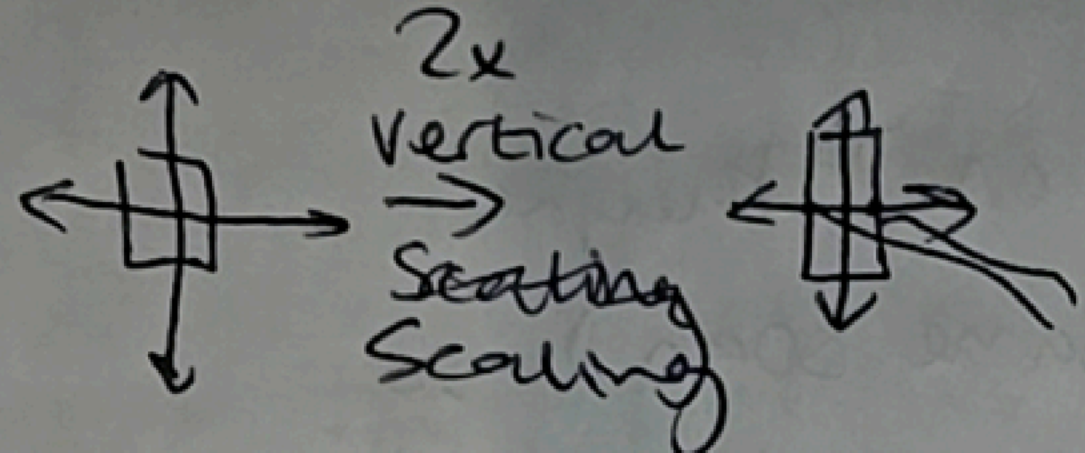
Module 5: with Dr Sam Cooper dyson school design 24/09/2025

What are eigenvalues and eigenvectors.

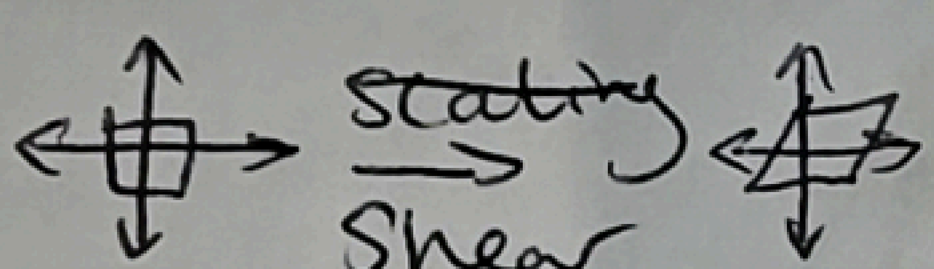
Over the last week we've learned expressing linear transformations in scalings rotations and shears.



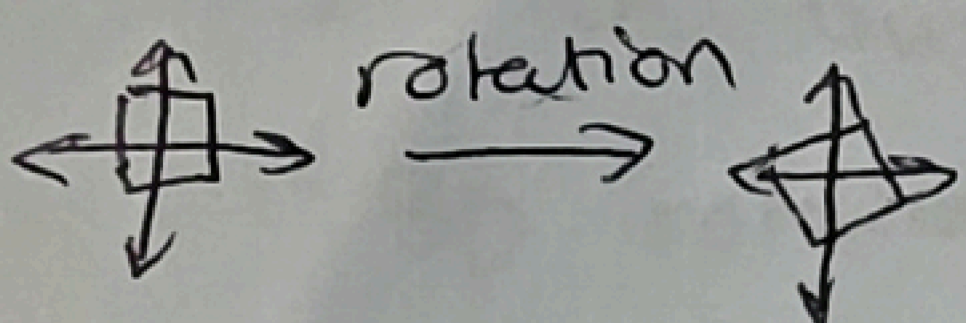
unchanged vectors

is ~~are~~ eigenvectors!
because horizontal ~~diagonal~~ is unchanged it has a corresponding eigen value 1, whereas because the vertical changed 2x it has eigenvalue 2.

so how much length has changed



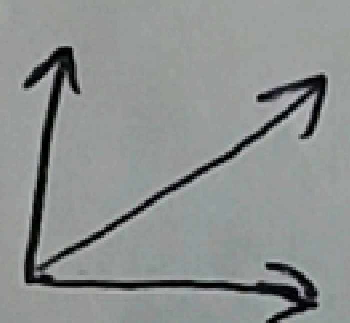
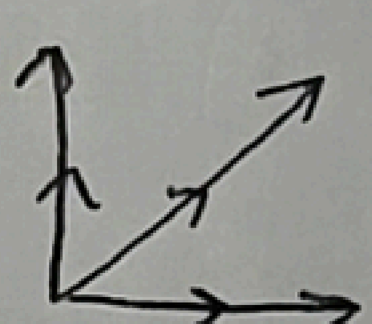
has 1 eigenvector ~~2~~ as only the horizontal ~~diagonal~~ has stayed same length



has 0 eigenvectors as all have been rotated off original span.

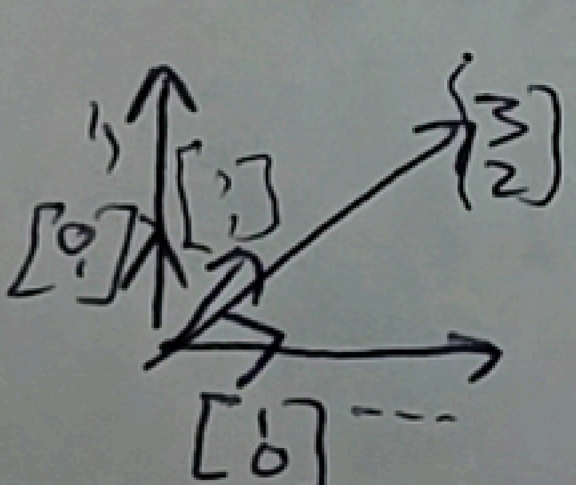
Problems/questions

1)

 $\longrightarrow T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} =$  doubles

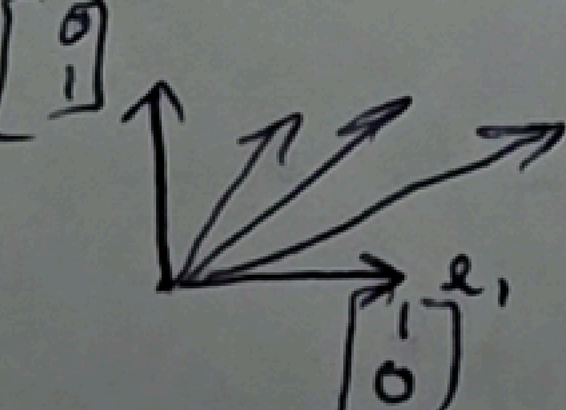
So all are ~~2~~ eigenvectors with value 2?

2)



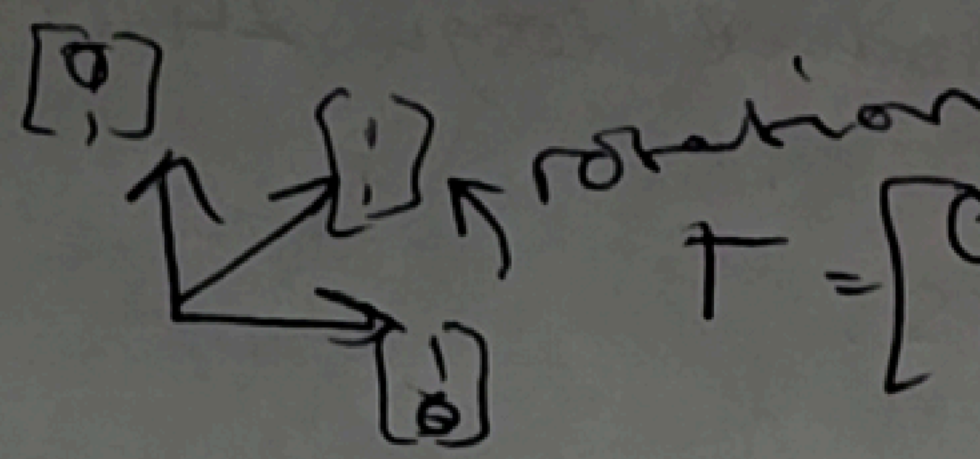
$T = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ eigenvectors only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3)



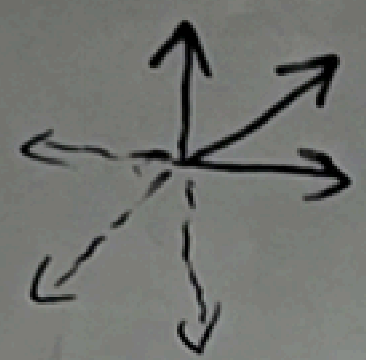
$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has stayed in same span so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is eigenvector

4)



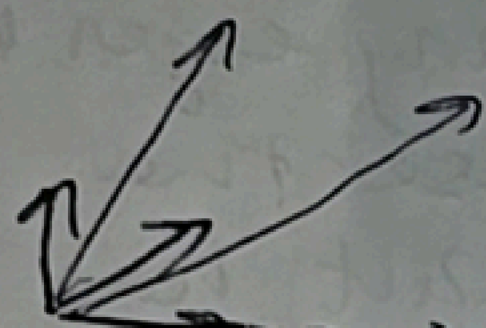
$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has 0 eigenvectors as no longer are any vectors in the span

5)



$T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ okay this one? they're still in the same span? so i think eigenvectors.

6)



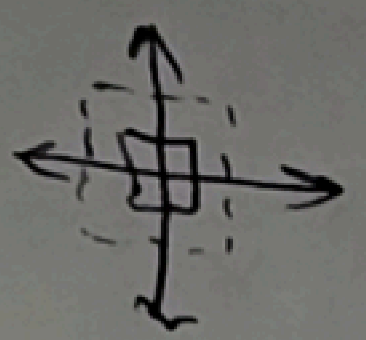
$T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is 2x so still eigenvector.

100% ✓.

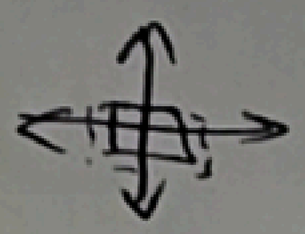
Special eigen cases

Eigenvectors → ~~how much~~ if the vector is in the same span

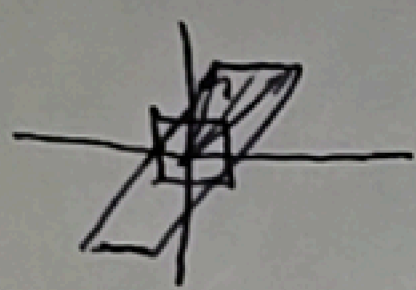
Eigenvalues → ~~y~~ by how much its changed e.g $\times 1 = 1$ $\times 2 = 2$



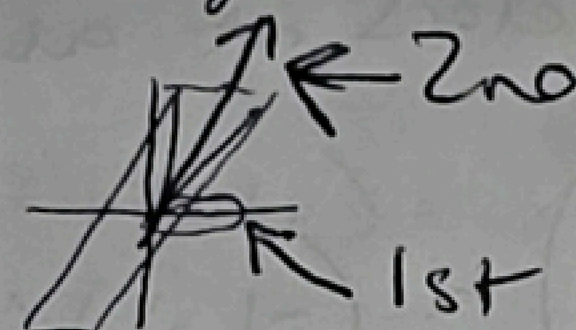
uniform scaling any vector would be an eigenvector.



180° rotation all vectors are eigenvectors. the eigen values would be −1.

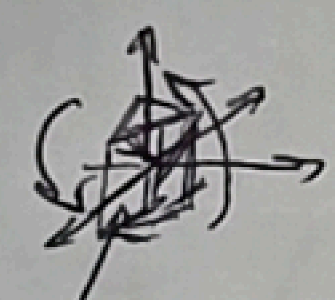


This transformation actually has 2 eigenvectors so.



In 3D this all gets much more complicated

A 3D rotation keeps an eigenvector



this vector stays.

Calculating eigenvectors.

$$A \underset{\text{eigenvector}}{x} = \underset{\text{lambda}}{\lambda} x$$

$(A - \lambda I)x = 0$ ← if it equals 0, one side must equal zero.

$\neq 0$ we can find x if 0 then no vector at all

~~A A~~ $A - \lambda I = 0$ or $\det(A - \lambda I) = 0$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bd = 0$$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = 0$

@ $\lambda = 1$: $\begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$

@ $\lambda = 2$: $\begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$

@ $\lambda = 1$: $x = \begin{pmatrix} t \\ 0 \end{pmatrix}$ any/any arbitrary. @ $\lambda = 2$: $x = \begin{pmatrix} 0 \\ t \end{pmatrix}$ any arbitrary

Lets now try with a rotation $90°$ anticlockwise, which should have no eigenvectors at all

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \det\begin{pmatrix} 0 & -1-\lambda \\ 1-\lambda & 0 \end{pmatrix} = (-1-\lambda)(1-\lambda) = 0 =$

$\lambda^2 + 1 = 0$

~~$\lambda^2 + \lambda - \lambda - \lambda\ \lambda^2 - \lambda = 0$~~

Practice questions

1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$: $\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = 0$

$2 - \lambda - 2\lambda + \lambda^2 = \lambda^2 - 3\lambda + 2 = 0$

$\lambda = 1 \quad \lambda = 2$

2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ select eigenvectors.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3) $A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$ $\quad \det\begin{pmatrix} 3-\lambda & 4-\lambda \\ 0 & 5-\lambda \end{pmatrix}$ $\quad (3-\lambda)(4-\lambda / 5-\lambda) = 12$ ~~$+15+3\lambda+3\lambda+\lambda^2$~~ $+\lambda^2$

~~$2\lambda^2 + 6\lambda + 27$~~

$\lambda^2 - \cancel{a+d}(3+5)\lambda + (ab-bc$ 3×4 ~~$= 4\times 5) = -$~~

$3\times 5 - 4\times 0 = 15$

also $(\lambda - 3)(\lambda - 5)$

$\lambda^2 \pm 8\lambda + 15$ $\quad \lambda^2 - 8\lambda + 15 \quad \lambda_1 = 3 \quad \lambda_2 = 5$

4) $A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$

eigen values are 3 & 5.

find eigen vectors by.

$\leftarrow \begin{pmatrix} 3-3 & \\ & 5-3 \end{pmatrix}$

For $\lambda = 3$. $A - 3I \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cancel{4} \\ \cancel{2} \end{pmatrix}$ $4y = 0$

$2y = 0$ $y = 0$

fo $\lambda = 5$ $\begin{pmatrix} 3-5 & \\ & 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cancel{2} \\ \cancel{0} \end{pmatrix}$ $-2x + 4y = 0$ x is free

$x = 2y$

$\frac{2y}{y} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} x \\ 0 \end{bmatrix}$ any number $x \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

5) $A \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$.

$\lambda^2 - 5\lambda + (4-0) = \lambda^2 - 5\lambda + 4$

or $(\lambda - 1)(\lambda - 4) = \lambda^2 - 5\lambda + 4$

$\lambda = 1$ $\lambda = 4$

6)

for $x = 1$ $\begin{pmatrix} 0 & 0 \\ -1 & 3 \end{pmatrix}$ $-x + 3y = 0$

$-x = -3y$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

for $x = 4$ $\begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix}$ $-3x - x = -4x$ y is free

to the pure

$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

7) $A = \begin{bmatrix} -3 & 8 \\ 2 & 3 \end{bmatrix}$

$\lambda^2 - \underbrace{\cancel{(-3+3)\lambda}}_{\text{cancels}} + (9-16)$

$\lambda^2 - 25$ $\lambda^2 - 5\lambda + 5$

I'm guessing from

$(\lambda - 5)(\lambda + 5) = \lambda^2 - 25$

8) $A = \begin{bmatrix} -3 & 8 \\ 2 & 3 \end{bmatrix}$

for $-5 \begin{pmatrix} -8 & 8 \\ 2 & -2 \end{pmatrix}$ $-8x + 8x - 2y + 2y = 0$?

$\frac{0}{0}$? $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ maybe

$\frac{-16}{-4}$? $\frac{4}{-1}$ $\frac{-4}{1}$

for $+5 \begin{pmatrix} 2 & 8 \\ 2 & 8 \end{pmatrix}$ $2x + 8x + 2y + 8y$

$10x + 10y$ $\frac{10}{10} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ maybe just the 1.

~~9) $A = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} = \lambda^2 - 2\lambda + (-15 + 16) = \lambda^2 - 2\lambda + 1$~~

~~$(x + 2)(x - 1)$~~ no 2 can x to 1

~~10) $A = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} = \lambda^2 - (-2 + 1)\lambda + (-2 + 3)$~~

$\lambda^2 - \cancel{2}\lambda + \cancel{5}1$

~~$(x - 1)(x - 1$~~

~~$x^2 - 2x$~~

Redo some answers as 60%

dont forget to select ALL eigenvalues

9) $\lambda^2 - 2\lambda + 1$ $(x - 1)(x - 1)$

$x - 1\lambda - 1\lambda + 1$

λ_1 and $\lambda_2 = 1$. $x - 2\lambda + 1$

• check $\begin{matrix} 4 & 4 \\ -4 & -4 \end{matrix} = 0$ for 1

$\begin{matrix} 6 & 4 \\ -4 & -2 \end{matrix} = 4 \neq 0$ for -1

10) $\lambda^2 - (-2 \underset{+1}{+} 1) + (-2 \underset{1}{\div} -3)$

$\lambda^2 + \lambda + 1$

Phew that was a tough one.

Eigenbasis Example.

$T = \begin{pmatrix}1&1\\0&2\end{pmatrix}$ eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\end{bmatrix}$ eigenvalues $\{\lambda=2, \lambda=1$

$\cancel{\#}$ $\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

$\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1+1\\0+2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

$T^2 = \begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}1&1\\0&2\end{pmatrix} = \begin{pmatrix}1&3\\0&4\end{pmatrix}$

Apply T again $\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} = \begin{matrix}0+2\\0+4\end{matrix} = \begin{pmatrix}2\\4\end{pmatrix}$ } same.

App T^2 to vector $\begin{pmatrix}1&3\\0&4\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{matrix}-1+3\\0+4\end{matrix} = \begin{pmatrix}2\\4\end{pmatrix}$

Apply through

$T^n = CD^nC^{-1}$

$C = \begin{pmatrix}1&1\\0&1\end{pmatrix}$ ← from eigenvectors

$C^{-1} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}$ simple ← vector coordinate ~~though~~

← diagonal only of T

$T^2 = CD^2C^{-1} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}^2\begin{pmatrix}1&-1\\0&1\end{pmatrix}$

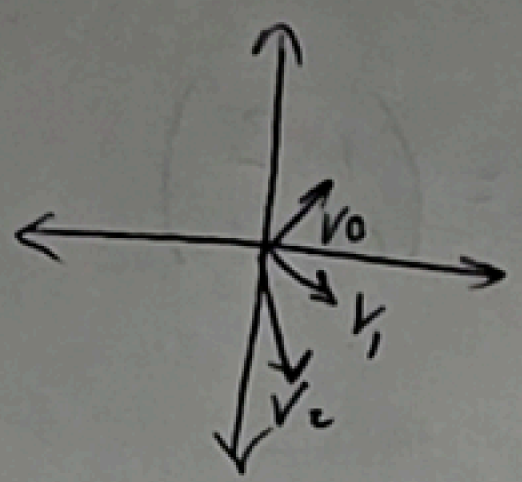
$= \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&4\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&-1\\0&4\end{pmatrix} = \begin{pmatrix}1&3\\0&4\end{pmatrix}$

$= \begin{pmatrix}1&3\\0&4\end{pmatrix}$

Applying this to vector $\begin{pmatrix}-1\\1\end{pmatrix}$, we get $= \begin{bmatrix}1&3\\0&4\end{bmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1+3\\0+4\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$ ✓

Changing to the EigenBasis

Diagonalisation.



$$T = \begin{pmatrix} 0.9 & 0.8 \\ -1 & 0.35 \end{pmatrix}$$

$$v_0 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$v_1 = T v_0$$

$$v_2 = T v_1 = T(T v_0) = T^2 v_0$$

$$v_n = T^n v_0$$

If all terms are zero except the diagonal, when raising matricies by powers it makes it alot easier e.g.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ calculate A^3

$A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$. $\quad T^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$

But if its not a diagonal matrix

Then the change to a basis where T becomes diagonal

$$C = \begin{pmatrix} x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$T = CDC^{-1}$$

$$T^2 = CDC^{-1}CDC^{-1} = CDDC^{-1} = CD^2C^{-1}$$

$$T^n = CD^nC^{-1}$$

$$\begin{tikzcd} \boxed{v} \arrow[r, "T^n"] \arrow[d, "C^{-1}"'] & \boxed{T^n v} \\ \boxed{[v]_E} \arrow[r, "D^n"] & \boxed{[T^n v]_E} \arrow[u, "C"'] \end{tikzcd}$$

$\leftarrow [T^n v]_E$

$$T^n = CD^nC^{-1}$$

CS50 Harvard 24/09/2025

Computational thinking

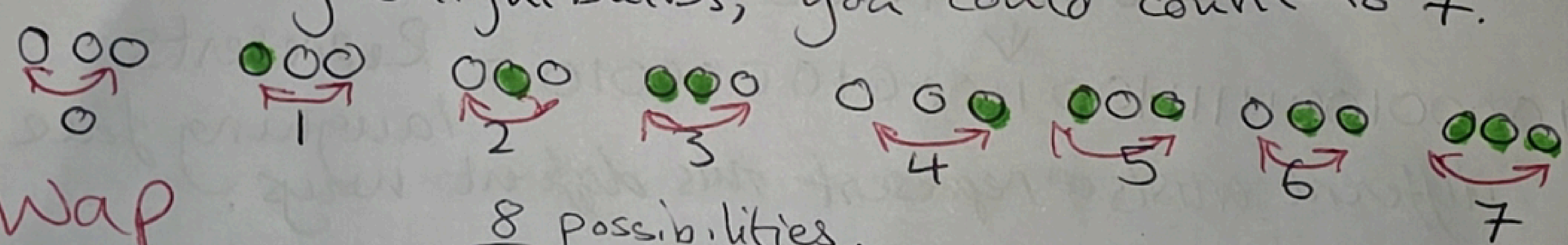
Input → ☐ → output

How to represent inputs & outputs.

Counting - Unary - Base-1
like counting on a hand - 5

- 31 ways to count on human hand → patterns of fingers etc.
but if we count including different states eg up down
base-2 AKA Binary.
0 1 → binary digits. 1 binary digit → bit.
e.g lightbulb on or off. the light switch that turns on & off is Transistor.

for lets say 3 lightbulbs, you could count to 7.



Swap

8 possibilities.

We're use to using base-10 - decimals

123 $1 = 10^0$ $10 = 10^1$ $100 = 10^2$
100 10 1

See high digit first

In computers, 2^0 2^1 2^2 because they only have 2 digits 0, 1. for transistors.
↓ 1 2 4 etc. ↓

If 1 2 4 / 0 0 0 = 0.

If 1 2 4 / 0 0 1 = 1 If we keep going... we get stuck at 7.

More commonly we use a byte which is 8 bits.

1 2 4 8 16 32 64 128 128 64 32 16 8 4 2 1
0 0 0 0 0 0 0 0 → Can count to 255. 256 possibilities

Using voltage to represent 0 & 1, so low voltage for 0, ~~large~~ 5 volt aka 5 for 1.

How could you represent A?

We select a pattern and assign it a pattern

	128	64	32	16	8	4	2	1	
A =	0	1	0	0	0	0	0	1	(this decimal would be 65)
a =									(97.)

ASCII only 8 bits on characters. only 255. (256 include 0)

If you recieved.

0100b00 01001001 00100001 = 72 73 33.

HI! 3 bytes. = H I !

quick questions

01000010 → B number 66. 01001111 = 15+64 = 79 = O

01010111 → ~~80~~ P add together 87. = W.

This is all okay for English but for other languages like chinese. so 8 bits nowadays for English 16 bits or maybe 24 or 32 for other perpoces. "fun fact" 32 bits = ~~Over~~ 4 billion.

Unicode Standardised - backwards compatible with ASCII.

e.g with 4 billion possible options have space for emojis.

Example

4036991106

↓

11110000100111111001100010000010 = Represents laughing face.

different artists ~~ep~~ represent this different ways.

How to represent Colour.

RGB. pixels. 3 bytes 24 bits per pixel.

~~[illegible]~~

R	G	B	
72	73	33	amounts Represent → Yellow.
med	med	little.	

Mega byte mb = 1000000 bytes, 3 bytes per Pixel. at least.

Compression can make less mb.