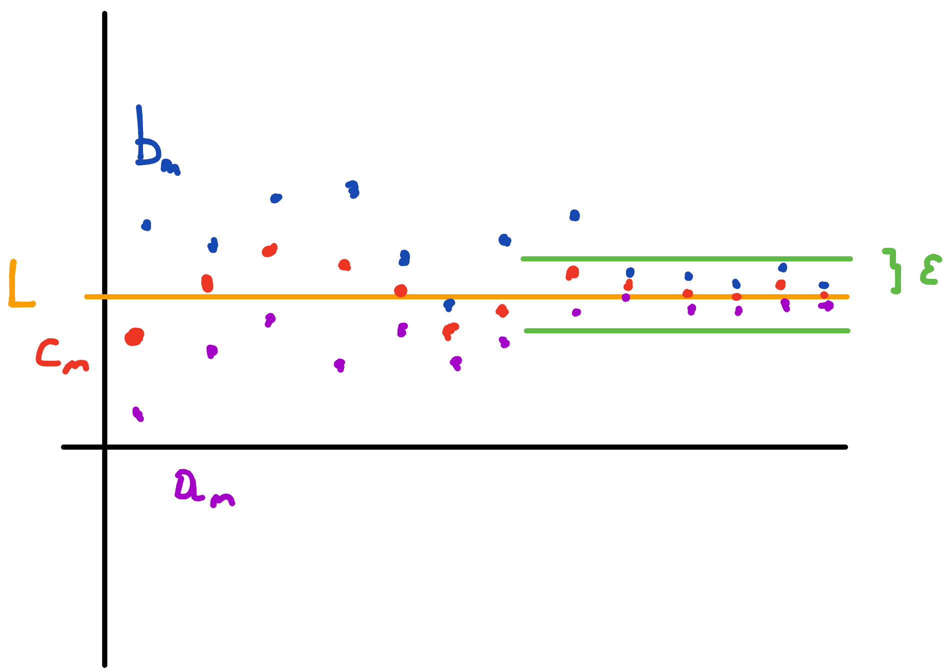
CLASE 27: LÍMITES
(Teorema del sándwich)



TEOREMA (Teorema del sándwich)

Sean $(a_n)_n, (b_n)_n$ y $(c_n)_n$ tres sucesiones.

Supongamos que:

i) $a_n \leq c_n \leq b_n$, $\forall n \geq 1$ (Puede ser para todo $n \geq N_0$)

ii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$

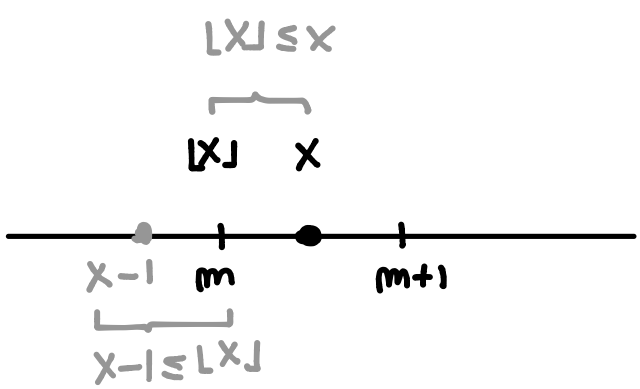
Luego, $\lim_{n\to\infty} c_n = L$

• **Ej**: $\lim_{n\to\infty} \frac{\lfloor n/2 \rfloor}{n} = ?$

Buscamos $(a_n)_n$ y $(b_n)_n$ tales que

i) $a_n \leq \frac{\lfloor \frac{n}{2} \rfloor}{n} \leq b_n$

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$



$$\Longrightarrow \quad x-1 \leq \lfloor x \rfloor \leq x \quad \forall x \in \mathbb{R}$$

Luego, $\frac{m}{2}-1 \le \left\lfloor \frac{m}{2} \right\rfloor \le \frac{m}{2}$

$$\Rightarrow \frac{\frac{m}{2}-1}{m} \le \frac{\left\lfloor \frac{m}{2} \right\rfloor}{m} \le \frac{\frac{m}{2}}{m}$$

$$\Rightarrow \frac{m-2}{2m} \le \frac{\left\lfloor \frac{m}{2} \right\rfloor}{m} \le \frac{1}{2}$$

- $b_m = \frac{1}{2} \Rightarrow \lim\limits_{m\to\infty} b_m = \frac{1}{2}$
- $a_m = \frac{m-2}{2m} \Rightarrow \lim\limits_{m\to\infty} a_m = \lim\limits_{m\to\infty} \frac{m-2}{2m}$

$$= \lim_{m\to\infty} \left(\frac{1}{2} - \frac{1}{m}\right)$$

$$= \frac{1}{2}$$

Luego,

$$\begin{array}{ccccc} \frac{m-2}{2m} & \le & \frac{\left\lfloor \frac{m}{2} \right\rfloor}{m} & \le & \frac{1}{2} \\ & \searrow & \downarrow & \swarrow & \\ & & \frac{1}{2} & & \end{array}$$

Es decir, $\lim\limits_{m\to\infty} \frac{\left\lfloor \frac{m}{2} \right\rfloor}{m} = \frac{1}{2}$

• <u>Ej</u>: $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = \ ?$

$$-1 \leq \sin n \leq 1$$

$$\Rightarrow \quad -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$$\searrow \qquad \downarrow \qquad \swarrow$$

$$0$$

Luego, $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$

• <u>Corolario</u>: Sean $(x_n)_n$ e $(y_n)_n$ dos sucesiones tales que:

i) $(x_n)_n$ es acotada

ii) $\lim\limits_{n\to\infty} y_n = 0$

Luego, $\lim\limits_{n\to\infty} x_n y_n = 0$

- **Obs**: En el ejemplo: $x_n = \sin n$

  $y_n = \dfrac{1}{n}$

- **Obs**: Volvemos al ejemplo.

$$0 \leq |\sin n| \leq 1$$

$$\Longrightarrow \quad 0 \leq \left| \frac{\sin n}{n} \right| \leq \frac{1}{n}$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

Ahora,
$$-\left|\frac{\sin n}{n}\right| \leq \frac{\sin n}{n} \leq \left|\frac{\sin n}{n}\right|$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

Moraleja: Supongamos que $\lim\limits_{n\to\infty} |x_n| = 0$

$$-|x_n| \leq x_n \leq |x_n| \quad \Longrightarrow \quad \lim_{n\to\infty} x_n = 0$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

• Obs./Advertencia: Esto solo funciona con el límite 0!

Ej: $x_n = (-1)^n$

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} 1 = 1$$

pero $\lim_{n\to\infty} x_n$ no existe.

• DEM (Teo. Sandwich)

Premisas: i) $a_n \leq c_n \leq b_n$

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$

Por demostrar: $\lim_{n\to\infty} c_n = L$.

Sea $\varepsilon > 0$. Buscamos $n_0 \geq 1$ tq

$$n \geq n_0 \Rightarrow |c_n - L| < \varepsilon$$

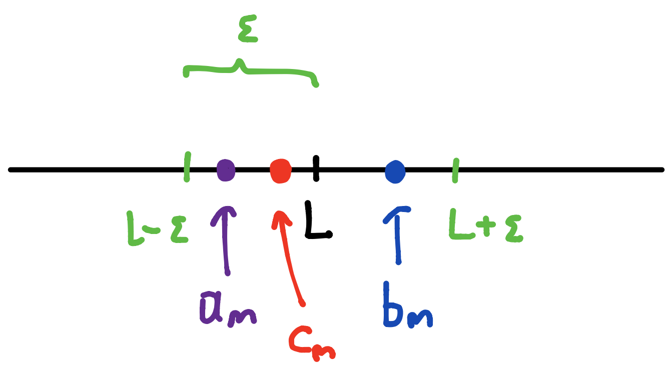
Sabemos que:

- $\exists\, n_1 \geq 1$ tq $|a_n - L| < \varepsilon$ si $n \geq n_1$

  $\Rightarrow L - \varepsilon < a_n$ si $n \geq n_1$

- $\exists\, n_2 \geq 1$ tq $|b_n - L| < \varepsilon$ si $n \geq n_2$

  $\Rightarrow b_n < L + \varepsilon$ si $n \geq n_2$



Sea $n_0 = \max\{n_1, n_2\}$. Luego, si $n \geq n_0$, entonces $n \geq n_1$, $n \geq n_2$ y

$$L - \varepsilon < a_n \leq c_n \leq b_n < L + \varepsilon$$

$\Rightarrow \quad L - \varepsilon < c_n < L + \varepsilon \quad$ si $n \geq n_0$

$\Rightarrow \quad |c_n - L| < \varepsilon \quad$ si $n \geq n_0$ $\quad \square$

• **DEM** (Corolario)

Premisas: i) $(x_n)_n$ acotada

ii) $\lim\limits_{n\to\infty} y_n = 0$

Por i), existe $K>0$ tal que $|x_n| \le K \quad \forall n \ge 1$

Luego, $\quad 0 \le |x_n y_n| = |x_n||y_n| \le K|y_n|$

Luego,
$$0 \le |x_n y_n| \le K|y_n|$$
$$0$$

Luego, por sándwich, $\lim\limits_{n\to\infty} |x_n y_n| = 0$

y, por lo tanto, $\lim\limits_{n\to\infty} x_n y_n = 0$

$\square$

• Ej: $\lim_{n\to\infty} \dfrac{n(\cos^2 n + \frac{1}{3}\sin(n^2) + 7)}{n^2+1}$

$X_n = \cos^2 n + \frac{1}{3}\sin(n^2) + 7$

$Y_n = \dfrac{n}{n^2+1}$

i) $|X_n| \leq |\cos^2 n| + \frac{1}{3}|\sin(n^2)| + 7$

$\leq 1 + \frac{1}{3} + 7$

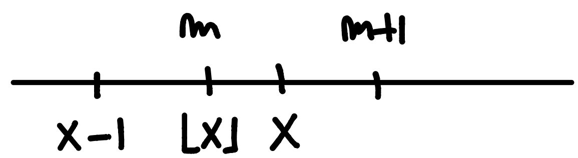
≤ 10

ii) $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \dfrac{n}{n^2+1} = 0$

Luego, por el corolario, $\lim_{n\to\infty} X_n Y_n = 0$

• **Ej:** $\lim\limits_{n\to\infty} \dfrac{\left\lfloor \frac{n^2}{9} \right\rfloor}{\left\lfloor \frac{n}{3} \right\rfloor^2} = ?$

Es decir, "comparar" $\lfloor y^2 \rfloor$ con $\lfloor y \rfloor^2$ $(x \geq 0)$



• $\underline{x = y^2}$: $\quad y^2 - 1 \leq \lfloor y^2 \rfloor \leq y^2 \qquad (*)$

• $\underline{x = y}$: $\quad y - 1 \leq \lfloor y \rfloor \leq y \qquad (**)$

Si $y - 1 \geq 0 \Rightarrow (y-1)^2 \leq \lfloor y \rfloor^2 \leq y^2$

En nuestro caso, $y = \frac{n}{3}$

$$(*) \Rightarrow \left(\frac{n}{3}\right)^2 - 1 \leq \left\lfloor \left(\frac{n}{3}\right)^2 \right\rfloor \leq \left(\frac{n}{3}\right)^2$$

$$\Rightarrow \frac{n^2 - 9}{9} \leq \left\lfloor \frac{n^2}{9} \right\rfloor \leq \frac{n^2}{9} \qquad (1)$$

$$(**) \Rightarrow \left(\frac{n}{3}-1\right)^2 \leq \left\lfloor \frac{n}{3} \right\rfloor^2 \leq \left(\frac{n}{3}\right)^2$$

$$\Rightarrow \frac{n^2-2n+1}{9} \leq \left\lfloor \frac{n}{3} \right\rfloor^2 \leq \frac{n^2}{9}$$

$$\Rightarrow \frac{9}{n^2} \leq \frac{1}{\left\lfloor \frac{n}{3} \right\rfloor^2} \leq \frac{9}{n^2-2n+1} \qquad (2)$$

Multiplicando (1) y (2), obtenemos

$$\frac{n^2-9}{9}\cdot\frac{9}{n^2} \leq \frac{\left\lfloor \frac{n^2}{9} \right\rfloor}{\left\lfloor \frac{n}{3} \right\rfloor^2} \leq \frac{n^2}{9}\cdot\frac{9}{n^2-2n+1}$$

$$\Rightarrow \frac{n^2-9}{n^2} \leq \frac{\left\lfloor \frac{n^2}{9} \right\rfloor}{\left\lfloor \frac{n}{3} \right\rfloor^2} \leq \frac{n^2}{n^2-2n+1}$$

$$1$$

Luego, $\displaystyle\lim_{n\to\infty} \frac{\left\lfloor \frac{n^2}{9} \right\rfloor}{\left\lfloor \frac{n}{3} \right\rfloor^2} = 1$