Universal monodromic Bezrukavnikov's equivalence

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Tame Betti local Langlands equiv.

$G/\mathbb{C}$ reductive, $\mathring{I} \subset I$ prounipotent radical of Iwahori.

$\widetilde{Fl} = G((t))/\mathring{I}$ enhanced affine flag, T-torsor over $Fl = G((t))/I$

$\mathcal{H} = Shv_{(I)}(\widetilde{Fl})$ weakly I-constructible sheaves on $\widetilde{Fl}$.

work in analytic topology and allow infinite dim'l stalks.

$\check{G}/k$ Langlands dual, char $k = 0$.

$\widetilde{\check{G}} = \check{G} \overset{\check{B}}{\times} \check{B}$, $St = \widetilde{\check{G}} \underset{\check{G}}{\times} \widetilde{\check{G}}$

<u>Theorem</u> (Dhillon-T.) There is a monoidal equiv.

$$\mathcal{H} \simeq \operatorname{IndCoh}_{\check{G}}(St)$$

(Family of equivalences over $\check{T}$, fiber at 1 recovers Bezrukavnikov's equivalence)

Case of a torus:

Betti $\quad Shv_{loc.\,const}(T \times \Lambda) \simeq QCoh(\check{T} \times pt/\check{T})$

de Rham $\quad DMod(T \times \Lambda) \simeq QCoh(\check{\mathfrak{t}}/\check{\Lambda} \times pt/\check{T})$

The big tilting $\quad \mathcal{H} \simeq \mathrm{IndCoh}_{\check{G}}(St)$

Ξ big tilting $\longleftrightarrow$ $\mathcal{O}$ structure sheaf

(supported on G/U)

<u>Thm</u> (T.) There exists $\Xi \in \mathrm{Shv}_{(B)}(G/U)$ s.t.

(1) $\mathrm{Hom}(\Xi, -) : \mathrm{Shv}_{(B)}(G/U) \longrightarrow \mathrm{QCoh}(\check{T} \times \check{T})$ is monoidal

and calculates certain vanishing cycles.

(2) the $!, *$ - restriction to any stratum is free over $\check{T}$, and concentrated in perverse degree 0

(3) $\mathrm{Hom}(\Xi, \Xi) \simeq \mathcal{O}(\check{T} \underset{\check{T}/\!/W}{\times} \check{T})$ $\quad$ (if G has connected center)

The Whittaker category.

By monoidality, define

$$ {}_{\chi}\mathcal{H} \simeq \mathrm{QCoh}(\check{T}) \underset{\mathrm{Shv}_{(B)}(G/U)}{\otimes} \mathcal{H} $$

where the finite Hecke category acts by

$$ \mathrm{Shv}_{(B)}(G/U) \xrightarrow{\mathrm{Hom}(\Xi, -)} \mathrm{QCoh}(\check{T} \times \check{T}) \circlearrowright \mathrm{QCoh}(\check{T}) $$

There are commuting actions $\quad {}_{\chi}\mathcal{H}_{\chi} \curvearrowright {}_{\chi}\mathcal{H} \curvearrowleft \mathcal{H}$

$$\updownarrow$$

$$ \mathrm{QCoh}_{\check{G}}(\check{\mathfrak{g}}) \curvearrowright \mathrm{QCoh}_{\check{G}}(\tilde{\check{\mathfrak{g}}}) \curvearrowleft \mathrm{IndCoh}_{\check{G}}(St) $$

The universal monodromic AB equivalence

Construct a functor

$$
\begin{tikzcd}
\mathrm{Rep}(\check{G} \times \check{T}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{pullback}"] & \mathrm{Perv}^{\mathrm{waki}}_{(I)}(\widetilde{Fl}) \arrow[r, "gr"] \arrow[d] & \mathrm{Coh}_{\check{T}}(\check{T}) \\
\mathrm{QCoh}_{\check{G}}(\tilde{\check{\mathfrak{g}}}) \arrow[r, "F"] & \mathcal{H} &
\end{tikzcd}
$$

Two approaches to fully faithfulness

(1) AB localize in the nilpotent directions

$$\mathrm{Perv}_{(I,\chi)}(Fl) \big/ \langle IC_w : \ell(w) > 0 \rangle \simeq \mathrm{QCoh}(\check{N}^{\mathrm{reg}} / \check{B})$$

(2) DT localize in semisimple direction

For $\lambda, \mu \in \Lambda^+$, need to check

$$
\begin{tikzcd}
\mathrm{Hom}_{\check{B}/\check{B}}(\mathcal{O}(-\mu), V_\lambda \otimes \mathcal{O}) \arrow[r, "F"] \arrow[dr, "i^*"', "\substack{\text{pullback along} \\ \check{T}/\check{T} \to \check{B}/\check{B}}"'] & \mathrm{Hom}_{\mathcal{H}}(\Delta_{-\mu}, Z_\lambda) \simeq \mathrm{Hom}_{\chi\mathcal{H}}({}_\chi\Delta_{-\mu}, {}_\chi Z_\lambda) \arrow[d, "gr", "\text{assoc. graded for Wakimoto filtration}"'] \\
& \mathrm{Hom}_{\check{T}/\check{T}}(\mathcal{O}(-\mu), V_\lambda \otimes \mathcal{O})
\end{tikzcd}
$$

Prove that images of i^* and gr coincide by characterizing both in terms of order vanishing conditions along the walls in $\check{T}$

Universal monodromic Bezrukavnikov equivalence

The commuting actions $\quad {}_xH_x \curvearrowright {}_xH \curvearrowleft H \quad$ get a monoidal functor

$$\begin{matrix} {}_xH_x & & {}_xH \\ \wr & & \wr \\ QCoh_{\check{h}}(\check{h}) & & QCoh_{\check{h}}(\tilde{\check{h}}) \end{matrix}$$

$$\iota^! : H \longrightarrow \mathrm{End}_{{}_xH_x}({}_xH) \simeq QCoh_{\check{h}}(St)$$

<u>Claims</u> (1) $\iota^!$ admits a fully faithful left adjoint $\iota_!$.

(2) $\ker \iota^! \simeq H^{\leq -\infty} := \bigcap_n H^{\leq -n}$ $\quad$ ($w_{\widetilde{Fl}}$ is an example)

(3) $\iota^! : H^c \xrightarrow{\sim} Coh_{\check{h}}(St)$

Claim (1) follows by a general observation of Ben-Zvi – Gunningham – Orem

(alternative proof Beraldo – Lin – Reeves)

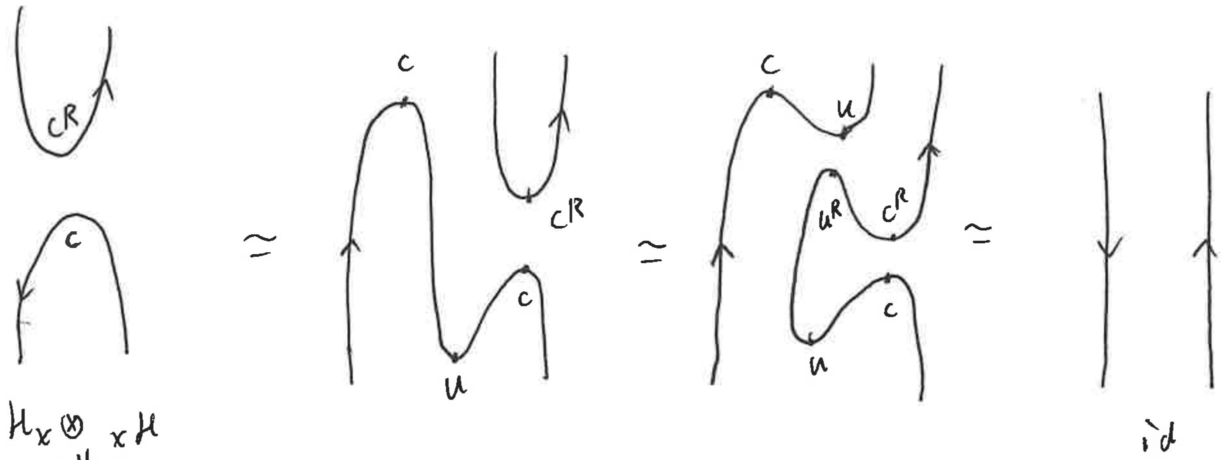
The ${}_xH \curvearrowleft H$ is dual to $H \curvearrowright H_x$

- the unit $u: {}_xH_x \xrightarrow{\sim} {}_xH \otimes_H H_x$

and counit $c: H_x \otimes_{{}_xH_x} {}_xH \longrightarrow H$ preserve compactness.

- the right adjoint to the unit u^R is fully faithful i.e. $uu^R \simeq id$

Want to prove $\quad c: H_x \otimes_{{}_xH_x} {}_xH \simeq QCoh_{\check{h}}(St) \longrightarrow H \quad$ fully faithful, i.e. $c^R c \simeq id$



$H_x \otimes_{{}_xH_x} {}_xH$

$c^R c$