# Ruminations about Langlands duality with generalized coefficients

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$G =$ conn. reductive gp $/ \mathbb{C}$

**Review.** $k =$ comm. $\mathbb{Q}$-alg

$G(\mathcal{O}) = G(\mathbb{C}[[t]])$, $G(F) = G(\mathbb{C}((t)))$, $\mathrm{Gr}_G = G(F)/G(\mathcal{O})$

$\mathrm{Shv}^{\mathrm{ren}}_{G(\mathcal{O})}(\mathrm{Gr}_G, k)$ $\quad \curvearrowleft S^1$

IS derived geom. Satake Bezrukavnikov-Finkelberg-(Mirković)-Ginzburg

$\mathrm{QCoh}((\check{\mathfrak{g}}^*_k)^\wedge_{\mathcal{N}}[2]/\check{G}_k) \underset{\rightsquigarrow\, U_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{G}_k}}{\overset{\text{simply laced}}{\simeq}} \mathrm{QCoh}((\mathfrak{g}_k)^\wedge_{\mathcal{N}}[2]/\check{G}_k)$

Simplification: $G =$ simply-laced, simple-conn'd $\quad G \curvearrowright \mathfrak{g}$, $\check{G} \curvearrowright$

$\check{\mathfrak{g}}^* \simeq \mathfrak{g}$, $\quad \check{G} \approx G/Z(G)$

**Rmk.** Whittaker approach:

$\mathrm{QCoh}((0 \overset{L}{\underset{\check{\mathfrak{g}}_k}{\times}} 0)/\check{G}_k) \hookrightarrow \mathrm{Shv}_{G(\mathcal{O})}(\mathrm{Gr}_G; k)$

**Q:** What if $k =$ comm. ring (spectrum)?

$X$ space $\rightsquigarrow C^*(X;\mathbb{Z}) \rightsquigarrow H^*(X;\mathbb{Z})$

Eilenberg-Steenrod axioms

Functors: $\mathrm{Spc} \longrightarrow \mathrm{GrAb}$

$* \longmapsto \mathbb{Z}$ in every even weight

$X \longmapsto KU^*(X)$ cpx K-theory

$$\begin{tikzcd} & X \mapsto k^*(X) & \\ \mathrm{Spc} \arrow[rr] \arrow[dr] & & \mathrm{GrAb} \\ & \mathrm{Sp} \arrow[ur] & \pi_* A \\ X \arrow[r, mapsto] & C^*(X;k) = \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, k) & A \arrow[u, mapsto] \end{tikzcd}$$

"Spectra linearize spaces"

Just as one can form $\mathrm{Shv}(X;\mathbb{Z})$, can also form $\mathrm{Shv}(X;k)$
$k$ comm. ring spectrum

Rmk. Subtle to extend to stacks.

Ex. $KU$ = cpx $K$-theory.

$$\begin{array}{ccc} KU^*(BS^1) & \simeq & \mathbb{Z}[[x-1]][\beta^{\pm 1}] \quad \text{Bott class in } KU^{-2}. \\ \| & & \uparrow \\ \mathbb{CP}^\infty & & \mathcal{O}(1) \\ \uparrow & & \\ \underbrace{\pi_*(KU_{S^1})}_{\text{equiv. } \mathbb{C}\text{-}K\text{-theory}} & \simeq & \underbrace{\mathbb{Z}[x^\pm]}_{R_{\mathbb{C}}(S^1)}[\beta^\pm] \end{array}$$

Thm (Atiyah-Segal) $KU_G(X) \to KU(X_{hG})$ is a completion.

Deep insight (Quillen, Morava, ...) $k$ = comm. ring spectrum s.t. $\pi_* C^*(BS^1;k) \cong \pi_*(k)[t]^\wedge$ cplx-oriented

$$BS^1 \times BS^1 \xrightarrow[\otimes]{} BS^1$$

$$\pi_*(k)[t_1,t_2]^\wedge \leftarrow \pi_*(k)[t]^\wedge$$

$$F(t_1,t_2) \longleftarrow\!\shortmid\; t = c_1^k(L)$$

$$\| $$

$$c_1^k(L_1 \otimes L_2)$$

$\rightsquigarrow$ $F$ is a (1-dim'l) formal gp law.

Eg. If $k=\mathbb{Z}$, $F(X,Y) = X+Y$

If $k = KU$, $c_1(L) = [L]-1$, $F(X,Y) = X+Y+XY$ $\left(t = \frac{x-1}{\beta}\right)$

$k =$ $\mathbb{C}$-oriented comm. ring spectrum

$G =$ torus $T$

$$\mathrm{Shv}_T(\underbrace{\mathrm{Gr}_T}_{\substack{\pi_1(T) \\ \shortparallel \\ X_*(T)}} ; k) = \bigoplus_{X_*(T)} \mathrm{Shv}_T(pt; k)$$

$$= \bigoplus_{X_*(T)} \mathrm{Mod}(\underbrace{k_T}_{\substack{\text{stand-in for} \\ \text{either "Borel" } C^*(BT; k) \\ \text{or "genuine" } T\text{-equiv. version of } k.}})$$

Take htpy gps of $k_T$:

$$A \qquad \begin{matrix} \tau_{\geq *} A \\ \wr\wr \\ \tau^{\leq -*} A \end{matrix} \qquad \begin{matrix} \pi_* A \\ \wr\wr \\ H_* A \end{matrix} \text{ if } A \in D(\mathbb{Z})$$

und $\rightsquigarrow$ filt. $\rightsquigarrow$ gr (+ shear)

So $\quad k_T \underset{\text{degeneration}}{\overset{\text{1-parameter}}{\rightsquigarrow}} \pi_*(k_T)$

(i.e. $\tau_{\geq *} k_T$)

$$\Rightarrow \mathrm{Shv}_T(\mathrm{Gr}_T; k) \underset{\substack{\text{1-par.} \\ \text{deg.}}}{\rightsquigarrow} \bigoplus_{X_*(T)} \mathrm{Mod}^{gr}(\pi_*(k_T))$$

If $\mathbb{H} = \underset{(\mathrm{Spf})}{\mathrm{Spec}}\ \pi_*(k_{S^1})$, 1-dim (formal) gp over $\underset{(\mathrm{Spf})}{\mathrm{Spec}}\ \pi_*(k_T) = \mathrm{Hom}(X^*(T), \mathbb{H}) =: T_{\mathbb{H}}$

$$\mathrm{QCoh}(T_{\mathbb{H}} \times B\check{T}) \simeq \bigoplus_{X^*(\check{T})} \mathrm{QCoh}^{gr}(T_{\mathbb{H}})$$

defined over $\pi_*(k)$

Summary: 1-par deg.

$$\mathrm{Shv}_{T(\mathcal{O})}(\mathrm{Gr}_T; k) \rightsquigarrow \mathrm{QCoh}^{gr}(T_{\mathbb{H}} \times B\check{T})$$

$k$-linear

(graded) $\pi_*(k)$-linear cat

Q. What replaces $T_{\mathbb{H}}$? (in general)

Def. $\mathbb{H}$ = 1-d (formal) gp scheme / $\pi_*(k)$

$X$ = stack / $\pi_*(k)$

$\mathbb{H}$-loop space $\quad L_{\mathbb{H}} X = \mathrm{Fun}^{\otimes, L}_{\pi_*(k)}\left(\mathrm{QCoh}(X)^{\otimes}, \mathrm{IndCoh}_0(\mathbb{H})^{conv}\right)$

Eg. $\mathbb{H} = \mathbb{G}_a$, $\quad L_{\mathbb{H}} X = T[-1](X)$

$\mathbb{H} = \mathbb{G}_m$, $\quad L_{\mathbb{H}} X = LX = X \times_{\substack{X \times X \\ \pi_*(k)}} X$

Def.

$$\begin{tikzcd} G_{\mathbb{H}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{\mathbb{H}}(BG) = G_{\mathbb{H}}/G \arrow[d] \\ \mathrm{Spec}\, \pi_*(k) \arrow[r] & BG \end{tikzcd}$$

Ex. $\mathbb{H} = \hat{\mathbb{G}}_a$, $\quad G_{\mathbb{H}} = \hat{\mathfrak{g}}_N$

$\mathbb{H} = \hat{\mathbb{G}}_m$, $\quad G_{\mathbb{H}} = \hat{G}_u$

$\mathbb{H} = \underset{\text{ell. curve}}{E}$, $\quad G_{\mathbb{H}} = \mathrm{Bun}_G^{(ss,0)}(E)$ trivialization at base pt

Conj. $k$ = $\mathbb{C}$-oriented comm. ring spectrum. $\quad \mathrm{Shv}_{G(\mathcal{O})}(\mathrm{Gr}_G, k) \overset{\text{1-par}}{\rightsquigarrow} \mathrm{QCoh}(G_{\mathbb{H}}/\check{G})$

$G$ simply-laced, simply conn'd $\quad$ $k$-linear $\quad$ $\pi_*(k)$-linear

Eg. $k = \mathbb{Z}$

$$\mathrm{Shv}^{\mathrm{ren}}_{G(\mathcal{O})}(\mathrm{Gr}_G;\mathbb{Z}) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathfrak{g}}/\check{G})$$

general $\quad \mathcal{C} \quad$ special $/B\mathbb{G}_m$

$\mathbb{A}^1/\mathbb{G}_m$

$$\mathrm{IndCoh}\left(\left(\mathcal{C} \times_{\mathrm{Def}^{\cdot}\mathrm{Lie}(\check{G})} \mathcal{C}\right)/\check{G}\right)$$

$k = KU$.

$$(\mathrm{Borel})\,(\,--\,; KU) \rightsquigarrow \mathrm{QCoh}(G(\tfrac{1}{u})/\check{G})$$

Thm. This is true if $G \neq E_8$, $k = \mathbb{Z}, KU$, elliptic cohomology

and you base-change the categories to ~~an~~ alg. closed field of large-enough char.

"Classically", $\exists$ nat'l object $\check{R} \in \mathrm{Shv}_{G(\mathcal{O}) \rtimes S^1}(\mathrm{Gr}_G;\mathbb{C})$ and a $\delta$-sheaf at the basepoint

s.t. $\mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{G(\mathcal{O})\rtimes S^1}}(\delta, \check{R}) \simeq U_\hbar(\check{\mathfrak{g}})$

Expectations. $\check{R}_k \in \mathrm{Shv}(\,--\,, k)$
$\quad\quad\quad\quad \delta$

$$\pi_* \, R\mathrm{Hom}_{\mathrm{Shv}}(\delta, \check{R}_k) = \, ? =: A$$

Expect $A$ sits inside an algebra; I'll describe for $SL_2$: $\quad \bar{h}$ = inverse of $h$ in $F$

$$U_{\mathbb{H}}(SL_2) = \underbrace{\pi_*(k)[t]^\wedge}_{\pi_* C^k(BS^1;k)} \langle e, f, h \rangle$$

$$fh = (h +_F t) f$$
$$eh = (h -_F t) e$$
$$[e,f] = h(t -_F h) - \bar{h}(h +_F t)$$

Eg. $\mathbb{H} = \mathbb{G}_m$, $U_{\mathbb{H}}(SL_2)$ basically $U_q(SL_2)$

$$A \hookrightarrow U_q(SL_2)$$

$$G \longleftarrow G^\circ$$

$$\begin{tikzcd}
\mathrm{LMod}_A & & \mathrm{LMod}^{gr}_{\pi_* A} \\
& \mathrm{LMod}^{fin}_A \arrow[ul] \arrow[ur, dash] &
\end{tikzcd}$$