Fontaine's theorem

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Thm (Fontaine) Let A be an ab. var. $/ \mathcal{O}_K$. Let $n \geq 1$, and let L be field gen. by coord. of pts in $A[p^n]$. Let $e = v_K(p)$, then $u_{L|K} \leq e\left(n + \frac{1}{p-1}\right)$

We write $m^t_{E|K} := \{x \in \mathcal{O}_E : v_K(x) \geq t\}$, so $m_E = m_{E|K}^{\frac{1}{e_{E|K}}}$

Def $P_m(L|K) \overset{def}{\Longleftrightarrow} \forall E|K$ finite, if $\exists$ hom. $\mathcal{O}_L \to \mathcal{O}_E / m^m_{E/K}$,

then $\exists$ hom. $\mathcal{O}_L \to \mathcal{O}_E$.

$L|K$ unr. $\Rightarrow$ Hensel's Lemma, $\begin{tikzcd} & R \arrow[d] \\ \mathcal{O}_L \arrow[r] \arrow[ur, dashed] & R/m \end{tikzcd}$ $\Rightarrow \forall \varepsilon > 0$, $P_\varepsilon(L|K)$ is true.

Thm a) $m > u_{L|K} \Rightarrow P_m(L|K)$ is true

b) $P_{u_{L|K} - \frac{1}{e_{L|K}}}(L|K)$ is false

Thm (Yoshida) b') $m < u_{L|K} \rightarrow P_m(L|K)$ is false.

Yoshida: Ramification of local fields and Fontaine's property (P_m).

Recall Krasner's lemma: Let $\alpha, \beta \in \overline{K}$ be s.t. $|\alpha - \beta| < |\alpha - \sigma\alpha|$, $\forall \sigma \in Gal(\overline{K}|K)$, $\sigma(\alpha) \neq \alpha$

then $K(\alpha) \subset K(\beta)$.

a) Let $P(X) \in \mathcal{O}_K[X]$ be min poly. of π_L over K. Let $\eta : \mathcal{O}_L \to \mathcal{O}_E / m_{E/K}^t$, $t > u_{L|K}$

Let $\beta \in \mathcal{O}_E$ be a lift of $\eta(\pi_L)$

$$p(\pi_L) = 0 \Rightarrow \quad p(\beta) \in m_{E|K}^t, \quad v_K(p(\beta)) \geq t > u_{L|K} \quad (*).$$

Let $\sigma_0 \in Gal(L|K)$ be such that $\sigma_0 \pi_L$ is closest to β

$$\beta - \sigma\pi_L = \beta - \sigma_0 \pi_L + \sigma_0 (\pi_L - \sigma_0^{-1}\sigma\pi_L)$$

$$\Rightarrow \quad v_K(\beta - \sigma\pi_L) = \min\left(v_K(\beta - \sigma_0\pi_L), v_K(\pi_L - \sigma_0^{-1}\sigma\pi_L)\right)$$

$$v_K(p(\beta)) = v_K\left(\prod_{\sigma \in Gal(L|K)} (\beta - \sigma\pi_L)\right)$$

$$= \varphi_{L|K}\left(v_K(\beta - \sigma_0\pi_L)\right) \geq t > u_{L|K}$$

$$v_K(\beta - \sigma_0\pi_L) > \varphi_{L|K}^{-1}(u_{L|K}) = i_{L|K} = \max_{\sigma \neq 1} v_K(\sigma(\pi_L) - \pi_L)$$

$$\overset{\text{Krasner}}{\Longrightarrow} L \subset k(\beta) \subset E.$$

b) Case 1 $L|K$ tamely ramified, $u_{L|K} = 1$. claim $P_1(L|K)$ is false.

take $E = K$, $\mathcal{O}_L \to \mathcal{O}_K / m_{K|K}^1$, but no map $L \to K$.

Case 2. $L|K$ wild. let $t := u_{L|K} - \frac{1}{e_{L|K}}$, write $t = r + \frac{s}{e_{L|K}}$ $\quad r \in \mathbb{Z}$, $0 \leq s < e_{L|K}$

Let $p(X) \in \mathcal{O}_K[X]$ min poly. of π_L over K, $Q(X) = p(X) - \pi_K^r X^s$.

$Q(X)$ is Eisenstein. Let β be a root of Q, $E = k(\beta)$ tot. ram. $/K$.

$$\mathcal{O}_L \longrightarrow \mathcal{O}_E / m_{E/K}^{t} \qquad , \quad P(\beta) = \pi_K^2 \beta^s, \quad v_K(p(\beta)) = 2 + s\, v_K(\beta) = 2 + \frac{s}{e_{L|K}} = t$$

$$\pi_L \longmapsto \beta$$

If $\exists\, L \to E$, then $L = E$

Then $v_K(\sigma\pi_L - \beta) \in \frac{1}{e_{L|K}}\mathbb{Z}$, $v_K\Big(\prod_{\sigma \in \mathrm{Gal}(L|K)} (\sigma\pi_L - \beta)\Big) = t$

$$e_{L|K} \sup_{\sigma \in G} \big(v_K(\sigma\pi_L - \beta)\big) \overset{\text{by a)}}{=} e_{L|K}\, \varphi_{L|K}^{-1}\big(v_K(P(\beta))\big) = e_{L|K}\, \varphi_{L|K}^{-1}(t) \in \mathbb{Z}$$

$$\varphi_{L|K}^{-1}\Big(u_{L|K} - \frac{1}{e_{L|K}}\Big) = i_{L|K} - \frac{1}{e_{L|K} \underbrace{|G_{i_{L|K}}|}_{>1}} \notin \frac{1}{e_{L|K}}\mathbb{Z} \quad . \text{ contradiction}$$

b') Take $K'|K$ tamely ram. of large degree

$L' = LK'$, take E to be Fontaine's example for L'

$$\mathcal{O}_{L'} \longrightarrow \mathcal{O}_E / m_{E/K}^{(u_{L|K} - \frac{1}{e_{L'|K}})}$$

$L' \not\hookrightarrow E$, $\quad L \hookrightarrow E$

$\qquad\qquad\quad K' \hookrightarrow E$

Prop. Let A be a finite flat $\mathcal{O}_K$-alg. of the form $A = \mathcal{O}_K[[X_1, \dots, X_m]]/(b_1, \dots, b_m)$

$X = \mathrm{Spec}\, A$. Suppose $\exists$ elt $a \in \mathcal{O}_K$, $a \neq 0$, $a\,\Omega^1_{A/\mathcal{O}_K} = 0$, $\Omega^1_{A/\mathcal{O}_K}$ is a free A/aA-mod.

Suppose that S is a finite flat $\mathcal{O}_K$-alg, $I \triangleleft S$ top. nilp. PD ideal. Then a map $A \longrightarrow S/aI$ lifts to $A \longrightarrow S$.

<u>Need to prove</u> $\left(A \text{ - ab. scheme} / \mathcal{O}_K, \quad L/K = K(A(\bar{K})) \right)$

$u_{L|K} \leq e\left(n + \frac{1}{p-1}\right) \Rightarrow$ Need to prove $P_{e(n+\frac{1}{p-1})}(L|K)$. $t = e\left(n + \frac{1}{p-1}\right)$

E/K finite

$$\begin{tikzcd} \mathcal{O}_L \arrow[r] \arrow[dr, dashed] & \mathcal{O}_E / m_{E/K}^t \\ & \mathcal{O}_E \arrow[u] \end{tikzcd}$$

$$\mathcal{A} = \mathcal{O}_K[A[p^n]], \quad a = p^n$$

<u>Thm</u> (Schoof) If k is a perfect field of char. p, and $G = \operatorname{Spec} A$ is a conn'd finite flat gp scheme, then $A \simeq k[x_1, \ldots, x_n] / (x_1^{p^{e_1}}, \ldots, x_n^{p^{e_n}})$ as a k-alg

<u>Cor</u> If R a complete dvr, G —||—, $A \simeq R[[x_1, \ldots, x_n]] / (f_1, \ldots, f_n)$.

$$A \otimes_R k = k[x_1, \ldots, x_n] / (x_1^{p^{e_1}}, \ldots, x_n^{p^{e_n}})$$

By NAK, lifts of x_i gen. A, $A = k[[x_1, \ldots, x_n]] / J$

By NAK, lifts of $x_i^{p^{e_i}}$ gen. J.

$$1 \to A[p^n] \to A \xrightarrow{p^n} A \to 1$$

$$\Omega^1_{A/\mathcal{O}_K} \xrightarrow{p^n} \Omega^1_{A/\mathcal{O}_K} \to \Omega^1_{A[p^n]/\mathcal{O}_K} \to 0$$

$$\Omega^1_{A[p^n]} \simeq \Omega^1_{A/\mathcal{O}_K} / p^n \Omega^1_{A/\mathcal{O}_K}.$$

$A = \mathcal{O}_K[\mathcal{A}[p^n]]$, $S = \mathcal{O}_E$, $a = p^n$, $m^t_{E/K} = p^n m_{E|K}^{t - v_K(p^n)} = p^n m_{E|K}^{t-ne}$

Claim $m_{E|K}^s$ is a PD ideal if $s > \frac{e_K}{p-1}$

$v_p(d!) = \lfloor \frac{d}{p} \rfloor + \lfloor \frac{d}{p^2} \rfloor + \cdots \le \frac{d}{p-1}$

Pick a pt $\alpha \in \mathcal{A}[p^n](L)$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[drr, dashed] & \mathcal{O}_L \arrow[r] & \mathcal{O}_E / m^t_{E/K} \\ & & \mathcal{O}_E \arrow[u] \end{tikzcd}$$

$\exists$ an $\mathcal{O}_E$-pt of $\mathcal{A}[p^n]$ ~~congruent to α mod $m^t_{E/K}$~~.

but any point of $\mathcal{A}[p^n](\bar{K})$ is already defined over L.

$v_K(\mathcal{D}_{L|K}) = u_{L|K} - i_{L|K} < u_{L|K} \le e_K\left(n + \frac{1}{p-1}\right)$

If $\mathcal{A}$ is an abelian scheme over $\mathbb{Z}$, then $L = \mathbb{Q}(\mathcal{A}[p^n](\bar{\mathbb{Q}}))$ is unram. away from p.

$\Delta_{L|\mathbb{Q}} = N_{L|\mathbb{Q}}(\mathcal{D}_{L|\mathbb{Q}}) \le \left(p^{n + \frac{1}{p-1}}\right)^{[L:\mathbb{Q}]}$

$\Delta_{L|\mathbb{Q}}^{\frac{1}{[L:\mathbb{Q}]}} \le p^{n + \frac{1}{p-1}}$ $\quad n = 1, p = 3, \quad \Delta_{L|\mathbb{Q}}^{\frac{1}{[L:\mathbb{Q}]}} \le 3^{3/2}$

Γ simple f. flat gp scheme $/\mathbb{Z}_p$, then either Γ is stupid or $[L:\mathbb{Q}] \ge p^2(p-1)$