Exercise 5.1

D 5.1.1 Prof: $\Sigma \hat{Y}_i E_i = 0$

(a) $\Sigma \hat{Y}_i E_i \quad (\hat{Y}_i = A + BX_i)$

$= \Sigma (A+BX_i) E_i = \Sigma A E_i + \Sigma B X_i E_i$

$= A \cdot \Sigma E_i + B \Sigma X_i E_i \quad$ ($\Sigma E_i = 0$, $\Sigma X_i E_i = 0$ from textbook)

$= A \cdot 0 + B \cdot 0 = 0 \quad \square$

(b) Prof: $\Sigma (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) = \Sigma E_i (\hat{Y}_i - \bar{Y}) = 0$

$\Sigma (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) \quad (E_i = Y_i - \hat{Y}_i)$

$= \Sigma E_i (\hat{Y}_i - \bar{Y})$

$= \Sigma E_i \hat{Y}_i - \Sigma E_i \bar{Y} = \Sigma E_i \hat{Y}_i - \bar{Y} \Sigma E_i \quad$ ($\Sigma E_i \hat{Y}_i = 0$ from (a)

$\Sigma E_i = 0$ from text book)

$= 0 - \bar{Y} \cdot 0 = 0 \quad \square$

D5.1.3 Prof: $A' = \bar{Y}$ minimizes the sum of squares $S(A') = \sum_{i=1}^{n} (Y_i - A')^2$

① Take Derivative: $\frac{dS}{dA'} = \frac{d}{dA'}\left[\sum_{i=1}^{n} (Y_i - A')^2\right]$

$= \sum_{i=1}^{n}\left[2 \cdot (Y_i - A')(-1)\right] = -2 \sum_{i=1}^{n} [Y_i - A']$

$= -2 \sum_{i=1}^{n} Y_i + 2A' \cdot n$

② Then, let $\frac{dS}{dA'} = 0$, S either has maximum or minimum at A'^*.

$\frac{dS}{dA'} = -2 \sum_{i=1}^{n} Y_i + 2A' n = 0$

$2 \cdot A' n = 2 \sum_{i=1}^{n} Y_i$

$A' n = \sum_{i=1}^{n} Y_i$

$A' = \frac{1}{n} \sum_{i=1}^{n} Y_i = \bar{Y}$

③ Check second derivative to check minimum or maximum

$\frac{d^2 S}{dA'^2} = \frac{d}{dA'}\left[-2 \sum_{i=1}^{n} (Y_i - A')\right] = -2 \sum_{i=1}^{n} (-1) = 2n > 0$

Since at $A' = \bar{Y}$, $\frac{dS}{dA'} = 0$ and $\frac{d^2S}{dA'^2} > 0$

Therefore $A' = \bar{Y}$ minimizes the sum of squares

$$S(A') = \sum_{i=1}^{n} (Y_i - A')^2 \quad \square$$

Exercise 6.1

D6.1 (a) Since $B = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$

①

$$= \frac{\sum_{i=1}^{n}[(X_i - \bar{X})Y_i - (X_i - \bar{X})\bar{Y}]}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$= \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i - \sum_{i=1}^{n}(X_i - \bar{X})\bar{Y}}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i - \bar{Y}\sum_{i=1}^{n}(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

Since $\sum_{i=1}^{n}(X_i - \bar{X}) = 0$

$(\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow n\bar{X} = \sum_{i=1}^{n} X_i$

$\Rightarrow 0 = \sum_{i=1}^{n} X_i - n\bar{X}$

$\Rightarrow \sum_{i=1}^{n}(X_i - \bar{X}) = 0$

$$= \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i - \bar{Y}\cdot 0}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

Let $m_i = \dfrac{(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$, $B = \sum_{i=1}^{n} m_i Y_i$

② Proof: $E(B) = \beta$

$$E[B] = E\left[\sum_{i=1}^{n} m_i Y_i\right] = \sum_{i=1}^{n} m_i E[Y_i] = \sum_{i=1}^{n} m_i(\alpha + \beta X_i)$$

$$= \sum_{i=1}^{n} \alpha \cdot m_i + \sum_{i=1}^{n} \beta \cdot X_i \cdot m_i$$

$$= \alpha \cdot \frac{\sum_{i=1}^{n}(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} + \beta \cdot \frac{\sum_{i=1}^{n}(X_i - \bar{X})\cdot X_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \qquad \left(\sum_{i=1}^{n}(X_i - \bar{X}) = 0\right)$$

$$= 0 + \beta \cdot \frac{\sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \qquad \left(\text{Since } \sum_{i=1}^{n}(X_i - \bar{X})\bar{X} = \bar{X}\cdot\sum_{i=1}^{n}(X_i - \bar{X}) = \bar{X}\cdot 0 = 0\right)$$

$$= \beta \cdot \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \beta \cdot 1 = \beta \quad \square$$

(b) Proof: $E[A]=\alpha$

Since $\bar{Y}=A+B\bar{x}$

$A=\bar{Y}-B\bar{x}$ and $\bar{Y}=\alpha+\beta\bar{x}$

$E[A]=E[\bar{Y}-B\bar{x}]=E[\alpha+\beta\bar{x}-B\bar{x}]=\alpha+\beta\bar{x}-E[B\bar{x}]$

$=\alpha+\beta\bar{x}-\bar{x}\cdot E[B]=\alpha+\beta\bar{x}-\beta\bar{x}=\alpha$ □

Db.2 Derive $Var[A]$, $Var[B]$ in simple regression

① From hint, $Var[B]=\sum_{i=1}^{n} m_i^2 Var[Y_i]$ where $m_i=\dfrac{x_i-\bar{x}}{\sum_{j=1}^{n}(x_j-\bar{x})^2}$

Since $m_i=\dfrac{x_i-\bar{x}}{\sum_{j=1}^{n}(x_j-\bar{x})^2}$ $\Rightarrow$ $\sum_{i=1}^{n} m_i^2=\sum_{i=1}^{n}\left[\dfrac{x_i-\bar{x}}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\right]^2$ constant consider as

$$=\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{j=1}^{n}(x_j-\bar{x})^4}=\frac{1}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

Therefore since Y_i, the sample response variable are independent of each other and $Var[Y_i]$ is σ^2

$Var[B]=\sum_{i=1}^{n} m_i^2\cdot Var[Y_i]$

$$=\sigma^2\cdot\sum_{i=1}^{n} m_i^2=\sigma^2\cdot\frac{1}{\sum_{i=1}^{n}(x_i-\bar{x})^2}=\frac{\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}=\frac{\sigma^2}{S_{xx}}$$

$\therefore Var[B]=\dfrac{\sigma^2}{S_{xx}}$

"always go through the mean point"

$V[Y]=E[(Y_i-\mu_i)^2]=E[(Y_i-\alpha-\beta x_i)^2]$ since $\mu=\alpha+\beta x_i$

$=E[\varepsilon_i^2]=\sigma_\varepsilon^2$

Also By Normality, $Y_i\sim N(\alpha+\beta x_i,\sigma_\varepsilon^2)$ and $\varepsilon_i\sim N(0,\sigma_\varepsilon^2)$

therefore $Var[Y_i]=\sigma_\varepsilon^2$ and $Var[B]=\dfrac{\sigma_\varepsilon^2}{S_{xx}}$ where $S_{xx}=\sum_{i=1}^{n}(x_i-\bar{x})^2$

② $Var[A]=Var[\bar{Y}-B\bar{x}]=Var[\bar{Y}]+Var[B\bar{x}]-2\bar{x}\cdot Cov(\bar{Y},B)$ (since $\bar{Y}=A+B\bar{x}$, $\bar{x}$ is considered as constant)

Part 1: $Var[\bar{Y}]=Var[\frac{1}{n}\sum_{i=1}^{n}Y_i]\overset{①}{=}\frac{1}{n^2}\sum_{i=1}^{n}Var[Y_i]\overset{②}{=}\frac{1}{n^2}\sum_{i=1}^{n}\sigma^2\overset{③}{=}\frac{n\sigma^2}{n^2}=\frac{\sigma^2}{n}$

$=\dfrac{\sigma_\varepsilon^2}{n}$

part 2: $Var[B\bar{x}]=\bar{x}^2\cdot Var[B]=\bar{x}^2\cdot\dfrac{\sigma_\varepsilon^2}{S_{xx}}$

Part 5: $Cov(\bar{Y}, b) = Cov\left(\frac{1}{n}\sum_{i=1}^{n} Y_i, \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right)$ ⊛

Since $\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y}) = \sum_{i=1}^{n}[(X_i-\bar{X})Y_i - (X_i-\bar{X})\cdot\bar{Y}]$

$= \sum_{i=1}^{n}(X_i-\bar{X})\cdot Y_i - \sum_{i=1}^{n}(X_i-\bar{X})\cdot\bar{Y}$ (since $\bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i \Rightarrow n\bar{X} = \sum_{i=1}^{n}X_i$

$= \sum_{i=1}^{n}(X_i-\bar{X})Y_i - \bar{Y}\cdot\sum_{i=1}^{n}(X_i-\bar{X})$ $\Rightarrow 0 = \sum_{i=1}^{n}X_i - n\bar{X}$)

$= \sum_{i=1}^{n}(X_i-\bar{X})Y_i - \bar{Y}[\sum_{i=1}^{n}X_i - n\bar{X}]$ → $=0$

$= \sum_{i=1}^{n}(X_i-\bar{X})Y_i - 0 = \sum_{i=1}^{n}(X_i-\bar{X})Y_i$

⊛ $= Cov\left[\frac{1}{n}\sum_{i=1}^{n}Y_i, \frac{\sum_{i=1}^{n}(X_i-\bar{X})\cdot Y_i}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]$

Since $\sum_{i=1}^{n}(X_i-\bar{X})^2$ is constant S_{xx}, $= \frac{1}{n}\frac{1}{\sum_{i=1}^{n}(X_i-\bar{X})^2} Cov\left[\sum_{i=1}^{n}Y_i, \sum_{i=1}^{n}(X_i-\bar{X})Y_i\right]$

$= \frac{1}{n\sum_{i=1}^{n}(X_i-\bar{X})^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(Y_i, (X_j-\bar{X})Y_j)\right]$ – by $Cov(\sum_{i=1}^{n}Y_i, \sum_{j=1}^{n}Z_j)$
$= \sum_{i=1}^{n}\sum_{j=1}^{n}Cov(Y_i, Z_j)$

$= \frac{1}{n\sum_{i=1}^{n}(X_i-\bar{X})^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(X_j-\bar{X})\cdot Cov(Y_i, Y_j)\right]$

If $i=j$, Y_i, Y_j are the same, $\Rightarrow Cov(Y_i, Y_j) = Var[Y_i] = \sigma^2 = \sigma_\epsilon^2$

$i \neq j$, Y_i, Y_j are independent $\Rightarrow Cov(Y_i, Y_j) = 0$

$= \frac{1}{n\sum_{i=1}^{n}(X_i-\bar{X})^2}\left[\sum_{i=1}^{n}\sum_{j\neq i}(X_j-\bar{X})\underbrace{Cov(Y_i, Y_j)}_{0} + \sum_{i=1}^{n}(X_i-\bar{X})\cdot\underbrace{Var[Y_i]}_{\sigma_\epsilon^2}\right]$

$= \frac{1}{n\sum_{i=1}^{n}(X_i-\bar{X})^2}\left[0 + \sigma^2\cdot\sum_{i=1}^{n}(X_i-\bar{X})\right]$

$= \frac{1}{n\sum_{i=1}^{n}(X_i-\bar{X})^2}\left[0 + \sigma^2\cdot\left(\sum_{i=1}^{n}X_i - \sum_{i=1}^{n}\bar{X}\right)\right]$

$= \frac{1}{n\sum_{i=1}^{n}(X_i-\bar{X})^2}\left[\sigma^2\cdot\left(\underbrace{\sum_{i=1}^{n}X_i - n\bar{X}}_{0 \text{ by previous}}\right)\right] = 0$

Misc

$Var[A] =$ part 1 + part 2 + part 3

$$= \frac{\sigma_\epsilon^2}{n} + \bar{x}^2 \frac{\sigma_\epsilon^2}{S_{xx}} + 0 = \sigma_\epsilon^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right]$$ where $S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$

Pb.3 $Var(A) = \frac{\sigma_\epsilon^2 \sum x_i^2}{n \cdot \sum (x_i - \bar{x})^2}$, to get this formula for $Var(A)$

$$Var[A] = \sigma_\epsilon^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] = \sigma_\epsilon^2 \cdot \left[\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 + n\bar{x}^2}{n \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

$$= \frac{\sigma_\epsilon^2}{n} \left[\frac{\sum_{i=1}^{n} (x_i^2 + \bar{x}^2 - 2x_i\bar{x}) + n\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] = \frac{\sigma_\epsilon^2}{n} \left[\frac{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \bar{x}^2 - 2\sum_{i=1}^{n} x_i\bar{x} + n\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

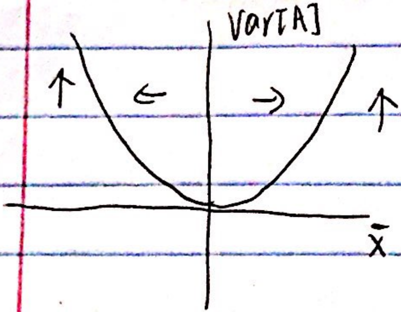
$$= \frac{\sigma_\epsilon^2}{n} \left[\frac{\sum_{i=1}^{n} x_i^2 + n\bar{x}^2 - 2\bar{x} \cdot (n\bar{x}) + n\bar{x}^2)}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

$$= \frac{\sigma_\epsilon^2}{n} \left[\frac{\sum_{i=1}^{n} x_i^2 + 2n\bar{x}^2 - 2n\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] = \frac{\sigma_\epsilon^2}{n} \left[\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

If x's mean => $\bar{x}$ deviates from 0, which means the X-values are not centered near 0, $\sum x_i^2$ will be much large than $\sum (x_i - \bar{x})^2$ since the power of 2 double the influence of spread x-values. Since $\sum (x_i - \bar{x})^2$ will not change a lot when x-values change, and all σ_ϵ^2, n can be taken as constant, $Var[A] = \frac{\sigma_\epsilon^2}{n} \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$ can be seen as a parabola with 2nd order.

$\downarrow$

the relationship between $Var[A]$ and $\bar{x}$.



Thus it's intuitively sensible that the variance of A is large when the mean of the x's is far away from 0.