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Week 6 补充题.

1. 对矩阵 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & -3 \\ 0 & 1 & 2 & 6 \\ 5 & 4 & 3 & 1 \end{bmatrix}$ 消元

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -6 \\ 0 & 1 & 2 & 6 \\ 0 & -1 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -6 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\therefore \alpha_1\alpha_2\alpha_3\alpha_4$ 的极大线性无关组为 $\{\alpha_1, \alpha_2, \alpha_4\}$.

(2) $\beta = x\alpha_1 + y\alpha_2 + z\alpha_4$

$$\begin{bmatrix} 1 \\ a \\ 3 \\ b \end{bmatrix} = \begin{bmatrix} x+y+z \\ 3x+2y-3z \\ y+6z \\ 5x+4y+z \end{bmatrix}$$

$a = 3x+2y-3z = 3(x+y+z)-(y+6z) = 0.$

由前3行无法解出 x.y.z. 且 R_4 不可由前3行表出，故 b 可以为任意值.

(3) 写为增广矩阵

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & -3 & 0 \\ 0 & 1 & 2 & 6 & 3 \\ 5 & 4 & 3 & 1 & b \end{array}\right] \xrightarrow[R_4-5R_1]{R_2-3R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -6 & -3 \\ 0 & 1 & 2 & 6 & 3 \\ 0 & -1 & -2 & -4 & b \end{array}\right]$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 6 & 3 \\ 0 & 0 & 0 & 2 & b+3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \cancel{\begin{cases} x_1+x_2+x_3+x_4 = 1 \\ x_2+2x_3+6x_4 = 3 \\ 2x_4 = b+3 \end{cases}}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & \frac{5b+11}{2} \\ 0 & 1 & 2 & 0 & -3b-6 \\ 0 & 0 & 0 & 1 & \frac{b+3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + \frac{5b+11}{2} \\ -(2x_3+3b+6) \\ x_3 \\ \frac{b+3}{2} \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5b+11}{2} \\ -3b-6 \\ 0 \\ \frac{b+3}{2} \end{bmatrix} \quad (x_3 \in R,\ b \text{为定值}).$$

为 $\sum_{i=1}^{4} x_i a_i = \beta$ 的解集.

2. 证明: $\because \operatorname{rank} AB \le \min(\operatorname{rank} A, \operatorname{rank} B)$

$\therefore \operatorname{rank} P^{-1}PA = \operatorname{rank} A \le \operatorname{rank} PA \le \operatorname{rank} A.$

$\therefore \operatorname{rank} PA = \operatorname{rank} A.$

$\operatorname{rank} PAQQ^{-1} = \operatorname{rank} PA \le \operatorname{rank} PAQ \le \operatorname{rank} PA = \operatorname{rank} A$

$\therefore \operatorname{rank} PAQ = \operatorname{rank} A.$

3. (1) 证明: 设 $A_{m\times n}$, $B_{m\times n}$. 将 A,B 写为增广矩阵 $[A \mid B]$. 则 $[A|B]\begin{bmatrix} I_n \\ I_n \end{bmatrix} = A+B.$

$\therefore \operatorname{rank}(A+B) \le \operatorname{rank}[A|B]$

而考虑式 $\begin{bmatrix} I_m & I_m \\ 0 & I_m \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & B \end{bmatrix}.$

由2题结论. $\operatorname{rank}\begin{bmatrix} A & B \\ 0 & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B).$

显然有 $\operatorname{rank}[A|B] \le \operatorname{rank}\begin{bmatrix} A & B \\ 0 & B \end{bmatrix}$

综上. $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B).$

(2) 证明: $\because A^2 - I = 0 \therefore (A+I)(A-I) = 0$

由提示: $\operatorname{rank}(A+I) + \operatorname{rank}(A-I) \le n.$

而 $\because \operatorname{rank}(A-I) = \operatorname{rank}(I-A)$

$\therefore \operatorname{rank}(A+I) + \operatorname{rank}(A-I) \ge \operatorname{rank}(A+I+I-A) = n.$

于是 $\operatorname{rank}(A+I) + \operatorname{rank}(A-I) = n.$

4. (1) $L(u_1) = \begin{bmatrix} -1 \\ +1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$

$L(u_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$ $\Rightarrow A = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$

$L(u_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(2) $L(u_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -5b_1 + 3b_2.$

$L(u_2) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -3b_1 + 3b_2.$ $\Rightarrow A = \begin{bmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{bmatrix}.$

$L(u_3) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = 4b_1 - 2b_2.$

(3) $L(u_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 2b_1 - b_2.$

$L(u_2) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} = -2b_1 + 3b_2$ $\Rightarrow A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}.$

$L(u_3) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -4b_1 + 3b_2.$

Week6 补充题

5. (1). 这是因为V中矩阵的加法和数乘都是封闭的.

(2). V的一组基为 $\left\{\underset{e_1}{\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0&1&0\\0&0&0\\0&1&0\end{bmatrix}}, \underset{e_3}{\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}, \underset{e_4}{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}\right\}$.

$\dim V = 4.$

(3). $Ae_1 = e_3.\quad Ae_2 = e_2.\quad Ae_3 = e_1.\quad Ae_4 = 2e_4.$

$$\therefore R = \begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&2\end{bmatrix}$$