1.- , $f:[a,b]\longrightarrow \mathbb{R} \uparrow$

i) f inv.

ii) $f^{-1}\uparrow$

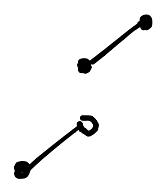
Sol.:

i) $x,y\in[a,b],\ x\neq y$

$$\left.\begin{array}{l} \bullet\ \underline{x<y} \longrightarrow f(x)<f(y) \\ \bullet\ y<x \longrightarrow f(y)<f(x) \end{array}\right] f(x)\neq f(y)$$

$\longrightarrow$ f es inyectiva

$\longrightarrow$ $f:[a,b]\longrightarrow \underset{[f(a),\,f(b)]}{\underset{\neq}{\underline{\operatorname{Rec} f}}}$ es inv.



ii) $x,y\in \operatorname{Rec} f,\ x<y$

pd $f^{-1}(x)<f^{-1}(y)$

Sup. $f^{-1}(x)\geq f^{-1}(y) \longrightarrow f(f^{-1}(x))\geq f(f^{-1}(y))$

$\longrightarrow\quad x\geq y \quad \rightarrow\!\!\times\!\!\leftarrow$

2.- $e^x + e^{-x} \geq 2$ $\quad y + \frac{1}{y} \geq 2 \quad y = e^x$

$e^x > 0$

$\Longleftrightarrow e^{2x} + 1 \geq 2e^x$

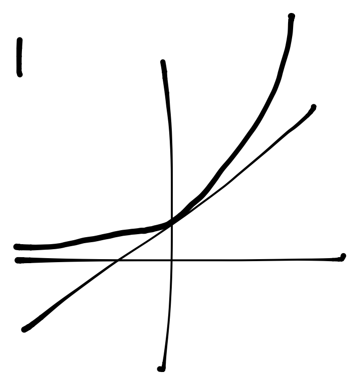
$\Longleftrightarrow e^{2x} - 2e^x + 1 \geq 0$

$\Longleftrightarrow (e^x - 1)^2 \geq 0$ ✓

Otras formas:

- $\frac{e^x + e^{-x}}{2} \overset{PA-PG}{\geq} \sqrt{e^x \cdot e^{-x}} = 1$

- $\left.\begin{array}{l} e^x \geq 1 + x \\ e^{-x} \geq 1 - x \end{array}\right\} e^x + e^{-x} \geq 2$



CLASE 21: SUMATORIAS (Cont.)

- **Sumas telescópicas:**

Sea $(a_n)_n$ una sucesión.

$$\sum_{k=1}^{n} \{a_k - a_{k-1}\} = ?$$

$$\sum_{k=1}^{3} \{a_k - a_{k-1}\} = \overset{k=1}{\{\cancel{a_1} - a_0\}} + \overset{k=2}{\{\cancel{a_2} - \cancel{a_1}\}} + \overset{k=3}{\{a_3 - \cancel{a_2}\}}$$

$$= a_3 - a_0$$

En general,

$$\sum_{k=1}^{n} \{a_k - a_{k-1}\} = \overset{k=1}{\{\cancel{a_1} - a_0\}} + \{\cancel{a_2} - \cancel{a_1}\} + \cancel{\cdots}$$

$$\cancel{\cdots} + \overset{k=n-1}{\{\cancel{a_{n-1}} - \cancel{a_{n-2}}\}} + \overset{k=n}{\{a_n - \cancel{a_{n-1}}\}}$$

$$= a_n - a_0$$

Conclusión:

$$\sum_{k=1}^{m} \{a_k - a_{k-1}\} = a_m - a_0$$

En general,

$$\sum_{k=n}^{m} \{a_k - a_{k-1}\} = a_m - a_{n-1}$$

$$\sum_{k=n}^{m} \{a_{k+1} - a_k\} = a_{m+1} - a_n$$

- Ej: $\displaystyle\sum_{k=1}^{m} \{\underbrace{(k+1)^2}_{a_{k+1}} - \underbrace{k^2}_{a_k = k^2}\} = \sum_{k=1}^{m} \{a_{k+1} - a_k\}$

$$= a_{m+1} - a_1$$

$$= (m+1)^2 - 1^2$$

$$= m^2 + 2m$$

• Ej: $\displaystyle\sum_{k=1}^{m} \frac{1}{k(k+1)} = ?$

$$\frac{1}{k(k+1)} \overset{?}{=} \frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\Rightarrow \sum_{k=1}^{m} \frac{1}{k(k+1)} = \sum_{k=1}^{m} \left\{ \frac{1}{k} - \frac{1}{k+1} \right\} = \sum_{k=1}^{m} \{a_k - a_{k+1}\}$$

$$a_k = \frac{1}{k}$$

$$= \underbrace{\frac{1}{1} - \frac{1}{m+1}}_{a_1 - a_{m+1}} = 1 - \frac{1}{m+1}$$

• Ej: $\displaystyle\sum_{k=1}^{m} \frac{1}{(2k-1)(2k+3)} = ?$

$$\frac{1}{(2k-1)(2k+3)} \overset{?}{=} \frac{1}{2k-1} - \frac{1}{2k+3}$$

$$= \frac{(2k+3)-(2k-1)}{(2k-1)(2k+3)} = \frac{4}{(2k-1)(2k+3)}$$

$$\Rightarrow \frac{1}{(2k-1)(2k+3)} = \frac{1}{4}\left\{\frac{1}{2k-1} - \frac{1}{2k+3}\right\}$$

$$\Rightarrow \sum_{k=1}^{m} \frac{1}{(2k-1)(2k+3)} = \frac{1}{4}\sum_{k=1}^{m}\left\{\frac{1}{2k-1} - \frac{1}{2k+3}\right\}$$

$a_k = \frac{1}{2k-1}$ $a_{k+1} = \frac{1}{2k+1}$

$a_{k+2} = \frac{1}{2k+3}$

$$= \frac{1}{4}\sum_{k=1}^{m}\left\{a_k - a_{k+2}\right\}$$

$$= \frac{1}{4}\sum_{k=1}^{m}\left\{a_k - a_{k+1} + a_{k+1} - a_{k+2}\right\}$$

$$= \frac{1}{4}\sum_{k=1}^{m}\left\{a_k - a_{k+1}\right\} + \frac{1}{4}\sum_{k=1}^{m}\left\{a_{k+1} - a_{k+2}\right\}$$

$$= \frac{1}{4}\left\{a_1 - a_{m+1}\right\} + \frac{1}{4}\left\{a_2 - a_{m+2}\right\}$$

$$= \frac{1}{4}\left\{1 - \frac{1}{2m+1}\right\} + \frac{1}{4}\left\{\frac{1}{3} - \frac{1}{2n+3}\right\}$$

- **Ej**: $S_m = \sum_{k=0}^{m} r^k$

$$rS_m = \sum_{k=0}^{m} r \cdot r^k = \sum_{k=0}^{m} r^{k+1}$$

$$\rightarrow (r-1)S_m = \sum_{k=0}^{m} \{r^{k+1} - r^k\} = r^{m+1} - r^0$$

$$= r^{m+1} - 1$$

$$\rightarrow \quad S_m = \frac{r^{m+1}-1}{r-1}$$

- **Obs**: $\sum_{k=1}^{m} r^k = r \sum_{k=1}^{m} r^{k-1}$ $\qquad j = k-1$

$$= r \sum_{j=0}^{m-1} r^j$$

$$= r \cdot \frac{r^m - 1}{r-1}$$

$$= \frac{r^{m+1} - r}{r-1}$$

• Ej: $\sum_{k=0}^{m} k \cdot k!$

$k! = k \cdot (k-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1$

$0! = 1$

$$= \sum_{k=0}^{m} (k+1-1) \cdot k!$$

$$= \sum_{k=0}^{m} \{(k+1)! - k!\}$$

$$= (m+1)! - 0!$$

$$= (m+1)! - 1$$

• Ej: $\sum_{k=0}^{m} k^2 = ? = S_m$

• $\sum_{k=0}^{m} \{(k+1)^3 - k^3\} = (m+1)^3 - 0^3 = (m+1)^3$

• $\sum_{k=0}^{m} \{(k+1)^3 - k^3\} = \sum_{k=0}^{m} \{k^3 + 3k^2 + 3k + 1 - k^3\}$

$$= \sum_{k=0}^{m} \{3k^2 + 3k + 1\}$$

$$= 3\sum_{k=0}^{m} k^2 + 3\sum_{k=0}^{m} k + \sum_{k=0}^{m} 1$$

$$= 3S_m + 3\frac{m(m+1)}{2} + (m+1)$$

$$\Rightarrow (m+1)^3 = 3S_m + \frac{3}{2}m(m+1) + (m+1)$$

$$\Rightarrow 3S_m = (m+1)^3 - \frac{3}{2}m(m+1) - (m+1)$$

$$= (m+1)\left[(m+1)^2 - \frac{3}{2}m - 1\right]$$

$$= (m+1)\left[m^2 + 2m + 1 - \frac{3}{2}m - 1\right]$$

$$= (m+1)\left[m^2 + \frac{1}{2}m\right]$$

$$= m(m+1)\left(m + \frac{1}{2}\right)$$

$$= \frac{1}{2}m(m+1)(2m+1)$$

$$\Rightarrow S_m = \sum_{k=0}^{m} k^2 = \frac{1}{6}m(m+1)(2m+1)$$

• Ejercicio: $\sum_{k=0}^{m} k^3 = ?$

• Ej: $S_m = \sum_{k=1}^{m} \sin(k) = ?$

$$2\sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\sin(k)\sin(1) = \cos(k-1) - \cos(k+1)$$

$$\rightarrow S_m = \frac{1}{2\sin(1)} \sum_{k=1}^{m} \left[\underbrace{\cos(k-1)}_{a_{k-1}} - \underbrace{\cos(k+1)}_{a_{k+1}} \right]$$

$$= \frac{1}{2\sin(1)} \sum_{k=1}^{m} \{a_{k-1} - a_k\}$$

$$+ \frac{1}{2\sin(1)} \sum_{k=1}^{m} \{a_k - a_{k+1}\}$$

$$= \frac{1}{2\sin(1)} \{a_0 - a_m\} + \frac{1}{2\sin(1)} \{a_1 - a_{m+1}\}$$

$$= \frac{1}{2\sin(1)} \left\{ \cos(0) - \cos(n) + \cos(1) - \cos(n+1) \right\}$$

- <u>Obs</u>: otra forma:

$$2\sin(k)\sin\left(\tfrac{1}{2}\right) = \cos\left(k-\tfrac{1}{2}\right) - \cos\left(k+\tfrac{1}{2}\right)$$

$$= \cos\left(\frac{2k-1}{2}\right) - \cos\left(\frac{2k+1}{2}\right)$$

$$= b_k - b_{k+1}$$

- <u>Obs</u>: otra forma:

$$\sum_{k=1}^{n} \sin k = \sum_{k=1}^{n} \operatorname{Im}\{e^{ik}\}$$

$$= \operatorname{Im}\Big\{ \underbrace{\sum_{k=1}^{n} e^{ik}}_{\text{geométrica: } e^{ik} = (e^i)^k,\ r = e^i} \Big\}$$

$$= \operatorname{Im}\left\{ \frac{e^{i(n+1)} - e^i}{e^i - 1} \right\} = \text{etc}$$