Arithmetic of the Knizhnik-Zamolodchikov equation

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$\Phi = (x-z_1)(x-z_2)\cdots(x-z_n)$ $\qquad z \in \mathbb{A}^n$, $z_i \neq z_j$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Spec} \mathbb{Z}[z_i, (z_i - z_j)^{-1}]$

$\mathbb{P}^1_Z = \mathbb{P}^1 \times Z$ $\qquad \mathbb{P}^1_Z \times \operatorname{Spec}\mathbb{Z}[h, h^{-1}] =: \mathbb{P}^1_W$, $W = Z \times \operatorname{Spec}\mathbb{Z}[h, h^{-1}]$

$$\underset{\substack{\uparrow \\ \text{local system on } \mathbb{P}^1_W}}{\Phi^h} = \Big(\mathcal{O}_{\mathbb{P}^1_W} \cdot \underset{\substack{\uparrow \\ \text{symbol.}}}{\Phi^h},\ \overset{\text{connection}}{\nabla}(f\Phi^h) = \big(df + h\tfrac{d\Phi}{\Phi}\big)\Phi^h\Big)$$

$V_i \subset \mathbb{P}^1_Z \supset V_\infty$, $x = \infty$

$x = z_i$

$T_i \subset \mathbb{P}^1_W \supset T_\infty$ $\qquad T = \bigcup T_i \cup T_\infty$

$\nabla : \mathcal{O}_{\mathbb{P}^1_W}\Phi^h \longrightarrow \Omega_{\mathbb{P}^1_W}(\log T)\Phi^h$ $\qquad \nabla^E$ - Gauß-Manin connection

$$\begin{array}{l} \mathbb{P}^1_W \\ \downarrow \\ W \end{array} \qquad E_h = E = H^1_{dR,\log}(\mathbb{P}^1_W/W, \Phi^h)$$

$$= H^1\Big(\Gamma(W, \mathcal{O}_W) \longrightarrow \bigoplus_{i=1}^n \Gamma(W, \mathcal{O}_W)\eta_i\Big) \qquad \eta_i := \frac{d(x-z_i)}{x-z_i}$$

$$f \longmapsto hf\sum \eta_i$$

<u>Explicit formula</u>. for ∇^E.

$$\nabla^E_{\frac{\partial}{\partial z_j}}([\eta_i]) = \operatorname{Lie}_{\frac{\partial}{\partial z_j}}(\eta_i \Phi^h)\Big/\Phi^h = -h\frac{d(x-z_i)}{(x-z_i)(x-z_j)}$$

$$\underset{i \neq j}{=} h\frac{[\eta_j] - [\eta_i]}{z_i - z_j}$$

$$H^1_{dR,\log}(\mathbb{P}^1_W/W, \phi^h) \longrightarrow H^1_{dR}(\mathbb{P}^1_W \setminus T/W; \phi^h)$$

is isom. for generic h.

$\Omega_{ij} \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$, $\Omega_{ij} = (c_{k,l})$, $c_{ii} = c_{jj} = -c_{ij} = -c_{ji} = 1$

all other entries 0

$$A = \sum_{1\le i\ne j\le n} \frac{\Omega_{ij}}{z_i - z_j}\, dz_i \in \mathrm{Mat}_n(\Gamma(Z, \Omega^1_Z))$$

$dA = 0$, $A \wedge A = 0$

$1 \mapsto \sum e_i$

$$\begin{tikzcd}
\Gamma(W,\mathcal{O}_W) \arrow[r] \arrow[dr, bend right, "n"'] & \left(\bigoplus_{i=1}^{n} \Gamma(W,\mathcal{O}_W)e_i,\ d+hA\right) \arrow[r] \arrow[d, "\Sigma"] & E \arrow[r] & 0 \\
& \Gamma(W,\mathcal{O}_W) & &
\end{tikzcd}$$

Cor. $E_h^*[\frac{1}{n}] \simeq E_{-h}[\frac{1}{n}]$

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p prime number, $p \nmid n$. $\mathbb{Z}[h,h^{-1}] \longrightarrow \overline{\mathbb{F}}_p$, $h \mapsto \bar{h}$

$$E_{\bar h} = E_h \otimes_{\mathbb{Z}[h,h^{-1}]} \overline{\mathbb{F}}_p$$

Thm. (a) $E_{\bar h}^{\nabla=0} = 0$ unless $\bar h \in \mathbb{F}_p$

(b) If $\bar h \in \mathbb{F}_p$, $1 \le \tilde h \le p-1$, $\tilde h \equiv \bar h \bmod p$,

$(E^*_{\bar h})^{\nabla=0}$ is a free module over $\Gamma(Z, \mathcal{O}_Z \otimes \overline{\mathbb{F}}_p)^p$

of rank $[\frac{n\tilde h}{p}]$.

$$E_{\tilde{h}} \longrightarrow \Gamma(Z, \mathcal{O}_Z \otimes \mathbb{F}_p)$$

Ψ

$[\eta_i]$

$$\eta_i \Phi^{\tilde{h}} = \frac{dx}{x - z_i} \Phi^{\tilde{h}} = \sum_{d=0}^{n\tilde{h}-1} c_{d,i}(z_1, \dots, z_n) x^d \, dx$$

$$I^{(\ell)}([\eta_i]) = c_{\ell p - 1, i}(z)$$

$$I^{(\ell)}(\Sigma \eta_i) = \text{coeff at } x^{\ell p - 1} \text{ of } (\phi^{\tilde{h}})'_x = 0$$

$$\int \phi^h \cdot f \, dx$$

$$\phi^h = \exp(h \log \phi).$$

$$E_h / \Gamma(Z, \mathcal{O}_Z) \otimes \mathbb{Z}[h, h^{-1}] \qquad h^{-1} \nabla = h^{-1} d + A$$

$$h^{-1} \frac{\partial}{\partial z_i}$$

$$\Phi^h = \mathcal{O}_{\mathbb{P}^1_Z \times \operatorname{Spec} \mathbb{Z}[h^{-1}]} \qquad h^{-1} \nabla = h^{-1} d + \frac{d\phi}{\phi}$$

$$E_h = H^1_{h^{-1}\text{-}dR, \log}(\mathbb{P}^1_Z \times \operatorname{Spec} \mathbb{Z}[h^{-1}] / Z \times \operatorname{Spec} \mathbb{Z}[h^{-1}], \Phi^h).$$

Crit ϕ — finite, étale

$$\begin{array}{ccccc} \text{Crit}_\phi & \hookrightarrow & \mathbb{A}^1_Z & \subset & \mathbb{P}^1_Z \\ \downarrow & & \downarrow & & \\ Z^\circ & \subset & Z & & \end{array} \qquad \phi'_x = 0 \quad (= n x^{n-1} + \dots)$$

Hessian^{-1}:

$$(\Omega^1_{\mathbb{P}^1_Z/Z})^{\otimes 2} \simeq \mathcal{O}_{\text{Crit}_\phi}$$

$$dx \otimes dx \mapsto \frac{2}{\phi''_{xx}}$$

$$M \ni a, \quad M \xrightarrow{\phi} \mathbb{R}$$

$$T_{M,a} \otimes T_{M,a} \longrightarrow \mathbb{R}$$

a connection on $\Omega^1_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}$

$$\begin{array}{c} \mathrm{Crit}\,\overset{\circ}{\phi} \times \operatorname{Spec} \mathbb{Z}[h^{-1}] \\ \big\downarrow {\scriptstyle \nu \times \mathrm{Id}} \\ Z^\circ \times \operatorname{Spec} \mathbb{Z}[h^{-1}]. \end{array}$$

$$\nabla(dx) = -\tfrac{1}{2}\, d \log \phi''_{xx} \otimes dx$$

$$\mathcal{L} = \phi^h|_{\mathrm{Crit}\,\overset{\circ}{\phi}} \otimes \Omega^1_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}}|_{\mathrm{Crit}\,\overset{\circ}{\phi}}$$

Thm. $\bar{E}|_{Z^\circ \times \mathrm{Spf}\, \mathbb{Z}[[h^{-1}]]} \simeq \mathcal{L}|_{Z^\circ \times \mathrm{Spf}\, \mathbb{Z}[[h^{-1}]]}$.

$p \nmid 2n$ $\quad \bar{E}|_{Z^\circ \times \operatorname{Spec} \mathbb{Z}/p^m\mathbb{Z}[h^{-1}, ((h^{-1})^{p-1}-1)^{-1}]} \simeq \mathcal{L}|$

$\varphi = 1 + x^2$

$$\begin{array}{ccccc} & & R & & \\ R[[x, h^{-1}]] & \xrightarrow{h^{-1}d + \frac{d\phi}{\phi}} & R[[x, h^{-1}]]\, dx & & 1\,dx \\ \uparrow & & A \downarrow \uparrow & & \uparrow \\ 0 & \longrightarrow & R[[h^{-1}]] & & 1 \end{array}$$

$\frac{d\phi}{\phi} = \frac{2x}{x^2+1}$

$A(1\,dx) = 1, \qquad A\left(h^{-1}f' + \frac{2x}{x^2+1} f\right) = 0$