CLASE 20: SUMATORIAS

• **DEF**: Sea $(a_n)_n$ una sucesión. Definimos

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

• Ej: $\displaystyle\sum_{k=1}^{4} \underbrace{(2k+1)}_{a_k} = a_1 + a_2 + a_3 + a_4$

$= 3 + 5 + 7 + 9$

$= 24$

• Ej: $\displaystyle\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n} = n$

$(a_k = 1)$

• Ej: $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$= \frac{1}{2} n(n+1)$

- **Obs**: Sea $(a_n)_n$ una sucesión y sea

$$S_n = \sum_{k=1}^{n} a_k$$

Luego, $(S_n)_n$ es una sucesión: es la sucesión de las sumas parciales de $(a_n)_n$.

$$\begin{cases} S_n = S_{n-1} + a_n, & n \geq 2 \\ S_1 = a_1 \end{cases}$$

- **Obs**: $\displaystyle\sum_{k=1}^{n} a_k = \sum_{j=1}^{n} a_j = \sum_{i=1}^{n} a_i$

$$\cancel{\sum_{n=1}^{n} a_n}$$

- **Obs**: $\displaystyle\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$

- **Obs**: $\displaystyle\sum_{k=m}^{n} 1 = n - m + 1$

$$\sum_{k=2}^{3} 1 = 1 + 1 = 3-2+1$$

- <u>Obs</u>: Sucesión: $f: \mathbb{N} \longrightarrow \mathbb{R}$

$$\rightsquigarrow a_n = f(n)$$

$$a_k = f(k)$$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) =$$ suma de los valores de f para la variable variando todos los valores entre 1 y k

En general, $f: \mathbb{Z} \longrightarrow \mathbb{R}$

$$\sum_{k=-27}^{11} a_k = a_{-27} + a_{-26} + \dots + a_9 + a_{10} + a_{11}$$

- <u>Ej</u>: $$\sum_{k=6}^{9} (2k+1) = (2\cdot 6+1) + (2\cdot 7+1) + (2\cdot 8+1) + (2\cdot 9+1)$$

$$= 13 + 15 + 17 + 19$$

$$= 64$$

• <u>Obs</u>: $\displaystyle\sum_{k=m}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{m-1} a_k$

• <u>Ej</u>: $\displaystyle\sum_{k=7}^{35} k = \sum_{k=1}^{35} k - \sum_{k=1}^{6} k$

$$= \frac{1}{2}\cdot 35\cdot 36 - \frac{1}{2}\cdot 6\cdot 7$$

• <u>Obs</u>:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

$$= a_1 + \cdots + a_{m-1} + a_m + \cdots + a_n$$

$$- (a_1 + \cdots + a_{m-1})$$

$$= \sum_{k=1}^{n} a_k - \sum_{k=1}^{m-1} a_k$$

- $$\sum_{k=0}^{n} k = \sum_{k=1}^{n} k$$ porque $a_0 = 0$ en este caso

- **PROPOSICIÓN**: Sean $(a_n)_n$ y $(b_n)_n$ dos sucesiones, $c \in \mathbb{R}$. Luego,

i) $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

ii) $$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$$

iii) $$\left(\sum_{k=1}^{n} a_k\right) \cdot \left(\sum_{j=1}^{m} b_j\right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{m} a_k b_j\right) = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_k b_j\right)$$

• Ej: $\sum_{k=1}^{100}(2k+1) \overset{i)}{=} \sum_{k=1}^{100} 2k + \sum_{k=1}^{100} 1$

$$\overset{ii)}{=} 2\sum_{k=1}^{100} k + \sum_{k=1}^{100} 1$$

$$= 2 \cdot \frac{1}{2} \cdot 100 \cdot 101 + 100$$

$$= 102 \cdot 100 = 10200$$

• DEM. PROPOSICIÓN:

i) $\sum_{k=1}^{m}(a_k + b_k) = \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} b_k$

Por inducción:

• $m=1$: $\sum_{k=1}^{1}(a_k + b_k) = a_1 + b_1$

$$= \sum_{k=1}^{1} a_k + \sum_{k=1}^{1} b_k$$

• Supongamos que

$$\sum_{k=1}^{m}(a_k + b_k) = \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} b_k$$

$$\sum_{k=1}^{m+1}(a_k+b_k)=\sum_{k=1}^{m}(a_k+b_k)+a_{m+1}+b_{m+1}$$

$$\overset{HI}{=}\sum_{k=1}^{m}a_k+\sum_{k=1}^{m}b_k+a_{m+1}+b_{m+1}$$

$$=\sum_{k=1}^{m+1}a_k+\sum_{k=1}^{m+1}b_k$$

- Luego, por inducción, la propiedad es válida para todo $n\geq 1$.

ii) $$\sum_{k=1}^{m}ca_k=c\sum_{k=1}^{m}a_k$$

- $\underline{n=1}$: $\sum_{k=1}^{1}ca_k=ca_1=c\sum_{k=1}^{1}a_k$

- Supongamos que $\sum_{k=1}^{m}ca_k=c\sum_{k=1}^{m}a_k$

$$\bullet \sum_{k=1}^{m+1} c a_k = \sum_{k=1}^{m} c a_k + c a_{m+1}$$

$$\overset{HI}{=} c \sum_{k=1}^{m} a_k + c a_{m+1}$$

$$= c \left(\sum_{k=1}^{m} a_k + a_{m+1} \right)$$

$$= c \sum_{k=1}^{m+1} a_k$$

- Luego, por inducción, la propiedad es verdadera para todo $m \geq 1$

iii) $$\left(\sum_{k=1}^{m} a_k \right) \cdot \left(\sum_{j=1}^{m} b_j \right) = \sum_{k=1}^{m} \left(\sum_{j=1}^{m} a_k b_j \right)$$

$$\left(\sum_{k=1}^{m} a_k \right) \underbrace{\left(\sum_{j=1}^{m} b_j \right)}_{C} \overset{ii)}{=} \sum_{k=1}^{m} \left(\underbrace{a_k}_{C'} \sum_{j=1}^{m} b_j \right)$$

$$\overset{ii}{=} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_k b_j \right)$$

La segunda identidad se demuestra de la misma forma

$\square$

- Ej: Sea $a_n = a + (n-1)d$

$$\sum_{k=1}^{n} \left(a + (k-1)d\right) = \sum_{k=1}^{n} (a + kd - d)$$

$$\overset{i)}{=} \sum_{k=1}^{n} a + \sum_{k=1}^{n} kd - \sum_{k=1}^{n} d$$

$$\overset{ii)}{=} a \sum_{k=1}^{n} 1 + d \sum_{k=1}^{n} k - d \sum_{k=1}^{n} 1$$

$$= an + d \cdot \frac{1}{2} n(n+1) - dn$$

$$= n\left(a + \frac{n+1}{2}\cdot d - d\right)$$

$$= n\left(a + \frac{n-1}{2}\, d\right)$$

$$= \frac{n}{2}\left(2a + (n-1)d\right)$$

- <u>Obs</u>: Otra forma:

$$\sum_{k=1}^{n}\left(a + (k-1)d\right) \overset{i)}{=} \sum_{k=1}^{n} a + \sum_{k=1}^{n}(k-1)d$$

$$\overset{ii)}{=} an + d\sum_{k=1}^{n}(k-1)$$

Ahora, $\displaystyle\sum_{k=1}^{n}(k-1) = \sum_{j=0}^{n-1} j = \sum_{j=1}^{n-1} j$

$j = k-1$

$1 \leq k \leq n$

$\rightarrow 0 \leq k-1 \leq n-1$

$\rightarrow 0 \leq j \leq n-1$

$$= \frac{1}{2}\cdot(n-1)\cdot n$$

Luego,

$$\sum_{k=1}^{n}\left(a+(k-1)d\right) = an + \frac{d}{2}(n-1)n$$

$$= n\left(a + \frac{n-1}{2}d\right)$$

$$= \frac{n}{2}\left(2a + (n-1)d\right)$$