Relative Langlands & Endoscopy.

F local or global field.

G reductive grp $/F$. $\check{G}$ Langlands dual.

$\Rightarrow \check{G} \rtimes \mathrm{Gal}_F = {}^L G$.

Functoriality: ${}^L H \to {}^L G$

$\Rightarrow$ F local, $\mathrm{Rep}(H(F)) \to \mathrm{Rep}(G(F))$

global $\mathrm{Rep}(H(\mathbb{A})) \to \mathrm{Rep}(G(\mathbb{A}))$

Basic examples.

H = trivial, Reciprocity

$\mathrm{Gal}_F \to {}^L G$ $\rightsquigarrow$ param. of reps of G

$s \in \check{G}^{ss}$, $\check{H} := \mathrm{Cent}_{\check{G}}(s)^\circ$.

Endoscopy: $H \dashrightarrow G$

endoscopy group

F global: fundamental in stabilization of trace formula for G

$$\underset{G(F)\text{-invariant}}{TF_G} = \sum_{\substack{\text{endoscopy}\\ H/\sim}} \underset{\text{geometrically invariant}}{STF_H}$$

Relative Langlands

(BZSV) extend $G \dashrightarrow \check{G}$

to $(G, M) \dashrightarrow (\check{G}, \check{M})$

Hamiltonian G-space $\quad\quad$ $\check{G}$-space

Arithmetic POV

$$H \subset G \longrightarrow X = \underset{\text{Dual group } \check{G}_X}{\overset{\text{spherical}}{G/H}} \longrightarrow M = T^*(X)$$

<u>Rep theoretically</u>.

(π, V) a G-rep that is M-distinguished

F - local $\Leftrightarrow$ $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0$.

F global $\Leftrightarrow$ Period integrals

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi_\pi(h) \neq 0, \quad \varphi_\pi \in \pi$$

(S-V): π, V M-dist. $\Leftrightarrow$

$$\begin{tikzcd} \mathrm{Gal}_F \arrow[r, "\varphi_\pi"] \arrow[dr, dashed] & {}^L G \\ & \check{G}_X \rtimes \mathrm{Gal}_F = {}^L G_X \arrow[u] \end{tikzcd}$$

& $L(s_0, \pi, X) \neq 0$

<u>Ex.</u> (Friedberg-Jacquet)

$G = GL_{2n}$, $H = GL_n^2$. $X = G/H$

(π, V) $T^*(X)$-distinguished $\Leftrightarrow$

$$\begin{tikzcd} \mathrm{Gal}_F \arrow[r] \arrow[dr, dashed] & GL_{2n} \\ & Sp_{2n} = \check{G}_X \arrow[u] \end{tikzcd}$$

& $L(\frac{1}{2}, \pi) \neq 0$

Conjectural $\check{M}$ is built from
- $\check{G}_X \times SL_2 \to \check{G}$
- S_X symplectic repn of $\check{G}_X$

In <u>Ex</u>.
- $Sp_{2n} \to GL_{2n}$
- $S_X = T^*(\mathbb{C}^{2n})$

In relative duality, relative functoriality

M_H - dist rep H $\rightsquigarrow$ M - dist rep of G

$\Rightarrow$ Lagrangian corresp.

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ \check{M}_H & & \check{M} \end{tikzcd}$$

Arithmetic Problem

$\check{M}$ does not determine $X = G/H$ up to isomorphism.

Ex. E/F quad. extension

$X_1 = GL_{2n}/GL_n^2$ $\qquad L(\frac{1}{2}, \pi)^2$

$X_2 = GL_{2n}/\mathrm{Res}_{E/F}(GL_n)$ $\qquad L(\frac{1}{2}, BC_E(\pi))$

G-outer forms

Losev. $\qquad X = G/H$ spherical

computes $\mathrm{Aut}^G(X)$
$\qquad = N_G(H)/H$

$\exists$ $\mathrm{Aut}^G(X) \to \mathrm{Aut}^{Geo}(X) = \prod \mu_2$

canonical quotient

encodes the action of $\varphi : X \to X$ on Borel orbits of X.

$X = GL_{2n}/GL_n^2$. $\quad \mathrm{Aut}^{Geo}(X) = \mu_2$

$\varphi \mapsto -1$ if φ permutes the colors of X.

$n = 1$. $\quad GL_2/GL_1^2 = \overset{\circ}{X} \sqcup D_1 \sqcup D_2$

Thm Let $X = G/H$ over F $\Rightarrow$ $\mathrm{Gal}_F \curvearrowright X(\bar{F})$

① $\exists$ a canonical 2-cocycle $c_X : \mathrm{Gal}_F \longrightarrow \mathrm{Aut}^{\mathrm{geo}}(X)(\bar{F})$

s.t. If $X_{\bar{F}}$ is "well-adapted"

$(T^*(X)$ hyper spherical $\Rightarrow X$ well-adapted$)$

& X_1, X_2 are 2 F-forms of $X_{\bar{F}}$,

then $\underset{G/H_1}{\overset{X_1}{\shortparallel}}$ is a G-inner form of $X_2 = G/H_2$

$\Leftrightarrow [c_{X_1}] \neq [c_{X_2}]$ in $H^1(F, \mathrm{Aut}^{\mathrm{geo}}(X))$

② Assume $X = G/G^\theta$, $\theta^2 = 1$, then if T^*X is hyperspherical, then

the data $c_X : \mathrm{Gal} \longrightarrow \mathrm{Aut}^{\mathrm{geo}}(X)$ is equivalent to

- a symplectic $\check{G}_X$-rep S_X
- compatible Gal_F-action on S_X.

$\quad\quad \downarrow$

$\quad G_X \curvearrowright S_X$.