Exercise: Collective Dynamics of Human Mobility

Summer term 2024 - TU Dresden

Randomization is too important to be left to chance.

J. D. Petruccelli

Random Walks

Random walks are the simplest model of movement in space. They are an ideal null model for human mobility to identify which patterns and properties emerge from complex dynamics and which are simply the result of the spatial constraints. However, random walks do not always have to represent diffusion with infinitesimally small Gaussian displacements.

Continuous-time random walks describe a general class of stochastic processes where a walker jumps at random times to random new locations. In the simplest one-dimensional case, the waiting time $\Delta t \in [\Delta t_{\text{min}}, \infty)$ between two jumps follows a distribution $\phi(\Delta t)$ and is independent of the displacements. The spatial displacements Δx of the jumps are symmetric around $\Delta x = 0$ and their magnitude $|\Delta x| \in [|\Delta x|_{\text{min}}, \infty)$ follows a distribution $\psi(|\Delta x|)$.

Observations of individual mobility patterns often show power law distributions over a large range of waiting times and displacements. Qualitatively, this is very intuitive: people often make many trips in quick succession but then stay at the same place for a long time and people often make many small trips close to their home but sometimes much longer trips, for example for vacation.

Typically, there will also be relevant correlations between the waiting times and spatial displacements, for example people making many small trips in quick succession but having longer waiting times before large trips.

The general structure of this movement can be modeled by waiting times Δt and displacements $|\Delta x|$ following distributions $\phi(\Delta t)$ and $\psi(|\Delta x|)$, respectively, with exponents of the cumulative distribution $\alpha, \beta \geq 0$ such that

$$\begin{array}{rclcl} \psi(|\Delta x|) & \propto & |\Delta x|^{-1-\alpha} & \text{for} & |\Delta x| \to \infty \\ \phi(\Delta t) & \propto & \Delta t^{-1-\beta} & \text{for} & \Delta t \to \infty \end{array}$$

A continuous time random walk then evolves as follows:

- Start the walker at the origin $x(t=0) = 0$.
- For each jump $k \in \{1, 2, 3 \dots\}$ do the following:
  - Draw the waiting time Δt_k before the k-th jump from the distribution $\phi(\Delta t)$.
  - Draw the displacement Δx_k of the k-th jump with a magnitude drawn randomly from the distribution $\psi(|\Delta x|)$.
  - Advance time by Δt_k and update the position of the walker.

  This is an event-based simulation approach. In contrast to, for example, solving differential equations by advancing time a constant time step Δt in every step, event-based simulations only consider the times when something actually happens in the simulation. This allows arbitrary precision in the timing of events in the simulation and at the same time avoids simulating long times when nothing happens since we just skip forward to the next event. However, if many events occur in a short time, we may not get very far in the simulation and would be better off approximating the state after a fixed, finite time step.

Task

Simulate continuous-time random walks for different combinations of $\alpha \in \{0.5, 1, 1.5, 2, 3\}$ and $\beta \in \{0.5, 1, 1.5\}$ and compute the mean-squared deviation of the position of the walker as a function of time.

Hand in:

- a figure illustrating a non-Gaussian random walk in space in two dimensions with $\alpha = 1.5$ and $\beta = 1$.
- a figure illustrating the mean squared displacement of the random walks in one spatial dimension and its asymptotic scaling for large times t

$$\ln\left[\left\langle x^2 \right\rangle\right] \propto 2\,\frac{\min(\beta, 1)}{\min(\alpha, 2)} \ln[t]$$

No more than 2 pages. Include short captions to explain your figures and tables.

Hints and tips

- You can find more information on random walks and Lévy flights in

  Brockmann et al.,
  The Scaling Laws of Human Travel,
  Nature, 439(7075):462, 2006

- Determine the displacement Δx for a jump by drawing a positive random variable $|\Delta x|$ for the magnitude of the jump and a uniformly random sign (in one dimension) or angle (in two dimensions).
- The exact distributions you choose for the waiting times and displacements should not impact the dynamics (for large times t) as long as the asymptotic behavior follows the power laws with exponents α and β, respectively.

  A simple choice are Pareto-distributions with a lower cut-off at 1

$$\begin{aligned}\psi\left(|\Delta x|\right) &= \begin{cases} 0 & \text{for} \quad |\Delta x| < 1 \\ \alpha\ |\Delta x|^{-1-\alpha} & \text{for} \quad |\Delta x| \geq 1 \end{cases} \\ \phi\left(\Delta t\right) &= \begin{cases} 0 & \text{for} \quad \Delta t < 1 \\ \beta\ \Delta t^{-1-\beta} & \text{for} \quad \Delta t \geq 1 \end{cases}\end{aligned}$$

- For $\alpha \geq 2$ and $\beta \geq 1$ the leading moments of both distributions exists (variance for the spatial displacement, mean for the waiting time), such that the central limit theorem or law of large numbers holds for the two quantities, respectively. In this case, you should recover standard diffusive dynamics

$$\ln\left[\left\langle x^2 \right\rangle\right] \propto 2\,\frac{\min(\beta, 1)}{\min(\alpha, 2)} \ln[t] = \ln[t]$$

malte.schroeder@tu-dresden.de