1a. $X_i \overset{iid}{\sim} \exp(\lambda) \Rightarrow M_{\sum_{i=k}^{n} X_i}(t) = \{ M_{X_1}(t) \}^{(n-k)}$, $k=1,\ldots,n-1$

$X_1 \sim \exp(\lambda) \Leftrightarrow M_{X_1}(t) = (1 - t/\lambda)^{-1}$, $t<\lambda$

$\Rightarrow M_{\sum_{i=k}^{n} X_i}(t) = (1 - t/\lambda)^{-(n-k)}$, $t < \lambda$

$\Leftrightarrow \sum_{i=k}^{n} X_i \sim$ gama$(n-k, \lambda)$

$\therefore \begin{cases} S = \sum_{i=2}^{n} X_i \sim \text{gama}(n-1, \lambda) \\ T = \sum_{i=1}^{n} X_i \sim \text{gama}(n, \lambda) \end{cases}$

1b. Note que $T = X_1 + S$, con $X_1 \perp\!\!\!\perp S = \sum_{i=2}^{n} X_i$

$\Rightarrow$ dado $X_1 = x_1$, $T = x_1 + S$. y continuo $\forall x_1$

$$\begin{aligned} \Rightarrow F_{T|X_1=x_1}(t) &= P(T \leq t \mid X_1 = x_1) \\ &= P(x_1 + S \leq t \mid X_1 = x_1) \\ &= P(S \leq t - x_1 \mid X_1 = x_1) \\ &= P(S \leq t - x_1) \quad \text{ya que } S \perp\!\!\!\perp X_1 \\ &= F_S(t - x_1) \quad \forall\, t - x_1 > 0 \ \forall x_1 > 0 \end{aligned}$$

$$\Rightarrow f_{T|X_1=x_1}(t) = \frac{\partial F_{T|X_1=x_1}(t)}{\partial t} = \frac{\partial F_S(t-x_1)}{\partial t} = f_S(t-x_1) \quad \forall\, t > x_1 > 0$$

1c. $$f_{X_1|T=t}(x_1) = \frac{f_{T|X_1=x_1}(t)\, f_{X_1}(x_1)}{f_T(t)}$$

$$= \begin{cases} \dfrac{f_S(t-x_1)\, f_{X_1}(x_1)}{f_T(t)} & \forall\, t > x_1 > 0 \\ 0 & \sim \end{cases}$$

$X_1 \sim Exp(\lambda)$, $S \sim Gamma(n-1,\lambda)$ y $T \sim Gamma(n,\lambda)$

$$\Rightarrow f_{X_1|T=t}(x_1) = \begin{cases} \dfrac{\frac{\lambda^{n-1}}{\Gamma(n-1)} (t-x_1)^{n-1-1} e^{-\lambda(t-x_1)} \times \lambda e^{-\lambda x_1}}{\frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}} & , t>x_1>0 \\ 0 & \sim \end{cases}$$

$$= \begin{cases} \dfrac{\Gamma(n)}{\Gamma(n-1)} \dfrac{(t-x_1)^{n-2}}{t^{n-1}} & , t>x_1>0 \\ 0 & \sim \end{cases}$$

$$= \begin{cases} (n-1) \frac{1}{t} \left(1-\frac{x_1}{t}\right)^{n-2} & , t>x_1>0 \\ 0 & \sim \end{cases}$$

$$\Rightarrow E(X_1|T=t) = \int_0^t (n-1) \frac{x_1}{t}\left(1-\frac{x_1}{t}\right)^{n-2} dx_1 \quad u = \frac{x_1}{t}$$

$$= (n-1)t \underbrace{\int_0^1 u(1-u)^{n-2} du}_{Beta(2,n-1) = \frac{\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)} = \frac{1}{(n-1)n}} = \frac{t}{n}$$

$\Rightarrow E(X_1|T=t) = t/n$ (no depende de λ)

$\left[\Rightarrow E(X_1|T) = \frac{T}{n} = \frac{1}{n}\sum_{i=1}^n X_i = \overline{X}\right]$ NO se puede!

Id. LDGN $\Rightarrow \frac{\sum_{i=1}^n X_i}{n} = \overline{X}_n \xrightarrow{P} E(X_1) = \frac{1}{\lambda}$ $\quad \underbrace{g(x) = \frac{1}{x}}_{\text{función continua}}$

$\Rightarrow \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\overline{X}_n} \xrightarrow{P} g(\frac{1}{\lambda}) = \lambda$

TCL $\Rightarrow \sqrt{n}(\overline{X}_n - \underset{\mu}{\frac{1}{\lambda}}) \xrightarrow{D} N(0, \underset{\sigma^2}{\frac{1}{\lambda^2}})$ $\quad g(x) = 1/x$, $g'(x) = -1/x^2$

$\xrightarrow{\text{Met. Delta}} \sqrt{n}\left(\frac{1}{\overline{X}_n} - \lambda\right) \xrightarrow{D} N(0, \underbrace{(g'(1/\lambda))^2 \frac{1}{\lambda^2}}_{\lambda^2})$

2a $X_i \overset{iid}{\sim} U(0,1)$, $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $Z_n = \max_{1\le i\le n}\{X_i\}$

LDGN $\Rightarrow Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \frac{1}{2} = E(X_1)$

$\Rightarrow 2Y_n \xrightarrow{P} 1$

$$F_{Z_n}(z) = P(\max_{1\le i\le n}\{X_i\} \le z) = (F_{X_1}(z))^n$$

$$= \begin{cases} 0 & \text{si } z < 0 \\ z^n & \text{si } 0 \le z < 1 \\ 1 & \text{si } z \ge 1 \end{cases} \qquad (*)$$

$$\Rightarrow f_{Z_n}(z) = \begin{cases} n z^{n-1} & \text{si } 0 < z < 1 \\ 0 & \sim \end{cases}$$

$$E(Z_n) = \int_0^1 n z^n \, dz = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1$$

$$E(Z_n^2) = \int_0^2 n z^{n+1} \, dz = \frac{n}{n+2} \xrightarrow[n\to\infty]{} 1$$

$$\Rightarrow Var(Z_n) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 \xrightarrow[n\to\infty]{} 0$$

$$\therefore E\{(Z_n - 1)^2\} \xrightarrow[n\to\infty]{} 0 \Rightarrow Z_n \xrightarrow{P} 1$$

~~Sea $g(y,z)=2$ ... $\Rightarrow g: \mathbb{R}^2 \to \mathbb{R}$ y continua~~

Sabemos que $\left.\begin{array}{l} Y_n \xrightarrow{P} a \\ Z_n \xrightarrow{P} b \end{array}\right\} \Rightarrow g(Y_n, Z_n) \xrightarrow{P} g(a,b)$ $\forall\, g: \mathbb{R}^2 \to \mathbb{R}$ continua

Sea $g(y,z) = 2yz$, lo cual es continua

$$\Rightarrow g(Y_n, Z_n) = 2Y_n Z_n \xrightarrow{P} 2 \cdot \frac{1}{2} \cdot 1 = 1$$

2b $\quad TCL \Rightarrow \sqrt{n}(\bar{Y}_n - 1/2) \xrightarrow{\mathcal{D}} N(0, \underbrace{1/12}_{\sigma^2})$

$\underset{(g(y)=2y)}{\xrightarrow{\text{Met. Delta}}} \quad \sqrt{n}(2\bar{Y}_n - 1) \xrightarrow{\mathcal{D}} N(0, \underbrace{4/12}_{1/3})$

$\Rightarrow \quad \Rightarrow \sqrt{3n}(2\bar{Y}_n - 1) \approx N(0,1)$

para "n lo suficientemente grande"

$\Rightarrow P(2\bar{Y}_n \leq 1) = P(2\bar{Y}_n - 1 \leq 0)$

$= P(\sqrt{3n}(2\bar{Y}_n - 1) \leq 0)$

$\approx \Phi(0) = 1/2$

2c) $G_n(t) = P(n(Z_n - 1) \leq t)$

$= P(Z_n \leq 1 + t/n)$

$$= \begin{cases} 0 & \text{si } 1 + t/n < 0 \Leftrightarrow t < -n \\ \left(1+\frac{t}{n}\right)^n & \text{si } 0 \leq 1 + \frac{t}{n} < 1 \Leftrightarrow -n \leq t < 0 \\ 1 & \text{si } 1 + \frac{t}{n} \geq 1 \Leftrightarrow t \geq 0 \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} G_n(t) = \left\{ \begin{array}{ll} e^t & \text{si } t \leq 0 \\ 0 & \text{si no} \end{array} \right\} = G(t)$$

2d. Sabemos que si $a_n(T_n - \theta) \xrightarrow{\mathcal{D}} X$

$\Rightarrow a_n(g(T_n) - g(\theta)) \xrightarrow{\mathcal{D}} g'(\theta) X$

$\forall g$ continua

Luego, de c) con $g(x) = 2x \Rightarrow g'(x) = 2$

$\Rightarrow n(Z_n^2 - 2) \xrightarrow{\mathcal{D}} 2X, \quad X \sim G$

$$F_{2X}(\omega) = G(\omega/2) = \begin{cases} e^{\omega/2} & \text{si } \omega \leq 0 \\ 0 & \sim \end{cases}$$