On the classification of $G((t))$-categories

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Local Langlands

G reductive gp / local field F. $G(F) \leftarrow$ topological group

Want to study: (smooth) rep'ns of $G(F)$

local langlands, as understood by a non-NTist,

Representations of $G(F)$ can be classified in terms of Langlands parameters

$$\mathrm{Gal}(\bar{F}/F) \to {}^L G$$

In particular:

$\exists$ map from irreps $\to$ { Langlands pars }

can describe fibers

or: $\mathrm{Rep}(G(F)) \simeq \mathrm{Coh}$ (Stack of Langlands pars)

Why categories?

If you have category C/k, endomorphism $\mathrm{Fr}: C \to C$ produce a vec. sp.

$$\mathrm{Vect} \to C \otimes C^\vee \xrightarrow{\mathrm{Fr} \otimes \mathrm{id}} C \otimes C^\vee \to \mathrm{Vect}.$$

$$k \longmapsto \mathrm{Tr}(\mathrm{Fr}, C)$$

Key example: $C = \mathrm{Shv}(X)$, $X/\mathbb{F}_q$, $\bar{X} = X_{\bar{\mathbb{F}}_q}$ (Shv = ℓ-adic sheaves)

Fr: pullback along Frobenius map $\bar{X} \to \bar{X}$, $\mathrm{Tr}(\mathrm{Fun}, C) \simeq \mathrm{Fun}(X(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell)$

(morally true, need Drinfeld lemma)

In particular. G alg gp over $\mathbb{F}_q$

A rep of $G(\mathbb{F}_q)$ / $\overline{\mathbb{Q}}_\ell$ is a module for $\mathrm{Fun}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$

$$\mathrm{Fun}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \overset{\text{is}}{=} \mathrm{Tr}(\mathrm{Fr}, \mathrm{Shv}(\overline{G}))$$

Thus: If C is a cat. w/ action of $\mathrm{Shv}(\overline{G})$, & there is a compatible $\mathrm{Fr}: C \to C$.

$$\mathrm{Tr}(\mathrm{Fr}, C) \circlearrowleft \mathrm{Tr}(\mathrm{Fr}, \mathrm{Shv}(\overline{G}))$$

Motivates: Want to classify G-categories.

Let's say $F = \mathbb{F}_q((t))$. $F^{unr} = \overline{\mathbb{F}}_q((t))$. Let G be a group / F, $G^{unr} = G((t))$ ind-scheme / $\overline{\mathbb{F}}_q$

Similarly, ~~G^{unr}~~ $\mathrm{Tr}(\mathrm{Fr}, \mathrm{Shv}(G^{unr})) \simeq \mathcal{H}$ (convolution alg of cptly supported locally constant functions on $G(F)$}

In short, from a cat. w/ action of $\mathrm{Shv}(G^{unr})$ & a suitable endo, get a rep'n of $G(F)$

$$\boxed{\text{Goal: classify } G^{unr} = G((t))\text{-categories.}}$$

Setting #1. Take $\mathrm{Shv}(X) = D(X)$, $k = \overline{k}$ char 0

#2. Take ℓ-adic sheaves, $k = \overline{\mathbb{F}}_q$

Take a gp $G / k((t))$.

On the arithmetic side: Exhaustion thm for rep'ns, Equalities of Hecke algebras [Alder - Fintzen - Mishra - Ohara]

Because $k = \bar{k}$, many things are simple.

More complicated: Five lemma fails.

$$\begin{tikzcd} 0 \arrow[r] & C_1 \arrow[r] \arrow[d, "\wr"] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & D_1 \arrow[r] & D_2 \arrow[r] & D_3 \arrow[r] & 0 \end{tikzcd}$$

$C_2 \to D_2$ NOT equiv.

Slogan: LGL holds "up to the Five Lemma" joint w/ G. Dhillon, Y. Varshavsky

A filtration on $\underset{\text{Cat. of } G((t))\text{-cats}}{\checkmark}$ indexed by a poset I:

assigns to each object $X \in \underset{G((t))\text{-Cat}}{\checkmark}$ a sequence of full subcats

$G((t)) \circlearrowright X_i \subset X$

w/ $X_i \subset X_j$ if $i < j$

$X_i \underset{\leftarrow}{\overset{\text{I}}{\rightleftarrows}} X_j$

For $i \in I$, can take $G((t))$-cat

of obj. X where $X_j = 0$ if j not $\geq i$

$X_j = X$ if $j \geq i$

LGL: Statement $G((t))$-Cat $\simeq$ QCoh(LocSys)-Cat

$\exists$ filtrations on bothsides s.t. assoc. graded are $\simeq$.

For every $G((t))$-cat. C,

for each rat'l no. r, will define full subcat. $C^{\leq r}$

$C^{\leq r}$ is the $G((t))$-cat. generated by all $C^{K_{x,r+}}$, $\forall x \in B(G)$, $K_{x,r+}$ Moy–Prasad subgp.

Enrich this slightly: Define a filtration indexed by pairs (r, G_0)

$$\begin{tikzcd} G_0 \arrow[r] & G \arrow[d] \\ & k((t)) \arrow[ul, phantom] \end{tikzcd}$$

is a twisted Levi subgp (i.e. becomes a Levi over $\overline{k((t))}$)

Thm. (Dhillon – Varshavsky – Y.)

$$r > 0$$

$$G((t))\text{-cat}^{=(r, G_0)} \simeq \left(G_0((t))\text{-cat}^{=r, G_0, gen} \right)^{W_{rel}}$$

$$W_{rel} = N_G(G_0)/G_0$$

Lets you reduce understanding pieces of $G((t))$-cat. to understanding pieces w/ $G = G_0$.

Step 1_r

If $G = G_0$:

$$G((t))\text{-cat}^{=(r,G)} \simeq G((t))\text{-cat}^{<r} \underset{Z\text{-cat}\ldots}{\otimes} Z\text{-cat}\ldots$$

Reduces understanding to understanding rep'ns of depth $< r$. Step 2_r

Applying 1_r, 2_r repeatedly reduces you to depth 0.

(Depth 0 study understood by Dhillon – Li – Yun – Zhu).

Step 3 + 4. Bezrukavnikov

On spectral side:

Thm

$$\mathrm{LocSys}_G^{=r, G} \simeq \mathrm{LocSys}_{G_0}^{=r, G_0, gen} / W_{rel}$$