Working notes for Alamo development

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Canonical derivation of FEM for reference

Solve

$$\mathbb{C}_{ijkl} u_{k,lj}(x) + b_i(x) = 0 \tag{1}$$

Discretization

$$u_i(x) = u_i^n \phi^n \qquad\qquad b_i(x) = b_i^m \phi^m \tag{2}$$

Substitute

$$\mathbb{C}_{ijkl} u_k^n \phi^n_{,lj} + b_i^m \phi^m = 0 \tag{3}$$

Weak form

$$\int_\Omega a^p \phi^p \mathbb{C}_{ijkl} u_k^n \phi^n_{,lj} dx + \int_\Omega a^p \phi^p \phi^m b_i^m dx = 0 \tag{4}$$

Integration by parts, factor out constants

$$a^p u_k^n \int_\Omega \phi^p_{,j} \mathbb{C}_{ijkl} \phi^n_{,l} dx + a^p b_i^m \int_\Omega \phi^p \phi^m dx = 0 \tag{5}$$

$\forall a^p$ to get locality

$$\underbrace{\Big(\int_\Omega \phi^p_{,j} \mathbb{C}_{ijkl} \phi^n_{,l} dx \Big)}_{K_{ik}^{pn}} u_k^n + \underbrace{\Big(\int_\Omega \phi^p \phi^n dx \Big)}_{M^{pm}} b_i^m = 0 \tag{6}$$

Stiffness matrix:

$$K_{ik}^{pn} = \mathbb{C}_{ijkl} \int_\Omega \phi^p_{,j} \phi^n_{,l} dx \tag{7}$$

In two dimensions – coordinate change $x^n \to 0$. **Require** $\Delta x = \Delta y = \Delta$

$$\phi^n = \frac{1}{\Delta^2} \begin{cases} (x_1 - \Delta)(x_2 - \Delta) & 0 < x_1 < \Delta, 0 < x_2 < \Delta \\ -(x_1 + \Delta)(x_2 - \Delta) & -\Delta < x_1 < 0, 0 < x_2 < \Delta \\ -(x_1 - \Delta)(x_2 + \Delta) & 0 < x_1 < \Delta, -\Delta < x_2 < 0 \\ (x_1 + \Delta)(x_2 + \Delta) & -\Delta 0 < x_1 < 0, -\Delta < x_2 < 0 \end{cases} \tag{8}$$

For $p = n$:

$$\int_\Omega \phi^p_{,i} \phi^n_{,j} dx = \begin{cases} 4/3 & i = j \\ 0 & i \neq j \end{cases} \tag{9}$$

For p to east or west of n

$$\int_\Omega \phi^p_{,i}\phi^n_{,j}dx = \begin{cases} -2/3 & i=j=1 \\ 1/3 & i=j=2 \\ 0 & \text{else} \end{cases} \tag{10}$$

For p to north or south of n

$$\int_\Omega \phi^p_{,i}\phi^n_{,j}dx = \begin{cases} 1/3 & i=j=1 \\ -2/3 & i=j=2 \\ 0 & \text{else} \end{cases} \tag{11}$$

For p to northeast / southwest of n

$$\int_\Omega \phi^p_{,i}\phi^n_{,j}dx = \begin{cases} -1/6 & i=j \\ -1/4 & i\neq j \end{cases} \tag{12}$$

For p to northwest / southeast of n

$$\int_\Omega \phi^p_{,i}\phi^n_{,j}dx = \begin{cases} -1/6 & i=j \\ 1/4 & i\neq j \end{cases} \tag{13}$$

Assuming linear isotropic plane strain:

$$\mathbb{C}_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\,\delta_{ij}\delta_{kl} \tag{14}$$

Substituting into stiffness matrix:

$$K^{pn}_{ik} = \mu\,\delta_{ik}\int_\Omega \phi^p_{,j}\phi^n_{,j}dx + (\mu+\lambda)\int_\Omega \phi^p_{,i}\phi^n_{,k}dx = \mu\,\delta_{ik}(\Phi^{pn}_{11} + \Phi^{pn}_{22}) + (\mu+\lambda)\Phi^{pn}_{ik} \tag{15}$$

Relative location	$i=1, k=1$	$i=1, k=2$	$i=2, k=1$	$i=2, k=2$
$n=p$	$(2\mu+\lambda)(4/3)$			