Einsteins summation convention ∧ Symmetry of dot product

22/09/2025

↓

writes down what the operations are on the elements of a matrix.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ \vdots & & & \\ a_{n1} & \vdots & & \end{pmatrix}} \overset{B}{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \ddots & & \\ b_{n1} & \cdots & & \end{pmatrix}} = \overset{AB}{\begin{pmatrix} & \\ & \end{pmatrix}}$$

row columb.

if I want $(ab)_{23}$ → take row 2 of a multiplied by columb 3 of b.

$= a_{21}b_{13} + a_{22}b_{23} + \dots a_{2n}b_{n3}$.

Einsteins convention $= \sum_j a_{ij}b_{jk} = a_{ij}b_{jk} = ab_{ik}$.

ab_{ik}

$AB = C$

$C_{ik} = a_{ij}b_{jk}$.

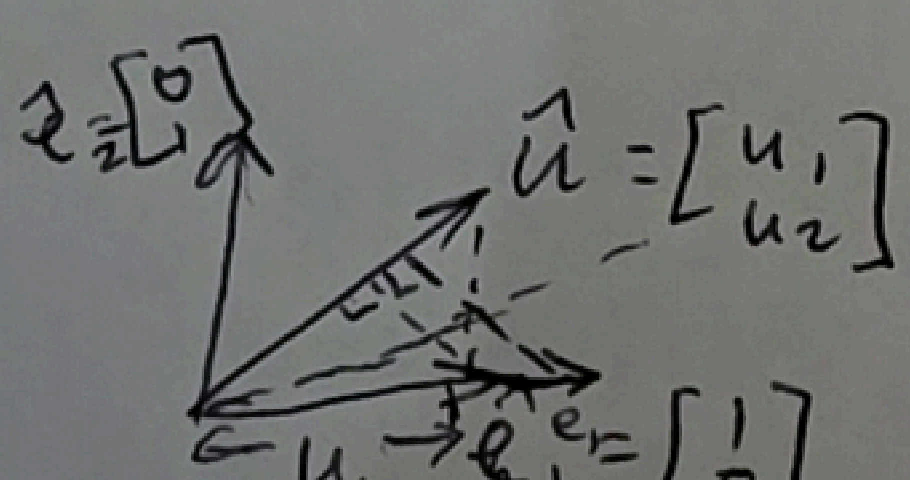
You end up with no. rows by left & no. colums by right.

$$\underset{2}{\overset{3}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}} \underset{3}{\overset{4}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}} = \underset{2}{\overset{4}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}}$$

Revisit dot product.

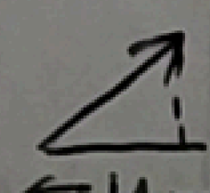
$\underset{(u_i)}{u} \cdot \underset{(v_i)}{v}$ $\quad [u_1, u_2 \dots u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$u_i v_i$ ← Einsteins sumation

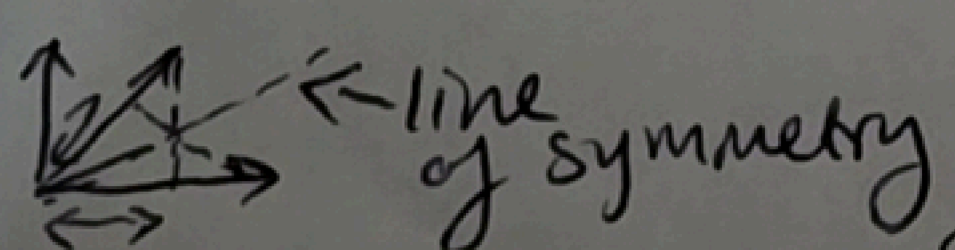


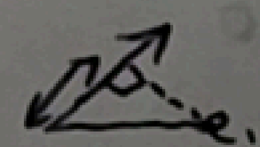
what happens when I dot $\hat{u}$ with $\hat{e}_1$

a.i... dot product, projection of $\hat{u}$ onto $\hat{e}_1$



what if project $\hat{e}_1$ onto $\hat{u}$...





Symmetry of dot product:

Assignment.

1).
$$\begin{array}{ccc} A & B & C \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = & \begin{bmatrix} \cdot & \cdot & \cdot \\ C_{21} & & \cdot \end{bmatrix} \end{array}$$

$C_{mn} = A_{mj} B_{jn}$.

$C_{21} = A_{2j} B_{j1}$ $\quad 4 \times 1 + 0 + 1 = 5$

$C_{21} = 5$

2) $\begin{bmatrix} 4 & 3 & 5 \\ 5 & 4 & 1 \end{bmatrix}$

1+2

1+2

3) $[2\ 4\ 5\ 6] \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

$2 + 12 + 10 + 6$

$= 30$

4) $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} [2\ 4\ 5\ 6]$

~~$2 + 12 + 10 + 6$~~

~~$= 30$~~

~~Same because symetrical.~~

$$\begin{array}{cc} & 4 \\ 4 & \begin{bmatrix} 2 & 4 & 5 & 6 \\ 6 & 12 & 15 & 18 \\ 4 & 8 & 10 & 12 \\ 2 & 4 & 5 & 6 \end{bmatrix} \end{array}$$

5)

$\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & -1 \\ -2 & 0 & 0 & 2 \end{bmatrix}$

$$\begin{array}{cc} & 4 \\ 3 & \begin{bmatrix} 2 & 2 & 8 & -4 \\ -6 & 0 & 0 & 6 \\ 0 & 1 & 4 & -1 \end{bmatrix} \end{array}$$

0+0

1

$\frac{0+2}{2}$ $\quad \frac{2+0}{2}$ $\quad \frac{8+0}{8}$ $\quad \frac{-2-2}{-4}$ $\quad \frac{0-6}{-6}$

6) $(5 \times 3 \to 3 \times 7)\ 7 \times 4$

1/ mrnnxb but 3?

$(5 \times 7 \to 7 \times 4)$

$= 5 \times 4$

7) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

doesn't change

I

~~8)~~

Practise Assignment
Using non-square matrices to do a projection.

1) $r' = r + \lambda\hat{s}$ $\quad r.\hat{e}_3 + \lambda s_3 = 0$

find r' in terms of r

$\lambda s_3 = -r.\hat{e}_3$

$\lambda = \dfrac{-r.\hat{e}_3}{s_3} = \dfrac{-r.\hat{e}_3}{\hat{s}.\hat{e}_3} = -\dfrac{r}{\hat{s}}$

~~$r' = r - \frac{r}{\hat{s}}$~~

~~$r' = r - \hat{s}\left(\frac{r}{\hat{s}}\right)$~~

~~$r' = r - r$~~ $\quad r' = r - \hat{s}(r.\hat{e}_3)/s_3$

2) r can be written as.

$r' = Ar$. for matrix A.

det with

$r'_i = r_i - s_i r_3/s_3$ $\quad \hat{e}_3 : r.\hat{e}_3 = r_j[\hat{e}_3]_j =$

$r'_i = I_{ij} - s_i[3/s_3)r_j$ $\quad I = I_{ij}r_j = r_i$

all of them.

3) $r'_i = r_i - s_i r_3/s_3$

goes to 2D. ~~stop~~
3D object 2D shadow.

$i = 1,2$.

$r'_i = \left(I_{ij} - \dfrac{s_i[\hat{e}_3]_j}{s_3}\right) r_j$. $\quad [\hat{e}_3]_j \to$ row 3, $s_3 \to$ row 3.

$$\begin{bmatrix} 1 & 0 & -s_1/s_3 \\ 0 & 1 & -s_2/s_3 \end{bmatrix}$$

4) 0, 0, 0

because 2D

5) $r' = r - \hat{s}\dfrac{r_3}{s_3}$

$\hat{s} = \begin{bmatrix} 4/13 \\ -3/13 \\ -12/13 \end{bmatrix}$ $\quad r = \begin{matrix} 6 \\ 2 \\ 3 \end{matrix}$

$s_3 = -\dfrac{12}{13}$

$r - s_1\dfrac{r_3}{s_3} =$

$6 - \dfrac{4}{13}\left(-\dfrac{13}{4}\right) = 6 + 1 = 7.$

$2 - \left(-\dfrac{3}{13}\right)\left(-\dfrac{13}{4}\right) = 2 - \dfrac{3}{4} = \dfrac{5}{4}$

$(7, 5/4)$

6) $\begin{bmatrix} 5 & -1 & -3 & 7 \\ 4 & -4 & 1 & -2 \\ 9 & 3 & 0 & 12 \end{bmatrix} \begin{bmatrix} 4/13 \\ -3/13 \\ -12/13 \end{bmatrix}$

P.T.O

$$A = \begin{bmatrix} 1 & 0 & -\frac{s_1}{s_3} \\ 0 & 1 & -\frac{s_2}{s_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{4} \end{bmatrix}$$

$$-\frac{s_1}{s_3} = \frac{-\frac{4}{13}}{-\frac{12}{13}} = \frac{1}{3}.$$

$$-\frac{s_2}{s_3} = \frac{\frac{3}{13}}{-\frac{12}{13}} = \frac{1}{4}.$$

$$[5,4,9] = 5+\tfrac{1}{3}\cdot 9,\ 4-\tfrac{1}{4}\cdot 9 = [8, \tfrac{7}{4}]$$

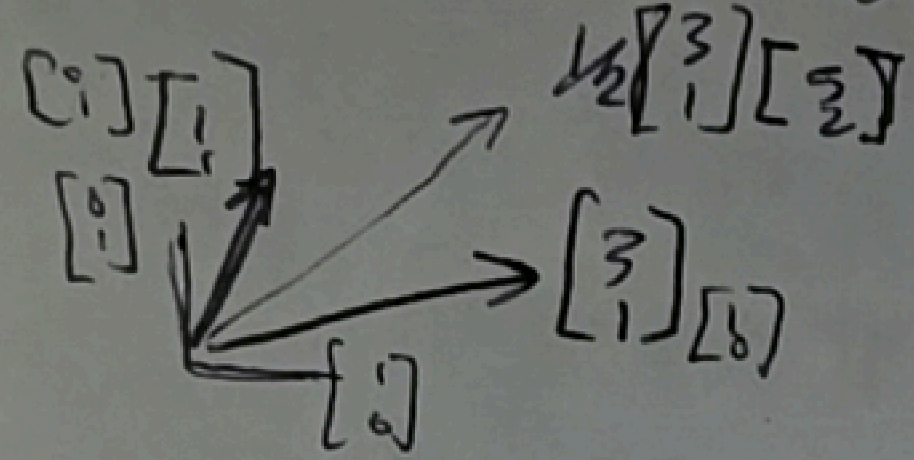
$$[-1\ -4\ 3] := [-1+1\ ;\ -4-\tfrac{1}{4}\cdot 3] = [0, -\tfrac{19}{4}]$$

$$[-3,1,0] := -3,\ 1 = [-3, 1]$$

$$[7,-2,12] :\ 7+4,\ -2-3 = [11,-5]$$

$$\begin{bmatrix} 8 & 0 & -3 & 11 \\ \frac{7}{4} & -\frac{19}{4} & 1 & -5 \end{bmatrix}$$

Matricies Changing Basis



The basis vectors for the bolder lines, $\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ in my frame.

Bold line transformation matrix $\begin{bmatrix}3&1\\1&1\end{bmatrix}$

$\underbrace{\begin{bmatrix}3&1\\1&1\end{bmatrix}}_{\cancel{\text{bold}}}\underbrace{\begin{bmatrix}3/2\\1/2\end{bmatrix}}_{\text{for bold}}$ ← thin line.

In my frame $= \underbrace{\begin{bmatrix}5\\2\end{bmatrix}}_{\text{for my.}}$

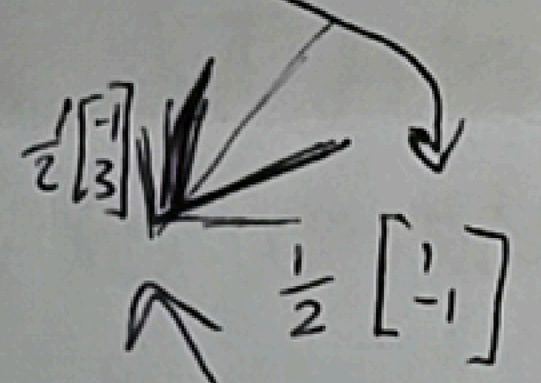
bold basis in my coordinate system. But usually want to go other way

To do reverse process want inverse.

B

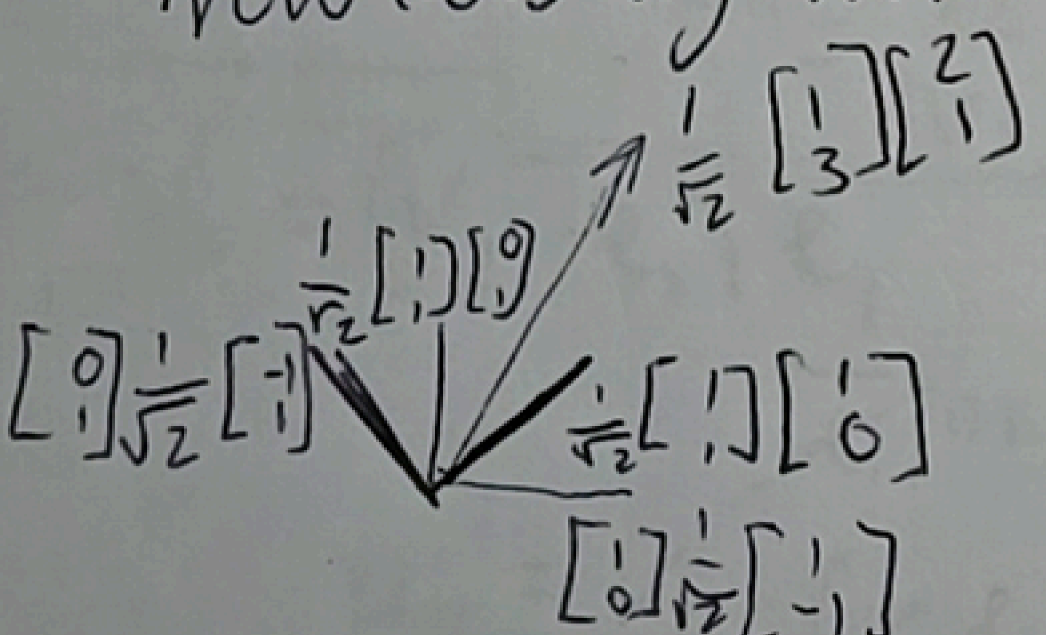
$B^{-1} \rightarrow \frac{1}{2}\begin{bmatrix}1&-1\\-1&3\end{bmatrix}$ ← my basis in bold world

minus flip



$\frac{1}{2}\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}3/2\\1/2\end{bmatrix}$

Now lets try with bold world is an orthonormal basis set.



$B\,\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix}$

$B^{-1}\,\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$

with projections

My version $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} \cdot$ Bold axis $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}4 = 2$

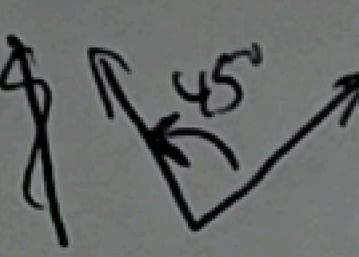
one axis $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{2}2 = 1$

$\begin{bmatrix}2\\1\end{bmatrix}$

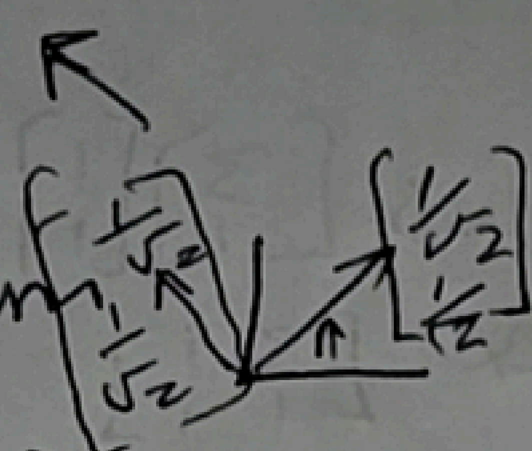
or try from earlier example, because not orthogonal then it doesn't work.

Doing a transformation in a changed basis.

$$\underset{B}{\begin{pmatrix}3&1\\1&1\end{pmatrix}}\ \underset{R\circlearrowleft 45^\circ}{\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}}$$



$\begin{bmatrix}x\\y\end{bmatrix}$ in 1001 system



$$\underset{R}{\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}}\ \underbrace{\underset{B}{\begin{pmatrix}3&1\\1&1\end{pmatrix}}\begin{bmatrix}x\\y\end{bmatrix}}_{\text{vector in my frame}}$$

$$\underbrace{\underset{B^{-1}}{\frac{1}{2}\begin{pmatrix}1&-1\\-1&3\end{pmatrix}}\ \underbrace{\underset{R\circlearrowleft 45}{\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}}\ \underset{B}{\begin{pmatrix}3&1\\1&1\end{pmatrix}}}_{\text{vector in my frame}}}_{\text{Vector in B frame.}}\begin{bmatrix}x\\y\end{bmatrix}$$

$\frac{1}{\sqrt{2}}\begin{pmatrix}2&0\\4&2\end{pmatrix}$

$\frac{1}{2\sqrt{2}}\begin{pmatrix}-2&-2\\10&6\end{pmatrix}$

$$\boxed{B^{-1}RB = R_B}$$

question

$$\frac{1}{2}\begin{bmatrix}3&-1\\-1&1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}2&0\\4&2\end{bmatrix}$$

$\cancel{6}\ 6-4=2$
$-2+4=2$

$2\ 2\ \frac{1}{\sqrt{2}}\begin{bmatrix}2&-2\\2&2\end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$

Orthogonal Matricies.

Transpose transformation

$A^T_{ij} = A_{ji}$.

interchange elements that make up.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

flip.

orthogonal.

A^T is a valid inverse

A^T $\quad A_{n \times n}$.

$A^T \cdot A = I$

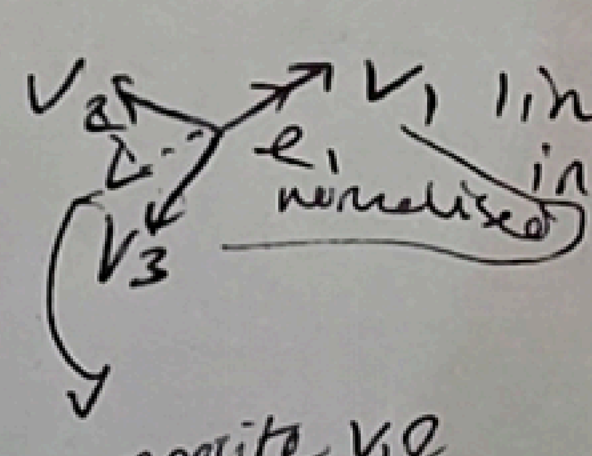
$$\begin{pmatrix} (a_1 \quad) \\ (a_2 \quad) \\ \vdots \\ (a_n \quad) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} a_1 \end{pmatrix} \begin{pmatrix} a_2 \end{pmatrix} \cdots \begin{pmatrix} a_n \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \cdots & & \\ 0 & 0 & 0 & 0 & 1 & \cdots & \end{pmatrix}$$

$$\left.\begin{array}{ll} a_i \cdot a_j = 0 & \text{if } i \neq j \\ \quad\quad = 1 & i = j \end{array}\right\} \text{orthonormal basis set.} \quad = I$$

The Gram - Schmidt process

how to construct an orthonormal basis

$V = \{v_1, v_2 \ldots v_n\}$



Reminder if determinant is 0 then is NOT independent.

$$\boxed{v_1 \qquad e_1 = \frac{v_1}{|v_1|}}$$

$$\boxed{\begin{array}{l} v_2 = (v_2 \cdot e_1) e_1 \\ \qquad\qquad\quad \cancel{|e_1|} \\ \qquad u_2 = v_2 - (v \cdot e_1) e_1 \\ \text{normalised } \dfrac{u_2}{|u_2|} = e_2 \end{array}}$$

vector projection v_2 onto e_1

$v_2 \cdot e_1 \quad \dfrac{v_2 \cdot e_1}{|e_1|}$

v_3 not in the plane.

$v_3 - (v_3 \cdot e_1) e_1 - (v_3 \cdot e_2) e_2 = u_3$

normalise u_3

$\dfrac{u_3}{|u_3|} = e_3$