Geometric Satake

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G conn'd reductive gp $/\mathbb{C}$, $\mathbb{k}$ a noetherian regular fin. dim.

$Gr_G/\mathbb{C}$ ind-projective.

$Perv_{L^+G}(Gr_G; \mathbb{k}) \subset D^b_{L^+G}(Gr_G, \mathbb{k})$.

$m: Gr_G \tilde{\times} Gr_G = LG \times^{L^+G} Gr_G \longrightarrow Gr_G$

For $\mathcal{A}, \mathcal{B} \in D^b_{L^+G}(Gr_G; \mathbb{k})$

$\mathcal{A} \star^{L^+G} \mathcal{B} := m_*(\mathcal{A} \tilde{\boxtimes}^L_{\mathbb{k}} \mathcal{B}) \in D^b_{L^+G}(Gr_G; \mathbb{k})$

<u>Fact</u>. m_* sends perverse to perverse.

$$\mathcal{A} \star^{L^+G}_0 \mathcal{B} := m_*\left({}^p\mathcal{H}^0(\mathcal{A} \tilde{\boxtimes}^L_{\mathbb{k}} \mathcal{B})\right)$$

$$= {}^p\mathcal{H}^0(\mathcal{A} \star^{L^+G} \mathcal{B}) \in Perv_{L^+G}(Gr_G, \mathbb{k})$$

when $\mathcal{A}, \mathcal{B}$ are.

<u>Thm</u> (Mirković–Vilonen) There's an equiv. of monoidal cat.

$$\left(Perv_{L^+G}(Gr_G; \mathbb{k}), \star^{L^+G}_0\right) \simeq \left(Rep(\hat{G}_{\mathbb{k}}), \otimes_{\mathbb{k}}\right).$$

first $\cong$ $(\mathrm{Rep}(\tilde{G}_{\Bbbk}), \otimes_{\Bbbk})$ for some $\tilde{G}_{\Bbbk}$, then show $\tilde{G}_{\Bbbk} \cong \hat{G}_{\Bbbk}$.

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§ Fusion product

$X = \mathbb{A}^1_{\mathbb{C}}$, $\quad \mathrm{Gr}_{G, X^I} \to X^I$

$$\begin{tikzcd}
\mathrm{Gr}_G \times X \arrow[r, "i"] \arrow[d] & \mathrm{Gr}_{G,X^2} \arrow[d] & \mathrm{Gr}_G \times \mathrm{Gr}_G \times (X^2 - \Delta_X) \arrow[l, "j"'] \arrow[d] \\
\Delta_X \arrow[r, hook] & X^2 & X^2 - \Delta_X \arrow[l, hook']
\end{tikzcd}$$

Prop. For $A, B \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G ; \Bbbk)$,

$$i^* j_{!*}\left({}^pH^0(A \boxtimes^L_{\Bbbk} B) \boxtimes \underline{\Bbbk}_{X^2-\Delta_X}[2]\right)[-1] \cong \left(A \star_0^{L^+G} B\right) \boxtimes \Bbbk_{\Delta_X}[1].$$

<u>Sketch</u>: Write $m_{X^2} : \widetilde{\mathrm{Gr}}_{G,X^2} \to \mathrm{Gr}_{G,X^2}$, then

$$j_{!*}(\cdots) \cong m_{X^2,*}\,{}^pH^0\Big(\underbrace{(A \boxtimes \underline{\Bbbk}_X[1])}_{\text{on } \mathrm{Gr}_X \cong \mathrm{Gr}_G \times X} \,\tilde{\boxtimes}^L_{\Bbbk}\, (B \boxtimes \underline{\Bbbk}_X[1])\Big)$$

(need m_{X^2} to be small)

and then use base change.

Switching A and B defines an isom.

$$\left(A \star_0^{L^+G} B\right) \boxtimes \Bbbk_{\Delta_X}[1] \cong \left(B \star_0^{L^+G} A\right) \boxtimes \Bbbk_{\Delta_X}[1].$$

Rmk:

Fact. $\boxtimes \underline{\mathbb{k}}_{\Delta X}$ is fully faithful.

$$\Rightarrow \quad \sigma^{Fus} : A \star_0^{L^+G} B \cong B \star_0^{L^+G} A$$

Def. $\sigma^{am} = (-1)^{ab} \sigma^{Fus}$, where a, b are the parities of $supp(A)$, $supp(B)$.

$$F : Perv_{L^+G}(Gr_G; \mathbb{k}) \longrightarrow Mod(\mathbb{k})$$

$$A \longmapsto H^\bullet(Gr_G, A) = \bigoplus_i H^i$$

Prop. Let $A, B \in Perv_{L^+G}(Gr_G; \mathbb{k})$, $f : Gr_{G, X^2} \longrightarrow X^2$ proj.

$\mathcal{C} := f_* j_{!*}\left({}^p\mathcal{H}^\circ(A \overset{\mathbb{L}}{\underset{\mathbb{k}}{\boxtimes}} B) \boxtimes \underline{\mathbb{k}}_{X^2 - \Delta X}[2]\right) \in D^b(X^2; \mathbb{k})$, then

① $x \in \Delta X$, $H^n(\mathcal{C}_x) = H^{n+2}(Gr_G, A \star_0^{L^+G} B)$

② $x \in X^2 - \Delta X$, $H^n(\mathcal{C}_x) = H^{n+2}(Gr_G \times Gr_G, {}^p\mathcal{H}^\circ(A \overset{\mathbb{L}}{\underset{\mathbb{k}}{\boxtimes}} B))$

③ Sheaves $\mathcal{H}^n(\mathcal{C})$ are locally constant

④ There's an isom.

$$H^\bullet(Gr_G \times Gr_G, {}^p\mathcal{H}^\circ(A \overset{\mathbb{L}}{\underset{\mathbb{k}}{\boxtimes}} B)) \simeq H^\bullet(Gr_G, A) \otimes_{\mathbb{k}} H^\bullet(Gr_G ; B).$$

①②③ $\Rightarrow$ $H^\bullet(Gr_G, A \star_0^{L^+G} B) \cong H^\bullet(Gr_G \times Gr_G, {}^p\mathcal{H}^\circ(A \overset{\mathbb{L}}{\underset{\mathbb{k}}{\boxtimes}} B))$

④ $\cong H^\bullet(Gr_G, A) \otimes_{\mathbb{k}} H^\bullet(Gr_G, B)$. $\Rightarrow$ F is symmetric monoidal.

① & ② : from previous facts + base change.

③ $$\mathcal{C} = \tilde{f}_*\Big(\underbrace{{}^p\mathcal{H}^0\big((A \boxtimes \underline{k}_X[1]) \overset{L}{\underset{k}{\boxtimes}} (B \boxtimes \underline{k}_X[1])\big)}_{D}\Big)$$

where
$$\begin{tikzcd} \widetilde{Gr}_{G,X^2} \arrow[r, "m"] \arrow[rd, bend right, "\tilde{f}"'] & Gr_{G,X^2} \arrow[d, "f"] \\ & X^2 \end{tikzcd}$$

D is constructible w.r.t. the stratification $\{Gr_{G,X,\lambda} \tilde{\times} Gr_{G,X,\mu}\}$

and $Gr_{G,X,\mu} \tilde{\times} Gr_{G,X,\lambda} \longrightarrow X^2$ are locally trivial fibration.

(better: use local acyclicity)

④ easily true if $H^*(Gr_G, A)$ is flat $/k$. In general need resolution.

Rmk. If k is a field of char. 0, then $\mathrm{Perv}_{L^+G}(Gr_G ; k)$ is semisimple

$\Rightarrow F$ is exact $\xRightarrow{\text{Tannakian}}$
$$\begin{tikzcd} (\mathrm{Perv}_{L^+G}(Gr_G, k), *) \arrow[r, "\cong"] \arrow[rd, "F"'] & (\mathrm{Rep}(\check{G}_k), \otimes_k) \arrow[d, "\text{forget}"] \\ & \mathrm{Vect}(k) \end{tikzcd}$$

There is supposed to be $\hat{T} \subset \hat{G} \leftrightsquigarrow$ a lift
$$\begin{tikzcd} \mathrm{Perv}_{L^+G}(Gr_G, k) \arrow[r, dashed] \arrow[rd, "F"'] & \bigoplus_{X_*(T)} \mathrm{Vect}(k) \arrow[d] \\ & \mathrm{Vect}(k) \end{tikzcd}$$

§ Decomposition of F

$T \hookleftarrow B \subset G$

$Gr_T \xleftarrow{p} Gr_B \xrightarrow{q} Gr_G$

where $(Gr_T)^{red} = \coprod_{\lambda \in X_*(T)} \mathrm{Spec}(\mathbb{C})$

& q is a bijection on geom. pts

$S_\lambda = q(p^{-1}(\lambda)) \subset Gr_G$ "semi-infinite orbits"

For $\mu \in X_*(T)^+$ strictly dominant $(\mu = 2\rho)$

a $\mathbb{G}_m$-action $\mathbb{G}_m \xrightarrow{\mu} T \curvearrowright Gr_G$

$S_\lambda = \{ x \in Gr_G : \lim_{a \to 0} a^\mu x = t^\lambda \}$

Similarly, define $T_\lambda \subset Gr_G$ using opposite Borel.

$T_\lambda = \{ x \in Gr_G : \lim_{a \to \infty} a^\mu x = t^\lambda \}$.

hyperbolic localization. for $A \in D^b_{L^+G}(Gr_G; \mathbb{k})$,

$F_\lambda(A) := H^\bullet_c(S_\lambda, A|_{S_\lambda}) \cong H^\bullet(T_\lambda, i^!_{T_\lambda} A)$

Thm. For $\mathcal{A} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G; \mathbb{k})$,

① $F_\lambda(\mathcal{A})$ is concentrated in deg $= \langle 2\rho, \lambda \rangle$

② there is an isom. $F(\mathcal{A}) = H^0(\mathrm{Gr}_G, \mathcal{A}) \cong \bigoplus_{\lambda \in X_*(T)} F_\lambda(\mathcal{A})$.

<u>Main geometric input</u>.

Prop For $\lambda, \mu \in X_*(T)$ w/ μ dominant.

① $S_\lambda \cap \mathrm{Gr}_{G, \le \mu}$ is nonempty iff $\lambda \in \mathrm{Conv}(W \cdot \mu) \cap (\mu + \mathbb{Z}\Phi^\vee)$

② $S_\lambda \cap \mathrm{Gr}_{G, \le \mu}$ has pure dim. $\langle \rho, \mu + \lambda \rangle$.

Pf ① $S_\lambda \cap \mathrm{Gr}_{G, \le \mu}$ is stable under $\mathbb{G}_m$

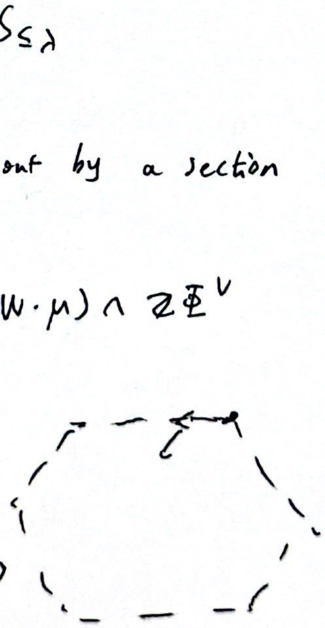
$\Rightarrow$ nonempty $\Leftrightarrow t^\lambda \in \mathrm{Gr}_{G, \le \mu} \Leftrightarrow \lambda \in \mathrm{Conv}(W \cdot \mu) \cap (\mu + \mathbb{Z}\Phi^\vee)$

② Fact: $\overline{S_\lambda} = \bigcup_{\lambda' \le \lambda} S_{\lambda'} = S_{\le \lambda}$

and $S_{< \lambda} \subset S_{\le \lambda}$ is cut out by a section of an ample l.b.

$S_{\le \lambda} \cap \mathrm{Gr}_{G, \le \mu}$ for $\lambda \in \mathrm{Conv}(W \cdot \mu) \cap \mathbb{Z}\Phi^\vee$

find a seq. $\underset{\mu}{\lambda_0} \geq \lambda_1 \geq \cdots \geq \underset{w_0(\mu)}{\lambda_m}$



steps of $\Delta^\vee$.

s.t. each $\lambda_i - \lambda_{i+1} \in \Delta^\vee$, $m = \langle 2\rho, \mu \rangle$

each $S_{\le \lambda_i} \cap Gr_{G,\le\mu}$

$\cup$ is cut out by one eqn.

$S_{\le\lambda_{i+1}} \cap Gr_{G,\le\mu}$

$S_{\le\mu} \cap Gr_{G,\le\mu} = Gr_{G,\le\mu}$. dim $= \langle 2\rho, \mu\rangle$

$S_{\le w_0(\mu)} \cap Gr_{G,\le\mu} = \{t^{w_0(\mu)}\}$ dim $= 0$

$\Rightarrow \dim(S_{\le\lambda_i} \cap Gr_{\le\mu}) = \langle \rho, \mu+\lambda\rangle$

$\Rightarrow \dim(S_{\lambda} \cap Gr_{\le\mu}) = \langle \rho, \mu+\lambda\rangle$ because $S_{<\lambda}$ meets every comp. of $S_{\le\lambda} \cap Gr_{\le\mu}$

Rmk. Also shows lci.

② $F_\lambda(A) = H_c^\cdot(S_\lambda, A|_{S_\lambda})$ can be filtered by $A|_{S_\lambda \cap Gr_{\le\mu}}$

$A \in Perv \Rightarrow A|_{Gr_\mu} \in D^{\le -\langle 2\rho,\mu\rangle}$

$\Rightarrow A|_{Gr_\mu \cap S_\lambda} \in D^{\le -\langle 2\rho,\mu\rangle}$

$\Rightarrow R\Gamma_c(S_\lambda \cap Gr_\mu, A) \in D^{\le \langle 2\rho,\lambda\rangle}$

$\Rightarrow R\Gamma_c(S_\lambda, A) \in D^{\le\langle 2\rho,\lambda\rangle}$.

Dual argument: $F_\lambda(A) = H^\cdot(T_\lambda, i^!_{T_\lambda} A) \in D^{\ge \langle 2\rho,\lambda\rangle}$

$\Rightarrow F_\lambda(A)$ is concentrated in deg $= \langle 2\rho, \lambda\rangle$.

② A is filtered by $A|_{S_{\le\lambda}}$, quotient $A|_{S_\lambda}$

$\Rightarrow$ spectral seq. $\quad H_c^\bullet(S_\lambda, A|_{S_\lambda}) \Rightarrow H^\bullet(\mathrm{Gr}_G, A)$

converges at E_1 page (parity vanishing)

$\Rightarrow$ filtration on $F(A)$ by $F_\lambda(A)$.

but opposite filtration working w/ B^-.

Cor. F is exact.

Pf. Each F_λ is exact because of degree.

Prop. The $X_*(T)$-grading $H^\bullet(\mathrm{Gr}_G, A) = F(A) \cong \bigoplus_{\lambda \in X_*(T)} F_\lambda(A)$

is preserved under $\star_0^{LG}$.

Pf. Interpret $\star_0^{LG}$ in terms of fusion.

$$\begin{array}{ccl}
\mathrm{Gr}_{G,X^2} \xleftarrow{\;S_\lambda\;} & S_{G,X^2,\lambda} = & (\Delta_X \times S_\lambda) \sqcup \coprod_{\lambda_1+\lambda_2=\lambda} (X^2 - \Delta_X) \times S_{\lambda_1} \times S_{\lambda_2} \\
& p_\lambda \downarrow \;\; \uparrow & \\
& C_{G,X^2,\lambda} = & (\Delta_X \times \{t^\lambda\}) \sqcup \coprod_{\lambda_1+\lambda_2=\lambda} (X^2 - \Delta_X) \times \{t^{\lambda_1}\} \times \{t^{\lambda_2}\}
\end{array}$$

are $\mathbb{G}_m$-fixed pts & attractors

$\Rightarrow$ use hyperbolic localization & look at

$$\tilde{\gamma}_* \left(\begin{array}{l} p_{\lambda !}\, s_\lambda^* \left(m_* \left({}^p\mathcal{H}^\circ \left((A \boxtimes \Bbbk[1]) \tilde{\boxtimes}^L_{\Bbbk} (B \boxtimes \Bbbk[1]) \right) \right) \right) \\ = q_{\lambda *}\, t_\lambda^! \left(\text{- - ——— ——— - -} \right) \end{array} \right) \text{ on } X^2$$

Now $F = \bigoplus_{\lambda \in X_*(T)} F_\lambda$ is an exact tensor functor.

§ When $\Bbbk$ is a field of char. 0

$\Rightarrow$ $\exists\ \tilde{G}_{\Bbbk}/\Bbbk$ and an equiv.

$$\left(\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G; \Bbbk), \star^{L^+G} \right) \cong \left(\mathrm{Rep}(\tilde{G}_{\Bbbk}), \otimes_{\Bbbk} \right).$$

Def. For general $\Bbbk$, $\mu \in X_*(T)^+$, $j_\mu : \mathrm{Gr}_{G,\mu} \hookrightarrow \mathrm{Gr}_G$

$$J_!(\mu; \Bbbk) = {}^p\mathcal{H}^\circ\left(j_{\mu !}\, \Bbbk[\langle 2\rho, \mu\rangle] \right)$$

$J_{!*}(\mu; \Bbbk) = \text{image}$

$$J_*(\mu; \Bbbk) = {}^p\mathcal{H}^\circ\left(j_{\mu *}\, \Bbbk[\langle 2\rho, \mu\rangle] \right)$$

Rmk. If $\Bbbk$ char. 0 field, $\Rightarrow J_! = J_* = J_{!*} = IC_\mu$.

Back to char. 0 field.

① Fix $\mu_1, \cdots, \mu_k$ gen. of monoid $X_*(T)^+$

$\Rightarrow$ every $\mu \in X_*(T)^+$ written as $\mu = \sum a_i \mu_i$

$\Rightarrow$ $J_{!*}(\mu, \Bbbk)$ is a direct summand of $J_{!*}(\mu_1, \Bbbk)^{\star a_1} \star \cdots \star J_{!*}(\mu_k, \Bbbk)^{\star a_k}$

$\Rightarrow$ $\tilde{G}$ is finite type.

② for every nontrivial $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G; \Bbbk)$,

the objects that are summands of $\mathcal{F}^{\oplus m}$ is not closed under $\circledast$.

($\because \mathcal{F}$ contains $IC_\mu \Rightarrow IC_{m\mu}$ has larger supports)

$\Rightarrow \tilde{G}$ is conn'd.

③ $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G; \Bbbk)$ semisimple $\Rightarrow \tilde{G}$ is reductive.

$\hat{T} \longrightarrow \tilde{G}$ from $F \cong \bigoplus F_\lambda$.

④ all $\lambda \in X_*(T)$ appears as nontrivial weights of IC_μ

$\Rightarrow \hat{T} \hookrightarrow \tilde{G}$ is a closed embedding.

⑤ There is a map

$$\begin{tikzcd} K_0(\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \Bbbk)) \arrow[r, "F"] \arrow[dr, dashed] & \mathbb{Z}[X_*(T)] \\ & \mathbb{Z}[X_*(T)]^W \arrow[u, phantom, "\cup"] \end{tikzcd}$$

isom. after $\otimes\, \mathbb{Q}$.

because simple objects are labelled by $X_*(T)^+$.

$\Rightarrow \mathrm{rk}\, \tilde{G} = \mathrm{rk}\, T \Rightarrow \hat{T} \hookrightarrow \tilde{G}$ is max'l.