Studying the decomposition theorem over the integers

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$X \subset \mathbb{P}^n$ proj. variety, X smooth:

$H^*(X;\mathbb{Z})$ satisfies (derived) Poincaré duality.

$H^*(X;\mathbb{Q})$ hard Lefschetz, pure Hodge structures, Hodge-Riemann relations

X singular, all statements above on $H^*(X)$ fail.

Instead, we can consider $IH^*(X)$ (Goresky MacPherson)

$IH^*(X,\mathbb{Q})$ PD, hL, pure Hodge str., Hodge-Riemann

GM: (Derived) Poincaré duality does not hold for $IH^*(X;\mathbb{Z})$

<u>Ex</u>. $C \subset \tilde{X}$ ← proj. smooth surface
$H^2(\tilde{X};\mathbb{Z})$ free

$\uparrow$
ADE
curve configuration.

$$\begin{array}{c}\tilde{X}\\ \downarrow \pi\\ X\end{array}$$ map contracting C to a point.

$$\det\left(\langle -,-\rangle_{\text{Poincaré}}, IH^2(X;\mathbb{Z})\right) = \det(\text{Cartan mat.})$$

$\Rightarrow$ PD fails over $\mathbb{Z}$.

<u>Local variant</u>. Determine p-torsion in the stalks and costalks of

$IC(X;\mathbb{Z})$ (integral IC sheaf)

<u>Rmk</u>. no p-torsion (in stalk or costalk) $IC(X;\mathbb{Z})$ $\Rightarrow$ $IH^*(X;\mathbb{Z}_p)$ satisfies PD. NOT CONVERSELY (probably).

$X = \bigsqcup X_\lambda$ stratification, $i_\lambda : X_\lambda \hookrightarrow X$.

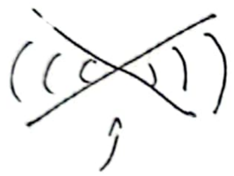
$IC(X;\mathbb{Z})$ satisfies

1) $i_\lambda^* \, IC(X;\mathbb{Z}) \in D^{<-\dim X_\lambda}(X_\lambda)$

2) $\mathcal{H}^j(i_\lambda^! \, IC(X;\mathbb{Z}))$ should be torsion for $j = \dim X_\lambda$ and vanish below $\dim X_\lambda$.

$$\begin{array}{c} IC(X;\mathbb{Z}) \\ \updownarrow \mathbb{D} \\ IC^+(X;\mathbb{Z}) \end{array}$$

<u>Exercise</u>. $\tilde{X} \to X = \{xy = z^2\} \subset \mathbb{C}^3$



$\mathcal{F} = \mathbb{Z}[2]$ satisfies

$$i_0^! \mathcal{F} = \begin{array}{|c|c|c|} 0 & 1 & 2 \\ \hline \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z} \end{array}$$

link is $S^3/\mu_2 \cong \mathbb{RP}^3$.

$IC(X;\mathbb{Q})$ has no self-dual $\mathbb{Z}$-form.

Minimal self-dual extension is the "parity sheaf", $f_* \mathbb{Z}[2]$.

<u>Rmk</u>. If X has isolated singularities $x_1, \dots, x_m$,

Q at p-torsion $\iff$ understanding p-torsion in the integral cohomology of the links at $x_1, \dots, x_m$.

So Q is very hard in general.

$$\begin{array}{c} \tilde{X} \\ f \downarrow \\ X \end{array}$$ resolution of singularities

(For dcM: assume f is projective, $\tilde{X}$ is projective.)

Choose a stratification of X, $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$. X_λ connected

For each X_λ, choose a point $x_\lambda \in X_\lambda$ and a normal slice to X_λ through x_λ.

$$\begin{tikzcd}
F_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \tilde{N}_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \tilde{X} \arrow[d] \\
\{x_\lambda\} \arrow[r, hook] & N_\lambda \arrow[r] & X
\end{tikzcd}$$

The embedding $F_\lambda \hookrightarrow \tilde{N}_\lambda$ equips $H_*(F_\lambda)$ w/ an intersection form

$$H_*(F_\lambda) \times H_*(F_\lambda) \xrightarrow{IF_\lambda} \mathbb{Z}.$$

<u>Basic observation</u>. Assume we know DT over $\mathbb{Q}$, k field

$f_* \underline{k}[\dim_{\mathbb{C}} \tilde{X}]$ splits into IC's as predicted by DT

$\iff$ $IF_\lambda \otimes k$ has the same rank as over $\mathbb{Q}$.

Basic observation says nothing about semi-simplicity of local systems occurring in direct image.

In the small case, no IF's except for open stratum.

In the semi-small case, there is only one piece of IF that can contribute for each stratum. Here

DT $\iff$ each IF is non-degenerate on top homology.

dCM prove this case as follows:

Assume $X_\lambda = \{x_\lambda\}$ is a pt stratum. $N_\lambda = X$, $\tilde{N}_\lambda = \tilde{X}$.

Semi-small $\Rightarrow$ $\dim F_\lambda \leq \frac{1}{2} \overset{=: d}{\dim \tilde{X}}$

Assume equality:

$$\begin{tikzcd}
H_d(F_\lambda) \times H_d(F_\lambda) \arrow[r, "IF_\lambda"] \arrow[d, "\wr\, cl"'] & \mathbb{Z} \\
H^d(\tilde{X}) \times H^d(\tilde{X}) \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

dCM : $H_d(F_\lambda) \to H^{d,d}(\tilde{X}) \cap \left(\begin{array}{c} \text{primitive} \\ \text{classes for} \\ \text{the pull back} \\ \text{of an ample lb.} \\ \text{from } X. \end{array} \right)$

Miracle: pull backs of ample classes under semismall maps satisfy hL and HR.

de Cataldo and Migliorini use the perverse filtration to

In general, predict the radical of $H^*(F_\lambda) \times H^*(F_\lambda) \to \mathbb{Z}$.

Rmk. 1) In the semismall case, relevant local systems factor through

$\pi_1(X_\lambda) \hookrightarrow \mathrm{Perm}\left(\begin{array}{c}\text{top dim. components} \\ \text{of fiber } F_\lambda\end{array}\right)$, hence is semisimple.

$\mathbb{Q}$ in general is much harder (use Deligne's theorem).

perverse filtration = wt filtration w.r.t. the action of an ample class

pulled back from X.

Schubert varieties.

$X = GL_n(\mathbb{C})/B$, $W = \mathfrak{S}_n$, $X_x = \overline{BxB/B}$.

Thm. The quantity $m_n := \max\limits_{x \in \mathfrak{S}_n} \left\{ \begin{array}{l} \text{there exists } p\text{-torsion in a} \\ \text{stalk or costalk of } IC(X_x; \mathbb{Z}) \end{array} \right\}$

grows at least as fast as c^n, where $c > 1$. (c is probably ~ 1.09).

Rmk 1) This is a new proof of an earlier result obtained using Soergel bimodule

techniques, partly joint with Xuhua He.

2) Many applications to representation theory.

a) Expected bounds for Lusztig character formula:

$\xRightarrow{\text{Soergel}}$ m_n grows linearly in n.

b) James conjecture

(simple $\mathbb{F}_p \mathfrak{S}_n$-modules) $\Rightarrow$ m_n grows quadratically in n.

$P_s \supset B$ minimal parabolic for s

$\underline{w} = s_1 s_2 \cdots s_m$ expression

$$BS(\underline{w}) = P_{s_1} \times P_{s_2} \times \cdots \times P_{s_m} / B^m$$

$T \curvearrowright BS(\underline{w})$, fixed pts $\longleftrightarrow$ $\underset{}{\overset{\text{subexpressions}}{\underline{e} \subset \underline{w}}}$

$\underline{e} = e_1 \cdots e_m, \quad e_i \in \{0,1\}$

Coh. class $f \in H_T^*(BS(\underline{w}))$

<u>combinatorial</u>: if $\exists$ $f_1, \dots, f_m \in H_T^*(pt)$ s.t.

$$f_{\underline{e}} = s_1^{e_1}(f_1 s_2^{e_2}(\cdots(s_m^{e_m} f_m)\cdots)).$$

$\uparrow$ local at $\underline{e} \in BS(\underline{w})$

<u>Ex.</u> V a B^m-module, $\quad L_V := (P_{s_1} \times \cdots \times P_{s_m}) \times_{B^m} V$

vector bundle on $BS(\underline{w})$. Then $\mathrm{Euler}(L_V)$ is combinatorial w/

$$f_i := \det\left(\mathrm{res}^{B \hookrightarrow \text{ith copy}}_{B^m} V\right).$$

<u>Euler class lemma</u>:

$p: BS(\underline{w}) \to pt$, $\qquad p_! f = \partial_{s_1}(f_1 \partial_{s_2}(\cdots \partial_{s_m} f_m)\cdots)).$

$\uparrow$ combinatorial