Geometric Satake for affine Lie algebras

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§1. Braverman-Finkelberg conjecture.

$Q = (Q_0, Q_1)$ valued quiver without edge loops.

$\rightsquigarrow$ $\mathfrak{g}^\vee$: symmetrizable Kac-Moody Lie algebra.

$\mathfrak{g}$ = Langlands dual.

V, W : Q_0 - graded vector spaces

$\lambda = \sum \dim W_i \cdot \Lambda_i$ dominant weight , $\mu = \lambda - \sum \dim V_i \cdot \alpha_i$ another weight.

[BFN, 2016] $\mathcal{M}_C \equiv \mathcal{M}_C(\lambda, \mu)$: Coulomb branch

$G := \prod GL(V_i)$

$\mathbb{N}$ = rep of G given by V, W.

Gr_G : affine Grassmannian $= G(\mathcal{K})/G(\mathcal{O})$. $\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$

$G(\mathcal{O}) \curvearrowright \mathcal{R}$ = variety of triples $= \{ (g(z), s(z)) \in G(\mathcal{K}) \times^{G(\mathcal{O})} \mathbb{N}(\mathcal{O}) : g(z)s(z) \in \mathbb{N}(\mathcal{O}) \}$

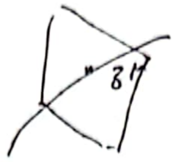
$H_*^{G(\mathcal{O})}(\mathcal{R})$ + convolution product $\rightsquigarrow$ commutative algebra

$$\mathcal{M}_C := \operatorname{Spec} H_*^{G(\mathcal{O})}(\mathcal{R})$$

Fact [BFN]

Q : finite type and μ dominant

$\mathcal{M}_C(\lambda, \mu)$ = slice in the affine Grassmannian for $G^\vee$. (adjoint)

$\overline{Gr_G^\lambda} \supset Gr_G^\mu \quad \to$ 

appears in the usual geometric Satake

$\mathcal{M}_C(\lambda,\mu)$ = moduli space of singular monopoles on $\mathbb{R}^3$.

Q: affine case

<u>expected</u> $\mathcal{M}_C(\lambda,\mu)$ = moduli space of instantons on $\mathbb{R}^4/\mathbb{Z}/\ell$,

(proved for type A [N.-Takayama]) $\qquad \ell$ = level of λ.

[BF] proposed geometric Satake for affine Lie algebra using $\mathcal{M}_C(\lambda,\mu)$.

$a_\mu : \{z^\mu\} \hookrightarrow \mathcal{M}_C(\lambda,\mu)$

unique $T^\vee$-fixed point in $\mathcal{M}_C(\lambda,\mu)$. (if exists)

$\pi_1(G)^\wedge$ = Pontryagin dual of $\pi_1(G)$

IC^λ_μ = intersection cohomology complex of $\mathcal{M}_C(\lambda,\mu)$

$= IC(\mathcal{M}_C(\lambda,\mu))$

<u>Conj</u> (1) $H^*(a_\mu^! IC^\lambda_\mu)$ (costalk) vanish in odd degrees.

(2) Euler characteristic = weight multiplicity $= \dim V_\mu(\lambda)$

$V(\lambda)$: integrable h.w. rep. of $\mathfrak{g}$

(3) tensor product

$\leftarrow$ convolution product.

<u>Rmk</u>. affine type A. $\mathcal{M}_C(\lambda,\mu) \xleftarrow{\exists} \mathcal{M}_C^X(\lambda,\mu)$ sympl. resolution

One can also consider $H_*(\underbrace{\pi^{-1}(z^\mu)})$

Lagrangian subvar.

Page?

<u>Thm</u> [N. 2008] True for affine type A.

- $\mathfrak{M}_C$ = a quiver variety of affine type A
- Use I. Frenkel's level-rank duality

One remaining statement of [BF]

Poincaré polynomial

$$P^\lambda_\mu(q) = \sum q^{\frac{i}{2}} \dim H^i(\mathcal{U}^!_\mu; IC^\lambda_\mu)$$

For Q: finite type, Lusztig: $P^\lambda_\mu(q) = C^\lambda_\mu(q)$

combinatorial polynomial
given by Kostant partition function.
Hall-Littlewood poly.

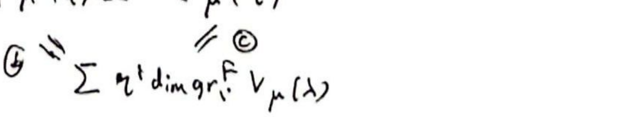
Moreover, $C^\lambda_\mu(q) = \sum q^i \dim gr_i^F V_\mu(\lambda)$ (Brylinski, Brown, Ginzburg)

F = Brylinski-Kostant filtration.

$$F^i V(\lambda) = \{ v \in V(\lambda) \,;\, e^{i+1} v = 0 \},\quad e = \text{principal nilpotent}.$$

e.g.

stict transf. of axis.



$\mathbb{C}^2/\mathbb{Z}/2 \longleftarrow T^*\mathbb{P}^1$

$\pi^{-1}(0_M) \simeq \mathbb{P}^1$.

<u>Conj.</u> (4) $P^\lambda_\mu(q) \overset{ⓐ}{=} C^\lambda_\mu(q)$

ⓑ $\Vert$ $\quad$ $\Vert$ ⓒ

$\sum q^i \dim gr_i^F V_\mu(\lambda)$

Slofstra 2010

ⓒ is not correct. but true if we replace F by $\mathring{F}$

$\mathring{F}_i V(\lambda) = \{ v \in V(\lambda) : x^{i+1} v = 0 \}$

$\forall x \in$ principal heisenberg $\cap\, \mathfrak{n}$.

(affine Lie algebra case)

Thm [Muthiah - N.]

(B) with F replaced by $\mathring{F}$ is true for affine type A

(We do not compute $P_\mu^\lambda(q)$)

§2. Arkhipov - Bezrukavnikov - Ginzburg $\quad Gr_{G^\vee}$ v.s. $T^*(G/B)$

(C) ← proof used $T^*(G/B)$ + fixed pt. formula.

Slodowy : Similar $T^*(G/B)$ for affine Lie group (secretly)

but algebraic approach. for rigorous treatment.

Thm [ABG]. G finite, μ dominant,

$$H^0(T^*(G/B), \mathcal{O}(\mu)) \simeq \bigoplus_{\lambda \text{ dominant}} V(\lambda)^* \otimes H^*(a_\mu^! IC_\mu^\lambda)$$

G-equivariant graded isomorphism

$\mathbb{C}^*$-action ⟷ cohomological grading.

Moreover, it is compatible with "product".

$$\mathcal{O}(\mu_1) \otimes \mathcal{O}(\mu_2) \simeq \mathcal{O}(\mu_1 + \mu_2) \quad \longleftrightarrow \quad \text{convolution.}$$

One can regard this statement as construction of T^*G/B from the topology of affine Grassmannian slices.

← similar to the construction of Coulomb branch.

Cf. [BFN 2017. Ring object ...]

$\llcorner D_{G(\mathcal{O})}(Gr_G)$

$\mathbb{C}[G^\vee]$ ring object in $Rep(G^\vee)$

$\updownarrow$ geometric Satake

$A_R = \bigoplus V(\lambda)^* \otimes IC(Gr^\lambda)$ ring object in $Perv_{G(\mathcal{O})}(Gr_G)$

i.e. RHS of [ABG] $\qquad H^*(\mathcal{A}_R)$

4. Gaiotto–Witten

$$T^*(G/B) = \text{Coulomb branch of } 3d\ \mathcal{N}=4 \text{ theory } T(G^\vee)$$

(resolved by flavor symmetry)

$$\text{type A} \qquad T(G^\vee) = \begin{array}{c} (n-1) - (n-2) - \cdots (1) \\ | \\ \boxed{n} \end{array}$$

[MN]: will be [ABG] for affine loop group.

$$T^*(G/B) \overset{?}{=} \text{Coulomb branch of } T(G^\vee) \text{ for } G^\vee = \text{loop group}$$

[Gukov – ⋯]

2005.05347

$$\begin{tikzcd} R \subset G(\mathcal{K}) \times^{G(\mathcal{O})} N(\mathcal{O}) \arrow[d] \arrow[dr, phantom] \\ \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, phantom, "\subset"] \arrow[dr, "p"'] & G(\mathcal{K}) \times^{G(\mathcal{O})} N(\mathcal{O}) \arrow[d] \\ & \mathrm{Gr}_G \end{tikzcd}$$

ω_R dualizing sheaf $\qquad p_! \omega_R$: a ring object in $D_{G(\mathcal{O})}(\mathrm{Gr}_G)$

§3. Ginzburg – Riche.

$$T^*(G/B) = G \times^B (\mathfrak{g}/\mathfrak{b})^*$$

$$H^0(T^*(G/B), \mathcal{O}(\mu)) = \mathrm{Ind}_B^G\, \mathbb{C}((\mathfrak{g}/\mathfrak{b})^* \otimes \mathbb{C}_{-\mu})$$

$$H^*(a_\mu^! IC_\mu^\lambda) = \left(V(\lambda) \otimes \mathrm{Ind}_B^G (\mathbb{C}[\mathfrak{g}/\mathfrak{b})^* \otimes \mathbb{C}_{-\mu})\right)^G$$

$$\simeq \Big(\mathrm{Res}_B^G V(\lambda) \otimes \underset{\substack{\uparrow \\ \text{algebraic, make sense for KM.}}}{\mathbb{C}[(\mathfrak{g}/\mathfrak{b})^*]} \otimes \mathbb{C}_{-\mu}\Big)^B$$

equivariant cohomology version

$$\underset{\substack{\uparrow \\ \text{cohomological} \\ \text{degree.}}}{\mathbb{C}^\times_{\text{loop}}} \times \underset{\substack{\| \\ \pi_1(G)^\wedge}}{T} \curvearrowright \mathcal{M}_C(\lambda,\mu)$$

$$\underset{\substack{\uparrow \\ \text{\& add } \times\, \mathbb{C}^\times_{\text{loop}}}}{H^*_T}\left(a_\mu^! IC^\lambda_\mu\right) \cong \left(\operatorname{Res}^G_B(V(\lambda)) \otimes \mathbb{C}[(\mathfrak{g}/\mathfrak{n})^*] \otimes \mathbb{C}_{-\mu}\right)^B$$

$$\begin{array}{ccc} T^*(G/B) & \subset & \widetilde{T^*(G/B)} \\ \downarrow & & \downarrow \\ 0 & \in & \mathfrak{t} \end{array}$$

quantization by universal Verma module.

* hyperbolic restriction (cf. Mirković–Vilonen's approach to geometric Satake)

$\chi : \mathbb{C}^\times \longrightarrow T^\vee \curvearrowright \mathcal{M}_C(\lambda,\mu)$.
regular dominant cocharacter

$$\begin{tikzcd}
\{z^\mu\} \arrow[rr, bend left, "a_\mu"] & & \mathcal{M}_C(\lambda,\mu) \\
& A_\chi \arrow[ul, "p_\mu"] \arrow[ur, "j_\mu"'] &
\end{tikzcd}$$

$\{z^\mu\}$: $T^\vee$-fixed pt

A_χ: attracting set for χ

j_μ: (inclusion)

hyp. restr. $\quad p_{\mu *} j_\mu^! IC^\lambda_\mu$

<u>Conjecture</u> (refinement of [BF])

(a) $p_{\mu *} j_\mu^! IC^\lambda_\mu$ is concentrated in deg. 0

(b) (+) " $\cong V(\lambda)$
$\quad\quad\quad\quad\quad\quad\quad\quad \curvearrowleft$
$\quad\quad\quad\quad\quad\quad\quad\quad e_i, f_i$ ⟶ realized by considering hyperbolic restriction w.r.t. χ_i a ~~generic~~ dominant coweight $\subset$ root hyperplane for α_i

$$H^*_{\check{T}\times\mathbb{C}^\times}(a^!_\mu IC^\lambda_\mu) \underset{?}{\cong} (\mathrm{Res}^G_B V(\lambda)\otimes M(\mu))^B$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$H^*_{\check{T}\times\mathbb{C}^\times}(p_{\mu *} j^!_\mu IC^\lambda_\mu) \cong V_\mu(\lambda)\otimes H^*_{\check{T}\times\mathbb{C}^\times}(pt)$$

cf. W-algebra rep.

universal Verma module

$\downarrow$

universal Wakimoto module $\qquad$ $W \subset$ Heis.

$\underset{?}{\cong}$ exists over Frac $H^*_{\check{T}\times\mathbb{C}^\times}(pt)$

enough to check that there are no poles.

$\uparrow$ appear at root hyperplanes

[GR]. only real root hyperplanes appear

$\uparrow$ W. simple roots

affine case: need to study χ_δ

$\uparrow$ character contained in $\delta = 0$ — imaginary root.

$\rightsquigarrow$ $M_C(\lambda,\mu)^{\chi_\delta}$ = Coulomb branch for Jordan quiver $\circled{n}$ $\circlearrowleft$

$\uparrow$ $\boxed{\ell}$

$\cong S^n(\mathbb{C}^2/(\mathbb{Z}/\ell))$

$\uparrow$

homog. Heisenberg alg.