

Gaitsgory's central functor

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Abstract

This is the note for a seminar talk in Tsinghua. My task is to introduce Gaitsgory's central functor.

1 Introduction

Let G be a connected reductive group, defined over \mathbb{Z} . For simplicity, let us assume that G is split. We fix standard notations B, T, N etc.

Temporarily, set $\mathcal{K} = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$. Let I be the Iwahori in $G(\mathcal{O})$. We have the affine Hecke algebra

$$\mathcal{H}^{\text{aff}} = (C_c(I \backslash G(\mathcal{K})/I), *) = C_{I,c}(\mathbf{Fl})$$

and the spherical affine Hecke algebra

$$\mathcal{H}^{\text{sph}} = (C_c(G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O})), *) = C_{G(\mathcal{O}),c}(\mathbf{Gr}).$$

Here $\mathbf{Fl} = G(\mathcal{K})/I$ and $\mathbf{Gr} = G(\mathcal{K})/G(\mathcal{O})$.

By integration along $G(\mathcal{O})/I$, I get a map

$$\mathcal{H}^{\text{aff}} = C_{I,c}(\mathbf{Fl}) \rightarrow C_{G(\mathcal{O}),c}(\mathbf{Fl}).$$

I also have a map

$$\mathcal{H}^{\text{sph}} = C_{G(\mathcal{O}),c}(\mathbf{Gr}) \rightarrow C_{G(\mathcal{O}),c}(\mathbf{Fl})$$

via pull-back.

Theorem 1.1 (Bernstein). The image of $Z(\mathcal{H}^{\text{aff}})$ and \mathcal{H}^{sph} in $C_{G(\mathcal{O}),c}(\mathbf{Fl})$ agree, and we have an isomorphism

$$Z(\mathcal{H}^{\text{aff}}) \simeq \mathcal{H}^{\text{sph}}.$$

From now on, let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. By the fonctions-faisceaux correspondence, a natural categorification of \mathcal{H}^{sph} is the Satake category $\text{Perv}_{G(\mathcal{O})}(\mathbf{Gr})$, while the affine Hecke category $\text{Perv}_I(\mathbf{Fl})$ is a categorification of \mathcal{H}^{aff} . Gaitsgory's central functor is a categorification of Bernstein's theorem above.

Theorem 1.2 (Gaitsgory). There is a functor

$$Z: \text{Perv}_{G(\mathcal{O})}(\mathbf{Gr}) \rightarrow \text{Perv}_I(\mathbf{Fl})$$

such that

1. For any $\mathcal{G} \in \text{Perv}_{G(\mathcal{O})}(\mathbf{Gr})$ and $\mathcal{F} \in \text{Perv}(\mathbf{Fl})$, the convolution $\mathcal{F} *_{\mathbf{Fl}} Z(\mathcal{G})$ is a perverse sheaf.
2. For any $\mathcal{G} \in \text{Perv}_{G(\mathcal{O})}(\mathbf{Gr})$ and $\mathcal{F} \in \text{Perv}_I(\mathbf{Fl})$, there is a canonical isomorphism

$$Z(\mathcal{G}) *_{\mathbf{Fl}} \mathcal{F} \simeq \mathcal{F} *_{\mathbf{Fl}} Z(\mathcal{G}).$$

3. $Z(\delta_{1_{\mathbf{Gr}}}) = \delta_{1_{\mathbf{Fl}}}$.

4. For any $\mathcal{G}^1, \mathcal{G}^2 \in \text{Perv}_{G(\mathcal{O})}(\mathbf{Gr})$, there is a canonical isomorphism

$$Z(\mathcal{G}^1) *_{\mathbf{Fl}} Z(\mathcal{G}^2) \simeq Z(\mathcal{G}^1 *_{\mathbf{Gr}} \mathcal{G}^2).$$

5. For any $\mathcal{G} \in \text{Perv}_{G(\mathcal{O})}(\mathbf{Gr})$, we have $\pi_!(Z(\mathcal{G})) \simeq \mathcal{G}$. Here π is the projection $\pi: \mathbf{Fl} \rightarrow \mathbf{Gr}$.

Here canonicity or naturality means certain higher compatibility isomorphisms.

2 Principal bundles

Moduli problems of principal G -bundles are ubiquitous in the study of affine grassmannians and affine flag varieties, so I feel like it's part of my duty to clarify what do we mean by principal bundles.

2.1 Grothendieck topologies

Let k be a commutative ring, $k\text{-Alg}$ the category of k -algebras. We know that $k\text{-Alg}^{\text{op}}$ is equivalent to the category of affine k -schemes, so a presheaf of set on the category affine k -schemes is the same as a functor $k\text{-Alg} \rightarrow \text{Set}$. Similarly, I have the category of presheaves of groups $\text{Fun}(k\text{-Alg}, \text{Grp})$, the category of presheaves of abelian groups $\text{Fun}(k\text{-Alg}, \text{Ab})$, etc.

Recall that for a topological space X , a sheaf on X is a presheaf on X satisfying the sheaf axiom, namely certain gluing property. More precisely, suppose $\bigcup_{i \in I} U_i$ is an open covering of some open subset U , we require the diagram

$$F(U) \rightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i, j \in I} F(U_i \cap U_j)$$

to be an equalizer. By specifying a collection of “open coverings” of objects $\text{Spec}(R) \in k\text{-Alg}^{\text{op}}$, I can define certain Grothendieck topology on $k\text{-Alg}^{\text{op}}$.

Definition 2.1 (fpqc topology). A collection

$$\{\text{Spec}(S_i) \rightarrow \text{Spec}(R)\}_{i \in I}$$

is an open covering of $\text{Spec}(R)$ in the *fpqc topology*, if

1. I is finite;
2. each $R \rightarrow S_i$ is flat;
3. $R \rightarrow \prod_{i \in I} S_i$ is faithfully flat.

Definition 2.2 (fppf topology). A collection

$$\{\text{Spec}(S_i) \rightarrow \text{Spec}(R)\}_{i \in I}$$

is an open covering of $\text{Spec}(R)$ in the *fppf topology*, if it is an open covering in the fpqc topology and each S_i is finitely presented over R .

Definition 2.3 (étale topology). A collection

$$\{\text{Spec}(S_i) \rightarrow \text{Spec}(R)\}_{i \in I}$$

is an open covering of $\text{Spec}(R)$ in the *étale topology*, if it is an open covering in the fppf topology and each S_i is étale over R .

Definition 2.4 (Zariski topology). A collection

$$\{\text{Spec}(S_i) \rightarrow \text{Spec}(R)\}_{i \in I}$$

is an open covering of $\text{Spec}(R)$ in the *Zariski topology*, if it is an open covering in the étale topology and each S_i is of the form R_f for some $f \in R$.

Now let $\tau \in \{\text{fpqc}, \text{fppf}, \text{ét}, \text{Zar}\}$ be one of these Grothendieck topologies.

Definition 2.5. A presheaf $F \in \text{Fun}(k\text{-Alg}, \text{Set})$ is a τ -sheaf, if for any τ -cover

$$\{\text{Spec}(S_i) \rightarrow \text{Spec}(R)\}_{i \in I},$$

the diagram

$$F(R) \rightarrow \prod_{i \in I} F(S_i) \rightrightarrows \prod_{i, j \in I} F(S_i \otimes_R S_j)$$

is an equalizer.

Let $\text{Shv}_\tau(k\text{-Alg}^{\text{op}})$ be the category of τ -sheaves on $k\text{-Alg}^{\text{op}}$.

By construction, we have

$$\mathrm{Shv}_{\mathrm{fpqc}}(k\text{-}\mathbf{Alg}^{\mathrm{op}}) \subset \mathrm{Shv}_{\mathrm{fppf}}(k\text{-}\mathbf{Alg}^{\mathrm{op}}) \subset \mathrm{Shv}_{\mathrm{\acute{e}t}}(k\text{-}\mathbf{Alg}^{\mathrm{op}}) \subset \mathrm{Shv}_{\mathrm{Zar}}(k\text{-}\mathbf{Alg}^{\mathrm{op}}).$$

Remark 2.1. From this point of view, Grothendieck's faithfully flat descent theorem tells us the presheaf $R \mapsto R$ is a fpqc-sheaf.

Remark 2.2. For any $\tau \in \{\mathrm{fppf}, \acute{e}t, \mathrm{Zar}\}$, the inclusion $\mathrm{Shv}_{\tau}(k\text{-}\mathbf{Alg}^{\mathrm{op}}) \hookrightarrow \mathrm{Fun}(k\text{-}\mathbf{Alg}, \mathrm{Set})$ has a left adjoint, the τ -sheafification. The fpqc-sheafification is more subtle for some set-theoretic issues. I will ignore these obstacles by just avoiding talking about fpqc-sheafification.

Remark 2.3. I can perform the same constructions over any base scheme S .

2.2 Yoneda embedding

By the Yoneda embedding, we have a faithful embedding

$$k\text{-}\mathbf{Alg}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(k\text{-}\mathbf{Alg}, \mathrm{Set}), \mathrm{Spec}(R) \mapsto [S \mapsto \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, S)].$$

For any $\tau \in \{\mathrm{fpqc}, \mathrm{fppf}, \acute{e}t, \mathrm{Zar}\}$, the image of this embedding lies in $\mathrm{Shv}_{\tau}(k\text{-}\mathbf{Alg}^{\mathrm{op}})$, the category of τ -sheaves, by Grothendieck's faithfully flat descent. Moreover, we have

Proposition 2.1. There is a faithful embedding

$$h: k\text{-}\mathbf{Sch} \hookrightarrow \mathrm{Shv}_{\tau}(k\text{-}\mathbf{Alg}^{\mathrm{op}}), X \mapsto [h_X: \mathrm{Spec}(S) \mapsto \mathrm{Hom}_{k\text{-}\mathbf{Sch}}(\mathrm{Spec}(S), X) = X(S)]$$

from the category of k -schemes to the category of τ -sheaves, for any $\tau \in \{\mathrm{fpqc}, \mathrm{fppf}, \acute{e}t, \mathrm{Zar}\}$.

Intuitively, this is an embedding because k -schemes are glued from Zariski open affine k -schemes.

A τ -sheaf in the essential image of the inclusion h is said to be representable in schemes. I do not distinguish a τ -sheaf representable in schemes with the representing scheme.

Since Set is cocomplete, I can talk about presheaves representable in ind-schemes. I also don't distinguish an ind-scheme with the presheaf it represents.

2.3 Principal bundles

Choose a Grothendieck topology $\tau \in \{\mathrm{fpqc}, \mathrm{fppf}, \acute{e}t, \mathrm{Zar}\}$ for $k\text{-}\mathbf{Alg}^{\mathrm{op}}$, and let \mathcal{G} be a τ -sheaf of groups.

Definition 2.6 (torsor). By a \mathcal{G} -torsor, I mean a τ -sheaf \mathcal{P} , endowed with a right action of \mathcal{G} (i.e. a morphism $\mathcal{P} \times \mathcal{G} \rightarrow \mathcal{P}$ satisfying the usual axioms), such that

1. for any $\mathrm{Spec}(R) \in k\text{-}\mathbf{Alg}^{\mathrm{op}}$, there exists a τ -cover $\{\mathrm{Spec}(S_i) \rightarrow \mathrm{Spec}(R)\}_{i \in I}$ such that $\mathcal{G}(S_i) \neq \emptyset$ for any $i \in I$;
2. the map

$$\mathcal{P} \times \mathcal{G} \rightarrow \mathcal{P} \times \mathcal{P}, (x, g) \mapsto (x, xg)$$

is an isomorphism of τ -sheaves.

Suppose now $\mathcal{G} = h_G$ is represented by a k -group scheme G .

Definition 2.7. By a *principal G -bundle*, I mean a k -scheme X endowed with a right action of G , such that

1. the morphism of schemes

$$X \times_k G \rightarrow X \times_k X, (x, g) \mapsto (x, xg)$$

is an isomorphism;

2. there exists a fpqc covering $\{\mathrm{Spec}(S_i) \rightarrow \mathrm{Spec}(k)\}_{i \in I}$ such that each $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(S_i)$ is isomorphic, as a G -scheme, to $G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(S_i)$.

By definition, any principal G -bundle is fpqc locally trivial. I say that this principal G -bundle is τ -locally trivial, if the covering can be chosen to be a τ -covering.

Let X be a k -scheme. By the fully faithfulness of the Yoneda embedding, the datum of a right h_G -action on h_X is equivalent to the datum of a right G -action on X . Moreover, X is a τ -locally trivial principal G -bundle if and only if the τ -sheaf h_X is an h_G -torsor.

Suppose G is smooth, then any principal G -bundle X is smooth since the property of being smooth is fpqc on the base. Surjective smooth morphisms admit sections étale locally, so now X is automatically étale locally trivial.

Suppose G is affine, then any h_G -torsor in the fpqc, fppf or étale topology is representable by a principal G -bundle. The basic idea is to use affine descent. Let \mathcal{P} be an h_G -torsor, $\{\mathrm{Spec}(S_i) \rightarrow \mathrm{Spec}(k)\}_{i \in I}$ be a covering (for the corresponding Grothendieck topology) over which \mathcal{P} is trivial. The restriction of \mathcal{P} to each $\mathrm{Spec}(S_i)$ is representable by a scheme P_i (noncanonically isomorphic to $G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(S_i)$). Each P_i is affine over $\mathrm{Spec}(S_i)$ because the property of being affine is fpqc on the base. These schemes P_i are naturally endowed with a descent datum relative to the covering $\{\mathrm{Spec}(S_i) \rightarrow \mathrm{Spec}(k)\}_{i \in I}$, which is effective by affine descent. So we can “glue” these P_i ’s to obtain a scheme P representing \mathcal{P} .

From above discussions, we know that the following notions coincide when G is smooth and affine:

- h_G -torsors for the fpqc topology;
- h_G -torsors for the fppf topology;
- h_G -torsors for the étale topology;
- principal G -bundles;
- fppf locally trivial principal G -bundles;
- étale locally trivial principal G -bundles.

From now on, I only consider the case in which G is smooth and affine, and I only use the term principal G -bundles.

Due to some mental block, I will set $k = \mathbb{C}$ to be the field of complex numbers.

Recall that $G_{\mathcal{K}}$ is the presheaf

$$G_{\mathcal{K}}: R \mapsto G(R((t)))$$

and $G_{\mathcal{O}}$ is the presheaf

$$G_{\mathcal{O}}: R \mapsto G(R[[t]]).$$

The affine grassmannian \mathbf{Gr} is defined to be the fppf sheafification of $G_{\mathcal{K}}/G_{\mathcal{O}}$. We know that $G_{\mathcal{O}}$ is representable in schemes, and $G_{\mathcal{K}}, \mathbf{Gr}$ are representable in ind-schemes.

Similarly, I define I to be the presheaf

$$I: R \mapsto \mathrm{ev}^{-1}(B(R)), \mathrm{ev}: G(R[[t]]) \rightarrow G(R),$$

\mathbf{Fl} to be the fppf sheafification of $G_{\mathcal{K}}/I$. It is known that \mathbf{Fl} is represented by an ind-scheme, called the *affine flag variety*.

3 Convolution product

Last time, Liangshi draw the fundamental convolution diagram

$$\mathbf{Gr} \times \mathbf{Gr} \xleftarrow{p} G_{\mathcal{K}} \times \mathbf{Gr} \xrightarrow{q} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathbf{Gr} \xrightarrow{m} \mathbf{Gr}$$

For $\mathcal{G}^1 \in \mathrm{Perv}(\mathbf{Gr}), \mathcal{G}^2 \in \mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, $p^*(\mathcal{G}^1 \boxtimes \mathcal{G}^2)$ is a $G_{\mathcal{O}}$ -equivariant perverse sheaf on $G_{\mathcal{K}} \times \mathbf{Gr}$, and hence descends to a perverse sheaf $\mathcal{G}^1 \tilde{\boxtimes} \mathcal{G}^2 \in \mathrm{Perv}(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathbf{Gr})$.

The convolution product is defined by

$$- *_{\mathbf{Gr}} -: \mathrm{Perv}(\mathbf{Gr}) \times \mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \rightarrow \mathrm{D}_c^b(\mathbf{Gr}), (\mathcal{G}^1, \mathcal{G}^2) \mapsto \mathcal{G}^1 *_{\mathbf{Gr}} \mathcal{G}^2 = m_!(\mathcal{G}^1 \tilde{\boxtimes} \mathcal{G}^2).$$

Miraculously, the convolution of two $G_{\mathcal{O}}$ -equivariant perverse sheaves is also perverse (and obviously $G_{\mathcal{O}}$ -equivariant). Moreover, the convolution product $*_{\mathbf{Gr}}$ on $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$ comes with a natural commutativity constraint, making $(\mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}), *_{\mathbf{Gr}})$ a symmetric monoidal category.

Similarly, I can draw the fundamental convolution diagram for the affine flag variety

$$\mathbf{Fl} \times \mathbf{Fl} \xleftarrow{p} G_{\mathcal{K}} \times \mathbf{Fl} \xrightarrow{q} G_{\mathcal{K}} \times^I \mathbf{Fl} \xrightarrow{m} \mathbf{Fl}$$

For $\mathcal{F}^1 \in \mathrm{Perv}(\mathbf{Fl})$, $\mathcal{F}^2 \in \mathrm{Perv}_I(\mathbf{Fl})$, $p^*(\mathcal{F}^1 \boxtimes \mathcal{F}^2)$ is I -equivariant, and hence descends to a perverse sheaf $\mathcal{F}^1 \widetilde{\boxtimes} \mathcal{F}^2 \in \mathrm{Perv}(G_{\mathcal{K}} \times^I \mathbf{Fl})$.

The convolution product is defined by

$$- *_{\mathbf{Fl}} - : \mathrm{Perv}(\mathbf{Fl}) \times \mathrm{Perv}_I(\mathbf{Fl}) \rightarrow \mathrm{D}_c^b(\mathbf{Fl}), (\mathcal{F}^1, \mathcal{F}^2) \mapsto \mathcal{F}^1 *_{\mathbf{Fl}} \mathcal{F}^2 = m_!(\mathcal{F}^1 \widetilde{\boxtimes} \mathcal{F}^2).$$

It restricts to a map

$$- *_{\mathbf{Fl}} - : \mathrm{Perv}_I(\mathbf{Fl}) \times \mathrm{Perv}_I(\mathbf{Fl}) \rightarrow \mathrm{D}_I^b(\mathbf{Fl}),$$

but the image does not lie in the heart $\mathrm{Perv}_I(\mathbf{Fl})$ in general.

4 Constructions

By finding a moduli interpretation of (ind) schemes arising before, I can study the global/factorisation analogue of these objects, with the aid of Beauville–Laszlo’s theorem.

4.1 Moduli interpretation

Let $D = \mathrm{Spec}(\mathcal{O})$, $D^* = \mathrm{Spec}(\mathcal{K})$. For a \mathbb{C} -algebra R , let $D_R = \mathrm{Spec}(R[[t]])$, $D_R^* = \mathrm{Spec}(R((t)))$. CAVEAT: $R[[t]] \neq R \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ and $R((t)) \neq R \otimes_{\mathbb{C}} \mathbb{C}((t))$ in general.

For a scheme X , let $\mathcal{E}_X^0 = X \times_{\mathbb{C}} G$ be the trivial principal G -bundle on X . Very often, I omit the subscript X for the sake of brevity.

Recall the moduli interpretation of (the presheaf represented by) \mathbf{Gr} :

$$\mathbf{Gr}(R) = \{(\mathcal{E}, \beta) : \mathcal{E} \text{ a principal } G\text{-bundle on } D_R, \beta : \mathcal{E}|_{D_R^*} \simeq \mathcal{E}_{D_R^*}^0 \text{ a trivialisation}\}.$$

Similarly, I have a moduli description of \mathbf{Fl} :

$$\mathbf{Fl}(R) = \{(\mathcal{E}, \beta, \epsilon) : (\mathcal{E}, \beta) \in \mathbf{Gr}(R), \epsilon \text{ a reduction of } \mathcal{E} \text{ to } B \text{ over } \mathrm{Spec}(R)\}.$$

Clearly, I have a natural projection $\pi : \mathbf{Fl} \rightarrow \mathbf{Gr}$ by forgetting ϵ . The fiber at 1 is G/B .

Let X be a pointed smooth geometrically connected curve. I have the global version of the affine grassmannian:

$$\mathbf{Gr}_X(R) = \{(y, \mathcal{E}, \beta) : y \in X(R), \mathcal{E} \text{ a principal } G\text{-bundle on } X(R), \beta : \mathcal{E}|_{(X \setminus y)(R)} \simeq \mathcal{E}_{(X \setminus y)(R)}^0 \text{ a trivialisation}\}.$$

I have a natural projection $\mathbf{Gr}_X \rightarrow X$. By Beauville–Laszlo’s theorem, the fiber $\mathbf{Gr}_{X,y} \simeq \mathbf{Gr}$ for any $y \in X$.

Now fix a closed point $x \in X$, I have the global version of the affine flag variety:

$$\mathbf{Fl}_{(X,x)}(R) = \{(y, \mathcal{E}, \beta, \epsilon) : (y, \mathcal{E}, \beta) \in \mathbf{Gr}_X(R), \epsilon \text{ a reduction of } \mathcal{E}_{x(R)} \text{ to } B\}.$$

I have a natural projection $\mathbf{Fl}_{(X,x)} \rightarrow X$. By Beauville–Laszlo’s theorem,

$$\mathbf{Fl}_{(X,x)}|_{X \setminus x} \simeq \mathbf{Gr}_X|_{X \setminus x} \times G/B, \mathbf{Fl}_{(X,x),x} \simeq \mathbf{Fl}.$$

4.2 Construction of the functor

Set $(X, x) = (\mathbb{A}^1, 0)$. Now we view the projection $\mathbf{Fl}_{(\mathbb{A}^1, 0)} \rightarrow \mathbb{A}^1$ as a regular function on the global affine flag variety $\mathbf{Fl}_{(\mathbb{A}^1, 0)}$. Associated is the nearby cycle functor

$$\Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}} : D_c^b(\mathbf{Fl}_{(\mathbb{A}^1, 0)}|_{\mathbb{G}_m}) \rightarrow D_c^b(\mathbf{Fl}_{(\mathbb{A}^1, 0), 0})$$

who has the virtue that

$$\Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}(\mathrm{Perv}(\mathbf{Fl}_{(\mathbb{A}^1, 0)}|_{\mathbb{G}_m})) \subset \mathrm{Perv}(\mathbf{Fl}_{(\mathbb{A}^1, 0), 0}).$$

Last time, Liangshi explained that any $G_{\mathcal{O}}$ -equivariant perverse sheaf \mathcal{G} on \mathbf{Gr} can be spread out to a $G_{\mathbb{A}^1, \mathcal{O}}$ -equivariant perverse sheaf $\mathcal{G}_{\mathbb{A}^1}$ on $\mathbf{Gr}_{\mathbb{A}^1}$. He used the global coordinate on \mathbb{A}^1 which enables him to make an identification $\mathbf{Gr}_{\mathbb{A}^1} \simeq \mathbf{Gr} \times \mathbb{A}^1$.

Remark 4.1. The spreading out procedure can be performed over any smooth algebraic curve X . Let's consider the pro-algebraic group $\mathrm{Aut}(\mathcal{O})$. We have a canonical $\mathrm{Aut}(\mathcal{O})$ -principal bundle $\mathrm{Aut}(X)$ over X . As Liangshi briefly explained, $\mathbf{Gr}_X \simeq \mathrm{Aut}(X) \times^{\mathrm{Aut}(\mathcal{O})} \mathbf{Gr}$, so any $\mathrm{Aut}(\mathcal{O})$ -equivariant perverse sheaf on \mathbf{Gr} can be spread out. We know that any $G_{\mathcal{O}}$ -equivariant perverse sheaf on \mathbf{Gr} is automatically $\mathrm{Aut}(\mathcal{O})$ -equivariant. This can be seen, for example, from the classification of simple objects in $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$ and the semisimplicity of $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$.

Now we can state Gaitsgory's construction of the central functor Z .

$$Z : \mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \rightarrow \mathrm{Perv}(\mathbf{Fl}), \mathcal{G} \mapsto \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}).$$

Proposition 4.1. We have $Z(\delta_{1_{\mathbf{Gr}}}) \simeq \delta_{1_{\mathbf{Fl}}}$.

Proof. I have a canonical section $1_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}} : \mathbb{A}^1 \rightarrow \mathbf{Fl}_{(\mathbb{A}^1, 0)}$ sending y to the quadruple $(y, \mathcal{E}^0, \beta^0, \epsilon^0)$ such that $1_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}|_{\mathbb{G}_m} = 1_{\mathbf{Gr}_{\mathbb{A}^1}}|_{\mathbb{G}_m} \times 1_{G/B}, 1_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}|_0 = 1_{\mathbf{Fl}}$. \square

Proposition 4.2. For any $\mathcal{G} \in \mathrm{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, $\pi_!(Z(\mathcal{G})) \simeq \mathcal{G}$. Here π is the projection $\pi : \mathbf{Fl} \rightarrow \mathbf{Gr}$.

Proof. This follows from the fact that nearby cycle commutes with proper pushforward. Consider the diagram

$$\begin{array}{ccccc} \mathbf{Fl}_{(\mathbb{A}^1, 0), 0} & \longrightarrow & \mathbf{Fl}_{(\mathbb{A}^1, 0)} & \longleftarrow & \mathbf{Fl}_{(\mathbb{A}^1, 0)}|_{\mathbb{G}_m} \\ \downarrow \pi_0 & & \downarrow \pi_{(\mathbb{A}^1, 0)} & & \downarrow \pi_{\mathbb{G}_m} \\ \mathbf{Gr}_{\mathbb{A}^1, 0} & \longrightarrow & \mathbf{Gr}_{\mathbb{A}^1} & \longleftarrow & \mathbf{Gr}_{\mathbb{A}^1}|_{\mathbb{G}_m} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \mathbb{G}_m \end{array}$$

We have

$$\pi_!(Z(\mathcal{G})) = \pi_0!(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})) \simeq \Psi_{\mathbf{Gr}_{\mathbb{A}^1}}(\pi_{\mathbb{G}_m!}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})) = \Psi_{\mathbf{Gr}_{\mathbb{A}^1}}(\mathcal{G}_{\mathbb{G}_m}).$$

We know that the vanishing cycle $\Phi_{\mathbf{Gr}_{\mathbb{A}^1}}(\mathcal{G}_{\mathbb{A}^1}) = 0$ (the support of the vanishing cycle lies in the singular locus), so $\Psi_{\mathbf{Gr}_{\mathbb{A}^1}}(\mathcal{G}_{\mathbb{G}_m}) = \mathcal{G}_{\mathbb{A}^1}|_0 = \mathcal{G}$, we are done. \square

4.3 Factorisation

Like the construction of the commutativity constraint in the geometric Satake equivalence explained by Liangshi last time, I need to construct a factorisation (Beilinson–Drinfeld) version of the affine grassmannian and the affine flag variety.

Let (X, x) be a pointed smooth geometrically connected curve. The following definition should be understood properly (using the functor of points point of view)

$$\mathbf{Gr}_{(X, x)}^{\mathrm{BD}} = \{(y, \mathcal{E}, \beta') : y \in X, \mathcal{E} \text{ a principal } G\text{-bundle on } X, \beta' \text{ a trivialisation of } \mathcal{E} \text{ away from } x \cup y\},$$

$$\mathbf{Fl}_{(X, x)}^{\mathrm{BD}} = \{(y, \mathcal{E}, \beta', \epsilon) : (y, \mathcal{E}, \beta') \in \mathbf{Gr}_{(X, x)}^{\mathrm{BD}}, \epsilon \text{ a reduction of } \mathcal{E} \text{ to } B \text{ at } x\}.$$

The factorisation affine grassmannian and the factorisation affine flag variety are representable in ind-schemes.

Using Beaville–Laszlo (type) theorem, I have

$$\mathbf{Gr}_{(X, x)}^{\mathrm{BD}}|_{X \setminus x} \simeq \mathbf{Gr}_X|_{X \setminus x} \times \mathbf{Gr}, \mathbf{Gr}_{(X, x), x}^{\mathrm{BD}} \simeq \mathbf{Gr},$$

$$\mathbf{Fl}_{(X, x)}^{\mathrm{BD}}|_{X \setminus x} \simeq \mathbf{Gr}_X|_{X \setminus x} \times \mathbf{Fl}, \mathbf{Fl}_{(X, x), x}^{\mathrm{BD}} \simeq \mathbf{Fl}.$$

4.4 Construction of the fusion

Now set $(X, x) = (\mathbb{A}^1, 0)$. Last time, Liangshi explained to us the construction of the commutativity constraint of $\text{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$:

$$C_{\mathbf{Gr}}(\cdot, \cdot): \text{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \times \text{Perv}(\mathbf{Gr}) \rightarrow \text{Perv}(\mathbf{Gr}), (\mathcal{G}^1, \mathcal{G}^2) \mapsto \Psi_{\mathbf{Gr}_{(\mathbb{A}^1, 0)}^{\text{BD}}}(\mathcal{G}_{\mathbb{G}_m}^1 \boxtimes \mathcal{G}^2).$$

Similarly, I can construct a fusion

$$C_{\mathbf{Fl}}(\cdot, \cdot): \text{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr}) \times \text{Perv}(\mathbf{Fl}) \rightarrow \text{Perv}(\mathbf{Fl}), (\mathcal{G}, \mathcal{F}) \mapsto \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \mathcal{F}).$$

Proposition 4.3. Let \mathcal{G} be an object of $\text{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, then

- For any $\mathcal{F} \in \text{Perv}_I(\mathbf{Fl})$, there is a canonical isomorphism $C_{\mathbf{Fl}}(\mathcal{G}, \mathcal{F}) \simeq Z(\mathcal{G}) *_{\mathbf{Fl}} \mathcal{F}$.
- For any $\mathcal{F} \in \text{Perv}(\mathbf{Fl})$, there is a canonical isomorphism $C_{\mathbf{Fl}}(\mathcal{G}, \mathcal{F}) \simeq \mathcal{F} *_{\mathbf{Fl}} Z(\mathcal{G})$.

Proof. I prove the first statement, the proof of the second one is similar. To do so, I need a global/factorisation version of the fundamental convolution diagram

$$\mathbf{Fl} \times \mathbf{Fl} \xleftarrow{p} G_{\mathcal{K}} \times \mathbf{Fl} \xrightarrow{q} G_{\mathcal{K}} \times^I \mathbf{Fl} \xrightarrow{m} \mathbf{Fl}$$

More precisely, I consider the diagram

$$\mathbf{Fl}_{(\mathbb{A}^1, 0)} \times_{\mathbb{A}^1} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}} \xleftarrow{p_{(\mathbb{A}^1, 0)}} G_{\mathcal{K}, \mathbb{A}^1} \times_{\mathbb{A}^1} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}} \xrightarrow{q_{(\mathbb{A}^1, 0)}} G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}} \xrightarrow{m_{(\mathbb{A}^1, 0)}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}$$

Notice that $m_{(\mathbb{A}^1, 0)}$ is ind-proper. From the diagram

$$\begin{array}{ccccccc} G_{\mathcal{K}} \times^I \mathbf{Fl} & \xlongequal{\quad} & G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}|_0 & \longrightarrow & G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}} & \longleftarrow & G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}|_{\mathbb{G}_m} \\ \downarrow m & & \downarrow m_{(\mathbb{A}^1, 0)}|_0 & & \downarrow m_{(\mathbb{A}^1, 0)} & & \downarrow m_{(\mathbb{A}^1, 0)}|_{\mathbb{G}_m} \\ \mathbf{Fl} & \xlongequal{\quad} & \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}|_0 & \longrightarrow & \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}} & \longleftarrow & \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}|_{\mathbb{G}_m} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \mathbb{G}_m \end{array}$$

I have that

$$\begin{aligned} C_{\mathbf{Fl}}(\mathcal{G}, \mathcal{F}) &= \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \mathcal{F}) = \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}}(m_{(\mathbb{A}^1, 0)!}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) \\ &= m_! \Psi_{G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})). \end{aligned}$$

By construction,

$$Z(\mathcal{G}) *_{\mathbf{Fl}} \mathcal{F} = m_!(Z(\mathcal{G}) \tilde{\boxtimes} \mathcal{F}) = m_!(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} \mathcal{F}),$$

so it suffices to show that

$$\Psi_{G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) = \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} \mathcal{F}.$$

Noticing that $q_{(\mathbb{A}^1, 0)}$ is smooth and that smooth pullback is conservative, it suffices to show that

$$q_{(\mathbb{A}^1, 0)}^* \Psi_{G_{\mathcal{K}, \mathbb{A}^1} \times^{I_{\mathbb{A}^1}} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}}((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) = q_{(\mathbb{A}^1, 0)}^*(\Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}}(\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \tilde{\boxtimes} \mathcal{F}).$$

Let $n_{(\mathbb{A}^1, 0)}$ be the projection $n_{(\mathbb{A}^1, 0)} : \mathcal{G}_{\mathcal{K}, \mathbb{A}^1} \rightarrow \mathbf{Fl}_{(\mathbb{A}^1, 0)}$, $n_{(\mathbb{A}^1, 0)}$ is smooth. Smooth pullback commutes with nearby cycle, so

$$\begin{aligned}
& q_{(\mathbb{A}^1, 0)}^* \Psi_{G_{\mathcal{K}, \mathbb{A}^1} \times_{\mathbb{A}^1} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}} ((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \widetilde{\boxtimes} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) \\
&= \Psi_{G_{\mathcal{K}, \mathbb{A}^1} \times_{\mathbb{A}^1} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}} (q_{(\mathbb{A}^1, 0)}^* ((\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \widetilde{\boxtimes} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) \\
&= \Psi_{G_{\mathcal{K}, \mathbb{A}^1} \times_{\mathbb{A}^1} \mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}} (n_{(\mathbb{A}^1, 0)}^* (\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \boxtimes (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}})) \\
&= \Psi_{G_{\mathcal{K}, \mathbb{A}^1}} (n_{(\mathbb{A}^1, 0)}^* (\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}})) \boxtimes \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}}) \\
&= n_{(\mathbb{A}^1, 0)}^* \Psi_{G_{\mathcal{K}, \mathbb{A}^1}} (\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \boxtimes \Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}^{\text{BD}}} (\mathcal{F} \boxtimes \delta_{1_{\mathbf{Gr}}}) \\
&= n_{(\mathbb{A}^1, 0)}^* \Psi_{G_{\mathcal{K}, \mathbb{A}^1}} (\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \boxtimes \mathcal{F} \\
&= q_{(\mathbb{A}^1, 0)}^* (\Psi_{\mathbf{Fl}_{(\mathbb{A}^1, 0)}} (\mathcal{G}_{\mathbb{G}_m} \boxtimes \delta_{1_{G/B}}) \widetilde{\boxtimes} \mathcal{F}).
\end{aligned}$$

We are done. \square

Proposition 4.4. For any $\mathcal{G}^1, \mathcal{G}^2 \in \text{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, there is a canonical isomorphism

$$Z(\mathcal{G}^1) *_{\mathbf{Fl}} Z(\mathcal{G}^2) \simeq Z(\mathcal{G}^1 *_{\mathbf{Gr}} \mathcal{G}^2).$$

The proof is analogous, using the fact that nearby cycle commutes with proper pushforward, external tensor product, and smooth pullback.

5 An example for $\text{SL}(2)$

Let $G = \text{SL}(2)$. Let's see what does Z do for $G_{\mathcal{O}}$ -equivariant perverse sheaves on \mathbf{Gr} supported on $\mathbb{P}^1 = \overline{\mathbf{Gr}_{\alpha^\vee/2}}$. To do so, it suffices to consider the following degeneration

$$\mathcal{Y} = \{([x : y : z], \lambda) \in \mathbb{P}^2 \times \mathbb{A}^1 : xy = \lambda z^2\} \rightarrow \mathbb{A}^1, ([x : y : z], \lambda) \mapsto \lambda.$$

We see that $\mathcal{Y}_\lambda \simeq \mathbb{P}^1$ for $\lambda \neq 0$ and \mathcal{Y}_0 is the transversal intersection of two \mathbb{P}^1 's.

I want to understand perverse sheaves on \mathcal{Y}_0 lisse along certain stratification. Let $Y = \{xy = 0\} \subset \mathbb{C}^2$, then $\mathcal{Y}_0 = \overline{Y}$. Let Λ be the following stratification of Y :

$$Y = \{0\} \sqcup \mathbb{C}_{x\text{-axis}}^\times \sqcup \mathbb{C}_{y\text{-axis}}^\times.$$

I have a description of $\text{Perv}_\Lambda(Y)$ using Beilinson's gluing. Namely, consider the regular function

$$f : \mathbb{C}^2 \rightarrow \mathbb{C}, (x, y) \mapsto x - y.$$

The zero locus of f restricts to $\{0\}$ on Y . Now using Beilinson's gluing, we see that

$$\begin{aligned}
\text{Perv}_\Lambda(Y) &= \left\{ \begin{array}{ll} \mathcal{F} \in \text{Perv}(\mathbb{C}_{x\text{-axis}}^\times \sqcup \mathbb{C}_{y\text{-axis}}^\times), & \mu \text{ monodromy of } \Psi_f(\mathcal{F}), \\ \mathcal{F} \text{ lisse along } \mathbb{C}_{x\text{-axis}}^\times, \mathbb{C}_{y\text{-axis}}^\times, & v : V_0 \rightarrow \Psi_f(\mathcal{F}), \\ V_0 \in \text{Perv}(\{0\}), & c : \Psi_f(\mathcal{F}) \rightarrow V_0, \\ & v \circ c = 1 - \mu. \end{array} \right\} \\
&= \left\{ \begin{array}{ll} \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \mu_y \\ \curvearrowright \\ V_y \end{array} & \begin{array}{c} \begin{array}{c} c_y \\ \swarrow \quad \searrow \\ V_0 \end{array} & \begin{array}{c} \begin{array}{c} \mu_x \\ \curvearrowright \\ V_x \end{array} \\ \begin{array}{c} v_y \quad v_x \\ \swarrow \quad \searrow \\ V_0 \end{array} \end{array} & \begin{array}{l} v_x \circ c_x = 1 - \mu_x, \\ v_y \circ c_y = 1 - \mu_y, \\ v_x \circ c_y = 0, \\ v_y \circ c_x = 0. \end{array} \end{array} \right\}
\end{aligned}$$

By requiring the lisse condition at ∞_x and ∞_y , i.e. requiring that $\mu_x = \text{id}, \mu_y = \text{id}$, I have

$$\text{Perv}_{\overline{\Lambda}}(\mathcal{Y}_0) = \left\{ \begin{array}{ll} \begin{array}{ccc} V_y & \begin{array}{c} \begin{array}{c} c_y \\ \swarrow \quad \searrow \\ V_0 \end{array} & V_x \\ \begin{array}{c} v_y \quad v_x \\ \swarrow \quad \searrow \\ V_0 \end{array} & & \end{array} & \begin{array}{l} v_x \circ c_x = 0, \\ v_y \circ c_y = 0, \\ v_x \circ c_y = 0, \\ v_y \circ c_x = 0. \end{array} \end{array} \right\}$$

Let us compute $Z(\mathbb{C}_{\mathbb{P}^1}[1]) = \Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ explicitly. I have a short exact sequence of perverse sheaves

$$0 \rightarrow \mathbb{C}_{\mathcal{Y}_0}[1] \rightarrow \Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2]) \rightarrow \Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2]) \rightarrow 0.$$

Noticing that $\bullet = [0 : 0 : 1]$ is the only singular point of the function $\mathcal{Y} \rightarrow \mathbb{A}^1$, $\Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ is supported on this point. Let us compute the stalk $\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_{\bullet}$, which is the same as the cohomology of the Milnor fiber at \bullet (up to some cohomological shift). Here the Milnor fiber is just

$$\{x, y \in \mathbb{C} : xy = \lambda\} \simeq S^1 \text{ (for sufficiently small } \lambda),$$

so

$$\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_{\bullet} = R\Gamma(S^1, \mathbb{C})[1] = \mathbb{C} \oplus \mathbb{C}[1].$$

Therefore I get $\Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_{\bullet} = \mathbb{C}$ and hence $\Phi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])_{\bullet} = \mathbb{C}_{\bullet} = \text{IC}_{\bullet}$. One now knows that $\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ is an extension of IC_{\bullet} by $\mathbb{C}_{\mathcal{Y}_0}[1]$. Moreover, there is a short exact sequence of perverse sheaves

$$0 \rightarrow \text{IC}_{\bullet} \rightarrow \mathbb{C}_{\mathcal{Y}_0}[1] \rightarrow \text{IC}_{\mathbb{C}_x}(\mathbb{C}_x^{\times}) \oplus \text{IC}_{\mathbb{C}_y}(\mathbb{C}_y^{\times}) \rightarrow 0,$$

so the Loewy diagram of $\Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2])$ is

IC_{\bullet}
$\text{IC}_{\mathbb{C}_x}(\mathbb{C}_x^{\times}) \oplus \text{IC}_{\mathbb{C}_y}(\mathbb{C}_y^{\times})$
IC_{\bullet}

Using the quiver description, I have

$$\begin{aligned} \text{IC}_{\bullet} &= \begin{array}{ccc} 0 & & 0 \\ & \swarrow \quad \searrow & \\ & \mathbb{C} & \end{array}, \\ \text{IC}_{\mathbb{C}_x}(\mathbb{C}_x^{\times}) &= \begin{array}{ccc} 0 & & \mathbb{C} \\ & \swarrow \quad \searrow & \\ & 0 & \end{array}, \\ \text{IC}_{\mathbb{C}_y}(\mathbb{C}_y^{\times}) &= \begin{array}{ccc} \mathbb{C} & & 0 \\ & \swarrow \quad \searrow & \\ & 0 & \end{array}, \\ \Psi_{\mathcal{Y}}(\mathbb{C}_{\mathcal{Y}}[2]) &= \begin{array}{ccc} \mathbb{C} & & \mathbb{C} \\ & \swarrow \quad \searrow & \\ & \mathbb{C} & \\ & \oplus & \\ & \mathbb{C} & \end{array}. \end{aligned}$$

6 Confession

I didn't explain the following.

6.1 I -equivariance

I want that for any $\mathcal{G} \in \text{Perv}_{G_{\mathcal{O}}}(\mathbf{Gr})$, $Z(\mathcal{G})$ is automatically I -equivariant. This is not obvious from Gaitsgory's original construction (and I cannot understand the argument in Gaitsgory's paper). In the book by Achar–Riche, they proposed another resolution of the problem. Instead of the constant group scheme $G_{\mathcal{O}} \times \mathbb{A}^1$ over \mathbb{A}^1 , they used a nonconstant group scheme $\mathcal{G} \rightarrow \mathbb{A}^1$ such that $\mathcal{G}|_{\mathbb{G}_m} \simeq G_{\mathcal{O}} \times \mathbb{G}_m$ and $\mathcal{G}_0 \simeq I$. Now from the construction of \mathcal{G} -equivariant nearby cycles, the image is automatically $\mathcal{G}_0 \simeq I$ -equivariant. Their construction of the nonconstant group scheme \mathcal{G} used Bruhat–Tits theory. Hope somebody can explain this to us.

6.2 Higher nearby cycles

To check higher compatibilities between isomorphisms constructed above, I need a theory of nearby cycles over \mathbb{A}^2 . This is explained in detail in a paper by Achar–Riche (also in their book). Hope somebody can explain it to us.

6.3 Monodromy

Constructed using nearby cycle, there is a natural monodromy action on Z . One can show that the monodromy action on Z is unipotent, hence inducing a monodromy weight filtration on Z as explained in Weil II.

6.4 Epitaph

Did you know: In one's afterlife, one is condemned to finding counterexamples to all false statements made in life?

Hence the advice: Start early!

I am still confused by the following issues:

1. I don't understand Gaitsgory's proof of I -equivariance. Can somebody help?

Confusion will be my epitaph.