Sliding Mode Control of a Two-Degree-of-Freedom Helicopter via Linear Quadratic Regulator

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Abstract - *This paper presents a novel design of flight control for a two-degree-of-freedom helicopter. The design is composed of the optimal Linear Quadratic Regulator (LQR) and the sliding mode control. The LQR is applied to control the pitch and yaw motion of the helicopter first. The sliding mode controller is employed to guarantee the robustness. Both simulation and experiment results demonstrate the proposed design possesses the optimal performance and robustness against external disturbance.*

Keywords: Sliding Mode Control, Linear Quadratic Regulator, Two-Degree-of-Freedom Helicopter.

1 Introduction

Helicopters are very important conveyances and used extensively to rescue disasters. Helicopters exhibit high levels of agility and maneuverability such as “climbing”, ”hovering”, and ”forward flight” [1]. A helicopter with superior performance is needed. With its high agility and performance, the dynamics of a helicopter are unstable and nonlinear. A nonlinear dynamic model is developed for a coaxial helicopter in hover condition [2]. Therefore, it is important to study the pitch and yaw control of a helicopter. In view of this, a more effective helicopter flight control strategy is developed in this paper to increase the stability and performance.

The robustness of a controller is quite crucial for the implementation. Sliding mode control has been applied to many engineering fields due to excellent robustness, for example, motor control, robotic control and flight control [3]. However, it is not easy to adapt the dynamics of the reaching phase in the design of sliding mode control [4]. To improve the transient response, it is necessary to shift the position of poles.

The linear quadratic (LQ) optimization has been utilized in the field of aerospace engineering. The LQ technology designs an optimal controller that minimizes a given performance index. The performance index is parameterized using state-weighting matrix Q and control-weighting matrix R. If the linear time-invariant system is controllable, the optimal control law will be obtained via solving the algebraic Riccati equation. The Kalman gain will be a constant matrix only [5]. After choosing the weighting matrices, the feedback gain is easily designed such that the required specifications are secured. Nevertheless, the robustness of the optimal gain is not satisfactory as similar as that of the sliding mode control. To take advantage of the robustness of sliding mode control and the facile implementation of LQR, it is necessary to put the concept of LQR design into the structure of sliding mode control. The advantages of this strategy are the required specifications could be easily met and the robustness is guaranteed [6]. Therefore, the technology is applied to the flight control of an experimental helicopter in this study.

In this paper, the sliding mode control via LQR design is submitted to track the pitch and yaw attitudes of a two-degree-of-freedom helicopter. Furthermore, the proposed technique is carried out through a personal computer, two power modules, a data acquisition card and a terminal board [7]. The merits of this control strategy are not only the optimal performance could be obtained but the robustness is guaranteed also. The optimal response of the two-degree-of-freedom helicopter is achieved by taking advantage of LQR design. On the other hand, sliding function could ensure the robustness of the two-degree-of-freedom helicopter against external disturbance.

2 The mathematical model

The mathematical model of a helicopter includes the propeller dynamics and the aerodynamic force. The pitch propeller is actuated by DC motor whose speed is controlled by way of input voltage V_p. The revolving speed causes to create a force acts normally to the helicopter fuselage at a distance R_p form the pitch axis. The revolving of propeller however also causes to create load torque T_p on the rotating motor observed at yaw axis (parallel axis theorem). Therefore the rotation of pitch propeller does not only affect about motion of the pitch axis, but also the yaw axis. At the same time, the yaw motor creates a force F_y to the helicopter fuselage at distance R_y from yaw axis, moreover, there is a torque T_y to the pitch axis.

The equations of motion of the two-degrees-of-freedom helicopter are given as follows

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$$J_{pp}\ddot{P} = R_p F_p + G_p(T_y) + Fg(p) \quad (1)$$

$$J_{pp}\ddot{y} = R_y F_y + G_y(T_p) \quad (2)$$

where "p" is the pitch angle relative to the horizontal axis, "y" is the yaw angle. The "F_p" and "F_y" are the forces produced by the propellers and are functions of V_p and V_y. The "R_p" is the horizontal interval between the center of mass and the pivot point. The "R_y" is the vertical interval between the center of mass and the pivot point. The "T_p" and "T_y" are torques at the propeller axes and also are the function of V_p and V_y. The "G_p" and "G_y" are the nonlinear function of the coupling. The "J_{pp}" and "J_{yy}" are the moment of inertia of the fuselage about the pitch and yaw axes.

The nonlinear dynamic equations (1) and (2) of the experimental helicopter could be linearized as a state-spaces model:

$$\begin{bmatrix} \dot{p} \\ \dot{y} \\ \ddot{p} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} p \\ y \\ \dot{p} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L\dfrac{K_{ff}}{J_{pp}} & L\dfrac{K_{tb}}{J_{pp}} \\ L\dfrac{K_{tf}}{J_{yy}} & L\dfrac{K_{fb}}{J_{yy}} \end{bmatrix} \times \begin{bmatrix} V_p \\ V_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G_d \\ 0 \end{bmatrix} \quad (3)$$

where V_p is the pitch motor voltage and V_y is the yaw motor voltage. The symbol "K_{ff}" and "K_{tf}" are the constants of front motor. The symbol "K_{tb}" and "K_{fb}" are the constants of back motor. These parameter values are listed in Table 1. The symbol "G_d" means the gravitational disturbance constant. Taking notice of a positive pitch voltage, it does not only result in a pitch but also a negative yaw.

To reduce the effects of the gravitational disturbance constant "G_d", the integrators have to be joined in the loop. Two new states α and ζ are defined as the integrations of pitch and yaw angles respectively. The new state-space model is derived as follows:

$$\begin{bmatrix} \dot{p} \\ \dot{y} \\ \ddot{p} \\ \ddot{y} \\ \dot{\alpha} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} p \\ y \\ \dot{p} \\ \dot{y} \\ \alpha \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ K_{pp} & K_{py} \\ K_{yp} & K_{yy} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} V_p \\ V_y \end{bmatrix} \quad (4)$$

Table 1. Parameters of the experimental helicopter

Symbol	Value	Unit
J_{pp}	0.0307	Kg-Meter^2
J_{yy}	0.0307	Kg-Meter^2
K_{ff}	0.8722	Newton/Voltage
K_{tf}	0.0200	Newton-Meter/Voltage
K_{fb}	0.4214	Newton/Voltage
K_{tb}	0.0100	Newton-Meter/Voltage
L	0.4064	Meter

3 Sliding mode control via LQR

To meet the desired specifications, the controller is designed using LQR methodology first [8]. Let the cost function be defined as

$$J = \frac{1}{2}\int_0^\infty \left(x^T Q x + u^T R u\right) dt \quad (5)$$

where Q is the weighting function of states and R is the weighting function of control variables.

To minimize the cost function, let the Hamiltonian

$$H = \frac{1}{2}\left(x^T Q x + u^T R u\right) + \lambda^T (Ax + Bu) \quad (6)$$

The necessary conditions for optimality are

$$\begin{cases} \dfrac{\partial H}{\partial u} = 0 = Ru + B^T\lambda \\ \dot{x} = \dfrac{\partial H}{\partial \lambda} = Ax + Bu \\ \dot{\lambda} = -\dfrac{\partial H}{\partial x} = -\left(Qx + A^T\lambda\right) \end{cases} \quad (7)$$

The optimal control input is solved in terms of the Lagrange multipliers λ and the linear homogeneous differential equations are yielded

$$u^*(t) = -R^{-1}B^T\lambda(t)$$
$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad (8)$$

After some calculations, the Lagrange multipliers can be written as

$$\lambda(t) = S(t)x(t) \quad (9)$$

where $S(t)$ satisfies the Riccati equation

$$\dot{S}(t) = -A^T S(t) - S(t)A + S(t)BR^{-1}B^T S(t) - Q \quad (10)$$

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Kalman has shown that if the system matrix (A, B) is controllable, $S(t)$ is a constant matrix. The optimal control law is stationary, that is, the Kalman gain is a constant matrix. In this paper, the optimal control input is written as

$$\begin{aligned} u^*(t) &= -R^{-1}B^T Sx(t) \\ &= -kx(t) \\ &= -\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \end{bmatrix} x(t) \end{aligned} \tag{11}$$

where the constant Kalman gain is acquired by choosing the appropriate weighting matrices

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_6 \end{bmatrix} \tag{12}$$

$$R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{13}$$

Then substituting (11) into (4)

$$\dot{x} = (A - Bk)x = A_P x \tag{14}$$

In the following, the sliding mode control is involved to make sure the robustness of the closed-loop system. Consider a linear time-invariant system as follow:

$$\dot{x} = Ax + Bu + d(x,t) \tag{15}$$

where the system states are $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$, control inputs are $u = [u_1 \quad u_2 \quad \cdots \quad u_m]^T$, $d(x,t)$ are the disturbance function and B is a full rank matrix.

Sliding functions have m numbers are defined as follows:

$$s_i(x) = c_i x = c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n, i = 1,2,...,m \tag{16}$$

where $c_i = [c_{i1} \quad c_{i2} \quad \cdots \quad c_{in}]$ is a row vector. Thus, the sliding vector is presented as follow:

$$s(x) = Cx \tag{17}$$

where $s = [s_1 \quad \cdots \quad s_m]^T$, $C \in R^{m \times n}$, and

$$\det(CB) \neq 0 \tag{18}$$

Form (15) and (17), the differentiated sliding vector is obtained

$$\dot{s} = C\dot{x} = CAx + CBu + Cd(x,t) \tag{19}$$

Since $\det(CB) \neq 0$, thus the control rule is designed as follow:

$$u = -(CB)^{-1}CAx - (CB)^{-1}(\gamma + \sigma) \cdot \frac{s}{\|s\|} \tag{20}$$

where $\gamma = \|C\|\delta(x,t)$.

Substituting u into (19) gets

$$\dot{s} = -(\gamma + \sigma) \cdot \frac{s}{\|s\|} + Cd(x,t) \tag{21}$$

Then, multiply s^T on both sides

$$\begin{aligned} s^T\dot{s} &= -\sigma\|s\| - \gamma \cdot \|s\| + s^T Cd(x,t) \\ &= -\sigma\|s\| - \gamma \cdot \|s\| \left(1 - \frac{s^T Cd(x,t)}{\gamma \cdot \|s\|}\right) \\ &< -\sigma\|s\| \end{aligned} \tag{22}$$

where $\gamma = \|C\|\delta(x,t) > \|Cd(x,t)\|$.

In order to assure the sliding condition

$$s^T\dot{s} < 0 \tag{23}$$

is satisfied, the constant σ should be positive. Therefore, the system will enter into the sliding mode $s = 0$ in a limited time. That is, the closed-loop system is stable such that the state error will be convergent.

To design the sliding mode controller via LQR, consider the sliding function [9]

$$s_T(x) = C(x - x(0)) - CA_P \int_0^t x d\tau \tag{24}$$

Obviously, $s_T(x)$ must be equal to zero. That is, there is a sliding surface in the feedback system. It is not necessary to consider whether the system can satisfy the approaching condition or not. The constant vector C is chosen to satisfy $CB \neq 0$.

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The states must be held on the sliding surface such that the feedback system is robust. Accordingly, an extra energy $-qsign(s_T)$ is added to the control effort. The total control energy will make the feedback system to satisfy the sliding condition $s_T \dot{s}_T < 0$ under external disturbance.

The total control energy u_T of sliding mode control with LQR is designed as

$$u_T = u^* - qsign(s_T) \tag{25}$$

From (4) and (25), gets

$$\dot{s}_T(x) = -qsign(s_T) \tag{26}$$

Then, multiply s_T on both sides

$$s_T \dot{s}_T = -q|s_T| \tag{27}$$

The constant q should be positive in order to satisfy the sliding condition

$$s_T \dot{s}_T < 0 \tag{28}$$

If the external disturbance d exists, (26) will be modified as

$$s_T \dot{s}_T = -q|s_T| + ds_T < 0 \tag{29}$$

The choice of q is $q > |d|$.

4 Computer simulation

According to equation (11), the Kalman gain could be acquired by choosing the state and control energy weighting matrices

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{31}$$

Thus, the optimal control gain K is obtained:

$$K = \begin{bmatrix} 0.196 & -0.002 & 0.177 & -0.001 & 0.095 & -0.0005 \\ 0.003 & 0.130 & 0.003 & 0.129 & 0.001 & 0.0302 \end{bmatrix} \tag{32}$$

The parameters of sliding mode control via LQR are

$$C = \begin{bmatrix} 0.6 & -2.4 & 0.1756 & 0.0205 & 3 & -0.1 \\ 0 & 2.5 & 0.0410 & 0.3634 & -1.5 & 2.5 \end{bmatrix} \tag{33}$$

and $q = 10$. In order to avoid the high gain and the chattering phenomena, the concept of the sliding layer is introduced in the control law [10]. Thus the saturation function $sat(s_S, \varepsilon)$ is

$$sat(s_S, \varepsilon) = \begin{cases} 1 & s_S > \varepsilon \\ s_S/\varepsilon & |s_S| \le \varepsilon \\ -1 & s_S > -\varepsilon \end{cases} \tag{34}$$

where the parameter ε is set as 0.01.

To demonstrate the effectiveness of the proposed methodology, two different conditions were discussed. These cases are system at nominal condition and the existence of disturbance respectively. Let the command be two step signals, one is the 45 degrees of pitch angle and the other is 15 degree of yaw angle. Figure 1 and Figure 2 show the step responses of pitch and yaw attitudes of the two-degree-of-freedom helicopter at nominal condition by two different controllers. Figure 3 and Figure 4 show the step responses of the helicopter subjected to the external disturbances during the 40^{th} sec to the 60^{th} sec, which the amplitude is equal to 2.5.

From Figure 1(a) and Figure 2(a), it is obvious that the settling time of the pitch and yaw angle is tardy in the case of LQR control. The overshoot of the pitch and yaw attitudes is small in the case of sliding mode control via LQR. Figure 3(a) and Figure 4(a) show the responses of pitch and yaw are shaken violently when external disturbance occurs in the case of LQR control. Apparently, the sliding mode control via LQR owns the optimal performance and excellent robustness.

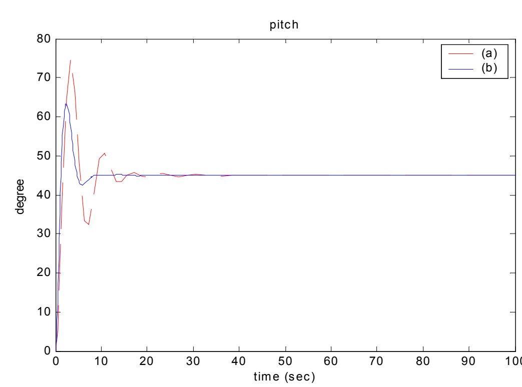

Figure 1. Simulated pitch responses at nominal condition: (a) LQR control (b) Sliding mode control via LQR

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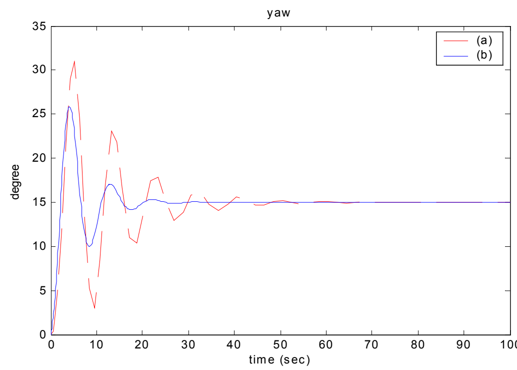

Figure 2. Simulated yaw responses at nominal condition: (a) LQR control (b) Sliding mode control via LQR

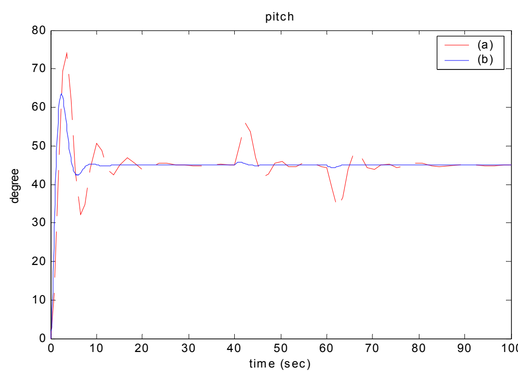

Figure 3. Simulated pitch responses under disturbance: (a) LQR control (b) Sliding mode control via LQR

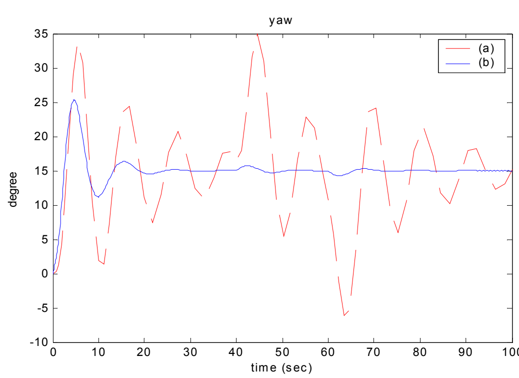

Figure 4. Simulated yaw responses under disturbance: (a) LQR control (b) Sliding mode control via LQR

5 Experimental results

In this study, an experimental helicopter is used as the platform to demonstrate the effectives of sliding mode control via LQR. The two-degree-of-freedom helicopter is made in Quanser Company of Canada. The components include a helicopter plant on a fixed base shown in Figure 5, two power modules (Universal Power Module UPM2405 and UPM1503), a data acquisition card (MultiQ-PCI data acquisition card) and a terminal board (MultiQ-PCI Terminal Board).

The MultiQ-PCI data acquisition card installed on computer supports 48 I/O channels. The MultiQ-PCI terminal board supports 4 analog output, 16 analog input, 6 encoder input and 48 digital I/O channels. The MATLAB Simulink is used as the interface for operation. Digital command signals are transmitted to the MultiQ-PCI terminal board via the MultiQ-PCI data acquisition. These signals are converted to two kinds of signals (analog and encoder). The analog signals are transmitted to the Universal Power Module. The encoder signals are transmitted to the helicopter plant. At the same time, the command signals are conveyed to the base of the helicopter plant from the Universal Power Module. Then the computer can receive the signals by way of the MultiQ-PCI terminal board and the MultiQ-PCI data acquisition, and the cycle of the experiment system is completed.

Applying the different control strategy to the Simulink window, the experimental results are obtained. Let the command be two step signals, one is the 45 degrees of pitch angle and the other is 15 degree of yaw angle. The helicopter is subjected to a sinusoidal disturbance during the 40th sec to the 60th sec, which the amplitude is equal to 1 and the frequency is 20 rad/sec. Figure 6 shows the experimental responses for the pitch angle and yaw angle of the two-degree-of-freedom helicopter using LQR controller only. Figure 7 shows the pitch and yaw attitudes of the two-degree-of-freedom helicopter using sliding mode control with LQR design.

From Figure 6(a) and Figure 7(a), it is obvious that the rising time of the pitch angle is fast in the case of sliding mode control via LQR. On the other hand, the settling time of the yaw angle is slow in the case of LQR control only. Apparently, the sliding mode control via LQR design possesses the excellent robustness against the external disturbance. The yaw response cannot meet the desired command in the case of LQR control under external disturbance.

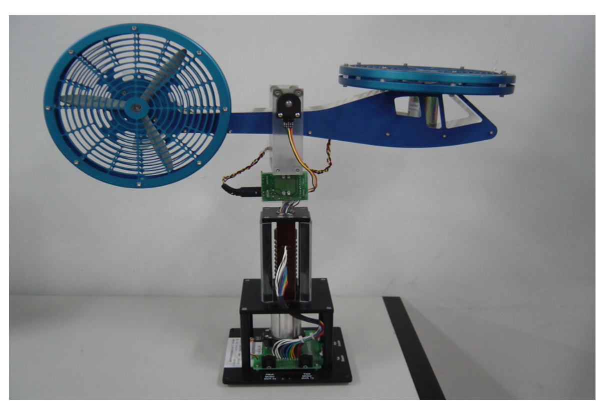

Figure 5. The helicopter plant

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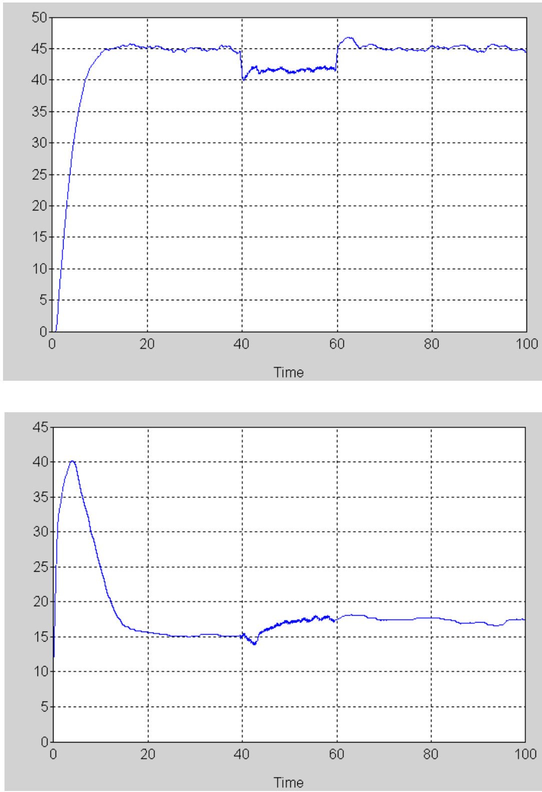

Figure 6. Experimental results using LQR controller under external disturbance: (a) Pitch responses (b) Yaw responses.

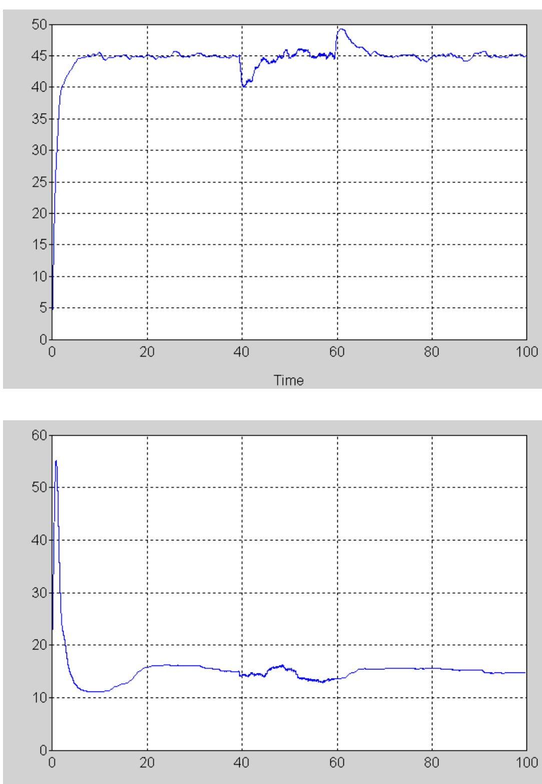

Figure 7. Experimental results using sliding mode control via LQR under disturbance: (a) Pitch angle (b) Yaw angle.

6 Conclusions

The design method of sliding mode control via LQR has been presented in this paper. Simulation results demonstrate the optimal performance and excellent robustness against external disturbance. Furthermore, the proposed control law is implemented on a personal computer to command the pitch and yaw angle of a two-degree-of-freedom helicopter. The experimental results have shown the effectiveness of the control strategy.

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