Perverse schobers and semi-orthogonal decompositions

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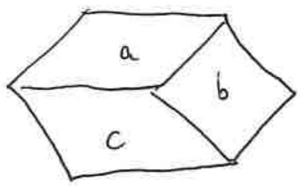
VV: "Higher dim Algebra"



categories (triangulated) — of motives; Fukaya cats

vector spaces — $H^\bullet$, $H_\bullet$

Levels of objects: numbers — X

Perverse sheaves

Perv (X, S) abelian cat. w/ duality $*$

X: $\mathbb{C}$-mfd, S: $\mathbb{C}$-strat.

$\cap$

$D^b_{\text{const}}(X, S)$ ← of k-vector spaces

? of triang. cats?

sheaves of cat: OK (stacks) ∞ — of complexes?

Ex. $\text{Perv}(\mathbb{C}, 0) \simeq \left\{ \phi \underset{b}{\overset{a}{\rightleftarrows}} \psi \ : \ \begin{matrix} 1-ab \\ 1-ba \end{matrix} \text{ invertible} \right\}$

ϕ: vanishing cycles; ψ: nearby cycles categorifies to

<u>Spherical functors</u> (R. Anno, T. Logvinenko)

$$\mathcal{B} \underset{f^*}{\overset{f}{\rightleftarrows}} \mathcal{C} \qquad \text{triang. w/ (dg) enhancement}$$

right adjoint

$$\mathrm{Cone}\left(f \circ f^* \xrightarrow{\epsilon} Id_{\mathcal{C}}\right) : \mathcal{C} \to \mathcal{C}$$

$$/, \ \mathrm{Cone}\left(Id_{\mathcal{B}} \xrightarrow{e} f^* \circ f\right) : \mathcal{B} \to \mathcal{B}$$

equivalences

Ex. $D^b Vect_k \xrightarrow{f} \mathcal{C}$

$k \longmapsto E \in \mathcal{C}$ called n-spherical if $Ext^j(E, E) = \begin{cases} k, & j = 0, n \\ 0, & \text{otherwise} \end{cases}$

$\Downarrow$

then f spherical

Perverse schober $\overset{?}{=}$ categorified perv. sheaf

On a stratified Riem. surf. (X, A)

A: finite subset

Naively:

- A "local sys" of cat. on $X - A$
- a spherical functor near $\forall x \in A$ (glued)

Localization on a skeleton

Prop. $\mathcal{F} \in D^b_{const}(X) \in Perv$

$\Leftrightarrow \forall$ graph $K \underset{i}{\subset} X$, $\underline{\mathbb{H}}^j_K(\mathcal{F}) = 0$, $j \neq 0$

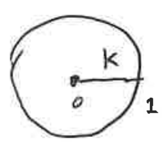
$R_K(\mathcal{F}) := i^! \mathcal{F} \simeq \underline{\mathbb{H}}^0_K(\mathcal{F})$ single sheaf

$R_K : Perv(X) \longrightarrow Sh_K$ exact functor of abelian cat.

Kontsevich : localization of Fukaya cat. on Lagrangian skeleton K

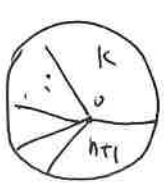
? gives a sheaf of cat. on K.

On $X =$ disk.



$K =$ diameter

$R_K(\mathcal{F})_0 = \Phi \to R_K(\mathcal{F})_1 = \Psi$



$R_K(\mathcal{F})_0 = \Phi_{z^{n+1}}(\mathcal{F})$

$\simeq \Psi^{\oplus n} \oplus \Phi$

$\circlearrowleft$

$\sqrt[n+1]{T}$

Want: cat. analog of this.

(Para) cyclic symmetry

$\Psi \xrightarrow{b} \Phi$ any linear map

$[\Psi \longrightarrow \Phi]$ Picard cat.

$\Psi \times \Phi \rightrightarrows \Phi = Ob$

Mor

$N_\bullet[\Psi \to \Phi]$ nerve

simplicial vect. space

$$N_n = \Psi^{\oplus n} \oplus \Phi$$

$$\Delta \longrightarrow Vect$$

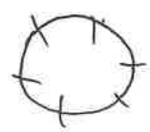
$[n], \partial_i, s_i$

(T_n^{n+1}) central.

T_n - isom.

para-cyclic $\qquad\qquad$ cyclic

$$\Lambda_\infty \longrightarrow \Lambda$$



$T_n^{n+1} = 1$

$[n] \circlearrowleft T_n$ $\quad$ (Connes)

Prop. Extending simpl. str. on $N[\Psi \xrightarrow{b} \Phi]$ $\longleftrightarrow$ maps $a : \Phi \to \Psi$ s.t. $\begin{matrix} 1-ab \\ 1-ba \end{matrix}$ inv.

I.e. $Perv(\mathbb{C}, 0) \simeq$ Paracyclic vec. sp. which as simplicial ones, are nerves of sth.

(Segal condition)

Categorifies to: Waldhausen S-construction

$\mathcal{B}$: tr. cat. w/ enhancement $\left\langle \begin{matrix} \text{dg} \\ \text{stable } \infty\text{-cat.} \end{matrix} \right.$

$S_\bullet(\mathcal{B})$ = "$K(\mathcal{B}, 1)$" simplicial dg-cat.

$S_n(\mathcal{B}) = \{$ diagrams of B_{ij}, $0 \le i < j \le n$ $\}$
famous

$\mathrm{Mor}[n] \longrightarrow \mathcal{B}$ s.t.

$\forall\ i<j<k$, $B_{ij} \longrightarrow B_{ik} \longrightarrow B_{jk}$ is "exact"

$S_n(\mathcal{B}) \xrightarrow{\sim} \{ B_{01} \longrightarrow B_{02} \longrightarrow \cdots \longrightarrow B_{0,n} \}$

size $\simeq \mathcal{B}^n$

Semiorthogonal decomp. $\langle \mathcal{B}, \mathcal{B}, \cdots, \mathcal{B} \rangle$.

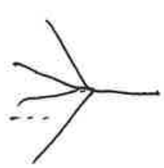
$f: \mathcal{B} \longrightarrow \mathcal{C}$ dg functor

$$\begin{tikzcd} S_n(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S_{n+1}(\mathcal{C}) \arrow[d, "\partial_{n+1}"] \\ S_n(\mathcal{B}) \arrow[r, "f_*"'] & S_n(\mathcal{C}) \end{tikzcd}$$

size $\simeq \mathcal{B}^n \times \mathcal{C}$

$\langle \mathcal{B}, \cdots, \mathcal{B}, \mathcal{C} \rangle$

Thm $S_n(f)$ can be made into a paracyclic object $\Leftrightarrow$ f is spherical.



Local model for $\mathcal{R}_k$ (a schober assoc. to f) (cf. MacPherson, Fary Sheaves)

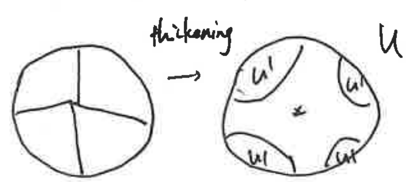
Perv. schobers as data on Milnor pairs

$X \supset U \supset U'$ // open disk, $\sqcup$ disks

$U' \simeq \{ \mathrm{Re}\, f(z) > \varepsilon \}$

$U - U'$ contractible $\Downarrow$ ≤ 1 elt of A

(X, A)



$\mathcal{F} \in \mathrm{Perv}(X, A)$

Observe: $\mathbb{H}^{\neq 0}(U, U', \mathcal{F}) = 0$

$E(U, U') := \mathbb{H}^0(U, U', \mathcal{F})$

$$\mathrm{Vect} \xleftarrow{E} (\mathrm{Mil}(X, A), \leq)^{op}$$

not Grothendieck topology

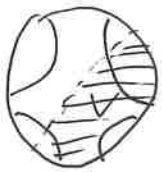
Prop. $\mathrm{Perv}(X, A) \simeq$ cat. of E satisfying

(1) Homotopy invariance Rel A

(2) Normalize

$E(U, U') = 0$

(3) Exactness



$0 \to E(U, U' \cup V) \to E(U, V) \to E(V, U' \cap V) \to 0$ exact

Def. A perv. schober on (X, A) is

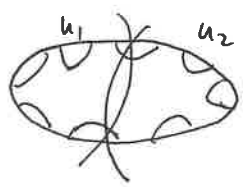
∞- functor $\mathfrak{S}: \mathrm{Mil}(X, A)^{op} \longrightarrow \mathrm{Cat}^{\infty}_{St}$ satisfying

(1), (2),

(3) Replaced by recollement (SOD)

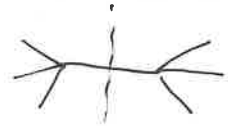
(with adjoints prescribed)

(4) Descent for



$(U, U'), (U_1, U_1 \cap U')$

$(U_2, U_2 \cap U''), (U_{12}, U_{12} \cap U'')$

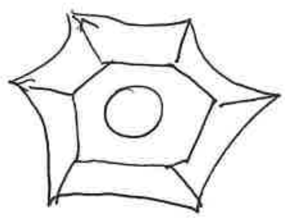


Thm 1) For $(\mathbb{C}, 0)$, this $\longleftrightarrow$ Sph. functors

2) $\forall$ graph $K \subset X$, such $\mathcal{F} \rightsquigarrow$ sheaf of (∞)-cat $R_K(\mathcal{F})$ on K

3) If K <u>spans</u> X ∂X possibly corners

then $\Gamma(K, R_K(\mathcal{F}))$ $\overset{\text{holim}}{\underset{\longleftarrow}{=}}$ is coherently indep. on K

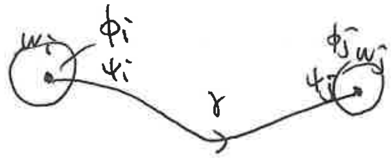


$\mathbb{F}(X, \mathcal{F})$ topological Fukaya cat.

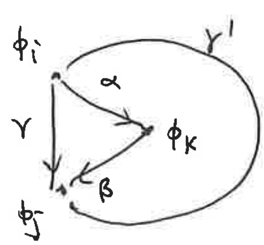
$\mathbb{H}^0(X, \partial X\text{-corner}, \mathcal{F})$

(Categorified) Picard-Lefschetz

$\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$
$A = \{w_1, \cdots, w_n\}$



$M_{ij}(\gamma)$ $\phi_i \to \psi_i \to \psi_j \to \phi_j$



$M_{ik}(\gamma')$
$= M_{ik}(\gamma) + M_{jk}(\beta) \cdot M_{ij}(\alpha)$

<u>Classical PL</u> for $\mathcal{F}$ assoc. to a Lefschetz pencil.

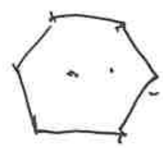
For $\mathfrak{F} \in \mathrm{Schob}(\mathbb{C}, A)$

Φ_i categ. $M_{ij}(\gamma)$ functors

PL exact triangles

"Algebra of infrared" Gaiotto - Moore - Witten

$$W : Y \longrightarrow \mathbb{C}$$
$$\cup$$
$$A$$ crit. values



convex hull of A

GMW: L_∞-algebra

acting by deformation on A_∞-alg.

joint w/ Soibelman (in progress)

$\forall\ \mathfrak{F}$, we have a similar formalism. & get by deformation

$\mathbb{F}(D, \mathfrak{F})$