gf-nishida-16: Simple and Fast $GF(2^{16})$ Library

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***Abstract*—This paper introduces a simple and fast $GF(2^{16})$ library that uses two step memory lookups and illustrates its mechanism. Our benchmark results show its superior performance over other available open source $GF(2^{n \geq 16})$ libraries. The source code of gf-nishida-16 is written in C and is released under a FreeBSD license.**

***Index Terms*—Galois Field, Finite Field, Random Network Coding**

I. Introduction

In some programs, the calculation in Galois (or Finite) Field greatly affects their overall performance. For example, encoding and decoding in Random Network Coding (RNC) repeatedly multiplies and divides given data in GF, and therefore the calculation in a GF determines the processing speeds of all programs based on it. Our distributed data system that uses RNC was not exceptional; our prototype was very slow and far from practical because of a slow GF library. This urged us to develop a fast 16 or greater bit GF library, and as a result a sufficiently fast 16bit library gf-nishida-16 was born. We have decided to release it under a FreeBSD license which is one of the least restricted licenses so that it helps many developers write their own code.

II. General $GF(2^n)$ Libraries

Fast 8bit GF multiplication and division can be easily achieved by simple memory lookups in the following way:

```
uint8_t GF8mulTbl[256][256];
uint8_t GF8divTbl[256][256];

// Returns a * b in GF(2^8)
void GF8mul(uint8_t a, uint8_t b)
{
    return GF8mulTbl[a][b];
}

// Returns a / b in GF(2^8)
void GF8div(uint8_t a, uint8_t b)
{
    return GF8divTbl[a][b];
}
```

assuming entries in `GF8mulTbl` and `GF8divTbl` are calculated beforehand. This requires only 2^{17} (128k) byte memory for the tables and is very efficient as long as the primitive polynomial is static. However, the same technique cannot be easily applied to $GF(2^{16})$ as the tables consumes 2^{34} (16G) byte memory in $GF(2^{16})$. The reason why we needed a 16 or greater bit GF library is simple. RNC encodes data by creating linear equations like below and decodes by solving a system of those linear equations.

$$\begin{cases} a_{0,0}x_0 + a_{0,1}x_1 + a_{0,2}x_2 = b_0 \\ a_{1,0}x_0 + a_{1,1}x_1 + a_{1,2}x_2 = b_1 \\ \cdots \end{cases} \quad (1)$$

Since $GF(2^8)$ space is too small to maintain the independency of all those linear equations, the probability of failure in solving a system of linear equations in $GF(2^8)$ is considerably high. Hence, we sought for a GF library with 16 or greater bits for our distributed data system. Note that randomly created $GF(2^{16})$ linear equations still produce duplications at a low probability. In our experiment, we had approximately four duplications out of randomly created 200,000 patterns of three linear equations. Although we regard it as low enough, our distributed data system uses by way of caution the combinations of coefficients that do not generate any duplications. The same approach is applicable to $GF(2^8)$ but there is still a much higher chance of duplication in regenerating a new linear equation from existing linear equations than $GF(2^{16})$ [1].

SSE provides probably the most ideal performance and parallel processing environment for GF calculation especially in high bits like 64. In fact, a 64bit multiplication function from GF-Complete created by Plank, et al. [2], which utilizes SSE, records superb benchmark results in our experiment (see Section V). However unfortunately, their program has been removed due to potential patent violation [3]. We regret it and at the same time decided to create an alternative library that does not use SSE.

Intel added Carry-less Multiplication (CLMUL) instruction set specialized in $GF(2^n)$ multiplication to their CPUs in 2010. There is also a patent on processing erasure coding by CLMUL [4]. However our benchmark does not show sufficient performance (see gf-solaris-128 in Section V).

There is another GF library by Plank [5] which covers 8 to 32bits. While our investigation, we noticed that its logarithmic table based functions used the same approach as gf-nishida-16. However, gf-nishida-16 is more optimized as a computer program and achieves further speedup. The benchmark results on Plank's library appear as gf-plank and gf-plank-logtable in Section V.

III. Mechanism of gf-nishida-16

We first prepare two memory spaces as follows:

```
uint16_t GF16memL[65536 * 4];
uint32_t GF16memIdx[65536];
```

gf-nishida-16 was originally developed as part of Random Network Coding Project in ASUSA Corporation

then input 2^i to `GF16memL[i]` for $0 \leq i \leq 65534$ in a $GF(2^{16})$ manner. Herein, a $GF(2^{16})$ manner means: if 2^i is equal to or greater than 65536, then we input $2^i \hat{} P$ instead, where P is a polynomial primitive like $2^{16}+2^{12}+2^3+2^1+1 =$ 0x1100b and $\hat{}$ is XOR. Consequently, we have the following code for initializing the first quarter space of `GF16memL`:

```
uint16_t t;
uint32_t i, n;

GF16memL[0] = t = 1;
for (i = 1; i <= 65534; i++) {
    n = (uint32_t)t << 1;
    GF16memL[i] = t =
          (uint16_t)((n >= 65536) ? n ^ P : n);
}
```

In this loop, we also input values to `GF16memIdx` as follows:

```
uint16_t t;
uint32_t i, n;

GF16memL[0] = t = 1;
for (i = 1; i <= 65534; i++) {
    n = (uint32_t)t << 1;
    GF16memL[i] = t =
          (uint16_t)((n >= 65536) ? n ^ P : n);
    GF16memIdx[t] = i;
}
```

so that we have

$$a = 2^{\texttt{GF16memIdx}[a]} = \texttt{GF16memL}[\texttt{GF16memIdx}[a]] \tag{2}$$

for $1 \leq a \leq 65535$. As a result, multiplication $a \times b$ in $GF(2^{16})$ can be expressed as:

$$\begin{aligned} a \times b &= 2^{\texttt{GF16memIdx}[a]} \times 2^{\texttt{GF16memIdx}[b]} \\ &= 2^{\texttt{GF16memIdx}[a]+\texttt{GF16memIdx}[b]} \\ &= \texttt{GF16memL}[\texttt{GF16memIdx}[a] + \texttt{GF16memIdx}[b]] \end{aligned} \tag{3}$$

Therefore, our multiplication function `GF16mul(a, b)` can be defined as:

```
#define GF16mul(a, b) \
          GF16memL[GF16memIdx[a] + GF16memIdx[b]]
```

However, we need to be careful of the value of `GF16memIdx[a] + GF16memIdx[b]` because it may exceed 65534. To deal with it, many programs check/change it as follows:

```
int idx = (GF16memL[GF16memIdx[a] + GF16memIdx[b
    ]]) % 65534;
return GF16memL[idx];
```

which adds extra code to the function. In order to avoid it, we utilize the second quarter space of `GF16memL`, aka `GF16memH`, by copying the content of `GF16memL` to it:

```
uint16_t *GF16memH = GF8memL + 65535;
memcpy(GF16memH, GF16memL, sizeof(uint16_t) *
    65535);
```

Thus, even if `GF16memIdx[a] + GF16memIdx[b] > 65534`, we are able to retrieve a correct value from one of `GF16memH[0-65534]`.

`GF16memH` also simplifies $GF(2^{16})$ division. Since

$$\begin{aligned} a/b &= 2^{\texttt{GF16memIdx}[a]}/2^{\texttt{GF16memIdx}[b]} \\ &= 2^{\texttt{GF16memIdx}[a]-\texttt{GF16memIdx}[b]} \\ &= \texttt{GF16memL}[\texttt{GF16memIdx}[a] - \texttt{GF16memIdx}[b]], \end{aligned} \tag{4}$$

we can define a $GF(2^{16})$ division function as:

```
#define GF16div(a, b) \
          GF16memL[GF16memIdx[a] - GF16memIdx[b]]
```

However, `GF16memIdx[a] - GF16memIdx[b] < 0` is possible and causes segmentation violation in this case. This is easily resolvable by replacing `GF16memL` with `GF16memH` because `GF16memH[[GF16memIdx[a] - GF16memIdx[b]]` always hits one of `GF16memL[0-65534]` or `GF16memH[0-65534]`. Therefore, we can define `GF16div(a, b)` as:

```
#define GF16div(a, b) \
          GF16memH[GF16memIdx[a] - GF16memIdx[b]]
```

Thus far, we haven't referred to the case `a = 0 || b = 0`. `GF16mul(0, b)`, `GF16mul(a, 0)` and `GF16div(0, b)` for `a, b != 0` should return 0, while `GF16div(a, 0)` is undefined. To satisfy the condition, we fill the rest of `GF16memL` (i.e., `GF16memL[13170-262143]`) with 0 and set `GF16memIdx[0]` as follows:

```
memset(GF16memL + (65535 * 2) - 2, 0, sizeof(
    uint16_t) * 65536 * 2 + 2);
GF16memIdx[0] = 65536 * 2 - 1
```

As

$$\begin{aligned} \texttt{GF16mul}(0,b) &= \texttt{GF16memL}[\texttt{GF16memIdx}[0] + \texttt{GF16memIdx}[b]] \\ &= \texttt{GF16memL}[(65536 * 2 - 1) + \texttt{GF16memIdx}[b]] \end{aligned} \tag{5}$$

and

$$0 \leq \texttt{GF16memIdx}[b] \leq 65534,$$

`GF16mul(0, b)` points to one of the numbers between `GF16memL[131071-196605]`, which is always 0. Similarly,

$$\begin{aligned} \texttt{GF16div}(0,b) &= \texttt{GF16memH}[\texttt{GF16memIdx}[0] - \texttt{GF16memIdx}[b]] \\ &= \texttt{GF16memL}[65535 + \texttt{GF16memIdx}[0] - \\ &\qquad \texttt{GF16memIdx}[b]] \\ &= \texttt{GF16memL}[196606 - \texttt{GF16memIdx}[b]] \end{aligned} \tag{6}$$

outputs one of the numbers between `GF16memL[131072-196606]`, which is also 0. Thus, our multiplication and division functions fulfill the above condition for the calculation using 0. Note the fourth quarter space of `GF16memL` is necessary only for `GF16mul(0, 0) = GF16memL[262142]`.

As for division by 0, `GF16div(a, 0)` for `a != 0` violates the memory segmentation because

$$\begin{aligned} \texttt{GF16div}(a,0) &= \texttt{GF16memH}[\texttt{GF16memIdx}[a] - \texttt{GF16memIdx}[0]] \\ &= \texttt{GF16memL}[65535 + \texttt{GF16memIdx}[a] - \\ &\qquad (65535 * 2 - 1)] \\ &= \texttt{GF16memL}[\texttt{GF16memIdx}[a] - 65534] \end{aligned} \tag{7}$$

and

$$-65534 \leq \texttt{GF16memIdx}[a] - 65534 \leq 0.$$

Programmers need to be aware of it.

IV. Faster Computation

Because of its simple mechanism, gf-nishida-16 can be faster at multiplying a region of data by a single number. For instance, RNC encoding repeats multiplication like:

```
uint16_t a, x[NUM], b[NUM];

for (i = 0; i < NUM; i++) {
        b[i] = GF16mul(a, x[i]);
}
```

Since we have

$$\begin{aligned}&\texttt{GF16mul}(a, x[i]) = \\ &\quad \texttt{GF16memL}[\texttt{GF16memIdx}[a] + \texttt{GF16memIdx}[x[i]]] \\ &= (\texttt{GF16memL + GF16memIdx}[a])[\texttt{GF16memIdx}[x[i]]],\end{aligned} \tag{8}$$

we can replace `(GF16memL + GF16memIdx[a])` with a static value like:

```
uint16_t *gf_a = GF16memL + GF16memIdx[a];
```

Thus, the above code can be rewritten as:

```
uint16_t a, x[NUM], b[NUM];
uint16_t *gf_a = GF16memL + GF16memIdx[a];

for (i = 0; i < NUM; i++) {
        b[i] = gf_a[x[i]];
}
```

which speeds up more than double (see Section V).

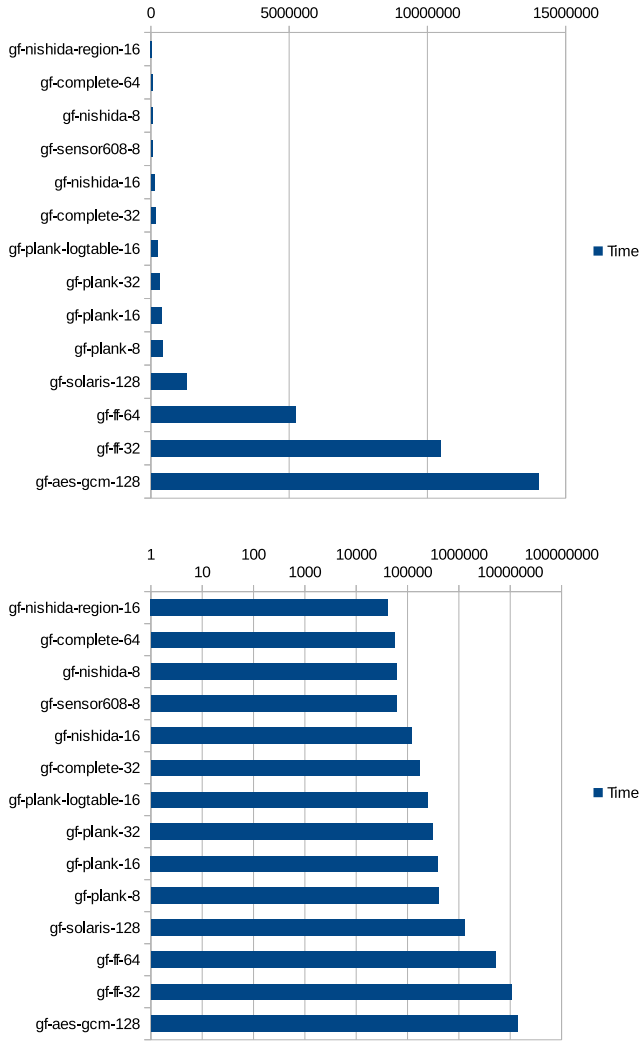
V. Benchmark Results

We measured elapsed times for multiplication and division with various open source GF libraries based on the program described in Section IV. The libraries used are:

- **gf-nishida-8**: An 8bit memory lookup type library described in Section II.
- **gf-nishida-16**: Our main library (see Section III).
- **gf-nishida-region-16**: A region computation version of gf-nishida-16 explained in Section IV.
- **gf-complete**: A library that uses SSE and has been removed by the author as mentioned in Section II (see [3] [2]).
- **gf-sensor608-8**: An 8bit library similar to gf-nishida-8.
- **gf-plank**: A libraray based on [5].
- **gf-plank-logtable-16**: A 16bit library similar to gf-nishida-16 which looks up two memory tables (see Section II and [5]).
- **gf-solaris-128**: A 128bit GF program retrieved from Solaris source code that uses CLMUL. See Section II.
- **gf-ff**: A library downloaded from [6] which provides arithmetic functions in $GF(2^n)$. Although we did not understand if it was applicable to 32 and 64bits, we benchmarked it based on the programs generated at their website.
- **gf-aes-gcm-128**: A 128bit GF program retrieved from FreeBSD source code that uses no special techniques.

Note that some libraries do not include a function for division and that some have a region computation function like gf-nishida-region-16, which we did not benchmark this time.

The benchmark results are shown in Fig. 1 and 2 (the shorter, the faster). As a consequence, gf-nishida-16, especially gf-nishida-region-16 shows excellent performance in both multiplication and division. Considering with the insufficiency of 8bit space for Random Network Coding and the possibility of patent violation by using other techniques, we believe gf-nishida-16 is the best solution for GF computation in RNC.



Library	Time
gf-nishida-region-16	41583
gf-complete-64	55106
gf-nishida-8	61171
gf-sensor608-8	61195
gf-nishida-16	118850
gf-complete-32	168429
gf-plank-logtable-16	244935
gf-plank-32	314016
gf-plank-16	391792
gf-plank-8	406299
gf-solaris-128	1281013
gf-ff-64	5231520
gf-ff-32	10464125
gf-aes-gcm-128	14013111

Fig. 1. Benchmark results in multiplication (standard and logarithm scales, the shorter, the faster)

References

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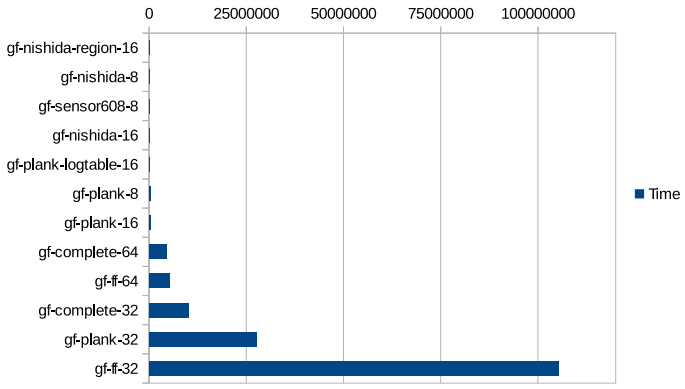

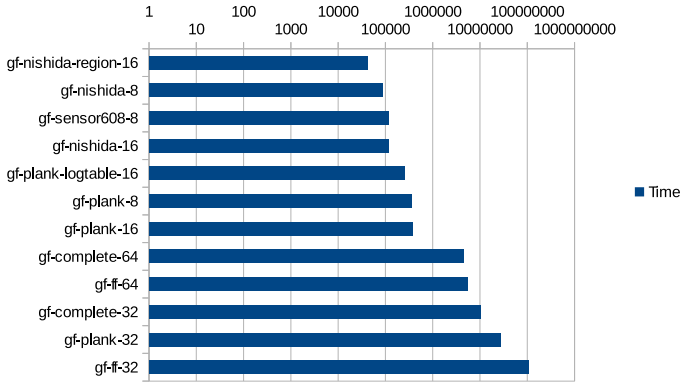

Library	Time
gf-nishida-region-16	41557
gf-nishida-8	86159
gf-sensor608-8	114292
gf-nishida-16	118973
gf-plank-logtable-16	251010
gf-plank-8	360553
gf-plank-16	369352
gf-complete-64	4424996
gf-ff-64	5393946
gf-complete-32	10053109
gf-plank-32	27664514
gf-ff-32	105356429

Fig. 2. Benchmark results in division (standard and logarithm scales, the shorter, the faster)

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