STA5103 Selected Topics in Frontiers of Statistics

Homework 2

12112627 李乐平

Question 1

Answer:

[a] The **2D Discrete Fourier Transform** of an $M \times N$ image is defined as:

$$F(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \cdot \mathrm{e}^{-2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\mathrm{i}},$$

where $f(x, y)$ is the pixel value at position (x, y) in the original image, $F(u, v)$ is the DFT coefficient at frequency (u, v) and $\mathrm{e}^{-2\pi\theta\,\mathrm{i}}$ is the complex exponential term (with i being the imaginary unit). Note that here we use the more symmetric form with coefficient $1/\sqrt{MN}$, and that might lead to differences in the result.

The Fourier transform matrix for 4×4 matrix is derived by

$$C_{\text{DFT}} = UCU^T,$$

and each entry of U is defined as

$$U(k,l) = \frac{1}{\sqrt{4}} \mathrm{e}^{-\frac{2\pi kl}{4}\mathrm{i}}.$$

Therefore,

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}.$$

[b] For the specifically given matrix

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix},$$

we can simply calculate corresponding C_{DFT} as

$$C_{\text{DFT}} = \begin{bmatrix} 30 & -2+2\mathrm{i} & -2 & -2-2\mathrm{i} \\ -8+8\mathrm{i} & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ -8-8\mathrm{i} & 0 & 0 & 0 \end{bmatrix}.$$

Question 2

Answer:

Shall we first consider calculating matrix-vector multiplication $A\boldsymbol{x}$ for a Toeplitz matrix A. Matrix A with entries a_{1-n} to a_{n-1} could be embedded into a circulant matrix

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \text{where } B_{n\times n} = \begin{bmatrix} 0 & \cdots & a_{2-n} & a_{1-n} \\ \vdots & \ddots & \ddots & a_{2-n} \\ a_{n-2} & \ddots & 0 & \vdots \\ a_{n-1} & a_{n-2} & \cdots & 0 \end{bmatrix}$$

that satisfies

$$C\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} \\ B\boldsymbol{x} \end{bmatrix}.$$

Considering calculating $C\boldsymbol{y}$ has a complexity of $O(n \log n)$, calculating $A\boldsymbol{x}$ could be done with a complexity of $O(n \log n)$. (Ignoring constant coefficient)

Based on the result above, we can easily derive the complexities of calculating matrix-vector multiplication for BCCB and BTTB matrices. Assuming the $n \times n$ matrix is made up of m^2 blocks of $k \times k$ sub Circulant matrices or Toeplitz matrices (i.e. $n = mk$), both complexities corresponding to BCCB and BTTB matrices are $O(m^2 k \log k)$.

Question 3

Answer:

In addition to zero and periodic boundary conditions, there are several other types of boundary conditions for convolution operators used in various applications, particularly in signal processing and image processing:

Dirichlet Boundary Conditions: These specify the values of the function at the boundary. For example, the boundary values might be set to a specific constant or to zero.

Neumann Boundary Conditions: These specify the values of the derivative of the function at the boundary, which can help control the gradient. This is useful in applications where the smoothness of the solution is desired.

Reflective (or Symmetric) Boundary Conditions: These create a mirror effect at the boundaries. The values outside the domain are set equal to the values inside the domain reflected at the boundary.

Antiperiodic Boundary Conditions: Similar to periodic conditions, but instead of matching values at the boundary, they require that the values are equal in magnitude but opposite in sign.

Mixed Boundary Conditions: This approach combines different types of boundary conditions on different edges of the domain, which can be useful in complex geometries.

Out-of-Bounds Handling: In some cases, specific methods like "clamping" (using the nearest valid point) or "border extension" (extending the boundary values) are employed.

Question 4

Answer:

(1971) Lightness and Retinex Theory: Proposes the Retinex theory which mimics human vision by estimating the perceived color and lightness of objects in a scene, offering a foundation for image enhancement, especially under varying illumination.

(1986) Adaptive Histogram Equalization and Its Variations: Proposes adaptive histogram equalization (AHE), which improves contrast by redistributing intensity values based on the image's local histogram, enhancing local details.

(2010) Single Image Haze Removal Using Dark Channel Prior: Proposes the dark channel prior (DCP), which assumes that haze-free images have at least one color channel with very low intensity in some pixels. It estimates the haze and enhances the image by restoring clear visibility.

(2016) LIME: Low-light Image Enhancement via Illumination Map Estimation: Introduces a method to enhance low-light images by estimating an illumination map and adjusting pixel brightness, while preserving details through structure-aware smoothing.

(2005) A Non-local Algorithm for Image Denoising: Proposes the non-local means (NLM) algorithm, which enhances images by averaging pixels based on their similarity, even if they are spatially far apart, improving denoising while preserving textures.

(1992) Nonlinear Total Variation Based Noise Removal Algorithms: Introduces a total variation (TV) minimization approach, which preserves edges while reducing noise in images, making it highly effective for denoising applications.

(1994) Ideal Spatial Adaptation by Wavelet Shrinkage: Introduces wavelet shrinkage for denoising images by transforming them into the wavelet domain and applying thresholding to reduce noise while retaining important features.

(1998) Bilateral Filtering for Gray and Color Images: Proposes bilateral filtering, which smooths images while preserving edges by considering both spatial distance and intensity differences, effectively reducing noise without blurring edges.

(2000) Image Impainting: Introduces a technique for filling in missing or corrupted parts of images using information from surrounding pixels, preserving texture and continuity.

(2002) Example-based super-resolution: Introduces an example-based approach to super-resolution that uses high-resolution patches from a training set to enhance low-resolution images.