Affine Springer fibers and representation of small quantum groups

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Lecture 1

A ring. $\quad Z(A) = \{ z \in A : zu = uz, \forall u \in A \}$.

Center of a category. $\mathcal{C}$

$$Z(\mathcal{C}) = \mathrm{End}(1_{\mathcal{C}}) = \left\{ (z_M)_{M \in \mathrm{ob}(\mathcal{C})} : \begin{tikzcd} M \arrow[r, "z_M"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[d, "f"] \\ N \arrow[r, "z_N"'] & N \end{tikzcd}, \forall f : M \to N \right\}$$

Commutative ring w/. the product $(z \cdot z')_M = z_M \cdot z'_M$.

If $\mathcal{C}$ is k-linear, then $Z(\mathcal{C})$ is a k-algebra.

Ex 1. $\mathcal{C} = A\text{-Mod}$, then

$$Z(A) \xrightarrow{\sim} Z(A\text{-Mod})$$
$$z \longmapsto [z_M : M \to M, m \mapsto z \cdot m]$$
$$z_A(1) \longleftarrow (z_M)$$

Ex 2. $\mathcal{C}$ abelian cat. w/. enough projectives,

$\mathcal{P}$ full additive subcat. consisting of proj.

$$Z(\mathcal{C}) \xrightarrow{\sim} Z(\mathcal{P})$$

Examples. 1) $\mathcal{O}_0$ = principal block for cat. $\mathcal{O}$ of a complex semisimple Lie alg $\mathfrak{g}$.

Thm (Soergel '90) $\quad Z(\mathcal{O}_0) = H^*(\underset{\mathcal{B}}{\text{Flag variety}} \text{ for } \mathfrak{g})$

2) Type A: $\mathfrak{g} = \mathfrak{gl}_n \supset \mathfrak{p} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ with blocks of sizes $\nu_1, \nu_2, \dots$ $\quad \nu \vdash n$ $\quad e$ nilpotent mat. whose JNF is of type ν.

<u>Thm</u> (Brundan, Stroppel). $Z(\mathcal{O}_0^\mu) = H^*(\mathcal{B}_e)$

$$\mathcal{B}_e = \{ \mathfrak{b} \in \mathcal{B} : e \in \mathfrak{b} \}.$$

3) $Z(u_q T\text{-mod}) \cong H^*(\text{affine Springer fibers})$

u_q small quantum group

4) $Z(\text{cyclotomic QHA}) \cong H^*(\text{Quiver var.})$ ADE

(Beliakova - Habiro - Lauda - Webster, S. - Varagnolo - Vasserot)

Lectures 1 & 2 : explain the case $\mathcal{O}_0$

Lecture 3 : Center of quantum groups at roots of unity

Lecture 4 . Example (3).

$\mathfrak{g}$ semisimple Lie algebra.

$\mathfrak{b}$ Borel subalg.

$\mathfrak{h}$ Cartan subalg.

$U\mathfrak{g}$ enveloping algebra

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$

$$\begin{tikzcd}
U\mathfrak{g} = U\mathfrak{n}^- \otimes U\mathfrak{h} \otimes U\mathfrak{n} \arrow[r, "pr"] & U\mathfrak{h} \arrow[d, equal] & \\
Z(U\mathfrak{g}) \arrow[u, phantom, sloped, "\subset"] \arrow[r] \arrow[dr, "\sim"] \arrow[drr, bend right, "HC"'] & S\mathfrak{h} & \\
& (S\mathfrak{h})^W \arrow[u, phantom, sloped, "\subset"] \arrow[r] & (S\mathfrak{h})^W \\
& h \arrow[r, mapsto] & h - h(\rho)
\end{tikzcd}$$

$$\mathrm{Cat}\,\mathcal{O} = \left\{ M \in \mathfrak{g}\text{-Mod} : \begin{array}{l} U\mathfrak{g}\text{- f.g.} \\ \mathfrak{h}\text{- semisimple} \\ \mathfrak{b}\text{- locally finite} \end{array} \right\}$$

$$\cap$$

$$\mathfrak{g}\text{-Mod}$$

abelian, Hom-finite.

<u>Verma module</u>. $\lambda \in \mathfrak{h}^*$, $\mathfrak{b} \to \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$, $M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$

$$\left(\begin{array}{ccc} Z & \to & \mathrm{End}_\mathfrak{g}(M(\lambda)) = \mathbb{C} \\ z & \longmapsto & pr(z)(\lambda) \end{array} \right) = \chi_\lambda$$

$L(\lambda)$ = unique simple quotient of $M(\lambda)$

proj. cover $P(\lambda) \twoheadrightarrow M(\lambda)$ has a filtration with subquotients $M(\mu)$

<u>BGG-reciprocity</u>: $\big(P(\lambda) : M(\mu)\big) = [M(\mu) : L(\lambda)]$.

<u>Block decomp.</u> $\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/W} \mathcal{O}_\lambda$, $\mathcal{O}_\lambda = \{ M \in \mathcal{O} : \exists N > 0, (\ker \chi_\lambda)^N$ acts by zero on $M \}$.

$$\mathcal{O}_\lambda \simeq \underbrace{\mathrm{End}\Big(\bigoplus_{w \in W} P(w.\lambda) \Big)^{op}}_{\text{f.d. algebra}}\text{-mod}$$

$\mathcal{O}_0$: anti-dom. proj. $P(w_0 . 0)$

each $M(w.0)$ appears exactly once.

<u>Thm (Soergel)</u>.

(1) $Z \to \mathrm{End}_\mathfrak{g}(P(w_0 . 0))$ is surjective. (3) $Z(\mathcal{O}_0) \simeq \mathrm{End}_\mathfrak{g}(P(w_0 . 0))$

(2) This induces an isom.

$C = \mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]^W_+ \xrightarrow{\sim} \mathrm{End}_\mathfrak{g}(P(w_0 . 0))$. $\boxed{C = H^*(\mathcal{B})}$

$\underline{sl_2}$: $0 \to M(-2) \to M(0) \to L(0) \to 0$

$$M(-2) = L(-2)$$

$$P(-2) = \begin{pmatrix} M(-2) \\ M(0) \end{pmatrix} = \begin{pmatrix} L(-2) \\ L(0) \\ L(-2) \end{pmatrix} \circlearrowleft \varepsilon$$

$$\mathrm{End}_{g}(P(-2)) = \frac{\mathbb{C}[\varepsilon]}{(\varepsilon^2)}$$

$Ub = \langle h, e \rangle$

$$P(-2) = Ug \otimes_{Ub} \Big(\frac{\mathbb{C}[e]}{(e^2)} \otimes \underset{\substack{\Downarrow \\ v_{-2}}}{\mathbb{C}_{-2}} \Big)$$

$$Z(g) \to \mathrm{End}_g(P(-2))$$

$$\Omega = ef + fe + \tfrac{1}{2}h^2 \mapsto \; ?$$

$$\Omega(1 \otimes v_{-2}) = 2f \otimes e v_{-2} + * \, v_{-2}$$

$$\mathrm{Im}(\Omega) = * \varepsilon + * .$$

Lecture 2.

$g = sl_2$, $\quad Z(Ug) \xrightarrow{HC} \mathbb{C}[y]^{S_2}$

$$\begin{array}{c} Z(Ug) \\ \| \\ \Omega \in \mathbb{C}[\Omega] \\ \downarrow \\ \mathrm{End}(P(-2)) \\ \| \\ \mathbb{C}[\varepsilon]/(\varepsilon^2) \end{array}$$

$\Omega \mapsto a\varepsilon$, $\quad a \in \mathbb{C}^*$

$$\widehat{\mathbb{C}[y]}^{S_2}_{y^2-1} = \left(\mathbb{C}[y]^{\wedge}_{1} \oplus \mathbb{C}[y]^{\wedge}_{-1} \right)^{S_2} \simeq \mathbb{C}[h]^{\wedge}_{0}$$

$$\mathrm{End}(P(-2)) = \frac{\mathbb{C}[\Omega]}{(\Omega^2)} = \frac{\widehat{\mathbb{C}[y]}^{S_2}_{y^2-1}}{(\Omega^4)} = \frac{\mathbb{C}[h]^{\wedge}_{0}}{(h^2)} = \frac{S}{S_+^{S_2}} = C$$

$\forall\, V$ f.d. rep. of $\mathfrak{g}$, $\quad V^{**} \simeq V$, $\; f \in \mathrm{End}(V)$

$$\left[\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{\eta} & V\otimes V^* & \xrightarrow{f\otimes 1_{V^*}} & V\otimes V^* & \xrightarrow{\varepsilon} & \mathbb{C} \\ 1 & \longmapsto & \sum v_i\otimes v_i^* & & v\otimes w & \longmapsto & w(v) \end{array}\right] = \mathrm{tr}(f)$$

Relative trace

$\forall\, M \in \mathfrak{g}\text{-Mod}$, $\quad f \in \mathrm{End}_{\mathfrak{g}}(M\otimes V)$

$$\left[M \xrightarrow{1_M\otimes\eta} M\otimes V\otimes V^* \xrightarrow{f\otimes 1_{V^*}} M\otimes V\otimes V^* \xrightarrow{1_M\otimes\varepsilon} M \right] = \mathrm{tr}_V(f)$$

$$\mathrm{tr}_V : \mathrm{End}_{\mathfrak{g}}(M\otimes V) \longrightarrow \mathrm{End}_{\mathfrak{g}}(M)$$

Ex. If M is finite dimensional, $f \in \mathrm{End}_{\mathfrak{g}}(M\otimes V)$

$$\mathrm{tr}(\mathrm{tr}_V(f)) = \mathrm{tr}_{M\otimes V}(f).$$

Rmk 1. $-\otimes V : \mathfrak{g}\text{-Mod} \to \mathfrak{g}\text{-Mod}$

$k \in \mathrm{End}(\cdot\otimes V)$, $\quad k_M \in \mathrm{End}(M\otimes V)$

$$\begin{aligned} \mathrm{tr}_V : \mathrm{End}(\cdot\otimes V) &\to \mathrm{End}(1) = Z(U\mathfrak{g}) \\ (k_M) &\longmapsto \mathrm{tr}_V(k_M) \end{aligned}$$

Further,

$$\begin{tikzcd} Z(U\mathfrak{g}) \arrow[rr, "\mathrm{tr}_V"] \arrow[dr] & & Z(U\mathfrak{g}) \\ & \mathrm{End}(-\otimes V) \arrow[ur, "\mathrm{tr}_V"'] & \end{tikzcd}$$

$$z \longmapsto z_M : M\otimes V \to M\otimes V$$

Rmk 2. If $M \in \mathcal{O}$, V f.d. rep., then $M\otimes V \in \mathcal{O}$.

$$\Rightarrow \mathrm{tr}_V : \mathrm{End}_{\mathcal{O}}(-\otimes V) \longrightarrow Z(\mathcal{O})$$

Rmk 3. $\mathcal{C} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{D}$, (E, F) biadjoint, $\quad \mathrm{tr}_{\mathcal{C}} : \mathrm{End}(E) \longrightarrow Z(\mathcal{D})$

V f.d. rep. of $\mathfrak{g}$, $P(V) = \{\text{wts of } V \text{ with mult.}\}$

ref. J. Bernstein, Traces in categories

$$\mathbb{C}[\mathfrak{h}^*] \longrightarrow \mathbb{C}[\mathfrak{h}^*]$$
$$f \longmapsto \Big[P(V)\cdot f : \lambda \mapsto \sum_{\mu\in P(V)} f(\lambda+\mu) \Big]$$

Set $\Lambda = \prod_{\alpha\in\Phi^\vee} \alpha^\vee = \mathbb{C}[\mathfrak{h}^*]$

Prop.

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "tr_V"] \arrow[d, "HC"', "S"] \arrow[dr, phantom, "\circlearrowleft"] & Z(\mathfrak{g}) \arrow[d, "HC", "S"'] \\ \mathbb{C}[\mathfrak{h}^*]^W \arrow[r, "\widetilde{tr_V}"] & \mathbb{C}[\mathfrak{h}^*]^W \end{tikzcd}$$
$$f \longmapsto \frac{1}{\Lambda}\big(P(V)\cdot(\Lambda f)\big)$$

pf. $\forall\, z\in Z(\mathfrak{g})$, $HC(tr_V(z)) \overset{?}{=} \widetilde{tr_V}(HC(z))$.

Enough to prove their evaluation at infinitely many λ coincide.

Pick λ s.t. $\lambda + P(V) \subset \underbrace{P^+}_{\{\text{dom. integral weights}\}}$

For each λ, $L(\lambda)\otimes V = \bigoplus_{\mu\in P(V)} L(\lambda+\mu)$

$$tr(z|_{L(\lambda)\otimes V}) = \sum_{\mu\in P(V)} \chi_{\lambda+\mu}(z)\dim L(\lambda+\mu)$$
$$\parallel$$
$$tr(tr_V(z)|_{L(\lambda)}) = \chi_\lambda(tr_V(z))\dim L(\lambda)$$

$$\dim L(\nu) = \frac{\Lambda(\nu+\rho)}{\Lambda(\rho)}, \quad \forall\, \nu\in P^+$$

$$\chi_\nu(z) = HC(z)(\nu+\rho)$$

Applications to category $\mathcal{O}$ $\qquad\qquad$ $P = \{\text{integral weights}\}$

<u>Translation functors</u> $\quad \forall \lambda, \mu \in \mathfrak{h}^*$, s.t. $\lambda - \mu \in P$

$V =$ f.d. module with extremal weights $w(\lambda-\mu)$

$$T_\lambda^\mu : \mathcal{O}_\lambda \hookrightarrow \mathcal{O} \xrightarrow{-\otimes V} \mathcal{O} \xrightarrow{pr_\mu} \mathcal{O}_\mu$$

$(T_\lambda^\mu, T_\mu^\lambda)$ biadjoint functors.

Tensor identity. $\quad M(\lambda) \otimes V = (U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}) \otimes V$

$$= U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathbb{C} \otimes V)$$

<u>Fact</u> $\quad T_{-\rho}^0 : \mathcal{O}_{-\rho} \to \mathcal{O}_0$

$P(-\rho) = M(-\rho)$ projective.

$T_{-\rho}^0 (M(-\rho)) = P(w_0 \cdot 0)$

<u>Thm.</u> (Soergel, Bernstein)

$$\underbrace{\mathbb{C}[\mathfrak{h}^*]^W \xrightarrow[HC]{} Z(\mathfrak{g}) \longrightarrow \mathrm{End}(P(w_0 \cdot 0))}_{a} \quad \text{is surjective.}$$

$\ker(a) = \{ f \in \mathbb{C}[\mathfrak{h}^*]^W : v(f\, \mathbb{C}[\mathfrak{h}^*]^W) = 0 \}$

$$v : \mathbb{C}[\mathfrak{h}^*]^W \longrightarrow \mathbb{C}$$

$$f \longmapsto \frac{\sum_{w \in W} T_{w\rho}(\Delta f)}{\Delta}(0)$$

$$T_{w\rho} : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]$$

$$f \longmapsto (\lambda \mapsto f(\lambda + w\rho))$$

$$\mathbb{C}[\mathfrak{h}^*]^W / \ker(a) \simeq \frac{\mathbb{C}[\mathfrak{h}^*]}{\mathbb{C}[\mathfrak{h}^*]^W_+} = C.$$

Lecture 3.

$$\mathrm{End}_{\mathcal{O}_0}(P(w_0\cdot 0)) = C = \mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]^W_+ \simeq H^*(G/B)$$

$\mathbb{V} = \mathrm{Hom}_{\mathfrak{g}}(P(w_0\cdot 0), -) : \mathcal{O}_0 \longrightarrow C\text{-Mod}$ fully faithful on projective objects.

$$Z(\mathcal{O}_0) = Z(\mathcal{O}_0^{\mathrm{proj}}) = Z(\mathbb{V}(\mathcal{O}_0^{\mathrm{proj}})) = Z(C) = C.$$

Deformed category $\mathcal{O}$ and its center

Basic facts. R noetherian normal domain (comm.), M reflexive R-module ($M \xrightarrow{\sim} M^{\vee\vee}$)
$M^\vee = \mathrm{Hom}_R(M,R)$

$K = \mathrm{Frac}(R)$, $M = \bigcap_{ht(\mathfrak{p})=1} M_{\mathfrak{p}} \subset M_K = M \otimes_R K$.

A R-alg. free of finite rk $/R$,

$$Z(A) = A \cap Z(A_K) \subset A_K$$
$$= \bigcap_{ht(\mathfrak{p})=1} A_{\mathfrak{p}} \cap Z(A_K) = \bigcap_{ht(\mathfrak{p})=1} Z(A_{\mathfrak{p}}) \subset Z(A_K)$$

R reg. local ring, $K = \mathrm{Frac}(R)$, F residue field

$$A_F \xleftarrow{-\otimes_R F} A_R \xrightarrow{-\otimes_R K} A_K$$

split semisimple

$$Z(A_F) \longleftarrow Z(A_R)$$

not necessarily surjective

$S = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, $R = S_{(0)}$, $\tau : S \longrightarrow R$

P = wt lattice of $\mathfrak{g}$

Def. $\mathcal{O}_R$ = cat. of $U(\mathfrak{g}\otimes R)$-modules which is f.g.

s.t. $M = \bigoplus_{\lambda\in P} M_\lambda$, $M_\lambda = \{m \in M : h.m = (\lambda+\tau)(h)\, m \ \ \forall h \in \mathfrak{h}\}$

$\mathfrak{n}$ acts locally nilpotently.

$M(\lambda)_R$, $P(\lambda)_R$ free over R

$$\begin{array}{ccc} \mathcal{O}_R = & \bigoplus_{\lambda\in P/W} & \mathcal{O}_{\lambda,R} \\ -\otimes_R F \downarrow & & \downarrow -\otimes_R F \\ \mathcal{O} = & \bigoplus_{\lambda\in P/W} & \mathcal{O}_\lambda \end{array}$$

$$\mathcal{O}_{\lambda,R} = \underbrace{\mathrm{End}\Big(\bigoplus_{w\in W} P(w.\lambda)_R\Big)^{op}}_{= A_R}\text{-mod}$$

① $K = \mathrm{Frac}(R)$, $\mathcal{O}_K$ is split semisimple.

$M(\lambda)_K$ $\quad \tilde{\lambda} : \mathfrak{h} \to K$, $\mathfrak{h} \subset \mathfrak{g}$

$h \mapsto \lambda(h) + \tau(h)$

$\chi_{\tilde{\lambda}} = \chi_{\tilde{\mu}} \iff \tilde{\mu} = w.\tilde{\lambda}$ for some $w \in W$

$\iff w = 1.$

$\Rightarrow \mathcal{O}_K$ semisimple, simples are $M(\lambda)_K$.

$$Z(\mathcal{O}_K) = \prod_{\lambda\in P} K$$

② $\forall \alpha \in \Phi_+$, $\alpha^\vee \in \mathfrak{h}$, $\mathfrak{p}_\alpha = (\alpha^\vee) \subset R$, $R_\alpha := R_{\mathfrak{p}_\alpha}$, k_α = residue field for R_α

$\mathcal{O}_{R_\alpha} \to \mathcal{O}_{k_\alpha}$, $\tilde{\lambda} : \mathfrak{h} \to k_\alpha$

$\chi_{\tilde{\lambda}} = \chi_{\tilde{\mu}} \iff \exists\, w \in W$ s.t. $\mu + \tau_\alpha = w.\lambda + w\tau_\alpha \mod \alpha^\vee$

$\tau_\alpha : S \to k_\alpha$

$\Rightarrow w = 1$ or s_α

$$\mathcal{O}_{R_\alpha} = \bigoplus_{\lambda \in P/\{1, s_\alpha\}} \mathcal{O}_{\lambda, R_\alpha}$$

1) $\langle \lambda+\rho, \check{\alpha} \rangle = 0$, then $P_{R_\alpha}(\lambda) = M_{R_\alpha}(\lambda)$

2) $\langle \lambda+\rho, \check{\alpha} \rangle > 0$, then

$$\begin{array}{c} 0 \to M(\lambda)_{R_\alpha} \to P(s_\alpha\cdot\lambda)_{R_\alpha} \to M(s_\alpha\cdot\lambda)_{R_\alpha} \to 0 \\ \hspace{-9em}\| \\ \hspace{-9em} P(\lambda)_{R_\alpha} \end{array}$$

$$A_{R_\alpha} = \mathrm{End}_{\mathfrak{g}_{R_\alpha}}\left(P(\lambda)_{R_\alpha} \oplus P(s_\alpha\cdot\lambda)_{R_\alpha}\right) = R_\alpha\left(\begin{tikzcd} \lambda \arrow[r, bend left, "i"] & s_\alpha\cdot\lambda \arrow[l, bend left, "j"] \end{tikzcd}\right) \Big/ j\circ i\, 1_\lambda = \check{\alpha}\, 1_\lambda$$

on k_α, $P(s_\alpha\cdot\lambda) = \begin{pmatrix} L(s_\alpha\cdot\lambda) \\ L(\lambda) \\ L(s_\alpha\cdot\lambda) \end{pmatrix} \longrightarrow \begin{array}{c} L(s_\alpha\cdot\lambda) \\ \| \\ M(s_\alpha\cdot\lambda) \end{array} \hookrightarrow \begin{array}{c} M(\lambda) \\ \| \\ P(\lambda) \end{array}$

$$\begin{array}{ccl} Z(A_{R_\alpha}) & = & \left\{ (a_\lambda) \in \prod_\lambda R : \begin{array}{l} a_\lambda \equiv a_{s_\alpha\cdot\lambda} \mod \alpha^\vee \\ \text{if } \lambda \neq s_\alpha\cdot\lambda \end{array} \right\} \\ \cap & & \\ Z(A_K) & = & \prod_\lambda K \end{array}$$

$\mathcal{O}_{0,R}$ $\qquad M(w.0)$

$$Z(\mathcal{O}_{0,R}) = \bigcap_{\alpha \in \Phi^+} Z(\mathcal{O}_{0,R_\alpha}) \subset Z(\mathcal{O}_{0,K}) = \prod_{w \in W} K$$

$$\| \qquad\qquad\qquad\qquad\qquad \|$$

$$\left\{ (a_w)_{w\in W} : \begin{array}{c} a_w \equiv a_{s_\alpha w} \\ \mod \alpha^\vee \end{array} \right\} \qquad H_T^*((G/B)^T)_{(0)} = \prod_{w\in W} H_T^*(pt)_{(0)}$$

$$\|$$

$$H_T^*(G/B)_{(0)}$$

$T \curvearrowright G/B$, $(G/B)^T = W$.

$\exists$ one-dim T-orbit between x, y $\Leftrightarrow$ $y = s_\alpha \cdot x$ $\qquad \alpha : T \to \mathbb{C}^\times$

$Z(\mathcal{O}_{0,R}) = \mathrm{End}_{\mathfrak{g}}(P(w_0\cdot 0)_R)$.

Summary:

$$\begin{array}{ccccc} Z(\mathcal{O}_R) & \simeq & \mathrm{End}(P(w_0\cdot 0)_R) & \simeq & H_T^*(G/B)_{(0)} \\ \downarrow & & \downarrow & & \downarrow -\otimes_R \mathbb{C} \\ Z(\mathcal{O}_R)\otimes_R \mathbb{C} & & & & \\ \downarrow\wr & & & & \\ Z(\mathcal{O}) & \simeq & \mathrm{End}_{\mathfrak{g}}(P(w_0\cdot 0)) & \simeq & H^*(G/B) \end{array}$$

Lecture 4

Quantum groups

$\mathfrak{g}$ simply-laced $\supset \mathfrak{h}$, $(\cdot,\cdot) : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$

P wt lattice

$F = \mathbb{C}(q) \supset A = \mathbb{C}[q^{\pm 1}]$

Def. U_q = F-alg. gen. by E_i, F_i, K_λ, $\lambda\in P$, $i\in I$

s.t. $K_\lambda K_\mu = K_{\lambda+\mu}$, $K_0 = 1$

$K_\lambda E_i K_\lambda^{-1} = q^{(\lambda,\alpha_i)} E_i$, $K_\lambda F_i K_\lambda^{-1} = q^{-(\lambda,\alpha_i)} F_i$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q-q^{-1}}$$

+ quantum Serre relations

Integral forms

$U_q \supset U_q^{DK} = A\langle E_i, F_i, K_\lambda : i\in I, \lambda\in P\rangle$

$\cap$

$U_q^{Lus} = A\langle E_i^{(n)}, F_i^{(n)}, K_\lambda : i\in I, \lambda\in P, n\in\mathbb{Z}_{\geq 0}\rangle$

$$E_i^{(n)} = \frac{E_i^n}{[n]_q!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]_q!}, \quad [n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}$$

$\zeta\in\mathbb{C}^\times$, $\xrightarrow{q\mapsto\zeta}$ U_ζ^{DK}
$\downarrow$
U_ζ^{Lus}

- If ζ not root of 1, then $U_\zeta^{DK} \simeq U_\zeta^{Lus}$.
- Fix ζ ℓ-th root of 1 (ℓ odd)
  $[\ell]_\zeta = 0$.

$$\Rightarrow \quad U_\zeta^{Lus} \ni E_i^\ell = [\ell]_\zeta!\, E_i^{(\ell)} = 0$$

Centers

1) Harish-Chandra / F

$$W \curvearrowright U_q^0 = \langle K_\lambda : \lambda \in P\rangle \supset U_q^{0,ev} = \langle K_{2\lambda} : \lambda \in P \rangle \simeq F[T] \xrightarrow{\xi} F[T]$$

$$w \cdot K_\lambda = q^{(w\lambda - \lambda, \rho)} K_{w(\lambda)} \qquad K_{2\lambda} \longmapsto e^\lambda \mapsto q^{-2(\lambda,\rho)} e^\lambda$$

$$Z(U_q) \simeq (U_q^{0,ev})^{W\cdot} \underset{\xi}{\simeq} F[T]^W$$

2) Center of U_ζ^{DK}

$$Z(U_q^{DK}) = Z(U_q) \cap U_q^{DK}$$

$$Z_{HC} := Z(U_q^{DK})\big|_{q=\zeta} = \mathbb{C}[T/W] \subset Z(U_\zeta^{DK})$$

$$Z_{Fr} := \langle F_\alpha^\ell, E_\alpha^\ell, K_{\ell\lambda} : \alpha \in \Phi^+, \lambda \in P \rangle$$

De Concini–Kac: $Z(U_\zeta^{DK}) = Z_{Fr} \underset{Z_{Fr}\cap Z_{HC}}{\otimes} Z_{HC}$, $\quad Z_{Fr} \cap Z_{HC} = \mathbb{C}\langle K_{2\ell\lambda} : \lambda \in P\rangle^W$

3) Small quantum group

$$u_\zeta = \langle E_i, F_i, K_\lambda \rangle \subset U_\zeta^{Lus} \qquad \begin{array}{c} U_\zeta^{DK} \\ \downarrow \varphi \\ U_\zeta^{Lus} \end{array}$$

$$= Im(\varphi)$$

$$= U_\zeta^{DK} \underset{Z_{Fr}}{\otimes} \mathbb{C}$$

where $Z_{Fr} \to \mathbb{C}$

$E_\alpha^\ell, F_\alpha^\ell \mapsto 0$

$K_\lambda^\ell \mapsto 1$ $\qquad T/W \to T/W,\ t \mapsto t^\ell$

$$Z(U_\zeta^{DK}) \longrightarrow Z(u_\zeta), \text{ its image is } \mathbb{C}[\underbrace{1 \underset{T/W}{\times} T/W}_{\Omega}]$$

$$\pi_0(\Omega) = (P/\ell P)/W = P/W_{\ell,ex}$$

$$W_{\ell,ex} = W \ltimes \ell P$$

$$\Omega = \coprod_{w \in P/W_{\ell,ex}} \Omega_w$$

$u_\zeta T\text{-mod} = \{ P\text{-graded f.g. } u_\zeta\text{-mod s.t. } P\text{-grading is compatible w/. } u_\zeta^0 \text{ action.} \}$

<u>Deformation</u> $\quad R = \mathbb{C}[T]^\wedge_1 \simeq \mathbb{C}[t]^\wedge_0$

$$= \mathbb{C}[K_{2\alpha}]^\wedge_1$$

$$u_{\zeta,R} = U_\zeta^{DK} \otimes_{Z_{Fr}} R, \qquad Z_{Fr} \to R$$

$$E_\alpha^\ell, F_\alpha^\ell \mapsto 0$$

$$K_{\ell\lambda} \mapsto K_{\ell\lambda}$$

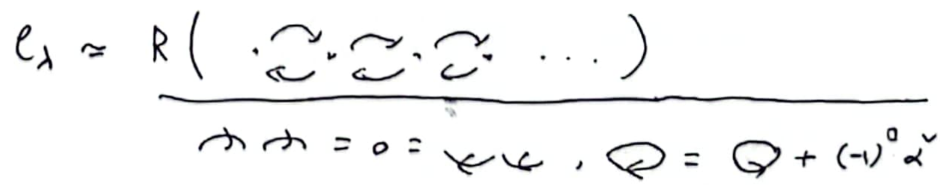
~~$u_{\zeta,R}\text{-mod}$~~

$u_{\zeta,R}T\text{-mod}$ $\quad$ base change to $\mathrm{Frac}(R)$, split semisimple

base change to R_{β_α}, $\beta_\alpha = (\alpha)$,

$$\simeq \bigoplus_{\lambda \in P/\ell P \rtimes \langle 1, s_\alpha \rangle} \mathcal{C}_\lambda$$

If $s_\alpha \lambda \not\equiv \lambda \mod \ell\alpha^\vee$, then



If $s_\alpha \lambda \equiv \lambda \mod \ell\alpha^\vee$,

$\mathcal{C}_\lambda = \cdot \quad \cdot \quad \cdot \quad \cdots$ $\qquad$ Semisimple

$$Z(\ell_\lambda) = H^*_{\mathbb{C}^\times}\left(\cdots \overset{\mathbb{P}^1\ \mathbb{P}^1\ \mathbb{P}^1\ \mathbb{P}^1}{\frown\frown\frown\frown} \cdots\right)$$

$$Z(U_{\zeta,R}T\text{-mod}) = \bigcap_{\alpha\in\Phi^+} Z(\ell_{\lambda}, P_\alpha) \subset \prod_{\lambda\in P} K$$

$G^\vee$ of adjoint type, $\mathbb{C}^\times \curvearrowright \bigcup_{\mu_\ell = \langle\zeta\rangle} Gr = G^\vee((\mathfrak{z}))/G^\vee[[\mathfrak{z}]]$

$$Gr^\zeta = \bigsqcup_{w\in P/W_{\ell,ex}} Fl_w, \quad Fl_w = G^\vee((\mathfrak{z}^\ell))/P_w$$

P_w parahoric subgroup

Affine Springer fiber

$\mathfrak{g}^\vee[[\mathfrak{z}]] \ni \gamma = \gamma_0 \cdot \mathfrak{z}^{\ell-1}$

$\gamma_0 \in \mathfrak{h}_{reg}$ $\quad T^\vee \curvearrowright Gr_\gamma = \{ gG^\vee[[\mathfrak{z}]] : Ad_{g^{-1}}(\gamma) \in \mathfrak{g}^\vee[[\mathfrak{z}]]\}$

$$\begin{array}{ccc} T^\vee & & \cup \\ \| & & \\ T^\vee \curvearrowright & Gr_\gamma^\zeta & = Gr_\gamma \cap Gr^\zeta \end{array}$$

$\left\{ (Gr^\zeta_\gamma)^{T^\vee} = P \right.$, $\exists$ at most one 1-dim'l $T^\vee$-orbit between two fixed pts

+ ~~lin.~~ pairwise indep. of char. at given fixed pt

GKM condition.

$$H^*_T(Gr_\gamma^\zeta) \subset H^*_{T^\vee}(Gr_\gamma^{T^\vee})$$

<u>Thm.</u> (Bezrukavnikov - Boixeda Alvarez - S. - Vasserot)

$$Z(U_{\zeta,R}T\text{-mod}) \simeq H^*_{T^\vee}(Gr_\gamma^\zeta)^\wedge_0 \qquad -\otimes_R \mathbb{C}$$

<u>Cor.</u> $H^*(Gr_\gamma^\zeta) \hookrightarrow Z(U_\zeta T\text{-mod})$.

$H^*(Gr_\gamma^\zeta)^P \hookrightarrow Z(U_\zeta)^T$

<u>Conj.</u> These are isomorphisms.

Cat $\mathcal{O}$ version

$$U_q^{DK} \subset U_q^{hb} = \langle E_i^{(n)}, F_i, K_\lambda \rangle \subset U_q^{Lus}$$

$$U_\zeta^{hb}$$

$\mathcal{O}_\zeta^{hb}$ = f.g. U_ζ^{hb}-module, P-graded s.t. $E_i^{(n)}$ acts locally nilpotent

Thm. (Situ) $Z(\mathcal{O}^{hb}_{\zeta,R}) \simeq H^*_{\check{T}}(\mathrm{Gr}^{\zeta})$

$Z(\mathcal{O}^{hb}_{\zeta}) \simeq H^*(\mathrm{Gr}^{\zeta})$

$$\begin{tikzcd}
H^*(\mathrm{Gr}^{\zeta}) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{res}"'] \arrow[rrdd, "a"] & Z(\mathcal{O}^{hb}_{\zeta}) \arrow[rd] & \\
 & & Z(\mathrm{Rep}\, U_\zeta^{Lus}) \\
H^*(\mathrm{Gr}^{\zeta}_{\gamma}) \arrow[r, phantom, "\curvearrowright"] \arrow[rrd, "b"] & & Z(u_\zeta)^{G^\vee} \arrow[u, phantom, sloped, "\subset"] \arrow[d, phantom, sloped, "\subset"] \\
 & & Z(u_\zeta)^{\check{T}}
\end{tikzcd}$$