# Generic character sheaves on parabolic subgroups, Charlotte Chan

Goal. Overview of developments towards explicit geometric model of RT

of p-adic gps .
nonarchimedean
local field

$\mathbb{Q}$ $\qquad$ $\mathbb{F}_q(t)$

$\mathbb{Q}_p$ $\qquad$ $\mathbb{F}_q((t))$

$\mathbb{Z}_p$ $\qquad$ $\mathbb{F}_q[[t]]$

Today. "supercuspidal reps"

easy.

Fact. Any irred. reps of a p-adic gp appears in the parabolic induction of a
hard

supercuspidal
medium

Constructing supercuspidal
- Moy - Prasad, Morris . depth 0
- Adler . pos depth
- Yu . pos depth

$SL_2 \mathbb{Q}_p$, $\quad SL_2 \mathbb{F}_p((t))$

① Take a repn $\pi$ of $SL_2 \mathbb{F}_p$

② Have a surj. $SL_2 \mathbb{Z}_p$ ; $SL_2 \mathbb{F}_p[[t]] \twoheadrightarrow SL_2 \mathbb{F}_p$

③ compact induction. $\mathrm{Ind}(\pi)$

It this is irred., then it is supercuspidal - depth zero...

To get pos. depth reps.

boost ② by replacing w/ alg constr. depends on some data about the gp.

e.g. $GL_8$ $\quad (\mathbb{C}^\times \hookrightarrow GL_2 \mathbb{R})$

(ALG) $\quad GL_2(\mathbb{F}_{p^4}((t))) \hookrightarrow GL_4(\mathbb{F}_{p^2}((t))) \hookrightarrow GL_8(\mathbb{F}_p((t)))$

(GEOM) boost ① by replacing $SL_2(\mathbb{F}_p)$ w/ $SL_2(\mathbb{Z}_p/p^{r+1})$ $\sim SL_2(\mathbb{F}_p[[t]]/t^{r+1})$

⌜ Yu's alg. recipe is geometrically simply parabolic induction. ⌟

RT. $\mathrm{pInd}^{GL_2(\mathbb{F}_p)}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}(\theta) = \{ f : GL_2 \mathbb{F}_p \to \mathbb{C} : f(gb) = \theta(pr(b)) f(g) \}$

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \downarrow \begin{pmatrix} * & \\ & * \end{pmatrix}$$

Class function.

$$g \mapsto \sum_{\substack{hB(\mathbb{F}_p) \in GL_2(\mathbb{F}_p)/B(\mathbb{F}_p) \\ \text{s.t. } h^{-1}gh \in B(\mathbb{F}_p)}} \theta(pr(h^{-1}gh))$$

alg. geom.

$$\begin{tikzcd} & \{(g, hB) \in G \times G/B : h^{-1}gh \in B\} \arrow[dl, "f"'] \arrow[dr, "\pi"] & \\ T & & GL_2 \end{tikzcd}$$

$(g, hB) \mapsto pr(h^{-1}gh)$, $\quad (g, hB) \mapsto g$

$\mathcal{L}_\theta$

$$\mathrm{pInd}(\mathcal{L}_\theta) := \pi_! f^* \mathcal{L}_\theta$$

Ex. $GL_2 \mathbb{F}_p \supset \begin{pmatrix} * & \\ & * \end{pmatrix} \simeq \mathbb{F}_p^\times \times \mathbb{F}_p^\times \qquad \# = (p-1)^2$

$\quad \rotatebox{90}{$\cong$}$

$GL_{\mathbb{F}_p}(\mathbb{F}_{p^2}) \supset \mathbb{F}_{p^2}^\times \qquad \# = p^2 - 1$

RT of $GL_2 \mathbb{F}_p$ splits into 2 comps.

- $\mathrm{pInd}^{GL_2(\mathbb{F}_p)}_{B(\mathbb{F}_p)}(\theta)$
- Deligne-Lusztig induction.

$GL_2 \mathbb{F}_p \curvearrowright \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_p^2 : x^p y - x y^p \in \mathbb{F}_p^\times \} \curvearrowleft \mathbb{F}_p^\times$

$$\rightsquigarrow$$

$R_T^G(\theta) = H^1_c(-)[\theta]$

Deligne-Lusztig induction

<u>Conj.</u> (Lusztig) This picture works for jet schemes, $\mathcal{L}_\theta$ sufficiently generic.

<u>Def</u> Let $X$ be a scheme over $\mathbb{F}_p$. Then the $r$th jet scheme $X_r$ is defined by

$$X_r(A) = \begin{cases} X(A[t]/t^{r+1}) \\ X(W_{r+1}(A)). \end{cases}$$

<u>Thm</u> (Bezrukavnikov - C. 2024) ✓.

<u>Difficulties</u>. • $\pi$ no longer proper
• $\pi$ no longer small.

Line of reasoning linking generic char. sh. to L-packets of pos. depth s.c.

$(T,\theta) \rightsquigarrow \mathcal{L}_\theta$ on $T_r$

$T \subset G$ ellipt. max. torus

$\theta$ smooth char depth $r$.

$\mathbb{F}_{p^2}((t))^\times \subset GL_2(\mathbb{F}_p((t)))$

$\mathcal{L}_\theta$ on $T_r \rightsquigarrow p\mathrm{Ind}_{B_r}^{G_r}(\mathcal{L}_\theta) \rightsquigarrow \chi_{p\mathrm{Ind}_{B_r}^{G_r}(\mathcal{L}_\theta)} : G_r(\mathbb{F}_q) \to \mathbb{C}$

irred. perverse
$G_r$-equiv. sheaf on $G_r$

$= \chi_{R_{T_r}^{G_r}(\theta)}$

$\rightsquigarrow \{ c\mathrm{Ind}(R_{T_r}^{G_r}(\theta)) : (T,\theta)$ varies over a stable conj. cl. of such pairs$\}$