Singular support for G-categories and applications to W-algebras

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Plan

1) Define singular support for G-categories

2) Whittaker coefficients for character sheaves

3) Applications to W-algebras

Let G be a connected reductive gp $/k = \bar{k}$, char $k = 0$.

Fix $(\cdot,\cdot) \in S^2 \mathfrak{g}^*$

Let $(D(G), *)$ (unbounded dg-) cat. of D-modules on G.

__Def__

A __G-category__ is a (dualizable) cat. $\mathcal{C}$ w/ an action $D(G) \otimes \mathcal{C} \to \mathcal{C}$.

__Recall__ $V \in \mathrm{Rep}(G) \rightsquigarrow \mathrm{End}(V) \xrightarrow{MC} \mathrm{Fun}(G)$

$$\varphi \longmapsto [g \mapsto \mathrm{tr}(\varphi \circ g)]$$

__Similarly__. Given G-cat. $\mathcal{C} \rightsquigarrow \mathrm{End}(\mathcal{C}) \xrightarrow{MC} D(G)$

$$\varphi \longmapsto (\text{fiber at } g \text{ is } \underbrace{\mathrm{tr}(\varphi \circ g)}_{\text{cat. trace}})$$

Given $Z \subset \mathfrak{g}^*$ closed, conical, Ad-invariant

$$\rightsquigarrow D_Z(G) = \{ \mathcal{F} \in D(G) : SS\,\mathcal{F} \subset G \times Z \subset T^*G \}$$

__Def'n__ (categorical singular support) Given G-cat. $\mathcal{C}$, we say $SS(\mathcal{C}) \subset Z \subset \mathfrak{g}^*$

if
$$\begin{tikzcd} MC: \mathrm{End}(\mathcal{C}) \arrow[r] \arrow[dr] & D(G) \\ & D_Z(G) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

(analogous to wave front cycle of p-adic reps)

__Examples.__ • $\mathcal{C} = D(G)$: $SS(\mathcal{C}) = \mathfrak{g}^*$

• $\mathcal{C} = \langle \text{cst sheaf} \rangle$: $SS(\mathcal{C}) = \{0\}$

• $\mathcal{C} = D(G/B)$: $SS(\mathcal{C}) = \mathcal{N} \subset \mathfrak{g}^*$

More generally, $G \curvearrowright X$ s.t. $\mu: T^*X \longrightarrow \mathcal{N} \subset \mathfrak{g}^*$. then $SS(D(X)) = \overline{\mathrm{Im}(\mu)}$.

• $\mathcal{C} = \{ M \in \mathfrak{g}\text{-mod}_0 : V(U\mathfrak{g}/\mathrm{Ann}_{U\mathfrak{g}}(M)) \subset \overline{\mathbb{O}} \}$

Then $SS(\mathcal{C}) = \overline{\mathbb{O}}_{sp} = \bigcup$ closure of special nilpotent orbits $\subset \overline{\mathbb{O}}$.

__Def'n.__ We say $\mathcal{C}$ __nilpotent__ if $SS(\mathcal{C}) \subset \mathcal{N}$. (main case of interest).

__Goal (§1)__ characterize $SS(\mathcal{C})$ in terms of vanishing of its Whittaker models.

__Whittaker models__:

Given $e \in \mathbb{O} \subset \mathcal{N}$, extend it to $\mathfrak{sl}_2$-triple $\{e, h, f\}$

$\rightsquigarrow \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$

• $\psi_e = (e, \cdot) \in \mathfrak{g}^*$

• $\mathfrak{u}_e = \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus \mathfrak{l}$, $\mathfrak{l} \subset \mathfrak{g}(-1)$ Lagrangian

- $\mathcal{U}_e \subset G$ subgp s.t. $\mathrm{Lie}(\mathcal{U}_e) = \mathfrak{u}_e$
- $\psi_e : \mathcal{U}_e \to \mathbb{A}^1$ char.

$$D(\mathcal{U}_e) \to \mathrm{Vect},\ \mathcal{F} \mapsto C_{dR}(\mathcal{U}_e, \mathcal{F} \overset{!}{\otimes} \psi_e^!(\exp))$$

Given G-cat. $\mathcal{C}$ $\rightsquigarrow$ $\mathrm{Whit}_{\mathbb{O}}(\mathcal{C}) := \mathcal{C}^{\mathcal{U}_e, \psi_e}$ $\left(= \mathrm{Hom}_{D(\mathcal{U}_e)}(\mathrm{Vect}_{\psi_e}, \mathcal{C})\right)$

__Thm.__ (Dhillon - F.) Let $\mathcal{C}$ be nilpotent G-cat.

1) If $\mathbb{O} \not\subset SS(\mathcal{C})$, then $\mathrm{Whit}_{\mathbb{O}}(\mathcal{C}) = 0$

2) If $\mathbb{O} \subset SS(\mathcal{C})$ max'l, then $\mathrm{Whit}_{\mathbb{O}}(\mathcal{C}) \neq 0$.

(analogous to results in p-adic rep. theory).

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§2. Whittaker coeff. of character sheaves

Consider $G/G = G \overset{Ad}{/} G$, $T^*(G/G) = \{(g, x) \in G \times \mathfrak{g}^* : \mathrm{Ad}_g(x) = x\}/G$

Let $\Lambda = T^*(G/G) \underset{\mathfrak{g}^*}{\times} \mathcal{N}$

For $\mathbb{O}$ nilp. orbit, then $\Lambda_{\overline{\mathbb{O}}} = \Lambda \underset{\mathcal{N}}{\times} \overline{\mathbb{O}}$, $\therefore \Lambda_{\overline{\mathbb{O}}} \subset T^*(G/G)$ lagr., but not conn'd

$\Lambda_{\mathbb{O}, 0} \subset \Lambda_{\overline{\mathbb{O}}}$ component containing $(1, \mathbb{O})$

Let $D_\Lambda(G/G)$ the category of character sheaves.

Given $\mathcal{F} \in D_{\Lambda_{\overline{\mathbb{O}}}}(G/G)$, write $c_{\mathbb{O}, \mathcal{F}} \in \mathbb{Z}$, multiplicity of $\Lambda_{\mathbb{O}, 0}$ in char. cycle of $\mathcal{F}$.

__Goal (§2)__ Access $c_{\mathbb{O}, \mathcal{F}}$ via Whittaker coeffs.

Define Whittaker coeff.

$\mathrm{Coeff}_{\mathbb{O}} : D(G/G) \longrightarrow \mathrm{Vect}$

$$\begin{tikzcd} U_e/U_e \arrow[r, "p"] \arrow[d, "\psi_e"'] & G/G \\ \mathbb{A}^1 & \end{tikzcd}$$

$$\mathrm{Coeff}_{\mathbb{O}}(\mathcal{F}) = C_{dR}\left(U_e/U_e,\ \psi_e^{!}(\exp) \overset{!}{\otimes} p^{!}(\mathcal{F})\right)$$

Thm (Dhillon – F.)

1) $\mathrm{coeff}_{\mathbb{O}} : D_{\Lambda_{\bar{\mathbb{O}}}}(G/G) \longrightarrow \mathrm{Vect}$ t-exact

2) $\chi(\mathrm{Coeff}_{\mathbb{O}}(\mathcal{F})) = c_{\mathbb{O},\mathcal{F}}$.

Rmk $\mathrm{Coeff}_{\mathbb{O}}$ is the microstalk at $(1,e) \in \Lambda_{\bar{\mathbb{O}}}$.

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§3. Applications to W-algebras

$e \in \mathbb{O} \rightsquigarrow$ W-alg. W_e

$U_\psi = \{x - \psi_e(x) : x \in u_e\}$, then $W_e = (Ug/Ug.U_\psi)^{\mathrm{ad}\, U_e}$

Alternatively, $W_e = \mathrm{End}_{Ug}(\mathrm{ind}_{U_e}^{g} \psi_e)$

$Z(Ug) \longrightarrow W_e$ isom. onto $Z(W_e)$

$W_{e,0} = W_e \otimes_{Z(Ug)} k$

$$W_{e,0}\text{-mod} \overset{\text{Skryabin}}{\simeq} \mathfrak{g}\text{-mod}_0^{U_e,\psi_e} \overset{BB}{\simeq} \mathrm{Whit}_{\mathbb{O}}(D(G/B))$$

Let $W_{e,0}\text{-mod}^{fin} \subset W_{e,0}\text{-mod}$ subcat. of finite-dim'l modules

<u>Conj.</u> (Theorem due to Losev–Ostrik and Bezrukavnikov–Losev)

$$K_0(W_{e,0}\text{-mod}^{fin})_k \simeq H^{top}(\mathcal{B}_e) \underset{W}{\otimes} [c] \quad \left(\text{iso. of } \underset{A_{\mathbb{O}}}{W\text{- reps}}\right)$$

Here • $\mathcal{B}_e$ Springer fiber

• $[c] \subset k[W]$ two-sided cell module corresponding to $\mathbb{O}$.

Categorical traces.

Let $\mathcal{C}$ dualizable cat. i.e. $\exists\, \mathcal{C}^\vee : \mathrm{Vect} \xrightarrow{u} \mathcal{C}\otimes\mathcal{C}^\vee$, $\mathcal{C}^\vee\otimes\mathcal{C} \xrightarrow{ev} \mathrm{Vect}$

Define $HH_*(\mathcal{C}) \in \mathrm{Vect}$ as

$$\underbrace{\mathrm{Vect} \xrightarrow{u} \mathcal{C}\otimes\mathcal{C}^\vee \xrightarrow{ev} \mathrm{Vect}}_{-\otimes HH_*(\mathcal{C})}$$

$S^1 \curvearrowright HH_*(\mathcal{C})$
$\rightsquigarrow HP_*(\mathcal{C})$

<u>For us</u>: categorical traces provide strong functoriality.

<u>Eg.</u> • $\mathcal{C} = D(B\backslash G/B)$, $HH_*(\mathcal{C}) = k[W] \otimes \mathrm{Sym}(\mathfrak{t}^*[-1] \oplus \mathfrak{t}^*[-2])$

• $\mathcal{C}$ G-cat. $\rightsquigarrow \chi_{\mathcal{C}} \in D(G/G)$

defined as image of $id_{\mathcal{C}}$ under $MC: \mathrm{End}(\mathcal{C}) \to D(G)$

Then $HH_*(\mathrm{Whit}_{\mathbb{O}}(\mathcal{C})) = \mathrm{Coeff}_{\mathbb{O}}(\chi_{\mathcal{C}})$

- $\mathcal{C} = D(G/B) \rightsquigarrow X_{\mathcal{C}} = \widetilde{Spr}$ Grothendieck - Springer

$$\begin{array}{ccc} HH_*(\mathrm{Whit}_{\mathbb{O}}(D(G/B))) & = & \mathrm{coh}_{\mathbb{O}}(\widetilde{Spr}) \\ \text{Sky. + BB} \ \wr\wr & & \| \\ HH_*(W_{e,0}) & = & H^*(\mathcal{B}_e)[2\dim \mathcal{B}_e] \end{array}$$

(Etingof - Schedler '10)

Consider $D_{\overline{\mathbb{O}}}(B\backslash G/B) \subset D(B\backslash G/B)$

$K_0(D_{\overline{\mathbb{O}}}(B\backslash G/B)) = [c]$ as W-reps

$k[W] = K_0(D(B\backslash G/B))$

Let $D(G/B)^{\overline{\mathbb{O}}} \subset D(G/B)$ biggest ~~smallest~~ G-stable subcat. s.t.

$SS(D(G/B)^{\overline{\mathbb{O}}}) \subset \overline{\mathbb{O}}$

<u>Main observation</u>:

$$\begin{array}{ccc} W_{e,0}\text{-mod} & = & \mathrm{Whit}_{\mathbb{O}}(D(G/B)) \\ \cup & & \cup \\ W_{e,0}\text{-mod}^{fin} & = & \mathrm{Whit}_{\mathbb{O}}(D(G/B)^{\overline{\mathbb{O}}}) \end{array}$$

$H^*(\mathcal{B}e) \otimes_W [c'] = HH_*(W_{e,0}\text{-mod}_{\overline{\mathbb{O}'}})$

$K_0(W_{e,0}\text{-mod}^{fin}) \longrightarrow K_0(W_{e,0}\text{-mod}) \xrightarrow{\text{chern}} HH_0(W_{e,0}\text{-mod}) = H^{top}(\mathcal{B}e)$

Compose w/ $W_{e,0}\text{-mod} \xrightarrow{i^L} W_{e,0}\text{-mod}^{fin}$

$$\rightsquigarrow \begin{tikzcd} K_0(W_{e,0}\text{-mod}^{fin}) \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & H^{top}(\mathcal{B}e) \arrow[r] & HP_0(W_{e,0}\text{-mod}^{fin}) \end{tikzcd}$$

In fact, $K_0(W_{e,0}\text{-mod}^{fin}) \hookrightarrow H^{top}(\mathcal{B}e) \otimes_W [c]$.