

Hecke operators for algebraic curves over local non-archimedean fields,

Sasha Braverman

Two parts

I. General things

II. Particular example + discuss to "usual" rep. theory of p -adic groups.

1. "Classical stuff"

X - smooth proj. curve / \mathbb{F}_q .

G - reductive alg. group.

Bun_G - stack of G -bundles on X

$\mathbb{C}_c(\text{Bun}_G(\mathbb{F}_q))$ functions with finite supp.

$\hat{\mathbb{L}^2(\text{Bun}_G(\mathbb{F}_q))}$

$\mathbb{C}(\text{Bun}_G(\mathbb{F}_q))$ all functions

$x \in X(\mathbb{F}_q)$

H_x - algebra of Hecke operators

$$H_x \cong^{\text{Satake}} \mathbb{C}[A^\vee]^{A^\vee} \cong k_0(\text{Rep } A^\vee)$$

Ex $G = \text{GL}(N)$

$$\{\Sigma_1 \hookrightarrow \Sigma_2$$

$$\Sigma_2 / \Sigma_1 = \mathbb{F}_{q,x} \}$$

p^{N-1}

$\text{Bun}_{\text{GL}(N)}$

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Correspondence $\longrightarrow [\mathbb{C}^N]$

Langlands (Replace \mathbb{C} by $\overline{\mathbb{C}}_q$)

Eigenvalues have to do with A^\vee local system on X .

(effective)
 D any divisor on X / \mathbb{F}_q

Hecke operators which "lie" at D .

Irreducible divisor = Galois orbit of $x \in X(\overline{\mathbb{F}}_q) \longrightarrow H_D$ is then $\mathbb{C}[A^\vee]^{A^\vee}$.

2. Want: replace \mathbb{F}_q by F local non-archimedean field.

Bun_G - alg. stack over F .

Y alg. var. / F , $\mathbb{C}_c^\infty(Y(F)) =$ locally const. functions w/ cpt support

!!
 $S(Y)$

L line bundle on Y ,

$$c \in \mathbb{C} \quad F^\times \longrightarrow \mathbb{R}_+ \xrightarrow{x \mapsto x^c} \mathbb{C}^*$$

$S(Y, L, c) =$ comp. supp. locally const. sections of $|L|^c$.

Y -stack

$$Y = \mathbb{Z}/G$$

\uparrow
scheme

Assume \mathbb{Q}/G is unimodular

② $\mathbb{Z}(\mathbb{F}) \rightarrow Y(\mathbb{F})$ surjective.

L line bundle on Y , $c \in \mathbb{C}$.

$$S(\mathbb{Z}, |L_{\mathbb{Z}}|^c)_{G(F)} = S(Y, |L|^c).$$

Lemma Independent of presentation. (Gaitsgory - Kazhdan)

Can generalize to Y is locally \mathbb{Z}/G .

In particular, can take $Y = \text{Bun}_G$

$$L = \Omega_{\text{Bun}_G} \text{ (some determinant)}$$

$$S_c(\text{Bun}_G) = S(\text{Bun}_G, |\Omega|^c)$$

Claim (B-Kazhdan, Frenkel - Frenkel - Kazhdan)

$$c = \frac{1}{2} (C_{\text{cst}})$$

Then we have Hecke operators.

$$x \in X(F).$$

$\forall \lambda$ - dom. weight of G

$$h_x^\lambda : S_{\frac{1}{2}} \rightarrow S_{\frac{1}{2}} \quad \text{they all commute}$$

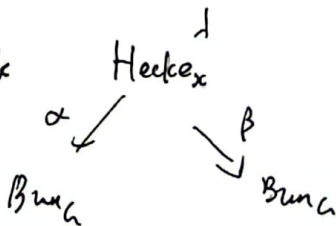
$$h_x^\lambda h_x^\mu = h_x^{\lambda+\mu}.$$

$\forall \lambda$

$\{\xi_1, \xi_2 - G\text{-bundles}, k: \xi_1 \rightsquigarrow \xi_2 \text{ isom. away from } x\}$

Hecke x

$\forall \lambda$, we have locally closed substack



Lemma. $\varphi \in S_{\frac{1}{2}}(\text{Bun}_G)$, $\lambda^* \varphi$ is a measure on the fibres of β .

h_x^λ commute, $\forall x, \lambda$.

Can take $x \in X(F)$, $E|F$ finite Galois extension.

Modification of $D = \text{Galois orbit of } x$.

All these things commute

Question: How to describe eigenfunctions & eigenvalues?
(or eigenfunctionals)

Back to \mathbb{F}_q

$$S(\text{Bun}_G(\mathbb{F}_q)) \subset \mathbb{Q}(\text{Bun}_G(\mathbb{F}_q))$$

Eigenfunctions of Hecke operators here are called cuspidal eigenfunctions (a semisimple)

Variant: Fix D divisor, we can consider bundles with some structures at D

(Hecke op. at $x \notin D$ still act.)

"Theorem": (B. - Kazhdan - Polishchuk)

(actual thm for $SL(2)$)

① genus $(X) \geq 2$. $S_{\frac{1}{2}}(\text{Bun}_G)$ embeds into $C_{\frac{1}{2}}^{\text{as}}(\text{Bun}_G^{\text{vs}}(F)) \cap L^2(\text{Bun}_G^{\text{vs}}(F))$

G adjoint

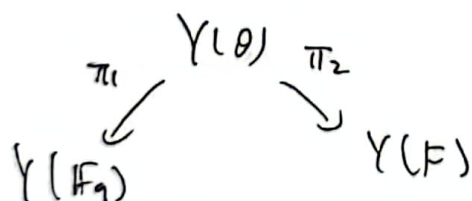
Bun_G^S - scheme, open & dense in Bun_G

\cup
 Bun_G^{vs}

Y - smooth stack $/ \mathcal{O} \rightarrow \text{pt}$

$\mathcal{L} = \Omega_Y$

$S(Y(\mathbb{F}_q)) \xrightarrow{\text{Eis}_Y} S_c(Y)$



$\pi_2^* |\Omega_Y|$ initializes $\boxed{X \text{ defined \& smooth } / \mathcal{O}}$

"Theorem 2" Apply to $c = \frac{1}{2}$, $Y = \text{Bun}_G$

$\text{Eis} : S(\text{Bun}_G(\mathbb{F})) \rightarrow S_{\frac{1}{2}}(\text{Bun}_G(F))$

① Unitary on cuspidal functions (in particular, injective)

② commutes with Hecke operators in the appropriate sense.

$x \in X(F) = X(\emptyset) \xrightarrow{\substack{\uparrow \\ X \text{ proj.}}} X(\mathbb{F}_q)$

$\exists \begin{array}{ccc} H_x^F & \xrightarrow{\sim} & H_{\bar{x}}^{\mathbb{F}_q} \\ \downarrow h & & \downarrow ? \end{array}$

Also same for $x \in X(F)$

$$h_x \cdot \text{Eis}(\varphi) = \text{Eis}(\eta_x(h_x)\varphi)$$

\Downarrow

φ - Hecke eigenfunctions

$\text{Eis}(\varphi)$ - Hecke eigenfunction

$$\text{Bun}(\mathbb{P}^1)_{0,\infty} > G$$

$$\forall c, S_c(G) \xrightarrow{i} S_c(\text{Bun}_G(\mathbb{P}^1)_{0,\infty})$$

$$\downarrow$$

$$S(G)$$

$G(F) \times G(F)$ acts

Lemma. i is an isom. on cuspidal part

Question: What are Hecke operators on $S(G)_{\text{cusp}}$?

Ex.

$G = \text{PGL}(N)$, $E|F$ ext. of degree n . $x \in E^*$

$\pi \in \text{Irr}_{\text{cusp}}(G(F))$

h_x - "fundamental" Hecke op. at x .

h - Hecke op

$$B^* \subset \text{GL}(N; F) \rightarrow \text{PGL}(N; F)$$

$$h(\pi) \in \mathbb{C}$$

Claim. $h(\pi) = \text{char}_{\pi}(x)$.