Singular support of coherent sheaves

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Monodromic Bezrukavnikov equiv:

$$
\begin{array}{ccc}
\mathcal{H}^{mon,mon} = D_{u\text{-}mon}(I^+\backslash LG/I^+) & \xrightarrow{\sim} & \mathrm{IndCoh}\left(\dfrac{\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right) \text{ set-theoretically supp. on } \tilde{\check{\mathcal{N}}}. \\
\circlearrowright \qquad\qquad \circlearrowleft & & \circlearrowright \qquad\qquad \circlearrowleft \\
D_{u\text{-}mon}(T) \qquad D_{u\text{-}mon}(T) & & \mathrm{IndCoh}(\check{\mathfrak{t}}^\wedge) \qquad \mathrm{IndCoh}(\check{\mathfrak{t}}^\wedge)
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{H}^{ea,mon} := \mathrm{Vect} \underset{D_{u\text{-}mod}(T)}{\otimes} \mathcal{H}^{mon,mon} & \xrightarrow{\sim} & \mathrm{Vect} \underset{\mathrm{IndCoh}(\check{\mathfrak{t}}^\wedge)}{\otimes} \mathrm{IndCoh}\left(\dfrac{\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right) \text{ set-theoretically supp. on } \tilde{\check{\mathcal{N}}} \\
\wr\wr & & \wr\wr \\
D(I\backslash LG/I^+) & \overset{(1)}{\cong} & \mathrm{IndCoh}\left(\dfrac{\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right)
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{H}^{ea,ea} = \mathcal{H}^{ea,mon} \underset{D_{u\text{-}mon}(T)}{\otimes} \mathrm{Vect} & \xrightarrow{\sim} & \mathrm{IndCoh}\left(\dfrac{\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right) \underset{\mathrm{IndCoh}(\check{\mathfrak{t}}^\wedge)}{\otimes} \mathrm{Vect} \\
\wr\wr & & \wr\wr \\
D(I\backslash LG/I) & \overset{(2)}{\cong} & \mathrm{IndCoh}_Z\left(\dfrac{\tilde{\check{\mathcal{N}}} \times^L_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}}}{\check{G}}\right) \subsetneq \mathrm{IndCoh}\left(\dfrac{\tilde{\check{\mathcal{N}}} \times^L_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}}}{\check{G}}\right) \\
 & & \uparrow \text{ some singular support condition} \\
\cup & & \cup \\
D^b_c(I\backslash LG/I) & \overset{(3)}{\cong} & \mathrm{Coh}\left(\dfrac{\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}}}{\check{G}}\right) \\
\not\supset & & \not\supset \\
D(I\backslash LG/I)^\omega & \xrightarrow{\sim} & \mathrm{Coh}_Z\left(\dfrac{\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}}}{\check{G}}\right)
\end{array}
$$

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X (derived lci) quasi-smooth scheme of finite type $/\Lambda$

$\mathbb{L}_{X/\Lambda} \in \mathrm{Perf}(X)$ locally can be represented by $\mathcal{E}^{-1} \to \mathcal{E}^{0}$

locally

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d] \\ 0 \arrow[r] & V \end{tikzcd}$$

smooth

$$f^* \mathbb{L}_V|_X \to \mathbb{L}_U|_X \to \mathbb{L}_X$$

$$\mathbb{T}_X = \mathbb{L}_X^\vee \in \mathrm{Perf}(X)$$

$$\mathrm{Sing}(X) = \mathrm{Sym}_{\pi_0 \mathcal{O}_X}(\mathcal{H}^1 \mathbb{T}_X) = \text{"total space of } \mathcal{H}^{-1}\mathbb{L}_X\text{"}$$

$$\downarrow$$

$$X_{cl}$$

locally free in deg 0

$Y \xrightarrow{f} X$ smooth morphism, $f^*\mathbb{L}_X \to \mathbb{L}_Y \to \mathbb{L}_{X/Y}$

$$f^* \mathcal{H}^1 \mathbb{T}_X \twoheadrightarrow \mathcal{H}^1 \mathbb{T}_Y.$$

An alg. stack X almost of finite presentation $/\Lambda$ is called quasi-smooth if

$\exists\ U \xrightarrow{\varphi} X$ atlas w/ U quasi-smooth

$$\varphi^* \mathbb{L}_X \to \mathbb{L}_U \to \mathbb{L}_{U/X}$$

$$\mathrm{Sing}(X) := \mathrm{Sym}_{X_{cl}}(\mathcal{H}^1 \mathbb{T}_X), \quad \mathrm{Sing}(X) \underset{X_{cl}}{\times} U_{cl} = \mathrm{Sing}(U)$$

Ex. $\mathrm{Sing}\left(\dfrac{\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right) \longrightarrow \dfrac{\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}$

$$\| $$

$$\check{\mathfrak{b}}/\check{B} \times_{\check{\mathfrak{g}}/\check{G}} \check{\mathfrak{b}}/\check{B} \qquad \left\{ \begin{matrix} (\xi, \check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2) \\ \xi \in \check{\mathfrak{b}}_1 \cap \check{\mathfrak{b}}_2 \end{matrix} \right\} / \check{G}$$

Claim.

$$\begin{tikzcd} x \arrow[r] \arrow[dr] & X \arrow[d] \\ & X/H \end{tikzcd} \qquad X \text{ smooth.} \qquad \mathfrak{h} \longrightarrow T_x X$$

$$(\mathbb{L}_{X/H})_x = \left(T_x^* X \longrightarrow \mathfrak{h}^*\right)$$

$$\begin{tikzcd} \mathbb{L}_{\check{\mathfrak{b}}/\check{B} \times_{\check{\mathfrak{g}}/\check{G}} \check{\mathfrak{b}}/\check{B}} & \mathbb{L}_{\check{\mathfrak{b}}/\check{B}} \arrow[l] & \check{\mathfrak{b}}_1^* \xrightarrow{ad^*\xi} \check{\mathfrak{b}}_1 \\ \mathbb{L}_{\check{\mathfrak{b}}/\check{B}} \arrow[u] & \mathbb{L}_{\check{\mathfrak{g}}/\check{G}} \arrow[l] \arrow[u] & (\mathbb{L}_{\check{\mathfrak{g}}/\check{G}})_\xi = \check{\mathfrak{g}}^* \xrightarrow{ad^*_\xi} \check{\mathfrak{g}}^* \arrow[u] \arrow[dll] \\ \check{\mathfrak{b}}_2^* \xrightarrow{ad^*\xi} \check{\mathfrak{b}}_2 & & \end{tikzcd}$$

$$\mu^{-1}\mathbb{L} = \left\{ \eta \in \check{\mathfrak{g}}^* : \begin{matrix} ad^*\xi(\eta) = 0 \\ \eta \in (\check{\mathfrak{g}}/\check{\mathfrak{b}}_1)^* \cap (\check{\mathfrak{g}}/\check{\mathfrak{b}}_2)^* \end{matrix} \right\}$$

$$\xrightarrow[\sim]{\text{Killing form}} \left\{ \eta \in \check{\mathfrak{g}} : [\xi, \eta] = 0, \ \eta \in \check{\mathfrak{n}}_2 \cap \check{\mathfrak{n}}_2 \right\}$$

$$\mathrm{Sing}\left(\frac{\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right) = \left\{ (\xi, \check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \eta) : \xi \in \check{\mathfrak{b}}_1 \cap \check{\mathfrak{b}}_2, \ \eta \in \check{\mathfrak{n}}_2 \cap \check{\mathfrak{n}}_2, \ [\xi, \eta] = 0 \right\} / \check{G}$$

$$\mathrm{Sing}\left(\frac{\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}}{\check{G}}\right) = \left\{ (\xi, \check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \eta) : \xi \in \check{\mathfrak{n}}_1 \cap \check{\mathfrak{b}}_2, \ \eta \in \check{\mathfrak{b}}_1 \cap \check{\mathfrak{n}}_2, \ [\xi, \eta] = 0 \right\} / \check{G}$$

$$\mathrm{Sing}\left(\frac{\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}}}{\check{G}}\right) = \left\{ (\xi, \check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \eta) : \xi \in \check{\mathfrak{n}}_1 \cap \check{\mathfrak{n}}_2, \ \eta \in \check{\mathfrak{b}}_1 \cap \check{\mathfrak{b}}_2, \ [\xi, \eta] = 0 \right\} / \check{G}$$

$X = \operatorname{Spec} A$

$$HH(X) = \mathcal{O}_X \overset{\mathbb{L}}{\underset{\mathcal{O}_{X\times X}}{\otimes}} \mathcal{O}_X = A \overset{\mathbb{L}}{\underset{A\otimes A}{\otimes}} A = \left(\cdots \substack{\to\\ \to\\ \to\\ \to} A\otimes A \otimes A \substack{\to\\ \to\\ \to} A \otimes A \rightrightarrows A \right)$$

$a\otimes b \mapsto ab - ba$

$a \otimes b \otimes c \mapsto ab\otimes c$, $a\otimes bc$, $b \otimes ca$

<u>HKR isom.</u> (in char. 0)

$$HH_\bullet A \to \cdots \tau^{\geq n-1} HH_\bullet(A) \longrightarrow \tau^{\geq n} HH_\bullet(A)$$

$$\frac{\tau^{\geq n} HH_\bullet(A)}{\tau^{\geq n-1} HH_\bullet(A)} \simeq \Omega_A^{-n}[-n] \qquad A \text{ smooth}$$

$$A \rightsquigarrow \left(\tau^{\geq \bullet} HH_\bullet\right) \longrightarrow gr$$

$$\begin{tikzcd}
\text{Pol alg.} \arrow[r] \arrow[rr, bend right] & (A, \text{ filtred } A\text{-mod}) \arrow[d, "gr^{-1}"] \arrow[r] & HH(A) \\
& \Omega_A = \mathbb{L}_A &
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Ani}(\mathrm{Pol}_\Lambda) \\
\mathrm{CAlg}_\Lambda \arrow[u, phantom, "\cong"] \arrow[r] \arrow[rr, bend left, "HH"] \arrow[dr, "\mathbb{L}"'] & \text{filtred mod} \arrow[r] \arrow[d, "gr^{-1}"] & \mathrm{Mod} \\
& \mathrm{Mod} &
\end{tikzcd}$$

$$HH^\bullet(A) = R\mathrm{Hom}_{A\otimes A}(A, A) = R\mathrm{Hom}_A\left(A \underset{A\otimes A}{\otimes} A, A\right)$$

<u>Lemma</u> $T_A[-1] \xrightarrow{\sim} gr^1 HH^\bullet(A)$

Lemma A quasi-smooth, $H^1 \mathbb{T}_A \hookrightarrow \mathrm{Ext}^2_{A\otimes A}(A,A) = H^2 HH^\cdot(A)$ injective.

$\mathcal{C}$ category, $\mathbb{E}_2$-center $Z(\mathcal{C})$ "$=$" $\mathrm{REnd}(\mathrm{Id}_{\mathcal{C}})$

$\mathcal{C}$ dualizable presentable $\searrow \simeq \mathrm{REnd}_{\mathcal{C}^\vee\otimes\mathcal{C}}(u_{\mathcal{C}})$

$\mathcal{C}^\vee\otimes\mathcal{C} = \mathrm{Fun}^{\text{colimit preserving}}(\mathcal{C},\mathcal{C})$

$\mathcal{C} = \mathrm{Mod}\,A \implies Z(\mathcal{C}) \simeq \mathrm{RHom}_{A\otimes A}(A,A)$

$Z(\mathcal{C}) \rightsquigarrow \mathrm{End}_{\mathcal{C}}(c)$, $\forall c \in \mathcal{C}$

$H^1 \mathbb{T}_A \longrightarrow \mathrm{Ext}^2_{A\otimes A}(A,A) \rightsquigarrow \mathrm{REnd}_A(M)$

$\searrow \mathrm{Ext}^2_A(M,M)$

$\mathrm{Sym}\, H^1 \mathbb{T}_A \curvearrowright \mathrm{Ext}^{2\cdot}_A(M,M)$ graded module

Prop. $\mathrm{Ext}^{2\cdot}_A(M,M)$ is f.g. over $\mathrm{Sym}\, H^1\mathbb{T}_A$. ($M$ f.g. A-module)

Def $\mathrm{Sing\,Supp}(M) := \mathrm{Supp}_{\mathrm{Sym}\, H^1\mathbb{T}_A}\left(\mathrm{Ext}^{2\cdot}_A(M,M)\right)$ M f.g. A-module.

Lemma $U \xrightarrow{\text{open}} X$, $U = \mathrm{Spec}\, A_f$, $X = \mathrm{Spec}\, A$

$\mathrm{Sing}(U) = \mathrm{Sing}(X) \times_X U$

$\mathrm{SingSupp}(M)|_U = \mathrm{SingSupp}(M|_U)$

$\rightsquigarrow$ SingSupp for coherent sheaves

$X \xrightarrow{f} Y$ smooth, $\mathrm{Sing\,Supp}(f^*\mathcal{F}) = \mathrm{SingSupp}(\mathcal{F}) \times_Y X$.

X quasi-smooth, $Z \subset \mathrm{Sing}(X)$ conic.

$$\mathrm{Coh}_Z(X) := \{ \mathcal{F} \in \mathrm{Coh}(X) : \mathrm{SingSupp}(\mathcal{F}) \subset Z \}$$

Prop $Z = X_{cl} \xrightarrow{\text{zero section}} \mathrm{Sing}(X)$, then $\mathrm{Coh}_Z(X) = \mathrm{Perf}(X)$.

Prop $i : Y \to X$ i quasi-smooth closed embedding, Y, X quasi-smooth

$$i^* \mathbb{L}_X \to \mathbb{L}_Y \to \underset{[-1,0]}{\mathbb{L}_{Y/X}}$$

$$\rightsquigarrow H^{-1}\mathbb{L}_X \hookrightarrow H^{-1}\mathbb{L}_Y.$$

$$\begin{tikzcd}
\mathrm{Sing}(X) & \mathrm{Sing}(X) \times_X Y \arrow[l] \arrow[r, hook] & \mathrm{Sing}(Y) \\
W \arrow[u, phantom, "\cup"] & Z \arrow[l] \arrow[u, phantom, "\cup"] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Coh}_W(X) \arrow[rr, "i^*"] \arrow[dr] & & \mathrm{Coh}(Y) \\
& \mathrm{Coh}_Z(Y) \arrow[ur] &
\end{tikzcd}$$

i^* factors through $\mathrm{Coh}_Z(Y)$ and $i^*(\mathrm{Coh}_W(X))$ gen. $\mathrm{Coh}_Z(Y)$ by cones & retracts.

e.g. Y scheme, $\mathcal{F}$ coh. sheaf on Y. $Y \hookrightarrow X$ (smooth),

$W = \mathrm{Sing}\, X = 0$, $Z = 0$, $\underset{\underset{\mathrm{Perf}(Y)}{\parallel}}{\mathrm{Coh}_{\{0\}} Y}$ is gen. under i^* by $\mathrm{Coh}(X) = \mathrm{Perf}(X)$.

e.g. $\mathrm{St}_{\mathrm{nilp}} = \dfrac{\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}}{G} \hookrightarrow \dfrac{\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}{G} = \mathrm{St}^*$,

$$Z = \mathrm{Sing}\left(\frac{\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}{G} \right) \Big|_{\mathrm{St}_{\mathrm{nilp}}} \hookrightarrow \mathrm{Sing}(\mathrm{St}_{\mathrm{nilp}}) = W$$

$$\mathrm{IndCoh}(St') \underset{\mathrm{IndCoh}(\check{t}^{\wedge})}{\otimes} \mathrm{Vect} \xrightarrow{gen} \mathrm{IndCoh}_Z(St_{nilp})$$

$$\mathrm{Sing}(St_{nilp}) = \{(\xi, \eta, \check{b}_1, \check{b}_2) : \xi \in \check{n}_1 \cap \check{n}_2,\ [\eta, \xi] = 0,\ \eta \in \check{b}_1 \cap \check{b}_2\} / \check{G}$$

$$\mathrm{Sing}(St') = \{(\xi, \eta, \check{b}_1, \check{b}_2) : \xi \in \check{n}_1 \cap \check{b}_2,\ [\eta, \xi] = 0,\ \eta \in \check{b}_1 \cap \check{n}_2\} / \check{G}$$

$$\Rightarrow Z = \{(\xi, \eta, \check{b}_1, \check{b}_2) : \xi \in \check{n}_1 \cap \check{n}_2,\ [\eta, \xi] = 0,\ \underbrace{\eta \in \check{b}_1 \cap \check{n}_2}_{\substack{\text{forces } \eta \text{ nilpotent,} \\ \eta \in \check{n}_1 \cap \check{n}_2}}\} / \check{G}$$

$\boxed{\text{"nilpotent singular support"}}$