Beilinson's notion of an f-category

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(Smooth) Variety X (points, lines ...)

$\Downarrow$

Coherent sheaves $\mathrm{Coh}(X)$

$\Downarrow$

Derived category $D^b(X)$

+ enhancement

Question: How much information about X survives in $D^b(X)$?

Answer: quite a few:

K-theory $K_\bullet(X)$

Hodge-de Rham sp. seq.

Thm. (Bondal-Orlov) 1995

K_X $(-K_X)$ ample $\Rightarrow$ X can be recovered from $D^b(X)$.

Hope for application to what MMP.

1. DG-categories

$$D^b(X) \cong D(A^\bullet)$$

$A^\bullet$ - DG algebra (associative)

Keep $A^\bullet$, prove that everything depends on $A^\bullet$ "only up to quasi isom."

2. Derivator approach

I small category ("a diagram")

$$\begin{array}{ccc} \mathcal{A} \text{ abelian category} & \Rightarrow & D(\mathcal{A}) \\ \Downarrow & & \not\Downarrow \\ \underset{\text{abelian}}{\mathrm{Fun}(I, \mathcal{A})} & \Rightarrow & D(\mathrm{Fun}(I, \mathcal{A})) \end{array}$$

$$D(\mathrm{Fun}(I, \mathcal{A})) \to \mathrm{Fun}(I, D(\mathcal{A}))$$

— not an equiv.

cf. Grothendieck "Pursuing stacks".

a derivator: $\mathbb{D}_I$ for every I (such as $D(\mathrm{Fun}(I, \mathcal{A}))$).

$I_1 \to I_2 \rightsquigarrow \mathbb{D}_{I_2} \to \mathbb{D}_{I_1}$ and natural compatibilities.

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f-categories in the sense of Beilinson

$\boxed{n}$ $\bullet \to \bullet \to \bullet \cdots \to \bullet$

$$D(\mathrm{Fun}(\boxed{n}, \mathcal{A})) = DF_{[0,n-1]}(\mathcal{A}).$$

<u>Def</u>. An f-category is

$(\mathcal{D}, F, s)$

$\mathcal{D}$ – triangulated cat,

$F: \mathcal{D} \to \mathcal{D}$ fully faithful + some conditions

$s: F \to \mathrm{Id}$

e.g. $\mathcal{D} = \mathcal{D}(\mathrm{Fun}(\mathbb{N}, \mathcal{A})) \cong DF_{\geq 0}(\mathcal{A})$

F: renumbering the filtration

$s: F \to \mathrm{Id}$

M, $W_\bullet M$

$$\begin{tikzcd} W_i FM = W_{i-1} M \arrow[dr, "s"'] \arrow[d, phantom, "\cap" rotate=-90] \\ & W_i M \end{tikzcd}$$

Given $(\mathcal{D}, F, s)$, we recover

$$\mathcal{D}_{\boxed{n}} \quad (\simeq DF_{[0,n-1]}(\mathcal{A}))$$

$\forall f : \boxed{n} \to \boxed{m}$, we can recover

$$f^* : \mathcal{D}_{\boxed{m}} \to \mathcal{D}_{\boxed{n}}$$

So, we get simplicial category.

$$\Delta^{opp} \longrightarrow Cat$$
$$\boxed{n} \longmapsto \mathcal{D}_{\boxed{n}}$$
$$f : \boxed{n} \to \boxed{m} \longmapsto f^*$$

Filtered objects in an ab. cat. $\mathcal{A}$ also form a simplicial category.

$$\boxed{\begin{array}{c} M_0 \overset{a}{\hookrightarrow} M_1 \\ \downarrow \\ M_0,\ M_1,\ M_1/M_0 = \operatorname{coker} a \end{array}}$$

Waldhausen's S-construction:

$S\mathcal{A}$ simpl. cat.

$$\boxed{n} \longmapsto F_{n-1}\mathcal{A}.$$

$$\begin{array}{c} M_0 \to M_1 \to \cdots \to M_j \to \cdots \to M_i \\ \downarrow \\ M_j / M_{j-1} \end{array}$$

Moreover, we have a simplicial cat. $S\mathcal{D}$ s.t. $S\mathcal{D} : \boxed{n} \to \mathcal{D}_{\boxed{n-1}}$.

$\mathcal{D}_{\boxed{2}}$

$$M_0 \xrightarrow{a} M_1 \to \mathrm{Cone}(a) \to M_0[1] \to M_1[1] \to \mathrm{Cone}(a)[1] \to$$

$T : \mathcal{D}_{\boxed{n}} \circlearrowleft$ s.t. $T^{n+1}(M) \simeq M[2]$

$\Delta \subset \Lambda_\infty$ (a version of Conne's cyclic category Λ).

$S\mathcal{D}$ extends to an Λ_∞-category.

<u>Def</u> (Waldhausen)

$S A : \Delta^{opp} \to \mathrm{Cat}.$

$\overline{SA} : \Delta^{opp} \to \mathrm{Groupoid}$

$K.(A) = \pi.(|\overline{SA}|)$

<u>Def</u> $\langle \mathcal{D}, F, s \rangle$

$S\mathcal{D} : \Delta^{opp} \to \mathrm{Cat}$

$\overline{S\mathcal{D}} : \Delta^{opp} \to \mathrm{Groupoid}$

$K.(\mathcal{D}) = \pi.(|\overline{S\mathcal{D}}|)$

<u>Thm 1</u>. Assume we have some t-structure on $\mathcal{D}_{\boxed{1}}$ with heart A.

Then $K.(A) \xrightarrow{\sim} K.(\mathcal{D})$

<u>Thm 2</u>. $\mathcal{D}_{\boxed{1}} \subset \mathcal{D}'_{\boxed{1}}$ localizing

$K.(\mathcal{D}) \to K.(\mathcal{D}') \to K.(\mathcal{D}'/\mathcal{D})$