DG categories of Quasi coherent sheaves

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QCoh(X) as a DG cat.

Last time: compactly generated DG cats.

Def. $A \subset B^c$ generates B if $A^\perp = 0$, and then $\text{Ind}(A) \hookrightarrow B$ is an equiv.

and then $B^c = (A^{\text{pre-tr}})^{\text{kar}}$

Conversely, if B is compactly generated, any $A \subset B^c$ with $B^c = (A^{\text{pre-tr}})^{\text{kar}}$ generates B.

Re (classically) X = scheme, $QCoh(X)$ $(= D_{qc}(X))$

Def. $QCoh(X)^{\heartsuit}$ (abelian) is defined by gluing.

$D_{qc}(X)$ is its derived cat.

Better. Consider full subcat. of $D(\mathcal{O}_X\text{-mod})$ with q. coh. cohomology.

- $QCoh(X)^{\heartsuit}$ doesn't have enough projectives.
- Are injectives in $QCoh(X)^{\heartsuit}$ injective as sheaves?

As a DG cat.

Plan. Define $QCoh(Y)$ directly by gluing

Def. $QCoh(X) = \lim\limits_{\text{Spec} R \to X} D(R)$ (works for any prestacks X)

for scheme: $\text{Spec} R \underset{\text{open}}{\hookrightarrow} X$

if X is classical, R's are classical

Rmk The lim is over ∞ - cat.

<u>Example</u>. $X =$ q.c. separated

$$X = \bigcup_{i=1} U_i \quad , U_i = \operatorname{Spec} R_i$$

$F \in QCoh(X)$ is 1) $F_i \in QCoh(U_i) = D(R_i)$

2) $F_i|_{U_i \cap U_j} \xrightarrow{\sim} F_j|_{U_i \cap U_j}$ in $D(R_{ij})$ $\quad U_i \cap U_j = \operatorname{Spec} R_{ij}$

3)
$$\begin{tikzcd}
F_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\sim"] \arrow[dr, "\wr"'] & \theta & F_j|\ldots \\
& F_k|\ldots \arrow[ur, "\sim"'] &
\end{tikzcd}$$

<u>Eg.</u> $X = U \cup V$, $\quad QCoh(X) = QCoh(U) \underset{QCoh(U \cap V)}{\times} QCoh(V)$

<u>Ex.</u> compute Hom's in this language.

<u>Corollary</u>. $(QCoh(X))^c = Perf(X)$

<u>Def</u>. $F \in QCoh(X)$ is perfect if $F|_{\operatorname{Spec} R}$ is perfect for all $\operatorname{Spec} R \hookrightarrow X$.

(suffices to check for an open cover)

$\boxed{\text{IMPORTANT}}$: Is $QCoh(X)$ compactly generated ?

<u>Thm</u> (Thomason-Trobaugh)

Yes, if X is q.c. sep. scheme.

$Perf(X)$ generates $QCoh(X)$

<u>Proof</u>. $\quad X = \underset{\text{"perfect"}}{U} \cup \underset{\text{affine}}{V}$

① Take generators $F_\alpha \in \mathrm{Perf}(U)$

and extend (how?) to $\bar{F}_\alpha \in \mathrm{Perf}(X)$.

② $\bar{F}_\alpha$ should generate $\mathrm{Perf}(X)$ "modulo" sheaves supported on $Y = X \setminus U \subseteq V$.

Take enough perfect objects G_β in $\mathrm{QCoh}(V)_Y := \{ F \in \mathrm{QCoh}(V) : F|_{U \cap V} = 0 \}$

$$\mathrm{QCoh}(V)_Y \subset \mathrm{QCoh}(V)$$

to generate it. (if $Y = V(f_1, \dots, f_k) \subset V$, can take

$$G = \bigotimes_{i=1}^{k} (\mathcal{O}_V \xrightarrow{f_i} \mathcal{O}_V))$$

Extend to $\bar{G}_\beta \in \mathrm{Perf}(X)$ s.t. $\bar{G}_\beta|_U = 0$

Operations on DG categories

Last time. $\mathrm{QCoh}(X) = \lim_{\mathrm{Spec} R \to X} D(R)$

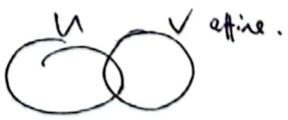
X qcompact separated.

Then 1) $\mathrm{Perf}(X) = \mathrm{QCoh}(X)^c$

$\mathrm{Perf}(X) = \lim \mathrm{Perf}(R)$

2) (Thomason–Trobaugh) $\mathrm{Perf}(X)$ generates $\mathrm{QCoh}(X)$.

Key step in proof.



$U \subset X = U \cup V$: needed to extend from $\mathrm{Perf}(U)$ to $\mathrm{Perf}(X)$. (the actual TT theorem)

More precisely, every $F \in \mathrm{Perf}(U)$ is a direct summand of $\tilde{F}|_U$ for some $\tilde{F} \in \mathrm{Perf}(X)$

Working with compactly gen. DG cats

① $\mathcal{B} = \mathrm{Ind}(\mathcal{A}) = D(\mathcal{A}^{op})$
$\ \ \ ''$
comp. gen.

Rmk. $\mathcal{B}$ has $\prod$ (and all limits) (computed pointwise on $\mathcal{A}$)

$\mathcal{B} = \mathrm{Funct}(\mathcal{A}^{op}, D(k))$

Prop (Brown Rep) $\phi : \mathcal{B}^{op} \to D(k)$ is representable iff it sends $\oplus$ to $\prod$.

Proof if) ϕ is determined by $\phi|_{\mathcal{A}^{op}}$. QED.

Corollary If $\mathcal{B}_1$ is c. gen., $\mathcal{B}_2$ is cocomplete, then any continuous functor $F: \mathcal{B}_1 \to \mathcal{B}_2$ has a right adjoint.

Proof $\forall x \in \mathcal{B}_2$, $\mathrm{Hom}(F(-), x) : \mathcal{B}_1^{op} \to D(k)$ is representable by $F^R(x)$. QED.

Rmk 1. F^R is continuous iff $F(\mathcal{B}_1^c) \subset \mathcal{B}_2^c$,

2. Assuming this, $F(\mathcal{B}_1^c)$ generates $\mathcal{B}_2^c$ iff F^R is conservative.
$[F^R(x) = 0 \Rightarrow x = 0]$

② **Quotients**.

Proposition (DG cats, Δ-cats) $\mathcal{B}$ = DG cat (pre-triangulated)
$\cup$ full subcat
$\mathcal{A}$

TFAE: 1) $i : \mathcal{A} \to \mathcal{B}$ admits a right adjoint.

2) $\exists$ full subcat. $\mathcal{C} \subset \mathcal{B}$ s.t. (a) $\mathcal{A} \perp \mathcal{C}$ ($\Leftrightarrow \mathrm{Hom}(\mathcal{A}, \mathcal{C}) = 0$)

(b) any $b \in \mathcal{B}$ fits into a Δ

$$\begin{array}{ccccc} a & \to & b & \to & c \\ \cap & & & & \cap \\ \mathcal{A} & & & & \mathcal{C} \end{array}$$

Properties 1) The Δ in 2 b) is unique

2) $\mathcal{A} = {}^{\perp}\mathcal{C}$, $\mathcal{C} = \mathcal{A}^{\perp}$.

3) For i^R, $\mathrm{Id} \to i^R \circ i$ is an isom. (because i is fully faithful)

4) $\mathcal{C} = \ker(i^R)$

5) $\mathcal{A} \xrightarrow[\sim]{i^R} \mathcal{B}/\mathcal{C}$. (also, $\mathcal{C} \simeq \mathcal{B}/\mathcal{A}$)

② Quotients

$\mathcal{A} \rightleftarrows \mathcal{B} \rightleftarrows \mathcal{C}$ is a short exact sequence

[$\mathcal{C}$ is a localization
$\mathcal{A}$ is a colocalization of $\mathcal{B}$]

Example

$$\begin{array}{ccc} & C(\mathcal{A}) & \\ \subset & & \cup \\ C(\mathcal{A})^{\text{semifree}} & & C(\mathcal{A})^{\text{acyclic}} \end{array}$$

which is why

$C(\mathcal{A})^{\text{semifree}} \simeq D(\mathcal{A})$

Now suppose $\mathcal{B}$ is cocomplete, $\mathcal{A}$ is generated by $\mathcal{A}^c \subset \mathcal{B}^c$.

Then $\mathcal{A} \subset \mathcal{B}$ has a continuous right adjoint, so $\mathcal{C} = \mathcal{A}^{\perp}$ gives

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}.$$

Exercise. $\mathcal{C}$ and all 4 functors preserve $\oplus$.

Application. $\mathcal{C} = 0 \iff (\mathcal{A}^c)^{\perp} = 0 \iff \mathcal{B} = \mathrm{Ind}(\mathcal{A})$.

Suppose $\mathcal{B}$ is cptly gen. as well.

Then 1)
$$\begin{array}{ccc} \mathcal{B}^c & \longrightarrow & (\mathcal{C})^c \\ & \searrow & \wedge \\ & & \mathcal{C} \end{array}$$
(r. adjoint is cont.)

2) $\mathcal{B}^c$ generates $\mathcal{C}$ (r. adjoint is conservative)

$\Downarrow$

$\mathcal{C} = \mathrm{Ind}(\mathcal{C}^c)$

$\mathcal{C}^c = ((\mathrm{Im}\, \mathcal{B}^c)^{\text{pretr}})^{\text{kar}} = \mathrm{Perf}(P(\mathcal{B}^c)^{op})$

Pages

$\underline{\text{Ex}}$. $\underset{Y}{\text{closed}\subset} X = \operatorname{Spec} R$

s.t. $U = X - Y$ is q. compact (or $Y = V(f_1, \dots, f_n)$ as a set).

$$\operatorname{QCoh}(X) = D(R)$$

$$\begin{array}{ccccc} \operatorname{QCoh}(X)_Y & \subset & \operatorname{QCoh}(X) = D(R) & \supset j_* & \operatorname{QCoh}(U) \\ \text{\rotatebox{90}{=}} & & \| & & \text{\rotatebox{90}{=}} \\ \operatorname{Ind}\left(\hat{\bigotimes}_{i=1}(\mathcal{O}_X \xrightarrow{f_i} \mathcal{O}_X)\right) & & \operatorname{Ind}(\mathcal{O}_X) & & \operatorname{Ind}(\mathcal{O}_U) \end{array}$$

Fourier - Mukai transform via Dh cats

$\underline{\text{Last time}}$ Quotients.

$\underline{\text{Large world}}$ $\quad A \rightleftarrows B \rightleftarrows \mathcal{C}$

$\underline{\text{Small world}}$ $\quad A^c \hookrightarrow B^c \to \mathcal{C}^c$

TT Thm measures surj.

$\underline{\otimes}$.

$\underline{\text{Recall}}$. small Dh cats. $\quad Ob(A \otimes B) = Ob(A) \times Ob(B)$

$$\operatorname{Hom}_{A\otimes B}(a_1 \otimes b_1, a_2 \otimes b_2) = \operatorname{Hom}_A(a_1, a_2) \otimes \operatorname{Hom}_B(b_1, b_2)$$

Has a universal property.

$\underline{\text{Rmk}}$. For Karoubian pre-Δ dg categories, take $\left((A \otimes B)^{\text{pre-tr}}\right)^{\text{kar}}$.

$\underline{\text{Key property}}$. $\quad \operatorname{Ind}(A) \hat{\otimes} \operatorname{Ind}(B) \overset{\text{Want}}{=} \operatorname{Ind}(A \otimes B)$

Large Dh cats

different., tensor prod. of cocomplete cats

Small $\otimes$

In general, $A \overset{\text{cocompleted}}{\otimes} B$ is defined by universal property using continuous bilinear functors.

$\leadsto$ Define $A \otimes B$ as $\mathrm{Ind}(A \overset{\text{plain}}{\otimes} B) /$ some objects

$$\mathrm{Cone}\left(\bigoplus_i (a_i \otimes b) \to \left(\bigoplus_i a_i\right) \otimes b\right)$$

$$\mathrm{Cone} \ \ldots \qquad a \otimes \left(\bigoplus_i b_i\right).$$

Dual category.

Small world. $A \mapsto A^{op}$

Key property. $(\mathrm{Ind}(A))^\vee = \mathrm{Ind}(A^{op}) = D(A) = \mathrm{Funct}(A, D(k))$

$$= \mathrm{Funct}(\mathrm{Ind}(A), D(k))^{\text{cont.}}$$

Better (In general). $\otimes$ on $(\mathrm{Dh\,Cats})^{\text{cocomplete}}$ $\leadsto$ Dual category may or may not exist.

All compactly generated cats are dualizable.

Geometrically:

X = q cpt, separated scheme.

Recall! $\mathrm{QCoh}(X) = \mathrm{Ind}(\mathrm{Perf}(X))$

$\mathrm{QCoh}(X)^\vee = \mathrm{Ind}(\mathrm{Perf}(X)^{op})$

(e.g. $X \overset{\text{Spec } R}{=}$ is affine,

$\mathrm{QCoh}(X) = D(R)$,

$\mathrm{QCoh}(X)^\vee = D(R^{op})$).

$\mathrm{Perf}(X)^{op} \xrightarrow{\sim} \mathrm{Perf}(X)$

$\mathrm{Hom}(-, \mathcal{O}_X)$ (ordinary dual)

Therefore. $\mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X)$; corresponds to $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to D(k)$

$\cup$

$\mathrm{Perf}(X) \otimes \mathrm{Perf}(X)$

X, Y q cpt. separated

$$QCoh(X) \otimes QCoh(Y) = \mathrm{Ind}(Perf(X) \otimes Perf(Y))$$

$$\begin{array}{c} F \otimes G \\ \downarrow \\ F \boxtimes G \end{array}$$

<u>answer</u>

$$QCoh(X \times Y) = \mathrm{Ind}(Perf(X \times Y))$$

Check that $(Perf(X) \otimes Perf(Y))^{\perp} \subset QCoh(X \times Y)$ vanishes.

<u>Corollary</u>. $\mathrm{Funct}(QCoh(X), QCoh(Y))^{cont.} = QCoh(X)^{\vee} \otimes QCoh(Y) = QCoh(X \times Y)$

$$\begin{aligned} &\| \\ &\mathrm{Funct}(D(Perf(X)^{op}), D(Perf(Y)^{op}))^{cont.} \\ &\| \\ &D(Perf(X) \otimes Perf(Y)^{op}) \\ &\| \\ &D(Perf(X \times Y)^{op}) = QCoh(X \times Y) \end{aligned}$$

$$\Phi_K = p_{2*}(p_1^*(-) \otimes K)$$

<u>Eg.</u> $\mathrm{Id}_{QCoh(X)} \in \mathrm{Funct}(QCoh(X), QCoh(X))^{cont.}$

$$\begin{array}{c} \mathrm{Id}_{QCoh(X)} \\ \updownarrow \\ \mathcal{O}_{\Delta} \in QCoh(X \times X). \end{array}$$

$$\mathrm{Ext}^{\bullet}(\mathrm{Id}_{QCoh(X)}, \mathrm{Id}_{QCoh(X)}) = \mathrm{Ext}^{\bullet}_{X \times X}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta})$$

Hochschild cohomology.