Ayudantía 12
Intro a Estadística

1) X: una plancha sea declarado de 2° clase
$X \sim Bernoulli(\theta)$
$\mathbb{P}(X=x) = \theta^{x} \cdot (1-\theta)^{1-x}$

a) Buscamos $\hat{\theta}_{EMV}$:

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} \left(\theta^{x_i} \cdot (1-\theta)^{1-x_i}\right)$$
$$= \theta^{\sum x_i} \cdot (1-\theta)^{n-\sum x_i}$$

$$\ell(\theta|\underline{x}) = \sum x_i \cdot \log\theta + (n - \sum x_i) \log(1-\theta) \quad / \frac{d}{d\theta}$$

$$\frac{d\,\ell(\theta|\underline{x})}{d\theta} = \frac{\sum x_i}{\hat{\theta}} - \frac{(n-\sum x_i)}{1-\hat{\theta}} = 0$$
$$= \frac{n\bar{x}}{\hat{\theta}} - \frac{n-n\bar{x}}{1-\hat{\theta}} = 0$$
$$\frac{n\bar{x}}{\hat{\theta}} = \frac{n-n\bar{x}}{1-\hat{\theta}}$$
$$n\bar{x} - \cancel{n\bar{x}\hat{\theta}} = n\hat{\theta} - \cancel{n\bar{x}\hat{\theta}}$$
$$\boxed{\bar{x} = \hat{\theta}}$$

b) Para n suf. grande:

$$\hat{\theta} = \bar{x} \overset{\cdot}{\sim} Normal\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$

Por el principio de invarianza (n grande)

$$\bar{x} \overset{\cdot}{\sim} Normal\left(\theta, \frac{\hat{\theta}(1-\hat{\theta})}{n}\right)$$

Ahora

$$Z = \frac{\bar{X} - \theta}{\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}} \overset{\sim}{\sim} N(0,1)$$

$$Z = \frac{(\bar{X} - \theta)}{\sqrt{\hat{\theta}(1-\hat{\theta})}} \cdot \sqrt{n} \overset{\sim}{\sim} \text{Normal}(0,1)$$

Luego Z es un pivote aproximado para θ.

c) Por def. un intervalo de confianza para θ basado en el pivote

$$\mathbb{P}(k_1 < Z < k_2) = 1-\alpha$$

$$\mathbb{P}(-Z_{1-\alpha/2} < Z < Z_{1-\alpha/2}) = 1-\alpha.$$

$$= \mathbb{P}\left(-Z_{1-\alpha/2} < \frac{(\bar{X}-\theta)}{\sqrt{\bar{X}(1-\bar{X})}} \cdot \sqrt{n} \leq Z_{1-\alpha/2}\right) = 1-\alpha$$

$$= \mathbb{P}\left(-Z_{1-\alpha/2} \cdot \frac{\sqrt{\bar{X}(1-\bar{X})}}{\sqrt{n}} \leq \bar{X} - \theta \leq Z_{1-\alpha/2} \cdot \frac{\sqrt{\bar{X}(1-\bar{X})}}{\sqrt{n}}\right) = 1-\alpha.$$

$$= \mathbb{P}\left(-\bar{X} - Z_{1-\alpha/2} \cdot \frac{\sqrt{\bar{X}(1-\bar{X})}}{\sqrt{n}} \leq -\theta \leq Z_{1-\alpha/2} \frac{\sqrt{\bar{X}(1-\bar{X})}}{\sqrt{n}} - \bar{X}\right) = 1-\alpha$$

$$= \mathbb{P}\Bigg(\underbrace{\bar{X} - Z_{1-\alpha/2}\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}}_{L(\underline{x})} \leq \theta \leq \underbrace{\bar{X} + Z_{1-\alpha/2} \sqrt{\frac{\bar{X}(1-\bar{X})}{n}}}_{U(\underline{x})}\Bigg) = 1-\alpha$$

Luego, el intervalo de confianza aproximado para θ al $(1-\alpha) \times 100\%$ es $IC(\theta) = \bar{x} \pm Z_{1-\alpha/2} \sqrt{\frac{\bar{x}(1-\bar{x})}{n}}$

d) $n = 1000$

Según los datos, la estimación puntual es:

$$\hat{\theta}_{E.M.V} = \bar{X} = \frac{3\cancel{0}}{100\cancel{0}} = 0.03$$

Luego, la estimación intervalar con 95% de confianza será:

$n = 1000$

$\alpha = 0.05$

$$IC(\theta) = 0.0\cancel{0}3 \pm \underbrace{Z_{1-\frac{0.05}{2}}}_{1.96} \sqrt{\left(\frac{0.0\cancel{0}3 \times 0.97}{1000}\right)}$$

$$= 0.03 \pm 1.96 \sqrt{\frac{0.03 \times 0.97}{1000}}$$

$$= [0.01942689;\ 0.04057311]$$

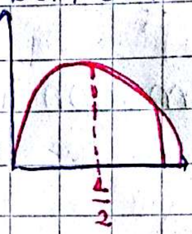
Luego, la ~~proporción~~ proporción de planchas de 2° clase está entre 1,94% y 4% con 95% de confianza aprox.

e) Nos piden n t.q.

$$Z_{1-\alpha/2}\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} \leq 0.02 \qquad \alpha = 0.05.$$

como la varianza máxima de una Bernoulli es 0.5

$$Var(x) = \theta(1-\theta)$$



$$\Rightarrow 1.96\sqrt{\frac{0.5\times 0.5}{n}} \leq 0.02$$

$$\frac{1.96\times 0.5}{0.02} \leq \sqrt{n}$$

$$49 \leq \sqrt{n}$$

$$\boxed{2401 \leq n}$$

Se recomendaría usar 2401 planchas para minimizar este error a lo pedido.

f. Esta cantidad se minimizaría estimando la varianza poblacional en base al estimador

$$1.96\times\sqrt{\frac{0.03\times 0.97}{n}} \leq 0.02 \Rightarrow n \geq 279.4764$$

$$\Rightarrow \boxed{n = 280 \text{ nos sirve.}}$$

(2) $Y \sim f_Y(y)$

$$f_Y(y) = \frac{1}{\theta \cdot y^{\frac{1}{\theta}+1}}, \quad y > 1, \quad 0 < \theta < 1$$

a) Para verificar si $Y^{\frac{1}{\theta}}$ es función pivote para θ, hay que ver que $f_W(w)$ no depende de θ, donde $W = Y^{\frac{1}{\theta}}$

Para encontrar $f_W(w)$ buscamos $F_W(w)$

$$F_W(w) = \mathbb{P}(W \leq w)$$
$$= \mathbb{P}(Y^{\frac{1}{\theta}} \leq w)$$
$$= \mathbb{P}(Y \leq w^{\theta})$$
$$= F_Y(w^{\theta}) \quad \Big/ \frac{d}{dw} \quad \text{~~donde $F_Y(y)$ =~~}$$

$$\frac{d}{dw} F_W(w) = f_W(w) = f_Y(w^{\theta}) \cdot \theta \cdot w^{\theta-1}$$
$$= \frac{1}{\cancel{\theta} \cdot (w^{\theta})^{\frac{1}{\theta}+1}} \cdot \cancel{\theta} \cdot w^{\theta-1}$$
$$= \frac{1}{w^{1+\theta}} \, w^{(\theta-1)} = w^{\cancel{\theta}-1-1-\cancel{\theta}} = w^{-2} \text{ no depende de } \theta \text{ !}$$

Luego, $Y^{\frac{1}{\theta}}$ es función pivote para θ.

b) $n = 1$

Buscamos $IC(\theta)$

$$\mathbb{P}(K_1 \leq W \leq K_2) = 1-\alpha$$

$$\mathbb{P}(w_{\alpha/2} \leq Y^{\frac{1}{\theta}} \leq w_{1-\alpha/2}) = 1-\alpha$$

$$\mathbb{P}\left(\log(w_{\alpha/2}) \leq \frac{1}{\theta} \cdot \log(Y) \leq \log(w_{1-\alpha/2})\right) = 1-\alpha.$$

$$\mathbb{P}\left(\frac{\log(w^{\alpha/2})}{\log(Y)} \leq \frac{1}{\theta} \leq \frac{\log(w_{1-\alpha/2})}{\log(Y)}\right) = 1-\alpha.$$

$$\mathbb{P}\left(\underbrace{\frac{\log(Y)}{\log(w_{1-\alpha/2})}}_{U(x)} \leq \theta \leq \underbrace{\frac{\log(Y)}{\log(w_{\alpha/2})}}_{V(x)}\right) = 1-\alpha.$$