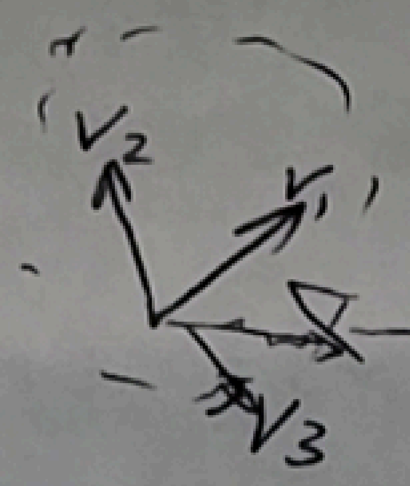
Reflecting in a plane 23/09/2025

Example

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\0\\1\end{pmatrix} \quad \begin{pmatrix}3\\1\\-1\end{pmatrix}$$
$$v_1 \qquad v_2 \qquad v_3$$



first start by finding orthonormal vectors describing this plane using Gram Schmidt process.

$\sqrt{1^2+1^2+1^2}$

$$e_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$u_2 = v_2 - (v_2 \cdot e_1)\,e_1 = \begin{pmatrix}2\\0\\1\end{pmatrix} - \left[\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$= \begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{1}{3}3\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

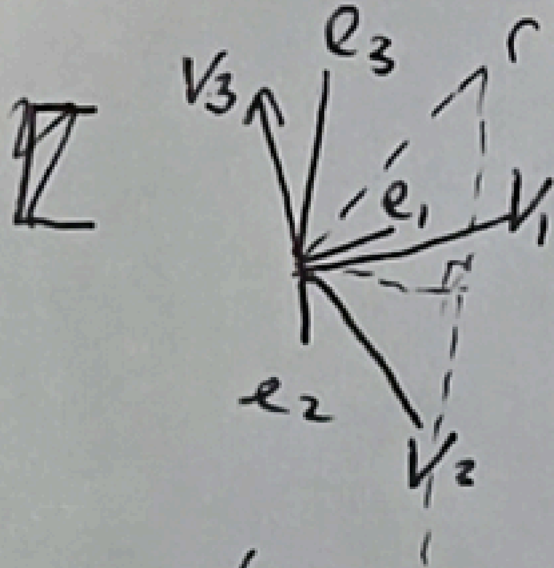
$$e_2 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$u_3 = v_3 - (v_3 \cdot e_1)\,e_1 - (v_3 \cdot e_2)\,e_2.$$
$$= \begin{pmatrix}3\\1\\-1\end{pmatrix} - \left[\begin{pmatrix}3\\1\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} - \left[\begin{pmatrix}3\\1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$
$$= \begin{bmatrix}3\\1\\-1\end{bmatrix} - \frac{1}{\cancel{3}}\cancel{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{\cancel{2}}\cancel{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$e_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$1^2 + -1^2 + 2^2 = 6.$

$r\begin{pmatrix}2\\3\\5\end{pmatrix}$ New transformation matrix



$$E = \big(\underbrace{(e_1)\,(e_2)}_{\text{contains plane}}\,\underbrace{(e_3)}_{\text{contains normal to plane}}\big) \qquad E = \left(\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)$$

$$r = \begin{pmatrix}2\\3\\5\end{pmatrix}$$

r=23 Think of r as a vector in the plane
Some vector composed of e_1 & e_1s
aka. some vector that's normal $\rightarrow e_3$
And some vector

1+
$-\frac{-1}{-1}$ -

this line will be some as in plane

$$T_E = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$
$$e_1 \; e_2 \; e_3$$

$$\begin{array}{ccc} \boxed{r} & \xrightarrow{\text{hard}} & \boxed{r'} \\ {\scriptstyle E^{-1}}\downarrow & & \uparrow{\scriptstyle E} \\ \boxed{r_E} & \xrightarrow{} & \boxed{r'_E} \end{array}$$

$$r' = E\,T_E\,E^{-1}\,r$$

$$r' = Tr = T\begin{pmatrix}2\\3\\5\end{pmatrix} = \frac{1}{3}\begin{pmatrix}11\\14\\5\end{pmatrix}.$$

$$T_E E^T = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}}(1,1,1)\\ \frac{1}{\sqrt{2}}(1,-1,0)\\ \frac{1}{\sqrt{6}}(1,1,-2)\end{pmatrix} \quad T_E = \begin{pmatrix}\frac{1}{\sqrt{3}}(1,1,1)\\ \frac{1}{\sqrt{2}}(1,-1,0)\\ \cancel{PTO}\ \frac{1}{\sqrt{6}}(1,-1,2)\end{pmatrix}$$

$$E T_E E^T = \begin{pmatrix} \frac{1}{3}+\frac{1}{2}+\frac{1}{6} & \frac{1}{3}-\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+0+\frac{3}{6} \\ \frac{1}{3}-\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+\frac{1}{2}+\frac{1}{6} & \frac{1}{3}+0+\frac{2}{6} \\ \frac{1}{3}+0+\frac{2}{6} & \frac{1}{3}+0+\frac{2}{6} & \frac{1}{3}+0-\frac{4}{6} \end{pmatrix} = \frac{1}{3}\begin{pmatrix}2&-1&2\\-1&2&2\\2&2&-1\end{pmatrix} = T$$

MIT Course Lecture 2 Problems set. 23/09/2025

Problem 1). Thm. ~~For every integer n, the number n is foolish precisely when n+1 is barsome.~~ Example

a) Thm: Every natural number is <u>tall</u> or <u>wide</u> but not both. (method proof by cases).

T=Tall W=wide

$\forall n \in \mathbb{N} = n(T \vee W)$ iff $n = T \,\&\, \overline{W}$ iff $n = W \& \overline{T}$.

Tall $\Rightarrow$ Not Wide $T = \overline{W}$

wide $\Rightarrow$ Not Tall $W = \overline{T}$

$\forall n \in \mathbb{N}: (T(n) \vee W(n)) \wedge \text{Not}(T(n) \wedge W(n))$

Suppose n is any natural number; we must show T(n) implies Not W(n) and that W(n) implies Not T(n).

① Assume T(n) is true, show $\overline{W}(n)$.

② Assume instead W(n) is true, wt WTS $\overline{T}(n)$

b) Thm. There exists a <u>happy</u> natural number between 10 and 100 that has no <u>happy</u> proper divisors. (proof method use 25 as the example)

H=happy. P=proper divisors.

$\exists n \in \mathbb{N}$ ~~nn H(n)~~ $\Rightarrow H(n)(10 \leq n \leq 100)$

$\exists n \in \mathbb{N}\ H(n)(10 \leq n \leq 100)\ \forall P(P(n) \rightarrow \overline{H}(p))$

For all proper divisors of n proper diviser is not a happy proper divisors

~~Take (n)~~.

There exists a ~~number~~ ~~natural~~ number equivalent to a natural number which is happy. H(25). I has no happy proper divisors $P \in \{1,5\}$. WTS between 10-100 no happy divisors $p \in \{1,5\} \Rightarrow P \notin H(p)$.

Therefore if P has no happy divisers H(n) exists.

c) Thm. "There are no devious natural numbers."

(proof method regular induction with base cases 0 and 1.)

D = devious.

$\exists n \in \mathbb{N} \quad n \neq D$

$P(n) := \exists n \in \mathbb{N} \neq D$. we'll show $\forall n \geq 0 . P(n)$

by induction. base case WTS $P(0) : \neq \bar{D}$

Assume $n \geq 0$, and assume $p(n)$ is true. WTS $P(0) \neq \bar{D}$

Assume $n \geq 1$, and assume $P(n)$ is true. WTS $P(1) = \bar{D}$

Show $P(0) \wedge P(1) \wedge \cdots \wedge P(n)$ (all numbers $\leq n$ are not devious).

$P(n+1)$: not devious.

We proved	want
$P(0)$	$P(0)$
$P(0) \to P(1)$	$P(1)$
$P(1) \to P \ldots$	$\vdots$
$\vdots$	

$P(0) \wedge \forall n \geq 0 . P(n) \to P(n+1) \Rightarrow$

?

$P \, \forall n \geq 0 . P(n)$.

~~No way~~ to work.

2 Find the Mistake. ← the question written down.

a) Proof. We'll use the stronger induction hypothesis $Q(i) :=$ "$x_i \geq 2^i$", using strong induction to show ~~that~~ $Q(i)$ holds for every $i \geq 1$.

before ~~$x_i \geq 1$~~

Base cases $Q(1)$ and $Q(2)$. $x_1 = 4 \geq 2^1$ and $x_2 = 13 \geq 2$

not sure? but i think proven problem here.

Inductive step: suppose $i \geq 2$. Assume $x_k \geq 2^k$ for $1 \leq k \leq i$ and lets prove that $x_{i+1} \geq 2^{i+1}$. ~~and $x_i \geq \ldots 2^{i-1}$. Plug these two inequalities into $x_{i+1} = 6x_i + 8x_{i-1}$~~

Lowering the bound

Consider $x_{i+1} = 6x_i - 8x_{i-1}$. By the inductive hypothesis know that $x_i \geq 2^i$ and $x_{i-1} \geq 2^{i-1}$. Plug those two inequalities into $x_{i+1} = 6x_i - 8x_{i-1}$ and you get $x_{i+1} \geq 6 \cdot 2^i - 8 \cdot 2^{i-1} = 4 \cdot 2^{i-1} = 2^{i+1}$. That completes the inductive step

end up with negative co-efficient

So $x_i \geq 2^i$ for all i.

Since $2^i \geq 1$ we have that $x_i \geq 1$ as desired.

b)

Proof: We'll use (regular) induction on the stronger hypothesis $R(i) := "x_i \geq 3x_{i-1}"$ proving that $R(i)$ holds for every $i \geq 2$

Base case $R(2)$: $x_2 = 13 \geq 12 = 3x_1$.

Inductive step: Suppose $i \geq 2$. Assuming that $x_i \geq 3x_{i-1}$ (ie. $R(i)$) is true, lets show that $x_{i+1} \geq 3x_i$ (ie $R(i+1)$) is also true. Consider $x_{i+1} = 6x_i - 8x_{i-1}$ By the inductive hypothesis we know that $x_i \geq 3x_{i-1}$, so $\frac{8}{3}x_i \geq 8x_{i-1}$. Plug that into ~~the inductive hypothesis~~ $x_{i+1} = 6x_i - 8x_{i-1} = \frac{10}{3}x_i + \frac{8}{3}x_i - 8x_{i-1}$ and you get $x_{i+1} \geq \frac{10}{3}x_i$. Since $x_i > 0$ $\frac{10}{3}x_i > 3x_i$ so $x_{i+1} \geq 3x_i$ That completes the inductive step, so $x_i \geq 3x_{i-1}$ for all $i \geq 2$. We know that $x_1 > 0$ and for all $i \geq 2$ $x_i \geq 3x_{i-1}$. Also $x_{i-1} > 0$, so $x_i > 0$ for all i.

I think here is assumption. If $x_i > 0$ ~~that~~ is part of the induction process then you need to prove it and ~~add~~ include this into the steps.

c) Proof. We'll use strong induction on the stronger induction hypothesis $S(i) := "x_i = 4^i"$, proving that $S(i)$ holds for every $i \geq 1$.

~~Base case~~ $S(1)$: $x_1 = 4 = 4^1$ Inductive step: Suppose $i \geq 1$. If $S(1) \ldots S(i)$ are all true, let's show that $S(i+1)$ is true, ie, $x_{i+1} = 4^{i+1}$. We know $x_{i+1} = 6x_i - 8x_{i-1}$. By the inductive hypothesis (specifically $S(i-1)$ and $S(i)$), we know that $x_{i-1} = 4^{i-1}$ and $x_i = 4^i$. Plug these (specifically $S(i-1)$ and $S(i)$) we know that x_{i+1}

$x_{i+1} = 4^{i-1} = 6 \cdot 4 \cdot 4^{i-1} - 8 \cdot 4^{i-1} = 16 \cdot 4^{i-1} = 4^2 4^{i-1} = 4^{i+1}$. That completes the inductive step, so $x_i = 4x_i$ for all $i \geq 1$

Since $4^i \geq 1$, we have that $x_i \geq 1$ as desired

Hint: Zachs claim is false → if $i = 2$, if you know base $i = 1$, what part of inductive step when $i = 1$ fails

Only one base case. Need to go between two to see previous or future terms.

- $x_i = 4^i$ is false for $i = 2$ since $x_2 = 13 \neq 16$.

d) Prove the claim yourself

Proof we'll use induction on a stronger hypothesis

$P(i) := x_i > 0$ and $x_i \geq 2x_{i-1}$ for $i \geq 2$

Base case, $P(1)$: $x_1 = 4 > 0$

$P(2)$ x_2 $13 > 0$ and $x_2 \geq 2x_1$ $13 \geq 8$

Inductive step: Suppose $i \geq 1$. Assume that $P(2), P(3)$ to $P(i)$. $i \geq 2$. Then $x_i > 0$, $x_{i-1} > 0$ and $x_i 2x_{i-1}$.

~~by the inductive process~~ $x_{i+1} = 6x_i - 8x_{i-1} \geq 6(2x_{i-1}) - 8x_{i-1}$

$= 4x_{i-1} > 0$. $\cancel{x_{i+1} \geq}$

Growth $x_{i+1} \geq$ from $x_{i-1} \leq \frac{1}{2} x_i$.

$x_{i+1} = 6x_i - 8x_{i-1} \geq 6x_i - 8(\frac{1}{2}x_i) = 2x_i$

By the inductive process $P(i+1)$ holds by $x_i > 0$ $\forall i \geq 1$.