Introduction to Symplectic duality

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Symplectic resolutions are Lie algebras of 21st century.

— Andrei Okounkov.

- Symplectic singularities (Beauville)

Y - smooth alg. var. $/\mathbb{C}$, symplectic $\omega \in \Omega^2(Y)$, $d\omega = 0$, ω non-deg.

$Y = \operatorname{Spec} A$ affine. sympl. str. induces Poisson bracket on A.

Y - normal alg. var., $\underset{\substack{\text{Smooth,} \\ \text{Symplectic}}}{U \subset Y}$, $\operatorname{codim}(Y \setminus U) \geq 2$

Def. Y singular symplectic, $\Leftrightarrow$ $\exists\, X \xrightarrow{\pi} Y$ resolution

s.t. $\pi^* \omega$ is defined on X.

(could be degenerate).

Lemma If this is true for some X, then it is true $\forall X$.

Y has finitely many Sympl. leaves

Best situation. $\pi^* \omega$ is non-degenerate.

Ex. $Y = \mathbb{C}^2 / \mathbb{Z}_2$ $\quad xy = z^2$ $\qquad X = T^* \mathbb{P}^1$

2 symplectic leaves

Conical: ① $\mathbb{C}^*$ -action on Y, $\lim_{t \to 0} t(y) = 0$, $t \in \mathbb{C}^*, y \in Y$

$0 \in Y$

② $\mathbb{C}^*$ dilates the symp. form, $t^* \omega = t^i \omega$, $i > 0$.

$X \to Y$ conical sympl. resolution if $\mathbb{C}^*$ acts on both X and Y subject to above cond.

Examples:

⓪ $Y = \mathbb{C}^2/\mathbb{Z}_2$, $X = T^*\mathbb{P}^1$

⓪ $\Gamma < SL(2;\mathbb{C})$ finite subgroup, $Y = \mathbb{C}^2/\Gamma$

$X = \widetilde{\mathbb{C}^2/\Gamma}$ - minimal resolution

$X \xrightarrow{\pi} Y$ Special fiber - tree of $\mathbb{P}^1$'s

① $\mathfrak{g}$ - simple Lie alg. $\simeq \mathfrak{g}^*$

$\mathcal{N}_\mathfrak{g}$ = nilpotent cone in $\mathfrak{g}$

$\mathcal{B}$ - flag var. of $\mathfrak{g}$

Springer resolution: $\underset{X}{\overset{\|}{T^*\mathcal{B}}} \xrightarrow{\pi} Y = \mathcal{N}_\mathfrak{g}$

② $Y = \mathrm{Sym}^n(\mathbb{C}^2) = (\mathbb{C}^2)^n/\mathfrak{S}_n$

$X = \mathrm{Hilb}^n(\mathbb{C}^2) = \{ I \trianglelefteq \mathbb{C}[x,y] : \dim \mathbb{C}[x,y]/I = n \}$

④ $Y = \mathbb{C}^4/\mathbb{Z}_2$ - sympl. singularity, but doesn't have a symp. resolution.

$\mathfrak{g}$ - simple, Y = closure of minimal nilpotent orbit

$\mathfrak{g} \neq sl(n)$ no symp. resolution

$\mathfrak{g} = sp(4)$, $Y = \mathbb{C}^4/\mathbb{Z}_2$

⑤ Nakajima quiver varieties - big source of conical sympl. resolutions

⑥ Hyper-toric varieties

$\mathbb{T} < (\mathbb{C}^*)^n \curvearrowright \mathbb{C}^n$, also acts on $T^*\mathbb{C}^n$

subtorus

$Y = T^*\mathbb{C}^n /\!/ \mathbb{T} = \mathrm{Spec}\, \mathbb{C}[\mu^{-1}(0)]^{\mathbb{T}}$

always have a symp. resolution

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Ginzburg, Kaledin, Bezrukavnikov produce a family of canonical quantizations of Y, $Y = \operatorname{Spec} A$

<u>Basic ex.</u> $Y = \mathcal{N}_{\mathfrak{g}}$, $\qquad U(\mathfrak{g})$ – quantization of $\mathfrak{g}^*$

$$U(\mathfrak{g})/\text{central character} = \text{quantization of } \mathbb{C}[\mathcal{N}_{\mathfrak{g}}]$$

Symplectic resolutions produce interesting non-commutative alg.

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$$\begin{array}{c} X \\ \downarrow \\ Y \end{array} \quad \text{Conical symp. resolution.}$$

X – property and not structure.

$$\mathfrak{h}_Y = H^2(X;\mathbb{C}) \supset \Lambda = H^2(X,\mathbb{Z}) \simeq \operatorname{Pic}(X)$$

<u>Claim</u>. $\Lambda = \operatorname{Pic}(X)$ is indep. of X.

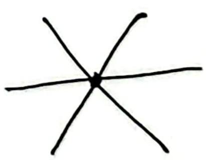
Namikawa defined $\mathfrak{h}_Y$ $\forall Y$ (i.e. not necessarily for those Y which have a sympl. resolution)

$\exists$ canonical deformation of Y (also of X) with base $\mathfrak{h}_Y$.

$$\begin{array}{c} \mathcal{Y} \\ \varphi \downarrow \\ \mathfrak{h}_Y \end{array} \qquad \begin{array}{l} \varphi^{-1}(0) = Y \\ \text{generic fiber is smooth } (\Leftrightarrow Y \text{ has a sympl. res.}) \end{array}$$

$\forall \lambda \in \Lambda$ defines partial sympl. resolution of Y

generic λ will define a resolution



Λ will decompose into chambers

in each chamber, resolution is the same.

<u>Rmk</u>. $\mathfrak{h}_Y$ also parametrizes quantizations of Y.

<u>Another space</u> $\mathfrak{s}_Y$ $\qquad A = \mathrm{Aut}_{\mathbb{C}^*}(Y)$

$\parallel$

Cartan subalgebra of A

$\underset{\text{torus}}{S \subset A}$, $\mathfrak{s}_Y := \mathrm{Lie}(S)$, $\mathfrak{s}_Y$ also has int. str.

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<u>Duality</u>. Conical sympl. singularities tend to come in pairs

(X, X^*) You can understand geometry of X^* in terms of X.

<u>Basic thing</u>

$\mathfrak{t}_Y = \mathfrak{s}_{Y^*}$, $\mathfrak{s}_Y = \mathfrak{t}_{Y^*}$ + many other properties

<u>Ex</u>. $Y = \mathcal{N}_{\mathfrak{g}}$. $\qquad \mathfrak{g}^\vee$ = Langlands dual

$Y^* = \mathcal{N}_{\mathfrak{g}^\vee}$

$T^*\mathcal{B} \longrightarrow \mathcal{N}_{\mathfrak{g}}$

$\mathfrak{t}_Y = H^2(T^*\mathcal{B}, \mathbb{C}) = \mathfrak{h}_{\mathfrak{g}}^*$, $\mathfrak{h}_{\mathfrak{g}}$ = Cartan of $\mathfrak{g}$

$A = G$ adjoint group, $\mathrm{Lie}\, G = \mathfrak{g}$

$\mathfrak{s}_Y = \mathfrak{h}_{\mathfrak{g}}$

By definition, $\mathfrak{g} \longrightarrow \mathfrak{g}^\vee$ sends $\mathfrak{h}_{\mathfrak{g}}$ to $\mathfrak{h}_{\mathfrak{g}}^*$.

<u>Example</u>. $Y = \mathbb{C}^2/\Gamma$, $\Gamma \subset SL(2, \mathbb{C})$

$\Gamma \leftrightarrow \mathfrak{g}$ - simple Lie alg. ADE type

$\dim \mathfrak{t}_Y = \mathrm{rank}\, \mathfrak{g}$

$\dim \mathfrak{s}_Y = 1$ if $\mathfrak{g} = \mathfrak{sl}(n)$, 0 if $\mathfrak{g} \neq \mathfrak{sl}(n)$

Y^* = closure of min. nilp. orbit in $\mathfrak{g}$.

Where does it appear in physics?

Has to do with $3d$ $\mathcal{N}=4$ SUSY QFT

$\forall$ such theory, has two special parts of its moduli space of vacua.

$\mathcal{M}_H$ - Higgs branch, $\mathcal{M}_C$ - Coulomb branch

Idea. $Y = \mathcal{M}_H$, $Y^* = \mathcal{M}_C$.

In this way physicists produce a lot of examples

$1 \to T \to (\mathbb{C}^*)^n \to T_F \to 1$ $\qquad$ $Y = T^*\mathbb{C}^n /\!/ T$

$1 \to T_F^\vee \to (\mathbb{C}^*)^n \to T^\vee \to 1$ $\qquad$ $Y^* = T^*\mathbb{C}^n /\!/ T_F^\vee$

How do you know if Y^* has a sympl. resolution?

$$\begin{array}{ccc} \mathbb{T}_{Y^*} & \supset & \Lambda_{Y^*} \\ \| & & \| \\ \mathbb{S}_Y & & \text{Hamiltonian coweights} \end{array}$$

$\lambda \in \Lambda_Y$, $\quad \lambda : \mathbb{C}^* \to \mathrm{Aut}(Y)$

Ex. $Y = \mathbb{C}^2/\Gamma$

if $\Gamma \neq \mathbb{Z}_n$, $A = \{1\}$, Y^* should not have a resolution.

$$\begin{array}{c} \mathbb{C}^* \curvearrowright X \quad \text{isolated fixed pts} \\ \downarrow \\ \mathbb{C}^* \curvearrowright Y \end{array}$$

another expectation.

$$X^{\mathbb{C}^*} = (X^*)^{\mathbb{C}^*}$$

Kaledin. $H^i(X;\mathbb{C}) = 0$ if i is odd.

$\# X^{\mathbb{C}^*} = \dim H^*(X;\mathbb{C})$

$\quad \|$

$\# (X^*)^{\mathbb{C}^*} = \dim H^*(X^*, \mathbb{C})$ $\qquad$ Remark

<u>Warning</u>: $H^*(X, \mathbb{C}) \not\cong H^*(X^*, \mathbb{C})$

$Y = \mathbb{C}^2/\mathbb{Z}_n$.

$X = \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ $\quad X^* = T^*\mathbb{P}^{n-1}$

<u>Question</u>: How to read off $H^*(X; \mathbb{C})$ from $Y^*, X^* \ldots$?

(Hashin: Hikita conjecture).

Work in progress of Aganagic - Okounkov.

(about quiver var. of finite a affine type A)

Relates quantum K-theory of X to quantum K-theory of X^*.

<u>Categorical duality</u>

· about categories of modules over quantizations of Y and Y^*

Braden - Licata - Proudfoot - Webster

Categories $\mathcal{O}$ for Y and Y^* are Koszul dual.

<u>Summary</u>

① You can formulate a lot of other relations between Y & Y^*.

How to prove them in interesting examples?

② Given Y how to construct Y^*?

$Y \longrightarrow Y^*$ should not be anything like a functor.