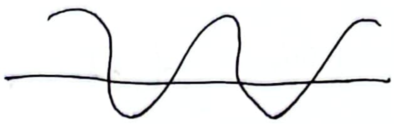
Koszul duality via dg Morita theory

Geordie Williamson

delta functions

Fourier $\longleftrightarrow$ transform



sine waves

("fundamental frequency")

extremely useful.

Examples: ① Fraunhofer lines

② Riemann ζ- function

One can think of derived categories as "categorical function spaces".

(Grothendieck function-sheaf correspondence)

$\xrightarrow{\text{suggests}}$ Is there a Fourier transform for derived categories?

Hope: skyscraper $\overset{\text{F.T.}}{\longleftrightarrow}$ vector bundle

simple object $\qquad$ projective object

Beautiful example $\quad A, A^\vee$ dual abelian varieties

P / Poincaré bundle

$$\begin{tikzcd} & A \times A^\vee \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & A^\vee \end{tikzcd}$$

$$FM := p_{1*}\left(p_2^*(\ ) \otimes P\right) : D^b(\mathrm{Coh}(A^\vee)) \to D^b(\mathrm{Coh}(A))$$

skyscraper $\longleftrightarrow$ deg zero line bundles / shift

Meta-mathematical $\mathcal{X}$ objects $\rightsquigarrow$ what can I build with $\mathcal{X}$?

① D^n + S^{n-1} $\rightsquigarrow$ CW complexes, glue: homotopy classes of maps

② $\mathcal{X} = \{\mathbb{Z}/p\mathbb{Z}\} \subset$ finite groups $\rightsquigarrow$ all p-groups. glue, very difficult.

③ $\mathcal{X}$ some objects inside $\subset \mathcal{A}$ abelian $\rightsquigarrow$ $\langle \mathcal{X} \rangle \subset \mathcal{A}$, glue: Ext's.

P projective $\rightsquigarrow$ $\langle P \rangle \hookleftarrow$ objects with a presentation

$$P^{\oplus n} \to P^{\oplus m} \twoheadrightarrow ?$$

$$\langle P \rangle \underset{\text{Hom}(P,-)}{\simeq} \text{f.p. right End}(P)\text{-modules}$$

(Morita theory)

④ $M \in \mathcal{T}$ triangulated cat. Want to understand $\langle M \rangle_\Delta \subset \mathcal{T}$.

Step 1: $\bigoplus_{i \in I} M[n_i] \xrightarrow{f} \bigoplus_{j \in J} M[n_j] \to C \xrightarrow{+1}$

Note f is determined by $\text{Ext}^*(M) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(M, M[i])$

Hope: $\langle M \rangle_\Delta$ is determined by $\text{Ext}^*(M)$

Problem. Non-functoriality of cones

Example. $n \geq 2$, $\Lambda_n = k[x]/(x^n)$. k simple Λ_n-module

$$\text{Ext}^*(k) = \begin{cases} k[x], \ n=2 & (x \text{ in deg } 1) \\ k[x,\eta]/(\eta^2), \ n>2 & (x \text{ in deg } 2,\ \eta \text{ in deg } 1) \end{cases}$$

But. $\langle k \rangle = D^b(\Lambda_n\text{-mod})$

Not equivalent for different n

Moral: need Ext algebra + ...

x x x

dg algebras & modules

dg-algebra: $A = \bigoplus_{i \in \mathbb{Z}} A^i$ graded algebra,

+ $d: A \to A[1]$ differential, s.t.

$$d(ab) = da \cdot b + (-1)^{|a|} a\, db$$

Ex ⓐ any graded algebra, $d=0$ ("formal")

ⓑ $\cdots \to C^j \to C^{j+1} \to C^{j+2} \to \cdots$ chain complex

$\downarrow \quad \downarrow$

$\cdots \to C^{j+i} \to C^{j+i+1} \to \cdots$

$\mathrm{End}^\bullet(C)$, $\mathrm{End}^j(C) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(C^j, C^{j+i})$

(usually enormous!)

alg. via composition.

$$(df)^k = d f^k - (-1)^i f^{k+1} \circ d$$

$\uparrow$ deg $= i$

right dg-module: $M = \bigoplus M^i$. graded right A-module

+ $d: M \to M[1]$

s.t. $d(m \cdot a) = dm \cdot a + (-1)^{|a|} m\, da$

Ex: ⓑ $\cdots \to E^j \to E^{j+1} \to E^{j+2} \to \cdots$

$\mathrm{Hom}^\bullet(C, M)$ is a right $\mathrm{End}^\bullet(C)$-dg module.

dg Morita theory:

$$A = \mathrm{End}^{\bullet}(C) \quad \text{dg-algebra}$$

$\mathcal{A}$ abelian cat.

$$\begin{tikzcd}
C(\mathcal{A}) \ni C \arrow[r, "\mathrm{Hom}(C,-)"] \arrow[d] & \text{dg-}C(A) \arrow[d] & \text{abelian cat. of dg modules} \\
K(\mathcal{A}) \ni \langle C\rangle_{\Delta} \arrow[r, "\mathrm{Hom}(C,-)"] \arrow[r, "\sim"'] \arrow[d, "\cup\!\!\!\mid"] & \text{dg-}K(A) \ni \langle A\rangle_{\langle\Delta\rangle} \arrow[d, "\cup\!\!\!\mid"] & \text{homotopy category} \\
D(\mathcal{A}) \ni \langle C\rangle_{\Delta} & \text{dg-}\mathrm{Der}(A) \ni \langle A\rangle_{\langle\Delta\rangle} & \text{localize quasi-isom. derived cat.}
\end{tikzcd}$$

(check) descends

both Δ^{ed}

Assumption: C is End acyclic, i.e.

e.g. C bounded above complex of projectives

$$(*) \quad \mathrm{Hom}_{K(\mathcal{A})}(C, C[m]) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(C, C[m])$$

Moral: All (algebraic) Δ^{ed}-cats are described by dg-Modules.

Formality. $A \xrightarrow{f} A'$ morphism of dg-algebras, q-iso if

$$H^{i}(f): H^{i}(A) \xrightarrow{\sim} H^{i}(A') \quad \rightsquigarrow \text{quasi-isomorphic.}$$

Thm A, A' are q-isomorphic, $\text{dg-}\mathrm{Der}\, A \underset{\text{q. equiv.}}{\simeq} \text{dg-}\mathrm{Der}\, A'$.

$$\begin{array}{ccc} \text{dg-}\mathrm{Der}\,A & & \text{dg-}\mathrm{Der}\,A' \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ A & \longleftrightarrow & A' \end{array}$$

A is formal if $A \underset{q.i.}{\simeq} (H^{\bullet}(A), d = 0)$.

In our example,

In our example from earlier, $\Lambda_n = k[x]/(x^n)$, $\langle k\rangle_\Delta = D^b(\Lambda_n\text{-mod})$

$P \twoheadrightarrow k$ proj. resolution.

$\mathrm{End}^\bullet(P)$ is formal $\Leftrightarrow$ $n=2$

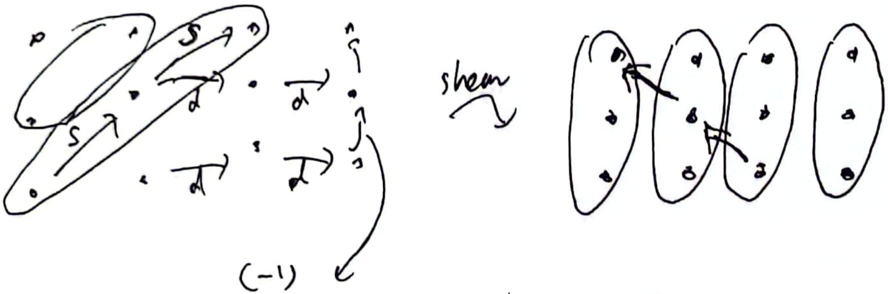
<u>Note</u>: $H^\bullet(\mathrm{End}^\bullet(P)) = \mathrm{Ext}^\bullet(k)$.

<u>Origins of Koszul duality</u>. V vector space f.d./k

$$\underset{\underset{\langle k\rangle_\Delta,\ \text{grading shifts}}{\cup}}{D^b(\Lambda^\bullet \overset{\leftarrow \deg -1}{V}\text{-gmod})} \xrightarrow[\text{Morita theory} + \text{formality}]{\text{dgg}} \text{dgg Der-}S(V^*) \qquad \begin{array}{l} k \longleftrightarrow S \\ \Lambda \longleftrightarrow k \end{array}$$

$\wr$ sheafy

$D^b(SV^*\text{-gmod})$ "Fourier transform"



Shift of grading $\longleftrightarrow$ $(-1)[1]$

"stable cat"

$$D^b(\Lambda\text{-gmod})/\langle \Lambda\rangle \simeq D^b(S\text{-gmod})/\langle k\rangle \simeq D^b(\mathrm{Coh}\ \mathbb{P}^n)$$

$\uparrow$ Serre's description

"Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra" Bernstein - Gelfand - Gelfand, Beilinson

<u>Koszul duality</u>:

$A = \bigoplus_{i\geq 0} A^i$ pos. graded w/ A^0 semi-simple

<u>Koszul</u> if $\mathrm{Ext}^i(A^0, A^0)$ is concentrated in deg. i.

Trick (Deligne) Any bigraded dg-algebra w/. $H^{\cdot}(A)$ concentrated in degrees on the diagonal (i,i) is formal.

Deligne's trick $\Rightarrow$ $\mathrm{End}^{\cdot}(A^0)$ is formal

$$D^b(A\text{-mod}) \simeq \mathrm{dgg}\text{-}\mathrm{Ext}^{\cdot}(A^0) \underset{\text{shear}}{\rightleftarrows} D^b(\mathrm{Ext}(A^0)\text{-gmod})$$

$$\text{Simples} \quad \longleftrightarrow \quad \text{projectives}$$