p-adic Hodge theory

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F field, Var_F reduced separated schemes of f.t. $/F$

F char 0, $\quad X \longmapsto R\Gamma_{dR}(X)$ filtered dga, $X \in \mathrm{Var}_F$

Hodge-Deligne filtration $F^\cdot$

X smooth, $R\Gamma_{dR}(X) = R\Gamma(X, \Omega^\cdot_{X/F})$

$F = \mathbb{C}$,

$$\begin{tikzcd} R\Gamma_{dR}(X) \arrow[rr, "\rho", "\sim"'] \arrow[dr] & & R\Gamma(X_{cl}, \mathbb{C}) \arrow[dl, "\wr"', "\text{Poincaré Lemma}"] \\ & R\Gamma^{an}_{dR}(X) & \end{tikzcd}$$

<u>p-adic setting</u> $\quad \mathrm{Var}_{\overline{K}} \ni X$

$\overline{K}$

$K = \mathrm{Frac}\, W(k)$

Fontaine: $\overline{K} \subset B_{dR}$ - complete d.v. field

B^+_{dR} - ring of int.

$B^+_{dR} / m_{dR} = \mathbb{C}_p$ (= p-adic completion of $\overline{K}$)

$\mathbb{Z}_p(1) \subset m_{dR}$

$B_{dR} \simeq \mathbb{C}_p((2\pi i))$

$$\rho: \quad R\Gamma_{dR}(X) \underset{\bar{K}}{\otimes} B_{dR} \xrightarrow{\sim} R\Gamma_{\bar{et}}(X, \mathbb{Q}_p) \underset{\mathbb{Q}_p}{\otimes} B_{dR} \qquad \text{nat'l filtered q-isom.}$$

$$F^n := m_{dR}^n$$

$$\begin{tikzcd}
 & R\Gamma_{dR}(X) \underset{\bar{K}}{\otimes} B_{dR}^+ \arrow[r] \arrow[d] & R\Gamma_{\bar{et}}(X, \mathbb{Q}_p) \underset{\mathbb{Q}_p}{\otimes} B_{dR}^+ \arrow[d] \\
R\Gamma_{dR}(X) \arrow[r] \arrow[rr, bend right] & R\Gamma_{dR}(X) \underset{\bar{K}}{\otimes} \mathbb{C}_p \arrow[r] & R\Gamma_{\bar{et}}(X, \mathbb{Q}_p) \underset{\mathbb{Q}_p}{\otimes} \mathbb{C}_p
\end{tikzcd}$$

<u>not</u> q-isom.

① <u>h-topology on Var_F</u>:

<u>Def</u>. $\begin{matrix} Y \\ \downarrow \\ X \end{matrix}$ is an h-covering iff

ramified finite cover

$$\begin{tikzcd}
S' \arrow[r, dashed] \arrow[d] & Y \arrow[d] \\
S \arrow[r] & X
\end{tikzcd}$$

henselian 1-dim.

$$R\Gamma(X_{\bar{et}}, R) \to R\Gamma(X_h, R)$$

<u>Th</u>. (Deligne) R is torsion, $R\Gamma(X_{\bar{et}}, R) \xrightarrow{\sim} R\Gamma(X_h, R)$.

② <u>Compactified varieties a.k.a. pairs</u>

<u>Geom. setting</u>: $(U, \bar{U})$ $\quad U \hookrightarrow \bar{U}$ proper

open dense

$$Var_F^{gc} \supset Var_F^{nc}$$

nc-pair: $\bar{U}$ is regular, $\bar{U} \setminus U$ is nc divisor

<u>p-adic arithmetic setting</u>: $\mathrm{Var}_K^{ar} \ni (U, \bar{U})$ $\quad U \underset{\text{open dense}}{\hookrightarrow} \bar{U}$ ← proper reduced $/ \mathcal{O}_K$

$\cup$

Var_K^{ss} $\quad$ ss-pair: - $\bar{U}$ regular

- $\bar{U} \setminus U$ is n.c.d.

- closed fiber of $\bar{U}$ is reduced

$\mathrm{Var}_{\bar{K}}^{ar}$

ss-pairs: base change of ss pair for K.

$$\begin{array}{c} \bar{U} \\ \downarrow \\ \mathrm{Spec}\,\Gamma(\bar{U}, \mathcal{O}_{\bar{U}}) \end{array}$$ ↙ closed fiber

h-topology: h-cover $(V, \bar{V}) \longrightarrow (U, \bar{U})$:

$V \to U$ is an h-cover.

$$\begin{array}{ccc} \mathrm{Var}_F^{?} & \longrightarrow & \mathrm{Var}_F \\ \Psi & & \Psi \\ (U, \bar{U}) & \longmapsto & U \end{array}$$

<u>Th.</u> (de Jong) This functor yields an eq. of toposes.

h-localization: $\mathrm{Presh}(\mathrm{Var}_F^{?}) \longrightarrow$ h-sheaves (Var_F)

③ Hodge - Deligne filt'n

$\mathrm{Var}_F^{nc} \ni (U, \bar{U}) \longmapsto R\Gamma(\bar{U}, \Omega^{\cdot}_{(U,\bar{U})/F})$

h-sheafify: $\mathcal{A}_{dR}$ - filtered dga

$R\Gamma_{dR}(X) = R\Gamma(X_h, \mathcal{A}_{dR})$. filtered dga.

The integral structure:

$$\mathrm{Var}^{ss}_{\bar{K}} \ni (U,\bar{U}) \longmapsto R\Gamma(\bar{U}, \Omega_{(U,\bar{U})/\mathcal{O}_{\bar{K}}})$$

h-sheafify: $\qquad \mathcal{A}^{\natural}_{dR} \qquad\qquad \mathcal{A}^{\natural}_{dR} \otimes \mathbb{Q} = \mathcal{A}_{dR}$

$$R\Gamma^{\natural}_{dR}(X) := R\Gamma(X_h, \mathcal{A}^{\natural}_{dR}).$$

Th. (Bhatt) $\quad H^0 \mathcal{A}^{\natural}_{dR} = \mathcal{O}_{\bar{K}}$, $\quad \tau_{>0} \mathcal{A}^{\natural}_{dR} \xrightarrow{\sim} \tau_{>0} \mathcal{A}_{dR}$ is a filtered q isom.

④ Construction of ρ^0. $\qquad \mathcal{A}^{\natural}_{dR} \overset{L}{\otimes} \mathbb{Z}/p^n \xleftarrow{\sim} \mathcal{O}_{\bar{K}} \overset{L}{\otimes} \mathbb{Z}/p^n$

$R\Gamma(X_h, -)$: $\qquad R\Gamma^{\natural}_{dR}(X) \overset{L}{\otimes} \mathbb{Z}/p^n \xleftarrow{\sim} R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) \overset{L}{\otimes} (\mathcal{O}_{\bar{K}}/p^n)$

$\underleftarrow{R\lim}$: $\qquad R\Gamma^{\natural}_{dR}(X) \hat{\otimes} \mathbb{Z}_p \xleftarrow{\sim} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes \mathcal{O}_{\mathbb{C}_p}$

Convention: $C \hat{\otimes} \mathbb{Z}_p := \underleftarrow{R\lim}\, C \overset{L}{\otimes} \mathbb{Z}/p^n$

$$R\Gamma^{\natural}_{dR}(X) \longrightarrow R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes \mathcal{O}_{\mathbb{C}_p}$$

$\otimes\, \mathbb{Q}$: $\qquad R\Gamma_{dR}(X) \longrightarrow R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes \mathbb{C}_p$

⑤ Derived de Rham cohomology (Illusie)

$$\begin{array}{ccc} A/B & & \Omega^{\cdot}_{A/B} \\ \uparrow & & \uparrow \\ P.(A)/B & & \Omega^{\cdot}_{P.(A)/B} \\ & & \| \\ & & L\Omega^{\cdot}_{A/B} \qquad F \end{array}$$

$$\mathrm{gr}^m_F L\Omega = L\Lambda^m(L\Omega^1)[-m]$$

⑥ Def. of B_{dR}

$$\mathcal{O}_{\bar K}/\mathcal{O}_K$$

$$\begin{tikzcd}
\Omega^1_{\mathcal{O}_{\bar K}/\mathcal{O}_K} & & \mathcal{O}_{\bar K} \otimes \mu_{p^\infty} \arrow[ll] = (\bar K/\mathcal{O}_{\bar K})(1) \arrow[dl] \\
& (\bar K/\mathfrak{a})(1) \arrow[ul, "\sim"] &
\end{tikzcd}$$

$$f\, d\log \varepsilon \longleftarrow\!\shortmid\ f \otimes \varepsilon$$

$$L\Omega^1 \xrightarrow{\sim} \Omega^1$$

$$\mathfrak{a} = p^{\frac{1}{p-1} - 1}\, \mathcal{O}_{\bar K}$$

$$\begin{tikzcd}
& (\bar K/\mathfrak{a})(1) \quad (\bar K/(2!)^{-1}\mathfrak{a}^2)(2) \quad \cdots \quad (\bar K/(m!)^{-1}\mathfrak{a}^m)(m) \quad \cdots \\
\mathcal{O}_{\bar K} \arrow[ur] &
\end{tikzcd}$$

$$B^+_{dR}/m^n_{dR} := \left(\left(L\Omega_{\mathcal{O}_{\bar K}/\mathcal{O}_K} / F^n\right) \hat\otimes \mathbb{Z}_p\right) \otimes \mathbb{Q}$$

⑦ Construction of ρ.

$$\mathrm{Var}^{ss}_{\bar K} \ni (U, \bar U) \mapsto R\Gamma(\bar U, L\Omega_{(U,\bar U)/\mathcal{O}_K})$$

h- loc. : $\mathcal{A}^{\#}_{dR}$

$$R\Gamma^{\#}_{dR}(X) = R\Gamma(X_h, \mathcal{A}^{\#}_{dR})$$

$$\begin{tikzcd}
(U, \bar U) \arrow[d, dash] \\
(\mathrm{Spec}\, \bar K, \mathrm{Spec}\, \mathcal{O}_{\bar K}) \arrow[d, dash] \\
\mathrm{Spec}\, \mathcal{O}_K
\end{tikzcd}$$