Wild ramification of schemes and sheaves

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Lecture 1

1. Formula for the Euler number
2. Filtration by ramification groups
3. Blow-up & characteristic class

1. Euler number

k field, $p = \mathrm{char}\, k$

U (smooth) separated k-scheme of finite type

ℓ. prime $\neq p$, $\mathcal{F}$ lisse ℓ-adic sheaf on the étale site of X

smooth ℓ-adic sheaf $\overset{(U\text{- conn'd})}{\longleftrightarrow}$ ℓ-adic rep'n of $\pi_1(U, \bar{x})$ geom pt

algebraic fundamental gp

profinite, a quotient of $G_K = \mathrm{Gal}(\bar{K}/K)$,

K = func. field of U

$H^i_c(U_{\bar{k}}, \mathcal{F})$ compact supp. cohomology

finite dim'l $\mathbb{Q}_\ell$ - vect. space

$= 0$ except for $0 \le i \le 2 \dim U$

$$\chi_c(U_{\bar{k}}, \mathcal{F}) = \sum_{i=0}^{2\dim U} (-1)^i \dim H^i_c(U_{\bar{k}}, \mathcal{F})$$

Euler #

2. Grothendieck – Ogg – Shafarevich formula

dim $U = 1$, k perfect
smooth curve

$\chi_c(U_{\bar{k}}, \mathcal{F}) - \mathrm{rk}\,\mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell)$
($\mathcal{F}$: lisse sheaf on U)

(char $k = 0 \Rightarrow \quad = 0$)

$p > 0$. GOS formula: $\chi_c(U_{\bar{k}}, \mathcal{F}) - \mathrm{rk}\,\mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell)$

$$= - \sum_{x \in X \setminus U} \mathrm{Sw}_x \mathcal{F} \cdot \deg x .$$

Swan conductor $\in \mathbb{Z}_{\geq 0}$

X smooth compactification of U

proper smooth $/k$ $\quad U \hookrightarrow X$ dense open

<u>Conductor</u>. $U \hookrightarrow X \ni x$

$\hat{\mathcal{O}}_{X,x}$ complete d.v.r.

$K_x = \mathrm{Frac}(\hat{\mathcal{O}}_{X,x})$ local field at x

$\pi_1(U, \bar{x}) \longleftarrow G_{K_x} = \mathrm{Gal}(\overline{K_x} \,|\, K_x)$

$\circlearrowleft$

ℓ-adic rep'n V

$\updownarrow$

$\mathcal{F}/U$

$K_{sep} = \bar{K} \supset K_{tr} \supset K_{ur} \supset K$

(K_{tr}: max'l tamely ram. ext'n; K_{ur}: max'l unram. ext'n)

$1 \subset P \subset I \subset G_K$

(P: wild inertia; I: inertia)

$K_{tr} = K_{ur}(\pi^{1/m} : p \nmid m)$

(π: unif. of K)

$$I/P \xrightarrow{\sim} \varprojlim_{p \nmid m} \mu_m \simeq \prod_{q \neq p} \mathbb{Z}_q(1)$$

$$I \simeq \underset{\substack{\uparrow \\ \text{pro-}p \text{ Sylow gp of } I}}{P} \rtimes I/P$$

ramification gps

- upper numbering
- lower numbering

$$P \supset \underset{\substack{\uparrow \\ \text{closed normal subgp} \\ \text{decreasing filtration}}}{G^{r}_{K,\log}} \qquad r > 0, \ r \in \mathbb{Q}$$

$$\underset{\substack{\uparrow \\ \text{pro-}p}}{P} = \overline{\bigcup_{r>0} G^{r}_{K,\log}} \curvearrowright V \quad \ell\text{-adic rep.}$$

acts through finite quotient $(p \neq \ell)$

$\exists!$ decomposition $V = \bigoplus_{\substack{r \geq 0 \\ r \in \mathbb{Q}}} V^{(r)}$

$$\left[\begin{array}{l} - \ V^{(r)} \text{ stable under } G_K \\ - \ G^{s}_{K,\log} \text{ acts trivially on } V^{(r)} \iff s > r. \end{array}\right.$$

$$\mathrm{Sw}_K V := \sum_{r \in \mathbb{Q}} r \dim V^{(r)} \qquad (\Rightarrow \ = 0 \iff P \text{ acts trivially on } V)$$

$\uparrow$ measure of wild ramification.

Lower numbering

For simplicity, assume G_K acts on V via finite quotient

$G = \mathrm{Gal}(L|K)$, L finite Galois $|K$.

$$G_i = \ker\left(G \to L^\times / 1 + m_L^i\right), \quad i \in \mathbb{N} \geq 1$$

Swan character $\quad S_{L|K}(\sigma) \qquad \sigma \in G \supset I \quad$ image of $I \subset G_K$

$$\begin{cases} = 0 & \text{if } \sigma \notin I \\ = -v_L\left(\dfrac{\sigma(\pi_L)}{\pi_L} - 1\right) & \pi_L \text{ unif. of } L,\ \sigma \in I, \neq 1 \end{cases}$$

normalized discrete valuation

$S_{L|K}(1)$ is defined by requiring $\sum_{\sigma \in G} S_{L|K}(\sigma) = 0$.

Swan character is a character of a rep'n of G (Fact).

$$\mathrm{Sw}_K V = \frac{1}{|I|} \sum_{\sigma \in I} S_{L|K}(\sigma) \cdot \mathrm{Tr}(\sigma ; V) \quad \in \mathbb{N}$$

Generalization to higher dim.

U smooth sep scheme of f.t. $/k$, $d = \dim U$ arbitrary.

$\mathcal{F}$ lisse ℓ-adic sheaf. $\quad \chi_c(U_{\bar{k}}, \mathcal{F}) - \mathrm{rk}\,\mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell) = ?$

Swan class $\quad Sw_U \mathcal{F} \in CH_0(X \setminus U)_{\mathbb{Q}(\zeta_{p^\infty})}$ $\quad$ generalization of the Swan conductor

ζ_{p^∞}: p-power th rt of 1

X <u>compactification</u> of U

proper $/k$ $\supset U$ dense open

$CH_0(S)$ Chow gp of 0-cycles

$$= \bigoplus_{\substack{s \in S \\ s \text{ closed pt}}} \mathbb{Z}[s] \Big/ \text{rat'l equiv.}$$

$$\Downarrow$$

$$Sw_U(\mathcal{F}) = \sum_{\sigma \in G} s_G(\sigma) \cdot Tr(\sigma : V)$$

V: l-adic rep. of $\pi_1(U, \bar{x})$ corresp. to $\mathcal{F}$

Simplifying assumption: $\pi_1 \longrightarrow G \curvearrowright V$

G: finite quotient

finite étale Galois covering

$$\begin{tikzcd} V \arrow[r, hook, "\text{cpth}"] \arrow[d, "G"'] & Y \arrow[d] \\ U \arrow[r, hook, "\text{cpt'n}"'] & X \end{tikzcd}$$

<u>Simplifying assump</u>

Y smooth, dim d, $V = Y \setminus D$

D: divisor of Y w/ <u>simple normal crossing</u>

$D = \bigcup_i D_i$, $\quad D_i$ smooth

$D_{i_1} \cap \dots \cap D_{i_j}$ transversal

<u>Lecture 2</u>

$-s_G(\sigma)$ = logarithmic modification of intersection product

$$(\Gamma_\sigma, \Delta_Y)$$

Γ_σ: graph of σ

log product.

$$\underset{\text{smooth}}{Y} \supset \underset{\text{div. SNC}}{D} = \bigcup_{i=1}^{n} D_i$$

$$\begin{tikzcd}
Y \times Y & Y * Y \arrow[l] \\
& Y \arrow[ul, "\Delta_Y"] \arrow[u, "\Delta_Y^{\log}"']
\end{tikzcd}$$

$\Delta_Y^{\log}$ ← logarithmic diagonal

$Y * Y$: blow up at $D_i \times D_i$, $i = 1, \dots, n$.

remove proper transform of $D \times Y \cup Y \times D$

Example $\mathbb{A}^d = \operatorname{Spec} k[T_1, \dots, T_d] \supset D = (T_1 \cdots T_n) = \bigcup_{i=1}^{n} D_i, \ D_i = (T_i)$

$$\begin{tikzcd}
\mathbb{A}^d \arrow[d, "\Delta"', "T_i = S_i"] \arrow[dd, bend right=60, "\Delta^{\log}"', "U_i = 1"] \\
\mathbb{A}^d \times \mathbb{A}^d = \operatorname{Spec} \underline{k[T_1, \dots, T_d, S_1, \dots, S_d]} = A \\
\mathbb{A}^d * \mathbb{A}^d = \operatorname{Spec} A[U_1^{\pm}, \dots, U_n^{\pm}] \Big/ \underset{i = 1, \dots, n}{(T_i - U_i S_i)} \arrow[u]
\end{tikzcd}$$

$$U_i = \left\{ \frac{T_i}{S_i} \right\}$$

$$\begin{tikzcd}
\underset{\text{smooth}}{Y * Y} & V \times V \arrow[l] & G \curvearrowright V \\
\overline{\Gamma}_\sigma \arrow[u, phantom, "\cup" description] & \Gamma_\sigma \arrow[u, phantom, "\cup" description] \arrow[l, phantom, "\supset"] & \sigma \neq 1 \\
Y \arrow[uu, bend left, "\Delta_Y^{\log}"] & &
\end{tikzcd}$$

Γ_σ ← graph

$$\Big(\underset{\dim d}{\overline{\Gamma}_\sigma}, \underset{= Y}{\Delta_Y^{\log}} \Big)_{\underset{\dim 2d}{Y * Y}} \in CH_0(\underset{= \Delta_Y^{\log}}{Y \setminus V})$$

$$s_G(\sigma) \overset{\text{def}}{:=} (\overline{\Gamma}_\sigma, \Delta_Y^{\log})$$

$\sigma = 1$, $s_G(1)$ by $\sum_{\sigma \in G} s_G(\sigma) = 0$

$S_G(\sigma) = 0$ unless order of σ = power of p

log modification $\Rightarrow$ wild ramification

Letschetz trace formula for open variety

$$\sigma \in G, \ \sigma \neq 1 \quad \sum_{q=0}^{2\dim V} (-1)^q \mathrm{Tr}\left(\sigma : H_c^q(V_{\bar{k}} : \mathbb{Q}_\ell)\right) = -\deg S_G(\sigma)$$

$$\deg : CH_0(Y \setminus V) \longrightarrow \mathbb{Z}$$
$$\sum n_y [y] \longmapsto \sum n_y \deg y$$

If $V = Y$ ($D = \phi$), usual LTF.

$$\dim V = 1, \ \sigma \neq 1, \quad S_G(\sigma) = \sum_{\substack{y \in Y \setminus V \\ \sigma \in I_y}} \mathrm{ord}_y \left(\frac{\sigma(\pi_y)}{\pi_y} - 1 \right) \cdot [y]$$

$$CH_0(Y \setminus V) = \bigoplus_{y \in Y \setminus V} \mathbb{Z} \cdot y$$

Def of Swan class.

$$\begin{array}{ccc} V & \to & Y \\ \pi \downarrow & G & \downarrow \bar{\pi} \\ U & \to & X \end{array}$$

$\mathcal{F}$ smooth $\mathbb{Q}_\ell$-sheaf on U, Simplifying assumption: $\pi^* \mathcal{F}$ constant. $\mathrm{Sw}_U \mathcal{F} = \frac{1}{|G|} \sum \bar{\pi}_* S_G(\sigma)$. $\mathrm{Tr}(\sigma : V)$

$CH_0(X \setminus U)_{\mathbb{Q}(\zeta_{p^\infty})}$

0 = unless ord σ = p power

$\mathbb{Z}[\zeta_{p^\infty}]$ rep. of G $\updownarrow$ $\mathcal{F}$.

Thm (Kato - S.)

U smooth $/k$, $\mathcal{F}$ smooth ℓ-adic sheaf $/U$,

$$\chi_c(U_{\bar{k}}, \mathcal{F}) - \mathrm{rk}\, \mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell) = -\deg \mathrm{Sw}(\mathcal{F})$$

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traditional method in ramification theory.

- kill ramification by ramified covering
- lower numbering filtration
- Swan class to compute Euler number

new method

- kill (partially) ramification by blow-up
- upper numbering filtration
- characteristic $\left[\begin{array}{l}\text{cycle}\\ \text{class}\end{array}\right.$

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2. Ramification groups of local field w/ non-perfect residue field

C/k curve over k = perfect $\rightsquigarrow$ res. field perfect

X/k variety dim > 1.

U irred
D divisor

$K = \mathrm{Frac}(\hat{\mathcal{O}}_{X,\xi})$ — residue field = func. field of D

$\uparrow$ gen. pt of D $\dim D \geq 1$

K complete discrete valuation ring, F res. field not necessarily perfect

$L|K$ finite Galois ext'n, $G = \mathrm{Gal}(L|K)$

G has two filtrations by ramification gps
- lower numbering (G_i) $i \in \mathbb{N}$
- upper numbering (G^r) $r \in \mathbb{Q}, r > 0$

$G_{i,\log} = \ker(G \longrightarrow \mathrm{Aut}(L^\times / 1 + m_L^i))$

$\left(G_i = \ker(G \longrightarrow \mathrm{Aut}(\mathcal{O}_L / m_L^i)) \right)$

rigid
geometric interpretation.

$\mathcal{O}_L = \mathcal{O}_K[X_1, \cdots, X_n] / (f_1, \cdots, f_n)$

$$\begin{array}{ccl} & & \tfrac{1}{\pi} \\ G = f^{-1}(0) & \subset & D^n \leftarrow \text{rigid analytic polydisks} \\ \downarrow & & \downarrow f \\ 0 & \in & D^n \end{array}$$

$$D^n = \{(x_1, \dots, x_n) : v(x_i) \geq 0\}$$

normalized valuation

$$\downarrow$$

$$\left(f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)\right)$$

$G_i = G \cap D(i, 1)$ ← small polydisc

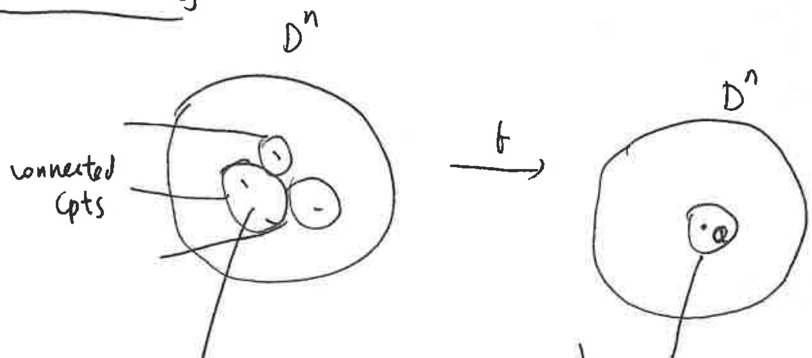
radius

center

$= \{\sigma \in G : d(\sigma, 1) \leq \|\pi^i_L\|\}$

uniformizer

Upper numbering



$G^2 = \{\sigma \in G : \sigma$ is in the same conn'd cpt as the identity$\}$

$\{x \in D^n : d(x, 0) \leq \|\pi\|^2\}$

unif. of K

rigid geometry v.s. algebraic geometry

shrinking the radius

blow-up

$L|K$ finite Galois ext'n, $G = \mathrm{Gal}(L|K)$

$\uparrow$

geometric origin

Assume X smooth $/k$ perfect, $D \subset X$ smooth irred div. $K = \mathrm{Frac}(\widehat{\mathcal{O}}_{X,\xi})$

$D \ni \xi$ gen. pt

$$\begin{tikzcd}
E \subset Y \arrow[d] & V \arrow[l] \arrow[d, "\text{finite étale Galois}"] & \mathrm{Spec}\, L \arrow[l] \arrow[d] \\
D \subset X & U = X \setminus D \arrow[l] & \mathrm{Spec}\, K \arrow[l]
\end{tikzcd}$$

$G = \mathrm{Gal}(V|U)$

Lecture 3

$$\begin{tikzcd}
U_K \arrow[r] \arrow[d] & P = X * S'' \arrow[r] \arrow[d] & X * X \arrow[r] \arrow[d, "pr_2"] & X \times X \arrow[dl, "pr_2"] \\
\mathrm{Spec}\, K \arrow[r] & \mathrm{Spec}\, \mathcal{O}_K = S \arrow[r] \arrow[u, bend left] & X \arrow[u, bend left, "\Delta_X^{\log}"] & \\
& & U \arrow[u, phantom, "\supset"] &
\end{tikzcd}$$

$\mathrm{Spec}\,\mathcal{O}_K$

$X \xrightarrow{\Delta_X^{\log}} X * X$, $X \xrightarrow{\mathcal{O}_X} X \times X$

$$\begin{tikzcd}
U \times_k \mathrm{Spec}\, K = U_K \arrow[r] \arrow[d] & U \times U \arrow[d] \\
\mathrm{Spec}\, K \arrow[r] & U
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spec}\,\mathcal{O}_L = T \arrow[r] \arrow[d] & Q \arrow[d] & V_K \arrow[l, phantom, "\supset"] \arrow[d] \\
\mathrm{Spec}\,\mathcal{O}_K = S \arrow[r] & P & U_K \arrow[l, phantom, "\supset"]
\end{tikzcd}$$

normalization

$\uparrow$ induced by $\Delta_X^{\log}$

Σ

$$D^n = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_K[x_1, \dots, x_n], \mathcal{O}_{\bar{K}})$$

$$F^{-1}(0) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\underbrace{\mathcal{O}_K[x_1, \dots, x_n]/(f_1, \dots, f_n)}_{\cong \mathcal{O}_L}, \mathcal{O}_{\bar{K}}) = G$$

algebraic construction corresponding to shrinking the radius

$r > 0$ rational number

$K' | K$ finite separable extension

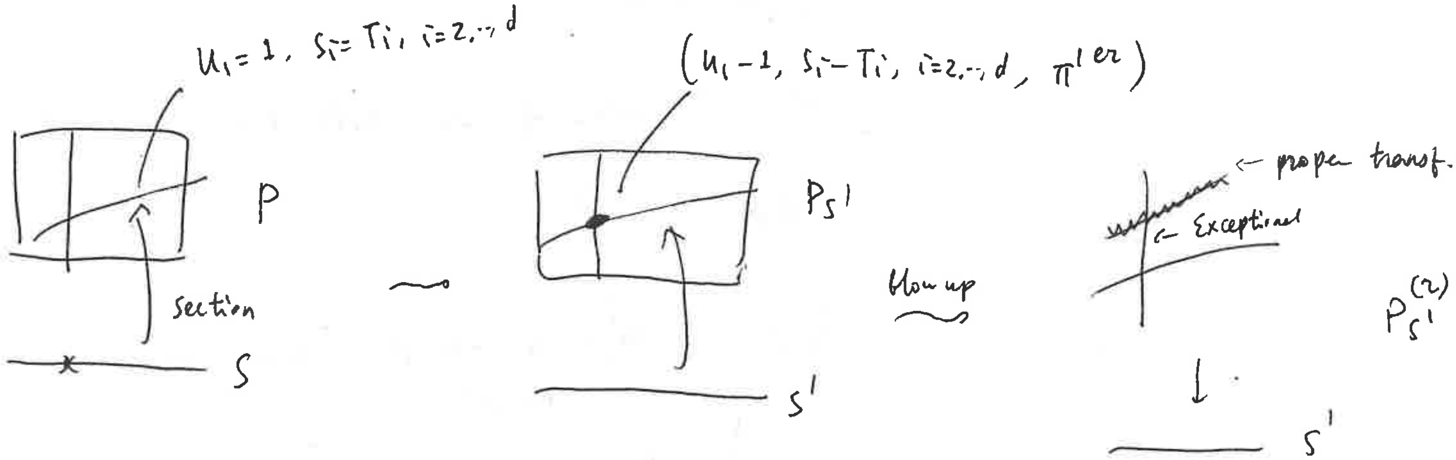
$e = e_{K'|K}$ ramification index.

assume $e \cdot r$ integer.

$S' = \operatorname{Spec} \mathcal{O}_{K'}$, $\quad P_{S'} = P \times_S S'$

$$P^{(r)}_{S'} = \text{blow up of } P_{S'} \text{ at } \operatorname{Spec} \mathcal{O}_{K'}/m_{K'}^{e \cdot r} \hookrightarrow S' = \operatorname{Spec} \mathcal{O}_{K'} \underset{\substack{\downarrow \\ \text{induced by } \Delta_X^{\log}}}{\subset} P_{S'}$$

remove the proper transform of the closed fiber.



<u>Example</u> $\quad X = \mathbb{A}^d_k = \operatorname{Spec} k[T_1, \dots, T_d]$

$\cup$

$D = (T_1 = 0)$

$K = k(T_2, \dots, T_d)((T_1))$

$X \times_k S = \operatorname{Spec} \mathcal{O}_K [S_1, \dots, S_d] \longleftarrow X * S = P$

$P = \operatorname{Spec} \mathcal{O}_K [U_1^{\pm 1}, S_1, S_2, \dots, S_d] / (S_1 = U_1 T_1)$

$$P_{S'} = \mathrm{Spec}\ \underbrace{\mathcal{O}_{K'}[U_1^{\pm 1}, S_2, \ldots, S_d]}_{A}$$

$$\uparrow$$

$$P_{S'}^{(2)} = \mathrm{Spec}\ \mathcal{O}_{K'}[V_1, \ldots, V_d]$$

$$= \mathrm{Spec}\ A\left[\frac{U_1 - L}{\pi'^{\,e.r}}, \frac{S_2 - T_2}{\pi'^{\,er}}, \ldots, \frac{S_d - T_d}{\pi'^{\,er}}\right]$$

$\uparrow$ unif. of $\mathcal{O}_{K'}$

$$\begin{array}{ccc} Q_{S'}^{(2)} & \supset & V_{K'} \\ \downarrow & & \downarrow \\ P_{S'}^{(2)} & \supset & U_{K'} \\ \downarrow & & \downarrow \\ S' & \longleftarrow & \mathrm{Spec}\, K' \end{array}$$

normalization

always smooth

$\exists K' | K$ s.t. (2) $e.r$ integer

(2) $L \subset K'$

(Epp's theorem) — (3) $Q_{S'}^{(r)}$ has geom. reduced closed fiber.

$\bar{F}$ alg. closure of the residue field F of K.

$$\mathcal{O}_{K'} \longrightarrow \bar{F}$$

$$Q_{\bar{F}}^{(2)} = Q_{S'}^{(2)} \times_{\mathrm{Spec}\, \mathcal{O}_{K'}} \mathrm{Spec}\, \bar{F} \quad \text{geom. fiber}$$

(3) $\uparrow$ reduced

$$\begin{array}{ccccccc} G & & & & & & \\ \| & & & & & & \\ \mathrm{Spec}(L \otimes K') & \to & V_{K'} & \longrightarrow & Q_{S'}^{(2)} & \longleftarrow & Q_{\bar{F}}^{(2)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\, K' & \to & U_{K'} & \longrightarrow & P_{S'}^{(2)} & & \\ & & & & \downarrow & & \\ & & \mathrm{Spec}\, K' & \longrightarrow & S' & \longleftarrow & \mathrm{Spec}\, \bar{F} \end{array}$$

induced by $\Delta_X^{\log}$

reduction map $G \to Q_{\bar{F}}^{(2)}$

$G_{\log}^{2} = \{ \sigma \in G :$ image of σ in $Q_{\bar{F}}^{(2)}$ lies in the same conn'd cpt as the image of id $\}$

Example : K as before . $p = \mathrm{char}\, k = \mathrm{char}\, K$
Artin - Schreier

$$\mathrm{Hom}(G_K, \mathbb{Z}/p\mathbb{Z}) = H^1(K, \mathbb{Z}/p\mathbb{Z}) = K/(a^p - a,\ a \in K)$$

$$\begin{array}{ccc} \cup & \chi_a \longleftarrow\!\shortmid a & \\ \mathrm{Fil}^n & = \ \text{Image of} & m_K^{-n} \end{array}$$

$a \in m_K^{-n}$ s.t. $\chi_a \in \mathrm{Fil}^n \notin \mathrm{Fil}^{n-1}$

$L|K \quad t^p - t = a.$

$G = \mathrm{Gal}(L|K) = \mathbb{Z}/p\mathbb{Z}$

$G_{\log}^r = 0 \iff r > n.$

Graded pieces (or last piece of fil)

$G = \mathrm{Gal}(L|K) \supset G_{\log}^r$

$G_{\log}^s = 1 \iff s > r$: last jump

$\bar{F} \leftarrow O_K = \hat{O}_{X,\xi} \leftarrow O_{X,\xi}$

$\bar{F}$ - vec. sp. of dim 1

$G_{\log}^r$ is abelian & p-torsion

Locally free on X

$$\mathrm{rsw} : \mathrm{Hom}(G_{\log}^r, \mathbb{F}_p) \hookrightarrow \mathrm{Hom}_{\bar{F}}\Big(m_{\bar{K}}^r / m_{\bar{K}}^{r+},\ \overbrace{\Omega_X^1 (\log D)}_{\xi} \otimes_{O_{X,\xi}} \bar{F}\Big)$$

canonical injection

$\bar{F}$ - vec. sp.

$\bar{K}$ = sep. closure of K.

$m_{\bar{K}}^{r+} = \{ - : v_K(a) > r \}$

$m_{\bar{K}}^r = \{ a \in \bar{K} : v_K(a) \geq r \}$

$v_K(\pi_K) = 1$
unif of K

3. Blow-up in global geometric situation

group scheme structure

$\downarrow$

differential form.

Lecture 4

rigid geom. $/K$

$$\begin{tikzcd} G \arrow[d] & \subset & D^n \arrow[d, "f"] & \supset & f^{-1}(D(0,r)) \arrow[d] \\ 0 & \in & D^n & \supset & D(0,r) \end{tikzcd}$$

scheme $/S = \operatorname{Spec} \mathcal{O}_K$

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_L \arrow[r] \arrow[d] & Q \arrow[d] & Q_{S'}^{(r)} \arrow[l] \arrow[d] \\ S \arrow[r] & P \arrow[d] & P_{S'}^{(r)} \arrow[l] \\ & S & \end{tikzcd}$$

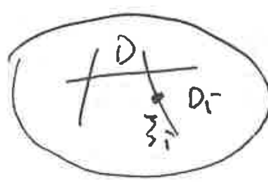
X smooth scheme $/k$, perfect of char $p > 0$

$\cup$

D divisor w/ SNC

$\|$

$\cup D_i$



X

$K_i = \operatorname{Frac}(\hat{\mathcal{O}}_{X,\xi_i})$

r_i

$R = \sum r_i D_i$

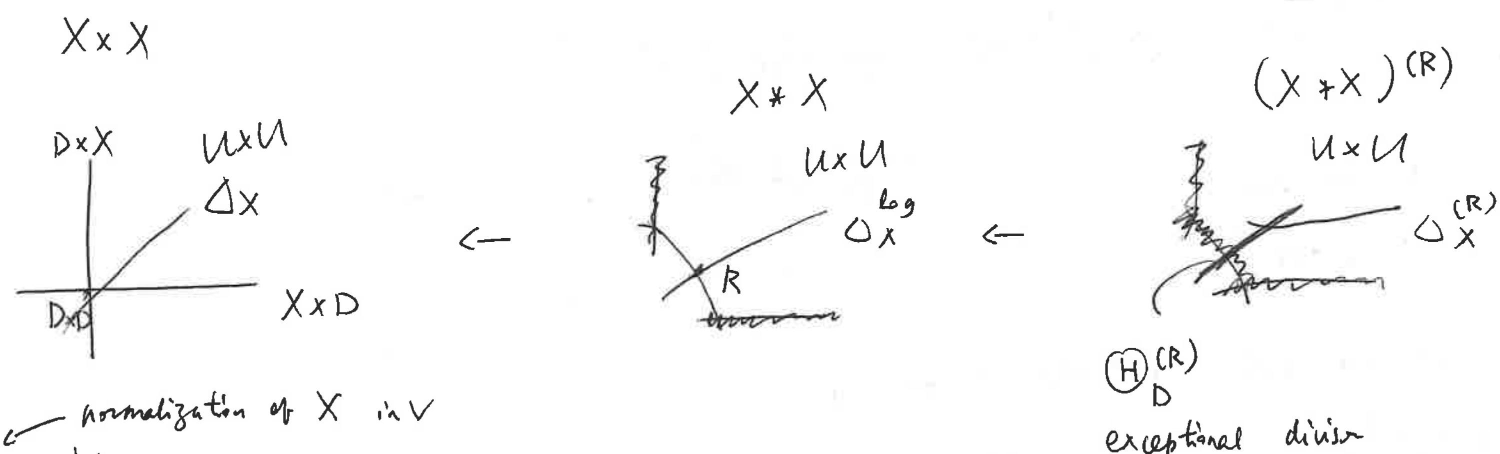
$r_i \geq 0$ rational

Simplifying assumption: $r_i > 0$, integer

$$\begin{tikzcd}
X \times X & X * X \arrow[l] & (X*X)^{(R)} \arrow[l] \\
X \arrow[u, "\Delta_X"] \arrow[ur, "\Delta_X^{\log}"] \arrow[urr, bend right, "\Delta_X^{(R)}"'] & & \\
\cup & & \\
R & &
\end{tikzcd}$$

effective Cartier divisor

blow-up at R & remove the proper transf. of $(X*X) \times D$ $\xrightarrow{pr_2} X$



$$\begin{tikzcd}
Y \arrow[d] & V \arrow[l, "\text{normalization of } X \text{ in } V"'] \arrow[d, "\text{finite étale Galois } G"] \\
X & U = X \setminus D \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
G \circlearrowright Y/G = X \arrow[r] \arrow[d, equal] & Z \arrow[d] & V \times V/\Delta G =: W \arrow[l, "\text{normalization}"'] \arrow[d, "\text{finite étale}"] \\
X \arrow[r, "\Delta^{(R)}"] \arrow[ur, bend left] & (X*X)^{(R)} & U \times U \arrow[l]
\end{tikzcd}$$

$G \times G \supset \Delta G \circlearrowright W$

<u>Example</u>. $X = \mathbb{A}^1 = \operatorname{Spec} k[T] \supset D = (0)$, $\quad p \nmid n$

$\cup$

$U = \mathbb{G}_m$

$R = nD$

$r = n$

$V \to U \quad , t^p - t = \frac{1}{T^n}, \quad G = \operatorname{Gal}(V|U) = \mathbb{F}_p$

$(X*X)^{(R)}$

$X * X = \operatorname{Spec} k[S, T, (\frac{S}{T})^{\pm 1}] = \operatorname{Spec} k[U^{\pm 1}, T]$

$\|$

$\operatorname{Spec} k[U^{\pm 1}, T, \frac{U-1}{T^n}]$

$$(X * X)^{(R)} = \mathrm{Spec}\, k[U^{\pm 1}, T, V] \big/ (U - (1 + VT^n))$$

$$\cup$$

$$U \times U = \mathrm{Spec}\, k[T^{\pm 1}, S^{\pm 1}]$$

$$W \longrightarrow U \times U \qquad t^p - t = \frac{1}{S^n} - \frac{1}{T^n}$$

$$\frac{1}{S^n} - \frac{1}{T^n} = \frac{1}{T^n}\left(U^{-n} - 1\right) = \frac{1}{T^n}\left((1 + VT^n)^{-n} - 1\right)$$

$$= \frac{1}{T^n}\left(1 - nVT^n + \cdots \quad -1\right)$$

↑ has no pole on $(X*X)^{(R)}$

We have killed ramification by blow up.

$$\begin{tikzcd}
\mathrm{Spec}\, k[V] = \Theta_D^{(R)} \arrow[d] & \subset & (X * X)^{(R)} = \mathrm{Spec}\, k[U^{\pm 1}, T, V] / \text{—} \arrow[d] \\
D & \subset & X = \mathrm{Spec}\, k[T]
\end{tikzcd}$$

$\Theta_D^{(R)} \cong \mathbb{A}^1$

D: $T = 0$

$$\begin{tikzcd}
Z \times_{(X*X)^{(R)}} \Theta_D^{(R)} \arrow[d] & \subset & Z \arrow[d, "t^p - t = -nV"] \\
\Theta_D^{(R)} & \subset & (X * X)^{(R)}
\end{tikzcd}$$

$\Theta_D^{(R)} \xleftarrow{T=0}$, $\Theta_D^{(R)} = \mathrm{Spec}\, k[V]$

$$t^p - t = -nV + \cdots$$

↑ divisible by T^n

$$V = \frac{U-1}{T^n} = \frac{1}{T^n} \cdot \frac{\overset{T\otimes 1}{S} - \overset{1 \otimes T}{T}}{T} = \frac{1}{T^n}\, d\log T$$

$$t^p - t = -n\left(\frac{1}{T^n}\, d\log T\right)$$

$$\begin{tikzcd}
\Theta_D^{(R)} \arrow[r] \arrow[d] & (X * X)^{(R)} \arrow[d] \\
D \arrow[r, hook] & X
\end{tikzcd}$$

group structure

vector bundle assoc. to

$$\Theta_D^{(R)} = \mathbb{V}\left(\Omega^1_X(\log D)(R) \otimes_{\mathcal{O}_X} \mathcal{O}_D\right)$$

locally free $\mathcal{O}_D$-module

$$= \operatorname{Spec} S^{\cdot}_{\mathcal{O}_D}\left(\text{———}\right)$$

<u>groupoid structure</u>

$$\begin{tikzcd}
(X * X) \times_X (X * X) \arrow[r, "pr_{13}"] \arrow[d, equal] & X \times X \\
X \times X \times X &
\end{tikzcd}$$

$pr_2 \searrow X \swarrow pr_1$

$$\begin{tikzcd}
(X * X)^{(R)} \times_X (X * X)^{(R)} \arrow[r, "\mu"] & (X * X)^{(R)} \\
\Theta_D^{(R)} \times_D \Theta_D^{(R)} \arrow[r, "+"] \arrow[u, phantom, "\cup" description] \arrow[ur, phantom, "\circlearrowright"] & \Theta_D^{(R)} \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

$$\begin{tikzcd}
W \times_U W \arrow[r] \arrow[d] & W = V \times V / \Delta G \arrow[d] \\
(U \times U) \times_U (U \times U) \arrow[r, "pr_{13}"] & U \times U
\end{tikzcd}$$

$$\begin{tikzcd} Z_0 \subset Z \arrow[d] \\ (X*X)^{(R)} \end{tikzcd}$$

max'l open subscheme étale over $(X*X)^{(R)}$

$$\begin{tikzcd} Z_0 \times Z_0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ (X*X)^{(R)} \times_X (X*X)^{(R)} \arrow[r, "\mu"] & (X*X)^{R} \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[d, "G"] \\ U \end{tikzcd}$$

$D = \bigcup D_i$ $\quad K_i$ local

$G \supset G_i$ decomposition gp

$\cup$

$G_i^{r_i}$ $\quad r_i$ the last jump

$G_i^s = 1 \iff s > r_i$

$R = \sum r_i D_i$

Assume

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r, "\Delta^{(R)}"] \arrow[ur] & (X*X)^{(R)} \end{tikzcd}$$

the image of $X \subset Z$ is inside Z_0

i.e. Z is étale on a nbhd of X

base change D

$$\begin{tikzcd} Z_{0,D} \times_D Z_{0,D} \arrow[r] \arrow[d] & Z_{0,D} \arrow[d, "\leftarrow \text{hom. of gp schemes}"] \\ \Theta_D^{(R)} \times_D \Theta_D^{(R)} \arrow[r, "+"] & \Theta_D^{(R)} \end{tikzcd}$$

$Z_{0,D}$ is a smooth group scheme, étale over $\Theta_D^{(R)}$

étale isogeny over a vector bundle $\Theta_D^{(R)}$.

$D = \bigcup D_i$, $\quad D_i \ni \xi_i$

$Z_{0,\xi_i} \longrightarrow \Theta_{\xi_i}^{(R)}$ ext'n of $\Theta_{\xi_i}^{(R)}$ by $\mathbb{F}_p$-v.s.

étale isogeny

vector space over $F_i = k(\xi_i)$

func. field of D_i

$\mathrm{Ker}(\longrightarrow) \simeq G^{r_i}$ ⇐ elementary p-group

$$\mathrm{Ext}^1\left(\Theta_{\xi_i}^{(R)}, \mathbb{F}_p\right) = \mathrm{Hom}\left(\Theta_{\xi_i}^{(R)}, G_a\right) = \{\text{linear forms on } \Theta_{\xi_i}^{(R)}\}$$

$$\mathrm{Ext}^1 = \mathrm{Hom}\left(m_{K_i}^{r_i} / m_{K_i}^{r_i+1}, \Omega_X^1(\log D) \underset{\mathcal{O}_{X,\xi_i}}{\otimes} F_i\right)$$