Cohomology of Shimura varieties

CJ Dowd

(G, X) Hodge type Shimura datum, reflex field $E = \mathbb{E}_v$, $v | p$

Have Shimura variety $\quad Sh_{K^p}(G,X)^\diamond$ over E

$K^p \subset G(\mathbb{A}_f^p)$. $\qquad \cup$

$\qquad Sh_{K^p}(G,X)^{\circ,\diamond}$

$\pi^\diamond_{HT} : Sh_{K^p}(G,X)^\diamond \longrightarrow Gr_{G,\mu^{-1}}$, μ is the Hodge cochar. (minuscule)

$$\begin{tikzcd}
Sh_{K^p}(G,X)^{\circ,\diamond} \arrow[r, "\pi^\diamond_{HT}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Gr_{G,\mu^{-1}} \arrow[d, "BL"] \\
Igs_{K^p}(G,X) \arrow[r, "\overline{\pi}_{HT}"] & Bun_{G,\mu^{-1}}
\end{tikzcd}$$

$\mathrm{Div}^1_E = [\mathrm{Spd}\, E/\phi^{\mathbb{Z}}]$

$\mathrm{Div}^1_E(\mathrm{Spa}(R,R^+)) = \left\{ \begin{array}{l} \text{untilt } R^\sharp \longrightarrow E \\ \text{mod Frob twisting} \end{array} \right\}$.

$\hookrightarrow \left\{ \begin{array}{l} \text{closed Cartier divisors} \\ \text{on } Y_{S,E} \end{array} \right\}$ via $\underset{\substack{\uparrow \\ \text{kernel.}}}{\theta} : W(R^+) \longrightarrow R^\sharp$

There is a 2-commutative diagram of v-stacks

$$\begin{tikzcd}
 & & & \mathrm{Bun}_{G,\mu^{-1}} \arrow[d] \\
\left[\mathrm{Gr}_{G,\mu^{-1}} / (\phi^{\mathbb{Z}} \times \underline{G(\mathbb{Q}_p)})\right] \arrow[rrru, dashed, bend left=10, "g"] \arrow[r, "\tilde{i}_1"] \arrow[d, "\tilde{h}_2"'] & \mathrm{Hck}_{G,\le\mu} = \mathrm{Hck}_{G,\mu} \arrow[rr] \arrow[d] & & \mathrm{Bun}_G \\
\mathrm{Bun}_G^{[1]} \times \mathrm{Div}_E^1 \arrow[r, "\text{inclusion}", "i_1"'] & \mathrm{Bun}_G \times \mathrm{Div}_E^1 & &
\end{tikzcd}$$

$$\{\mathcal{E}_0 \dashrightarrow \mathcal{E}_1\} \mapsto \mathcal{E}_1$$

$$\{\mathcal{E}_0 \dashrightarrow \mathcal{E}_1\} \mapsto (\mathcal{E}_0, D_S)$$

$\mathrm{Bun}_G(S) = \{ G\text{- bundles on } X_S \}$

$\mathrm{Bun}_G^{[1]} \hookrightarrow \mathrm{Bun}_G$ $\qquad\qquad$ $(\mathrm{Bun}_G^{[b]},\ b \in B(G))$

$\uparrow$

trivial bundle

$$\left[\mathrm{Gr}_{G,\mu^{-1}} / \phi^{\mathbb{Z}}\right] \Big(\underset{\substack{\| \\ S \\ \downarrow\, = \text{"}D_S\text{"} \\ \mathrm{Div}_E^1}}{\mathrm{Spa}\,(R,R^+)} \Big) = \left\{ \begin{array}{l} \text{pairs } \mathcal{E}_1 \overset{\text{mero, defined away from } D_S}{\dashrightarrow} \text{triv} \\ \text{on } X_S \text{ bounded by} \\ \quad \mu^{-1} \end{array} \right\}$$

$\tilde{i}_1$ is the inverse of this

$$\mathrm{Hck}_{G,\mu}\big(\underset{\text{"}\|\text{"} \atop D_S}{S \longrightarrow} \mathrm{Div}_E^1\big) = \left\{ \mathcal{E}_0 \overset{\text{mero}}{\dashrightarrow} \mathcal{E}_1,\ \text{away from } D_S \right\}$$

$$T_\mu^{[1]} : D(\mathrm{Bun}_{G,\mu^{-1},k} ; \Lambda) \longrightarrow \underset{\substack{D([\mathrm{Div}_E^1 / \underline{G(\mathbb{Q}_p)}] ; \Lambda) \\ \|}}{D(\mathrm{Bun}^{[1]}_{G,k} \times_k \mathrm{Div}^1_{E,k} ; \Lambda)}$$

$$A \longmapsto R\tilde{h}_{2,k,*}\left(g_k^* A[d](d/2)\right)$$

$R\Gamma(Sh_{K^p}(G, X) ; \Lambda)$. $\quad C$ is a completion of alg. closure of E. w/ res. field k.

$C = \mathbb{C}_p$

$\mathcal{F} = R\bar{\pi}_{HT,k,*}\Lambda$ for Λ a torsion $\mathbb{Z}_\ell$-sheaf w/ $\mathcal{J}p$.

$\uparrow$

$D(Bun_{G,\mu^{-1},k} ; \Lambda)$

Thm. There is an isom. $R\Gamma(Sh_{K^p,C} ; \Lambda) \simeq T_\mu^{[1]}(\mathcal{F}[-d])(-d/2)$

$\boxed{\text{Daniels, van Hoften, Kim, Zhang}}$ equiv. w.r.t. prime-to-p Hecke action.

Lemma. The map $R\Gamma(Sh_{K^p,C}, \Lambda) \longrightarrow R\Gamma(Sh^\circ_{K^p,C}, \Lambda)$ is an isom.

Pf. Omitted, but can be done at the finite level

$$R\Gamma(Sh_{K^pK_p,C} ; \Lambda) \xrightarrow{\sim} R\Gamma(S^\circ_{K^pK_p,C} ; \Lambda)$$

$$\begin{tikzcd}
{[S^\circ_{K^p}/G(\mathbb{Q}_p) \times \phi^{\mathbb{Z}}]} \arrow[r, "\tilde{s}"] \arrow[d, "\pi_{HT}"] & Igs_{K^p} \arrow[d, "\bar{\pi}_{HT}"] \\
{[Gr_{G,\mu^{-1}}/G(\mathbb{Q}_p) \times \phi^{\mathbb{Z}}]} \arrow[r, dashed] & Bun_{G,\mu^{-1}}
\end{tikzcd}$$

Quotient entire diagram by ϕ

Prop For any conn'd reductive $G/\mathbb{Q}_p$, there is a 2-isom. $\phi_{Bun_G} \simeq id_{Bun_G}$.

Pf. A G-bundle on X_S is the same as a G-bundle P on $Y_{(0,\infty)}(S)$ w/ automorphism $\phi^* P \xrightarrow{\sim} P$. This automorphism supplies exactly the 2-iso. we want. $\square$

Recall Ig_b is the sheafification of the presheaf sending $\mathrm{Spa}(R, R^+)$ to the groupoid,

- Objects: Pairs $(S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+}), x)$

$$x : \mathrm{Spf}\, R^{\sharp+} \to \hat{\mathcal{I}}(\mathcal{G}, X)$$

- Morphisms: $f : (S^{\sharp_1}, x) \to (S^{\sharp_2}, y)$ is a formal q-isogeny $f : (A_x, \lambda_x) \dashrightarrow (A_y, \lambda_y)$

$$A_x \otimes R^{\sharp_1 +}/\varpi_1 \longrightarrow A_y \otimes R^{\sharp_2 +}/\varpi_2$$

satisfying

$$\begin{tikzcd} V_x^p \arrow[r, "f"] \arrow[d] & V_y^p \arrow[d] \\ V \otimes \mathbb{A}_f^p \arrow[r, equal] & V \otimes \mathbb{A}_f^p \end{tikzcd}$$

$V^p = R^1 \pi_* \underline{\mathbb{A}}_f^p$

π map from universal ab. scheme.

If we regard $R^\sharp$ as two different untilts of R, $(R^{\sharp, b} \Rightarrow R,)$
$(R^{\sharp, b} \xrightarrow{\phi} R^{\sharp, b} \Rightarrow R)$

then we get a formal q-isogeny

$$(A_x, \lambda_x) \longrightarrow (A_x, \lambda_x) \text{ satisfying the diagram, etc.}$$

Same as finding a (genuine) q-isogeny

$$A_x \big|_{\mathrm{Spec}\, R^{\sharp, +}/\varpi} \longrightarrow \phi^* A_x \big|_{\mathrm{Spec}\, R^{\sharp +}/\varpi}$$

But such an isogeny is given by rel. Frob. on A_x. $\square$

$\phi \curvearrowright \mathrm{Spa}(R, R^+)$

$$\begin{tikzcd}
\left[S^{\circ}_{K^p} / G(\mathbb{Q}_p) \times \phi^{\mathbb{Z}} \right] \arrow[rr, "\tilde{g}"] \arrow[dd, "\pi_{HT}"'] & & \mathrm{Igs}_{K^p} \arrow[d, "\bar{\pi}_{HT}"] \\
& & \mathrm{Bun}_{G, \mu^{-1}} \arrow[d, "j_\mu"] \\
\left[\mathrm{Gr}_{G, \mu^{-1}} / \phi^{\mathbb{Z}} \times \underline{G(\mathbb{Q}_p)} \right] \arrow[urr, "g"] \arrow[r, hook] \arrow[d, "\tilde{h}_2"] & \mathrm{Hck}_{G,\mu} \arrow[r] \arrow[d] & \mathrm{Bun}_G \\
\mathrm{Bun}_G^{[1]} \times \mathrm{Div}^1_E & \mathrm{Bun}_G \times \mathrm{Div}^1_E \arrow[l, hook', "i_1"'] &
\end{tikzcd}$$

$$T^{[1]}_{\mu}(F[-d])\left(-\tfrac{d}{2}\right) = R\tilde{h}_{2,k,*}\, g^* F = R\tilde{h}_{2,k,*}\, g^* R\bar{\pi}_{HT,k,*} \Lambda$$

(qcqs base change) $\quad = R\tilde{h}_{2,k,*}\, R\pi_{HT,k,*}\, \tilde{g}^* \Lambda$

$\bar{\pi}_{HT,k}$ is qcqs $\quad = R(\tilde{h}_2 \circ \pi_{HT})_{k,*} \Lambda$

$$= R\Gamma(S^{\circ}_{K^p, C}, \Lambda) \in D_{ét}(\mathrm{Bun}_G^{[1]} \times \mathrm{Div}^1_{E}, \Lambda)$$

$$\simeq D_{ét}([\mathrm{Div}^1_E / G(\mathbb{Q}_p)], \Lambda) \quad \square$$