Differential operators on the base affine space and quantized Coulomb branches

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The base affine space

G = simple alg. grp $/\mathbb{C}$, B Borel.

$U = [B, B]$, $T = B/U$.

Ex. $G = SL_n$.

G/U - base affine space, a fundamental object in repr theory.

- G/U is a <u>quasi-affine variety</u>. i.e. the map $G/U \to \operatorname{Spec} \mathbb{C}[G/U] = \overline{G/U}$ is open embedding.

- The multiplicative action of $G \times G \curvearrowright G$ descends to an action
$$G \times T \curvearrowright G/U.$$

- G/U is a T-bundle over the (projective) flag variety G/B.
$$G/U \to G/B \simeq (G/U)/T.$$

- As a G-rep. $\mathbb{C}[G/U] = \bigoplus$ all simple G-reps.

Ex. $G = SL_2$, $\{\text{simple } SL_2\text{-reps}\} = \{\operatorname{Sym}^n \mathbb{C}^2\}$

$\rightsquigarrow \mathbb{C}[SL_2/U] \simeq \bigoplus_n \operatorname{Sym}^n \mathbb{C}^2$

$\rightsquigarrow \overline{SL_2/U} \simeq \mathbb{C}^2$ and $SL_2/U \simeq \mathbb{C}^2 \setminus \{0\}$.

$(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^* \simeq \mathbb{P}^1 = SL_2/B$.

Warning: $G = SL_2$ is the only case where $\overline{G/U}$ is smooth.

- Equally fundamental is the ring $D_\hbar(G/U)$ of differential operators on G/U.
- Gelfand-Graev: hidden Weyl group action on $D_\hbar(G/U)$

Ex. $D_\hbar(SL_2/U) \simeq T^\bullet(x, y, \partial x, \partial y, \hbar) / \langle x\partial x - \partial x x = \hbar, \ldots \rangle$

$\mathfrak{S}_2$ acts via Fourier transform

$$x \longleftrightarrow \partial y, \quad y \longleftrightarrow -\partial x.$$

- action for general G is generated by partial Fourier transforms, but relations are mysterious.

Coulomb branches

- $\mathcal{J}$ = 3d $\mathcal{N}=4$ QFT

$\rightsquigarrow$ Coulomb branch $\mathcal{M}_C(\mathcal{J})$

- parametrizes certain values of $\mathcal{J}$.

need to define various expectation values on $\mathbb{R}^3$

- affine variety w/ natural quantization.

$\mathbb{C}_\hbar[\mathcal{M}_C(\mathcal{J})]$

- Ex. G as before, N a repn of G.

$\longrightarrow$ 3d $\mathcal{N}=4$ gauge theory $\mathcal{J}_{G,N}$

BFN: $\mathbb{C}_\hbar[\mathcal{M}_C(\mathcal{J}_{G,N})] = H_*^{G \times \mathbb{C}^\times}(\mathcal{R}_{G,N})$ the equivariant BM homology of a space $\mathcal{R}_{G,N}$ w/ a convolution structure.

Rem.)

- Here $G_{\mathcal{O}} = G[[t]]$, w/ $\mathbb{C}^\times$ acting by scaling t

- $R_{G,N}$ lies over the affine Grassmannian $Gr_G = G_K / G_{\mathcal{O}}$

  and for $g \in G_K$, the fiber over $[g]$ is $N[[t]] \cap g N[[t]] \subset N((t))$.

<u>Ex.</u> Q a quiver, w/ function $d : \{\text{vertices}\} \to \mathbb{N}$

$$\rightsquigarrow \quad G_Q = \Big(\prod_v GL_{d(v)}\Big) / \mathbb{C}^\times$$

$$N_Q = \bigoplus_{v \to w} \operatorname{Hom}\big(\mathbb{C}^{d(v)}, \mathbb{C}^{d(w)}\big).$$

- we call $\mathcal{T}_{G_Q, N_Q}$ a quiver gauge theory

<u>Thm</u> (Gannon – W.) Let $\mathcal{T}_n$ be the gauge theory associated to

$$\textcircled{1} - \textcircled{2} - \cdots - \textcircled{n-2} - \textcircled{n-1} < \left.\begin{matrix} \textcircled{1} \\ \textcircled{1} \\ \vdots \\ \textcircled{1} \end{matrix}\right\} n \ \textcircled{1}\text{'s}$$

Then there is an alg isom. $\mathbb{C}_\hbar[\mathcal{M}_C(\mathcal{T}_n)] \simeq D_\hbar(SL_n/U)$,

which identifies the Gelfand – Graev action w/ the $\mathfrak{S}_n$ action induced by permuting the right-hand vertices.

<u>Cor.</u> (proposal of Dancer – Hanany – Kirwan) There is an algebraic symplectic isomorphism

$$\overline{T^*(SL_n/U)} \simeq \mathcal{M}_C(\mathcal{T}_n).$$

<u>Generalization</u>

- let $(n_1, \cdots, n_k)$ be an ordered tuple of natural numbers s.t. $n_1 + \cdots + n_k = n$.
  These tuples are in bijection w/ functions $\psi : \{e_1, \cdots, e_n\} \to \{0, 1\}$.

(e.g. ψ constantly 0 $\leftrightarrow$ $(1, \cdots, 1)$.)

ψ constantly 1 $\leftrightarrow$ (n).

any such ψ extends to a Lie algebra character $\psi \in \mathfrak{u}^*$.

The action $U \curvearrowright SL_n$ induces a Hamiltonian action $U \curvearrowright T^*SL_n$

w/ moment map $\mu : T^*SL_n \to \mathfrak{u}^*$

Given $\psi \in \mathfrak{u}^*$, we have the Hamiltonian reduction

$$T^*SL_n /\!\!/_\psi U = \mu^{-1}(\psi)/U$$

<u>Ex.</u> $T^*SL_n /\!\!/_0 U \simeq \overline{T^*(SL_n/U)}$

<u>Thm.</u> Let $(n_1, \cdots, n_k)$ and ψ be as above,

and J_ψ the gauge theory of

$$\begin{tikzcd}[column sep=small, row sep=small] & & & & & \textcircled{n_1} \\ \textcircled{1} \arrow[r, dash] & \textcircled{2} \arrow[r, dash, dashed] & \textcircled{$n-2$} \arrow[r, dash] & \textcircled{$n-1$} \arrow[urr, dash] \arrow[rr, dash] \arrow[drr, dash, dotted] \arrow[ddrr, dash] & & \textcircled{n_2} \\ & & & & & \vdots \\ & & & & & \textcircled{n_k} \end{tikzcd}$$

Then there is an isomorphism

$$T^*SL_n /\!\!/_\psi U \simeq \mathcal{M}_C(J_\psi)$$

<u>Key ingredient</u>. the <u>regular sheaf</u> $\mathcal{A}^G_{reg} \in P^{G_O}(Gr_G)$ corresponding to the regular representation $\mathbb{C}[G^\vee]$ of $G^\vee$ under the geometric Satake equiv.

- Ginzburg – Riche: $D_\hbar(G^\vee/U^\vee)$ is the $T_O \rtimes \mathbb{C}^*$-equiv. cohomology of $i^! \mathcal{A}^G_{reg}$, where $i : Gr_T \hookrightarrow Gr_G$.

- BFN: $\lambda^{\mathrm{PhLa}}_{\mathrm{reg}} \simeq \pi_* \underset{\substack{\uparrow \\ \text{dualizing complex}}}{\omega_R}$, where R is associated to $1 - 2 - \cdots - (n-1) - (n)$

- Macerato: generalization of GR to the case where T is replaced by a Levi.