Riemann-Hilbert correspondence

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Lecture 1.

Exp. I (Overview, historical)

Exp. II (on the notion of sing. point of DE)

Exp. III (Gen. result, Key points, strategy of proofs)

R-H correspondence.

19 th century Riemann

20 th century Grothendieck

<u>Def.</u> (Top. covering) X top space. $f: Y \to X$ is a top. covering if it is a bundle with <u>discrete fiber</u>.

Suppose X diff. / complex mfd., $\rightsquigarrow$ str. lifts to Y.

<u>Thm.</u> (Riemann, 1850's) $X/\mathbb{C}$ is a smooth alg. curve, $f: X' \to X$ is a top. cover

$\Rightarrow X'$ is alg. and $f: X' \to X$ is algebraic. (Riemann used Dirichlet problem.)

<u>1950</u>

Cartan - Serre . local analytic geometry.

<u>Thm</u> (Grant and Remmert 1958)

X is smooth alg. var., then any topological covering is smooth.

(didn't use anything about Dirichlet problem, but only local analytic geometry.)

Grothendieck <u>1955</u>

$f: Y \to X$ <u>algebraic covering</u> ?

1959 May $\Rightarrow$ <u>182 Bourbaki</u>.

<u>Def</u>. (182 Bourbaki)

$f: Y \to X$ morphism between locally noetherian scheme, of finite type,

is <u>algebraic covering</u> if it is étale + <u>finite</u>.

étale : local condition on Y. $y \in Y$, $x = f(y)$, $\mathcal{O}_x \to \mathcal{O}_y$ flat, <u>unramified</u>.

① $\mathcal{O}_y \cdot m(x) = m(y)$

② $k(x) \to k(y)$ finite separable.

finite : U affine on X, $f^{-1}(U)$ affine on Y, $\mathcal{O}(f^{-1}(U))$ finite over $\mathcal{O}(U)$.

$$x \in X, \quad n(x) := \sum_{y \in f^{-1}(x)} \dim_{k(x)} k(y) < +\infty .$$

<u>Thm</u> : $f: Y \to X$ is a covering $\Leftrightarrow$ $n(x)$ is locally constant.

$f: Y \to X$ étale topology $\Rightarrow H_{ét}^{\cdot}(X)$

étale + finite $\Rightarrow \pi_1(X, x)$.

SGA I LN244
Exp 13 – [□ GAGA X scheme $/\mathbb{C}$, locally of finite type
□ Riemann existence of covering.
↑
after SGA IV, 63-64

$X \longrightarrow X^{an}$

$X' \to X.$

Thm. X scheme (locally of finite type, as always) over $\mathbb{C}$, Alg. Cov. (X)

(*) Alg. Cov. $(X) \longrightarrow$ Top. Cov. (X^{an})

(*) : equiv. of cat.

fully faithful : easy

essentially surjective : X local Zariski
- reduced
- normal
- Smooth

$\longrightarrow$ Hironaka's thm.

Thm $X/\mathbb{C}$ smooth alg. var. G finite ab. group.

$$H^i_{\acute{e}t}(X, G) \xrightarrow[\text{SGA 4}]{\sim} H^i(X(\mathbb{C}), G)$$

Artin fibration.

Riemann 1826 – 1866) < 40

Grothendieck 1928 – 14/11/2014, 86

De Rham Cohomology comparison thm and Riemann's existence for local system.

Thm. (Grothendieck 1963) $X/\mathbb{C}$ smooth alg. var. X^{an}

$$H_{DR}(X/\mathbb{C}) := \mathbb{H}(X, \Omega^{\cdot}_{X/\mathbb{C}}) \xrightarrow{\sim} H_{DR}(X^{an}/\mathbb{C}) := \mathbb{H}(X^{an}, \Omega^{\cdot}_{X^{an}/\mathbb{C}})$$

X affine

$X \subset \bar{X} \supset Z$ Hironaka compactification

1964 Deligne <u>Paris</u> $\mathcal{M}$ locally free $\mathcal{O}_X$-modules with ∇

LN <u>163</u> $\mathcal{M}$ $\mathcal{O}_X - \nabla$, $\mathcal{E}$

$X \subset \bar{X}$ $\mathcal{L}(*\bar{x} \setminus X)$

$\underbrace{\hspace{10em}}$

Local cohomology and $\mathcal{D}_X$-mod. theory.

<u>1972</u> Sato, Malgrange PDE

<u>1971</u>.

<u>Thm</u>. (Hamm(?), Lihmann(?)) $X \supset Y$, $\mathcal{O}_X(*Y) \subset \mathcal{D}'_X$

<u>Thm</u>. (Bernstein, 1972) $f \in \mathbb{C}[x_1, \cdots, x_n]$,

$\mathbb{C}[x_1, \cdots, x_n, \frac{1}{f}]$ - $D_{\mathbb{C}[x_1, \cdots, x_n]}$-mod. of finite type.

<u>Thm</u>. (1974-75) $\mathcal{O}_X(*Y) = \mathcal{D}_X$-mod. coherent, holonomic.

<u>Thm</u> (Grothendieck 63) $Y \subset X \supset U$. $DR(\mathcal{O}_X(*Y)) \xrightarrow{\sim} Rj_* j^{-1} DR(\mathcal{M}(*Y))$. <u>Perverse sheaf</u>.

<u>Thm</u> (1975) <u>Kashiwara</u>. $\mathcal{M}$ $\mathcal{D}_X$-module, <u>holonomic</u>, $DR(\mathcal{M})$ is constructible - perverse sheaf.

<u>Thm</u> (1975). $\mathcal{M}$ $DR(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{M})$

$Sol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$

$\mathcal{M} = \mathcal{D}_X$-mod., holonomic. $DR(\mathcal{M}) \xrightarrow{\sim} R\mathcal{H}om_{\mathbb{C}_X}(Sol(\mathcal{M}), \mathbb{C})$

Used Poincaré duality & Serre duality

$$\begin{array}{ccc} D^b_h(\mathcal{D}_X) & \longrightarrow & D^b_c(\mathbb{C}_X) \\ \uparrow & & \uparrow \\ \text{Sato school} & & \text{Grothendieck school.} \end{array}$$

complete intersection

$Y \subset X$

Thm. $DR(R_{alg}\Gamma_Y(\mathcal{O}_X))$
$\xrightarrow{\sim} DR(R\Gamma_Y(\mathcal{O}_X))$.

DR. Sol.

$Sol(R_{alg}\Gamma_Y(\mathcal{O}_X)) \xrightarrow{\sim} \mathbb{C}_Y$

regularity condition. $\quad D^b_{hr}(\mathcal{D}_X) \xrightarrow[Sol]{DR} D^b_c(\mathbb{C}_X)$

equiv. of cat.

"t-structure"

$M_{hr}(\mathcal{D}_X) \longrightarrow Perv(\mathbb{C}_X)$.

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Lecture 2

Singular points of DE.

§1. Local

$\mathbb{Q} \subset k = \bar{k}$, $K = k((x))$, $\quad MLC(K) = \{ M, K\text{-v.sp.}, \dim_K M < \infty, \nabla : M \to M \}$

M $\mathcal{D}_K$-mod, $\quad D_K = K[\frac{d}{dx}]$

$e = (e_1, \cdots, e_m)$, $\quad m = \dim_K M$.

$A \in M_m(K)$

$M \simeq D_K / D_K P$, $\quad P = \frac{d^m}{dx^m} + a_{m-1} \frac{d^{m-1}}{dx^{m-1}} + \cdots + a_0$, $\; a_j \in K$

regularity. $\quad \dim_K M = 1$, $\quad M = D_K / D_K(\frac{d}{dx} + a)$, $\quad \underset{\substack{\uparrow \\ \text{valuation}}}{\mathcal{O}}(a) \geq -1$

Thm (Fuchs, 1870) $M \in MLC(K)$.

① M_j irr. factor of M $\Rightarrow \dim_K M_j = 1$, M_j is regular.

② $M \simeq D_K / D_K P$, $o(a_{m-1}) \geq -1, \cdots, o(a_0) \geq -m$

③ M, c. G $\exists H \in GL(K)$, $(x \frac{d}{dx} H + HG) H^{-1} \in M_m(k) \subset M_m(K)$

④ $\dim_k \mathrm{Hom}_{D_K}(M, \Psi_0(K)) = \dim_K M$.

$$\Psi_0(K) = \{ \text{finite sum } \sum a_{\alpha j} x^{\alpha} (\log x)^j \}$$

M regular $\Leftrightarrow$ ① $\Leftrightarrow$ ② $\Leftrightarrow$ ③ $\Leftrightarrow$ ④

Def. M. $\mathrm{irr}_0(M) := \mathrm{Max}\left(0, -\left(o(a_j) + m - j\right)\right) \geq 0$. (Fuchs number)

M is reg. $\Leftrightarrow$ $\mathrm{irr}_0(M) = 0$.

Thm (Turrittin 1954) M $k((x)) \longrightarrow k((\sqrt[q]{x}))$.

irr. factor of $M(\sqrt[q]{x})$ has rank 1.

2. Complex situation, $k = \mathbb{C}$.

$K = \mathcal{O}_D(* 0)$ ③ replace by ③': $H \in M_m(\mathcal{O}_D(* D))$.

Index Theorem. $D \subset \mathbb{C}, 0$, $D^* = D \setminus \{0\}$.

$\mathcal{D}_D$-mod. $\mathcal{M}$ hol. $\mathcal{D}_D$-coh. $\mathcal{M}_{D^*}$. $\mathcal{O}_{D^*}$-loc. fun.

Thm. (Malgrange 1972) $\mathcal{M}$, $\mathcal{D}_D$-hol.

① $\dim_{\mathbb{C}} \mathrm{hom}_{\mathcal{D}_D}(\mathcal{M}, \mathcal{O}_D)_x < +\infty$, $x \in D$

$\mathcal{E}xt^1_{\mathcal{D}_D}(\mathcal{M}, \mathcal{O}_D)_x < +\infty$

② $\mathrm{hom}_{\mathcal{D}_D}(\mathcal{M}, \mathbb{C}[[x]]) < +\infty$, $\mathcal{E}xt^1_{\mathcal{D}_D}(\mathcal{M}, \mathbb{C}[[x]]) < +\infty$

③ $\hom_{\mathcal{D}_0}(\mathcal{M}, \mathbb{C}[[x]]/\mathbb{C}\{x\}) < \infty$, $\mathcal{E}xt^1(\mathcal{M}, \mathbb{C}[[x]]/\mathbb{C}\{x\}) = 0$

④ $\dim_k(\quad \downarrow \quad) = \mathrm{irr}_0(\mathcal{M})$.

$\dim X = 1$, $\quad Y \subset X \supset U$.

$MLC(\mathcal{O}_X(*Y)) \quad y \in Y, \quad \mathrm{irr}_y(M) = 0$.

$\underset{\substack{\uparrow \\ \text{regular}}}{MLR}(\mathcal{O}_X(*Y))$

<u>Thm</u> (21st Hilbert problem).

$$MLR(\mathcal{O}_X(*Y)) \xrightarrow[\sim]{DR} Loc(U)$$

equiv. of cat.

<u>Bolibruch Thm. 1994</u>

$X = \mathbb{P}^1_{\mathbb{C}} \quad U, Y$, alg. set.

$MLC(\mathcal{O}_U) \quad e_1, \cdots, e_m$

$G \in M_m(\mathcal{O}(U))$.

$\underset{\substack{\uparrow \\ \text{simple pole}}}{MSP}(\mathcal{O}_U) \subset MLR(\mathcal{O}_U)$

, G has only simple pole

$rk \geq 3$, the inclusion is strict!

<u>§3</u>. p-adic local thy. 1995.

$\dim X \geq 2$.

$X/\mathbb{C}$ smooth alg. $\quad$ many problems are local of Zariski topology

$\rightsquigarrow X$ affine. $\quad \bar{X} \subset \mathbb{P}^N \longrightarrow \mathbb{P}^{N\,an} \xrightarrow{GAGA}$ come back

X = analytic mfd. $= X^{an}$

$\mathcal{D}_X$-mod Theory (4 main point)

① Constructibility

X, $\mathcal{F}$ $\mathbb{C}_X$-sheaf $\Leftrightarrow$ $\forall x \in X$, $\dim_{\mathbb{C}} \mathcal{F}_x < \infty$, $X = \bigcup_{i \in I} X_i$ whitney strat.

$\mathcal{F}|_{X_i}$ locally constant.

$\rightsquigarrow D^b_c(X)$ triangulated cat

$D^b_c(\mathbb{C}_X)$ étale SGA 4, exp 9.

Verdier 1974

holonomy.

X, $\mathcal{O}_X$ coh. sheaf (Oka), $\mathcal{D}_{X/\mathbb{C}} = \mathcal{D}_X$.

$M_{coh}(\mathcal{D}_X)$ $D^b_{coh}(\mathcal{D}_X)$

$M \in \mathrm{Mod}(\mathcal{D}_X)$ $Ch(M) \subset T^*X$, $n \leq \dim Ch(M) \leq 2n$

M is holonomic if $\dim Ch(M) = n$.

t-str. $\begin{cases} Mh(\mathcal{D}_X) & \text{abelian.} \\ D^b_h(\mathcal{D}_X) & \text{hol. } M \quad h^i(M) \in Mh(\mathcal{D}_X) \end{cases}$

$DR(M) = \mathbb{R}hom_{\mathcal{D}}(\mathcal{O}_X, M)$

$Sol(M) = \mathbb{R}hom_{\mathcal{D}_X}(M, \mathcal{O}_X)$.

$M \in D^b_h(\mathcal{D}_X) \Rightarrow DR(M), Sol(M) \in D^b_c(\mathbb{C}_X)$

③ Duality.

$DR(M) \simeq Rhom_{\mathbb{C}_X}(Sol(M), \mathbb{C}_X)$

Def $F \in D^b_c(\mathbb{C}_X)$. F is a perverse sheaf

① $h^i(F) = 0$ if $i \notin [-\dim X, 0]$

② $\dim \operatorname{supp}(h^i(F)) \leq \dim X - i$

③ co. supp. $F^\vee$

Cor. $M \in Mh(\mathcal{D}_X)$, $DR(M)$, $Sol(M)$ a perv. sheaf.

$$\begin{tikzcd} Mh(\mathcal{D}_X) \arrow[r, "{DR, S}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Perv(\mathbb{C}_X) \arrow[d] \\ D^b_h(\mathcal{D}_X) \arrow[r, "{DR, S}"] & D^b_c(\mathbb{C}_X) \end{tikzcd}$$

④ Regularity.

Local coh.

$Z \subset T$, $R_{alg.}\Gamma_Z(F) := R \varinjlim_k hom_{\mathcal{O}_T}(\mathcal{O}_T/\mathcal{J}^k_Z, F) \longrightarrow R\Gamma_Z(F)$.

$M \in D^b_q(\mathcal{D}_X)$ $\quad R_{alg}\Gamma_Y(M) \to M \to RM(*Y) \xrightarrow{+1}$

$$\begin{tikzcd} DR(R_{alg}\Gamma_Y(M)) \arrow[r] \arrow[d] & DR(M) \arrow[r] \arrow[d, equal] & DR(RM(*Y)) \arrow[r, "+1"] \arrow[d, "a_Y"] & {} \\ R\Gamma_Y DR(M) \arrow[r] & DR(M) \arrow[r] & Rj_*j^{-1}DR(M) \arrow[r, "+1"] & {} \end{tikzcd}$$

$M = \mathcal{O}_X \Rightarrow a_Y(M)$ is isom.

Def M is regular w.r.t. Y if $a_Y(M)$ is an isom.

M is regular if $a_Y(M)$ is isom. $\forall Y$.

Dévissage and resolution of singularity, 1970's

① $D^b_{hr}(\mathcal{D}_X) \subsetneq D^b_h(\mathcal{D}_X)$

② $D_{hr}(\mathcal{D}_X) \neq 0$. $\mathcal{O}_X$ regular.

③ $\mathcal{N} \subset \underset{\text{reg.}}{\overset{?}{\mathcal{M}}} \overset{?}{\Rightarrow} \mathcal{N}$ regular

④ $\mathcal{M}$ reg. $\overset{?}{\Rightarrow} \mathcal{M}^*$ reg.

$D^b_{hr}(\mathcal{D}_X)$ triangulated.

M.V. sequence. $\mathcal{M} \in D^b_h(\mathcal{D}X)$

$a_Y(\mathcal{M})$ iso. $\forall$ Y hypersurface $\Rightarrow \mathcal{M}$ regular.

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Lecture 3

X, Y $\quad \mathcal{M} \in D^b_h(\mathcal{D}_X)$, $\quad U \overset{j}{\subset} X \overset{i}{\supset} Y$

$DR(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$.

$Sol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$.

$F^\vee = R\mathcal{H}om_{\mathbb{C}_X}(F, \mathbb{C}_X)$.

$$\begin{tikzcd}
DR(R_{alg}\Gamma_Y(\mathcal{M})) \arrow[r] \arrow[d] & DR(\mathcal{M}) \arrow[r] \arrow[d, equal] & DR(R\mathcal{M}(*Y)) \arrow[r, "+1"] \arrow[d, "a_Y(\mathcal{M})"] & {} \\
R\Gamma_Y(DR(\mathcal{M})) \arrow[r] & DR(\mathcal{M}) \arrow[r] & Rj_*j^{-1}DR(R\mathcal{M}(*Y)) \arrow[r, "+1"] & {}
\end{tikzcd}$$

Duality

$$\begin{tikzcd}
{} & i^{-1}(S(\mathcal{M})) \arrow[l, "+1"'] & S(\mathcal{M}) \arrow[l] & j_!j^{-1}S(R\mathcal{M}(*Y)) \arrow[l] \\
{} & S(R_{alg}\Gamma_Y(\mathcal{M})) \arrow[l, "+1"'] \arrow[u] & S(\mathcal{M}) \arrow[l] \arrow[u, equal] & S(R\mathcal{M}(*Y)) \arrow[l] \arrow[u, "b_Y(\mathcal{M})"']
\end{tikzcd}$$

$\mathcal{M} = \mathcal{O}_X \quad 0 \leftarrow \mathbb{C}_Y \leftarrow \mathbb{C}_X \leftarrow j_!\mathbb{C}_U \leftarrow 0$

Poincaré lemma

$\Updownarrow$

Grothendieck comp Thm. $\quad \mathbb{C}_Y \xrightarrow{\sim} S(R_{alg}\Gamma_Y(\mathcal{O}_X))$

$D^b_{hr}(\mathcal{D}_X) \subset D^b_h(\mathcal{D}_X)$

$\boxed{\text{IRREGULARITY}}$

$$DR(RM(*Y)) \to Rj_* j^{-1} DR(RM(*Y)) \to \underbrace{\mathrm{Cone}(a_Y(M))}_{\text{irregularity}} \xrightarrow{+1}$$

$$R\Gamma_Y DR(RM(*Y)) \to R\Gamma_Y Rj_* j^{-1} DR(RM(*Y)) \to$$
$$\hphantom{R\Gamma_Y DR(RM(*Y)) \to R\Gamma_Y R} \| $$
$$\hphantom{R\Gamma_Y DR(RM(*Y)) \to R\Gamma_Y R} 0$$

$$\mathrm{Cone}(a_Y(M)) \xrightarrow{\sim} R\Gamma_Y(DM(*Y))[1]$$

Def (X, Y, M). $\quad Irr_Y(M) := R\Gamma_Y DR\, RM(*Y)$

$\quad Irr_Y^*(M) := i^{-1} S(RM(*Y))$

Thm. (MIRACLE, March 1986)

(X, Y, M) $\quad Y$ hypersurface. $\quad M \in M_h(\mathcal{D}_X)$.

$Irr_Y(M)$, $Irr_Y^*(M)$ are perverse sheaves.

dim $X = 1$. $\quad Irr_Y(M) \quad y \in Y$

$hom_{\mathcal{D}}(M(*Y), \mathcal{O}_X)|_Y = 0$

$\mathcal{E}xt^1_{\mathcal{D}}(M(*Y), \mathcal{O}_X)_y = Irr_y(M) \leftarrow$ Fuchs number

Cor $\quad 0 \to M_1 \to M \to M_2 \to 0 \quad \mathcal{D}_X$-mod

$\rightsquigarrow 0 \to Irr_Y(M_1) \to Irr_Y(M) \to Irr_Y(M_2) \to 0$.

$\Rightarrow$ the cat. of regular $\mathcal{D}$-modules is abelian.

$\mathcal{M} \in D^b_h(\mathcal{D}_X)$, $\mathrm{Irr}_Y(\mathcal{M}) = 0 \iff \mathrm{Irr}_Y(h^i(\mathcal{M})) = 0$.

$\mathcal{M} \to \mathcal{M}^* = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \otimes_{\mathcal{O}_X} \Omega_X [\dim X]$

$\mathrm{Irr}_Y(\mathcal{M})$ is <u>not</u> $\mathrm{Irr}_Y(\mathcal{M}^*)$

$\mathcal{M} \in D^b_h(\mathcal{D})$, $\chi(\mathrm{Irr}_Y(\mathcal{M})) = \chi(\mathrm{Irr}_Y(\mathcal{M}^*))$
$\forall Y$

$\mathrm{Irr}_Y(\mathcal{M}) = 0 \iff \chi(\mathrm{Irr}_Y(\mathcal{M})) = \chi(\mathrm{Irr}_Y(\mathcal{M}^*)) = 0 \iff \mathrm{Irr}_Y(\mathcal{M}^*) = 0$

$\mathcal{M}$ is smooth on U

$U \subset X \supset Y$

$CCh(\mathcal{M}(*Y)) - 2CCh(\mathcal{O}_X(*Y)) \in T^*_X$

≥ 0

$\mathrm{Irr}_Y(\mathcal{O}_X)$ $\qquad$ $\mathrm{Supp}(\ ) \subset \mathrm{Sing}(Y)$

Thm $(X, Y, \mathcal{M})$ $\quad$ $\mathcal{M}$ is smooth on U

$\dim \mathrm{supp}(\mathrm{Irr}_Y(\mathcal{M})) < \dim Y \Rightarrow \mathrm{Irr}_Y(\mathcal{M}) = 0$

Cor. Grothendieck $\mathrm{Irr}_Y(\mathcal{O}_X)$.

$\dim X = 1$. $\quad \mathrm{Irr}_Y(\mathcal{M}) = 0$, ϕ

$\dim X = 2$ $\quad \dim \mathrm{supp}\, \mathrm{Irr}_Y(\mathcal{M}) < 1$ $\quad \mathcal{E}xt^2_{\mathcal{D}}(\mathcal{M}(*Y), \mathcal{O}_X)_y$, $y \in Y$

$n \geq 3$ $\quad$ singularity Y $\quad \pi_1(U, *) \longrightarrow GL(\mathbb{C}^M)$

$n = 2$, $\pi_1(U, *)$ complicated

$n \geq 3$.

$f : X \to X$ $\qquad M \in D^b_{hr}(\mathcal{D}_X) \Rightarrow f^*_{dR} M^+ \in D^b_{hr}(\mathcal{D}_X)$

D^b_{hr} $\quad f_* f^*$ $\qquad f^{-1} S(M) \to S(f^*_{dR} M)$

$\Leftrightarrow S(M_1, M_2)$ $\quad R\,hom_{\mathcal{D}_X}(M_1, M_2) \Rightarrow R\,hom_{\mathbb{C}_X}(S(M_2), S(M_1))$

$\Rightarrow$ DR, S: fully faithful

essentially surjectivity is a problem of local analytic geometry

$\mathcal{O}_X$-mod and not of $\mathcal{D}_X$-mod

$\dim X = n$

X. F $\mathcal{O}_X$-coh

$x \in X$, $dp(F_x) = \dim X - pd(F_x) \in [0, \dim X]$

$m \geq 0$, $S_m(F) := \{x \in X : dp(F_x) \leq m\}$.

Thm. $m \geq 0$. $S_m(F)$ closed subset $\subset X$. $\dim S_m(F) \leq m$.

Thm (Trautman, Frisch, Guenot Siu, 1969)

$U \subset X \supset Z$ $\qquad S_k$, $S_k(F) \leq k-2$

$F_U = \mathcal{O}_U$-coh. $\qquad k \leq \dim Z + 2$

$j_* F_U$ - coh.