Spectrum estimation using Periodogram, Bartlett and Welch

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Slides follow closely chapter 8 in the book „Statistical Digital Signal Processing and Modeling“ by Monson H. Hayes and most of the figures and formulas are taken from there

Introduction

- We want to estimate the power spectral density of a wide-sense stationary random process

- Recall that the power spectrum is the Fourier transform of the autocorrelation sequence

$$P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k) e^{-jk\omega}$$

- For an ergodic process the following holds

$$\lim_{N\to\infty} \left\{ \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+k)x^*(n) \right\} = r_x(k)$$

Introduction

- The main problem of power spectrum estimation is
  - The data x(n) is always finite!
- Two basic approaches
  - Nonparametric (Periodogram, Bartlett and Welch)
    - These are the most common ones and will be presented in the next pages
  - Parametric approaches
    - not discussed here since they are less common

Nonparametric methods

- These are the most commonly used ones
- x(n) is only measured between n=0,..,N-1

$$\hat{r}_x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n+k)x^*(n)$$

- Ensures that the values of x(n) that fall outside the interval [0,N-1] are excluded, where for negative values of k we use conjugate symmetry

$$\hat{r}_x(k) = \frac{1}{N}\sum_{n=0}^{N-1-k} x(n+k)x^*(n) \quad ; \quad k = 0, 1, \ldots, N-1$$

Periodogram

- Taking the Fourier transform of this autocorrelation estimate results in an estimate of the power spectrum, known as the Periodogram

- This can also be directly expressed in terms of the data x(n) using the rectangular windowed function $x_N(n)$

$$\hat{P}_{per}(e^{j\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_x(k) e^{-jk\omega}$$

$$x_N(n) = \begin{cases} x(n) & ; \quad 0 \le n < N \\ 0 & ; \quad \text{otherwise} \end{cases}$$

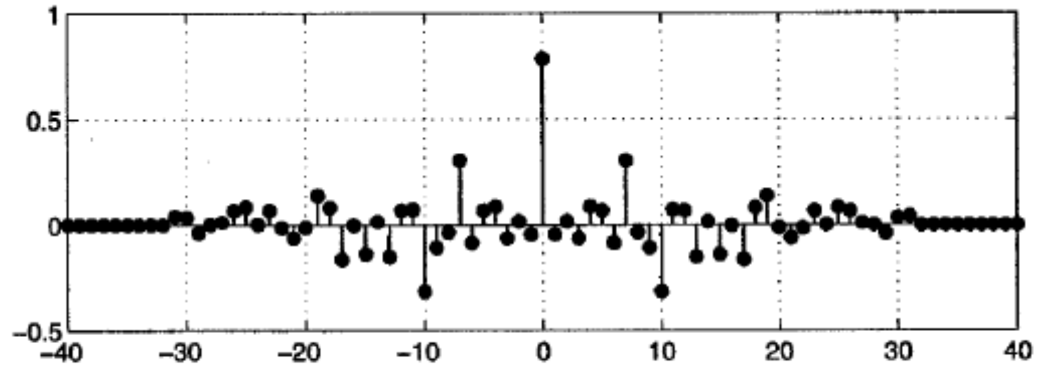
$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x_N(n+k) x_N^*(n) = \frac{1}{N} x_N(k) * x_N^*(-k)$$

$$\boxed{\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} X_N(e^{j\omega}) X_N^*(e^{j\omega}) = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2}$$

$$X_N(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_N(n) e^{-jn\omega} = \sum_{n=0}^{N-1} x(n) e^{-jn\omega}$$

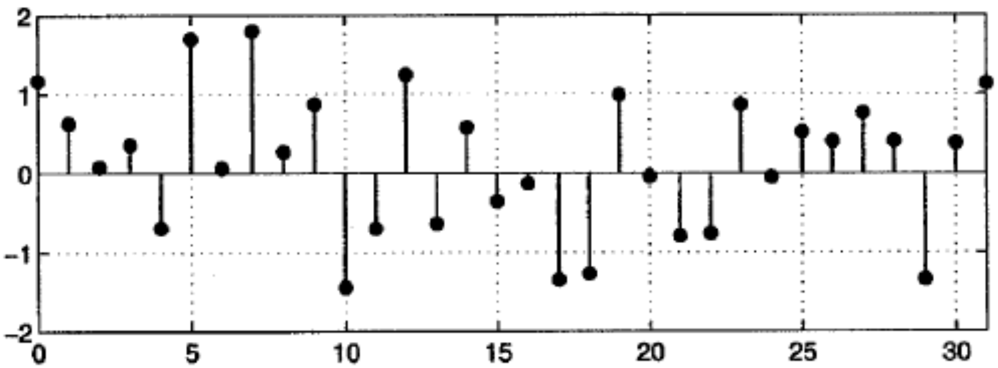
$$x_N(n) \xrightarrow{\text{DFT}} X_N(k) \longrightarrow \frac{1}{N} |X_N(k)|^2 = \hat{P}_{per}(e^{j2\pi k/N})$$

Periodogram of white noise

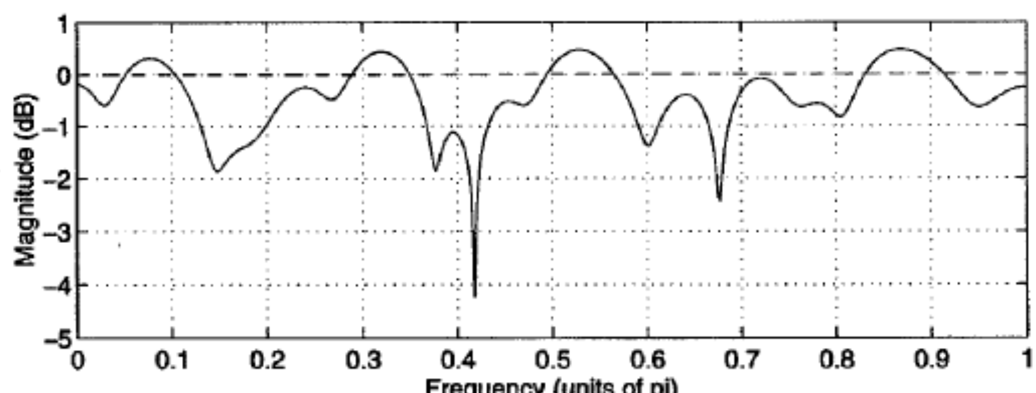


$r_x(k) = \sigma_x^2 \delta(k)$

32 samples



$P_x(e^{j\omega}) = \sigma_x^2$



Performance of the Periodogram

- If N goes to infinity, does the Periodogram converge towards the power spectrum in the mean squared sense?

$$\lim_{N\to\infty} E\left\{\left[\hat{P}_{per}(e^{j\omega}) - P_x(e^{j\omega})\right]^2\right\} = 0$$

- Necessary conditions
  - asymptotically unbiased:

$$\lim_{N\to\infty} E\left\{\hat{P}_{per}(e^{j\omega})\right\} = P_x(e^{j\omega})$$

  - variance goes to zero:

$$\lim_{N\to\infty} \mathrm{Var}\left\{\hat{P}_{per}(e^{j\omega})\right\} = 0$$

- In other words, it must be a consistent estimate of the power spectrum

Recall: sample mean as estimator

- Assume that we measure an iid process x[n] with mean μ and variance σ^2
- The sample mean is m=(x[0]+x[1]+x[2]+..+x[N-1])/N
- The sample mean is unbiased

  $$\begin{aligned} E[m] &= E[(x[0]+x[1]+x[2]+..+x[N-1])/N] \\ &= (E[x[0]]+E[x[1]]+E[x[2]]+..+E[x[N-1]])/N \\ &= N\mu/N \\ &= \mu \end{aligned}$$

- The variance of the sample mean is inversely proportional to the number of samples

  $$\begin{aligned} VAR[m] &= VAR[(x[0]+x[1]+x[2]+..x[N-1])/N]= \\ &= (VAR[x[0]]+VAR[x[1]]+VAR[x[2]]+..+VAR[x[N-1]])/N^2 \\ &= N\sigma^2/N^2 \\ &= \sigma^2/N \end{aligned}$$

Periodogram bias

- To compute the bias we first find the expected value of the autocorrelation estimate

$$E\{\hat{r}_x(k)\} = \frac{1}{N}\sum_{n=0}^{N-1-k} E\left\{x(n+k)x^*(n)\right\}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1-k} r_x(k) = \frac{N-k}{N} r_x(k)$$

- Hence the estimate of the autocorrelation is biased with a triangular window (Bartlett)

$$E\{\hat{r}_x(k)\} = w_B(k) r_x(k)$$

$$w_B(k) = \begin{cases} \dfrac{N-|k|}{N} & ; \quad |k| \le N \\ 0 & ; \quad |k| > N \end{cases}$$

Periodogram bias

- The expected value of the Periodogram can now be calculated:

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = E\left\{\sum_{k=-N+1}^{N-1} \hat{r}_x(k)e^{-jk\omega}\right\}$$

$$= \sum_{k=-N+1}^{N-1} E\left\{\hat{r}_x(k)\right\} e^{-jk\omega}$$

$$= \sum_{k=-\infty}^{\infty} r_x(k) w_B(k) e^{-jk\omega}$$

- Thus the expected value of the Periodogram is the convolution of the power spectrum with the Fourier transform of a Bartlett window

$$\boxed{E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})}$$

$$W_B(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right]^2$$

Example: Periodogram of a Sinusoidal in Noise

- Consider a a random process consisting of a sinusoidal in white noise, where the phase of the sinusoidal is uniformly [$-\pi$, π] distributed and
  A=5, $\omega_0=0.4\pi$
- N=64 on top
  and N=256 on the bottom
- Overlay of 50 Periodogram on the left and average on the right

$$x(n) = A\sin(n\omega_0 + \phi) + v(n)$$

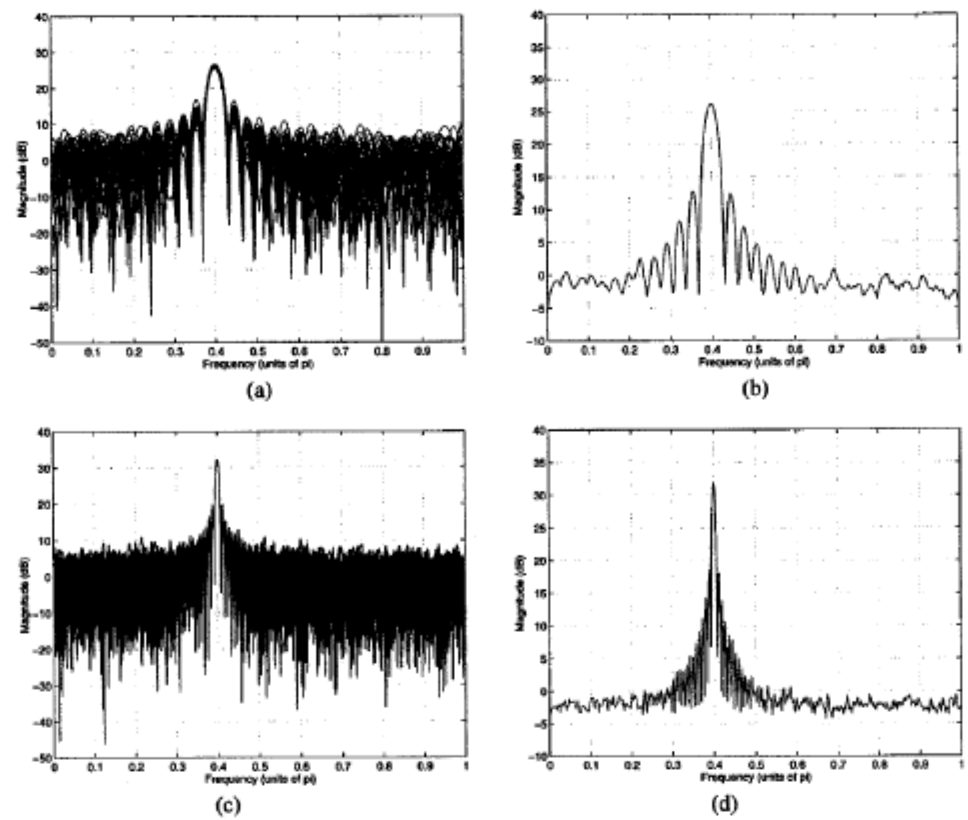


Figure 8.6 *The periodogram of a sinusoid in white noise. (a) Overlay plot of 50 periodograms using $N = 64$ data values and (b) the periodogram average. (c) Overlay plot of 50 periodograms using $N = 256$ data values and (d) the periodogram average.*

Periodogram resolution

- In addition to biasing the Periodogram, the spectral smoothing that is introduced by the Bartlett window also limits the ability of the Periodogram to resolve closely-spaced narrowband components

- Consider this random process consisting of two sinusoidal in white noise where the phases are again uniformly distributed and uncorrelated with each other

$$x(n) = A_1 \sin(n\omega_1 + \phi_1) + A_2 \sin(n\omega_2 + \phi_2) + v(n)$$

Periodogram resolution

- The power spectrum of the above random process is

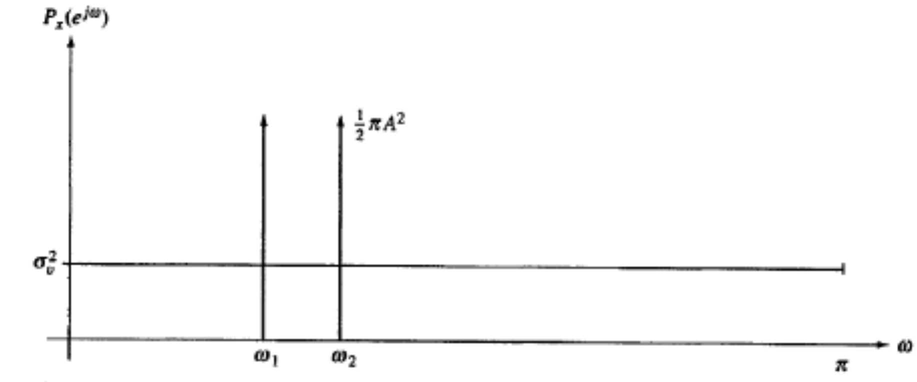
$$P_x(e^{j\omega}) = \sigma_v^2 + \frac{1}{2}\pi A_1^2\Big[u_0(\omega-\omega_1) + u_0(\omega+\omega_1)\Big] + \frac{1}{2}\pi A_2^2\Big[u_0(\omega-\omega_2) + u_0(\omega+\omega_2)\Big]$$

- And the expected value of the Periodogram is

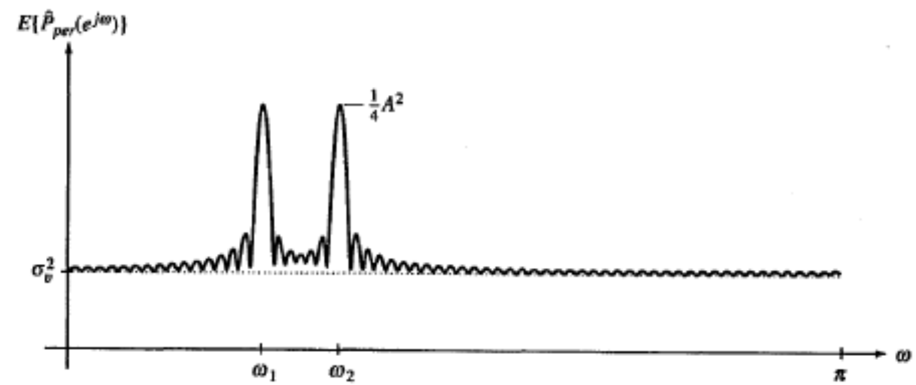
$$\begin{aligned} E\left\{\hat{P}_{per}(e^{j\omega})\right\} &= \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega}) \\ &= \sigma_v^2 + \frac{1}{4}A_1^2\left[W_B(e^{j(\omega-\omega_1)}) + W_B(e^{j(\omega+\omega_1)})\right] \\ &\quad + \frac{1}{4}A_2^2\left[W_B(e^{j(\omega-\omega_2)}) + W_B(e^{j(\omega+\omega_2)})\right] \end{aligned}$$

Periodogram resolution

- Since the width of the main lobe increases as N decreases, for a given N there is a limit on how closely two sinusoidal may be located before they can no longer be resolved
- This is usually defined as the bandwidth of the window at its half power points (-6dB), which is for the Bartlett window at 0.89*2π/N
- This is just a rule of thumb!



(a)



$$\boxed{\mathrm{Res}\left[\hat{P}_{per}(e^{j\omega})\right] = 0.89\frac{2\pi}{N}}$$

Example: Periodogram of two Sinusoidal in Noise

- Consider a a random process consisting of two sinusoidal in white noise, where the phases of the sinusoidal are uniformly [-π, π] distributed and
  $A=5$, $\omega_1=0.4\pi$, $\omega_2=0.45\pi$

- N=40 on top
  and N=64 on the bottom

- Overlay of 50 Periodogram on the left and average on the right

$$x(n) = A\sin(n\omega_1 + \phi_1) + A\sin(n\omega_2 + \phi_2) + v(n)$$

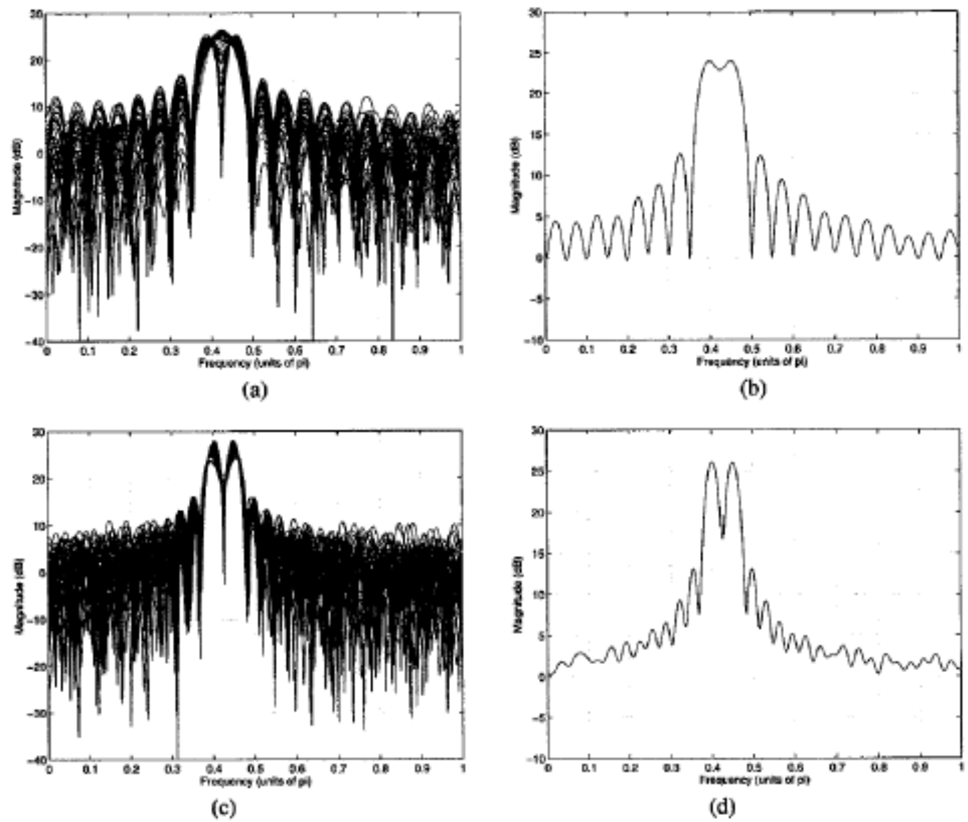


Figure 8.8 *The periodogram of two sinusoids in white noise with $\omega_1 = 0.4\pi$ and $\omega_2 = 0.45\pi$. (a) Overlay plot of 50 periodograms using $N = 40$ data values and (b) the ensemble average. (c) Overlay plot of 50 periodograms using $N = 64$ data values and (d) the ensemble average.*

Variance of the Periodogram

- The Periodogram is an asymptotically unbiased estimate of the power spectrum

$$\lim_{N \to \infty} E\left\{\hat{P}_{per}(e^{j\omega})\right\} = P_x(e^{j\omega})$$

- To be a consistent estimate, it is necessary that the variance goes to zero as N goes to infinity

$$\lim_{N \to \infty} \mathrm{Var}\left\{\hat{P}_{per}(e^{j\omega})\right\} = 0$$

- This is however hard to show in general and hence we focus on a white Gaussian noise, which is still hard, but can be done

Periodogram summary

Table 8.1 ***Properties of the Periodogram***

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-jn\omega}\right|^2$$

Bias

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$

Resolution

$$\Delta\omega = 0.89\frac{2\pi}{N}$$

Variance

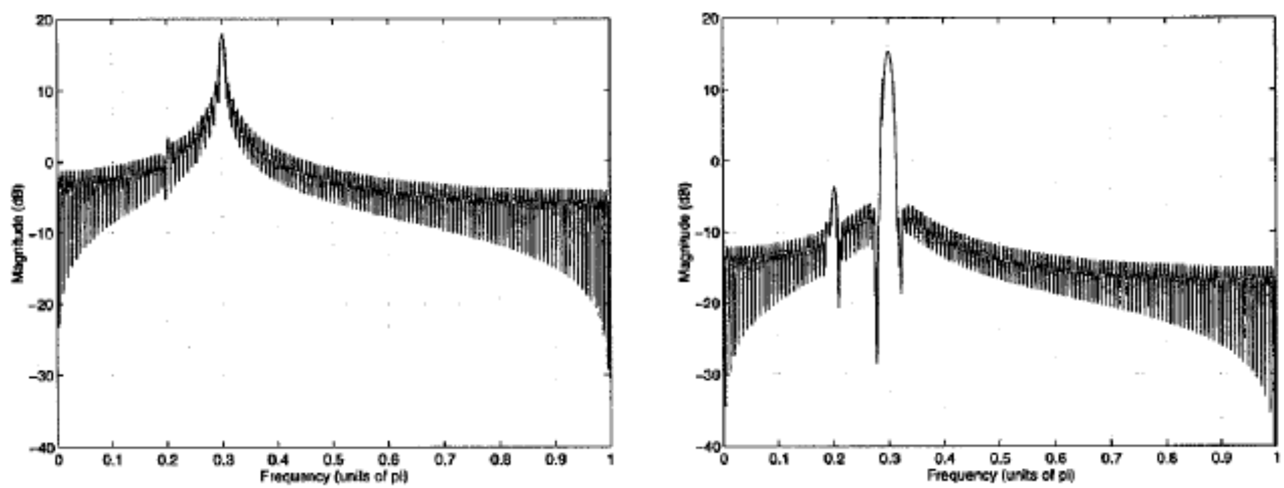
$$\text{Var}\left\{\hat{P}_{per}(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})$$

The modified Periodogram

- Smoothing is determined by the window that is applied to the data
- While the rectangular window as the smallest main lobe of all windows, its sidelobes fall off rather slowly

$$x(n) = 0.1\sin(n\omega_1 + \phi_1) + \sin(n\omega_2 + \phi_2) + v(n)$$

Rectangular data window   Hamming data window



(a) (b)

Figure 8.10 *Spectral analysis of two sinusoids in white noise with sinusoidal frequencies of $\omega_1 = .2\pi$ and $\omega_2 = .3\pi$ and a data record length of $N = 128$ points. (a) The expected value of the periodogram. (b) The expected value of the modified periodogram using a Hamming data window.*

The modified Periodogram

- Nothing is free. As you notice, the Hamming window has a wider main lobe
- The Periodogram of a process that is windowed with a general window is called modified Periodogram
- N is the length of the window and U is a constant that is needed so that the modified Periodogram is asymptotically unbiased

$$\boxed{\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} x(n)w(n)e^{-jn\omega}\right|^2}$$

$$U = \frac{1}{N}\sum_{n=0}^{N-1}|w(n)|^2$$

The modified Periodogram

- For evaluating the Bias we take the expected value of the modified Periodogram, where $W(e^{j\omega})$ is the Fourier transform of the data window

$$\boxed{E\left\{\hat{P}_M(e^{j\omega})\right\} = \frac{1}{2\pi NU} P_x(e^{j\omega}) * |W(e^{j\omega})|^2}$$

- Using the Parseval theorem, it follows that U is the energy of the window divided by N

$$U = \frac{1}{N}\sum_{n=0}^{N-1} |w(n)|^2 = \frac{1}{2\pi N}\int_{-\pi}^{\pi} |W(e^{j\omega})|^2 d\omega$$

$$\frac{1}{2\pi NU}\int_{-\pi}^{\pi} |W(e^{j\omega})|^2 d\omega = 1$$

- With an appropriate window, $|W(e^{j\omega})|^2/NU$ will converge to an impulse of unit area and hence the modified Periodogram will be asymptotically unbiased

Variance of the modified Periodogram

- Since the modified Periodogram is simply the Periodogram of a windowed data sequence, not much changes

$$\boxed{\mathrm{Var}\left\{\hat{P}_M(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})}$$

- Hence the estimate is still not consistent

- Main advantage is that the window allows a tradeoff between spectral resolution (main lobe width) and spectral masking (sidelobe amplitude)

Resolution versus masking of the modified Periodogram

- The resolution of the modified Periodogram defined to be the 3dB bandwidth of the data window

$$\text{Res}\left[\hat{P}_M(e^{j\omega})\right] = (\Delta\omega)_{3\text{dB}}$$

- Note that when we used the Bartlett lag window before, the resolution was defined as the 6dB bandwidth. This is consistent with the above definition, since the 3dB points of the data window transform into 6dB points in the Periodogram

Table 8.2 ***Properties of a Few Commonly Used Windows. Each Window is Assumed to be of Length N.***

Window	Sidelobe Level (dB)	3 dB BW $(\Delta\omega)_{3\text{dB}}$
Rectangular	-13	$0.89(2\pi/N)$
Bartlett	-27	$1.28(2\pi/N)$
Hanning	-32	$1.44(2\pi/N)$
Hamming	-43	$1.30(2\pi/N)$
Blackman	-58	$1.68(2\pi/N)$

Modified periodogram summary

Table 8.3 ***Properties of the Modified Periodogram***

$$\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} w(n)x(n)e^{-jn\omega}\right|^2$$

$$U = \frac{1}{N}\sum_{n=0}^{N-1} |w(n)|^2$$

Bias	$E\left\{\hat{P}_M(e^{j\omega})\right\} = \frac{1}{2\pi NU} P_x(e^{j\omega}) * |W(e^{j\omega})|^2$
Resolution	Window dependent
Variance	$\mathrm{Var}\left\{\hat{P}_M(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})$

Bartlett's method

- Still have not a consistent estimate of the power spectrum!
- Nevertheless, the periodogram is asymptotically unbiased

$$\lim_{N \to \infty} E\{\hat{P}_{per}(e^{j\omega})\} = P_x(e^{j\omega})$$

- Hence if we can find a consistent estimate of the mean, then this estimate would also be a consistent estimate of the power spectrum

Bartlett’s method

- Averaging (sampe mean) a set of uncorrelated measurements of a random variable results in a consistent estimate of its mean
- In other words: Variance of the sample mean is inversely proportional to the number of measurements
- Hence this should also work here, by averaging Periodograms

This suggests that we consider estimating the power spectrum of a random process by periodogram averaging. Thus, let $x_i(n)$ for $i = 1, 2, \ldots, K$ be K uncorrelated realizations of a random process $x(n)$ over the interval $0 \leq n < L$. With $\hat{P}_{per}^{(i)}(e^{j\omega})$ the periodogram of $x_i(n)$,

$$\hat{P}_{per}^{(i)}(e^{j\omega}) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x_i(n)e^{-jn\omega}\right|^2 \quad ; \quad i = 1, 2, \ldots, K$$

Bartlett's method

- Averaging these Periodograms

$$\hat{P}_x(e^{j\omega}) = \frac{1}{K}\sum_{i=1}^{K}\hat{P}_{per}^{(i)}(e^{j\omega})$$

- This results in an asymptotically unbiased estimate of the power spectrum

$$E\left\{\hat{P}_x(e^{j\omega})\right\} = E\left\{\hat{P}_{per}^{(i)}(e^{j\omega})\right\} = \frac{1}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega})$$

- Since we assume that the realizations are uncorrelated, it follows, that the variance is inversely proportional to the number of measurements K

$$\mathrm{Var}\left\{\hat{P}_x(e^{j\omega})\right\} = \frac{1}{K}\mathrm{Var}\left\{\hat{P}_{per}^{(i)}(e^{j\omega})\right\} \approx \frac{1}{K}P_x^2(e^{j\omega})$$

- Hence this is a consistent estimate of the power spectrum, if L and K go to infinity

Bartlett’s method

- There is still a problem: we usually do not have uncorrelated data records!
- Typically there is only one data record of length N available
- Hence Bartlett proposes to partition the data record into K nonoverlapping sequences of the length L, where N=K*L

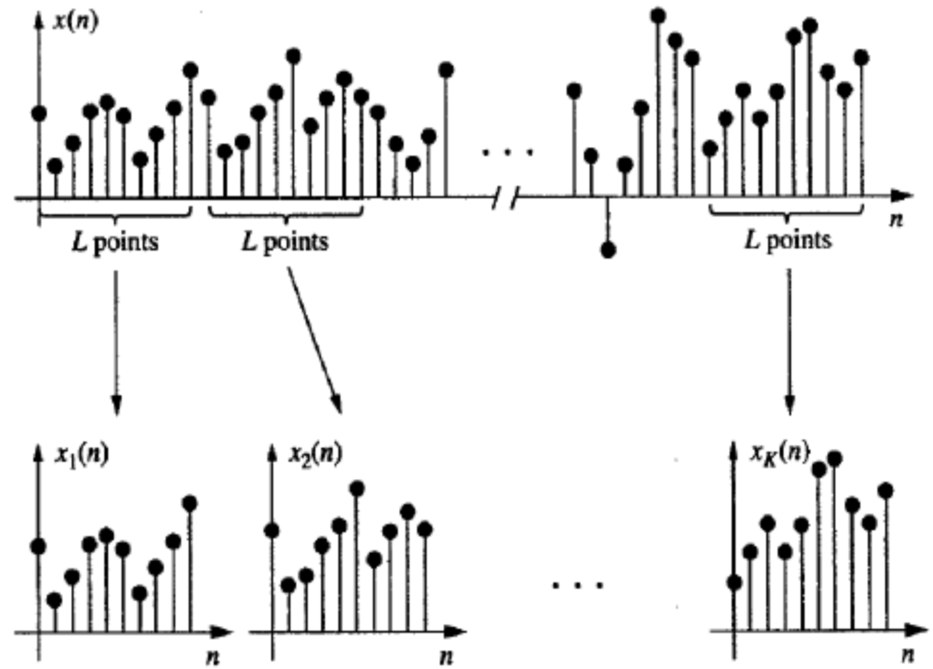


Figure 8.12 *Partitioning $x(n)$ into nonoverlapping subsequences.*

$$x_i(n) = x(n + iL) \qquad \begin{array}{l} n = 0, 1, \dots, L-1 \\ i = 0, 1, \dots, K-1 \end{array}$$

Thus, the Bartlett estimate is

$$\boxed{\hat{P}_B(e^{j\omega}) = \frac{1}{N} \sum_{i=0}^{K-1} \left| \sum_{n=0}^{L-1} x(n + iL) e^{-jn\omega} \right|^2}$$

Bartlett's method

- Each expected value of the periodogram of the subsequences are identical hence the process of averaging subsequences Periodograms results in the same average value => asymptotically unbiased

$$\boxed{E\left\{\hat{P}_B(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})}$$

- Note that the data length used for the Periodograms are now L and not N anymore, the spectral resolution becomes worse (this is the price we are paying)

$$\mathrm{Res}\left[\hat{P}_B(e^{j\omega})\right] = 0.89\frac{2\pi}{L} = 0.89K\frac{2\pi}{N}$$

Bartlett's method

- Now we reap the reward: the variance is going to zero as the number of subsequences goes to infinity
- If both, K and L go to infinity, this will be a consistent estimate of the power spectrum
- In addition, for a given N=K*L, we can trade off between good spectral resolution (large L) and reduction in variance (Large K)

$$\mathrm{Var}\left\{\hat{P}_B(e^{j\omega})\right\} \approx \frac{1}{K}\mathrm{Var}\left\{\hat{P}_{per}^{(i)}(e^{j\omega})\right\} \approx \frac{1}{K}P_x^2(e^{j\omega})$$

Table 8.4 ***Properties of Bartlett's Method***

$$\hat{P}_B(e^{j\omega}) = \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2$$

Bias

$$E\left\{\hat{P}_B(e^{j\omega})\right\} = \frac{1}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega})$$

Resolution

$$\Delta\omega = 0.89K\frac{2\pi}{N}$$

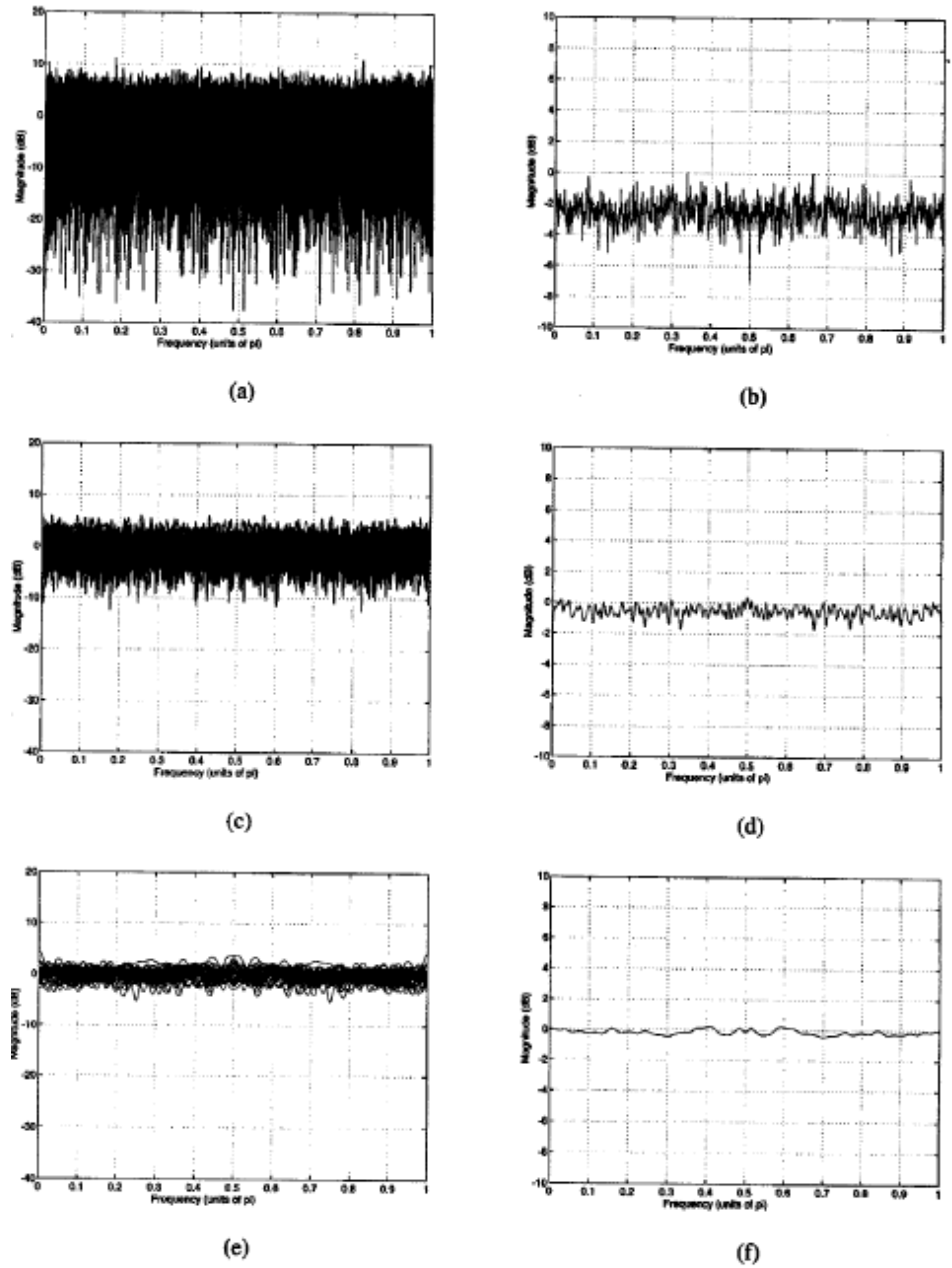
Variance

$$\mathrm{Var}\left\{\hat{P}_B(e^{j\omega})\right\} \approx \frac{1}{K}P_x^2(e^{j\omega})$$

Bartlett's method: White noise

- a) Periodogram with N=512
- b) Ensemble average
- c) Overlay of 50 Bartlett estimates with K=4 and L=128
- d) Ensemble average
- e) Overlay of 50 Bartlett estimates with K=8 and L=64
- f) Ensemble average

Overlay of 50 estimates   Ensemble Average



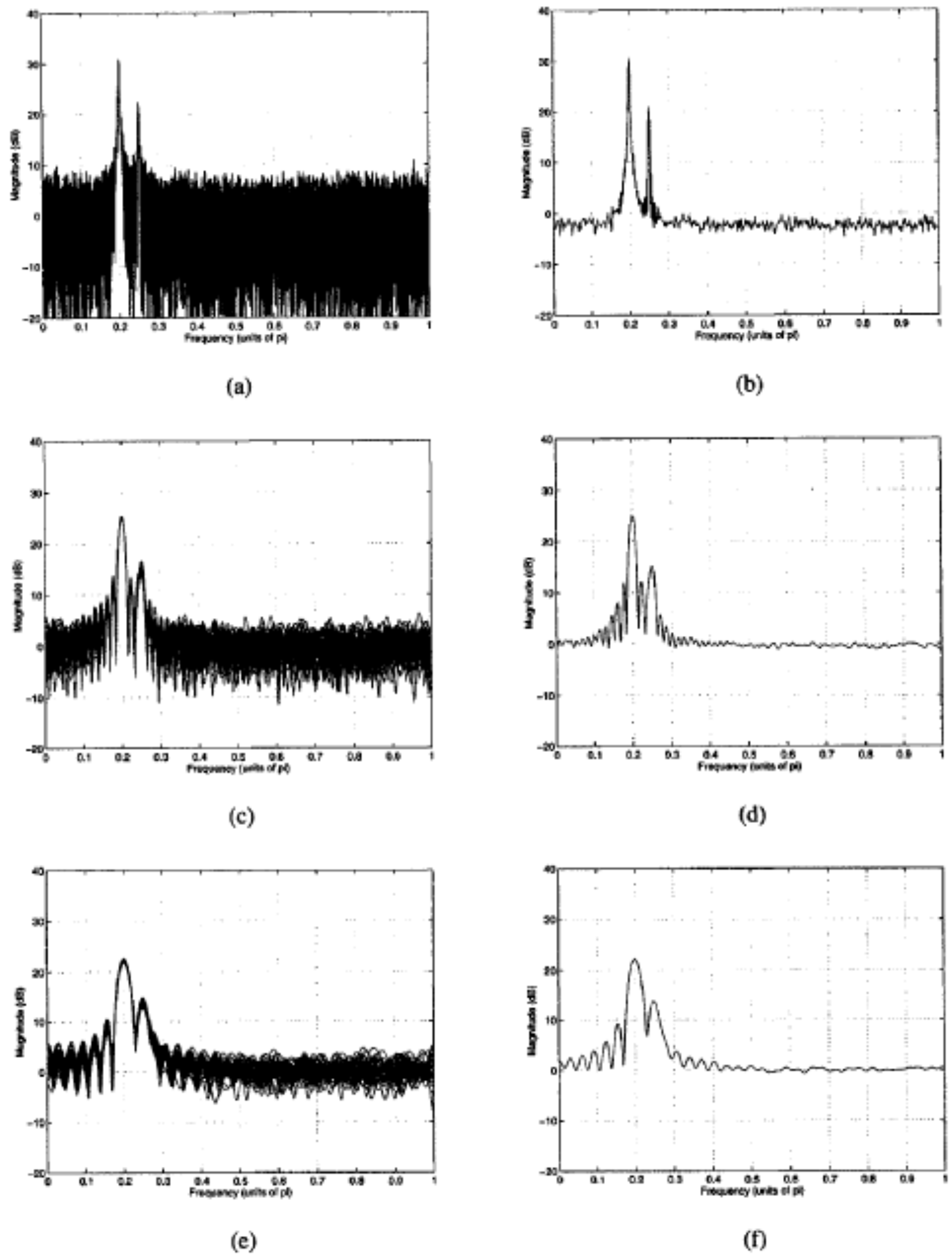
Bartlett’s method: Two sinusoidal in white noise

- a) Periodogram with N=512
- b) Ensemble average

- c) Overlay of 50 Bartlett estimates with K=4 and L=128
- d) Ensemble average

- e) Overlay of 50 Bartlett estimates with K=8 and L=64
- f) Ensemble average

Note how larger K results in shorter L and hence in less spectral resolution



(a) (b) (c) (d) (e) (f)

Welch's method

- Two modifications to Bartlett's method
  - 1) the subsequences are allowed to overlap
  - 2) instead of Periodograms, modified Periodograms are averaged

- Assuming that successive sequences are offset by D points and that each sequence is L points long, then the i^{th} sequence is

$$x_i(n) = x(n + iD) \quad ; \quad n = 0, 1, \ldots, L-1$$

- Thus the overlap is L-D points and if K sequences cover the entire N data points then

$$N = L + D(K-1).$$

Welch's method

- For example, with no overlap (D=L) there are K=N/L subsequences of length L
- For a 50% overlap (D=L/2) there is a tradeoff between increasing L or increasing K
  - If L stays the same then there are more subsequences to average, hence the variance of the estimate is reduced

$$K = 2\frac{N}{L} - 1$$

  - If subsequences are doubled in length and hence the spectral resolution is then doubled

$$K = \frac{N}{L} - 1$$

Performance of Welch's method

- Welch's method can be written in terms of the data record as follows

$$\hat{P}_W(e^{j\omega}) = \frac{1}{KLU} \sum_{i=0}^{K-1} \left| \sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega} \right|^2$$

- Or in terms of modified Periodograms

$$\hat{P}_W(e^{j\omega}) = \frac{1}{K} \sum_{i=0}^{K-1} \hat{P}_M^{(i)}(e^{j\omega})$$

- Hence the expected value of Welch's estimate is

$$E\{\hat{P}_W(e^{j\omega})\} = E\{\hat{P}_M(e^{j\omega})\} = \frac{1}{2\pi LU} P_x(e^{j\omega}) * |W(e^{j\omega})|^2$$

- Where $W(e^{j\omega})$ is the Fourier transform of the L-point data window w(n)

Performance of Welch's method

- Welch's method is asymptotically unbiased estimate of the power spectrum

- The variance is much harder to compute, since the overlap results in a correlation

- Nevertheless for an overlap of 50% and a Bartlett window it has been shown that

$$\mathrm{Var}\{\hat{P}_W(e^{j\omega})\} \approx \frac{9}{8K} P_x^2(e^{j\omega})$$

- Recall Bartlett's Method results in

$$\mathrm{Var}\left\{\hat{P}_B(e^{j\omega})\right\} \approx \frac{1}{K}\mathrm{Var}\left\{\hat{P}_{per}^{(i)}(e^{j\omega})\right\} \approx \frac{1}{K} P_x^2(e^{j\omega})$$

Performance of Welch's method

- For a fixed number of data N, with 50% overlap, twice as many subsequences can be averaged, hence expressing the variance in terms of L and N we have

$$\mathrm{Var}\{\hat{P}_W(e^{j\omega})\} \approx \frac{9}{16}\frac{L}{N}P_x^2(e^{j\omega})$$

- Since N/L is the number of subsequences K used in Bartlett's method it follows

$$\mathrm{Var}\{\hat{P}_W(e^{j\omega})\} \approx \frac{9}{16}\mathrm{Var}\{\hat{P}_B(e^{j\omega})\}$$

- In other words, and not surprising, with 50% overlap (and Bartlett window), the variance of Welch's method is about half that of Bartlett's method

Welch's method summary

Table 8.5 *Properties of Welch's Method*

	$$\hat{P}_W(e^{j\omega}) = \frac{1}{KLU}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega}\right|^2$$
	$$U = \frac{1}{L}\sum_{n=0}^{L-1}\lvert w(n)\rvert^2$$
Bias	
	$$E\left\{\hat{P}_W(e^{j\omega})\right\} = \frac{1}{2\pi LU} P_x(e^{j\omega}) * \lvert W(e^{j\omega})\rvert^2$$
Resolution	Window dependent
Variance†	
	$$\mathrm{Var}\left\{\hat{P}_W(e^{j\omega})\right\} \approx \frac{9}{16}\frac{L}{N} P_x^2(e^{j\omega})$$

† Assuming 50% overlap and a Bartlett window.