Technische Univesität Dresden
Computational Tools for QMBP

Exercise Sheet 4: One dimensional XXZ spin chain

1 Mapping to spinless fermions and qualitative phase diagram

$$H = -J\sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right) + \Delta \sum_j S_j^z S_{j+1}^z \tag{1}$$

We then apply the Jordan-Wigner transformations.

$$\sigma_j^z = 2c_j^\dagger c_j - 1, \quad \sigma_j^+ = F_1 \dots F_{j-1} c_j^\dagger, \quad \sigma_j^- = F_1 \dots F_{j-1} c_j \tag{2}$$

For the term with S^x, this will look as follows

$$\begin{aligned}\left(S_j^x S_{j+1}^x\right) = \left(\sigma_j^+ + \sigma_j^-\right)\left(\sigma_{j+1}^+ + \sigma_{j+1}^-\right) = \left[\sigma_j^+\sigma_{j+1}^+ + \sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+ + \sigma_j^-\sigma_{j+1}^-\right] = \\ \left[c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} + c_j c_{j+1} + c_j c_{j+1}\right]\end{aligned} \tag{3}$$

And for S^y, it will look as

$$\begin{aligned}\left(S_j^y S_{j+1}^y\right) = (-1)\left(\sigma_j^+ - \sigma_j^-\right)\left(\sigma_{j+1}^+ - \sigma_{j+1}^-\right) = (-1)\left[\sigma_j^+\sigma_{j+1}^+ - \sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+ + \sigma_j^-\sigma_{j+1}^-\right] = \\ \left[-c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} + c_j c_{j+1} - c_j c_{j+1}\right]\end{aligned} \tag{4}$$

And so, the first term of the Hamiltonian transforms into

$$\sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right) = 2c_j^\dagger c_{j+1} + \mathrm{h.c} \tag{5}$$

Now, transforming the term with S^z, we will get

$$S_j^z S_{j+1}^z = \left(2c_j^\dagger c_j + 1\right)\left(2c_{j+1}^\dagger c_{j+1} + 1\right) \tag{6}$$

So, the Hamiltonian reads

$$H = -J\sum_j^{L-1}\left(2c_j^\dagger c_{j+1} + \mathrm{h.c}\right) + \Delta\sum_j^{L-1}\left(2c_j^\dagger c_j - 1\right)\left(2c_{j+1}^\dagger c_{j+1} - 1\right) \tag{7}$$

which we can transform into

$$H = -\frac{J}{2}\sum_j^{L-1}\left(c_j^\dagger c_{j+1} + \mathrm{h.c}\right) + \Delta\sum_j^{L-1}\left(c_j^\dagger c_j - \frac{1}{2}\right)\left(c_{j+1}^\dagger c_{j+1} - \frac{1}{2}\right) \tag{8}$$

We can check that the Hamiltonian commutes with the total number of particles $N = \sum_j c_j^\dagger c_j$, looking first at just one occupation number n_j

$$\left[\sum_{j=1}^{L-1} c_j^\dagger c_{j+1}, n_j\right] = \left[c_j^\dagger c_{j+1} + c_{j-1}^\dagger c_j, n\right] = \left[c_j^\dagger c_{j+1}, n\right] + \left[c_{j-1}^\dagger c_j, n\right] = -c_j^\dagger c_{j+1} + c_{j-1}^\dagger c_j \tag{9}$$

In a similar fashion, we see that for the *h.c.* term

$$\left[\sum_{j=1}^{L-1} c_{j+1}^\dagger c_j, n\right] = c_j^\dagger c_{j+1} - c_{j-1}^\dagger c_j \quad \Rightarrow \quad \left[\sum_{j=1}^{L-1} c_j^\dagger c_{j+1} + \text{h.c.}, n\right] = 0 \tag{10}$$

And for the second term involving the spin in z

$$\begin{aligned}\left[\left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right), n_j\right] &= [n_j n_{j+1}, n_j] - \frac{1}{2}[n_j, n_j] - \frac{1}{2}[n_{j+1}, n_j] + \left[\frac{1}{4}, n_j\right] \\ &= n_j[n_{j+1}, n_j] + [n_j, n_j]n_{j+1} = 0\end{aligned} \tag{11}$$

So we see that for the total number of particles, we have $[H, N] = 0$. And we can see that this is equivalent for the total spin conservation in the z direction.

$$\begin{aligned}[H, S_z] = -J\left[\sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y, \sum_k S_k^z\right] + \Delta\underbrace{\left[\sum_j S_j^z S_{j+1}^z, \sum_k S_k^z\right]}_{=0} \\ = -J\sum_k\left[\sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y, S_k^z\right] = -J\sum_k \left[S_k^x S_{k+1}^x + S_{k-1}^x S_k^x, S_k^z\right] + \left[S_k^y S_{k+1}^y + S_{k-1}^y S_k^y, S_k^z\right] \\ = -J\sum_k [S_k^x, S_k^z]S_{k+1}^x + S_{k-1}^x[S_k^x, S_k^z] + [S_k^y, S_k^z]S_{k+1}^y + S_{k-1}^y[S_k^y, S_k^z] \\ = -iJ\sum_k\left[-S_k^y S_{k+1}^x - S_{k-1}^x S_k^y + S_k^x S_{k+1}^y + S_{k-1}^y S_k^x\right] = 0\end{aligned} \tag{12}$$

We can see that after the transformation, we can describe the spin conservation in z in a way similar to what we do with the conservation of particles, since there is a direct correspondence between the two operators given by the Jordan-Wigner transformation.

Finally, we can consider the limits given by $\Delta \to \pm\infty$. In this cases we can see that the dominating term in the Hamiltonian is going to be $\Delta \sum_j S_j^z S_{j+1}^z$. The ground states will minimize the Hamiltonian, and thus we see that for the limit $\Delta \to +\infty$, we will have antiparallel spins, resulting in an antiferromagnetic system. And in a similar way, in the limit $\Delta \to -\infty$, we will have parallel spins, resulting in a ferromagnetic phase of the system. For the case $\Delta \to 0$, we will transition into an isotropic system in the xy plane, describing something known as a "Gapless Luttinger Liquid Phase".