Tolya Kirillov and fermionic formulas

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$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n}}\ , \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} \qquad (q)_n = (1-q)\cdots(1-q^n)$$

1) $q \sim$ modular form τ, $q = e^{2\pi i \tau}$

2) ch Vir : $2/5$

3) $a(z) = \sum a_i z^{-i}$

a_i commutes with each other

$a(z)^2 = 0 \Rightarrow \sum_{\alpha+\beta=j} a_\alpha a_\beta = 0$

Vacuum vector v, killed by $a_0, a_1, a_2, \cdots$

$$W = \mathbb{C}[a_{-1}, a_{-2}\cdots] \Big/ a_{-1}^2, \ldots$$

W q-grading : $\deg_q a_i = -i$

z-grading: $\deg_z a_i = 1$

$$\operatorname{ch} W = \sum \frac{q^{n^2} z^n}{(q)_n}$$

<u>Algebra</u>; $a_1(x)$, $a_2(x)$

$a_i(x)$ gen. algebra

$a(x)$, $a^2(x) = 0$.

R rep. $f(x_1, x_2) = \langle v^\vee | a(x_1) a(x_2) | v \rangle$

1) Symm. $f(x_1, x_2) = x \cdot (x_1 - x_2)^2$

0 1

$a_1(x),\ a_2(x)$ $\qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

R rep. $\qquad f(x_1, x_2) = \langle v^\vee | a_i(x_1) a_j(x_2) | v \rangle$

$$= (x_1 - x_2)^{a_{ij}}$$

$\to a_{ij}(x)$: quad. relations

R rep, v. $\qquad a_i[m] \cdot v = 0, \quad m \geq 0$

$a_i[-1], [-2], \ldots$

q-grading, z_i-grading as before.

character $\quad f(q, z_1, \cdots, z_n)$

a, (2) : $\displaystyle\sum \frac{q^n z^n}{(q)_n}$

(1) : $\displaystyle\sum \frac{q^{\frac{n^2-n}{2}} z^n}{(q)_n}$

$a_1, a_2 \qquad \displaystyle\sum_{n_1, n_2 \geq 0} \frac{q^{\boxed{}} z_1^{n_1} z_2^{n_2}}{(q)_{n_1} (q)_{n_2}}$ $\qquad \gamma = (n_1, n_1)$

$\langle \gamma. \gamma \rangle / 2$

$\mathbb{C}[a_{-1}, a_{-2}, \cdots] / (a_{-N}, a_{-N-1}, \cdots)$

$\displaystyle\frac{q^{\langle\ \rangle}}{(q)(q)} \longrightarrow$

$\to \mathfrak{g}: \quad V_1 \otimes \cdots \otimes V_n$

$V_i \sim q_i \qquad v_i : \quad \mathrm{Tr}\, H$

Kirillov – Reshetikhin

$[V_1 \otimes \cdots \otimes V_n] = \bigoplus \quad \square \quad W_\alpha.$

$\uparrow$ irred.

Ex. $U_q(sl_2)$ $\quad , q^N = 1 \,, \; N = 5$

fusion product $\quad \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N} = \bigoplus \cdots$

$\dim_q$

$N \to \infty \qquad \mathrm{ch} \qquad \mathrm{Vir}\ 2/5$

$\to$ 1) Rep Vir R, ch

$$\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}. \qquad N \to \infty.$$

2) link, M^3, $\to f_g(q; a, t)$

$\rightsquigarrow$ (a) $\underline{\text{fer}}$.

moduli space. $\quad T\Omega \,, \quad \vartheta \,, \quad \underline{\chi(\cdots)}$

$\to$ ferm. formula.

$\mathfrak{gl}_n$: Zastava, transversal slice in affine grassmannian

$\{\mathbb{C}P^1 \to \underset{F}{\text{flag mfd}}\} \quad H^2(F).$

$$\rightsquigarrow \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

C

Cartan matrix

G_∞

$$\begin{pmatrix} 2 & 2 & \cdots & 2 \\ 2 & 4 & \cdots & \\ 2 & 6 & 6 & \cdots \\ \vdots & & \ddots & \\ & & & \ddots \end{pmatrix}$$

$G_\infty \otimes C$

$\rightsquigarrow \sum$ ________

(Γ) $(n_1, \dots, n_m)$ $m \to \infty$

$\uparrow$

huge lattice $\mathbb{C}P^1 \to F$

1) Vir $\mathfrak{gl}_n$

2) link

3) moduli space

4) $U_q(\mathfrak{gl}_n)$ $W_1 \otimes W_2 \otimes \cdots \otimes W_n$

$U_q(\mathfrak{gl}_n)$

Whittaker W

$(W \quad W^\vee)$