Weighted projective degenerations of projective space

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<u>Question</u>: what are the klt degenerations of $\mathbb{P}^n_{\mathbb{C}}$?
(normal,
mild singularity)

<u>Def'n</u> A <u>klt degeneration</u> of $\mathbb{P}^n_{\mathbb{C}}$ is a family $\begin{array}{c}\mathcal{X}\\ \downarrow \text{ flat}\\ 0\in T\end{array}$ smooth pted curve

- $\mathcal{X}_t \simeq \mathbb{P}^n$ general $t\in T$
- $\mathcal{X}_0$ klt (mild sing.) proj.
- $K_{\mathcal{X}/T}$ is $\mathbb{Q}$-Cartier.

(control singularities of $\mathcal{X}$)

<u>Example</u> $n=1$. klt degeneration of $\mathbb{P}^1$: $\begin{array}{c}\mathcal{X}\\ \downarrow\\ T\end{array}$ flat family

$\mathcal{X}_0$ normal $\Rightarrow$ smooth $\Rightarrow \mathcal{X}_0$ has genus 0, $\mathcal{X}_0 \simeq \mathbb{P}^1$.

<u>Example</u> $n=2$. <u>Construction</u>. $\mathbb{P}^2 \xrightarrow[v]{\text{Veronese}} \mathbb{P}^5$, $V = v(\mathbb{P}^2)$ image

$Y =$ cone over $V \subset \mathbb{P}^6$

$Y \subset \mathbb{P}^6$

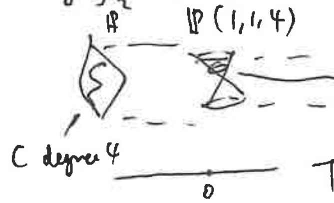
Hyperplane sections : generic $H\cap Y \simeq V$

$H\cap Y \simeq \mathbb{P}^2$

Special hyperplane, $H\cap Y \simeq$ cone over hyperplane section of V

$=$ cone over rat'l curve

Varying hyperplane section gives a degeneration of $\mathbb{P}^2$ to cone / rat'l curve $= \mathbb{P}(1,1,4)$



Specialization of deg 4 plane curve (canonical genus 3) , hyperelliptic

Motivation (degenerations of $\mathbb{P}^n$)

① Moduli problems

if we understand all klt degenerations of $\mathbb{P}^n$, explicit description of varieties (or pairs) appearing in certain moduli spaces. $\rightsquigarrow$ $\mathbb{P}^n$, hypersurfaces in $\mathbb{P}^n$, complete intersections.

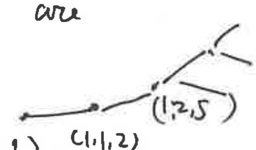
② Vector bundles on $\mathbb{P}^n$, derived cat. question

$\rightsquigarrow$ formalism relating ~~the~~ klt deg. to these "areas".

③ Arise naturally in mirror symmetry.

Theorem. (Hacking–Prokhorov, Manetti) The klt degenerations of $\mathbb{P}^2$ are

① $\mathbb{P}(a^2, b^2, c^2)$ where $\boxed{a^2+b^2+c^2 = 3abc}$ ← Markov equation



solutions: $(1,1,1)$,

$(a,b,c) \rightsquigarrow (a,b,3ab-c)$

② A partial smoothing of $\mathbb{P}(a^2, b^2, c^2)$.

Weighted projective space

Def'n. $a_0, a_1, \cdots, a_n \in \mathbb{Z}_{>0}$,
$$\mathbb{P}(a_0, a_1, \cdots, a_n) = \frac{\mathbb{A}^{n+1} - \{0\}}{(x_0, \cdots, x_n) \sim (\lambda^{a_0} x_0, \cdots, \lambda^{a_n} x_n)}$$
$\lambda \in \mathbb{C}^\times$

$\mathbb{P}(a_0, a_1, \cdots a_n) = \operatorname{Proj} \mathbb{C}[x_0, x_1, \cdots x_n]$, grading: $\deg(x_i) = a_i$.

- assume a_i's have no common factors
- assume "well-formedness" $\forall i$, $\gcd(a_0, \cdots, \hat{a_i}, \cdots, a_n) = 1$.

Rmk These are singular unless $(a_0, \cdots, a_n) = (1, 1, \cdots, 1)$.

Ex

$$\mathbb{P}(1,1,2) \longrightarrow \mathbb{P}(1,1,1,1) = \mathbb{P}^3$$

$$[x:y:z] \longmapsto [\underset{x_0}{x^2} : \underset{x_1}{xy} : \underset{x_2}{y^2} : \underset{x_3}{z}]$$

image $= V(x_0 x_2 - x_1^2) \subset \mathbb{P}^3$ singular quadric cone

Sketch of ideas of proof.

- If X_0 is a klt degeneration of X_t, $\Rightarrow (K_{X_0})^n = (K_{X_t})^n$
- If $X_t = \mathbb{P}^2$, $(K_{X_0})^2 = (K_{\mathbb{P}^2})^2 = 9$

$\rightarrow$ if $X_0 = \mathbb{P}(a_0, a_1, a_2)$ is a weighted projective space,

$$9 = (K_{X_0})^2 = \frac{(a_0+a_1+a_2)^2}{a_0 a_1 a_2}$$

$\Rightarrow$ $a_0 + a_1 + a_2 = 3\sqrt{a_0 a_1 a_2}$ $\Rightarrow$ $\sqrt{a_0 a_1 a_2} \in \mathbb{Z}$

all a_i's are relatively prime $\Rightarrow$ $a_0 = a^2$, $a_1 = b^2$, $a_2 = c^2$,

$a^2 + b^2 + c^2 = 3abc$, $\mathbb{P}(a^2, b^2, c^2)$

Key! - Every singularity on $\mathbb{P}(a^2, b^2, c^2)$ is smoothable. ($\mathbb{Q}$-Gorenstein)

- No local-to-global obstructions. $\rightarrow$ $\mathbb{P}(a^2, b^2, c^2)$ is smoothable. (to $\mathbb{P}^2$)

$\mathbb{P}^1 \times \mathbb{P}^1$ klt degenerations w/ $\rho(X) = 1$ are $\mathbb{P}(a^2, b^2, 2c^2)$, $a^2 + b^2 + 2c^2 = 4abc$

or partial smoothings.

Notation.

$$\frac{1}{m}(b_1, b_2, \dots, b_n) = \mathbb{A}^n / \mu_m, \quad \zeta_m (x_1, \dots, x_n) = (\zeta_m^{b_1} x_1, \dots, \zeta_m^{b_n} x_n).$$

Cyclic quotient surface singularity : $\frac{1}{n}(1,a)$

- $\mathbb{Q}$- Gorenstein smoothable $\iff n \mid (a+1)^2$

- Smoothable always.

Two questions.

① When are cyclic quotient singularities $\mathbb{Q}$- Gorenstein smoothable ? [in codim ≥ 3, always rigid]

(Schlessinger)

$\rightarrow \frac{1}{27}(1,4,16)$ rigid!

② Can we describe solutions to $\mathbb{P}(a_0, a_1, \cdots, a_n)$.

$$\left(K_{\mathbb{P}(a_0,\cdots,a_n)}\right)^n = \left(K_{\mathbb{P}^n}\right)^n \quad ?$$

$$\Uparrow$$

$$(a_0 + \cdots + a_n)^n = (n+1)^n a_0 \cdots a_n \quad ?$$

Thm (D-Li-Torres) If $\begin{matrix} \mathcal{X} \\ \downarrow \\ T \end{matrix}$ klt degeneration of $\mathcal{X}_t$ to $\mathcal{X}_0$,

and $\mathcal{L}$ a relatively ample $\mathbb{Q}$- Cartier divisor on $\mathcal{X}$,

$\Rightarrow \exists$ klt degeneration $C(\mathcal{X}_t, \mathcal{L}_t)$ to $C(\mathcal{X}_0, \mathcal{L}_0)$.

cone

L ample, V variety, $C(V,L) := \mathrm{Proj}\left(\sum_{m\geq 0}\left(\sum_{r=0}^{m} H^0(X, L^{[r]})\, z^{m-r}\right)\right)$.

If $V = \mathbb{P}^n$, $L = \mathcal{O}(1)$, $C(V,L) \simeq \mathbb{P}^{n+1}$.

Thm (D-L-T) Infinitely many smoothable weighted proj. spaces

$\mathbb{P}(a^2, b^2, c^2, abc, abc, \cdots, abc)$ $\qquad a^2+b^2+c^2 = 3abc$

$\mathbb{P}(a^2, b^2, 2c^2, 4abc, 2abc, 2abc, \cdots, 2abc)$ $\qquad a^2+b^2+2c^2 = 4abc$

that smooth to $\mathbb{P}^n$.

Also! There are weighted proj. degens of $\mathbb{P}^n$ <u>not</u> of this form.

(Sporadic)

$\mathbb{P}(2,3,3,4,18)$

$\mathbb{P}(2,12,21,49,126)$