B_{dR} affine grassmannian

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Want analogue of the following:

X smooth proper curve $/\mathbb{C}$, $x \in X$, geom. pt

for any $\mathbb{C}$-alg. R, X_R base change, x_R Cartier divisor

complete local ring: $R[[t]]$

Let $D = \mathrm{Spec}\, R[[t]]$, $\mathring{D} = \mathrm{Spec}\, R((t))$ punctured local disk.

Affine grassmannian: Gr_G is the étale sheafification of the functor

$$\{\mathbb{C}\text{-alg.}\} \longrightarrow \mathrm{Sets}, \quad R \longmapsto G(R((t)))/G(R[[t]]) \qquad G/\mathbb{C} \text{ reductive gp}$$

this parametrizes (P, γ): P is a G-bundle on D, $\gamma: P|_{D^\circ} \simeq P^\circ|_{D^\circ}$, P°: trivial G-bundle.

<u>Thm</u> (Beauville - Laszlo): $\exists$ equiv. of cats

$$\{(M_{\mathring{X}}, M_D, \beta)\} \xrightarrow{\sim} \{M_X\}$$

M°_X = v.b. on $\mathring{X} = X \setminus \{x\}$, M_D = v.b. on D, $\beta: M_{\mathring{X}}|_{\mathring{D}} \xrightarrow{\sim} M_D|_{D^\circ}$.

M_X = v.b. on X

"glue vector bundle defined on $\mathring{X}$ w/ v.b. defined on D"

$R \longrightarrow \hat{R}$ is not flat if R is not noetherian.

Restate in terms of rings: If $f \in R$ non-zero divisor, $\hat{R} = f$-adic completion,

then cat. of R-modules M where f is not a zero divisor is equiv. to

$(M_{\hat{R}}, M[f^{-1}], \beta)$.

$\uparrow$ f not zero divisor β isom.

M finite proj. $\Leftrightarrow$ $M_{\hat{R}}, M[f^{-1}]$ finite proj.

Can consider map $Gr_G \xrightarrow{BL} Bun_G$

by gluing the trivial bundle on $X_R \setminus \{x_R\}$ to $g \in Gr_G(R)$.

Want this for $X = X_{FF}$ adic curve.

Def. B_{dR} affine grassmannian Gr_G is the étale sheafification of the functor on Perf

$S = Spa(R, R^+)$ w/ a map $\underbrace{S \to Spd\, \mathbb{C}_p}_{\text{defines an untilt } R^\# \text{ + map to } Spa\, \mathbb{C}_p}$ to $G(B_{dR}(R^\#))/G(B_{dR}^+(R^\#))$

What is $B_{dR}^+(R^\#)$?

If $(R^\#, R^{\#+})$ perfectoid Tate-Huber pair,

$\theta: W(R^+) \longrightarrow R^{\#+}$, kernel is (ξ) ξ non-zero divisor

$W(R^+) = A_{inf}(R^{\#+})$ ϖ uniformizer of R

$B_{dR}^+(R^\#) = \xi$-adic completion of $W(R^+)[[\varpi]^{-1}]$

this is a filtered ring w/ filtration $Fil^i B_{dR}^+(R^\#) = \xi^i B_{dR}^+(R^\#)$

when $R^\# = \mathbb{C}_p$, $B_{dR}^+(\mathbb{C}_p) = B_{dR}^+$ Fontaine's period ring.

is a complete DVR, abstractly isom. to $\mathbb{C}_p[[\xi]]$

$B_{dR}(R^\#) = B_{dR}^+(R^\#)[\frac{1}{\xi}]$

$(R^\#, R^{\#+})$ defines an untilt of (R, R^+), and hence defines a divisor $S^\#$ on X_S.

Prop The complete local ring $\hat{\mathcal{O}}_{X_S, S^\#}$ is $B_{dR}^+(R^\#)$.

Pf. $B_{dR}^+(R^\#)$ is the completion $\hat{\mathcal{O}}_{Y_S, S^\#}$, passing to quotient get prop.

Prop. Gr_G is the functor taking $S \in Perf$ w/ untilt $S^\#$ to

$$\{(P, \gamma) : P \text{ } G\text{-torsor on } \operatorname{Spec} B_{dR}^+(R^\#),\ \gamma \text{ trivialization on } \operatorname{Spec} B_{dR}(R^\#)\}$$

[Prop: any G-torsor $\mathcal{E}$ on $\operatorname{Spec} B_{dR}^+(R^\#)$ is locally trivial for étale top. on $\operatorname{Spa}(R^\#, R^{\#+})$]

Get: $BL : Gr_G \longrightarrow Bun_G$

by applying Beauville-Laszlo to glue P to trivial bundle on $X_S \setminus S^\#$.

Prop Gr_G is a v-stack.

$G/\mathbb{C}_p$ reductive gp, Schubert varieties $\overset{\text{fix}}{T} \subset B \subset G$

$\uparrow$ max. torus

Have a decomposition

$$Gr_G(C^b) = \coprod_{\mu \in X_*^+(T)} G(B_{dR}^+(C))\, \mu(\xi)$$

C complete non-arch. field, alg. closed

from the Cartan decomposition, via the isom. $B_{dR}(C) \simeq C((\xi))$.

Define $\mu \in X_*^+(T)$, have subfunctor $Gr_\mu : S \in Perf \longmapsto$ subset of $Gr_G(S)$ s.t. $\forall$ geom. pts $x = \operatorname{Spa}(C(x), C(x)^+)$,

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the $C(x)$-valued pt lies in $G(B^+_{dR}(C(x)^\sharp))\cdot\mu(\xi)$.

$Gr_{\le\mu} : S \in Perf \longmapsto$ ———''——— lies in $\coprod_{\mu'\le\mu} G(B^+_{dR}(C(x)^\sharp))\cdot\mu(\xi)$.

<u>Prop</u>. $Gr_{\le\mu}$ is closed subfunctor, $Gr_\mu \subset Gr_{\le\mu}$ is open.

Moreover, $Gr_{\le\mu}$ is proper, locally spatial diamond.

<u>Prop</u> (Bialynicki-Birula map)

$\forall \mu \in X_*^+(T)$, $\exists$ map $\pi_\mu : Gr_\mu \to Fl^\diamond_{G,\mu}$ $\leftarrow$ diamond assoc. w/ flag variety G/P_μ

When μ is minuscule, this is an isom.

(Sketch): reduce to $G = GL_n$ by tannakian formalism.

$Gr_\mu(R,R^+) \ni L$ lattice in $B_{dR}(R^\sharp)^n$ w/ rel. pos. $\mu = (m_1,\dots,m_n)$

Look at $Fil^i_L = \xi^i L \cap B^+_{dR}(R^\sharp)^n / (\xi^i L \cap \xi B^+_{dR}(R^\sharp)^n)$ on $(R^\sharp)^n$.

$\uparrow$

Each of these is fin. proj. $R^\sharp$-module.

and filtration type given by μ.

When μ is minuscule, $Gr_\mu = Gr_{\le\mu}$ proper.

$Fl^\diamond_{G,\mu}$ is proper.

to show isom., suffices to show bijection on (C,C^+)-points.

follows from $B^+_{dR}(C) \cong C[[\xi]]$ as in the classical case.

Come back to BL

<u>Prop</u>. $|BL| : |\mathrm{Gr}_\mu| \longrightarrow |\mathrm{Bun}_G|$ has image in $B(G,\mu)$.

$$B(G,\mu) = \{ b \in B(G) : \nu_b \le \bar{\mu} ,\ \kappa(b) = \mu^\flat \}$$

image of μ in $\pi_1(G)_\Gamma$.

(sketch): I: $\nu_b \le \bar{\mu}$: reduce to $G = GL_n$.

Statement becomes: for v.b. $\mathcal{E}$, we have smallest newton slope of $\wedge^m(\mathcal{E})$

$\geq$ smallest Hodge slope of $\wedge^m(\mathcal{E})$,

and newton and Hodge slopes of $\wedge^n(\mathcal{E})$ agree.

$$\wedge^n \mathcal{E} = \mathcal{O}_X(d)$$

Newton slope is d

$\mathcal{O}_X(d)$ is the modification given by lattice $\xi^{-d} B^+_{dR}$, so it has slope d.

II. $\kappa(b) = \mu^\flat$: reduce to G = torus.