CLASE 23: TEOREMA DEL BINOMIO (Cont.)

- Recordamos:

- $$(a+b)^n = \underbrace{b^n}_{c_0 a^0 b^n} + c_1 a b^{n-1} + \cdots + c_{n-1} a^{n-1} b + \underbrace{a^n}_{c_n a^n b^0}$$

$$= \sum_{k=0}^{n} c_k^{(n)} a^k b^{n-k}$$

- Coeficientes binomiales:

$$c_k^{(n)} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- TEOREMA DEL BINOMIO:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad n \geq 1$$

DEM: pendiente $\square$

- $\binom{m}{k} + \binom{m}{k+1} = \binom{m+1}{k+1}$

$$\begin{array}{ccccccc} & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \end{array}$$

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \iddots & & & & & & & & \ddots \end{array}$$

- <u>Obs</u>: $\binom{m}{k} = \frac{m!}{k!\,(m-k)!} = \frac{m!}{(m-k)!\,k!} = \binom{m}{m-k}$

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} = \sum_{k=0}^{m} \binom{m}{m-k} a^k b^{m-k}$$

$$\underset{j=m-k}{=} \sum_{j=0}^{m} \binom{m}{j} a^{m-j} b^j$$

• Ej: $(a+b)^7 = \sum_{k=0}^{7} \binom{7}{k} a^k b^{7-k}$

$\binom{7}{0} = 1 \quad \binom{7}{1} = \frac{7!}{1! \cdot 6!} = \frac{7 \cdot 6!}{1! \cdot 6!} = 7 \quad \left(\underline{\text{Obs}}: \binom{m}{1} = m \right)$

$\binom{7}{2} = \frac{7!}{2! \cdot 5!} = \frac{7 \cdot 6 \cdot 5!}{2 \cdot 5!} = \frac{7 \cdot 6}{2} = 21$

$\binom{7}{3} = \frac{7!}{3!\, 4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 4!} = 7 \cdot 5 = 35$

$\binom{7}{4} = \binom{7}{7-4} = \binom{7}{3} = 35$

$\binom{7}{5} = \binom{7}{2} = 21 \qquad \binom{7}{6} = \binom{7}{1} = 7 \quad \binom{7}{7} = 1$

$$(a+b)^7 = b^7 + 7ab^6 + 21a^2b^5 + 35a^3b^4 + 35a^4b^3 + 21a^5b^2 + 7a^6b + a^7$$

• Ej: Encuentre el coeficiente que acompaña a x^{-2} en el desarrollo de $\left(x+\frac{1}{x^2}\right)^{10}$

Sol:

$$\left(x+\frac{1}{x^2}\right)^{10} = \sum_{k=0}^{10} \binom{10}{k} x^{10-k} \left(\frac{1}{x^2}\right)^{k}$$

$$= \sum_{k=0}^{10} \binom{10}{k} x^{10-k} \, x^{-2k}$$

$$= \sum_{k=0}^{10} \binom{10}{k} x^{10-3k}$$

Queremos $10-3k=-2 \Rightarrow k=4$

Luego, el coeficiente que acompaña x^{-2}

es $\binom{10}{4}$.

• **Obs**: i) $$\sum_{k=0}^{n}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}\underset{\overset{\uparrow}{a}}{1^{k}}\;\underset{\overset{\uparrow}{b}}{1^{n-k}}$$

$$= (1+1)^{n} = 2^{n}$$

ii) $$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}\underset{\overset{\uparrow}{a}}{(-1)^{k}}\;\underset{\overset{\uparrow}{b}}{1^{n-k}}$$

$$= (-1+1)^{n} = 0$$

• **DEM** (Teorema del binomio)

• Queremos demostrar que

$$(a+b)^{n} = \sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}, \quad n \geq 1$$

• Procedemos por inducción:

• **$n=1$**: $(a+b)^{1} = b + a$

$$= \binom{1}{0}a^{0}b^{1-0} + \binom{1}{1}a^{1}b^{1-1} \checkmark$$

• Supongamos que

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} \qquad (HI)$$

Luego,

$$(a+b)^{m+1} = (a+b)(a+b)^m$$

$$\overset{HI}{=} (a+b) \cdot \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} a^{k+1} b^{m-k} + \sum_{k=0}^{m} \binom{m}{k} a^k b^{m+1-k}$$

$$\qquad (j=k+1) \qquad\qquad\qquad (j=k)$$

$$= \sum_{j=1}^{m+1} \binom{m}{j-1} a^j b^{m+1-j} + \sum_{j=0}^{m} \binom{m}{j} a^j b^{m+1-j}$$

$$= \binom{m}{m} a^{m+1} b^0 + \sum_{j=1}^{m} \left[\binom{m}{j-1} + \binom{m}{j} \right] a^j b^{m+1-j}$$

$$+ \binom{m}{0} a^0 b^{m+1}$$

Ahora,
$$\binom{m}{m} = 1 = \binom{m+1}{m+1}$$
$$\binom{m}{0} = 1 = \binom{m+1}{0}$$
$$\binom{m}{j-1} + \binom{m}{j} = \binom{m+1}{j}$$

Luego,

$$(a+b)^{m+1} = \binom{m+1}{0} a^0 b^{m+1-0}$$
$$+ \sum_{j=1}^{m} \binom{m+1}{j} a^j b^{m+1-j}$$
$$+ \binom{m+1}{m+1} a^{m+1} b^{m+1-(m+1)}$$
$$= \sum_{j=0}^{m+1} \binom{m+1}{j} a^j b^{m+1-j}$$

Luego, el teorema queda demostrado por inducción $\square$

• **Obs.**:

$$\overbrace{\textcircled{1}\ \textcircled{2}\ \bigcirc \cdots\cdots \bigcirc\ \textcircled{m}}^{m}$$

Elijo k

$\binom{m}{k}$ = número de grupo de k elementos entre m elementos

= número de subconjunto de $\{1,2,\dots,m\}$ de k elementos

Número de subconjunto de $\{1,2,\dots,m\}$

$$= \sum_{k=0}^{m} (\text{número de subconjunto de tamaño } k)$$

$$= \sum_{k=0}^{m} \binom{m}{k} = 2^m$$

• Ej: Demostrar que $\displaystyle\sum_{k=0}^{m} \binom{m}{k}^2 = \binom{2m}{m}$

Sol:

$$\overbrace{O\ O\ O \cdots\cdots O\ O}^{2m} \quad : \quad \binom{2m}{m}$$

Elijo m

$$\overbrace{O\ O \cdots O}^{m}\ \overbrace{O \cdots\cdots O\ O}^{m} \quad : \quad \binom{m}{k}\binom{m}{m-k}$$

Elijo k Elijo m-k

$$\Rightarrow \binom{2m}{m} = \sum_{k=0}^{m} \binom{m}{k}\binom{m}{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k}\binom{m}{k}$$

$$= \sum_{k=0}^{m} \binom{m}{k}^2$$

$\square$