p-adic integration, buildings and BPS-invariants

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p-adic integration = integration theory of $\mathbb{R}$-valued functions (on $\mathbb{Q}_p$) / $\mathbb{F}_q, \mathbb{Q}_p$

motivic integration = $\widehat{K(\mathrm{Var})}[\mathbb{L}^{-1}]$-valued integration / $\mathbb{C}$

$$\mathbb{Q}_p \longleftrightarrow \mathbb{C}((t)) \qquad \mathbb{Z}_p \longleftrightarrow \mathbb{C}[[t]]$$

<u>Why?</u>

	$\mathbb{C}$	$\mathbb{F}_q$	$F \mid \mathbb{Q}_p$
X smooth proj.	$H^*(X,\mathbb{Q})$	$\# X(\mathbb{F}_q)$ point-counts	$\int_{X(\mathbb{Z}_p)} \mu_{can} \leftarrow X/\mathbb{Z}_p$
$\mathfrak{X} = [Y/G]$, G finite gp; $\alpha \in Br(\mathfrak{X})$; $\alpha \in H^2_{et}(\mathfrak{X}, \mu_\infty)$	$H^*_{orb}(\mathfrak{X}, \alpha) = \bigoplus_{g \in G/G} H^*(Y^g/C(g), \alpha)$ (Tate twist $w(g)$)	$\# \mathfrak{X}(\mathbb{F}_q) = \sum_{g \in G/G} \frac{Y^g(\mathbb{F}_q)}{\lvert C(g) \rvert}$	$\int_{Y(\mathbb{Z}_p)} \kappa_{can}\, h$; $h : \mathfrak{X}(\mathbb{Q}_p) \to \mathbb{C}^\times$, $x \longmapsto e^{2\pi i \,\mathrm{Hasse}(\alpha)_x}$

Weil

p-adic Hodge theory

<u>Batyrev</u> X, Y birat'l CY $\Rightarrow h^{p,q}(X) = h^{p,q}(Y)$

<u>Hasse</u>: $Br(F) = \mathbb{Q}/\mathbb{Z} \subset S^1[\mathrm{tors}]$, e.g. $\mathbb{Q}_p$

$$\begin{tikzcd} \hat{M} \arrow[dr] & & \check{M} \arrow[dl] \\ & A & \end{tikzcd}$$

<u>Assumption</u>:
- smooth DM stack
- tameness

<u>Thm</u> (GWZ) $$\int_{\hat{M}(\mathcal{O}_F)} h_\alpha = \int_{\check{M}(\mathcal{O}_F)} h_{\check{\alpha}}$$

$$\Rightarrow \#^{\hat{\alpha}}_{orb} \hat{M} = \#^{\check{\alpha}}_{orb} \check{M}$$

Cf $\exists$ Hitchin-Kostant section $\mathrm{vol}\,\lvert \hat{M}(\mathcal{O}_F) \rvert = \mathrm{vol}\,\lvert \check{M}(\mathcal{O}_F) \rvert$

Prop (GWZ)

$$I_{\mathbb{R}} \tilde{\hat{M}}_G = \coprod_{\substack{{}^L H \\ \in \mathrm{End}({}^L G)}} \hat{M}_H$$

What happens if $\mathcal{X}$ is not a smooth DM stack?

$$\begin{array}{ll} \mathcal{X} & \text{smooth alg stack } / \mathcal{O}_F \\ \downarrow & \\ X & \text{coarse moduli space} \end{array}$$

↑ singular

$\mathrm{vol}(X(\mathcal{O}_F)) = ?$

point-count $/ \mathbb{F}_q$

cohomology $/ \mathbb{C}$

Eg $\mathcal{X} = \mathcal{M}_{2,d}(C) =$ Higgs bundles on C (semistable)

$$\begin{array}{l} \downarrow \\ X = M_{2,d}(C) \end{array}$$

$\int_{X/\mathcal{O}_F} h_\alpha$ is degree-independent $\left(\begin{array}{l} 2,d \text{ coprime} \\ \text{general} \end{array} \right.$

$\rightsquigarrow$ BPS cohomology is degree ($\sim \chi$) independent

Kinjo - Koseki proved χ- indep. & describe BPS cohom. using IC.

Setting $\mathcal{M}$ = smooth moduli stack of semistable (same slope) objects in an abelian cat. $/ \mathcal{O}_F$ $/ \mathbb{F}_q$

$\oplus : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$

$CF(\mathcal{M}) = \{ f : \mathcal{M}(\mathbb{F}_q) \longrightarrow V = \prod_{n \in \mathbb{Z}} \mathbb{C} \}$

$$\psi_n(f)(x) = \begin{cases} \psi_n(f(x)) & , x = n x' \\ 0 \end{cases}$$

$$\psi_m(c_n) = (c_{mn})$$

Exp: $CT_0 \longrightarrow CT_0$

$$f \mapsto \exp\left(\sum \frac{\psi_n(f)}{n} \right)$$

$$\mathrm{Log}(1+f) = \sum_{n \geq 1} \frac{\mu(n)}{n} \psi_n\left(\log(1+f)\right)$$

Thm (GWZ) $\exists$ specialization map $sp : M(\mathcal{O}_F) \longrightarrow M(\mathbb{F}_q)^{\text{iso}}$

$\forall E \in \mathcal{M}(\mathbb{F}_q)$, $F(E) = \left(\int_{sp^{-1}(E)(\mathcal{O}_F)} h_\alpha \right)$, then

$$F(E) = \mathrm{Log}\left(\frac{\mathbb{L}^{-}}{\mu} \right)$$

Compare this w/

$$\underset{\mathrm{Sym}}{\mathrm{Exp}}\left(H^*(BG_m) \otimes BPS_{\mu}\right) = \bigoplus_{\frac{m}{p} = \mu} H^{BM}(\mathcal{M}(\cdot\cdot)) \mathbb{L}^{\cdot\cdot(g-)}$$