Martes 7/06/16

Estadística
Ayudantía 13

(1) $X \sim \text{Exponencial}(\lambda)$ $E(X) = 800 = \frac{1}{\lambda}$ $\Rightarrow$ $\boxed{\lambda = \frac{1}{800}}$

$Var(X) = 800^2$

Luego por T.C.L.

$\bar{X} \overset{\cdot}{\sim} \text{Normal}\left(E(x), \frac{Var(x)}{n}\right)$

$\left(800, \frac{800^2}{n}\right)$

$\Rightarrow \mathbb{P}(\bar{X} < 775)$

$\Rightarrow \mathbb{P}\left(Z < \frac{775 - 800}{800/\sqrt{T6}}\right)$

$\Rightarrow \Phi\left(-\frac{1}{8}\right) = 1 - \Phi\left(\frac{1}{8}\right)$

$= 1 - \Phi(0.125)$

$= 1 - 0.5478 = 0.4522$ aproximado.

(2) X: ingreso de habitantes de un país.

$X \sim \text{Uniforme}(200.000, 500.000)$

$X_1, \dots, X_{100} \sim \text{Uniforme}(200.000, 500.000)$

$E(X) = \frac{700.000}{2} = 350.000$

$Var(x) = \frac{(300.000)^2}{12} = \cancel{\frac{3000000}{12}}$

Luego, por T.C.L

$\sum_{i=1}^{100} X_i \overset{\cdot}{\sim} \text{Normal}(n\, E(X), n \cdot Var(x))$

$\left(100 \cdot 350000, 100 \cdot \frac{300000^2}{12}\right)$

Nos piden $\mathbb{P}(\sum_i X_i > 30.000)$

$$= \mathbb{P}\left(Z > \frac{30.000.000 - 35.000.000}{10.300.000} \cdot \sqrt{12}\right)$$

$-5.000.000$

$$= \mathbb{P}(Z > -5{,}77335)$$

$$= 1 - \mathbb{P}(Z \leq -5{,}77335)$$

$$= 1 - (1 - \mathbb{P}(Z \leq 5{,}77335)) = \Phi(5{,}77335) \approx 1$$

3) La función de verosimilitud

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta)$$

0	1	2	3
2	3	3	2

$$= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^1 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2$$

$$= \frac{2^5}{3^{10}} \theta^5 (1-\theta)^5$$

$$\ell(\theta|x) = 5\log 2 - 10\log 3 + 5\cdot\log\theta + 5\cdot\log(1-\theta)$$

$$\frac{d\,\ell(\theta|x)}{d\theta} = \frac{5}{\hat\theta} - \frac{5}{1-\hat\theta} = 0$$

$$\frac{5}{\hat\theta} = \frac{5}{1-\hat\theta} \Rightarrow \hat\theta = 1-\hat\theta \Rightarrow \boxed{\hat\theta = \frac{1}{2}} \leftarrow \text{E.M.V.}$$

0.5

Verificamos que sea máximo:

$$\frac{d^2\,\ell(\theta|x)}{d\theta^2} = -\frac{5}{\theta^2} - \frac{5}{(1-\theta)^2} < 0 \Rightarrow \text{luego, es máximo.}$$

④ $X_1, \dots, X_n \overset{iid}{\sim} f_X(x) = e^{-x+\theta}, x \geq \theta$

$f_X(x) = e^{-x+\theta}, x \geq \theta.$

a) $\mathcal{L}(\theta|\underline{x}) = \prod_{i=1}^{n} e^{-x_i+\theta} I\{x \geq \theta\}$

$= e^{-\sum x_i + n\theta} \cdot I(\min x_i \geq \theta)$

luego $\hat{\theta}_{E.M.V.} = \min x_i$ pues es el valor de θ que (*) me maximiza la ~~verosimilitud~~ verosimilitud.

b) Por método de momentos

$E(X) = \theta + 1 = \frac{\sum_i x_i}{n} = \overline{X}$

$\Rightarrow \hat{\theta} = \overline{X} - 1$

c) Prefiero $\hat{\theta}_{E.M.V.}$ pues toma en cuenta los valores para los cuales la función toma valores $\neq 0$.

⑤ $X_1, \dots, X_n \overset{iid}{\sim} \text{Uniforme}(0, \theta)$ $c > 0$

y $\hat{\theta} = c \cdot \overline{X}$

a) $ECM(\hat{\theta}) = \text{Sesgo}^2(\hat{\theta}) + Var(\hat{\theta})$

$\text{Sesgo}(\hat{\theta}) = E(\hat{\theta}) - \theta$

$= E(c \cdot \overline{X}) - \theta = c \cdot E(\overline{X}) - \theta$

$= c \cdot \frac{\theta}{2} - \theta$ //

$Var(\hat{\theta}) = Var(c\overline{X}) = c^2 \cdot Var(\overline{X})$

$= c^2 \cdot \frac{\theta^2}{12n}$

$\Rightarrow \boxed{ECM(\hat{\theta}) = \left(\frac{c\theta}{2} - \theta\right)^2 + \frac{c^2\theta^2}{12n}}$

$= \frac{c^2\theta^2}{4} - \frac{2c\theta^2}{2} + \theta^2 + \frac{c^2\theta^2}{12n}$

$\star E(\overline{X}) = E\left(\frac{\sum x_i}{n}\right) = \frac{E(\sum X_i)}{n} = \frac{\sum E(X_i)}{n} = \frac{\cancel{n}}{\cancel{n}} E(X_i) = \frac{\theta}{2}$

$\star Var(\overline{X}) = \frac{1}{n^2} Var(\sum_i X_i) = \frac{\sum Var(X_i)}{n^2} = \frac{n\theta^2}{12n^2} = \frac{\theta^2}{12n}$

b) $ECM(\hat{\theta}) = \theta^2\left[\frac{c^2}{4} - c + 1 + \frac{c^2}{12n}\right]$ → debo minimizar esto.

$c^2 - 4c + 4 = 0$

$(c-2)^2 = 0$

2

$= \theta^2\left[\frac{3nc^2 - 12cn + 12n + c^2}{12n}\right]$

$\frac{d\,ECM(\hat{\theta})}{dc} = \theta^2\left[\frac{2c}{4} - 1 + \frac{2c}{12n}\right]$

$= \theta^2\left[\frac{c}{2} - 1 + \frac{c}{6n}\right]$

$\frac{c}{2} - 1 = 0 \quad \boxed{c = 2}$

$\frac{d^2 ECM(\hat{\theta})}{dc^2} = \theta^2\left[\frac{1}{2} + \frac{1}{6n}\right] > 0 \Rightarrow c = 2$ minimizará el valor de $ECM(\hat{\theta})$!

6 tamaño $2n$

$E(X) = \mu$

$Var(X) = \sigma^2$

$\overline{X}_1 = \frac{1}{2n}\sum_{i=1}^{2n} X_i \qquad \overline{X}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i$ est. de μ.

a) $Sesgo(\overline{X}_1) = E(\overline{X}_1) - \mu$

$= \frac{1}{2n}\sum_{i=1}^{2n} E(X_i) - \mu$

$= \frac{1}{\cancel{2n}}\cancel{2n}\cdot\mu - \mu = 0$ insesgado!

$Sesgo(\overline{X}_2) = E(\overline{X}_2) - \mu$

$= \frac{1}{n}\sum_{i=1}^{n} E(X_i) - \mu$

$= \frac{n}{n}\mu - \mu = 0$ insesgado!

b) $ECM(\overline{X}_1) = Var(\overline{X}_1) + \cancel{Sesgo^2(\overline{X}_1)}^{\,0}$ X_i independientes.

$= Var(\overline{X}_1) = Var\left(\frac{1}{2n}\sum_{i=1}^{2n} X_i\right) = \frac{1}{4n^2}\sum_{i=1}^{2n} Var(X_i)$

$= \frac{1}{4n^2}\, 2n\,\sigma^2 = \frac{\sigma^2}{2n}$

$ECM(\overline{X}_2) = Var(\overline{X}_2) + \cancel{Sesgo^2(\overline{X}_2)}^{\,0}$

$= Var(\overline{X}_2) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2 n}{n^2} = \frac{\sigma^2}{n}$

Prefiero $\overline{X}_1$ pues se va ligeramente más rápido a cero.

(7) $X_1, \dots, X_n \overset{iid}{\sim}$ Normal(μ, σ^2)

$$\hat{\mu} = \bar{X}$$
$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

a) $E(\hat{\mu}) = E(\bar{X}) = \mu \Rightarrow$ Sesgo$(\hat{\mu}) = 0 \rightarrow$ insesgado!

$$E(\hat{\sigma}^2) = \cancel{\frac{1}{n} \sum_{i=1}^{n} E(X_i - \bar{X})^2} =$$

$$\cancel{= \frac{1}{n} E\left(\sum X_i - \bar{X} n\right)^2 = \frac{1}{n} E\left(\sum X_i\right)^2}$$

$$= \cancel{\frac{1}{n} E\left(\sum (X_i - \bar{X})^2\right)} = \frac{1}{n} E\left(\sum X_i^2 - n\bar{X}^2\right)$$

$$\cancel{= \frac{1}{n} \sum E(X_i^2)}$$

$$\sum X_i^2 - 2\underbrace{\sum X_i \bar{X}}_{n\bar{X}} + \bar{X}^2 n$$
$$\sum (X_i)^2 - 2n\bar{X}^2 + n\bar{X}$$
$$\sum (X_i^2) - n\bar{X}^2$$

$$E(\hat{\sigma}^2) = \frac{1}{n} E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right)$$

$$= \frac{1}{n} E\left(\sum X_i^2 - n\bar{X}^2\right)$$

$$= \frac{1}{n}\left(\sum E(X_i^2) - n \cdot E(\bar{X}^2)\right)$$

$$= \frac{1}{n}\left\{\sum \left(Var(X_i) + E^2(X_i)\right) - n \cdot E(\bar{X}^2)\right\}$$

$$= \frac{1}{n}\left\{\sum_{i=1}^{n} (\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right\} = \sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2}$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \left(\frac{n-1}{n}\right)\sigma^2$$

$$= \sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2} = 1 - \frac{1}{n}\left(\sigma^2\right)$$

$$\cancel{= \frac{n-1}{n}(\sigma^2) - \sigma^2 = -\frac{\sigma^2}{n} \neq 0 \text{ si } \sigma \neq 0}$$

$$\Rightarrow \text{Sesgo}(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2$$
$$= \cancel{\sigma^2} - \frac{\sigma^2}{n} - \cancel{\sigma^2} = -\frac{\sigma^2}{n} < 0$$

b)

Propongo estimar $\hat{\sigma} = S_{n-1} = \frac{n}{n-1} S_n$

$$\Rightarrow \mathbb{E}(S_{n-1}) = \mathbb{E}\left(\frac{n}{n-1} S_n\right)$$

$$= \mathbb{E}(S_n) \cdot \frac{n}{n-1}$$

$$= \left(\sigma^2 - \frac{\sigma^2}{n}\right)\left(\frac{n}{n-1}\right)$$

$$= \sigma^2\left(1 - \frac{1}{n}\right)\left(\frac{n}{n-1}\right)$$

$$= \sigma^2\left(\frac{n-1}{n}\right)\left(\frac{n}{n-1}\right) = \sigma^2$$

$$\Rightarrow \text{Sesgo}(S_{n-1}) = \mathbb{E}(S_{n-1}) - \sigma^2$$

$$= \sigma^2 - \sigma^2 = 0$$ es insesgado!