Algebraic Cycles

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$X/\mathbb{C}$ smooth projective

i $\quad Z_i(X) \twoheadrightarrow CH_i(X)$

cycles mod rational equiv.

$S \quad \mathbb{A}^1$

$$CH_i(X) \to H_{2i}(X;\mathbb{Z})$$

- $\dim X = 1$ $\qquad CH_0(X) \xrightarrow{\deg} \mathbb{Z}$

$$\Omega^1(X) \oplus \bar{\Omega}^1(X) \xrightarrow{\sim} H^1(X;\mathbb{C})$$

$$\begin{tikzcd} H_1(X;\mathbb{C}) \arrow[r] & \Omega^1(X)^* \arrow[r, two heads] & J(X) \\ H_1(X;\mathbb{Z}) \arrow[u, phantom, "\cup"] \arrow[ur, hook] & & \end{tikzcd}$$

the quot. w.r.t. $H_1(X;\mathbb{Z})$

$$\begin{tikzcd} CH(X)^{\deg=0} \arrow[r] & J(X) \\ \sum a_i x_i \arrow[r, leftrightarrow] & \int_\gamma \in \Omega^1(X)^* \end{tikzcd}$$

$\sum a_i x_i = \partial\gamma$, $\gamma :$

$\dim \geq 2 :$ $\quad CH(X)^{\deg=0} \twoheadrightarrow Alb(X)$

Albanese kernel

Thm $\dim X = 2$, $H^0(X, \Omega^2_X) \neq 0$. Then the Alb. kernel is not trivial.

Pf (S. Bloch)

Lemma 1. $\quad$ implies that for every Zar. open $\emptyset \neq U \subset X$, the map

$$\underset{[\Delta]}{\underset{\cup}{H^4(X\times X;\mathbb{Q})}} \longrightarrow H^4(U\times U) \quad \text{sends } [\Delta] \text{ to a nonzero class.}$$

Pf $[\Delta] = \sum \alpha_i \otimes \alpha_i^{\vee}$ $\qquad H^2(X)\otimes H^2(X) \longrightarrow H^2(U)\otimes H^2(U)$

$C \subset X$

$$CH_0(C)^{\deg=0} \longrightarrow CH_0(X)^{\deg=0} \longrightarrow \mathrm{Alb}(X)$$

(Reformulation of Mumford's thm)

For any curve $C \subset X$, the map $CH_0(C) \to CH_0(X)$ is

NOT surjective.

Lemma 2 $C \subset X$ is any curve s.t. $CH_0(C) \to CH_0(X)$ is surjective,

then for a suff. small Zar. open $U \subset X$, the image of $[\Delta]$ in

$H^4((X\setminus C)\times U)$ is 0.

Pf. "Spreading".

Is there a "linear structure" on the coh. of an alg. var that determines

the $CH_*(X)$?

? Test question. Suppose a finite group G acts on X. Suppose that an irred. reps

of G does not occur in $H_*(X;\mathbb{Q})$; Then it does not occur in

$CH_*(X)\otimes\mathbb{Q}$.

A dreamy picture what "linear structure" means:

We look for a ~~topological~~ space M s.t. to every alg. var. X, there corresponds

a fibration $\begin{array}{c}\mathfrak{X}\\ \downarrow\\ M\end{array}$

We also want a base point $\mu \in M$ and an identification

$\mathfrak{X}_\mu \xrightarrow{\sim} X$ top.

Conj:

X_K $\quad (CH_0(X_K))^{\deg 0} \xrightarrow{\sim} Alb(X)(K)$

$\}$ number field

Ex X is an abel. ~~var.~~ surface $/K$ number field

$a, b \in X$ $\quad [a+b]-[a]-[b]+[0]$ is rat. equiv. to 0.