Singular Value Decomposition (SVD)

Predictive Modeling & Statistical Learning

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Introduction

Two Special Decompositions

Last time we talked about the Eigen Value Decomposition (EVD).

In these slides, we'll talk about a closely related decomposition of EVD: the so-called Singular Value Decomposition (SVD)

Recap

Matrix decompositions, also known as matrix factorizations

$$\mathbf{M} = \mathbf{AB} \qquad \text{or} \qquad \mathbf{M} = \mathbf{ABC}$$

are a means of expressing a matrix as a product of usually two or three simpler matrices.

Types of matrices

Two types of matrices

We said that in data analysis we typically concentrate on two types of matrices:

- general **rectangular** matrices used to represent data tables.
- **positive semi-definite** matrices used to represent covariance matrices, correlation matrices, and any matrix that results from a crossproduct.

SVD

Singular Value Decomposition

- One of the most important decompositions in matrix algebra
- Can be applied to **any** rectangular matrix
- ANY: rectangular or square, singular or nonsigular.

Singular Value Decomposition

An $n \times p$ matrix $\mathbf{M}$ can be decomposed as:

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathsf{T}}$$

where

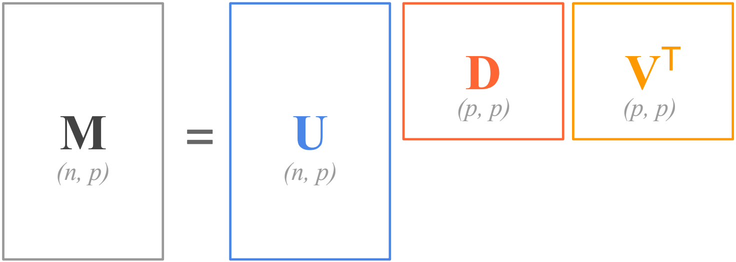
- $\mathbf{U}$ is a $n \times p$ column *orthonormal* matrix containing the **left singular vectors**
- $\mathbf{D}$ is a $p \times p$ **diagonal** matrix containing the **singular values** of $\mathbf{M}$
- $\mathbf{V}$ is a $p \times p$ column **orthonormal** matrix containing the **right singular vectors**

SVD

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\mathsf{T}$$

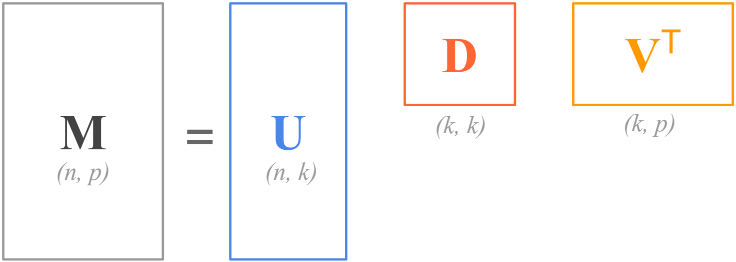
$$\mathbf{M} = \begin{bmatrix} u_{11} & \cdots & u_{1p} \\ u_{21} & \cdots & u_{2p} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{np} \end{bmatrix} \begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_p \end{bmatrix} \begin{bmatrix} v_{11} & \cdots & v_{p1} \\ \vdots & \ddots & \vdots \\ v_{1p} & \cdots & v_{pp} \end{bmatrix}$$

SVD Diagram



When $\mathbf{M}$ is of full rank p

SVD Diagram



When $\mathbb{M}$ is of rank $k < p$

Singular Value Decomposition

We can think of the SVD structure as *the basic structure of a matrix*. What do we mean by "basic"? Well, this has to do with what each of the matrices $\mathbf{UDV}^{\mathsf{T}}$ represent.

- $\mathbf{U}$ is the orthonormalized matrix which is the most basic component. It's like the skeleton of the matrix.
- $\mathbf{D}$ is referred to as the *spectrum* and it is a scale component.
- $\mathbf{V}$ is an orientation component, also referred to as the *rotation* matrix.

SVD

- $\mathbf{U}$ is unitary, and its columns form a basis for the space spanned by the columns of $\mathbf{M}$.

$$\mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}_p$$

- $\mathbf{V}$ is unitary, and its columns form a basis for the space spanned by the rows of $\mathbf{M}$.

$$\mathbf{V}^{\mathsf{T}}\mathbf{V} = \mathbf{I}_p$$

- $\mathbf{D}$ has non-negative real numbers on the diagonal (assuming $\mathbf{M}$ is real).

SVD in R

`svd()` in R

`svd()` function

R provides the function `svd()` to perform a singular value decomposition of a given matrix

`svd()` output

A list with the following components

- `d` a vector containing the singular values
- `u` a matrix whose columns contain the left singular vectors
- `v` a matrix whose columns contain the right singular vectors

SVD example in R

```
# X matrix
set.seed(22)
X = matrix(rnorm(20), 5, 4)

# singular value decomposition
SVD = svd(X)

# elements returned by svd()
names(SVD)

## [1] "d" "u" "v"

# vector of singular values
(d = SVD$d)

## [1] 3.9516353 2.0223602 1.4748193 0.4324292
```

SVD example in R (con't)

```
# matrix of left singular vectors
(U = SVD$u)

##             [,1]        [,2]       [,3]        [,4]
## [1,] -0.4251177 -0.53913435 -0.7232572  0.00979433
## [2,]  0.5268694 -0.76862769  0.2860048  0.05610045
## [3,]  0.5752546  0.04999546 -0.4421464  0.13107213
## [4,]  0.2215220  0.05272644 -0.1702161 -0.95123359
## [5,] -0.4021114 -0.33655016  0.4130778 -0.27337073

# matrix of right singular vectors
(V = SVD$v)

##             [,1]       [,2]        [,3]       [,4]
## [1,]  0.5708354 -0.7406782  0.33862988  0.1042716
## [2,] -0.2741800 -0.5295008 -0.76797328  0.2338189
## [3,]  0.2772481  0.3206239 -0.04462207  0.9046229
## [4,]  0.7225689  0.2611992 -0.54180782 -0.3407543
```

SVD example in R (con't)

```
# U orthonormal  (U'U = I)
t(U) %*% U
```

```
##                [,1]          [,2]          [,3]          [,4]
## [1,] 1.000000e+00  1.387779e-16  2.775558e-17  0.000000e+00
## [2,] 1.387779e-16  1.000000e+00 -2.775558e-17 -8.326673e-17
## [3,] 2.775558e-17 -2.775558e-17  1.000000e+00  5.551115e-17
## [4,] 0.000000e+00 -8.326673e-17  5.551115e-17  1.000000e+00
```

```
# V orthonormal  (V'V = I)
t(V) %*% V
```

```
##                 [,1]          [,2]          [,3]          [,4]
## [1,]  1.000000e+00 -1.110223e-16 -5.551115e-17  1.110223e-16
## [2,] -1.110223e-16  1.000000e+00  8.326673e-17  1.942890e-16
## [3,] -5.551115e-17  8.326673e-17  1.000000e+00 -8.326673e-17
## [4,]  1.110223e-16  1.942890e-16 -8.326673e-17  1.000000e+00
```

SVD example in R (con't)

```
# X equals U D V'
U %*% diag(d) %*% t(V)

##             [,1]        [,2]        [,3]       [,4]
## [1,] -0.5121391  1.85809239 -0.76390728 -0.9221536
## [2,]  2.4851837 -0.06602641  0.08196190  0.8615624
## [3,]  1.0078262 -0.16276495  0.74302828  2.0029422
## [4,]  0.2928146 -0.19986068 -0.08402219  0.9365510
## [5,] -0.2089594  0.30056173 -0.79289452 -1.6157349

# compare to X
X

##             [,1]        [,2]        [,3]       [,4]
## [1,] -0.5121391  1.85809239 -0.76390728 -0.9221536
## [2,]  2.4851837 -0.06602641  0.08196190  0.8615624
## [3,]  1.0078262 -0.16276495  0.74302828  2.0029422
## [4,]  0.2928146 -0.19986068 -0.08402219  0.9365510
## [5,] -0.2089594  0.30056173 -0.79289452 -1.6157349
```

SVD and Cross-products

Data Matrix

Data

The analyzed data can be expressed in matrix format $\mathbf{X}$:

$$\underset{n\times p}{\mathbf{X}} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}$$

- n objects in the rows
- p variables in the columns

Relation of SVD and Cross-Product Matrices

The cross-product matrix of columns of $\mathbf{X}$ can be expressed as:

$$\boxed{\mathbf{X}^\top \mathbf{X} = \mathbf{V}\mathbf{D}^2\mathbf{V}^\top}$$

Relation of SVD and Cross-Product Matrices

The cross-product matrix of columns can be expressed as:

$$\begin{aligned}
\mathbf{X}^\top\mathbf{X} &= (\mathbf{UDV}^\top)^\top(\mathbf{UDV}^\top) \\
&= (\mathbf{VDU}^\top)(\mathbf{UDV}^\top) \\
&= \mathbf{VD}(\mathbf{U}^\top\mathbf{U})\mathbf{DV}^\top \\
&= \mathbf{VD^2V}^\top
\end{aligned}$$

Relation of SVD and Cross-Product Matrices

The cross-product matrix of rows of $\mathbf{X}$ can be expressed as:

$$\mathbf{X}\mathbf{X}^{\top} = \mathbf{U}\mathbf{D}^{2}\mathbf{U}^{\top}$$

Relation of SVD and Cross-Product Matrices

The cross-product matrix of rows can be expressed as:

$$
\begin{aligned}
\mathbf{X}\mathbf{X}^\mathsf{T} &= (\mathbf{U}\mathbf{D}\mathbf{V}^\mathsf{T})(\mathbf{U}\mathbf{D}\mathbf{V}^\mathsf{T})^\mathsf{T} \\
&= (\mathbf{U}\mathbf{D}\mathbf{V}^\mathsf{T})(\mathbf{V}\mathbf{D}\mathbf{U}^\mathsf{T}) \\
&= \mathbf{U}\mathbf{D}(\mathbf{V}^\mathsf{T}\mathbf{V})\mathbf{D}\mathbf{U}^\mathsf{T} \\
&= \mathbf{U}\mathbf{D}^\mathbf{2}\mathbf{U}^\mathsf{T}
\end{aligned}
$$

Relation of SVD and Cross-Product Matrices

One of the interesting things about SVD is that $\mathbf{U}$ and $\mathbf{V}$ are matrices whose columns are eigenvectors of product moment matrices that are *derived* from $\mathbf{X}$. Specifically,

- $\mathbf{U}$ is the matrix of eigenvectors of (symmetric) $\mathbf{X}\mathbf{X}^\intercal$ of order $n \times n$
- $\mathbf{V}$ is the matrix of eigenvectors of (symmetric) $\mathbf{X}^\intercal\mathbf{X}$ of oreder $p \times p$

Of additional interest is the fact that $\mathbf{D}$ is a diagonal matrix whose main diagonal entries are the square roots of $\mathbf{\Lambda}$, the *common* matrix of eigenvalues of $\mathbf{X}\mathbf{X}^\intercal$ and $\mathbf{X}^\intercal\mathbf{X}$.

Relation between EVD and SVD

The EVD of the cross-product matrix of columns (or minor product moment) $\mathbf{X}^\top\mathbf{X}$ can be expressed as:

$$\mathbf{X}^\top\mathbf{X} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$$

in terms of the SVD factorization of $\mathbf{X}$:

$$\mathbf{X}^\top\mathbf{X} = \mathbf{V}\mathbf{D}^2\mathbf{V}^\top$$

Relation between EVD and SVD

The EVD of the cross-product matrix of rows (or major product moment) $\mathbf{X}\mathbf{X}^\intercal$ can be expressed as:

$$\mathbf{X}\mathbf{X}^\intercal = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\intercal$$

in terms of the SVD factorization of $\mathbf{X}$:

$$\mathbf{X}\mathbf{X}^\intercal = \mathbf{U}\mathbf{D}^2\mathbf{U}^\intercal$$

Rank Reduction

SVD Rank-Reduction Theorem

In terms of the diagonal elements $l_1, l_2, \ldots, l_r$ of $\mathbf{D}$, the columns $\mathbf{u_1}, \ldots, \mathbf{u_r}$ of $\mathbf{U}$, and the columns $\mathbf{v_1}, \ldots, \mathbf{v_r}$ of $\mathbf{V}$, the basic structure of $\mathbf{X}$ may be written as

$$\mathbf{X} = l_1 \mathbf{u_1} \mathbf{v_1}^\top + l_2 \mathbf{u_2} \mathbf{v_2}^\top + \cdots + l_p \mathbf{u_p} \mathbf{v_p}^\top$$

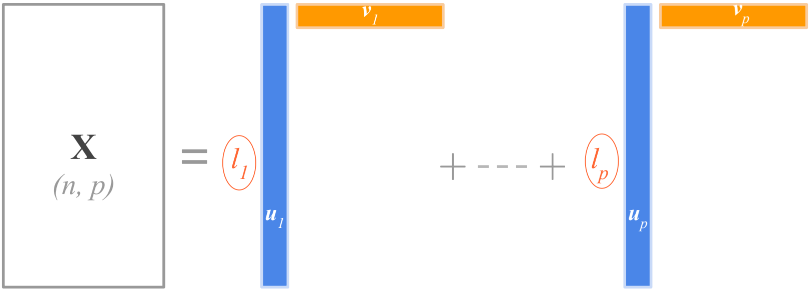
which shows that the matrix $\mathbf{X}$ of rank p is a linear combination of r matrices of rank 1.

SVD Rank-Reduction Theorem

A very interesting and alternative way to represent the SVD is with the following formula:

$$\mathbf{X} = \sum_{k=1}^{p} l_k \mathbf{u_k} \mathbf{v_k}^\mathsf{T}$$

SVD Diagram



SVD as sum of rank one matrices

SVD Rank-Reduction Theorem

SVD alternative formula:

$$\boxed{\mathbf{X} = \sum_{k=1}^{p} l_k \mathbf{u_k} \mathbf{v_k}^\mathsf{T}}$$

- This expresses the SVD as a sum of p rank 1 matrices.
- This result is formalized in what is known as the **SVD theorem** described by Carl Eckart and Gale Young in 1936, and it is often referred to as the Eckart-Young theorem.
- This theorem applies to practily any arbitrary rectangular matrix.

SVD Rank-Reduction Theorem

What if you take $r < p$ terms?

$$\hat{\mathbf{X}} = \sum_{k=1}^{r} l_k \mathbf{u_k} \mathbf{v_k}^\mathsf{T}$$

How would $\hat{\mathbf{X}}$ compare to $\mathbf{X}$?

SVD Rank-Reduction Theorem

The SVD theorem of Eckart and Young is related to the important problem of approximating a matrix.

The basic result says that if $\mathbf{X}$ is an $n \times p$ rectangular matrix, then the best r-dimensional approximation $\hat{\mathbf{X}}$ to $\mathbf{X}$ is obtained by minimizing:

$$min \quad \|\mathbf{X} - \hat{\mathbf{X}}\|^2$$

SVD Rank-Reduction Theorem

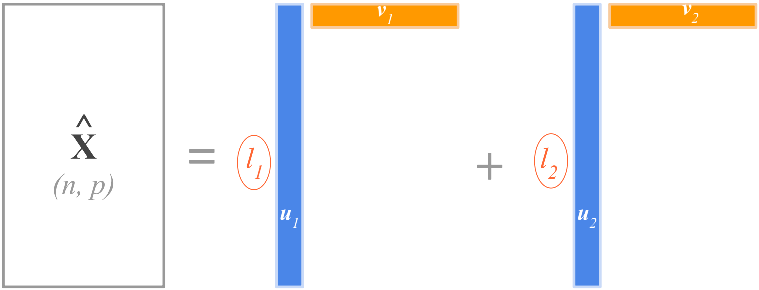
The minimization problem:

$$min \quad \|\mathbf{X} - \hat{\mathbf{X}}\|^2$$

is a special type of approximation: a least squares approximation.

The solution is obtained by taking the first r elements of matrices $\mathbf{U}, \mathbf{D}, \mathbf{V}$ so that $\hat{\mathbf{X}} = \mathbf{U_r D_r V_r^\mathsf{T}}$

SVD rank-two approximation



SVD as sum of two rank one matrices

SVD Rank-Reduction Theorem

The best 2-rank approximation $\hat{\mathbf{X}}$ of $\mathbf{X}$ is given by:

$$\hat{\mathbf{X}} = l_1 \mathbf{u_1} \mathbf{v_1}^\top + l_2 \mathbf{u_2} \mathbf{v_2}^\top$$

We can say that the “information” contained in $n \times p$ values is compressed into $n \times 2$ values.

References

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