Graded sheaves and applications to link homology

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1. Graded sheaves (enhancement of ℓ-adic sheaves)

Let X be a scheme / stack (Artin stack of finite type) over $\overline{\mathbb{F}_q}$.

$D^b_c(X; \overline{\mathbb{Q}_\ell})$ bounded derived category of constr. ℓ-adic sheaves
(dg cat. / ∞-cat.)

$X/\mathbb{F}_q$

BBDG: $D^b_m(X, \overline{\mathbb{Q}_\ell}) \subset D^b_c(X, \overline{\mathbb{Q}_\ell})$

$$X = pt = \operatorname{Spec} \mathbb{F}_q$$

$$D^b_c(pt, \overline{\mathbb{Q}_\ell}) = \text{Fr-mod} = \{(V, \varphi) : V \text{ chain complex} / \overline{\mathbb{Q}_\ell},\ \varphi : V \to V \text{ automorphism} + \cdots\}$$

$$\cup$$

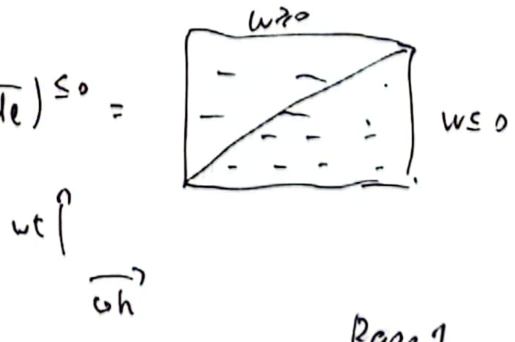
Fix $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$, $\text{Fr-mod}_m = \{(V, \varphi) :$ eigenvalues of φ has weight $\in \mathbb{Z}\}$
i.e. $|\text{eigenvalue}| \in q^{\mathbb{Z}/2}$

For general X, $i_x : \{x\} \to X$, $\{x\} = \operatorname{Spec} \mathbb{F}_q$

$$\mathcal{F} \in D^b_m(X, \overline{\mathbb{Q}_\ell}) \Leftrightarrow i_x^* \mathcal{F} \in \text{Fr-mod}_m \subset \text{Fr-mod}$$

<u>Def</u>. $D^b_m(X; \overline{\mathbb{Q}_\ell})^{w \le 0} = \{\mathcal{F} \in D^b_m(X, \overline{\mathbb{Q}_\ell}) : i_x^* \mathcal{F} \in D^b_m(pt, \overline{\mathbb{Q}_\ell})^{w \le 0}\}$

where $D^b_m(pt; \overline{\mathbb{Q}_\ell})^{\le 0} =$



$D^b_m(X; \overline{\mathbb{Q}_\ell})^{w \ge 0} = \{\mathcal{F} \in D^b_m(X, \overline{\mathbb{Q}_\ell}) : i_x^! \mathcal{F} \in D^b_m(pt, \overline{\mathbb{Q}_\ell})^{w \ge 0}\}$

$D^b_m(X, \overline{\mathbb{Q}_\ell})^{w=0} = (\quad)^{w\geq 0} \cap (\quad)^{w\leq 0}$ pure objects

$D^b_m(X, \overline{\mathbb{Q}_\ell})$ has the perverse t-structure.

<u>Decomposition theorem</u>.

$X/\mathbb{F}_q$ finite type stack, $\mathcal{F} \in D^b_m(X, \overline{\mathbb{Q}_\ell})^{w=0}$,

then $\mathcal{F} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is semisimple

need $- \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ even when $X = \operatorname{Spec} \mathbb{F}_q$

$$\begin{tikzcd} D^b_m(X, \overline{\mathbb{Q}_\ell}) \arrow[r, "?"] & D^b_c(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell}) \\ \mathcal{F} \text{ pure} \arrow[r, mapsto] & \mathcal{F} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \text{ semisimple} \end{tikzcd}$$

$D^b_{gr}(X, \overline{\mathbb{Q}_\ell})$

Question, Can I construct sth. in the middle so that

- $D^b_{gr}(X, \overline{\mathbb{Q}_\ell})$ has a notion of a weight str.
  has a t-str.
- Pure objects are semisimple in $D^b_{gr}(-)$

<u>$\mathcal{C}$ dg cat.</u>

weight str.	t-str.
$\mathcal{C}^{w\leq 0}, \mathcal{C}^{w\geq 0}$	$\mathcal{C}^{t\leq 0}, \mathcal{C}^{t\geq 0}$ s.t.
(1) $\mathcal{C}^{w\geq 0}[1] \subset \mathcal{C}^{w\geq 0}$ $\mathcal{C}^{w\leq 0}[-1] \subset \mathcal{C}^{w\leq 0}$	(1) $\mathcal{C}^{t\leq 0}[1] \subset \mathcal{C}^{t\leq 0}$ $\mathcal{C}^{t\geq 0}[-1] \subset \mathcal{C}^{t\geq 0}$
(2)	(2) $\operatorname{Hom}_{\mathcal{C}}(X, Y[-1]) = 0$, $X \in \mathcal{C}^{t\leq 0}$, $Y \in \mathcal{C}^{t\geq 0}$
(3) } the same	(3) $\exists\ X \to A \to Y \to X[1]$, $\forall A \in \mathcal{C}$ $X \in \mathcal{C}^{t\leq 0}$, $Y \in \mathcal{C}^{t\geq 1}$

$\mathcal{C}^{w=0}$ additive cat.

$K^b(B) = Ch^b(B)/\text{homotopy}$

B: additive

$\mathrm{Ext}^*(X,Y)$ — nonpositive degree

$X, Y \in \mathcal{C}^{w=0}$

Bondarko weight complex

$$wt : \mathcal{C} \to K^b(\mathcal{C}^{w=0})$$

$\mathcal{C}^{t=0}$ abelian cat.

$D^b(A)$

A: abelian cat.

$\mathrm{Ext}^*(X,Y)$ lives in nonnegative degree

$X, Y \in \mathcal{C}^{t=0}$

Beilinson realization

$$real : D^b(\mathcal{C}^{t=0}) \to \mathcal{C}$$

For $\mathcal{F}, \mathcal{G} \in D^b_m(X; \overline{\mathbb{Q}}_\ell)$, $\mathrm{Ext}^1(\mathcal{F},\mathcal{G})$ could be non-zero

$\Rightarrow (D^{b, w\leq 0}_m, D^{b, w\geq 0}_m)$ does not define a weight structure

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What is $D^b_{gr}(pt) = ?$

$Vect^{gr}$ = the category of graded vector spaces

$D^b_m(pt) \to Vect^{gr}$ symmetric monoidal

$(V, \varphi) \mapsto \bigoplus V_i$. $V_i = \{\text{wt } i \text{ summand of } V\}$

<u>Def</u> $D^b_{gr}(X) = D^b_m(X) \otimes_{D^b_m(pt)} Vect^{gr}$ Lurie's tensor product.

<u>Thm</u> $D^b_m(X) \xrightarrow{gr} D^b_{gr}(X) \xrightarrow{oblv} D^b_c(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$

<u>Rmk</u>. $D^b_{gr}(X)$ only depends on $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

$\exists$ a weight str. w, t-str. t on $D^b_{gr}(X)$

s.t. gr preserves weights, gr, oblv t-exact (any object in <u>t-heart</u> has a canonical weight truncation.)

w and t transverse.

($\Rightarrow$ pure objects are semisimple)

2. Soergel bimodules

Let G be a reductive gp $/\mathbb{F}_q$, B Borel subgp.

<u>Thm</u>. There is an equiv. of cats

$$D^b_{gr}(B\backslash G/B) \cong K^b(\underbrace{SBim}_{\text{additive cat. of Soergel bimodules}})$$

<u>Def</u>. $F : (\mathcal{C}, w) \to (\mathcal{D}, t)$

$$\mathcal{C}^{w=0} \to \mathcal{D} \xrightarrow{H^*} \bigoplus_{\mathbb{Z}} \mathcal{D}^{t=0}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "wt"] \arrow[ddrrrr, bend right, "\tilde{F}"] & K^b(\mathcal{C}^{w=0}) \arrow[r] & K^b(\mathcal{D}) \arrow[r] & K^b(\bigoplus_{\mathbb{Z}} \mathcal{D}^{t=0}) = & \bigoplus_{\mathbb{Z}} K^b(\mathcal{D}^{t=0}) \arrow[d] \\
 & & & & \bigoplus_{\mathbb{Z}} D^b(\mathcal{D}^{t=0}) \arrow[d, "\text{Beilinson realization}"] \\
 & & & & \bigoplus_{\mathbb{Z}} \mathcal{D}
\end{tikzcd}$$

<u>Def</u>. Let γ be a Braid, $\gamma = \sigma_{i_1} \cdots \sigma_{i_n}$

$$R : Br \to D^b_{gr}(B\backslash G/B)$$

$$\sigma_i \mapsto \Delta_{s_i} = j_! \, \overline{\mathbb{Q}}_{\ell\, B\backslash Bs_iB/B}$$

<u>Def</u>. $HHH(\gamma) = \widetilde{R\Gamma}(R(\gamma))$

$$R\Gamma : \underset{\mathcal{H}}{\overset{}{D^b_{gr}(B\backslash G/B)}} \to D^b_{gr}(pt) = Vect^{gr}$$

$$\widetilde{R\Gamma} : D^b_{gr}(B\backslash G/B) \to Vect^{gr, gr}$$

$$\downarrow H^*$$

$$Vect^{t=0, gr, gr, gr}$$

<u>Fact</u>. $HHH(\gamma)$ only depends on the closure of γ.

<u>Thm</u>. (a) the categorical trace of $\mathcal{H}$ is canonically equiv.

$$\text{to} \quad \mathrm{Coh}_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}_n(\mathbb{A}^2)) \qquad (G = GL_n)$$

(b) $\mathrm{HHH}(\gamma) = H^*(\mathrm{Hilb}_n, \mathcal{F}_\gamma)$. $\mathcal{F}_\gamma \in \mathrm{Coh}(\mathrm{Hilb}_n)$

(GNR-conjecture) some canonically defined coherent sheaf.