

Differential operators on the base affine space and quantized Coulomb branches

Harold Williams

The base affine space

G = simple alg. grp / \mathbb{C} , B Borel.

$$U = [B, B], \quad T = B/U$$

Ex. $G = SL_n$.

G/U - base affine space, * fundamental object in repr theory.

- G/U is a quasi-affine variety. i.e. the map $G/U \rightarrow \text{Spec } \mathbb{C}[G/U] = \overline{G/U}$
is open embedding.

- The multiplicative action of $G \times G \curvearrowright G$ descends to an action

$$G \times T \curvearrowright G/U.$$

- G/U is a T -bundle over the (projective) flag variety G/B .

$$G/U \rightarrow G/B \simeq (G/U)/T.$$

- As a G -rep, $\mathbb{C}[G/U] = \bigoplus$ all simple G -reps.

Ex. $G = SL_2$, $\{\text{simple } SL_2\text{-reps}\} = \{\text{Sym}^n \mathbb{C}^2\}$

$$\leadsto \mathbb{C}[SL_2/U] \simeq \bigoplus_n \text{Sym}^n \mathbb{C}^2$$

$$\leadsto \overline{SL_2/U} \simeq \mathbb{C}^2 \quad \text{and} \quad SL_2/U \simeq \mathbb{C}^2 \setminus \{0\}.$$

$$(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^* \simeq \mathbb{P}^1 = SL_2/B.$$

Warning: $G = SL_2$ is the only case where $\overline{G/U}$ is smooth.

- Equally fundamental is the ring $D_{\hbar}(G/U)$ of differential operators on G/U .
- Gelfand - Graev: hidden Weyl group action on $D_{\hbar}(G/U)$

Ex. $D_{\hbar}(SL_2/U) \cong T^*(x, y, \partial_x, \partial_y, \hbar) / \langle x\partial_x - \partial_x x = \hbar, \dots \rangle$

\hat{S}_2 acts via Fourier transform

$$x \mapsto \partial_y, \quad y \mapsto -\partial_x.$$

- action for general G is generated by partial Fourier transforms, but relations are mysterious.

Coulomb branches

- $J = 3d \quad N=4 \quad \text{QFT}$

\leadsto Coulomb branch $\mathcal{M}_c(J)$

- parametrizes certain vacua of J .

need to define various expectation values on \mathbb{R}^3

- affine variety w/ natural quantization.

$$\mathbb{C}_{\hbar}[\mathcal{M}_c(J)]$$

- Ex. G as before, N a repn of G .

\leadsto 3d $N=4$ gauge theory $T_{G,N}$

BFN: $\mathbb{C}_{\hbar}[\mathcal{M}_c(T_{G,N})] = H_{*}^{\hbar \times \mathbb{C}^*}(R_{G,N})$ the equivariant BM homology of a space $R_{G,N}$ w/ a convolution structure.

Van der

- Here $G_0 = G[[t]]$, w/ \mathbb{C}^\times acting by scaling t

- $R_{G,N}$ lies over the affine Grassmannian $Gr_G = G_K / G_0$

and for $g \in G_K$, the fiber over $[g]$ is $N[[t]] \cap g N[[t]] \subset N(\mathbb{C}(t))$.

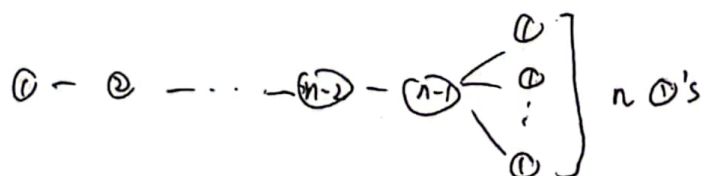
Ex. Q a quiver, w/ function $d: \{\text{vertices}\} \rightarrow \mathbb{N}$

$$G_Q = \left(\prod_v GL_{d(v)} \right) / \mathbb{C}^\times$$

$$N_Q = \bigoplus_{v \rightarrow w} \text{Hom}(\mathbb{C}^{d(v)}, \mathbb{C}^{d(w)}).$$

- we call J_{G_Q, N_Q} a quiver gauge theory

Thm (Casson-W.) let J_n be the gauge theory associated to



Then there is an isom. $C_h[\mathcal{M}_c(J_n)] \cong D_h(SL_n/U)$,

which identifies the Gelfand-Graev action w/ the \hat{G}_n action induced by permuting the right-hand vertices.

Cor. (proposal of Dargatzis-Hamann-Kirwan) There is an algebraic symplectic isomorphism

$$\overline{T^*(SL_n/U)} \cong \mathcal{M}_c(J_n).$$

Generalization

- let (n_1, \dots, n_k) be an ordered tuple of natural numbers s.t. $n_1 + \dots + n_k = n$.

These tuples are in bijection w/ functions $\psi: \{e_1, \dots, e_n\} \rightarrow \{0, 1\}$.

(e.g. ψ constantly 0 \hookrightarrow $(1, \dots, 1)$)
 ψ constantly 1 \hookrightarrow (n) .

any such ψ extends to a Lie algebra character $\psi \in \mathfrak{u}^*$.

The action $U \curvearrowright SL_n$ induces a Hamiltonian action $U \curvearrowright T^*SL_n$

w/ moment map $\mu: T^*SL_n \rightarrow \mathfrak{u}^*$

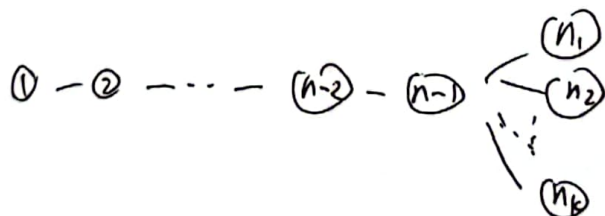
Given $\psi \in \mathfrak{u}^*$, we have the Hamiltonian reduction

$$T^*SL_n //_{\psi} U = \mu^{-1}(\psi) / U$$

Ex. $T^*SL_n //_0 U \cong \overline{T^*(SL_n/U)}$

Thm. Let (n_1, \dots, n_k) and ψ be as above,

and \mathcal{I}_{ψ} the gauge theory of



Then there is an isomorphism

$$T^*SL_n //_{\psi} U \cong \mathcal{M}_c(\mathcal{I}_{\psi})$$

key ingredients . the regular sheaf $A_{\text{reg}}^G \in P^{G_0}(Gr_G)$ corresponding to the regular representation $\mathbb{C}[G^v]$ of G^v under the geometric Satake equiv.

- Lusztig - Riche : $D_*(G^v/U^v)$ is the $T_0 \times \mathbb{C}^*$ -equiv. cohomology
 of $i^! A_{\text{reg}}^G$, where $i: Gr_T \hookrightarrow Gr_G$.

- BFN: $\Lambda_{\text{reg}}^{\text{pHLn}} \cong \pi_* \omega_R$, where R is associated to $\textcircled{1} - \textcircled{2} \dots - \textcircled{n-1} - \textcircled{n}$
 \uparrow
 dualizing complex

- Macratis: generalization of GR to the case where T is replaced by a Len' .