STA5103 Selected Topics in Frontiers of Statistics

Homework 1

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Question 1

Answer:

Fourier Slice Theorem: the Fourier transform of the Radon transform of a function $f\,(x,\ y)$ corresponds to the two-dimensional Fourier transform of that function evaluated along specific radial lines in the frequency domain.

Proof:

Now we are going to prove that

$$\int_{-\infty}^{+\infty} Rf(\theta,t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \hat{f}(\omega\cos\theta,\omega\sin\theta),$$

where

$$\begin{aligned} Rf(\theta,t) &= \int_{-\infty}^{+\infty} f(t\cos\theta + s\sin\theta, t\sin\theta - s\cos\theta)\mathrm{d}s \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(t-(x\cos\theta + y\sin\theta))\mathrm{d}x\mathrm{d}y, \end{aligned}$$

and δ is the Dirac's delta function.

We know that 2-dimensional Fourier transformation is defined as

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{e}^{-\mathrm{i}(ux+vy)}\mathrm{d}x\mathrm{d}y\cdot$$

With the substitution of $x = a\cos\theta + b\sin\theta, y = a\sin\theta - b\cos\theta$, we could transform the coordination such that variable a runs along the the direction of projection line. Hereby the Fourier transform of Radon transform $Rf\,(\theta,\ t)$ becomes

$$\begin{aligned} \int_{-\infty}^{+\infty} Rf(\theta,t)\mathrm{e}^{-\omega t\,\mathrm{i}}\mathrm{d}t &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(t-(x\cos\theta + y\sin\theta))\mathrm{d}x\mathrm{d}y\right)\mathrm{e}^{-\omega t\,\mathrm{i}}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(a\cos\theta + b\sin\theta, a\sin\theta - b\cos\theta)\delta(t-a)\mathrm{d}a\mathrm{d}b\right)\mathrm{e}^{-\omega t\,\mathrm{i}}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t\cos\theta + b\sin\theta, t\sin\theta - b\cos\theta)\mathrm{e}^{-\omega t\,\mathrm{i}}\mathrm{d}t\mathrm{d}b \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{e}^{-\omega(x\cos\theta + y\sin\theta)\,\mathrm{i}}\mathrm{d}x\mathrm{d}y \\ &= \hat{f}(\omega\cos\theta,\omega\sin\theta), \end{aligned}$$

which is just the Fourier transform along radial lines.

Q.E.D.

In CT imaging, this theorem allows for the reconstruction of the original image from its projections. By taking multiple projections at various angles, the Radon transform captures the essential frequency information. The inverse Radon transform (or filtered back-projection) can then be applied to retrieve the original function, effectively reconstructing the cross-sectional image from the collected data. This principle is foundational for modern imaging techniques used in medical diagnostics.

Question 2

Answer:

In PCA, the goal is to reduce dimensionality while retaining as much variance as possible from the data. The feature vectors, or principal components, correspond to the largest q eigenvalues of the covariance matrix because these eigenvalues represent the directions in which the data varies the most. By selecting the eigenvectors associated with the largest eigenvalues, PCA identifies the axes that capture the greatest variance, effectively projecting the data onto a lower-dimensional space while preserving essential information. This process allows for efficient data representation and visualization.

Question 3

Answer:

The Conjugate Gradient Least Squares (CGLS) algorithm is an iterative method used to solve problems where you need to find an unknown image based on measured data, like in CT reconstruction. The core idea behind CGLS is to gradually improve an initial guess of the image by leveraging both the measured data and the properties of the imaging system.

In each iteration, CGLS evaluates how well the current image approximation fits the observed data and adjusts the image to minimize the difference between the predicted and actual measurements. It does this by exploring directions in the solution space that are "conjugate" to each other, meaning they are independent in a specific mathematical sense. This allows the algorithm to efficiently navigate towards the optimal solution, balancing between fitting the data and maintaining a smooth image.

Pseudo-code:

```
vector conjugate_gradient(matrix A, vector b){
    initialize x_0 = 0;
    assert matrix A is a real, symmetric, and positive definite matrix;
    r_0 = b - Ax_0;
    if(norm(r_0) < ε){ // Indicating the residual is sufficiently small.
        return x_0;
    }
    p_0 = r_0;
    k = 0;
    while(true){
        α_k = r_k^T r_k / p_k^T A p_k;
        x_k+1 = x_k + α_k p_k;
        r_k+1 = r_k - α_k A p_k;
        if(norm(r_0) < ε){
            break;
        }
        β_k = r_k+1^T r_k+1;
        p_k+1 = r_k+1 + β_k p_k;
        k = k + 1;
    }
    return x_k+1;
}
```

The Conjugate Gradient Least Squares (CGLS) algorithm is crucial in computed tomography (CT) reconstruction because it addresses the problem of recovering an image from incomplete and noisy data, typically acquired through projections. The main challenge in CT reconstruction is to solve an ill-posed inverse problem, where the goal is to find a high-dimensional image that best fits the limited measurements while minimizing artifacts.

CGLS is particularly beneficial because it efficiently handles large-scale linear systems, making it suitable for high-resolution images. Unlike traditional methods such as filtered back projection, which can introduce significant artifacts and require ideal conditions, CGLS iteratively refines the solution, converging to a more accurate representation of the original object. This results in improved image quality, reduced noise, and better handling of incomplete data. Additionally, CGLS is advantageous in terms of computational efficiency and scalability, allowing it to be applied effectively in modern CT imaging applications.

Question 4

Answer:

In addition to TSVD and Tikhonov regularizations, two notable techniques are Lasso regularization and Ridge regression. Lasso regularization is particularly effective in regression problems where feature selection is important, as it encourages sparsity in the model coefficients, making it suitable for high-dimensional datasets. Ridge regression, on the other hand, is used to address multicollinearity in regression models by adding a penalty equal to the square of the magnitude of coefficients, which stabilizes the estimates when predictors are highly correlated. Both techniques help improve model performance by preventing overfitting in various applications, such as predictive modeling and machine learning.