From affine Hecke category to invariant distributions, Roman Bezrukavnikov

G - p-adic group

$G = \underline{G}(F)$, $\underline{G}$ (split) reductive. $F = \mathbb{F}_q((t))$

G simply connected

Do harmonic analysis using ℓ-adic sheaves.

Unipotent invariant distributions

$S = C_c^\infty(G)$ - algebra under convolution.

An invariant distribution:

$$\mathrm{Dist}^G(G) = (S^*)^G = (S_G)^* = (S/[S,S])^*$$

$S/[S,S]$ - cocenter

I : Iwahori subgroup

$S \supset H = C_c^\infty(I \backslash G / I)$.

1) $\mathrm{Rep}(H) \longleftrightarrow \{$rep-s of G generated by an I-inv. ut vectors$\}$

2) $H/[H,H]$ - direct summand in the cocenter.

Irr. reps as in 1) were classified by KL (1987)

they are in bijection with (s, u, ψ).

$G^\vee \ni s$ semisimple

$G^\vee \ni u$ - unipotent.

$su = us$

ψ - a repr of $\pi_0(Z_{G^\vee}(s,u))$, s.t. ....

This was extended by Lusztig

Irr - unipotent reps $\longleftrightarrow$ (s, u, ψ) (no condition)

The set of unipotent reps is a union of <u>L-packets</u>

Fact. $S/[S,S] = H/[H,H] \oplus$ complement

$\overset{\shortparallel}{C}$

$= C_{unip} \oplus C_{non\text{-}unip}$

orthogonal to the span of unip. characters.

Characters of unip. reps lie in C_{unip}^*

C_{unip} - ?

H admits a cat-n $\mathcal{H} = D(\underline{I}\backslash LG/\underline{I})$, $I = I(\mathbb{F}_q)$

ℓ-adic sheaves category

Technical point. $Fr \curvearrowright \mathcal{H}$

$$H = K_q(\mathcal{H}) := K(\mathcal{H}^{Fr}) \underset{K(\mathbb{Z})}{\otimes} \mathbb{C}$$

$$\mathcal{H} \supset Rep(\mathbb{Z})$$

$$K(Rep(\mathbb{Z})) = \mathbb{Z}(\mathbb{C}^\times) \to \mathbb{C}$$

$$Z \subset \mathfrak{g}^\vee \times G^\vee$$

$$\shortparallel$$

$$\{(e,g) : ge = eg,\ e \in \mathcal{N}\}$$

Thm (Ben-Zvi, Nadler, Preygel) $\mathcal{H} \to D(Coh^{G^\vee}(Z))$ - universal commutator functor

based on coherent realization of $\mathcal{H}$

$$\mathcal{H} \simeq D^b Coh^{G^\vee}(\tilde{\mathcal{N}} \underset{\mathfrak{g}}{\times} \tilde{\mathcal{N}}), \quad \tilde{\mathcal{N}} - \text{Springer resolution.}$$

b, g, x $\qquad \{(b, g, x) : x \in rad(b),\ gx = xg\}$

$(b, g(b), x)$ $\qquad \{b_1, b_2, x\}$ $\qquad g, x$

$x \in rad(b_1) \cap rad(b_2)$ $\qquad gx = xg$

<u>Thm</u> (B, C, K, V) $C_{uni} \simeq K_q(\mathrm{Coh}^{G^\vee}(Z))$
in progress

$$q : (e, g) \longmapsto (qe, g)$$

Rewrite the RHS 1) $K_q(\mathrm{Coh}^{G^\vee}(Z)) \simeq \bigoplus_{N/\sim} K(\mathrm{Coh}^{Z_e}(Z_e))$

Z_e : reductive centralizer

H reductive group 2) $K(\mathrm{Coh}^H(H)) \simeq \mathcal{O}_\ell(\mathrm{Com}(H))$

$$\mathrm{Com}(H) = \{(x, y) \in H : xy = yx\}$$

$$\mathcal{O}_\ell = \{ f \in \mathcal{O}(\mathrm{Com}(H))^H : \forall y,\ x \longmapsto f(x,y) \text{ is locally constant} \}$$

$$[F] \longmapsto f_F(x, y) = \mathrm{Tr}(y, F_x)$$

<u>Rmk</u>. The splitting indexed by $N/\sim$.
on $H/[\,,\,]$ can be obtained in another way.

$$H \xrightarrow{\phi} J \quad \text{(Lusztig)}$$

asymptotic Hecke algebra

$$H/[H, H] \xrightarrow{\sim} J/[J, J].$$

$$J = \bigoplus_{e \in N/\sim} J_e.$$

$$J_e \simeq K(\mathrm{Coh}^{Z_e}((B_e^{G^*})^2))$$

$e \in N$, B_e - preimage of e in $\tilde{N}$.

<u>Conjectures on compatibility</u> with (almost) characters.

Given $(s, u, \psi) \rightsquigarrow$ (standard) repn. $R_{s,u,\psi}$, its character is a linear functional on C_{uni}.

$$C_{uni} \simeq \bigoplus_e \mathcal{O}_\ell(\mathrm{Com}(Z_e)) \ni f = (f_e)$$

Conj. $f \mapsto (fe|_{y=s.}, \psi)$

Thm. $C_{uni} \supset C^c_{uni}$ – image of functions supported on G_c.

$G \supset G_c$ - set of compact elements.

Thm. $C^c_{uni} = \bigoplus_{e \in N/\sim} \Theta_{\ell r}(\mathrm{Coh}(Z_e))$

$\Theta_{\ell r} \supseteq i$

$i(x,y) = (y,x)$

Conj 2 $i(\chi_{R(S,E,\psi)})$ is the functional con-g to unipotent character sheaf on the loop group (Lusztig).

Approach to Conj. 2. A character sheaf on a parahoric in G defines an object in $\mathrm{Coh}^{G^\vee}(Z)$

Prop This object is Hodge graded of a CS on $G^\vee$.

The picture has a conj. gen-n. to all depth 0 repns

In particular, for generic cuspidal depth 0, L-packets follows from Thm B.. Varshavsky.