Ayudantía 10
Intro A Estadística.

(1) $f_X(x) = \dfrac{1}{b-a} \quad a < x < b \qquad \theta = (a,b)$

$= \dfrac{1}{b-a} \, I\{a < x < b\}(x)$

$\Rightarrow f_\theta(x) = L(\theta \mid \underline{x}) = \prod_{i=1}^{n} \dfrac{1}{b-a} \, I\{a < x < b\}(x)$

$= \underbrace{\dfrac{1}{(b-a)^n} \cdot I\{\min\{x_i\} > a\} \cdot I\{\max\{x_i\} < b\}}_{g(T(\underline{x}), \theta)}$

Luego, por teo de factorización un estadístico suficiente para $\theta = \{\min\{x_i\}, \max\{x_i\}\}$

(2) $X_1, \dots, X_n \sim Normal(\mu, \sigma^2)$

$L(\mu, \sigma^2 / \underline{x}) = \prod_{i=1}^{n} \dfrac{1}{(2\pi\sigma^2)^{1/2}} \cdot \exp\left\{ \dfrac{-1}{2\sigma^2} (x-\mu)^2 \right\}$

$= \dfrac{1}{(2\pi\sigma^2)^{n/2}} \cdot \exp\left\{ -\dfrac{1}{2\sigma^2} \cdot \sum_{i=1}^{n} (X_i - \mu)^2 \right\}$

$= \underbrace{\dfrac{1}{(2\pi\sigma^2)^{n/2}}}_{h(\underline{x})} \underbrace{\exp\left\{ \dfrac{-1}{2\sigma^2} \cdot \sum x_i^2 - 2\mu \sum x_i + n \cdot \mu^2 \right\}}_{g(T(\underline{x}), \theta)}$

Luego, por criterio de factorización $\hat{\theta} = \left\{ \sum_{i=1}^{n} x_i^2, \sum_{i=1}^{n} x_i \right\}$ es estimador de θ.

③ $f_X(x) = \theta \cdot x^{\theta-1}, \; 0 < x < 1$

a) Método de momentos.

$$\mathbb{E}(X) = \int_0^1 x \cdot \theta \cdot x^{\theta-1}\,dx = \int_0^1 \theta \cdot x^{\theta}\,dx = \theta \cdot \frac{x^{\theta+1}}{\theta+1}\Big]_0^1 = \frac{\theta}{\theta+1}$$

$$\Rightarrow \frac{\hat{\theta}}{\hat{\theta}+1} = \bar{x} \Rightarrow \hat{\theta} = \bar{x}\cdot\hat{\theta} + \bar{x} \Rightarrow \hat{\theta}(1-\bar{x}) = \bar{x} \Rightarrow \boxed{\hat{\theta} = \frac{\bar{x}}{1-\bar{x}}}$$

Luego $\bar{x} = \frac{0.4+0.7+0.9}{3} = \frac{2}{3}$

$$\Rightarrow \hat{\theta} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

nuestra estimación es 2.

b) Método de Máx. Verosimilitud.

$$f_\theta(\underline{x}) = L(\theta \mid \underline{x}) = \prod_{i=1}^{n} \theta \cdot x_i^{\theta-1}$$
$$= \theta^n \cdot \prod_{i=1}^{n} x_i^{\theta-1} \quad / \log.$$

$$\Rightarrow n \cdot \log\theta + (\theta-1)\cdot\log\left(\prod_{i=1}^{n} x_i\right)$$
$$= n\cdot\log\theta + (\theta-1)\sum_{i=1}^{n}\log(x_i)$$

Luego, $\frac{dL(\theta \mid \underline{x})}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0$

$$\frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0 \Rightarrow \frac{n}{\theta} = -\sum \log x_i$$

$$\Rightarrow \hat{\theta} = \frac{n}{-\sum\log x_i} \Rightarrow \hat{\theta} = \frac{3}{-(\log 0.4 + \log 0.7 + \log 0.9)}$$

$\therefore \hat{\theta}_{E.M.V} \cong 2.17.$

④ Sea $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Poisson}(\lambda)$

$$f_x(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$\mathcal{L}(\lambda|\underline{x}) = \prod_{i=1}^{n} \frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum x_i} \cdot e^{-n\lambda}}{\prod_{i=1}^{n} x_i!} \;/\log$$

$$\log(\mathcal{L}(\lambda|\underline{x})) = \sum_i x_i \cdot \log\lambda - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{d \log(\mathcal{L}(\lambda|\underline{x}))}{d\cancel{\theta}} = \frac{\sum x_i}{\lambda} - n$$

$$\therefore \frac{\sum x_i}{\lambda} - n = 0 \Rightarrow \hat{\lambda} = \frac{\sum x_i}{n} = \bar{x}$$

$$\therefore \hat{\lambda}_{E.M.V} = \bar{x} = \frac{0 \cdot 17 + 22 \cdot 1 + 7 \cdot 2 + 3 \cdot 3 + 4 \cdot 1}{50} = \frac{49}{50} = 0.98$$

Luego, nos pregunten

$$P(X = 0 / \hat{\lambda}_{E.M.V} = 0.98) = \frac{0.98^0 \, e^{-0.98}}{0!} = e^{-0.98} = 0.37$$

⑤ $X_1, \ldots, X_n \overset{i.i.d}{\sim} \text{Weibull}(\lambda, 2)$

$$f_x(x) = \frac{2x}{\lambda} \exp\left\{\frac{-x^2}{\lambda}\right\}, \quad x > 0, \; \lambda > 0$$

a. $$\mathcal{L}(\lambda|\underline{x}) = \prod_{i=1}^{n} \frac{2x_i}{\lambda} \exp\left\{\frac{-x_i^2}{\lambda}\right\}$$

$$= \frac{2^n \cdot \prod_{i=1}^{n} x_i}{\lambda^n} \cdot \exp\left\{-\frac{1}{\lambda} \sum_i x_i^2\right\}$$

c $$\log(L(\lambda|\underline{x})) = n \cdot \log 2 + \sum_{i=1}^{n} \log(x_i) - n \cdot \log(\lambda) - \frac{1}{\lambda} \sum_i x_i^2$$

$$\frac{d \log(L(\lambda|\underline{x}))}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2} \sum x_i^2 =$$

$$\Rightarrow \quad \frac{-n}{\hat{\lambda}} + \frac{1}{\hat{\lambda}^2} \sum x_i^2 = 0$$

$$\Rightarrow \quad \frac{n}{\hat{\lambda}} = \frac{1}{\hat{\lambda}^2} \sum x_i^2$$

$$\Rightarrow \quad \boxed{\hat{\lambda} = \frac{\sum x_i^2}{n}}$$

b: $\mathbb{E}(\hat{\lambda}_{e.M.V.}) = \mathbb{E}\left(\frac{\sum_{i=1}^{n} X_i^2}{n}\right) \quad \left(Y = x^2 \sim exp\left(\frac{1}{\lambda}\right)\right)$

$$= \frac{1}{n} \mathbb{E}\left(\sum Y_i\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(Y_i)$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} \lambda = \frac{1}{\cancel{n}} \cdot \cancel{n}\lambda = \lambda$$

• $Var(\hat{\lambda}_{E.M.P}) = Var\left(\frac{\sum X_i^2}{n}\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Y_i\right)$ Y_i independientes.

$$= \frac{1}{n^2} \sum Var(y_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \lambda^2 = \frac{n \cdot \lambda^2}{n^2} = \frac{\lambda^2}{n}$$

⑥ $X_1, \dots, X_n \sim \text{Gamma}(\alpha, \beta)$

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\beta x}, \quad x > 0$$

$$\Rightarrow \mathcal{L}(\alpha, \beta \mid \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x_i^{\alpha-1} \cdot e^{-\beta x_i}$$

$$= \frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \cdot \prod_{i=1}^{n} x_i^{\alpha-1} \cdot e^{-\beta \sum x_i} \quad / \log$$

$$\ell(\alpha, \beta \mid \underset{\sim}{x}) = n\alpha \cdot \log\beta - n \cdot \log(\underbrace{\Gamma(\alpha)}_{(\alpha-1)!}) + (\alpha-1) \sum_{i=1}^{n} \log(x_i) - \beta \sum_{i=1}^{n} x_i$$

$$\frac{d\, \ell(\alpha, \beta \mid \underset{\sim}{x})}{d\beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i$$

$$= \frac{n\hat{\alpha}}{\hat{\beta}} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\beta} = \frac{n\hat{\alpha}}{\sum_{i=1}^{n} x_i}$$

$$\mathbb{E}(X) = \frac{\alpha}{\beta} \qquad \text{Var}(X) = \frac{\alpha}{\beta^2}$$