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1.

(1) 记 $M_n = \begin{vmatrix} 2\cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2\cos\theta \end{vmatrix}_{n\times n}$

$= 2\cos\theta M_{n-1} - M_{n-2}$.

α, β 为方程 $x^2 = 2\cos\theta x - 1$ 的两根.

$$x_{1,2} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cancel{\cos^2\theta} - \sin^2\theta}$$

$$= \cos\theta \pm i\sin\theta.$$

※: 若 $\sin\theta = 0$. 则

$M_n \pm M_{n-1} = \pm M_{n-1} - M_{n-2}$

$\cos\theta = 1$ 时 $M_n = n+1$

$\cos\theta = -1$ 时, $M_n = (n+1)(-1)^n$

$$M_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$= \frac{1}{2i\sin\theta}\left[(\cos\theta + i\sin\theta)^{n+1} - (\cos\theta - i\sin\theta)^{n+1}\right]$$

$D_n = \cos\theta \cdot M_{n-1} - M_{n-2}$.

$$= \frac{1}{2i\sin\theta}\Big[\cos\theta\big((\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\big) - \big((\cos\theta + i\sin\theta)^{n-1} - (\cos\theta - i\sin\theta)^{n-1}\big)\Big]$$

$$= \frac{1}{2i\sin\theta}\left(\cos\theta(e^{in\theta} - e^{-in\theta}) - (e^{i(n-1)\theta} - e^{-i(n-1)\theta})\right)$$

$$= \frac{1}{2i\sin\theta}\left(\cos\theta \cdot 2i\sin n\theta - 2i\sin((n-1)\theta)\right)$$

$$= \frac{1}{\sin\theta}\left(\cos\theta(\sin(n-1)\theta\cos\theta + \cos(n-1)\theta\sin\theta - \sin(n-1)\theta\right)$$

$$= \frac{\cos\theta\cos(n-1)\theta\sin\theta - \sin^2\theta\sin(n-1)\theta}{\sin\theta}$$

$$= \cos\theta\cos(n-1)\theta - \sin\theta\sin(n-1)\theta = \cos n\theta \quad (\sin\theta \neq 0 \text{时})$$

(2) $$D_n = \sum_{i=1}^{n} x_i^n \Big(\prod_{\substack{1\le j<k\le n-1 \\ j\neq i, k\neq i}} (x_k - x_j)\Big)(-1)^{n+i}$$

2. $|A| = -1\times 5 = -5$.

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{5} & -\frac{3}{5} & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \end{bmatrix}.$$

$$A^* = |A|A^{-1} = \begin{bmatrix} 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 0 & -5 \\ -2 & 3 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

3. (1) 证:

$$\left|\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix}\right| = \left|\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix}\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}\right| = \left|\begin{bmatrix} I_m & 0 \\ B & I_n - BA \end{bmatrix}\right|$$

$= |I_n - BA|$.

$$\left|\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix}\right| = \left|\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix}\right| = \left|\begin{bmatrix} I_m - AB & 0 \\ B & I_n \end{bmatrix}\right|$$

$= |I_m - AB|$.

$\therefore |I_m - AB| = |I_n - BA|$

(2) $$|kI - uv^T| = \begin{vmatrix} kI - uv^T & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} I_n & u \\ 0 & 1 \end{vmatrix}\begin{vmatrix} kI - uv^T & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} I & 0 \\ v^T & 1 \end{vmatrix}$$

$$= \begin{vmatrix} kI - uv^T & u \\ 0 & 1 \end{vmatrix}\begin{vmatrix} I & 0 \\ v^T & 1 \end{vmatrix} = \begin{vmatrix} kI_n & u \\ v^T & 1 \end{vmatrix} = \begin{vmatrix} I_n & 0 \\ -\frac{v^T}{k} & 1 \end{vmatrix}\begin{vmatrix} kI_n & u \\ v^T & 1 \end{vmatrix}$$

$$= \begin{vmatrix} kI & u \\ 0 & 1 - \frac{v^Tu}{k} \end{vmatrix} = k^{n-1}(k - v^Tu) \quad (k=0\text{时亦成立}).$$

4. 证明: ∵ A的非零子式最高阶数为r.

∴ 任意阶数大于r的A的子式均为0. 不可逆.

∴ $\mathrm{rank}\,A \le r$. 而A存在阶数为r的非零子式.

意味着该子式内向量线性无关. 对应A的向量亦线性无关.

∴ $\mathrm{rank}\,A \ge r$. ∴ $\mathrm{rank}\,A = r$.

5. (1) ① $\mathrm{rank}\,A = n$ 时. ∵ $AA^* = |A|I \neq 0$.

∴ $A^* = A^{-1}|A|$ $\mathrm{rank}\,A^* = n$.

② $\mathrm{rank}\,A = n-1$ 时. $AA^* = |A|I = 0$.

∵ $0 = \mathrm{rank}\,AA^* \ge \mathrm{rank}\,A + \mathrm{rank}\,A^* - n$

~~rank AA* = rank~~ ∴ $\mathrm{rank}\,A^* \le 1$.

而 ∵ $\mathrm{rank}\,A = n-1$. ∴ 至少存在一个子式行列式不为0. ∴ $\mathrm{rank}\,A^* = 1$

③ $\mathrm{rank}\,A < n-1$ 时. 任意n−1阶子式行列式均为0 ∴ $\mathrm{rank}\,A^* = 0$.

5. (2) ∵ $|M_{12}| \neq 0$ ∴ $\mathrm{rank}\,A = 3$, $\mathrm{rank}\,A^* = 1$. $\mathrm{rank}\,N(A^*) = 3$.

而 ∵ $A^*A = |A|I = 0$. ∴ $\{a_1, a_3, a_4\} \subset N(A^*)$.

A^*的一组基为 $\{a_1, a_3, a_4\}$. (∵ $M_{12} \neq 0$. 其线性无关).

∴ 通解为 $ua_1 + va_3 + wa_4$ $(u, v, w \in R)$.