Hecke operators for algebraic curves over local non-archimedian fields,

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<u>Two parts</u>

I. General things

II. Particular example + discuss to "usual" rep. theory of p-adic groups.

1. "Classical stuff"

X - smooth proj. curve $/\mathbb{F}_q$.

G - reductive alg. group.

Bun_G - stack of G-bundles on X

$\mathbb{C}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ functions with finite supp.

$\cap$

$L^2(\mathrm{Bun}_G(\mathbb{F}_q))$

$\cap$

$\mathbb{C}(\mathrm{Bun}_G(\mathbb{F}_q))$ all functions

$x \in X(\mathbb{F}_q)$

H_x - algebra of Hecke operators

$H_x \overset{\text{Satake}}{\simeq} \mathbb{C}[G^\vee]^{G^\vee} \simeq K_0(\mathrm{Rep}\, G^\vee)$

<u>Ex</u> $G = GL(N)$

$$\begin{tikzcd} & \{\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \quad \mathcal{E}_2/\mathcal{E}_1 = \mathbb{F}_q . x\}_{\mathbb{P}^{N-1}} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_{GL(N)} & & \mathrm{Bun}_{GL(N)} \end{tikzcd}$$

Correspondence $\longrightarrow [\mathbb{C}^N]$

<u>Langlands.</u> (Replace $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$)

Eigenvalues have to do with $G^\vee$ local system on X.

D any (effective) divisor on $X/\mathbb{F}_q$

Hecke operators which "live" at D.

Irreducible divisor = Galois orbit of $x \in X(\overline{\mathbb{F}}_q)$ $\longrightarrow$ H_D is still $\mathbb{C}[G^\vee]^{G^\vee}$.

2. Want: replace $\mathbb{F}_q$ by F local non-archimedean field.

Bun_G - alg. stack over F.

Y alg. var. $/F$, $C_c^\infty(Y(F))$ = locally const. functions w/ cpt support

$$C_c^\infty(Y(F)) = S(Y)$$

$\mathcal{L}$ line bundle on Y,

$c \in \mathbb{C}$ $\quad F^* \longrightarrow \mathbb{R}_+ \xrightarrow{x \mapsto x^c} \mathbb{C}^*$

$S(Y, \mathcal{L}, c) =$ comp. supp. locally const. sections of $|\mathcal{L}|^c$.

Y - stack

$Y = Z/G$

Z scheme

<u>Assume</u> ① G is unimodular

② $Z(F) \longrightarrow Y(F)$ surjective.

$\mathcal{L}$ line bundle on Y, $c \in \mathbb{C}$.

$$S(Z, |\mathcal{L}_Z|^c)_{G(F)} = S(Y, |\mathcal{L}|^c).$$

<u>Lemma</u> Independent of presentation. (Gaitsgory - Kazhdan)

Can generalize to Y is locally Z/G.

In particular, can take $Y = Bun_G$

$\mathcal{L} = \Omega_{Bun_G}$ (some determinant)

$S_c(Bun_G) = S(Bun_G, |\Omega|^c)$

Claim (B-Kazhdan, Etingof-Frenkel-Kazhdan)

$c = \frac{1}{2}(c_{crit})$

Then we have Hecke operators.

$x \in X(F)$.

$\forall \lambda$ - dom. coweight of G

$h_x^\lambda : S_{\frac{1}{2}} \to S_{\frac{1}{2}}$ they all commute

$h_x^\lambda h_x^\mu = h_x^{\lambda+\mu}$.

$\forall \lambda$

$\underbrace{\{ \mathcal{E}_1, \mathcal{E}_2 \text{ - } G\text{-bundles}, k: \mathcal{E}_1 \rightsquigarrow \mathcal{E}_2 \text{ isom. away from } x \}}_{\text{Hecke}_x}$

$\forall \lambda$, we have locally closed substack

$$\begin{tikzcd} & \mathrm{Hecke}_x^\lambda \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd}$$

Lemma. $\varphi \in S_{\frac{1}{2}}(\mathrm{Bun}_G)$, $\alpha^* \varphi$ is a measure on the fibers of β.

h_x^λ commute, $\forall x, \lambda$.

Can take $x \in X(E)$, $E|F$ finite Galois extension.

Modification of D = Galois orbit of x.

All these things commute

Question: How to describe eigenfunctions & eigenvalues?
(or eigenfunctionals)

Back to $\mathbb{F}_q$

$S(\mathrm{Bun}_G(\mathbb{F}_q)) \subset \mathbb{C}(\mathrm{Bun}_G(\mathbb{F}_q))$

Eigenfunctions of Hecke operators here are called Cuspidal eigenfunctions (G semisimple)

Variant: Fix D divisor, we can consider bundles with some structures at D

Hecke op. at $x \notin D$ still act.

"Theorem": (B. – Kazhdan – Polishchuk)
(actual thm for $SL(2)$)

① genus $(X) \geq 2$. $S_{\frac{1}{2}}(Bun_G)$ embeds into $C^{\infty}_{\frac{1}{2}}(Bun_G^{vs}(F)) \cap L^2(Bun_G^{vs}(F))$

G adjoint

$$\begin{array}{l} Bun_G^{s} \text{ - scheme, open \& dense in } Bun_G \\ \cup \\ Bun_G^{vs} \end{array}$$

Y - smooth stack $/ \mathcal{O} \to \mathbb{F}_q$

$\mathcal{L} = \Omega_Y$

$$S(Y(\mathbb{F}_q)) \xrightarrow{Eis_Y} S_c(Y)$$

$$\begin{tikzcd} & Y(\mathcal{O}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y(\mathbb{F}_q) & & Y(F) \end{tikzcd}$$

$\pi_2^* |\Omega_Y|$ trivializes $\quad$ X defined & smooth $/ \mathcal{O}$

"Theorem 2" Apply to $c = \frac{1}{2}$, $Y = Bun_G$

$$Eis: S(Bun_G(\mathbb{F})) \to S_{\frac{1}{2}}(Bun_G(F))$$

① Unitary on cuspidal functions (in particular, injective)

② commutes with Hecke operators in the appropriate sense.

$$x \in X(F) = X(\mathcal{O}) \to X(\mathbb{F}_q) \ni \bar{x}$$

X proj.

$$\exists\ H_x^F \xrightarrow{\sim} H_{\bar{x}}^{\mathbb{F}_q} \qquad \text{Also same for } x \in X(E)$$

$$h^\lambda \longrightarrow ?$$

$h_x \cdot \mathrm{Eis}(\varphi) = \mathrm{Eis}(\eta_x(h_x)\varphi)$

$\Downarrow$

φ - Hecke eigen functions

$\mathrm{Eis}(\varphi)$ - Hecke eigen function

$\mathrm{Bun}(\mathbb{P}^1)_{0,\infty} \supset G$

$\forall c, \; S_c(G) \overset{i}{\hookrightarrow} S_c(\mathrm{Bun}_G(\mathbb{P}^1)_{0,\infty})$

$\quad \parallel$

$\quad S(G)$

$G(F) \times G(F)$ acts

<u>Lemma</u>. i is an isom. on cuspidal part

<u>Question</u>: What are Hecke operators on $S(G)_{\text{cusp}}$?

<u>Ex</u>.

$G = PGL(N)$, $E|F$ ext. of degree n. $x \in E^*$

h_x - "fundamental" Hecke op. at x.

$E^* \subset GL(N;F) \longrightarrow PGL(N;F)$

<u>Claim</u>. $h(\pi) = \mathrm{char}_\pi(x)$.

$\pi \in \mathrm{Irr}_{\text{cusp}}(G(F))$

h - Hecke op

$h(\pi) \in \mathbb{C}$