Geometric Representations of affine W-algebras

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0. introduction

$\mathfrak{g}$ simple Lie algebra $\rightsquigarrow$ $\hat{\mathfrak{g}}$ affine Lie algebra

$\rightsquigarrow$ $\mathcal{W}$ affine W-algebra (principal)

$\mathcal{W}$ arises in quite a few contexts (e.g. quantum geometric Langlands)

h.w. reps are closely related to reps of quantum grps.

h.w. reps are fairly well understood:

1. (Feigin-Frenkel, FKW, Arakawa)

$$\Psi^- : \hat{\mathfrak{g}}\text{-mod}^{h.w.} \longrightarrow \mathcal{W}\text{-mod}^{h.w.}$$

1. t-exact
2. Standard/Verma $\rightarrow$ Standard/Verma
3. simple $\longrightarrow$ simple or zero

2. $\Psi^+ : \hat{\mathfrak{g}}\text{-mod} \rightarrow \mathcal{W}\text{-mod}$

① sends KM vertex algebra to $\mathcal{W}$

② slightly more complicated

(FKW): can show $\Psi^+\left(\begin{matrix}\text{certain}\\ \text{simple h.w.}\\ \text{modules}\end{matrix}\right) \longrightarrow$ zero or shifts of simples

3. also have a geometric picture of reps

G s.c. group, $\mathrm{Lie}(G) = \mathfrak{g}$

$\mathcal{O} = \mathbb{C}[[t]]$, $\mathcal{F} = \mathbb{C}((t))$.

the regular block at integral level

affine flag variety

$$\mathcal{W}\text{-mod}^{h.w.}_0 \simeq \text{D-mod}(\mathring{I}, \psi \backslash G_{\mathcal{F}}/I)$$

character of $\mathring{I}$ pulled back from generic char of N

$$\begin{tikzcd} \mathring{I} \arrow[r] \arrow[d, "\text{Iwahori}"'] & I \arrow[r] \arrow[d] & G_{\mathcal{O}} \arrow[d, "ev_{t=0}"] \\ N \arrow[r, hook] & B \arrow[r, "\text{Borel}"] & G \end{tikzcd}$$

$$\simeq \text{D-mod}(N, \psi \backslash G/B) \underset{D(B\backslash G/B)}{\otimes} D(I \backslash G_{\mathcal{F}}/I).$$

Grothendieck group:

$$\mathrm{sgn} \underset{\mathbb{Z}W_{fin}}{\otimes} \mathbb{Z}W_{aff}.$$

What about other nilp. conj. classes $G\cdot e \in \mathfrak{g}$? $\rightsquigarrow$ W_e

again for applications, one wants a good understanding of $W_e\text{-mod}^{\text{h.w.}}$

(e.g. Gaiotto conjectures, CL, BFGT, TY, ...)

1. recall / reformulate some facts for finite W-algebras

basic picture of nilp orbits: have a Harish-Chandra "cusp. form":

$e \in \mathfrak{g}$, $\mathfrak{l}$ minimal Levi containing it,

$\mathfrak{l} \subsetneq \mathfrak{g}$, control things $(\mathfrak{g}, e)$ via $(\mathfrak{l}, e)$ via induction.

if $\mathfrak{l} = \mathfrak{g}$, e distinguished (analogs of cuspidal reps).

e.g. $\mathfrak{g} = \mathfrak{sl}_n$ only principal nilpotent conj. class

$$\sum \#\text{ dist in } \mathfrak{sl}_n \cdot q^n = \frac{1}{1-q}$$

Symplectic groups: $\sum \#\text{ dist in } \mathfrak{sp}_{2n}\ q^n = \dfrac{1}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots}$

$$\sum \mathfrak{so}(n) - \frac{(1-q^2)(1-q^4)\cdots}{(1-q)(1-q^3)\cdots}$$

Q. What do we mean by $W\text{-mod}_e^{\text{h.w.}}$, finite?
(e dist)

A. just means f.d. rep.

$f \quad h \quad e$ (dist)

$\mathfrak{g}^{<0} \quad \mathfrak{g}^0 \quad \mathfrak{g}^{>0}$ h-grading.

Prop (Bezrukavnikov - Braverman - Mirković)

$$D\text{-mod}(U^-,\psi\backslash G/P) \xrightarrow[\sim]{!\text{-fiber at identity}} \mathrm{Vect}$$

i.e. "only object on LHS is "e^ψ" on big cell $U^-P \hookrightarrow G$.

analog.:

$$\mathrm{Fun}\left(U^-(\mathbb{F}_q),\psi\backslash G(\mathbb{F}_q)/P(\mathbb{F}_q)\right)$$

1-dim'l

$$\mathrm{Fun}(P\backslash G(\mathbb{F}_q)/P)$$

i.e. $\exists!$ irred $G(\mathbb{F}_q)$ $\mathcal{H}(p) \hookrightarrow \mathrm{Fun}(G/P)$

$$\begin{array}{c}\mathcal{H}(p)\\ \downarrow\\ \mathrm{Fun}(G/U^-,\psi)\end{array}$$

$$D(G/U^-,\psi,st) := D\text{-mod}(G/P) \otimes_{D(P\backslash G/P)} D(P\backslash G/U^-,\psi)$$