Fachhochschule Vorarlberg

Master in Mechatronics

Higher Mathematics III

Homework

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1 First Homework

1.1 Linear Least Squares

1.1.1 Script Applied Numerical Computing - Exercises 8.1

Formulate the following problems as least-squares problems. For each problem, give a matrix A and a vector b, such that the problem can be expressed as

$$\text{minimize } \|Ax - b\|^2 .$$

(You do not have to solve the problems.)

(a) Minimize
following Equation

$$x_1^2 + 2 * x_2^2 + 3 * x_3^2 + (x_1 - x_2 + x_3 - 1)^2 + (-x_1 - 4 * x_2 + 2)^2 \tag{1.1}$$

According to the lecture the equation

$$\|Ax - b\|^2 \tag{1.2}$$

can be written as

$$\left\| \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ \vdots \\ a^m \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \right\|^2 = \tag{1.3}$$

$$= (a_{11} * x_1 + a_{12} * x_2 + a_{13} * x_3 - b_1)^2 + (a_{21} * x_1 + a_{22} * x_2 + a_{23} * x_3 - b_2)^2 + \dots \tag{1.4}$$

To find the single parts of the A matrix and the b vector a comparison between the terms of equation 1.1 and the terms of equation 1.4.

$$x_1^2 = (a_{11} * x_1 + a_{12} * x_2 + a_{13} * x_3 - b_1)^2$$

$$a_{11} = 1,\ a_{12} = 0,\ a_{13} = 0,\ b_1 = 0$$

Comparison of the second term from equation 1.1

$$a_{21} = 0,\ a_{22} = \sqrt{2},\ a_{23} = 0,\ b_2 = 0$$

same steps for the rest of the terms leads to following matrix and vector:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \\ 1 & -1 & 1 \\ -1 & -4 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}$$

(b) Minimize
following Equation

$$(-6*x_2+4)^2+(-4*x_1+3*x_2-1)^2+(x_1+8*x_2-3)^2 \tag{1.5}$$

To minimize the equation a factorization with the following equation is necessary:

$$(a_{11}*x_1+a_{12}*x_2+a_{13}*x_3-b_1)^2+(a_{21}*x_1+a_{22}*x_2+a_{23}*x_3-b_2)^2+\dots$$

That leads to a Matrix A and a b of:

$$A=\begin{pmatrix} 0 & -6 & 0 \\ -4 & 3 & 0 \\ 1 & 8 & 0 \end{pmatrix} \quad b=\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

(c) Minimize
following Equation

$$2*(-6*x_2+4)^2+3*(-4*x_1+3*x_2-1)^2+4*(x_1+8*x_2-3)^2 \tag{1.6}$$

The equation can be rewritten as

$$(-6*\sqrt{2}*x_2+\sqrt{2}*4)^2+(-4*\sqrt{3}*x_1+3*\sqrt{3}*x_2-\sqrt{3})^2+(2*x_1+16*x_2-6)^2$$

To minimize the equation a factorization with the following equation is necessary:

$$(a_{11}*x_1+a_{12}*x_2-b_1)^2+(a_{21}*x_1+a_{22}*x_2-b_2)^2+\dots$$

That leads to a Matrix A and a b of:

$$A=\begin{pmatrix} 0 & -6*\sqrt{2} \\ -4*\sqrt{3} & 3*\sqrt{3} \\ 2 & 16 \end{pmatrix} \quad b=\begin{pmatrix} -\sqrt{2}*4 \\ \sqrt{3} \\ 6 \end{pmatrix}$$

2 Second Homework - Least Norm Problems

In todays lecture (October 5^{th} , 2016) an example of a simple control problem was presented. Details can be found in the script of Prof. Vandenberghe (Ex. 1.3 p18 and Example on p150).

Work on the following:

2.1 Compute a solution for the control problem with x_i= 0 for 2 $\leq$ i $\leq$ 9.

System of linear Equation fo control problem:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{19}{2} & \frac{17}{2} & \frac{15}{2} & \frac{13}{2} & \frac{11}{2} & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{19}{2} & \frac{17}{2} & \frac{15}{2} & \frac{13}{2} & \frac{11}{2} & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix} * \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1 + x_{10} = 0 \rightarrow x_1 = -x_{10}$$

$$19 * x_1 + 1 * x_{10} = 2$$

$$-19 * x_{10} + 1 * x_{10} = 2 \rightarrow x_{10} = \frac{-2}{18} = \frac{-1}{9}$$

$$x_1 = -x_{10} = \frac{1}{9}$$

2.2 Compute a solution for the control problem with $x_1 = x_2 = x_3 = x_4 = x_5$ and $x_6 = x_7 = x_8 = x_9 = x_{10}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{19}{2} & \frac{17}{2} & \frac{15}{2} & \frac{13}{2} & \frac{11}{2} & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix} * \begin{pmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \\ x_1 \\ x_{10} \\ x_{10} \\ x_{10} \\ x_{10} \\ x_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$5 * x_1 + 5 * x_{10} = 0 \rightarrow x_1 = -x_{10}$$

$$(19 + 17 + 15 + 13 + 11) * x_1 + (9 + 7 + 5 + 3 + 1) * x_{10} = 2$$

$$-75 * x_{10} + 25 * x_{10} = 2 \longrightarrow x_{10} = -\frac{2}{50} = -\frac{1}{25}$$

$$x_1 = -x_{10} = \frac{1}{25}$$

2.3 Find the "minimal energy"solution with $|x|^2 \rightarrow min$

The Energie can be calculated with following formula:

$$Energie \approx \int_{t=0}^{10} F(t)^2 dt \hat{=} |x|^2 \rightarrow min$$

According to the Theorem:
Solution of the Least Norm Problem is $x = A^T A(AA^T)^{-1}b$ where $A^T A(AA^T)^{-1} = right\, inverse\, of\, A$.

With the MATLAB code:

```
x = A'*((A*A')\[0;1])
```

The x with the minimal energy is:

$$\begin{pmatrix} 0.0545 \\ 0.0424 \\ 0.0303 \\ 0.0182 \\ 0.0061 \\ -0.0061 \\ -0.0182 \\ -0.0303 \\ -0.0424 \\ -0.0545 \end{pmatrix}$$

2.4 Compute the "energy demand" $|x|^2$ of these three solutions

$$|x|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2$$

2.4.1 Solutionx_i= 0 for 2 ≤ i ≤ 9

$$|x|^2 = \left(\frac{1}{9}\right)^2 + \left(\frac{-1}{9}\right)^2 = \frac{2}{81} = 0,0247$$

2.4.2 Solution $x_1 = x_2 = x_3 = x_4 = x_5$ and $x_6 = x_7 = x_8 = x_9 = x_{10}$

$$|x|^2 = 5 * \left(\frac{1}{25}\right)^2 + 5 * \left(\frac{-1}{25}\right)^2 = \frac{10}{625} = 0,0160$$

2.4.3 Solution minimal Energie

```
x = A'*((A*A')\[0;1])
sum(x.^2)
```

$$|x|^2 = 0.0121$$