consider $\underset{\theta_\Lambda :=}{\theta}(\pi_* \omega_{X/S}, \Lambda)$ rat'l section of $\mathrm{Det}(\Lambda)^{-1} \otimes \mathrm{Ber}(R^1\pi_* \omega_X)^{-1}$

Thm 2. 1) Locally, $\theta_\Lambda^{-1} = f^2$. f regular

equation of theta null divisor

2) If $R^1 \pi_* \mathbb{C}_{X/S} = \Lambda \oplus \Lambda'$

Lagrangian splitting

$\pi_* \omega_{X/S}|_U = \mathrm{graph}(\Omega: \Lambda' \to \Lambda)$, then $f \cdot \Omega$ is regular.

Lecture 2 cplx analytic $\pi: X \longrightarrow S$ smooth supercurve

for generic $s \in S$, $H^0(C_s, L_s) = 0$
$U \subset S$

$\Lambda \subset R^1 \pi_* \mathbb{C}_{X/S}$ Lagrangian subbundle

$$\rightsquigarrow \quad \theta_\Lambda := \theta\left(\pi_* \omega_{X/S}\big|_U, \Lambda\right)$$

merom. section of $\mathrm{Det}(\Lambda)^{-1} \otimes \mathrm{Ber}_1^{-1}$

$$\mathrm{Ber}_1 := \mathrm{Ber}\left(R\pi_* \omega_{X/S}\right)$$

$$\mathrm{Ber}\,(A \longrightarrow B) = \mathrm{Ber}(A) \otimes \mathrm{Ber}(B)^{-1}$$

Theorem 1) Locally $\theta_\Lambda^{-1} = f^2 \cdot$ trivializing section, where f is regular

$$Z(f) = \{ s : H^0(C_s, L_s) \neq 0 \}$$

2) $R^1 \pi_* \mathbb{C}_{X/S} = \Lambda \oplus \Lambda'$

$$\pi_* \omega|_U = \mathrm{graph}(\Omega : \Lambda' \longrightarrow \Lambda)$$

$f\Omega$ regular

Setup for Proof: Can pick relative divisor

$$\begin{array}{ccc} p & \subset & X \\ & {\scriptstyle 0|1}\searrow & \downarrow \\ & & S \end{array}$$

locally, $p : (z = 0)$

$$\tilde{V} = \pi_*\Big(\underbrace{\mathcal{O}_X(np)}_{\text{functions w/ pole of order} \le n} \big/ \mathcal{O}_X(-(n+1)p)\Big)$$

skew-sym. form on $\tilde{V}$ $\quad B(f, g) = \mathrm{Res}_p\,(f \cdot \delta(g))$, $\quad \delta: \mathcal{O}_X \longrightarrow \omega_{X/S}$

$$\delta: \mathcal{O}_X(np) \longrightarrow \omega_{X/S}((n+1)p)$$

Grothendieck duality for $X \to S$

$$R^1 \pi_* (\omega_{X/S}) \longrightarrow \mathcal{O}_S$$

$$0 \to \omega_{X/S} \longrightarrow \omega_{X/S}(Np) \longrightarrow \omega_{X/S}(Np)/\omega_{X/S} \longrightarrow 0$$

$$\pi_* \left(\omega_{X/S}(Np)/\omega_{X/S}\right) \to R^1\pi_*(\omega_{X/S}) \longrightarrow \mathcal{O}_S$$

$$\underbrace{\qquad\qquad\qquad}_{\mathrm{Res}_p}$$

(z, θ) s.t. $p : z = 0$

$$\mathrm{Res}_{z=0}\left(f_0(z) + f_1(z)\theta\right)[dz/d\theta] = \mathrm{Res}_{z=0}\left(f_1(z)\, dz\right)$$

Exer check that B is skew-symmetric

$$V := \tilde{V}/\ker(B)$$

local basis of $\tilde{V}$: $z^{-n}, \cdots, z^{-1}, 1, z, \cdots, z^n$

$z^{-n}\theta, \cdots \cdots, z^n\theta$

Exer $\ker B = \langle 1, z^n\theta \rangle$.

Want to construct: two max'l isotropic subbundles.

$L_{can} \subset V$, no pole at p

$$\left\langle \begin{matrix} z, \cdots, z^n \\ \theta, z\theta, \cdots, z^{n-1}\theta \end{matrix} \right\rangle$$

2nd isotropic subbundle will depend on $\Lambda \subset R^1\pi_* \underline{\mathbb{C}}_{X/S}$.

Use exact seq. of sheaves on X:

$$0 \to \underline{\mathbb{C}}_{X/S} \longrightarrow \mathcal{O}_X(np) \longrightarrow \mathcal{O}_X(np)/\underline{\mathbb{C}}_{X/S} \longrightarrow 0$$

$$\begin{tikzcd} \rightsquigarrow \quad \pi_*\left(\mathcal{O}_X(np)/\underline{\mathbb{C}}_{X/S}\right) \arrow[r, two heads, "\gamma"] \arrow[d] & R^1\pi_* \underline{\mathbb{C}}_{X/S} \supset \Lambda \\ V & \end{tikzcd}$$

symplectic reduction

$$L_\Lambda = \gamma^{-1}(\Lambda)$$

$$0 \to \langle 1, z^n\theta \rangle \to \mathcal{O}_X(np)/\mathcal{O}_X(-(n+1)p) \xrightarrow{\delta} \omega_{X/S}((n+1)p)^{ex}/\omega_{X/S}(-np) \longrightarrow 0$$

$$\Rightarrow V \simeq \pi_*\left(\omega_{X/S}((n+1)p)^{ex}/\omega_{X/S}(-np)\right)$$

$$\pi_*\left(\mathcal{O}_X(np)/\underline{\mathbb{C}}_{X/S}\right) \xrightarrow[\sim]{\delta} \pi_*\left(\omega_{X/S}((n+1)p)^{ex}\right)$$

this is injective for $n \gg 0$.

<u>Key computation</u>: two skew-symmetric forms on $\pi_*\left(\mathcal{O}_X(np)/\underline{\mathbb{C}}_{X/S}\right)$ agree $\begin{cases} \gamma^*(\ ,\ ) \\ B| . \end{cases}$

$\to$ get isotropic intersection setup

$$L_{can} \subset V \supset L_\Lambda$$

$$C^\cdot(L_\Lambda, L_{can}) \sim [L_\Lambda \to V/L_{can}]$$

Over $U \subset S$,

$$C^\cdot(L_\Lambda, L_{can}) \underset{q\text{ism.}}{\simeq} C^\cdot(\Lambda, \pi_* \omega_{X/S}|_U)$$

$$\{\ \} \subset R^1\pi_* \mathbb{C}$$

$$\Rightarrow \theta(\Lambda, \pi_* \omega_{X/S}|_U) = \theta(L_\Lambda, L_{can})|_U$$ Apply Theorem 1.

<u>Def. of Supermeasure</u>

Two ingredients

1. Mumford ism.

$$\Psi : Ber_1^5 \xrightarrow{\sim} \omega_{S_g}$$

$\left(S_g = \text{moduli of supercurves of genus } g \right.$ corr. spin-str. <u>even</u> $\left. \right)$

2. Herm. pairing on Ber_1 over $U \subset S_g \supset D = S_g - U$.

$$U = \{ s : H^0(C_s, L_s) = 0 \}$$

Over U, $F = \pi_* \omega_{X/S} \subset R^1\pi_*(\mathbb{C}_{X/S})$

has real str.

$$h(s,t) = (s, \bar{t})$$

Consider $S_g \times S_g^{op}$ ← cplx conj.

$\cup$

$U \times U^{op}$

$$F \boxtimes \bar{F} \xrightarrow{h} \mathcal{O}$$ nondegenerate near quasi-diagonal

$(C_1, L_1), (C_2, L_2), C_1 = C_2$

$$h: p_1^* F \xrightarrow{\sim} p_2^* \bar{F}^\vee$$

$$\det(h)^{-1} \in p_1^* \mathrm{Ber}(F) \otimes p_2^* \mathrm{Ber}(\bar{F})$$

$$\simeq \mathrm{Ber}_1 \boxtimes \overline{\mathrm{Ber}_1}$$

<u>Def.</u> μ meromorphic section of $\omega_{S_g \times S_g^{op}} = \omega_{S_g} \boxtimes \overline{\omega_{S_g}}$

$$\Psi \boxtimes \bar{\Psi} : \mathrm{Ber}_1^5 \boxtimes \overline{\mathrm{Ber}_1^5} \xrightarrow{\sim} \omega_{S_g} \boxtimes \overline{\omega_{S_g}}$$

$$\mu := (\Psi \boxtimes \bar{\Psi})(\det(h)^{-5})$$

<u>Thm</u> Assume $g \le 11$, then μ is regular near quasi-diagonal in $S_g \times S_g^{op}$.

<u>Proof.</u> $\overset{\det(h)^{-5}}{\Longleftrightarrow}$ as a section of $\mathrm{Ber}_1^5 \boxtimes \overline{\mathrm{Ber}_1}^5$ is regular near q-diag.

Over $U \times U^{op}$, near q-diag, have two isotropic subbundles of

$$\begin{array}{c} p_1^* F \\ \cap \\ p_1^* R^1\pi_*(\underline{\mathbb{C}}_{X/S}) \\ \simeq p_2^* R^1\pi_*(\underline{\mathbb{C}}_{X/S}) \\ \cup \\ p_2^* \bar{F} \end{array}$$

$$U \xleftarrow{p_1} U \times U^{op} \xrightarrow{p_2} U^{op}$$

$$\det(h)^{-1} = \theta(p_1^* F, p_2^* \bar{F})^{-1}$$

Fix Lagr. splitting $R^1\pi_*(\underline{\mathbb{C}}_{X/S}) = W \oplus W'$

<u>Abstract statement</u>

$L_1, L_2 \subset V = W \oplus W'$

max. isotropic (V symplectic; $W \oplus W'$ isotropic splitting)

L_1 & L_2 transversal to W

$\Rightarrow \quad L_i = \text{graph}(\tau_i : W' \longrightarrow W)$

Identify (exer.) $\quad \theta(L_1, L_2) = \theta(L_1, W) \cdot \theta(L_2, W) \cdot \text{ber}(\tau_1 - \tau_2)$

$$\tau_1 - \tau_2 : W' \longrightarrow W$$

In our case, $\quad L_1 = p_1^* F, \quad L_2 = p_2^* \bar{F}$

$$\theta(F, \bar{F})^{-1} = p_1^* \theta(F, W)^{-1} \; p_2^* \theta(\bar{F}, \bar{W})^{-1} \cdot \text{ber}(\tau - \bar{\tau})^{-1}$$

Recall $\quad \theta(F, W)^{-1} = f^2 \quad$, f loc. eq'n of D.

τ , know: $f\tau$ is regular.

Super period matrix

$\tau = A + \Omega, \quad \Omega \in N^2$ over U $\quad N =$ (odd coord.)

classical period

$$\det(A - \bar{A}) \neq 0$$

$\Omega \in \frac{N^2}{f} \quad$ Consider $\left(\mathcal{O}_{S \times \bar{S}} \text{-alg. gen. by } p_1^* \frac{N^2}{f}, \; p_2^* \frac{\bar{N}^2}{\bar{f}} \right) = A$

$$\text{ber}(\tau - \bar{\tau})^{-1} \in A$$

$$\Rightarrow \theta(F, \bar{F})^{-5} \in f^{10} \cdot \bar{f}^{10} \cdot A$$

Enough: $\quad \frac{N^{22}}{f^{11}} = 0$

Enough: $\quad N^{22} = 0$

$22 > 2g - 2$

$g \geq 11$

Lecture 3. Stable Supercurves

Over a pt. X, $\mathcal{O}_X = \mathcal{O}_C \oplus L$

C stable curve

L generalized spin structure : (torsion-free) coherent sheaf on C, $\mathrm{rk}=1$ on smooth locus

w/ isom. $L \xrightarrow{\sim} \mathcal{H}om(L, \omega_C)$

$\uparrow$ dualizing sheaf

on smooth locus, L is a line bundle, $L^2 \xrightarrow{\sim} \omega_C$.

Torsion free modules on the node $xy=0$

$R = \mathbb{C}[[x,y]]/(xy)$ indecomposable ones: R, $R/(x)$, $R/(y)$.

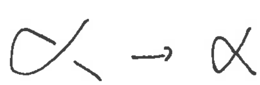
rk 1 on smooth locus $\rightsquigarrow$ R or $R/(x) \oplus R/(y) \simeq (x,y)$.

L locally free at the node — Ramond node

not loc. free — NS node

Ex C has 1 node

$\tilde{C} \xrightarrow{\nu} C$ normalization



$\tilde{C} \ni p_1, p_2 \longmapsto$ node

1) NS node for L

$\iff L \simeq \nu_* \tilde{L}$

$\tilde{L}^2 \simeq \omega_{\tilde{C}}$

2) Ramond node : L line bundle on C

$\tilde{L} := \nu^* L$, $\tilde{L}^2 \simeq \omega_{\tilde{C}}(p_1+p_2)$

X/pt stable supercurve, $\mathcal{O}^+ = \mathcal{O}_C$, $\mathcal{O}^- = L$

$\exists$ dualizing sheaf on X

$$\omega_X = \underset{\text{even}}{\omega_C} \oplus \underset{\text{odd}}{\underline{Hom}(L, \omega_C)} \overset{\simeq}{} L$$

On smooth locus: ω_X is a line bundle of rk $(0|1)$

loc. free near Ramond node, not loc. free near NS node

<u>Deformations of nodes</u>:

Base of deformation: even affine line $\mathbb{A}^1$, coord. t

<u>def. of Ramond node</u>: z_1, z_2, θ
$z_1 z_2 = t$ + formulas for $\delta: \mathcal{O} \to \omega$

<u>def of NS node</u>: $z_1, z_2, \theta_1\ \theta_2$

$$z_1 z_2 = t^2, \quad z_1\theta_2 = t\theta_1, \quad z_2\theta_1 = t\theta_2, \quad \theta_1\theta_2 = 0$$

<u>Def</u>. (Deligne)

X/S stable supercurve over S. + $\delta: \mathcal{O}_X \longrightarrow \omega_{X/S}$
flat family
rel derivation $(\mathcal{O}_S \to 0)$

Require: $\forall s \in S(\mathbb{C})$,

$$\begin{array}{ccc} \mathcal{O}_{X_s} & = & \mathcal{O}_{C_s} \oplus L \\ \delta \downarrow & & \downarrow \text{induced by } \delta \\ \omega_{X_s} & = & \omega_{C_s} \oplus \mathcal{H}om(L, \omega_{C_s}) \end{array}$$

Def of stable supercurves

1. Smooth case $\quad X/S \quad$ smooth supercurve

$$D = \partial_\theta + \theta \partial_z$$

$$\underset{0|1}{D} \subset \mathcal{T}_{X/S}$$

not $\mathcal{O}$-submod $\rightarrow \mathcal{T}_{sc} \subset \mathcal{T}_{X/S}$

$$\mathcal{T}_{sc} = \{v : [v, D] \subset D\}$$

Lemma. composition $\mathcal{T}_{sc} \to \mathcal{T}_{X/S} \to \mathcal{T}_{X/S}/D$ is $\sim$

Proof. $\quad v = a\,\partial_z + b \cdot D$

$v \in \mathcal{T}_{sc} \iff b = \pm D(a)$

$v \bmod D \longleftrightarrow a$

$\Rightarrow \mathcal{T}_{sc} \simeq \mathcal{T}/D \simeq \omega_{X/S}^{-2}$

$\square$

2. X/point $\quad$ sheaf of inf. symmetries

$$\mathcal{A}_X \subset \mathcal{T}_X = \mathrm{Der}(\mathcal{O}_X)$$

$$\mathcal{A}_X = \{v : v \text{ preserves } D \text{ on smooth locus}\}$$

Lemma. X stable $\Rightarrow H^0(X, \mathcal{A}_X) = 0$.

$\square$

X/S good supercurve. $\quad \forall$ node $\in X_s$, $\quad S \xrightarrow{\text{smooth}} \mathrm{Def}(\text{node})$

$$\begin{array}{ccc} U & \overset{j}{\hookrightarrow} & X \\ & {\scriptstyle smooth}\searrow & \swarrow \\ & S & \end{array}$$ complement to nodes.

Thm X/S good, The sheaf $j_*(\omega^2_{U/S})$ is locally free. (Same for $\omega^{2n}_{X/S}$)

It is relative ample.

$\omega_X = \omega_C \oplus L$, $\omega_X^2 = \underbrace{\omega_C \oplus \omega_C \otimes L}_{\otimes L} = p^*\omega_C$

$\otimes L$ $\quad$ $p: X \to C$

Thm $\overline{S_g}$ moduli stack of stable supercurves of genus g is a DM stack, smooth proper$/\mathbb{C}$.

1) $$\begin{array}{ccc} \mathfrak{X} & & \mathcal{Y} \\ & \searrow \; \swarrow & \\ & S & \end{array}$$ $\mathrm{Isom}(\mathfrak{X}, \mathcal{Y})$ is representable by superscheme

uses theory of Hilbert schemes.

2) $\exists$ étale atlas

3) smoothness follows from def. theory

4) properness from $\overline{S_{g,Bos}}$

classify generalized Spin-curves

(Cornalba, Jarvis)

Mumford isomorphisms

1. Kodaira - Spencer map.

a. Classical case (even)

$$\begin{array}{ccc} C & \xrightarrow[\pi]{\text{stable curve}} & S \\ \cup & & \cup \\ C_0 & \longrightarrow & S_0 \end{array}$$ nodal curves divisor $(t=0)$

$t\partial_t, \partial_x, \cdots \qquad t, x, \cdots$

$$\begin{array}{ccc} \mathcal{T}_{S,S_0} & \subset & \mathcal{T}_S \\ \| & & \\ \{ v : v(t) \in (t) \} & & \end{array}$$

$$KS : \mathcal{T}_{S,S_0} \longrightarrow R^1\pi_*(\mathcal{T}_{C/S}) \qquad \mathcal{T}_{C/S} \cong \omega_{C/S}^{-1}$$

isom. for universal family

$$0 \to \mathcal{T}_{C/S} \longrightarrow \mathcal{T}_{C,C_0} \longrightarrow \pi^* \mathcal{T}_{S,S_0} \to 0$$

KS is obtained as coboundary map.

6.
$$\begin{array}{ccc} X & \xrightarrow{\pi} & S \\ \cup & & \cup \quad (t=0) \\ X_0 & \longrightarrow & S_0 \end{array}$$

$$0 \to A_{X/S} \longrightarrow A_{X,X_0} \longrightarrow \pi^{-1}\mathcal{T}_{S,S_0} \to 0 \qquad A_{X/S} \cong \omega_{X/S}^{-2}$$

$$\rightsquigarrow KS : \mathcal{T}_{S,S_0} \to R^1\pi_*(\omega_{X/S}^{-2})$$

isom. for univ. family

Thm. $\exists$ nat. Cartier divisor $\Delta \subset \overline{S}_g$

KS induces $\omega_{\overline{S}_g} \simeq \mathrm{Ber}\left(R\pi_*(\omega_{\mathcal{X}/\overline{S}_g}^{-2})\right)(-\Delta)$

Mumford isom.: need to rewrite $\uparrow$ in terms of $\mathrm{Ber}_1 = \mathrm{Ber}(R\pi_* \omega_{\mathcal{X}/S})$

$\simeq \mathrm{Ber}(R\pi_* \mathcal{O}_{\mathcal{X}/S})$

Thm $\qquad \omega_{\overline{S}_g} \simeq \mathrm{Ber}_1^5(-2\Delta_{NS} - \Delta_R)$

$(\Delta = \Delta_{NS} + \Delta_R)$

How to find relations between det. bundles?

Deligne "Le det. de la cohomologie".

$$\begin{array}{c} C \\ \downarrow \pi \\ S \end{array} \qquad L \text{ line bundle on } C$$

$$d(L) = \mathrm{Det}(R\pi_* L)$$

$$\begin{array}{c} \wr \\ \| \quad [\underset{0}{A} \to \underset{1}{B}] \end{array}$$

$$\mathrm{Det}(A) \otimes \mathrm{Det}(B)^{-1}$$

$$d(L_1 \otimes L_2) \simeq d(L_1) \otimes d(L_2) \otimes d(\mathcal{O})^{-1} \otimes \langle L_1, L_2 \rangle$$

$\uparrow$ bilinear in L_1, L_2

$D \subset C$, finite $\searrow$ S

$$\langle \mathcal{O}_C(D), L \rangle = \mathrm{Det}(\pi_*(L|_D)) \otimes \mathrm{Det}(\pi_* \mathcal{O}_D)^{-1}$$

In supercase, $\begin{array}{c} X \\ \downarrow \\ S \end{array}$

$$B(L) = \mathrm{Ber}(R\pi_* L)$$

$\langle L_1, L_2 \rangle \simeq \mathcal{O}_S$ canonical

$$B(\overset{1|0}{L_1} \otimes \overset{1|0}{L_2}) \simeq B(L_1) \otimes B(L_2) \otimes B(\mathcal{O})^{-1}$$

$$\mathrm{Ber}\, \pi_*(L_1|_D) \longrightarrow \mathrm{Ber}(\pi_* L_2|_D)$$

$\uparrow$ locally choose $L_1 \xrightarrow{f} L_2$,

$$\mathrm{Ber}\begin{pmatrix} f & \\ & f \end{pmatrix} = 1.$$

$B(\omega^{-2})$ in terms of $B(\omega) \overset{SD}{=} B(0)$

SD | S

$$B(\omega^3) \simeq B(\overset{1|0}{\Pi\omega} \otimes \overset{1|0}{\omega^2})^{-1}$$

$$\simeq \underbrace{B(\Pi\omega)^{-1}}_{Ber_1} \otimes B(\omega^2)^{-1} \otimes \underbrace{B(0)}_{Ber_1}$$

$$\simeq B(\omega^2)^{-1} \otimes Ber_1^2$$

$$B(\omega^2)^{-1} = B(\underset{1|0}{\Pi\omega} \otimes \underset{1|0}{\Pi\omega})^{-1} \simeq B(\Pi\omega)^{-2} \cdot B(0) \simeq Ber_1^3$$