1a) Sea $\left.\begin{array}{l} Z = Y - X, \\ X \in \{1,2\} \\ Y \in \{1,2,3\} \end{array}\right\} \Rightarrow Z \in \{-1, 0, 1, 2\}$

$$f_Z(z) = P(Z=z) = \sum_{\{(x,y) \in \{1,2\}\times\{1,2,3\}:\ y-x=z\}} f_{X,Y}(x,y)$$

$$= \begin{cases} f_{X,Y}(2,1) = 3/21 & \text{si } z = -1 \\ f_{X,Y}(1,1) + f_{X,Y}(2,2) = 6/21 & \text{si } z = 0 \\ f_{X,Y}(1,2) + f_{X,Y}(2,3) = 8/21 & \text{si } z = 1 \\ f_{X,Y}(1,3) = 4/21 & \text{si } z = 2 \\ 0 & \sim \end{cases}$$

$$\Rightarrow P(|Z| \le 1) = P(-1 \le Z \le 1)$$
$$= P(Z=-1) + P(Z=0) + P(Z=1)$$
$$= \frac{3}{21} + \frac{6}{21} + \frac{8}{21} = \frac{17}{21}$$

$$P(|Z|<1) = P(Z=0) = 6/21$$

Nota:
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
(Esto no lo pueden dar!

1b) $E(Z) = -1 \cdot \frac{3}{21} + 0 \cdot \frac{6}{21} + 1 \cdot \frac{8}{21} + 2 \cdot \frac{4}{21} = \frac{16-3}{21} = \frac{13}{21}$

Alternativa 1. $E(Z) = E(Y-X) = \sum_{x=1}^{2}\sum_{y=1}^{3} (y-x)\frac{(y+x)}{21} = \sum_{x=1}^{2}\sum_{y=1}^{3} \frac{(y^2-x^2)}{21}$

$$= \frac{1}{21}\left\{ \sum_{x=1}^{2}\sum_{y=1}^{3} y^2 - \sum_{x=1}^{2} x^2 \sum_{y=1}^{3} \right\} = \frac{1}{21}\left\{ 2 \cdot \frac{3 \cdot 4 \cdot 7}{6} - \frac{2 \cdot 3 \cdot 5}{6} \cdot 3 \right\}$$

$$= \frac{1}{21}\{28 - 15\} = \frac{13}{21}$$

Alt. 2 $E(Z) = E(Y-X) = E(Y) - E(X) \Rightarrow$ marginalizar o usar nuevamente la cjta.

1c) ~~Parlaxx~~ $P(Y - X = z \mid X = 1) = P(Y = z+1 \mid X=1) = f_{Y|X=1}(z+1), \quad z \in \{-1,0,1,2\}$

Pero,

$$f_{Y|X=1}(y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} & \text{si } y = 1,2,3, \text{ con } x = 1,2 \\ 0 & \sim \end{cases}$$

$$f_X(x) = \sum_{y=1}^{3} \frac{x+y}{21} = \frac{1}{21}\left(3x + \frac{3 \cdot 4}{2}\right) = (x+2)/7 \quad \text{si } \underline{x = 1,2}$$

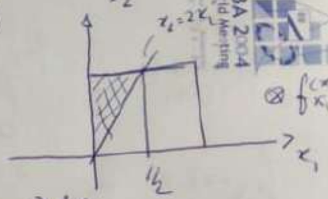
$$\Rightarrow f_{Y|X=1}(y) = \begin{cases} \dfrac{(x+y)/21}{(x+2)/7} = \dfrac{1}{3}\,\dfrac{y+x}{2+x}\Big|_{x=1} & \text{si } y = 1,2,3 \\ 0 & \sim \end{cases}$$

$$= \begin{cases} \dfrac{1}{9}(y+1) & \text{si } y = 1,2,3 \\ 0 & \sim \end{cases}$$

1d) $$E(Y|X=1) = \sum_{y=1}^{3} y\, f_{Y|X=1}(y) = \frac{1}{9}\sum_{y=1}^{3} y(y+1)$$

$$= \frac{1}{9}\left\{ \sum_{y=1}^{3} y^2 + \sum_{y=1}^{3} y \right\} = \frac{1}{9}\left\{ \frac{3 \cdot 4 \cdot 7}{6} + \frac{3 \cdot 4}{2} \right\} = \frac{14+6}{9} = \frac{20}{9}$$

2a) $P(2X_1 \le X_2 \mid 2X_1 \le 1) = \dfrac{P(2X_1 \le X_2,\ 2X_1 \le 1)}{P(2X_1 \le 1)}$



$\otimes\ f_{X_1,X_2}(x_1,x_2)$

$$= \frac{\int_0^{1/2}\int_{2x_1}^{1} 4x_1x_2\,dx_2\,dx_1}{\int_0^{1/2}\int_0^{1} 4x_1x_2\,dx_2\,dx_1}$$

$$= \frac{\int_0^{1/2} 2x_1\,(x_2^2\big|_{2x_1}^{1})\,dx_1}{\int_0^{1/2} 2x_1\,(x_2^2\big|_0^1)\,dx_1} = \frac{\int_0^{1/2} 2x_1(1-4x_1^2)\,dx_1}{\int_0^{1/2} 2x_1\,dx_1}$$

$$= \frac{x_1^2\big|_0^{1/2} - 2x_1^4\big|_0^{1/2}}{x_1^2\big|_0^{1/2}} = \frac{\frac{1}{4}-\frac{2}{16}}{1/4} = \frac{1/8}{1/4} = \frac{1}{2}$$

2b) X_1, X_2 son va's iid $f(x) = \begin{cases} 2x & \text{si } 0<x<1 \\ 0 & \sim \end{cases}$

$F(x) = \begin{cases} 0 & x<0 \\ x^2 & \text{si } 0 \le x < 1 \\ 1 & \text{si } x \ge 1 \end{cases}$

Sabemos que

$f_{X_{(1)}}(x) = n\,(1-F(x))^{n-1} f(x)$

$= 2\,(1-F(x))\,f(x)$ si $\underline{n=2}$

$$f_{X_{(n)}}(x) = n\,(F(x))^{n-1} f(x)$$

$$= 2\,F(x)\,f(x) \quad \text{si } \underline{n=2}$$

$$\Rightarrow f_{X_{(1)}}(x) = \begin{cases} 4(1-x^2)\,x & \text{si } 0<x<1 \\ 0 & \sim \end{cases}$$

$$f_{X_{(2)}}(x) = \begin{cases} 4\,x^3 & \text{si } 0<x<1 \\ 0 & \sim \end{cases}$$

$$\Rightarrow E(X_{(1)}) = \int_0^1 4\,(1-x^2)\,x^2\,dx = \frac{4}{3}x^3\Big|_0^1 - \frac{4}{5}x^5\Big|_0^1 = \frac{4}{3} - \frac{4}{5}$$

$$4\left(\frac{5-3}{15}\right)$$

$$= \frac{8}{15}$$

$$E(X_{(2)}) = \int_0^1 4\,x^4\,dx = \frac{4}{5}$$

$$\Rightarrow E(R) = E(X_{(2)} - X_{(1)}) = E(X_{(2)}) - E(X_{(1)})$$

$$= \frac{4}{5} - \left(\frac{4}{3} - \frac{4}{5}\right) = \frac{8}{5} - \frac{4}{3} = \frac{20}{15}$$

2c) Es claro que $X_1 \perp\!\!\!\perp X_2 \Rightarrow Y_1 = -\log X_1 \perp\!\!\!\perp Y_2 = -\log X_2$

Además $X_1 \overset{d}{=} X_2 \Rightarrow Y_1 \overset{d}{=} Y_2$

(X_1, X_2 iid) (Y_1, Y_2 iid)

$$\Rightarrow f_{Y_1, Y_2}(y_1, y_2) = f_{Y_1}(y_1)\, f_{Y_2}(y_2)$$

Basta con determinar la marginal común

$$Y = -\log X, \quad f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \sim \end{cases}$$

$$\Rightarrow X = e^{-Y} \quad \frac{dX}{dY} = -e^{-Y}$$

$0 < x < 1$
$\Rightarrow -\log x \in (0, \infty)$

$$\Rightarrow f_Y(y) = |-e^{-y}|\, f_X(e^{-y})$$

$$= \begin{cases} e^{-y}\, 2e^{-y} = 2e^{-2y} & \text{si } y > 0 \\ 0 & \sim \end{cases}$$

$$\Rightarrow Y \sim \mathcal{E}xp(2) \Rightarrow \underline{Y_1, Y_2 \text{ iid } \mathcal{E}xp(2)}$$

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 4\, e^{-2(y_1+y_2)} & \text{si } y_1>0,\ y_2>0 \\ 0 & \text{c.c.} \end{cases}$$

2d) Sea $Y = -\log(X_1 X_2) = -\log X_1 - \log X_2$

$= Y_1 + Y_2$; Y_1, Y_2 i.i.d. $\underline{Exp(2)}$

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) \\
&= E(e^{t(Y_1+Y_2)}) \\
&= E(e^{tY_1} e^{tY_2}) \quad Y_1 \perp\!\!\!\perp Y_2 \\
&= E(e^{tY_1})\, E(e^{tY_2}) \\
&= M_{Y_1}(t)\, M_{Y_2}(t) \\
&= (1-t/2)^{-1}(1-t/2)^{-1}, \quad t/2<1 \\
&= (1-t/2)^{-2}, \quad t<2 \\
&= \text{f.g.m de una Gamma}(2,2)
\end{aligned}$$

$\rightarrow \underline{Y \sim \text{Gamma}(2,2)}$

3a) $X_1 \sim N(0,1) \rightarrow E(X_1) = 0$ y $Var(X_1) = 1$

$X_2 | X_1 = x_1 \sim N(\rho x_1, 1-\rho^2) \rightarrow E(X_2|X_1 = x_1) = \rho x_1$ y $Var(X_2|X_1=x_1) = 1-\rho^2$ $\forall x_1$

$\rightarrow E(X_2) = E\{E(X_2|X_1)\} = E(\rho X_1) = \rho E(X_1) = 0$

$Var(X_2) = Var\{E(X_1|X_2)\} + E\{Var(X_2|X_1)\}$
$= Var(\rho X_1) + E(1-\rho^2) = \rho^2 \underbrace{Var(X_1)}_{1} + 1 - \rho^2 = 1$

$\rho_{X_1,X_2} = E(X_1 X_2)$ ya que X_1 y X_2 son variables estandarizadas
$= E\{E(X_1 X_2 | X_1)\} = E\{X_1 E(X_2|X_1)\} = E\{\rho X_1^2\} = \rho \underbrace{E(X_1^2)}_{1} = \rho$

3b) Sí, pues $f_{X_1,X_2}(x_1,x_2) = f_{X_2|X_1=x_1}(x_2) \, f_{X_1}(x_1)$ por el Teor. Mult.

Aquí, $f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x_2-\rho x_1)^2}{2(1-\rho^2)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}}$, $-\infty < x_1 < \infty$
$-\infty < x_2 < \infty$

3c) $E(Y_1) = E(X_1 + X_2) = E(X_1) + E(X_2) = 0$
$E(Y_2) = E(X_1 - X_2) = E(X_1) - E(X_2) = 0$
$\Rightarrow Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = E(Y_1 Y_2)$
$= E\{(X_1 + X_2)(X_1 - X_2)\} = E(X_1^2 - X_2^2)$

$\Rightarrow Cov(Y_1, Y_2) = E(X_1^2) - E(X_2^2) = 1 - 1 = 0$ ya que $E(X_i^2) = Var(X_i) = 1$ (*) $i = 1, 2$.

$\Rightarrow \rho_{Y_1, Y_2} = 0 \Rightarrow Y_1$ e Y_2 no están correlacionadas!

3d) Sí, pues tenemos la conjunta de (X_1, X_2) y la transformación

$(X_1, X_2) \to (X_1 + X_2, X_1 - X_2) = (Y_1, Y_2)$

es 1 a 1 $\Rightarrow$ por el teor. del Jacobiano

$f_{Y_1, Y_2}(y_1, y_2) = |J| \, f_{X_1, X_2}(h_1(y_1, y_2), h_2(y_1, y_2))$,

donde $x_1 = h_1(y_1, y_2) = \frac{y_1 + y_2}{2}$ y $x_2 = h_2(y_1, y_2) = \frac{y_1 - y_2}{2}$

son las inversas, y $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

$\Rightarrow f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y_1 - y_2}{2} - \rho \frac{y_1 + y_2}{2}\right)^2} \frac{e^{-\frac{1}{2}\left(\frac{y_1 + y_2}{2}\right)^2}}{\sqrt{2\pi}}$, $\begin{matrix} -\infty < y_1 < \infty \\ -\infty < y_2 < \infty \end{matrix}$

$= \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2 \cdot 2} y_1^2} \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2 \cdot 2} y_2^2}$, $\begin{matrix} -\infty < y_1 < \infty \\ -\infty < y_2 < \infty \end{matrix}$

$\Rightarrow Y_1, Y_2$ son v.a.'s i.i.d. $\underline{N(0, 2)}$