PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
FACULTAD DE MATEMATICAS
DEPARTAMENTO DE MATEMATICA
Primer semestre 2022

Ayudantía 14 - MAT1610

1. Determine:

   (a) $\displaystyle\int e^{-x}\ln(1+e^x)\,dx$

   (b) $\displaystyle\int_0^{\frac{1}{2}} \frac{xe^{\operatorname{arcsen}(x)}}{\sqrt{1-x^2}}dx$

   Solución:

   (a) Entonces, considerando $u=\ln(1+e^x)$ y $dv=e^{-x}dx$ se tiene que $du=\frac{e^x}{1+e^x}dx$ y $v=\displaystyle\int dv=\int e^{-x}dx=-e^{-x}$. Así, aplicando integración por partes

$$\begin{aligned}
\int e^{-x}\ln(1+e^x)\,dx &= -\ln(1+e^x)\,e^{-x}+\int e^{-x}\frac{e^x}{1+e^x}dx\\
&= -\ln(1+e^x)\,e^{-x}+\int \frac{1}{1+e^x}dx\\
&= -\ln(1+e^x)\,e^{-x}+\int \frac{1+e^x-e^x}{1+e^x}dx\\
&= -\ln(1+e^x)\,e^{-x}+\int \frac{1+e^x}{1+e^x}dx-\int\frac{e^x}{1+e^x}dx\\
&= -\ln(1+e^x)\,e^{-x}+\int 1dx-\int\frac{e^x}{1+e^x}dx\\
&= -\ln(1+e^x)\,e^{-x}+x-\ln(|1+e^x|)+C\\
&= -\ln(1+e^x)\,e^{-x}+x-\ln(1+e^x)+C\\
&= -\ln(1+e^x)\left(e^{-x}+1\right)+x+C
\end{aligned}$$

   (b) Notar que $\frac{d}{dx}\operatorname{arcsen}(x)=\frac{1}{\sqrt{1-x^2}}$, entonces haciendo la sustitución $t=\operatorname{arcsen}(x)$, se tiene que $dt=\frac{1}{\sqrt{1-x^2}}dx$, $x=\operatorname{sen}(t)$, si $x=0$ entonces $t=\operatorname{arcsen}(0)=0$ y si $x=\frac{1}{2}$ entonces $t=\operatorname{arcsen}\left(\frac{1}{2}\right)=\frac{\pi}{6}$ y $\displaystyle\int_0^{\frac{1}{2}}\frac{xe^{\operatorname{arcsen}(x)}}{\sqrt{1-x^2}}dx=\int_0^{\frac{\pi}{6}}\operatorname{sen}(t)e^t dt$ cuyo valor puede obtenerse integrando por partes.
   Considerando la integral indefinida, aplicando integración pr partes dos veces:
   $u=\operatorname{sen}(t)$, $dv=e^t$, $du=\cos(t)dt$, $v=e^t$

$u = \cos(t)$, $dv = e^t$, $du = -\text{sen}(t)dt$, $v = e^t$

$$\begin{aligned}
\int \text{sen}(t)e^t dt &= e^t\text{sen}(t) - \int \cos(t)e^t dt \\
&= e^t\text{sen}(t) - \left(\cos(t)e^t + \int \text{sen}(t)e^t dt\right) \\
&= e^t\text{sen}(t) - \cos(t)e^t - \int \text{sen}(t)e^t dt \\
&= e^t\left(\text{sen}(t) - \cos(t)\right) - \int \text{sen}(t)e^t dt
\end{aligned}$$

Entonces, $\displaystyle\int \text{sen}(t)e^t dt = \frac{e^t}{2}\left(\text{sen}(t) - \cos(t)\right) + C$ y

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} \text{sen}(t)e^t dt &= \frac{e^t}{2}\left(\text{sen}(t) - \cos(t)\right)\Big|_0^{\frac{\pi}{6}} \\
&= \frac{e^{\frac{\pi}{6}}}{4}\left(1 - \sqrt{3}\right) + \frac{1}{2}
\end{aligned}$$

2. Determine $\underbrace{\int \cos^2(8\pi x)\text{sen}^2(5x)dx}_{I}$.

Solución:
Reescribir usando primero la identidad $\cos(\alpha)\text{sen}(\beta) = \frac{\text{sen}(\alpha+\beta)-\text{sen}(\alpha-\beta)}{2}$ y después que $\text{sen}^2(\theta) = \frac{1-\cos(2\theta)}{2}$ y $\text{sen}(\alpha)\text{sen}(\beta) = \frac{cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$. Entonces,

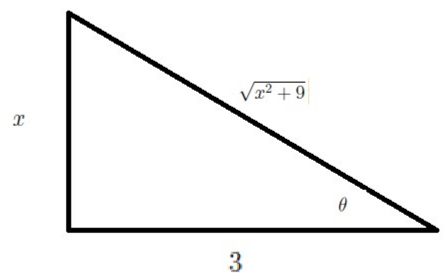
$$\begin{aligned}
I &= \int \left(\cos(8\pi x)\text{sen}(5x)\right)^2 dx \\
&= \int \left(\frac{\text{sen}((8\pi+5)x) - \text{sen}((8\pi-5)x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int \left[\text{sen}^2((8\pi+5)x) - 2\text{sen}((8\pi+5)x)\text{sen}((8\pi-5)x) + \text{sen}^2((8\pi-5)x)\right] dx \\
&= \frac{1}{4}\int \text{sen}^2((8\pi+5)x)dx - \frac{1}{2}\int \text{sen}((8\pi+5)x)\text{sen}((8\pi-5)x)dx \\
&+ \frac{1}{4}\int \text{sen}^2((8\pi-5)x)dx \\
&= \frac{1}{8}\int \left(1 - \cos(2(8\pi+5)x)\right) dx - \frac{1}{4}\int \left(\cos(10x) - \cos(16\pi x)\right) dx \\
&+ \frac{1}{8}\int \left(1 - \cos(2(8\pi-5)x)\right) dx \\
&= \frac{1}{8}\left(x - \frac{\text{sen}(2(8\pi+5)x)}{2(8\pi+5)}\right) - \frac{1}{4}\left(\frac{\text{sen}(10x)}{10} - \frac{\text{sen}(16\pi x)}{16\pi}\right) \\
&+ \frac{1}{8}\left(x - \frac{\text{sen}(2(8\pi-5)x)}{2(8\pi-5)}\right) + C
\end{aligned}$$

3. Determine $\underbrace{\int \frac{x^2}{\sqrt{x^2+9}}dx}_{I}$

Solución:

Considerar la sustitución $x = 3\tan(\theta)$ o $\frac{x}{3} = \tan(\theta)$ con $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ entonces $dx = 3\sec^2(\theta)d\theta$ y $\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = \sqrt{9(\tan^2(\theta)+1)} = 3|\sec(\theta)| = 3\sec(\theta)$

$\sec(\theta) > 0$ porque $\cos(\theta) > 0$ para $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.



$$\begin{aligned} I = \int \frac{x^2}{\sqrt{x^2+9}}dx &= \int \frac{9\tan^2(\theta)3\sec^2(\theta)d\theta}{\sqrt{9(\tan^2(\theta)+1)}} \\ &= 9\underbrace{\int \tan^2(\theta)\sec(\theta)d\theta}_{I_1} \end{aligned}$$

Una forma para determinar I_1 es aplicar integración por partes

$$\begin{aligned} I_1 &= \int \tan^2(\theta)\sec(\theta)d\theta \\ &= \int \underbrace{\tan(\theta)}_{u}\underbrace{\tan(\theta)\sec(\theta)d\theta}_{dv} \\ &= \tan(\theta)\sec(\theta) - \int \sec(\theta)\sec^2(\theta)d\theta \\ &= \tan(\theta)\sec(\theta) - \int \sec(\theta)\left(\tan^2(\theta)+1\right)d\theta \\ &= \tan(\theta)\sec(\theta) - \underbrace{\int \sec(\theta)\tan^2(\theta)d\theta}_{I_1} - \int \sec(\theta)d\theta \\ &= \tan(\theta)\sec(\theta) - I_1 - \ln\left(|\sec(\theta)+\tan(\theta)|\right) \end{aligned}$$

Por lo tanto,

$$2I_1 = \tan(\theta)\sec(\theta) - \ln\left(|\sec(\theta) + \tan(\theta)|\right)$$

es decir,

$$I_1 = \frac{\tan(\theta)\sec(\theta) - \ln\left(|\sec(\theta) + \tan(\theta)|\right)}{2} + K$$

Así,

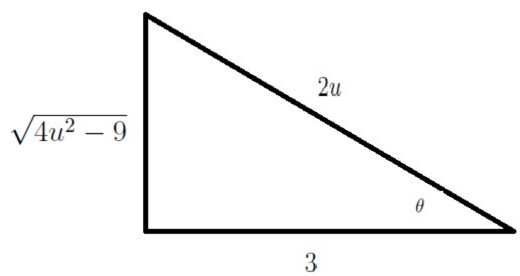
$$\begin{aligned}
I &= 9I_1 \\
&= \frac{9}{2}\left(\tan(\theta)\sec(\theta) - \ln\left(|\sec(\theta) + \tan(\theta)|\right)\right) + C \\
&= \frac{9}{2}\left(\frac{x}{3}\frac{\sqrt{x^2+9}}{3} - \ln\left(\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right)\right) + C \\
&= \frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln\left(\left|\frac{x + \sqrt{x^2+9}}{3}\right|\right) + C
\end{aligned}$$

4. Determine $\underbrace{\int \frac{1}{\sqrt{(4x^2-24x+27)^3}}dx}_{I}$.

Solución:
Note que, $\int \frac{1}{\sqrt{(4x^2-24x+27)^3}}dx = \int \frac{1}{\sqrt{(4(x-3)^2-9)^3}}dx$. Entonces, consideranto la sustitución $u = x-3$ se tiene que $du = dx$ y

$$\int \frac{1}{\sqrt{(4x^2-24x+27)^3}}dx = \int \frac{1}{\sqrt{(4u^2-9)^3}}du$$

Ahora, haciendo la sustitución $\frac{2}{3}u = \sec(\theta)$, $0 \leq \theta < \frac{\pi}{2}$ o $\pi \leq \theta < \frac{3\pi}{2}$, entonces $du = \frac{3}{2}\sec(\theta)\tan(\theta)d\theta$



$$
\begin{aligned}
I &= \int \frac{1}{\sqrt{(4u^2-9)^3}}du \\
&= \int \frac{1}{\sqrt{9^3\left(\left(\frac{2}{3}u\right)^2-1\right)^3}}du \\
&= \frac{1}{27}\int \frac{1}{\sqrt{(\sec^2(\theta)-1)^3}}\frac{3}{2}\sec(\theta)\tan(\theta)d\theta \\
&= \frac{1}{18}\int \frac{\sec(\theta)\tan(\theta)d\theta}{\tan^3(\theta)} \\
&= \frac{1}{18}\int \frac{\sec(\theta)d\theta}{\tan^2(\theta)} \\
&= \frac{1}{18}\int \frac{\cos(\theta)d\theta}{\operatorname{sen}^2(\theta)} \\
&= -\frac{1}{18}\frac{1}{\operatorname{sen}(\theta)}+C \\
&= -\frac{1}{18}\frac{2u}{\sqrt{4u^2-9}}+C \\
&= -\frac{1}{18}\frac{2(x-3)}{\sqrt{4(x-3)^2-9}}+C \\
&= -\frac{1}{9}\frac{(x-3)}{\sqrt{4(x-3)^2-9}}+C
\end{aligned}
$$

5. Considere la sustitución $u = \sqrt{\frac{1-x}{1+x}}$ en

$$\underbrace{\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx}_{I}$$

y resuelva.
Solución:
Sea $u = \sqrt{\frac{1-x}{1+x}}$ entonces,
$du = \frac{1}{2\sqrt{\frac{1-x}{1+x}}}\left(\frac{1-x}{1+x}\right)' dx = \frac{1}{2\sqrt{\frac{1-x}{1+x}}}\left(-1+\frac{2}{1+x}\right)' dx = \frac{1}{\sqrt{\frac{1-x}{1+x}}}\left(-\frac{1}{(1+x)^2}\right) dx$
Note que $u = \sqrt{\frac{1-x}{1+x}} = \sqrt{-1+\frac{2}{1+x}}$, entonces $u^2 = -1+\frac{2}{1+x}$, $\frac{u^2+1}{2} = \frac{1}{1+x}$ ó $\frac{(u^2+1)^2}{4} = \frac{1}{(1+x)^2}$, $x = \frac{2}{u^2+1} - 1 = \frac{1-u^2}{u^2+1}$ y $x^2 = \frac{(1-u^2)^2}{(u^2+1)^2}$. Por lo tanto,
$du = -\frac{1}{u}\frac{(u^2+1)^2}{4}dx$ o $dx = -\frac{4u}{(u^2+1)^2}du$ y

$$\begin{aligned}
I = &= -\int u\frac{(u^2+1)^2}{(1-u^2)^2}\frac{4u}{(u^2+1)^2}du \\
&= -4\int \frac{u^2}{(1-u^2)^2}du \\
&= -4\int \frac{u^2}{(1-u)^2(1+u)^2}du \\
&= -4\left[\int \frac{A}{1-u}du + + \int \frac{B}{(1-u)^2}du + \int \frac{C}{1+u}du + \int \frac{D}{(1+u)^2}du\right] \\
&= -4\left[-A\ln(|1-u|) + B\frac{1}{1-u} + C\ln(|1+u|) - D\frac{1}{1+u}\right] + K \\
&= -4\left[\frac{1}{4}\ln(|1-u|) + \frac{1}{4(1-u)} - \frac{1}{4}\ln(|1+u|) - \frac{1}{4(1+u)}\right] + K \\
&= -4\left[\frac{1}{4}\ln(|1-u|) + \frac{1}{4(1-u)} - \frac{1}{4}\ln(|1+u|) - \frac{1}{4(1+u)}\right] + K \\
&= -\ln(|1-u|) - \frac{1}{1-u} + \ln(|1+u|) + \frac{1}{1+u} + K \\
&= -\ln(|1-u|) + \frac{1}{u-1} + \ln(|1+u|) + \frac{1}{1+u} + K \\
&= -\ln\left(\left|1-\sqrt{\frac{1-x}{1+x}}\right|\right) + \frac{1}{\sqrt{\frac{1-x}{1+x}}-1} + \ln\left(\left|1+\sqrt{\frac{1-x}{1+x}}\right|\right) + \frac{1}{1+\sqrt{\frac{1-x}{1+x}}} + K
\end{aligned}$$

El valor de las constantes A, B, C y D se obtiene de plantear que para todo u:

$$u^2 = A(1-u)(1+u)^2 + B(1+u)^2 + C(1-u)^2(1+u) + D(1-u)^2$$

o

$$u^2 = A(1-u^2)(1+u) + B(1+u)^2 + C(1-u)(1-u^2) + D(1-u)^2$$

o

$$u^2 = A(1+u-u^2-u^3) + B(1+2u+u^2) + C(1-u-u^2+u^3) + D(1-2u+u^2)$$

es decir,

$$u^2 = (-A+C)u^3 + (-A+B-C+D)u^2 + (A+2B-C-2D)u + (A+B+C+D)$$

lo cual implica que:

$$\begin{aligned}
-A+C &= 0 \\
-A+B-C+D &= 1 \\
A+2B-C-2D &= 0 \\
A+B+C+D &= 0
\end{aligned}$$

De la primera ecuación se tiene que $A = C$, sustituyendo en la tercera se obtiene que $B-D = 0$, es decir, $B = D$. Usando dichas relaciones en la cuarta ecuación se obtiene que $2A+2B = 0$ ó $B = -A$ y por ello, $B = -C$

Ahora sustituyendo en la segunda ecuación se tiene que $-A-A-A-A = 1$, esto es, $-4A = 1$ y en consecuencia, $A = -\frac{1}{4}$, $C = -\frac{1}{4}$, $B = \frac{1}{4} = D$