Algebraic K-theory

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Grothendieck K_0. $\mathcal{A}$ abelian cat. $K_0(\mathcal{A}) = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[A] \big/ [A_1]+[A_2]-[A_3]$

for $0 \to A_1 \to A_3 \to A_2 \to 0$

Quillen. Define an invariant $K(\mathcal{A})$: Spectra (as a "top. space")

$$\pi_i K(\mathcal{A}) =: K_i(\mathcal{A})$$

One can specialize to $\mathcal{A} = \mathrm{Coh}(A)$, A is a Noetherian comm. ring,

$K_i(A) := K_i(\mathrm{Coh}(A))$. $\rightsquigarrow$ extend to X Noetherian scheme

$$K_i(X) := K_i(\mathrm{Coh}(X)).$$

$U \underset{\text{open}}{\subset} X \underset{\text{complement}}{\hookleftarrow} Z \quad \rightsquigarrow$ LES

$$\cdots \to K_{i+1}(U) \to K_i(Z) \to K_i(X) \to K_i(U) \to \cdots$$

Today equivariant version $\underset{\text{Noetherian scheme}}{X} \circlearrowleft G$ affine alg group.

$$K_i^G(X) := K_i(\mathrm{Coh}^G(X))$$
$$\mathrm{Coh}^G(X) = \mathrm{Coh}([X/G])$$

Eg. $K_0([*/G_c]) = K_0(\mathrm{Rep}_{f.d.}(G))$, $\quad G = T$ split torus.

G reductive $\quad K_0([*/T]) = \mathbb{Z}X^*(T)$.

F field, $K_0(\mathrm{Spec}\, F) = \mathbb{Z}$

<u>Fact</u>: $K_1(\mathrm{Spec}\, F) = F^\times$. $K_2(\mathrm{Spec}\, F) = F^\times \otimes F^\times / \langle a \otimes (1-a) : a \neq 0, 1 \rangle$

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/(q^j - 1), & i = 2j-1 \\ 0, & i \text{ even}, i \neq 0 \end{cases}$$

[Aside: $K_\cdot(\mathrm{Spec}\, F) \otimes_{\mathbb{Z}} \mathbb{F}_\ell$, $\ell \neq \mathrm{char}\, F$ is described by Bloch-Kato conj.]

In general, it is hard.

<u>Property</u>:

- projective bundle formula. $K_i(\mathbb{P}^n_X) = \bigoplus_{j=0}^{n} K_i(X)$
- Thom isom. (homotopy invariance) $K_i(\mathbb{A}^n_X) = K_i(X)$
- Computations $\left[\begin{array}{l} \text{cellular fibration} \\ \text{Künneth formula} \quad (\text{NOT always true!}) \end{array}\right.$

Modern treatment: replace abelian cat. by small stable ∞-cat., idempotent complete

$\rightsquigarrow$ the cat. of such cat. : $\mathrm{Cat}^{\mathrm{Perf}}$. Functors are exact functors.

$$\mathrm{Cat}_\infty^{\mathrm{stable}} : \text{small } \infty\text{-cat.} \longrightarrow \mathrm{Cat}_{cg}^{\mathrm{stable}} \text{ cptly gen. cat}$$

$$\mathcal{C} \longmapsto \mathrm{Ind}(\mathcal{C})$$

$$\mathrm{Cat}^{\mathrm{Perf}} \ni \mathcal{C}^\omega \longleftarrow\!\shortmid \mathcal{C} \qquad \rightsquigarrow \mathrm{Cat}^{\mathrm{Perf}} \simeq \mathrm{Cat}_{cg}^{\mathrm{stable}}$$

presentable cat.
obj. cptly gen. stable ∞-cat.,
mor: functors preserving colimits + cpt objects

<u>Fact</u>: Cat_{cg} is compactly gen. by <u>one generator</u>, called <u>Sp</u>
the cat. of spectra.

$\mathcal{S}$: the cat. of spaces $\simeq$ CW cpxes / homotopy

$\simeq$ Top. Spaces / weak homotopy

$\downarrow$

Sp = stabilization of $\mathcal{S}$ $\subset$ $\mathrm{Fun}(\mathcal{S}_*^{fin}, \mathcal{S}_*)$ satisfies $F(*) = *$

gen. by one objs under colim.

finite CW cpxes pointed

and F pull back diagram to pushout diagram.

Classify all cohomology theory.

$\mathrm{Cat}^{\mathrm{Perf}}$ presentable

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & \mathcal{B} \arrow[d] \\ 0 \arrow[r] & \mathcal{C} \end{tikzcd}$$

$\mathcal{A} \subset \mathcal{B}$ full subset

(1) pushout diagram

then we say $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a Karoubi sequence.

denote $\mathcal{C} = \mathcal{B}/\mathcal{A}$, Karoubi quotient (localization)

Weird thing: $\mathcal{B} \to \mathcal{B}/\mathcal{A}$ may not be essentially surj.

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{A}) \arrow[r, "i"] \arrow[d] & \mathrm{Ind}(\mathcal{B}) \arrow[d, "j"] \\ 0 \arrow[r] & \mathrm{Ind}(\mathcal{C}) \end{tikzcd}$$

in Pr^L colim in Pr^L

$\simeq$ limit w/ right adjoint as transition maps in Pr^R

= limit as cat.

$\mathrm{Ind}(\mathcal{C}) \simeq \ker(i^R : \mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{A}))$, $\mathcal{C} \simeq (\mathrm{Ind}\,\mathcal{C})^{\omega}$

Eg. $U \sqcup Z = X$. $\mathrm{Perf}(X \text{ on } Z) \xrightarrow{i_*} \mathrm{Perf}(X) \xrightarrow{j^*} \mathrm{Perf}(U)$

X qcqs scheme.

X noetherian, $\mathrm{Coh}(X^\wedge_Z) \to \mathrm{Coh}(X) \to \mathrm{Coh}(U)$

Def. • a functor $\mathcal{E}$ presentable stable cat.

$F: \mathrm{Cat}^{\mathrm{Perf}} \to \mathcal{E}$.

We say F is a localizing invariant, if $F(0) = 0$,

F sends Karoubi seq. to cofiber seq. in $\mathcal{E}$.

• $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ Karoubi seq. $\rightsquigarrow$ $F(\mathcal{A}) \to F(\mathcal{B}) \to F(\mathcal{C})$ cofiber seq.

$\mathcal{C} \in \mathrm{Cat}^{\mathrm{Perf}}$, $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ full subcat. ($\in \mathrm{Cat}^{\mathrm{Perf}}$) We say $\mathcal{C}$ is a semiorthogonal decomp. $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ if $\mathrm{Hom}(A, B) = 0$, $B \in \mathcal{B}$, $A \in \mathcal{A}$,

and for any $C \in \mathcal{C}$, can find cofiber seq.

$A \to C \to B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$

eg. $\mathcal{A} \underset{i^R}{\overset{i}{\rightleftarrows}} \mathcal{C} \underset{j^R}{\overset{j}{\rightleftarrows}} \mathcal{B}$ Karoubi quotient.

(i fully faithful, j^R fully faithful)

$\langle i(\mathcal{A}), j^R(\mathcal{B}) \rangle$ semiorthogonal

$\langle i(\mathcal{A}), \ker i^R \rangle$ semiorth.

If $\mathcal{B} = \langle \mathcal{A}, \mathcal{C} \rangle$

$F(\mathcal{B}) = F(\mathcal{A}) \oplus F(\mathcal{C})$. $\quad$ F additive invariant.

Fact. $\mathcal{C}$ admits countable $\oplus$, F preserves filtered colimit,

then $F(\mathcal{C}) = 0$.

- We can also define $\tilde{F}: \mathrm{Cat}_{cg,st} \to \mathcal{E}$ by $\tilde{F}(\mathcal{C}) := F(\mathcal{C}^{\omega})$

- $\ker(\mathrm{Ind}\, \mathcal{A} \to \mathrm{Ind}\, \mathcal{B})$ may not be cptly gen.

  but always dualizable.

- (Efimov): $F^{\mathrm{cont}}: \mathrm{Cat}^{\mathrm{dual}} \to \mathcal{E}$.

(connective) $\qquad$ (additive invariant)

Thm / Def: K-theory is the initial localizing invariant

$F: \mathrm{Cat}^{\mathrm{Perf}} \to \mathrm{Sp}$ commuted filtered colimit

together with $\mathcal{C} \to \Omega^{\infty} K(\mathcal{C})$.

$$\begin{array}{c} \uparrow \\ S \end{array}$$

$\mathrm{ob}\, \mathcal{C} \to K_0(\mathcal{C})$

$X \longmapsto [X]$

$\pi_i K(\mathcal{C})$ may not be connected.

$A \in \mathrm{CRing}$ (can be derived)

$\mathrm{Perf}(A) \subset D(A)$ spanned by finite colimit + retraction under A.

$\quad \shortparallel$

$D(A)^{\omega} \simeq D(A)^{\text{dualisable}}$

- A Noetherian, $D_{coh}(A) \subset D(A)$ consisting of $X \in D(A)$ bounded

and $\pi_i X$ is f.g. A-module.

$D(A)$ has a t-str. inducing a t-str. on $D_{coh}(A)$.

$D_{coh}(A)^{\heartsuit} \simeq \mathrm{Coh}(\pi_0 A)$.

Usually, $K(\mathrm{Perf}(A))$ algebraic K-theory of A

A noetherian $K(D_{coh}(A))$ algebraic G-theory of A.

$\simeq K(\mathrm{Coh}(\pi_0 A))$ (Quillen)

$\mathcal{A}$ abelian, $K(\mathcal{A}) \simeq K(D^b(\mathcal{A}))$

$K(D_{coh}(A))$ is connective

extend to $\mathcal{X}$ algebraic stack

$$\mathrm{Perf}(\mathcal{X}) = \lim_{\substack{\mathrm{Spec}\, A \to \mathcal{X} \\ \text{smooth}}} \mathrm{Perf}(A)$$

$$D\mathrm{Coh}(\mathcal{X}) = \lim_{\substack{\mathrm{Spec}\, A \to \mathcal{X} \\ \text{smooth}}} D_{coh}(A).$$

$\rightsquigarrow$

$K^{\mathrm{Perf}}(\mathcal{X})$

$\shortparallel$

$K(\mathrm{Perf}(\mathcal{X}))$ Zariski descent.

$K(D_{coh}(\mathcal{X}))$ (Nisnevich descent)

$\shortparallel$

$K(\mathcal{X})$

But $K(D_{coh}(X))_{\mathbb{Q}}$ satisfies fppf descent.

X restricts to qcqs scheme.

étale sheafification $K^{ét}(D_{coh}(X))_{\mathbb{Q}}$

(Quillen) open closed

Thm. $X = U \sqcup Z$

$K(Z) \to K(X) \to K(U)$ cofiber seq

In particular, $K(Z) = K(Z_{red})$

Thm. $\mathcal{E}/\mathcal{X}$ is a locally free sheaf of finite rank $n+1$

$K(\mathbb{P}_{\mathcal{E}}(\mathcal{X})) \simeq \bigoplus_{i=0}^{n} K(\mathcal{X})$

$D_{coh}(\mathbb{P}_{\mathcal{E}}(\mathcal{X})) = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n) \rangle$

· Cor. $K(\mathbb{V}_{\mathcal{E}}(\mathcal{X})) = K(\mathcal{X})$] not true for K^{Perf} in general.

$\mathbb{P}_{\mathcal{E}}(\mathcal{X}) \hookrightarrow \mathbb{P}_{\mathcal{E}\oplus\mathcal{O}}(\mathcal{X}) \hookleftarrow \mathbb{V}_{\mathcal{E}}(\mathcal{X})$.

made it true: KH: homotopy inv. K-thry.

X regular scheme (classic), $\mathrm{Perf}(X) \overset{(\Leftrightarrow)}{\simeq} D_{coh}(X)$.

$K^G(X) := K_0^G(X) = K_0([X/G])$.

$H \subset G$, $\quad X/H \simeq G \times_H X / G$,

$$K^H(X) \simeq K^G(G \times_H X).$$

$f: \mathcal{X} \to \mathcal{Y}$, $f^*: D_{coh}(\mathcal{Y}) \to D_{coh}(\mathcal{X})$.

$$Rf_*: D_{coh}(\mathcal{X}) \to D_{coh}(\mathcal{Y}).$$

to make f^* well-defined, we want it to have finite Tor dim.

Rf_* : we want it to have finite cohomological dim + proper.

$K(\mathcal{X})$ is a module over $K^{Perf}(\mathcal{X})$.

$$Perf(\mathcal{X}) \otimes D_{coh}(\mathcal{X}) \xrightarrow{-\otimes-} D_{coh}(\mathcal{X}).$$

<u>Convolution</u> $\quad D_{coh}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \qquad \mathcal{X} \xrightarrow{f} \mathcal{Y}$.

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow f & & \\ \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & & \end{array}$$

$$f_* \Delta^*: D_{coh}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to D_{coh}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})$$

$$D_{coh}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \otimes D_{coh}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \nearrow$$

G affine algebraic group, $G = U \rtimes R$, U unipotent, R is reductive.
$K^G(X) \simeq K^R(X)$,

Pf $K^R(X) = K^G(G \times_R X)$ $\quad G \times_R X \simeq G/R \times X$

$$\begin{aligned} &\simeq K^G(G/R \times X) \\ &\simeq K^G(X) \end{aligned}$$

$F \xrightarrow{\pi} X$ is called a cellular fibration, if $F \supset F^i$, $F^i \supset F^{i+1}$, s.t.

(1) $F^i \to X$ is a locally trivial fibration

(2) $\underset{\overset{\shortparallel}{E^i}}{F^i \setminus F^{i+1}} \to X$ is affine bundle.

then $0 \to K^G(F^{i+1}) \to K^G(F^i) \to K^G(E^i) \to 0$.

$$\begin{tikzcd} K_1^G(F^i) \arrow[r, two heads] & K_1^G(E^i) \arrow[r] & K^G(F^{i+1}) \\ & K_1^G(X) \arrow[ul, "\pi^*"] \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

Thm (5.6 from Chriss–Ginzburg) G reductive, X smooth projective, TFAE

(a) $\pi : K^G(X) \otimes_{R(G)} K^G(Y) \longrightarrow K^G(X \times Y)$

$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G}$ is an isom. for any Y.

(b) above, $\Delta_* \mathcal{O}_X$, $\Delta : X \to X \times X$ lies in the image of π

(c) $K^G(X)$ f.g. proj. $R(G)$-module, G-variety Y, $K^G(Y \times X) \xrightarrow{\sim} \mathrm{Hom}_{R(G)}(K^G(X), K^G(Y))$

h. .

(d) $K^G(X)$ f.g. proj. , $K^G(X\times X)$ f.g. proj.

$\mathrm{rk}\, K^G(X\times X) = (\mathrm{rk}\, K^G(X))^2$.

$p: X \to pt$

$\langle -,- \rangle : K^G(X)\times K^G(X) \longrightarrow R(G)$

$(F, G) \longmapsto p_*\big([F]\otimes[G]\big)$ is non degenerate

Thm. G s.c., $K^G(X) \longrightarrow K^T(X)$ $\quad$ G red $\supset$ T red.

① $K^G(X)\otimes_{R(G)} R(T) \xrightarrow{\sim} K^T(X)$

② $K^G(X) \simeq K^T(X)^W$.

$$\begin{tikzcd}
K_0^{et}(\mathfrak{X})_{\mathbb{Q}} \arrow[r, phantom, "\simeq"] & A_*(\mathfrak{X};\mathbb{Q}) \\
K_0(\mathfrak{X})_{\mathbb{Q}} \arrow[u] \arrow[ur, "\mathrm{ch}_*"']
\end{tikzcd}$$

$T \curvearrowright X$, $K^T(X)$ is a $R(T)$-module, $a\in T$

$K^T(X) \xleftarrow{i_*} K^T(X^a)$

$K^T(X)[\Sigma^{-1}] \xleftarrow[\sim]{i_*} K^T(X^a)[\Sigma^{-1}]$