Theta correspondence and relative Langlands

$/\mathbb{F}_q$. char $\neq 2$.

V symplectic v.s. dim $2n$

$G = Sp(V)$ finite gp. has Weil representation ω

Analogue of $GL_n(\mathbb{F}_q) \curvearrowright C(\mathbb{F}_q^n)$

$\dim_{\mathbb{C}} \omega = q^n = \sqrt{|V|}$

Dual pair,

V_1, V_2

sympl. quadratic

$G_1 = Sp(V_1)$, $G_2 = O(V_2)$

$V = V_1 \otimes V_2$ symplectic

$G_1 \times G_2 \longrightarrow Sp(V)$

Understand how $\omega|_{G_1 \times G_2}$ decomposes.

Complicated: Not a bijection between subsets of $Irr(G_1)$ and $Irr(G_2)$

<u>True</u>: mult. 1 holds (Waldspurger) $\pi_1 \boxtimes \pi_2$ appears at most once in $\omega|_{G_1 \times G_2}$.

Aubert - Michel - Rouquier conjecture.

A-M-R: prove conj. for unitary gps

S-Y. Pan 2019

J. Ma, C. Qiu, J. Zou 2022

Geometric approach. (split, $\dim V_2 =$ even)

restrict to unipotent principal series.

$H_{G_1} \otimes H_{G_2} \curvearrowright \omega^{B_1 \times B_2}$

$$H_{G_i} := \mathbb{C}(B_i \backslash G_i / B_i) \simeq H_q(W_{n_i})$$

$$n_i = \operatorname{rk} G_i = \frac{\dim V_i}{2}$$

$$W_n = (\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n$$

$W_{n_1} \times W_{n_2}$ "$\curvearrowright$" $\omega^{B_1 \times B_2}$

Answer
$$\omega^{B_1 \times B_2} \simeq \bigoplus_{\ell} \operatorname{Ind}_{W_{n_1-\ell} \times \Delta W_\ell \times W_{n_2-\ell}}^{W_{n_1} \times W_{n_2}} \left(\begin{matrix}\text{1-dim'l} \\ \text{char.}\end{matrix}\right)$$

Reprove

answer in terms of Springer corresp.

Springer corresp. G

$$\operatorname{Irr}(W) \longleftrightarrow \{(\mathcal{O}, \rho) : \mathcal{O} \text{ nilp. orbit of } \mathfrak{g},\ \rho \in \operatorname{Irr}(A_{\mathcal{O}})\}$$

$$A_{\mathcal{O}} = \pi_0(C_G(e)),\ e \in \mathcal{O}.$$

Ortho-symplectic Steinberg var.

(Braverman - Finkelberg - Travkin)

$$V = V_1 \otimes V_2 = \operatorname{Hom}(V_1, V_2)$$

$$V_1 \underset{a^*}{\overset{a}{\rightleftarrows}} V_2$$

a is nilp. if aa^* is nilpotent.

$G_1 \times G_2 \curvearrowright \mathcal{N}_V$ nilp. cone in $V = V_1 \otimes V_2$

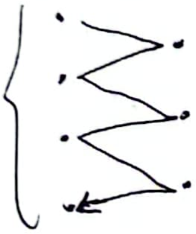
finitely many orbits

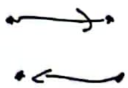
$$\widetilde{\mathcal{N}}_V = \{(a, F_1, F_2) : aa^* \text{ preserves } F_2,\ a^*a \text{ preserves } F_1\}$$

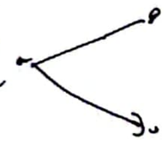
F_1, F_2: full isot. flags in V_1, V_2

$$\begin{tikzcd} \widetilde{\mathcal{N}}_V \arrow[d, "\pi"] & (a, F_1, F_2) \arrow[d, mapsto] \\ \mathcal{N}_V & a \end{tikzcd}$$

Relevant nilp. orbits in $\mathcal{N}_V$

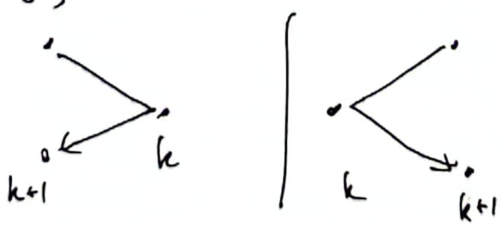
even 

 come in pairs

odd  should appear with even mult.

$\mathcal{O} \subset \mathcal{N}_V$ is <u>relevant</u> if $\pi^{-1}(\mathcal{O})$ has top. dim. in $\widetilde{\mathcal{N}}_V$.

Concretely,



don't both appear.

$$\begin{tikzcd} & \mathcal{N}_V \arrow[dl] \arrow[dr] & & & a \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mathcal{N}_{\mathfrak{g}_1} & & \mathcal{N}_{\mathfrak{g}_2} & a^*a & & aa^* \\ \mathcal{O}_1 \arrow[u, phantom, "\cup"] & & \mathcal{O}_2 \arrow[u, phantom, "\cup"] & & & \end{tikzcd}$$

$(\mathcal{O}_1, \mathcal{O}_2)$, there's at most one relevant $\mathcal{O} \subset \mathcal{N}_V$ mapping to them.

$$\left\{\begin{array}{l}\text{Relevant nilp}\\ \text{orb in } \mathbb{W}_V\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{"relevant"}\\ \text{pairs}\\ (\mathcal{O}_1, \mathcal{O}_2)\end{array}\right\},$$

$$\begin{array}{ccc} 3 & | & 4 \\ 3 & | & 4 \\ 2 & | & 1 \\ 1 & | & 2 \\ 1 & | & 1 \end{array}$$

$$\begin{tikzcd} & A_{\mathcal{O}} \arrow[dl] \arrow[dr] & \\ A_{\mathcal{O}_1} & & A_{\mathcal{O}_2} \end{tikzcd}$$

gaps ≤ 1

<u>Thm.</u> (M-Q-Y-Z)

$\omega^{B_1 \times B_2}$ as $W_{n_1} \times W_{n_2}$-mod is

$$\bigoplus_{\substack{\text{relev. nilp}\\ \text{orbit } \mathcal{O}\\ \swarrow\ \searrow\\ \mathcal{O}_1\quad \mathcal{O}_2}} \bigoplus_{\substack{\rho_1 \in \operatorname{Irr} A_{\mathcal{O}_1} \text{ app. in Spr. corr.}\\ \rho_2 \in \operatorname{Irr} A_{\mathcal{O}_2} \text{ app. in Spr. corr.}\\ \rho_1|_{A_{\mathcal{O}}} = \rho_2|_{A_{\mathcal{O}}}}} E_{(\mathcal{O}_1, \rho_1)} \boxtimes E_{(\mathcal{O}_2, \rho_2)}$$

<u>Pf idea.</u>

$$\omega^{B_1 \times B_2} \underset{\text{as } W_{n_1} \times W_{n_2}\text{-mod}}{\cong} H_{top}(\tilde{\mathbb{W}}_V)$$

Schrödinger model

$L \subset V$

$\omega \simeq \mathbb{C}(V/L)$

$V = \operatorname{Hom}(V_1, V_2)$

$L_1 \subset V_1$ Lag.

$\omega \simeq \mathbb{C}(\underbrace{\operatorname{Hom}(L_1, V_2)}_{\mathbb{L}_1})$

$$\omega^{B_1 \times B_2} \simeq \mathbb{C}(\mathcal{N}_{\mathbb{L}_1})^{\bar{B}_1 \times B_2}$$

$$\mathcal{N}_{\mathbb{L}_1} = \{a : L_1 \to V_2 : a^*(\text{quad. form}) = 0\}$$

$$B_1 \to \bar{B}_1 \subset GL(L_2)$$

Hecke cat. for $G_1 \times G_2$

$$\curvearrowright D_{\bar{B}_1 \times B_2}(\mathcal{N}_{\mathbb{L}_1}) \xrightarrow{cc} \widetilde{\mathcal{N}_V} / G \times G_2.$$

"character sheaves" for theta corresp.

$G \curvearrowright X_1, X_2$

$\mathbb{C}(X_1) \to \mathbb{C}(X_2)$
G-equiv.

$\mathbb{C}(X_1 \times X_2)^G$

categorify $\in \mathrm{End}_{G_1 \times G_2}(\omega)$

projectors

$\mathrm{End}_{\mathbb{C}}(\omega) \simeq \mathbb{C}(V)$

$\in \mathbb{C}(V)^{G_1 \times G_2}$

these sh. $\rightsquigarrow$ almost projectors
$\Updownarrow$
projectors

construction of a class of simple perverse sheaves $\in \mathrm{Perv}_{G_1 \times G_2}(V)$

Lusztig: char. shv. $\rightsquigarrow$ almost char.
$\Updownarrow$ NFT
char.

Ex V_1 arbitrary sympl.
$\dim V_2 = 2$
$V_2 = \langle e, f \rangle$
$Sp(V_1) \times SO(V_2)$ - equiv. perv. sh. on
$V = V_1 \otimes V_2 \simeq V_1 \oplus V_1$

6 Simple perv. sh.

$$V_1 \times V_1 \xrightarrow{\langle \cdot , \cdot \rangle} \mathbb{A}^1$$

const $(0 \times V_1)$
const $(V_1 \times 0)$

$\langle \cdot , \cdot \rangle^* \mathcal{L}_\psi$

$\langle \cdot , \cdot \rangle^* \mathcal{L}_{\overline{\psi}}$

IC_D $\qquad D = \{(x,y) : x \parallel y\}$

IC_Q $\qquad Q = \{(x,y) : \langle x,y \rangle = 0\}$