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3.2

5.

夹角 $\theta = \arccos \frac{1}{\sqrt{n}}$.

投影矩阵

$$P = \frac{aa^T}{a^Ta} = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}_{(n\times n)}$$

10. 不可逆.

这是因为投影矩阵所代表的映射不是一一对应的.

16. (a) $\because P = P^T$

$\therefore x^T P y = (x^T P^T)\cdot y = (Px)^T \cdot y$

$\therefore \langle x, Py\rangle = \langle Px, y\rangle$.

(b) 不等于。在这给条件下.

$$P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$Px = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad Py = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\cos(x, Py) = \frac{x^T P y}{\|x\|\|Py\|} = \frac{1}{\sqrt{15}}.$$

$$\cos(y, Px) = \frac{y^T P x}{\|y\|\|Px\|} = \frac{\sqrt{3}}{9}.$$

(c) 显然 Px 与 Py 的夹角为 0.

这说明 $x-y \perp a$.

20. $P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$P_2 = \frac{a_2 a_2^T}{a_2^T a_2} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

$$P_1 P_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \neq 0.$$

$\therefore (P_1+P_2)^2 \neq P_1 + P_2$.

23.

$$P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}.$$

$$P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$P_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

$$P_1 + P_2 + P_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = I.$$

3.3

3. $A^TA\hat{x} = A^Tb$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\hat{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$b - p = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$ 这显然垂直于A的列向量。

4.

$$E^2 = \|Ax - b\|^2 = \left\|\begin{bmatrix} u \\ v \\ u+v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}\right\|^2$$

$$= (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

$$= 2u^2 + 2v^2 + 2uv - 10u - 14v + 26$$

$$\begin{cases} \frac{\partial E^2}{\partial u} = 4u + 2v - 10 = 0 \\ \frac{\partial E^2}{\partial v} = 4v + 2u - 14 = 0 \end{cases} \Rightarrow \begin{cases} u = 1 \\ v = 3 \end{cases}$$

$A^TA\hat{x} = A^Tb$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\hat{x} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\hat{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} u = 1 \\ v = 3 \end{cases}$$

$$p = A\hat{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b.$$

$p = b$ 是因为 $b \subset C(A)$. $\mathrm{Proj}_A(b) = b$.

6.

$$p = A(A^TA)^{-1}A^Tb$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$= \frac{1}{44}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} -11 \\ 27 \end{bmatrix}.$$

$$= \frac{1}{44}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 18 \\ 7\cancel{8}4 \end{bmatrix}.$$

$$= \frac{1}{44}\begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 23 \\ -14 \\ 65 \end{bmatrix}$$

$$q = b - p = \frac{1}{11}\begin{bmatrix} -12 \\ 36 \\ 12 \end{bmatrix}.$$

$\because q \perp C(A)$

$\therefore q \subset N(A^T)$.

8. $C(P) = S$.

$\mathrm{rank} P = k$

12. (a).

$V^\perp = \mathrm{Span}\{(1,-1,0,0), (1,0,0,-1)\}$.

(b) $P = A(A^TA)^{-1}A^T$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

(c) $p = Pb = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ 离b最近.

13. 设所求直线为

$y = kt + m$

令 $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

$A^TA \begin{bmatrix} m \\ k \end{bmatrix} = A^T \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$

$= \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$

$= \frac{1}{35} \begin{bmatrix} 61 \\ -36 \end{bmatrix}$

$\therefore$ 最佳拟合直线为

$y = -\frac{36}{35} t + \frac{61}{35}$

$P = A \begin{bmatrix} m \\ k \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 133 \\ 97 \\ 61 \\ -11 \end{bmatrix}$

19. 在行满秩的前提下

（若非行满秩可去掉线性相关行使行满秩）

$P_R = A^T(AA^T)^{-1}A$

21. x_1, x_2 满足

$\begin{bmatrix} a_1^T \\ -a_2^T \end{bmatrix} \begin{bmatrix} a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1^T \\ -a_2^T \end{bmatrix} b.$

$\Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

25. $\underset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \\ E \end{bmatrix}} = \underset{b.}{\begin{bmatrix} 2 \\ 0 \\ -3 \\ 5 \end{bmatrix}}$

36.

$\underset{A}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix}} = \underset{b.}{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}$ 注意到 A 可逆.

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 1 & 2 & 4 & 8 & 8 \\ 1 & 3 & 9 & 27 & 20 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 2 & 6 & -8 \\ 0 & 0 & 0 & 6 & 20 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{56}{3} \\ 0 & 0 & 1 & 0 & -14 \\ 0 & 0 & 0 & 1 & \frac{10}{3} \end{array}\right]$

$\therefore b = 0 + \frac{56}{3}t - 14t^2 + \frac{10}{3}t^3$

$P = b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ $e = 0.$

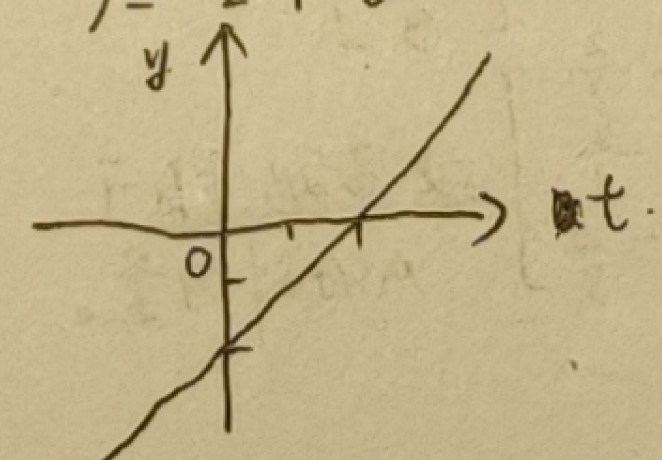
34.

1. $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ -1 \\ 0 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = 0$

(b) $D = 1$ $C = -2.$

$y = -2 + t.$



因为b在直线上

$\therefore E^2 = 0.$

(c) 略.

3. $p_1 = \frac{a_1a_1^T}{a_1^Ta_1} b = \begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{bmatrix}$

$p_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{bmatrix}$ $p_3 = \begin{bmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$

计算可知

$P = a_1a_1^T + a_2a_2^T + a_3a_3^T = I$

这是因为 $\{a_1, a_2, a_3\}$ 是标准正交的

4. 这是因为

$(Q_1Q_2)^TQ_1Q_2 = Q_2^TQ_1^TQ_1Q_2$

$= Q_2^TQ_2 = I.$

Q_1Q_2 代表逆时针旋转 $\theta+\varphi$ 角.

$Q_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $Q_2 = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

$Q_1Q_2 = \begin{bmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \cos\varphi\sin\theta \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{bmatrix}$

$= \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix}.$

5. 证明. $\because$ u是单位向量 $\therefore u^Tu = 1.$

$QQ^T = (I - 2uu^T)(I - 2(uu^T)^T)$

$= I - 2uu^T - 2uu^T + 4uu^Tuu^T$

$= I$

$\therefore Q$ 是正交矩阵.

$u^T = [\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}]$ 时,

$Q = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - 2\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$

6. 第三列

$x = \pm(\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}})^T \times (\frac{1}{\sqrt{14}}\ \frac{2}{\sqrt{14}}\ \frac{-3}{\sqrt{14}})^T$

$= \pm\begin{bmatrix} \frac{-5}{\sqrt{42}}, & \frac{4}{\sqrt{42}}, & \frac{1}{\sqrt{42}} \end{bmatrix}^T$

验证过程十分显然, 在此省略.