Rigidity method for automorphic forms over function fields

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Setup.

Lecture 1. • $k = \mathbb{F}_q$, X proj. sm. geom. conn'd curve $/k$

$k(X) = F$, $|X|$ = closed pts on X

$$\begin{array}{cccccc} |X| & & & & & \\ \Downarrow & & & & & \\ x & , & F_x & \supset & \mathcal{O}_x & \twoheadrightarrow k_x \\ & & \wr\wr & & \wr\wr & \\ & & k_x((t_x)) & & k_x[[t_x]] & \end{array}$$

• G/k split semisimple gp SL_n, PGL_n, Sp_{2n}, G_2, E_8, ...

• $\mathbb{A} = \prod'_{x\in|X|} F_x$, $G(\mathbb{A}) = \prod'_{x\in|X|} G(F_x)$ (almost all components $\in G(\mathcal{O}_x)$)

Level gps $K = \prod_{x\in|X|} K_x$, $K_x \subset G(F_x)$ compact open

almost all $K_x = G(\mathcal{O}_x)$

$$K^{\natural} = \prod_{x\in|X|} G(\mathcal{O}_x)$$

• $\mathcal{A}_K = C(G(F)\backslash G(\mathbb{A})/K, \mathbb{C})$

$\circlearrowleft$

$H_K = \{ h : K\backslash G(\mathbb{A})/K \to \mathbb{C}$, cptly supported$\}$

Hecke algebra (unit $= 1_K$)

$$f : G(F) \backslash G(\mathbb{A}) / K \longrightarrow \mathbb{C}$$

$$h : K \backslash G(\mathbb{A}) / K \longrightarrow \mathbb{C}$$

$$\underset{\substack{\uparrow \\ G(\mathbb{A})}}{(f * h)(x)} = \sum_{\substack{g \in G(\mathbb{A})/K \\ \text{(finite sum)}}} f(xg)\, h(g^{-1})$$

Study $\mathcal{A}_K$ as an H_K-module.

- $\mathcal{A}_{K,c} :=$ cptly supported fncs $\in \mathcal{A}_K$

$\mathcal{A}_{K,\text{cusp}} = \{ f \in \mathcal{A}_{K,c} : \dim_{\mathbb{C}} (H_K \cdot f) < \infty \}$

eigenforms f: for almost all x $\quad (K_x = G(\mathcal{O}_x))$

f is an eigenvector under $H_{K_x} = C_c(K_x \backslash G(F_x) / K_x)$

Bun$_G$

$$\begin{array}{c} G(F) \backslash G(\mathbb{A}) / K^{\text{G}} = \prod_{x \in |X|} G(\mathcal{O}_x) \\ \updownarrow \\ G\text{-bdles on } X \end{array}$$

$G = GL_n$ $\qquad$ G-bdles $\longleftrightarrow$ vector bdles of rk n

$$\underset{\substack{(\\ \text{“principal } GL_n\text{-bdle over } X\text{''}}}{\mathrm{Isom}(\mathcal{O}^{\oplus n}, V)} \longleftarrow\!\shortmid V$$

$\mathrm{Isom}(\mathcal{O}^{\oplus n}, V) \circlearrowleft GL_n$

Pr...

<u>Weil</u> :

$$GL_n(F) \backslash GL_n(\mathbb{A}) / K^{q} \longleftrightarrow \mathrm{Vect}_n(X)$$

stabilizers $\quad \simeq \quad$ automorphisms

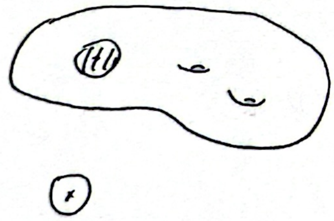
$g = (g_x) \in GL_n(\mathbb{A})$

Assume $\quad g_x = 1 \quad \forall\, x \neq x_0$

$$\Lambda_{x_0} = g_{x_0} \mathcal{O}_{x_0}^{\oplus n} \subset F_{x_0}^{\oplus n}$$

$\mathcal{O}_{x_0}$ - submod. of rank n

Then glue $\quad \Lambda_{x_0}$ w/ $\quad \mathcal{O}_{X \backslash x_0}^{\oplus n}$



$j : X \backslash x_0 \hookrightarrow X$ $\qquad$ $U \subset X$ affine open

$j_* \mathcal{O}_{X\backslash x_0}^{\oplus n}$

$$U \longmapsto \Gamma(U \backslash x_0, \mathcal{O}^{\oplus n}) \cap \Lambda_{x_0}$$

inside $F_{x_0}^{\oplus n}$

e.g. $g_{x_0} = \begin{pmatrix} t_{x_0} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \rightsquigarrow \mathcal{O}(-x_0) \oplus \mathcal{O}^{\oplus n-1}$

$$\mathrm{Vect}_n(X) \longrightarrow GL_n(F) \backslash GL_n(\mathbb{A}) / K^G$$

$$V \longmapsto \exists\, U \subset X \text{ s.t. } V|_U \simeq \mathcal{O}_U^{\oplus n}$$

$$\Lambda_x = V|_{\mathrm{Spec}\, \mathcal{O}_x} = g_x \mathcal{O}_x^{\oplus n}$$

$$(g_x)$$

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general G (split)

$$G(F) \backslash G(\mathbb{A}) / K^G \longleftrightarrow \{ G\text{-bdles on } X \}$$

$G = Sp_{2n}$, G-bdles are the same as (V, ω) : V rk $2n$ vec. bdle

$\omega : V \otimes_{\mathcal{O}_X} V \longrightarrow \mathcal{O}_X$ symplectic

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$\mathrm{Bun}_G(k) = \{ G\text{-bdles on } X \}$

$\mathrm{Bun}_G(R) = \{ G\text{-bdles on } X_R \}$

Bun_G : Artin stack

Ex $X = \mathbb{P}^1$

$\mathrm{Bun}_G(k)/\cong$

$G = GL_n$

$\mathrm{Vect}_n(\mathbb{P}^1)/\cong \quad \longleftrightarrow \quad (d_1 \geq d_2 \geq \cdots \geq d_n)$
$d_i \in \mathbb{Z}$

$\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n)$

In general, $T \subset G$ max torus, W,

$\mathrm{Bun}_{G,\mathbb{P}^1}(k)/\cong \quad \longleftrightarrow \quad X_*(T)/W$

$\mathcal{A}_{K^G}$ = functions on $\mathrm{Bun}_G(k)$

H_K - action ?

Ex. $G = GL_n$

$h_x = \mathbb{1}_{K_x \begin{pmatrix} t_{x} & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} K_x} \in H_{K_x}$

$f : \mathrm{Bun}_G(k) \longrightarrow \mathbb{C}$

$f * h_x : \underset{\underset{\mathcal{V}}{\Psi}}{\mathrm{Bun}_G(k)} \longrightarrow \mathbb{C}$

$(f * h_x)(\mathcal{V}) = \sum_{\substack{\mathcal{V} \overset{i}{\hookrightarrow} \mathcal{V}' \\ \mathrm{coker}(i) \simeq k_x}} f(\mathcal{V}')$ (elementary upper modification of $\mathcal{V}$)

$$h_x = \mathbb{1}_{K_x \begin{pmatrix} t_x^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t_x^{\lambda_n} \end{pmatrix} K_x} \qquad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$(f * h_x)(\nu) = \sum_{\substack{\nu \dashrightarrow \nu' \\ \lambda, x}} f(\nu')$$

Level structures. Fix $x \in |X|$

Parahoric subgps $\subset G(F_x)$

Iwahori

$$\begin{array}{ccc} I_x & \subset & G(\mathcal{O}_x) \\ \downarrow & \ulcorner & \downarrow \\ B(k_x) & \subset & G(k_x) \end{array}$$

$G = GL_2$, $I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} a, b, c, d \in \mathcal{O}_x \\ c \in \mathfrak{m}_x \end{array} \right\}$

Analogy:

G/k	G/F_x
Borel subgp	Iwahori
parabolic subgp	parahoric
Dynkin	affine Dynkin

$G = GL_n$, $\Lambda_x \subset F_x^{\oplus n}$

$$\mathrm{Stab}_{G(F_x)}(\Lambda_x) = \{ g \in GL_n(F_x) : g\Lambda_x = \Lambda_x \}$$

$\Lambda_x = \mathcal{O}_x^{\oplus n} \rightsquigarrow GL_n(\mathcal{O}_x)$

- $\Lambda_0 \underset{1}{\subsetneq} \Lambda_1$, $\quad \Lambda_1 / \Lambda_0 \simeq k_x$, $\quad \mathrm{Stab}(\Lambda_0, \Lambda_1)$ parahoric

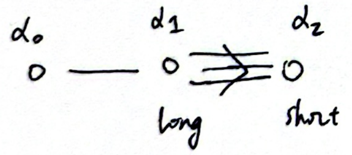
- $$\begin{array}{ccccccc} \Lambda_0 \subset \Lambda_{a_1} \subset \cdots \subset & \Lambda_n & \subset & \Lambda_{n+a_1} & \subset \cdots \\ & \parallel & & \parallel & \\ & t_x^{-1}\Lambda_0 & & t_x^{-1}\Lambda_{a_1} & \end{array}$$

$\mathrm{Stab}_{GL_n(F_x)}(\Lambda_\bullet)$ give all parahoric subgps in $GL_n(F_x)$.

Iwahori: $$\begin{array}{cc} \subset \Lambda_0 \underset{1}{\subset} \Lambda_1 \underset{1}{\subset} \cdots \underset{1}{\subset} & \Lambda_n \subset \cdots \\ & \parallel \\ & t^{-1}\Lambda_0 \end{array}$$

Affine Dynkin diagram

G_2: 



ϕ: Iwahori

$\{\alpha_1, \alpha_2\}$, $\quad G(\mathcal{O}_x)$

$\{\alpha_0, \alpha_1\}$: $\quad P \twoheadrightarrow SL_3$

$\{\alpha_0, \alpha_2\}$: $\quad Q \twoheadrightarrow SO_4 = (SL_2 \times SL_2)/\Delta(\pm 1)$

Lecture 2 Rigidity

Automorphic data: • $S \subset |X|$ finite
$\quad$ └ all $k = \mathbb{F}_q$- pts

(e.g. $X = \mathbb{P}^1$, $S = \{0, \infty\}$, $\{0,1,\infty\}, \cdots$)

• $x \in S \rightsquigarrow K_x \subset G(F_x)$ cpt open

• $x \in S$,

$$\begin{tikzcd} K_x \arrow[rr, "\chi_x"] \arrow[dr] & & \mathbb{C}^* \\ & L_x = \text{finite} \arrow[ur] & \end{tikzcd}$$

(K_S, χ_S)

Given (K_S, χ_S),

(K_S, χ_S) - typical automorphic forms

If $\chi_S = 1$, these are $f \in A_{K,c}$, $K = K_S \times \prod_{x \notin S} G(\mathcal{O}_x)$.

In general, $f : G(F) \backslash G(\mathbb{A}) / \prod_{x \notin S} G(\mathcal{O}_x) \xrightarrow{\text{cpt. supp}} \mathbb{C}$

is (K_S, χ_S) - typical, if $f(g k_x) = \chi_x(k_x) f(g)$, $\forall x \in S$,
$k_x \in K_x$, $g \in G(\mathbb{A})$)

Want to make $\dim A_c(K_S, \chi_S) = 1$.

If $\dim \mathcal{A}_c(K_S, \chi_S) = 1$, $\forall\, y \notin S$, $f \in \mathcal{A}_c(K_S, \chi_S) \circlearrowleft H_{K_y}$

f is Hecke eigen.

Ex. $X = \mathbb{P}^1$, $S = \{0, 1, \infty\}$, $G = SL_2$, $K_x = I_x$, $\forall\, x \in S$

$$I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} a, b, c, d \in \mathcal{O}_x \\ c \in \mathfrak{m}_x \end{array} \right\}.$$

$$\begin{array}{ccc} I_x & \ni & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \downarrow & & \downarrow \\ k^\times & \ni & \bar{a} \\ \downarrow{\scriptstyle \chi_x} & & \\ \mathbb{C}^* & & \end{array}$$

$\chi_0, \chi_1, \chi_\infty$ "generic" position

$\Rightarrow \mathcal{A}_c(K_S, \chi_S)$ has dim 1.

Generic means $\chi_0^{\pm 1} \chi_1^{\pm 1} \chi_\infty^{\pm 1} \neq 1$.

$f \in \mathcal{A}_c(K_S, \chi_S) \rightsquigarrow$ 2-dim'l local system on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

"hypergeometric"

$\underline{G = PGL_2}$: $\chi_0 = \chi_1 = 1$, χ_∞ = quadratic

$$\rightsquigarrow \text{local system } \{E_t\} : y^2 = x(x-1)(x-t)$$

$$\downarrow$$

$$\mathbb{P}^1 \setminus \{0, 1, \infty\}$$

$\{H^1(E_t)\}$ rank 2 local sys. on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Ex. $X = \mathbb{P}^1$, $S = \{0, \infty\}$, $G = SL_2$, $K_0 = I_0$, $\chi_0 = 1$,

τ = uniformizer at ∞

$$\begin{tikzcd} K_\infty = I_\infty^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a, d \equiv 1 \ (m_x) \\ b \in \mathcal{O}_x, \quad c \in m_x \end{array} \right\} \arrow[d] & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[d, mapsto] \\ k \arrow[d, "\psi"'] & b + \frac{c}{\tau} \mod \tau \\ \mathbb{C}^* & \end{tikzcd}$$

$\chi_\infty : K_\infty \longrightarrow k \xrightarrow{\psi} \mathbb{C}^* \Rightarrow \dim A_c(K_S, \chi_S) = 1$

$\rightsquigarrow$ Kloosterman loc. sys. on $\mathbb{P}^1 \setminus \{0, \infty\}$.

$$Kl(a) = \sum_{x \in k^\times} \psi\left(x + \frac{a}{x}\right).$$

Naive rigidity of (K_S, χ_S): $\dim A_c(K_S, \chi_S) = 1$.

Issue: • $G \neq$ simply connected, Bun_G has several components,

• not clear same holds after base change $k \rightsquigarrow k'$.

<u>Base change of auto. data</u>. $k'|k$ finite ext'n, $X' = X \otimes_k k'$

$S \rightsquigarrow S'$ preimage of S in X'.

$K_x \otimes_k k'$.

$\chi'_x : K_x(k') \overset{Nm}{\dashrightarrow} K_x \xrightarrow{\chi_x} \mathbb{C}^*$

Ex 1. $I_x \longrightarrow k^\times \xrightarrow{\chi} \mathbb{C}^*$

$I_x' = I_x(k') \longrightarrow k'^\times \xrightarrow{Nm} k^\times \xrightarrow{\chi} \mathbb{C}^*$

$a, b, c, d \in \mathcal{O}_x \hat{\otimes} k' \simeq k'[[t_x]]$

Ex 2. $I_x^+ \longrightarrow k \xrightarrow{\psi} \mathbb{C}^*$

$I_x^+(k') \longrightarrow k' \xrightarrow{Tr} k \xrightarrow{\psi} \mathbb{C}^*$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto b + \frac{c}{t} \mod t$

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characters $\underset{\text{geom.}}{\rightsquigarrow}$ char. sheaves (rk 1 local systems on K_x)

$(K_S, \chi_S) \rightsquigarrow (K_S', \chi_S')$
$/k \quad \underset{\text{b.c.}}{\rightsquigarrow} \quad /k'$

$\mathcal{A}_c(k'; K_S', \chi_S')$

Def. (K_S, χ_S) is <u>weakly rigid</u> if $\dim \mathcal{A}_c(k'; K_S', \chi_S')$ is uniformly bounded $\forall\, k'|k$ finite ext'n.

Relevant points.

$G(F)\backslash G(\mathbb{A})/K \longleftrightarrow Bun_G(K)(k)$

$f \in \mathcal{A}_c(K_S, \chi_S)$ are functions on $Bun_G(K_S^+)(k)$

..

where $K_x^+ \triangleleft K_x$ $(x \in S)$ s.t. $\chi_x|_{K_x^+} = 1$

K_x / K_x^+ = k-points of a finite dim'l gp L_x.

Ex1. $I_x^+ \triangleleft \underset{K_x}{\overset{}{I_x}} \longrightarrow \mathbb{G}_m(k)$, where $K_x = I_x$

Ex2. $I_\infty^{++} \triangleleft I_\infty^+ \longrightarrow k \oplus k$, where $I_\infty^{++} = K_\infty^+$, $I_\infty^+ = K_\infty$

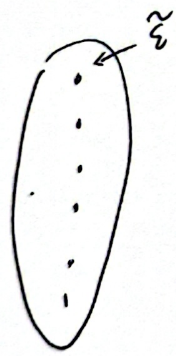
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(b, \frac{c}{t}\right) \bmod t$$

$$C(\mathrm{Bun}_G(K_S^+)(k)) \circlearrowleft \prod_{x \in S} L_x(k)$$

eigenfunctions w. eigenval. $(\chi_x)_{x \in S}$

$= A_c(K_S, \chi_S)$

$\mathrm{Aut}(\mathcal{E})$ $\curvearrowright$



$\subset \mathrm{Bun}_G(K_S^+)$

$\circlearrowleft \prod L_x(k)$

$\mathrm{Aut}(\mathcal{E}) \curvearrowright \underset{\mathcal{E}}{\bullet} \in \mathrm{Bun}_G(K_S)$

$\Rightarrow \mathrm{Aut}(\mathcal{E}) \xrightarrow{ev_{\tilde{\mathcal{E}}}} \prod L_x(k)$

alg. gp — well-defined up to conjugacy.

Def. A k-point $\mathcal{E} \in \mathrm{Bun}_G(K_S)(k)$ is <u>(K_S, χ_S) - relevant</u> if

$$ev_{\mathcal{E}}^{*}\left(\prod_{x\in S} \chi_x\right)\Big|_{\mathrm{Aut}(\mathcal{E})^{\circ}} = 1.$$

Similarly, <u>relevant k'-pts</u>

$\bar{k}$ - pts

Fact: $\dim A_c(k'; K_S, \chi_S) \le \#\ (K'_S, \chi'_S)$- relevant k'-points of $\mathrm{Bun}_G(K_S)$

Cor. (K_S, χ_S) is weakly rigid iff there are finitely many (K_S, χ_S)—rel. $\bar{k}$- points of $\mathrm{Bun}_G(K_S)$.

Ex1. $G = SL_2$

$K_x = I_x, \qquad x \in \{0, 1, \infty\} = S$

$$\mathrm{Bun}_G(K_S)(k) = \left\{\left(V,\ \iota : \wedge^2 V \xrightarrow{\sim} \mathcal{O}_X,\ \{\ell_x \subset V_x\}_{x\in S}\right)\right\}$$

$$I_x \longrightarrow \underset{\parallel}{\overset{L_x}{}}\!\!\mathbb{G}_m, \quad \forall x \in S$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a \bmod t$

$\mathcal{E} = (V, \iota, \ell_0, \ell_1, \ell_\infty)$

$$ev : \mathrm{Aut}(\mathcal{E}) \longrightarrow \prod_{x\in S} \mathbb{G}_m$$

$\gamma : V \to V$, $\gamma_0 \curvearrowright V_0$ preserves ℓ_0. $\rightsquigarrow$ scalar of $\gamma_0 \curvearrowright \ell_0$

$$\left.\prod_{x \in S} \chi_x \right|_{\mathrm{Aut}(\mathcal{E})^\circ} \overset{?}{=} 1$$

$V = \mathcal{O}^2$, $\quad \ell_0, \ell_1, \ell_\infty \subset k^2$ in generic pos.

$\mathrm{Aut}(\mathcal{E}) = \{\pm 1\}$

relevant.

Other pts are irrelevant.

$$V = \overset{\curvearrowleft \lambda}{L} \oplus \overset{\curvearrowleft \lambda^{-1}}{L'} \text{ s.t. } \ell_x \in L_x \text{ or } \ell_x \in L'_x$$

$$\mathbb{G}_m \subset \mathrm{Aut}(V, \cdots) \longrightarrow \prod_{x \in S} \mathbb{G}_m$$

$$\chi_0^{\pm 1} \chi_1^{\pm 1} \chi_\infty^{\pm 1} \neq 1$$

$$\Rightarrow \left.\left(ev_{\mathcal{E}}^* \prod \chi_x\right)\right|_{\mathbb{G}_m} \neq 1.$$

Lecture 3.

Automorphic G	Galois $G^\vee$
$A_c(K_S, \chi_S)$ $\ni$ f eigenform	$\pi_1(X \setminus S) \to G^\vee(\overline{\mathbb{Q}}_\ell)$
rigid auto. data	$\mapsto$ rigid local systems (Katz)

Designing rigid auto. data.

① Numerical rigidity

$\underline{\mathrm{Bun}_G(K_S)}(k)$

alg. stack should have dim. 0.

$\dim \mathrm{Bun}_G(K_S) = 0$

$\Updownarrow$

$$\sum_{x\in S} [G(\mathcal{O}_x) : K_x] = (1-g)\dim G$$

relative dim.

e.g. $K_x = I_x$, $[G(\mathcal{O}_x) : I_x] = \dim \left(G(\mathcal{O}_x)/I_x\right) = \dim (G/B) = \# \Phi^+$

if $K_x \not\subset G(\mathcal{O}_x)$, $\overset{[G(\mathcal{O}_x):K_x]}{=} \dim G(\mathcal{O}_x)/G(\mathcal{O}_x)\cap K_x - \dim K_x/G(\mathcal{O}_x)\cap K_x$

RHS ≥ 0 only when $g = 0, 1$. (very special)

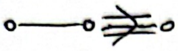
$\Downarrow$ 0 $\qquad K_x \sim G(\mathcal{O}_x)$.

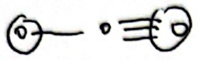
Ex. $S = \{0, 1, \infty\} \subset \mathbb{P}^1$, $K_x =$ parahoric subgp.

$$\sum_{x=0,1,\infty} [G(\mathcal{O}_x) : K_x] = \dim G$$

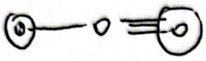
$K_x \twoheadrightarrow L_x =$ reductive qt of K_x.

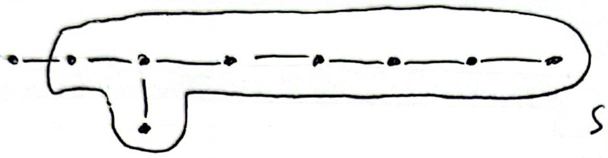
$$[G(\mathcal{O}_x) : K_x] = \frac{\dim G - \dim L_x}{2} \qquad \sum_{x=0,1,\infty} \dim L_x = \dim G.$$

$\underline{\underline{G_2}}$. 

L_0  SO_4 dim 6

L_1 Iwahori T (2dim'l)

L_∞  SO_4 dim 6

$\underline{\underline{E_8}}$ 

$Spin(16)/\{\pm 1\}$

$L_0 = L_\infty$ $\quad Spin(16)/\{\pm 1\}$ $\quad$ dim 120

L_1 $\quad$ Iwahori $\twoheadrightarrow T$ $\quad$ dim 8

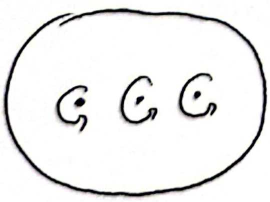
$\Big]$ $248 = \dim E_8$

$$\chi_0 : k_0 \longrightarrow L_0(k) \twoheadrightarrow L_0(k)/Spin(16)(k) = \mathbb{Z}/2\mathbb{Z} \longrightarrow \{\pm 1\}$$

$\chi_1 = 1$, $\chi_\infty = 1$.

$\rightsquigarrow$ rigid auto. datum.

② Matching auto. data w/ local monodromy



$$\begin{array}{l} \text{loc sys} \\ \quad \downarrow \\ \rho_x : \mathrm{Gal}(\overline{F_x} \mid F_x) \longrightarrow G^\vee(\overline{\mathbb{Q}}_\ell) \\ \qquad\qquad \cup \\ \text{inertia} \quad In_x \end{array}$$

Ex. $K_x = I_x \longrightarrow T(k) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$

$\Rightarrow \rho_x$ is tamely ramified

$$I n_x \longrightarrow k_x^\times \underset{\hat{\imath}}{\longrightarrow} \hat{T}(\overline{\mathbb{Q}}_\ell^\times)$$

given by χ

This is $(\rho_x|_{In_x})^{ss}$

Ex. $K_x = I_x^+ \longrightarrow k \xrightarrow{\psi} \mathbb{C}^*$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto b + \frac{c}{t} \bmod t$

$a, d \equiv 1\ (t)$

$c \equiv 0\ (t)$

$\rho_x : \mathrm{Gal}(\overline{F}_x | F_x) \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell)$

wildly ramif.

$\mathrm{Sw}(\rho_x) = 1 = \frac{1}{2} + \frac{1}{2}$

I_x^+

Moy-Prasad filtration on I_x

indexed by $\frac{1}{h}\mathbb{Z}$, h = Coxeter of G $(=2$ if $G = SL_2)$

$I_x = I_x(0) \supset I_x(\tfrac{1}{2}) = I_x^+ \supset I_x(1) \supset I_x(\tfrac{3}{2}) \supset \cdots$

If $K_x \subset P_x(r)$ depth
$\quad\quad\quad\quad \llcorner_{\in \mathbb{Q}}$

$\updownarrow$

Then all slopes of ρ_x are $\leq r$ slopes

<u>local numerical condition</u>.

(K_S, χ_S) rigid. $\rightsquigarrow \rho : \pi_1 \longrightarrow G^\vee$

$$[G(\mathcal{O}_x) : K_x] = \frac{1}{2}\, \underset{\substack{\uparrow \\ \text{Artin conductor}}}{a}(\mathrm{Ad}(\rho_x))$$

$\underline{Ex}$. (epipelagic auto. data)

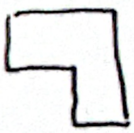
$S = \{0, \infty\}$

$K_0 = P_0$ parahoric , $\chi_0 = 1$

$K_\infty = P_\infty^+ \xrightarrow{*} k \xrightarrow{\psi} \mathbb{C}^\times$

$G = Sp_{2n} = Sp(V)$

Siegel parabolic

 $= P_{sieg}$

$\shortparallel$

Stab. of a Lagrangian $\subset V$

$$\begin{array}{ccc} P_0 & \subset & G(\mathcal{O}_0) \\ \downarrow & & \downarrow \\ P_{sieg} & \subset & G \end{array}$$

$$\begin{array}{ccc} P_\infty & \subset & G(\mathcal{O}_\infty) \\ \downarrow & & \downarrow \\ P_{sieg} & \subset & G \end{array}$$

Lag. $= L \subset V$

$$\begin{tikzcd} P^+_\infty \arrow[d] & = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathcal{O}_\infty) : A, D \equiv I_n \bmod \tau , C \equiv 0 \bmod \tau \right\} \arrow[d, mapsto] \\ W & \ni \left(B \bmod \tau, \frac{C}{\tau} \bmod \tau\right) \end{tikzcd}$$

$\bar{A} \in GL(L)$, $\bar{D} \in GL(L^\vee)$

$\bar{B} : L^\vee \to L$, $\bar{B}^\vee = \bar{B}$

$\frac{C}{\tau} : L \to L^\vee$ self-adjoint

$$\begin{tikzcd} W = \mathrm{Sym}^2(L) \oplus \mathrm{Sym}^2(L^\vee) \arrow[d, "{(S,T)}"'] & (X,Y) \arrow[d, mapsto] \\ k & \ni \mathrm{Tr}(XT) + \mathrm{Tr}(YS) \end{tikzcd}$$

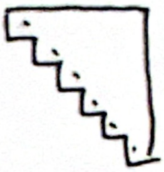
$$P^+_\infty \longrightarrow W \xrightarrow{(S,T)} k \xrightarrow{\psi} \mathbb{C}^*.$$

For "stable" (S,T) we will get rigid auto. datum.

Stable means: $ST \in \mathrm{End}(L)$ has distinct $\neq 0$ eigenval. in $\bar{k}$.

Epipelagic reps of $G(F_\infty)$

(Reeder – J.K. Yu)

Sp_{2n}  equal sized block

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<u>Lecture 4</u>. (K_S, χ_S)

$f \in \mathcal{A}_c(K_S, \chi_S) \circlearrowleft H_{K_x}^{\text{spherical}}$ $x \in |X| \setminus S$

Satake isom. $\{H_{K_x} \longrightarrow \bar{\mathbb{Q}}_\ell\} \longleftrightarrow \{\text{ss conj. class in } \hat{G}(\bar{\mathbb{Q}}_\ell)\}$

$\forall\ x \notin S \rightsquigarrow \quad \sigma_x \in \hat{G}_{ss}(\bar{\mathbb{Q}}_\ell)/\sim$

Langlands corr. $\rightsquigarrow \quad \exists\ \rho : \pi_1(X \setminus S) \to \hat{G}(\bar{\mathbb{Q}}_\ell)$

s.t. $\rho(\mathrm{Fr}_x)^{ss} \sim \sigma_x$

Construct ρ from f.

Geometrize (Drinfeld, Laumon, ...)

$G = GL_n$,

$T_x \in H_{K_x}$

$\mathbb{1}_{K_x \begin{pmatrix} t_x & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} K_x}$

$f : \mathrm{Bun}_G(k) \to \bar{\mathbb{Q}}_\ell$

$$(T_x f)(\mathcal{E}) = \sum_{\substack{\mathcal{E}' \hookrightarrow \mathcal{E} \\ \text{length } 1 \\ \text{at } x}} f(\mathcal{E}')$$

$\cong \mathbb{P}(\mathcal{E}_x)$

$$\begin{tikzcd} & \mathrm{Hk}^1 \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (X \setminus S) \end{tikzcd}$$

$\mathrm{Hk}^1 = \{ \mathcal{E}' \underset{\text{length } 1}{\hookrightarrow} \mathcal{E} \}$

T_x geometrized into $\mathcal{F}$ sheaf on Bun_G, $T_1 \mathcal{F} = h_{2!}(h_1^* \mathcal{F})$: sheaf on $\mathrm{Bun}_G \times (X \setminus S)$

B...

Eigensheaf: $T_x f = \lambda_x f$, $\lambda_x \in \overline{\mathbb{Q}_\ell}$

$$T_1 \mathcal{F} = \mathcal{F} \boxtimes E$$
$\hookrightarrow$ on $X \setminus S$

Goal: compute E.

$$\mathrm{Tr}(\mathrm{Fr}_x, E) = \lambda_x, \quad \forall x \in X\setminus S$$

Ex. (Kloosterman auto. datum)

Gross: constructed Kl auto. datum

showed rigidity (trace formula)

prediction on ρ

(Kloosterman local system)

Heinloth - Ngô - Y.: construct ρ

$G = PGL_n$, $K_0 = I_0$, $\chi_0 = 1$

x_∞: $K_\infty = I_\infty^+$, $\longrightarrow k \xrightarrow{\psi} \overline{\mathbb{Q}_\ell}^\times$

$$\begin{bmatrix} a_{11} & a_{12} & & \\ & \ddots & \ddots & \\ & & \ddots & a_{n-1,n} \\ a_{n1} & & & a_{nn} \end{bmatrix} \longmapsto \underbrace{\sum_{i=1}^{n-1} a_{i,i+1} + \frac{a_{n1}}{\tau}}_{\in k} \mod \tau$$

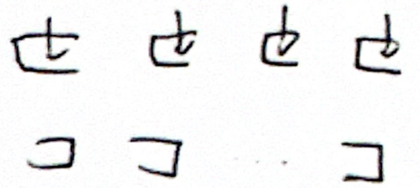
$\mathrm{Bun}_G(K_0, K_\infty) = \{ V$: rk n v.b. on $\mathbb{P}^1$, $F^n \supset F^{n-1} \supset \cdots \supset F^1$ full flag V_0

$0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V_\infty$, e_i a basis of F_i/F_{i-1} $\} / \mathrm{Pic}$

$\exists!$ relevant pt on $\underset{\deg V \bmod n}{\underline{\text{each comp.}}}$ of $\mathrm{Bun}_G(K_0, K_\infty)$.

$\mathcal{E}_0$: $\mathcal{O}^n = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_n$

at 0, F^*

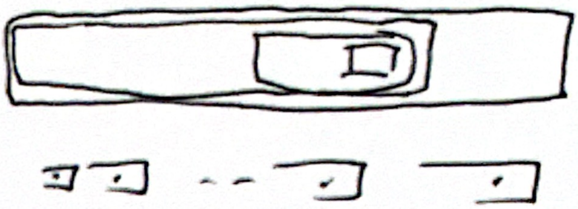


at ∞ F_*

$\mathrm{Aut}(\mathcal{E}_0) = 1$

$\mathcal{E}_1$: $\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_{n-1} \oplus \mathcal{O}(1)e_n$

at 0, F^*



at ∞, F_*

$\mathrm{Hk} \supset \{ \mathcal{E}_0 \xhookrightarrow{\varphi} \mathcal{E}_1$ preserving $F^*, F_*, \cdots \}$

$\mathrm{ev}_\infty \swarrow \qquad \searrow \pi$

$\mathbb{A}^n \cong I^+_\infty / I^{++}_\infty$ $\qquad (-)^*$ $\qquad$ curve $\pi_!$ $\qquad \mathbb{P}^1 \setminus \{0,\infty\}$

E $\qquad$ supp (coker (φ))

$\downarrow$ sum

$\mathbb{A}^1 \qquad AS_\psi$

$$\begin{array}{ccc} & (\mathbb{G}_m)^n & \\ \sigma = \text{sum} \swarrow & & \searrow \pi = \text{prod.} \\ \mathbb{A}^1 & & \mathbb{G}_m \end{array}$$

$E = R^{n-1}\pi_! \, \sigma^* AS_\psi$ is a rk n local system

$=$ Deligne's Kloosterman sheaf

General G

H_{K_x} has a Kazhdan–Lusztig basis C_λ

$$\lambda \in X_*(T)^+ \longleftrightarrow \text{irr. rep. of } \hat{G}$$

T_x for $GL_n \rightsquigarrow$ std rep. of $\hat{G} = GL_n$

$$\begin{tikzcd}
 & \mathrm{Hk}_\lambda \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_G(K_S) & & \mathrm{Bun}_G(K_S) \times (X \setminus S)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, bend left, mapsto, "\otimes IC_\lambda"] & T_\lambda \mathcal{F}
\end{tikzcd}$$

$\mathcal{F}$ eigensheaf

$$T_\lambda \mathcal{F} = \mathcal{F} \boxtimes \underbrace{E_\lambda}_{\hookleftarrow X \setminus S}$$

Fact: $\lambda \longmapsto E_\lambda$ comes from a $\hat{G}$-loc. sys. on $X \setminus S$

$$\begin{tikzcd}
\rho : \pi_1(X \setminus S) \arrow[r] \arrow[dr, "E_\lambda"'] & \hat{G} \arrow[d] \\
 & GL(V_\lambda)
\end{tikzcd}$$

Applications. (K_S, x_S) "tame" auto. datum

$(K_x$ parahoric$)$

makes sense over any k. Can construct $\hat{G}$-local systems on $\mathbb{P}^1_k \setminus S$.

$\rightsquigarrow$ E_8 local system on $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$

<u>motivic</u>.

$\rightsquigarrow$ inverse Galois problem

$$\mathrm{Gal}(K|\mathbb{Q}) \cong E_8(\mathbb{F}_\ell), \quad \ell \gg 0$$

"rigidity method"

Open problems:

- <u>Classification</u> of rigid auto. data

$G = GL_n$ $\quad\quad\quad\updownarrow$

rigid local systems

$\quad\quad\quad\quad$ ↖ Katz, Arinkin

- <u>Checking rigidity</u>

- Weak rigidity case.

$$\dim \mathcal{A}_C(K_S, \chi_S) > 1$$