Cohomology of BG (via derived Satake)

Dimitry Kubrak

Let G be a split reductive group scheme over $\mathbb{Z}$

$$G \rightsquigarrow \underset{\substack{\uparrow \\ \text{Lie gp}}}{G(\mathbb{C})} \rightsquigarrow H^*_{sing}(BG(\mathbb{C}), \underset{\substack{\uparrow \\ \text{field}}}{k})$$

Prop (Borel) $H^*_{sing}(BG(\mathbb{C}); \mathbb{Q}) \simeq (\mathrm{Sym}\, \underset{\substack{\uparrow \\ \deg 2}}{\mathfrak{g}^*_{\mathbb{Q}}})^{G_{\mathbb{Q}}}$

Proof (w/ $\mathbb{C}$-coeff) Using Hodge-theory

$$R\Gamma_{sing}(BG(\mathbb{C}), \mathbb{C}) \overset{\text{Grothendieck}}{\simeq} R\Gamma_{dR}(BG/\mathbb{C})$$

$$\rightsquigarrow \bigoplus_{q \geq 0} R\Gamma(BG_{\mathbb{C}}, \Lambda^q \mathbb{L}_{BG/\mathbb{C}}[-q]) =: R\Gamma_{Hdg}(BG/\mathbb{C})$$

$$\mathbb{L}_{BG} \simeq \mathfrak{g}^*[-1]$$

$$\Lambda^q(-) \simeq (\mathrm{Sym}^q \mathfrak{g}^*)[-q]$$

$$\bigoplus_{q \geq 0} \mathrm{Sym}(\mathfrak{g}^*)^G$$

(By parity vanishing, s.s. automatically deg.)

Question: is there a similar formula for $H^*_{sing}(BG(\mathbb{C}), \mathbb{F}_p)$?

Indeed, $H^*_{sing}(G(\mathbb{C}), \mathbb{Z})$ is p-torsion free

no higher cohom.

poly. ring in even generators

Prop. Let p be a non-torsion prime for G, then $H^*_{sing}(BG(\mathbb{C}); \mathbb{F}_p) \simeq (\mathrm{Sym}\, \mathfrak{g}^*_{\mathbb{F}_p})^{G_{\mathbb{F}_p}}$

(Borel, Totaro)

Ex. $G = GL_n$, $H^*_{\mathrm{sing}}(BGL_n, \mathbb{F}_p) \simeq \mathbb{F}_p[c_1, \dots, c_n]$

$\deg c_i = 2i$, Chern classes

Q. What happens for torsion primes?

Ex. 1) $H^*_{\mathrm{sing}}(BSO_n, \mathbb{F}_2) \simeq \mathbb{F}_2[w_2, w_3, \dots, w_n]$ Stiefel-Whitney classes

$p=2$ is torsion $\qquad \deg w_i = i$

2) For $G = PGL_n$, torsion primes are primes p that divide n.

$H^*_{\mathrm{sing}}(BPGL_n, \mathbb{F}_p)$ is not known.

Naive attempt, check whether the same formula works.

(note that $\left(\mathrm{Sym}_{\mathbb{F}_p} \mathfrak{g}^*_{\mathbb{F}_p}\right)^{G_{\mathbb{F}_p}} \simeq R\Gamma_{\mathrm{Hdg}}(BG_{\mathbb{F}_p})$)

Totaro. $\dim H^{32}_{\mathrm{sing}}(BSpin_{11}, \mathbb{F}_2) < \dim H^{32}_{dR}(BSpin_{11})$

Thm (K.-Prikhodko) $\dim H^n_{dR}(BG_{\mathbb{F}_p}) \geq \dim H^n_{\mathrm{sing}}(BG(\mathbb{C}); \mathbb{F}_p)$.

and the difference is controlled by $H^{n+1}_{\Delta/p}(BG)[u]$.

$\mathbb{F}_p[\![u]\!]$-module

(Less naive) attempt. Derived Satake

Expectation: there is a fully faithful embedding

$$D^b_{\mathrm{constr.}}(G(\mathcal{O})\backslash \mathrm{Gr}_G, \mathbb{F}_p) \hookrightarrow \mathrm{QCoh}(\mathrm{Map}(S^2, B\check{G}_{\mathbb{F}_p}))$$

derived stack $\qquad$ $\mathbb{E}_3$-monoidal

$$G(\mathcal{O})\backslash G(\mathcal{O})/G(\mathcal{O}) \xhookrightarrow{i} G(\mathcal{O})\backslash \mathrm{Gr}_G$$

$$\begin{array}{c} \| \\ BG(\mathcal{O}) \\ \simeq \\ BG \end{array}$$

$$R\mathrm{Hom}(i_* \mathbb{F}_p, i_* \mathbb{F}_p) \simeq R\Gamma_{sing}(BG(\mathbb{C}), \mathbb{F}_p)$$

$$\mathrm{Map}(S^2, B\check{G}_{\mathbb{F}_p}) \simeq \underbrace{(e \overset{R}{\underset{\check{G}}{\times}} e)}_{\mathcal{O}(-) = B} / \check{G}$$

$$\pi_*(B) = \underset{\substack{\uparrow \\ \deg 1}}{\Lambda^*}(\check{\mathfrak{g}})^*$$

$$\begin{array}{l} B \longrightarrow \mathbb{F}_p = \pi_0(B) \\ \circlearrowleft \\ G^\vee \end{array}$$

$$\underline{R\mathrm{Hom}_{B}(\mathbb{F}_p, \mathbb{F}_p)^{h\check{G}}} \overset{\text{deforms}}{\rightsquigarrow} (R\mathrm{Hom}_{\Lambda^*(\mathfrak{g}^\vee)^*}(\mathbb{F}_p, \mathbb{F}_p))^{h\check{G}} \simeq (\underset{\substack{\uparrow \\ \text{in deg } 2}}{\mathrm{Sym}}(\mathfrak{g}^\vee))^{h\check{G}}$$

For $(\mathrm{Spin}_{11})^\vee = PSp_{10}$, no clear way to relate Hodge coh. and "Satake approximation".

Rmk. 1) For simply laced case, Hodge coh. is the same as "Satake approx".

2) If $\hat{\mathfrak{g}}^\vee_0 \simeq G^\vee_e$, $G^\vee$-equiv.

then deformation above is trivial.

$(\mathrm{Spin}_{12})^\vee \simeq PSO_{12}$

and $\mathfrak{g}^\vee \simeq \mathfrak{g}^*$ as PSO_{12}-rep'n $\qquad B\mu_2 \longrightarrow B\mathrm{Spin}_n \longrightarrow BSO_n$

<u>Thm.</u> (K.- Chua) Let $n \equiv \overset{n \geq 11}{3, 4, 5} \mod 8$, then

$$\dim H^k_{sing}(B\mathrm{Spin}_n, \mathbb{F}_2) < \dim H^k_{dR}(B\mathrm{Spin}_n, \mathbb{F}_2) \quad \text{for } k \gg 0.$$