Quantizations of nilpotent orbits
& Lagrangian subvarieties

1) Harish-Chandra modules

2) Quantizations of conical sympl. sing.

3) Quantizations of lag. subvar.

1) Setting: $\mathfrak{g}$ ss Lie alg./ $\mathbb{C}$, $\sigma : \mathfrak{g} \to \mathfrak{g}$, $\sigma^2 = 1$. $\mathfrak{k} = \mathfrak{g}^\sigma$

Ex. 1) $\mathfrak{g} = \mathfrak{sl}_n$, $\sigma(x) = -x^t$. $\mathfrak{k} = \mathfrak{so}_n$

2) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$, $\sigma(x,y) = (y,x)$ $\mathfrak{k} = \mathfrak{g}_0$, diag

Def. A HC $(\mathfrak{g},\mathfrak{k})$-module is fin. gen. $U\mathfrak{g}$-module w/ loc. finite action of $\mathfrak{k}$.

Ex. $U\mathfrak{g} / U\mathfrak{g}\cdot\mathfrak{k}$ is HC.

Unitary $G_{\mathbb{R}}$-irreps $\subset$ Irred HC-modules

$G_{\mathbb{R}}$ q split. quantizations of certain lag. subvar.

2) Quantizations of conical sympl. sing.

$\mathcal{A}$ filt assoc. alg. / $\mathbb{C}$

$\bigcup_{i \geq 0} \mathcal{A}_{\leq i}$, assume $\operatorname{gr}\mathcal{A}$ is commutative.

$\{\cdot,\cdot\}$

$\{a + \mathcal{A}_{\leq i-1}, b + \mathcal{A}_{\leq j-1}\} := [a,b] + \mathcal{A}_{\leq i+j-2}$

Say $\mathcal{A}$ is quantization of $\mathrm{gr}\,\mathcal{A}$.

Ex. $U\mathfrak{g}$ is quantization of $S\mathfrak{g}$.

Ex. $\mathbb{O} \subset \mathfrak{g}$ nilpotent orbit $\rightsquigarrow$ $\mathbb{C}[\mathbb{O}]$

graded

Poisson

fin. gen'd $\rightsquigarrow X = \mathrm{Spec}\,\mathbb{C}[\mathbb{O}]$

(affine, singular, Poisson var.)

Classification Thm:

$$\{\text{quant. of } \mathbb{C}[X]\} \xleftrightarrow{\sim} \mathfrak{P}_X \ (\text{fin. dim. vec. space}).$$

$$\mathcal{A}_\lambda \longleftarrow\!\shortmid \lambda \in \mathfrak{P}_X$$

Rmk. $X = \mathrm{Spec}\,\mathbb{C}[\mathbb{O}] \curvearrowleft G$.

$\rightsquigarrow G \curvearrowright \mathcal{A}_\lambda$, $\quad U\mathfrak{g} \xrightarrow[G]{} \mathcal{A}_\lambda$ (often surjective)

Ex. $\mathbb{O}$ - principal nilp. orbit, $\overline{\mathbb{O}} = \mathcal{N}$ - nilp cone
$$\overline{\mathbb{O}} = X$$

Quantizations: $U\mathfrak{g}/U\mathfrak{g}\cdot$(max. ideal in center of $U\mathfrak{g}$)

$\mathfrak{P}_X = \mathfrak{h}^*$ (Cartan)

3) Quantizations of Lagrangian subvarieties.

Setting: $\mathcal{A}$ is filtered alg. as in beginning of 2).

$\theta: \mathcal{A} \to \mathcal{A}$ anti-involution. $(\theta(ab) = \theta(b)\theta(a))$

Ex 1) $(\mathfrak{g},\sigma) \rightsquigarrow \theta = -\sigma$ (on $\mathfrak{g}$) $\rightsquigarrow \theta \curvearrowright \mathcal{A} := \mathcal{U}\mathfrak{g}$.

2) $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0^{opp}$, $\theta(a\otimes b) = b\otimes a$

Def. • <u>Good filtration</u> on $\mathcal{A}$-module $\mathcal{M} = \bigcup_{j\geq 0} \mathcal{M}_{\leq j}$

$$\mathcal{A}_{\leq i}\mathcal{M}_{\leq j} \subset \mathcal{M}_{\leq i+j} \ \& \ \operatorname{gr}\mathcal{M} \text{ is fin. gen. } \operatorname{gr}\mathcal{A}\text{-module}$$

• If $\forall a \in \mathcal{A}_{\leq i}$, $\theta(a) = -a \Rightarrow a\mathcal{M}_{\leq j} \subset \mathcal{M}_{i+j-1}$.

Say filtration is compatible with θ

• $\mathcal{M}$ is HC $(\mathcal{A},\theta)$-module if it admits such filtration.

In Ex 1, good filtration is compatible w/ θ

$\Leftrightarrow$ K-stable

So HC $(\mathcal{U}\mathfrak{g},\theta)$-module is HC $(\mathfrak{g},K)$-module.

$\operatorname{gr}\theta$ is anti-Poisson invol.n of $\operatorname{gr}\mathcal{A}$

if $\operatorname{gr}\mathcal{A} = \mathbb{C}[X]$, X is conic. $\underbrace{\text{sympl. sing.}}$

$\Rightarrow X^{reg}$ is sympl.

$Y = X^{\theta} \Rightarrow Y\cap X^{reg}$ is $\underset{\text{Smooth}}{\wedge}$ Lagrangian.

<u>Observation:</u> If M is HC $(\mathcal{A},\theta)$-module, then

• $\operatorname{gr}M$ is supported on Y.

• $(\operatorname{gr}M)|_{X^{reg}}$ = twisted local system on $Y\cap X^{reg}$.

Q: Can we recover irr. $\mathcal{M}$ from $(\mathrm{gr}\mathcal{M})|_{X^{reg}}$

<u>Assumption.</u> $\mathrm{codim}_Y (Y \setminus X^{reg}) \geq 2$

Essentially Thm (Losev-Yu' 18): if $\mathcal{M}$ is irred.

$\Rightarrow$ so $(\mathrm{gr}\mathcal{M})|_{X^{reg}}$ is

& $(\mathrm{gr}\mathcal{M})|_{X^{reg}}$ recovers $\mathcal{M}$ uniquely.

Existence results (for which twisted local systems, there's quantization).

Special case: $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0^{opp}$, $X = X_0 \times X_0^{opp}$

$Y = X_0$, diag.

I.L. 18: Almost always we can completely describe "quantizable" local systems.

Special case 2 (jt. w. S. Yu, in preparation).

$\mathbb{O} \subset \mathfrak{g}$ nilp. orbits, $\mathrm{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} \geq 4$ $(\Rightarrow \mathrm{codim}_Y Y \setminus X^{reg} \geq 2)$

$\mathcal{A}$ is quantization of $\mathbb{C}[\mathbb{O}]$, θ comes from $-\sigma$ on $\mathfrak{g}$.

$Y \cap \mathbb{O} = \coprod_{\text{finite}} K\text{-orbits}$ $(K = (G^{\sigma})^{\circ})$

<u>Result</u>: Every K-equiv. twisted local system on K-orbit is quantizable.