Symplectic geometry related to G/U and Sicilian theories.

Victor Ginzburg.

G cpx reductive group $\qquad \mathfrak{g} = \mathrm{Lie}(G)$

U max. unipotent subgp.

W = Weyl grp

$\mathcal{D}(G/U)$ algebraic diff. operators on G/U.

<u>Thm</u> (Gelfand–Graev (1960's))

$\exists$ natural action of W on $\mathcal{D}(G/U)$ by algebra automorphisms.

<u>E.g.</u> $G = SL_2$, $U = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ = stabilizer of elements in $\mathbb{C}^2$

$G/U = \mathbb{C}^2 \setminus 0$

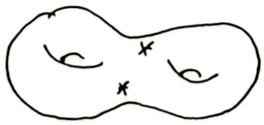
$\mathcal{D}(G/U) = \mathcal{D}(\mathbb{C}^2)$

$W = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{D}(\mathbb{C}^2)$ by Fourier transform

$x \mapsto \frac{\partial}{\partial x}, \; y \mapsto -\frac{\partial}{\partial y}$

<u>Moore – Tachikawa (2011)</u>

TQFT with values in holomorphic symplectic mfds.

$\Sigma =$  punctured Riem. surf.

$\longrightarrow$ Higgs branch $\mathcal{M}(\Sigma)$

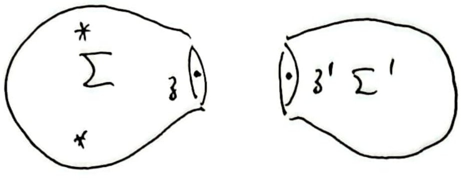
$\mathcal{M}(\Sigma)$ holomorphic symplectic with a Hamiltonian action of $\prod G$

index by punctures.

Moment map

$$\mathcal{M}(\Sigma) \longrightarrow \prod_{\text{punctures}} \mathfrak{g}^*$$

Gluing axioms:



$$\Sigma \underset{\gamma}{*} \Sigma'$$

$$\mu * \mu : \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma') \to \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$$

Axiom: $\mathcal{M}(\Sigma \underset{\gamma}{*} \Sigma') \simeq \left(\mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma')\right) /\!/\!/ G$

$$= \{ m \in \mathcal{M}(\Sigma),\ m' \in \mathcal{M}(\Sigma') : \mu(m) = \mu(m') \} / G$$

Boils down

$$\mathcal{M}_n = \mathcal{M}(S^2, n\text{-marked pts}), \quad n = 1, 2, \ldots$$

should satisfy

$$(\mathcal{M}_m \times \mathcal{M}_n) /\!/\!/ G \simeq \mathcal{M}_{m+n-2}$$

Should have a natural action of the symmetric group $\mathfrak{S}_n \curvearrowright \mathcal{M}_n$

$\mathcal{M}_1$ = principal Nahm pole.

Fix a nondegenerate character $\psi : U \to \mathbb{C}^*$

e.g.
$$\begin{pmatrix} 1 & u_1 & & * \\ & 1 & \ddots & \\ & & \ddots & u_{n-1} \\ 0 & & & 1 \end{pmatrix} \longmapsto \exp(c_1 u_1 + \cdots + c_{n-1} u_{n-1})$$
for any fixed $c_1, \ldots, c_{n-1}$ s.t. $c_i \neq 0, \forall i$

$D^\psi(G/U)$ operators which act on funcs $f : G \to \mathbb{C}$

$f(gu) = \psi(u) f(g), \ \forall u \in U.$

$\cup\!\!\!\!\mid$

quantization of $\mathbb{C}[\mathcal{M}_1]$

$\mathcal{D}(G/U)$ = quantization of func.s on $T^*(G/U) = G \times_U \mathfrak{b}$

$\mathcal{D}^\psi(G/U)$ = — — " — — $T^\psi(G/U) = G \times_U (\psi + \mathfrak{b})$.

Quantum Hamiltonian Reduction

$$A_1 \xleftarrow{\quad} Z \xrightarrow{\text{alg. homo.}} A_2$$

assoc. algebras

$$(A_1 \otimes_Z A_2)^Z := \{ b \in A_1 \otimes_Z A_2 : zb = bz,\ \forall z \in Z \}$$

Claim. $(A_1 \otimes_Z A_2)^Z$ inherits an algebra str.

$U\mathfrak{g}$ enveloping alg.

Z = center of $U\mathfrak{g}$

$Z \hookrightarrow U\mathfrak{g} \to \mathcal{D}(G/U)$, $U\mathfrak{g} \to \mathcal{D}^\psi(G/U)$

$Z \simeq (U\mathfrak{t})^W \hookrightarrow U\mathfrak{t} = \mathrm{Sym}\,\mathfrak{t} \hookrightarrow \mathcal{D}(T)$

Harish-Chandra

T max. torus in G

$\mathfrak{t} = \mathrm{Lie}\, T$ Cartan

Thm 1. $\exists$ alg. isom.

$$\boxed{\mathcal{D}(G/U) = \left(\mathcal{D}^\psi(G/U) \otimes_Z \mathcal{D}(T)\right)^Z}$$

The Gelfand-Graev action in LHS $\longleftrightarrow$ W action on $\mathcal{D}(T)$.

$\mathcal{D}^\psi = \mathcal{D}^\psi(G/U)$ = quantization of princ. Nahm pole.

Quantization of $M_n := \left(\mathcal{D}^\psi \otimes_Z \mathcal{D}^\psi \otimes_Z \mathcal{D}^\psi \cdots \otimes_Z \mathcal{D}^\psi\right)^{Z \times \cdots \times Z}$

$= \{ u_1 \otimes u_2 \otimes \underbrace{\cdots}_{i\ \ i+1} \otimes u_n : z(u_i \otimes u_{i+1}) = (u_i \otimes u_{i+1})z,\ \forall i, \forall z \}$

$u_i \otimes u_{i+1}$

Geometry

$$A_{\mathfrak{g}} = \{ (g,x) \in G \times \mathfrak{g}^* : \mathrm{Ad}_g(x) = x \}$$
$$\downarrow$$
$$\mathfrak{g}^*$$

Centralizer group scheme.

Let G act Hamiltonian on M

moment map $M \to \mathfrak{g}^*$

Have an A-action.

regular
$\mathfrak{g}_r \subset \mathfrak{g}^* \to \mathfrak{g}^* /\!/ G = \mathfrak{t}/W =: \mathfrak{a}$

Kostant: $\pi : \mathfrak{g}_r \to \mathfrak{t}/W$ smooth, each fiber is t-orbit

- $A_{\mathfrak{g}_r}$ descends to a commutative flat group scheme $A \to \mathfrak{a}$

(A is connected if G has no center)

$\mathrm{Lie}\, A = T^* \mathfrak{a}$

Classical version of M_n

$$\ker\left(A \times_{\mathfrak{a}} A \times_{\mathfrak{a}} \cdots \times_{\mathfrak{a}} A \xrightarrow{\text{mult.}} A. \right)$$
$$\|$$
$$\ker_n$$

$$M_n := M_1^n /\!/\!/ \ker_n = \left(M_1 \times_{\mathfrak{a}} M_1 \times_{\mathfrak{a}} \cdots \times_{\mathfrak{a}} M_1 \right)$$

Decorated version [MT]

$\rho_i : sl_2 \to \mathfrak{g}$

$M(S^2, \rho_1, \cdots, \rho_n)$

Let $\rho, \rho' : sl_2 \to \mathfrak{g}$, $M(\rho, \rho') =$ sol's of the Nahm equation $(0,1)$ with pole ρ' at 0, ρ at 1

Bielawski

Let $\rho :=$ principal sl_2

$\exists\ \mathcal{M}(\rho', \rho) \xrightarrow{A} \mathfrak{a}$

$$\mathcal{M}(S^2, \rho_1, \cdots, \rho_n) = \left(\mathcal{M}(\rho_1, \rho) \times \mathcal{M}(\rho_2, \rho) \times \cdots \times \mathcal{M}(\rho_n, \rho)\right) /\!/\!/ \ker n$$