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1. $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix}$.

$= \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 2.\ \lambda_2 = 3.$

For $\lambda_1 = 2$:

$(A-\lambda_1 I)x = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} x = 0$

特征向量为 $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 3$:

$(A-\lambda_2 I)x = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} x = 0.$

特征向量为 $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\lambda_1 + \lambda_2 = 5 = trace(A).$

$\lambda_1 \lambda_2 = 6 = |A|.$

3. $\lambda_{B1} = -5$ $\lambda_{B2} = -4.$

$= \lambda_{A1} - 7$ $= \lambda_{A2} - 7.$

特征向量不变.

5. $(A-\lambda I) = \begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix}$.

$|A-\lambda I| = -\lambda(1-\lambda)(3-\lambda) = 0.$

$\lambda_1 = 0$ $\lambda_2 = 1.$ $\lambda_3 = 3.$

$\lambda_1 + \lambda_2 + \lambda_3 = 4 = trace(A).$

$\lambda_1 \lambda_2 \lambda_3 = 0 = |A|.$

$|B-\lambda I| = \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix}.$

$= \lambda^2(2-\lambda) + 4(2-\lambda) = 0.$

$\lambda_1 = \lambda_2 = 2$ $\lambda_3 = -2.$

$\lambda_1 + \lambda_2 + \lambda_3 = 2 = trace A.$

$\lambda_1 \lambda_2 \lambda_3 = -8 = |A|.$

7. (a) $Ax = \lambda x.$

$Ax - 7Ix = \lambda x - 7x.$

$(A-7I)x = (\lambda-7)x$

∴ x 也是 $B = A-7I$ 的特征向量

B的特征值可由A的特征值减7求得

(b) $A^{-1}(Ax) = A^{-1}\lambda x.$

$Ix = \lambda A^{-1} x.$

$A^{-1}x = \frac{1}{\lambda} x.$ ∴ x 也是 A^{-1} 的特征向量.

A^{-1} 的特征值为 $\frac{1}{\lambda}$.

9 ~~8~~. $|A-\lambda I|$

$= \prod_{i=1}^{n} (a_{ii} - \lambda) + C$

where C is irrelative to λ^{n-1}.

∴ the coefficient of λ^{n-1} is

$\sum_{i=1}^{n} a_{ii} = trace(A).$

At the same time, we note that

$|A-\lambda I| = \prod_{i=1}^{n} (\lambda_i - \lambda)$

∴ the coefficient of λ^{n-1} is

$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = trace(A).$

8. $\det(A - 0I) = |A|$

$= \prod_{i=1}^{n} \lambda_i.$

10. 如 $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

$\lambda_{A1} = \lambda_{A2} = 1.$ $\lambda_{B1} = \lambda_{B2} = 1.$

$AB = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}.$

$(2-\lambda_{AB})^2 + 2 = 0$

$2 - \lambda_{AB} = \pm 2i$

$\lambda_{AB} = 2 \pm 2i \neq \lambda_A$ or λ_B.

$AB = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}.$

$(-1-\lambda)(1-\lambda) + 2 = 0.$

$\lambda^2 + 1 = 0.$ $\lambda = \pm i \neq \lambda_A$ or λ_B.

(b) Verifying:

$\lambda_{A+B,1} + \lambda_{A+B,2} = 2+2i+2-2i = 4$

$\lambda_{A1} + \lambda_{A2} + \lambda_{B1} + \lambda_{B2} = 4.$

$\lambda_{AB,1} \cdot \lambda_{AB,2} = i \cdot (-i) = 1.$

$\lambda_{A1} \cdot \lambda_{A2} \cdot \lambda_{B1} \cdot \lambda_{B2} = 1.$

14. $A = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}$. $rank A = 1.$

$|A-\lambda I| = \lambda^4 - 4\lambda^3 = 0.$

$\lambda_1 = \lambda_2 = \lambda_3 = 0$ $\lambda_4 = 4.$ → 对应 $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$C = \begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{bmatrix}$ $rank C = 2.$

$|C-\lambda I| = \lambda^4 - 4\lambda^2 = 0$

$\lambda_1 = \lambda_2 = 0.$ $\lambda_3 = 2.$ $\lambda_4 = -2.$

对应 $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ 对应 $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

15. $rank A = 1.$ $rank C = 2$

$\lambda_{A1} = \lambda_{A2} = \cdots = \lambda_{An-1} = 0.$ $\lambda_{An} = n.$

(n为偶数) $\lambda_{C1} = \lambda_{C2} = \cdots = \lambda_{Cn-2} = 0.$ $\lambda_{Cn-1} = \frac{n}{2}$

$\lambda_{Cn} = -\frac{n}{2}.$

(n为奇数) $\lambda_{C1} = \lambda_{C2} = \cdots = \lambda_{Cn-2} = 0$ $\lambda_{Cn-1} = \frac{1}{2}\sqrt{n^2-1}.$

$\lambda_{Cn} = -\frac{1}{2}\sqrt{n^2-1}.$

18(a) $N(A) = Span\{u\}.$

$C(A) = Span\{v, w\}.$

(b) Considering that

$(A-3I)v = 0.$ $(A-5I)w = 0.$

∴ ~~$A(v+w) = 3v+5w$~~.

$A(5v+3w) = 15(v+w).$

∴ 特解为 $\frac{v}{3} + \frac{w}{5}$.

通解为 $ku + \frac{v}{3} + \frac{w}{5}$.

(c). 这是因为 $u \in N(A)$. $N(A) \cap C(A) = \emptyset$.

19. A^2 确实等于 $\frac{1}{2}(A + A^{\infty})$.

24. (a) $A(Ax) = A\lambda x = \lambda(Ax) = \lambda^2 x.$

∴ A^2 的特征值是 λ^2.

(b) $A^{-1}Ax = A^{-1}\lambda x = \lambda A^{-1} x = x.$

∴ $A^{-1}x = \frac{x}{\lambda}.$

∴ A^{-1} 的特征值是 $\frac{1}{\lambda}$.

(c). 这是显然的.

25.

$$P=uu^T=\begin{bmatrix} \frac{1}{36} & \frac{1}{36} & \frac{3}{36} & \frac{5}{36} \\ \frac{1}{36} & \frac{1}{36} & \frac{3}{36} & \frac{5}{36} \\ \frac{3}{36} & \frac{3}{36} & \frac{9}{36} & \frac{15}{36} \\ \frac{5}{36} & \frac{5}{36} & \frac{15}{36} & \frac{25}{36} \end{bmatrix}.$$

(a) 由简单计算易知 $Pu=u$.

(b) ~~Pv~~.

$Pv=uu^Tv=u(u^Tv)=0$.

(c). $[1\ {-1}\ 0\ 0]^T$.

$[3\ 0\ {-1}\ 0]^T$

$[5\ 0\ 0\ {-1}]^T$.

26. ①对 $\lambda_1=\cos\theta+i\sin\theta$.

$$(Q-\lambda_1 I)x=\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$=-i\sin\theta(a+b)+\sin\theta(a-b)$.

$=0$.

则 $\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$ 是一个特征向量.

~~同理~~ $\begin{bmatrix} 1-i \\ -1-i \end{bmatrix}$ 是另一个.

29. (a) $\operatorname{rank} B=2$.

(b) $|B^TB|=0$.

(d). $B+I$ 的特征值为 1.2.3.

$(B+I)^{-1}$ 的为 $1.\frac{1}{2}.\frac{1}{3}$.

39. ~~trace A = -1. |A|~~

$\operatorname{trace} A=e^{\frac{2\pi i}{3}}+e^{-\frac{2\pi i}{3}}=-1$

$|A|=1$.

如 $\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}^3=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5.2.

2. 由题意.

$S^{-1}AS=\Lambda$ where

Λ is diagonal.

$S=\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $S^{-1}=\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

$$\begin{aligned} A&=S\Lambda S^{-1} \\ &=\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\ &=\begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\ &=\begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}. \end{aligned}$$

6. $A^2x=x$.

$\therefore \lambda^2=1$.

(a) $\lambda=1$ 或 $\lambda=-1$

(b) $\operatorname{trace} A=1-1=0$.

$|A|=1\times(-1)=-1$.

(c) $A=\begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}$.

7. A的特征值为

$\lambda_1=1$. $\lambda_2=5$.

对应的特征向量为

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$、$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$\therefore A=\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

$\cancel{=\frac{1}{4}\begin{bmatrix} 1 & 15 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}}$

$\cancel{=\frac{1}{4}\begin{bmatrix} 16 & 12 \\ 4 & 8 \end{bmatrix}}$

$\cancel{=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}}$

$=S\Lambda S^{-1}$.

$A^{100}=S\Lambda^{100}S^{-1}$

$=\frac{1}{4}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

8 (a) $Au=uv^Tu=(v^Tu)u$.

对应的特征值为 v^Tu.

(b) 其余特征值均为0.

(c). $\sum a_{ii}=\sum u_iv_i=v^Tu$.

$\sum\lambda_i=v^Tu$.

10.

(b) $\operatorname{trace} A^2=\sum\lambda_i^2=21$.

$|A|=\prod\lambda_i=8$.

$|(A^{-1})^T|=|(A^{-1})|=\frac{1}{8}$.