Statistics for Number Fields, Function fields & 3-Manifolds

§1 Class Group Distributions

K number field $\rightsquigarrow$ Cl_K class gp (finite ab. gp)

Ques. As K varies, how is Cl_K distributed?

Fix G finite group, K varies over $\mathrm{Gal}(K|K_0) \simeq G$ for fixed K_0

(w/ fixed behavior at ∞ places)

Ex $G = \mathbb{Z}/2\mathbb{Z}$, $K_0 = \mathbb{Q}$, real quadratic fields

$\mathbb{Q}(\sqrt{D})$, $D > 0$

80's - 90's: Cohen, Lenstra & Martinet made conjectures for these distributions

Conj
$$\mathrm{Prob}\left(Cl_K^{odd} \simeq A\right) = \frac{\prod_{p \text{ prime} > 2} \prod_{i=2}^{\infty} (1 - p^{-i})}{|A|\,|\mathrm{Aut}\,A|}$$

'08, '10 Malle: data suggesting conj's wrong for p-Sylow subgroups when

$p \mid |\mu(K_0)|$

\# roots of unity.

§2 Function field analog.

$\mathbb{Q} \rightsquigarrow \mathbb{F}_q(t)$

$$\begin{array}{ccc} K & & \\ & \diagdown & \\ & & K_0 \\ & \diagup & \\ \mathbb{F}_q(t) & & \end{array} \qquad C \to C_0 \quad \text{Curves} / \mathbb{F}_q$$

- use alg. geom. of maps of curves.
- can use many q & send $q \to \infty$.

Class group is the abelization of a natural (generally non-abelian) group

$\mathrm{Gal}(K^{un}|K)$ $\qquad$ $\pi_1^{et}(C)$

max. unram. extn

More generally, we ask about the statistics of these groups.

Achter '06

H_g moduli space of hyperelliptic curves (quad. extns of $\mathbb{F}_q(t)$)

$$\begin{array}{c} \mathrm{Jac}(\mathcal{U}_g)[\ell^k] \text{ Jacobian of universal curve} \\ \downarrow \\ H_g \end{array}$$

Used monodromy i.e. image of $\pi_1(H_g) \longrightarrow GL_{2g}(\mathbb{Z}/\ell^k)$

As $q \to \infty$, as long as $\ell \nmid q-1$, we get distribution matching conj.

'16 Ellenberg - Venkatesh - Westerland

Proved & used much stronger topological information on these moduli spaces to show this match when $g \to \infty$ first & $q \to \infty$ later.

Liu-W.-Zureick-Brown Show a match w/ conjectures for general G, $q \to \infty$ first, while avoiding roots of unity

Alg. geom. / top gives certain averages.

Moments

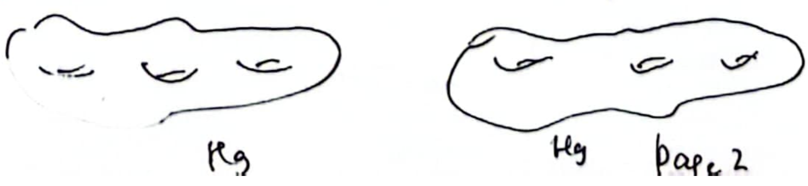
$$\mathbb{E}(\# \underset{\substack{\uparrow \\ \text{B th moment}}}{} \overset{\text{surjective homomorphisms}}{\mathrm{Sur}}(\underset{\substack{\uparrow \\ \text{random} \\ \text{group}}}{Cl_K}, \underset{\text{fixed}}{B}))$$

Knew moments of conj. dists & determined by moments

§3 3-mfds

3-mfds analogous to # fields and function fields
(Artin-Verdier duality 3 dim'l cohomological duality)

Dunfield - Thurston '06 random Heegaard splitting



get random 3-mfd M by gluing surfaces w/ a random element of mapping class group,. let $g \to \infty$.

asked questions about dist. of $\pi_1(M)$

could compute moments $\mathbb{E}(\# \mathrm{Sur}(\pi_1(M), B))$.

Sawin - Wood '23 Moment Problem work to describe a distribution from these averages

Turned out much easier to study $(\pi_1(M), [M] \in H_3(\pi_1(M); \mathbb{Z}))$.

This is a pair $(G, c \in H_3(G; \mathbb{Z}))$

Moments of pairs $\mathbb{E}(\# \mathrm{Sur}((\pi_1(M), [M]), (B, c)))$.

$\rightsquigarrow$ gives dist. of $(\pi_1(M), [M])$.

Some properties of gps are prob 0.

The followings never occur:

Ex If $\pi_1(M) \curvearrowright V$ sympl. irrep. $/\mathbb{F}_q$, q odd

$\dim H^1(\pi_1(M); V)$ is even

Then characterize $\overline{\{\pi_1(M)\}}$

Moral: Prob 0 $\rightsquigarrow$ Prove never occurs

Back to function fields

realize need to use fundamental class (AV duality class)

Function field obtain distribution w/ any # of roots of unity
as $q \to \infty$

When probs are 0, can prove never occur over function fields or number fields.