Central sheaves

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Nearby cycles.

X separated f.t. $/\mathbb{C}$, $\mathbb{k}$ = coeff ring

$f : X \longrightarrow \mathbb{A}^1$

$X^\times = f^{-1}(\mathbb{G}_m)$, $X_0 = f^{-1}(0)$

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] & X \arrow[d, "f"] & X^\times \arrow[l] \arrow[d, "f^\times"] & \widetilde{X^\times} \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] & \mathbb{C} \arrow[l, "\exp"'] \end{tikzcd}$$

Def $\Psi_X = \Psi_f : D^b_c(X^\times) \longrightarrow D^b(X_0)$

$$\Psi_f(\mathcal{F}) = i^* j_* \exp_{X*} \exp_X^*(\mathcal{F})[-1]$$

Prop. (1) Ψ_f preserves constructible sheaves

(2) Ψ_f preserves perverse sheaves

Eg. $X = \mathbb{C}$, $f = id$.

$\mathcal{L}$ local sys. on $\mathbb{C}^\times$, $\mathcal{L}_1 \circlearrowleft \pi_1(\mathbb{C}^\times, 1) = \mathbb{Z}$

$\Psi_{id}(\mathcal{L}[1]) = \mathcal{L}_1$

Prop. $X \xrightarrow{g} Y \xrightarrow{f} \mathbb{C}$

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] & X^\times \arrow[l, hook'] \arrow[d] \\ Y_0 \arrow[r] \arrow[d] & Y \arrow[d] & Y^\times \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l, hook'] \end{tikzcd}$$

(1) g proper, $\mathcal{F} \in D^b_c(X^\times)$,

$$\Psi_f((g^\times)_* \mathcal{F}) \xrightarrow{\sim} (g_0)_* \Psi_{f\circ g}(\mathcal{F})$$

(2) g smooth, $\mathcal{F} \in D^b_c(Y)$

$$(g_0)^* \Psi_f(\mathcal{F}) \xrightarrow{\sim} \Psi_{f\circ g}((g^\times)^* \mathcal{F}).$$

Prop (1) $\Psi_f \circ \mathbb{D}_{X^\times}(\mathcal{F}) \xrightarrow{\sim} \mathbb{D}_{X_0} \circ \Psi_f(\mathcal{F})$

(2) $X \xrightarrow{f} \mathbb{C}$, $Y \xrightarrow{g} \mathbb{C}$

$X \times_{\mathbb{C}} Y \xrightarrow{f \times_{\mathbb{C}} g} \mathbb{C}$, $\mathcal{F} \in D^b_c(X)$, $\mathcal{G} \in D^b_c(Y)$.

$$\Psi_f(\mathcal{F}) \boxtimes \Psi_g(\mathcal{G}) \xrightarrow{\sim} \Psi_{f \times_{\mathbb{C}} g}(\mathcal{F} \boxtimes_{\mathbb{C}} \mathcal{G}).$$

<u>Monodromy</u>: $\mathcal{F} \in D^b_c(X^\times)$, $\exp_X^* \mathcal{F} \in D^b(\widetilde{X^\times})$ is $\pi_1(\mathbb{C}^\times, 1)$-equiv.

$$\exp_{X*} \exp_X^* \mathcal{F} \circlearrowleft \pi_1(\mathbb{C}^\times, 1)$$

Choose a generator $\circlearrowleft \in \pi_1(\mathbb{C}^\times, 1)$

$\rightsquigarrow \quad m_F : \Psi_f(F) \xrightarrow{\sim} \Psi_f(F)$ monodromy automorphism.

<u>Prop</u> $F \in D^b_c(X^\times)$

$$i^* j_* F[-1] \to \Psi_f(F) \xrightarrow{m_F - 1} \Psi_f(F) \xrightarrow{+1} \qquad \text{dist. } \triangle$$

<u>Monodromic complex</u>

T alg. torus $/\mathbb{C}$, $\quad \tilde{T} = X_*(T) \otimes \mathbb{C} \xrightarrow{\exp} T$

$X \circlearrowleft T$

<u>Def</u>. $D^b_{T\text{-mon}}(X) := D^b_{\tilde{T}, c}(X)$

$= \{\tilde{T}\text{- equiv. cons. sheaves on } X\}$

$\tilde{T}$ contractible, $D^b_{T\text{-mon}}(X) \to D^b_c(X)$ fully faithful.

<u>Def</u>. $D^b(X /\!\!/ T) :=$ sub $\triangle$-cat. gen. by image of $D^b_T(X) \to D^b_{T\text{-mon}}(X)$

cat. of unipotent monodromic sheaves.

<u>Prop</u>. $F \in D^b_{T\text{-mon}}(X)$ equipped w/ an action $\mu_F : X_*(T) \to \mathrm{Aut}(F)$.

Pf. $\quad act,\ pr : \tilde{T} \times X \longrightarrow X$

$$\theta : pr^* \mathcal{F} \xrightarrow{\sim} act^* \mathcal{F}$$

$\gamma \in X_*(T)$, $\gamma \curvearrowright X$ trivially

$$act^* \mathcal{F}\big|_{\{\gamma\} \times X} \cong \mathcal{F}.$$

$$\mu_{\mathcal{F}}(\gamma) = \theta\big|_{\{\gamma\} \times X} : \mathcal{F} \xrightarrow{\sim} \mathcal{F}. \qquad \square$$

Prop. $\quad f : X \longrightarrow Y \quad T$-equiv.

f_*, f^*, $f_!$, $f^!$, $\otimes$, $\underline{Hom}$ preserve $D^b_{T\text{-mon}}(-)$.

compatible w/ $\mu_{\mathcal{F}}$.

Eg. (1) $\mathcal{F} \in D^b_T(X)$, then $\mu_{\mathcal{F}} = id$.

(2) $X = T$, $D^b_{T\text{-mon}}(T) = \{$locally const. sheaves on $T\}$ $\ni \mathcal{L}$

$$\begin{tikzcd} X_*(T) \arrow[r, "\mu_{\mathcal{L}}"] \arrow[dr, "taut"', "\sim"] & Aut(\mathcal{L}) \arrow[d, "\wr"] \\ & Aut_{X_*(T)}(\mathcal{L}_1) \end{tikzcd}$$

$\mathcal{L} \in D^b(T /\!\!/ T) \iff \mu_{\mathcal{L}}$ is unipotent.

Def. $\mathrm{Perv}_{T\text{-mon}}(X) = \mathrm{Perv}(X) \cap D^b_{T\text{-mon}}(X)$

Prop. $\mathcal{F} \in \mathrm{Perv}_{T\text{-mon}}(X)$, $\mathcal{F}$ is T-equiv. $\Leftrightarrow$ $\mu_{\mathcal{F}}(\lambda) = id$, $\forall \lambda \in X_*(T)$.

<u>Monodromic complexes and nearby cycles</u>

Prop. $X \longrightarrow \mathbb{C}$, $\mathbb{C}^\times$-equivariant,

Ψ_f lifts to $\Psi_f : D^b_{\mathbb{C}^\times}(X^\times) \longrightarrow D^b_{\mathbb{C}^\times\text{-mon}}(X_0)$

w/ $m_{\mathcal{F}} = \mu_{\Psi_f(\mathcal{F})}(-1)$.

Pf. $X_0 \xrightarrow{i} X \xhookleftarrow{j} X^\times \xleftarrow{\exp_X} \widetilde{X^\times}$ $\mathbb{C}$-equiv.

$\rightarrow \Psi_f$ sends $D^b_{\mathbb{C}^\times}(X^\times)$ to $D^b_{\mathbb{C}^\times\text{-mon}}(X_0)$.

$$\Psi_f(\mathcal{F}) = i^* j_* \exp_{X*} \exp_X^*(\mathcal{F})[-1]$$

$$\cong i^* j_* \exp_{X*} \underline{\mathrm{Hom}}(\Bbbk_{\widetilde{X^\times}}, \exp_X^* \mathcal{F})[-1]$$

$$\cong i^* j_* \underline{\mathrm{Hom}}(\exp_{X!} \Bbbk_{\widetilde{X^\times}}, \mathcal{F})[-1]$$

$$||$$

$$(f^\times)^* \exp_! \Bbbk_{\mathbb{C}^\times}$$

$$\circlearrowleft$$

$m_{\mathcal{F}}$ defined by $\mu_{\exp_! \Bbbk_{\mathbb{C}^\times}}(1)$. $\mu_{(\exp)_! \Bbbk_{\mathbb{C}}}$

étale case

A henselian discrete valuation ring, k = res. field, K = Frac(A)

$s = \operatorname{Spec} k$, $\bar{s} = \operatorname{Spec} \bar{k}$

$\eta = \operatorname{Spec} K$, $\bar{\eta} = \operatorname{Spec} \bar{K}$, $\bar{S} = \operatorname{Spec} \bar{A}$, $\bar{A}$ normalization of A in $\bar{K}$.

$X \xrightarrow{f} S$

$$\begin{tikzcd} X_{\bar{s}} \arrow[r, "\bar{i}"] \arrow[d] & X_{\bar{S}} \arrow[d] & X_{\bar{\eta}} \arrow[l, hook', "\bar{j}"'] \arrow[r, "\rho"] \arrow[d] & X_{\eta} \arrow[d] \\ \bar{s} \arrow[r] & \bar{S} & \bar{\eta} \arrow[l] \arrow[r] & \eta \end{tikzcd}$$

Def $\Psi_f : D^b_c(X_\eta) \longrightarrow D^b_c(X_{\bar{s}})$

$\Psi_f(\mathcal{F}) := \bar{i}^* \bar{j}_* \rho^* \mathcal{F}[-1]$

$\circlearrowleft$

$\mathrm{Gal}(\bar{K}|K)$

eg. C curve $/k$, $c \in C$, A = henselization of $\mathcal{O}_{C,c}$

$X \xrightarrow{f} C$: $\Psi_f : D^b_c(X^\times) \to D^b_c(X_{\bar{c}})$

Gaitsgory's central functors

$G/\mathbb{Q}_p$ split reductive gp. $K \subset G(\mathbb{Q}_p)$ hyperspecial, $I \subset K$ Iwahori

$\mathcal{H}_I$, $\mathcal{H}_{sph} = \mathcal{H}_K$.

Thm (Bernstein) $Z(\mathcal{H}_I) \xrightarrow{\sim} \mathcal{H}_{sph}$

$$\begin{array}{ccc} \psi & & \\ f & \longmapsto & f * 1_K = 1_K * f \end{array}$$

Goal: Construct an inverse geometrically.

$G/\mathbb{C}$ reductive gp.

Last time $\left(\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G), \overset{L^+G}{*}\right) \simeq \left(\mathrm{Rep}(\hat{G}), \otimes\right)$

I = preimage of B along $L^+G \to G$.

$$\begin{array}{l} \mathcal{F}\ell_G := LG/I \quad \text{affine flag var.} \\ \downarrow \pi \\ \mathrm{Gr}_G \end{array}$$

Eg. $G = GL_n$, $\mathcal{F}\ell_G = \left\{ \begin{array}{c} \Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset t\Lambda_1 \\ \text{lattices in } K^{\oplus n} \text{ s.t. } \dim \Lambda_i/\Lambda_{i+1} = 1 \end{array} \right\}$

$K = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$

$$\begin{array}{ccc} \mathcal{F}\ell_G & \ni & (\Lambda_1 \supset \cdots) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_G & \ni & \Lambda_1 \end{array}$$

- $(D_I(\mathcal{F}\ell_G), \overset{I}{*})$
  - $\overset{I}{*}$ not commutative
  - $P_I(\mathcal{F}\ell_G)$ not closed under $\overset{I}{*}$.

Goal: Construct a central functor

$$Z: \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G) \to D_I(\mathcal{F}\ell_G)$$

I Iwahori gp scheme $/\mathcal{O}$

affine smooth w/ $I_K = G$, $I(\mathcal{O}) = I(k)$

Eg. GL_n, $R/\mathcal{O}$

$$I(R) = \left\{ \begin{array}{l} (g_1, \cdots, g_n) \in GL_n(R) \\ \begin{tikzcd} R^{\oplus n} \arrow[r, "\varepsilon_n^\vee(t)"] \arrow[d, "g_1"] & R^{\oplus n} \arrow[r, "\varepsilon_{n-1}^\vee(t)"] \arrow[d, "g_2"] & \cdots & R^{\oplus n} \arrow[r, "\varepsilon_1^\vee(t)"] \arrow[d, "g_n"] & R^{\oplus n} \arrow[d, "g_1"] \\ R^{\oplus n} \arrow[r, "\varepsilon_n^\vee(t)"] & R^{\oplus n} \arrow[r, "\varepsilon_{n-1}^\vee(t)"] & \cdots & R^{\oplus n} \arrow[r, "\varepsilon_1^\vee(t)"] & R^{\oplus n} \end{tikzcd} \end{array} \right\}$$

$I \to G_{\mathcal{O}}$ isom. over K.

$C = \mathbb{A}^1_{\mathbb{C}} \ni 0$, $C^\circ = C \setminus \{0\}$,

$\mathcal{G}$ affine smooth gp scheme $/C$, defined by glueing $G \times C^\circ$ w/ I

- $L\mathcal{G}$, $L^+\mathcal{G}$ over C. $(L^+\mathcal{G})_y = \begin{cases} L^+G, & y \neq 0 \\ I, & y = 0 \end{cases}$

Def $\quad \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}} = \left\{ (y, \mathcal{E}, \beta) : \begin{array}{l} y \in C, \\ \mathcal{E} \ \mathcal{G}\text{-torsor on } \widehat{\Gamma}_y, \\ \beta : \mathcal{E}|_{\widehat{\Gamma}_y^{\circ}} \to \mathcal{E}^{\mathrm{triv}}|_{\widehat{\Gamma}_y^{\circ}} \end{array} \right\}$

$$\widehat{\Gamma}_y^{\circ} = \widehat{\Gamma}_y \setminus \Gamma_y$$

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}}|_{C^{\circ}} \simeq \mathrm{Gr}_G \times C^{\circ}$$

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}}|_{\bullet} \simeq \mathcal{F}\ell_G$$

Def. $\quad Z : D^b_c(\mathrm{Gr}_G) \to D^b_c(\mathcal{F}\ell_G)$

$$Z(\mathcal{F}) = \Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}}}(\mathcal{F} \boxtimes \mathbb{k}_{C^{\circ}}[1])$$

Prop. $\quad Z$ lifts to

$$Z : D^b_{L^+G}(\mathrm{Gr}_G) \to D^b_I(\mathcal{F}\ell_G)$$

Sketch: $\quad \mathrm{act}, \mathrm{pr} : L^+\mathcal{G} \times \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}} \to \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}}$

$$\mathrm{pr}^* Z(\mathcal{F}) \simeq \Psi_{L^+\mathcal{G} \times \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}}}(\mathrm{pr}^*(\mathcal{F} \boxtimes \mathbb{k}_{C^{\circ}}[1]))$$

$$\simeq \Psi_{L^+\mathcal{G} \times \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Ran}}}(\mathrm{act}^*(\mathcal{F} \boxtimes \mathbb{k}_{C^{\circ}}[1]))$$

$$\simeq \mathrm{act}^* Z(\mathcal{F}). \qquad \square$$

Thm. $Z : \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G) \longrightarrow D_I(\mathrm{Fl}_G)$

(1) Z sends $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ to $\mathrm{Perv}_I(\mathrm{Fl}_G)$

(2) $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$, $\mathcal{G} \in D_I(\mathrm{Fl}_G)$

$\exists$ nat. isom.

$$\sigma_{\mathcal{F},\mathcal{G}} : Z(\mathcal{F}) \overset{I}{*} \mathcal{G} \xrightarrow{\sim} \mathcal{G} \overset{I}{*} Z(\mathcal{F})$$

\+ compatibilities

(3) $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$

$$Z(\mathcal{F}) \overset{I}{*} Z(\mathcal{G}) \xrightarrow{\sim} Z(\mathcal{F} \overset{L^+G}{*} \mathcal{G})$$

\+ compatibilities

Z is a central functor.

(4) $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$, $\mathcal{G} \in \mathrm{Perv}_I(\mathrm{Fl}_G)$.

then $Z(\mathcal{F}) \overset{I}{*} \mathcal{G}$ is perverse

(5) $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$

$\exists$ unipotent monodromy action $m_{\mathcal{F}} : Z(\mathcal{F}) \xrightarrow{\sim} Z(\mathcal{F})$

$$\begin{tikzcd}
Z(\mathcal{F}) \overset{I}{*} Z(\mathcal{G}) \arrow[r, "m_{\mathcal{F}} \overset{I}{*} m_{\mathcal{G}}"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & Z(\mathcal{F}) \overset{I}{*} Z(\mathcal{G}) \arrow[d, "\wr"] \\
Z(\mathcal{F} \overset{L^+G}{*} \mathcal{G}) \arrow[r, "m_{\mathcal{F} \overset{L^+G}{*} \mathcal{G}}"] & Z(\mathcal{F} \overset{L^+G}{*} \mathcal{G})
\end{tikzcd}$$

(6) $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$, $\pi_* Z(\mathcal{F}) \simeq \mathcal{F}$.

Proof. (1) $\Phi_{\mathrm{Gr}_G^{\mathrm{cen}}}$ is perverse t-exact.

(2) $\mathrm{Gr}_G^{BD} := \left\{ (y, \mathcal{E}, \beta) : \begin{array}{l} y \in C \\ \mathcal{E} \ G\text{-torsor on } \widehat{\Gamma_0 \cup \Gamma_y} \\ \beta : \mathcal{E}|_{\widehat{\Gamma_0\cup\Gamma_y} \setminus \Gamma_0 \cup \Gamma_y} \xrightarrow{\sim} \mathcal{E}^{\mathrm{triv}}|_{\widehat{\Gamma_0\cup\Gamma_y} \setminus \Gamma_0\cup\Gamma_y} \end{array} \right\}$

$$\mathrm{Gr}_G^{BD}|_{C^\circ} = \mathrm{Gr}_G \times \mathrm{Fl}_G \times C^\circ$$

$$\mathrm{Gr}_G^{BD}|_0 = \mathrm{Fl}_G$$

$$L^+ G^{BD} = \mathrm{Aut}\left(\mathcal{E}^{\mathrm{triv}}|_{\widehat{\Gamma_0\cup\Gamma_y}} \right), \quad \begin{array}{l} L^+G^{BD}|_{C^\circ} = L^+G \times I \times C^\circ \\ L^+G^{BD}|_0 = I \end{array}$$

Def. $\mathcal{F} \in D_{L^+G}(\mathrm{Gr}_G)$, $\quad \mathcal{G} \in D_I(\mathrm{Fl}_G)$

$$\underset{\substack{\uparrow \\ D_I(\mathrm{Fl}_G)}}{C(\mathcal{F}, \mathcal{G})} := \Phi_{\mathrm{Gr}_G^{BD}} \big(\underbrace{\mathcal{F} \boxtimes \mathcal{G} \boxtimes \underline{k}_{C^\circ}[1]}_{L^+G^{BD}\text{- equiv}} \big).$$

Lemma. $Z(\mathcal{F}) \overset{I}{*} \mathcal{G} \simeq C(\mathcal{F}, \mathcal{G}) \simeq \mathcal{G} \overset{I}{*} Z(\mathcal{F})$

$$\mathrm{Gr}_G(\underline{0}, \underline{y}) = \left\{ (y, \underbrace{\mathcal{E}^1, \mathcal{E}^2}_{\text{torsors on } \widehat{\Gamma_0\cup\Gamma_y}}, \beta_1, \beta_2) : \begin{array}{l} \beta_1 : \mathcal{E}^1|_{\widehat{\Gamma_0\cup\Gamma_y}\setminus\Gamma_0} \xrightarrow{\sim} \mathcal{E}^{\mathrm{triv}}|_{\widehat{\Gamma_0\cup\Gamma_y}\setminus\Gamma_0}, \\ \beta_2 : \mathcal{E}^2|_{\widehat{\Gamma_0\cup\Gamma_y}\setminus\Gamma_y} \xrightarrow{\sim} \mathcal{E}^1|_{\widehat{\Gamma_0\cup\Gamma_y}\setminus\Gamma_y} \end{array} \right\}$$

$$\mathrm{Gr}_{\mathcal{G}}(\underline{y}, \underline{o}) = \left\{ (y, \mathcal{E}^1, \mathcal{E}^2, \beta_1, \beta_2) : \begin{array}{l} \beta_1 \text{ over } \widehat{\Gamma_o \cup \Gamma_y} \setminus \Gamma_y \\ \beta_2 \text{ over } \widehat{\Gamma_o \cup \Gamma_y} \setminus \Gamma_o \end{array} \right\}$$

$$\mathrm{Gr}_{\mathcal{G}}(\underline{o}, \underline{y}) \xrightarrow{m_1} \mathrm{Gr}_{\mathcal{G}}^{BD} \xleftarrow{m_2} \mathrm{Gr}_{\mathcal{G}}(\underline{y}, \underline{o})$$

fibers over C°: all $\mathrm{Gr}_G \times \mathrm{Fl}_G \times C^\circ$

fibers over 0: $LG \times^I \mathrm{Fl}_G \xrightarrow{m} \mathrm{Fl}_G \xleftarrow{m} LG \times^I \mathrm{Fl}_G$.

Claim. $\Psi_{\mathrm{Gr}_{\mathcal{G}}(\underline{o}, \underline{y})} (\mathcal{F} \boxtimes \mathcal{G} \boxtimes \underline{k}_{C^\circ}[1]) \simeq \mathcal{G} \tilde{\boxtimes} Z(\mathcal{F})$

$\Psi_{\mathrm{Gr}_{\mathcal{G}}(\underline{y}, \underline{o})} (\mathcal{F} \boxtimes \mathcal{G} \boxtimes \underline{k}_{C^\circ}[1]) \simeq Z(\mathcal{F}) \tilde{\boxtimes} \mathcal{G}$.

Sketch. $\mathrm{Gr}_{\mathcal{G}}(\underline{o}, \underline{y}) \simeq \mathrm{Gr}_{\mathcal{G}}^{(\infty)}(\underline{o}) \overset{L^+\mathcal{G}^{BD}}{\times} \mathrm{Gr}_{\mathcal{G}}(\underline{y})$

trivialize $\mathcal{E}^2$

$\mathrm{Gr}_{\mathcal{G}}(\underline{o}) = \mathrm{Fl}_G \times C$

$\mathrm{Gr}_{\mathcal{G}}(\underline{y}) = \mathrm{Gr}_{\mathcal{G}}^{cen}$

$$\mathrm{Gr}_{\mathcal{G}}(\underline{o}, \underline{y}) \longleftarrow \mathrm{Gr}_{\mathcal{G}}^{(\infty)}(\underline{o}) \times \mathrm{Gr}_{\mathcal{G}}(\underline{y}) \longrightarrow \mathrm{Gr}_{\mathcal{G}}(o) \times \mathrm{Gr}_{\mathcal{G}}(\underline{y})$$

$$\mathcal{G} \tilde{\boxtimes} Z(\mathcal{F}) \longleftarrow \tilde{\mathcal{G}} \boxtimes Z(\mathcal{F}) \longleftarrow \mathcal{G} \boxtimes Z(\mathcal{F})$$

(3). similarly

(4) $C(\mathcal{F}, \mathcal{G})$ perverse if $\mathcal{F}, \mathcal{G}$ perverse.

(6) $L^+\mathcal{G} \to L^+G \times C$

$\rightsquigarrow \pi : Gr_{\mathcal{G}}^{\mathrm{Cen}} \to Gr_G \times C$

$\rightsquigarrow \pi_* Z(\mathcal{F}) = \Psi_{Gr_G \times C}(\mathcal{F} \boxtimes \Bbbk_{C^\circ}[1]) = \mathcal{F}$.

(5) $\mathcal{F} \in \mathrm{Perv}_{L^+G}(Gr_G)$

$m_{\mathcal{F}} : Z(\mathcal{F}) \to Z(\mathcal{F})$ monodromy.

WTS $m_{\mathcal{F}}$ unipotent

· $L^+\mathcal{G}$ carries an action of $\mathbb{G}_m$

also on $L\mathcal{G}$, $Gr_{\mathcal{G}}^{\mathrm{Cen}}$.

. $\mathcal{F} \boxtimes \Bbbk_{C^\circ}[1]$ is $\mathbb{G}_m$-equiv.

$\cap$

$D_{L^+\mathcal{G} \rtimes \mathbb{G}_m}(Gr_{\mathcal{G}}^{\mathrm{Cen}}|_{C^\circ})$

$\Rightarrow Z(\mathcal{F}) \in D_{I, \mathbb{G}_m\text{-mon}}(Fl_G)$ (Verdier)

<u>Lemma</u>. $D_{I, \mathbb{G}_m\text{-mon}}(Fl_G) = D_I(Fl_G /\!/ \mathbb{G}_m) = D_I(Fl_G)$

Pf. $D_I(Fl_G)$ gen. by $\Delta_w = j_{w!}\,\underline{k}[\ell(w)]$; $j_w : IwI/I \hookrightarrow Fl_G$
$w \in \widetilde{W}$

j_w is $\mathbb{G}_m$-equiv.

$\to \Delta_w \in D_{I \rtimes \mathbb{G}_m}(Fl_G)$. $\square$.