Appendix to GRT

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Appendix

In the appendix, we review some classical constructions and results in geometric representation theory.

A Harish-Chandra modules

The notion of Harish-Chandra modules originated from Harich-Chandra's study on unitary representations of real Lie groups. Here we present a review of the algebraic theory of Harish-Chandra modules.

Definition A.1 (Harish-Chandra pair)**.** By a *Harish-Chandra pair*, we mean a pair $(\mathfrak{g}, K)$ consisting of a finite-dimensional Lie algebra $\mathfrak{g}$ and an algebraic group K, whose Lie algebra is denoted by $\mathfrak{k}$, together with an "adjoint" action $\mathrm{Ad}\colon K \to \mathrm{Aut}(\mathfrak{g})$ and an inclusion of Lie algebra $i\colon \mathfrak{k} \to \mathfrak{g}$ such that the inclusion i is K-equivariant and that the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$ via the inclusion i coincides with the derivation of Ad.

Definition A.2 (Harish-Chandra module)**.** Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. A $(\mathfrak{g}, K)$-module, or a *Harish-Chandra module*, is a $\mathbb{C}$-vector space M equipped with $\mathfrak{g}$-module and K-module structures such that

- for $k \in K, x \in \mathfrak{g}$ and $m \in M$, one has $\mathrm{Ad}_k(x) \cdot m = kxk^{-1} \cdot m$;
- the two $\mathfrak{k}$-actions on M, namely the one comes from $\mathfrak{g}$-action via i and the one comes from the differential of K-action, coincide.

Here is the standard example of a Harish-Chandra module.

Example A.1. Let G be an algebraic group with K being an algebraic subgroup. Let $\mathfrak{g}$ be the Lie algebra of G, then $(\mathfrak{g}, K)$ is a Harish-Chandra pair with the adjoint action $K \to G \to \mathrm{Aut}(\mathfrak{g})$ and the obvious inclusion $\mathfrak{k} \to \mathfrak{g}$. Now a $(\mathfrak{g}, K)$-module is the same as a $\mathfrak{g}$-module such that the action of $\mathfrak{k}$ integrates to a K-action. Therefore, we also call a $(\mathfrak{g}, K)$-module a $\mathfrak{g}$-module integrable with respect to K.

Let us now focus on the case where G is an reductive group. Let $B \subset G$ be a Borel subgroup and H be the corresponding Cartan torus.

Example A.2. A $(\mathfrak{g}, G)$-module is nothing but a G-module.

Example A.3. A $(\mathfrak{g}, H)$-module is a weight $\mathfrak{g}$-module whose weights are integral.

Example A.4. A finitely generated $(\mathfrak{g}, B)$-module lies in the BGG category $\mathcal{O}$. Conversely, a module from the category $\mathcal{O}$ whose weights are integral is a $(\mathfrak{g}, B)$-module.

Remark A.1. The theory of Harish-Chandra modules extends, mutatis mutandis, to the setting of infinite dimensional Lie algebras and pro-algebraic groups, so that we are allowed to treat Kac–Moody algebras and Kac–Moody groups, cf. [BFM91] and [Kum12].

B Borel–Weil–Bott theorem

Let G be a reductive algebraic group over $\mathbb{C}$, with B a fixed Borel subgroup. Let $X = G/B$ be the flag variety. It is known that X is projective and smooth.

Let U be the unipotent radical of B and $H = B/U$ be the Cartan torus associated to B.

Example B.1. For $G = \mathrm{SL}_2$, $X = G/B$ is isomorphic to the projective space $\mathbb{P}^1$.

B.1 Equivariant vector bundles

Notice that X carries a G-action on the left. Denote by $a\colon G\times X\to X$ the action map and $p_2\colon G\times X\to X$ the projection map. Let $p_{23}\colon G\times G\times X\to G\times X$ be the projection to the second and third factor.

Definition B.1 (Equivariant vector bundle)**.** Let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module of finite rank (i.e. an algebraic vector bundle). We say $\mathcal{M}$ is G*-equivariant* if we are given an isomorphism

$$\phi\colon a^*\mathcal{M}\xrightarrow{\sim}p_2^*\mathcal{M}$$

satisfying the *cocycle condition*

$$p_{23}^*\phi\circ(1_G\times a)^*\phi=(m\times 1_X)^*\phi,$$

where $m\colon G\times G\to G$ is the multiplication map.

On the stalk level, ϕ specifies isomorphisms $\mathcal{M}_{g\cdot x}\xrightarrow{\sim}\mathcal{M}_x$ for all $(g,x)\in G\times X$. The cocycle condition means that the following diagram

$$\begin{tikzcd}
\mathcal{M}_{g_1\cdot(g_2\cdot x)} \arrow[r, "\sim"] \arrow[dr, equal] & \mathcal{M}_{g_2\cdot x} \arrow[r, "\sim"] & \mathcal{M}_x \\
 & \mathcal{M}_{(g_1g_2)\cdot x} \arrow[ur, "\sim"'] &
\end{tikzcd}$$

is commutative for any $(g_1,g_2,x)\in G\times G\times X$.

Let us denote by $\mathbb{C}[G]=\mathrm{H}^0(G,\mathcal{O}_G)$ the coordinate ring of (the affine algebraic variety) G. Since G is affine, we have $\mathrm{H}^i(G,\mathcal{O}_G)=0$ for all $i>0$. Therefore

$$\mathrm{H}^i(G\otimes X,p_2^*\mathcal{M})=\mathrm{H}^i(G\times X,\mathcal{O}_G\boxtimes\mathcal{M})=\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}).$$

The map $\sigma\colon G\times X\to G\times X,(g,x)\mapsto(g,g\cdot x)$ is an automorphism of $G\times X$ and $a=p_2\circ\sigma$. Hence we also have

$$\mathrm{H}^i(G\otimes X,a^*\mathcal{M})=\mathrm{H}^i(G\times X,\sigma^*p_2^*\mathcal{M})=\mathrm{H}^i(G\times X,p_2^*\mathcal{M})=\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}).$$

Now the isomorphism $\phi^{-1}\colon p_2^*\mathcal{M}\to a^*\mathcal{M}$ induces a linear map

$$\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M})\to\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}).$$

It restricts to a map

$$f\colon\ \mathrm{H}^i(X,\mathcal{M})=1\otimes\mathrm{H}^i(X,\mathcal{M})\to\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}),$$

which leads to a morphism $b\colon G\times\mathrm{H}^i(X,\mathcal{M})\to\mathrm{H}^i(X,\mathcal{M})$.

Proposition B.1 (Equivariant vector bundle produces representations)**.** The morphism

$$b\colon G\times\mathrm{H}^i(X,\mathcal{M})\to\mathrm{H}^i(X,\mathcal{M})$$

makes $\mathrm{H}^i(X,\mathcal{M})$ a representation of G.

Proof. To show the morphism $b\colon G\times\mathrm{H}^i(X,\mathcal{M})\to\mathrm{H}^i(X,\mathcal{M})$ defines an action, we need to show that the following diagram

$$\begin{tikzcd}
G\times G\times\mathrm{H}^i(X,\mathcal{M}) \arrow[r, "m\times 1"] \arrow[d, "1_G\times b"'] & G\times\mathrm{H}^i(X,\mathcal{M}) \arrow[d, "b"] \\
G\times\mathrm{H}^i(X,\mathcal{M}) \arrow[r, "b"] & \mathrm{H}^i(X,\mathcal{M})
\end{tikzcd}$$

is commutative. This is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{C}[G]\otimes\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}) & \mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}) \arrow[l, "m^*\otimes 1"'] \\
\mathbb{C}[G]\otimes\mathrm{H}^i(X,\mathcal{M}) \arrow[u, "1\otimes f"] & \mathrm{H}^i(X,\mathcal{M}) \arrow[l, "f"] \arrow[u, "f"']
\end{tikzcd}$$

Notice that $\mathrm{H}^i(G, \mathcal{O}_G) = 0$ for $i > 0$, so the above diagram is equivalent to

$$
\begin{tikzcd}
 & \mathrm{H}^i(G \times G \times X, (1_G \times a)^* a^* \mathcal{M}) \arrow[r, equal] & \mathrm{H}^i(G \times G \times X, (m \times 1_X)^* a^* \mathcal{M}) \\
\mathrm{H}^i(G \times G \times X, p_{23}^* a^* \mathcal{M}) \arrow[r, equal] & \mathrm{H}^i(G \times G \times X, (1_G \times a)^* p_2^* \mathcal{M}) \arrow[u, "\mathrm{H}^i((1_G \times a)^* \phi^{-1})"] & \mathrm{H}^i(G \times X, a^* \mathcal{M}) \arrow[u, "(m \times 1_X)^*"'] \\
\mathrm{H}^i(G \times X, a^* \mathcal{M}) \arrow[u, "p_{23}^*"] & & \mathrm{H}^i(G \times X, p_2^* \mathcal{M}) \arrow[ll, "\mathrm{H}^i(\phi^{-1})"'] \arrow[u, "\mathrm{H}^i(\phi^{-1})"']
\end{tikzcd}
$$

whose commutativity follows from the cocycle condition. The linearity of the action is easy to see. □

Remark B.1. The above illustration makes sense for any projective variety X carrying an algebraic G-action.

Intuitively, one might think that G-equivariant vector bundle on $X = G/B$ should coincide with B-equivariant vector bundle on a point, which is nothing but finite-dimensional representations of B. This heuristic guess is justified by the following proposition.

Proposition B.2 (Equivariant vector bundles on the flag variety)**.** The category of G-equivariant vector bundles on $X = G/B$ is equivalent to the category of finite-dimensional representations of B. In particular, equivariant line bundles on X are parametrized by the character lattice of G.

Proof. First, for any G-equivariant vector bundle $\mathcal{M}$ on X, its stalk $\mathcal{M}_B$ over B/B carries a B-action because B fixes $B/B \in G/B$.

Conversely, for a finite-dimensional B-representation M, we form a G-equivariant vector bundle $\mathcal{M}$ on G/B as follows: the total space of $\mathcal{M}$ is $G \times_B M = G \times M / \langle (gb, b^{-1}m) \sim (g, m) \text{ for any } b \in B \rangle$. The projection $\pi\colon G \times_B M \to G/B, [(g, m)] \mapsto [g]$ gives rise to a vector bundle on G/B. One can check that $\mathcal{M}$ is G-equivariant. Moreover, the above two constructions are quasi-inverse to each other, and hence give the desired equivalence of categories.

Since one-dimensional representation of B is parametrized by the character lattice of G, equivariant line bundles are parametrized by the same data. □

For $\lambda \in P$, the character lattice of G, denote by $\mathcal{L}^\lambda$ the corresponding equivariant line bundle on X. Notice that we can identify a global section of $\mathcal{L}^\lambda$ as a regular function $f\colon G \to \mathbb{A}^1$ satisfying $f(xb) = \lambda(b)^{-1} f(x)$ for $x \in G$ and $b \in B$. The action of G is given by $(gf)(x) = f(g^{-1}x)$. One can see that $\mathcal{L}^\lambda \otimes_{\mathcal{O}_X} \mathcal{L}^\mu = \mathcal{L}^{\lambda+\mu}$.

Example B.2. For $G = \mathrm{SL}_2$, one can check that $\mathcal{L}^{n\rho} \simeq \mathcal{O}_X(-n)$, where $\mathcal{O}_X(1)$ denotes the Serre twisted line bundle on $X \simeq \mathbb{P}^1$.

B.2 Borel–Weil–Bott theorem

Theorem B.1 (Borel–Weil–Bott)**.** Let $\lambda \in P$.

- If $-\lambda$ is not ρ-regular, then all the cohomology groups $\mathrm{H}^i(X, \mathcal{L}^\lambda)$ vanish.
- If $-\lambda$ is ρ-regular, there exists a unique element w in the Weyl group W such that $w \cdot (-\lambda)$ is ρ-dominant. Then there is a unique non-vanishing cohomology group $\mathrm{H}^{\ell(w)}(X, \mathcal{L}^\lambda)$, where $\ell(w)$ is the length of w, and $\mathrm{H}^{\ell(w)}(X, \mathcal{L}^\lambda) = L(w \cdot (-\lambda))^*$.

The proof we present here (which we learn from [Dem76] and [Lur07]) is in the spirit of parabolic induction. We will prove by induction on the cohomological degree.

Here is the base of the induction.

Lemma B.1. If $-\lambda$ is not dominant, then $\mathrm{H}^0(X, \mathcal{L}^\lambda) = 0$. If $-\lambda$ is dominant, then $\mathrm{H}^0(X, \mathcal{L}^\lambda) = L(-\lambda)^*$.

Proof. Since X is projective, $M = \mathrm{H}^0(X, \mathcal{L}^\lambda)$ is a finite-dimensional representation of G. In particular, it is completely reducible as a representation of G.

Let U^- be the opposite unipotent radical. Consider the subspace

$$M^{U^-} = \{f \in M \colon u \cdot f = f \text{ for any } u \in U^-\}$$

of “lowest weight” vectors. For any $f \in M^{U^-}$, we have

$$f(ub) = f(b) = \lambda(b)^{-1} f(1)$$

for any $u \in U^-$ and $b \in B$. Since U^-B is dense in G, f is completely determined by its value $f(1)$ at 1. Thus M^{U^-} is at most one-dimensional, and M is either zero or irreducible.

Suppose $M^{U^-} \neq 0$, pick a nonzero element $f \in M^{U^-}$, then

$$(hf)(1) = f(h^{-1}) = \lambda(h^{-1})^{-1} f(1) = \lambda(h) f(1)$$

for any $h \in H$. So $hf = \lambda(h)f$ and f has weight λ. As a consequence, M is a irreducible representation with lowest weight λ, i.e. $M \simeq L(-\lambda)^*$. In particular, $-\lambda$ should be dominant.

It remains to show that M^{U^-} is indeed nonzero when $-\lambda$ is dominant. This is done by an explicit computation. Let $B' = U'H$ be an opposite Borel, and consider the Bruhat decomposition $B'\backslash G/B$. Define a function f on the big cell $U'B$ by the formula $f(ub) = \lambda(b)^{-1}$ for $u \in U'$ and $b \in B$. We want to extend f to other Bruhat cells. Since $X = G/B$ is normal, we only need to extend f to all cells with codimension 1.

Let C be such a cell, corresponding to a simple root α. If f has a pole along C, then it approaches infinity around every point of C since it does not vanish on $U'B$. Let P' be the opposite parabolic subgroup corresponding to α, $C \cup U'B = P'B$. There must exist a Levi factor L of P' which has a Cartan torus contained in H such that f has a pole on LB.

By passing to universal cover, it suffices to prove the statement in the case $G = \mathrm{SL}_2$. Let us pick the usual choice of Borel B as upper triangular matrices, then H is identified with diagonal matrices. The character λ restricts to

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^k, \text{ where } k = \langle \lambda, \alpha^\vee \rangle \leq 0.$$

Now the function f is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a^{-k}$$

on the big cell. Clearly it has no poles, a contradiction. We are done. □

Our induction process is based on the following observation, whose proof will be sketched later.

Lemma B.2. Let E, S be smooth projective varieties and $\pi\colon E \to S$ be a $\mathbb{P}^1$-bundle. Let K be the relative canonical bundle. Suppose $\mathcal{L}$ is a line bundle on E with degree $n \geq -1$ on each fiber of S, then there are natural isomorphisms

$$\mathrm{H}^i(E, \mathcal{L}) \simeq \mathrm{H}^{i+1}(E, \mathcal{L} \otimes_{\mathcal{O}_E} K^{n+1}).$$

Moreover, if E and S carry algebraic G-action and everything is G-equivariant, then the isomorphisms also preserve G-actions.

Now we can prove the Borel–Weil–Bott theorem.

Proof. Suppose for some simple root α, we have $\langle -\lambda + \rho, \alpha^\vee \rangle \leq 0$. Let P be the minimal standard parabolic subgroup associated to α, then we have a $\mathbb{P}^1$-bundle $\pi\colon G/B \to G/P$. The restriction of $\mathcal{L}^\mu$ to each fiber has degree $-\langle \mu, \alpha^\vee \rangle$ and the relative canonical bundle is $K = \mathcal{L}^\alpha$. In particular, the degree of $\mathcal{L}^{-s_\alpha \cdot (-\lambda)}$ restricted to each fiber, where s_α denotes the simple refletcion corresponding to α, is

$$n = \langle s_\alpha \cdot (-\lambda), \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle - 2 \geq -1.$$

We also have

$$\mathcal{L}^{-s_\alpha \cdot (-\lambda)} \otimes_{\mathcal{O}_X} K^{n+1} = \mathcal{L}^{-s_\alpha \cdot (-\lambda) + (\langle \lambda, \alpha^\vee \rangle - 1)\alpha} = \mathcal{L}^\lambda.$$

By Lemma B.2, we get a an isomorphism

$$\mathrm{H}^i(X, \mathcal{L}^{-s_\alpha \cdot (-\lambda)}) \simeq \mathrm{H}^{i+1}(X, \mathcal{L}^\lambda).$$

Now by induction, we can conclude that $\mathrm{H}^i(X, \mathcal{L}^\lambda)$ all vanish if $-\lambda$ is not ρ-regular. If λ is ρ-regular and $w \cdot (-\lambda)$ is ρ-dominant, then we can easily conclude that $\mathrm{H}^{\ell(w)}(X, \mathcal{L}^\lambda) = L(w \cdot (-\lambda))^*$ and $\mathrm{H}^i(X, \mathcal{L}^\lambda) = 0$ for $i < \ell(w)$. Notice that $\mathcal{L}^{2\rho}$ is the canonical bundle of X, so by Serre duality,

$$\mathrm{H}^i(X, \mathcal{L}^\lambda) = \mathrm{H}^{\dim X - i}(X, \mathcal{L}^{-\lambda + 2\rho})^* = 0$$

for $i > \ell(w)$ by the corresponding vanishing result for the weight $-\lambda + 2\rho$. We are done. □

Finally we sketch the proof of Lemma B.2.

Proof. We first consider the case $E = \mathbb{P}^1$ and S is a point. For any integer n, we *fix* once and for all a line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$ of degree n in its isomorphism class. By Serre duality, the vector spaces $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n))$ and $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} K^{n+1})$ are dual to each other. We *fix* once and for all an isomorphism

$$\phi_n \colon \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n)) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} K^{n+1})$$

between these vector spaces.

Now for any line bundle $\mathcal{L}$ on $\mathbb{P}^1$ of degree n, the isomorphism

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{P}^1, \mathcal{L} \otimes_{\mathcal{O}_{\mathbb{P}^1}} K^{n+1})$$

is functorial in $\mathcal{L}$ because the isomorphism $\mathcal{L} \simeq \mathcal{O}(n)$ appears in the same way on both sides. So now for any $\mathbb{P}^1$-bundle $\pi \colon E \to S$ and any line bundle $\mathcal{L}$ on E having degree n on fibers, by descent theory for locally trivial fibration, we obtain a natural isomorphism

$$\pi_* \mathcal{L} \simeq R^1 \pi_* (\mathcal{L} \otimes_{\mathcal{O}_E} K^{n+1}).$$

Consider Leray spectral sequences

$$E_2^{pq} = \mathrm{H}^p(S, R^q \pi_* \mathcal{L}) \Rightarrow \mathrm{H}^{p+q}(E, \mathcal{L})$$

and

$$E_2^{pq} = \mathrm{H}^p(S, R^{q+1} \pi_* (\mathcal{L} \otimes_{\mathcal{O}_E} K^{n+1})) \Rightarrow \mathrm{H}^{p+q+1}(E, \mathcal{L} \otimes_{\mathcal{O}_E} K^{n+1}).$$

Since $n \geq -1$, $\pi_* \mathcal{L} = R^0 \pi_* \mathcal{L}$ and $R^1 \pi_* (\mathcal{L} \otimes_{\mathcal{O}_E} K^{n+1})$ are the only non-vanishing direct images. So the Leray spectral sequences degenerate and we obtain the desired isomorphisms

$$\mathrm{H}^i(E, \mathcal{L}) \simeq \mathrm{H}^{i+1}(E, \mathcal{L} \otimes_{\mathcal{O}_E} K^{n+1})$$

that are functorial in $\mathcal{L}$. □

B.3 Ampleness

Proposition B.3 (Ampleness)**.** If $\lambda \in P$ is anti-dominant, then $\mathcal{L}^\lambda$ is generated by global sections. Furthermore, if λ is anti-dominant regular, then $\mathcal{L}^\lambda$ is very ample.

Proof. For λ anti-dominant, clearly $\mathcal{L}^\lambda$ is basepoint-free, so it is generated by global sections. Now suppose λ is anti-dominant regular. We have to show that the morphism

$$X = G/B \to \mathbb{P}(\mathrm{H}^0(X, \mathcal{L}^\lambda)^*) = \mathbb{P}(L(-\lambda))$$

is a closed immersion. Let $\mathbb{C}_{\text{high}}$ denote the highest weight subspace of $L(-\lambda)$, considered as a point in $\mathbb{P}(L(-\lambda))$. By the regularity assumption, the stabilizer of $\mathbb{C}_{\text{high}} \in \mathbb{P}(L(-\lambda))$ under the G-action is B, hence the G-orbit $\mathbb{O} = G \cdot \mathbb{C}_{\text{high}} \simeq G/B$. It remains to show that the orbit $\mathbb{O}$ is closed. For any closed orbit $\mathbb{O}'$ in the complement of $\mathbb{O}$, $\mathbb{O}'$ must have a B-fixed point because B is solvable. This point will correspond to another "highest weight" subspace in $L(-\lambda)$ that is different from $\mathbb{C}_{\text{high}}$, which is impossible. Thus $\mathbb{O}$ is a closed orbit. □

C BGG resolution

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight, then there is an exact sequence of the form

$$0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to L(\lambda) \to 0,$$

where $m = \#$ of positive roots,

$$C_k = \bigoplus_{w \in W^{(k)}} M(w \cdot \lambda), W^{(k)} = \{w \in W : \ell(w) = k\}.$$

In particular, $C_0 = M(\lambda)$ and $C_m = M(w_0 \cdot \lambda)$ for w_0 being the unique longest element in W. This is the so-called BGG resolution.

In this appendix, we present a geometric construction of the BGG resolution, which was realized by George Kempf [Kem78]. This elegant construction can be generalized to Kac–Moody or semiinfinite cases.

C.1 Grothendieck–Cousin complex

Let X be a topological space. For any closed subset $Z \subset X$ and any abelian sheaf $\mathcal{F}$ on X, we have the abelian group of global section supported on Z, denoted by $\Gamma_Z(X, \mathcal{F})$. The corresponding derived functors, $\mathrm{H}^i_Z(X, \mathcal{F}) := R^i\Gamma_Z(X, \mathcal{F})$, are called local cohomology groups. More generally, for two closed subsets $Z_2 \subset Z_1 \subset X$, we define $\Gamma_{Z_1/Z_2}(X, \mathcal{F})$ to be the quotient $\Gamma_{Z_1}(X, \mathcal{F})/\Gamma_{Z_2}(X, \mathcal{F})$, and define $\mathrm{H}^i_{Z_1/Z_2}(X, \mathcal{F})$ to be $R^i\Gamma_{Z_1/Z_2}(X, \mathcal{F})$.

Let $X = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ be a decreasing filtration by closed subsets.

Definition C.1. We have a complex of the following form

$$0 \to \Gamma(X, \mathcal{F}) \to \mathrm{H}^0_{Z_0/Z_1}(X, \mathcal{F}) \to \mathrm{H}^1_{Z_1/Z_2}(X, \mathcal{F}) \to \mathrm{H}^2_{Z_2/Z_3}(X, \mathcal{F}) \to \cdots .$$

It is called the *Grothendieck–Cousin complex* associated to the filtration $Z_0 \supset Z_1 \supset \cdots$.

Definition C.2. An abelian sheaf $\mathcal{F}$ on X is said to be *Cohen–Macaulay* with respect to the filtration $Z_0 \supset Z_1 \supset \cdots$ if the corresponding Grothendieck–Cousin complex is exact.

Now let X be a locally noetherian scheme.

Definition C.3. A filtration $X = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ is *innocent*, if

- the codimension of Z_i in X is greater or equal to i;
- for each i, the inclusion $Z_i \backslash Z_{i+1} \subset X$ is an affine morphism;
- for each i, $Z_i \backslash Z_{i+1}$ is an affine scheme.

Kempf proved the following useful theorem:

Theorem C.1 (cf. [Kem78] Theorem 10.9)**.** Let X be a locally noetherian scheme, $X = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ be an innocent filtration. Let $\mathcal{F}$ be a Cohen–Maucalay $\mathcal{O}_X$-module with vanishing higher cohomology groups. Assume that the support of $\mathcal{F}$ is X, then $\mathcal{F}$ is Cohen–Maucalay with respect to the filtration $Z_0 \supset Z_1 \supset \cdots$.

C.2 Schubert filtration

Let G be the 1-connected semisimple algebraic group whose Lie algebra is $\mathfrak{g}$. Let B be a fixed Borel subgroup of G, with B^- being the opposite Borel subgroup. For any element $w \in W$, let us fix a lifting of w in G, which, by abuse of notation, is still denoted by w.

Let $X = G/B$ be the flag variety. It is a disjoint union of Bruhat cells

$$X = \bigsqcup_{w \in W} B^- wB/B.$$

Notice that the codimension of B^-wB/B in X is $\ell(w)$. The closure of these cells are called Schubert varieties.

Now let Z_i be the union of all codimenison i Schubert varieties. This gives rise to a filtration of X by closed subvarieties. It is well known that this filtration is innocent. We call it the Schubert filtration.

Using the notation from Appendix B, let us consider the line bundle $\mathcal{L}^{-\lambda}$ on X. By the Borel–Weil–Bott theorem, we know that $\Gamma(X, \mathcal{L}^{-\lambda}) = L(\lambda)^\vee$ and there are no higher cohomologies. Since X is smooth, the line bundle $\mathcal{L}^{-\lambda}$ is a Cohen–Macaulay coherent sheaf with support equal to X. Now by using Theorem C.1, we see that $\mathcal{L}^{-\lambda}$ is Cohen–Macaulay with respect to the Schubert filtration. In other words, we have an exact sequence

$$0 \to \Gamma(X, \mathcal{L}^{-\lambda}) \to \mathrm{H}^0_{Z_0/Z_1}(X, \mathcal{L}^{-\lambda}) \to \cdots \to \mathrm{H}^m_{Z_m/Z_{m+1}}(X, \mathcal{L}^{-\lambda}) \to 0.$$

Theorem C.2 (BGG resolutionn)**.** The above exact sequence is the same as

$$0 \to L(\lambda)^\vee \to M(\lambda)^\vee \to \cdots \to M(w_0 \cdot \lambda)^\vee \to 0.$$

By taking restricted dual, we obtain the original BGG resolution

$$0 \to M(w_0 \cdot \lambda) \to \cdots \to M(\lambda) \to L(\lambda) \to 0.$$

Proof. We have already seen that $\Gamma(X, \mathcal{L}^{-\lambda}) = L(\lambda)^\vee$. For any i, by "excision", we have

$$\mathrm{H}^i_{Z_i/Z_{i+1}}(X, \mathcal{L}^{-\lambda}) = \mathrm{H}^i_{Z_i\setminus Z_{i+1}}(X\setminus Z_{i+1}, \mathcal{L}^{-\lambda}) = \bigoplus_{w\in W, \ell(w)=i} \mathrm{H}^i_{B^-wB/B}(X\setminus Z_{i+1}, \mathcal{L}^{-\lambda}).$$

It is an important fact of life (for instance, in the proof of the Beilinson–Bernstein localization) that

$$\mathrm{H}^i_{B^-wB/B}(X\setminus Z_{i+1}, \mathcal{L}^{-\lambda}) = M(w\cdot\lambda)^\vee.$$

We are done. □

D Beilinson–Bernstein localization

A huge part of geometric representation theory is dealing with "localization" of representations, that is, trying to establish equivalences between certain category of representations and certain category of sheaves (which are of local nature). In this appendix, we review the very first work [BB81], namely the Beilinson–Bernstein localization, in this field.[1]

D.1 $\mathcal{D}$-affinity

A module over a commutative ring is the same as a quasi-coherent sheaf over the corresponding affine scheme. This point of view is one of the starting point of scheme theory. Meanwhile, it provides geometric tools, like the localization technique, to deal with pure algebraic problems. We can do the same for any ringed space. Namely, let $(X, \mathcal{A}_X)$ be any ringed space, then we have a natural functor

$$\Gamma(X, -)\colon \mathcal{A}_X\text{-}\mathsf{Mod} \to \Gamma(X, \mathcal{A}_X)\text{-}\mathsf{Mod}, \mathcal{M} \to \Gamma(X, \mathcal{M}).$$

Now we set $\mathcal{A}_X$ to be the sheaf of differential operators $\mathcal{D}_X$.

Definition D.1 ($\mathcal{D}$-affinity)**.** Let X be a smooth algebraic variety. We call X *$\mathcal{D}$-affine*, if the functor

$$\Gamma(X, -)\colon \mathcal{D}_X\text{-}\mathsf{Mod}_{qc} \to \Gamma(X, \mathcal{D}_X)\text{-}\mathsf{Mod}, \mathcal{M} \to \Gamma(X, \mathcal{M})$$

induces an equivalence of abelian category. Here $\mathcal{D}_X\text{-}\mathsf{Mod}_{qc}$ means the full subcategory of $\mathcal{D}_X\text{-}\mathsf{Mod}$ consisting of $\mathcal{D}_X$-modules that are quasi-coherent as $\mathcal{O}_X$-modules.

In practice, one can check $\mathcal{D}$-affinity using the following proposition.

Proposition D.1 (Criterion for $\mathcal{D}$-affinity)**.** Let X be a smooth algebraic variety. Suppose the global section functor

$$\Gamma(X, -)\colon \mathcal{D}_X\text{-}\mathsf{Mod}_{qc} \to \Gamma(X, \mathcal{D}_X)\text{-}\mathsf{Mod}$$

is exact and conservative, then X is $\mathcal{D}$-affine.

Proof. The global section functor $\Gamma(X, -)$ admits a left-ajoint $\mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} -$. So we only need to show that for any $\mathcal{M} \in \mathcal{D}_X\text{-}\mathsf{Mod}_{qc}$ and $M \in \Gamma(X, \mathcal{D}_X)\text{-}\mathsf{Mod}$, the adjunctions

$$\alpha_{\mathcal{M}}\colon \mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} \Gamma(X, \mathcal{M}) \to \mathcal{M},$$

$$\beta_M\colon M \to \Gamma(X, \mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} M)$$

are isomorphisms.

We first show that $\alpha_{\mathcal{M}}$ is an isomorphism. Let $\mathcal{M}'$ be the image of $\alpha_{\mathcal{M}}$ in $\mathcal{M}$. Since $\Gamma(X, -)$ is exact, we get a short exact sequence

$$0 \to \Gamma(X, \mathcal{M}') \to \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{M}/\mathcal{M}') \to 0.$$

[1] In Chinese mythology, there was nothing and the universe was in a featureless, formless primordial state in the very beginning. Pangu created the world by separating yin from yang with a swing of his giant axe, creating the earth (murky yin) and the sky (clear yang). Unfortunately for later human beings living on earth, the vast sky is beyond their reach.

By definition, the second arrow $\Gamma(X,\mathcal{M}') \to \Gamma(X,\mathcal{M})$ is an isomorphism, whence $\Gamma(X,\mathcal{M}/\mathcal{M}') = 0$. By the conservatism of $\Gamma(X,-)$, we see $\mathcal{M}/\mathcal{M}' = 0$ and hence $\alpha_{\mathcal{M}}$ is surjective. Similarly, let $\mathcal{M}''$ be the kernel of $\alpha_{\mathcal{M}}$. Then we have a short exact sequence

$$0 \to \Gamma(X,\mathcal{M}'') \to \Gamma(X,\mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} \Gamma(X,\mathcal{M})) \to \Gamma(X,\mathcal{M}) \to 0.$$

By definition, the third arrow $\Gamma(X,\mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} \Gamma(X,\mathcal{M})) \to \Gamma(X,\mathcal{M})$ is an isomorphism, whence $\Gamma(X,\mathcal{M}'') = 0$. Again by the conservatism of $\Gamma(X,-)$, we see that $\mathcal{M}'' = 0$. Now we conclude that $\alpha_{\mathcal{M}}$ is an isomorphism.

To show that β_M is an isomorphism, we choose a two-step free resolution

$$\Gamma(X,\mathcal{D}_X)^{\oplus I} \to \Gamma(X,\mathcal{D}_X)^{\oplus J} \to M \to 0$$

of M. Applying the right exact functor $\Gamma(X,\mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} -)$, we obtain a commutative diagram

$$\begin{tikzcd}
\Gamma(X,\mathcal{D}_X)^{\oplus I} \arrow[r] \arrow[d, equal] & \Gamma(X,\mathcal{D}_X)^{\oplus J} \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\beta_M"] & 0 \\
\Gamma(X,\mathcal{D}_X)^{\oplus I} \arrow[r] & \Gamma(X,\mathcal{D}_X)^{\oplus J} \arrow[r] & \Gamma(X,\mathcal{D}_X \otimes_{\Gamma(X,\mathcal{D}_X)} M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By the Five Lemma, we see that β_M is an isomorphism. $\square$

Remark D.1. What do we mean by local? While, in the above discussions, we always equip the space X with the Zariski topology, a coarse topology. In fact, one can show that a $\mathcal{D}_X$-module on X is always a sheaf on $X_{\text{ét}}$, where X is equipped with the étale topology (cf. [BB93]).

Remark D.2. All discussions above hold, mutatis mutandis, for twisted differential operators.

Example D.1 (Affine variety)**.** As an easy corollary of Proposition D.1, we see that any smooth affine algebraic variety is $\mathcal{D}$-affine.

Here is the standard non-trivial example of a $\mathcal{D}$-affine algebraic variety.

Example D.2 (Projective space)**.** The projective space $\mathbb{P}^n$ is $\mathcal{D}$-affine.

Proof. Let $\pi\colon Y = \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the standard projection. For any $\mathcal{M} \in \mathcal{D}_{\mathbb{P}^n}\text{-}\mathsf{Mod}_{qc}$, let $\pi^*\mathcal{M}$ denote its $\mathcal{O}$-module theoretic inverse image. The sheaf $\pi^*\mathcal{M}$ carries a natural $\mathcal{D}_Y$ action. Moreover, it carries a standard $\mathbb{G}_m$-action induced by dilation on fibers. Thus the space of sections $\Gamma(Y,\pi^*\mathcal{M})$ is a graded vector space

$$\Gamma(Y,\pi^*\mathcal{M}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(Y,\pi^*\mathcal{M})_n.$$

Explicitly, let $x_0, \cdot, x_n$ be a choice of coordinate functions on $\mathbb{A}^{n+1}$, $\partial_i = \frac{\partial}{\partial x_i}$, then the grading is given by the Euler vector field

$$L = \sum_{i=0}^{n} x_i \partial_i.$$

Namely, $\Gamma(Y,\pi^*\mathcal{M})_n$ is the eigenspace of L with respect to the eigenvalue $n \in \mathbb{Z}$. Notice that

$$\Gamma(\mathbb{P}^n,\mathcal{M}) = \Gamma(Y,\pi^*\mathcal{M})_0.$$

Let $j\colon Y \hookrightarrow \mathbb{A}^{n+1}$ be the open embedding. In this case, the sheaf theoretic, $\mathcal{O}$-module theoretic, and $\mathcal{D}$-module theoretic pushforward all coincide, which we denote by j_*. We can decompose $\Gamma(Y,-)$ as $\Gamma(Y,-) = \Gamma(\mathbb{A}^{n+1}, j_*-)$, where $\Gamma(\mathbb{A}^{n+1},-)$ is exact. Let

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence in $\mathcal{D}_{\mathbb{P}^n}\text{-}\mathsf{Mod}_{qc}$. The complex

$$0 \to j_*\pi^*\mathcal{M}_1 \to j_*\pi^*\mathcal{M}_2 \to j_*\pi^*\mathcal{M}_3 \to 0$$

is exact upon restricting to Y, so its cohomologies are $\mathcal{D}_{\mathbb{A}^{n+1}}$-modules supported on $\{0\} \subset \mathbb{A}^{n+1}$. By Kashiwara's theorem, they are direct sums of delta-function $\mathcal{D}$-modules supported on $\{0\}$. In terms of global differential operators, the delta-function $\mathcal{D}$-module (or more precisely, its global section) is

$$\delta_{\{0\}} = \mathbb{C}[\partial_0, \cdots, \partial_n]\delta, \text{ where } x_i\delta = 0.$$

From this we can see that the eigenvalues of L on $\delta_{\{0\}}$ are $-(n+1), -(n+2), \cdots$. As a consequence, the sequence

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\mathbb{P}^n, \mathcal{M}_1) \arrow[r] \arrow[d, equal] & \Gamma(\mathbb{P}^n, \mathcal{M}_2) \arrow[r] \arrow[d, equal] & \Gamma(\mathbb{P}^n, \mathcal{M}_3) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Gamma(\mathbb{A}^{n+1}, j_*\pi^*\mathcal{M}_1)_0 \arrow[r] & \Gamma(\mathbb{A}^{n+1}, j_*\pi^*\mathcal{M}_2)_0 \arrow[r] & \Gamma(\mathbb{A}^{n+1}, j_*\pi^*\mathcal{M}_3)_0 \arrow[r] & 0 \end{tikzcd}$$

is exact. This shows that $\Gamma(\mathbb{P}^n, -)\colon \mathcal{D}_{\mathbb{P}^n}\text{-}\mathsf{Mod}_{qc} \to \Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n})\text{-}\mathsf{Mod}$ is exact.

Now suppose $\Gamma(\mathbb{P}^n, \mathcal{M}) = 0$ for some $0 \neq \mathcal{M} \in \mathcal{D}_{\mathbb{P}^n}\text{-}\mathsf{Mod}_{qc}$. Then $\pi^*\mathcal{M} \neq 0$ and hence

$$\Gamma(Y, \pi^*\mathcal{M}) = \Gamma(\mathbb{A}^{n+1}, j_*\pi^*\mathcal{M}) \neq 0.$$

This means that there exists $n \neq 0$ such that $\Gamma(Y, \pi^*\mathcal{M})_n \neq 0$.

If $n > 0$, we can pick n to be the smallest positive integer such that $\Gamma(Y, \pi^*\mathcal{M})_n \neq 0$. Pick a nonzero section $s \in \Gamma(Y, \pi^*\mathcal{M})_n$. There must exist some i such that $\partial_i s \neq 0$, otherwise $Ls = 0 \neq ns$. It is easy to compute that $\partial_i s \in \Gamma(Y, \pi^*\mathcal{M})_{n-1}$, a contradiction to the minimality of n.

If $n < 0$, we can pick n to be the largest negative integer such that $\Gamma(Y, \pi^*\mathcal{M})_n \neq 0$. Pick a nonzero section $s \in \Gamma(Y, \pi^*\mathcal{M})_n$. If $x_i s = 0$ for all $i = 0, \cdots, n$, then s is supported on $\{0\}$ and s equals to the zero section in $\Gamma(Y, \pi^*\mathcal{M})$. Hence there exists i such that $x_i s \neq 0$. It is easy to compute that $x_i s \in \Gamma(Y, \pi^*\mathcal{M})_{n+1}$, a contradiction to the maximality of n.

Now we can conclude that $\Gamma(\mathbb{P}^n, -)$ is conservative on $\mathcal{D}_{\mathbb{P}^n}\text{-}\mathsf{Mod}_{qc}$. Therefore $\mathbb{P}^n$ is $\mathcal{D}$-affine by Proposition D.1. □

Let G be a 1-connected semisimple algebraic group, B be a Borel, and $X = G/B$ be the flag variety. In a similar manner, we can prove that the flag variety G/B is $\mathcal{D}$-affine.[2]

Theorem D.1 ($\mathcal{D}$-affinity of G/B)**.** The flag variety $X = G/B$ is $\mathcal{D}$-affine.

Let N be the unipotent radical of B, $H = B/N$ be the corresponding Cartan torus, and $Y = G/N$ be the base affine space. The idea is to pullback a $\mathcal{D}$-module $\mathcal{M}$ on X along $\pi\colon Y \to X$. The morality to do this is that $\Gamma(Y, \pi^*\mathcal{M})$ admits a weight decomposition coming from the right action of H on the fibers. Here we collect some facts about differential operators on the base affine space, in particular, the Fourier transform due to Kazhdan–Laumon. For more details, consult [Sha74], [BGG75], [KL88], [Pol01], [BBP02] and references therein.

It is known that Y is quasi-affine, and its affine closure $\overline{Y}$ is normal. The right H-action on Y induces a weight decomposition on $\mathcal{D}(Y)$ and $\mathcal{O}(Y)$, which we denote by lower indices, i.e. $\mathcal{D}(Y)_\lambda$ (resp. $\mathcal{O}(Y)_\lambda$) means the λ-eigenspace in $\mathcal{D}(Y)$ (resp. $\mathcal{O}(Y)$) with respect to this right H-action. It is known that $\mathcal{O}(Y)_\lambda$ is the irreducible G-module of highest weight λ if λ is a dominant weight for G. Otherwise, $\mathcal{O}(Y)_\lambda = 0$. The defining ideal of $\overline{Y} - Y$ is generated by $\mathcal{O}(Y)_\rho$.

There is another left G-action on Y commuting with the right H-action. This yields another wight decomposition on $\mathcal{D}(Y)$ and $\mathcal{O}(Y)$, which we denote by super indices, i.e. $\mathcal{D}(Y)^\mu$ (resp. $\mathcal{O}(Y)^\mu$) means the μ-eigenspace in $\mathcal{D}(Y)$ (resp. $\mathcal{O}(Y)$) with respect to this left G-action. The right H-action induces an algebra embedding $U\mathfrak{h} \hookrightarrow \mathcal{D}(Y)$, whose image equals to $\mathcal{D}(Y)^0_0$ (cf. [Sha74]).

For any simple root α of G, denote by P_α the standard minimal parabolic subgroup of type α. Let $B_\alpha = [P_\alpha, P_\alpha]$ be the commutator subgroup, $Y_\alpha = G/B_\alpha$. The standard projection

$$\pi_\alpha\colon Y = G/N \to Y_\alpha = G/B_\alpha$$

is a fibration with fiber $B_\alpha/N \simeq \mathbb{A}^2 - \{0\}$.

[2]We learn this proof from Roman Bezrukavnikov.

Let $\overline{\pi}_\alpha \colon \overline{Y}^\alpha \to Y_\alpha$ be the affine completion of π_α, that is, the affine morphism corresponding to the sheaf of algebras $\pi_{\alpha*}\mathcal{O}_Y$ on Y_α. Then $\overline{\pi}_\alpha$ is a rank 2 vector bundle, and Y is identified with the complement of the zero section in $\overline{Y}^\alpha$.

Suppose we are given a symplectic vector space E. The ring $\mathcal{D}(E)$ of global differential operators on E has an automorphism, that is, the Fourier transform $\mathcal{F}_E$ (cf. [Yu20][3]). Let $\omega\colon E\times E\to\mathbb{C}$ be the symplectic form, $\mathrm{pr}_i\colon E\times E\to E$ $(i=1,2)$ be the projection to the ith factor. On the function level, the Fourier transform is defined by

$$\mathcal{F}_{\mathrm{fun}}\colon \mathcal{O}(E)\to\mathcal{O}(E), f\mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1{}^! f\cdot\omega^!(\exp(2\pi\sqrt{-1}\dagger))).$$

Here $\exp(2\pi\sqrt{-1}\dagger)$ is the (Tate twisted) exponential function on $\mathbb{C}$ and $\cdot$ is the pointwise multiplication of functions. This will induce an automorphism of $\mathcal{D}(E)$, defined by

$$\mathcal{F}_E\colon \mathcal{D}(E)\to\mathcal{D}(E), D\mapsto [f\mapsto \mathcal{F}_{\mathrm{fun}}^{-1}(D\mathcal{F}_{\mathrm{fun}}(f))].$$

Explicitly, let $\{q^1,\cdots,q^n,p_1,\cdots,p_n\}$ be a Darboux coordinate on E^*. Then $\mathcal{F}_E$ is the automorphism on $\mathcal{D}(E)$ exchanging q^i and $\sqrt{-1}\frac{\partial}{\partial p_i}$, p_i and $\sqrt{-1}\frac{\partial}{\partial q^i}$.[4,5]

This construction comes in family. Let $\pi\colon E\to S$ be a symplectic vector bundle, then the sheaf $\pi_*\mathcal{D}_E$ carries an automorphism that is induced by the Fourier transform on fibers. In particular, this leads to an automorphism of the algebra of global differential operators $\mathcal{D}(E)$.

The vector bundle $\overline{\pi}_\alpha$ carries a unique (up to a nonzero constant) G-invariant symplectic form (cf. [KL88] for more details). This allows us to define a Fourier transform $\mathcal{F}_\alpha$ on the ring of global differential operators $\mathcal{D}(\overline{Y}^\alpha)=\mathcal{D}(Y)$ (this equality follows from the fact that $\overline{Y}^\alpha - Y$ has codimension 2 in $\overline{Y}^\alpha$).

Theorem D.2 (cf. [BBP02])**.** Let s_α be the simple reflection corresponding to the simple root α in the Weyl group W. Then the assignment $s_\alpha\mapsto\mathcal{F}_\alpha$ extends to an action of W on $\mathcal{D}(Y)$.

Let $\mathcal{F}_w$ denote the automorphism of $\mathcal{D}(Y)$ given by $w\in W$. It is also calculated in [BBP02] that $\mathcal{F}_w$ commutes with the left G-action on $\mathcal{D}(Y)$ and $\mathcal{F}_w(\mathcal{D}(Y)^\lambda_\mu)=\mathcal{D}(Y)^\lambda_{w(\mu)}$.

Let w_0 be the longest element in W. For a dominant weight μ, let $\mu^*=-w_0(\mu)$, then $L(\mu^*)\simeq L(\mu)^*$. Consider the map

$$m\colon \mathcal{O}(Y)_\mu\otimes\mathcal{O}(Y)_{\mu^*}\to\mathcal{D}(Y), f\otimes g\mapsto f\cdot\mathcal{F}_{w_0}(g).$$

The image of m lands in $\mathcal{D}(Y)_{\mu+\omega_0(\mu^*)}=\mathcal{D}(Y)_0$.

Let $C_\mu\in\mathcal{O}(Y)_\mu\otimes\mathcal{O}(Y)_{\mu^*}\simeq L(\mu)\otimes L(\mu)^*$ be the unique (up to a nonzero constant) nonzero G-invariant element, $P_\mu=m(C_\mu)\in\mathcal{D}(Y)^0_0=U\mathfrak{h}$. Similar to the calculation of the determinant of the Shapovalov form on Verma modules, the element P_μ is calculated explicitly in [BBP02].

Proposition D.2. Up to a nonzero constant, we have

$$P_\mu=\prod_{\alpha^\vee\in R_+^\vee}\prod_{i=1}^{\langle\alpha^\vee,\mu\rangle}(\alpha^\vee+\langle\alpha^\vee,\rho\rangle-i).$$

Here $R_+^\vee$ means the set of positive coroots.

Now we present Bezrukavnikov's proof of Theorem D.1.

Proof. For $\mathcal{M}\in\mathcal{D}_X\text{-}\mathsf{Mod}_{qc}$, $\Gamma(Y,\pi^*\mathcal{M})$ admits a weight decomposition coming from the right action of H, which, to keep our notation stable, is denoted by lower indices. Notice that $\Gamma(X,\mathcal{M})=\Gamma(Y,\pi^*\mathcal{M})_0$. We are going to prove that the functor

$$\mathcal{D}_X\text{-}\mathsf{Mod}_{qc}\to\Gamma(X,\mathcal{D}_X)\text{-}\mathsf{Mod}, \mathcal{M}\mapsto\Gamma(Y,\pi^*\mathcal{M})_0$$

is exact and conservative.

First, for the conservatism part, suppose $\mathcal{M}\in\mathcal{D}_X\text{-}\mathsf{Mod}_{qc}$ is nonzero, but $\Gamma(Y,\pi^*\mathcal{M})_0=0$. Since π is smooth and Y is quasi-affine, $\Gamma(Y,\pi^*\mathcal{M})\neq 0$. Thus there exists some weight ν such that $\Gamma(Y,\pi^*\mathcal{M})_\nu\neq 0$. Pick $w\in W$ such that $w(\nu)$ is dominant. By the explicit formula from Proposition D.2, we see that $\mathcal{F}_{w^{-1}}(P_{w(\nu)})$

[3]Available at the Banana Space.

[4]I copy the signs from a cheatsheet created by Jinyi Wang.

[5]A famous slogan: taking derivative in the position space is the same as multiplication in the frequency space, vice versa.

acts as a nonzero constant on $\Gamma(Y, \pi^*\mathcal{M})_\nu$. Notice that by construction, $\mathcal{F}_{w^{-1}}(P_{w(\nu)}) \in \mathcal{D}(Y)_\nu \cdot \mathcal{D}(Y)_{-\nu}$. This shows that the action of $\mathcal{D}(Y)_{-\nu}$ on $\Gamma(Y, \pi^*\mathcal{M})_\nu$ is nonzero. However,

$$\mathcal{D}(Y)_{-\nu} \cdot \Gamma(Y, \pi^*\mathcal{M})_\nu \subset \Gamma(Y, \pi^*\mathcal{M})_0 = 0,$$

a contradiction. This finishes the proof of conservatism.

To show the exactness, let

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence in $\mathcal{D}_X\text{-}\mathsf{Mod}_{qc}$. Let $j\colon Y \to \overline{Y}$ be the affine closure. The complex

$$0 \to j_*\pi^*\mathcal{M}_1 \to j_*\pi^*\mathcal{M}_2 \to j_*\pi^*\mathcal{M}_3 \to 0$$

is exact upon restricting to Y, so its cohomologies are supported on $\overline{Y} - Y$. By Kashiwara's theorem, they are extended from $\mathcal{D}$-modules on $\overline{Y} - Y$. However, notice that the defining ideal of $\overline{Y} - Y$ is generated by $\mathcal{O}(Y)_\rho$, the weights (coming from the right H-action) of these cohomologies are greater or equal to ρ. Taking the weight 0 part, we get a short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(X, \mathcal{M}_1) & \longrightarrow & \Gamma(X, \mathcal{M}_2) & \longrightarrow & \Gamma(X, \mathcal{M}_3) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \Gamma(\overline{Y}, j_*\pi^*\mathcal{M}_1)_0 & \longrightarrow & \Gamma(\overline{Y}, j_*\pi^*\mathcal{M}_2)_0 & \longrightarrow & \Gamma(\overline{Y}, j_*\pi^*\mathcal{M}_3)_0 & \longrightarrow & 0
\end{array}$$

This finishes the proof of exactness. □

Remark D.3. Instead of pulling back to G/N, one can also pull back to G. This idea, due to Frenkel–Gaitsgory [FG04], gives rise to another proof of the exactness part. In short, they decompose the functor $\Gamma(X, -)\colon \mathcal{D}_X\text{-}\mathsf{Mod}_{qc} \to U\mathfrak{g}\text{-}\mathsf{Mod}$ into the composition of

$$\Gamma(G, \pi^*-)\colon \mathcal{D}_X\text{-}\mathsf{Mod}_{qc} \to U\mathfrak{g}\text{-}\mathsf{biMod}, \mathcal{M} \mapsto \Gamma(G, \pi^*\mathcal{M})$$

and

$$\mathrm{Hom}_{U\mathfrak{g}}(M(0), -)\colon U\mathfrak{g}\text{-}\mathsf{biMod} \to U\mathfrak{g}\text{-}\mathsf{Mod}, M \mapsto \mathrm{Hom}_{U\mathfrak{g}}(M(0), M).$$

Here $\pi\colon G \to X$ is the projection, $U\mathfrak{g}\text{-}\mathsf{biMod}$ is the category of $U\mathfrak{g}$-bimodules and $M(0)$ is the Verma module with highest weight 0. The first step is exact because π is smooth and G is affine. Moreover, its image lies in the ind-completion of the category $\mathcal{O}$. But it is well-known that $M(0)$ is projective in $\mathcal{O}$ (so its ind-completion), the exactness follows.

D.2 Global differential operators on the flag variety

In this section, we are going to calculate the ring of global differential operators $\mathcal{D}(X)$ on the flag variety X.

Notice that the left G-action on X induces an algebra homomorphism $\Psi\colon U\mathfrak{g} \to \mathcal{D}(X)$.

Proposition D.3. The map $\Psi\colon U\mathfrak{g} \to \mathcal{D}(X)$ is surjective.

Proof. Filtration plays the key role. On the left hand side, there is the PBW filtration, which we denote by $F^\bullet_{\mathrm{PBW}}$. It is an increasing filtration, whose associated graded ring is isomorphic to $\mathrm{Sym}^\bullet \mathfrak{g} = \mathcal{O}(\mathfrak{g}^*)$. On the right hand side, there is the filtration given by the order of differential operators, which we denote by $F^\bullet_{\mathrm{ord}}$. It is also an increasing filtration, whose associated graded ring is isomorphic to $\mathcal{O}(T^*X)$. The map Ψ is a morphism of filtered rings, so it suffices to show that the associated graded map

$$\mathrm{gr}\,\Psi\colon \mathcal{O}(\mathfrak{g}^*) \to \mathcal{O}(T^*X)$$

is surjective. This map is induced from the moment map $\mu\colon T^*X \to \mathfrak{g}^*$. Recall that the Springer map $T^*X \to \mathcal{N}$ is a resolution of singularity, where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone, so $\mathcal{O}(T^*X) = \mathcal{O}(\mathcal{N})$. Using the nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Since $\mathcal{N}$ is a closed subvariety of $\mathfrak{g}$, the map $\mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathcal{N})$ is surjective. This finishes the proof. □

Now it remains to calculate the kernel of Ψ. From the proof above, we already see that the kernel of $\mathrm{gr}\,\Psi$ is the defining ideal of $\mathcal{N}$ inside $\mathfrak{g} \simeq \mathfrak{g}^*$.

Proposition D.4. Let $\mathfrak{z}$ denote the center of $U\mathfrak{g}$, $\chi_0 \colon \mathfrak{z} \to \mathbb{C}$ be the central character corresponding to the simple module $L(0)$ (the trivial representation). Then $\ker \Psi$ is generated by $\ker \chi_0$.

Proof. Notice that $\Psi \colon U\mathfrak{g} \to \mathcal{D}(X)$ is G-equivariant, so it induces a map $\Psi^G \colon \mathfrak{z} = (U\mathfrak{g})^G \to \mathcal{D}(X)^G$.

We first show that $\mathcal{D}(X)^G = \mathbb{C} \cdot 1$. Recall that there is an open dense G-orbit in $\mathcal{N}$ (the regular orbit),

$$\mathcal{O}(T^*X)^G = \mathcal{O}(\mathcal{N})^G = \mathbb{C} \cdot 1.$$

In particular, $\mathcal{O}(T^*X)^G$ is concentrated in the degree 0 part. Since the G-action on $\mathcal{D}(X)$ is compatible with the order filtration, we have $\mathcal{D}(X)^G = \mathbb{C} \cdot 1$.

This shows that $\Psi(z) = \chi_0(z) \cdot 1$. In particular, $\ker \chi_0 \subset \ker \Psi$. It remains to show that $\ker \Psi$ is generated by $\ker \chi_0$. Again, it remains to show that the associated graded ideal $\operatorname{gr}(\ker \chi_0)$ equals to $\operatorname{gr} \ker \Psi$, the defining ideal of $\mathcal{N}$ inside $\mathfrak{g}$. It is known that the defining ideal of $\mathcal{N}$ inside $\mathfrak{g}$ is generated by $\left(\bigoplus_{\bullet > 0} \operatorname{Sym}^\bullet \mathfrak{g}\right)^G$. Now the proposition follows from the Harish-Chandra isomorphism and the Chevalley restriction theorem $\mathfrak{z} \simeq (\operatorname{Sym} \mathfrak{h})^W \simeq (\operatorname{Sym} \mathfrak{g})^G$. □

Combining the discussions in the above two sections, we obtain the Beilinson–Bernstein localization.

Theorem D.3 (Beilinson–Bernstein localization)**.** The global section functor induces an equivalence of categories

$$\Gamma(X, -) \colon \mathcal{D}_X\text{-}\mathsf{Mod}_{qc} \to U\mathfrak{g}/(\ker \chi_0)\text{-}\mathsf{Mod} = \mathfrak{g}\text{-}\mathsf{Mod}_{\chi_0},$$

where $\mathfrak{g}\text{-}\mathsf{Mod}_{\chi_0}$ denotes the category of $\mathfrak{g}$-modules with central character χ_0. A quasi-inverse to this functor is given by the localization functor

$$\mathcal{D}_X \otimes_{\mathcal{D}(X)} - \colon \mathfrak{g}\text{-}\mathsf{Mod}_{\chi_0} \to \mathcal{D}_X\text{-}\mathsf{Mod}_{qc}.$$

D.3 Some generalizations

Recall that we have introduced the line bundle $\mathcal{L}^\lambda$ on $X = G/B$ corresponding to a weight λ in Appendix B.1. Let $\mathcal{D}_X^\lambda = \mathcal{L}^\lambda \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} (\mathcal{L}^\lambda)^*$ be the sheaf of $\mathcal{L}^\lambda$-twisted differential operators. We have the following twisted version of Beilinson–Bernstein localization.

Theorem D.4 (Twisted Beilinson–Bernstein localiztaion)**.** Suppose $w_0(\lambda)$ is ρ-dominant, then the flag variety X is $\mathcal{D}_X^\lambda$-affine. Moreover, $\Gamma(X, \mathcal{D}_X^\lambda) = U\mathfrak{g}/(\ker \chi_{w_0(\lambda)})$, where $\chi_{w_0(\lambda)} \colon \mathfrak{z} \to \mathbb{C}$ is the central character corresponding to $L(w_0(\lambda))$. So there is an equivalence of categories

$$\Gamma(X, -) \colon \mathcal{D}_X^\lambda\text{-}\mathsf{Mod} \to U\mathfrak{g}/(\ker \chi_{w_0(\lambda)})\text{-}\mathsf{Mod} = \mathfrak{g}\text{-}\mathsf{Mod}_{\chi_{w_0(\lambda)}},$$

where $\mathfrak{g}\text{-}\mathsf{Mod}_{\chi_{w_0(\lambda)}}$ denotes the category of $\mathfrak{g}$-modules with central character $\chi_{w_0(\lambda)}$. A quasi-inverse is given by the localization functor

$$\mathcal{D}_X^\lambda \otimes_{\Gamma(X, \mathcal{D}_X^\lambda)} - \colon \mathfrak{g}\text{-}\mathsf{Mod}_{\chi_{w_0(\lambda)}} \to \mathcal{D}_X^\lambda\text{-}\mathsf{Mod}.$$

Let K be a closed subgroup of G. Akin to our discussion of equivariant vector bundles in Appendix B.1, we can talk about K-equivariant $\mathcal{D}$-modules on X. The formulae are exactly the same, except we need to take the extraordinary inverse image $^!$ of $\mathcal{D}$-modules in the cocycle condition. We have the following equivariant localization theorem.

Theorem D.5 (Equivariant localization)**.** Inside the Beilinson–Bernstein localization $\mathcal{D}_X\text{-}\mathsf{Mod} \simeq \mathfrak{g}\text{-}\mathsf{Mod}_{\chi_0}$, we can identify the subcategory of K-equivariant objects, namely

$$\mathcal{D}_X\text{-}\mathsf{Mod}^K \simeq (\mathfrak{g}, K)\text{-}\mathsf{Mod}_{\chi_0}.$$

Here $\mathcal{D}_X\text{-}\mathsf{Mod}^K$ is the category of K-equivariant $\mathcal{D}$-modules on X, $(\mathfrak{g}, K)\text{-}\mathsf{Mod}$ is the category of Harish-Chandra modules introduced in Appendix A.

One remarkable point is, if there are only finitely many K-orbits on X (this is the case if $K = B$), then every $\mathcal{D}$-module from $\mathcal{D}_X\text{-}\mathsf{Mod}^K$ is automatically regular holonomic. This allows us, under the Riemann–Hilbert correspondence, to treat them as perverse sheaves, so that techniques and constructions from topology and algebraic geometry, like the intersection homology and the weight filtration, can be utilized to solve purely representation theoretic problems, including the famous Kazhdan–Lusztig conjecture.

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