AGT conjecture and q-vertex operators

E Vasserot.

$\mathcal{M}_{r,m}$ = moduli space of torsion free sheaves of rank r, and $c_2 = m$ on $\mathbb{P}^2$

+ framing on $\mathbb{P}^1_\infty$.

$T = (\mathbb{C}^\times)^r \times (\mathbb{C}^\times)^2 \times \mathbb{C}^\times$

vector bdle on $\mathcal{M}_r \times \mathcal{M}_r$, $\mathcal{M}_r = \bigsqcup_{m \geq 0} \mathcal{M}_{r,m}$

(Carlsson - Okounkov)

$\mathcal{E}, \mathcal{F} \in \mathcal{M}_{r,m}$,

$\mathrm{Ext}^i_{\mathbb{P}^2}(\mathcal{E}, \mathcal{F}(-\mathbb{P}^1_\infty)) \underset{\text{Serre duality}}{=} 0$ unless $i = 1$.

$\exists$ bdle $E^{(r)}$ T-equivariant / $E^{(r)}_{\mathcal{E},\mathcal{F}} = \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{E}, \mathcal{F}(-\mathbb{P}^1_\infty))$.

$\forall m \in \mathbb{Z}$, $\mathbb{C}_m$ = character of $\mathbb{C}^\times$

$eu(E^{(r)} \otimes \mathbb{C}_m)$ = equiv. Euler class

$F^{(r)}_n = H^*_T(\mathcal{M}_{r,m})_{\text{loc}}$

$\otimes_{H^*_T(pt)} \mathrm{Frac}(H^*_T(pt))$

$F^{(r)} = \bigoplus_{n \geq 0} F^{(r)}_n$

operator $W^{(r)}$ on $F(r)$ s.t

$W^{(r)}(z) = \sum_{l \in \mathbb{Z}} W^{(r)}_l z^l \in \mathrm{End}(F^{(r)})[[z, z^{-1}]]$

$(\cdot,\cdot)$ = Poincaré pairing.

$\forall \alpha, \beta \in F^{(r)}$, $(W^{(r)}_m(\alpha), \beta) = (\alpha \otimes \beta, eu(E^{(r)} \otimes \mathbb{C}_m))$.

NB $\exists$ K-theoretic version $F_n^{(r)} = K_T(M_{r,m})_{loc}$

$$(W_m^{(r)}(\alpha), \beta) = (\alpha \otimes \beta, \Lambda_{-1}(E^{(r)} \otimes \mathbb{C}_m)), \quad \Lambda_{-1} = \sum_i (-1)^i \Lambda^i.$$

Prob. Compute $W^{(r)}(z)$, vertex operators

Cohomological version (Alday - Tachikawa)

$(r=1)$. $\mathcal{H}$ = Heisenberg algebra

$\alpha_l,\ l \in \mathbb{Z}$. $[\alpha_l, \alpha_k] = l\,\delta_{k,-l}$ (level = 1)

Fock = simple lowest weight module / $\mathcal{H}$

$a, b \rightsquigarrow V_{a,b}(z) \in (\mathrm{End}\ \mathrm{Fock})[[z, z^{-1}]]$

$$V_{a,b}(z) = \exp\left(a \sum_{m>0} z^m \alpha_m / m\right) \exp\left(-b \sum_{m>0} z^{-m} \alpha_{-m}/m\right)$$

$(r \geq 1)$ $\quad \mathcal{H}^{\otimes r} \curvearrowright \mathrm{Fock}^{\otimes r}$

$$\parallel$$

$\mathcal{H} \otimes W(\mathfrak{sl}_r) \curvearrowright \mathrm{Fock} \otimes \mathrm{Fock}(\mathfrak{sl}_r)$

(Miura transform)

$\exists$ representation of $W(\mathfrak{sl}_r) \otimes \mathcal{H}$ on $F^{(r)}$ s.t.

(1) $F^{(r)} \simeq \mathrm{Fock}(\mathfrak{sl}_2) \otimes \mathrm{Fock}$ as $W(\mathfrak{sl}_r) \otimes \mathcal{H}$-module

(2) (conjecture): $W^{(r)}(z)$ is identified with $V_{rm\lambda}(z) \otimes V_{m+x+y,\, -\frac{rm}{xy}}(z)$.

$F^{(r)}$ vector space over $\mathrm{Frac}(H_T^*(pt))$

$$\parallel$$

$$\mathbb{C}(\underbrace{x, y}_{\text{coh. } (\mathbb{C}^\times)^2}, \underbrace{e_1, \dots, e_r}_{\text{cohomology } (\mathbb{C}^\times)^r}, m)$$

λ = explicit weight of $\mathfrak{sl}_r$.

NB. $r=1$ (Carlson - Okounkov), $m=0$ ✓

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K-theoretic version Algebraic characterization.

$\mathbb{C}_{q,t} = \mathbb{C}(q,t)$ ~ Elliptic Hall algebra

$\mathbb{Z}^2$- graded $\mathbb{C}_{q,t}$- algebra $\widetilde{SH}$

(a) $A = \mathbb{C}_{q,t}[k_1^{\pm 1}, k_2^{\pm 1}] \subset Z(\widetilde{SH})$

(b) $\widetilde{SL_2}(\mathbb{Z}) \curvearrowright \widetilde{SH}, A$

(c) $\widetilde{SH}$ is generated by u_x, $x \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$\tilde{\gamma} \in \widetilde{SL_2}(\mathbb{Z})$ takes u_x to $k_1^{m_1} k_2^{m_2} u_{\gamma(x)}$

($\gamma \in SL_2(\mathbb{Z})$ projection of $\tilde{\gamma}$).

(d) $SH^{(r)} / (k_1 = (qt)^{r/2}, k_2 = 1)$ acts on $F^{(r)}$

(e) $\alpha_k = u_{k,0}$, $k \neq 0$ generate a q-Heis algebra

$$[\alpha_k, \alpha_\ell] = \delta_{k,\ell} \cdot \ell (k_2^\ell - k_2^{-\ell}) / (1-q^{-\ell})(1-t^{-\ell})(1-(qt)^{-\ell})$$

(f) $\widetilde{SH}$ has a topological coproduct

$$\Delta : \widetilde{SH} \to \widetilde{SH} \hat{\otimes} \widetilde{SH}$$

explicit on $u_{k,0}$, $u_{k,1}$, $u_{k,-1}$ (any k)

($r=1$)

Lemma $F^{(1)}$ irreducible as a module over $\langle u_{k,0}\text{'s} \rangle$.

Thm. (Feigin & Co., Carlsson - Nekrasov - Okounkov)

$$W^{(1)}(z) = a^{L_0} \exp\Big(\sum_{\ell \geq 1} b_\ell u_{\ell,0} z^\ell / \ell\Big) \exp\Big(\sum_{\ell \geq 1} c_\ell u_{-\ell,0} z^{-\ell}/\ell\Big)$$

a, b_ℓ, c_ℓ explicit constants.

L_0 grading operator

($r \geq 1$)

<u>Lemma</u> $F^{(r)} \simeq (F^{(1)})^{\hat{\otimes} r}$ as a $SH^{(r)}$-module

$F^{(r)}$ = "lowest weight module" (x kill any element if its L_0-degree is $\ll 0$).

$|\phi\rangle \in F_0^{(r)}$ (vacuum vector)

<u>Thm</u> $\exists$ continuous automorphisms T, S of $\widehat{SH}^{(r)}$ (completion of $SH^{(r)}$)

(a) $T(SH^{(r)})(|\phi\rangle) = F^{(r)}$

(b) $W^{(r)}$ intertwines T and S

$$W^{(r)} \circ T(x) = S(x) \circ W^{(r)} \quad \text{in } \mathrm{End}(F^{(r)})$$
$$x \in SH^{(r)}$$

(c) $W^{(r)}$ determined by the following "Whittaker type" condition

$$W^{(r)}(|\phi\rangle) =: w \in \hat{F}^{(r)} = \prod_n F_n^{(r)}$$

$$\left(\sum_{l \in \mathbb{Z}} u_{1,l} z^l \right) \cdot |\psi\rangle = \Gamma_0(w)(\psi)$$

$$\Gamma_0(w) \in k[u_{0l} ; l \in \mathbb{Z} \setminus 0][[w, w^{-1}]]$$

<u>NB</u>. $\hat{F}^{(r)} = (F^{(1)})^{\hat{\otimes} r}$.

q-vertex operators $\phi_1, \cdots, \phi_r$

$\phi_1 \otimes \cdots \otimes \phi_r$ acts on RHS

One can normalise s.t. (b) is satisfied by $\phi_1(x) \cdots \otimes \phi_r(x)$

<u>Prob./Conj.</u> The Whittaker condition is also satisfied by $\phi_1 \otimes \cdots \otimes \phi_r$

<u>Prob</u>. to know Δ of u_{1k}

$\widetilde{SH}$ = Drinfeld double of subalgebra SH^+ generated by $\{a_{1k};\ k \in \mathbb{Z}\}$

SH^+ = shuffle algebra (Feigin - Odesski)

↳ 3-dim'l analogue

(r=1) SH^+ acts on the K-theory $Hilb(\mathbb{C}^2)$

Derived Hilbert scheme of $\mathbb{C}^3$

dg-scheme (smooth, bounded)

$K(dg\ Hilb(\mathbb{C}^3)) \circlearrowleft$ Shuffle algebra

$(\mathbb{C}^\times)^3 \curvearrowright$

$$q(z) = \frac{(1-q_1q_2q_3 z)\cdot \prod_{i=1}^{3}(1-q_i z)}{(1-z)\prod_{i<j}(1-q_iq_j z)}$$