运筹与优化 Homework1

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1.(a) $f(x,y)=\sqrt{x^2+y^2}$

$$\nabla f_{(x,y)} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$Hf(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} & \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}} \\ \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}} & \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} \end{bmatrix}$$

(b) $f(x,y) = \frac{4y}{x^2+1}$.

$$\nabla f(x,y) = \begin{bmatrix} \frac{-8xy}{(x^2+1)^2} \\ \frac{4}{x^2+1} \end{bmatrix}$$

$$Hf(x,y) = \begin{bmatrix} \frac{8y(3x^2-1)}{(x^2+1)^3} & -\frac{8x}{(x^2+1)^2} \\ -\frac{8x}{(x^2+1)^2} & 0 \end{bmatrix}$$

2.(a) Proof. ~~Suppose~~ For $\forall(\vec{x}_1, t_1)\in A$, $\forall(\vec{x}_2, t_2)\in A$.

We have $\|\vec{x}_1\| \le t_1$, $\|\vec{x}_2\| \le t_2$.

Then for any $\theta\in[0,1]$ and ~~vector~~ parameters $(\theta\vec{x}_1+(1-\theta)\vec{x}_2, \theta t_1+(1-\theta)t_2)$

There has $\|\theta\vec{x}_1+(1-\theta)\vec{x}_2\| \le \|\theta\vec{x}_1\|+\|(1-\theta)\vec{x}_2\|$

$= \theta\|\vec{x}_1\|+(1-\theta)\|\vec{x}_2\| \le \theta t_1+(1-\theta)t_2$.

Therefore, $(\theta\vec{x}_1+(1-\theta)\vec{x}_2, \theta t_1+(1-\theta)t_2)\in A$.

$\therefore$ A is a convex set.

(b) Proof. For $\forall\ \vec{x}_1, \vec{x}_2 \in C$. We have

$\|\vec{x}_1-\vec{x}_c\|_2 \le r$, $\vec{x}_1^T\vec{x}_b > 0$, $\|\vec{x}_2-\vec{x}_c\|_2 \le r$, $\vec{x}_2^T\vec{x}_b > 0$.

Then for $\forall\ \theta\in[0,1]$ and corresponding vector $\theta\vec{x}_1+(1-\theta)\vec{x}_2$, there has

$$\|\theta\vec{x}_1+(1-\theta)\vec{x}_2-\vec{x}_c\|_2 = \|\theta(\vec{x}_1-\vec{x}_c)+(1-\theta)(\vec{x}_2-\vec{x}_c)\|_2$$
$$\le \theta\|\vec{x}_1-\vec{x}_c\|_2+(1-\theta)\|\vec{x}_2-\vec{x}_c\|_2$$
$$\le r$$

and $(\theta\vec{x}_1+(1-\theta)\vec{x}_2)^T\vec{x}_b = \theta\vec{x}_1^T\vec{x}_b+(1-\theta)\vec{x}_2^T\vec{x}_b > 0$.

Therefore, $\theta\vec{x}_1+(1-\theta)\vec{x}_2 \in C$.

$\therefore$ C is a convex set.

3.(a). Proof. Denote A° as the interior of convex set A.

Then for any $\vec{x}, \vec{y} \in A^\circ$. $\theta \in [0,1]$.

There exists a neibourhood of $\vec{x}$ $B_{\vec{x}} \subset A^\circ$.

$\because$ A is a convex set.

$\therefore$ set $\theta B_{\vec{x}}+(1-\theta)\vec{y} \subseteq A$. which is a neibourhood of $\theta\vec{x}+(1-\theta)\vec{y}$.

$\therefore$ $\theta\vec{x}+(1-\theta)\vec{y}\in A^\circ$. A° is a convex set. Q.E.D.

(b). Proof. Denote $\overline{A}$ as the enclosure of convex set A.

Then for any $\vec{x}, \vec{y} \in \overline{A}$, $\theta\in[0,1]$

There exist series $\{\vec{x}_n\}$, $\{\vec{y}_n\}$ such that

$\lim_{n\to\infty}\vec{x}_n = \vec{x}$, $\lim_{n\to\infty}\vec{y}_n = \vec{y}$, $\vec{x}_n \in A$, $\vec{y}_n \in A$. $n=0,1,2,\cdots$

Denote $\vec{c}_n = \theta\vec{x}_n+(1-\theta)\vec{y}_n$, then $\vec{c}_n \in A$, also.

$\therefore \lim_{n\to\infty}\vec{c}_n = \theta\vec{x}+(1-\theta)\vec{y}\in\overline{A}$ $\therefore$ $\overline{A}$ is a convex set.

Q.E.D. | P.S. The problem is trivial in the case of A is a single point set or an empty set.

4. Proof:

"$\Rightarrow$": $\because$ Cone C is convex $\therefore$ For any $\vec{x}, \vec{y}\in C$. $\theta\in[0,1]$.

$\theta\vec{x}+(1-\theta)\vec{y}\in C$.

let $\theta=\frac{1}{2} \Rightarrow \frac{1}{2}(\vec{x}+\vec{y})\in C$

$\because$ C is a cone $\therefore$ $\vec{x}+\vec{y}\in C$. $\therefore$ $C+C\subseteq C$

"$\Leftarrow$": $\because$ $C+C\subseteq C$ $\therefore$ For any $\vec{x}, \vec{y}\in C$. $\vec{x}+\vec{y}\in C$.

$\because$ C is a cone. $\vec{x}\in C$, $\vec{y}\in C$

$\therefore$ For any $\theta\in(0,1)$, $\theta\vec{x}\in C$, $(1-\theta)\vec{y}\in C$.

$\because$ $C+C\subseteq C$ $\therefore$ $\theta\vec{x}+(1-\theta)\vec{y}\in C$. $\therefore$ Cone C is convex.

Q.E.D.

5. Assume in 1 ton of final mixture. there has c ton of mixture A, d ton of mixture B and e ton of road salt. Then the linear model can be formulated as follow:

$$\min\ 5c+12d+100e.$$

$$\text{s.t}\begin{cases} c+d+e=1 \\ 0.75c+0.6d\le 0.7 \\ 0.02c+0.06d+e\ge 0.1 \\ c\ge 0 \\ d\ge 0 \\ e\ge 0 \end{cases}$$