Geometric Langlands

X/k smooth proj. curve, char $(k)=0$

Thm.

$$\begin{tikzcd} \text{D-mod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G", "\sim"'] \arrow[d] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[d] \\ \text{D-mod}(\mathrm{Bun}_G)_{\mathrm{temp}} \arrow[r, "\mathbb{L}_{G,\mathrm{temp}}", "\sim"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

Variant. Betti and restricted variants also hold

Nice feature. $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \xrightarrow{\mathbb{L}_{G,\mathrm{temp}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{Betti}/\mathrm{restr.}})$

is t-exact.

Concretely.

$\sigma \in \mathrm{LS}_{\check{G}}^{\mathrm{irred}}$, $k_\sigma \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}})$

$\longleftrightarrow \mathcal{F}_\sigma \in \text{D-mod}(\mathrm{Bun}_G)$

What we know.

a) $\mathcal{F}_\sigma$ is perverse.

b) it has $SS(\mathcal{F}_\sigma) \subset \mathrm{Nilp} \subset T^*\mathrm{Bun}_G$

c) it's semisimple

d) $\mathcal{F}_\sigma = \bigoplus_\rho \mathcal{F}_{\sigma,\rho}^{\mathrm{disp}}$ $\quad \rho \in \mathrm{Irrep}(\mathrm{Cent}(\sigma))$

$\mathcal{F}_{\sigma,\rho} \leftarrow$ irred. perverse.

Big picture

- construct $\mathbb{L}_G$
- prove some structural facts
- conclude via tricks

Main player 1:

$$\mathrm{Coeff}: \mathcal{D}\text{-mod}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$$

\begin{tikzcd}
 & \mathrm{Bun}_N^{\tilde{P}(\Omega_X^{\frac12})} \arrow[dl, "p"'] \arrow[dr, "\psi"] & \\
\mathrm{Bun}_G & & \mathbb{A}^1
\end{tikzcd}

$\psi \leftarrow$ non deg. char.

$$\mathrm{Coeff}(\mathcal{F}) = C^\cdot(p^!(\mathcal{F}) \otimes \psi^*(\exp))$$

$$= \mathrm{Hom}(\mathrm{Poinc}_!, \mathcal{F})$$

Hecke functors:

$x \in X$

$\mathrm{Rep}\,\check{G} \curvearrowright D(\mathrm{Bun}_G)$

Thm (Gaitsgory - Drinfeld)

$\exists!\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \curvearrowright D(\mathrm{Bun}_G)$ extending the Hecke action.

Thm: a) (Taut)

\begin{tikzcd}
\exists!\ \mathbb{L}_G^{\mathrm{temp}}: D\text{-mod}(\mathrm{Bun}_G) \arrow[rr] \arrow[dr, "\mathrm{Coeff}"'] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[dl, "\Gamma"] \\
 & \mathrm{Vect} &
\end{tikzcd}

b) $\mathbb{L}_G^{\mathrm{temp}}$ has finite coh. amplitude.

c) cpt's in $D(\mathrm{Bun}_G)$ are bounded below.

d) $\exists!\ \mathbb{L}_G$ fitting into commutative diagram w/ $\mathbb{L}_{G,\mathrm{temp}}$. So

$\mathbb{L}_G(\mathrm{cpt}) \leftarrow$ bdded below.

Main player 2. $\quad \mathcal{A}_G := \mathbb{L}_G(\mathrm{Poinc}_!) \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

"automorphic multiplicity sheaf".

$\mathcal{A}_G$ is an algebra.

WTS: $\mathcal{O}_{\mathrm{LS}_{\check{G}}} \xrightarrow{\sim} \mathcal{A}_G$ is an isom.

Thm. (Faergeman – R., '22) $\mathbb{L}_{G,\mathrm{temp}}$ is conservative.

So u is an iso. $\iff$ GLC.

Structural features of GL & $\mathfrak{a}_G$

Input:
$$\begin{tikzcd} \mathrm{KL}(G) \arrow[r, "\simeq", "\mathrm{FLE}"'] \arrow[d, "\mathrm{Loc}"'] & \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mf}}_{\check{G}}) \arrow[d, "\text{pull push}"] \\ D(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_{G,\mathrm{temp}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

Deduce.
$$\check{M} \leftarrow \check{P} \rightarrow \check{G}$$
$$M \leftarrow P \rightarrow G$$

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\mathrm{CT}_*"] \arrow[d, "\mathbb{L}_G"'] & D(\mathrm{Bun}_M) \arrow[d, "\mathbb{L}_M"] \\ \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[r, "\mathrm{CT}^{\mathrm{Spec}}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \end{tikzcd}$$

For us. $\rightarrow \mathfrak{a}_G|_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}} \simeq \mathcal{O}$.

Second str. thing.

Thm. $\sigma \in \mathrm{LS}^{\mathrm{irred}}_{\check{G}}$. $\exists$ perfect pairing
$$\underbrace{\mathfrak{a}_{G,\sigma}}_{\text{algebra}} \otimes \underbrace{H_*(\mathrm{Op}^{\mathrm{mf}}_{\sigma}(X))}_{\text{coalgebra.}} \longrightarrow k$$

Upshots $\mathfrak{a}_G^{\mathrm{irred}} \simeq (\mathfrak{a}_G^{\mathrm{irred}})^\vee$

$\mathfrak{a}_G^{\mathrm{irred}}$ is in degs ≤ 0

it's commutative, its fibers are reduced.

$$\mathrm{Spec}(\mathfrak{a}_G^{\mathrm{irred}}) \xrightarrow{\text{fin., ét}} \mathrm{LS}^{\mathrm{irred}}_{\check{G}}$$

<u>Tricks</u>: Simplify. $g \geq 2$. G adj

not $G = \cancel{PGL_2}, g = 2$

$\cancel{SL_2}$.

PGL_2

<u>Main input</u>: $\Gamma(\mathfrak{a}_G) \overset{\text{def'n}}{=} \mathrm{End}(\mathrm{Poinc}_!) = k$

$LS_{\check{G}}^{\mathrm{irred}} \leftarrow$ simply-connected

$\mathfrak{a}_G^{\mathrm{irred}} = \mathcal{O}_{LS^{\mathrm{irred}}}^{\oplus n}$

$$H^0\Gamma(\mathfrak{a}_G) \xrightarrow{\sim} H^0\Gamma(\mathfrak{a}_G^{\mathrm{irred}})$$

Local coh. $\to \mathfrak{a}_G \to j_* \mathfrak{a}_G^{\mathrm{irred}}$

?! $\mathfrak{a}_G$

$\leftarrow$ Eis

?! $\mathcal{O}$

$\deg \geq 2$

$$k = \Gamma(\mathfrak{a}_G) = H^0\Gamma(\mathfrak{a}_G^{\mathrm{irred}}) = \underbrace{H^0\Gamma(\mathcal{O}_{LS^{\mathrm{irred}}}^{\oplus n})}_{\dim \geq n}$$

$\to n = 1.$