Rigid spaces and their étale cohomology

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Lecture 1

§0. Intro

Alg. geom. $/k$ field

locally, $\{a=(a_i)\in k^n : f_1(a)=\cdots=f_m(a)=0\}$

$f_i\in k[T_1,\dots,T_n]$

$\longleftrightarrow k[T_1,\dots,T_n]/(f_1,\dots,f_m)$, ring

Rigid geometry $/k=\mathbb{C}_p$

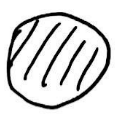
locally, $\{a=(a_i)\in \mathbb{C}_p^n : |a_i|\le 1,\ \forall i,\ f_1(a)=\cdots=f_m(a)=0\}$

$$f_i\in \mathbb{C}_p\langle T_1,\dots,T_n\rangle=\Big\{\sum_{I\in(\mathbb{Z}_{\ge0})^n} a_I T^I : |a_I|\to 0\ (|I|\to\infty)\Big\},\quad a_I\in\mathbb{C}_p$$

ring of convergent power series

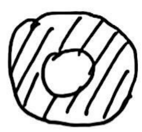
$\longleftrightarrow \mathbb{C}_p\langle T_1,\dots,T_n\rangle/(f_1,\dots,f_m)$ topological ring

Ex. $|z|\le 1$



disk w/ boundary

$|p|\le|z|\le 1$ $\cong\{(x,y)\in\mathbb{C}_p^2 : |x|\le 1,\ |y|\le 1,\ xy=p\}$



annulus

$|z|<1$

disk w/o boundary

$=\bigcup_{n\ge1}$  radius $|p|^{\frac{1}{n}}$

Consider mod p map

$$sp : \mathbb{Z}_p \xrightarrow{\text{mod } p} \underset{\substack{\Psi \\ 0}}{\mathbb{F}_p} \quad \text{(specialization map)}$$

$$(|z| < 1) = sp^{-1}(0)$$

Natural topology on $\mathbb{C}_p^n$ is totally disconnected

$\to$ difficult to do geometry (glueing, ...)

<u>Tate</u>: decrease open subsets $\rightsquigarrow$ Tate's theory of rigid analytic spaces (classical)

add more points $\bigg\langle$ Berkovich's theory / Huber's theory of <u>adic spaces</u>

points $\longleftrightarrow$ cts valuations

<u>§1</u>. Def. of adic spaces

<u>Def</u>. A : topological ring,

A f-adic if
- $\exists A_0 \subset A$ open subring (ring of definition)
- $\exists I \subset A_0$ fin. gen. ideal (ideal of def. of A_0)

s.t. topology of A_0 is I-adic (i.e. $\{I^n\}$ fund. system of open nbhds of 0)

<u>Ex</u>. $\mathbb{C}_p\langle T_1, \cdots, T_n\rangle$ f-adic

$$\begin{array}{c} \mathbb{C}_p\langle T_1, \cdots, T_n\rangle \\ \cup \\ \mathbb{Z}_p\langle T_1, \cdots, T_n\rangle = \left\{ \sum a_I T^I \in \mathbb{C}_p\langle T_1, \cdots, T_n\rangle : a_I \in \mathbb{Z}_p \right\} \\ \| \\ \varprojlim_m \mathbb{Z}_p[T_1, \cdots, T_n]/(p^m) \end{array}$$

p-adic topology

Def A : f-adic top. ring

$a \in A$ power-bounded $\overset{\text{def}}{\iff}$ $\{a^n\}_{n \geq 1}$ is bounded

$\iff$ $\forall\, U$ open nbhd of $0 \in A$.

$\exists\, U'$ open nbhd of $0 \in A$ s.t.

$\forall\, n \geq 1$, $\forall\, u \in U'$, $a^n u \in U$.

$A^\circ := \{ a \in A \cdot\ a$ power-bounded $\}$

open subring of A $\quad$ ($\because I \subset A^\circ$ for $I \subset A_0$ ideal of def.)

Def. (affinoid ring) $\quad A = (A^\triangleright, A^+)$: affinoid ring if

- $A^\triangleright$: f-adic topological ring
- $A^+ \subset (A^\triangleright)^\circ$ open subring s.t. A^+ is integrally closed in $A^\triangleright$.

Ex. $\left(\mathbb{Q}_p \langle T_1, \cdots, T_n \rangle,\ \mathbb{Z}_p \langle T_1, \cdots, T_n \rangle \right)$

$(\mathbb{Z}_p, \mathbb{Z}_p)$

Def A ring

Valuation of A is a pair (Γ, v)

- Γ : totally ordered comm. gp. (written multiplicatively)
- $v : A \longrightarrow \Gamma \cup \{0\}$
  - $v(ab) = v(a) v(b)$
  - $v(a+b) \leq \max\{v(a), v(b)\}$
  - $v(0) = 0$, $v(1) = 1$.

$$\left[\begin{array}{l} \forall\, \gamma \in \Gamma,\ 0 \cdot \gamma = 0 \\ 0 \cdot 0 = 0 \\ \forall\, \gamma \in \Gamma,\ 0 < \gamma \end{array}\right.$$

$\mathrm{supp}\, v := v^{-1}(0)$ prime ideal of A

$\Gamma' :=$ subgp of Γ gen. by $v(A)\setminus\{0\}$.

$(\Gamma_1, v_1) \sim (\Gamma_2, v_2) \overset{\text{def}}{\iff} \exists f : \Gamma_1' \xrightarrow{\sim} \Gamma_2'$ order-preserving isom. s.t.

$$v_2 = f \circ v_1$$

$$\iff \forall a, b \in A, \left[v_1(a) \le v_1(b) \iff v_2(a) \le v_2(b) \right]$$

A topological ring, (Γ, v) cts if $\forall \gamma \in \Gamma'$, $\{a \in A : v(a) < \gamma\}$ is open.

<u>Def</u> $A = (A^{\triangleright}, A^+)$ affinoid ring.

$$\mathrm{Spa}\, A := \{ v : A^{\triangleright} \to \Gamma \cup \{0\} \text{ cts val.} : v(a) \le 1, \ \forall a \in A^+ \} / \sim$$

For $t_1, \dots, t_n, s \in A^{\triangleright}$ s.t. $t_1 A^{\triangleright} + \dots + t_n A^{\triangleright}$ is open in $A^{\triangleright}$.

$$R\left(\frac{t_1, \dots, t_n}{s}\right) := \{ v \in \mathrm{Spa}\, A : v(t_i) \le v(s) \ne 0 \} \quad \text{rat'l subset}$$

Top. on $\mathrm{Spa}\, A$: open basis $= \{$rat'l subsets$\}$

<u>Rmk</u>. If $A^{\triangleright}$ has a top. nilp. unit (such a top. ring is called "Tate")

then $t_1 A^{\triangleright} + \dots + t_n A^{\triangleright}$ is open $\iff t_1 A^{\triangleright} + \dots + t_n A^{\triangleright} = A^{\triangleright}$

(a top. nilp. $\iff a^n \to 0 \ (n \to \infty)$)

e.g. $A^{\triangleright} = \mathbb{Q}_p\langle T_1, \dots, T_n \rangle$, $p \in A^{\triangleright}$ is top. nilp. unit.

<u>Thm</u>. $\mathrm{Spa}\, A$: spectral space, i.e.

i) quasi-cpt, <u>quasi-separated</u> (U, V q-cpt open, $U \cap V$ q-cpt)

ii) $\exists$ open basis consisting of q-cpt open subsets.

iii) sober, i.e. $\forall$ irred. closed subset has unique generic pt.

Ex Points of $\mathbb{D}^1 = \mathrm{Spa}\left(\mathbb{C}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle\right)$

$\mathbb{D}^1$

- $a \in \mathbb{C}_p$, $|a| \leq 1$

$$\rightsquigarrow v_a : \mathbb{C}_p\langle T\rangle \longrightarrow \mathbb{C}_p \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0} = \mathbb{R}_{>0} \sqcup \{0\}$$
$$f \longmapsto f(a)$$

$v_a \in \mathbb{D}^1$ (classical point)

- $r \in |\mathbb{C}_p^\times|$, $r \leq 1$

$$w_r : \mathbb{C}_p\langle T\rangle \longrightarrow \mathbb{R}_{\geq 0} \qquad \text{rank } 1 \quad (\text{ht } 1)$$
$$f = \sum_{n=0}^{\infty} a_n T^n \mapsto \max_n \{|a_n| r^n\}$$

($r = 1$, Gauß norm)

$w_r \in \mathbb{D}^1$

- $w_r^\pm : \mathbb{C}_p\langle T\rangle \longrightarrow \underbrace{(\mathbb{R}_{>0} \times \mathbb{Z})}_{\text{lexicographic order}} \cup \{0\}$

$$f = \sum_{n=0}^{\infty} a_n T^n \mapsto \max\{(|a_n| r^n, \pm n)\}$$

$w_r^- \in \mathbb{D}^1$

$w_r^+ \in \mathbb{D}^1 \Leftrightarrow r < 1$

$\left(r = 1, \quad \underset{\mathbb{Z}_p\langle T\rangle}{w_1^+(T)} = \overset{\mathbb{R}_{>0} \times \mathbb{Z}}{(1, 1) > (1, 0)} \right)$

$v(a) \leq 1$, $a \in A^+$

$w_r^+, w_r^- \in \overline{\{w_r\}}$.

In particular, for $a \in \mathbb{C}_p$ w/ $|a| < 1$, $R\left(\frac{T}{a}\right)$ is open but not closed!

$R\left(\frac{T}{a}\right) =$ "$|T| \leq |a|$"

$w_{|a|}, w_{|a|}^- \in R\left(\frac{T}{a}\right)$, $w_{|a|}^+ \notin R\left(\frac{T}{a}\right)$

in fact, $\partial R\left(\frac{I}{a}\right) = \{w_{|a|}^{+}\}$.

$\rightsquigarrow$ $\mathbb{D}^1$ is $\underline{\text{NOT}}$ tot. disconnected.

$\underline{\text{Ex}}$. $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) = \{s, \eta\}$

$$s : \mathbb{Z}_p \to \mathbb{F}_p \to \{0,1\}$$
$$a \mapsto \begin{cases} 0, & (a=0) \\ 1, & (a \neq 0) \end{cases}$$

$$\eta : \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$$

topology: s closed, not open

Structure sheaf.

$\underline{\text{Def}}$. $A = (A^{\triangleright}, A^{+})$ affinoid ring, take $I \subset A_0 \subset A^{\triangleright}$

$R\left(\frac{t_1, \dots, t_n}{s}\right)$ rat'l subset of $\mathrm{Spa}\, A$, define $\Gamma\left(R\left(\frac{t_1,\dots,t_n}{s}\right), \mathcal{O}\right) := \underline{A^{\triangleright}\left\langle \frac{t_1,\dots,t_n}{s}\right\rangle}$

completion of $A^{\triangleright}[\frac{1}{s}]$ w.r.t. the top. gen. by $\left\{I^m A_0[\frac{t_1}{s}, \dots, \frac{t_n}{s}]\right\}$.

$$\Gamma\left(R\left(\frac{t_1,\dots,t_n}{s}\right), \mathcal{O}^{+}\right) := A^{+}\left\langle \frac{t_1,\dots,t_n}{s}\right\rangle$$

$\underline{\text{Thm}}$. $\mathcal{O}, \mathcal{O}^{+}$ are sheaves of complete topological rings on $\mathrm{Spa}\, A$ if $A^{\triangleright}$ has noetherian ring of def., or $A^{\triangleright}$ is strongly noetherian & Tate.

- $A^{\triangleright}$ strongly noetherian $\overset{\text{def}}{\iff}$ $\forall n \geq 0$, $A^{\triangleright}\langle T_1, \dots, T_n\rangle$ noetherian

e.g. $k = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$

$\underline{\text{Eg}}$. k na field, (i.e. field w/ val $k \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$, topology defined by $|\cdot|$).

$\Rightarrow k\langle T_1, \dots, T_n\rangle / (f_1, \dots, f_m)$ strongly noetherian & Tate.

For $x \in \operatorname{Spa} A$, $\mathcal{O}_x$ carries a natural val. v_x.

$\mathcal{O}_x$ is a local ring w/ max. ideal $\operatorname{supp} v_x$

$\mathcal{O}_x^+ := \{a \in \mathcal{O}_x : v_x(a) \le 1\}$

<u>Def</u> $\mathcal{V}$: cat. of triples $(X, \mathcal{O}_X, (v_x)_{x \in X})$

- X top. sp.
- $\mathcal{O}_X$ sheaf of complete top. ring
- v_x : val. of $\mathcal{O}_{X,x}$

mor: $f : X \longrightarrow Y$ cts map,

$$f^*\mathcal{O}_Y \to \mathcal{O}_X \quad \text{s.t.} \quad \forall x \in X, \quad \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x} \xrightarrow{v_x} \Gamma \cup \{0\}$$

$\sim$ up to equiv.

$v_{f(x)}$

<u>Def.</u> Adic space is an obj. of $\mathcal{V}$ loc. isom. to $\operatorname{Spa} A$ for some affinoid ring A.

Adic space isom. to $\operatorname{Spa} A$ is called <u>affinoid</u>.

<u>Rmk</u> X adic space. $\mathcal{O}_X^+ \subset \mathcal{O}_X$

$$\mathcal{O}_X^+(U) = \{s \in \mathcal{O}_X(U) : v_x(s) \le 1, \forall x \in U\}$$

$\to$ can recover $\mathcal{O}^+$

<u>Thm</u> A aff. ring, X adic sp., $\to$ $\operatorname{Hom}_{\text{adic}}(X, \operatorname{Spa} A) \simeq \operatorname{Hom}_{\text{cont}}((A^\triangleright, A^+), (\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{O}_X^+)))$

In particular, for $B = (B^\triangleright, B^+)$ complete aff. ring,

$$\operatorname{Hom}(\operatorname{Spa} B, \operatorname{Spa} A) \simeq \operatorname{Hom}_{\text{cont}}(A, B).$$

Lecture 2. §2. Adic spaces and formal schemes

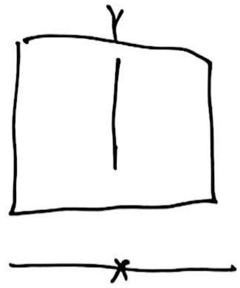
Formal schemes

V: CDVR, $X \to \operatorname{Spec} V$, X_s : special fiber

$Y \subset X_s$ closed subscheme $\rightsquigarrow$ formal completion $X^\wedge_{/Y}$ formal scheme

$(Y = X_s, \quad X^\wedge := X^\wedge_{/X_s})$

as a top space, $X^\wedge_{/Y} = Y$; str. sheaf is different



$W \subset Y$ open, what is $\Gamma(W, \mathcal{O}_{X^\wedge_{/Y}})$?

W : affine

U : affine open of X s.t. $U \cap Y = W$.
$U \cong \operatorname{Spec} A$

defined by ideal $I \lhd A$

$\Gamma(W, \mathcal{O}_{X^\wedge_{/Y}}) = \hat{A}$ = completion of A by I-adic topology.

What is good?

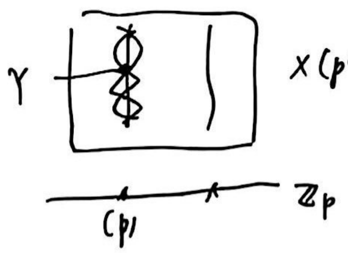
1. $X^\wedge_{/Y}$ is simpler than X.

If $X \to \operatorname{Spec} V$ is smooth, Y closed pt,

$X^\wedge_{/Y} \simeq \operatorname{Spf} V[[T_1, \dots, T_d]]$, for some d.

2. New symmetry appears

Ex. Integral model of modular curve $X(p^m)$



$X(p^m)^\wedge_{/Y} \curvearrowleft \mathcal{O}_D^\times$

gen. fib. $\{$ D : quat. div. alg. / $\mathbb{Q}_p$

Lubin-Tate space for $GL(2)$.

Relation w/ p-adic uniformization theory. (Ito)

$\mathfrak{X}$: locally noetherian formal scheme $\xrightarrow{\text{attach}}$ • $t(\mathfrak{X})$ adic space

• $\left(t(\mathfrak{X}), \mathcal{O}^{+}_{t(\mathfrak{X})}\right) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$

morphism of topologically ringed space

Locally, $\mathfrak{X} = \operatorname{Spf} A \rightsquigarrow t(\mathfrak{X}) = \operatorname{Spa}(A, A)$

$$t(\mathfrak{X}) \longrightarrow \mathfrak{X}$$
$$v \longmapsto \{a \in A : v(a) < 1\}$$

open prime ideal of A

$t: \mathfrak{X} \longmapsto t(\mathfrak{X})$ is fully faithful.

universal property of t:

$$\begin{tikzcd} & t(\mathfrak{X}) \arrow[d] \\ S \arrow[ur, dashed, "\exists!"] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \mathfrak{X} \end{tikzcd}$$

adic space

$$\operatorname{Hom}\left(\operatorname{Spa}(A^{\triangleright}, A^{+}), \operatorname{Spa}(B^{\triangleright}, B^{+})\right) = \operatorname{Hom}\left((B^{\triangleright}, B^{+}), (A^{\triangleright}, A^{+})\right)$$

V: CDVR, $\mathfrak{X}$: formal scheme $/\operatorname{Spf} V$

$$\rightsquigarrow \quad t(\mathfrak{X}) \longrightarrow t(\operatorname{Spf} V) = \operatorname{Spa}(V, V) = \{s, \eta\}$$

$\rightsquigarrow$ $t(\mathfrak{X})_{\eta}$: adic space over $\operatorname{Spa}(K, V)$, $K = \operatorname{Frac}(V)$

$\uparrow$ rigid generic fiber

$\hookrightarrow$ adic space version of $\operatorname{Spec} K$

Ex. $V = \mathbb{Z}_p$, $\mathfrak{X} = \mathrm{Spf}\ \underbrace{\mathbb{Z}_p\langle T\rangle}_{p\text{-adic}} = (\mathbb{A}^1_V)^\wedge$

$$t(\mathfrak{X})_\eta = \{ v \in \mathrm{Spa}(\mathbb{Z}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) : \underbrace{v(p) \neq 0}_{v(p) \leq v(p) \neq 0} \}$$

$= $ rat'l subset $R\left(\frac{p}{p}\right)$

$= \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) = \mathbb{D}^1$ (q.cpt)

$\mathfrak{Y} = \mathrm{Spf}\ \mathbb{Z}_p[[T]] = (\mathbb{A}^1_V)^\wedge_{/0}$, $0 \in \mathbb{A}^1_S$

$\mathbb{Z}_p[[T]]$: not p-adic, but (p,T)-adic

$$t(\mathfrak{Y})_\eta = \{ v \in \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) : v(p) \neq 0 \}$$

$v(p) \leq v(p) \neq 0$

$p\mathbb{Z}_p[[T]]$ is NOT open!

$\Downarrow$

$t(\mathfrak{Y})_\eta$ is NOT rat'l subset

v cont. $\Rightarrow$ $v(T^n) \leq v(p)$ for $n \gg 0$.

$$\Rightarrow t(\mathfrak{Y})_\eta = \bigcup_{n \geq 1} \left\{ v \in \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) : \begin{array}{l} v(p) \leq v(p) \neq 0 \\ v(T^n) \leq v(p) \neq 0 \end{array} \right\}$$

(p, T^n) open ideal of $\mathbb{Z}_p[[T]]$

rat'l subset!

$R\left(\frac{T^n, p}{p}\right)$

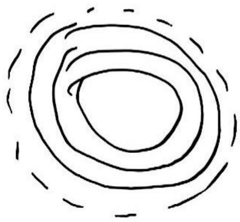
$$\mathbb{Z}_p[[T]]\left\langle \frac{T^n, p}{p} \right\rangle = \text{completion of } \mathbb{Z}_p[T, \tfrac{1}{p}] \text{ w.r.t. } \left\{ (T^n, p)^m \mathbb{Z}_p[T, \tfrac{T^n}{p}] \right\}_m$$

$$= \mathbb{Q}_p\langle T\rangle\left\langle \frac{T^n}{p} \right\rangle \qquad (T^n,p)^m \mathbb{Z}_p[T, \tfrac{T^n}{p}] = p^m \mathbb{Z}_p[T, \tfrac{T^n}{p}]$$

$\rightsquigarrow R\left(\frac{T^n, p}{p}\right) = \mathrm{Spa}\left(\mathbb{Q}_p\langle T\rangle\langle \frac{T^n}{p}\rangle, \mathbb{Z}_p\langle T\rangle\langle\frac{T^n}{p}\rangle\right)$

" $|T| \leq |p|^{\frac{1}{n}}$ "

$\rightsquigarrow t(\mathcal{Y})_\eta$: disk w/o boundary



$t(\mathcal{Y})_\eta$ = interior of $\{v \in \mathbb{D}^1 : v(T) < 1\}$
closed subset

NOT q-cpt.

<u>Def.</u> $sp_{\mathfrak{X}} : t(\mathfrak{X})_\eta \hookrightarrow t(\mathfrak{X}) \longrightarrow \mathfrak{X}$ specialization map

"mod p map"

<u>Prop</u> $\mathfrak{X} = X^\wedge_{/Y}$, $X \longrightarrow \mathrm{Spec}\, V$

$$\begin{array}{c} X \\ \cup \\ X_s \\ \cup \\ Y \end{array}$$

$sp_{X^\wedge} : t(X^\wedge)_\eta \longrightarrow X^\wedge \overset{\text{as top. space}}{=} X_s$

$t(\mathfrak{X})_\eta = t(X^\wedge_{/Y})_\eta \simeq sp^{-1}_{X^\wedge}(Y)^\circ$ ← interior

<u>eg.</u>

$$\begin{array}{ccc} sp_{\mathbb{A}^{1\wedge}_V} : t((\mathbb{A}^1_V)^\wedge) & \longrightarrow & \mathbb{A}^1_s \\ \cup & & \cup \\ sp^{-1}(\{0\}) & & \{0\} \\ \cup & & \\ sp^{-1}(\{0\})^\circ = t(\mathcal{Y})_\eta & & \end{array}$$

Raynaud's thm $K = \mathrm{Frac}\, V$, $\varpi \in V$ uniformizer

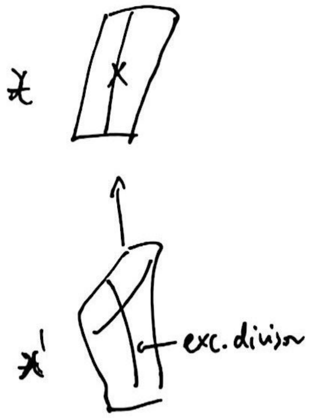
$$\left(\text{formal schemes of } \underline{\text{f. type}} / \mathrm{Spf}\, V\right) \xrightarrow[\sim]{t(-)_\eta} \left(\text{adic sp. of f. type} / \mathrm{Spa}(K,V)\right)$$

ϖ-adic   admissible blow-up   cat. equiv. $\| \; (\text{rigid spaces} / K)$

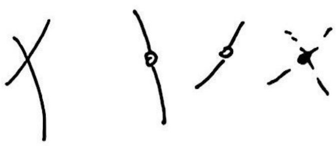
$\uparrow$ blow-up along open ideal

Ex. $\mathfrak{X} = \mathrm{Spf}\, \mathbb{Z}_p\langle T \rangle$, $\mathfrak{X}' \longrightarrow \mathfrak{X}$

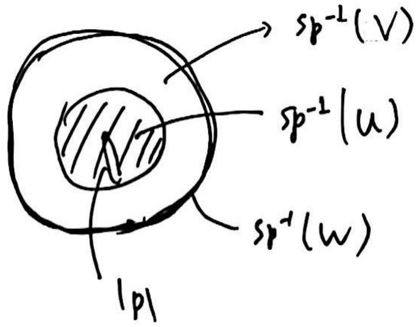
blow-up along (p, T).



$$\mathfrak{X}' = U \amalg V \amalg W$$



$$t(\mathfrak{X}')_\eta = \mathbb{D}^1 = t(\mathfrak{X})_\eta$$



~ blow-up $\longleftrightarrow$ open-coverings

§3. Comparison functor

V, K as above. construct a functor $(\text{Sch.}/K) \to (\text{adic spaces} / \mathrm{Spa}(K,V))$

<u>Prop-Def</u> $X \to Y$ locally of finite type morphism of schemes

S : adic space. $(S, \mathcal{O}_S) \to Y$ morphism of locally ringed spaces

$\rightsquigarrow$ $\exists!$ adic space $X \times_Y S$ satisfying

$$\begin{tikzcd}
\text{adic sp. } T \arrow[r, dashed, "\text{adic}", "\exists!"'] \arrow[rr, bend left, "\text{locally ringed}"] \arrow[dr, bend right, "\text{adic}"'] & X \times_Y S \arrow[r, "\text{locally}", "\text{ringed}"'] \arrow[d, "\text{adic}"] & X \text{ scheme} \arrow[d, "\text{sch}"] \\
& \underset{\text{adic sp.}}{S} \arrow[r, "\text{locally ringed}"'] & Y \text{ scheme}
\end{tikzcd}$$

Apply it to $Y = \operatorname{Spec} K$, $S = \operatorname{Spa}(K, V)$

get $(\text{sch. loc. of finite type} / K) \longrightarrow (\text{adic sp.} / \operatorname{Spa}(K, V))$

$$X \longmapsto X^{ad} = X \times_{\operatorname{Spec} K} \operatorname{Spa}(K, V)$$

<u>Ex.</u> $X = \mathbb{A}^1_K = \operatorname{Spec} K[T]$.

$$\Rightarrow X^{ad} = \bigcup_{n \geq 1} \underbrace{\operatorname{Spa}\left(K\langle \varpi^n T \rangle, V\langle \varpi^n T \rangle\right)}_{\text{disk w/ radius } |\varpi|^{-n}}$$

<u>NOT q-cpt</u>.

<u>Prop</u> X proper $/K$,

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bar{X} \arrow[d, "\text{proper}"] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} V
\end{tikzcd}$$

$X \hookrightarrow \bar{X}$: compactification over V

$$\Rightarrow t(\bar{X}^\wedge)_\eta \simeq X^{ad}.$$

§4. Étale / smooth morphisms of adic spaces.

Def. X adic sp.

$x \in X$: analytic pt $\overset{\text{def}}{\iff}$ $\exists\ x \in U \subset X$ aff. open

s.t. $\mathcal{O}_X(U)$ Tate.

X analytic $\overset{\text{def}}{\iff}$ $\forall x \in X$ is analytic.

Rmk. $X = \mathrm{Spa}\, A$ affinoid,

$v \in \mathrm{Spa}\, A$ is analytic iff $\mathrm{supp}(v) \subset A^{\triangleright}$ is not open.

Ex. $\mathrm{Spa}(V, V) = \{s, \eta\}$

$\mathrm{supp}(s) = \varpi V \subset V$ open $\Rightarrow s$ is not analytic.

$\mathrm{supp}(\eta) = 0$ not open $\to \eta$ analytic.

$A = (A^{\triangleright}, A^+)$ aff. ring $\rightsquigarrow A\langle T_1, \dots, T_n\rangle = (A^{\triangleright}\langle T_1, \dots, T_n\rangle, ///)$

Def. $f : Y \longrightarrow X$ morphism of analytic adic space

f loc. of finite type $\iff$ $\forall y \in Y$, $\exists\ \underset{y}{\underset{\Psi}{V}} \subset Y$ affinoid open, $\exists\ \underset{f(y)}{\underset{\Uparrow}{U}} \subset X$ affinoid open

s.t. • $f(V) \subset U$

• $\underset{A}{\underset{\|}{(\mathcal{O}_X(U), \mathcal{O}_X^+(U))}} \longrightarrow \underset{B}{\underset{\|}{(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))}}$ is finite type.

$B \simeq A\langle T_1, \dots, T_n\rangle / (\text{ideal})$ as topological ring

define $\left[\begin{array}{l}\text{finite type} \\ \text{loc. of fin. presentation} \\ \text{fin. presentation}\end{array}\right.$

f étale $\overset{\text{def}}{\iff}$ $\forall y \in Y$, $\exists\, y \in V \overset{\subset}{} \operatorname{Spa} B$, $\exists\, f(y) \in U \overset{\subset}{} \operatorname{Spa} A$

s.t. $B^{\triangleright} \simeq A^{\triangleright}\langle T_1, \dots, T_n \rangle / (f_1, \dots, f_n)$ as t.f.p. $A^{\triangleright}$-alg

s.t. $\det\left(\frac{\partial f_i}{\partial T_j}\right)$ is invertible in RHS.

f smooth $\overset{\text{def}}{\iff}$ $\forall y \in Y$, $\exists\, y \in V \subset Y$ open nbhd of y

$\exists\, f(y) \in U \subset X$ open nbhd of $f(y)$, $U = \operatorname{Spa} A$

s.t. $f(V) \subset U$, $\exists\, n$,

$$\begin{tikzcd} V \arrow[r, "\exists\ \text{étale}"] \arrow[d, "f"'] & \operatorname{Spa} A\langle T_1, \dots, T_n \rangle \arrow[dl] \\ U = \operatorname{Spa} A & \end{tikzcd}$$

Def k non-arch. field. Rigid space over k is an adic space locally of finite type over $\operatorname{Spa}(k, k^{\circ})$.

k°: valuation ring of k

Rigid space over k is locally isom. to $\operatorname{Spa}(A, A^{\circ})$, $A \simeq k\langle T_1, \dots, T_n \rangle / (f_1, \dots, f_m)$.

Lecture 3 §5. Étale coh. of adic spaces

Def X: analytic adic sp.

$X_{\text{ét}}$: étale site of X

- as cat, $X_{\text{ét}}$: cat. of étale mor. $Y \to X$

  mor. = mor. $/X$.

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\text{ét}"'] & & Y' \arrow[dl, "\text{ét}"] \\ & X & \end{tikzcd}$$

automatically étale

$\{Y_\alpha \xrightarrow{f_\alpha} Y\}_\alpha$ (over X) is a covering $\overset{\text{def}}{\iff}$ $Y = \bigcup_\alpha \underbrace{f_\alpha(Y_\alpha)}_{\text{open}}$

Def (abelian) sheaf on $X_{\bar{e}t}$ is a contravariant functor

$F : X_{\bar{e}t} \longrightarrow Ab = (\text{abel. gps})$

s.t. $\forall$ covering $(Y_\alpha \to Y)$,

$$0 \longrightarrow F(Y) \longrightarrow \prod_\alpha F(Y_\alpha) \rightrightarrows \prod_{\alpha, \beta} F(Y_\alpha \times_Y Y_\beta) \quad \text{is exact.}$$

(fiber product $X \times_S Y$ exists if $X \to S$ is loc. of finite type)

Cat. of abelian sheaves is an abelian cat. w/ enough injectives.

$\Rightarrow$ can construct the right derived functor of $\mathcal{F} \mapsto \Gamma(X; \mathcal{F}) := \mathcal{F}(X \xrightarrow{id} X)$

$0 \to \mathcal{F} \to \mathcal{I}^\bullet$ injective resolution. $R\Gamma(X; \mathcal{F}) := \Gamma(X, \mathcal{I}^\bullet) \in D(Ab)$

$H^i(X; \mathcal{F}) =$ i-th coh. of $R\Gamma(X; \mathcal{F})$

$=$ i-th étale coh.

We can also define [inverse image f^*
(derived) pushforward Rf_*

w.r.t. mor. $f : X \to Y$.

Q. How do we compute ét coh. of adic spaces?

--- use [formal nearby cycles
comparison theorem.

§6. formal nearby cycles

Nearby cycles for schemes V: CDVR w/ sep. closed residue field.

$K = \operatorname{Frac} V$, $S = \operatorname{Spec} V$, $\bar{K}$ = sep. closure of K

$$s \longrightarrow S \longleftarrow \eta \longleftarrow \bar{\eta}$$

closed pt $\qquad$ $\eta = \operatorname{Spec} K$ $\qquad$ $\bar\eta = \operatorname{Spec} \bar{K}$

classical analogue $(/\mathbb{C})$ $\longleftrightarrow$

$X \longrightarrow S$ scheme $\rightsquigarrow$

$$X_s \xrightarrow{i} X \xleftarrow{j} X_\eta \xleftarrow{\pi} X_{\bar\eta}, \qquad \bar{j} : X_{\bar\eta} \to X$$

Def $\mathcal{F}$ (abelian étale) sheaf on X_η, $\quad R\Psi \mathcal{F} := i^* R\bar{j}_* (\pi^* \mathcal{F})$ nearby cycle complex $\in D(X_s)$

Properties. ① $X = S \Rightarrow R\Psi \mathcal{F} = \mathcal{F}_{\bar\eta}$

∵ $R\Gamma(s, R\Psi\mathcal{F}) = (R\Psi\mathcal{F})_s = (R\bar{j}_*(\pi^*\mathcal{F}))_s = R\Gamma(S, R\bar{j}_*(\pi^*\mathcal{F}))$

$= R\Gamma(\bar\eta, \pi^*\mathcal{F}) = \mathcal{F}_{\bar\eta}$.

② $X \xrightarrow{f} Y$ over S $\qquad$ f proper, $\mathcal{F}$ torsion

$$\Rightarrow Rf_{s*} R\Psi_X \mathcal{F} \xrightarrow{\sim} R\Psi_Y Rf_{\eta *} \mathcal{F}.$$

(proper base change)

Put $Y = S$, If $X \to S$ proper, $R\Gamma(X_s, R\Psi\mathcal{F}) \simeq R\Gamma(X_{\bar\eta}, \pi^*\mathcal{F})$

Under properness assumption, (coh. of $R\Psi$) = (coh. of geom. generic fiber)

③ f smooth, $\mathcal{F}$ - $\mathbb{Z}/n\mathbb{Z}$ sheaf, n invertible in S

$$\Rightarrow f_s^* R\Psi_Y \mathcal{F} \xrightarrow{\sim} R\Psi_X f_\eta^* \mathcal{F} \quad \text{(smooth base change)}$$

Put $Y = S$, If $X \to S$ is smooth.

$$M \simeq R\Psi M \quad , \quad M \text{ ab. gp} \quad (\text{constant sheaf})$$
$(\mathbb{Z}/n\mathbb{Z}\text{-mod})$

By ② & ③, $f : X \to S$ proper, smooth.

$$R\Gamma(X_s, M) \simeq R\Gamma(X_{\bar\eta}, M)$$

<u>Formal nearby cycles</u>. $\mathfrak{X}$ formal scheme over $\mathrm{Spf}\, V$.

Want to construct $R\Psi$ for $\mathfrak{X}$.

$$t(\mathfrak{X})_{\bar\eta} := t(\mathfrak{X}) \times_{\mathrm{Spa}(K,V)} \mathrm{Spa}(\bar K, \bar V)$$
$\bar V$ val. ring of $\bar K$

$$\begin{tikzcd} (\mathfrak{X}_{\mathrm{red}})_{\acute{e}t} \arrow[r, "i", "\sim"'] & \mathfrak{X}_{\acute{e}t} & t(\mathfrak{X})_{\eta, \acute{e}t} \arrow[l, hook', "\lambda"'] & t(\mathfrak{X})_{\bar\eta, \acute{e}t} \arrow[l, "\pi"'] \arrow[ll, bend right, "\bar\lambda"'] \end{tikzcd}$$
diagram of sites.

$$\lambda : (\mathcal{Y} \xrightarrow{\acute{e}t} \mathfrak{X}) \mapsto (t(\mathcal{Y})_\eta \xrightarrow{\acute{e}t} t(\mathfrak{X})_\eta)$$

Def. $\mathcal{F}$ sheaf on $t(\mathfrak{X})_{\eta, \acute{e}t}$, $R\Psi \mathcal{F} := i^* R\bar\lambda_* \pi^* \mathcal{F}$ formal nearby cycle complex

Prop. $R\Gamma(\mathfrak{X}_{\mathrm{red}}, R\Psi\mathcal{F}) \simeq R\Gamma(t(\mathfrak{X})_{\bar\eta}, \pi^*\mathcal{F})$ (no assumption, because i is an isom of sites)

Thm. X scheme of finite type $/V$, $Y \subset X_s$ closed subscheme.

$\mathfrak{X} := X^\wedge_{/Y}$. $\rightsquigarrow$ $\varepsilon : (t(\mathfrak{X}), \mathcal{O}_{t(\mathfrak{X})}) \to (X, \mathcal{O}_X)$ mor. of loc. ringed space.

ε induces mor. of sites $t(\mathfrak{X})_{\eta, \acute{e}t} \to X_{\eta, \acute{e}t}$

- $Y = X_s \Rightarrow$ for torsion sheaf $\mathcal{F}$ on X_η, $\underset{/\mathfrak{X}_{red}}{R\Psi(\varepsilon^* \mathcal{F})} \simeq \underset{/X_s}{R\psi \mathcal{F}}$.

- for general Y, n invertible in V, $\mathcal{F}$ constructible $\mathbb{Z}/n\mathbb{Z}$-sheaf on X_η,

$$\to R\Psi(\varepsilon^* \mathcal{F}) \simeq (R\psi \mathcal{F})|_Y .$$

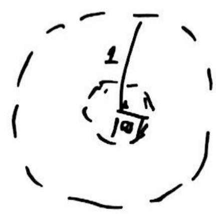
Ex. $X = \mathbb{A}^1_S$, $Y = X_s$, $\rightsquigarrow t(\mathfrak{X})_{\bar\eta} = \mathbb{D}^1_{\bar K}$

$$\begin{aligned} H^i(\mathbb{D}^1_{\bar K}; \mathbb{Z}/n\mathbb{Z}) &\overset{\text{Prop}}{\simeq} H^i(\mathbb{A}^1_s, R\Psi(\mathbb{Z}/n\mathbb{Z})) \\ (n \text{ inv. in } V) \quad &\overset{\text{Thm}}{\simeq} H^i(\mathbb{A}^1_s, R\psi(\mathbb{Z}/n\mathbb{Z})) \\ &\simeq H^i(\mathbb{A}^1_s, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & i=0 \\ 0, & i \geq 1 \end{cases} \end{aligned}$$

By Čech spectral seq., we can also compute $H^i((\mathbb{A}^1_{\bar K})^{ad}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & i=0 \\ 0, & i \geq 1 \end{cases}$

Ex. $X = \operatorname{Spec} V[T,S]/(TS - \varpi)$

$Y = 0 \in X_s = \times$ $\rightsquigarrow t(X)_{\bar\eta} = \{|\varpi| < |T| < 1\}^\circ$



By Thm, $H^i(t(\mathfrak{X})_{\bar\eta}, \mathbb{Z}/n\mathbb{Z}) \simeq H^i(Y, R\Psi\, \mathbb{Z}/n\mathbb{Z})$

$$\begin{aligned} &\simeq H^i(\underset{\text{pt}}{\overset{\shortparallel}{Y}}, (R\psi\, \mathbb{Z}/n\mathbb{Z})|_Y) \\ &= (R^i\psi\, \mathbb{Z}/n\mathbb{Z})_0 \end{aligned}$$

Should compute $R^i\psi\, \mathbb{Z}/n\mathbb{Z}$. If $i > 1$, $R^i\psi\, \mathbb{Z}/n\mathbb{Z} = 0$

$(R^1\psi\, \mathbb{Z}/n\mathbb{Z})|_{X_s \setminus \{0\}} = 0$ (0 is the only sing. pt of X_s)

$Z :=$ blow-up of $\mathbb{P}^1_S$ at $0 \in \mathbb{P}^1_s$.

$X \subset Z$ open $\overset{?}{\shortparallel}\ (R^1\psi_Z\, \mathbb{Z}/n\mathbb{Z})_0$

$$\mathbb{Z}/n\mathbb{Z} \to R\Psi_{\mathbb{Z}}\, \mathbb{Z}/n\mathbb{Z} \to \left(R^1\Psi\, \mathbb{Z}/n\mathbb{Z}\right)_0[-1] \xrightarrow{+1}$$ dist. triangle.

$$\rightsquigarrow H^1(Z_s, R\Psi_Z\, \mathbb{Z}/n\mathbb{Z}) \to H^0(0, R^1\Psi\, \mathbb{Z}/n\mathbb{Z}) \to H^2(Z_s, \mathbb{Z}/n\mathbb{Z}) \to H^2(Z_s, R\Psi_Z\, \mathbb{Z}/n\mathbb{Z})$$

$$H^1(Z_s, R\Psi_Z\, \mathbb{Z}/n\mathbb{Z}) \cong H^1(Z_{\bar\eta}; \mathbb{Z}/n\mathbb{Z}) \cong H^1(\mathbb{P}^1_{\bar\eta}; \mathbb{Z}/n\mathbb{Z}) = 0$$

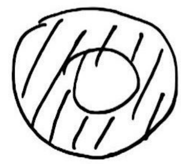
$$H^2(Z_s, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z}(-1))^2, \qquad H^2(Z_s, R\Psi_Z\, \mathbb{Z}/n\mathbb{Z}) \cong H^2(\mathbb{P}^1_{\bar\eta}: \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}(-1)$$

<u>Consequence</u>

$$\left(R^1\Psi\, \mathbb{Z}/n\mathbb{Z}\right)_0 \simeq \mathbb{Z}/n\mathbb{Z}(-1)$$

$$\left(R^0\Psi\, \mathbb{Z}/n\mathbb{Z}\right)_0 \simeq \mathbb{Z}/n\mathbb{Z}$$

$$H^i(t(*)_{\bar\eta}; \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & i=0 \\ \mathbb{Z}/n\mathbb{Z}(-1), & i=1 \\ 0, & i\geq 2 \end{cases}$$



Similarly, we can compute coh. of annulus w/ boundary. $\{|\varpi| \leq |T| \leq 1\}$

$(X = $ as above, $Y = X_s)$.

<u>Cor.</u> X proper $/K$. $\mathcal{F}$: torsion sheaf on X

$$\Rightarrow H^i(X_{\bar K}; \mathcal{F}) \simeq H^i(X^{ad}_{\bar K}, \mathcal{F}^{ad})$$

<u>Proof</u>.
$$\begin{array}{ccc} X & \hookrightarrow & \bar X \\ \downarrow & & \downarrow \text{proper} \\ \eta & \longrightarrow & S \end{array}$$
take compactification $/S$

$$\Rightarrow X^{ad} \simeq t(\bar X^\wedge)_\eta$$

$$\therefore H^i(X^{ad}_{\bar K}, \mathcal{F}^{ad}) \simeq H^i(t(\bar X^\wedge)_{\bar\eta}, \varepsilon^*\mathcal{F})$$

$$= H^i(X_s, R\Psi\mathcal{F}) = H^i(X_{\bar K}, \mathcal{F}). \quad \square$$

§7. Comparison theorem

V, K as above. $f: X \to Y$ mor. of schemes of finite type $/K$.

$\rightsquigarrow$ comm. diag.

$$\begin{array}{ccc} X^{ad}_{\acute{e}t} & \xrightarrow{f^{ad}} & Y^{ad}_{\acute{e}t} \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ X_{\acute{e}t} & \xrightarrow{f} & Y_{\acute{e}t} \end{array} \quad \rightsquigarrow \quad \varepsilon^* Rf_* \mathcal{F} \longrightarrow Rf^{ad}_* \varepsilon^* \mathcal{F}$$

comparison map.

<u>Thm.</u> Comparison map is isom. if one of the following holds:

- f proper, $\mathcal{F}$: torsion sheaf on $X_{\acute{e}t}$
- n invertible in $\underline{K}$, $\mathcal{F}$: constr. $\mathbb{Z}/n\mathbb{Z}$-sheaf.

<u>Cor.</u> X scheme of finite type $/K$, $\mathcal{F}$ sheaf on $X_{\acute{e}t}$.

$\Rightarrow H^i(X_{\bar{K}}; \mathcal{F}) \simeq H^i(X^{ad}_{\bar{K}}; \mathcal{F}^{ad})$ if

- X proper $/K$, $\mathcal{F}$ torsion
- n invertible in K, $\mathcal{F}$ constr. $\mathbb{Z}/n\mathbb{Z}$-sheaf.

§8. Finiteness result

<u>Thm.</u> k sep. closed non-arch. field, X rigid space over k. n : invertible in k° (val. ring of k)

$\mathcal{F}$ constr. $\mathbb{Z}/n\mathbb{Z}$-sheaf on X $\Rightarrow H^i(X; \mathcal{F})$ fin. gen. $\mathbb{Z}/n\mathbb{Z}$-mod if char $k=0$

or X smooth.

but little is known about finiteness of $Rf_* \mathcal{F}$.

<u>Lecture 4.</u> <u>§9. Compactly supported cohomology</u>

$$\mathcal{M}^{rig} = \coprod_{h \in \mathbb{Z}} \mathcal{M}^{rig}_h \rightsquigarrow H^i(\mathcal{M}^{rig}) = \prod_{h \in \mathbb{Z}} H^i(\mathcal{M}^{rig}_h) \text{ too large.}$$

$$H^i_c(\mathcal{M}^{rig}) = \bigoplus_h H^i_c(\mathcal{M}^{rig}_h)$$

Def. $f : (A^\triangleright, A^+) \to (B^\triangleright, B^+)$ its hom. of complete Tate rings.

f : weakly finite type $\overset{\text{def}}{\iff}$ $B^\triangleright \simeq A^\triangleright \langle T_1, \cdots, T_n \rangle / (\text{ideal})$ as top. $A^\triangleright$-algs.

f : $^+$weakly fin. type $\overset{\text{def}}{\iff}$ w. finite type $+$ $\exists\ E \subset B^+$ finite set

s.t. $B^+ =$ minimal int. closed open subring of $B^\triangleright$

containing $f(A^+)$ and E.

$\left(= \text{int. closure of } \ f(A^+)\left[(B^\triangleright)^{\circ\circ} \cup E \right] \right)$

Def. $f : X \to Y$ mor. of analytic adic sp.

- f separated $\overset{\text{def}}{\iff}$ f loc. w. fin. type and the image of $\Delta : X \to X \times_Y X$ is closed.
- f partially proper $\overset{\text{def}}{\iff}$ f separated & loc. $^+$weakly fin. type & <u>universal specializing</u>.

<u>Specializing</u> : $\forall x \in X$, $\forall y' \in \overline{\{f(x)\}}$, $\exists\ x' \in \overline{\{x\}}$, s.t. $y' = f(x')$.

$$\begin{tikzcd} x \arrow[r, rightsquigarrow, "\text{sp.}"] \arrow[d, mapsto] & x' \arrow[d, mapsto] \\ y \arrow[r, rightsquigarrow, "\text{sp.}"] & y' \end{tikzcd}$$

- f proper $\overset{\text{def}}{\iff}$ $^+$weakly fin. type, separated, univ. closed

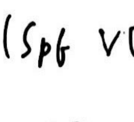
$\iff$ quasi-compact $+$ partially proper.

Ex. V : CDVR, $K = \operatorname{Frac} V$.

X scheme, sep. of fin. type $/K$ $\Rightarrow X^{ad} \to \operatorname{Spa}(K, V)$ partially proper.

If moreover, X is proper $/K$, then $X^{ad} \to \operatorname{Spa}(K, V)$ is proper.

$t(\operatorname{Spf} V[[T]])_\eta \longrightarrow \operatorname{Spa}(K, V)$ partially proper.



$t(\operatorname{Spf} V\langle T \rangle)_\eta \longrightarrow \operatorname{Spa}(K, V)$ <u>NOT</u> partially proper.

$\mathbb{D}^1$

Def $f : X \to Y$ partially proper mor., $\mathcal{F}$ sheaf on $X_{\text{ét}}$.

- $\Gamma_c(X/Y ; \mathcal{F}) := \{ s \in \Gamma(X, \mathcal{F}) : \operatorname{supp} s \text{ is q-cpt} \}$
$$\begin{array}{c}\operatorname{supp} s \\ \downarrow \\ Y\end{array}$$

- $f_! \mathcal{F}$: sheaf on $Y_{\text{ét}}$, defined by $(f_! \mathcal{F})(V \to Y) = \Gamma_c(V \times_Y X / V ; \mathcal{F})$.

$R\Gamma_c(X/Y, -)$, $Rf_!$: right derived functors.

If f is not partially proper, we need <u>compactification</u>.

<u>Universal compactification</u>

Def. X adic sp. is taut

$\overset{\text{def}}{\Longleftrightarrow}$ X is quasi-separated & $\forall\, U \subset X$ q-cpt open, $\overline{U}$ is q-cpt.

$f : X \to Y$ is taut $\overset{\text{def}}{\Longleftrightarrow}$ $\forall\, U \subset Y$ taut open, $f^{-1}(U)$ taut.

Ex. q-sep, q-cpt $\to$ taut

partially proper $\Rightarrow$ taut

$\mathfrak{X}$ q-sep., formally of finite type $/V$ $\Rightarrow$ $t(\mathfrak{X})_\eta \to \operatorname{Spa}(K, V)$: taut.

Thm. $f : X \to Y$ loc. $^+$weakly fin. type, sep., taut.

$\Rightarrow$ $X \overset{j}{\hookrightarrow} \exists\, \overline{X}$ universal compactification of f.

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & \overline{X} \arrow[dl, "\overline{f}"] \\ Y & \end{tikzcd}$$

i.e. $\overline{f}$: partially proper, j : q-cpt open immersion

and $\forall$ diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[dl, "f'"] \\ Y & \end{tikzcd}$$
s.t. f' is partially proper,

$$\begin{tikzcd} & \overline{X} \arrow[dr] \arrow[dd, dashed, "\exists!"] & \\ X \arrow[ur] \arrow[dr] & & Y \\ & X' \arrow[ur] & \end{tikzcd}$$

Ex k na field, $A = (A, A^\circ)$ affinoid ring of fi. type over (k, k°)

$A^+ :=$ int. closure of $\underset{\substack{\uparrow \\ \text{top. nilp. elts}}}{k^\circ[A^{\circ\circ}]}$ in A.

$\Rightarrow \underset{\substack{\uparrow \\ \text{rig. sp.}}}{\mathrm{Spa}(A, A^\circ)} \longrightarrow \underset{\substack{\uparrow \\ \text{not rig. sp.}}}{\mathrm{Spa}(A, A^+)}$ is universal compactification.

If $A = k\langle T\rangle$, $A^+ = k^\circ\left[\sum a_i T^i \in k^\circ\langle T\rangle : |a_i| < 1\right]^{\text{int. clos.}}$

$$\underset{\substack{\| \\ \mathbb{D}^1}}{\mathrm{Spa}(A, A^\circ)} = \underbrace{\left\{ v \in \mathrm{Spa}(A, A^+) : \begin{array}{l} v(T) \le v(1) \\ v(p) \le v(1) \end{array} \right\}}_{\text{rat'l subset}} \subset \underset{\substack{\| \\ \overline{\mathbb{D}^1}}}{\mathrm{Spa}(A, A^+)}.$$

$\overline{\mathbb{D}^1} \setminus \mathbb{D}^1 = \{w_1\}$, $w_1 : \sum a_i T^i \mapsto \max_i \{ |a_i| \}$

Def $f : X \to Y$ loc. $^+$weakly fin. type, sep., taut.

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$ univ. compactification.

put $Rf_! := R\overline{f}_! \circ \underbrace{j_!}_{\text{ext'n by zero}}$.

If $Y = \mathrm{Spa}(k, k^\circ)$ w/ sep. cl. non-arch. field k,

sheaf on $Y_{\text{ét}}$ $\longleftrightarrow$ abel. gps.

$Rf_! \longleftrightarrow R\Gamma_c(X, -)$.

$H^i_c(X, -) =$ i-th coh. of $R\Gamma_c(X, -)$.

Rmk. For torsion sheaves

$$\left[\begin{array}{l} R(g\circ f)_! = Rg_! \circ Rf_! \\ \\ Rf_! \text{ can be computed using arbitrary compactification.} \end{array}\right.$$

Properties.

Prop. $X \longrightarrow \mathrm{Spa}(k, k^\circ)$ loc. + weakly fin. type, sep., taut.

$X = \bigcup_{\lambda \in \Lambda} U_\lambda$: filtered union of taut opens (e.g. q-cpt opens)

$\rightarrow H^i_c(X, \mathcal{F}) \simeq \varinjlim_{\lambda} H^i_c(U_\lambda ; \mathcal{F}).$

$U \hookrightarrow V$ open immersion

$\rightarrow H^i_c(U) \rightarrow H^i_c(V)$

Thm V: CDVR w/ sep. closed res. field.

$X \longrightarrow \mathrm{Spec}\, V$ separated of fin. type mor. of schemes.

$\mathfrak{X} = X^\wedge$, $\mathcal{F}$: torsion sheaf on X_η,

$\rightarrow H^i_c(t(\mathfrak{X})_{\bar{\eta}} ; \varepsilon^* \mathcal{F}) \simeq H^i_c(X_s, R\Psi \mathcal{F}).$

Ex. $H^i_c(\mathbb{D}^1, \mathbb{Z}/n\mathbb{Z}) \simeq H^i_c(\mathbb{A}^1 ; \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}(-1), & i=2 \\ 0, & i \neq 2 \end{cases}$

Rmk. Thm above cannot be extended to the case $\mathfrak{X} = \hat{X}_{/Y}$, $Y \underset{\text{closed}}{\subset} X_s$!

If $X = \mathbb{A}^1_V$, $Y = pt \subset X_s$.

$H^i_c(Y, R\Psi\, \mathbb{Z}/n\mathbb{Z}|_Y) \not\simeq H^i_c(t(\mathfrak{X})_{\bar{\eta}}, \mathbb{Z}/n\mathbb{Z})$

$\qquad \| \qquad\qquad\qquad\qquad \not\simeq \longleftarrow$ NOT isom. if $i=0,2$.

$H^i(Y, R\Psi\, \mathbb{Z}/n\mathbb{Z}|_Y) \simeq H^i(t(\mathfrak{X})_{\bar{\eta}} ; \mathbb{Z}/n\mathbb{Z})$

Thm. (Comparison thm)

V : CDVR, $K = \mathrm{Frac}(V)$, $f : X \to Y$ sep. mor. of schemes of fin. type $/K$.

$$\begin{tikzcd} X^{ad}_{\acute{e}t} \arrow[r, "f^{ad}"] \arrow[d, "\varepsilon"'] & Y^{ad}_{\acute{e}t} \arrow[d, "\varepsilon"] \\ X_{\acute{e}t} \arrow[r, "f"'] & Y_{\acute{e}t} \end{tikzcd}$$

For torsion sheaf $\mathcal{F}$ on $X_{\acute{e}t}$,

$$\varepsilon^* Rf_! \mathcal{F} \xrightarrow{\sim} Rf^{ad}_! \varepsilon^* \mathcal{F}.$$

Cor. X scheme sep. of finite type $/K$, $\mathcal{F}$ torsion sheaf on X

$$\Rightarrow H^i_c(X_{\bar{K}} ; \mathcal{F}) \simeq H^i_c(X^{ad}_{\bar{K}} ; \mathcal{F}^{ad}).$$

Thm (finiteness) k sep. closed n.a. field, X : sep. q-cpt. rigid space $/k$,

n : invertible in k°, $\mathcal{F}$: constr. $\mathbb{Z}/n\mathbb{Z}$-sheaf on X.

$$\Rightarrow H^i_c(X ; \mathcal{F}) : \text{fin. gen. } \mathbb{Z}/n\mathbb{Z}\text{-mod}.$$

Thm (Poincaré duality). k sep. closed n.a. field, X purely d-dim'l rigid space $/k$.

sep., smooth, taut $/k$. n : invertible in k°, $\mathcal{F}$: loc. constant $\mathbb{Z}/n\mathbb{Z}$-sheaf of fin. type.

$$\Rightarrow H^{2d-i}(X, \mathcal{F}^\vee(d)) \simeq H^i_c(X, \mathcal{F})^\vee.$$

§10. ℓ-adic coh.

Want to consider $H^i_c(X, \mathbb{Z}_\ell)$

$$H^i_c(X, \mathbb{Q}_\ell) := H^i_c(X ; \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Def. $\left\{ \begin{array}{l} \text{proj. sys. of sheaves } (\mathcal{F}_n) \\ \text{s.t. } \mathcal{F}_n : \mathbb{Z}/\ell^{n+1}\mathbb{Z}\text{-mod} \end{array} \right\} \longrightarrow Ab$

$$(\mathcal{F}_n) \longmapsto \varprojlim \Gamma_c(X, \mathcal{F}_n)$$

X partially proper

$\Rightarrow H_c^i(X, (\mathcal{F}_n)) :=$ i-th derived functor of it

X general

$\Rightarrow H_c^i(X, (\mathcal{F}_n)) := H_c^i(\bar{X}; (j_! \mathcal{F}_n))$

$j : X \hookrightarrow \bar{X}$: univ. compactification.

- X q-cpt. rig. sp. $/ k = \bar{k}$, sep. taut

$\Rightarrow H_c^i(X; \mathbb{Z}_\ell) = \varprojlim_n H_c^i(X, \mathbb{Z}/\ell^n\mathbb{Z})$

- X rig. sp. $/ k = \bar{k}$, sep., taut.

$\Rightarrow H_c^i(X; \mathbb{Z}_\ell) = \varinjlim_{\substack{U \subset X \\ \text{q-cpt open}}} H_c^i(U, \mathbb{Z}_\ell)$

$K = \mathrm{CDVF}$, X: rig. sp. , sep. taut $/K$.

$\Rightarrow H_c^i(X_{\bar{K}}, \mathbb{Q}_\ell) := H_c^i(X_{\bar{K}}; \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

$\circlearrowleft$

$\mathrm{Gal}(\bar{K}|K)$ action is cont.

If p-adic gp G acts on X "continuously" $\Rightarrow$ then $G \curvearrowright H_c^i(X_{\bar{K}}; \mathbb{Q}_\ell)$ smooth. action.

get nice rep!