How to enhance (triangulated) categories, and why

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Commutative square of categories

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right, "(*)"'] & & & \\
& \mathcal{C}_{0,1} \arrow[r, "\gamma^0_{0,1}"] \arrow[d, "\gamma^1_{0,1}"'] & \mathcal{C}_0 \arrow[d, "\gamma_0"] \arrow[ddr, bend left] & \\
& \mathcal{C}_1 \arrow[r, "\gamma_1"'] \arrow[drr, bend right] & \mathcal{C} \arrow[dr, dotted] & \\
& & & \mathcal{C}'
\end{tikzcd}$$

$$\gamma_1 \circ \gamma^1_{01} \xrightarrow{\sim} \gamma_0 \circ \gamma^0_{0,1}$$

$$\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1 \qquad \langle c_0, c_1, \alpha \rangle$$

$$c_0 \in \mathcal{C}_0,\ c_1 \in \mathcal{C}_1, \quad \alpha : \gamma_0(c_0) \xrightarrow{\sim} \gamma_1(c_1)$$

Cocartesian square. $[1]$ simple arrow category

$$0 \longrightarrow 1$$

$$\begin{tikzcd}
W \times [1] \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
W \arrow[r] & h^W(\mathcal{C})
\end{tikzcd}$$

W class (or set)

(localization)

<u>Example</u>: $\mathcal{C}at$: the category of small categories

W equivalences

Then $h^W(\mathcal{C}at) =: \mathcal{C}at^\circ$ exists. small cat's, iso. classes of functors.

$\mathcal{C}at$ has cartesian and cocartesian square, but they are not $(*)$

In Cat°, $(*)$ is commutative. but Cartesian square has no univ. property.

Considered simply as a category, Cat° is bad.

In general, a category obtained by localization need to be enhanced.

At the very least, it has objects, and

$$\mathcal{H}^w(\mathcal{C})(c,c') \quad \text{, a homotopy type of morphisms for any } c,c' \in \mathcal{C},$$

$$h^w(\mathcal{C})(c,c') = \pi_0\, \mathcal{H}^w(\mathcal{C})(c,c')$$

E.g. a topological cat. $\in$ Top. Cat

The idea goes back to Grothendieck: ("derivator")

For any $\mathcal{C} \in \mathrm{Top\,Cat}$, we have $\pi_0(\mathcal{C})$, but also, for any small I, we have

$I^\circ\mathcal{C}$ – the category of functors from $I^{op} \to \mathcal{C}$.

$\rightsquigarrow \pi_0(I^\circ\mathcal{C})$

Q: Is this enough? A. Yes (with some modifications).

What is a "collection of categories indexed by some $\mathcal{E}$"?
↑ Some cat.

$e \in \mathcal{E}$, $\mathcal{C}_e$

$f: e \to e'$, $f^*: \mathcal{C}_{e'} \to \mathcal{C}_e$

f, g composable, $(f\circ g)^* \simeq g^* f^*$ subject to conditions.

$$\left(\begin{array}{l} \mathcal{C}_e \xrightarrow{r_e} \mathcal{C}'_e \\ f^*\circ r_e \simeq r_{e'}\circ f^* \\ \vdots \end{array} \right)$$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\ \mathcal{E} \arrow[r, "\mathrm{Id}"] & \mathcal{E} \end{tikzcd}$$

+ a condition (Cartesian functors over $\mathcal{E}$)

Grothendieck construction (SGA 1)

Construct $\mathcal{C}$ of $(e,\ c \in \mathcal{C}_e)$

$\mathcal{C} \longrightarrow \mathcal{E}$ (Grothendieck fibration)

Let Pos be the category of partially ordered sets.

$X \in$ Top Cat, define $k(X)_J = \pi_0 (J^* X)$

Theorem. $X, X' \in$ fibered Top Cat, then any cartesian functor

$\gamma: k(X) \to k(X')$ over Pos

comes from some $f: X \to X'$, and $\gamma \simeq \gamma'$ iff f is homotopic to f'.

Informally, $h^W(\text{Top. Cat}) \longrightarrow (\text{Cat} /^{\#} \text{Pos})^{\circ}$

↑

this is fully faithful embedding

Moreover, can describe the essential image

("an enhanced category": $\mathcal{C} \to \text{Pos}$, 6 axioms or so)

$\mathcal{C}_{pt} \to pt$

We have Cat^h, the cat. of small enhanced categories

It has a natural enhancement. $\mathcal{C}at^h_J = \mathcal{C} \to \text{Pos}$ enhanced

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ k(J) \end{tikzcd}$$

Grothendieck fibration

Prop. For any small enhanced category $\mathcal{C}$, we have $J \in \text{Pos}$, $W \times [1] \to J$

s.t.

$$\begin{tikzcd} k(W \times [1]) \arrow[r] \arrow[d] & k[J] \arrow[d] \\ k[W] \arrow[r] & \mathcal{C} \end{tikzcd}$$

Cor: Cat^h is cartesian closed.

i.e. $\text{Fun}^h(\mathcal{C}, \mathcal{C}')$ with the usual univ. prop.

Cor 2. Assume given a semicartesian square

$$\begin{tikzcd} \mathcal{C}_{01} \arrow[r, "\gamma_{01}^{\circ}"] \arrow[d, "\gamma_{01}'"'] & \mathcal{C}_0 \arrow[d, "\gamma_0"] \\ \mathcal{C}_1 \arrow[r, "\gamma_1"] & \mathcal{C} \end{tikzcd}$$

of enhanced cat. and functors

$\mathcal{C}_{01} \longrightarrow \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1$ is epivalence (i.e. full, ess. surj., conservative).

then for any

$$\begin{tikzcd}
e'_{01} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& e_{01} \arrow[r] \arrow[d] & e_0 \arrow[d] \\
& e_1 \arrow[r] & e
\end{tikzcd}$$

there exists $\dashrightarrow$

unique, up to a non-unique iso.

<u>Lemma</u>. We always have

$$\begin{tikzcd}
e_1 \times^h_e e_0 \arrow[r] \arrow[d] & e_0 \arrow[d] \\
e_1 \arrow[r] & e
\end{tikzcd}$$

Some things are literally inherited from the usual category theory.

1. fully faithful embedding.

$$\begin{tikzcd}
e \arrow[r] \arrow[d] & e' \arrow[d] \\
Pos \arrow[r, "id"] & Pos
\end{tikzcd}$$

– fully faithful in the usual sense.

2. Adjoint pair of functors

$$\begin{tikzcd}
e \arrow[r, shift left, "r"] & e' \arrow[l, shift left, "r^{\dagger}"]
\end{tikzcd}$$

$r^{\dagger} \circ r \to id$, $id \to r \circ r^{\dagger}$ + condition

In the enhanced setting, the same.

$$\begin{tikzcd}
e \arrow[r, shift left] \arrow[d] & e' \arrow[l, shift left] \arrow[d] \\
Pos \arrow[r, "id"] & Pos
\end{tikzcd}$$